University of Ioannina
Department of Mathematics

PhD Dissertation

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# Representation Dimension, Cohen-Macaulay Modules and Triangulated Categories

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The present dissertation was carried out under the PhD program of the Department of Mathematics of the University of Ioannina in order to obtain the degree of Doctor of Philosophy.

Accepted by the seven members of the evaluation committee on September 11th, 2013:

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The acceptance of this dissertation by the Department of Mathematics of the University of Ioannina does not imply the approval of the opinions of the writer (section 202 par. 2 Law 5343/1932 and section 50 par. 8 Law 1268/1982).

## Statutory Declaration

I declare herewith statutorily that the present dissertation was carried out under the international ethical and academical rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis.

To my teacher Apostolos Beligiannis and to my wife Ioanna Katrakazou

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Chrysostomos Psaroudakis
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# Περίληψη

Σε αυτήν τη διδακτορική διατριβή μελετάμε ομολογικές διαστάσεις που εμφανίζονται στη Θεωρία Αναπαραστάσεων Αλγεβρών του Artin. Η βασική μας έρευνα επικεντρώνεται στη διάσταση αναπαράστασης (representation dimension) και στην περατοκρατική και ολική διάσταση (finitistic and global dimension) για Artin άλγεβρες, στην κλάση των προτύπων Cohen-Macaulay και στη διάσταση Rouquier τριγωνισμένων κατηγοριών. Το κατάλληλο εννοιολογικό πλαίσιο, από τη σκοπιά μας, για τη μελέτη αυτή είναι οι συγκολλήσεις αβελιανών κατηγοριών (recollements of abelian categories), μια θεμελειώδης έννοια που έχει εμφανιστεί στην Άλγεβρα, στη Γεωμετρία και στην Τοπολογία, και η ευρέως γενική κλάση των δακτυλίων Morita. Πριν εξηγήσουμε αναλυτικότερα το κίνητρο αυτής της μελέτης συνοψίζουμε παρακάτω τα βασικά της στοιχεία:

- Ταξινομούμε συγκολλήσεις κατηγοριών προτύπων (Κεφάλαιο 1).
- Αναπτύσσουμε μια γενικευμένη ομολογική θεωρία για συγκολλήσεις αβελιανών κατηγοριών (Κεφάλαιο 2).
- Δίνουμε εφαρμογές για τη διάσταση αναπαράστασης και την περατοκρατική διάσταση για άλγεβρες του Artin (Κεφάλαιο 5).
- Κατασκευάζουμε μια αβελιανή κατηγορία που υλοποιεί ένα αφηρημένο μοντέλο της κατηγορίας προτύπων υπεράνω ενός δακτυλίου Morita και μελετάμε συνθήκες περατότητας συγκεκριμένων υποκατηγοριών της (Κεφάλαιο 3).
- Δίνουμε ικανές συνθήκες για το πότε οι δακτύλιοι Morita, ως άλγεβρες του Artin, είναι Gorenstein και παρουσιάζουμε εφαρμογές σε πρότυπα Cohen-Macaulay (Κεφάλαιο 4).
- Δίνουμε φράγματα για τη διάσταση Rouquier τριγωνισμένων κατηγοριών (Κεφάλαιο 5).

Το 1971 ο Auslander [10] όρισε τη διάσταση αναπαράστασης για μια άλγεβρα του Artin $\Lambda$ ως μια αριθμητική αναλλοίωτη μέτρησης της πολυπλοκότητας της θεωρίας αναπαράστασης της άλγεβρας $\Lambda$. Από την άλλη πλευρά μια σημαντική αναλλοίωτη της μέτρησης της ομολογικής πολυπλοκότητας της άλγεβρας $\Lambda$ είναι η περατοκρατική διάσταση της $\Lambda$ η οποία ορίσθηκε από τους Bass, Rosenberg, Zelinsky [21] στα τέλη της δεκαετίας του 50. Σημειώνουμε ότι η διάσταση αναπαράστασης συνδέεται άμεσα με την εικασία περατοκρατικής διάστασης. Συγκεκριμένα οι Igusa και Todorov [66] έδειξαν ότι κάθε άλγεβρα του Artin με διάσταση αναπαράστασης το πολύ 3 ικανοποιεί την εικασία περατοκρατικής διάστασης, δηλαδή η περατοκρατική διάσταση είναι πεπερασμένη. Μετά από αυτό το αποτέλεσμα ήταν εύλογο το ερώτημα αν κάθε άλγεβρα του Artin έχει διάσταση αναπαράστασης το πολύ 3. Στο πλαίσιο αυτό σημαντικές κλάσεις αλγεβρών αποδείχθηκαν ότι έχουν διάσταση αναπαράστασης το πολύ 3. Επίσης, είναι γνωστό από τον Iyama [67] ότι η διάσταση αναπαράστασης μιας άλγεβρας του Artin είναι πάντα πεπερασμένη. Το 2006 ο Rouquier έδειξε ότι η εξωτερική άλγεβρα $\Lambda(k^n)$, όπου το $k$ είναι σώμα και $n \geq 1$, έχει διάσταση αναπαράστασης $n+1$. Το βασικό εργαλείο για αυτό το αποτέλεσμα ήταν η διάσταση τριγωνισμένων κατηγοριών, η οποία ονομάζεται σήμερα διάσταση Rouquier, που εισήχθηκε από τον ίδιο [113]. Τα παραπάνω εξηγούν το σημαντικό ρόλο που διαδραματίζει η διάσταση αναπαράστασης στη δομή και στην πολυπλοκότητα της κατηγορίας των προτύπων καθώς και τις αλληλεπιδράσεις της με εργαλεία από τον «κόσμο» των τριγωνισμένων κατηγοριών. Από την άλλη πλευρά όμως υπάρχουν κλάσεις προτύπων που ο τύπος αναπαράστασής τους

διαδραματίζει σημαντικό ρόλο, όχι μόνο για την κατηγορία των προτύπων, αλλα στη δομή του δακτυλίου. Το 1967 ο Auslander [9], [13] όρισε την κλάση των προτύπων Cohen-Macaulay γενικεύοντας την κλάση των πεπερασμένα παραγόμενων προβολικών προτύπων. Η δομή και η (ομολογική) πολυπλοκότητα της κατηγορίας $\mathsf{Gproj}\,\Lambda$ των πεπερασμένα παραγόμενων προτύπων Cohen-Macaulay (γνωστά και ως Gorenstein-προβολικά πρότυπα) χαρακτηρίζει, υπό μια έννοια, το αν ο δακτύλιος είναι Gorenstein. Συγκεκριμένα η $\Lambda$ είναι Gorenstein αν, και μόνο αν, η ευσταθής κατηγορία των Gorenstein-προβολικών προτύπων είναι τριγωνικά ισοδύναμη με την κατηγορία ιδιομορφιών της $\Lambda$, όπως αυτή έχειορισθεί από τους Buchweitz [31] και Orlov [103].

Σκοπός της παρούσας διατριβής είναι να αναπτύξουμε μια γενικευμένη ομολογική θεωρία για συγκολλήσεις αβελιανών κατηγοριών και να μελετήσουμε δακτύλιους Morita από τη σκοπιά κυρίως της θεωρίας αναπαραστάσεων αλγεβρών και της ομολογικής άλγεβρας Gorenstein, έχοντας ως στόχο εφαρμογές στις ομολογικές διαστάσεις κατηγοριών και δακτυλίων. Για συγκολλήσεις αβελιανών κατηγοριών μελετάμε πως σχετίζονται διάφορες ομολογικές αναλλοίωτες και διαστάσεις των εμπλεκόμενων κατηγοριών και επιπλέον ταξινομούμε συγκολλήσεις από κατηγορίες προτύπων, αποδεικνύοντας έτσι μια εικασία του Kuhn. Το βασικό κίνητρο για την ενασχόληση μας με τις συγκολλήσεις αβελιανών κατηγοριών είναι ερωτήματα και προβλήματα αναφορικά με τη διάσταση αναπαράστασης καθώς και με την περατοκρατική διάσταση. Από την άλλη το έναυσμα για τη μελέτη των δακτυλίων Morita πηγάζει από τη συχνή εμφάνιση αυτών των δακτυλίων στη θεωρία αναπαραστάσεων αλγεβρών και τη σχέση τους με συγκολλήσεις.

Οι συγκολλήσεις αβελιανών κατηγοριών είναι ακριβείς ακολουθίες αβελιανών κατηγοριών $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$ έτσι ώστε ο συναρτητής έγκλεισης $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ και ο συναρτητής πηλίκο $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ να έχουν αριστερά και δεξιά συζυγείς συναρτητές. Διαγραμματικά μια συγκόλληση αβελιανών κατηγοριών συμβολίζεται ως εξής:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{i}"] & \mathscr{B} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \qquad \mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$$

Γενικά οι συγκολλήσεις ορίσθηκαν πρώτα στο πλαίσιο των τριγωνισμένων κατηγοριών από τους Beilinson, Bernstein και Deligne [22] για τη μελέτη των παραγόμενων κατηγοριών από δράγματα (sheaves) σε κατάλληλους τοπολογικούς χώρους. Το πρώτο σημαντικό παράδειγμα συγκολλήσεων στο πλαίσιο των αβελιανών κατηγοριών εμφανίστηκε από τους MacPherson και Vilonen [91] για την κατασκευή των perverse sheaves. Επίσης ο Kuhn χρησιμοποίησε συγκολλήσεις κατηγοριών για τη μελέτη της generic θεωρίας αναπαραστάσεων της γενικευμένης γραμμικής ομάδας. Σημειώνουμε ακόμα το έργο των Cline, Parshall και Scott που συνέδεσαν συγκολλήσεις παραγόμενων κατηγοριών με κατηγορίες υψίστου βάρους (highest weight) και ημι-κληρονομικές (quasi-hereditary) άλγεβρες. Από την άλλη οι Happel [61, 63], Beligiannis, Reiten [30], και πρόσφατα οι Hugel, Koenig, Liu [2–4], και Chen, Xi [33–35] μελέτησαν σχέσεις μεταξύ συγκολλήσεων τριγωνισμένων κατηγοριών και θεωρίας Tilting, ομολογικών εικασιών, stratifications παραγόμενων κατηγοριών και αλγεβρικής K-θεωρίας. Οι συγκολλήσεις αβελιανών κατηγοριών εμφανίζονται με φυσιολογικό τρόπο και στη θεωρία αναπαραστάσεων. Για παράδειγμα κάθε ταυτοδύναμο στοιχείο σε ένα δακτύλιο επάγει μια συγκόλληση μεταξύ των κατηγοριών προτύπων υπεράνω των δακτυλίων $R$, $R/ReR$ και $eRe$. Πιο γενικά ένας δακτύλιος Morita, δηλαδή δακτύλιος πινάκων της μορφής:

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$$

όπου $A$, $B$ δακτύλιοι και ${}_AN_B$, ${}_BM_A$ διπρότυπα, επάγει δύο συγκολλήσεις κατηγοριών προτύπων. Γενικότερα έστω $\mathscr{C}$ μια προσθετική κατηγορία και $X$, $Y$ δυο αντικείμενα της $\mathscr{C}$. Τότε ο δακτύλιος ενδομορφισμών $\mathsf{End}_{\mathscr{C}}(X \oplus Y)$ είναι δακτύλιος Morita, και μάλιστα κάθε δακτύλιος Morita προκύπτει με αυτόν τον τρόπο. Τα παραπάνω παραδείγματα καθώς και άλλες καταστάσεις όπου εμφανίζονται συγκολλήσεις αβελιανών κατηγοριών που σχετίζονται με δακτύλιους Morita, αποτελούν ένα ισχυρό κίνητρο για τη διατριβή.

Η αναλυτική περιγραφή των Κεφαλαίων της διατριβής, καθώς και των κυριότερων αποτελεσμάτων της, έχει ως εξής:

**Κεφάλαιο 1**: Στο Κεφάλαιο 1 ταξινομούμε συγκολλήσεις κατηγοριών προτύπων. Ο Kuhn [82] είχε εικάσει ότι αν οι εμπλεκόμενες κατηγορίες σε μια συγκόλληση είναι κατηγορίες προτύπων υπεράνω $k$-αλγεβρών πεπερασμένης διάστασης ($k$ σώμα), τότε η συγκόλληση αυτή είναι ισοδύναμη (με κάποια κατάλληλη έννοια), με μια συγκόλληση που επάγεται από ένα ταυτοδύναμο στοιχείο. Το κεντρικό αποτέλεσμα του κεφαλαίου αυτού, Θεώρημα 1.3.6, είναι η απόδειξη της εικασίας του Kuhn, και μάλιστα γενικά για δακτυλίους. Η δομή του Κεφαλαίου 1 είναι η εξής: Στην ενότητα 1.1 συγκεντρώνουμε βασικές έννοιες περί των συγκολλήσεων αβελιανών κατηγοριών και παρουσιάζουμε πληθώρα παραδειγμάτων συγκολλήσεων που θα χρησιμοποιηθούν σε όλη τη διατριβή. Στην ένοτητα 1.2 ταξινομούμε συγκολλήσεις αβελιανών κατηγοριών αντιστοιχίζοντας κατάλληλες $\mathsf{TTF}$-τριάδες. Στην ενότητα 1.3 ταξινομούμε συγκολλήσεις κατηγοριών προτύπων υπεράνω δακτυλίων αποδεικνύοντας έτσι την εικασία του Kuhn, και τέλος στην ενότητα 1.4 παρουσιάζουμε ένα παράδειγμα συγκόλλησης που δεν επάγεται από κάποιο ταυτοδύναμο στοιχείο. Τα αποτελέσματα των ενοτήτων 1.2, 1.3, 1.4 είναι σε συνεργασία με τον Jorge Vitória στο άρθρο [109] με τίτλο *Recollements of Module Categories*.

**Κεφάλαιο 2**: Το κύριο αντικείμενο μελέτης του Κεφαλαίου 2 είναι η ομολογική συμπεριφορά των συγκολλήσεων αβελιανών κατηγοριών. Ιδιαίτερα εξετάζουμε τις ομάδες επεκτάσεων $\mathsf{Ext}$, τις ολικές ομολογικές διαστάσεις και τις περατοκρατικές διάστασεις των εμπλεκόμενων κατηγοριών σε μια συγκόλληση, και κυρίως πως σχετίζονται μεταξύ τους. Το βασικό πρόβλημα για τις ομάδες επεκτάσεων $\mathsf{Ext}$ είναι το πότε ο συναρτητής $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ είναι μια $k$-*ομολογική εμφύτευση*, δηλαδή πότε επάγει φυσικούς ισομορφισμούς $\mathsf{i}^m\colon \mathsf{Ext}^m_{\mathscr{A}}(-,-) \longrightarrow \mathsf{Ext}^m_{\mathscr{B}}(\mathsf{i}(-),\mathsf{i}(-))$ για κάθε $0 \leq m \leq k \leq \infty$, και το πότε ο συναρτητής $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ επάγει, περιορισμένος σε κάποιες κατάλληλες υποκατηγορίες, φυσικούς ισομορφισμούς $\mathsf{e}^m\colon \mathsf{Ext}^m_{\mathscr{B}}(-,-) \longrightarrow \mathsf{Ext}^m_{\mathscr{B}}(\mathsf{e}(-),\mathsf{e}(-))$ για $0 \leq m \leq k$, όπου $0 \leq k \leq \infty$. Στα Θεωρήματα 2.1.10 και 2.1.11 λύνουμε τα παραπάνω δυο προβλήματα δίνοντας ικανές και αναγκαίες συνθήκες ώστε να έχουμε τους ζητούμενους ισομορφισμούς. Σημειώνουμε ότι τα αποτελέσματα αυτά γενικεύουν και επεκτείνουν αποτελέσματα των Auslander-Platzeck-Todorov [14] και Geigle-Lenzing [56]. Στις ενότητες 2.2 και 2.3 δίνουμε φράγματα για την ολική και περατοκρατική διάσταση κάτω από ορισμένες φυσιολογικές υποθέσεις. Ενδεικτικά, αποδεικνύουμε ότι η ολική ομολογική διάσταση της $\mathscr{B}$ είναι πάντα φραγμένη από την ολική ομολογική διάσταση της $\mathscr{A}$ και την ολική ομολογική διάσταση της $\mathscr{C}$ και από το supremum των προβολικών διαστάσεων στην $\mathscr{B}$ των προβολικών αντικειμένων της $\mathscr{A}$. Τα βασικά αποτελέσματα σε αυτές τις δυο ενότητες είναι τα Θεωρήματα 2.2.1, 2.2.8, 2.2.9, 2.3.2 και 2.3.6. Τέλος στην ενότητα 2.4 δίνουμε εφαρμογές στη Θεωρία Δακτυλίων χρησιμοποιώντας τα παραδείγματα συγκολλήσεων της ενότητας

1.1. Επισημαίνουμε ότι τα αποτελέσματα των ενοτήτων 2.2, 2.3 και 2.4, αναφορικά με τις ομολογικές διαστάσεις, γενικεύουν πληθώρα αποτελεσμάτων στη βιβλιογραφία, όπως των Auslander-Platzeck-Todorov [14] και Fossum-Griffith-Reiten [49]. Τα αποτελέσματα του κεφαλαίου αυτού βρίσκονται στο άρθρο [108] με τίτλο *Homological Theory of Recollements of Abelian Categories*.

**Κεφάλαιο 3**: Στο Κεφάλαιο 3 κατασκευάζουμε μια αβελιανή κατηγορία που αναπαριστά ένα αφηρημένο μοντέλο για την κατηγορία προτύπων υπεράνω ενός δακτυλίου Morita. Στις πρώτες τρεις ενότητες εξετάζουμε διάφορες ιδιότητες αυτής της κατασκευής, μελετώντας κυρίως δομικές ιδιότητες. Στην τέταρτη ενότητα μελετάμε συνθήκες περατότητας συγκεκριμμένων υποκατηγοριών της. Στις ενότητες που ακολουθούν μελετάμε ομολογικές ιδιότητες αυτής της αβελιανής κατηγορίας, προετοιμάζοντας ουσιαστικά το έδαφος για το Κεφάλαιο 4 αφού πολλά από τα αποτελέσματα αυτού του κεφαλαίου χρησιμοποιούνται εκεί.

**Κεφάλαιο 4**: Κύριο αντικείμενο μελέτης του Κεφαλαίου 4 είναι οι δακτύλιοι Morita κυρίως από τη σκοπια της θεωρίας αναπαράστασεων αλγεβρών. Στην ενότητα 4.1 συγκεντρώνουμε τις απαραίτητες έννοιες και παράδειγματα που θα μας χρειαστούν στο κεφαλαίο αυτό. Στην ενότητα 4.2 περιγράφουμε τα προβολικά, τα ενέσιμα και τα απλά πρότυπα υπεράνω ενός δακτυλίου Morita που είναι άλγεβρα του Artin. Χρησιμοποιώντας αυτήν την περιγραφή χαρακτηρίζουμε πότε ένας δακτύλιος Morita είναι άλγεβρα selfinjective και ως άμεση συνέπεια αυτού δίνουμε ένα άνω φράγμα για τη διάσταση αναπαράστασης του δακτύλιου Morita $\Lambda_{(\phi,\psi)}$ όπου $A = B = N = M$ είναι άλγεβρες selfinjective και $\phi = \psi = 0$. Στις ενότητες 4.3 και 4.4 μελετάμε συνθήκες περατότητας για υποκατηγορίες της κατηγορίας προτύπων ενός δακτυλίου Morita και δίνουμε φράγματα για την ομολογική διάσταση ενός Morita δακτυλίου, στην περίπτωση όπου $\phi = \psi = 0$. Τα βασικά αποτελέσματα των ενοτήτων αυτών είναι τα Θεωρήματα 4.3.4, 4.3.6 και 4.4.9, 4.4.14 αντίστοιχα. Σημειώνουμε ότι τα αποτελέσματα της ενότητας 4.3 έχουν αποδειχτεί στο γενικό πλαίσιο των κατηγοριών Morita στην ενότητα 3.4 του Κεφαλαίου 3. Στην τελευταία ενότητα 4.5 εξετάζουμε το πότε ένας δακτύλιος Morita, που είναι άλγεβρα του Artin, είναι Gorenstein. Το κύριο αποτέλεσμα της παραγάφου αυτής είναι το Θεώρημα 4.5.3 που δίνει μια ικανή συνθήκη ώστε ένας δακτύλιος Morita να είναι Gorenstein. Χρησιμοποιώντας το αποτέλεσμα αυτό χαρακτηρίζουμε τα Gorenstein-προβολικά πρότυπα υπεράνω του δακτυλίου Morita με $A = B = N = M$ και όπου η $A$ είναι μια Gorenstein Artin άλγεβρα, βλέπε Θεώρημα 4.5.10. Τα αποτελέσματα του κεφαλαίου αυτού είναι σε συνεργασία με τον Edward L. Green στο άρθρο [60] με τίτλο *On Artin Algebras Arising from Morita Contexts*.

**Κεφάλαιο 5**: Το τελευταίο Κεφάλαιο 5 ασχολείται με τη διάσταση Rouquier τριγωνισμένων κατηγοριών και τη διάσταση αναπαράστασης αλγεβρών του Artin. Στην ενότητα 5.1 μελετάμε τη διάσταση Rouquier σε σχέση και με το πρόβλημα ανύψωσης μιας συγκόλλησης αβελιανών κατηγοριών σε μια συγκόλληση των αντίστοιχων παραγόμενων κατηγοριών. Ιδιαίτερα στο Θεώρημα 5.1.6, δίνουμε ικανές και αναγκαίες συνθήκες ώστε, μια συγκόλληση αβελιανών κατηγοριών να επάγει μια συγκόλληση τριγωνισμένων κατηγοριών στις αντίστοιχες παραγόμενες κατηγορίες των φραγμένων συμπλόκων. Στη συνέχεια δίνουμε φράγματα για τη διάσταση Rouquier μιας τριγωνισμένης κατηγορίας $\mathcal{T}$ σε μια συγκόλληση τριγωνισμένων κατηγοριών $(\mathcal{U}, \mathcal{T}, \mathcal{V})$. Συγκεκριμένα στο Θεώρημα 5.1.10 αποδεικνύουμε ότι $\max\{\operatorname{dim}\mathcal{U}, \operatorname{dim}\mathcal{V}\} \leq \operatorname{dim}\mathcal{T} \leq \operatorname{dim}\mathcal{U} + \operatorname{dim}\mathcal{V} + 1$, όπου με $\operatorname{dim}\mathcal{T}$ συμβολίζουμε τη διάσταση Rouquier της $\mathcal{T}$. Τέλος εφαρμόζουμε τα παραπάνω σε δακτυλίους με ταυτοδύναμα στοιχεία και ιδιαίτερα σε τριγωνικούς δακτύλιους πινάκων. Στην τελευταία ενότητα αυτού του κεφαλαίου μελετάμε το πως συμπεριφέρεται η διάσταση αναπαράστασης σε καταστάσεις συγκολλήσεων αβελιανών κατηγοριών $(\mathscr{A}, \mathscr{B}, \mathscr{C})$. Καταρχήν ορίζουμε την κατάλληλη έννοια της διάστασης αναπαράστασης σε αβελιανές κατηγορίες. Στη συνέχεια αποδεικνύουμε ότι αν, η διάσταση αναπαράστασης της $\mathscr{B}$ είναι το πολύ 3, τότε και η διάσταση αναπαράστασης της $\mathscr{C}$ είναι επίσης το πολύ 3. Επίσης, κάτω από κάποιες υποθέσεις, συγκρίνουμε τη διάσταση αναπαράστασης της $\mathscr{B}$ με αυτή της $\mathscr{A}$ και εξετάζουμε γενικά το πότε η διάσταση αναπαράστασης της $\mathscr{C}$ είναι μικρότερη ή ίση από τη διάσταση αναπαράστασης της $\mathscr{B}$. Τα παραπάνω αποτελέσματα αποτελούν τα Θεωρήματα 5.2.2 και 5.2.3. Στη συνέχεια δίνουμε εφαρμογές για τη διάσταση αναπαράστασης αλγεβρών του Artin, βλ. Πορίσματα 5.2.5 - 5.2.14. Το τελευταίο βασικό αποτέλεσμα αυτής της διατριβής, βλ. Θεώρημα 5.2.15, δίνει συνθήκες ώστε, η περατοκρατική διάσταση μιας άλγεβρας του Artin να είναι πεπερασμένη και σχετίζει με γόνιμο τρόπο τη διάσταση αναπαράστασης με την περατοκρατική διάσταση. Τα αποτελέσματα του κεφαλαίου αυτού βρίσκονται στο άρθρο [**108**] με τίτλο *Homological Theory of Recollements of Abelian Categories.*

## Contents

# Introduction

In this thesis we investigate homological invariants arising in the representation theory of Artin algebras. The main focus of our study is on the representation, finitistic and global dimension of Artin algebras, the class of Cohen-Macaulay modules and the Rouquier dimension of triangulated categories. The proper conceptual framework, from our perspective, for this study is the general setting of recollements of abelian categories, a concept which is fundamental in algebra, geometry and topology, and the closely related omnipresent class of Morita rings. Before we explain the motivation of our work we summarize below the main features of this thesis:

- Classify recollements of module categories.
- Develop a general homological theory of recollements of abelian categories.
- Construct an abelian category which represents an abstract model for the category of modules over a Morita ring and study finiteness conditions on certain subcategories.
- Give sufficient conditions for Gorensteinness of Morita rings, in the context of Artin algebras, and provide applications to Cohen-Macaulay modules.
- Provide bounds for the Rouquier dimension of triangulated categories.
- Give applications to representation and finitistic dimension of Artin algebras.

## 1. Background and Overview

The representation dimension of an Artin algebra has been introduced by Auslander [10] with the expectation that: "this notion gives a reasonable way of measuring how far an Artin algebra is from being of finite representation type". This hope was based on his famous result, that an Artin algebra $\Lambda$ is of finite representation type if and only if the representation dimension of $\Lambda$ is at most two. It should be noted that the representation dimension is strongly connected with the finitistic dimension conjecture explicitly formulated by Bass [21]. In fact Igusa and Todorov [66] have shown that the finitistic dimension conjecture is valid for any Artin algebra of representation dimension at most three. Several classes of algebras have representation dimension less than or equal to 3 and it is known from Iyama that the representation dimension of an Artin algebra is always finite [67]. Recently these important invariants were combined with tools coming from the world of triangulated categories. Rouquier [112] has shown that there is no upper bound for the representation dimension of Artin algebras. More precisely Rouquier proved that the exterior algebra $\Lambda(k^n)$, where $k$ is a field and $n \geq 1$ an integer, has representation dimension $n + 1$. This was the first example where the representation dimension is greater or equal to 4. The main tool for this result was the notion of dimension for triangulated categories introduced by

Rouquier [113]. In conclusion representation dimension provides a homological criterion for finite representation type and in some sense measures the homological complexity of the module category. On the other hand the representation type of certain subcategories of the module category gives information on the structure of the ring. In the mid-sixties Auslander [9], [13] introduced the class of Cohen-Macaulay modules (also known as: totally reflexive modules [70], modules of $G$-dimension zero [13, 130], maximal Cohen-Macaulay modules [16, 30] or Gorenstein-projective modules [47]) as a natural generalization of finitely generated projective modules. The structure of the category $\mathsf{Gproj}\,\Lambda$ of finitely generated Gorenstein-projective modules measures how far the ring $\Lambda$ is from being Gorenstein. It is well known that the stable category $\underline{\mathsf{Gproj}}\,\Lambda$ is triangulated and there is an equivalence with the singularity category of $\Lambda$, in the sense of Buchweitz [31] and Orlov [103], if and only if $\Lambda$ is Gorenstein. But for the ring itself, the representation type of the category $\mathsf{Gproj}\,\Lambda$ has significant consequences. For example consider a Noetherian commutative, local, complete Gorenstein ring $\Lambda$. If $\Lambda$ is of finite Cohen-Macaulay type [27], [29] ($\mathsf{Gproj}\,\Lambda$ is of finite representation type, i.e. the set of isomorphism classes of indecomposable Gorenstein-projective modules is finite), then $\Lambda$ is a simple singularity. For more details see Yoshino's book [129] and the recent book of Leuschke and Wiegand [86]. We refer also to the work of Beligiannis [29] for a representation-theoretic study of Gorenstein-projective modules and algebras of finite Cohen-Macaulay type.

Our aim in this thesis is to investigate homological aspects of recollements of abelian categories and to study Morita rings in the context of Artin algebras, concentrating mainly at representation-theoretic and homological aspects. In the context of recollements of abelian categories we investigate how various homological invariants and dimensions of the categories involved in a recollement situation are related. Moreover we classify recollements of abelian categories whose terms are module categories, thus solving a conjecture by Kuhn. Our interest in recollements is motivated from questions and problems on representation and finitistic dimension of Artin algebras and the interrelation between them. On the other hand our interest in Morita rings is motivated by the frequent occurrence of this class of matrix rings in the representation theory of Artin algebras and elsewhere, and the interpretation of their module categories via suitable recollements.

## 2. Motivation and Examples

In this part of the introduction we give motivation for the study of recollements of abelian categories and Morita rings, we present illuminating examples, and we explain also the connections between them.

**2.1. Recollements of Abelian Categories.** We start by explaining how naturally recollements of abelian categories arise/appear in different branches of mathematics.

• Recollements of Abelian Categories in Algebra: Let $R$ be a ring and $e$ an idempotent element of $R$. Associated with the data $(R, e)$ are the rings $eRe$, $R/ReR$ and their module categories $\mathsf{Mod}\text{-}R$, $\mathsf{Mod}\text{-}eRe$, $\mathsf{Mod}\text{-}R/ReR$. Since we have a surjective ring homomorphism $R \longrightarrow R/ReR$ it follows that the restriction functor from the module category of $R/ReR$ to the module category of $R$ is fully faithful. On the other hand the assignment $X \mapsto eX$, where $X$ is a (left) $R$-module, induces an exact functor from the module category of $R$ to the module category of $eRe$. Moreover these two functors have left and right adjoints. Hence any pair $(R, e)$ induces the following situation between the module categories over the rings $R/ReR$, $R$ and $eRe$:

$$\begin{tikzcd}
\mathsf{Mod}\text{-}R/ReR \arrow[r, "\mathsf{inc}"] & \mathsf{Mod}\text{-}R \arrow[l, bend right, "R/ReR\otimes_R -"'] \arrow[l, bend left, "\mathsf{Hom}_R(R/ReR,-)"] \arrow[r, "e(-)"] & \mathsf{Mod}\text{-}eRe \arrow[l, bend right, "Re\otimes_{eRe}-"'] \arrow[l, bend left, "\mathsf{Hom}_{eRe}(eR,-)"]
\end{tikzcd}$$

The situation described above is a typical example of a recollement situation, namely *a recollement induced by an idempotent element*, and it should be considered as the universal example concerning recollements of module categories. This statement will be explained in Chapter 1.

• Recollements of Abelian Categories in Geometry: Let $X$ be a topological space and $U$ an open subspace of $X$. Denote by $F$ the complement of $U$ in $X$. Let $\mathcal{O}_X$ be a sheaf of commutative rings on $X$ and denote by $\mathcal{O}_U$, resp. $\mathcal{O}_F$, the restricted sheaves of rings on $U$, resp. $F$. Then the abelian categories $\mathsf{Mod}\text{-}\mathcal{O}_F$, $\mathsf{Mod}\text{-}\mathcal{O}_X$, $\mathsf{Mod}\text{-}\mathcal{O}_U$, of sheaves of $\mathcal{O}_i$-modules, where $i = F, X$ or $U$, are related via the following recollement:

$$\begin{tikzcd}
\mathsf{Mod}\text{-}\mathcal{O}_F \arrow[r, "\mathsf{i}"] & \mathsf{Mod}\text{-}\mathcal{O}_X \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathsf{Mod}\text{-}\mathcal{O}_U \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"]
\end{tikzcd}$$

It should be noted that the above situation is the classical example of a recollement of abelian categories arising in topology and in algebraic geometry, and has appeared in the work of Beilinson, Bernstein and Deligne [22]. There are also various derived versions of the above recollement at the level of triangulated categories. We refer to the book [71] for more details.

• Recollements of Abelian Categories in Topology: Let $\mathbb{F}_q$ be a finite field of characteristic $p$ and order $|\mathbb{F}_q| = q = p^s$, where $p$ is a fixed prime. We denote by $\mathsf{Fun}\,(\mathbb{F}_q)$ the functor category with objects the functors from finite dimensional $\mathbb{F}_q$-vector spaces to $\mathbb{F}_q$-vector spaces, and with morphisms the natural transformations between such functors. It is well known that $\mathsf{Fun}\,(\mathbb{F}_q)$ is an abelian category. Let $F \in \mathsf{Fun}\,(\mathbb{F}_q)$. Then for every finite dimensional $\mathbb{F}_q$-vector space $V$ (say $\dim_{\mathbb{F}_q} V = n$) the object $F(V)$ becomes a module over the group algebra $\mathbb{F}_q[\mathsf{GL}_n(V)]$. This observation shows that we can view the objects of $\mathsf{Fun}\,(\mathbb{F}_q)$ as generic representations of the general linear group over $\mathbb{F}_q$. Then $\mathsf{Fun}\,(\mathbb{F}_q)$ is called the category of generic representations [81]. Denote by $M_k(\mathbb{F}_q)$ the multiplicative semigroup of $k \times k$ matrices over $\mathbb{F}_q$, and consider the categories of modules $\mathsf{Mod}\text{-}\mathbb{F}_q[\mathsf{GL}_k(\mathbb{F}_q)]$ and $\mathsf{Mod}\text{-}\mathbb{F}_q[M_k(\mathbb{F}_q)]$, where $\mathbb{F}_q[M_k(\mathbb{F}_q)]$ is the semigroup algebra. Then from basic results of Kuhn [81, 82] we have the following diagram:

$$\begin{tikzcd}
 & \mathsf{Mod}\text{-}\mathbb{F}_q[\mathsf{GL}_3(\mathbb{F}_q)] \arrow[d, bend right] \arrow[d] \arrow[d, bend left] & \mathsf{Mod}\text{-}\mathbb{F}_q[\mathsf{GL}_2(\mathbb{F}_q)] \arrow[d, bend right] \arrow[d] \arrow[d, bend left] & & \\
\cdots \arrow[r] & \mathsf{Mod}\text{-}\mathbb{F}_q[M_3(\mathbb{F}_q)] \arrow[r] \arrow[d, bend right] \arrow[d] \arrow[d, bend left] & \mathsf{Mod}\text{-}\mathbb{F}_q[M_2(\mathbb{F}_q)] \arrow[r] \arrow[d, bend right] \arrow[d] \arrow[d, bend left] & \mathsf{Mod}\text{-}\mathbb{F}_q[M_1(\mathbb{F}_q)] \arrow[r] & 0 \\
 & \mathsf{Mod}\text{-}\mathbb{F}_q[M_2(\mathbb{F}_q)] & \mathsf{Mod}\text{-}\mathbb{F}_q[M_1(\mathbb{F}_q)] & &
\end{tikzcd}$$

where the vertical diagrams are recollements of module categories and the inverse limit of the horizontal sequence of $\mathsf{Mod}\text{-}\mathbb{F}_q[M_k(\mathbb{F}_q)]$, $k \geq 1$, is the category $\mathsf{Fun}\,(\mathbb{F}_q)$. The category of generic representations $\mathsf{Fun}\,(\mathbb{F}_q)$ is a fundamental object of study and has strong connections with several different topics, like: Symmetric groups, Schur algebras, Topological Hochschild Homology, Steenrod algebra. We refer to the important work of Kuhn [81–83] for more details on the above subject and the use of recollements of abelian categories in this area of mathematics.

We continue now by describing the notion of recollements for abelian categories and we give also some more motivation.

Recollements of abelian categories are exact sequences of abelian categories

$$0 \longrightarrow \mathscr{A} \xrightarrow{\mathsf{i}} \mathscr{B} \xrightarrow{\mathsf{e}} \mathscr{C} \longrightarrow 0$$

where the inclusion functor $\mathsf{i}$ represents $\mathscr{A}$ as a Serre subcategory of $\mathscr{B}$ and the functor $\mathsf{e}$ represents $\mathscr{C}$ as the quotient category $\mathscr{B}/\mathscr{A}$, enjoying the additional property that the inclusion functor $\mathsf{i}$ admits a left adjoint $\mathsf{q}$ and a right adjoint $\mathsf{p}$, and the quotient functor $\mathsf{e}$ admits a left adjoint $\mathsf{l}$ and a right adjoint $\mathsf{r}$. Such a recollement situation of abelian categories is denoted throughout this thesis by the following diagram:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{i}"] & \mathscr{B} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \qquad \mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$$

Recollement situations were introduced first in the context of triangulated categories by Beilinson, Bernstein and Deligne, see [22], in their axiomatization of the Grothendieck's six functors for derived categories of sheaves obtained from stratifications of spaces. In this connection a fundamental example of a recollement situation of abelian categories appeared in the construction of perverse sheaves, by MacPherson and Vilonen [91]. In the late eighties, Cline, Parshall and Scott, working in the context of representation theory, indicated the connections between recollements of derived categories, highest weight categories, and quasi-hereditary algebras, see [38], [107], and later Kuhn used recollements of certain abelian categories in his study of generic representation theory of the general linear group, see [82]. On the other hand, Happel [61,63], and recently Hugel, Koenig, Liu [2–4], and Chen, Xi [33–35] studied connections between recollements of triangulated categories in connection with tilting theory, homological conjectures, stratifications of derived categories of rings, and algebraic K-theory. It is well known that recollements and $\mathsf{TTF}$-triples of triangulated categories are in bijection [22], [30], [99]. In this connection Beligiannis-Reiten investigates hereditay torsion pairs in triangulated categories and discuss applications on homological conjectures and tilting theory, see [30, Chapter IV] for more details.

It should be noted that recollements of abelian or module categories appear quite naturally in various settings and are omnipresent in representation theory. Typically recollement situations arise from the endomorphism ring of a direct sum of two objects in an additive category. For instance, as we discussed above, any idempotent element $e$ in a ring $R$ induces a recollement situation between the module categories over the rings $R$, $R/ReR$ and $eRe$. More generally the ring associated to a Morita context induces two recollements situations, and many categories arising from natural constructions in ring and module theory, for example comma categories, are related via a recollement situation. This includes module categories over one sided Artinian rings and module categories which play an important role in Galois theory of rings. Finally we mention that any full subcategory $\mathcal{X}$ of an additive category $\mathscr{C}$ induces a recollement situation between the module categories over $\mathscr{C}$, $\mathcal{X}$, and $\mathscr{C}/\mathcal{X}$, considered as rings with several objects, and where the latter denotes the stable category of $\mathscr{C}$ modulo the subcategory $\mathcal{X}$.

The examples mentioned above and the important role they play in various contexts suggests naturally the following two problems:

- The general study of homological invariants associated to the abelian categories $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ of the recollement $\mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$, and how they are related.
- The classification of recollements whose terms are module categories.

Our main aim in this thesis is to investigate in detail the above problems concentrating in the behavior of global dimension, finitistic dimension and representation dimension under recollement of abelian categories and then to give applications to ring theory. We also study when a recollement situation between abelian categories induces a recollement situation between the corresponding bounded derived categories, and we give applications to Rouquier's dimension of the triangulated categories involved in a recollement situation. Finally we classify recollements whose terms are module categories, thus answering a conjecture by Kuhn.

**2.2. Morita Rings.** Morita contexts, also known as pre-equivalence data, have been introduced by Bass in [20], see also [41], in his exposition of the Morita Theorems on equivalences of module categories. Let $A$ and $B$ be unital associative rings. Recall that a *Morita context* over $A$, $B$, is a 6-uple $\mathcal{M} = (A, N, M, B, \phi, \psi)$, where ${}_BM_A$ is a $B$-$A$-bimodule, ${}_AN_B$ is an $A$-$B$-bimodule, and $\phi\colon M \otimes_A N \longrightarrow B$ is a $B$-$B$-bimodule homomorphism, and $\psi\colon N \otimes_B M \longrightarrow A$ is an $A$-$A$-bimodule homomorphism, satisfying the following associativity conditions, $\forall m, m' \in M$, $\forall n, n' \in N$:

$$\phi(m \otimes n)m' = m\psi(n \otimes m') \qquad \text{and} \qquad n\phi(m \otimes n') = \psi(n \otimes m)n'$$

Associated to any Morita context $\mathcal{M}$ as above, there is an associative ring, the *Morita ring* of $\mathcal{M}$, which incorporates all the information involved in the 6-uple $\mathcal{M}$, defined to be the formal $2 \times 2$ matrix ring

$$\Lambda_{(\phi,\psi)}(\mathcal{M}) = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$$

where $\Lambda_{(\phi,\psi)}(\mathcal{M}) = A \oplus N \oplus M \oplus B$ as an abelian group, and the formal matrix multiplication is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \cdot \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(n \otimes m') & an' + nb' \\ ma' + bm' & bb' + \phi(m \otimes n') \end{pmatrix}$$

The Morita ring of a Morita context, not to be confused with the notion of a (right or left) Morita ring appearing in Morita duality, has been studied explicitly by various authors in ring, module, or representation, theoretic framework; in this connection we refer to the papers by Amitsur [1], Muller [97], Green [59], Cohen [40], Loustaunau [89], and Buchweitz [32], among others. We refer also to the classical textbooks [85], [92], [116] for the terminology of Morita rings.

It should be noted that Morita rings form an omnipresent class of rings, providing sources of many important examples and the proper conceptual framework for the study of many problems in several different contexts in ring theory. We describe briefly some important examples and situations where Morita rings are involved.

• Examples of Morita Rings: Let $A$ be a ring and $M_A$ be a right $A$-module. If $B = \mathsf{End}_A(M)$ is the endomorphism ring of $A$, then viewing $M$ as a $B$-$A$-bimodule and setting $N = \mathsf{Hom}_A(M, A)$, it is easy to see that there exist naturally induced bimodule homomorphisms $\phi$ and $\psi$ and a Morita context $\mathcal{M} = (A, M, N, B, \phi, \psi)$. Hence any pair $(A, M_A)$, where $A$ is a ring and $M_A$ is a right $A$-module induces a Morita context.

An important special case is when $M = eA$, where $e^2 = e$ is an idempotent element of $A$. Clearly then $N = \mathsf{Hom}_A(M, A) = Ae$ and $B = eAe$, and the Morita ring $\Lambda_{(\phi,\psi)}(\mathcal{M})$ takes the form

$$\Lambda_{(\phi,\psi)}(\mathcal{M}) = \begin{pmatrix} A & Ae \\ eA & eAe \end{pmatrix}$$

On the other hand if $e^2 = e \in A$ is an idempotent element of $A$ and $f = 1_A - e$, then the Pierce decomposition of $A$ with respect to the idempotents $e, f$ induces a Morita context $\mathcal{M}(e, f) = (eAe, eA, fA, fAf, \alpha, \beta)$, and the ring $A$ is isomorphic to

$$A \;\cong\; \Lambda_{(\phi,\psi)}(\mathcal{M}(e, f)) = \begin{pmatrix} eAe & eAf \\ fAe & fAf \end{pmatrix}$$

Note that since any Morita ring $\Lambda_{\phi,\psi}(\mathcal{M})$ contains the idempotents $e = \left(\begin{smallmatrix} 1_A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = 1_\Lambda - e = e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_B \end{smallmatrix}\right)$, it is not difficult to see that there is a ring isomorphism $\Lambda_{\phi,\psi}(\mathcal{M}) \cong \Lambda_{\phi,\psi}(\mathcal{M}(e, f))$. It follows that any Morita ring is isomorphic to the Morita ring arising from the Pierce decomposition of a ring $A$ with respect to two orthogonal idempotents whose sum is the identity of $A$. We mention that, as a consequence, any upper or lower triangular matrix ring is a Morita ring.

As another important example, in a more general context, let $\mathscr{C}$ be an additive category and $X, Y$ be arbitrary objects of $\mathscr{C}$. We view $M := \mathsf{Hom}_{\mathscr{C}}(X, Y)$ as an $A$-$B$-bimodule and $N := \mathsf{Hom}_{\mathscr{C}}(Y, X)$ as a $B$-$A$-bimodule in a natural way, where $A = \mathsf{End}_{\mathscr{C}}(X)$ and $B = \mathsf{End}_{\mathscr{C}}(Y)$. It is easy to see that there is a Morita context $\mathcal{M} = (A, M, N, B, \phi, \psi)$ and an isomorphism of rings

$$\mathsf{End}_{\mathscr{C}}(X \oplus Y) \;\cong\; \Lambda_{\phi,\psi}(\mathcal{M})$$

i.e. Morita rings appear as endomorphism rings of a direct sum of objects in any additive category. This is the universal example of a Morita ring since it is not difficult to see that any Morita ring arises in this way. On the other hand the above construction gives the well-known bijective correspondence $\mu\colon \mathscr{C} \longrightarrow \mathcal{M}$ between pre-additive categories $\mathscr{C}$ with two objects $X, Y$ and Morita Contexts $\mathcal{M} = (A, N, M, B, \phi, \psi)$. Under this correspondence $\mu(X) = \mathsf{End}_{\mathscr{C}}(X) = A$, $\mu(Y) = \mathsf{End}_{\mathscr{C}}(Y) = B$, $M = \mathsf{Hom}_{\mathscr{C}}(X, Y)$, $N = \mathsf{Hom}_{\mathscr{C}}(Y, X)$ and the maps $\phi$ and $\psi$ are given by the composition of maps in $\mathscr{C}$. As a consequence the study of Morita rings subsumes the study of pre-additive categories with two objects and can be considered as 2-dimensional ring theory.

• Morita Rings and Extensions of Grothendieck Categories: Let $0 \longrightarrow \mathscr{D} \longrightarrow \mathscr{C} \longrightarrow \mathscr{E} \longrightarrow 0$ $(*)$ be an exact sequence of Grothendieck categories. The sequence $(*)$ is called an extension of $\mathscr{C}/\mathscr{D}$ $(\simeq \mathscr{E})$ by $\mathscr{D}$. Already Gabriel in his thesis [54] has proved that any locally Noetherian category (i.e. a Grothendieck category with a set of Noetherian generators), for instance the category of quasi-coherent sheaves over a Noetherian scheme or the module category over a left Noetherian ring, can be obtained by successive extensions of locally finite categories (i.e. a Grothendieck category with a set of generators of finite Jordan-Hölder length). It is a natural problem to classify the extensions $(*)$ of locally finite categories or more generally of locally noetherian categories. Roos has classified such extensions in the stable case [114]. Recall that an extension

$(*)$ is called stable if the injective envelope of any object of $\mathscr{D}$ in $\mathscr{C}$ lies in $\mathscr{D}$. Suppose that the extension $(*)$ is stable and the categories $\mathscr{D}, \mathscr{C}$ and $\mathscr{E}$ are locally Noetherian. Then there are equivalences of categories:

$$\mathscr{D} \xrightarrow{\simeq} \mathsf{TC}(A_0)^{\circ}\ , \quad \mathscr{E} \xrightarrow{\simeq} \mathsf{TC}(A_1)^{\circ} \quad \text{and} \quad \mathscr{C} \xrightarrow{\simeq} \mathsf{TC}(A)^{\circ}\ ,\ A = \left(\begin{smallmatrix} A_0 & M \\ 0 & A_1 \end{smallmatrix}\right)$$

where the rings $A_0$, $A_1$ and $A$ are topologically coherent, topologically coperfect and complete, and $\mathsf{TC}(A_0)^{\circ}$ (resp. $\mathsf{TC}(A_1)^{\circ}$, $\mathsf{TC}(A)^{\circ}$) is the dual category of the category of left linearly topologized modules over $A_0$ (resp. $A_1$, $A$) which are topologically coherent and complete. The solution of the above problem in the nonstable case is not known. Roos suggests that in order to attack the nonstable case one has to develop a theory about Morita rings with a linear topology and try also to determine how the homological properties of a Morita ring $\Lambda_{(\phi,\psi)}$ are related to those of $A$, $B$, $N$, $M$, $\phi$ and $\psi$. We refer to [104], [114] for more background and details on the above problem.

The examples and situations mentioned above and the important role they play in various different contexts provide a motivation for studying Morita rings in a general context using homological and representation-theoretic tools. Our second aim in this thesis is to study Morita rings, mainly in the context of Artin algebras, concentrating at representation-theoretic and homological aspects.

## 3. Organization and the Main Results

The organization and the main results of this thesis are as follows. In Chapter 1 we classify recollements of abelian categories whose terms are module categories. Kuhn conjectured in [82] that if the categories of a recollement are equivalent to categories of modules over finite dimensional algebras over a field, then the recollement is equivalent, in a certain sense, to one induced by an idempotent element in the way we described in section 2.1 above. We solve this conjecture for general rings. More precisely we have the following first main result of this thesis, see Theorem 1.3.6.

**Theorem A.** (Joint with J. Vitória [109]) *Let $A$ be a ring. A recollement of* $\mathsf{Mod}$-$A$ *is equivalent to a recollement in which the categories involved are module categories if and only if it is equivalent to a recollement induced by an idempotent element of a ring $S$, Morita equivalent to $A$.*

As a consequence we get that if $A$ is semiprimary, in particular a finite dimensional algebra as in Kuhn's situation, then any recollement of $\mathsf{Mod}$-$A$ is equivalent to a recollement induced by an idempotent element of $A$, see Corollary 1.3.8. Chapter 1 is structured as follows. Section 1.1 collects preliminaries notions and results on recollements of abelian categories that will be useful throughout the thesis and we fix notation. We also give a variety of model examples of recollements that will be used in the sequel as illustrations of our main results. In section 1.2, we discuss $\mathsf{TTF}$-triples in abelian categories and we use them to classify recollements of abelian categories. In section 1.3 we focus on recollements of module categories, proving Kuhn's conjecture. Finally in section 1.4 we give an example of a recollement not induced by an idempotent element.

In Chapter 2 we investigate several homological aspects of recollements of abelian categories. In section 2.1 we study the homological behavior of the six functors involved in the adjoint triples $(\mathsf{q}, \mathsf{i}, \mathsf{p})$ and $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ of a recollement situation $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ of abelian categories. More precisely we are interested in finiteness conditions of the derived functors of $\mathsf{p}$, $\mathsf{q}$, $\mathsf{l}$ and $\mathsf{r}$, as well as in the problem of when the exact functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ induces natural isomorphisms $\mathsf{i}^m\colon \mathsf{Ext}^m_{\mathscr{A}}(-,-) \longrightarrow \mathsf{Ext}^m_{\mathscr{B}}(\mathsf{i}(-),\mathsf{i}(-))$, for

$0 \le m \le k \le \infty$, i.e. when $\mathsf{i}$ is a *a $k$-homological embedding*, and also in the problem of when the exact functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ induces, restricted to suitable subcategories, natural isomorphisms $\mathsf{e}^m\colon \mathsf{Ext}^m_{\mathscr{B}}(-,-) \longrightarrow \mathsf{Ext}^m_{\mathscr{B}}(\mathsf{e}(-),\mathsf{e}(-))$, up to some steps, i.e. for $0 \le m \le k$, where $0 \le k \le \infty$. It turns out that the answer to the above problems are closely related and depend on the behavior of the *$k$-generalized perpendicular subcategory* of $\mathscr{B}$ associated to $\mathscr{A}$, see Definition 2.1.1, and the subcategory $\mathfrak{X}_k$ (or $\mathfrak{Y}_k$) of $\mathscr{B}$ consisting of all objects admitting a truncated projective (or injective) resolution by projectives (or injectives) coming from the quotient category $\mathscr{C}$ via the section functor $\mathsf{l}$ (or $\mathsf{r}$). Note that the following second main result of this thesis, see Theorems 2.1.10 and 2.1.11, extend and generalize related results of Auslander-Platzeck-Todorov [14], formulated in the setting of finitely generated modules over an Artin algebra equipped with an idempotent ideal, and Geigle-Lenzing [56], formulated in the context of classical perpendicular categories, i.e. $k = 1$, and homological epimorphisms of rings. In the next result $\mu\colon \mathsf{le} \longrightarrow \mathrm{Id}_{\mathscr{B}}$ is the counit of the adjoint pair $(\mathsf{l},\mathsf{e})$ and $\nu\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow \mathsf{re}$ is the unit of the adjoint pair $(\mathsf{e},\mathsf{r})$.

**Theorem B.** ( [108] ) *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects.*

(i) *The following statements are equivalent.*
   (a) *The functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a $k$-homological embedding.*
   (b) $\mathsf{Im}\,\mu_P \in \mathfrak{X}_{k-1}$, $\forall P \in \mathsf{Proj}\,\mathscr{B}$.
   (c) $\mathsf{Im}\,\nu_I \in \mathfrak{Y}_{k-1}$, $\forall I \in \mathsf{Inj}\,\mathscr{B}$.

(ii) *The following statements are equivalent.*
   (a) *The map $\mathsf{e}^n_{Z,W}\colon \mathsf{Ext}^n_{\mathscr{B}}(Z,W) \longrightarrow \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W))$ is invertible, $\forall W \in \mathscr{B}$, (resp. $\forall Z \in \mathscr{B}$), and $0 \le n \le k$.*
   (b) $Z \in \mathfrak{X}_{k+1}$ *(resp. $W \in \mathfrak{Y}_{k+1}$).*

In section 2.2 and 2.3 of Chapter 2, which form a unit, we study the problem of how various dimensions, including global and finitistic dimension, of the abelian categories involved in a recollement situation $(\mathscr{A},\mathscr{B},\mathscr{C})$ are related. Although one can not expect in general a precise relation, see Example 2.4.8, we prove the existence of several bounds concerning the global and finitistic dimension under natural assumptions. In section 2.2 we first show that the global dimension of $\mathscr{B}$ is always bounded by the sum of the global dimensions of $\mathscr{A}$ and $\mathscr{C}$ plus the supremum of the projective dimension in $\mathscr{B}$ of the projectives of $\mathscr{A}$, see Theorem 2.2.1. On the other hand if the functor $\mathsf{i}$ is a homological embedding, meaning that $\mathsf{i}$ is a $k$-homological embedding $\forall k \ge 0$, then in Theorem 2.2.8 we give lower and upper bounds for the global dimension of $\mathscr{B}$ in terms of the global dimensions of $\mathscr{A}$ and $\mathscr{C}$ and related projective or injective dimensions in $\mathscr{B}$ of objects coming from $\mathscr{A}$. It follows in particular that if $\mathscr{B}$ is hereditary, then so are $\mathscr{A}$ and $\mathscr{C}$ (Theorem 2.2.9). Under the assumption that any projective object of $\mathscr{A}$ has, via the inclusion $\mathsf{i}$, projective dimension at most one, we show that finiteness of global dimension of $\mathscr{B}$ is equivalent to finiteness of the global dimension of both $\mathscr{A}$ and $\mathscr{C}$ (Corollary 2.2.18). Finally we give precise formulas for projective or injective dimension for objects lying in special subcategories of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, and we apply our results to stratified abelian categories, see Corollary 2.2.22. Our results in this section extend and generalize several results of the literature, including those of Auslander-Platzeck-Todorov [14]. In section 2.3 we study the finitistic dimension, denoted by $\mathsf{FPD}\,\mathcal{M}$ for an abelian category $\mathcal{M}$, and how it behaves in a recollement situation $(\mathscr{A},\mathscr{B},\mathscr{C})$. After introducing the notion of the *locally bounded homological dimension*, $\mathsf{l.b.hom.dim}F$, of a right exact functor $F$ between abelian categories, see Definition 2.3.2, we prove our first main result of the section, namely: if the section functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ has locally bounded homological dimension, then the finitistic projective dimension of $\mathscr{C}$ is bounded by the finitistic projective dimension of $\mathscr{B}$ plus the locally bounded homological dimension of the section functor $\mathsf{l}$ (Theorem 2.3.2). Under various natural conditions, we also give lower and upper bounds for the finitistic dimension of $\mathscr{B}$ in terms of the finitistic dimension of $\mathscr{A}$ and $\mathscr{C}$, and related homological invariants. The results of this section, applied to comma categories, generalize and extend results of Fossum-Griffith-Reiten [49]. Indicatively we state the following main result on global and finitistic dimension of the categories involved in a recollement situation.

**Theorem C.** ( [108] ) *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects.*

(i) *We have:*

$$\mathsf{gl.dim}\,\mathscr{B} \ \leq \ \mathsf{gl.dim}\,\mathscr{A} + \mathsf{gl.dim}\,\mathscr{C} + \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} + 1$$

(ii) *If the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ has locally bounded homological dimension, then:*

$$\mathsf{FPD}\,(\mathscr{C}) \ \leq \ \mathsf{FPD}\,(\mathscr{B}) + \mathsf{l.b.hom.dim}\,\mathsf{l}$$

The final section 2.4 of Chapter 2 is devoted to applications in Ring Theory, guided by the examples analyzed in section 1.1 of Chapter 1. More precisely we apply our results of sections 2.2 and 2.3 to recollements of module categories arising from $(\alpha)$ pairs $(R, e)$ consisting of a ring $R$ and an idempotent $e \in R$, $(\beta)$ triangular matrix rings or more generally rings of Morita Contexts, $(\gamma)$ quasi-hereditary rings. Our results give a nice interplay between global and finitistic dimension and generalize and extend a host of related results in the literature.

In Chapter 3 we construct an abelian category which represents an abstract model for the category of modules over a Morita ring $\Lambda_{(\phi,\psi)}$. Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories, $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ two additive functors, and $\phi\colon FG \longrightarrow \mathrm{Id}_{\mathscr{B}}$ and $\psi\colon GF \longrightarrow \mathrm{Id}_{\mathscr{A}}$ two natural transformations such that $F(\psi) = \phi_F$ and $G(\phi) = \psi_G$. Then from this data we define the *Morita category* $\mathcal{M}(\phi,\psi)$, see section 3.1, and we examine when the latter is an abelian category. The first three sections collects several structural properties of this category. In the fourth section we study finiteness conditions on subcategories of the Morita category $\mathcal{M}(0,0)$ and in the remaining sections we investigate some homological aspects of this construction. This chapter serves as an introductory step for Chapter 4 and several of its results are used later on.

In Chapter 4 we discuss Artin algebras arising from Morita contexts. More precisely we study Morita rings $\Lambda_{(\phi,\psi)}$ in the context of Artin algebras concentrating mainly at representation-theoretic and homological aspects. In section 4.1 we collect preliminary notions and results on Morita rings that will be useful throughout this chapter and we fix notation. In particular we describe the module category over a Morita ring and also we analyze the connections with recollement situations between the involved module categories. In section 4.2 we describe the projective, injective and simple modules in case the Morita ring is an Artin algebra. Using this description we characterize when the Morita ring is selfinjective and then, as an immediate consequence of this we give an upper bound for the representation dimension of the Morita ring $\Lambda_{0,0}(\mathcal{M})$ arising from the Morita context $\mathcal{M}$ where $A = B = M = N = \Lambda$ is a selfinjective Artin algebra and $\phi = 0 = \psi$.

In sections 4.3 and 4.4 we study finiteness conditions on subcategories of the module category of a Morita ring $\Lambda_{(\phi,\psi)}(\mathcal{M})$ arising from a Morita context $\mathcal{M} = (A, N, M, B, \phi, \psi)$, and also we investigate the global dimension of $\Lambda_{(\phi,\psi)}(\mathcal{M})$, in the special case when the bimodule homomorphisms $\phi$ and $\psi$ are zero. One advantage for working in this setting is that the module categories $\mathsf{Mod}\text{-}A$, $\mathsf{Mod}\text{-}B$, $\mathsf{Mod}\text{-}\left(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\right)$, $\mathsf{Mod}\text{-}\left(\begin{smallmatrix} A & {}_AN_B \\ 0 & B \end{smallmatrix}\right)$ are fully embedded into the module category $\mathsf{Mod}\text{-}\Lambda_{(0,0)}(\mathcal{M})$ as functorially finite subcategories. Indeed we prove in Theorem 3.4.9 that the above natural subcategories of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$ are bireflective (i.e. the full embedding functors have left and right adjoints) and therefore they are functorially finite in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$, closed under isomorphic images, direct sums, direct products, kernels and cokernels. More generally we show in Theorem 3.4.11 that if $\mathcal{U}$ is a functorially finite subcategory of $\mathsf{Mod}\text{-}A$ and $\mathcal{V}$ is a functorially finite subcategory of $\mathsf{Mod}\text{-}B$, then under some additional conditions the subcategories $\mathcal{U}$ and $\mathcal{V}$ induce a functorially finite subcategory of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}(\mathcal{M})$, thus generalizing some well known results of the literature, see [120], in the setting of triangular matrix rings. In section 4.4, after introducing the notion of an *$A$-tight* (resp. *$B$-tight*) *projective $\Lambda_{(0,0)}$-resolution*, see Definition 4.4.2, we give an upper bound for the global dimension of $\Lambda_{(0,0)}$ in terms of the global dimension of $A$ and $B$. In particular in Theorem 4.4.9 we show the following main result of this Chapter.

**Theorem D.** (Joint with E.L. Green [60]) *Let $\Lambda_{(0,0)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring which is an Artin algebra and suppose that $M$ has a $B$-tight projective $\Lambda_{(0,0)}$-resolution and $N$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution. Then*:

$$\mathsf{gl.dim}\,\Lambda_{(0,0)} \ \leq\ \mathsf{gl.dim}\,A + \mathsf{gl.dim}\,B + 1$$

Further in Theorem 4.4.14 we deal with the case where either $M$ or $N$ does not have a tight projective $\Lambda_{(0,0)}$-resolution and we present some formulas for the global dimension of $\Lambda_{(0,0)}$. We provide examples which show that the inequalities of our bounds are sharp and that the inequalities can be proper, see Examples 4.4.10, 4.4.11 and 4.4.15.

The final section 4.5 of Chapter 4 is devoted to investigate when a Morita ring $\Lambda_{(\phi,\psi)}$, which is an Artin algebra, is Gorenstein and discuss applications of this result. The following main result of Chapter 4, which is Theorem 4.5.3, gives a sufficient condition for a Morita ring to be Gorenstein. For an Artin algebra $\Lambda$ we denote by $(\mathsf{proj}\,\Lambda)^{<\infty}$, resp. $(\mathsf{inj}\,\Lambda)^{<\infty}$, the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of the $\Lambda$-modules of finite projective, resp. injective, dimension.

**Theorem E.** (Joint with E.L. Green [60]) *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring which is an Artin algebra such that the adjoint pair of functors $(M \otimes_A -, \mathsf{Hom}_B(M, -))$ induces an equivalence*

$$M \otimes_A - \colon (\mathsf{proj}\,A)^{<\infty} \ \underset{\longrightarrow}{\overset{\simeq}{\longleftarrow}}\ (\mathsf{inj}\,B)^{<\infty} \ :\ \mathsf{Hom}_B(M, -)$$

*and the adjoint pair of functors $(N \otimes_B -, \mathsf{Hom}_A(N, -))$ induces an equivalence*

$$N \otimes_B - \colon (\mathsf{proj}\,B)^{<\infty} \ \underset{\longrightarrow}{\overset{\simeq}{\longleftarrow}}\ (\mathsf{inj}\,A)^{<\infty} \ :\ \mathsf{Hom}_A(N, -)$$

*Then the Morita ring $\Lambda_{(\phi,\psi)}$ is Gorenstein.*

For example if $\Lambda$ is a Gorenstein Artin algebra then the matrix algebra $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$ is Gorenstein, see Corollary 4.5.5. As a consequence of the above result we characterize the Gorenstein-projective modules over the Morita ring $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$, see Corollary 4.5.10, in case $\Lambda$ is a Gorenstein Artin algebra.

The final Chapter 5 of this thesis is devoted to Rouquier's dimension for triangulated categories and to representation dimension of Artin algebras. In section 5.1 of chapter 5 we study the Rouquier dimension of triangulated categories in connection with finding conditions ensuring that a recollement of abelian categories induces a recollement of triangulated categories at the level of the corresponding bounded derived categories. In fact we give several equivalent conditions for the existence of a triangulated recollement of the bounded derived categories of the abelian categories involved in a recollement situation, in terms of homological finiteness properties of the functors $\mathsf{q}$, $\mathsf{p}$, $\mathsf{l}$, and $\mathsf{r}$ (see Definition 5.1.5). More precisely our first main result of this section is the following, see Theorem 5.1.6 for more details.

**Theorem F.** ( [108] ) *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects.*

(i) *The following statements are equivalent:*
   (a) *The functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding, and the functor $\mathsf{q}\colon \mathscr{B} \to \mathscr{A}$, resp. $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$, is of locally finite homological, resp. cohomological, dimension.*
   (b) *There exists a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}^b(\mathscr{A}) \arrow[r, "\mathbf{D}^b(\mathsf{i})"] & \mathbf{D}^b(\mathscr{B}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{q}"'] \arrow[l, bend left, "\mathbf{R}^b\mathsf{p}"] \arrow[r, "\mathbf{D}^b(\mathsf{e})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right, "\mathsf{l}'"'] \arrow[l, bend left, "\mathsf{r}'"] \end{tikzcd}$$

(ii) *The following statements are equivalent:*
   (a) *The functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding, and the functor $\mathsf{l}\colon \mathscr{C} \to \mathscr{B}$, resp. $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$, is of locally finite homological, resp. cohomological, dimension.*
   (b) *There exists a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}^b(\mathscr{A}) \arrow[r, "\mathbf{D}^b(\mathsf{i})"] & \mathbf{D}^b(\mathscr{B}) \arrow[l, bend right, "\mathsf{q}'"'] \arrow[l, bend left, "\mathsf{p}'"] \arrow[r, "\mathbf{D}^b(\mathsf{e})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{l}"'] \arrow[l, bend left, "\mathbf{R}^b\mathsf{r}"] \end{tikzcd}$$

Next we give bounds for the dimension of a triangulated category $\mathcal{T}$, in the sense of Rouquier [113] (see Definition 5.1.7), in a recollement situation $(\mathcal{U},\mathcal{T},\mathcal{V})$ of triangulated categories. In this context we have our second main result of this section, see Theorem 5.1.10 for more details.

**Theorem G.** ( [108] ) *Let $(\mathcal{U},\mathcal{T},\mathcal{V})$ be a recollement of triangulated categories. Then:*

$$\max\{\dim\mathcal{U},\dim\mathcal{V}\} \;\leq\; \dim\mathcal{T} \;\leq\; \dim\mathcal{U}+\dim\mathcal{V}+1$$

As a consequence if one of the four functors $\mathsf{q}$, $\mathsf{p}$, $\mathsf{l}$, and $\mathsf{r}$ of a recollement of abelian categories $(\mathscr{A},\mathscr{B},\mathscr{C})$ has finite (co)homological dimension (in an appropriate sense, see Definition 5.1.5), then we get Theorem 5.1.12: $\max\{\dim\mathbf{D}^b(\mathscr{A}),\dim\mathbf{D}^b(\mathscr{C})\} \leq \dim\mathbf{D}^b(\mathscr{B}) \leq \dim\mathbf{D}^b(\mathscr{A})+\dim\mathbf{D}^b(\mathscr{C})+1$. Finally we give applications to Rouquier's dimension of bounded derived categories of rings. In fact if $\Lambda = \left(\begin{smallmatrix} R & {}_RN_S \\ 0 & S \end{smallmatrix}\right)$ is a triangular matrix ring then as a consequence of the above we get that: $\max\{\dim \mathbf{D}^b(R), \dim \mathbf{D}^b(S)\} \leq \dim \mathbf{D}^b(\Lambda) \leq \dim \mathbf{D}^b(R) + \dim \mathbf{D}^b(S) + 1$, see Corollary 5.1.15.

In the last section section of Chapter 5 we study the representation dimension, denoted by $\operatorname{rep.dim} \mathcal{M}$ for an abelian category $\mathcal{M}$, and how it behaves in a recollement situation $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ between abelian categories. Firstly we review Auslander's notion of representation dimension in the context of abelian categories, see Definition 5.2.1. Note that if $\mathcal{M} = \operatorname{mod-}\Lambda$, where $\Lambda$ is an Artin algebra, then our definition coincides with the one given by Auslander, namely: $\operatorname{rep.dim} \Lambda = \min\{\operatorname{gl.dim} \operatorname{End}_\Lambda(X \oplus \Lambda \oplus \mathsf{D}(\Lambda)) \mid X \in \operatorname{mod-}\Lambda\}$. Recall also that a $\Lambda$-module $Y = X \oplus \Lambda \oplus \mathsf{D}(\Lambda)$ that realizes the minimum is called an Auslander generator of $\operatorname{mod-}\Lambda$. Our first main result shows that if the representation dimension of $\mathscr{B}$ is at most three, then the same holds for the representation dimension of $\mathscr{C}$, and, under an additional condition, the representation dimension of $\mathscr{A}$. Finally we show that $\operatorname{rep.dim} \mathscr{C} \leq \operatorname{rep.dim} \mathscr{B}$, provided that there exists an Auslander generator of $\mathscr{B}$ enjoying special properties. These are Theorems 5.2.2 and 5.2.3 respectively. As an application we derive several results on representation and finitistic dimension of Artin algebras, see Corollaries 5.2.5 - 5.2.14. The final main result, Theorem 5.2.15, of this thesis gives an interesting interplay between representation dimension and finitistic dimension, and presents situations where the finitistic dimension of an Artin algebra is finite using a basic construction of Auslander. We summarize next some of the above main results.

**Theorem H.** ( [108] ) *Let $\Lambda$ be an Artin algebra and $e$ an idempotent element of $\Lambda$.*

(i) *If* $\operatorname{rep.dim} \Lambda \leq 3$ *then:*
$$\operatorname{rep.dim} e\Lambda e \leq 3$$
*In particular, we have:*
$$\operatorname{rep.dim} \Lambda \leq 3 \iff \operatorname{rep.dim} \operatorname{End}_\Lambda(P) \leq 3$$
*for any finitely generated projective $\Lambda$-module $P$.*

(ii) *Let $\Lambda \oplus X$ be an Auslander generator of $\Lambda$. If the functor* $\operatorname{Hom}_\Lambda(X, \Lambda) \otimes_\Lambda - \colon \operatorname{mod-}\Lambda \longrightarrow \operatorname{mod-}\Gamma$ *has locally bounded homological dimension, then:*
$$\operatorname{fin.dim} \Lambda \leq \operatorname{rep.dim} \Lambda + \operatorname{l.b.hom.dim} \operatorname{Hom}_\Lambda(X, \Lambda) \otimes_\Lambda - < \infty$$

**Conventions and Notations.** We compose morphisms $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$ in a given category in a diagrammatic order, i.e the composition of $f$, $g$ is denoted by $f \circ g$. We use the usual anti-diagrammatic order when we compose functors and when we apply elements to morphisms in categories. For a ring $R$ we work usually with left $R$-modules and the corresponding category is denoted by $\operatorname{Mod-}R$. Our additive categories are assumed to have finite direct sums and our subcategories are assumed to be closed under isomorphisms and direct summands. For all unexplained notions and results concerning the representation theory of Artin algebras we refer to the book [18], see also [7], [131]. For the homological algebra used throughout this thesis we refer to the books [65], [115], [123], and for the theory of triangulated categories see [98].

CHAPTER 1

# Recollements of Module Categories

In this Chapter we give definitions and useful properties of recollements of abelian categories. We also give a variety of model examples that will be useful throughout this thesis and we fix notation. We define when two recollements of abelian categories are equivalent and we establish a correspondence between recollements of abelian categories up to equivalence and certain $\mathsf{TTF}$-triples. For module categories we show also a correspondence with idempotent ideals, recovering a theorem of Jans [69]. Finally, we show that a recollement of abelian categories whose terms are module categories is equivalent to one induced by an idempotent element, thus proving a conjecture by Kuhn [82]. The results of sections 1.2, 1.3 and 1.4 are included in the paper entitled: Recollements of Module Categories [109] which is joint work with Jorge Vitória.

## 1.1. Preliminaries on Recollements of Abelian Categories

In this section we give the definition of recollement between abelian categories and we fix notation. We also include a variety of examples that will be used throught in this thesis.

To begin with, we recall the definition of a recollement situation in the context of abelian categories, see for instance [50, 61, 82]. For an additive functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ between additive categories, we denote by $\mathsf{Im}\, F = \{B \in \mathscr{B} \mid B \cong F(A) \text{ for some } A \in \mathscr{A}\}$ the essential image of $F$ and by $\mathsf{Ker}\, F = \{A \in \mathscr{A} \mid F(A) = 0\}$ the kernel of $F$.

Definition 1.1.1. A recollement situation between abelian categories $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ is a diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{i}"] & \mathscr{B} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \qquad \mathsf{R}_{\mathsf{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$$

henceforth denoted by $(\mathscr{A}, \mathscr{B}, \mathscr{C})$, satisfying the following conditions:

**1.** $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ is an adjoint triple.
**2.** $(\mathsf{q}, \mathsf{i}, \mathsf{p})$ is an adjoint triple.
**3.** The functors $\mathsf{i}$, $\mathsf{l}$, and $\mathsf{r}$ are fully faithful.
**4.** $\mathsf{Im}\,\mathsf{i} = \mathsf{Ker}\,\mathsf{e}$.

In this case the category $\mathscr{B}$ is said to be the recollement of $\mathscr{A}$ and $\mathscr{C}$.

Throughout this chapter we fix a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ of abelian categories as in Definition 1.1.1. Then we have the adjoint pairs $(\mathsf{l}, \mathsf{e})$, $(\mathsf{e}, \mathsf{r})$, $(\mathsf{q}, \mathsf{i})$, and $(\mathsf{i}, \mathsf{p})$. We always denote by $\mu\colon \mathsf{le} \longrightarrow \mathrm{Id}_{\mathscr{B}}$, resp. $\kappa\colon \mathsf{ip} \longrightarrow \mathrm{Id}_{\mathscr{B}}$, the counit of the adjoint pair $(\mathsf{l}, \mathsf{e})$, resp. $(\mathsf{i}, \mathsf{p})$, and by $\lambda\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow \mathsf{iq}$, resp. $\nu\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow \mathsf{re}$, the unit of the adjoint pair $(\mathsf{q}, \mathsf{i})$, resp. $(\mathsf{e}, \mathsf{r})$. In the following remarks we isolate some easily established properties of a recollement situation which will be useful later.

REMARK 1.1.2. (i) The functors $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ and $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ are exact, since from the Definition 1.1.1 the functors $\mathsf{e}$ and $\mathsf{i}$ have left and right adjoints.

(ii) The composition of functors $\mathsf{ql} = \mathsf{pr} = 0$. Indeed for every $A \in \mathscr{A}$ we have:

$$\mathsf{Hom}_{\mathscr{A}}(\mathsf{ql}(C), A) \simeq \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(C), \mathsf{i}(A)) \simeq \mathsf{Hom}_{\mathscr{C}}(C, \mathsf{ei}(A)) = 0$$

We infer that $\mathsf{ql}(C) = 0$ for every $C \in \mathscr{C}$ and then our claim follows. Similarly we show that $\mathsf{pr} = 0$.

(iii) Since the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is fully faithful it follows that the counit $\mathsf{qi} \xrightarrow{\simeq} \mathrm{Id}_{\mathscr{A}}$ of the adjoint pair $(\mathsf{q}, \mathsf{i})$ is invertible and the unit $\mathrm{Id}_{\mathscr{A}} \xrightarrow{\simeq} \mathsf{pi}$ of the adjoint pair $(\mathsf{i}, \mathsf{p})$ is invertible (see [90]).

(iv) Since the functor $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is fully faithful the counit $\mathsf{er} \xrightarrow{\simeq} \mathrm{Id}_{\mathscr{C}}$ of the adjoint pair $(\mathsf{e}, \mathsf{r})$ is invertible and since the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is fully faithful the unit $\mathrm{Id}_{\mathscr{C}} \xrightarrow{\simeq} \mathsf{el}$ of the adjoint pair $(\mathsf{l}, \mathsf{e})$ is invertible ( see [90] ).

(v) The functor $\mathsf{i}$ induces an equivalence between $\mathscr{A}$ and the Serre subcategory (i.e. closed under subobjects, quotients and extensions) $\mathsf{Ker}\,\mathsf{e} = \mathsf{Im}\,\mathsf{i}$ of $\mathscr{B}$. In the sequel we shall view the above equivalence as an identification.

(vi) Since the exact functor $\mathsf{e}$ admits a fully faithful left adjoint and a fully faithful right adjoint, it follows that $\mathscr{A}$ is a localizing and colocalizing subcategory of $\mathscr{B}$ and there is an equivalence $\mathscr{B}/\mathscr{A} \simeq \mathscr{C}$ [54]. In particular any recollement $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$ induces a short exact sequence of abelian categories $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$.

In section 1.2 we will discuss in more detail the notions of Serre, localizing and colocalizing subcategories.

REMARK 1.1.3. Let $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ be an exact functor between abelian categories which is part of an adjoint triple $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ where $\mathsf{l}$ or $\mathsf{r}$ is fully faithful. Note that $\mathsf{l}$ is fully faithful if and only if $\mathsf{r}$ is fully faithful. Denote by $\mu\colon \mathsf{le} \longrightarrow \mathrm{Id}_{\mathscr{B}}$ the counit of the adjoint pair $(\mathsf{l}, \mathsf{e})$ and $\nu\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow \mathsf{re}$ the unit of the adjoint pair $(\mathsf{e}, \mathsf{r})$. It is easy to see that for any object $B \in \mathscr{B}$, the assignments $\mathsf{Ker}\,\nu_B =: \mathsf{p}(B) \leftarrow\!\shortmid B \mapsto \mathsf{q}(B) := \mathsf{Coker}\,\mu_B$ induce functors $\mathsf{q}\colon \mathscr{B} \longrightarrow \mathsf{Ker}\,\mathsf{e}$ and $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathsf{Ker}\,\mathsf{e}$. It is straightforward to check that $(\mathsf{q}, \mathsf{i}, \mathsf{p})$ is an adjoint triple, where $\mathsf{i}\colon \mathsf{Ker}\,\mathsf{e} \longrightarrow \mathscr{B}$ is the inclusion, and so $(\mathsf{Ker}\,\mathsf{e}, \mathscr{B}, \mathscr{C})$ is a recollement of abelian categories.

REMARK 1.1.4. Let $\mathscr{B}$ be an abelian category and $\mathscr{A}$ a Serre subcategory of $\mathscr{B}$. Then we have the exact sequence of abelian categories $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{B}/\mathscr{A} \longrightarrow 0$. Assume that $\mathscr{B}$ is a Grothendieck category with projective covers and $\mathscr{A}$ is a Serre subcategory of $\mathscr{B}$ which is closed under products and coproducts. Then from [78] the quotient functor $\mathscr{B} \longrightarrow \mathscr{B}/\mathscr{A}$ has a left and right adjoint. Therefore from Remark 1.1.3 we have the recollement of abelian categories $(\mathscr{A}, \mathscr{B}, \mathscr{B}/\mathscr{A})$.

REMARK 1.1.5. Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and suppose that $\mathscr{B}$ has enough projective and injective objects. Then the functor $\mathsf{q}\colon \mathscr{B} \longrightarrow \mathscr{A}$ preserves projective objects since we have the adjoint pair $(\mathsf{q}, \mathsf{i})$ and the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is exact. Similarly the functor $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ preserves injective objects. Let $A$ be an object of $\mathscr{A}$. Then $\mathsf{i}(A) \in \mathscr{B}$ and there exists an epimorphism $P \longrightarrow \mathsf{i}(A)$ with $P \in \mathsf{Proj}\,\mathscr{B}$. Applying the functor $\mathsf{q}$ we obtain the epimorphism $\mathsf{q}(P) \longrightarrow A$ and $\mathsf{q}(P) \in \mathsf{Proj}\,\mathscr{A}$. Hence the category $\mathscr{A}$ has enough projective objects and $\mathsf{Proj}\,\mathscr{A} = \mathsf{add}\,\mathsf{q}(\mathsf{Proj}\,\mathscr{B})$. Note that if $\mathfrak{X}$ is a full subcategory of an abelian category $\mathscr{A}$ then $\mathsf{add}\,\mathfrak{X}$ denotes the full subcategory of $\mathscr{A}$ consisting of all direct summands of finite coproducts of objects of $\mathcal{X}$. Similarly the category $\mathscr{A}$ has enough injective objects and $\mathsf{Inj}\,\mathscr{A} = \mathsf{add}\,\mathsf{p}(\mathsf{Inj}\,\mathscr{B})$. Suppose that $\mathscr{C}$ has enough projective and injective objects. Then the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ preserves projective objects since we have the adjoint pair $(\mathsf{l}, \mathsf{e})$ and the functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ is exact. Similarly the functor $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ preserves injective objects.

We continue with the following result which we will need in the sequel, see also [50].

PROPOSITION 1.1.6. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories.*

(i) *There are equivalences of categories:*

$$\mathsf{p} \ : \ \{B \in \mathscr{B} \mid \mathsf{ip}(B) \simeq B\} \xrightarrow{\ \simeq\ } \mathscr{A} \xleftarrow{\ \simeq\ } \{B \in \mathscr{B} \mid \mathsf{iq}(B) \simeq B\} \ : \ \mathsf{q}$$

*In particular, an object $B \in \mathscr{B}$ belongs to $\mathsf{i}(\mathscr{A})$ if and only if $B \simeq \mathsf{iq}(B)$ if and only if $B \simeq \mathsf{ip}(B)$.*

(ii) *Let $B \in \mathscr{B}$. Then we have the exact sequences*

$$0 \longrightarrow \mathsf{Ker}\,\mu_B \longrightarrow \mathsf{le}(B) \xrightarrow{\mu_B} B \xrightarrow{\lambda_B} \mathsf{iq}(B) \longrightarrow 0$$

*and*

$$0 \longrightarrow \mathsf{ip}(B) \xrightarrow{\kappa_B} B \xrightarrow{\nu_B} \mathsf{re}(B) \longrightarrow \mathsf{Coker}\,\nu_B \longrightarrow 0$$

*where the objects $\mathsf{Ker}\,\mu_B$ and $\mathsf{Coker}\,\nu_B$ lie in $\mathsf{i}(\mathscr{A})$.*

PROOF. Part (i) is straightforward since we have isomorphisms $\mathsf{qi} \simeq \mathrm{Id}_{\mathscr{A}}$ and $\mathrm{Id}_{\mathscr{A}} \simeq \mathsf{pi}$. For part (ii) we prove only the existence of the first exact sequence. Let $B$ be an object of $\mathscr{B}$. Then from the counit $\mu_B\colon \mathsf{le}(B) \longrightarrow B$ of the adjoint pair $(\mathsf{l}, \mathsf{e})$ we have the exact sequence $0 \longrightarrow \mathsf{Ker}\,\mu_B \longrightarrow \mathsf{le}(B) \longrightarrow B \longrightarrow \mathsf{Coker}\,\mu_B \longrightarrow 0$ in $\mathscr{B}$. Applying the exact functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ we infer that $\mathsf{e}(\mathsf{Ker}\,\mu_B) = 0 = \mathsf{e}(\mathsf{Coker}\,\mu_B)$, i.e. $\mathsf{Ker}\,\mu_B$ and $\mathsf{Coker}\,\mu_B$ are objects of $\mathsf{i}(\mathscr{A})$. Since $\mathsf{Coker}\,\mu_B = \mathsf{i}(A)$ for some $A \in \mathscr{A}$ it follows from (i) that $\mathsf{iq}(\mathsf{Coker}\,\mu_B) \simeq \mathsf{Coker}\,\mu_B$. Since the functor $\mathsf{iq}$ is right exact and $\mathsf{ql} = 0$, the assertion follows from the following commutative diagram:

$$\begin{tikzcd}
\mathsf{le}(B) \arrow[r, "\mu_B"] \arrow[d] & B \arrow[r] \arrow[d, "\lambda_B"] & \mathsf{Coker}\,\mu_B \arrow[r] \arrow[d, "\lambda_{\mathsf{Coker}\,\mu_B}", "\simeq"'] & 0 \\
0 \arrow[r] & \mathsf{iq}(B) \arrow[r, "\simeq"] & \mathsf{iq}(\mathsf{Coker}\,\mu_B) \arrow[r] & 0
\end{tikzcd}$$

The proof that the sequence $0 \longrightarrow \mathsf{ip}(B) \xrightarrow{\kappa_B} B \xrightarrow{\nu_B} \mathsf{re}(B) \longrightarrow \mathsf{Coker}\,\nu_B \longrightarrow 0$ is exact is similar. □

**1.1.1. Examples of Recollements of Abelian Categories.** We continue by giving a variety of examples of recollements of abelian categories which will be used throughout in the next Chapters.

EXAMPLE 1.1.7. (Idempotents) Let $R$ be a ring and $e^2 = e \in R$ an idempotent. Then we have a recollement

$$\begin{tikzcd}
\mathsf{Mod}\text{-}R/ReR \arrow[r, "\mathsf{inc}"] & \mathsf{Mod}\text{-}R \arrow[l, bend right, "R/ReR\otimes_R -"'] \arrow[l, bend left, "\mathsf{Hom}_R(R/ReR,-)"] \arrow[r, "e(-)"] & \mathsf{Mod}\text{-}eRe \arrow[l, bend right, "Re\otimes_{eRe}-"'] \arrow[l, bend left, "\mathsf{Hom}_{eRe}(eR,-)"]
\end{tikzcd}$$

where the category $\mathsf{Mod}\text{-}R/ReR$ of (left) modules over $R/ReR$ is the kernel of the functor $e(-)\colon \mathsf{Mod}\text{-}R \longrightarrow \mathsf{Mod}\text{-}eRe$. Note that if $R$ is semiprimary, then any idempotent ideal $I$ is of the form $ReR$ for an idempotent $e \in R$, see [45]. Hence idempotent ideals in semiprimary rings induce recollements of module categories. The recollement $\mathsf{R}(\mathsf{Mod}\text{-}R/ReR, \mathsf{Mod}\text{-}R, \mathsf{Mod}\text{-}eRe)$ is said to be the recollement of $\mathsf{Mod}\text{-}R$ **induced by the idempotent element** $e$. In the last section of this Chapter we show that a recollement whose terms are module categories is equivalent to one induced by an idempotent element, thus answering a question by Kuhn in [82].

We now turn our attention to recollement arising from a ring with two idempotent elements.

EXAMPLE 1.1.8. (Morita Contexts) Let $R$, $S$ be rings, $M$ a $S$-$R$-bimodule and $N$ a $R$-$S$-bimodule. Let $\phi\colon M \otimes_R N \longrightarrow S$ be a $S$-$S$-bimodule homomorphism and let $\psi\colon N \otimes_S M \longrightarrow R$ be a $R$-$R$-bimodule homomorphism. Then the above data allow us to define the **Morita ring**:

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} R & {}_RN_S \\ {}_SM_R & S \end{pmatrix}$$

where the addition of elements of $\Lambda$ is componentwise and multiplication is given by

$$\begin{pmatrix} r & n \\ m & s \end{pmatrix} \cdot \begin{pmatrix} r' & n' \\ m' & s' \end{pmatrix} = \begin{pmatrix} rr' + \psi(n \otimes m') & rn' + ns' \\ mr' + sm' & ss' + \phi(m \otimes n') \end{pmatrix}$$

The $\Lambda_{(\phi,\psi)}$-modules are tuples $(X, Y, f, g)$ where $X \in \mathsf{Mod}\text{-}R$, $Y \in \mathsf{Mod}\text{-}S$, $f\colon M \otimes_R X \longrightarrow Y$ and $g\colon N \otimes_S Y \longrightarrow X$ such that the following diagrams are commutative:

$$\begin{tikzcd} N \otimes_S M \otimes_R X \arrow[r, "1_N \otimes f"] \arrow[d, "\psi \otimes 1_X"'] & N \otimes_S Y \arrow[d, "g"] \\ R \otimes_R X \arrow[r, "\simeq"] & X \end{tikzcd} \qquad \begin{tikzcd} M \otimes_R N \otimes_S Y \arrow[r, "1_M \otimes g"] \arrow[d, "\phi \otimes 1_Y"'] & M \otimes_R X \arrow[d, "f"] \\ S \otimes_S Y \arrow[r, "\simeq"] & Y \end{tikzcd}$$

If $(X, Y, f, g)$ and $(X', Y', f', g')$ are $\Lambda_{(\phi,\psi)}$-modules, then a morphism $(X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is a pair of homomorphisms $(a, b)$ where $a\colon X \longrightarrow X'$ is a morphism in $\mathsf{Mod}\text{-}R$ and $b\colon Y \longrightarrow Y'$ is a morphism in $\mathsf{Mod}\text{-}S$ such that the following diagrams are commutative:

$$\begin{tikzcd} M \otimes_R X \arrow[r, "f"] \arrow[d, "1_M \otimes a"'] & Y \arrow[d, "b"] \\ M \otimes_R X' \arrow[r, "f'"] & Y' \end{tikzcd} \qquad \begin{tikzcd} N \otimes_S Y \arrow[r, "g"] \arrow[d, "1_N \otimes b"'] & X \arrow[d, "a"] \\ N \otimes_S Y' \arrow[r, "g'"] & X' \end{tikzcd}$$

Then from Example 1.1.7 using the idempotent elements $e_1 = \left(\begin{smallmatrix} 1_R & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_S \end{smallmatrix}\right)$ of $\Lambda_{(\phi,\psi)}$, we derive the following recollements of abelian categories:

$$\begin{tikzcd}[column sep=large] \mathsf{Mod}\text{-}\Lambda/\Lambda e_1\Lambda \arrow[r, "\mathsf{inc}"] & \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \arrow[l, bend right, "\Lambda/\Lambda e_1\Lambda \otimes_\Lambda -"'] \arrow[l, bend left, "\mathsf{Hom}_\Lambda(\Lambda/\Lambda e_1\Lambda, -)"] \arrow[r, "e_1(-)"] & \mathsf{Mod}\text{-}e_1\Lambda e_1 \arrow[l, bend right, "\Lambda e_1 \otimes_{e_1\Lambda e_1} -"'] \arrow[l, bend left, "\mathsf{Hom}_{e_1\Lambda e_1}(e_1\Lambda, -)"] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
\mathsf{Mod}\text{-}\Lambda/\Lambda e_2\Lambda \arrow[r, "\mathsf{inc}"] & \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \arrow[l, bend right, "\Lambda/\Lambda e_2\Lambda\otimes_\Lambda -"'] \arrow[l, bend left, "\mathsf{Hom}_\Lambda(\Lambda/\Lambda e_2\Lambda,-)"] \arrow[r, "e_2(-)"] & \mathsf{Mod}\text{-}e_2\Lambda e_2 \arrow[l, bend right, "\Lambda e_2\otimes_{e_2\Lambda e_2}-"'] \arrow[l, bend left, "\mathsf{Hom}_{e_2\Lambda e_2}(e_2\Lambda,-)"]
\end{tikzcd}$$

where $\mathsf{Mod}\text{-}\Lambda/\Lambda e_1\Lambda \simeq \mathsf{Mod}\text{-}S/\operatorname{Im}\phi$, $\mathsf{Mod}\text{-}e_1\Lambda e_1 \simeq \mathsf{Mod}\text{-}R$, $\mathsf{Mod}\text{-}\Lambda/\Lambda e_2\Lambda \simeq \mathsf{Mod}\text{-}R/\operatorname{Im}\psi$ and $\mathsf{Mod}\text{-}S \simeq \mathsf{Mod}\text{-}e_2\Lambda e_2$.

Typically rings of the form $\Lambda_{(\phi,\psi)}$ arise as endomorphism rings $\Lambda = \mathsf{End}_{\mathcal{M}}(X \oplus Y)$, where $X$ and $Y$ are arbitrary objects in an additive category $\mathcal{M}$: clearly then we have an isomorphism

$$\Lambda \simeq \begin{pmatrix} \mathsf{End}_{\mathcal{M}}(X) & \mathsf{Hom}_{\mathcal{M}}(X,Y) \\ \mathsf{Hom}_{\mathcal{M}}(Y,X) & \mathsf{End}_{\mathcal{M}}(Y) \end{pmatrix}$$

Then setting $e_X = \left(\begin{smallmatrix} \mathrm{Id}_X & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_Y = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Id}_Y \end{smallmatrix}\right)$, as above we obtain the recollements of abelian categories:

$$\begin{tikzcd}
\mathsf{Mod}\text{-}\mathsf{End}_{\mathcal{M}}(Y)/\operatorname{Im}\phi \arrow[r] & \mathsf{Mod}\text{-}\Lambda \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathsf{Mod}\text{-}\mathsf{End}_{\mathcal{M}}(X) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)/\operatorname{Im}\psi \arrow[r] & \mathsf{Mod}\text{-}\Lambda \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathsf{Mod}\text{-}\mathsf{End}_{\mathcal{M}}(Y) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

For a general study of Morita rings via extensions of abelian categories see Chapter 3. In Chapter 4 we study the representation theory of such rings under some restrictions.

The following example is a special case of the above recollement situation and arises in the Galois theory of noncommutative rings.

Example 1.1.9. (Skew Group Rings and Invariant Subrings) Let $R$ be a ring and $G$ be a finite group of automorphisms of $R$. Associated with $R$ and $G$ we have the fixed ring $R^G = \{r \in R \mid g(r) = r \text{ for every } g \in G\}$ and the skew group ring $R * G = \{\sum_{g\in G} r_g g \mid r_g \in R\}$ with multiplication given by the rule $(rg)(sh) = rg^{-1}(s)gh$ for every $r, s \in R$ and $g, h \in G$. Then the ring $R$ is a left and right $R^G$-module, $R$ is a left and right $R * G$-module and we have bimodule homomorphisms $\phi\colon R \otimes_{R^G} R \longrightarrow R * G$ and $\psi\colon R \otimes_{R*G} R \longrightarrow R^G$, see [88]. Then the Morita ring $\Lambda_{(\phi,\psi)} = \left(\begin{smallmatrix} R^G & R \\ R & R*G \end{smallmatrix}\right)$ is defined and the following diagrams:

$$\begin{tikzcd}
\mathsf{Mod}\text{-}R*G/\operatorname{Im}\phi \arrow[r] & \mathsf{Mod}\text{-}\left(\begin{smallmatrix} R^G & R \\ R & R*G \end{smallmatrix}\right) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathsf{Mod}\text{-}R^G \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Mod}\text{-}R^G/\operatorname{Im}\psi \arrow[r] & \mathsf{Mod}\text{-}\left(\begin{smallmatrix} R^G & R \\ R & R*G \end{smallmatrix}\right) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathsf{Mod}\text{-}R*G \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

are recollements of module categories by Example 1.1.8.

The homological theory of the next example was studied in [14].

Example 1.1.10. (Trace Ideals) Let $\Lambda$ be an Artin algebra and $\mathcal{U}$ a two-sided idempotent ideal in $\Lambda$, so $\mathcal{U} = \Lambda e\Lambda$ for some idempotent element of $\Lambda$. Then we know from [14] that $\mathcal{U} = \tau_P(\Lambda)$ for some projective $\Lambda$-module $P$, where $\tau_P(\Lambda)$ is the trace of $P$ in $\Lambda$. Recall that $\tau_P(\Lambda)$ is the ideal $\sum \operatorname{Im} f$ where $f$ ranges over $\mathsf{Hom}_\Lambda(P,\Lambda)$. Also associated with $\Lambda$ we have the Artin algebras $\Lambda/\mathfrak{U}$ and $\Gamma = \mathsf{End}_\Lambda(P)$. Then we have the following recollement of abelian categories:

$$\begin{tikzcd} \mathsf{mod}\text{-}\Lambda/\mathfrak{U} \arrow[r, "\mathsf{inc}"] & \mathsf{mod}\text{-}\Lambda \arrow[l, bend right, "\Lambda/\mathfrak{U}\otimes_\Lambda -"'] \arrow[l, bend left, "\mathsf{Hom}_\Lambda(\Lambda/\mathfrak{U},-)"] \arrow[r, "\mathsf{Hom}_\Lambda(P,-)"] & \mathsf{mod}\text{-}\Gamma \arrow[l, bend right, "P\otimes_\Gamma -"'] \arrow[l, bend left, "\mathsf{Hom}_\Gamma(\mathsf{Hom}_\Lambda(P,\Lambda),-)"] \end{tikzcd}$$

More generally, let $R$ be a ring and $P$ a finitely generated projective $R$-module. Then as above we have a recollement $(\mathsf{Ker}\,\mathsf{Hom}_R(P,-), \mathsf{Mod}\text{-}R, \mathsf{Mod}\text{-}\Gamma)$, where $\Gamma = \mathsf{End}_R(P)$. Note that $\mathsf{Ker}\,\mathsf{Hom}_R(P,-)$ is a module category: by a result of Auslander there exist a uniquely determined idempotent ideal $\underline{\alpha}$ of $R$ such that $\mathsf{Ker}\,\mathsf{Hom}_R(P,-) = \mathsf{Mod}\text{-}R/\underline{\alpha}$, see [12], where we refer for the explicit description of the ideal $\underline{\alpha}$.

The following example shows that the module category of a left Artinian ring can be placed always in the right part of a recollement.

EXAMPLE 1.1.11. (Artinian Rings) Let $\Lambda$ be a left Artinian ring. Then by a basic result of Auslander, see [10], there exists a semiprimary ring $\Gamma$ (i.e. the Jacobson radical $\mathfrak{r}_\Gamma$ is nilpotent and $\Gamma/\mathfrak{r}_\Gamma$ is semisimple) of finite global dimension and an idempotent element $e \in \Gamma$ such that $\Lambda \simeq e\Gamma e$. Therefore, we have the recollement $(\mathsf{Mod}\text{-}\Gamma/\Gamma e\Gamma, \mathsf{Mod}\text{-}\Gamma, \mathsf{Mod}\text{-}\Lambda)$.

EXAMPLE 1.1.12. (Comma Categories) Let $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ be a right exact functor between abelian categories. The objects of the *comma-category* $\mathscr{C} = (G, \mathscr{B}, \mathscr{A})$, denoted also as $\mathscr{C} = (\mathrm{Id} \downarrow G)$, are triples $(A, B, f)$ where $f\colon G(B) \longrightarrow A$ is a morphism in $\mathscr{A}$. A morphism $\gamma\colon (A, B, f) \longrightarrow (A', B', f')$ in $\mathscr{C}$ consists of two morphisms $\alpha\colon A \longrightarrow A'$ in $\mathscr{A}$ and $\beta\colon B \longrightarrow B'$ in $\mathscr{B}$ such that $f \circ \alpha = G(\beta) \circ f'$. It is well known that the comma category $\mathscr{C}$ is abelian since the functor $G$ is right exact. Assume that $G$ has a right adjoint $G'\colon \mathscr{A} \longrightarrow \mathscr{B}$ and let $\epsilon\colon GG' \longrightarrow \mathrm{Id}_\mathscr{A}$ be the counit and $\eta\colon \mathrm{Id}_\mathscr{B} \longrightarrow G'G$ be the unit of the adjoint pair $(G, G')$. We define the following functors:

(i) The functor $\mathsf{T}_\mathscr{B}\colon \mathscr{B} \longrightarrow \mathscr{C}$ is defined by $\mathsf{T}_\mathscr{B}(Y) = (G(Y), Y, \mathrm{Id}_{GY})$ on objects $Y \in \mathscr{B}$ and given a morphism $\beta\colon Y \longrightarrow Y'$ in $\mathscr{B}$ then $\mathsf{T}_\mathscr{B}(\beta) = (G(\beta), \beta)$ is a morphism in $\mathscr{C}$.

(ii) The functor $\mathsf{U}_\mathscr{B}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is defined on objects $(A, B, f) \in \mathscr{C}$ by $\mathsf{U}_\mathscr{B}(A, B, f) = B$ and given a morphism $(\alpha, \beta)\colon (A, B, f) \longrightarrow (A', B', f')$ in $\mathscr{C}$ then $\mathsf{U}_\mathscr{B}(\alpha, \beta) = \beta$ is a morphism in $\mathscr{B}$. Similarly we define the functor $\mathsf{U}_\mathscr{A}\colon \mathscr{C} \longrightarrow \mathscr{A}$.

(iii) The functor $\mathsf{Z}_\mathscr{B}\colon \mathscr{B} \longrightarrow \mathscr{C}$ is defined on objects $Y \in \mathscr{B}$ by $\mathsf{Z}_\mathscr{B}(Y) = (0, Y, 0)$ and given a morphism $\beta\colon Y \longrightarrow Y'$ in $\mathscr{B}$ then $\mathsf{Z}_\mathscr{B}(\beta) = (0, \beta)$ is a morphism in $\mathscr{C}$. Similarly we define the functor $\mathsf{Z}_\mathscr{A}\colon \mathscr{A} \longrightarrow \mathscr{C}$.

(iv) The functor $\mathsf{q}\colon \mathscr{C} \longrightarrow \mathscr{A}$ is defined on objects $(A, B, f) \in \mathscr{C}$ by $\mathsf{q}(A, B, f) = \mathsf{Coker}\, f$ and if $(\alpha, \beta)\colon (A, B, f) \longrightarrow (A', B', f')$ is a morphism in $\mathscr{C}$ then we have the induced morphism $\mathsf{q}(\alpha, \beta)\colon\ \mathsf{Coker}\, f \longrightarrow \mathsf{Coker}\, f'$.

(v) The functor $\mathsf{H}_\mathscr{A}\colon \mathscr{A} \longrightarrow \mathscr{C}$ is defined on objects $X \in \mathscr{A}$ by $\mathsf{H}_\mathscr{A}(X) = (X, G'(X), \epsilon_X)$ and given a morphism $\alpha\colon X \longrightarrow X'$ in $\mathscr{A}$ then $\mathsf{H}_\mathscr{A}(\alpha) = (\alpha, G'(\alpha))$ is a morphism in $\mathscr{C}$.

(vi) The functor $\mathsf{p}\colon \mathscr{C} \to \mathscr{B}$ is defined on objects $(A, B, f) \in \mathscr{C}$ by $\mathsf{p}(A, B, f) = \mathsf{Ker}\,(\eta_B \circ G'(f))$ and if $(\alpha, \beta)\colon (A, B, f) \longrightarrow (A', B', f')$ is a morphism in $\mathscr{C}$ then we get the morphism $\mathsf{p}(\alpha, \beta)\colon\ \mathsf{Ker}\,(\eta_B \circ G'(f)) \longrightarrow \mathsf{Ker}\,(\eta_{B'} \circ G'(f'))$.

It is easy to check that the above data define the following recollements:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{Z}_{\mathscr{A}}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{U}_{\mathscr{A}}"] \arrow[r, "\mathsf{U}_{\mathscr{B}}"] & \mathscr{B} \arrow[l, bend right, "\mathsf{T}_{\mathscr{B}}"'] \arrow[l, bend left, "\mathsf{Z}_{\mathscr{B}}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{B} \arrow[r, "\mathsf{Z}_{\mathscr{B}}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{U}_{\mathscr{B}}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{U}_{\mathscr{A}}"] & \mathscr{A} \arrow[l, bend right, "\mathsf{Z}_{\mathscr{A}}"'] \arrow[l, bend left, "\mathsf{H}_{\mathscr{A}}"] \end{tikzcd}$$

Example 1.1.13. (Subcategories) Let $\mathscr{C}$ be a skeletally small additive category and let $\mathcal{X}$ a full subcategory of $\mathscr{C}$. Let $\mathsf{i}\colon \mathcal{X} \longrightarrow \mathscr{C}$ be the inclusion functor. We denote by $\mathsf{Mod}\text{-}\mathscr{C}$ the category of additive functors $\mathscr{C}^{\mathsf{op}} \longrightarrow \mathfrak{Ab}$ to the category $\mathfrak{Ab}$ of abelian groups. It is well-known that the restriction functor $\mathsf{res}\colon \mathsf{Mod}\text{-}\mathscr{C} \longrightarrow \mathsf{Mod}\text{-}\mathcal{X}$, $G \mapsto \mathsf{res}\,(G) = G \circ \mathsf{i}$ is exact and admits a fully faithful left and a fully faithful right adjoint, see [12] and [79]. Its kernel $\mathsf{Ker}\,\mathsf{res}$ is identified with the full subcategory of $\mathsf{Mod}\text{-}\mathscr{C}$ consisting of all functors $F\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathfrak{Ab}$ vanishing on $\mathcal{X}$ and this last category is equivalent to the module category $\mathsf{Mod}\text{-}\mathscr{C}/\mathcal{X}$ over the stable category $\mathscr{C}/\mathcal{X}$. It follows from Remark 1.1.3 that the triple $(\mathsf{Mod}\text{-}\mathscr{C}/\mathcal{X}, \mathsf{Mod}\text{-}\mathscr{C}, \mathsf{Mod}\text{-}\mathcal{X})$ is a recollement of abelian categories. An interesting special case of the above recollement situation is the following. Let $\Lambda$ be an Artin algebra and let $\mathcal{X}$ be a full subcategory of $\mathsf{mod}\text{-}\Lambda$ containing the projectives. Then we have the recollement $(\mathsf{Mod}\text{-}\underline{\mathcal{X}}, \mathsf{Mod}\text{-}\mathcal{X}, \mathsf{Mod}\text{-}\Lambda)$ where $\underline{\mathcal{X}}$ is the stable category of $\mathcal{X}$ modulo projectives.

We continue with the following classical example of a recollement of abelian categories arising in Algebraic Geometry. We refer to [71], [72], and to the stacks project [122] for more details on sheaves over topological spaces.

Example 1.1.14. (Sheaves over topological spaces) Let $X$, $Y$ be two topological spaces and $f\colon X \longrightarrow Y$ a continuous map. Let $\mathcal{F}$ be a presheaf of sets on $X$. Then for every open subset $V$ of $Y$ the pushforward of $\mathcal{F}$ is defined by

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

where $f^{-1}(V)$ is an open subset of $X$ since the map $f$ is continuous. Then it is easy to see that we have a functor $f_*\colon \mathsf{PSh}(X) \longrightarrow \mathsf{PSh}(Y)$, where $\mathsf{PSh}(X)$, resp. $\mathsf{PSh}(Y)$, is the category of presheaves of sets on $X$, resp. $Y$. The functor $f_*$ has a left adjoint defined by

$$f_p\colon \mathsf{PSh}(Y) \longrightarrow \mathsf{PSh}(X),\ \mathcal{G} \ \mapsto \ f_p\mathcal{G}(U) = \mathsf{colim}_{f(U)\subseteq V}\mathcal{G}(V)$$

where the colimit runs over the open neighbourhoods $V$ of $f(U)$ in $Y$. Consider now a sheaf $\mathcal{F}$ of sets on $X$. We write $\mathsf{Sh}(X)$ for the category of sheaves of sets on $X$. Since $f$ is continuous it follows that the pushforward $f_*\mathcal{F}$ of $\mathcal{F}$ is a sheaf of sets on $Y$. Then we obtain a functor

$$\mathsf{r}\colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(Y),\ \mathcal{F} \ \mapsto \ \mathsf{r}\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

which has a left adjoint as well. If $\mathcal{G}$ is a sheaf of sets on $Y$, then the left adjoint is defined by

$$\mathsf{e}\colon \mathsf{Sh}(Y) \longrightarrow \mathsf{Sh}(X),\ \mathcal{G} \ \mapsto \ \mathsf{e}\mathcal{G} = (f_p\mathcal{G})^{\mathsf{Sheaf}}$$

where $(f_p\mathcal{G})^{\mathsf{Sheaf}}$ is the sheafification of the presheaf $f_p\mathcal{G}$ and is called the pullback sheaf. In other words the functor $\mathsf{e}$ is the following composition:

$$\begin{tikzcd} \mathsf{Sh}(X) \arrow[r, "\mathsf{inc}"] & \mathsf{PSh}(X) \arrow[l, bend right, "(-)^{\mathsf{Sheaf}}"'] \arrow[r, "f_*"] & \mathsf{PSh}(Y) \arrow[l, bend right, "f_p"'] & \mathsf{Sh}(Y) \arrow[l, "\mathsf{inc}"'] \arrow[lll, bend left, "\mathsf{e}=\mathsf{inc}\circ f_p\circ(-)^{\mathsf{Sheaf}}"] \end{tikzcd}$$

where sheafification $(-)^{\mathsf{Sheaf}}$ is a left adjoint of the inclusion functor $\mathsf{Sh}(X) \longrightarrow \mathsf{PSh}(X)$ and $f_p$ is the left adjoint of the pushforward $f_*$ on presheaves.

Let $X$ be a topological space as above and consider now $U$ an open subspace of $X$. Denote by $F$ the complement of $U$ in $X$. For simplicity we denote again by $\mathsf{Sh}(X)$ the category of sheaves of abelian groups on $X$. Recall that an abelian sheaf on $X$ is an abelian presheaf on $X$ such that the underlying presheaf of sets is a sheaf. In the setting of abelian sheaves we also have the pullback functor $\mathsf{e}\colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(U)$, the pushforward functor $\mathsf{r}\colon \mathsf{Sh}(U) \longrightarrow \mathsf{Sh}(X)$ and $(\mathsf{e}, \mathsf{r})$ is an adjoint pair. But moreover in this case $(U \subset X)$ the functor $\mathsf{e}$ has a simplest description. In particular we have

$$\mathsf{e}\colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(U), \mathcal{G} \ \mapsto \ \mathsf{e}\mathcal{G}(V) = \mathcal{G}(V)$$

for $V$ open subset of $U$, and the sheaf $\mathsf{e}\mathcal{G}$ is called the restriction of $\mathcal{G}$ to $U$. It turns out that for an open immersion $U \longrightarrow X$ the functor $\mathsf{e}$ has a left adjoint defined as follows, first on presheaves:

$$\mathsf{l}_p\colon \mathsf{PSh}(U) \longrightarrow \mathsf{PSh}(X), \ \mathcal{F} \ \mapsto \begin{cases} \mathcal{F}(V) & V \subset U \\ \\ 0 & V \not\subset U \end{cases}$$

and then if $\mathcal{F}$ is an abelian sheaf on $U$ we define $\mathsf{l}\colon \mathsf{Sh}(U) \longrightarrow \mathsf{Sh}(X)$ to be $(\mathsf{l}_p\mathcal{F})^{\mathsf{Sheaf}}$, i.e. the sheafification of the abelian presheaf $\mathsf{l}_p\mathcal{F}$. Thus so far we have the adjoint triple $(\mathsf{l}, \mathsf{e}, \mathsf{r})$:

$$\begin{tikzcd} \mathsf{Sh}(X) \arrow[r, "\mathsf{e}"] & \mathsf{Sh}(U) \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

and since $\mathsf{el} \simeq \mathrm{Id}_{\mathsf{Sh}(U)}$ we infer that the functors $\mathsf{l}$ and $\mathsf{r}$ are fully faithful. The kernel of the pullback functor $\mathsf{e}$ consists of all sheaves $\mathcal{F}$ of abelian groups on $X$ such that for every open subset $V$ of $U$ we have $\mathcal{F}(V) = 0$. Hence $\mathsf{Ker}\,\mathsf{e}$ consists of all sheaves of abelian groups on the complement of $U$, i.e. on $F = X\backslash U$. Then from Remark 1.1.3 we have the following recollement of abelian categories:

$$\begin{tikzcd} \mathsf{Sh}(F) \arrow[r, "\mathsf{i}"] & \mathsf{Sh}(X) \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathsf{Sh}(U) \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

Consider again a topological space $X$ and $U$ an open subset. Let $\mathcal{O}_X$ be a sheaf of commutative rings on $X$ and denote by $\mathcal{O}_U$, resp. $\mathcal{O}_F$, the restricted sheaves of rings on $U$, resp. $F$. Then the abelian categories $\mathsf{Mod}\text{-}\mathcal{O}_F$, $\mathsf{Mod}\text{-}\mathcal{O}_X$, $\mathsf{Mod}\text{-}\mathcal{O}_U$, of sheaves of modules are related via the following recollement:

$$\begin{tikzcd} \mathsf{Mod}\text{-}\mathcal{O}_F \arrow[r, "\mathsf{i}"] & \mathsf{Mod}\text{-}\mathcal{O}_X \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathsf{Mod}\text{-}\mathcal{O}_U \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

where the above functors are defined in a similar way. For more information on Sheaves on Spaces and the use of the above recollements in Algebraic Geometry we refer to the book [43], see also Chapter 6 of the Stacks project [122].

EXAMPLE 1.1.15. (Generic Representations, Kuhn [81–83]) Let $\mathbb{F}_q$ be a finite field of characteristic $p$ and order $|\mathbb{F}_q| = q = p^s$, where $p$ is a fixed prime. Denote by $M_k(\mathbb{F}_q)$ the multiplicative semigroup of $k \times k$ matrices over $\mathbb{F}_q$ and by $\mathsf{GL}_k(\mathbb{F}_q)$ the general linear group over $\mathbb{F}_q$. Consider the categories of modules $\mathsf{Mod}\text{-}\mathbb{F}_q[\mathsf{GL}_k(\mathbb{F}_q)]$ and $\mathsf{Mod}\text{-}\mathbb{F}_q[M_k(\mathbb{F}_q)]$, where $\mathbb{F}_q[\mathsf{GL}_k(\mathbb{F}_q)]$ is a group algebra and $\mathbb{F}_q[M_k(\mathbb{F}_q)]$ is a semigroup algebra. Then for $R = \mathbb{F}_q[M_k(\mathbb{F}_q)]$ and

$$e = \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}$$

we derive from Example 1.1.7 that the following diagram:

$$\begin{tikzcd} \mathsf{Mod}\text{-}\mathbb{F}_q[\mathsf{GL}_k(\mathbb{F}_q)] \arrow[r, "\mathsf{i}"] & \mathsf{Mod}\text{-}\mathbb{F}_q[M_k(\mathbb{F}_q)] \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathsf{Mod}\text{-}\mathbb{F}_q[M_{k-1}(\mathbb{F}_q)] \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

is a recollement of abelian categories. Note that the ideal $ReR$ consists of all noninvertible matrices and therefore $R/ReR = \mathbb{F}_q[\mathsf{GL}_k(\mathbb{F}_q)]$. We refer to Kuhn [81–83] for more details on the above recollements and their uses in generic representation theory.

The last example shows a natural way that recollements of abelian categories arise from recollements of triangulated categories.

EXAMPLE 1.1.16. (Recollements of Hearts, [22, Beilinson-Bernstein-Deligne]) Let $\mathsf{R}_{\mathsf{tr}}(\mathcal{U}, \mathcal{T}, \mathcal{V})$ be a recollement of triangulated categories, i.e. we have three triangulated categories $\mathcal{U}$, $\mathcal{T}$ and $\mathcal{V}$ and the following triangulated functors between them:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mathsf{i}"] & \mathcal{T} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathcal{V} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \qquad \mathsf{R}_{\mathsf{tr}}(\mathcal{U}, \mathcal{T}, \mathcal{V})$$

satisfying the analogues axioms as in the abelian case, that is $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ and $(\mathsf{q}, \mathsf{i}, \mathsf{p})$ are adjoint triples, the functors $\mathsf{i}$, $\mathsf{l}$, and $\mathsf{r}$ are fully faithful and $\mathsf{Im}\,\mathsf{i} = \mathsf{Ker}\,\mathsf{e}$, see Definition 5.1.1 for more details. Recall from [22] that a **t-structure** in a triangulated category $\mathcal{T}$ is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of full subcategories such that setting $\mathcal{T}^{\leq n} = \Sigma^{-n}(\mathcal{T}^{\leq 0})$ and $\mathcal{T}^{\geq n} = \Sigma^{-n}(\mathcal{T}^{\geq 0})$, $\forall n \in \mathbb{Z}$, the following conditions are satisfied:

(i) $\mathsf{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$, i.e. $\mathsf{Hom}_{\mathcal{T}}(X, Y) = 0 \quad \forall X \in \mathcal{T}^{\leq 0}$ and $\forall Y \in \mathcal{T}^{\geq 1}$.
(ii) $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$.
(iii) For every $D \in \mathcal{T}$ there exists a triangle $X \longrightarrow D \longrightarrow Y \longrightarrow \Sigma(X)$ such that $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$.

Then from [22] it follows that the **heart** $\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ of a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is an abelian category and we have the full embedding $\epsilon\colon \mathcal{H} \longrightarrow \mathcal{T}$.

Let $\mathsf{T}_{\mathcal{U}} = (\mathcal{U}^{\leq 0}, \mathcal{U}^{\geq 0})$ be a t-structure in $\mathcal{U}$ and $\mathsf{T}_{\mathcal{V}} = (\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ be a t-structure in $\mathcal{V}$. Then from [22] we have the $t$-structure $\mathsf{T}_{\mathcal{T}} = (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq})$ in $\mathcal{T}$ defined by

$$\mathcal{T}^{\leq 0} = \big\{x \in \mathcal{T} \mid \mathsf{e}(X) \in \mathcal{V}^{\leq 0} \ \text{ and } \ \mathsf{q}(X) \in \mathcal{U}^{\leq 0}\big\}$$

and

$$\mathcal{T}^{\geq 0} = \big\{x \in \mathcal{T} \mid \mathsf{e}(X) \in \mathcal{V}^{\geq 0} \ \text{ and } \ \mathsf{p}(X) \in \mathcal{U}^{\geq 0}\big\}$$

Consider the hearts $\mathcal{H}_{\mathcal{U}}$, $\mathcal{H}_{\mathcal{T}}$ and $\mathcal{H}_{\mathcal{V}}$ of the corresponding t-structures $\mathsf{T}_{\mathcal{U}}$, $\mathsf{T}_{\mathcal{T}}$ and $\mathsf{T}_{\mathcal{V}}$. Then we claim that the following diagram:

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{U}} \arrow[r, "\mathsf{i}'"] & \mathcal{H}_{\mathcal{T}} \arrow[l, bend right, "\mathsf{q}'"'] \arrow[l, bend left, "\mathsf{p}'"] \arrow[r, "\mathsf{e}'"] & \mathcal{H}_{\mathcal{V}} \arrow[l, bend right, "\mathsf{l}'"'] \arrow[l, bend left, "\mathsf{r}'"]
\end{tikzcd} \qquad \mathsf{R}_{\mathsf{ab}}(\mathcal{H}_{\mathcal{U}}, \mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{V}})$$

is a recollement of abelian categories. To show this we need to recall some basic facts about t-structures in triangulated categories.

Let $\mathcal{T}$ be a triangulated category with suspension functor $\Sigma$ and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a $t$-structure in $\mathcal{T}$. Associated to the t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ are the truncation functors $\tau^{\leq n}\colon \mathcal{T}^{\leq n} \longrightarrow \mathcal{T}$ and $\tau^{\geq n}\colon \mathcal{T}^{\geq n} \longrightarrow \mathcal{T}$. Let $D$ be an object of $\mathcal{T}$. Then from (iii) there is a triangle $X \longrightarrow D \longrightarrow Y \longrightarrow \Sigma(X)$ such that $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$. Thus we have the triangle $\Sigma^n(X) \longrightarrow \Sigma^n(D) \longrightarrow \Sigma^n(Y) \longrightarrow \Sigma^{n+1}(X)$ with $\Sigma^n(X) \in \mathcal{T}^{\leq 0}$ and $\Sigma^n(Y) \in \mathcal{T}^{\geq 1}$, and therefore we obtain the triangle:

$$X \longrightarrow D \longrightarrow Y \longrightarrow \Sigma(X)$$

with $X \in \mathcal{T}^{\leq n}$ and $Y \in \mathcal{T}^{\geq n+1}$. Let $D' \in \mathcal{T}$, $f\colon D \longrightarrow D'$ a morphism and $X' \longrightarrow D' \longrightarrow Y' \longrightarrow \Sigma(X')$ a triangle as above with $X' \in \mathcal{T}^{\leq n}$ and $Y' \in \mathcal{T}^{\geq n+1}$. Since $\mathsf{Hom}_{\mathcal{T}}(X, Y') = 0$ we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "g"] & D \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "h"] & \Sigma(X) \arrow[d, "\Sigma(g)"] \\
X' \arrow[r] & D' \arrow[r] & Y' \arrow[r] & \Sigma(X')
\end{tikzcd}$$

and since $\mathsf{Hom}_{\mathcal{T}}(X, \Sigma^{-1}(Y')) = 0$ it follows that the maps $g$ and $h$ are unique. Thus for $D' = D$ and $f = \mathrm{Id}_D$ we infer that the objects $X$ and $Y$ are unique up to a unique isomorphism. Then the truncations functors are defined on objects $D \in \mathcal{T}$ by $\tau^{\leq n}\colon \mathcal{T}^{\leq n} \longrightarrow \mathcal{T}$, $\tau^{\leq n}(D) = X$, and $\tau^{\geq n+1}\colon \mathcal{T}^{\geq n+1} \longrightarrow \mathcal{T}$, $\tau^{\geq n+1}(D) = Y$, and given a morphism $f\colon D \longrightarrow D'$ then $\tau^{\leq n}(f) = g$ and $\tau^{\geq n+1}(f) = h$. Hence for any object $D$ of $\mathcal{T}$ there is a triangle

$$\tau^{\leq n}(D) \longrightarrow D \longrightarrow \tau^{\geq n+1}(D) \longrightarrow \Sigma(\tau^{\leq n}(D))$$

where $\tau^{\leq n}(D) \in \mathcal{T}^{\leq n}$, $\tau^{\geq n+1}(D) \in \mathcal{T}^{\geq n+1}$ and the morphism $\tau^{\geq n+1}(D) \longrightarrow \Sigma(\tau^{\leq n}(D))$ is uniquely determined. Note that the truncation functor $\tau^{\leq n}$ is a right adjoint of the inclusion functor $\mathcal{T}^{\leq n} \longrightarrow \mathcal{T}$ and also the truncation functor $\tau^{\geq n+1}$ is a left adjoint of the inclusion functor $\mathcal{T}^{\geq n+1} \longrightarrow \mathcal{T}$. To proceed we need the following functor:

$$\mathsf{H}^0\colon \mathcal{T} \longrightarrow \mathcal{H}_{\mathcal{T}} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}, \ X \ \mapsto \mathsf{H}^0(X) = \tau^{\geq 0}\tau^{\leq 0}(X) \simeq \tau^{\leq 0}\tau^{\geq 0}(X)$$

It is not difficult to check that the object $\mathsf{H}^0(X) = \tau^{\geq 0}\tau^{\leq 0}(X)$ lies in the heart $\mathcal{H}_{\mathcal{T}}$ and the functors $\tau^{\geq 0}\tau^{\leq 0}$ and $\tau^{\leq 0}\tau^{\geq 0}$ are isomorphic.

Let $\mathcal{T}$ and $\mathcal{V}$ be two triangulated categories with t-structures $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ respectively. A functor $\mathsf{e}\colon \mathcal{T} \longrightarrow \mathcal{V}$ is called

(i) left t-exact if $\mathsf{e}(\mathcal{T}^{\geq 0}) \subseteq \mathcal{V}^{\geq 0}$,
(ii) right t-exact if $\mathsf{e}(\mathcal{T}^{\leq 0}) \subseteq \mathcal{V}^{\leq 0}$ and
(iii) t-exact if $\mathsf{e}$ is both left t-exact and right t-exact.

Assume that the functor $\mathsf{e}$ has a right adjoint $\mathsf{r}\colon \mathcal{V} \longrightarrow \mathcal{T}$. Then from [22] the functor $\mathsf{e}$ is right t-exact if and only if $\mathsf{r}$ is left t-exact and moreover if these conditions are satisfied then we have the adjoint pair $(\mathsf{H}^0_{\mathcal{V}} \circ \mathsf{e} \circ \epsilon_{\mathcal{T}}, \mathsf{H}^0_{\mathcal{T}} \circ \mathsf{r} \circ \epsilon_{\mathcal{V}})$ between the hearts $\mathcal{H}_{\mathcal{T}}$ and $\mathcal{H}_{\mathcal{V}}$. For more details on t-structures we refer to [22], [57], [72, Chapter X] and the lecture notes of Milicic [93].

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "\mathsf{i}"] \arrow[d, "\mathsf{H}^0_{\mathcal{U}}"'] & & \mathcal{T} \arrow[ll, bend right, "\mathsf{q}"'] \arrow[ll, bend left, "\mathsf{p}"] \arrow[rr, "\mathsf{e}"] \arrow[d, "\mathsf{H}^0_{\mathcal{T}}"'] & & \mathcal{V} \arrow[ll, bend right, "\mathsf{l}"'] \arrow[ll, bend left, "\mathsf{r}"] \arrow[d, "\mathsf{H}^0_{\mathcal{V}}"'] \\
\mathcal{H}_{\mathcal{U}} \arrow[u, bend right, "\epsilon_{\mathcal{U}}"'] \arrow[rr, "\mathsf{i}'"] & & \mathcal{H}_{\mathcal{T}} \arrow[u, bend right, "\epsilon_{\mathcal{T}}"'] \arrow[ll, bend right, "\mathsf{q}'"'] \arrow[ll, bend left, "\mathsf{p}'"] \arrow[rr, "\mathsf{e}'"] & & \mathcal{H}_{\mathcal{V}} \arrow[u, bend right, "\epsilon_{\mathcal{V}}"'] \arrow[ll, bend right, "\mathsf{l}'"'] \arrow[ll, bend left, "\mathsf{r}'"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{R}_{\mathsf{tr}}(\mathcal{U}, \mathcal{T}, \mathcal{V}) \arrow[d, "\mathsf{H}^0"'] \\
\mathsf{R}_{\mathsf{ab}}(\mathcal{H}_{\mathcal{U}}, \mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{V}}) \arrow[u, bend right, "\epsilon"']
\end{tikzcd}$$

where the functors of $\mathsf{R}_{\mathsf{ab}}(\mathcal{H}_{\mathcal{U}}, \mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{V}})$ are defined as follows:

$$\mathsf{q}' = \mathsf{H}^0_{\mathcal{U}} \circ \mathsf{q} \circ \epsilon_{\mathcal{T}} \qquad \mathsf{l}' = \mathsf{H}^0_{\mathcal{T}} \circ \mathsf{l} \circ \epsilon_{\mathcal{V}}$$

$$\mathsf{i}' = \mathsf{H}^0_{\mathcal{T}} \circ \mathsf{i} \circ \epsilon_{\mathcal{U}} \qquad \mathsf{e}' = \mathsf{H}^0_{\mathcal{V}} \circ \mathsf{e} \circ \epsilon_{\mathcal{T}}$$

$$\mathsf{p}' = \mathsf{H}^0_{\mathcal{U}} \circ \mathsf{p} \circ \epsilon_{\mathcal{T}} \qquad \mathsf{r}' = \mathsf{H}^0_{\mathcal{T}} \circ \mathsf{r} \circ \epsilon_{\mathcal{V}}$$

We have:

(i) $(\mathsf{l}', \mathsf{e}', \mathsf{r}')$ is an adjoint triple.
From the above and since $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ is an adjoint triple, it remains to show that $\mathsf{l}$ is right t-exact, $\mathsf{e}$ is left t-exact or right t-exact and $\mathsf{r}$ is left t-exact. Let $X \in \mathcal{V}^{\leq 0}$. Then $\mathsf{l}(X) \in \mathcal{T}^{\leq 0}$ since $\mathsf{e}(\mathsf{l}(X)) \simeq X \in \mathcal{V}^{\leq 0}$ and $\mathsf{q}(\mathsf{l}(X)) = 0 \in \mathcal{U}^{\leq 0}$. Thus $\mathsf{l}$ is right t-exact and in the same way we get that $\mathsf{e}$ and $\mathsf{r}$ are left t-exact.
(ii) $(\mathsf{q}', \mathsf{i}', \mathsf{p}')$ is an adjoint triple.
Similarly as in (i) our statement follows.
(iii) The functors $\mathsf{i}'$, $\mathsf{l}'$ and $\mathsf{r}'$ are fully faithful.

We first show that the functor $\mathsf{l}'\colon \mathcal{H}_{\mathcal{V}} \longrightarrow \mathcal{H}_{\mathcal{T}}$ is fully faithful, equivalently the unit $\mathrm{Id}_{\mathcal{H}_{\mathcal{V}}} \longrightarrow \mathsf{e}'\mathsf{l}'$ is an isomorphism. Let $Y$ be an object of $\mathcal{H}_{\mathcal{V}}$. Then we have

$$\begin{array}{rcll}
\mathsf{e}'\mathsf{l}'(Y) & = & \mathsf{H}^0_{\mathcal{V}}\,\mathsf{e}\,\epsilon_{\mathcal{T}}\,\mathsf{H}^0_{\mathcal{T}}\,\mathsf{l}\,\epsilon_{\mathcal{V}}(Y) & \\
& = & \mathsf{H}^0_{\mathcal{V}}\,\mathsf{e}\,\epsilon_{\mathcal{T}}\,\mathsf{H}^0_{\mathcal{T}}(\mathsf{l}(Y)) & \\
& = & \mathsf{H}^0_{\mathcal{V}}\,\mathsf{e}\,\epsilon_{\mathcal{T}}(\tau^{\geq 0}(\mathsf{l}(Y))) & \mathsf{l}\colon\ \text{right t-exact} \\
& = & \mathsf{H}^0_{\mathcal{V}}\,\mathsf{e}\,\tau^{\geq 0}(\mathsf{l}(Y)) & \\
& \simeq & \mathsf{H}^0_{\mathcal{V}}\,\tau^{\geq 0}\,\mathsf{e}\,\mathsf{l}(Y) & \mathsf{e}\colon\ \text{t-exact} \\
& \simeq & \mathsf{H}^0_{\mathcal{V}}\,\tau^{\geq 0}(Y) & \\
& = & Y &
\end{array}$$

Thus the functor $\mathsf{l}'$ is fully faithful. Then since $(\mathsf{l}', \mathsf{e}', \mathsf{r}')$ is an adjoint triple we infer that the functor $\mathsf{r}'$ is fully faithful as well. Similarly, using that the functor $\mathsf{i}\colon \mathcal{U} \longrightarrow \mathcal{T}$ is t-exact we derive an isomorphism $\mathsf{q}'\mathsf{i}'(X) \simeq X$ for every $X \in \mathcal{H}_{\mathcal{U}}$. Hence the functor $\mathsf{i}'\colon \mathcal{H}_{\mathcal{U}} \longrightarrow \mathcal{H}_{\mathcal{T}}$ is fully faithful.

(iv) $\mathsf{Im}\,\mathsf{i}' = \mathsf{Ker}\,\mathsf{e}'$.
Let $X \in \mathcal{H}_{\mathcal{U}}$. Since the functor $\mathsf{i}$ is t-exact it follows that $\mathsf{i}(X) \in \mathcal{H}_{\mathcal{T}}$. Thus $\mathsf{i}' = \mathsf{i}$ restricting on the heart $\mathcal{H}_{\mathcal{U}}$ and similarly since $\mathsf{e}$ is t-exact we have $\mathsf{e}' = \mathsf{e}$ on the heart $\mathcal{H}_{\mathcal{T}}$. Then $\mathsf{i}' \circ \mathsf{e}' = \mathsf{i} \circ \mathsf{e} = 0$.

Finally, after the above, we infer that $\mathsf{R}_{\mathsf{ab}}(\mathcal{H}_{\mathcal{U}}, \mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{V}})$ is a recollement of abelian categories, called the recollement of hearts.

## 1.2. Recollements of Abelian Categories and TTF-triples

Throughout, $\mathscr{A}$ denotes an abelian category and all subcategories considered are strict (i.e. closed under isomorphisms). In this section we discuss some aspects of TTF-triples in abelian categories and our aim is to establish a correspondence between recollements of abelian categories and certain TTF-triples. We start by defining torsion pairs.

Definition 1.2.1. [42, 121] A **torsion pair** in $\mathscr{A}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories satisfying the following conditions:

(i) $\mathsf{Hom}_{\mathscr{A}}(\mathcal{X}, \mathcal{Y}) = 0$, i.e. $\mathsf{Hom}_{\mathscr{A}}(X, Y) = 0\ \forall X \in \mathcal{X}, \forall Y \in \mathcal{Y}$;

(ii) For every object $A \in \mathscr{A}$, there are objects $X_A$ in $\mathcal{X}$ and $Y^A$ in $\mathcal{Y}$ and a short exact sequence

$$0 \longrightarrow X_A \longrightarrow A \longrightarrow Y^A \longrightarrow 0$$

Given a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathscr{A}$, we say that $\mathcal{X}$ is a **torsion class** and $\mathcal{Y}$ is a **torsion-free class**. It follows easily from the definition that

$$\begin{array}{rcl}
\mathcal{X} & = & {}^{\circ}\mathcal{Y} := \{A \in \mathscr{A} : \mathsf{Hom}_{\mathscr{A}}(A, Y) = 0, \forall Y \in \mathcal{Y}\} \\
\mathcal{Y} & = & \mathcal{X}^{\circ} := \{A \in \mathscr{A} : \mathsf{Hom}_{\mathscr{A}}(X, A) = 0, \forall X \in \mathcal{X}\}
\end{array}$$

and that the assignment $\mathsf{R}_{\mathcal{X}}(A) = X_A$ (respectively, $\mathsf{L}_{\mathcal{Y}}(A) = Y^A$) yields an additive functor $\mathsf{R}_{\mathcal{X}}\colon \mathscr{A} \longrightarrow \mathcal{X}$ (respectively, $\mathsf{L}_{\mathcal{Y}}\colon \mathscr{A} \longrightarrow \mathcal{Y}$) which is right (respectively, left) adjoint of the inclusion functor $\mathsf{i}_{\mathcal{X}}\colon \mathcal{X} \longrightarrow \mathscr{A}$ (respectively, $\mathsf{i}_{\mathcal{Y}}\colon \mathcal{Y} \longrightarrow \mathscr{A}$). Hence, $\mathcal{X}$ (respectively, $\mathcal{Y}$) is a reflective (respectively, coreflective) subcategory of $\mathscr{A}$. Moreover, the endofunctors $\mathsf{i}_{\mathcal{X}}\mathsf{R}_{\mathcal{X}}$ and $\mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}$ satisfy:

- $\mathsf{i}_{\mathcal{X}}\mathsf{R}_{\mathcal{X}}$ is a **radical functor**, i.e., there is $\mu\colon \mathsf{i}_{\mathcal{X}}\mathsf{R}_{\mathcal{X}} \longrightarrow \mathrm{Id}_{\mathscr{A}}$ a natural transformation such that $\mu_A$ is a monomorphism and $\mathsf{i}_{\mathcal{X}}\mathsf{R}_{\mathcal{X}}(\mathsf{Coker}\,\mu_A) = 0$.

- $\mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}$ is a **coradical functor**, i.e., there is $\nu\colon \mathrm{Id}_{\mathscr{A}} \longrightarrow \mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}$ a natural transformation such that $\mu_A$ is an epimorphism and $\mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}(\mathsf{Ker}\,\nu_A) = 0$.
- Both $\mathsf{i}_{\mathcal{X}}\mathsf{R}_{\mathcal{X}}$ and $\mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}$ are **idempotent**, i.e., both $\mu_{\mathsf{i}_{\mathcal{X}}\mathsf{R}_{\mathcal{X}}(A)}$ and $\nu_{\mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}(A)}$ are isomorphisms, for all $A$ in $\mathscr{A}$.

In fact, there are bijections between torsion pairs in $\mathscr{A}$, idempotent radical functors $F\colon \mathscr{A} \longrightarrow \mathscr{A}$ and idempotent coradical functors $G\colon \mathscr{A} \longrightarrow \mathscr{A}$. Thus, the endofunctors $\mathsf{i}_{\mathcal{X}}\mathsf{R}_{\mathcal{X}}$ and $\mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}$ determine the torsion pair uniquely, for more details see [**42**, Theorem 2.8], [**100**, Theorem 1.2].

We continue by illustrating the notion of torsion pair with two standard examples in the category of abelian groups.

Example 1.2.2. Consider the category of abelian groups $\mathsf{Mod}\text{-}\mathbb{Z}$ and let $\mathcal{T}$ (respectively, $\mathcal{F}$) denote its full subcategory of abelian groups whose elements have finite (respectively, infinite) order. Let $G$ be an abelian group and denote by $T(G) = \{g \in G \mid \exists n \in \mathbb{N} : g^n = e\}$ its torsion subgroup, i.e. the subgroup of $G$ formed by its elements of finite order. Note that $T(G)$ is a normal subgroup of $G$ and the quotient group $F(G) = G/T(G)$ is the torsion-free quotient group of $G$. Then we have the following exact sequence of abelian groups:

$$0 \longrightarrow T(G) \longrightarrow G \longrightarrow F(G) \longrightarrow 0$$

and clearly $\mathsf{Hom}_{\mathbb{Z}}(\mathcal{T},\mathcal{F}) = 0$. Thus $(\mathcal{T},\mathcal{F})$ is a torsion pair in $\mathsf{Mod}\text{-}\mathbb{Z}$. Moreover, the right adjoint $R_{\mathcal{T}}$ to the inclusion functor $i_{\mathcal{T}}$ is given by associating to an abelian group its torsion subgroup. It is clear that $i_{\mathcal{T}}R_{\mathcal{T}}$ is an idempotent radical functor and that it determines $\mathcal{T}$ as the class of objects $X$ such that $i_{\mathcal{T}}R_{\mathcal{T}}(X) \cong X$.

More generally, for a domain $R$ and $M$ an $R$-module consider its torsion submodule:

$$tM = \{m \in M \mid rm = 0 \ \text{ for some nonzero } \ r \in R\}$$

Then $M$ is a torsion module if $tM = M$ and is torsion-free if $tM = \{0\}$. Note that if $R$ is not a domain, $tM$ is not necessary a submodule of $M$. We denote by $\mathcal{X}_{\mathsf{t}}$ the full subcategory of $\mathsf{Mod}\text{-}R$ consisting of all torsion $R$-modules and by $\mathcal{Y}_{\mathsf{tf}}$ the full subcategory of $\mathsf{Mod}\text{-}R$ formed by the torsion-free $R$-modules. Then it is easy to see that for every $R$-module $M$ there is an exact sequence $0 \longrightarrow tM \longrightarrow M \longrightarrow M/tM \longrightarrow 0$ where $tM \in \mathcal{X}_{\mathsf{t}}$ and $M/tM \in \mathcal{Y}_{\mathsf{tf}}$, and $\mathsf{Hom}_R(X,Y) = 0$ for every $X \in \mathcal{X}_{\mathsf{t}}$ and $Y \in \mathcal{Y}_{\mathsf{tf}}$. Hence $(\mathcal{X}_{\mathsf{t}},\mathcal{Y}_{\mathsf{tf}})$ is a torsion pair in $\mathsf{Mod}\text{-}R$.

Example 1.2.3. Let $\mathsf{Mod}\text{-}\mathbb{Z}$ be the category of abelian groups. Let $\mathscr{D}$ be the full subcategory of divisible groups and $\mathcal{R}$ the full subcategory of reduced groups. Recall that an abelian group $G$ is called divisible if for every $x \in G$ and every positive integer $n$ there exists an element $y \in G$ such that $ny = x$. Typical examples of divisible abelian groups are the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$. Another example of a divisible group is $\mathbb{Z}/p^{\infty}$. Denote by $D(G)$ the subgroup of $G$ generated by all divisible subgroups of $G$. Note that $D(G)$ is divisible subgroup of $G$ and $G$ is called reduced if $D(G) = 0$. We will show that $(\mathscr{D},\mathcal{R})$ is a torsion pair in $\mathsf{Mod}\text{-}\mathbb{Z}$.

(i) Let $f\colon D \longrightarrow R$ be a homomorphism of abelian groups with $D \in \mathscr{D}$ and $R \in \mathcal{R}$. First note that $\mathsf{Im}\,f$ is divisible. Indeed, let $y = f(x)$ and $n > 0$. Since $x \in D$ and $D$ is divisible there exists an element $d \in D$ such that $nd = x$. Then $nf(d) = f(nd) = f(x) = y$ and thus $\mathsf{Im}\,f$ is a divisible subgroup of $R$. But since $R$ is reduced it follows that $f = 0$. Hence $\mathsf{Hom}_{\mathbb{Z}}(\mathscr{D},\mathcal{R}) = 0$.

(ii) Recall that a group $G$ is divisible if and only if it is an injective object in the category of abelian groups. This implies easily that any divisible subgroup $D$ of $G$ is a direct summand of $G$. Consider the divisible subgroup $D(G)$ of $G$. Then we have

$$G = D(G) \oplus H$$

and we claim that $H$ is reduced. Since the divisible subgroup $D(H)$ of $H \leq G$ is contained in $D(G)$ and $D(G) \cap H = 0$ it follows that $D(H) = 0$, i.e. $H$ is reduced. Thus every group $G$ splits as a direct sum $G = D(G) \oplus H$ where $D(G)$ is the divisible subgroup of $G$ and $H$ is a reduced subgroup of $G$. Then we have the short exact sequence of abelian groups:

$$0 \longrightarrow D(G) \longrightarrow G \longrightarrow H \longrightarrow 0$$

Hence $(\mathscr{D}, \mathcal{R})$ is a torsion pair in $\mathsf{Mod}\text{-}\mathbb{Z}$. In fact $(\mathscr{D}, \mathcal{R})$ is a split torsion pair since for every abelian group $G$ the torsion subobject $D(G)$ of $G$ is a direct summand of $G$.

Often, torsion and torsion-free classes can be identified by closure properties. Recall that $\mathscr{A}$ is said to be **well-powered** ( [121] ) if the class of subobjects of any given object forms a set.

PROPOSITION 1.2.4. *[42, Theorem 2.3] Let $\mathscr{A}$ be a well-powered, complete and cocomplete abelian category. A full subcategory $\mathcal{X}$ is a torsion (respectively, torsion-free) class if and only if it is closed under quotients, extensions and coproducts (respectively, subobjects, extensions and products).*

Recall from [121] that a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathscr{A}$ is **hereditary** if $\mathcal{X}$ is closed under subobjects and **cohereditary** if $\mathcal{Y}$ is closed under quotients. We will be interested in classes which are both torsion and torsion-free.

DEFINITION 1.2.5. [121] A triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories of $\mathscr{A}$ is called a **torsion torsion-free triple** (**TTF-triple**, for short) (and $\mathcal{Y}$ is a **TTF-class**) if $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs.

Clearly, if $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF-triple in $\mathscr{A}$, then the torsion pair $(\mathcal{X}, \mathcal{Y})$ is cohereditary and $(\mathcal{Y}, \mathcal{Z})$ is hereditary. By Proposition 1.2.4, when $\mathscr{A}$ is well-powered, complete and cocomplete, a full subcategory $\mathcal{Y}$ of $\mathscr{A}$ is a TTF-class if and only if it is closed under products, coproducts, extensions, subobjects and quotients. We refer to [30] for further details on torsion theories and TTF-triples in both abelian and triangulated categories. In ring theory, TTF-triples are well understood due to the following result of Jans, which establish a bijection between TTF-triples and idempotent ideals and it will be proved in section 1.3 (Proposition 1.3.4) using our results on TTF-triples of abelian categories. Note that by an idempotent ideal of a ring $A$, we mean a two-sided ideal $I$ of $A$ such that $I^2 = I$.

THEOREM 1.2.6. *[69, Corollary 2.2] There is a bijection between TTF-triples in $\mathsf{Mod}\text{-}A$ and idempotent ideals of the ring $A$.*

We start with an adaptation of the classical bijection between torsion pairs and idempotent radicals to TTF-triples. This will, later, yield a proof for Jans' correspondence of Theorem 1.2.6.

PROPOSITION 1.2.7. *Let $\mathscr{A}$ be a well-powered, complete and cocomplete abelian category $\mathscr{A}$. There are bijections between the following classes:*

(i) *TTF-triples in $\mathscr{A}$;*

(ii) *Left exact radical functors $F\colon \mathscr{A} \longrightarrow \mathscr{A}$ preserving products;*
(iii) *Right exact coradical functors $G\colon \mathscr{A} \longrightarrow \mathscr{A}$ preserving coproducts.*

PROOF. We show a bijection between (i) and (ii) (a bijection with (iii) can be obtained dually). Let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a $\mathsf{TTF}$-triple in $\mathscr{A}$. The $\mathsf{TTF}$-class $\mathcal{Y}$ is clearly bireflective and, hence, the inclusion functor $\mathsf{i}_\mathcal{Y}$ of $\mathcal{Y}$ in $\mathscr{A}$ admits a left adjoint $\mathsf{L}_\mathcal{Y}$ and a right adjoint $\mathsf{R}_\mathcal{Y}$. Thus, $\mathsf{i}_\mathcal{Y}$ is exact and $\mathsf{i}_\mathcal{Y}\mathsf{R}_\mathcal{Y}$ is a left exact radical functor preserving products. We define a correspondence:

$$\Phi\colon (\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \longmapsto (\mathsf{i}_\mathcal{Y}\mathsf{R}_\mathcal{Y}\colon \mathscr{A} \longrightarrow \mathscr{A}).$$

Given a left exact radical functor $F\colon \mathscr{A} \longrightarrow \mathscr{A}$ preserving products, it is easy to see that $F$ is idempotent and $(\mathcal{Y}_F := \{A \in \mathscr{A} \mid F(A) = A\}, \mathcal{Y}_F^\circ = \mathsf{Ker}\, F)$ is a hereditary torsion pair (see also [121, Proposition VI.1.7]). Since $F$ preserves products, $\mathcal{Y}_F$ is closed under products and thus, Proposition 1.2.4 shows that $\mathcal{Y}_F$ is a $\mathsf{TTF}$-class. Hence, we can associate a $\mathsf{TTF}$-triple to $F$ as follows.

$$\Psi\colon F \longmapsto ({}^\circ\mathcal{Y}_F, \mathcal{Y}_F, \mathcal{Y}_F^\circ)$$

Finally, it easily follows that $\Phi$ and $\Psi$ are inverse correspondences. □

REMARK 1.2.8. To be left exact and to preserve products is the same as to commute with limits and dually to be right exact and to preserve coproducts is the same as to commute with colimits, see [94].

For a subcategory $\mathcal{Y}$ of $\mathscr{A}$ closed under subobjects, quotients and extensions, Gabriel constructed in [54] an abelian category $\mathscr{A}/\mathcal{Y}$ with morphisms

$$\mathsf{Hom}_{\mathscr{A}/\mathcal{Y}}(\mathsf{e}(M), \mathsf{e}(N)) = \varinjlim_{\substack{N' \leq N : N' \in \mathcal{Y} \\ M' \leq M : M/M' \in \mathcal{Y}}} \mathsf{Hom}_{\mathscr{A}}(M', N/N'). \tag{1.2.1}$$

Such a subcategory $\mathcal{Y}$ is called a **Serre subcategory** and it yields an exact and essentially surjective **quotient functor** $\mathsf{e}\colon \mathscr{A} \longrightarrow \mathscr{A}/\mathcal{Y}$. A Serre subcategory $\mathcal{Y}$ is said to be **localising** (respectively, **colocalising**) if $j^*$ admits a right (respectively, left) adjoint. Moreover, it is said to be **bilocalising** if it is both localising and colocalising. These properties are related to the structure of subcategories orthogonal to $\mathcal{Y}$ with respect to the pairings $\mathsf{Hom}_{\mathscr{A}}(-,-)$ and $\mathsf{Ext}^1_{\mathscr{A}}(-,-)$ (in the sense of Yoneda), i.e.,

$${}^\perp\mathcal{Y} := \{A \in \mathscr{A} : \mathsf{Hom}_{\mathscr{A}}(A, Y) = 0 = \mathsf{Ext}^1_{\mathscr{A}}(A, Y), \forall Y \in \mathcal{Y}\} \text{ and}$$

$$\mathcal{Y}^\perp := \{A \in \mathscr{A} : \mathsf{Hom}_{\mathscr{A}}(Y, A) = 0 = \mathsf{Ext}^1_{\mathscr{A}}(Y, A), \forall Y \in \mathcal{Y}\}.$$

THEOREM 1.2.9. *[56, Lemma 2.1, Proposition 2.2] The following hold for a Serre subcategory $\mathcal{Y}$ of $\mathscr{A}$.*
(i) *The quotient functor $\mathsf{e}$ induces fully faithful functors $\mathcal{Y}^\perp \longrightarrow \mathscr{A}/\mathcal{Y}$ and ${}^\perp\mathcal{Y} \longrightarrow \mathscr{A}/\mathcal{Y}$.*
(ii) *The functor $\mathsf{e}\colon \mathcal{Y}^\perp \longrightarrow \mathscr{A}/\mathcal{Y}$ is an equivalence if and only if $\mathcal{Y}$ is localising, in which case a quasi-inverse for $\mathsf{e}$ is its right adjoint $\mathsf{r}$.*
(iii) *The functor $\mathsf{e}\colon {}^\perp\mathcal{Y} \longrightarrow \mathscr{A}/\mathcal{Y}$ is an equivalence if and only if $\mathcal{Y}$ is colocalising, in which case a quasi-inverse for $\mathsf{e}$ is its left adjoint $\mathsf{l}$.*

Localisations and colocalisations with respect to a torsion pair $(\mathcal{X},\mathcal{Y})$ in $\mathscr{A}$ first appeared in [54] (see also [100], [121]). As in [100], we say that $(\mathcal{X},\mathcal{Y})$ is **strongly hereditary**, (respectively **strongly cohereditary**), if there is a functor $\mathsf{L}\colon \mathscr{A}\longrightarrow\mathscr{A}$ (respectively, $\mathsf{C}\colon \mathscr{A}\longrightarrow\mathscr{A}$), the **localisation**, (respectively the **colocalisation**) **functor with respect to** $(\mathcal{X},\mathcal{Y})$, and a natural transformation $\phi\colon \mathrm{Id}_{\mathscr{A}}\longrightarrow \mathsf{L}$ (respectively, $\psi\colon \mathsf{C}\longrightarrow \mathrm{Id}_{\mathscr{A}}$) such that, for all $A$ in $\mathscr{A}$:

(i) $\mathsf{Ker}\,\phi_A, \mathsf{Coker}\,\phi_A\in\mathcal{X}$ (respectively, $\mathsf{Ker}\,\psi_A, \mathsf{Coker}\,\psi_A\in\mathcal{Y}$);
(ii) $\mathsf{L}(A)\in\mathcal{Y}$ (respectively, $\mathsf{C}(A)\in\mathcal{X}$);
(iii) $\mathsf{L}(A)$ is **$\mathcal{X}$-divisible** (respectively, $\mathsf{C}(A)$ is **$\mathcal{Y}$-codivisible**), meaning that the functor $\mathsf{Hom}_{\mathscr{A}}(-,\mathsf{L}(A))$ (respectively, $\mathsf{Hom}_{\mathscr{A}}(\mathsf{C}(A),-)$) is exact on exact sequences, $0\to K\to M\to N\to 0$, with $N\in\mathcal{X}$ (respectively, $K\in\mathcal{Y}$).

The embedding in $\mathscr{A}$ of $\mathsf{Im}\,\mathsf{L}$, the **Giraud subcategory** of $\mathscr{A}$ associated with $(\mathcal{X},\mathcal{Y})$, admits an exact left adjoint such that $\mathsf{L}$ is given by the composition of the functors and $\phi$ is the unit of this adjunction ( [100]). Also, $\mathsf{Im}\,\mathsf{L}$ is the full subcategory of $\mathcal{X}$-divisible objects of $\mathcal{Y}$. Dual statements holds for $\mathsf{Im}\,\mathsf{C}$.

Definition 1.2.10. We say that a $\mathsf{TTF}$-triple $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ in $\mathscr{A}$ is **strong** if $(\mathcal{X},\mathcal{Y})$ is strongly cohereditary and $(\mathcal{Y},\mathcal{Z})$ is strongly hereditary.

If $\mathscr{A}$ has enough projectives (respectively, injectives), then by [100, Theorem 1.8-1.8*], a torsion pair is cohereditary (respectively, hereditary) if and only if it is strongly cohereditary (respectively, strongly hereditary)

The next result identifies $\mathsf{TTF}$-classes which are localising and colocalising.

Lemma 1.2.11. *Let $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ be a $\mathsf{TTF}$-triple in $\mathscr{A}$. Then $\mathcal{Y}$ is a localising subcategory of $\mathscr{A}$ if and only if $(\mathcal{Y},\mathcal{Z})$ is a strongly hereditary torsion pair. Dually, $\mathcal{Y}$ is a colocalising subcategory if and only if $(\mathcal{X},\mathcal{Y})$ is a strongly cohereditary torsion pair.*

Proof. It is enough to prove the first statement. Suppose that the torsion pair $(\mathcal{Y},\mathcal{Z})$ is strongly hereditary. Let $\mathsf{L}$ and $\phi\colon \mathrm{Id}_{\mathscr{A}}\longrightarrow\mathsf{L}$ be the associated localisation functor and natural transformation, respectively, and $\mathsf{e}\colon \mathscr{A}\longrightarrow\mathscr{A}/\mathcal{Y}$ the quotient functor. Recall that $\mathsf{L}=il$, where $i\colon \mathcal{G}\longrightarrow\mathscr{A}$ is the inclusion functor of the Giraud subcategory $\mathcal{G}:=\mathsf{Im}\,\mathsf{L}$ in $\mathscr{A}$ and $l$ is its left adjoint. We observe that $\mathsf{e}i\colon \mathcal{G}\longrightarrow\mathscr{A}/\mathcal{Y}$ is an equivalence. It is essentially surjective since it is easy to check that $\mathsf{e}\phi$ is a natural equivalance between $\mathsf{e}$ and $\mathsf{eL}$. On the other hand, it is fully faithful by the description (1.2.1) of morphisms in $\mathscr{A}/\mathcal{Y}$. Indeed, given $M$ and $N$ in $\mathcal{G}$, there are no subobjects of $N$ lying in $\mathcal{Y}$ and, for all subobjects $M'$ of $M$ such that $M/M'$ lies in $\mathcal{Y}$, $\mathcal{Y}$-divisibility guarantees that $\mathsf{Hom}_{\mathscr{A}}(M',N)=\mathsf{Hom}_{\mathscr{A}}(M,N)$. Since both $\mathsf{e}i$ and $l$ have right adjoints then so does $(\mathsf{e}i)l\cong\mathsf{eL}\cong\mathsf{e}$.

Conversely, suppose that $\mathcal{Y}$ is a localising subcategory of $\mathscr{A}$ and let $\mathsf{r}\colon \mathscr{A}/\mathcal{Y}\longrightarrow\mathscr{A}$ be the right adjoint of the quotient $\mathsf{e}\colon \mathscr{A}\longrightarrow\mathscr{A}/\mathcal{Y}$. Given any object $A$ in $\mathscr{A}$, consider the map given by the unit of the adjunction $\phi_A\colon A\longrightarrow\mathsf{re}(A)$. We will show that this is a localisation with respect to $(\mathcal{Y},\mathcal{Z})$. By Theorem 1.2.9, $\mathsf{re}(A)$ lies in $\mathcal{Y}^{\perp}$ and, thus, in $\mathcal{Z}=\mathcal{Y}^{\circ}$. Since $\mathsf{e}$ is exact and $\mathsf{er}\cong\mathrm{Id}_{\mathscr{A}/\mathcal{Y}}$, it is also clear that $\mathsf{e}(\mathsf{Ker}\,\phi_A)=0=\mathsf{e}(\mathsf{Coker}\,\phi_A)$ and, thus, both $\mathsf{Ker}\,\phi_A$ and $\mathsf{Coker}\,\phi_A$ lies in $\mathcal{Y}$. Finally, since $\mathsf{Ext}^1_{\mathscr{A}}(Y,\mathsf{re}(A))=0$ for all $Y$ in $\mathcal{Y}$, $\mathsf{re}(A)$ is $\mathcal{Y}$-divisible, as wanted. ☐

Remark 1.2.12. In Lemma 1.2.11 we in fact prove that, if $\mathcal{Y}$ is a localising subcategory and $\mathsf{r}$ is the right adjoint of the quotient functor $\mathsf{e}\colon \mathscr{A}\longrightarrow\mathscr{A}/\mathcal{Y}$, then the Giraud subcategory $\mathcal{G}$ associated to the strongly hereditary torsion pair $(\mathcal{Y},\mathcal{Z})$ coincides with

$\mathcal{Y}^{\perp}$. Similarly, the Co-Giraud subcategory $\mathcal{H}$ of $\mathscr{A}$ induced by the strongly cohereditary torsion pair $(\mathcal{X},\mathcal{Y})$ (formed by the $\mathcal{Y}$-codivisible objects of $\mathcal{X}$) coincides with the subcategory ${}^{\perp}\mathcal{Y}$.

Our aim is to establish a correspondence between recollements of abelian categories up to equivalence and strong $\mathsf{TTF}$-triples. For this reason we define an equivalence relation on the class of recollements of $\mathscr{B}$. Although seemingly artificial, Lemma 1.2.14 shows that Definition 1.2.13 is natural.

Definition 1.2.13. Two recollements $\mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$ and $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}',\mathscr{B}',\mathscr{C}')$ are **equivalent** if there are equivalence functors $\Phi\colon \mathscr{B}\longrightarrow\mathscr{B}'$ and $\Theta\colon\mathscr{C}\longrightarrow\mathscr{C}'$ such that the diagram below commutes up to natural equivalence, i.e. there is a natural equivalence of functors between $\Theta\mathsf{e}$ and $\mathsf{e}'\Phi$.

$$\begin{tikzcd} \mathscr{B} \arrow[r, "\mathsf{e}"] \arrow[d, "\Phi"', "\simeq"] & \mathscr{C} \arrow[d, "\Theta", "\simeq"'] \\ \mathscr{B}' \arrow[r, "\mathsf{e}'"] & \mathscr{C}' \end{tikzcd}$$

Lemma 1.2.14. *Two recollements $\mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$ and $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}',\mathscr{B}',\mathscr{C}')$ are equivalent if and only if there are equivalences $\Phi\colon\mathscr{B}\longrightarrow\mathscr{B}'$, $\Psi\colon\mathscr{A}\longrightarrow\mathscr{A}'$ and $\Theta\colon\mathscr{C}\longrightarrow\mathscr{C}'$ such that the six diagrams associated to the six functors of the recollements commute up to natural equivalences.*

Proof. The condition in the lemma is clearly sufficient to get an equivalence of recollements. Conversely, suppose that we have an equivalence of recollements as in Definition 1.2.13. Recall that left (or right) adjoints of naturally equivalent functors are naturally equivalent. Thus, the left (or right) adjoints of $\Theta\mathsf{e}$ and of $\mathsf{e}'\Phi$ are equivalent. Such adjoints can be obtained by choosing a quasi-inverse of the equivalences $\Phi$ and $\Theta$. Using then the fact that the composition of two quasi-inverse functors is naturally equivalent to the identity functor, we easily get the desired natural equivalences between $\Phi\mathsf{l}$ and $\mathsf{l}'\Theta$ and between $\Phi\mathsf{r}$ and $\mathsf{r}'\Theta$. Up to equivalence, the two recollements are uniquely determined by these functors (see Remark 1.1.3). Let $\Psi$ be the restriction of $\Phi$ to $\operatorname{Ker}\mathsf{e}$ (which is equivalent to $\mathscr{A}$), where $\mathsf{e}\colon\mathscr{B}\longrightarrow\mathscr{C}$. Then, the diagram associated with the inclusion functor $\mathsf{i}\colon\operatorname{Ker}\mathsf{e}\longrightarrow\mathscr{B}$ clearly commutes and so do the other two, by an adjunction argument analogous to the one above. $\square$

We summarize next the above discussion about equivalent recollements:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{i}"] & \mathscr{B} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \xrightarrow[(\Psi,\Phi,\Theta)]{\simeq} \begin{tikzcd} \mathscr{A}' \arrow[r, "\mathsf{i}"] & \mathscr{B}' \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{C}' \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

$$\overset{\text{Def.}}{\underset{1.2.13}{\Longleftrightarrow}} \begin{tikzcd} \mathscr{B} \arrow[r, "\mathsf{e}"] \arrow[d, "\Phi"', "\simeq"] & \mathscr{C} \arrow[d, "\Theta", "\simeq"'] \\ \mathscr{B}' \arrow[r, "\mathsf{e}'"] & \mathscr{C}' \end{tikzcd}$$

$$\overset{\text{Lemma}}{\underset{1.2.14}{\Longleftrightarrow}} \begin{tikzcd} \mathscr{B} \arrow[d, "\Phi"', "\simeq"] & \mathscr{C} \arrow[l, "\mathsf{l}"'] \arrow[d, "\Theta", "\simeq"'] \\ \mathscr{B}' & \mathscr{C}' \arrow[l, "\mathsf{l}'"] \end{tikzcd} \quad \begin{tikzcd} \mathscr{B} \arrow[d, "\Phi"', "\simeq"] & \mathscr{C} \arrow[l, "\mathsf{r}"'] \arrow[d, "\Theta", "\simeq"'] \\ \mathscr{B}' & \mathscr{C}' \arrow[l, "\mathsf{r}'"] \end{tikzcd} \quad \begin{tikzcd} \mathscr{B} \arrow[r, "\mathsf{e}"] \arrow[d, "\Phi"', "\simeq"] & \mathscr{C} \arrow[d, "\Theta", "\simeq"'] \\ \mathscr{B}' \arrow[r, "\mathsf{e}'"] & \mathscr{C}' \end{tikzcd} \quad \begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{i}"] \arrow[d, "\Psi"', "\simeq"] & \mathscr{B} \arrow[d, "\Phi", "\simeq"'] \\ \mathscr{A}' \arrow[r, "\mathsf{i}'"] & \mathscr{B}' \end{tikzcd} \quad \begin{tikzcd} \mathscr{A} \arrow[d, "\Psi"', "\simeq"] & \mathscr{B} \arrow[l, "\mathsf{q}"'] \arrow[d, "\Theta", "\simeq"'] \\ \mathscr{A}' & \mathscr{B}' \arrow[l, "\mathsf{q}'"] \end{tikzcd} \quad \begin{tikzcd} \mathscr{A} \arrow[d, "\Psi"', "\simeq"] & \mathscr{B} \arrow[l, "\mathsf{p}"'] \arrow[d, "\Phi", "\simeq"'] \\ \mathscr{A}' & \mathscr{B}' \arrow[l, "\mathsf{p}'"] \end{tikzcd}$$

Equivalences of recollements whose outer equivalence functors ($\Psi$ and $\Theta$ in the lemma) are the identity functor have been studied in [50]. Equivalences of recollements of triangulated categories also appear in [107, Theorem 2.5].

In the following theorem, we use the fact that structural properties of $\mathscr{B}$, such as $\mathsf{TTF}$-triples, are preserved under equivalence.

THEOREM 1.2.15. *Let $\mathscr{B}$ be an abelian category. The following are in bijection.*

(i) *Equivalence classes of recollements of abelian categories $\mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$;*
(ii) *Strong $\mathsf{TTF}$-triples $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ in $\mathscr{B}$;*
(iii) *Bilocalising $\mathsf{TTF}$-classes $\mathcal{Y}$ of $\mathscr{B}$;*
(iv) *Bilocalising Serre subcategories $\mathcal{Y}$ of $\mathscr{B}$.*

PROOF. Let $\mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of $\mathscr{B}$. Firstly, $(\mathsf{Ker}\,\mathsf{q},\mathsf{i}(\mathscr{A}),\mathsf{Ker}\,\mathsf{p})$ is a $\mathsf{TTF}$-triple in $\mathscr{B}$. The adjoint triple $(\mathsf{q},\mathsf{i},\mathsf{p})$ ensures that

$$\mathsf{Hom}_{\mathscr{B}}(\mathsf{Ker}\,\mathsf{q},\mathsf{i}(\mathscr{A}))=0=\mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(\mathscr{A}),\mathsf{Ker}\,\mathsf{p})$$

Let $B$ be an object of $\mathscr{B}$. From Proposition 1.1.6, we have an exact sequence

$$0\longrightarrow \mathsf{Ker}\,\mu_B\longrightarrow \mathsf{le}(B)\xrightarrow{\mu_B} B\longrightarrow \mathsf{iq}(B)\longrightarrow 0$$

where $\mathsf{Ker}\,\mu_B$ lies in $\mathsf{i}(\mathscr{A})$. Applying the right exact functor $\mathsf{q}$ to the sequence, we see that $\mathsf{q}(\mathsf{Im}\,\mu_B)=0$. Thus, the sequence

$$0\longrightarrow \mathsf{Im}\,\mu_B\longrightarrow B\longrightarrow \mathsf{iq}(B)\longrightarrow 0$$

shows that $(\mathsf{Ker}\,\mathsf{q},\mathsf{i}(\mathscr{A}))$ is a torsion pair. Similarly, the exact sequence induced by $\nu_B\colon B\longrightarrow \mathsf{re}(B)$ can be used to show that $(\mathsf{i}(\mathscr{A}),\mathsf{Ker}\,\mathsf{p})$ is a torsion pair in $\mathscr{B}$. Since $\mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$ is a recollement, $\mathsf{i}(\mathscr{A})$ is a bilocalising subcategory of $\mathscr{B}$. Hence, by Lemma 1.2.11, the torsion pairs $(\mathsf{Ker}\,\mathsf{q},\mathsf{i}(\mathscr{A}))$ and $(\mathsf{i}(\mathscr{A}),\mathsf{Ker}\,\mathsf{p})$ are, respectively, strongly cohereditary and strongly hereditary and $(\mathsf{Ker}\,\mathsf{q},\mathsf{i}(\mathscr{A}),\mathsf{Ker}\,\mathsf{p})$ is a strong $\mathsf{TTF}$-triple. Note that this $\mathsf{TTF}$-triple depends only on the equivalence class of the recollement. Indeed, if $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}',\mathscr{B}',\mathscr{C}')$ is a recollement equivalent to $\mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{A},\mathscr{C})$ via an equivalence $\Phi\colon\mathscr{B}'\longrightarrow\mathscr{B}$, then the corresponding $\mathsf{TTF}$-class of $\mathscr{B}$ associated to it is given by $\Phi\mathsf{i}'(\mathscr{A}')$ which coincides, by Lemma 1.2.14, with $\mathsf{i}(\mathscr{A})$.

We construct now an inverse correspondence (see also [100, Theorem 4.5]). Let $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ be a strong $\mathsf{TTF}$-triple in $\mathscr{B}$. Since $\mathcal{Y}$ is a $\mathsf{TTF}$-class, it follows that it is bireflective and hence the embedding $\mathsf{i}$ of $\mathcal{Y}$ in $\mathscr{B}$ admits a left adjoint $\mathsf{q}$ and a right adjoint $\mathsf{p}$. It is also a Serre subcategory, and we consider the quotient functor $\mathsf{e}\colon\mathscr{B}\longrightarrow\mathscr{B}/\mathcal{Y}$. Since the triple is strong it follows from Lemma 1.2.11 that $\mathcal{Y}$ is bilocalising. Thus, $\mathsf{e}$ has both left and right adjoints, $\mathsf{l}$ and $\mathsf{r}$ respectively, which are fully faithful (because $\mathsf{er}$ and $\mathsf{el}$ are naturally equivalent to $\mathrm{Id}_{\mathscr{B}/\mathcal{Y}}$, see [54]). Hence, we have a recollement $\mathsf{R}_{\mathsf{ab}}(\mathcal{Y},\mathscr{B},\mathscr{B}/\mathcal{Y})$. Clearly these correspondences are inverse to each other, up to equivalence of recollements.

Finally, since $\mathsf{i}(\mathscr{A})$ is a bilocalising $\mathsf{TTF}$-class as well as a (bireflective) Serre subcategory, the bijection between (i) and (ii) easily implies the bijections between (i), (iii) and (iv). □

Under some conditions on $\mathscr{B}$, the above bijection becomes more clear.

COROLLARY 1.2.16. *If $\mathscr{B}$ has enough projectives and injectives, then the equivalence classes of recollements of $\mathscr{B}$ are in bijection with the $\mathsf{TTF}$-triples in $\mathscr{B}$.*

PROOF. Let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a $\mathsf{TTF}$-triple in $\mathscr{B}$. Since $\mathscr{B}$ has enough projectives and injectives it follows from [100, Theorem 1.8/1.8*] that every $\mathsf{TTF}$-triple in $\mathscr{B}$ is strong. The result then is a consequence of Theorem 1.2.15. $\square$

Given a recollement, we then have the following notable equivalences.

COROLLARY 1.2.17. *Let $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of $\mathscr{B}$, $\mathcal{G}$ be the Giraud subcategory associated to the torsion pair $(\mathsf{i}(\mathscr{A}), \operatorname{Ker} \mathsf{p})$ and $\mathcal{H}$ the Co-Giraud subcategory associated to the torsion pair $(\operatorname{Ker} \mathsf{q}, \mathsf{i}(\mathscr{A}))$. Then, the quotient functor $\mathsf{e}$ induces:*

(i) *an equivalence* $\mathsf{i}(\mathscr{A})^{\perp} = \mathcal{G} = \operatorname{Im} \mathsf{r} \xrightarrow{\cong} \mathscr{B}/\mathsf{i}(\mathscr{A})$ ,

(ii) *an equivalence* ${}^{\perp}\mathsf{i}(\mathscr{A}) = \mathcal{H} = \operatorname{Im} \mathsf{l} \xrightarrow{\cong} \mathscr{B}/\mathsf{i}(\mathscr{A})$ *and*

(iii) *an equivalence* $\operatorname{Ker} \mathsf{q} \cap \operatorname{Ker} \mathsf{p} \xrightarrow{\cong} \mathscr{B}/\mathsf{i}(\mathscr{A})$ .

PROOF. Statements (i) and (ii) follow immediately from the fact that the $\mathsf{TTF}$-triple $(\operatorname{Ker} \mathsf{q}, \mathsf{i}(\mathscr{A}), \operatorname{Ker} \mathsf{p})$ is strong and from Remark 1.2.12. Statement (iii) is well-known for $\mathsf{TTF}$-triples, see [58, Theorem 1.9] or [30, Proposition 1.3]. $\square$

## 1.3. Kuhn's Conjecture on Recollements of Module Categories

Let $A$ be a unitary ring and $\mathsf{Mod}\text{-}A$ the category of right $A$-modules. Our purpose in this section is to classify recollements of $\mathsf{Mod}\text{-}A$ whose terms are module categories. Kuhn conjectured in [82] that if the categories of a recollement are equivalent to categories of modules over finite dimensional algebras over a field, then it is equivalent to one arising from an idempotent element. In this section we prove this conjecture for general rings.

To study recollements of $\mathsf{Mod}\text{-}A$ we look at its bireflective subcategories, which are classified by epimorphisms in the category of unitary rings. A ring homomorphism $f\colon A \longrightarrow B$ is an epimorphism if and only if the restriction functor $f_*\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}A$ is fully faithful ( [121]). Theorem 1.3.1 states that all bireflective subcategories of $\mathsf{Mod}\text{-}A$ arise in this way. Two ring epimorphisms $f\colon A \longrightarrow B$ and $g\colon A \longrightarrow C$ lie in the same **epiclass of** $A$, if there is a ring isomorphism $h\colon B \longrightarrow C$ such that $g = hf$.

THEOREM 1.3.1. *[68, Theorem 1.6.3] [56] [55, Theorem 1.2] There is a bijection between epiclasses of $A$ and bireflective subcategories of $\mathsf{Mod}\text{-}A$, defined by assigning to an epimorphism $f\colon A \longrightarrow B$, the subcategory $\mathcal{X}_B := \operatorname{Im} f_*$. Moreover, a full subcategory $\mathcal{X}$ of $\mathsf{Mod}\text{-}A$ is bireflective if and only if it is closed under products, coproducts, kernels and cokernels.*

Given a ring epimorphism $f\colon A \longrightarrow B$, let $\psi_M\colon M \longrightarrow M \otimes_A B$ denote the unit of the adjoint pair $(- \otimes_A B, f_*)$ at a right $A$-module $M$ (given by $\psi_M(m) = m \otimes 1_B$, for all $m$ in $M$). Note that $\psi_N$ is an isomorphism for all $N$ in $\mathcal{X}_B$. In fact, $\psi_M$ is the $\mathcal{X}_B$-reflection of the right $A$-module $M$. In particular, $f\colon A \longrightarrow B$, regarded as a morphism in $\mathsf{Mod}\text{-}A$, is the $\mathcal{X}_B$-reflection $\psi_A$.

Some properties of a ring epimorphism $f\colon A \longrightarrow B$ can be seen from the bireflective subcategory $\mathcal{X}_B$. In particular we have the following result.

LEMMA 1.3.2. *[119, Theorem 4.8] Let $f\colon A \longrightarrow B$ be a ring epimorphism. Then $\mathcal{X}_B$ is extension-closed if and only if $\operatorname{Tor}_1^A(B, B) = 0$.*

PROOF. Suppose that $\operatorname{Tor}_1^A(B, B) = 0$ and let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an exact sequence of $A$-modules, where $X$ and $Z$ are $B$-modules. Let $M$ be a right

$B$-module and let $0 \longrightarrow K_0 \longrightarrow F \longrightarrow M \longrightarrow 0$ be an exact sequence of $B$-modules with $F$ free. Then from the long exact sequence:

$$0 \longrightarrow \mathsf{Tor}_1^A(M,B) \longrightarrow K_0 \otimes_A B \longrightarrow F \otimes_A B \longrightarrow M \otimes_A B \longrightarrow 0$$

we get that

$$\mathsf{Tor}_1^A(M,B) = 0 \tag{$*$}$$

since the unit of the adjoint pair $(- \otimes_A B, f_*)$ is an isomorphism for all objects of $\mathcal{X}_B$. Let $N$ be a left $B$-module and $0 \longrightarrow L_0 \longrightarrow F' \longrightarrow N \longrightarrow 0$ be an exact sequence of $B$-modules where $F'$ is free. Then from $(*)$ we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Tor}_1^A(M,N) \arrow[r] \arrow[d] & M \otimes_A L_0 \arrow[r] \arrow[d, "\simeq"] & M \otimes_A F' \arrow[r] \arrow[d, "\simeq"] & M \otimes_A N \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathsf{Tor}_1^B(M,N) \arrow[r] & M \otimes_B L_0 \arrow[r] & M \otimes_B F' \arrow[r] & M \otimes_B N \arrow[r] & 0
\end{tikzcd}$$

and this implies that

$$\mathsf{Tor}_1^A(M,N) \simeq \mathsf{Tor}_1^B(M,N) \tag{$**$}$$

Then using the relation $(**)$ we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\simeq"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & X \otimes_A B \arrow[r] & Y \otimes_A B \arrow[r] & Z \otimes_A B \arrow[r] & 0
\end{tikzcd}$$

Hence $Y \simeq Y \otimes_A B$ and so $Y$ is a $B$-module. We infer that the subcategory $\mathcal{X}_B$ is closed under extensions in $\mathsf{Mod}\text{-}A$.

Suppose that $\mathcal{X}_B$ is extension closed and let $0 \longrightarrow K_0 \longrightarrow F \longrightarrow B \longrightarrow 0$ be an exact sequence of right $A$-modules with $F$ free. Then applying the functor $- \otimes_A B$ we obtain the following long exact sequence:

$$0 \longrightarrow \mathsf{Tor}_1^A(B,B) \longrightarrow K_0 \otimes_A B \longrightarrow F \otimes_A B \longrightarrow B \otimes_A B \longrightarrow 0$$

From the following pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_0 \otimes_A B \arrow[r] & N \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

it follows that $N$ is a $B$-module since $\mathcal{X}_B$ is closed under extensions. Then we have the exact commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[r] & K_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathsf{Tor}_1^A(B,B) \arrow[r] & K_0 \otimes_A B \arrow[r] \arrow[d, equal] & F \otimes_A B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & K_0 \otimes_A B \arrow[r] & N \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

and this implies that $\mathsf{Tor}_1^A(B,B) = 0$. □

Since $\mathsf{Mod}\text{-}A$ has enough projectives and injectives, by Corollary 1.2.16, there is a bijection between equivalence classes of recollements of $\mathsf{Mod}\text{-}A$ and $\mathsf{TTF}$-triples in $\mathsf{Mod}\text{-}A$. Moreover, there is a bijection between $\mathsf{TTF}$-classes and bireflective Serre subcategories, since the closure conditions for both types of subcategories are the same (see Proposition 1.2.4 and Theorem 1.3.1). In particular, any bireflective Serre subcategory of $\mathsf{Mod}\text{-}A$ is bilocalising by Lemma 1.2.11. In the following result we describe these categories in terms of ring epimorphisms. Similar results can be found in [11, Section 7] and in [56, Proposition 5.3].

Proposition 1.3.3. *Let $\mathcal{Y}$ be a bireflective Serre subcategory of $\mathsf{Mod}\text{-}A$. Then there is an idempotent ideal $I$ of $A$ such that $\mathcal{Y}$ is the essential image of the restriction functor induced by the ring epimorphism $f\colon A \longrightarrow A/I$.*

Proof. Since $\mathcal{Y}$ is a bireflective subcategory of $\mathsf{Mod}\text{-}A$, by Theorem 1.3.1, there is a ring epimorphism $f\colon A \longrightarrow B$, for some ring $B$, such that $f_*(\mathsf{Mod}\text{-}B) = \mathcal{Y}$. We will now prove that $f$ is surjective. Since $\mathcal{Y}$ is a $\mathsf{TTF}$-class, $({}^{\circ}\mathcal{Y}, \mathcal{Y})$ is a torsion pair and the composition $\mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}$ ($\mathsf{i}_{\mathcal{Y}}$ being the inclusion $\mathcal{Y} \longrightarrow \mathsf{Mod}\text{-}A$ and $\mathsf{L}_{\mathcal{Y}}$ its left adjoint) is the idempotent coradical functor sending a module $M$ to its torsion-free part. In particular the unit of this adjunction is surjective on every $A$-module. On the other hand, since $f_*(\mathsf{Mod}\text{-}B) = \mathcal{Y}$, it follows that $\mathsf{i}_{\mathcal{Y}}\mathsf{L}_{\mathcal{Y}}$ is naturally equivalent to $f_*(- \otimes_A B)$. Thus, $\psi_M\colon M \longrightarrow M \otimes_A B$ is surjective for every $A$-module $M$. In particular, $f = \psi_A$ is surjective. Since $f_*(\mathsf{Mod}\text{-}B)$ is closed under extensions, for $I = \mathsf{Ker}\, f$ we have by Lemma 1.3.2

$$0 = \mathsf{Tor}_1^A(B,B) = \mathsf{Tor}_1^A(A/I, A/I) = I/I^2$$

and, thus, we infer that $\mathcal{Y} = f_*(\mathsf{Mod}\text{-}A/I)$, with $I^2 = I$. $\square$

We now recover Jans' bijection between $\mathsf{TTF}$-triples and idempotent ideals (Theorem 1.2.6) and classifiy equivalence classes of recollements of $\mathsf{Mod}\text{-}A$.

Proposition 1.3.4. *There is a bijection between equivalence classes of recollements of $\mathsf{Mod}\text{-}A$, $\mathsf{TTF}$-triples in $\mathsf{Mod}\text{-}A$ and idempotent two-sided ideals of $A$. In particular the bijections are given as follows:*

- $\{$*equivalence classes of recollements of* $\mathsf{Mod}\text{-}A\} \longleftrightarrow \{$*$\mathsf{TTF}$-triples in* $\mathsf{Mod}\text{-}A\}$

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{i}"] & \mathsf{Mod}\text{-}A \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \longmapsto (\mathsf{Ker}\,\mathsf{q}, \mathsf{i}(\mathsf{Mod}\text{-}B), \mathsf{Ker}\,\mathsf{p})$$

$$(\mathcal{Y}, \mathsf{Mod}\text{-}A, (\mathsf{Mod}\text{-}A)/\mathcal{Y}) \longleftarrow\!\!\shortmid (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$$

- $\{$*idempotent two-sided ideals of $A$*$\} \longleftrightarrow \{$*$\mathsf{TTF}$-triples in* $\mathsf{Mod}\text{-}A\}$

$$I \trianglelefteq A,\ I^2 = I \longmapsto ({}^{\circ}\mathcal{Y}_F, \mathcal{Y}_F := \{X \in \mathsf{Mod}\text{-}A \mid F(X) = X\}, \mathcal{Y}_F^{\circ})$$

$$\textit{and} \quad F = f_*(- \otimes_A A/I)\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}A$$

$$I \longleftarrow\!\!\shortmid (\mathcal{X}, \mathcal{Y}, \mathcal{Z}),\ \mathcal{Y} = f_*(\mathsf{Mod}\text{-}A/I)$$

PROOF. The bijection between equivalence classes of recollements and TTF-triples follows from Corollary 1.2.16, since Mod-$A$ has enough projectives and injectives. The bijection between TTF-triples and idempotent ideals of $A$ can be seen as a consequence of Proposition 1.2.7. Indeed, the bijection in that proposition assigns to a TTF-triple a functor which, by Proposition 1.3.3, is precisely $f_*(- \otimes_A A/I)$ for some idempotent ideal $I$ and $f\colon A \longrightarrow A/I$ the canonical projection, thus uniquely determined by the ideal $I$. Conversely, given an idempotent ideal $I$ and the quotient map $f\colon A \longrightarrow A/I$ it is easy to check that $f_*(- \otimes_A A/I)$ is a right exact idempotent coradical endofunctor of Mod-$A$ preserving coproducts. Indeed, for any module $X$ in Mod-$A$ we have the isomorphism $X \otimes_R R/I \simeq X/XI$. Thus we have the exact sequence $0 \longrightarrow XI \longrightarrow X \longrightarrow X/XI \longrightarrow 0$ in Mod-$A$ and since the ideal $I$ is idempotent it follows that $XI \otimes_A A/I \simeq XI/XI^2 \simeq XI/XI = 0$. Hence the functor $f_*(- \otimes_A A/I)$ is right exact coradical and preserves coproducts since $f_*$ and $- \otimes_A A/I$ are left adjoints. □

DEFINITION 1.3.5. We say that a recollement of Mod-$A$ is a **recollement by module categories** if it is equivalent to a recollement in which the categories involved are module categories.

We recall the conjecture made by Kuhn in [82].

**Conjecture** [82] *Let $A$ be a finite dimensional algebra over a field. Then any recollement of* Mod-$A$ *by module categories is equivalent to a recollement induced by an idempotent element.*

Indeed, we prove Kuhn's conjecture for any ring $A$.

THEOREM 1.3.6. *A recollement of* Mod-$A$ *is a recollement by module categories if and only if it is equivalent to a recollement induced by an idempotent element of a ring $S$, Morita equivalent to $A$.*

PROOF. By Proposition 1.3.4, any recollement of Mod-$A$ is equivalent to

$$\begin{tikzcd} \mathsf{Mod}\text{-}A/I \arrow[rr, "\mathsf{inc}"] & & \mathsf{Mod}\text{-}A \arrow[ll, bend right, "-\otimes_A A/I"'] \arrow[ll, bend left, "\mathsf{Hom}_A(A/I,-)"] \arrow[rr, "\mathsf{e}"] & & \mathscr{C}_I, \arrow[ll, bend right, "\mathsf{l}"'] \arrow[ll, bend left, "\mathsf{r}"] \end{tikzcd} \qquad (1.3.1)$$

for some idempotent ideal $I$ of $A$ and $\mathscr{C}_I$ the corresponding quotient category. Clearly, if $I$ is generated by an idempotent element $e$ in $A$, then $\mathscr{C}_I$ is equivalent to Mod-$eAe$, see Example 1.1.7.

Conversely, assume that (1.3.1) is equivalent to a recollement by module categories. Let $P$ be a small (i.e., $\mathsf{Hom}_{\mathscr{C}_I}(P,-)$ commutes with coproducts) projective generator of $\mathscr{C}_I$, which exists since we assume that $\mathscr{C}_I$ is equivalent to a module category. Let us denote by $C$ the ring $\mathsf{End}_{\mathscr{C}_I}(P)$ and by $\Theta$ the equivalence $\mathsf{Hom}_{\mathscr{C}_I}(P,-)\colon \mathscr{C}_I \longrightarrow$ Mod-$C$. The object $\mathsf{l}(P)$ is projective since we have the adjoint pair $(\mathsf{l},\mathsf{e})$ and the functor $\mathsf{e}$ is exact. It is also small since $\mathsf{e}$ commutes with coproducts and $P$ is small. Since a projective object is small in a module category if and only if it is finitely generated, there is a surjective map $p\colon A^{\oplus n} \longrightarrow \mathsf{l}(P)$, for some $n$ in $\mathbb{N}$. This surjective map splits since $\mathsf{l}(P)$ is a projective $A$-module, i.e., there is an injective map $h\colon \mathsf{l}(P) \longrightarrow A^{\oplus n}$ such that $ph = \mathrm{Id}_{\mathsf{l}(P)}$. Let $S$ denote the endomorphism ring of $A^{\oplus n}$, i.e., $S = \mathsf{End}_A(A^{\oplus n})$, and let $\Phi := \mathsf{Hom}_A(A^{\oplus n},-)$ denote the Morita equivalence between Mod-$A$ and Mod-$S$.

Then we have a surjection $\Phi(p)\colon S = \Phi(A^{\oplus n}) \longrightarrow \Phi(\mathsf{I}(P))$ which splits via $\Phi(h)$, i.e., $\Phi(\mathsf{I}(P))$ is a direct summand of $S$. Moreover, it is precisely generated by the idempotent $\Phi(h)\Phi(p)$ in $\mathsf{End}_S(S)$, which, under the isomorphism $\mathsf{End}_S(S) \cong S$ is identified with $hp$. Denote this element by $e$. Clearly, $eS$ is the image of $\Phi(h)\Phi(p)$ and it is isomorphic to $\Phi(\mathsf{I}(P))$ in $\mathsf{Mod}\text{-}S$. Since both $\mathsf{I}$ and $\Phi$ are fully faithful, we have the following chain of ring isomorphisms:

$$\begin{aligned} C &= \mathsf{Hom}_{\mathscr{C}_I}(P,P) \cong \mathsf{Hom}_A(\mathsf{I}(P),\mathsf{I}(P)) \cong \mathsf{Hom}_S(\Phi(\mathsf{I}(P)),\Phi(\mathsf{I}(P))) \\ &\cong \mathsf{Hom}_S(eS,eS) \cong eSe \end{aligned}$$

The last isomorphism is $\alpha\colon eSe \longrightarrow \mathsf{End}_S(eS)$, sending an element in $eSe$ to the endomorphism given by left multiplication with it. This is clearly an injective ring homomorphism. Given an endomorphim $g$ of $eS$, $g$ is given by left multiplication with $g(e)$. Since $g(e)$ lies in $eS$ and $g(e)e = g(e^2) = g(e)$, $g(e)$ lies in $eSe$. Thus, $\alpha$ is surjective. Now, the functors $\Theta\colon \mathscr{C}_I \longrightarrow \mathsf{Mod}\text{-}eSe$ and $\Phi$ form an equivalence of recollements from $\mathsf{R}(\mathsf{Mod}\text{-}A/I, \mathsf{Mod}\text{-}A, \mathscr{C}_I)$ to $\mathsf{R}(\mathsf{Mod}\text{-}S/SeS, \mathsf{Mod}\text{-}S, \mathsf{Mod}\text{-}eSe)$. Indeed, we have natural isomorphisms

$$\begin{aligned} \Theta\mathsf{e}(M) &= \mathsf{Hom}_{\mathscr{C}_I}(P,\mathsf{e}(M)) \cong \mathsf{Hom}_A(\mathsf{I}(P),M) \\ &\cong \mathsf{Hom}_S(\Phi(\mathsf{I}(P)),\Phi(M)) = \mathsf{Hom}_S(eS,\Phi(M)) \end{aligned}$$

Since the functor $\mathsf{Hom}_S(eS,-)$ is the quotient functor $\mathsf{Mod}\text{-}S \longrightarrow \mathsf{Mod}\text{-}eSe$, $\Phi$ and $\Theta$ form an equivalence of recollements, as wanted. $\square$

Under additional conditions, we can say more about the ideal $I$ of $A$. If $A$ admits the Krull-Schmidt property for finitely generated projective $A$-modules (i.e., $A$ is semiperfect), we can simplify the statement of the theorem.

COROLLARY 1.3.7. *Let $A$ be a semiperfect ring. Then a recollement of* $\mathsf{Mod}\text{-}A$ *is a recollement by module categories if and only if the associated idempotent ideal $I$ is generated by an idempotent element.*

PROOF. This follows from the proof of Theorem 1.3.6. Let $P$ be a basic (i.e., every indecomposable summand occurs with multiplicity one) small projective generator of $\mathscr{C}_I$. Then $\mathsf{I}(P)$ is also basic. Since $A$ satisfies the Krull-Schmidt property for projective modules, $\mathsf{I}(P)$ is a direct summand of $A^{\oplus n}$ if and only if it is a direct summand of $A$. Using the arguments in the proof of Theorem 1.3.6 for $S = A$, we see that there is an equivalence of recollements induced by $\Phi$, $\mathrm{Id}_{\mathsf{Mod}\text{-}A}$ and $\Psi$ from the recollement (1.3.1) to the recollement induced by the idempotent element $e$ (see Example 1.1.7). Thus, the essential image of the embeddings $\mathsf{Mod}\text{-}A/I \longrightarrow \mathsf{Mod}\text{-}A$ and $\mathsf{Mod}\text{-}A/AeA \longrightarrow \mathsf{Mod}\text{-}A$ coincide. By Theorem 1.3.1, the epimorphisms $f\colon A \longrightarrow A/I$ and $g\colon A \longrightarrow A/AeA$ must then lie in the same epiclass, i.e., there is an isomorphism $h\colon A/I \longrightarrow A/AeA$ such that $hf = g$. Note now that, since $h$ is an isomorphism, we have $I = \mathsf{Ker}(f) = \mathsf{Ker}(g) = AeA$, thus showing that $I = AeA$, as wanted. $\square$

Recall that $A$ is semiprimary if the Jacobson radical $\mathfrak{r}$ of $A$ is nilpotent and $A/\mathfrak{r}$ is semisimple. Indeed, semiprimary rings are semiperfect (see, for example, [85, Corollary 23.19]) and every idempotent ideal is generated by an idempotent element of $A$ ( [45]). Finite dimensional algebras over a field are well-known examples of semiprimary rings. The following corollary provides an answer to Kuhn's question in the context where it appeared.

COROLLARY 1.3.8. *Let $A$ be a semiprimary ring. Then any recollement of* $\mathsf{Mod}\text{-}A$ *is equivalent to a recollement induced by an idempotent element of $A$. In particular, any recollement of* $\mathsf{Mod}\text{-}A$ *is a recollement by module categories.*

PROOF. Let $I$ be the idempotent ideal associated to a recollement of $\mathsf{Mod}\text{-}A$, as in the proof of Theorem 1.3.6. Since $A$ is semiprimary, $I = AeA$ for some $e$ idempotent element of $A$. Therefore, the equivalent recollement (1.3.1) is induced by the ring epimorphism $f\colon A \longrightarrow A/AeA$, thus finishing the proof. $\square$

## 1.4. A recollement not induced by an idempotent element

In this section we present an example of a recollement of $\mathsf{Mod}\text{-}R$ which is not induced by an idempotent element.

We start by recalling some basics for von Neuman regular rings.

DEFINITION 1.4.1. A ring $R$ is called **von Neuman regular** if for every $a \in R$ there exists an element $x \in R$ such that $a = axa$.

A von Neuman regular ring is also called **absolutely flat**. This is due to the characterization that over a von Neuman regular ring every left $R$-module is flat [84, Theorem 4.21]. In what follows we need the following result.

LEMMA 1.4.2. *[124, Proposition 2.1] Let $R$ be a von Neuman regular ring and $P$ a projective $R$-module. Then every finitely generated submodule of $P$ is a direct summand of $P$, i.e. $P$ is regular.*

From now on and until the end of this section we fix a field $\mathbb{K}$. Set

$$R := \prod_{i=1}^{\infty} \mathbb{K} \qquad \text{and} \qquad I := \bigoplus_{i=1}^{\infty} \mathbb{K}$$

We have the following easy observation.

LEMMA 1.4.3. *Let $\mathbb{K}$ be a field and $R$, $I$ as above. Then $R$ is a unital commutative von Neumann regular ring and $I$ is a two-sided idempotent ideal of $R$.*

PROOF. Since $\mathbb{K}$ is a field it follows that for every $a \in \mathbb{K}$ there exists an element $x = a^{-1} \in \mathbb{K}$ such that $axa = a$. Thus $\mathbb{K}$ is a von Neumann regular ring. Then since the multiplication of elements in $R$ is componentwise we infer that $R$ is a unital commutative von Neumann regular ring. It is easy to verify that $I$ is a two-sided idempotent ideal of $R$. Note that in a von Neumann regular ring every ideal is idempotent. $\square$

Let $\pi\colon R \longrightarrow R/I$ denote the canonical quotient map and $\pi_*\colon \mathsf{Mod}\text{-}R/I \longrightarrow \mathsf{Mod}\text{-}R$ the corresponding (fully faithful) restriction functor. As in Theorem 1.3.6 consider the recollement of $\mathsf{Mod}\text{-}R$ induced by $I$:

$$\begin{tikzcd}[column sep=huge]
\mathsf{Mod}\text{-}R/I \arrow[r, "\pi_*"] & \mathsf{Mod}\text{-}R \arrow[l, bend right, "-\otimes_R R/I"'] \arrow[l, bend left, "\mathsf{Hom}_R(R/I,-)"] \arrow[r, "\mathsf{e}"] & \mathscr{C}_I, \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"]
\end{tikzcd}$$

The statement of this section is as follows.

PROPOSITION 1.4.4. *The recollement of* $\mathsf{Mod}\text{-}R$ *induced by the idempotent ideal $I$ is not equivalent to any recollement induced by an idempotent element.*

Proof. We will show that the category $\mathscr{C}_I$ is not equivalent to a category of modules. By Theorem 1.3.6 the recollement cannot be equivalent to a recollement by module categories and, thus, it cannot be induced by an idempotent element.

Recall that the recollement above satisfies $\mathsf{Im}\,\mathsf{l} \subseteq \mathsf{Ker}(-\otimes_R R/I)$. For any module $M$ in $\mathsf{Mod}\text{-}R$, $M \otimes_R R/I$ is isomorphic to $M/MI$. Therefore, we have

$$\mathscr{C}_I \cong \mathsf{Im}\,\mathsf{l} \subseteq \{M \in \mathsf{Mod}\text{-}R \mid MI = M\}$$

Let $P$ be a small projective generator in $\mathscr{C}_I$. Then $Q := \mathsf{l}(P)$ is a small (i.e., finitely generated) projective $R$-module such that $QI = Q$. We write $S_j$ for the one dimensional simple module corresponding to the copy of the field in position $j$, with $j \in \mathbb{N}$.

First we claim that there is $k$ in $\mathbb{N}$ such that $\mathsf{Hom}_R(S_k, Q) \neq 0$. Indeed, by the Hom-tensor adjunction

$$\mathsf{Hom}_R(Q \otimes_R I, Q) \cong \mathsf{Hom}(Q, \mathsf{Hom}_R(I, Q)),$$

and since, for $Q$ projective with $Q \otimes_R R/I = 0$, we conclude that $Q \otimes_R I \cong Q$ and, thus, $\mathsf{Hom}_R(I, Q) \neq 0$. Since $I = \bigoplus_{i=1}^{\infty} S_i$, we have the claim.

Secondly, since $Q$ is regular (see Lemma 1.4.2), every simple $S_k$ embedded in $Q$ is a summand of $Q$. Therefore, if we define the set

$$J := \{i \in \mathbb{N} \mid \mathsf{Hom}(S_i, Q) \neq 0\},$$

it is then clear that, since each $S_i$ is simple, $\oplus_{i \in J} S_i$ is a summand of $Q$. Since $Q$ is a finitely generated $R$-module, this implies that $J$ is a finite set. Observe now that, for all $k$, $S_k$ lies in $\mathsf{Im}\,\mathsf{l}$ and that $S_k$ is projective since

$$R_R := \prod_{i=1}^{k} S_i \oplus S_k \oplus \prod_{i=k+1}^{\infty} S_i$$

Choose now $k$ in $\mathbb{N} \setminus J$. Clearly $\mathsf{Hom}(Q, S_k) = 0$, since any map would have to be surjective, thus splitting and forcing $k$ to lie in $J$. This, however, implies that $\mathsf{Hom}_{\mathscr{C}_I}(P, \mathsf{e}(S_k)) = 0$, which is a contradiction with the assumption that $P$ is a projective generator for $\mathscr{C}_I$. $\square$

Remark 1.4.5. More examples of this nature occur in the class of rings considered in [117].

CHAPTER 2

# Homological Theory of Recollements of Abelian Categories

In this Chapter we investigate several homological aspects of recollements of abelian categories. First we compare the extension groups between the categories involved in a recollement of abelian categories. Further we investigate the global and finitistic dimension of the categories involved in a recollement situation. Note that these results extend and generalize related results of Auslander-Platzeck-Todorov [14], formulated in the setting of finitely generated modules over an Artin algebra equipped with an idempotent ideal. We also give applications to ring theory. In Chapter 5 we will discuss applications of these results on finitistic and representation dimension of Artin algebras. The results of this Chapter are included in the paper entitled: Homological Theory of Recollements of Abelian Categories, see [108].

## 2.1. Generalized Perpendicular Categories and Homological Embeddings

Let as before $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{i}"] & \mathscr{B} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \qquad \mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$$

Our purpose in this section is to study the derived functors of the six functors involved in a recollement as well as the relationship and the interplay between the extension functors of the abelian categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$. Note that since the functors $\mathsf{i}$ and $\mathsf{e}$ are exact, they induce natural maps:

$$\mathsf{i}^n_{X,Y}\colon \mathsf{Ext}^n_{\mathscr{A}}(X,Y) \longrightarrow \mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(X),\mathsf{i}(Y)) \quad \text{and} \quad \mathsf{e}^n_{Z,W}\colon \mathsf{Ext}^n_{\mathscr{B}}(Z,W) \longrightarrow \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W))$$

In this connection we are interested in finding necessary and sufficient conditions such that the induced homomorphisms $\mathsf{i}^n_{X,Y}$ and/or $\mathsf{e}^n_{Z,W}$ are invertible for $0 \leq n \leq k$.

**2.1.1. Generalized Perpendicular Categories.** Let $\mathcal{M}$ be an abelian category and $\mathcal{U} \subseteq \mathcal{M}$ be a full subcategory. For integers $0 \leq i \leq k$ we denote by $\mathcal{U}^{i\perp_k}$ the full subcategory of $\mathcal{M}$ which is defined by

$$\mathcal{U}^{i\perp_k} = \{M \in \mathcal{M} \mid \mathsf{Ext}^n_{\mathcal{M}}(\mathcal{U},M)=0,\ \forall\ i \leq n \leq k\}$$

We also denote by $\mathcal{U}^{1\perp_\infty}$ the full subcategory of $\mathcal{M}$ defined by

$$\mathcal{U}^{1\perp_\infty} = \{M \in \mathcal{M} \mid \mathsf{Ext}^n_{\mathcal{M}}(\mathcal{U},M)=0,\ \forall n \geq 1\}$$

Similarly the full subcategories ${}^{i\perp_k}\mathcal{U}$ and ${}^{i\perp_\infty}\mathcal{U}$ are defined. Note that $\mathcal{U}^{0\perp_1}$ and ${}^{0\perp_1}\mathcal{U}$ are the right and left perpendicular categories of $\mathcal{U}$ as defined by Geigle-Lenzing, see [56].

Definition 2.1.1. For $0 \leq k \leq \infty$, the right k-perpendicular subcategory $\mathsf{i}(\mathscr{A})^{0\perp_k}$ of $\mathscr{A}$ in $\mathscr{B}$ is defined by

$$\mathsf{i}(\mathscr{A})^{0\perp_k} = \{B \in \mathscr{B} \mid \mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(A),B)=0,\ \forall A \in \mathscr{A} \text{ and } 0 \leq n \leq k\}$$

and dually the left k-perpendicular subcategory ${}^{0\perp_k}\mathsf{i}(\mathscr{A})$ of $\mathscr{A}$ in $\mathscr{B}$ is defined by

$$^{0\perp_k}\mathsf{i}(\mathscr{A}) = \{B \in \mathscr{B} \mid \mathsf{Ext}^n_{\mathscr{B}}(B, \mathsf{i}(A)) = 0, \ \forall A \in \mathscr{A} \text{ and } 0 \leq n \leq k\}$$

In order to study the main properties of the generalized perpendicular categories of $\mathscr{A}$ in $\mathscr{B}$, it will be useful to introduce, for $0 \leq n \leq \infty$, the following full subcategories of $\mathscr{B}$ which describe objects which admit (truncated) projective or injective resolutions by objects from the quotient category $\mathscr{C}$:

$$\mathcal{X}_n = \{B \in \mathscr{B} \mid \exists \text{ exact sequence } \ \mathsf{l}(P_n) \longrightarrow \cdots \longrightarrow \mathsf{l}(P_0) \longrightarrow B \longrightarrow 0, \ \ P_i \in \mathsf{Proj}\,\mathscr{C}, \ 0 \leq i \leq n\}$$

$$\mathcal{Y}_n = \{B \in \mathscr{B} \mid \exists \text{ exact sequence } \ 0 \longrightarrow B \longrightarrow \mathsf{r}(I_0) \longrightarrow \cdots \longrightarrow \mathsf{r}(I_n), \ \ I_i \in \mathsf{Inj}\,\mathscr{C}, \ 0 \leq i \leq n\}$$

Note that $\mathsf{l}(P_i) \in \mathsf{Proj}\,\mathscr{B}$ and $\mathsf{r}(I_i) \in \mathsf{Inj}\,\mathscr{B}$, see Remark 1.1.5.

We begin with the following result which describes the quotient category $\mathscr{C}$ of a recollement.

Proposition 2.1.2. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{C}$ has enough projective and injective objects.*

(i) *We have:* ${}^{0\perp_1}\mathsf{i}(\mathscr{A}) = \mathcal{X}_1$ *and* $\mathcal{Y}_1 = \mathsf{i}(\mathscr{A})^{0\perp_1}$.

(ii) *There are equivalences*:

$$\mathsf{e}|_{\mathcal{X}_1} \ : \ \mathcal{X}_1 \xrightarrow{\ \simeq\ } \mathscr{C} \xleftarrow{\ \simeq\ } \mathcal{Y}_1 \ : \ \mathsf{e}|_{\mathcal{Y}_1}$$

Proof. (i) Let $B \in \mathcal{X}_1$. Then there exists an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r] & \mathsf{l}(P_1) \arrow[r] \arrow[d] & \mathsf{l}(P_0) \arrow[r] & B \arrow[r] & 0 \\ & & K_0 \arrow[ur] & & & \end{tikzcd}$$

with $P_1, P_0 \in \mathsf{Proj}\,\mathscr{C}$. Applying the functor $\mathsf{Hom}_{\mathscr{B}}(-, \mathsf{i}(A))$ and the adjoint pair $(\mathsf{l}, \mathsf{e})$, we infer that $\mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A)) = 0$ and $\mathsf{Ext}^1_{\mathscr{B}}(B, \mathsf{i}(A)) \simeq \mathsf{Hom}_{\mathscr{B}}(K_0, \mathsf{i}(A)) = 0$. Hence $B \in {}^{0\perp_1}\mathsf{i}(\mathscr{A})$. Conversely suppose that $B \in {}^{0\perp_1}\mathsf{i}(\mathscr{A})$. Then the object $\mathsf{e}(B) \in \mathscr{C}$ and since $\mathscr{C}$ has enough projective objects there exists an epimorphism $a_0 \colon P_0 \longrightarrow \mathsf{e}(B)$ with $P_0 \in \mathsf{Proj}\,\mathscr{C}$. Then from Proposition 1.1.6 we have the exact sequence

$$\mathsf{le}(B) \xrightarrow{\mu_B} B \xrightarrow{0} \mathsf{iq}(B) \longrightarrow 0$$

and so the morphism $\mu_B$ is an epimorphism since $\mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A)) = 0$ for every $A \in \mathscr{A}$. Hence we have the epimorphism $\mathsf{l}(a_0) \circ \mu_B \colon \mathsf{l}(P_0) \longrightarrow B$ which induces an exact sequence $0 \longrightarrow K_0 \longrightarrow \mathsf{l}(P_0) \longrightarrow B \longrightarrow 0$, i.e. $B \in \mathcal{X}_0$. From the long exact sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A)) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_0), \mathsf{i}(A)) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(K_0, \mathsf{i}(A)) \longrightarrow \mathsf{Ext}^1_{\mathscr{B}}(B, \mathsf{i}(A)) \longrightarrow 0$$

it follows that $\mathsf{Hom}_{\mathscr{B}}(K_0, \mathsf{i}(A)) \simeq \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_0), \mathsf{i}(A))$ since $\mathsf{Ext}^1_{\mathscr{B}}(B, \mathsf{i}(A)) = \mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A)) = 0$. But then $\mathsf{Hom}_{\mathscr{B}}(K_0, \mathsf{i}(A)) = 0$ for every object $A \in \mathscr{A}$ since $\mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_0), \mathsf{i}(A)) \simeq \mathsf{Hom}_{\mathscr{C}}(P_0, \mathsf{ei}(A)) = 0$. Therefore repeating the same argument as above it follows that $K_0 \in \mathcal{X}_0$ and then we infer that the object $B \in \mathcal{X}_1$. Hence $\mathcal{X}_1 = {}^{0\perp_1}\mathsf{i}(\mathscr{A})$ and similarly we obtain that $\mathcal{Y}_1 = \mathsf{i}(\mathscr{A})^{0\perp_1}$.

(ii) Clearly, $\mathsf{le}(B) \simeq B$ for every $B \in \mathcal{X}_1$. This implies that the categories $\mathscr{C}$ and $\mathcal{X}_1$ are equivalent. See also Corollary 1.2.17. Finally, similar arguments prove the other equivalence. □

The following result characterizes when an object in $\mathscr{B}$ belongs to the subcategory $\mathcal{X}_n$.

PROPOSITION 2.1.3. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories, where $\mathscr{B}$ has enough projective and injective objects and $\mathscr{C}$ has enough projective objects. Then for any $B \in \mathscr{B}$, the following are equivalent:*

(i) $B \in \mathcal{X}_k$.
(ii) $\mathsf{Ext}^n_{\mathscr{B}}(B, \mathsf{i}(A)) = 0$ *for every* $A \in \mathscr{A}$ *and* $0 \leq n \leq k$.
(iii) $\mathsf{Ext}^n_{\mathscr{B}}(B, \mathsf{i}(I)) = 0$ *for every* $I \in \mathsf{Inj}\,\mathscr{A}$ *and* $0 \leq n \leq k$.

*Then*:

$$ {}^{0\perp_k}\mathsf{i}(\mathscr{A}) = \mathcal{X}_k = {}^{0\perp_k}\mathsf{i}(\mathsf{Inj}\,\mathscr{A}) $$

PROOF. (i) $\Rightarrow$ (ii) By hypothesis there exist an exact sequence $\mathsf{l}(P_k) \longrightarrow \cdots \longrightarrow \mathsf{l}(P_1) \longrightarrow \mathsf{l}(P_0) \longrightarrow B \longrightarrow 0$ which is part of a projective resolution of $B$ since $P_i \in \mathsf{Proj}\,\mathscr{C}$ for every $i = 0, \cdots, k$. Let $A$ be an object of $\mathscr{A}$. Then from the complex $0 \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A)) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_0), \mathsf{i}(A)) \longrightarrow \cdots \longrightarrow \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_k), \mathsf{i}(A))$ it follows that $\mathsf{Ext}^n_{\mathscr{B}}(B, \mathsf{i}(A)) = 0$ for every $0 \leq n \leq k$ since $\mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_i), \mathsf{i}(A)) \simeq \mathsf{Hom}_{\mathscr{C}}(P_i, \mathsf{ei}(A)) = 0$.

(ii) $\Rightarrow$ (i) Let $k = 0$. Then as in the proof of Proposition 2.1.2 we infer that $B \in \mathcal{X}_0$ since $\mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A)) = 0$ for every $A \in \mathscr{A}$. The case $k = 1$ was done explicitly in Proposition 2.1.2. We continue inductively. Assume that the result holds for every $0 \leq n \leq k-1$. Let $\mathsf{Ext}^n_{\mathscr{B}}(B, \mathsf{i}(A)) = 0$ for every $A \in \mathscr{A}$ and $0 \leq n \leq k$. Then we have the following exact sequence:

$$ 0 \longrightarrow K_{k-1} \longrightarrow \mathsf{l}(P_{k-1}) \longrightarrow \cdots \longrightarrow \mathsf{l}(P_0) \longrightarrow B \longrightarrow 0 $$

and from the exact sequence

$$ 0 \to \mathsf{Hom}_{\mathscr{B}}(K_{k-2}, \mathsf{i}(A)) \to \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_{k-1}), \mathsf{i}(A)) \to \mathsf{Hom}_{\mathscr{B}}(K_{k-1}, \mathsf{i}(A)) \to \mathsf{Ext}^1_{\mathscr{B}}(K_{k-2}, \mathsf{i}(A)) \to 0 $$

we obtain the isomorphism

$$ \mathsf{Ext}^1_{\mathscr{B}}(K_{k-2}, \mathsf{i}(A)) \simeq \mathsf{Hom}_{\mathscr{B}}(K_{k-1}, \mathsf{i}(A)) $$

since $\mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_{k-1}), \mathsf{i}(A)) = 0$. Then from the following isomorphisms:

$$ \mathsf{Ext}^k_{\mathscr{B}}(B, \mathsf{i}(A)) \simeq \mathsf{Ext}^{k-1}_{\mathscr{B}}(K_0, \mathsf{i}(A)) \simeq \cdots \simeq \mathsf{Ext}^1_{\mathscr{B}}(K_{k-2}, \mathsf{i}(A)) \simeq \mathsf{Hom}_{\mathscr{B}}(K_{k-1}, \mathsf{i}(A)) = 0 $$

we infer that $K_{k-1} \in \mathcal{X}_0$. Thus the object $B \in \mathcal{X}_k$.

(iii) $\Leftrightarrow$ (ii) The implication (ii) $\Rightarrow$ (iii) is obvious. Let $A \in \mathscr{A}$ and $0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$ be an exact sequence with $P_0 \in \mathsf{Proj}\,\mathscr{B}$. From the exact sequence

$$ 0 \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, -) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(P_0, -) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(K_0, -) \longrightarrow \mathsf{Ext}^1_{\mathscr{B}}(B, -) \longrightarrow 0 \tag{2.1.1} $$

we have $\mathsf{Hom}_{\mathscr{B}}(P_0, \mathsf{i}(I)) \simeq \mathsf{Hom}_{\mathscr{B}}(K_0, \mathsf{i}(I))$ since $\mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(I)) = 0 = \mathsf{Ext}^1_{\mathscr{B}}(B, \mathsf{i}(I))$ for every $I \in \mathsf{Inj}\,\mathscr{A}$. Hence we obtain the isomorphism

$$ \mathsf{Hom}_{\mathscr{A}}(\mathsf{q}(P_0), I) \simeq \mathsf{Hom}_{\mathscr{A}}(\mathsf{q}(K_0), I) \tag{2.1.2} $$

for every $I \in \mathsf{Inj}\,\mathscr{A}$. Let $0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1$ be the start of an injective coresolution of $A$. Then from the isomorphism (2.1.2) we have $\mathsf{Hom}_{\mathscr{A}}(\mathsf{q}(P_0), A) \simeq \mathsf{Hom}_{\mathscr{A}}(\mathsf{q}(K_0), A)$ and hence $\mathsf{Hom}_{\mathscr{B}}(P_0, \mathsf{i}(A)) \simeq \mathsf{Hom}_{\mathscr{B}}(K_0, \mathsf{i}(A))$. Then from the exact sequence (2.1.1) it follows that $\mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A)) = 0 = \mathsf{Ext}^1_{\mathscr{B}}(B, \mathsf{i}(A))$ for every $A \in \mathscr{A}$. Then the result follows from the long exact sequence

$$ \cdots \longrightarrow \mathsf{Ext}^1_{\mathscr{B}}(B, \mathsf{i}(I)) \longrightarrow \mathsf{Ext}^1_{\mathscr{B}}(B, \mathsf{i}(\Sigma(A))) \longrightarrow \mathsf{Ext}^2_{\mathscr{B}}(B, \mathsf{i}(A)) \longrightarrow \cdots $$

obtained from the short exact sequence $0 \longrightarrow \mathsf{i}(A) \longrightarrow \mathsf{i}(I) \longrightarrow \mathsf{i}(\Sigma(A)) \longrightarrow 0$ with $I \in \mathsf{Inj}\,\mathscr{A}$. $\square$

We state now the dual result which characterizes the objects of $\mathscr{B}$ lying in $\mathcal{Y}_n$. The proof is similar with the proof of Proposition 2.1.3 and is left to the reader.

PROPOSITION 2.1.4. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories, where $\mathscr{B}$ has enough projective and injective objects and $\mathscr{C}$ has enough injective objects. Then for any $B \in \mathscr{B}$, the following are equivalent:*

(i) $B \in \mathcal{Y}_k$.
(ii) $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(A), B) = 0$ *for every* $A \in \mathscr{A}$ *and* $0 \leq n \leq k$.
(iii) $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(P), B) = 0$ *for every* $P \in \mathsf{Proj}\,\mathscr{A}$ *and* $0 \leq n \leq k$.

*Then:*

$$\mathsf{i}(\mathscr{A})^{0\perp_k} = \mathcal{Y}_k = \mathsf{i}(\mathsf{Proj}\,\mathscr{A})^{0\perp_k}$$

**2.1.2. Idempotent Functors.** Let $\mathsf{F}\colon \mathscr{B} \longrightarrow \mathscr{B}$ be an endofunctor for an abelian category $\mathscr{B}$. We recall that if $\mathsf{F}$ is a subfunctor of $\mathrm{Id}_B$, say via natural monic $\mu\colon \mathsf{F} \longrightarrow \mathrm{Id}_{\mathscr{B}}$, then $\mathsf{F}$ is called $\mathsf{radical}$ if $\mathsf{F}(\mathsf{Coker}\,\mu) = 0$, and is called $\mathsf{idempotent}$, if $\mu_{\mathsf{F}}\colon \mathsf{F}^2 \longrightarrow \mathsf{F}$ is invertible. Dually if $\mathsf{F}$ is a quotient functor of $\mathrm{Id}_{\mathscr{B}}$, say via a natural epic $\nu\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow \mathsf{F}$, then $F$ is called $\mathsf{coradical}$ if $\mathsf{F}(\mathsf{Ker}\,\nu) = 0$ and is called $\mathsf{idempotent}$, if $\nu_{\mathsf{F}}\colon \mathsf{F} \longrightarrow \mathsf{F}^2$ is invertible.

Now we use the natural maps $\mu\colon \mathsf{le} \longrightarrow \mathrm{Id}_{\mathscr{B}}$ and $\nu\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow \mathsf{re}$ to define idempotent radical subfunctors and idempotent coradical quotient functors of the identity functor of the middle part $\mathscr{B}$ of a recollement:

(i) The functor $\mathsf{F}\colon \mathscr{B} \longrightarrow \mathscr{B}$ is defined by $\mathsf{F}(B) = \mathsf{Im}\,\mu_B$ on the objects $B \in \mathscr{B}$ and given any morphism $b\colon B \longrightarrow B'$ in $\mathscr{B}$ then we get the morphism $\mathsf{F}(b)\colon \mathsf{Im}\,\mu_B \longrightarrow \mathsf{Im}\,\mu_{B'}$.
(ii) The functor $\mathsf{G}\colon \mathscr{B} \longrightarrow \mathscr{B}$ is defined by $\mathsf{G}(B) = \mathsf{Im}\,\nu_B$ on the objects $B \in \mathscr{B}$ and given any morphism $b\colon B \longrightarrow B'$ in $\mathscr{B}$ then we have the morphism $\mathsf{G}(b)\colon \mathsf{Im}\,\nu_B \longrightarrow \mathsf{Im}\,\nu_{B'}$.

PROPOSITION 2.1.5. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories. Then the functors $\mathsf{F},\mathsf{ip}\colon \mathscr{B} \longrightarrow \mathscr{B}$ are idempotent radical subfunctors of $\mathrm{Id}_{\mathscr{B}}$ and the functors $\mathsf{G},\mathsf{iq}\colon \mathscr{B} \longrightarrow \mathscr{B}$ are idempotent coradical quotient functors of $\mathrm{Id}_{\mathscr{B}}$.*

PROOF. Clearly the functor $\mathsf{F}\colon \mathscr{B} \longrightarrow \mathscr{B}$ is a subfunctor of $\mathrm{Id}_{\mathscr{B}}$ and from the following diagram

$$\begin{tikzcd}
\mathsf{lele}(B) \arrow[rr, "\mathsf{le}(\mu_B)", "\simeq"'] \arrow[dr, "\mathsf{le}(\rho_B)"'] & & \mathsf{le}(B) \arrow[rr, "\mu_B"] \arrow[dr, "\rho_B"] & & B \\
& \mathsf{le}(\mathsf{F}(B)) \arrow[ur, "\mathsf{le}(\xi_B)"] \arrow[rr, dashed, "\mu_{\mathsf{F}(B)}"'] & & \mathsf{F}(B) \arrow[ur, "\xi_B"'] &
\end{tikzcd}$$

it follows that the map $\mathsf{le}(\xi_B)$ is epic and $\mu_{\mathsf{F}(B)} = \mathsf{le}(\xi_B) \circ \rho_B$. Since the map $\mu_{\mathsf{F}(B)}$ is epic we have $\mathsf{F}^2(B) = \mathsf{Im}\,\mu_{\mathsf{F}(B)} \simeq \mathsf{F}(B)$, $\forall B \in \mathscr{B}$, i.e. the functor $\mathsf{F}$ is idempotent. Clearly $\mathsf{F}$ is radical since $\mathsf{F}(\mathsf{Coker}\,\xi_B) = \mathsf{F}(\mathsf{iq}(B)) = 0$. Similarly we prove the other claims for the functors $\mathsf{ip}, \mathsf{G}$ and $\mathsf{iq}$. $\square$

**2.1.3. Homological Embeddings.** To state our first main result of this section we need the following notion which will play an important role in the sequel.

Definition 2.1.6. An exact functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ between abelian categories is called a $k$-homological embedding, $k \geq 0$, if the map $\mathsf{i}^n_{X,Y}\colon \mathsf{Ext}^n_{\mathscr{A}}(X,Y) \longrightarrow \mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(X),\mathsf{i}(Y))$ is invertible, $\forall X, Y \in \mathscr{A}$ and $0 \leq n \leq k$. The functor $\mathsf{i}$ is called a homological embedding, if $\mathsf{i}$ is a $k$-homological embedding, $\forall k \geq 0$.

Remark 2.1.7.

(i) Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories. Then the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is 1-homological embeding since $\mathsf{i}^n_{X,Y}\colon \mathsf{Ext}^n_{\mathscr{A}}(X,Y) \longrightarrow \mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(X),\mathsf{i}(Y))$ is an isomorphism for every $X, Y \in \mathscr{A}$ and $n = 0, 1$. This is a well-known result due to Oort, see [101], and can be easily proved using that $\mathscr{A}$ is a Serre subcategory of $\mathscr{B}$.

(ii) If $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a $k$-homological embedding, then the map
$$\mathsf{i}^{k+1}_{X,Y}\colon \mathsf{Ext}^{k+1}_{\mathscr{A}}(X,Y) \longrightarrow \mathsf{Ext}^{k+1}_{\mathscr{B}}(\mathsf{i}(X),\mathsf{i}(Y))$$
is a monomorphism for every $X, Y \in \mathscr{A}$, see [101].

The following example shows that there are examples of functors which are $k$-homological embedding but fail to be $(k+1)$-homological embeddings.

Example 2.1.8. Let $\Lambda$ be the path algebra of the quiver
$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow k \longrightarrow k+1 \longrightarrow k+2$$
modulo the ideal generated by the paths of lenght 2. Then, as in Example 1.1.10, consider the recollement of abelian categories $(\mathsf{mod}\text{-}\Lambda/\mathfrak{U}, \mathsf{mod}\text{-}\Lambda, \mathsf{mod}\text{-}\mathsf{End}_{\Lambda}(P_3 \oplus \cdots \oplus P_{k+1}))$, where the idempotent ideal $\mathfrak{U} = \tau_{P_3 \oplus \cdots \oplus P_{k+1}}(\Lambda)$ is the trace of the projective $\Lambda$-module $P_3 \oplus \cdots \oplus P_{k+1}$ in $\Lambda$. Auslander, Platzeck and Todorov proved in [14] that $\mathfrak{U}$ is $k$-idempotent but not $k+1$-idempotent, i.e. the map $\mathsf{i}^n_{X,Y}\colon \mathsf{Ext}^n_{\Lambda/\mathfrak{U}}(X,Y) \longrightarrow \mathsf{Ext}^n_{\Lambda}(X,Y)$ is invertible, $\forall X, Y \in \mathsf{mod}\text{-}\Lambda/\mathfrak{U}$ and $n = 0, 1, \cdots, k$, but it is not invertible for $n = k+1$. It follows that the functor $\mathsf{i}\colon \mathsf{mod}\text{-}\Lambda/\mathfrak{U} \longrightarrow \mathsf{mod}\text{-}\Lambda$ is a $k$-homological embedding but not a $(k+1)$-homological embedding. For more details see [14].

We need the following well known lemma.

Lemma 2.1.9. *Let $\mathscr{A}$ be an abelian category and $A \xrightarrow{g} B \xrightarrow{f} C$ a sequence of morphisms such that*
$$\mathsf{Hom}_{\mathscr{A}}(X,A) \xrightarrow{g_*} \mathsf{Hom}_{\mathscr{A}}(X,B) \xrightarrow{f_*} \mathsf{Hom}_{\mathscr{A}}(X,C)$$
*is exact for every $X \in \mathscr{A}$. Then $A \xrightarrow{g} B \xrightarrow{f} C$ is exact.*

Proof. For $X = A$ we have the exact sequence
$$\mathsf{Hom}_{\mathscr{A}}(A,A) \xrightarrow{g_*} \mathsf{Hom}_{\mathscr{A}}(A,B) \xrightarrow{f_*} \mathsf{Hom}_{\mathscr{A}}(A,C)$$
and consider the morphisms $\mathrm{Id}_A \in \mathsf{Hom}_{\mathscr{A}}(A,A)$ and $g \in \mathsf{Hom}_{\mathscr{A}}(A,B)$. Then since $g_* \circ f_* = 0$ it follows that $f_*(g) = 0$, i.e. $g \circ f = 0$, and therefore we have $\mathsf{Im}\, g \subseteq \mathsf{Ker}\, f$. Let $a\colon X \longrightarrow B$ be a morphism such that $a \circ f = 0$. Then $f_*(a) = 0$ and thus there exists a morphism $b\colon X \longrightarrow A$ such that $g_*(b) = a$, i.e. $b \circ g = a$. We infer that $\mathsf{Ker}\, f \subseteq \mathsf{Im}\, g$. Hence the sequence $A \longrightarrow B \longrightarrow C$ is exact. □

We are ready now to prove our first main result which gives characterizations for the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ to be a $k$-homological embedding.

THEOREM 2.1.10. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ has enough projective and injective objects. Then the following statements are equivalent.*

(i) *The functor $\mathsf{i} \colon \mathscr{A} \longrightarrow \mathscr{B}$ is a $k$-homological embedding.*
(ii) $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(Y)) = 0$, *for every $P \in \mathsf{Proj}\,\mathscr{A}$, $Y \in \mathscr{A}$ and $1 \leq n \leq k$.*
(iii) $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(X), \mathsf{i}(I)) = 0$, *for every $I \in \mathsf{Inj}\,\mathscr{A}$, $X \in \mathscr{A}$ and $1 \leq n \leq k$.*
(iv) $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(I)) = 0$, *for every $P \in \mathsf{Proj}\,\mathscr{A}$, $I \in \mathsf{Inj}\,\mathscr{A}$ and $1 \leq n \leq k$.*
(v) $\mathsf{F}|_{\mathsf{Proj}\,\mathscr{B}} \in \mathcal{X}_{k-1}$.
(vi) $\mathsf{G}|_{\mathsf{Inj}\,\mathscr{B}} \in \mathcal{Y}_{k-1}$.
(vii) $\mathsf{R}^n\mathsf{p}(\mathsf{i}(I)) = 0$ *for every $I \in \mathsf{Inj}\,\mathscr{A}$ and $1 \leq n \leq k$.*
(viii) $\mathsf{R}^n\mathsf{p}(\mathsf{i}(A)) = 0$ *for every $A \in \mathscr{A}$ and $1 \leq n \leq k$.*
(ix) $\mathsf{L}_n\mathsf{q}(\mathsf{i}(P)) = 0$ *for every $P \in \mathsf{Proj}\,\mathscr{A}$ and $1 \leq n \leq k$.*
(x) $\mathsf{L}_n\mathsf{q}(\mathsf{i}(A)) = 0$ *for every $A \in \mathscr{A}$ and $1 \leq n \leq k$.*

PROOF. (i) $\Rightarrow$ (ii) Let $P \in \mathsf{Proj}\,\mathscr{A}$. Then $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(Y)) \simeq \mathsf{Ext}^n_{\mathscr{A}}(P, Y) = 0$ for all $1 \leq n \leq k$ and $Y \in \mathscr{A}$.

(ii) $\Rightarrow$ (i) Let $X, Y \in \mathscr{A}$ and let $0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be an exact sequence with $P_0 \in \mathsf{Proj}\,\mathscr{A}$. From Remark 2.1.7(i) we know that the map $\mathsf{i}^1_{X,Y} \colon \mathsf{Ext}^1_{\mathscr{A}}(X, Y) \longrightarrow \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{i}(X), \mathsf{i}(Y))$ is an isomorphism for every $X, Y \in \mathscr{A}$. Then from the following exact commutative diagram:

$$\begin{tikzcd}
0 = \mathsf{Ext}^1_{\mathscr{A}}(P_0, Y) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Ext}^1_{\mathscr{A}}(K_0, Y) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathsf{Ext}^2_{\mathscr{A}}(X, Y) \arrow[r] \arrow[d, "\mathsf{i}^2_{X,Y}"'] & \mathsf{Ext}^2_{\mathscr{A}}(P_0, Y) = 0 \arrow[d] \\
0 = \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{i}(P_0), \mathsf{i}(Y)) \arrow[r] & \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{i}(K_0), \mathsf{i}(Y)) \arrow[r, "\simeq"] & \mathsf{Ext}^2_{\mathscr{B}}(\mathsf{i}(X), \mathsf{i}(Y)) \arrow[r] & \mathsf{Ext}^2_{\mathscr{B}}(\mathsf{i}(P_0), \mathsf{i}(Y)) = 0
\end{tikzcd}$$

it follows that the map $\mathsf{i}^2_{X,Y}$ is invertible. Continuing the above long exact $\mathsf{Ext}$-sequences we infer that $\mathsf{i}^3_{X,Y}$ is invertible since so is the map $\mathsf{i}^2_{K_0,Y}$. Then the result follows by induction on $n$.

(i) $\Leftrightarrow$ (iii) This is similar to the proof of the equivalence (i) $\Leftrightarrow$ (ii).

(ii) $\Leftrightarrow$ (iv) The direction (ii) $\Rightarrow$ (iv) is clear, so it remains to prove that (iv) $\Rightarrow$ (ii). Let $Y \in \mathscr{A}$ and $0 \longrightarrow Y \longrightarrow I \longrightarrow \Sigma Y \longrightarrow 0$ be exact, where $I \in \mathsf{Inj}\,\mathscr{A}$. Then from the exact commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathscr{A}}(P, Y) \arrow[r, tail] \arrow[d, "\simeq"] & \mathsf{Hom}_{\mathscr{A}}(P, I) \arrow[r, two heads] \arrow[d, "\simeq"] & \mathsf{Hom}_{\mathscr{A}}(P, \Sigma(Y)) \arrow[d, "\simeq"] & \\
\mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(Y)) \arrow[r, tail] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(I)) \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(\Sigma(Y))) \arrow[r] & \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(Y)) \arrow[r] & 0
\end{tikzcd}$$

we have $\mathsf{Ext}^1_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(Y)) = 0$ for every $Y \in \mathscr{A}$. Then the result follows from the long exact $\mathsf{Ext}$-sequence of the above diagram using induction on $n$.

(ii) $\Leftrightarrow$ (v) Let $Y \in \mathscr{A}$ and $P \in \mathsf{Proj}\,\mathscr{B}$. From Proposition 1.1.6 and Proposition 2.1.5 we have the short exact sequence $0 \longrightarrow \mathsf{F}(P) \longrightarrow P \longrightarrow \mathsf{iq}(P) \longrightarrow 0$. Since the functor $\mathsf{i}$ is fully faithful and $(\mathsf{q}, \mathsf{i})$ is an adjoint pair we have $\mathsf{Hom}_{\mathscr{B}}(\mathsf{iq}(P), \mathsf{i}(Y)) \simeq \mathsf{Hom}_{\mathscr{B}}(P, \mathsf{i}(Y))$ and then we get the isomorphism:

$$\mathsf{Ext}^{n+1}_{\mathscr{B}}(\mathsf{iq}(P), \mathsf{i}(Y)) \simeq \mathsf{Ext}^n_{\mathscr{B}}(\mathsf{F}(P), \mathsf{i}(Y))$$

for every $n \geq 0$. Then since $\mathsf{Proj}\,\mathscr{A} = \mathsf{add}\,\mathsf{q}(\mathsf{Proj}\,\mathscr{B})$ we infer that (ii) holds if and only if $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{iq}(P), \mathsf{i}(Y)) = 0$ for every $P \in \mathsf{Proj}\,\mathscr{B}$, $Y \in \mathscr{A}$ and $1 \leq n \leq k$. This is equivalent to $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{F}(P), \mathsf{i}(Y)) = 0$ for every $P \in \mathsf{Proj}\,\mathscr{B}$, $Y \in \mathscr{A}$ and $0 \leq n \leq k-1$. Then from Proposition 2.1.3 this is equivalent to $\mathsf{F}(P) \in \mathcal{X}_{k-1}$ and so we are done.

(iii) $\Leftrightarrow$ (vi) The proof of this equivalence is dual to (ii) $\Leftrightarrow$ (v) and is left to the reader.

(i) $\Rightarrow$ (vii) Let $I \in \mathsf{Inj}\,\mathscr{A}$ and $0 \longrightarrow \mathsf{i}(I) \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^k$ an injective coresolution of $\mathsf{i}(I)$ in $\mathscr{B}$. Since $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(X), \mathsf{i}(I)) = 0$, for any $X \in \mathscr{A}$ and $1 \leq n \leq k$, the complex

$$0 \longrightarrow \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(X), \mathsf{i}(I)) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(X), I^0) \longrightarrow \cdots \longrightarrow \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(X), I^k)$$

is exact and clearly it is isomorphic to the complex

$$(*)\colon 0 \longrightarrow \mathsf{Hom}_{\mathscr{A}}(X, I) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(X, \mathsf{p}(I^0)) \longrightarrow \cdots \longrightarrow \mathsf{Hom}_{\mathscr{A}}(X, \mathsf{p}(I^k))$$

Then from Lemma 2.1.9 it follows that the sequence

$$0 \longrightarrow I \longrightarrow \mathsf{p}(I^0) \longrightarrow \mathsf{p}(I^1) \longrightarrow \cdots \longrightarrow \mathsf{p}(I^k)$$

is exact since the complex $(*)$ is exact for every $X \in \mathscr{A}$. We infer that $\mathsf{R}^n\mathsf{p}(\mathsf{i}(I)) = 0$ for every $1 \leq n \leq k$.

(vii) $\Rightarrow$ (viii) Let $A \in \mathscr{A}$ and

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & I^k \\
& & \Sigma(A) \arrow[ur] & \Sigma^2(A) \arrow[ur] & &
\end{tikzcd}$$

an injective coresolution of $A$ in $\mathscr{A}$. Then if we apply the functor $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ to the short exact sequence $0 \longrightarrow \mathsf{i}(A) \longrightarrow \mathsf{i}(I^0) \longrightarrow \mathsf{i}(\Sigma(A)) \longrightarrow 0$ we get the following long exact sequence

$$0 \to A \to I^0 \to \Sigma(A) \to \mathsf{R}^1\mathsf{p}(\mathsf{i}(A)) \to \mathsf{R}^1\mathsf{p}(\mathsf{i}(I^0)) \to \mathsf{R}^1\mathsf{p}(\mathsf{i}(\Sigma(A))) \to \mathsf{R}^2\mathsf{p}(\mathsf{i}(A)) \to \mathsf{R}^2\mathsf{p}(\mathsf{i}(I^0))$$

Since $\mathsf{R}^1\mathsf{p}(\mathsf{i}(I^0)) = 0 = \mathsf{R}^2\mathsf{p}(\mathsf{i}(I^0))$ it follows that $\mathsf{R}^1\mathsf{p}(\mathsf{i}(A)) = 0$ and $\mathsf{R}^1\mathsf{p}(\mathsf{i}(\Sigma(A))) \simeq \mathsf{R}^2\mathsf{p}(\mathsf{i}(A))$ $(*)$. On the other hand from the short exact sequence $0 \longrightarrow \mathsf{i}(\Sigma(A)) \longrightarrow \mathsf{i}(I^1) \longrightarrow \mathsf{i}(\Sigma^2(A)) \longrightarrow 0$ we have the exact sequence $0 \longrightarrow \Sigma(A) \longrightarrow I^1 \longrightarrow \Sigma^2(A) \longrightarrow \mathsf{R}^1\mathsf{p}(\mathsf{i}(\Sigma(A))) \longrightarrow 0$ and $(*)$ gives $\mathsf{R}^2\mathsf{p}(\mathsf{i}(A)) = 0$. Then continuing with the same procedure we infer that $\mathsf{R}^n\mathsf{p}(\mathsf{i}(A)) = 0$ for every $1 \leq n \leq k$.

(viii) $\Rightarrow$ (i) Let $X, Y \in \mathscr{A}$ and $0 \longrightarrow \mathsf{i}(Y) \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^k$ an injective coresolution of $\mathsf{i}(Y)$ in $\mathscr{B}$. Then the sequence $0 \longrightarrow Y \longrightarrow \mathsf{p}(I^0) \longrightarrow \mathsf{p}(I^1) \longrightarrow \cdots \longrightarrow \mathsf{p}(I^k)$ is an injective coresolution of $Y$ in $\mathscr{A}$ since $\mathsf{R}^n\mathsf{p}(\mathsf{i}(Y)) = 0$ for every $1 \leq n \leq k$. Therefore from the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(X), \mathsf{i}(Y)) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(X), I^0) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(X), I^1) \arrow[r] \arrow[d, "\simeq"'] & \cdots \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(X), I^k) \arrow[d, "\simeq"'] \\
0 \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(X, Y) \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(X, \mathsf{p}(I^0)) \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(X, \mathsf{p}(I^1)) \arrow[r] & \cdots \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(X, \mathsf{p}(I^k))
\end{tikzcd}$$

it follows that the map $\mathsf{i}^n_{X,Y}\colon \mathsf{Ext}^n_{\mathscr{A}}(X, Y) \longrightarrow \mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(X), \mathsf{i}(Y))$ is invertible for every $1 \leq n \leq k$. Therefore the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a $k$-homological embedding.

Dually we prove the implications (i) $\Rightarrow$ (ix) $\Rightarrow$ (x) $\Rightarrow$ (i) and so the proof is finished. $\square$

Closely related to $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ being a $k$-homological embedding, also in connection with the behavior of perpendicular categories, is the question of when the map $\mathsf{e}^n_{Z,W}\colon \mathsf{Ext}^n_{\mathscr{B}}(Z,W) \longrightarrow \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W))$, induced from the exact quotient functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$, is invertible for every $0 \leq n \leq k$. In this respect we have the following result.

THEOREM 2.1.11. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Then the following statements are equivalent.*

(i) *The map $\mathsf{e}^n_{Z,W}\colon \mathsf{Ext}^n_{\mathscr{B}}(Z,W) \longrightarrow \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W))$ is invertible, $\forall\, W \in \mathscr{B}$, (resp. $\forall\, Z \in \mathscr{B}$), and $0 \leq n \leq k$.*

(ii) *$Z \in \mathcal{X}_{k+1}$ (resp. $W \in \mathcal{Y}_{k+1}$).*

PROOF. (ii) $\Rightarrow$ (i) Let $k = 0$. Since $Z \in \mathcal{X}_1$ there exists an exact sequence $\mathsf{l}(P_1) \longrightarrow \mathsf{l}(P_0) \longrightarrow Z \longrightarrow 0$ with $P_1, P_0 \in \mathsf{Proj}\,\mathscr{C}$. Then $P_1 \longrightarrow P_0 \longrightarrow \mathsf{e}(Z) \longrightarrow 0$ is exact in $\mathscr{C}$ and since $(\mathsf{l},\mathsf{e})$ is an adjoint pair we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(Z,W) \arrow[r] \arrow[d, dashed] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_0),W) \arrow[r] \arrow[d, "\simeq"] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_1),W) \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W)) \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(P_0,\mathsf{e}(W)) \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(P_1,\mathsf{e}(W))
\end{tikzcd}$$

Hence the map $\mathsf{e}^0_{Z,W}\colon \mathsf{Hom}_{\mathscr{B}}(Z,W) \longrightarrow \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W))$ is invertible. Suppose now that $k = 1$, i.e. $Z \in \mathcal{X}_2$. Then we have the exact sequence $\mathsf{l}(P_2) \longrightarrow \mathsf{l}(P_1) \longrightarrow \mathsf{l}(P_0) \longrightarrow Z \longrightarrow 0$ where $P_0, P_1, P_2 \in \mathsf{Proj}\,\mathscr{C}$ and let $K_0$ be the kernel of the morphism $\mathsf{l}(P_0) \longrightarrow Z$. Then $K_0 \in \mathcal{X}_1$ and the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(Z,W) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(P_0),W) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathscr{B}}(K_0,W) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Ext}^1_{\mathscr{B}}(Z,W) \arrow[r] \arrow[d, "e^1_{Z,W}"'] & 0 \\
0 \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W)) \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(P_0,\mathsf{e}(W)) \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(K_0),\mathsf{e}(W)) \arrow[r] & \mathsf{Ext}^1_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W)) \arrow[r] & 0
\end{tikzcd}$$

shows that the map $\mathsf{e}^1_{Z,W}$ is invertible. Finally suppose that $Z \in \mathcal{X}_{k+1}$ and that the result holds for every object $B \in \mathcal{X}_m$ and $m < k+1$, i.e. the map $\mathsf{e}^{m-1}_{B,W}$ is invertible. Then the object $K_0 \in \mathcal{X}_k$ and hence from our induction hypothesis the map $\mathsf{e}^{k-1}_{K_0,W}$ is invertible. Therefore from the commutative diagram

$$\begin{tikzcd}
\mathsf{Ext}^{k-1}_{\mathscr{B}}(K_0,W) \arrow[r, "\simeq"] \arrow[d, "\mathsf{e}^{k-1}_{K_0,W}"', "\simeq"] & \mathsf{Ext}^k_{\mathscr{B}}(Z,W) \arrow[d, "\mathsf{e}^k_{Z,W}"] \\
\mathsf{Ext}^{k-1}_{\mathscr{C}}(\mathsf{e}(K_0),\mathsf{e}(W)) \arrow[r, "\simeq"] & \mathsf{Ext}^k_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W))
\end{tikzcd}$$

we deduce that the map $\mathsf{e}^k_{Z,W}\colon \mathsf{Ext}^k_{\mathscr{B}}(Z,W) \longrightarrow \mathsf{Ext}^k_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W))$ is invertible for every $W \in \mathscr{B}$.

(i) $\Rightarrow$ (ii) Assume that the map $\mathsf{e}^n_{Z,W}$ is invertible for every $W \in \mathscr{B}$ and $0 \leq n \leq k$. We will prove that $Z \in \mathcal{X}_{k+1}$. Hence from Proposition 2.1.3 we have to show that $\mathsf{Ext}^n_{\mathscr{B}}(Z,\mathsf{i}(I)) = 0$ for every $I \in \mathsf{Inj}\,\mathscr{A}$ and $0 \leq n \leq k+1$. Let us suppose first that $k = 0$. This means that we have the isomorphism $\mathsf{Hom}_{\mathscr{B}}(Z,W) \simeq \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(W))$ for every $W \in \mathscr{B}$ and we have to prove that $Z \in \mathcal{X}_1$. Let $I \in \mathsf{Inj}\,\mathscr{A}$ and

$$0 \longrightarrow \mathsf{i}(I) \longrightarrow J \longrightarrow \Sigma(\mathsf{i}(I)) \longrightarrow 0 \qquad (*)$$

be an exact sequence with $J \in \mathsf{Inj}\,\mathscr{B}$. Then $e(J) \simeq e(\Sigma(\mathsf{i}(I)))$ and from the following long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(Z,\mathsf{i}(I)) \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(Z,J) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathscr{B}}(Z,\Sigma(\mathsf{i}(I))) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Ext}^1_{\mathscr{B}}(Z,\mathsf{i}(I)) \arrow[r] & 0 \\ & & \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(J)) \arrow[r, "\cong"] & \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(\Sigma(\mathsf{i}(I)))) & & \end{tikzcd}$$

we derive that $\mathsf{Hom}_{\mathscr{B}}(Z,\mathsf{i}(I)) = 0 = \mathsf{Ext}^1_{\mathscr{B}}(Z,\mathsf{i}(I))$. Thus $Z \in \mathfrak{X}_1$. Assume now that the map $\mathsf{e}^n_{Z,W}$ is invertible for every $W \in \mathscr{B}$ and $0 \leq n \leq k$. Then we have $\mathsf{Ext}^n_{\mathscr{B}}(Z,\mathsf{i}(I)) \simeq \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{ei}(I))$ for every $0 \leq n \leq k$ and therefore we obtain that $\mathsf{Ext}^n_{\mathscr{B}}(Z,\mathsf{i}(I)) = 0$ for every $0 \leq n \leq k$. Hence it remains to prove that $\mathsf{Ext}^{k+1}_{\mathscr{B}}(Z,\mathsf{i}(I)) = 0$. From the long exact homology sequence obtained from the exact sequence $(*)$ we have the following isomorphisms:

$$\begin{aligned} \mathsf{Ext}^{k+1}_{\mathscr{B}}(Z,\mathsf{i}(I)) &\simeq \mathsf{Ext}^{k}_{\mathscr{B}}(Z,\Sigma(\mathsf{i}(I))) \simeq \mathsf{Ext}^{k}_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(\Sigma(\mathsf{i}(I)))) \\ &\simeq \mathsf{Ext}^{k}_{\mathscr{C}}(\mathsf{e}(Z),\mathsf{e}(J)) \simeq \mathsf{Ext}^{k}_{\mathscr{B}}(Z,J) \end{aligned}$$

This clearly implies that $Z \in \mathfrak{X}_{k+1}$ since $\mathsf{Ext}^{k+1}_{\mathscr{B}}(Z,\mathsf{i}(I)) = 0$. □

As a consequence of Theorem 2.1.10 and Theorem 2.1.11 we have the following.

Corollary 2.1.12. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Then the following statements are equivalent.*

(i) *The functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a $(k+1)$-homological embedding.*
(ii) $\mathsf{F}|_{\mathsf{Proj}\,\mathscr{B}} \in \mathfrak{X}_k$.
(iii) *The map $\mathsf{e}^n_{\mathsf{F}(P),-}\colon \mathsf{Ext}^n_{\mathscr{B}}(\mathsf{F}(P),-) \longrightarrow \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(\mathsf{F}(P)),\mathsf{e}(-))$ is invertible, for every $P \in \mathsf{Proj}\,\mathscr{B}$ and $0 \leq n \leq k-1$.*
(iv) $\mathsf{G}|_{\mathsf{Inj}\,\mathscr{B}} \in \mathfrak{Y}_k$.
(v) *The map $\mathsf{e}^n_{-,\mathsf{G}(I)}\colon \mathsf{Ext}^n_{\mathscr{B}}(-,\mathsf{G}(I)) \longrightarrow \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(-),\mathsf{e}(\mathsf{G}(I)))$ is invertible, for every $I \in \mathsf{Inj}\,\mathscr{B}$ and $0 \leq n \leq k-1$.*

Let $R$ be a ring with an idempotent element $e \in R$. In this case the full subcategory $\mathfrak{X}_n$ of $\mathsf{Mod}\text{-}R$ consists of all objects $B \in \mathscr{B}$ such that there exists an exact sequence $Re \otimes_{eRe} P_n \longrightarrow \cdots \longrightarrow Re \otimes_{eRe} P_0 \longrightarrow B \to 0$ with $P_i \in \mathsf{Proj}\,(eRe)$, and dually we have an analogous description for $\mathfrak{Y}_n$.

The next consequences of Theorems 2.1.10 and 2.1.11 generalize results of Auslander-Platzeck-Todorov [14].

Corollary 2.1.13. *Let $(\mathsf{Mod}\text{-}R/ReR, \mathsf{Mod}\text{-}R, \mathsf{Mod}\text{-}eRe)$ be a recollement of rings. Then the following statements are equivalent.*

(i) *The functor $\mathsf{i}\colon \mathsf{Mod}\text{-}R/ReR \longrightarrow \mathsf{Mod}\text{-}R$ is a $(k+1)$-homological embedding.*
(ii) $\mathsf{Ext}^n_R(R/ReR,Y) = 0$ *for every $Y \in \mathsf{Mod}\text{-}R/ReR$ and $1 \leq n \leq k+1$.*
(iii) $\mathsf{Ext}^n_R(X,I) = 0$ *for every $X \in \mathsf{Mod}\text{-}R/ReR$, $I \in \mathsf{Inj}\,(R/ReR)$ and $1 \leq n \leq k+1$.*
(iv) $\mathsf{Ext}^n_R(R/ReR,I) = 0$ *for every $I \in \mathsf{Inj}\,(R/ReR)$ and $1 \leq n \leq k+1$.*
(v) $ReR \in \mathfrak{X}_k$.
(vi) *The map $e^n_{ReR,W}\colon \mathsf{Ext}^n_R(ReR,W) \longrightarrow \mathsf{Ext}^n_{eRe}(eR,eW)$ is invertible for every $W \in \mathsf{Mod}\text{-}R$ and $0 \leq n \leq k-1$.*
(vii) $\mathsf{Tor}^R_n(R/ReR,Y) = 0$ *for every $Y \in \mathsf{Mod}\text{-}R/ReR$ and $1 \leq n \leq \kappa+1$.*

(viii) $\mathsf{Tor}_n^R(R/ReR, R/ReR) = 0$ *for every* $1 \leq n \leq \kappa + 1$.

Part (ii) of the following consequence of Corollary 2.1.13 gives the connections between homological embeddings and homological epimorphisms in the sense of Geigle-Lenzing [56].

Corollary 2.1.14. *Let $R$ be a ring and $e^2 = e$ an idempotent element of $R$.*

(i) *The map $e^n_{Z,W}\colon \mathsf{Ext}^n_R(Z, W) \longrightarrow \mathsf{Ext}^n_{eRe}(eZ, eW)$ is invertible for every $0 \leq n \leq k$ and for every $W \in \mathsf{Mod}\text{-}R$ (resp. $Z \in \mathsf{Mod}\text{-}R$) if and only if $Z \in \mathcal{X}_{k+1}$ (resp. $W \in \mathcal{Y}_{k+1}$).*

(ii) *The following are equivalent:*
   (a) *The natural map $R \longrightarrow R/ReR$ is a homological epimorphism of rings.*
   (b) *$ReR \in \mathcal{X}_\infty$.*
   (c) *The functor $\mathsf{i}\colon \mathsf{Mod}\text{-}R/ReR \longrightarrow \mathsf{Mod}\text{-}R$ is a homological embedding.*

Example 2.1.15. Let $R$ be a ring and $e$ an idempotent element of $R$ such that the idempotent ideal $ReR$ is a projective $R$-module. Let $Y$ be a $R/ReR$-module. Then $\mathsf{Ext}^1_R(R/ReR, Y) = 0$ and since $\mathsf{pd}_R\, R/ReR \leq 1$ we have $\mathsf{Ext}^n_R(R/ReR, Y) = 0$ for every $n \geq 1$. Therefore from Corollary 2.1.13 it follows that the functor $\mathsf{i}\colon \mathsf{Mod}\text{-}R/ReR \longrightarrow \mathsf{Mod}\text{-}R$ is a homological embedding.

**2.1.4. Syzygies and Extensions.** Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. In this subsection we investigate objects $B \in \mathscr{B}$ such that there exists a projective resolution of the form:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathsf{l}(Q_2) \arrow[r] & \mathsf{l}(Q_1) \arrow[r] & \mathsf{l}(Q_0) \arrow[r, dashed] \arrow[d] & P_{k-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & B \arrow[r] & 0 \\
& & & \Omega^k_{\mathscr{B}}(B) \arrow[ur, hook] & & & & &
\end{tikzcd}$$

In other words, we are interested for objects $B \in \mathscr{B}$ with a syzygy $\Omega^k_{\mathscr{B}}(B)$, for some $k \gg 0$, such that $\Omega^k_{\mathscr{B}}(B) \in \mathcal{X}_\infty$. If such an object exists then from Theorem 2.1.11 we have the isomorphism $\mathsf{Ext}^n_{\mathscr{B}}(\Omega^k_{\mathscr{B}}(B), W) \simeq \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(\Omega^k(B)), \mathsf{e}(W))$ for every $n \geq 0$ and $W \in \mathscr{B}$. Hence we can ask the following natural question:

Can we obtain an isomorphism $\mathsf{Ext}^n_{\mathscr{B}}(B, W) \xrightarrow{\simeq} \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(B), \mathsf{e}(W))$ for some $n \geq \lambda$ when the object $B$ has a syzygy such that $\Omega^k(B) \in \mathcal{X}_\infty$ for some $k \gg 0$ ?

The motivation for this question is the finite generation hypothesis on Hochschild cohomology, see [110] for more details. Indeed, we will show, under a condition, that the above question has an affirmative answer. For the purpose of our application we need to work with extensions.

Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories and $e\colon \mathscr{C} \longrightarrow \mathscr{D}$ an exact functor. If

$$\xi\colon 0 \longrightarrow X_n \xrightarrow{f_n} X_{n-1} \cdots \to X_1 \xrightarrow{f_1} X_0 \longrightarrow 0$$

is an exact sequence in $\mathscr{C}$, then we denote by $e(\xi)$ the following exact sequence

$$e(\xi)\colon 0 \longrightarrow e(X_n) \xrightarrow{e(f_n)} e(X_{n-1}) \longrightarrow \cdots \longrightarrow e(X_1) \xrightarrow{e(f_1)} e(X_0) \longrightarrow 0$$

in $\mathscr{D}$. It is clear that this operation commutes with Yoneda product; that is, if $\xi$ and $\zeta$ are composable exact sequences in $\mathscr{C}$, then $e(\xi\zeta) = e(\xi) \cdot e(\zeta)$. For every pair of objects $X$ and $Y$ in $\mathscr{C}$ and every nonnegative integer $i$, we define a group homomorphism

$$e^i_{X,Y}\colon \mathsf{Ext}^i_{\mathscr{C}}(X, Y) \longrightarrow \mathsf{Ext}^i_{\mathscr{D}}(e(X), e(Y))$$

by

$$e^0_{X,Y}(f) = e(f) \qquad \text{for a morphism } f\colon X \longrightarrow Y;$$
$$e^i_{X,Y}([\eta]) = [e(\eta)] \qquad \text{for an } i\text{-fold extension } \eta \text{ of } X \text{ by } Y, \text{ where } i > 0.$$

To proceed further we need the following easy result.

Lemma 2.1.16. *Let $n$ be an integer, and let*

$$\epsilon\colon 0 \longrightarrow X \longrightarrow E_{m-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow Y \longrightarrow 0$$

*be an exact sequence where $\mathsf{pd}\, E_i \leq n$ for every $i$. Then for every $i > n$, the map*

$$\epsilon^*\colon \mathsf{Ext}^i_{\mathscr{A}}(X, Z) \longrightarrow \mathsf{Ext}^{i+m}_{\mathscr{A}}(Y, Z)$$

*given by $\epsilon^*([\eta]) = [\eta\epsilon]$ is an isomorphism.*

Proof. If $0 \longrightarrow X \longrightarrow E_0 \longrightarrow Y \longrightarrow 0$ is an exact sequence with $\mathsf{pd}_{\mathscr{A}}\, E_0 \leq n$ then from the long exact homology sequence we obtain that $\mathsf{Ext}^i_{\mathscr{A}}(X, Z) \simeq \mathsf{Ext}^{i+1}_{\mathscr{A}}(Y, Z)$ for every $i > n$ and $Z \in \mathscr{A}$. Then continuing inductively on the length of $\epsilon$ we get the isomorphism $\epsilon^*$ for every $i > n$. □

Now we are ready to prove the following result which answers the question stated above.

Theorem 2.1.17. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories, where $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Let $n$ be an integer, and assume that $\mathsf{pd}_{\mathscr{C}}\, e(P) \leq n$ for every projective object $P$ in $\mathscr{B}$. Let $B$ be an object of $\mathscr{B}$ which has a projective resolution of the form*

$$\cdots \longrightarrow \mathsf{l}(Q_2) \longrightarrow \mathsf{l}(Q_1) \longrightarrow \mathsf{l}(Q_0) \longrightarrow P_{m-1} \longrightarrow P_{m-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow B \longrightarrow 0,$$

*where every $Q_i$ is a projective object of $\mathscr{C}$. Then for every $i > m + n$ and every object $B'$ of $\mathscr{B}$, the map*

$$e^i_{B,B'}\colon \mathsf{Ext}^i_{\mathscr{B}}(B, B') \xrightarrow{\cong} \mathsf{Ext}^i_{\mathscr{C}}(e(B), e(B'))$$

*is an isomorphism.*

Proof. Let

$$\pi\colon 0 \longrightarrow M_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

be the beginning of the chosen projective resolution of $B$, where $M_m$ denotes the $m$th syzygy of $B$. Consider the following group homomorphisms:

$$\mathsf{Ext}^i_{\mathscr{B}}(B, B') \xleftarrow{\pi^*} \mathsf{Ext}^{i-m}_{\mathscr{B}}(M_m, B') \xrightarrow{e^{i-m}_{M_m,B'}} \mathsf{Ext}^{i-m}_{\mathscr{C}}(e(M_m), e(B')) \xrightarrow{(e(\pi))^*} \mathsf{Ext}^i_{\mathscr{C}}(e(B), e(B')) \tag{2.1.3}$$

Here, the maps $\pi^*$ and $(e(\pi))^*$ are isomorphisms by Lemma 2.1.16 (for $(e(\pi))^*$ we use the fact that $\mathsf{pd}_{\mathscr{C}}\, e(P) \leq n$ for every projective object $P$ in $\mathscr{B}$). The map $e^{i-m}_{M_m,B'}$ is an isomorphism by Theorem 2.1.11. Thus, we have an isomorphism

$$(e(\pi))^* \circ e^{i-m}_{M_m,B'} \circ (\pi^*)^{-1}\colon \mathsf{Ext}^i_{\mathscr{B}}(B, B') \longrightarrow \mathsf{Ext}^i_{\mathscr{C}}(e(B), e(B'))$$

We want to show that this is the same map as $e^i_{B,B'}$. We consider an element $[\eta] \in \mathsf{Ext}^{i-m}_{\mathscr{B}}(M_m, B')$, and follow it through the maps (2.1.3). We then get the following elements:

$$\begin{array}{ccccccc}
\mathsf{Ext}^i_{\mathscr{B}}(B,B') & \xleftarrow[\cong]{\pi^*} & \mathsf{Ext}^{i-m}_{\mathscr{B}}(M_m,B') & \xrightarrow[\cong]{e^{i-m}_{M_m,B'}} & \mathsf{Ext}^{i-m}_{\mathscr{C}}(e(M_m),e(B')) & \xrightarrow[\cong]{(e(\pi))^*} & \mathsf{Ext}^i_{\mathscr{C}}(e(B),e(B')) \\
[\eta\pi] & \longleftarrow\!\shortmid & [\eta] & \longmapsto & [e(\eta)] & \longmapsto & [e(\eta)\cdot e(\pi)] \\
 & & & & & & \| \\
 & & & & & & [e(\eta\pi)]
\end{array}$$

This shows that our isomorphism takes any element $[\zeta] \in \mathsf{Ext}^i_{\mathscr{B}}(B,B')$ to the element $[e(\zeta)] \in \mathsf{Ext}^i_{\mathscr{C}}(e(B),e(B'))$. Thus, our isomorphism is $e^i_{B,B'}$. □

## 2.2. Global Dimension

In this section we study the connections between the global dimension of the categories involved in a recollement $(\mathscr{A},\mathscr{B},\mathscr{C})$ of abelian categories.

Our first aim is to prove, and analyze the consequences, of the following result which gives an upper bound for the global dimension of $\mathscr{B}$ in terms of the global dimension of $\mathscr{A}$ and $\mathscr{C}$.

THEOREM 2.2.1. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects. Then:*

$$\mathsf{gl.dim}\,\mathscr{B} \ \leq\ \mathsf{gl.dim}\,\mathscr{A} + \mathsf{gl.dim}\,\mathscr{C} + \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} + 1$$

For the proof we shall need the following auxiliary results which are interesting in their own right. To begin with, we recall the following well known result (c.f. [123]).

LEMMA 2.2.2. *Let $0 \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \longrightarrow 0$ be an exact sequence in an abelian category $\mathscr{A}$ with enough projective objects.*

(i) *If $A_4 = 0$, then:*
  (a) *If $\mathsf{pd}_{\mathscr{A}} A_1 < \mathsf{pd}_{\mathscr{A}} A_2$, then $\mathsf{pd}_{\mathscr{A}} A_3 = \mathsf{pd}_{\mathscr{A}} A_2$.*
  (b) *If $\mathsf{pd}_{\mathscr{A}} A_1 > \mathsf{pd}_{\mathscr{A}} A_2$, then $\mathsf{pd}_{\mathscr{A}} A_3 = \mathsf{pd}_{\mathscr{A}} A_1 + 1$.*
  (c) *If $\mathsf{pd}_{\mathscr{A}} A_1 = \mathsf{pd}_{\mathscr{A}} A_2$, then $\mathsf{pd}_{\mathscr{A}} A_3 \leq \mathsf{pd}_{\mathscr{A}} A_1 + 1$.*
  (d) *$\mathsf{pd}_{\mathscr{A}} A_3 \leq \max\{\mathsf{pd}_{\mathscr{A}} A_1 + 1, \mathsf{pd}_{\mathscr{A}} A_2\}$.*

(ii) *If $A_4 \neq 0$, then: $\mathsf{pd}_{\mathscr{A}} A_3 \leq \max\{\mathsf{pd}_{\mathscr{A}} A_1 + 1, \mathsf{pd}_{\mathscr{A}} A_2, \mathsf{pd}_{\mathscr{A}} A_4\}$.*

PROOF. Part (i) is standard, see [65], [115], [123]. For part (ii) consider the short exact sequences

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \mathsf{Im}\,a_2 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathsf{Im}\,a_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow 0$$

By virtue of part (i)(d), we obtain that $\mathsf{pd}_{\mathscr{A}}\,\mathsf{Im}\,a_2 \leq \max\{\mathsf{pd}_{\mathscr{A}} A_1 + 1, \mathsf{pd}_{\mathscr{A}} A_2\}$. Then the result follows from the following long exact sequence:

$$\cdots \longrightarrow \mathsf{Ext}^{n-1}_{\mathscr{A}}(\mathsf{Im}\,a_2, A') \longrightarrow \mathsf{Ext}^n_{\mathscr{A}}(A_4, A') \longrightarrow \mathsf{Ext}^n_{\mathscr{A}}(A_3, A') \longrightarrow \mathsf{Ext}^n_{\mathscr{A}}(\mathsf{Im}\,a_2, A') \longrightarrow \cdots$$

where $A'$ lies in $\mathscr{A}$. □

LEMMA 2.2.3. *Let $\mathscr{A}$ be an abelian category with enough projective objects and let $A \in \mathscr{A}$. If there exists an exact sequence:*

$$0 \longrightarrow X_m \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

*with $\mathsf{pd}_{\mathscr{A}} X_i \leq n$ for all $0 \leq i \leq m$, then $\mathsf{pd}_{\mathscr{A}} A \leq m + n$.*

PROOF. If $0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$ is exact with $\mathsf{pd}_{\mathscr{A}} X_i \leq n$ then from the long exact $\mathsf{Ext}$-sequence we get that $\mathsf{pd}_{\mathscr{A}} A \leq n+1$. Then by induction on $m$ the result follows. $\square$

To proceed further we need the notion of the **$\mathscr{A}$-relative global dimension** of $\mathscr{B}$ which is defined by:

$$\mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} := \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(A) \mid A \in \mathscr{A}\}$$

LEMMA 2.2.4. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{C}$ has enough projective objects. Let $C \in \mathscr{C}$. Then:*

$$\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C) \ \leq\ \mathsf{pd}_{\mathscr{C}}\, C + \mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} + 1$$

PROOF. If $\mathsf{gl.dim}_{\mathscr{A}}\,\mathscr{B} = \infty$ or $\mathsf{pd}_{\mathscr{C}}\, C = \infty$ there is nothing to prove. Assume that $\mathsf{gl.dim}_{\mathscr{A}}\,\mathscr{B} = n$. The proof will proceed by induction on the projective dimension of $C$. If $C$ is a projective object of $\mathscr{C}$, then $\mathsf{l}(C)$ is a projective object of $\mathscr{B}$ and so the result follows. Suppose now that $\mathsf{pd}_{\mathscr{C}}\, C = m$ and that the result holds for every object of $\mathscr{C}$ with projective dimension less than of $m$, i.e., $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C') \leq \mathsf{pd}_{\mathscr{C}}\, C' + n + 1$ for every object $C' \in \mathscr{C}$ with $\mathsf{pd}\, C' < m$. Since $\mathsf{pd}_{\mathscr{C}}\, C = m$ we have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & P_m \arrow[r, "a_m"] & \cdots \arrow[r] & P_2 \arrow[r, "a_2"] & P_1 \arrow[r, "a_1"] \arrow[d, "j_0"'] & P_0 \arrow[r, "a_0"] & C \arrow[r] & 0 \\
 & & & & K_0 \arrow[ur, "i_0"'] & & &
\end{tikzcd}$$

where the objects $P_i \in \mathsf{Proj}\,\mathscr{C}$ and $K_0 = \mathsf{Ker}\, a_0$. Because $\mathsf{pd}_{\mathscr{C}}\, K_0 < m$, from the induction hypothesis we deduce that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(K_0) \leq \mathsf{pd}_{\mathscr{C}}\, K_0 + n + 1 \leq m - 1 + n + 1 = m + n$. Since the functor $\mathsf{l}\colon \mathscr{B} \longrightarrow \mathscr{C}$ is right exact we have the following exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{L}_1\mathsf{l}(C) \arrow[r] & \mathsf{l}(K_0) \arrow[r, "\mathsf{l}(i_0)"] \arrow[d] & \mathsf{l}(P_0) \arrow[r, "\mathsf{l}(a_0)"] & \mathsf{l}(C) \arrow[r] & 0 \\
 & & \mathsf{Ker}\,\mathsf{l}(a_0) \arrow[ur] & & &
\end{tikzcd} \tag{2.2.1}$$

We claim that the object $\mathsf{L}_1\mathsf{l}(C) \simeq \mathsf{Ker}\,\mathsf{l}(i_0) \in \mathsf{i}(\mathscr{A})$. Since the functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ is exact and the unit $\mathrm{Id}_{\mathscr{C}} \xrightarrow{\simeq} \mathsf{el}$ is invertible we have $\mathsf{e}(\mathsf{Ker}\,\mathsf{l}(i_0)) \simeq \mathsf{Ker}\, i_0$ and so $\mathsf{e}(\mathsf{Ker}\,\mathsf{l}(i_0)) = 0$. Hence $\mathsf{pd}_{\mathscr{B}}\,\mathsf{L}_1\mathsf{l}(C) \leq n$. Then from the short exact sequence $0 \longrightarrow \mathsf{L}_1\mathsf{l}(C) \longrightarrow \mathsf{l}(K_0) \longrightarrow \mathsf{Ker}\,\mathsf{l}(a_0) \longrightarrow 0$ we have $\mathsf{pd}_{\mathscr{B}}\,\mathsf{Ker}\,\mathsf{l}(a_0) \leq m + n$. Therefore from the exact sequence (2.2.1) it follows that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C) \leq m + n + 1$ since $\mathsf{l}(P_0) \in \mathsf{Proj}\,\mathscr{B}$. $\square$

The following result gives some upper bounds for the global dimension of $\mathscr{B}$, $\mathscr{C}$ and completes the proof of Theorem 2.2.1.

PROPOSITION 2.2.5. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects. Then:*

(i) $\mathsf{gl.dim}\,\mathscr{B} \ \leq\ \mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} + \mathsf{gl.dim}\,\mathscr{C} + 1$.
(ii) $\mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} \ \leq\ \mathsf{gl.dim}\mathscr{A} + \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\}$.
(iii) $\mathsf{gl.dim}\mathscr{C} \ \leq\ \mathsf{gl.dim}\mathscr{B} + \sup\{\mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(P) \mid P \in \mathsf{Proj}\,\mathscr{B}\}$.

*In particular if either $(\alpha)$ the functor $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ is exact, or $(\beta)$ the categories $\mathscr{B}$ and $\mathscr{C}$ have enough injectives and the functor $\mathsf{q}\colon \mathscr{B} \longrightarrow \mathscr{A}$ is exact, then:*

$$\mathsf{gl.dim}\,\mathscr{B} \ \leq\ \mathsf{gl.dim}\,\mathscr{A} + \mathsf{gl.dim}\,\mathscr{C} + 1$$

PROOF. (i) Let $B$ be an object of $\mathscr{B}$. Suppose that $\mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} = n < \infty$ and $\mathsf{gl.dim}\,\mathscr{C} = m < \infty$. From Proposition 1.1.6 there exists the exact sequence $0 \longrightarrow \mathsf{Ker}\,\mu_B \longrightarrow \mathsf{le}(B) \xrightarrow{\mu_B} B \longrightarrow \mathsf{Coker}\,\mu_B \longrightarrow 0$ where the objects $\mathsf{Ker}\,\mu_B$ and $\mathsf{Coker}\,\mu_B$ belong to $\mathsf{i}(\mathscr{A})$. Hence $\mathsf{pd}_{\mathscr{B}}\,\mathsf{Ker}\,\mu_B \leq n$ and $\mathsf{pd}_{\mathscr{B}}\,\mathsf{Coker}\,\mu_B \leq n$. Then from Lemma 2.2.2 and Lemma 2.2.4 we have the following bound:

$$\begin{aligned} \mathsf{pd}_{\mathscr{B}}\,B &\leq \mathsf{max}\,\{n+1, \mathsf{pd}_{\mathscr{B}}\,\mathsf{le}(B), n\} \\ &\leq \mathsf{max}\,\{n+1, \mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(B) + n + 1, n\} \\ &= \mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(B) + n + 1 \end{aligned}$$

Since $\mathsf{e}(B)$ is an object of $\mathscr{C}$ we infer that $\mathsf{pd}_{\mathscr{B}}\,B \leq m + n + 1$ and the result follows.

(ii) Let $A$ be an object of $\mathscr{A}$ and suppose that $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} = n < \infty$. We will prove that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(A) \leq \mathsf{pd}_{\mathscr{A}}\,A + n$. If $A$ is a projective object of $\mathscr{A}$ then $\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(A) \leq n$ and so our result holds. Suppose now that $\mathsf{pd}_{\mathscr{A}}\,A = m$. Then we have the exact sequence

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

where the objects $P_i \in \mathsf{Proj}\,\mathscr{A}$ for $0 \leq i \leq m$. Hence $\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P_i) \leq n$ and since the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is exact we infer from Lemma 2.2.3 that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(A) \leq m + n$.

(iii) Suppose that $\mathsf{gl.dim}\,\mathscr{B} = m < \infty$ and $\sup\{\mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(P) \mid P \in \mathsf{Proj}\,\mathscr{B}\} = n < \infty$. Let $C$ be an arbitrary object of $\mathscr{C}$. Then $\mathsf{l}(C) \in \mathscr{B}$ and so there exists an exact sequence

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathsf{l}(C) \longrightarrow 0$$

with $P_i \in \mathsf{Proj}\,\mathscr{B}$ for $0 \leq i \leq m$. Then applying the functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ we obtain the exact sequence

$$0 \longrightarrow \mathsf{e}(P_m) \longrightarrow \cdots \longrightarrow \mathsf{e}(P_1) \longrightarrow \mathsf{e}(P_0) \longrightarrow C \longrightarrow 0$$

where $\mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(P_i) \leq n$ for all $0 \leq i \leq m$. Therefore from Lemma 2.2.3 we conclude that $\mathsf{pd}_{\mathscr{C}}\,C \leq m + n$.

Finally if the functor $\mathsf{p}$ is exact then the inclusion $\mathsf{i}$ preserve projectives as a left adjoint of $\mathsf{p}$. In this case we have $\mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} \leq \mathsf{gl.dim}\mathscr{A}$ and the assertion follows from Theorem 2.2.1. Case $(\beta)$ is dual. □

We continue with some consequences of the above results.

COROLLARY 2.2.6. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects.*

(i) *If $\mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} < \infty$ and $\mathsf{gl.dim}\,\mathscr{C} < \infty$ then $\mathsf{gl.dim}\,\mathscr{B} < \infty$.*
(ii) *If $\mathsf{gl.dim}\,\mathscr{A} < \infty$ then $\mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} < \infty$ if and only if $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} < \infty$.*
(iii) *If $\mathsf{gl.dim}\,\mathscr{B} < \infty$ then $\mathsf{gl.dim}\,\mathscr{C} < \infty$ if and only if $\sup\{\mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(P) \mid P \in \mathsf{Proj}\,\mathscr{B}\} < \infty$.*

PROOF. The result follows immediately from Propostion 2.2.5. □

REMARK 2.2.7. If $0 \longrightarrow B \xrightarrow{b_0} I^0 \xrightarrow{b_1} I^1 \longrightarrow \cdots$ is an injective coresolution of $B \in \mathscr{B}$ then we denote by $\Sigma^n(B)$ the $n$th *cosyzygy* of $B$, that is the image of the morphism $b_n\colon I^{n-1} \longrightarrow I^n$. Also, we denote by $\Sigma^n(\mathscr{B})$ the full subcategory of $\mathscr{B}$ consisting of the $n$th cosyzygy objects of $\mathscr{B}$. Then since

$$\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(P),\Sigma^m(B)) \simeq \mathsf{Ext}^{n+m}_{\mathscr{B}}(\mathsf{i}(P),B) \quad \text{and} \quad \mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(P),\Sigma^m(C)) \simeq \mathsf{Ext}^{n+m}_{\mathscr{C}}(\mathsf{e}(P),C)$$

for every $n \geq 1$, we have the following:

(i) $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} \leq m$ if and only if $\Sigma^m(\mathscr{B}) \subseteq \mathsf{i}(\mathsf{Proj}\,\mathscr{A})^{1\perp_\infty}$.

(ii) $\sup\{\mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(P) \mid P \in \mathsf{Proj}\,\mathscr{B}\} \leq m$ if and only if $\Sigma^m(\mathscr{C}) \subseteq \mathsf{e}(\mathsf{Proj}\,\mathscr{B})^{1\perp_\infty}$.

The following result gives an upper bound for the global dimension of $\mathscr{B}$ provided that the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding.

Theorem 2.2.8. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. If the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding, then:*

$$\begin{aligned}\mathsf{gl.dim}\,\mathscr{A} \;\leq\; \mathsf{gl.dim}\,\mathscr{B} \;\leq\; & \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\}+ \\ & \max\{\sup\{\mathsf{id}_{\mathscr{B}}\mathsf{i}(I) \mid I \in \mathsf{Inj}\,\mathscr{A}\}+\mathsf{gl.dim}\,\mathscr{A},\mathsf{gl.dim}\,\mathscr{C}\}\end{aligned}$$

Proof. Let $B \in \mathscr{B}$ and suppose that $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} = m < \infty$. Then from Remark 2.2.7 we have $\Sigma^m(B) \subseteq \mathsf{i}(\mathsf{Proj}\,\mathscr{A})^{1\perp_\infty}$ and from Proposition 1.1.6 we have the following exact sequence:

$$0 \longrightarrow \mathsf{ip}(\Sigma^m(B)) \longrightarrow \Sigma^m(B) \longrightarrow \mathsf{Im}\,\nu_{\Sigma^m(B)} \longrightarrow 0 \tag{2.2.2}$$

Then since the map $\mathsf{i}^n_{X,Y}\colon \mathsf{Ext}^n_{\mathscr{A}}(X,Y) \longrightarrow \mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(X),\mathsf{i}(Y))$ is invertible for every $X,Y \in \mathscr{A}$ and $n \geq 0$, $\Sigma^m(B) \subseteq \mathsf{i}(\mathsf{Proj}\,\mathscr{A})^{1\perp_\infty}$, and $\mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(P),\mathsf{ip}(\Sigma^m(B))) \simeq \mathsf{Hom}_{\mathscr{B}}(\mathsf{i}(P),\Sigma^m(B))$ we have

$$\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(P),\mathsf{Im}\,\nu_{\Sigma^m(B)}) = 0$$

for every $n \geq 0$ and $P \in \mathsf{Proj}\,\mathscr{A}$. Thus from Proposition 2.1.4 we infer that $\mathsf{Im}\,\nu_{\Sigma^m(B)} \in \mathscr{Y}_\infty$ and therefore from Theorem 2.1.11 we have the isomorphism

$$\mathsf{Ext}^n_{\mathscr{B}}(Z,\mathsf{Im}\,\nu_{\Sigma^m(B)}) \simeq \mathsf{Ext}^n_{\mathscr{C}}(e(Z),e(\mathsf{Im}\,\nu_{\Sigma^m(B)}))$$

for every $n \geq 0$ and $Z \in \mathscr{B}$. This implies that

$$\mathsf{id}_{\mathscr{B}}\,\mathsf{Im}\,\nu_{\Sigma^m(B)} \leq \mathsf{gl.dim}\,\mathscr{C}$$

Similarly as in the proof of Proposition 2.2.5(ii) we obtain

$$\mathsf{id}_{\mathscr{B}}\mathsf{ip}(\Sigma^m(B)) \;\leq\; \mathsf{id}_{\mathscr{A}}\mathsf{p}(\Sigma^m(B)) + \sup\{\mathsf{id}_{\mathscr{B}}\mathsf{i}(I) \mid I \in \mathsf{Inj}\,\mathscr{A}\}$$

Hence from the above inequalities, the exact sequence (2.2.2) yields

$$\mathsf{id}_{\mathscr{B}}\Sigma^m(B) \;\leq\; \max\{\sup\{\mathsf{id}_{\mathscr{B}}\mathsf{i}(I) \mid I \in \mathsf{Inj}\,\mathscr{A}\} + \mathsf{gl.dim}\,\mathscr{A},\mathsf{gl.dim}\,\mathscr{C}\}$$

and the assertion follows since $\mathsf{id}_{\mathscr{B}}\,B \leq \mathsf{id}_{\mathscr{B}}\,\Sigma^m(B) + m$. □

The following result shows that if $\mathscr{B}$ is hereditary then the categories $\mathscr{A}$ and $\mathscr{C}$ are hereditary as well.

Theorem 2.2.9. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories and suppose that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects. If $\mathsf{gl.dim}\,\mathscr{B} \leq 1$, then:*

$$\mathsf{gl.dim}\,\mathscr{A} \;\leq\; 1 \quad \textit{and} \quad \mathsf{gl.dim}\,\mathscr{C} \;\leq\; 1$$

*Conversely, if $\mathsf{gl.dim}\,\mathscr{A} \leq 1$ and $\mathsf{gl.dim}\,\mathscr{C} \leq 1$ then:*

$$\mathsf{gl.dim}\,\mathscr{B} \;\leq\; 3 + \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\}$$

PROOF. Let $C$ be an object of $\mathscr{C}$ and $\cdots \longrightarrow P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} C \longrightarrow 0$ a projective resolution of $C$. Since the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is right exact we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Ker}\,\mathsf{l}(a_1) \arrow[r] & \mathsf{l}(P_1) \arrow[r, "\mathsf{l}(a_1)"] \arrow[d] & \mathsf{l}(P_0) \arrow[r, "\mathsf{l}(a_0)"] & \mathsf{l}(C) \arrow[r] & 0 \\ & & \mathsf{Ker}\,\mathsf{l}(a_0) \arrow[ur] & & & \end{tikzcd}$$

where the objects $\mathsf{l}(P_1)$ and $\mathsf{l}(P_0)$ are projective in $\mathscr{B}$. Since $\mathsf{gl.dim}\,\mathscr{B} \leq 1$, the object $\mathsf{Ker}\,\mathsf{l}(a_0)$ is projective and therefore the short exact sequence $0 \longrightarrow \mathsf{Ker}\,\mathsf{l}(a_1) \longrightarrow \mathsf{l}(P_1) \longrightarrow \mathsf{Ker}\,\mathsf{l}(a_0) \longrightarrow 0$ splits. Applying the functor $\mathsf{e}$ we get the split exact sequence $0 \longrightarrow \mathsf{e}(\mathsf{Ker}\,\mathsf{l}(a_1)) \longrightarrow \mathsf{el}(P_1) \longrightarrow \mathsf{e}(\mathsf{Ker}\,\mathsf{l}(a_0)) \longrightarrow 0$. Since $\mathsf{e}$ is exact and the unit $\mathrm{Id}_{\mathscr{C}} \xrightarrow{\simeq} \mathsf{el}$ is invertible we have $\mathsf{e}(\mathsf{Ker}\,\mathsf{l}(a_0)) \simeq \mathsf{Ker}\,a_0$ and $\mathsf{e}(\mathsf{Ker}\,\mathsf{l}(a_1)) \simeq \mathsf{Ker}\,a_1$. Hence the exact sequence $0 \longrightarrow \mathsf{Ker}\,a_1 \longrightarrow P_1 \longrightarrow \mathsf{Ker}\,a_0 \longrightarrow 0$ splits, so $\mathsf{Ker}\,a_0$ is projective as a direct summand of $P_1$. Hence $\mathsf{pd}_{\mathscr{C}}\,C \leq 1$ and therefore $\mathsf{gl.dim}\,\mathscr{C} \leq 1$. Let $A_1, A_2 \in \mathscr{A}$ and $0 \longrightarrow K \longrightarrow P \longrightarrow A_1 \longrightarrow 0$ be a short exact sequence with $P \in \mathsf{Proj}\,\mathscr{A}$. Then we have the exact commutative diagram:

$$\begin{tikzcd} \mathsf{Ext}^1_{\mathscr{A}}(P, A_2) \arrow[r] \arrow[d, "\mathsf{i}^1_{P,A_2}"', "\simeq"] & \mathsf{Ext}^1_{\mathscr{A}}(K, A_2) \arrow[r, "\simeq"] \arrow[d, "\mathsf{i}^1_{K,A_2}"', "\simeq"] & \mathsf{Ext}^2_{\mathscr{A}}(A_1, A_2) \arrow[r] \arrow[d, "\mathsf{i}^2_{A_1,A_2}"'] & \mathsf{Ext}^2_{\mathscr{A}}(P, A_2) \arrow[d] \\ \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(A_2)) \arrow[r] & \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{i}(K), \mathsf{i}(A_2)) \arrow[r, "\simeq"] & \mathsf{Ext}^2_{\mathscr{B}}(\mathsf{i}(A_1), \mathsf{i}(A_2)) \arrow[r] & \mathsf{Ext}^2_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(A_2)) \end{tikzcd}$$

where, by Remark 2.1.7(i), the vertical maps $\mathsf{i}^1_{P,A_2}$ and $\mathsf{i}^1_{K,A_2}$ are invertible. Since $\mathsf{gl.dim}\,\mathscr{B} \leq 1$ we have $\mathsf{Ext}^2_{\mathscr{B}}(\mathsf{i}(A_1), \mathsf{i}(A_2)) = 0 = \mathsf{Ext}^2_{\mathscr{B}}(\mathsf{i}(P), \mathsf{i}(A_2))$ and thus $\mathsf{Ext}^2_{\mathscr{A}}(A_1, A_2) = 0$. This implies that $\mathsf{gl.dim}\,\mathscr{A} \leq 1$. The converse follows directly from Theorem 2.2.1. □

REMARK 2.2.10. The implication $\mathsf{gl.dim}\,\mathscr{B} \leq 1 \Rightarrow \mathsf{gl.dim}\,\mathscr{A} \leq 1$ holds without assuming enough projectives. Indeed let $A_1, A_2 \in \mathscr{A}$; since the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is 1-homological embeding, see Remark 2.1.7(i), it follows from [101] that the map $\mathsf{i}^2_{A_1,A_2}\colon \mathsf{Ext}^2_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathsf{Ext}^2_{\mathscr{B}}(\mathsf{i}(A_1), \mathsf{i}(A_2))$ is a monomorphism. Since $\mathsf{gl.dim}\,\mathscr{B} \leq 1$ we have $\mathsf{Ext}^2_{\mathscr{B}}(\mathsf{i}(A_1), \mathsf{i}(A_2)) = 0$ and therefore $\mathsf{Ext}^2_{\mathscr{A}}(A_1, A_2) = 0$. Hence $\mathsf{gl.dim}\,\mathscr{A} \leq 1$.

Now we turn our attention to the study of the structure of the categories $\mathfrak{X}_n$ and $\mathfrak{Y}_n$ in connection with the behaviour of the homological dimensions of the categories $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$. We begin with the following observation.

COROLLARY 2.2.11. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects.*

(i) *If $B \in \mathfrak{X}_\infty$ then $\mathsf{pd}_{\mathscr{B}}\,B = \mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(B)$.*
(ii) *If $B \in \mathfrak{Y}_\infty$ then $\mathsf{id}_{\mathscr{B}}\,B = \mathsf{id}_{\mathscr{C}}\,\mathsf{e}(B)$.*
(iii) *If $\mathfrak{X}_1 = \mathfrak{X}_\infty$ or $\mathfrak{Y}_1 = \mathfrak{Y}_\infty$ then $\mathsf{gl.dim}\,\mathscr{C} \leq \mathsf{gl.dim}\,\mathscr{B}$.*

PROOF. Parts (i) and (ii) follow from Theorem 2.1.11 using that the quotient functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ is surjective on objects. Let $\mathfrak{X}_1 = \mathfrak{X}_\infty$ and suppose that $\mathsf{gl.dim}\,\mathscr{B} = n < \infty$. Let $C \in \mathscr{C}$ and $P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0$ be an exact sequence with $P_1, P_0 \in \mathsf{Proj}\,\mathscr{C}$. Then $\mathsf{l}(C) \in \mathfrak{X}_\infty$ and from (i) we have $\mathsf{pd}_{\mathscr{C}}\,C = \mathsf{pd}_{\mathscr{C}}\,\mathsf{el}(C) = \mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C) \leq n$. Hence $\mathsf{gl.dim}\,\mathscr{C} \leq \mathsf{gl.dim}\,\mathscr{B}$. Similarly using the other hypothesis we obtain the same result. □

REMARK 2.2.12. Suppose that the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding and let $I \in \mathsf{Inj}\,\mathscr{B}$. By Theorem 2.1.10 we have $\mathsf{G}(I) \in \mathcal{Y}_\infty$, hence $\mathsf{id}_{\mathscr{B}}\,\mathsf{G}(I) = \mathsf{id}_{\mathscr{C}}\,\mathsf{e}(\mathsf{G}(I)) \leq \mathsf{gl.dim}\,\mathscr{C}$ by Corollary 2.2.11. Then by Proposition 1.1.6 we have $\mathsf{id}_{\mathscr{B}}\,\mathsf{ip}(I) \leq \mathsf{gl.dim}\,\mathscr{C} + 1$. This shows that Theorem 2.2.8 improves Theorem 2.2.1.

The following result characterizes when the inclusions $\mathcal{X}_1 \supseteq \mathcal{X}_\infty$ or $\mathcal{Y}_1 \supseteq \mathcal{Y}_\infty$ are equalities, in terms of properties of the quotient functor of the recollement.

PROPOSITION 2.2.13. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Then:*

(i) *$\mathcal{Y}_1 = \mathcal{Y}_\infty$ if and only if the functor $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is exact, equivalently the functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ preserves projectives.*

(ii) *$\mathcal{X}_1 = \mathcal{X}_\infty$ if and only if the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is exact, equivalently the functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ preserves injectives.*

PROOF. (i) Let $Y \in \mathcal{Y}_1$ and $k \geq 1$. We will prove first that $Y \in \mathcal{Y}_k$ if and only if $\mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(P), \mathsf{e}(Y)) = 0$ for every $P \in \mathsf{Proj}\,\mathscr{B}$ and $1 \leq n \leq k-1$. Then our claim follows from this. Indeed, let $Y \in \mathcal{Y}_1$ and suppose that the functor $\mathsf{e}$ preserves projectives. Then $\mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(P), \mathsf{e}(Y)) = 0$ for every $n \geq 1$ and therefore $Y \in \mathcal{Y}_\infty$. Assume conversely that $\mathcal{Y}_1 = \mathcal{Y}_\infty$ and let $P$ be a projective object of $\mathscr{B}$. Since the object $\mathsf{r}(C) \in \mathcal{Y}_1$ for every $C \in \mathscr{C}$, it follows that the group $\mathsf{Ext}^1_{\mathscr{C}}(\mathsf{e}(P), \mathsf{er}(C)) = 0$. Thus the object $\mathsf{e}(P)$ is projective in $\mathscr{C}$.

We show now our initial statement. If $Y \in \mathcal{Y}_k$ then from Theorem 2.1.11 we have $\mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(P), \mathsf{e}(Y)) \simeq \mathsf{Ext}^n_{\mathscr{B}}(P, Y)$ for all $0 \leq n \leq k-1$ and so $\mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(P), \mathsf{e}(Y)) = 0$ for every $P \in \mathsf{Proj}\,\mathscr{B}$ and $1 \leq n \leq k-1$.

Let's make some remarks before we prove the converse. Let $P \in \mathsf{Proj}\,\mathscr{B}$. Then from Proposition 1.1.6 we have the exact sequence $\mathsf{le}(P) \xrightarrow{\mu_P} P \longrightarrow \mathsf{iq}(P) \longrightarrow 0$ and $\mathsf{e}(\mathsf{F}(P)) \simeq \mathsf{e}(P)$, where $\mathsf{F}(P) = \mathsf{Im}\,\mu_P$. Also $\mathsf{F}(P) \in \mathcal{X}_0$ since there exists an epimorphism $\mathsf{l}(Q) \longrightarrow \mathsf{le}(P)$ for some $Q \in \mathsf{Proj}\,\mathscr{C}$. From Remark 1.1.5 we know that $\mathsf{Proj}\,\mathscr{A} = \mathsf{add}\,\mathsf{q}(\mathsf{Proj}\,\mathscr{B})$. By Proposition 2.1.4 in order to prove that $B \in \mathcal{Y}_k$ it is enough to show that $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{iq}(P), B) = 0$ for all $P \in \mathsf{Proj}\,B$ and $0 \leq n \leq k$. Suppose for the converse that $\mathsf{Ext}^n_{\mathscr{C}}(\mathsf{e}(P), \mathsf{e}(Y)) = 0$ for every $P \in \mathsf{Proj}\,\mathscr{B}$ and $1 \leq n \leq k-1$. Since $\mathsf{F}(P) \in \mathcal{X}_0$ we have the short exact sequences $0 \longrightarrow K \longrightarrow \mathsf{l}(Q) \longrightarrow \mathsf{F}(P) \longrightarrow 0$ and $0 \longrightarrow \mathsf{e}(K) \longrightarrow Q \longrightarrow \mathsf{e}(\mathsf{F}(P)) \longrightarrow 0$ where $Q \in \mathsf{Proj}\,\mathscr{C}$. Then from the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{F}(P), Y) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(Q), Y) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Hom}_{\mathscr{B}}(K, Y) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{F}(P), Y) \arrow[r] \arrow[d, "\mathsf{e}^1_{\mathsf{F}(P),Y}"'] & 0 \\
0 \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(\mathsf{F}(P)), \mathsf{e}(Y)) \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(Q, \mathsf{e}(Y)) \arrow[r] & \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(K), \mathsf{e}(Y)) \arrow[r] & \mathsf{Ext}^1_{\mathscr{C}}(\mathsf{e}(\mathsf{F}(P)), \mathsf{e}(Y)) \arrow[r] & 0
\end{tikzcd}$$

it follows that the map $\mathsf{e}^1_{\mathsf{F}(P),Y}$ is invertible, where the first and third vertical isomorphisms follow from Theorem 2.1.11 since $Y \in \mathcal{Y}_1$. Therefore we have

$$\mathsf{Ext}^2_{\mathscr{B}}(\mathsf{iq}(P), Y) \simeq \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{F}(P), Y) \simeq \mathsf{Ext}^1_{\mathscr{C}}(\mathsf{e}(\mathsf{F}(P)), \mathsf{e}(Y)) \simeq \mathsf{Ext}^1_{\mathscr{C}}(\mathsf{e}(P), \mathsf{e}(Y)) = 0$$

Hence from Proposition 2.1.4 we infer that $Y \in \mathcal{Y}_2$. Moreover, we have the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Ext}^1_{\mathscr{B}}(K, Y) \arrow[r, "\simeq"] \arrow[d, "\mathsf{e}^1_{K,Y}", "\simeq"'] & \mathsf{Ext}^2_{\mathscr{B}}(\mathsf{F}(P), Y) \arrow[r] \arrow[d, "\mathsf{e}^2_{\mathsf{F}(P),Y}"'] & 0 \\
0 \arrow[r] & \mathsf{Ext}^1_{\mathscr{C}}(\mathsf{e}(K), \mathsf{e}(Y)) \arrow[r, "\simeq"] & \mathsf{Ext}^2_{\mathscr{C}}(\mathsf{e}(\mathsf{F}(P)), \mathsf{e}(Y)) \arrow[r] & 0
\end{tikzcd}$$

and from Theorem 2.1.11 it follows that the map $\mathsf{e}^1_{K,Y}$ is an isomorphism since $Y \in \mathcal{Y}_2$. Hence the map $\mathsf{e}^2_{\mathsf{F}(P),Y}$ is invertible and then we have isomorphisms

$$\mathsf{Ext}^3_{\mathscr{B}}(\mathsf{iq}(P), Y) \simeq \mathsf{Ext}^2_{\mathscr{B}}(\mathsf{F}(P), Y) \simeq \mathsf{Ext}^2_{\mathscr{C}}(\mathsf{e}(\mathsf{F}(P)), \mathsf{e}(Y)) \simeq \mathsf{Ext}^2_{\mathscr{C}}(\mathsf{e}(P), \mathsf{e}(Y)) = 0.$$

Proposition 2.1.4 then implies that $Y \in \mathcal{Y}_3$. Then by induction it is easy to see that $Y \in \mathcal{Y}_k$.

(ii) The proof is similar and is left to the reader. □

COROLLARY 2.2.14. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories such that the functor $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is exact or the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is exact. Then:*

$$\mathsf{gl.dim}\,\mathscr{C} \ \leq\ \mathsf{gl.dim}\,\mathscr{B} \ \leq\ \max\{\mathsf{gl.dim}\,\mathscr{A} + \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} + 1, \mathsf{gl.dim}\,\mathscr{C}\}$$

PROOF. The lower bound is a consequence of Corollary 2.2.11 and Proposition 2.2.13. Note that if the functor $\mathsf{r}$ is exact then the functor $\mathsf{e}$ preserves projectives and therefore $\mathsf{gl.dim}\,\mathscr{C} \leq \mathsf{gl.dim}\,\mathscr{B}$ by Proposition 2.2.5(iii). Now since the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is exact we claim that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C) = \mathsf{pd}_{\mathscr{C}}\,C$ for every $C \in \mathscr{C}$. Let $C$ be an object of $\mathscr{C}$ with $\mathsf{pd}_{\mathscr{C}}\,C = n$ and let $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow C \longrightarrow 0$ be the projective resolution of $C$. Then if we apply the exact functor $\mathsf{l}$ we get that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C) \leq n = \mathsf{pd}_{\mathscr{C}}\,C$ since the functor $\mathsf{l}$ preserves projectives. Conversely suppose that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C) = m < \infty$. Let $0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow C \longrightarrow 0$ be an exact sequence with $P_0 \in \mathsf{Proj}\,\mathscr{C}$ and $K_0$ the kernel of $a_0\colon P_0 \longrightarrow C$. Since the functor $\mathsf{l}$ is exact the sequence $0 \longrightarrow \mathsf{l}(K_0) \longrightarrow \mathsf{l}(P_0) \longrightarrow \mathsf{l}(C) \longrightarrow 0$ is exact. Continuing with the same way, after $m$-steps we obtain the exact sequence:

$$0 \longrightarrow \mathsf{l}(K_{m-1}) \longrightarrow \mathsf{l}(P_{m-1}) \longrightarrow \cdots \longrightarrow \mathsf{l}(P_0) \longrightarrow \mathsf{l}(C) \longrightarrow 0$$

where $\mathsf{l}(K_{m-1})$ is projective since $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C) = m$. Then if we apply the functor $\mathsf{e}$ we get the exact sequence:

$$0 \longrightarrow K_{m-1} \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow C \longrightarrow 0$$

and we claim that $\Omega^m(X) = K_{m-1} \in \mathsf{Proj}\,\mathscr{C}$. But this follows easily since $\mathsf{l}(K_{m-1})$ is projective and the functor $\mathsf{l}$ is fully faithful. Thus we have $\mathsf{pd}_{\mathscr{C}}\,C \leq m = \mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C)$ and then we conclude that $\mathsf{pd}_{\mathscr{C}}\,C = \mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(C)$. Let $B$ be an object in $\mathscr{B}$. Then working as in the proof of Proposition 2.2.5(i) combined with Proposition 2.2.5(ii) we obtain that

$$\begin{aligned}
\mathsf{pd}_{\mathscr{B}}\,\mathsf{F}(B) &\leq \mathsf{max}\,\{\mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} + 1, \mathsf{pd}_{\mathscr{B}}\,\mathsf{le}(B)\} \\
&\leq \mathsf{max}\,\{\mathsf{gl.dim}\,\mathscr{A} + \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} + 1, \mathsf{gl.dim}\,\mathscr{C}\}
\end{aligned}$$

Then from the exact sequence $0 \longrightarrow \mathsf{F}(B) \longrightarrow B \longrightarrow \mathsf{iq}(B) \longrightarrow 0$ we infer that $\mathsf{pd}_{\mathscr{B}}\,B \leq \mathsf{max}\{\mathsf{gl.dim}\,\mathscr{A} + \sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} + 1, \mathsf{gl.dim}\,\mathscr{C}\}$. The dual for the upper bound using that the functor $\mathsf{r}$ is exact is left to the reader. □

Corollary 2.2.15. *Let $\mathscr{B}$ be an abelian category such that the diagrams*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathsf{i}"] & \mathscr{B} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{A}' \arrow[r, "\mathsf{i}'"] & \mathscr{B} \arrow[l, bend right, "\mathsf{q}'"'] \arrow[l, bend left, "\mathsf{p}'"] \arrow[r, "\mathsf{e}'"] & \mathscr{C}' \arrow[l, bend right, "\mathsf{l}'"'] \arrow[l, bend left, "\mathsf{r}'"] \end{tikzcd}$$

*are recollements of abelian categories. Suppose that the functors $\mathsf{l}$, $\mathsf{l}'$ are exact and $\mathsf{i}(\mathsf{Proj}\,\mathscr{A}) \in \mathcal{X}'_1$. Then:*

$$\max\{\mathsf{gl.dim}\,\mathscr{C}, \mathsf{gl.dim}\,\mathscr{C}'\} \ \leq \ \mathsf{gl.dim}\,\mathscr{B} \ \leq \ \max\{\mathsf{gl.dim}\,\mathscr{A} + \\ \sup\{\mathsf{pd}_{\mathscr{C}'}\,\mathsf{e}'\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} + 1, \mathsf{gl.dim}\,\mathscr{C}\}$$

Proof. Let $P \in \mathsf{Proj}\,\mathscr{A}$ and assume that $\mathsf{pd}_{\mathscr{C}'}\,\mathsf{e}'\mathsf{i}(P) = n < \infty$. Then there exists an exact sequence $0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow \mathsf{e}'\mathsf{i}(P) \longrightarrow 0$ with $Q_i \in \mathsf{Proj}\,\mathscr{C}'$. Since the functor $\mathsf{l}'$ is exact and preserves projective objects we have the exact sequence $0 \longrightarrow \mathsf{l}'(Q_n) \longrightarrow \cdots \longrightarrow \mathsf{l}'(Q_0) \longrightarrow \mathsf{l}'\mathsf{e}'\mathsf{i}(P) \longrightarrow 0$ with $\mathsf{l}'(Q_i) \in \mathsf{Proj}\,\mathscr{B}$, i.e. $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}'\mathsf{e}'\mathsf{i}(P) \leq n$. Since $\mathsf{i}(P) \in \mathcal{X}'_1$ it follows from Proposition 2.1.2 that $\mathsf{l}'\mathsf{e}'\mathsf{i}(P) \simeq \mathsf{i}(P)$. Therefore we infer that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \leq n = \mathsf{pd}_{\mathscr{C}'}\,\mathsf{e}'\mathsf{i}(P)$ and then the result follows from Corollary 2.2.14. □

The following result characterizes when $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} \leq 1$. We leave to the reader its dual version concerning the condition $\sup\{\mathsf{id}_{\mathscr{B}}\,\mathsf{i}(I) \mid I \in \mathsf{Inj}\,\mathscr{A}\} \leq 1$.

Proposition 2.2.16. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ has enough projective and injective objects. Then the following are equivalent.*

(i) *The idempotent functor $\mathsf{F}\colon \mathscr{B} \longrightarrow \mathscr{B}$ preserves projective objects.*
(ii) $\Sigma(\mathscr{B}) \subseteq \mathsf{i}(\mathsf{Proj}\,\mathscr{A})^{1\perp_\infty}$.
(iii) $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} \leq 1$.
(iv) *The functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding and $\mathcal{Y}_1 = \mathcal{Y}_\infty$.*
(v) *The functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a 2-homological embedding and $\mathcal{Y}_1 = \mathcal{Y}_\infty$.*

*If one of the above equivalent statements holds, then:*

$$\max\{\mathsf{gl.dim}\,\mathscr{A}, \mathsf{gl.dim}\,\mathscr{C}\} \ \leq \ \mathsf{gl.dim}\,\mathscr{B} \ \leq \ \mathsf{gl.dim}\,\mathscr{A} + \mathsf{gl.dim}\,\mathscr{C} + 2$$

Proof. (i) ⇒ (ii) Let $Y$ be an object of $\mathscr{B}$ and $0 \longrightarrow Y \longrightarrow I \longrightarrow \Sigma(Y) \longrightarrow 0$ (1) be an exact sequence with $I \in \mathsf{Inj}\,\mathscr{B}$. From Proposition 1.1.6 we have the exact sequence $0 \longrightarrow \mathsf{F}(P) \longrightarrow P \longrightarrow \mathsf{iq}(P) \longrightarrow 0$, where $P \in \mathsf{Proj}\,\mathscr{B}$. Hence $\mathsf{pd}_{\mathscr{B}}\,\mathsf{iq}(P) \leq 1$ and so we have $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{iq}(P), \Sigma(Y)) = 0$ for every $n \geq 1$. This implies that the cosygygy object $\Sigma(Y) \in \mathsf{i}(\mathsf{Proj}\,\mathscr{A})^{1\perp_\infty}$ since $\mathsf{Proj}\,\mathscr{A} = \mathsf{add}\,\mathsf{q}(\mathsf{Proj}\,\mathscr{B})$.

(iii) ⇔ (ii) ⇒ (i) The equivalence (iii) ⇔ (ii) follows from Remark 2.2.7. Assuming (ii), let $P \in \mathsf{Proj}\,\mathscr{B}$ and consider the extension (1) for an object $Y \in \mathscr{B}$. Then from the isomorphism $\mathsf{Ext}^2_{\mathscr{B}}(\mathsf{iq}(P), Y) \simeq \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{iq}(P), \Sigma(Y))$ it follows that $\mathsf{Ext}^2_{\mathscr{B}}(\mathsf{iq}(P), Y) = 0$ since $\Sigma(Y) \in \mathsf{i}(\mathsf{Proj}\,\mathscr{A})^{1\perp_\infty}$. This implies that $\mathsf{Ext}^1_{\mathscr{B}}(\mathsf{F}(P), Y) = 0$ for every $Y \in \mathscr{B}$ since $\mathsf{Ext}^2_{\mathscr{B}}(\mathsf{iq}(P), Y) \simeq \mathsf{Ext}^1_{\mathscr{B}}(\mathsf{F}(P), Y)$. Hence $\mathsf{F}(P) \in \mathsf{Proj}\,\mathscr{B}$.

(i) ⇒ (iv) Since the idempotent functor $\mathsf{F}\colon \mathscr{B} \longrightarrow \mathscr{B}$ preserves projective objects, as above we have $\mathsf{pd}_{\mathscr{B}}\,\mathsf{iq}(P) \leq 1$. Hence $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{iq}(P), \mathsf{i}(A)) = 0$ for every $A \in \mathscr{A}$ and $n \geq 1$. Then from Theorem 2.1.10 we infer that the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding since $\mathsf{Proj}\,\mathscr{A} = \mathsf{add}\,\mathsf{q}(\mathsf{Proj}\,\mathscr{B})$. The inclusion $\mathcal{Y}_\infty \subseteq \mathcal{Y}_1$ always holds. Let $Y \in \mathcal{Y}_1$. Then from Proposition 2.1.4 we have $\mathsf{Ext}^n_{\mathscr{B}}(\mathsf{i}(Q), Y) = 0$ for $n = 0, 1$ and for every $Q \in \mathsf{Proj}\,\mathscr{A}$. Since $\mathsf{pd}_{\mathscr{B}}\,\mathsf{iq}(P) \leq 1$ we conclude that $Y \in \mathcal{Y}_\infty$ and so $\mathcal{Y}_1 = \mathcal{Y}_\infty$.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (i) The implication (iv) $\Rightarrow$ (v) is clear. Assuming (v), let $P \in \mathsf{Proj}\,\mathscr{B}$. Since the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a 2-homological embedding it follows from Theorem 2.1.10 that $\mathsf{F}(P) \in \mathfrak{X}_1$. Thus there exists an exact sequence

$$\mathsf{l}(P_1) \longrightarrow \mathsf{l}(P_0) \longrightarrow \mathsf{F}(P) \longrightarrow 0 \tag{$*$}$$

with $P_1, P_0 \in \mathsf{Proj}\,\mathscr{C}$ and from Proposition 2.1.2 we have $\mathsf{le}(\mathsf{F}(P)) \simeq \mathsf{F}(P)$. Also from Proposition 2.2.13 we deduce that the object $\mathsf{e}(\mathsf{F}(P)) \simeq \mathsf{e}(P)$ is projective since $\mathcal{Y}_1 = \mathcal{Y}_\infty$. Then if we apply the functor $\mathsf{e}$ to the sequence $(*)$ we obtain the split exact sequence

$$P_1 \longrightarrow P_0 \longrightarrow \mathsf{e}(\mathsf{F}(P)) \longrightarrow 0 \tag{$**$}$$

Therefore, applying the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ to $(**)$ we get that the sequence $(*)$ splits and so $\mathsf{F}(P) \in \mathsf{Proj}\,\mathscr{B}$.

If one of the conditions (i)-(v) holds, then the lower bound for the global dimension of $\mathscr{B}$ follows from statement (iv) using Corollary 2.2.11 and the upper bound follows from Theorem 2.2.1. $\square$

As a consequence of Theorem 2.2.8 and Proposition 2.2.16 we have the following.

Corollary 2.2.17. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories where $\mathscr{B}$, $\mathscr{C}$ have enough projective and injective objects. If* $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} \leq 1$ *and* $\sup\{\mathsf{id}_{\mathscr{B}}\mathsf{i}(I) \mid I \in \mathsf{Inj}\,\mathscr{A}\} \leq 1$*, then:*

$$\mathsf{max}\{\mathsf{gl.dim}\,\mathscr{A}, \mathsf{gl.dim}\,\mathscr{C}\} \ \leq \ \mathsf{gl.dim}\,\mathscr{B} \ \leq \ \mathsf{max}\{\mathsf{gl.dim}\,\mathscr{A} + 2, \mathsf{gl.dim}\,\mathscr{C} + 1\}$$

The following consequence of Propostion 2.2.16 gives a sufficient condition such that the finiteness of the global dimension of $\mathscr{B}$ is equivalent to the finiteness of the global dimension of $\mathscr{A}$ and $\mathscr{C}$.

Corollary 2.2.18. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories, where $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. If* $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} \leq 1$*, then the following are equivalent.*

(i) $\mathsf{gl.dim}\,\mathscr{B} \ < \ \infty$.
(ii) $\mathsf{gl.dim}\,\mathscr{A} \ < \ \infty$ *and* $\mathsf{gl.dim}\,\mathscr{C} \ < \ \infty$.

The following is a well known result for comma categories, see [49], and follows directly from our results.

Example 2.2.19. Let $\mathscr{C} = (G, \mathscr{B}, \mathscr{A}) = (\mathrm{Id} \downarrow G)$ be a comma category. From Example 1.1.12 we have the recollements of abelian categories $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ and $(\mathscr{B}, \mathscr{C}, \mathscr{A})$ where the functors $\mathsf{Z}_{\mathscr{A}}\colon \mathscr{A} \longrightarrow \mathscr{C}$, $\mathsf{Z}_{\mathscr{B}}\colon \mathscr{B} \longrightarrow \mathscr{C}$ and $\mathsf{U}_{\mathscr{A}}\colon \mathscr{C} \longrightarrow \mathscr{A}$, $\mathsf{U}_{\mathscr{B}}\colon \mathscr{C} \longrightarrow \mathscr{B}$ are exact. Then from Proposition 2.2.5 and Corollary 2.2.14 we have

$$\mathsf{max}\{\mathsf{gl.dim}\,\mathscr{A}, \mathsf{gl.dim}\,\mathscr{B}\} \ \leq \ \mathsf{gl.dim}\,\mathscr{C} \ \leq \ \mathsf{gl.dim}\,\mathscr{A} + \mathsf{gl.dim}\,\mathscr{B} + 1$$

We close this section by introducing the notion of the stratification dimension for abelian categories.

Definition 2.2.20. Let $\mathscr{B}$ be an abelian category. A **stratification** of $\mathscr{B}$ is a sequence of recollements of abelian categories of the following form:

$$\begin{tikzcd} \mathscr{A}_0 \arrow[r, "\mathsf{i}_0"] & \mathscr{B} \arrow[l, bend right, "\mathsf{q}_0"'] \arrow[l, bend left, "\mathsf{p}_0"] \arrow[r, "\mathsf{e}_0"] & \mathscr{C}_0 \arrow[l, bend right, "\mathsf{l}_0"'] \arrow[l, bend left, "\mathsf{r}_0"] \end{tikzcd}, \qquad \begin{tikzcd} \mathscr{A}_1 \arrow[r, "\mathsf{i}_1"] & \mathscr{A}_0 \arrow[l, bend right, "\mathsf{q}_1"'] \arrow[l, bend left, "\mathsf{p}_1"] \arrow[r, "\mathsf{e}_1"] & \mathscr{C}_1 \arrow[l, bend right, "\mathsf{l}_1"'] \arrow[l, bend left, "\mathsf{r}_1"] \end{tikzcd}, \qquad \cdots$$

We say that the above sequence is a *non trivial stratification* of $\mathscr{B}$ if the first recollement in the above sequence is not trivial, i.e. it is not of the form $(0, \mathscr{B}, \mathscr{C}_0)$ or $(\mathscr{A}_0, \mathscr{B}, 0)$.

The stratification dimension of $\mathscr{B}$, denoted by $\mathsf{str.dim}\,\mathscr{B}$, is defined inductively as follows. The stratification dimension of $\mathscr{B}$ is $\mathsf{str.dim}\,\mathscr{B} = 0$ if we have a trivial recollement $(0, \mathscr{B}, \mathscr{C}_0)$ or $(\mathscr{A}_0, \mathscr{B}, 0)$ and there is no non trivial recollement for $\mathscr{B}$. We define $\mathsf{str.dim}\,\mathscr{B} \leq 1$ if there exists a non trivial recollement $(\mathscr{A}_0, \mathscr{B}, \mathscr{C}_0)$ such that $\mathsf{str.dim}\,\mathscr{A}_0 = 0$. Then inductively the stratification dimension of $\mathscr{B}$ is $\mathsf{str.dim}\,\mathscr{B} \leq n$ if there exists a non trivial recollement $(\mathscr{A}_0, \mathscr{B}, \mathscr{C}_0)$ such that $\mathsf{str.dim}\,\mathscr{A}_0 \leq n-1$. If no such integer exists then we set $\mathsf{str.dim}\,\mathscr{B} = \infty$ and we make the convention that $\mathscr{A}_{-1} = \mathscr{B}$.

EXAMPLE 2.2.21. Let $\Lambda$ be an Artin algebra and $0 = \mathfrak{U}_0 \subseteq \mathfrak{U}_1 \subseteq \mathfrak{U}_2 \subseteq \cdots$ a chain of idempotent ideals. Then there exists a positive integer $n$ such that $\mathfrak{U}_n = \mathfrak{U}_{n+1}$. Assume that the quotient algebra $\Lambda/\mathfrak{U}_n$ has only the trivial idempotents 0 and 1. Recall that $\mathfrak{U}_i = \tau_{P_i}(\Lambda)$ is the trace ideal of a projective $\Lambda$-module $P_i$. Then we have the following stratification of $\mathsf{mod}\text{-}\Lambda$:

$$\begin{array}{rl}
(\mathsf{mod}\text{-}\Lambda/\mathfrak{U}_1, \mathsf{mod}\text{-}\Lambda, \mathsf{mod}\text{-}\Gamma_1) & \mathfrak{U}_1 : \text{ idempotent ideal in } \Lambda \\
(\mathsf{mod}\text{-}\Lambda/\mathfrak{U}_2, \mathsf{mod}\text{-}\Lambda/\mathfrak{U}_1, \mathsf{mod}\text{-}\Gamma_2) & \mathfrak{U}_2/\mathfrak{U}_1 : \text{ idempotent ideal in } \Lambda/\mathfrak{U}_1 \\
\vdots & \vdots \\
(\mathsf{mod}\text{-}\Lambda/\mathfrak{U}_{n-1}, \mathsf{mod}\text{-}\Lambda/\mathfrak{U}_{n-2}, \mathsf{mod}\text{-}\Gamma_{n-1}) & \mathfrak{U}_{n-1}/\mathfrak{U}_{n-2} : \text{ idempotent ideal in } \Lambda/\mathfrak{U}_{n-2} \\
(\mathsf{mod}\text{-}\Lambda/\mathfrak{U}_n, \mathsf{mod}\text{-}\Lambda/\mathfrak{U}_{n-1}, \mathsf{mod}\text{-}\Gamma_n) & \mathfrak{U}_n/\mathfrak{U}_{n-1} : \text{ idempotent ideal in } \Lambda/\mathfrak{U}_{n-1} \\
(\mathsf{mod}\text{-}\Lambda/\mathfrak{U}_n, \mathsf{mod}\text{-}\Lambda/\mathfrak{U}_n, 0) &
\end{array}$$

Therefore we have $\mathsf{str.dim}\,\mathsf{mod}\text{-}\Lambda \leq n$. Similarly if we have a chain of idempotent ideals $0 = \mathfrak{U}_0 \subseteq \mathfrak{U}_1 \subseteq \cdots \subseteq \mathfrak{U}_{n-1} \subseteq \mathfrak{U}_n = \Lambda$ such that $\Lambda/\mathfrak{U}_{n-1}$ has only the trivial idempotents 0 and 1, then we have a stratification of $\mathsf{mod}\text{-}\Lambda$ as above and $\mathsf{str.dim}\,\mathsf{mod}\text{-}\Lambda \leq n-1$.

The following result is a consequence of Theorem 2.2.1 for a stratified abelian category.

COROLLARY 2.2.22. *Let $\mathscr{B}$ be an abelian category with $\mathsf{str.dim}\,\mathscr{B} \leq n$. Suppose that $\sup\{\mathsf{pd}_{\mathscr{A}_k} \mathsf{i}_{k+1}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}_{k+1}\} \leq 1$ for every $-1 \leq k \leq n-2$ and the categories $\mathscr{C}_0, \mathscr{C}_1, \cdots, \mathscr{C}_{n-1}$ have finite global dimension. Let $\max\{\mathsf{gl.dim}\,\mathscr{C}_0, \cdots, \mathsf{gl.dim}\,\mathscr{C}_{n-1}\} = m$. Then*:

$$\mathsf{gl.dim}\,\mathscr{B} \;\leq\; (2m+2) + (n-1)(m+2)$$

PROOF. For $n = 0$ we have the trivial stratification $(0, \mathscr{B}, \mathscr{C}_0)$ and therefore we infer that $\mathsf{gl.dim}\,\mathscr{B} \leq m$. Suppose that $n = 1$. This means that we have a sequence of recollements $(\mathscr{A}_0, \mathscr{B}, \mathscr{C}_0)$ and $(0, \mathscr{A}_0, \mathscr{C}_1)$. Thus $\mathsf{gl.dim}\,\mathscr{A}_0 \leq m$ and then from Theorem 2.2.1 it follows that $\mathsf{gl.dim}\,\mathscr{B} \leq m + m + 1 + 1 = 2m + 2$. Continuing inductively the result follows. $\square$

## 2.3. Finitistic Dimension

In this section we turn our attention to the behavior of the finitistic dimension of the categories involved in a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ of abelian categories. Recall that the finitistic projective dimension $\mathsf{FPD}(\mathscr{B})$ of $\mathscr{B}$ is defined by

$$\mathsf{FPD}\,(\mathscr{B}) := \sup\{\mathsf{pd}_{\mathscr{B}} B \mid \mathsf{pd}_{\mathscr{B}} B < \infty\}$$

To this end it is convenient to introduce the following notion.

Definition 2.3.1. Let $\mathsf{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a right exact functor between abelian categories, where we assume that $\mathscr{A}$ has enough projectives. We say that $\mathsf{F}$ has locally bounded homological dimension, if there exists $n \geq 0$ such that whenever $\mathsf{L}_m\mathsf{F}(A) = 0$ for $m \gg 0$ then $\mathsf{L}_m\mathsf{F}(A) = 0$ for every $m \geq n+1$. The minimum such $n$ (if it exists) is called the *locally bounded homological dimension* of $\mathsf{F}$ and is denoted by $\mathsf{l.b.hom.dim}\,\mathsf{F}$.

For instance $\mathsf{F}$ has locally bounded homological dimension if it has finite homological dimension, i.e. $\mathsf{L}_k\mathsf{F} = 0$ for some $k \geq 1$. Dually one defines the *locally bounded cohomological dimension* of a left exact functor $\mathsf{G}\colon \mathscr{A} \longrightarrow \mathscr{B}$.

We begin with the following result which provides a general bound for the finitistic projective dimension of $\mathscr{B}$ in terms of the finitistic projective dimension of $\mathscr{C}$ and the locally bounded homological dimension of the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$.

Theorem 2.3.2. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects. If the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ has locally bounded homological dimension, then*:

$$\mathsf{FPD}\,(\mathscr{C}) \;\leq\; \mathsf{FPD}\,(\mathscr{B}) + \mathsf{l.b.hom.dim}\,\mathsf{l}$$

Proof. Assume that $\mathsf{l.b.hom.dim}\,\mathsf{l} = n$ and $\mathsf{FPD}\,(\mathscr{B}) = k < \infty$. Let $C \in \mathscr{C}$ has finite projective dimension, so the $n$th syzygy $\Omega^n(C)$ of $C$ has finite projective dimension as well. Since clearly $\mathsf{L}_m\mathsf{l}(C) = 0$ for $m \gg 0$, it follows that $\mathsf{L}_m\mathsf{l}(C) = 0$ for every $m \geq n+1$. We infer that:

$$\mathsf{L}_m\mathsf{l}(\Omega^n(C)) = 0 \quad \text{for every} \quad m \geq 1 \tag{2.3.1}$$

Consider a (finite) projective resolution

$$\cdots \longrightarrow Q_k \longrightarrow Q_{k-1} \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \Omega^n(C) \longrightarrow 0$$

of $\Omega^n(C)$. Applying the right exact functor $\mathsf{l}$ and using (2.3.1), we then have an exact sequence:

$$\cdots \longrightarrow \mathsf{l}(Q_k) \longrightarrow \mathsf{l}(Q_{k-1}) \longrightarrow \cdots \longrightarrow \mathsf{l}(Q_1) \longrightarrow \mathsf{l}(Q_0) \longrightarrow \mathsf{l}(\Omega^n(C)) \longrightarrow 0$$

which, since $\mathsf{l}$ preserves projectives, is a projective resolution of $\mathsf{l}(\Omega^n(C))$ and then we have $\mathsf{pd}_{\mathscr{B}}\,\mathsf{l}(\Omega^n(C)) < \infty$. Since $\mathsf{FPD}\,(\mathscr{B}) = k$, it follows that $\mathsf{l}(\Omega^{n+k}(C)) = \Omega^k(\mathsf{l}(\Omega^n(C)))$ lies in $\mathsf{Proj}\,\mathscr{B}$. Let $f\colon Q \longrightarrow \Omega^{n+k}(C)$ be an epimorphism with $Q \in \mathsf{Proj}\,\mathscr{C}$. Then we have the split exact sequence $0 \longrightarrow K \longrightarrow \mathsf{l}(Q) \longrightarrow \mathsf{l}(\Omega^{n+k}(C)) \longrightarrow 0$ and if we apply the exact functor $\mathsf{e}$ we deduce that $\Omega^{n+k}(C)$ is a direct summand of the projective object $Q$, i.e. the object $\Omega^{n+k}(C)$ is projective and therefore $\mathsf{pd}_{\mathscr{C}}\,C \leq n+k$. We infer that $\mathsf{FPD}\,(\mathscr{C}) \leq \mathsf{FPD}\,(\mathscr{B}) + n$. □

Our aim in the remainder of this section is to improve on the bound of Theorem 2.3.2 under natural conditions concerning the homological behavior of the functors $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ and $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$. To this end we need some preparations.

Lemma 2.3.3. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects, and assume that the functors $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ and $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ are exact.*

(i) *For every $B \in \mathscr{B}$ we have the exact sequence*:

$$0 \longrightarrow \mathsf{ip}(B) \xrightarrow{\kappa_B} B \xrightarrow{\nu_B} \mathsf{re}(B) \longrightarrow 0$$

(ii) *For every $A \in \mathscr{A}$ we have: $\mathsf{L}_m\mathsf{q}(\mathsf{i}(A)) = 0$, $\forall m \geq 1$. In particular: $\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(A) = \mathsf{pd}_{\mathscr{A}}\,A$.*

(iii) *There exists a natural map* $\mathsf{L}_m\mathsf{q}(-) \longrightarrow \mathsf{L}_m\mathsf{q}(\mathsf{re}(-))$ *which is a monomorphism for* $m = 1$ *and an isomorphism for every* $m \geq 2$.

(iv) *If* $C \in \mathscr{C}$ *with* $\mathsf{pd}_{\mathscr{C}}\, C \leq n$ *then* $\mathsf{L}_m\mathsf{q}(\mathsf{r}(C)) = 0$ *for every* $m \geq n+2$.

(v) *Let* $B \in \mathscr{B}$ *such that* $\mathsf{L}_1\mathsf{q}(B) = 0$. *Then the sequence*

$$0 \longrightarrow \mathsf{le}(B) \xrightarrow{\mu_B} B \xrightarrow{\lambda_B} \mathsf{iq}(B) \longrightarrow 0$$

*is exact.*

(vi) *Let* $B \in \mathscr{B}$. *Then* $B \in \mathsf{Proj}\,\mathscr{B}$ *if and only if* $\mathsf{q}(B) \in \mathsf{Proj}\,\mathscr{A}$, $\mathsf{e}(B) \in \mathsf{Proj}\,\mathscr{C}$ *and* $\mathsf{L}_1\mathsf{q}(B) = 0$.

PROOF. (i) Let $B \in \mathscr{B}$. Then from Proposition 1.1.6 there exists an exact sequence $0 \longrightarrow \mathsf{ip}(B) \xrightarrow{\kappa_B} B \xrightarrow{\nu_B} \mathsf{re}(B) \longrightarrow \mathsf{Coker}\,\nu_B \longrightarrow 0$ in $\mathscr{B}$, where $\mathsf{Coker}\,\nu_B \in \mathsf{i}(\mathscr{A})$ and thus $\mathsf{Coker}\,\nu_B = \mathsf{i}(A)$ for some $A \in \mathscr{A}$. Since $\mathsf{pr} = 0$ and $\mathsf{p}$ is exact we infer that $\mathsf{p}(\mathsf{i}(A)) = A = 0$, and then $\mathsf{i}(A) = \mathsf{Coker}\,\nu_B = 0$.

(ii) Since the functor $\mathsf{p}$ is exact it follows that the functor $\mathsf{i}$ preserves projectives. Hence if $P^\bullet \longrightarrow A$ is a projective resolution of $A$ in $\mathscr{A}$ then $\mathsf{i}(P^\bullet) \longrightarrow \mathsf{i}(A)$ is a projective resolution of $\mathsf{i}(A)$ in $\mathscr{B}$. Since $\mathsf{qi} \simeq \mathrm{Id}_{\mathscr{A}}$, this clearly implies that $\mathsf{L}_m\mathsf{q}(\mathsf{i}(A)) = 0$, $\forall m \geq 1$, and therefore $\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(A) = \mathsf{pd}_{\mathscr{A}}\, A$.

(iii) Applying the functor $\mathsf{q}$ to the exact sequence in (i) we have the long exact sequence:

$$\cdots \to \mathsf{L}_1\mathsf{q}(\mathsf{ip}(B)) \to \mathsf{L}_1\mathsf{q}(B) \to \mathsf{L}_1\mathsf{q}(\mathsf{re}(B)) \to \mathsf{q}(\mathsf{ip}(B)) \to \mathsf{q}(B) \to \mathsf{q}(\mathsf{re}(B)) \to 0$$

and the result follows from (ii).

(iv) Let $Q \in \mathsf{Proj}\,\mathscr{C}$. From (iii) we have the isomorphism $\mathsf{L}_m\mathsf{q}(\mathsf{l}(Q)) \simeq \mathsf{L}_m\mathsf{q}(\mathsf{re}(\mathsf{l}(Q)))$ for every $m \geq 2$. Since $\mathsf{el} \simeq \mathrm{Id}_{\mathscr{C}}$ and $\mathsf{l}(Q) \in \mathsf{Proj}\,\mathscr{B}$ it follows that $\mathsf{L}_m\mathsf{q}(\mathsf{r}(Q)) = 0$ for every $m \geq 2$. Suppose that $n = 1$, so we have an exact sequence $0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow C \longrightarrow 0$ with $Q_i \in \mathsf{Proj}\,\mathscr{C}$. Applying the functor $\mathsf{r}$ we obtain the exact sequence $0 \longrightarrow \mathsf{r}(Q_1) \longrightarrow \mathsf{r}(Q_0) \longrightarrow \mathsf{r}(C) \longrightarrow 0$. Then from the following long exact sequence:

$$\cdots \to \mathsf{L}_1\mathsf{q}(\mathsf{r}(Q_1)) \to \mathsf{L}_1\mathsf{q}(\mathsf{r}(Q_0)) \to \mathsf{L}_1\mathsf{q}(\mathsf{r}(C)) \to \mathsf{q}(\mathsf{r}(Q_1)) \to \mathsf{q}(\mathsf{r}(Q_0)) \to \mathsf{q}(\mathsf{r}(C)) \to 0$$

we derive that $\mathsf{L}_m\mathsf{q}(\mathsf{r}(C)) = 0$ for every $m \geq 3$. Then the result follows by induction on $n$.

(v) Consider the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Ker}\,\mu_B \arrow[r] & \mathsf{le}(B) \arrow[r, "\mu_B"] \arrow[d] & B \arrow[r, "\lambda_B"] & \mathsf{iq}(B) \arrow[r] & 0 \\ & & \mathsf{F}(B) \arrow[ur] & & & \end{tikzcd} \tag{2.3.2}$$

where, since $\mathsf{Ker}\,\mu_B \in \mathsf{i}(\mathscr{A})$, we have $\mathsf{Ker}\,\mu_B = \mathsf{i}(A)$ for some $A \in \mathscr{A}$. Applying the functor $\mathsf{q}\colon \mathscr{B} \longrightarrow \mathscr{A}$ to the sequence (2.3.2) we have the following long exact sequence:

$$\cdots \longrightarrow \mathsf{L}_2\mathsf{q}(\mathsf{iq}(B)) \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{F}(B)) \longrightarrow \mathsf{L}_1\mathsf{q}(B) \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{iq}(B)) \longrightarrow \cdots$$

Since $\mathsf{L}_1\mathsf{q}(B) = 0$, by (ii) we deduce that $\mathsf{L}_1\mathsf{q}(\mathsf{F}(B)) = 0$. Hence the long exact sequence

$$\cdots \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{F}(B)) \longrightarrow \mathsf{q}(\mathsf{i}(A)) \longrightarrow \mathsf{q}(\mathsf{le}(B)) \longrightarrow \mathsf{q}(\mathsf{F}(B)) \longrightarrow 0$$

implies that $A = 0$ since $\mathsf{ql} = 0$ and $\mathsf{qi} \simeq \mathrm{Id}_{\mathscr{A}}$. We infer that (2.3.2) becomes a short exact sequence:

$$0 \longrightarrow \mathsf{le}(B) \xrightarrow{\mu_B} B \xrightarrow{\lambda_B} \mathsf{iq}(B) \longrightarrow 0 \tag{2.3.3}$$

(vi) Let $B \in \mathscr{B}$ be such that $\mathsf{q}(B) \in \mathsf{Proj}\,\mathscr{A}$, $\mathsf{e}(B) \in \mathsf{Proj}\,\mathscr{C}$ and $\mathsf{L}_1\mathsf{q}(B) = 0$. Since $\mathsf{i}$ and $\mathsf{l}$ preserve projectives, it follows that both $\mathsf{le}(B)$ and $\mathsf{iq}(B)$ are projective in $\mathscr{B}$. Then the sequence (2.3.3) is split exact, so $B \simeq \mathsf{le}(B) \oplus \mathsf{iq}(B)$ and $B \in \mathsf{Proj}\,\mathscr{B}$. The converse implication is clear. □

LEMMA 2.3.4. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects, and assume that the functors $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ and $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ are exact. Let $B \in \mathscr{B}$ be a left $\mathsf{q}$-acyclic object, i.e. $\mathsf{L}_m\mathsf{q}(B) = 0$ for every $m \geq 1$. Then:*

$$\mathsf{pd}_{\mathscr{B}}\, B = \max\{\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B), \mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(B)\}$$

PROOF. If $\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B) = \infty$ or $\mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(B) = \infty$, then $\mathsf{pd}_{\mathscr{B}}\, B \leq \max\{\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B), \mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(B)\}$. Hence we may assume that $\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B) = \kappa < \infty$ and $\mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(B) = \lambda < \infty$, and let $m = \max\{\kappa, \lambda\}$. If $m = 0$, then $\mathsf{e}(B)$ is projective in $\mathscr{C}$ and $\mathsf{q}(B)$ is projective in $\mathscr{A}$. Then from Lemma 2.3.3(vi) it follows that $B$ is projective in $\mathscr{B}$ since $\mathsf{L}_1\mathsf{q}(B) = 0$. Suppose that $m \neq 0$. Let $0 \longrightarrow \Omega(B) \longrightarrow P_0 \longrightarrow B \longrightarrow 0$ be an exact sequence with $P_0 \in \mathsf{Proj}\,\mathscr{B}$. Then from Lemma 2.3.3(v) and since $\mathsf{L}_1\mathsf{q}(B) = 0$ we have the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathsf{le}(\Omega(B)) \arrow[r] \arrow[d] & \mathsf{le}(P_0) \arrow[r] \arrow[d] & \mathsf{le}(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega(B) \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathsf{iq}(\Omega(B)) \arrow[r] \arrow[d] & \mathsf{iq}(P_0) \arrow[r] \arrow[d] & \mathsf{iq}(B) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then from the exact sequence $0 \longrightarrow \mathsf{le}(\Omega(B)) \longrightarrow \mathsf{le}(P_0) \longrightarrow \mathsf{le}(B) \longrightarrow 0$ it follows that

$$\mathsf{L}_m\mathsf{l}(\mathsf{e}(B)) = 0 \quad \forall m \geq 1 \tag{2.3.4}$$

Also the middle vertical exact sequence splits since $\mathsf{iq}(P_0) \in \mathsf{Proj}\,\mathscr{B}$ and therefore $P_0 \simeq \mathsf{le}(P_0) \oplus \mathsf{iq}(P_0)$. Continuing in this way we construct a projective resolution of $B$ as follows:

$$0 \to \Omega^m(B) \to \mathsf{le}(P_{m-1}) \oplus \mathsf{iq}(P_{m-1}) \to \cdots \to \mathsf{le}(P_1) \oplus \mathsf{iq}(P_1) \to \mathsf{le}(P_0) \oplus \mathsf{iq}(P_0) \to B \to 0 \tag{2.3.5}$$

where the $m$th syzygy fits in the following exact sequence:

$$0 \longrightarrow \mathsf{le}(\Omega^m(B)) \longrightarrow \Omega^m(B) \longrightarrow \mathsf{iq}(\Omega^m(B)) \longrightarrow 0 \tag{2.3.6}$$

From the relation (2.3.4) it follows that $\mathsf{l}(\Omega^m(\mathsf{e}(B))) \simeq \mathsf{le}(\Omega^m(B))$ and since $\mathsf{L}_m\mathsf{q}(B) = 0$ for every $m \geq 1$ we have $\mathsf{i}(\Omega^m(\mathsf{q}(B))) \simeq \mathsf{iq}(\Omega^m(B))$. Since $m$ is the maximum of the projective dimension of $\mathsf{e}(B)$ and $\mathsf{q}(B)$ it follows that $\Omega^m(\mathsf{q}(B)) \in \operatorname{Proj}\mathscr{A}$ and $\Omega^m(\mathsf{e}(B)) \in \operatorname{Proj}\mathscr{C}$. Since $\mathsf{i}$ and $\mathsf{l}$ preserve projectives we derived that the objects $\mathsf{i}(\Omega^m(\mathsf{q}(B)))$ and $\mathsf{l}(\Omega^m(\mathsf{e}(B)))$ are projective in $\mathscr{B}$. Hence the exact sequence (2.3.6) splits and therefore the $m$th syzygy $\Omega^m(B) \simeq \mathsf{le}(\Omega^m(B)) \oplus \mathsf{iq}(\Omega^m(B))$ is projective in $\mathscr{B}$. Then from the sequence (2.3.5) we infer that

$$\mathsf{pd}_{\mathscr{B}}\, B \leq m = \max\{\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B), \mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(B)\}$$

Conversely, suppose that $\mathsf{pd}_{\mathscr{B}}\, B = m$. Then $\Omega^m(B) \in \operatorname{Proj}\mathscr{B}$ and therefore $\mathsf{e}(\Omega^m(B)) \in \operatorname{Proj}\mathscr{C}$ and $\mathsf{q}(\Omega^m(B)) \in \operatorname{Proj}\mathscr{A}$. But since $\Omega^m(\mathsf{e}(B)) = \mathsf{e}(\Omega^m(B))$ and $\Omega^m(\mathsf{q}(B)) = \mathsf{q}(\Omega^m(B))$ (because $\mathsf{L}_m\mathsf{q}(B) = 0$ for every $m \geq 1$) it follows that $\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B) \leq m$ and $\mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(B) \leq m$. Hence

$$\max\{\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B), \mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(B)\} \leq \mathsf{pd}_{\mathscr{B}}\, B$$

and therefore we conclude that $\mathsf{pd}_{\mathscr{B}}\, B = \max\{\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B), \mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(B)\}$. ☐

LEMMA 2.3.5. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects, and assume that the functors $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ and $\mathsf{l}, \mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ are exact. Then:*

(i) $\mathsf{L}_m\mathsf{q}(-) = 0, \ \ \forall m \geq 2$.
(ii) *For any object $C \in \mathscr{C}$:* $\mathsf{pd}_{\mathscr{B}}\, \mathsf{r}(C) = \max\{\mathsf{pd}_{\mathscr{C}}\, C, \mathsf{pd}_{\mathscr{A}}\, \mathsf{pl}(C) + 1\}$.

PROOF. (i) Let $B \in \mathscr{B}$. Then from Lemma 2.3.3(i) we have the exact sequence $0 \longrightarrow \mathsf{iple}(B) \longrightarrow \mathsf{le}(B) \longrightarrow \mathsf{re}(B) \longrightarrow 0$. Let $0 \longrightarrow K_0 \longrightarrow Q_0 \longrightarrow \mathsf{e}(B) \longrightarrow 0$ be an exact sequence with $Q_0 \in \operatorname{Proj}\mathscr{C}$. Then since $\mathsf{l}$ is exact we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{l}(K_0) \arrow[r] \arrow[d] & \mathsf{l}(Q_0) \arrow[r] \arrow[d, equal] & \mathsf{le}(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & \mathsf{l}(Q_0) \arrow[r] & \mathsf{re}(B) \arrow[r] & 0
\end{tikzcd}$$

and from the Snake Lemma we derive the exact sequence

$$0 \longrightarrow \mathsf{l}(K_0) \longrightarrow K \longrightarrow \mathsf{iple}(B) \longrightarrow 0$$

Then we have the following long exact sequence:

$$\cdots \longrightarrow \mathsf{L}_2\mathsf{q}(\mathsf{re}(B)) \longrightarrow \mathsf{L}_1\mathsf{q}(K) \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{l}(Q_0)) \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{re}(B)) \longrightarrow \cdots$$

and therefore

$$\mathsf{L}_2\mathsf{q}(\mathsf{re}(B)) \simeq \mathsf{L}_1\mathsf{q}(K) \tag{$*$}$$

Hence from the following long exact sequence:

$$\cdots \longrightarrow \mathsf{L}_2\mathsf{q}(\mathsf{iple}(B)) \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{l}(K_0)) \longrightarrow \mathsf{L}_1\mathsf{q}(K) \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{iple}(B)) \longrightarrow \cdots$$

we deduce, using Lemma 2.3.3(ii), that $\mathsf{L}_1\mathsf{q}(K) \simeq \mathsf{L}_1\mathsf{q}(\mathsf{l}(K_0))$. But since the functor $\mathsf{l}$ is exact and $\mathsf{ql} = 0$ it follows that $\mathsf{L}_m\mathsf{q}(\mathsf{l}(K_0)) = 0$ for every $m \geq 1$. Thus from the isomorphism $(*)$ it follows that $\mathsf{L}_2\mathsf{q}(\mathsf{re}(B)) = 0$ and therefore $\mathsf{L}_2\mathsf{q}(B) = 0$. Continuing in this way we infer that $\mathsf{L}_m\mathsf{q}(\mathsf{re}(B)) = 0$ for every $m \geq 2$. By Lemma 2.3.3(iii), this implies that $\mathsf{L}_m\mathsf{q}(B) = 0, \forall m \geq 2$.

(ii) Let $C \in \operatorname{Proj}\mathscr{C}$. Then from the exact sequence $0 \longrightarrow \mathsf{ipl}(C) \longrightarrow \mathsf{l}(C) \longrightarrow \mathsf{r}(C) \longrightarrow 0$ (1) and since $\mathsf{pd}_{\mathscr{B}}\,\mathsf{ipl}(C) = \mathsf{pd}_{\mathscr{A}}\,\mathsf{pl}(C)$, see Lemma 2.3.3(ii), our result holds. Suppose that $\mathsf{pd}_{\mathscr{C}}\,C = \infty$. Then $\mathsf{pd}_{\mathscr{C}}\,C = \mathsf{pd}_{\mathscr{C}}\,\mathsf{er}(C) \leq \mathsf{pd}_{\mathscr{B}}\,\mathsf{r}(C)$ and therefore $\mathsf{pd}_{\mathscr{B}}\,\mathsf{r}(C) = \infty$. Let $\mathsf{pd}_{\mathscr{C}}\,C = n < \infty$ and let $0 \longrightarrow K \longrightarrow Q_0 \longrightarrow C \longrightarrow 0$ be exact with $Q_0 \in \operatorname{Proj}\mathscr{C}$. From Lemma 2.3.3(i) and since the functor $\mathsf{r}$ is exact we have the following exact commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{ipl}(Q_0) \arrow[r] \arrow[d, tail] & \mathsf{l}(Q_0) \arrow[r] \arrow[d, equal] & \mathsf{r}(Q_0) \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & K \arrow[r] & \mathsf{l}(Q_0) \arrow[r] & \mathsf{r}(C) \arrow[r] & 0 \end{tikzcd} \tag{2.3.7}$$

Then we have the following long exact sequence:

$$\cdots \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{l}(Q_0)) \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{r}(C)) \longrightarrow \mathsf{q}(K) \longrightarrow \mathsf{q}(\mathsf{l}(Q_0)) \longrightarrow \mathsf{q}(\mathsf{r}(C)) \longrightarrow 0$$

and thus we have $\mathsf{q}(K) \simeq \mathsf{L}_1\mathsf{q}(\mathsf{r}(C))$ since $\mathsf{ql} = 0$. From the exact sequence (1) we obtain the following long exact sequence:

$$\cdots \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{l}(C)) \longrightarrow \mathsf{L}_1\mathsf{q}(\mathsf{r}(C)) \longrightarrow \mathsf{pl}(C) \longrightarrow \mathsf{q}(\mathsf{l}(C)) \longrightarrow \mathsf{q}(\mathsf{r}(C)) \longrightarrow 0$$

Since the functor $\mathsf{l}$ is exact and $\mathsf{ql} = 0$ it follows that $\mathsf{L}_1\mathsf{q}(\mathsf{l}(C)) = 0$. Hence we infer that $\mathsf{L}_1\mathsf{q}(\mathsf{r}(C)) \simeq \mathsf{pl}(C)$ and therefore we have the isomorphism:

$$\mathsf{q}(K) \simeq \mathsf{pl}(C) \tag{2.3.8}$$

From the lower exact sequence of diagram (2.3.7) we have $\mathsf{pd}_{\mathscr{B}}\,\mathsf{r}(C) = 1 + \mathsf{pd}_{\mathscr{B}}\,K$. Since $\mathsf{L}_m\mathsf{q}(-) = 0$, $\forall m \geq 2$, it follows that $\mathsf{L}_m\mathsf{q}(K) = 0$ for every $m \geq 1$. Then from Lemma 2.3.4 we have $\mathsf{pd}_{\mathscr{B}}\,K = \max\{\mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(K), \mathsf{pd}_{\mathscr{A}}\,\mathsf{q}(K)\}$ and thus $\mathsf{pd}_{\mathscr{B}}\,\mathsf{r}(C) = \max\{1 + \mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(K), 1 + \mathsf{pd}_{\mathscr{A}}\,\mathsf{q}(K)\}$. Using relation (2.3.8) we infer that $\mathsf{pd}_{\mathscr{B}}\,\mathsf{r}(C) = \max\{\mathsf{pd}_{\mathscr{C}}\,C, 1 + \mathsf{pd}_{\mathscr{A}}\,\mathsf{pl}(C)\}$. □

After these preparations we can prove the second main result of this section which provides bounds for the finitistic projective dimension of $\mathscr{B}$ in terms of the finitistic projective dimension of $\mathscr{A}$ and $\mathscr{C}$.

THEOREM 2.3.6. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects.*

(i) *If* $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{i}(P) \mid P \in \operatorname{Proj}\mathscr{A}\} \leq 1$, *then:*

$$\mathsf{FPD}\,(\mathscr{A}) \;\leq\; \mathsf{FPD}\,(\mathscr{B})$$

(ii) *If* $\sup\{\mathsf{pd}_{\mathscr{C}}\,\mathsf{e}(P) \mid P \in \operatorname{Proj}\mathscr{B}\} < \infty$, *then:*

$$\mathsf{FPD}(\mathscr{B}) \;\leq\; \mathsf{FPD}(\mathscr{C}) + \mathsf{gl.dim}_{\mathscr{A}}\mathscr{B} + 1$$

(iii) *If the functors* $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ *and* $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ *are exact, then:*

$$\mathsf{FPD}(\mathscr{A}) \;\leq\; \mathsf{FPD}(\mathscr{B}) \;\leq\; \mathsf{FPD}(\mathscr{A}) + \mathsf{FPD}(\mathscr{C}) + 1$$

*Suppose in addition that the functor* $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ *is exact and* $\mathsf{pd}_{\mathscr{A}}\,\mathsf{pl}(C) < \infty$, $\forall C \in \mathscr{C}$. *Then:*

(a) *If* $\mathsf{FPD}(\mathscr{C}) \;\leq\; \mathsf{FPD}(\mathscr{A}) + 1$, *then* $\mathsf{FPD}(\mathscr{B}) \;\leq\; \mathsf{FPD}(\mathscr{A}) + 1$.

(b) *If* $\mathsf{FPD}(\mathscr{C}) \;>\; \mathsf{FPD}(\mathscr{A}) + 1$, *then* $\mathsf{FPD}(\mathscr{B}) \;\leq\; \mathsf{FPD}(\mathscr{C})$.

PROOF. (i) Assume that $\mathsf{FPD}\,(\mathscr{B}) = k < \infty$ and let $A \in \mathscr{A}$ with $\mathsf{pd}_{\mathscr{A}}\, A < \infty$. Since $\sup\{\mathsf{pd}_{\mathscr{B}}\, \mathsf{i}(P) \mid P \in \mathsf{Proj}\,\mathscr{A}\} \leq 1$ it follows from Proposition 2.2.16 that the functor $\mathsf{i} \colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding. Thus $\mathsf{pd}_{\mathscr{A}}\, A \leq \mathsf{pd}_{\mathscr{B}}\, \mathsf{i}(A)$. Then from Proposition 2.2.5 we infer that $\mathsf{i}(A)$ is an object of finite projective dimension since $\mathsf{pd}_{\mathscr{B}}\, \mathsf{i}(A) \leq \mathsf{pd}_{\mathscr{A}}\, A + 1$. This implies that $\mathsf{pd}_{\mathscr{B}}\, \mathsf{i}(A) \leq k$ and therefore $\mathsf{pd}_{\mathscr{A}}\, A \leq k$. Hence $\mathsf{FPD}\,(\mathscr{A}) \leq \mathsf{FPD}\,(\mathscr{B})$.

(ii) The upper bound for the $\mathsf{FPD}\,(\mathscr{B})$ follows from Proposition 2.2.5(i) since the functor $\mathsf{e} \colon \mathscr{B} \longrightarrow \mathscr{C}$ preserves objects of finite projective dimension if and only if $\sup\{\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(P) \mid P \in \mathsf{Proj}\,\mathscr{B}\} < \infty$.

(iii) Let $B \in \mathscr{B}$ with $\mathsf{pd}_{\mathscr{B}}\, B < \infty$ and let

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, two heads] & P_n \arrow[r] \arrow[d, two heads] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & B \arrow[r] & 0 \\
& \Omega^{n+1}(B) \arrow[ur, tail] & \Omega^n(B) \arrow[ur, tail] & & & & & &
\end{tikzcd}$$

be a finite projective resolution of $B$. Since the functor $\mathsf{r}$ is exact it follows that the quotient functor $\mathsf{e}$ preserves projective objects. Thus we have the following exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathsf{e}(P_{n+1}) \arrow[r] \arrow[d, two heads] & \mathsf{e}(P_n) \arrow[r] \arrow[d, two heads] & \mathsf{e}(P_{n-1}) \arrow[r] & \cdots \arrow[r] & \mathsf{e}(P_0) \arrow[r] & \mathsf{e}(B) \arrow[r] & 0 \\
& \mathsf{e}(\Omega^{n+1}(B)) \arrow[ur, tail] & \mathsf{e}(\Omega^n(B)) \arrow[ur, tail] & & & & &
\end{tikzcd}$$

which is a finite projective resolution of $\mathsf{e}(B)$ in $\mathscr{C}$ since $\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B) \leq \mathsf{pd}_{\mathscr{B}}\, B < \infty$ and $\mathsf{e}(P_i) \in \mathsf{Proj}\,\mathscr{C}$. Let $\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B) = n < \infty$. Then the $n$th syzygy $\Omega^n(\mathsf{e}(B)) = \mathsf{e}(\Omega^n(B))$ of $\mathsf{e}(B)$ is projective and thus $\Omega^{n+1}(\mathsf{e}(B)) = \mathsf{e}(\Omega^{n+1}(B)) \in \mathsf{Proj}\,\mathscr{B}$. From Lemma 2.3.3(iv) we have $\mathsf{L}_m\mathsf{q}(\mathsf{r}(\mathsf{e}(B))) = 0$ for every $m \geq n+2$ and using the isomorphism of Lemma 2.3.3(iii) we infer that $\mathsf{L}_m\mathsf{q}(B) = 0$ for every $m \geq n+2$. Therefore it follows that:

$$\mathsf{L}_m\mathsf{q}(\Omega^{n+1}(B)) = 0 \ \text{ for every } \ m \geq 1$$

Since $\mathsf{e}(\Omega^{n+1}(B))$ is projective, applying Lemma 2.3.4 we have

$$\mathsf{pd}_{\mathscr{B}}\, \Omega^{n+1}(B) = \mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(\Omega^{n+1}(B))$$

As a consequence we infer that

$$\mathsf{pd}_{\mathscr{B}}\, B \leq n+1+\mathsf{pd}_{\mathscr{B}}\, \Omega^{n+1}(B) = \mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B) + \mathsf{pd}_{\mathscr{A}}\, \mathsf{q}(\Omega^{n+1}(B)) + 1 < \infty$$

This yields that $\mathsf{FPD}(\mathscr{B}) \leq \mathsf{FPD}(\mathscr{A}) + \mathsf{FPD}(\mathscr{C}) + 1$. For the lower bound assume that $\mathsf{FPD}\,(\mathscr{B}) = k < \infty$ and let $A \in \mathscr{A}$ with $\mathsf{pd}_{\mathscr{A}}\, A < \infty$. Since $\mathsf{p}$ is exact we have $\mathsf{pd}_{\mathscr{B}}\, \mathsf{i}(A) < \infty$ and therefore $\mathsf{pd}_{\mathscr{B}}\, \mathsf{i}(A) \leq k$. By Theorem 2.1.10 it follows that $\mathsf{pd}_{\mathscr{A}}\, A \leq \mathsf{pd}_{\mathscr{B}}\, \mathsf{i}(A)$ and therefore $\mathsf{pd}_{\mathscr{A}}\, A \leq k$. Hence $\mathsf{FPD}(\mathscr{A}) \leq \mathsf{FPD}(\mathscr{B})$. Finally we prove the last statement of (iii). Assume that $\mathsf{FPD}\,(\mathscr{C}) = m < \infty$ and $\mathsf{FPD}\,(\mathscr{A}) = n < \infty$. Let $B \in \mathscr{B}$ with $\mathsf{pd}_{\mathscr{B}}\, B < \infty$. From Lemma 2.3.3(i),(ii) we have an extension $0 \longrightarrow \mathsf{ip}(B) \longrightarrow B \longrightarrow \mathsf{re}(B) \longrightarrow 0$ and $\mathsf{pd}_{\mathscr{B}}\, \mathsf{ip}(B) = \mathsf{pd}_{\mathscr{A}}\, \mathsf{p}(B)$. Since $\mathsf{pd}_{\mathscr{A}}\, \mathsf{ple}(B) < \infty$, Lemma 2.3.5 implies that $\mathsf{pd}_{\mathscr{B}}\, \mathsf{re}(B) = \max\{\mathsf{pd}_{\mathscr{C}}\, \mathsf{e}(B), \mathsf{pd}_{\mathscr{A}}\, \mathsf{ple}(B)+1\} \leq \max\{m, n+1\}$. As a consequence we have $\mathsf{pd}_{\mathscr{B}}\, B \leq \max\{n, \max\{m, n+1\}\}$ and the assertions (a) and (b) follow. $\square$

REMARK 2.3.7. There is a dual version of Theorems 2.3.2 and 2.3.6 concerning the finitistic injective dimension $\mathsf{FID}\,(\mathscr{B}) := \sup\{\mathsf{id}_{\mathscr{B}} B \mid \mathsf{id}_{\mathscr{B}} B < \infty\}$ of $\mathscr{B}$. For instance assuming that the functor $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ has locally bounded cohomological dimension then we can show that $\mathsf{FID}\,(\mathscr{C}) \leq \mathsf{FID}\,(\mathscr{B}) + \mathsf{l.b.cohom.dim}\,\mathsf{r}$. We leave the dual formulation of Theorems 2.3.2 and 2.3.6 to the reader.

Applying Theorem 2.3.6 and Theorem 2.3.2 to comma categories $\mathscr{C} = (G, \mathscr{B}, \mathscr{A}) = (\mathrm{Id} \downarrow G)$, see Example 1.1.12, we have the following result. Recall that the functor $\mathsf{Z}_{\mathscr{B}}\colon \mathscr{B} \longrightarrow \mathscr{C}$ is defined on objects $Y \in \mathscr{B}$ by $\mathsf{Z}_{\mathscr{B}}(Y) = (0, Y, 0)$.

COROLLARY 2.3.8. *Let $\mathscr{C} = (G, \mathscr{B}, \mathscr{A})$ be a comma category.*

(i) *[49, Theorem 4.20] We have:*

$$\mathsf{FPD}(\mathscr{A}) \leq \mathsf{FPD}(\mathscr{C}) \leq \mathsf{FPD}(\mathscr{A}) + \mathsf{FPD}(\mathscr{B}) + 1$$

(ii) *If the functor $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ has locally bounded homological dimension, then:*

$$\mathsf{FPD}\,(\mathscr{C}) \leq \mathsf{FPD}\,(\mathscr{B}) + \mathsf{l.b.hom.dim}\,G$$

(iii) *If $\sup\{\mathsf{pd}_{\mathscr{C}}\,\mathsf{Z}_{\mathscr{B}}(P) \mid P \in \mathsf{Proj}\,\mathscr{B}\} \leq 1$, then:*

$$\mathsf{FPD}\,(\mathscr{B}) \leq \mathsf{FPD}\,(\mathscr{C})$$

PROOF. From Example 1.1.12 we have the recollement $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ and the functors $\mathsf{U}_{\mathscr{A}}\colon \mathscr{C} \longrightarrow \mathscr{A}$ and $\mathsf{Z}_{\mathscr{B}}\colon \mathscr{B} \longrightarrow \mathscr{C}$ are exact. Thus (i) follows immediately from Theorem 2.3.6. Since $\mathsf{L}_n\mathsf{T}_{\mathscr{B}}(B) = (\mathsf{L}_nG(B), 0, 0)$ $\forall n \geq 1$ and $B \in \mathscr{B}$, statement (ii) follows from Theorem 2.3.2. Finally, the last statement follows from Theorem 2.3.6 using now the recollement $(\mathscr{B}, \mathscr{C}, \mathscr{A})$. □

## 2.4. Applications to Ring Theory

In this section we apply several of the results of the previous sections, mainly those concerning global and finitistic dimension, to ring theory, building on the Examples 1.1.7, 1.1.8, 1.1.10, and 1.1.12.

**2.4.1. Global Dimension.** Let $R$ be a ring and $e$ an idempotent element of $R$. The following summarizes some of the results of section 2 of chapter 2 applied to the recollement of rings $(\mathsf{Mod}\text{-}R/ReR, \mathsf{Mod}\text{-}R, \mathsf{Mod}\text{-}eRe)$, see Example 1.1.7. We denote by $\mathsf{gl.dim}_{R/ReR}R = \sup\{\mathsf{pd}_R X \mid X \in \mathsf{Mod}\text{-}R/ReR\}$. For (i) see also Corollary 1.5 of [53] and for (iii),(iv) and (viii) see Proposition 2.2 and Proposition 2.6 of [75]. Note that the formula (iv) applied to Example 1.1.10 implies directly Theorem 5.4 of [14]. Also compare (x) with statement 5 of [45].

COROLLARY 2.4.1. *Let $R$ be a ring and $e$ an idempotent element of $R$. Then the following hold.*

(i) $\mathsf{gl.dim}\,R \leq \mathsf{gl.dim}_{R/ReR}R + \mathsf{gl.dim}\,eRe + 1$.
(ii) $\mathsf{gl.dim}_{R/ReR}R \leq \mathsf{gl.dim}\,R/ReR + \mathsf{pd}_R\,R/ReR$.
(iii) $\mathsf{gl.dim}\,eRe \leq \mathsf{gl.dim}\,R + \mathsf{pd}_{eRe}\,eR$.
(iv) *We have:*

$$\mathsf{gl.dim}\,R \leq \mathsf{gl.dim}\,R/ReR + \mathsf{gl.dim}\,eRe + \mathsf{pd}_R\,R/ReR + 1$$

(v) *If $R/ReR$ is a flat right $R$-module or $R/ReR$ is a projective left $R$-module, then:*

$$\mathsf{gl.dim}\,R \leq \mathsf{gl.dim}\,R/ReR + \mathsf{gl.dim}\,eRe + 1$$

(vi) *If $Re$ is a flat right $R$-module or $eR$ is a projective left $eRe$-module, then*:
$$\operatorname{gl.dim} eRe \ \leq\ \operatorname{gl.dim} R$$
(vii) *Let* $\operatorname{gl.dim} R/ReR < \infty$. *Then*:
$$\operatorname{gl.dim}_{R/ReR} R < \infty \quad \text{if and only if} \quad \operatorname{pd}_R R/ReR < \infty$$
(viii) *Let* $\operatorname{gl.dim} R < \infty$. *Then*:
$$\operatorname{gl.dim} eRe < \infty \quad \text{if and only if} \quad \operatorname{pd}_{eRe} eR < \infty$$
(ix) *If* $\operatorname{pd}_R ReR = n$, *then*
$$\operatorname{gl.dim} R \ \leq\ \operatorname{gl.dim} R/ReR + \operatorname{gl.dim} eRe + n + 2$$
(x) *If $ReR$ is a projective $R$-module and $eRe$ is a semisimple ring, then*
$$\operatorname{gl.dim} R \ \leq\ \operatorname{gl.dim} R/ReR + 2$$

The following is a consequence of Theorem 2.2.8.

Corollary 2.4.2. *If the natural map $R \longrightarrow R/ReR$ is a homological epimorphism, i.e. $ReR \in \mathcal{X}_\infty$, then*:
$$\operatorname{gl.dim} R/ReR \ \leq\ \operatorname{gl.dim} R \ \leq\ \operatorname{pd}_R R/ReR + \max\{\sup\{\operatorname{id}_R \mathrm{i}(I) \mid I \in \operatorname{Inj} R/ReR\} + \operatorname{gl.dim} R/ReR, \operatorname{gl.dim} eRe\}$$

By Theorem 2.2.9 we have the following result. Note that the implication $R$: right hereditery $\Rightarrow eRe$: right hereditary is due to Sandomierski [118].

Corollary 2.4.3. *If $R$ is a right hereditary ring and $e^2 = e \in R$, then the rings $eRe$ and $R/ReR$ are right hereditary. Conversely, if the rings $eRe$ and $R/ReR$ are right hereditary then*
$$\operatorname{gl.dim} R \ \leq\ 3 + \operatorname{pd}_R R/ReR$$

The next statement is due to Fossum-Griffith-Reiten [49] and follows from Example 2.2.19.

Corollary 2.4.4. *[49] If $e_1$ and $e_2$ are idempotent elements of $R$ such that $1_R = e_1 + e_2$ and $e_1Re_2 = 0$, then*:
$$\max\{\operatorname{gl.dim} e_1Re_1, \operatorname{gl.dim} e_2Re_2\} \ \leq\ \operatorname{gl.dim} R \ \leq\ \operatorname{gl.dim} e_1Re_1 + \operatorname{gl.dim} e_2Re_2 + 1$$

The next result follows from Corollary 2.2.18 and generalizes Corollary 5.6 of [14].

Corollary 2.4.5. *Let $R$ be a ring and $e^2 = e \in R$ an idempotent element. If the ideal $ReR \in \operatorname{Proj} R$, then*:
$$\max\{\operatorname{gl.dim} R/ReR, \operatorname{gl.dim} eRe\} \ \leq\ \operatorname{gl.dim} R \ \leq\ \operatorname{gl.dim} R/ReR + \operatorname{gl.dim} eRe + 2$$
*and*
$$\operatorname{gl.dim} R < \infty \quad \Longleftrightarrow \quad \operatorname{gl.dim} R/ReR < \infty \quad \text{and} \quad \operatorname{gl.dim} eRe < \infty$$

Working in the setting of Morita rings, see Example 1.1.8, we have the following consequence which is the main result of Loustaunau and Shapiro [88].

Corollary 2.4.6. *[88, Theorem 1.8] Let $\Lambda_{(\phi,\psi)}$ be a Morita ring. Assume that $M$ is a right flat $R$-module, $N$ is a right flat $S$-module and $N \operatorname{Im}\phi = N$. Then*:
$$\max\{\operatorname{gl.dim} R, \operatorname{gl.dim} S\} \ \leq\ \operatorname{gl.dim} \Lambda_{(\phi,\psi)} \ \leq\ \max\{\operatorname{gl.dim} S/\operatorname{Im}\phi + \operatorname{pd}_S S/\operatorname{Im}\phi + 1, \operatorname{gl.dim} R\}$$

PROOF. From Example 1.1.8 we have the recollements of module categories

$$(\mathsf{Mod}\text{-}S/\operatorname{Im}\phi, \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}, \mathsf{Mod}\text{-}R) \quad \text{and} \quad (\mathsf{Mod}\text{-}R/\operatorname{Im}\psi, \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}, \mathsf{Mod}\text{-}S)$$

Since $N$ is a right flat $S$-module it follows that the functor $\Lambda e_2 \otimes_S - \colon \mathsf{Mod}\text{-}S \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is exact. Thus from the exact sequence $0 \longrightarrow \operatorname{Im}\phi \longrightarrow S \longrightarrow S/\operatorname{Im}\phi \longrightarrow 0$ we have the exact sequence $0 \longrightarrow \Lambda e_2 \otimes_S \operatorname{Im}\phi \longrightarrow \Lambda e_2 \longrightarrow \Lambda e_2 \otimes_S S/\operatorname{Im}\phi \longrightarrow 0$. Hence using that $N\operatorname{Im}\phi = N$ we deduce the following isomorphisms

$$\Lambda e_2 \otimes_S e_2(S/\operatorname{Im}\phi) \simeq \Lambda e_2 \otimes_S S/\operatorname{Im}\phi \simeq \Lambda e_2/(\Lambda e_2 \operatorname{Im}\phi) \simeq \left(\begin{smallmatrix} 0 & N \\ 0 & S \end{smallmatrix}\right)/\left(\begin{smallmatrix} 0 & N \\ 0 & \operatorname{Im}\phi \end{smallmatrix}\right) \simeq S/\operatorname{Im}\phi$$

as $\Lambda_{(\phi,\psi)}$-modules. This shows that the $\Lambda_{(\phi,\psi)}$-module $S/\operatorname{Im}\phi$ belongs to the corresponding subcategory $\mathcal{X}_1$ of $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$, see Proposition 2.1.2, which is defined from the recollement of module categories $(\mathsf{Mod}\text{-}R/\operatorname{Im}\psi, \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}, \mathsf{Mod}\text{-}S)$. Then the assertion follows from Corollary 2.2.15. □

Setting $m = 0$ in Corollary 2.2.22 and by Example 2.2.21, we obtain directly part (i) in the following. Part (ii) follows easily from Theorem 2.2.1.

COROLLARY 2.4.7. (i) *[45, Statement 9] Let $\Lambda$ be a semiprimary quasi-hereditary ring with a heredity chain of length $n$. Then $\mathsf{gl.dim}\,\Lambda \leq 2n-2$.*
(ii) *[14, Corollary 6.6] Let $\Lambda$ be a quasi-hereditary algebra and let $0 = \mathcal{U}_0 \subseteq \mathcal{U}_1 = \tau_{P_1}(\Lambda) \subseteq \cdots \subseteq \mathcal{U}_n = \Lambda$ be a chain of idempotent ideals such that $\mathcal{U}_i/\mathcal{U}_{i-1} \in \mathsf{proj}\,\Lambda/\mathcal{U}_{i-1}$ and $\mathsf{End}_{\Lambda/\mathcal{U}_{i-1}}(P_i/\mathcal{U}_{i-1}P_i)$ is hereditary for every $1 \leq i \leq n$. Then $\mathsf{gl.dim}\,\Lambda \leq 3n-2$.*

The next examples illustrate that in general one cannot expect exact relations between the global dimensions of the abelian categories involved in a recollement.

EXAMPLE 2.4.8. (i) Let $\Lambda$ be the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[r, shift right, "\alpha"'] & 2 \arrow[l, shift right, "\beta"'] \end{tikzcd}$$

modulo the relation $\alpha \circ \beta = 0$. Let $e$ be the idempotent associated to the vertex 1. Then we have the recollement $(\mathsf{mod}\text{-}\Lambda/\Lambda e\Lambda, \mathsf{mod}\text{-}\Lambda, \mathsf{mod}\text{-}e\Lambda e)$ and it is not so difficult to observe the following values for the global dimension of the algebras involved in the above recollement:

$$\begin{cases} \mathsf{gl.dim}\,\Lambda/\Lambda e\Lambda = 0 \\ \mathsf{gl.dim}\,\Lambda = 2 \\ \mathsf{gl.dim}\,e\Lambda e = \infty \end{cases}$$

(ii) Let $\mathbb{K}$ be a field and consider the ring $\mathbb{K}[X]/(X^2)$. Then we can view $\mathbb{K}$ as a $\mathbb{K}[X]/(X^2)$-$\mathbb{K}$-bimodule and as a $\mathbb{K}$-$\mathbb{K}[X]/(X^2)$-bimodule. Also we have the homomorphisms $\Phi\colon \mathbb{K} \otimes_{\mathbb{K}} \mathbb{K} \longrightarrow \mathbb{K}[X]/(X^2)$, $k \mapsto kx$ and the zero map $\Psi\colon \mathbb{K} \otimes_{\mathbb{K}[X]/(X^2)} \mathbb{K} \longrightarrow \mathbb{K}$. Then we have the Morita ring $\left(\begin{smallmatrix} \mathbb{K}[X]/(X^2) & \mathbb{K} \\ \mathbb{K} & \mathbb{K} \end{smallmatrix}\right)$ and from [105] we have

$$\mathsf{gl.dim}\left(\begin{smallmatrix} \mathbb{K}[X]/(X^2) & \mathbb{K} \\ \mathbb{K} & \mathbb{K} \end{smallmatrix}\right) = 2$$

From Example 1.1.8, see also Proposition 3.3.1 and Proposition 4.1.4, since $\Psi = 0$ we have the following recollement of abelian categories

$$\begin{tikzcd} \mathsf{Mod}\text{-}\mathbb{K}[X]/(X^2) \arrow[r, "\mathsf{Z}_{\mathbb{K}[X]/(X^2)}"] & \mathsf{Mod}\text{-}\begin{pmatrix} \mathbb{K}[X]/(X^2) & \mathbb{K} \\ \mathbb{K} & \mathbb{K} \end{pmatrix} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{U}_{\mathbb{K}}"] & \mathsf{Mod}\text{-}\mathbb{K} \arrow[l, bend right, "\mathsf{T}_{\mathbb{K}}"'] \arrow[l, bend left, "\mathsf{H}_{\mathbb{K}}"] \end{tikzcd}$$

and then

$$\begin{cases} \mathsf{gl.dim}\,\mathbb{K}[X]/(X^2) = \infty \\ \mathsf{gl.dim}\begin{pmatrix} \mathbb{K}[X]/(X^2) & \mathbb{K} \\ \mathbb{K} & \mathbb{K} \end{pmatrix} = 2 \\ \mathsf{gl.dim}\,\mathbb{K} = 0 \end{cases}$$

(iii) Let $\Lambda$ be an Artin algebra and ${}_\Lambda M_\Lambda$ be a $\Lambda$-$\Lambda$-bimodule. Then we have the upper triangular matrix Artin algebra $\begin{pmatrix} \Lambda & {}_\Lambda M_\Lambda \\ 0 & \Lambda \end{pmatrix}$ and the recollement of module categories $(\mathsf{mod}\text{-}\Lambda, \mathsf{mod}\text{-}\begin{pmatrix} \Lambda & {}_\Lambda M_\Lambda \\ 0 & \Lambda \end{pmatrix}, \mathsf{mod}\text{-}\Lambda)$, see Example 1.1.12. If $\mathsf{gl.dim}\,\Lambda < \infty$ or $\mathsf{gl.dim}\,\Lambda = \infty$ then from Example 2.2.19 it follows that all the involved algebras in the recollement have finite global dimension or all of them have infinite global dimension.

(iv) Let $\Lambda$ be a left Artinian ring with $\mathsf{gl.dim}\,\Lambda = n$, allowing $n = \infty$, and with Loewy length $\ell\ell(\Lambda) = m$. Then by Example 1.1.11, we have a recollement of module categories $(\mathsf{Mod}\text{-}\Gamma/\Gamma e\Gamma, \mathsf{Mod}\text{-}\Gamma, \mathsf{Mod}\text{-}\Lambda)$, where $\Gamma$ is a semiprimary ring of finite global dimension $\mathsf{gl.dim}\,\Gamma \leq m$. It follows that the difference $|\,\mathsf{gl.dim}\,\mathscr{B} - \mathsf{gl.dim}\,\mathscr{C}|$ of the global dimensions of the categories $\mathscr{B}$ and $\mathscr{C}$ involved in a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ can be arbitrary large.

**2.4.2. Finitistic Dimension.** We continue with various applications for the finitistic dimension of rings. Applying Theorem 2.3.6(i) and (ii) to the recollement of module categories $(\mathsf{Mod}\text{-}R/ReR, \mathsf{Mod}\text{-}R, \mathsf{Mod}\text{-}eRe)$, induced by an idempotent element $e$ of $R$, we have the following consequence which is due to Fuller-Saorin and Kirkman-Kuzmanovich.

Corollary 2.4.9. *Let $R$ be a ring, $e^2 = e \in R$ an idempotent and $e' = 1 - e$.*

(i) *[53, Corollary 1.5] If $\mathsf{pd}_{eRe}\, eRe' < \infty$, then:*

$$\mathsf{Fin.dim}\,R \;\leq\; \mathsf{Fin.dim}\,eRe + \mathsf{gl.dim}_{R/ReR}R + 1$$

(ii) *[76, Theorem 1.7] If the left $R$-module $ReR$ is projective, then:*

$$\mathsf{Fin.dim}\,R/ReR \;\leq\; \mathsf{Fin.dim}\,R$$

The next result follows from Theorem 2.3.2. First note that if $M$ is a right $R$-module over a ring $R$, then the functor $M \otimes_R -$ has locally bounded homological dimension $n$ if and only if $n$ is the minimal bound on the vanishing of $\mathsf{Tor}^R_*(M, -)$ in the sense of Kirkman-Kuzmanovich, see [77].

Corollary 2.4.10. (i) *Let $R$ be a ring and $e$ an idempotent element of $R$. If the functor $Re \otimes_{eRe} -$ has locally bounded homological dimension, then*

$$\mathsf{Fin.dim}\,eRe \;\leq\; \mathsf{Fin.dim}\,R + \mathsf{l.b.hom.dim}\,Re \otimes_{eRe} -$$

(ii) *[77, Proposition 10] Let $\Lambda_{(\phi,\psi)} = \begin{pmatrix} R & {}_RN_S \\ {}_SM_R & S \end{pmatrix}$ be a Morita ring. If the functor $M \otimes_R -$ has locally bounded homological dimension, then:*

$$\mathsf{Fin.dim}\,R \;\leq\; \mathsf{Fin.dim}\,\Lambda_{(\phi,\psi)} + \mathsf{l.b.hom.dim}\,M \otimes_R -$$

PROOF. Part (i) follows immediately from Theorem 2.3.2. For part (ii) we use the notation introduced in Example 1.1.8. If $X$ is a left $R$-module, then it is easy to see that the $S$-module $\mathsf{Tor}_n^R(M,X)$ is annihilated by the ideal $\mathsf{Im}\,\phi$ of $S$: $\mathsf{Im}\,\phi \cdot \mathsf{Tor}_n^R(M,X) = 0$, see also Proposition 3.7.1. Thus $\mathsf{Tor}_n^R(M,X)$ is a module over the ring $S/\mathsf{Im}\,\phi$ and therefore it is a $\Lambda_{(\phi,\psi)}$-module in a natural way. Since there is an isomorphism $\mathsf{Tor}_n^R(\Lambda e_1,-) \simeq \mathsf{Tor}_n^R(M,-)$, $\forall n \geq 1$, as functors $\mathsf{Mod}\text{-}R \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$, see also Proposition 3.7.1 and Lemma 4.5.2, the assertion is a direct consequence of Theorem 2.3.2. □

It is well known from [49], see also [18, Chapter 3, section 2], that the module category over a triangular matrix ring is a comma category (Example 1.1.12). The next well known result of Fossum-Griffith-Reiten follows immediately from Theorem 2.3.6(iii).

COROLLARY 2.4.11. *[49, Corollary 4.21] Let $\Lambda = \left(\begin{smallmatrix} R & {}_R N_S \\ 0 & S \end{smallmatrix}\right)$ be a triangular matrix ring. Then*:

$$\mathsf{Fin.dim}\, R \;\leq\; \mathsf{Fin.dim}\, \Lambda \;\leq\; \mathsf{Fin.dim}\, R + \mathsf{Fin.dim}\, S + 1$$

CHAPTER 3

# The Morita Category

In this Chapter we introduce the concept of the Morita category. This category should be considered as a general abstract model for the category of modules over a Morita ring. We investigate several structural properties of this category, for instance we examine when it is an abelian category, and we also study also finiteness conditions on subcategories. Further we investigate some homological aspects of this construction. Note that a similar construction has appeared in [37]. We mention that this Chapter serves as an introductory step for Chapter 4. At the time of writing this thesis the results of this Chapter are unpublished.

## 3.1. Morita Extensions of Abelian Categories

Let $\mathscr{A}$ and $\mathscr{B}$ be two additive categories, $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ two additive functors, and $\phi\colon FG \longrightarrow \mathrm{Id}_{\mathscr{B}}$ and $\psi\colon GF \longrightarrow \mathrm{Id}_{\mathscr{A}}$ two natural transformations such that the diagrams

$$\begin{tikzcd} FGF \arrow[r, "\phi_F"] \arrow[d, "F\psi"'] & F \\ F \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} GFG \arrow[r, "\psi_G"] \arrow[d, "G\phi"'] & G \\ G \arrow[ur, equal] & \end{tikzcd}$$

are commutative. We define the category $\mathcal{M} = (\mathscr{A}, \mathscr{B}, F, G, \phi, \psi)$ with respect to $\mathscr{A}, \mathscr{B}$, $F, G, \phi, \psi$, which has as objects tuples $(X, Y, f, g)$ where $X \in \mathscr{A}$, $Y \in \mathscr{B}$, $f\colon F(X) \longrightarrow Y$ is a morphism in $\mathscr{B}$ and $g\colon G(Y) \longrightarrow X$ is a morphism in $\mathscr{A}$ such that the following diagrams are commutative:

$$\begin{tikzcd} GF(X) \arrow[r, "G(f)"] \arrow[d, "\psi_X"'] & GY \arrow[dl, "g"] \\ X & \end{tikzcd} \qquad \begin{tikzcd} FG(Y) \arrow[r, "F(g)"] \arrow[d, "\phi_Y"'] & FX \arrow[dl, "f"] \\ Y & \end{tikzcd}$$

A morphism $(X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathcal{M}$ is a pair of morphisms $(a, b)$ where $a \in \mathsf{Hom}_{\mathscr{A}}(X, X')$ and $b \in \mathsf{Hom}_{\mathscr{B}}(Y, Y')$ such that the following diagrams are commutative:

$$\begin{tikzcd} F(X) \arrow[r, "f"] \arrow[d, "F(a)"'] & Y \arrow[d, "b"] \\ F(X') \arrow[r, "f'"] & Y' \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "g"] \arrow[d, "G(b)"'] & X \arrow[d, "a"] \\ G(Y') \arrow[r, "g'"] & X' \end{tikzcd}$$

If $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ and $(a', b')\colon (X', Y', f', g') \longrightarrow (X'', Y'', f'', g'')$ are morphisms in $\mathscr{C}$ then the composition $(a, b) \circ (a', b')\colon (X, Y, f, g) \longrightarrow (X'', Y'', f'', g'')$

is the morphism $(a \circ a', b \circ b')$ since the following diagrams are commutative:

$$\begin{tikzcd} F(X) \arrow[r, "F(a)"] \arrow[d, "f"'] & F(X') \arrow[r, "F(a')"] \arrow[d, "f'"'] & F(X'') \arrow[d, "f''"] \\ Y \arrow[r, "b"] & Y' \arrow[r, "b'"] & Y'' \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "G(b)"] \arrow[d, "g"'] & G(Y') \arrow[r, "G(b')"] \arrow[d, "g'"'] & G(Y'') \arrow[d, "g''"] \\ X \arrow[r, "a"] & X' \arrow[r, "a'"] & X'' \end{tikzcd}$$

DEFINITION 3.1.1. The category $\mathcal{M} = (\mathscr{A}, \mathscr{B}, F, G, \phi, \psi)$ is said to be the $\mathsf{Morita}$-$\mathsf{extension}$ of $\mathscr{A}$ and $\mathscr{B}$ by the natural transformations $\phi$ and $\psi$.

From now on we will denote this category by $\mathcal{M}(\phi, \psi)$ and for simplicity we call $\mathcal{M}(\phi, \psi)$ the $\mathsf{Morita\ category}$ of $\mathscr{A}$ and $\mathscr{B}$.

In what follows, our purpose is to investigate when a Morita category $\mathcal{M}(\phi, \psi)$ is abelian. The following result describes the kernels in $\mathcal{M}(\phi, \psi)$.

LEMMA 3.1.2. *Let $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ be a morphism in $\mathcal{M}(\phi, \psi)$. Then there are maps $h\colon F(\mathsf{Ker}\, a) \longrightarrow \mathsf{Ker}\, b$ and $h'\colon G(\mathsf{Ker}\, b) \longrightarrow \mathsf{Ker}\, a$ such that the kernel of $(a, b)$ is the object $\mathsf{Ker}\,(a, b) = (\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, h') \in \mathcal{M}(\phi, \psi)$.*

PROOF. Let $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ be a morphism in $\mathcal{M}(\phi, \psi)$. Let $r\colon \mathsf{Ker}\, a \longrightarrow X$ be the kernel of $a\colon X \longrightarrow X'$ in $\mathscr{A}$ and $s\colon \mathsf{Ker}\, b \longrightarrow Y$ be the kernel of $b\colon Y \longrightarrow Y'$ in $\mathscr{B}$. Then the following commutative diagrams

$$\begin{tikzcd} & F(\mathsf{Ker}\, a) \arrow[r, "F(r)"] \arrow[d, "h"'] & F(X) \arrow[r, "F(a)"] \arrow[d, "f"'] & FX' \arrow[d, "f'"] \\ 0 \arrow[r] & \mathsf{Ker}\, b \arrow[r, "s"] & Y \arrow[r, "b"] & Y' \end{tikzcd} \qquad \begin{tikzcd} & G(\mathsf{Ker}\, b) \arrow[r, "G(s)"] \arrow[d, "j"'] & G(Y) \arrow[r, "G(b)"] \arrow[d, "g"'] & G(Y') \arrow[d, "g'"] \\ 0 \arrow[r] & \mathsf{Ker}\, a \arrow[r, "r"] & X \arrow[r, "a"] & X' \end{tikzcd}$$

imply that $(r, s)\colon (\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, j) \longrightarrow (X, Y, f, g)$ is a morphism in $\mathcal{M}(\phi, \psi)$. From the following diagrams

$$\begin{tikzcd} GF(\mathsf{Ker}\, a) \arrow[rr, "G(h)"] \arrow[dr, "\psi_{\mathsf{Ker}\, a}"'] \arrow[dd, "GF(r)"'] & & G(\mathsf{Ker}\, b) \arrow[dl, "j"] \arrow[dd, "G(s)"] \\ & \mathsf{Ker}\, a \arrow[dd, "r"] & \\ GF(X) \arrow[rr, "G(f)"] \arrow[dr, "\psi_X"'] & & G(Y) \arrow[dl, "g"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} FG(\mathsf{Ker}\, b) \arrow[rr, "F(j)"] \arrow[dr, "\phi_{\mathsf{Ker}\, b}"'] \arrow[dd, "FG(s)"'] & & F(\mathsf{Ker}\, a) \arrow[dl, "h"] \arrow[dd, "F(r)"] \\ & \mathsf{Ker}\, b \arrow[dd, "s"] & \\ FG(Y) \arrow[rr, "F(g)"] \arrow[dr, "\phi_Y"'] & & F(X) \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

we have

$$\psi_{\mathsf{Ker}\, a} \circ r = GF(r) \circ \psi_X = GF(r) \circ G(f) \circ g = G(h) \circ G(s) \circ g = G(h) \circ j \circ r$$

and therefore $\psi_{\mathsf{Ker}\, a} = G(h) \circ j$ since the map $r$ is a monomorphism. Similarly we get that $\phi_{\mathsf{Ker}\, b} = F(j) \circ h$. Hence the the object $(\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, j) \in \mathcal{M}(\phi, \psi)$. We claim that the object $(\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, j)$ is the kernel of the morphism $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathcal{M}(\phi, \psi)$. First note that $(r, s) \circ (a, b) = (r \circ a, s \circ b) = (0, 0)$. Let $(\mu, \nu)\colon (X'', Y'', f'', g'') \longrightarrow (X, Y, f, g)$ be a morphism in $\mathcal{M}(\phi, \psi)$ such that $(\mu, \nu) \circ (a, b) = 0$. Then there exists a unique morphism $\kappa\colon X'' \longrightarrow \mathsf{Ker}\, a$ in $\mathscr{A}$ such that $\kappa \circ r = \mu$ and a unique morphism $\lambda\colon Y'' \longrightarrow \mathsf{Ker}\, b$ in $\mathscr{B}$ such that $\lambda \circ s = \nu$. Then the following diagram

$$\begin{tikzcd}
(\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, j) \arrow[r, "{(r,s)}"] & (X, Y, f, g) \arrow[r, "{(a,b)}"] & (X', Y', f', g') \\
(X'', Y'', f'', g'') \arrow[u, "{(\kappa,\lambda)}"] \arrow[ur, "{(\mu,\nu)}"'] & &
\end{tikzcd}$$

is commutative. It remains to show that $(\kappa, \lambda)\colon (X'', Y'', f'', g'') \longrightarrow (\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, h')$ is a morphism in $\mathcal{M}(\phi, \psi)$. Since $(\mu, \nu)$ and $(r, s)$ are morphism in $\mathcal{M}(\phi, \psi)$ we have $f'' \circ \nu = F(\mu) \circ f$, $h \circ s = F(r) \circ f$ and $g'' \circ \mu = G(\nu) \circ g$, $j \circ r = G(s) \circ g$. Then from the following diagrams

$$\begin{tikzcd}
F(X'') \arrow[r, "f''"] \arrow[d, "F(\kappa)"'] \arrow[dd, bend right=60, "F(\mu)"'] & Y'' \arrow[d, "\lambda"] \arrow[dd, bend left=60, "\nu"] \\
F(\mathsf{Ker}\, a) \arrow[r, "h"] \arrow[d, "F(r)"'] & \mathsf{Ker}\, b \arrow[d, "s"] \\
F(X) \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
G(Y'') \arrow[r, "g''"] \arrow[d, "G(\lambda)"'] \arrow[dd, bend right=60, "G(\nu)"'] & X'' \arrow[d, "\kappa"] \arrow[dd, bend left=60, "\mu"] \\
G(\mathsf{Ker}\, b) \arrow[r, "j"] \arrow[d, "G(s)"'] & \mathsf{Ker}\, a \arrow[d, "r"] \\
G(Y) \arrow[r, "g"] & X
\end{tikzcd}$$

we have $f'' \circ \lambda \circ s = F(\kappa) \circ F(r) \circ f = F(\kappa) \circ h \circ s$ and hence $f'' \circ \lambda = F(\kappa) \circ h$. Similarly we have $g'' \circ \kappa = G(\lambda) \circ j$. Hence $(\kappa, \lambda)\colon (X'', Y'', f'', g'') \longrightarrow (\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, h')$ is a morphism in $\mathcal{M}(\phi, \psi)$. We infer that the kernel of the morphism $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ is the object $(\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, j) \in \mathcal{M}(\phi, \psi)$. □

Corollary 3.1.3. *Let $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ be a morphism in $\mathcal{M}(\phi, \psi)$. Then $(a, b)$ is a monomorphism in $\mathcal{M}(\phi, \psi)$ if and only if $a\colon X \longrightarrow X'$ is a monomorphism in $\mathscr{A}$ and $b\colon Y \longrightarrow Y'$ is a monomorphism in $\mathscr{B}$.*

Proof. Suppose that $(a, b)$ is a monomorphism in $\mathcal{M}(\phi, \psi)$. Then $\mathsf{Ker}\,(a, b) = 0$ and from Lemma 3.1.2 it follows that $\mathsf{Ker}\, a = 0$ and $\mathsf{Ker}\, b = 0$. Hence $a\colon X \longrightarrow X'$ is a monomorphism in $\mathscr{A}$ and $b\colon Y \longrightarrow Y'$ is a monomorphism in $\mathscr{B}$. Similalry by Lemma 3.1.2 we get easily the converse and so our claim follows. □

The following result describes the cokernels in $\mathcal{M}(\phi, \psi)$ and as a consequence we characterize when a morphism in $\mathcal{M}(\phi, \psi)$ is an epimorphism. Note that for the cokernels we need the functors $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ to be right exact.

Lemma 3.1.4. *Let $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ be a morphism in $\mathcal{M}(\phi, \psi)$ and assume that the functors $F$ and $G$ are right exact. Then the cokernel of $(a, b)$ is the object $\mathsf{Coker}\,(a, b) = (\mathsf{Coker}\, a, \mathsf{Coker}\, b, f'', g'') \in \mathcal{M}(\phi, \psi)$.*

Proof. Let $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ be a morphism in $\mathcal{M}(\phi, \psi)$. Let $\delta\colon Y' \longrightarrow \mathsf{Coker}\, b$ be the cokernel of $b\colon Y \longrightarrow Y'$ in $\mathscr{B}$ and let $\gamma\colon X' \longrightarrow \mathsf{Coker}\, a$ be the cokernel of $a\colon X \longrightarrow X'$ in $\mathscr{A}$. Since $F$ and $G$ are right exact we have the following exact commutative diagrams:

$$\begin{tikzcd}
F(X) \arrow[r, "F(a)"] \arrow[d, "f"'] & F(X') \arrow[r, "F(\gamma)"] \arrow[d, "f'"] & F(\mathsf{Coker}\, a) \arrow[r] \arrow[d, "f''"] & 0 \\
Y \arrow[r, "b"] & Y' \arrow[r, "\delta"] & \mathsf{Coker}\, b \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
G(Y) \arrow[r, "G(b)"] \arrow[d, "g"'] & G(Y') \arrow[r, "G(\delta)"] \arrow[d, "g'"] & G(\mathsf{Coker}\, b) \arrow[r] \arrow[d, "g''"] & 0 \\
X \arrow[r, "a"] & X' \arrow[r, "\gamma"] & \mathsf{Coker}\, a \arrow[r] & 0
\end{tikzcd}$$

We claim that the object $(\mathsf{Coker}\, a, \mathsf{Coker}\, b, f'', g'')$ is the cokernel of the morphism $(a, b)$. First note that $(a, b) \circ (\gamma, \delta) = (a \circ \gamma, b \circ \delta) = (0, 0)$. Let $(\mu, \nu)\colon (X', Y', f', g') \longrightarrow (X''', Y''', f''', g''')$ be a morphism in $\mathcal{M}(\phi, \psi)$ such that $(a, b) \circ (\mu, \nu) = 0$. Since $a \circ \mu = 0$ and $b \circ \nu = 0$ there exists a unique morphism $\lambda\colon \mathsf{Coker}\, a \longrightarrow X'''$ in $\mathscr{A}$ such that $\gamma \circ \lambda = \mu$ and a unique morphism $k\colon \mathsf{Coker}\, b \longrightarrow Y'''$ in $\mathscr{B}$ such that $\delta \circ k = \nu$. Hence we have the following commutative diagram:

$$\begin{tikzcd}
(X, Y, f, g) \arrow[r, "{(a,b)}"] & (X', Y', f', g') \arrow[r, "{(\gamma,\delta)}"] \arrow[dr, "{(\mu,\nu)}"'] & (\mathsf{Coker}\, a, \mathsf{Coker}\, b, f'', g'') \arrow[d, "{(\lambda,k)}"] \\
& & (X''', Y''', f''', g''')
\end{tikzcd}$$

and we have to check that $(\lambda, k)\colon (\mathsf{Coker}\, a, \mathsf{Coker}\, b, f'', g'') \longrightarrow (X''', Y''', f''', g''')$ is a morphism in $\mathcal{M}(\phi, \psi)$. Since $(\mu, \nu)$ and $(\gamma, \delta)$ are morphism in $\mathcal{M}(\phi, \psi)$ we have $f' \circ \delta = F\gamma \circ f''$, $f' \circ \nu = F\mu \circ f'''$ and $g' \circ \gamma = G\delta \circ g''$, $g' \circ \mu = G\nu \circ g'''$. Then from the following diagrams:

$$\begin{tikzcd}
F(X') \arrow[r, "f'"] \arrow[d, "F(\gamma)"'] \arrow[dd, bend right=60, "F(\mu)"'] & Y' \arrow[d, "\delta"] \arrow[dd, bend left=60, "\nu"] \\
F(\mathsf{Coker}\, a) \arrow[r, "f''"] \arrow[d, "F(\lambda)"'] & \mathsf{Coker}\, b \arrow[d, "k"] \\
F(X''') \arrow[r, "f'''"] & Y'''
\end{tikzcd}
\qquad
\begin{tikzcd}
G(Y') \arrow[r, "g'"] \arrow[d, "G(\delta)"'] \arrow[dd, bend right=60, "G(\nu)"'] & X' \arrow[d, "\gamma"] \arrow[dd, bend left=60, "\mu"] \\
G(\mathsf{Coker}\, b) \arrow[r, "g''"] \arrow[d, "G(k)"'] & \mathsf{Coker}\, a \arrow[d, "\lambda"] \\
G(Y''') \arrow[r, "g'''"] & X'''
\end{tikzcd}$$

we deduce that $F(\gamma) \circ f'' \circ \delta \circ k = f' \circ \delta \circ k = f' \circ \nu = F(\mu) \circ f''' = F(\gamma) \circ F(\lambda) \circ f'''$ and hence $f'' \circ k = F(\lambda) \circ f'''$ since $F(\gamma)$ is an epimorphism. Similaly we have $G(\delta) \circ g'' \circ \lambda = g' \circ \gamma \circ \lambda = g' \circ \mu = G(\nu) \circ g'' = G(\delta) \circ G(k) \circ g'''$ and so $g'' \circ \lambda = G(k) \circ g'''$ since $G(\delta)$ is an epimorphism. Therefore $(\lambda, k)\colon (\mathsf{Coker}\, a, \mathsf{Coker}\, b, f'', g'') \longrightarrow (X''', Y''', f''', g''')$ is a morphism in $\mathcal{M}(\phi, \psi)$. In order to finish we have to show that the tuple $(\mathsf{Coker}\, a, \mathsf{Coker}\, b, f'', g'')$ lies in $\mathcal{M}(\phi, \psi)$. Thus we have to prove that the following diagrams:

$$\begin{tikzcd}
GF(\mathsf{Coker}\, a) \arrow[r, "G(f'')"] \arrow[d, "\psi_{\mathsf{Coker}\, a}"'] & G(\mathsf{Coker}\, b) \arrow[dl, "g''"] \\
\mathsf{Coker}\, a &
\end{tikzcd}
\qquad
\begin{tikzcd}
FG(\mathsf{Coker}\, b) \arrow[r, "F(g'')"] \arrow[d, "\phi_{\mathsf{Coker}\, b}"'] & F(\mathsf{Coker}\, a) \arrow[dl, "f''"] \\
\mathsf{Coker}\, b &
\end{tikzcd}$$

are commutative. From the following commutative diagrams:

$$\begin{tikzcd}
GF(X') \arrow[rr, "G(f')"] \arrow[dr, "\psi_{X'}"] \arrow[dd, "GF(\gamma)"'] & & G(Y') \arrow[dl, "g'"'] \arrow[dd, "G(\delta)"] \\
& X' \arrow[dd, "\gamma"] & \\
GF(\mathsf{Coker}\, a) \arrow[rr, "G(f'')"] \arrow[dr, "\psi_{\mathsf{Coker}\, a}"'] & & G(\mathsf{Coker}\, b) \arrow[dl, "g''"] \\
& \mathsf{Coker}\, a &
\end{tikzcd}
\qquad
\begin{tikzcd}
FG(Y') \arrow[rr, "F(g')"] \arrow[dr, "\phi_{Y'}"] \arrow[dd, "FG(\delta)"'] & & F(X') \arrow[dl, "f'"'] \arrow[dd, "F(\gamma)"] \\
& Y' \arrow[dd, "\delta"] & \\
FG(\mathsf{Coker}\, b) \arrow[rr, "F(g'')"] \arrow[dr, "\phi_{\mathsf{Coker}\, b}"'] & & F(\mathsf{Coker}\, a) \arrow[dl, "f''"] \\
& \mathsf{Coker}\, b &
\end{tikzcd}$$

we have

$$GF(\gamma)\circ\psi_{\mathsf{Coker}\,a}=\psi_{X'}\circ\gamma=G(f')\circ g'\circ\gamma=G(f')\circ G(\delta)\circ g''=GF(\gamma)\circ G(f'')\circ g''$$

and so $\psi_{\mathsf{Coker}\,a}=G(f'')\circ g''$ since $GF(\gamma)$ is an epimorphism. Similarly since $FG(\delta)$ is an epimorphism we have

$$FG(\delta)\circ\phi_{\mathsf{Coker}\,b}=\phi_{Y'}\circ\delta=F(g')\circ f'\circ\delta=F(g')\circ F(\gamma)\circ f''=FG(\delta)\circ F(g'')\circ f''$$

and so $\phi_{\mathsf{Coker}\,b}=Fg''\circ f''$. Hence the cokernel of the morphism $(a,b)\colon(X,Y,f,g)\longrightarrow(X',Y',f',g')$ is the object $(\mathsf{Coker}\,a,\mathsf{Coker}\,b,f'',g'')\in\mathcal{M}(\phi,\psi)$. $\square$

COROLLARY 3.1.5. *Let $(a,b)\colon(X,Y,f,g)\longrightarrow(X',Y',f',g')$ be a morphism in $\mathcal{M}(\phi,\psi)$ and assume that the functors $F$ and $G$ are right exact. Then $(a,b)$ is an epimorphism in $\mathcal{M}(\phi,\psi)$ if and only if $a\colon X\longrightarrow X'$ is an epimorphism in $\mathscr{A}$ and $b\colon Y\longrightarrow Y'$ is an epimorphism in $\mathscr{B}$.*

PROOF. This follows immediately from the above description of cokernels. $\square$

We continue by describing the finite coproduct of objects of $\mathcal{M}(\phi,\psi)$.

LEMMA 3.1.6. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category and $(X_i,Y_i,f_i,g_i)$, $1\leq i\leq n$, a finite number of objects of $\mathcal{M}(\phi,\psi)$. Then the direct sum is the object:*

$$(X_1\oplus\cdots\oplus X_n,Y_1\oplus\cdots\oplus Y_n,f,g)\quad\textit{where}\quad f=\begin{pmatrix}f_1&\cdots&0\\&\ddots&\\0&\cdots&f_n\end{pmatrix},\ g=\begin{pmatrix}g_1&\cdots&0\\&\ddots&\\0&\cdots&g_n\end{pmatrix}$$

PROOF. It is enough to show that the direct sum of $(X_1,Y_1,f_1,g_1)$ and $(X_2,Y_2,f_2,g_2)$ is the object $X=(X_1\oplus X_2,Y_1\oplus Y_2,\left(\begin{smallmatrix}f_1&0\\0&f_2\end{smallmatrix}\right),\left(\begin{smallmatrix}g_1&0\\0&g_2\end{smallmatrix}\right))$. First it easy to check that $X$ is an object of $\mathcal{M}(\phi,\psi)$. We define the maps $a_1=\big((1\ 0),(1\ 0)\big)\colon(X_1,Y_1,f_1,g_1)\longrightarrow X$, $b_1=\big(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\big)\colon X\longrightarrow(X_1,Y_1,f_1,g_1)$ and $a_2=\big((0\ 1),(0\ 1)\big)\colon(X_2,Y_2,f_2,g_2)\longrightarrow X$, $b_2=\big(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\big)\colon X\longrightarrow(X_2,Y_2,f_2,g_2)$. Note that $a_1,a_2,b_1,b_2$ are morphisms in $\mathcal{M}(\phi,\psi)$. Then we have $a_1\circ b_1=\mathrm{Id}_{(X_1,Y_1,f_1,g_1)}$, $a_2\circ b_2=\mathrm{Id}_{(X_2,Y_2,f_2,g_2)}$ and $b_1\circ a_1+b_2\circ a_2=\mathrm{Id}_X$. Hence the object $X$ is the direct sum of $(X_1,Y_1,f_1,g_1)$ and $(X_2,Y_2,f_2,g_2)$. $\square$

The following main result of this section gives a sufficient condition for a Morita category to be abelian.

THEOREM 3.1.7. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category of the abelian categories $\mathscr{A}$ and $\mathscr{B}$. If the functors $F\colon\mathscr{A}\longrightarrow\mathscr{B}$ and $G\colon\mathscr{B}\longrightarrow\mathscr{A}$ are right exact then the category $\mathcal{M}(\phi,\psi)$ is abelian.*

PROOF. Since the tuple $(0,0,0,0)$ is the zero object of $\mathcal{M}(\phi,\psi)$, every finite family of objects has a coproduct (Lemma 3.1.6), and every morphism in $\mathcal{M}(\phi,\psi)$ has a kernel and a cokernel (Lemmas 3.1.2 and 3.1.4), it follows from Freyd [51] that in order to show that $\mathcal{M}(\phi,\psi)$ is an abelian category we must prove that monomorphisms are kernels and epimorphisms are cokernels.

Claim (i): Monomorphisms are kernels. Let $(a,b)\colon(X,Y,f,g)\longrightarrow(X',Y',f',g')$ be a monomorphism in $\mathcal{M}(\phi,\psi)$. Let $(\mathsf{Coker}\,a,\mathsf{Coker}\,b,f'',g'')$ be the cokernel of $(a,b)$ and so we have the exact sequence:

$$0\longrightarrow(X,Y,f,g)\xrightarrow{(a,b)}(X',Y',f',g')\xrightarrow{(c,d)}(\mathsf{Coker}\,a,\mathsf{Coker}\,b,f'',g'')\longrightarrow 0$$

and the following commutative diagrams:

$$\begin{tikzcd} F(X) \arrow[r, "f"] \arrow[d, "F(a)"'] & Y \arrow[d, "b"] \\ F(X') \arrow[r, "f'"] \arrow[d, "F(c)"'] & Y' \arrow[d, "d"] \\ F(\mathsf{Coker}\, a) \arrow[r, "f''"] & \mathsf{Coker}\, b \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "g"] \arrow[d, "G(b)"'] & X \arrow[d, "a"] \\ G(Y') \arrow[r, "g'"] \arrow[d, "G(d)"'] & X' \arrow[d, "c"] \\ G(\mathsf{Coker}\, b) \arrow[r, "g''"] & \mathsf{Coker}\, a \end{tikzcd}$$

Since $(a, b)$ is a monomorphism it follows from Corollary 3.1.3 that the maps $a\colon X \longrightarrow X'$ and $b\colon Y \longrightarrow Y'$ are monomorphisms. Then the map $b\colon Y \longrightarrow Y'$ is the kernel of $d\colon Y' \longrightarrow \mathsf{Coker}\, b$ and the map $a\colon X \longrightarrow X'$ is the kernel of $c\colon X' \longrightarrow \mathsf{Coker}\, a$. We claim that the morphism $(a, b)$ is the kernel of $(c, d)$. First we have $(a, b) \circ (c, d) = (0, 0)$. Now let $(a', b')\colon (X''', Y''', f''', g''') \longrightarrow (X', Y', f', g')$ be a morphism in $\mathcal{M}(\phi, \psi)$ such that $(a', b') \circ (c, d) = 0$, i.e. $a' \circ c = 0$ and $b' \circ d = 0$. Then there exists a unique morphism $k\colon X''' \longrightarrow X$ in $\mathscr{A}$ such that $k \circ a = a'$ and a unique morphism $\lambda\colon Y''' \longrightarrow Y$ in $\mathscr{B}$ such that $\lambda \circ b = b'$. Then we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & (X, Y, f, g) \arrow[r, "{(a,b)}"] & (X', Y', f', g') \arrow[r, "{(c,d)}"] & (\mathsf{Coker}\, a, \mathsf{Coker}\, b, f'', g'') \arrow[r] & 0 \\ & (X''', Y''', f''', g''') \arrow[u, "{(k,\lambda)}"] \arrow[ur, "{(a',b')}"'] & & & \end{tikzcd}$$

Since $k$ and $\lambda$ are unique it follows that $(k, \lambda)$ is the unique morhism such that $(k, \lambda) \circ (a, b) = (a', b')$. It remains to check that $(k, \lambda)$ is a morphism in $\mathcal{M}(\phi, \psi)$. Since $(a, b)$ and $(a', b')$ are morphism in $\mathcal{M}(\phi, \psi)$ we have $f \circ b = F(a) \circ f'$, $f''' \circ b' = F(a') \circ f'$ and $g \circ a = G(b) \circ g'$, $g''' \circ a' = G(b') \circ g'$. Then from the following diagrams

$$\begin{tikzcd} F(X''') \arrow[r, "f'''"] \arrow[d, "F(k)"'] \arrow[dd, bend right=60, "F(a')"'] & Y''' \arrow[d, "\lambda"] \arrow[dd, bend left=60, "b'"] \\ F(X) \arrow[r, "f"] \arrow[d, "F(a)"'] & Y \arrow[d, "b"] \\ F(X') \arrow[r, "f'"] & Y' \end{tikzcd} \qquad \begin{tikzcd} G(Y''') \arrow[r, "g'''"] \arrow[d, "G(\lambda)"'] \arrow[dd, bend right=60, "G(b')"'] & X''' \arrow[d, "k"] \arrow[dd, bend left=60, "a'"] \\ G(Y) \arrow[r, "g"] \arrow[d, "G(b)"'] & X \arrow[d, "a"] \\ G(Y') \arrow[r, "g'"] & X' \end{tikzcd}$$

we deduce that $f''' \circ \lambda \circ b = F(k) \circ F(a) \circ f' = F(k) \circ f \circ b$. Hence $f''' \circ \lambda = F(k) \circ f$ since $b$ is monomorphism. Similarly using that $a$ is monomorphism we get that $g''' \circ k = G(\lambda) \circ g$. Thus the map $(a, b)$, which is a monomorphism, is the kernel of $(c, d)$.

Claim (ii): Epimorphisms are cokernels. Let $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ be an epimorphism in $\mathcal{M}(\phi, \psi)$ and let $(\mathsf{Ker}\, a, \mathsf{Ker}\, b, f'', g'')$ be the kernel of $(a, b)$. Thus we have the exact sequence:

$$0 \longrightarrow (\mathsf{Ker}\, a, \mathsf{Ker}\, b, f'', g'') \xrightarrow{(\delta,\gamma)} (X, Y, f, g) \xrightarrow{(a,b)} (X', Y', f', g') \longrightarrow 0$$

From Corollary 3.1.5 we have that $a\colon X \longrightarrow X'$ is an epimorphism in $\mathscr{A}$ and $b\colon Y \longrightarrow Y'$ is an epimorphism in $\mathscr{B}$. Hence the map $b\colon Y \longrightarrow Y'$ is the cokernel of $\gamma\colon \mathsf{Ker}\, b \longrightarrow Y$ and the map $a\colon X \longrightarrow X'$ is the cokernel of $\delta\colon \mathsf{Ker}\, a \longrightarrow X$. We claim that the morphism $(a, b)$ is the cokernel of $(\delta, \gamma)$. First note that $(\delta, \gamma) \circ (a, b) = (0, 0)$. Now let $(a', b') \colon (X, Y, f, g) \longrightarrow (X'', Y'', f'', g'')$ be a morphism in $\mathcal{M}(\phi, \psi)$ such that $(\delta, \gamma) \circ (a', b') = 0$. Then there exists a unique morphism $k \colon X' \longrightarrow X'''$ in $\mathscr{A}$ such that $a \circ k = a'$ and a unique morphism $\lambda \colon Y' \longrightarrow Y'''$ in $\mathscr{B}$ such that $b \circ \lambda = b'$. Hence the diagram

$$\begin{tikzcd} 0 \arrow[r] & (\mathsf{Ker}\, a, \mathsf{Ker}\, b, f'', g'') \arrow[r, "(\delta,\gamma)"] & (X, Y, f, g) \arrow[r, "(a,b)"] \arrow[dr, "(a',b')"'] & (X', Y', f', g') \arrow[r] \arrow[d, "(k,\lambda)"] & 0 \\ & & & (X'', Y'', f'', g'') & \end{tikzcd}$$

is commutative. Note that $(k, \lambda)$ is the unique morphism such that $(a, b) \circ (k, \lambda) = (a', b')$. It remains to check that $(k, \lambda) \colon (X', Y', f', g') \longrightarrow (X'', Y'', f'', g'')$ is a morphism in $\mathcal{M}(\phi, \psi)$. Since $(a, b)$ and $(a', b')$ are morphism in $\mathcal{M}(\phi, \psi)$ we have $f \circ b' = F(a') \circ f''$, $f \circ b = F(a) \circ f'$ and $g \circ a' = G(b') \circ g''$, $g \circ a = G(b) \circ g'$. Then from the following diagrams:

$$\begin{tikzcd} F(X) \arrow[r, "f"] \arrow[d, "F(a)"'] \arrow[dd, bend right=60, "F(a')"'] & Y \arrow[d, "b"] \arrow[dd, bend left=60, "b'"] \\ F(X') \arrow[r, "f'"] \arrow[d, "F(k)"'] & Y' \arrow[d, "\lambda"] \\ F(X'') \arrow[r, "f''"] & Y'' \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "g"] \arrow[d, "G(b)"'] \arrow[dd, bend right=60, "G(b')"'] & X \arrow[d, "a"] \arrow[dd, bend left=60, "a'"] \\ G(Y') \arrow[r, "g'"] \arrow[d, "G(\lambda)"'] & X' \arrow[d, "k"] \\ G(Y'') \arrow[r, "g''"] & X'' \end{tikzcd}$$

we deduce that $F(a) \circ F(k) \circ f'' = F(a') \circ f'' = f \circ b \circ \lambda = F(a) \circ f' \circ \lambda$ and hence $f' \circ \lambda = F(k) \circ f'''$ since $F(a)$ is an epimorphism. Similarly we get that $g' \circ k = G(\lambda) \circ g''$ since $G$ is right exact. Therefore the map $(a, b)$, which was an epimorphism, is the cokernel of $(\delta, \gamma)$. We infer that the Morita category $\mathcal{M}(\phi, \psi)$ is abelian. $\square$

Let $\mathscr{A}$ and $\mathscr{B}$ be two additive categories and $F \colon \mathscr{A} \longrightarrow \mathscr{B}$, $G \colon \mathscr{B} \longrightarrow \mathscr{A}$ two additive functors. Also let $\phi \colon \mathrm{Id}_{\mathscr{A}} \longrightarrow FG$ and $\psi \colon \mathrm{Id}_{\mathscr{B}} \longrightarrow GF$ be two natural transformations such that the diagrams

$$\begin{tikzcd} F \arrow[r, "\phi_F"] \arrow[d, equal] & FGF \\ F \arrow[ur, "F_\psi"'] & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\psi_G"] \arrow[d, equal] & GFG \\ G \arrow[ur, "G_\phi"'] & \end{tikzcd}$$

are commutative. We define the dual Morita category of $\mathcal{M}(\phi, \psi)$, which we denote by $Co\mathcal{M}(\phi, \psi)$. The category $Co\mathcal{M}(\phi, \psi) = (\mathscr{A}, \mathscr{B}, F, G, \phi, \psi)$ has as objects tuples $(X, Y, f, g)$ where $X \in \mathscr{A}$, $Y \in B$ and $f \colon Y \longrightarrow FX$ is a morphism in $\mathscr{B}$, $g \colon X \longrightarrow GY$ is a morphism in $\mathscr{A}$, such that the following diagrams are commutative:

$$\begin{tikzcd} G(Y) \arrow[r, "G(f)"] & GF(X) \\ X \arrow[u, "g"] \arrow[ur, "\psi_X"'] & \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "F(g)"] & FG(Y) \\ Y \arrow[u, "f"] \arrow[ur, "\phi_Y"'] & \end{tikzcd}$$

A morphism $(X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $Co\mathcal{M}(\phi, \psi)$ is a pair of morphisms $(a, b)$ where $a \in \mathsf{Hom}_{\mathscr{A}}(X, X')$ and $b \in \mathsf{Hom}_{\mathscr{B}}(Y, Y')$ such that the following diagrams commute:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "b"'] & F(X) \arrow[d, "F(a)"] \\ Y' \arrow[r, "f'"] & F(X') \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "a"'] & G(Y) \arrow[d, "Gb"] \\ X' \arrow[r, "g'"] & G(Y') \end{tikzcd}$$

We close this section with the following result which gives a sufficient condition such that the dual Morita category $Co\mathcal{M}(\phi, \psi)$ is abelian. The proof is dual to Theorem 3.1.7 and is left to the reader.

THEOREM 3.1.8. *Let $Co\mathcal{M}(\phi, \psi)$ be the dual Morita category of the abelian categories $\mathscr{A}$ and $\mathscr{B}$. If the functors $F$ and $G$ are left exact then $Co\mathcal{M}(\phi, \psi)$ is abelian.*

**3.1.1. The Morita Category with $\phi = \psi = 0$.** Let $\mathcal{M}(\phi, \psi)$ be a Morita category of the abelian categories $\mathscr{A}$ and $\mathscr{B}$ with $\phi = \psi = 0$. We denote this category with $\mathcal{M}(0,0)$. We assume that the functors $F$ and $G$ are right exact, thus $\mathcal{M}(0,0)$ is an abelian category. In this subsection we will describe $\mathcal{M}(0,0)$. In particular we prove that there is an equivalence of categories between $\mathcal{M}(0,0)$ and $(\mathscr{A} \times \mathscr{B}) \ltimes H$, for a suitable endofunctor $H \colon \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{A} \times \mathscr{B}$, where $(\mathscr{A} \times \mathscr{B}) \ltimes H$ is the trivial extension of $\mathscr{A} \times \mathscr{B}$ by an endofunctor $H$, see [49].

We define the functor

$$H \colon \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{A} \times \mathscr{B}, \;\; H(A, B) = (GB, FA)$$

and given a morphism $(a, b) \colon (A, B) \longrightarrow (A', B')$ then $H(a, b) = (Gb, Fa)$. Then we can define the trivial extension $(\mathscr{A} \times \mathscr{B}) \ltimes H$, where the objects are morphisms $\alpha \colon H(A, B) \longrightarrow (A, B)$ such that $H(\alpha) \circ \alpha = 0$, and if $\alpha \colon H(A, B) \longrightarrow (A, B)$ and $\beta \colon H(A', B') \longrightarrow (A', B')$ are two objects in $(\mathscr{A} \times \mathscr{B}) \ltimes H$, then a morphism $\gamma \colon \alpha \longrightarrow \beta$ is a morphism $\gamma \colon (A, B) \longrightarrow (A', B')$ such that the diagram

$$\begin{tikzcd} H(A,B) \arrow[r, "H(\gamma)"] \arrow[d, "\alpha"'] & H(A',B') \arrow[d, "\beta"] \\ (A,B) \arrow[r, "\gamma"] & (A',B') \end{tikzcd}$$

is commutative, where $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ and $\gamma = (c_1, c_2)$. Also, since the functors $F$ and $G$ are right exact it follows that the endofunctor $H$ is right exact. Hence from [49] the trivial extension $(\mathscr{A} \times \mathscr{B}) \ltimes H$ is an abelian category.

PROPOSITION 3.1.9. *Let $\mathcal{M}(0,0)$ be a Morita category of $\mathscr{A}$ and $\mathscr{B}$ with $\phi = \psi = 0$. Then there is an equivalence:*

$$\mathcal{M}(0,0) \xrightarrow{\simeq} (\mathscr{A} \times \mathscr{B}) \ltimes H$$

PROOF. Let $(X, Y, f, g)$ be an object of $\mathcal{M}(0,0)$. We define the functor

$$\mathcal{F} \colon \mathcal{M}(0,0) \longrightarrow (\mathscr{A} \times \mathscr{B}) \ltimes H \;, \;\; \mathcal{F}(X, Y, f, g) = \;\; H(X, Y) \xrightarrow{(g,f)} (X, Y)$$

and given a morphism $(a, b) \colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathcal{M}(0,0)$ then $\mathcal{F}(a, b) = (a, b)$. The functor $\mathcal{F}$ is well defined since the following composition

$$(GF(X), FG(Y)) \xrightarrow{(G(f), F(g))} (G(Y), F(X)) \xrightarrow{(g,f)} (X, Y)$$

is zero, i.e. the object $\mathcal{F}(X,Y,f,g) \in (\mathscr{A} \times \mathscr{B}) \ltimes H$. It is clear that the functor $\mathcal{F}$ is faithful. Let

$$\big[(G(Y),F(X)) \xrightarrow{(g,f)} (X,Y)\big] \xrightarrow{(a,b)} \big[(G(Y'),F(X')) \xrightarrow{(g',f')} (X',Y')\big]$$

be a morphism in $(\mathscr{A} \times \mathscr{B}) \ltimes H$. Then we have the commutative diagram

$$\begin{tikzcd}
(G(Y),F(X)) \arrow[r, "{(G(b),F(a))}"] \arrow[d, "{(g,f)}"'] & (G(Y'),F(X')) \arrow[d, "{(g',f')}"] \\
(X,Y) \arrow[r, "{(a,b)}"] & (X',Y')
\end{tikzcd}$$

which implies that $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ is a morphism in $\mathcal{M}(0,0)$ and $\mathcal{F}(a,b) = (a,b)$. Thus the functor $\mathcal{F}$ is full. Let $(a_1,a_2)\colon H(X,Y) \longrightarrow (X,Y)$ be an object of $(\mathscr{A} \times \mathscr{B}) \ltimes H$. Since $H(a_1,a_2) \circ (a_1,a_2) = 0$ we have the commutative diagrams:

$$\begin{tikzcd}
GF(X) \arrow[r, "G(a_2)"] \arrow[d, "\psi_X=0"'] & G(Y) \arrow[dl, "a_1"] \\
X &
\end{tikzcd}
\qquad
\begin{tikzcd}
FG(Y) \arrow[r, "F(a_1)"] \arrow[d, "\phi_Y=0"'] & F(X) \arrow[dl, "a_2"] \\
Y &
\end{tikzcd}$$

and so the object $(X,Y,f,g) \in \mathcal{M}(0,0)$. Then $\mathcal{F}(X,Y,f,g) \simeq [H(X,Y) \to (X,Y)]$ and this shows that the functor $\mathcal{F}$ is essentially surjective. Hence the categories $\mathcal{M}(0,0)$ and $(\mathscr{A} \times \mathscr{B}) \ltimes H$ are equivalent. □

Remark 3.1.10. Let $\mathcal{M}(\phi,\psi)$ be a Morita category and denote by $\mathscr{C}(0,0)$ the full subcategory consisting of all objects $(X,Y,f,g)$ of $\mathcal{M}(\phi,\psi)$ such that $\psi_X = 0$ and $\phi_Y = 0$. Then from Proposition 3.1.9 we infer that $\mathscr{C}(0,0)$ is equivalent with $(\mathscr{A} \times \mathscr{B}) \ltimes H$. Hence we always have a full embedding from the trivial extension $(\mathscr{A} \times \mathscr{B}) \ltimes H$ to the Morita category $\mathcal{M}(\phi,\psi)$.

**3.1.2. Functors and Adjoint Pairs.** Let $\mathcal{M}(\phi,\psi)$ be the Morita category of the abelian categories $\mathscr{A}$ and $\mathscr{B}$ by the natural transformations $\phi$ and $\psi$. From now on we assume that the functors $F$ and $G$ are right exact, i.e. $\mathcal{M}(\phi,\psi)$ is abelian. In this section we introduce several connecting functors between $\mathscr{A}$, $\mathscr{B}$ and $\mathcal{M}(\phi,\psi)$. Furthermore we are interested in the following two subcategories

$$\mathsf{Ker}\,\psi = \{X \in \mathscr{A} \mid \psi_X = 0\} \quad \text{and} \quad \mathsf{Ker}\,\phi = \{Y \in \mathscr{B} \mid \phi_Y = 0\}$$

of $\mathscr{A}$ and $\mathscr{B}$, respectively, and we also introduce functors between $\mathsf{Ker}\,\psi$, $\mathsf{Ker}\,\phi$ and $\mathcal{M}(\phi,\psi)$. In other words we will study a diagram of categories and functors of the following form:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, shift right, "\mathsf{T}_{\mathscr{A}}"'] & \mathcal{M}(\phi,\psi) \arrow[l, shift right, "\mathsf{U}_{\mathscr{A}}"'] \arrow[r, shift right, "\mathsf{U}_{\mathscr{B}}"'] \arrow[d, equal] & \mathscr{B} \arrow[l, shift right, "\mathsf{T}_{\mathscr{B}}"'] \\
\mathsf{Ker}\,\psi \arrow[u, hook] \arrow[r, shift right, "\mathsf{Z}_{\mathscr{A}}"'] & \mathcal{M}(\phi,\psi) \arrow[l, shift right, "\mathsf{C}_{\mathscr{A}}"'] \arrow[r, shift right, "\mathsf{C}_{\mathscr{B}}"'] & \mathsf{Ker}\,\phi \arrow[u, hook] \arrow[l, shift right, "\mathsf{Z}_{\mathscr{B}}"']
\end{tikzcd}$$

We define the following functors:

(i) The functor $\mathsf{T}_{\mathscr{A}}\colon \mathscr{A} \longrightarrow \mathcal{M}(\phi,\psi)$ is defined by $\mathsf{T}_{\mathscr{A}}(X) = (X, F(X), \mathrm{Id}_{F(X)}, \psi_X)$ on the objects $X \in \mathscr{A}$ and given a morphism $a\colon X \longrightarrow X'$ in $\mathscr{A}$ then $\mathsf{T}_{\mathscr{A}}(a) = (a, F(a))$.

(ii) The functor $\mathsf{U}_{\mathscr{A}}\colon \mathcal{M}(\phi,\psi) \longrightarrow \mathscr{A}$ is defined by $\mathsf{U}_{\mathscr{A}}(X,Y,f,g) = X$ on the objects $(X,Y,f,g) \in \mathcal{M}(\phi,\psi)$ and given a morphism $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ in $\mathcal{M}(\phi,\psi)$ then $\mathsf{U}_{\mathscr{A}}(a,b) = a$.

(iii) The functor $\mathsf{T}_{\mathscr{B}}\colon \mathscr{B} \longrightarrow \mathcal{M}(\phi,\psi)$ is defined by $\mathsf{T}_{\mathscr{B}}(Y) = (G(Y),Y,\phi_Y,\mathrm{Id}_{G(Y)})$ on the objects $Y \in \mathscr{B}$. On a morphism $b\colon Y \longrightarrow Y'$ in $\mathscr{B}$ the functor $\mathsf{T}_{\mathscr{B}}$ is defined by $\mathsf{T}_{\mathscr{B}}(b) = (G(b),b)$.

(iv) The functor $\mathsf{U}_{\mathscr{B}}\colon \mathcal{M}(\phi,\psi) \longrightarrow \mathscr{B}$ is defined on the objects $(X,Y,f,g) \in \mathcal{M}(\phi,\psi)$ by $\mathsf{U}_{\mathscr{B}}(X,Y,f,g) = Y$ and given a morphism $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ in $\mathcal{M}(\phi,\psi)$ then $\mathsf{U}_{\mathscr{B}}(a,b) = b$.
It is easy to check that $\mathsf{T}_{\mathscr{A}}(X), \mathsf{T}_{\mathscr{B}}(Y) \in \mathcal{M}(\phi,\psi)$ and that $\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}}, \mathsf{T}_{\mathscr{B}}$, $\mathsf{U}_{\mathscr{B}}$ are functors.

(v) For $X \in \mathsf{Ker}\,\psi$ we define the functor $\mathsf{Z}_{\mathscr{A}}\colon \mathsf{Ker}\,\psi \longrightarrow \mathcal{M}(\phi,\psi)$ by $\mathsf{Z}_{\mathscr{A}}(X) = (X,0,0,0)$ and if $a\colon X \longrightarrow X'$ is a morphism in $\mathsf{Ker}\,\psi$ then $\mathsf{Z}_{\mathscr{A}}(a) = (a,0)$. Since $\psi_X = 0$ it follows that $\mathsf{Z}_{\mathscr{A}}(X) \in \mathcal{M}(\phi,\psi)$ and one can easily check that $\mathsf{Z}_{\mathscr{A}}\colon \mathsf{Ker}\,\psi \longrightarrow \mathcal{M}(\phi,\psi)$ is a functor.

(vi) For $(X,Y,f,g) \in \mathcal{M}(\phi,\psi)$ we define the functor $\mathsf{C}_{\mathscr{A}}\colon \mathcal{M}(\phi,\psi) \longrightarrow \mathsf{Ker}\,\psi$ by $\mathsf{C}_{\mathscr{A}}(X,Y,f,g) = \mathsf{Coker}\,g$ and if $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ is a morphism in $\mathcal{M}(\phi,\psi)$ then $\mathsf{C}_{\mathscr{A}}(a,b) = \xi$ where $\xi\colon \mathsf{Coker}\,g \longrightarrow \mathsf{Coker}\,g'$ is the unique morphism which makes the following diagram commutative

$$\begin{tikzcd}
G(Y) \arrow[r, "g"] \arrow[d, "G(b)"'] & X \arrow[r, two heads, "\pi_X"] \arrow[d, "a"] & \mathsf{Coker}\,g \arrow[d, "\xi"] \\
G(Y') \arrow[r, "g'"] & X' \arrow[r, two heads, "\pi_{X'}"] & \mathsf{Coker}\,g'
\end{tikzcd}$$

We verify that $\mathsf{C}_{\mathscr{A}}(X,Y,f,g) \in \mathsf{Ker}\,\psi$. Let $(X,Y,f,g) \in \mathcal{M}(\phi,\psi)$. Then from the following commutative diagram

$$\begin{tikzcd}
GFG(Y) \arrow[r, "GF(g)"] \arrow[d, "\psi_{GY}"'] & GF(X) \arrow[r, two heads, "GF\pi_X"] \arrow[d, "\psi_X"] \arrow[dl, "G(f)"'] & GF(\mathsf{Coker}\,g) \arrow[d, "\psi_{\mathsf{Coker}\,g}"] \\
G(Y) \arrow[r, "g"] & X \arrow[r, two heads, "\pi_X"] & \mathsf{Coker}\,g
\end{tikzcd}$$

we infer that $\psi_{\mathsf{Coker}\,g} = 0$. Hence we deduce that $\mathsf{C}_{\mathscr{A}}(X,Y,f,g) = \mathsf{Coker}\,g \in \mathsf{Ker}\,\psi$ and it is straightforward to check that $\mathsf{C}_{\mathscr{A}}\colon \mathcal{M}(\phi,\psi) \longrightarrow \mathsf{Ker}\,\psi$ is a functor.

(vii) The functor $\mathsf{Z}_{\mathscr{B}}\colon \mathsf{Ker}\,\phi \longrightarrow \mathcal{M}(\phi,\psi)$ is defined on the objects $Y \in \mathsf{Ker}\,\phi$ by $\mathsf{Z}_{\mathscr{B}}(Y) = (0,Y,0,0)$, which lies in $\mathcal{M}(\phi,\psi)$ since $\phi_Y = 0$, and if $b\colon Y \longrightarrow Y'$ is a morphism in $\mathscr{B}$ then $\mathsf{Z}_{\mathscr{B}}(b) = (0,b)$.

(viii) The functor $\mathsf{C}_{\mathscr{B}}\colon \mathcal{M}(\phi,\psi) \longrightarrow \mathsf{Ker}\,\phi$ is defined by $\mathsf{C}_{\mathscr{B}}(X,Y,f,g) = \mathsf{Coker}\,f$ on objects $(X,Y,f,g)$ of $\mathcal{M}(\phi,\psi)$ and if $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ is a morphism in $\mathcal{M}(\phi,\psi)$ then $\mathsf{C}_{\mathscr{B}}(a,b) = \zeta$, where $\zeta\colon \mathsf{Coker}\,f \longrightarrow \mathsf{Coker}\,f'$ is the unique morphism which makes the following diagram commutative:

$$\begin{tikzcd}
F(X) \arrow[r, "f"] \arrow[d, "F(a)"'] & Y \arrow[r, two heads, "\pi_Y"] \arrow[d, "b"] & \mathsf{Coker}\,f \arrow[d, "\zeta"] \\
F(X') \arrow[r, "f'"] & Y' \arrow[r, two heads, "\pi_{Y'}"] & \mathsf{Coker}\,f'
\end{tikzcd}$$

Similarly with (vi) we get that $\mathsf{C}_{\mathscr{B}}(X,Y,f,g) = \mathsf{Coker}\, f$ belongs to $\mathcal{M}(\phi,\psi)$ and then it follows that $\mathsf{C}_{\mathscr{B}}\colon \mathcal{M}(\phi,\psi) \longrightarrow \mathsf{Ker}\,\phi$ is a functor.

Next we collect several useful properties of the above functors.

**Proposition 3.1.11.** (i) *The pairs $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}})$ and $(\mathsf{T}_{\mathscr{B}}, \mathsf{U}_{\mathscr{B}})$ are adjoint pairs of additive functors.*

(ii) *The pairs $(\mathsf{C}_{\mathscr{A}}, \mathsf{Z}_{\mathscr{A}})$ and $(\mathsf{C}_{\mathscr{B}}, \mathsf{Z}_{\mathscr{B}})$ are adjoint pairs of additive functors.*

(iii) *The functors $\mathsf{T}_{\mathscr{A}}$ and $\mathsf{T}_{\mathscr{B}}$ are fully faithful.*

(iv) *We have $\mathsf{U}_{\mathscr{A}}\mathsf{T}_{\mathscr{A}} = \mathrm{Id}_{\mathscr{A}}$ and $\mathsf{U}_{\mathscr{B}}\mathsf{T}_{\mathscr{B}} = \mathrm{Id}_{\mathscr{B}}$.*

(v) *The functors $\mathsf{U}_{\mathscr{A}}$ and $\mathsf{U}_{\mathscr{B}}$ are surjective on objects.*

(vi) *The functors $\mathsf{U}_{A}$ and $\mathsf{U}_{\mathscr{B}}$ are exact.*

(vii) *The functors $\mathsf{T}_{\mathscr{A}}$ and $\mathsf{T}_{\mathscr{B}}$ are right exact.*

**Proof.** (i) Let $(a,b)\colon \mathsf{T}_{\mathscr{A}}(X) \longrightarrow (X',Y',f',g')$ be a morphism in $\mathcal{M}(\phi,\psi)$. We define the map

$$\mathcal{F}\colon \mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X),(X',Y',f',g')) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(X,X'),\ \mathcal{F}((a,b)) = a$$

It is straightforward that $\mathcal{F}$ is a homomorphism of abelian groups. Since $(a,b)$ is a morphism in $\mathcal{M}(\phi,\psi)$ then we have the following commutative diagrams:

$$\begin{tikzcd}
F(X) \arrow[r, "\mathrm{Id}_{F(X)}"] \arrow[d, "F(a)"'] & F(X) \arrow[d, "b"] \\
F(X') \arrow[r, "f'"] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
GF(X) \arrow[r, "\psi_X"] \arrow[d, "G(b)"'] & X \arrow[d, "a"] \\
G(Y') \arrow[r, "g'"] & X'
\end{tikzcd}$$

It follows easily from the above first diagram that $\mathcal{F}$ is a monomorphism. Let $a\colon X \longrightarrow X'$ be a morphism in $\mathscr{A}$. Then $\mathcal{F}(a, F(a)\circ f') = a$ and $(a, Fa\circ f')$ is a morphism in $\mathcal{M}(\phi,\psi)$. Indeed we have that the following diagrams

$$\begin{tikzcd}
F(X) \arrow[r, "\mathrm{Id}_{F(X)}"] \arrow[d, "F(a)"'] & F(X) \arrow[d, "F(a)\circ f'"] \\
F(X') \arrow[r, "f'"] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
GF(X) \arrow[r, "\psi_X"] \arrow[d, "GF(a)\circ G(f')"'] & X \arrow[d, "a"] \\
G(Y') \arrow[r, "g'"] & X'
\end{tikzcd}$$

are commutative since $GF(a)\circ G(f')\circ g' = GF(a)\circ \psi_{X'} = \psi_X \circ a$. Thus $\mathcal{F}$ is an epimorphism. Also it is not difficult to check that the isomorphism $\mathcal{F}$ is natural. Hence the pair $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}})$ is an adjoint pair of functors and similarly we get that $(\mathsf{T}_{\mathscr{B}}, \mathsf{U}_{\mathscr{B}})$ is an adjoint pair.

(ii) Let $k\colon \mathsf{Coker}\, g \longrightarrow X'$ be a morphism in $\mathscr{A}$. We define

$$\mathcal{F}\colon \mathsf{Hom}_{\mathscr{A}}(\mathsf{Coker}\, g, X') \longrightarrow \mathsf{Hom}_{\mathcal{M}(\phi,\psi)}((X,Y,f,g),\mathsf{Z}_{\mathscr{A}}(X')),\ \mathcal{F}(k) = (\pi_X\circ k, 0)$$

It is easy to see that $\mathcal{F}$ is a homomorphism of abelian groups and that $\mathcal{F}(k) = (\pi_X\circ k, 0)$ is a morphism in $\mathcal{M}(\phi,\psi)$. Let $k\colon \mathsf{Coker}\, g \longrightarrow X'$ be a morphism in $\mathscr{A}$ such that $\mathcal{F}(k) = 0$. Then $\pi_X\circ k = 0$ and since $\pi_X$ is an epimorphism it follows that $k = 0$. Hence $\mathcal{F}$ is a monomorphism. Let $(a,0)\colon (X,Y,f,g) \longrightarrow (X',0,0,0)$ be a morphism in $\mathcal{M}(\phi,\psi)$. Since $g\circ a = 0$ there exist a map

$k\colon \mathsf{Coker}\, g \longrightarrow X'$ such that the following diagram is commutative:

$$\begin{tikzcd} G(Y) \arrow[r, "g"] & X \arrow[r, two heads, "\pi_X"] \arrow[d, "a"'] & \mathsf{Coker}\, g \arrow[dl, dashed, "k"] \\ & X' & \end{tikzcd}$$

Then $\mathcal{F}(k) = (\pi_X \circ k, 0) = (a, 0)$ and so $\mathcal{F}$ is an epimorphism. Also it is easy to check that $\mathcal{F}$ is a natural isomorphism. Therefore the pair of functors $(\mathsf{C}_{\mathscr{A}}, \mathsf{Z}_{\mathscr{A}})$ is an adjoint pair. Similarly we show that $(\mathsf{C}_{\mathscr{B}}, \mathsf{Z}_{\mathscr{B}})$ is an adjoint pair.

(iii) Let $a\colon X \longrightarrow X'$ be a morphism in $\mathscr{A}$ such that $\mathsf{T}_{\mathscr{A}}(a) = 0$. Then $(a, F(a)) = (0, 0)$ and thus $a = 0$. Hence the functor $\mathsf{T}_{\mathscr{A}}$ is faithful and similarly we deduce that $\mathsf{T}_{\mathscr{B}}$ is faithful. Let $(a, b)\colon \mathsf{T}_{\mathscr{A}}(X) \longrightarrow \mathsf{T}_{\mathscr{A}}(X')$ be a morphism in $\mathcal{M}(\phi, \psi)$. Then we have the commutative diagrams

$$\begin{tikzcd} F(X) \arrow[r, "\mathrm{Id}_{F(X)}"] \arrow[d, "F(a)"'] & F(X) \arrow[d, "b"] \\ F(X') \arrow[r, "\mathrm{Id}_{F(X')}"] & F(X') \end{tikzcd} \qquad \begin{tikzcd} GF(X) \arrow[r, "\psi_X"] \arrow[d, "G(b)"'] & X \arrow[d, "a"] \\ GF(X') \arrow[r, "\psi_{X'}"] & X' \end{tikzcd}$$

and so $b = F(a)$. Thus we have $\mathsf{T}_{\mathscr{A}}(a) = (a, F(a)) = (a, b)$. We infer that the functor $\mathsf{T}_{\mathscr{A}}$ is full and similarly we prove that $\mathsf{T}_{\mathscr{B}}$ is full.

(iv) Let $X \in \mathscr{A}$. Then $\mathsf{T}_{\mathscr{A}}(X) = (X, F(X), \mathrm{Id}_{F(X)}, \psi_X) \in \mathcal{M}(\phi, \psi)$ and $\mathsf{U}_{\mathscr{A}}\mathsf{T}_{\mathscr{A}}(X) = X$. Therefore $\mathsf{U}_{\mathscr{A}}\mathsf{T}_{\mathscr{A}} = \mathrm{Id}_{\mathscr{A}}$ and similarly we get that $\mathsf{U}_{\mathscr{B}}\mathsf{T}_{\mathscr{B}} = \mathrm{Id}_{\mathscr{B}}$.

(v) This follows immediately from (iv).

(vi) Let

$$0 \longrightarrow (X_1, Y_1, f_1, g_1) \xrightarrow{(a,b)} (X_2, Y_2, f_2, g_2) \xrightarrow{(c,d)} (X_3, Y_3, f_3, g_3) \longrightarrow 0$$

be an exact sequence in $\mathcal{M}(\phi, \psi)$. Then from Corollaries 3.1.3 and 3.1.5 we have the monomorphisms $\mathsf{U}_{\mathscr{A}}(a, b) = a$ and $\mathsf{U}_{\mathscr{B}}(a, b) = b$, and the epimorphisms $\mathsf{U}_{\mathscr{A}}(c, d) = c$ and $\mathsf{U}_{\mathscr{B}}(c, d) = d$. Since $\mathsf{Im}\,(a, b) = (\mathsf{Im}\, a, \mathsf{Im}\, b, \kappa, \lambda) = (\mathsf{Ker}\, c, \mathsf{Ker}\, d, h, j) = \mathsf{Ker}\,(c, d)$ it follows that $\mathsf{U}_{\mathscr{A}}(\mathsf{Im}\,(a, b)) = \mathsf{U}_{\mathscr{A}}(\mathsf{Ker}\,(c, d))$ and $\mathsf{U}_{\mathscr{B}}(\mathsf{Im}\,(a, b)) = \mathsf{U}_{\mathscr{B}}(\mathsf{Ker}\,(c, d))$. Then $\mathsf{Im}\, a = \mathsf{Ker}\, c$ and $\mathsf{Im}\, b = \mathsf{Ker}\, d$ and so the sequences $0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$ and $0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow 0$ are exact. Hence the functors $\mathsf{U}_{\mathscr{A}}$ and $\mathsf{U}_{\mathscr{B}}$ are exact.

(vii) Since from (i) the functor $\mathsf{T}_{\mathscr{A}}$ is left adjoint it follows that it is right exact. Similarly from the adjoint pair $(\mathsf{T}_{\mathscr{B}}, \mathsf{U}_{\mathscr{B}})$ we get that $\mathsf{T}_{\mathscr{B}}$ is right exact.

□

We close this section with the following results where we investigate when the categories $\mathscr{A}$ and $\mathscr{B}$ of a Morita category $\mathcal{M}(\phi, \psi)$ are equivalent.

LEMMA 3.1.12. *Let $\mathcal{M}(\phi, \psi)$ be a Morita category.*

(i) *If the natural transformation $\psi\colon GF \longrightarrow \mathrm{Id}_{\mathscr{A}}$ is an epimorphism then it is an isomorphism.*

(ii) *If the natural transformation $\phi\colon FG \longrightarrow \mathrm{Id}_{\mathscr{B}}$ is an epimorphism then it is an isomorphism.*

(iii) *If $\phi$ and $\psi$ are epimorphisms then the categories $\mathscr{A}$ and $\mathscr{B}$ are equivalent.*

PROOF. (i) Suppose that the natural tranformation $\psi_X \colon GF(X) \longrightarrow X$ is an epimorphism for every $X \in \mathscr{A}$. Then since the functor $GF$ is right exact we have the following exact commutative diagram:

$$\begin{tikzcd} GF(\mathsf{Ker}_{\psi_X}) \arrow[r, "GF(i)"] \arrow[d, "\psi_{\mathsf{Ker}\,\psi_X}"'] & GFGF(X) \arrow[r, "GF(\psi_X)"] \arrow[d, "\psi_{GF(X)}"'] & GF(X) \arrow[r] \arrow[d, "\psi_X"] & 0 \\ 0 \arrow[r] & \mathsf{Ker}\,\psi_X \arrow[r, "i"] & GF(X) \arrow[r, "\psi_X"] & X \arrow[r] & 0 \end{tikzcd}$$

and $GF(i)\circ\psi_{GF(X)} = GF(i)\circ GF(\psi_X) = 0$. Then $\psi_{\mathsf{Ker}\,\psi_X} = 0$ and so $\mathsf{Ker}\,\psi_X = 0$. Hence the morphism $\psi_X \colon GF(X) \longrightarrow X$ is an isomorphism for every $X \in \mathscr{A}$. Similarly we prove (ii). Finally for (iii), if $\phi$ and $\psi$ are epimorphisms then it follows from (i) and (ii) that the functor $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ is an equivalence of categories with inverse $F^{-1} = G$. $\square$

PROPOSITION 3.1.13. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category.*

(i) *The natural transformation $\psi \colon GF \longrightarrow \mathrm{Id}_{\mathscr{A}}$ is an epimorphism if and only if the categories $\mathcal{M}(\phi,\psi)$ and $\mathscr{B}$ are equivalent via the functor $\mathsf{T}_{\mathscr{B}}$.*
(ii) *The natural transformation $\phi \colon FG \longrightarrow \mathrm{Id}_{\mathscr{B}}$ is an epimorphism if and only if the categories $\mathcal{M}(\phi,\psi)$ and $\mathscr{A}$ are equivalent via the functor $\mathsf{T}_{\mathscr{A}}$.*
(iii) *The categories $\mathscr{A}$ and $\mathscr{B}$ are equivalent if and only if the natural transformations $\psi_X$ and $\phi_Y$ are epimorphisms for every $X \in \mathscr{A}$ and $Y \in \mathscr{B}$.*

PROOF. (i) Suppose that the natural transformation $\psi_X \colon GF(X) \longrightarrow X$ is an epimorphism for every $X \in \mathscr{A}$. From Proposition 3.1.11 we know that the functor $\mathsf{T}_{\mathscr{B}} \colon \mathscr{B} \longrightarrow \mathcal{M}(\phi,\psi)$ is fully faithful. So we have to show that $\mathsf{T}_{\mathscr{B}}$ is surjective on objects. Let $(X,Y,f,g)$ be an object of $\mathcal{M}(\phi,\psi)$. Then we have the morphism $(g,\mathrm{Id}_Y) \colon \mathsf{T}_{\mathscr{B}}(Y) \longrightarrow (X,Y,f,g)$. Also from the following commutative diagram

$$\begin{tikzcd} GF(\mathsf{Ker}\,g) \arrow[r, "GF(i)"] \arrow[d, "\psi_{\mathsf{Ker}\,g}"'] & GFG(Y) \arrow[r, two heads, "GF(g)"] \arrow[d, "\psi_{G(Y)}"'] & GF(X) \arrow[d, "\psi_X"] \\ \mathsf{Ker}\,g \arrow[r, tail, "i"] & G(Y) \arrow[r, two heads, "g"] & X \end{tikzcd}$$

it follows that $\psi_{\mathsf{Ker}\,g} = 0$ since $\psi_{\mathsf{Ker}\,g} \circ i = GF(i) \circ \psi_{G(Y)} = G(F(i) \circ \phi_Y) = G(F(i) \circ F(g) \circ f) = 0$ and $i$ is a monomorphism. But then $\mathsf{Ker}\,g = 0$ since $\psi_{\mathsf{Ker}\,g}$ is an epimorphism. Hence the morphism $g \colon G(Y) \longrightarrow X$ is an isomorphism and then the morphism $(g,\mathrm{Id}_Y)$ is also an isomorphism. We infer that the categories $\mathscr{B}$ and $\mathcal{M}(\phi,\psi)$ are equivalent via the functor $\mathsf{T}_{\mathscr{B}}$. Conversely, suppose that the functor $\mathsf{T}_{\mathscr{B}} \colon \mathscr{B} \longrightarrow \mathcal{M}(\phi,\psi)$ is an equivalence of categories. Let $X$ be an arbitrary object of $\mathscr{A}$ and consider the object $\mathsf{T}_{\mathscr{A}}(X) = (X, F(X), \mathrm{Id}_{F(X)}, \psi_X) \in \mathcal{M}(\phi,\psi)$. Then we get that $(\psi_X, \mathrm{Id}_{F(X)}) \colon \mathsf{T}_{\mathscr{B}}(F(X)) \longrightarrow \mathsf{T}_{\mathscr{A}}(X)$ is an isomorphism. But this implies that the morphism $\psi_X \colon GF(X) \longrightarrow X$ is an isomorphism, and so we are done. The equivalence of (ii) is dual with (i).

(iii) If the functor $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ is an equivalence with inverse $G$ then the result follows immediately. If conversely the natural transformations $\psi_X$ and $\phi_Y$ are epimorphisms for every $X \in \mathscr{A}$ and $Y \in \mathscr{B}$ then from (i) and (ii) we infer that the categories $\mathscr{A}$, $\mathcal{M}(\phi,\psi)$ and $\mathscr{B}$ are equivalent. $\square$

## 3.2. Morita Categories Isomorphic With Their Duals

Let $\mathcal{M}(\phi,\psi) = (\mathscr{A},\mathscr{B},F,G,\phi,\psi)$ be a Morita category. In this section we assume that the functors $F$ and $G$ have right adjoints $F'$ and $G'$ respectively. Using $\phi$ and $\psi$, we will prove the existence of natural transformations $\phi'\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow G'F'$ and $\psi'\colon \mathrm{Id}_{\mathscr{A}} \longrightarrow F'G'$ such that the Morita category $\mathcal{M}(\phi,\psi)$ is isomorphic to the dual Morita category $Co\mathcal{M}(\phi',\psi')$.

Let $(X,Y,f,g)$ be an object of $\mathcal{M}(\phi,\psi)$. Since $(F,F')$ and $(G,G')$ are adjoint pairs, we have the following natural isomorphism:

$$\mathsf{Hom}_{\mathscr{B}}(F(X),Y) \underset{\eta_{X,Y}}{\overset{\theta_{X,Y}}{\rightleftarrows}} \mathsf{Hom}_{\mathscr{A}}(X,F'(Y)) \qquad \mathsf{Hom}_{\mathscr{A}}(G(Y),X) \underset{\eta'_{X,Y}}{\overset{\theta'_{X,Y}}{\rightleftarrows}} \mathsf{Hom}_{\mathscr{B}}(Y,G'(X))$$

and let $\epsilon\colon FF' \longrightarrow \mathrm{Id}_{\mathscr{B}}$, resp. $\epsilon'\colon GG' \longrightarrow \mathrm{Id}_{\mathscr{A}}$, and $\delta\colon \mathrm{Id}_{\mathscr{A}} \longrightarrow F'F$, resp. $\delta'\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow G'G$ be the unit and the counit of the adjoint pair $(F,F')$, resp. $(G,G')$. For any object $X \in A$, we define the morphism $\psi'_X = \eta(\eta'(\psi_X))\colon X \longrightarrow F'G'X$, and for any object $Y \in B$ we define the morphism $\phi'_Y = \eta'(\eta(\phi_Y))\colon Y \longrightarrow G'F'Y$.

Lemma 3.2.1. (i) *Let $X$ be an object in $\mathscr{A}$. Then $\psi'_X = \delta_X \circ F'(\delta'_{F(X)}) \circ F'G'(\psi_X)$.*
(ii) *Let $Y$ be an object in $\mathscr{B}$. Then $\phi'_Y = \delta'_Y \circ G'(\delta_{G(Y)}) \circ G'F'(\phi_Y)$.*

Proof. From well known results concerning adjoints functors, see the classical book [90], we have the following equalities:

$$\psi'_X = \eta\circ\eta'(\psi_X) = \eta(\delta'_{F(X)}\circ G'(\psi_X)) = \eta(\delta'_{F(X)})\circ F'G'(\psi_X) = \delta_X\circ F'(\delta'_{F(X)})\circ F'G'(\psi_X)$$

The second equality follows easily like the first one. □

From the above description we have immediately the following result.

Corollary 3.2.2. *The family of morphisms $\psi'_X = \delta_X\circ F'(\delta'_{F(X)})\circ F'G'(\psi_X)\colon X \longrightarrow F'G'(X)$, for every $X \in \mathscr{A}$, determines a natural transformation $\psi'\colon \mathrm{Id}_{\mathscr{A}} \longrightarrow F'G'$, and the family of morphisms $\phi'_Y = \delta'_Y\circ G'(\delta_{G(Y)})\circ G'F'(\phi_Y)\colon Y \longrightarrow G'F'(Y)$, for every $Y \in \mathscr{B}$, determines a natural transformation $\phi'\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow G'F'$.*

Since $(F,F')$ and $(G,G')$ are adjoint pairs then we have the adjoint pair $(GF,F'G')$ with unit $G(\varepsilon_{G'})\circ\varepsilon'\colon GFF'G' \longrightarrow \mathrm{Id}_{\mathscr{A}}$ and counit $\delta\circ F'(\delta'_F)\colon \mathrm{Id}_{\mathscr{A}} \longrightarrow F'G'GF$, and also we have the adjoint pair $(FG,G'F')$ with unit $F(\varepsilon'_{F'})\circ\varepsilon\colon FGG'F' \longrightarrow \mathrm{Id}_{\mathscr{B}}$ and counit $\delta'\circ G'(\delta_G)\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow G'F'FG$.

Proposition 3.2.3. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category. If the functors $F$ and $G$ have right adjoints, $F'$ and $G'$ respectively, then there exist natural transformations $\phi'\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow G'F'$ and $\psi'\colon \mathrm{Id}_{\mathscr{A}} \longrightarrow F'G'$ such that $Co\mathcal{M}(\phi',\psi') = (\mathscr{A},\mathscr{B},G',F',\psi',\phi')$ is a dual Morita category of $\mathscr{A}$ and $\mathscr{B}$ by $\phi'$ and $\psi'$.*

Proof. We are going to show that the following diagrams are commutative

$$\begin{tikzcd} F' \arrow[r, "\psi'_{F'}"] \arrow[d, equal] & F'G'F' \\ F' \arrow[ur, "F'\phi'"'] & \end{tikzcd} \qquad \begin{tikzcd} G' \arrow[r, "\phi'_{G'}"] \arrow[d, equal] & G'F'G' \\ G' \arrow[ur, "G'\psi'"'] & \end{tikzcd}$$

Let $(X,Y,f,g)$ be an object in $\mathcal{M}(\phi,\psi)$. Since $\delta\circ F'(\delta'_F)\colon \mathrm{Id}_{\mathscr{A}}\longrightarrow F'G'GF$ is the counit of the adjoint pair $(GF,F'G')$ we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\delta_X\circ F'(\delta'_{F(X)})"] \arrow[d, "\delta_X\circ F'(f)"'] & F'G'GF(X) \arrow[d, "F'G'GF(\delta_X\circ F'(f))"] \\ F'(Y) \arrow[r, "\delta_{F'(Y)}\circ F'(\delta'_{FF'(Y)})"'] & F'G'GFF'(Y) \end{tikzcd}$$

Then

$$\begin{aligned}
\eta(f)\circ\psi'_{F'(Y)} &= \delta_X\circ F'(f)\circ\delta_{F'(Y)}\circ F'(\delta'_{FF'(Y)})\circ F'G'(\psi_{F'(Y)})\\
&= \delta_X\circ F'(\delta'_{F(X)})\circ F'G'GF(\delta_X)\circ F'G'GFF'(f)\circ F'G'(\psi_{F'(Y)})\\
&= \delta_X\circ F'(\delta'_{F(X)})\circ F'G'GF(\delta_X)\circ F'G'(\psi_{F'F(X)})\circ F'G'F'(f)\\
&= \delta_X\circ F'(\delta'_{F(X)})\circ F'G'(\psi_X)\circ F'G'(\delta_X)\circ F'G'F'(f)\\
&= \delta_X\circ F'(\delta'_{F(X)})\circ F'G'(\delta_{GF(X)})\circ F'G'F'F(\psi_X)\circ F'G'F'(f)\\
&= \eta(\delta'_{F(X)}\circ G'(\delta_{GF(X)})\circ G'F'F(\psi_X)\circ G'F'(f))\\
&= \eta(\eta'(\delta_{GF(X)}\circ F'F(\psi_X)\circ F'(f)))\\
&= \eta(\eta'(\eta(F(\psi_X)\circ f)))
\end{aligned}$$

and thus the following diagram is commutative:

$$\begin{tikzcd} \mathsf{Hom}_{\mathscr{B}}(F(X),Y) \arrow[r, "\eta"] \arrow[d, "(F(\psi_X){,}Y)"'] & \mathsf{Hom}_{\mathscr{A}}(X,F'(Y)) \arrow[d, "(X{,}\psi'_{F'(Y)})"] \\ \mathsf{Hom}_{\mathscr{B}}(FGF(X),Y) \arrow[r, "\eta\circ\eta'\circ\eta"] & (X,F'G'F'(Y)) \end{tikzcd}$$

Similarly as above it follows that the following diagram is commutative:

$$\begin{tikzcd} \mathsf{Hom}_{\mathscr{B}}(F(X),Y) \arrow[r, "\eta"] \arrow[d, "(\phi_{F(X)}{,}Y)"'] & \mathsf{Hom}_{\mathscr{A}}(X,F'(Y)) \arrow[d, "(X{,}F'(\phi'_Y))"] \\ \mathsf{Hom}_{\mathscr{B}}(FGF(X),Y) \arrow[r, "\eta\circ\eta'\circ\eta"] & \mathsf{Hom}_{\mathscr{A}}(X,F'G'F'(Y)) \end{tikzcd}$$

Since $F(\psi_X)=\phi_{F(X)}$ it follows from the above commutative diagrams that $\psi'_{F'(Y)}=F'(\phi'_Y)$. Similarly we show that $\phi'_{G'}=G'(\psi')$ and therefore we infer that $Co\mathcal{M}(\phi',\psi')=(\mathscr{A},\mathscr{B},G',F',\psi',\phi')$ is a dual Morita category of $\mathscr{A}$ and $\mathscr{B}$ by $\phi'$ and $\psi'$. $\square$

Now we are ready to prove the main result of this section which gives a sufficient condition for a Morita Category to be equivalent with its dual.

THEOREM 3.2.4. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category. If the functors $F$ and $G$ have right adjoints then the Morita category $\mathcal{M}(\phi,\psi)$ is isomorphic to the dual Morita category $Co\mathcal{M}(\phi',\psi')$.*

PROOF. Let $(X,Y,f,g)$ be an object of $\mathcal{M}(\phi,\psi)$ and consider the tuple $(X,Y,\delta'_Y\circ G'(g),\delta_X\circ F'(f))$. We claim that $(X,Y,\delta'_Y\circ G'(g),\delta_X\circ F'(f))$ belongs to $Co\mathcal{M}(\phi',\psi')$.

We need to show that the following diagrams are commutative:

$$
\begin{tikzcd}
F'(Y) \arrow[r, "F'(\delta'_Y)\circ F'G'(g)"] & F'G'(X) \\
X \arrow[u, "\delta_X\circ F'(f)"] \arrow[ur, "\psi'_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G'(X) \arrow[r, "G'(\delta_X)\circ G'F'(f)"] & G'F'(Y) \\
Y \arrow[u, "\delta'_Y\circ G'(g)"] \arrow[ur, "\phi'_Y"'] &
\end{tikzcd}
$$

Indeed we have $\psi'_X = \delta_X \circ F'(\delta'_{F(X)}) \circ F'G'(\psi_X) = \delta_X \circ F'(\delta'_{F(X)}) \circ F'G'G(f) \circ F'G'(g) = \delta_X \circ F'(f) \circ F'(\delta'_Y) \circ F'G'(g)$ and similarly we derive that $\delta'_Y \circ G'(g) \circ G'(\delta_X) \circ G'F'(f) = \phi'_Y$. Then we can define a functor

$$\mathcal{F}\colon \mathcal{M}(\phi,\psi) \longrightarrow Co\mathcal{M}(\phi',\psi')$$

by $\mathcal{F}(X,Y,f,g) = (X,Y,\delta'_Y \circ G'(g), \delta_X \circ F'(f))$ on objects $(X,Y,f,g) \in \mathcal{M}(\phi,\psi)$ and given a morphism $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ in $\mathcal{M}(\phi,\psi)$ then $\mathcal{F}(a,b) = (a,b)\colon (X,Y,\delta'_Y \circ G'(g), \delta_X \circ F'(f)) \longrightarrow (X',Y',\delta'_{Y'} \circ G'(g'), \delta_{X'} \circ F'(f'))$ is a morphism in $Co\mathcal{M}(\phi',\psi')$. We define also the functor

$$\mathcal{F}'\colon Co\mathcal{M}(\phi',\psi') \longrightarrow \mathcal{M}(\phi,\psi)$$

by $\mathcal{F}'(X,Y,f,g) = (X,Y,F(g)\circ\varepsilon_Y, G(f)\circ\varepsilon'_X)$ on objects $(X,Y,f,g) \in Co\mathcal{M}(\phi',\psi')$ and by $\mathcal{F}'(a,b) = (a,b)$ on morphisms $(a,b)$ of $Co\mathcal{M}(\phi',\psi')$. Then it is easy to check that $\mathcal{F}'\mathcal{F} = \mathrm{Id}_{\mathcal{M}(\phi,\psi)}$ and $\mathcal{F}\mathcal{F}' = \mathrm{Id}_{Co\mathcal{M}(\phi',\psi')}$. Hence the categories $\mathcal{M}(\phi,\psi)$ and $co\mathcal{M}(\phi',\psi')$ are equivalent. $\square$

We define the functor

$$
\begin{tikzcd}
\mathscr{A} \arrow[r, dashed, "\mathsf{H}_{\mathscr{A}}=\mathcal{F}'\circ\mathsf{T}'_{\mathscr{A}}"] \arrow[d, "\mathsf{T}'_{\mathscr{A}}"'] & \mathcal{M}(\phi,\psi) \\
Co\mathcal{M}(\phi',\psi') \arrow[ur, "\mathcal{F}'"'] &
\end{tikzcd}
$$

by $\mathsf{H}_{\mathscr{A}}(X) = (X, G'(X), F(\psi'_X)\circ\varepsilon_{G'(X)}, \varepsilon'_X)$ on objects $X \in \mathscr{A}$, where $\mathsf{T}'_{\mathscr{A}}(X)$ is the tuple $(X, G'(X), \psi'_X, \mathrm{Id}_{G'(X)}) \in Co\mathcal{M}(\phi',\psi')$, and given a morphism $c\colon X \longrightarrow X'$ in $\mathscr{A}$ then $\mathsf{H}_{\mathscr{A}}(c) = (c, G'(c))$. Similarly we can define the functor $\mathsf{H}_{\mathscr{B}}\colon \mathscr{B} \longrightarrow \mathcal{M}(\phi,\psi)$.

The next result shows that we have more adjoint pairs under the existence of right adjoints for $F$ and $G$.

PROPOSITION 3.2.5. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category such that the functors $F$ and $G$ have right adjoints. Then the pairs $(\mathsf{U}_{\mathscr{A}}, \mathsf{H}_{\mathscr{A}})$ and $(\mathsf{U}_{\mathscr{B}}, \mathsf{H}_{\mathscr{B}})$ are adjoint pairs of functors.*

PROOF. We show that $(\mathsf{U}_{\mathscr{A}}, \mathsf{H}_{\mathscr{A}})$ is an adjoint pair of functors. Let $(X,Y,f,g)$ be an object of $\mathcal{M}(\phi,\psi)$. We define the morphism $\rho_{(X,Y,f,g)}\colon (X,Y,f,g) \longrightarrow \mathsf{H}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}}(X,Y,f,g)$, $\rho_{(X,Y,f,g)} = (\mathrm{Id}_X, \delta'_Y \circ G'(g))$. First we show that $\rho_{(X,Y,f,g)}$ is a morphism in $\mathcal{M}(\phi,\psi)$.

Since $G(\delta'_Y) \circ GG'(g) \circ \varepsilon'_X = G(\delta'_Y) \circ \varepsilon'_{G(Y)} \circ g = \mathrm{Id}_{G(Y)} \circ g = g$ and

$$\begin{aligned}
f \circ \delta'_Y \circ G'(g) &= \delta'_{F(X)} \circ G'G(f) \circ G'(g) \\
&= \delta'_{F(X)} \circ G'(\psi_X) \\
&= \mathrm{Id}_{F(X)} \circ \delta'_{F(X)} \circ G'(\psi_X) \\
&= F(\delta_X) \circ \varepsilon_{F(X)} \circ \delta'_{F(X)} \circ G'(\psi_X) \\
&= F(\delta_X) \circ FF'(\delta'_{F(X)}) \circ \varepsilon_{G'GF(X)} \circ G'(\psi_X) \\
&= F(\delta_X) \circ FF'(\delta'_{F(X)}) \circ FF'G'(\psi_X) \circ \varepsilon_{G'(X)} \\
&= F(\delta_X \circ F'(\delta'_{F(X)}) \circ F'G'(\psi_X)) \circ \varepsilon_{G'(X)} \\
&= F(\psi'_X) \circ \varepsilon_{G'(X)}
\end{aligned}$$

it follows that the following diagrams are commutative:

$$\begin{tikzcd}
G(Y) \arrow[r, "g"] \arrow[d, "G(\delta'_Y)\circ GG'(g)"'] & X \arrow[d, "\mathrm{Id}_X"] \\
GG'(X) \arrow[r, "\varepsilon'_X"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "f"] \arrow[d, "\mathrm{Id}_{F(X)}"'] & Y \arrow[d, "\delta'_Y\circ G'(g)"] \\
F(X) \arrow[r, "F(\psi'_X)\circ\varepsilon_{G'(X)}"] & G'(X)
\end{tikzcd}$$

Hence the map $\rho_{(X,Y,f,g)}$ is indeed a morphism in $\mathcal{M}(\phi,\psi)$. If $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ is a morphism in $\mathcal{M}(\phi,\psi)$, then the following diagram

$$\begin{tikzcd}
(X,Y,f,g) \arrow[r, "(\mathrm{Id}_X,\delta'_Y\circ G'(g))"] \arrow[d, "(a,b)"'] & \mathsf{H}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}}(X,Y,f,g) \arrow[d, "(a,G'(a))"] \\
(X',Y',f',g') \arrow[r, "(\mathrm{Id}_{X'},\delta'_{Y'}\circ G'(g'))"] & \mathsf{H}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}}(X,Y,f,g)
\end{tikzcd}$$

is commutative since $b \circ \delta'_{Y'} \circ G'(g') = \delta'_Y \circ G'G(b) \circ G'(g') = \delta'_Y \circ G'(g) \circ G'(a)$. Thus the morphism $\rho_{(X,Y,f,g)}$ is a natural transformation. We have also the natural transformation $\pi_X\colon \mathsf{U}_{\mathscr{A}}\mathsf{H}_{\mathscr{A}}(X) \longrightarrow X$ defined by $\pi_X = \mathrm{Id}_X$. Then it is easy to check that the following diagrams are commutative:

$$\begin{tikzcd}
\mathsf{U}_{\mathscr{A}} \arrow[r, "\mathsf{U}_{\mathscr{A}}\rho"] \arrow[dr, "\mathrm{Id}_{\mathsf{U}_{\mathscr{A}}}"'] & \mathsf{U}_{\mathscr{A}}\mathsf{H}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}} \arrow[d, "\pi\mathsf{U}_{\mathscr{A}}"] \\
& \mathsf{U}_{\mathscr{A}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{H}_{\mathscr{A}} \arrow[r, "\rho\mathsf{H}_{\mathscr{A}}"] \arrow[dr, "\mathrm{Id}_{\mathsf{H}_{\mathscr{A}}}"'] & \mathsf{H}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}}\mathsf{H}_{\mathscr{A}} \arrow[d, "\mathsf{H}_{\mathscr{A}}\pi"] \\
& \mathsf{H}_{\mathscr{A}}
\end{tikzcd}$$

We infer that $(\mathsf{U}_{\mathscr{A}},\mathsf{H}_{\mathscr{A}})$ is an adjoint pair of functors where $\rho\colon \mathrm{Id}_{\mathscr{C}(\phi,\psi)} \longrightarrow \mathsf{H}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}}$ is the unit and $\pi\colon \mathsf{U}_{\mathscr{A}}\mathsf{H}_{\mathscr{A}} \longrightarrow \mathrm{Id}_{\mathscr{A}}$ is the counit of $(\mathsf{U}_{\mathscr{A}},\mathsf{H}_{\mathscr{A}})$. $\square$

## 3.3. Recollements and Morita Categories

Let $\mathcal{M}(\phi,\psi)$ be a Morita category and suppose that the functors $F$ and $G$ have right adjoints. In this section we show that the Morita category $\mathcal{M}(\phi,\psi)$ admits a recollement of abelian categories.

Consider the following diagrams:

$$\begin{tikzcd}[column sep=large]
\mathsf{Ker}\,\mathsf{U}_{\mathscr{A}} \arrow[r, "\mathsf{i}_{\mathscr{A}}"] & \mathcal{M}(\phi,\psi) \arrow[l, bend right=40, "\mathsf{q}_{\mathscr{A}}"'] \arrow[l, bend left=40, "\mathsf{p}_{\mathscr{A}}"] \arrow[r, "\mathsf{U}_{\mathscr{A}}"] & \mathscr{A} \arrow[l, bend right=40, "\mathsf{T}_{\mathscr{A}}"'] \arrow[l, bend left=40, "\mathsf{H}_{\mathscr{A}}"]
\end{tikzcd} \tag{3.3.1}$$

$$\begin{tikzcd}[column sep=large]
\mathsf{Ker}\,\mathsf{U}_{\mathscr{B}} \arrow[r, "\mathsf{i}_{\mathscr{B}}"] & \mathcal{M}(\phi,\psi) \arrow[l, bend right=40, "\mathsf{q}_{\mathscr{B}}"'] \arrow[l, bend left=40, "\mathsf{p}_{\mathscr{B}}"] \arrow[r, "\mathsf{U}_{\mathscr{B}}"] & \mathscr{B} \arrow[l, bend right=40, "\mathsf{T}_{\mathscr{B}}"'] \arrow[l, bend left=40, "\mathsf{H}_{\mathscr{B}}"]
\end{tikzcd} \tag{3.3.2}$$

We will show that the above diagrams are recollements of abelian categories. Consider the subcategories $\mathsf{Ker}\,\phi = \{Y \in \mathscr{B} \mid \phi_Y = 0\}$ and $\mathsf{Ker}\,\psi = \{X \in \mathscr{A} \mid \psi_X = 0\}$ of $\mathscr{B}$ and $\mathscr{A}$ respectively. Then $\mathsf{Ker}\,\phi$ is equivalent to $\mathsf{Ker}\,\mathsf{U}_{\mathscr{A}} = \{(0,Y,0,0) \in \mathcal{M}(\phi,\psi) \mid \phi_Y = 0\}$ via the functor $\mathsf{Z}_{\mathscr{B}}$, and $\mathsf{Ker}\,\psi$ is equivalent to $\mathsf{Ker}\,\mathsf{U}_{\mathscr{B}} = \{(X,0,0,0) \in \mathcal{M}(\phi,\psi) \mid \psi_X = 0\}$ via the functor $\mathsf{Z}_{\mathscr{A}}$.

We define the following functors:

(i) The functor $\mathsf{q}_{\mathscr{A}} \colon \mathcal{M}(\phi,\psi) \longrightarrow \mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$ is defined by $\mathsf{q}_{\mathscr{A}}(X,Y,f,g) = (0,\mathsf{Coker}\,f,0,0)$ on the objects $(X,Y,f,g) \in \mathcal{M}(\phi,\psi)$ and given a morphism $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ in $\mathcal{M}(\phi,\psi)$ then $\mathsf{q}_{\mathscr{A}}(a,b) = (0,\zeta)$ where $\zeta\colon \mathsf{Coker}\,f \longrightarrow \mathsf{Coker}\,f'$ is the unique morphism which makes the following diagram commutative:

$$\begin{tikzcd}
F(X) \arrow[r, "f"] \arrow[d, "F(a)"'] & Y \arrow[r, two heads, "\pi_Y"] \arrow[d, "b"'] & \mathsf{Coker}\,f \arrow[d, "\zeta"] \\
F(X') \arrow[r, "f'"] & Y' \arrow[r, two heads, "\pi_{Y'}"] & \mathsf{Coker}\,f'
\end{tikzcd}$$

Let us explain why $\mathsf{q}_{\mathscr{A}}(X,Y,f,g)$ lies in $\mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$. From the following commutative diagram:

$$\begin{tikzcd}
FGF(X) \arrow[r, "FG(f)"] \arrow[d, "\phi_{F(X)}"'] & FG(Y) \arrow[r, two heads, "FG(\pi_Y)"] \arrow[d, "\phi_Y"'] \arrow[dl, "F(g)"'] & FG(\mathsf{Coker}\,f) \arrow[d, "\phi_{\mathsf{Coker}\,f}"] \\
F(X) \arrow[r, "f"] & Y \arrow[r, two heads, "\pi_Y"] & \mathsf{Coker}\,f
\end{tikzcd}$$

and the fact that $F(g) \circ f = \phi_Y$ it follows that $\phi_{\mathsf{Coker}\,f} = 0$ and thus $\mathsf{q}_{\mathscr{A}}(X,Y,f,g) \in \mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$. Then it is straightforward to check that $\mathsf{q}_{\mathscr{A}} \colon \mathcal{M}(\phi,\psi) \longrightarrow \mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$ is a functor and dually we have the functor $\mathsf{q}_{\mathscr{B}} \colon \mathcal{M}(\phi,\psi) \longrightarrow \mathsf{Ker}\,\mathsf{U}_{\mathscr{B}}$.

(ii) Let $(X,Y,f,g) \in \mathcal{M}(\phi,\psi)$. Then we have the object $(X,Y,\delta'_Y \circ G'(g), \delta_X \circ F'(f)) \in Co\mathcal{M}(\phi',\psi')$ and we define the functor $\mathsf{p}_{\mathscr{A}}(X,Y,f,g) = (0,\mathsf{Ker}\,(\delta'_Y \circ G'(g)),0,0)$ on objects. If $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ is a morphism in $\mathcal{M}(\phi,\psi)$ then $\mathsf{p}_{\mathscr{A}}(a,b) = (0,\xi)$ where $\xi\colon \mathsf{Ker}\,(\delta'_Y \circ G'(g)) \longrightarrow \mathsf{Ker}\,(\delta'_{Y'} \circ G'(g'))$ is the unique morphism which makes the following diagram commutative:

$$\begin{tikzcd}
\mathsf{Ker}\,(\delta'_Y \circ G'(g)) \arrow[r, tail, "i"] \arrow[d, "\xi"'] & Y \arrow[r, "\delta'_Y \circ G'(g)"] \arrow[d, "b"'] & G'(X) \arrow[d, "G'(a)"] \\
\mathsf{Ker}\,(\delta'_{Y'} \circ G'(g')) \arrow[r, tail, "i'"] & Y' \arrow[r, "\delta'_{Y'} \circ G'(g')"] & G'(X')
\end{tikzcd}$$

From the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Ker}\,(\delta'_Y \circ G'(g)) \arrow[r, tail, "i"] \arrow[d, "\phi'_{\mathsf{Ker}\,(\delta'_Y \circ G'(g))}"'] & Y \arrow[r, "\delta'_Y \circ G'(g)"] \arrow[d, "\phi'_Y"'] & G'(X) \arrow[d, "\phi'_{G'(X)}"] \\
G'F'(\mathsf{Ker}\,(\delta'_Y \circ G'g)) \arrow[r, tail, "G'F'(i')"] & G'F'(Y) \arrow[r] & G'F'G'(X)
\end{tikzcd}$$

we get that $\phi'_{\mathsf{Ker}\,(\delta'_Y \circ G'(g))} = 0$ since $\delta'_Y \circ G'(\delta_{G(Y)}) \circ G'F'(\phi_Y) = \phi'_Y$. Hence the object $\mathsf{p}_{\mathscr{A}}(X, Y, f, g) \in \mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$ and it follows easily that $\mathsf{p}_{\mathscr{A}} \colon \mathcal{M}(\phi, \psi) \longrightarrow \mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$ is a functor. Dually we have the functor $\mathsf{p}_{\mathscr{B}} \colon \mathcal{M}(\phi, \psi) \longrightarrow \mathsf{Ker}\,\mathsf{U}_{\mathscr{B}}$.

PROPOSITION 3.3.1. *Let $\mathcal{M}(\phi, \psi)$ be a Morita category such that the functors $F$ and $G$ have right adjoints. Then the diagrams (3.3.1) and (3.3.2) are recollements of abelian categories.*

PROOF. Since the functors $F$ and $G$ have right adjoints it follows from Proposition 3.1.11 and Proposition 3.2.5 that $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}}, \mathsf{H}_{\mathscr{A}})$ and $(\mathsf{T}_{\mathscr{B}}, \mathsf{U}_{\mathscr{B}}, \mathsf{H}_{\mathscr{B}})$ are adjoint triples. Also the functors $\mathsf{T}_{\mathscr{A}}$ and $\mathsf{T}_{\mathscr{B}}$ are fully faithful from Propostion 3.1.11, and therefore the functors $\mathsf{H}_{\mathscr{A}}$ and $\mathsf{H}_{\mathscr{B}}$ are fully faithful as well. Then we are done since the remaining adjoint pairs follow immediately, see Remark 1.1.3. Hence the diagrams (3.3.1) and (3.3.2) are recollements of abelian categories. $\square$

COROLLARY 3.3.2. *Let $\mathcal{M}(\phi, \psi)$ be a Morita category.*

(i) *If $\phi = 0$ then we have the following recollement of abelian categories:*

$$\begin{tikzcd}
\mathscr{B} \arrow[r] & \mathcal{M}(\phi, \psi) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{A} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

(ii) *If $\psi = 0$ then we have the following recollement of abelian categories:*

$$\begin{tikzcd}
\mathscr{A} \arrow[r] & \mathcal{M}(\phi, \psi) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{B} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

(iii) *If $\phi = 0 = \psi$ then the category $(\mathscr{A} \times \mathscr{B}) \ltimes H$ admits a symmetric recollement: $(\mathscr{A}, (\mathscr{A} \times \mathscr{B}) \ltimes H, \mathscr{B})$ and $(\mathscr{B}, (\mathscr{A} \times \mathscr{B}) \ltimes H, \mathscr{A})$.*

PROOF. Since $\phi_Y = 0$ for every $Y \in \mathscr{B}$ and $\psi_X = 0$ for every $X \in \mathscr{A}$, it follows that the categories $\mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$ and $\mathscr{B}$ are equivalent, and that the categories $\mathsf{Ker}\,\mathsf{U}_{\mathscr{B}}$ and $\mathscr{A}$ are equivalent. Then the result follows from Proposition 3.3.1 and Proposition 3.1.9. $\square$

**3.3.1. The MacPherson-Vilonen Category.** Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories, $G, F \colon \mathscr{A} \longrightarrow \mathscr{B}$ two functors such that $F$ is right exact and $G$ is left exact and let $\xi \colon F \longrightarrow G$ be a natural transformation. Then the MacPherson-Vilonen category, denoted by $\mathsf{MV}(\xi)$, is defined as follows, see [91]. The objects of $\mathsf{MV}(\xi)$ are tuples $(A, B, f, g)$, where $A \in \mathscr{A}$, $B \in \mathscr{B}$, $f \colon F(A) \longrightarrow B$ and $g \colon B \longrightarrow G(A)$ are morphisms in $\mathscr{B}$ such that the diagram

$$\begin{tikzcd}
F(A) \arrow[rr, "\xi_A"] \arrow[dr, "f"'] & & G(A) \\
& B \arrow[ur, "g"'] &
\end{tikzcd}$$

is commutative. Let $(A, B, f, g)$ and $(A', B', f', g')$ be two objects of $\mathsf{MV}(\xi)$. Then a morphism from $(A, B, f, g) \longrightarrow (A', B', f', g')$ is a pair $(\alpha, \beta)$, where $\alpha\colon A \longrightarrow A'$ is a morphism in $\mathscr{A}$ and $\beta\colon B \longrightarrow B'$ is a morphism in $\mathscr{B}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[rr, bend left, "\xi_A"] \arrow[d, "F(\alpha)"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & G(A) \arrow[d, "G(\alpha)"] \\ F(A') \arrow[r, "f'"] \arrow[rr, bend right, "\xi_{A'}"'] & B' \arrow[r, "g'"] & G(A') \end{tikzcd}$$

is commutative. Then it follows from [91] that the category $\mathsf{MV}(\xi)$ is abelian and there is a recollement of abelian categories:

$$\begin{tikzcd} \mathscr{B} \arrow[r, "\mathsf{i}"] & \mathsf{MV}(\xi) \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathscr{A} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

and $\mathsf{q}(A, B, f, g) = \mathsf{Coker}\, f$, $\mathsf{i}(B) = (0, B, 0, 0)$, $\mathsf{p}(A, B, f, g) = \mathsf{Ker}\, g$, $\mathsf{e}(A, B, f, g) = A$, $\mathsf{l}(A) = (A, F(A), \mathrm{Id}_{F(A)}, \xi_A)$, and $\mathsf{r}(A) = (A, G(A), \xi_A, \mathrm{Id}_{G(A)})$.

The following shows that there are two MacPherson-Vilonen categories associated to a Morita category.

Proposition 3.3.3. *Let $\mathcal{M}(\phi, \psi)$ be a Morita category such that the functors $F$ and $G$ have right adjoints $F'$ and $G'$ respectively. Then we have the following recollements of abelian categories:*

$$\begin{tikzcd} \mathscr{B} \arrow[r] & \mathsf{MV}(\xi) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{A} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \arrow[r] & \mathsf{MV}(\theta) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{B} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

*where $\mathsf{MV}(\xi)$ and $\mathsf{MV}(\theta)$ are MacPherson-Vilonen categories.*

Proof. From our setting we have the functors $G', F\colon \mathscr{A} \longrightarrow \mathscr{B}$ where $F$ is right exact and $G'$ is left exact. Let $(X, Y, f, g)$ be an object of $\mathcal{M}(\phi, \psi)$. Then we define the natural transformation $\xi\colon F \longrightarrow G'$, where $\xi_X = \delta'_{F(X)} \circ G'(\psi_X)$, and consider the object $(X, Y, f, \delta'_Y \circ G'(g))$. Then we have that the following diagram is commutative

$$\begin{tikzcd} F(X) \arrow[rr, "\xi_X"] \arrow[dr, "f"'] & & G'(X) \\ & Y \arrow[ur, "\delta'_Y \circ G'(g)"'] & \end{tikzcd}$$

since $f \circ \delta'_Y \circ G'(g) = \delta'_{F(X)} \circ G'G(f) \circ G'(g) = \xi_X$. Hence the above data defines the MacPherson-Vilonen category $\mathsf{MV}(\xi)$. Similarly using the functors $F', G\colon \mathscr{B} \longrightarrow \mathscr{A}$, where $G$ is right exact and $F'$ is left exact, we define the natural transformation $\theta\colon G \longrightarrow F'$ by $\theta_Y = \delta_{G(Y)} \circ F'(\phi_Y)$ and as objects we take the tuples $(Y, X, g, \delta_X \circ F'(f))$. Then we have the MacPherson-Vilonen category $\mathsf{MV}(\theta)$. Finally the recollement situation of the categories $\mathsf{MV}(\xi)$ and $\mathsf{MV}(\theta)$ follows from the general construction of a MacPherson-Vilonen category. $\square$

Proposition 3.3.4. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories. Then there is a Morita category $\mathcal{M}(\phi,\psi)$ such that $(\mathscr{A},\mathscr{B},\mathscr{C})$ and $(\mathsf{Ker}\,\mathsf{U}_{\mathscr{C}},\mathcal{M}(\phi,\psi),\mathscr{C})$ are equivalent recollements.*

Proof. From the recollement $(\mathscr{A},\mathscr{B},\mathscr{C})$ we define the Morita category:

$$\mathcal{M}(\phi,\psi)=(\mathscr{B},\mathscr{C},\mathsf{e}:\mathscr{B}\longrightarrow\mathscr{C},\mathsf{l}\colon\mathscr{C}\longrightarrow\mathscr{B},\phi\colon\mathsf{el}\longrightarrow\mathrm{Id}_{\mathscr{C}},\psi\colon\mathsf{le}\longrightarrow\mathrm{Id}_{\mathscr{B}})$$

Note that the natural transformation $\phi$ is an isomorphism. Then from Proposition 3.1.13 there is an equivalence of categories between $\mathscr{B}$ and $\mathcal{M}(\phi,\psi)$ via the functor $\mathsf{T}_{\mathscr{B}}$. Since the following diagram:

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "\mathsf{e}"] \arrow[d, "\mathsf{T}_{\mathscr{B}}"', "\simeq"] & \mathscr{C} \arrow[d, "\mathrm{Id}_{\mathscr{C}}"] \\
\mathcal{M}(\phi,\psi) \arrow[r, "\mathsf{U}_{\mathscr{C}}"] & \mathscr{C}
\end{tikzcd}$$

commutes up to natural equivalence, see Definition 1.2.13, we infer that the recollements $(\mathscr{A},\mathscr{B},\mathscr{C})$ and $(\mathsf{Ker}\,\mathsf{U}_{\mathscr{C}},\mathcal{M}(\phi,\psi),\mathscr{C})$ are equivalent. $\square$

## 3.4. Functorially Finite Subcategories

In this section our purpose is to study finiteness conditions on subcategories of $\mathcal{M}(0,0)$. Throughout this section we assume that the right exact functors $F$ and $G$ have right adjoints $F'$ and $G'$ respectively. We denote by $(F\downarrow\mathrm{Id})=(F,\mathscr{A},\mathscr{B})$ and $(\mathrm{Id}\downarrow G)=(G,\mathscr{B},\mathscr{A})$ the comma categories, see Example 1.1.12, associated to the Morita category $\mathcal{M}(0,0)$. The reason for restricting to the case where $\phi=\psi=0$ is that we have full embeddings from the abelian categories $\mathscr{A}$, $\mathscr{B}$, $(F\downarrow\mathrm{Id})$ and $(\mathrm{Id}\downarrow G)$ to $\mathcal{M}(0,0)$. In particular we show that the above **natural** subcategories of $\mathcal{M}(0,0)$ are bireflective and therefore functorially finite.

We start by defining the following full subcategories of $\mathcal{M}(0,0)$:

$$\begin{cases}
\mathcal{X}=\big\{(X,Y,f,0)\mid f\colon F(X)\longrightarrow Y \text{ is an epimorphism}\big\}\\
\mathcal{Y}=\big\{(0,Y,0,0)\mid Y\in\mathscr{B}\big\}=\mathsf{Im}\,\mathsf{Z}_{\mathscr{B}}\\
\mathcal{Z}=\big\{(X,Y,0,g)\mid \rho(g)\colon Y\longrightarrow G'(X) \text{ is a monomorphism}\big\}\\
\mathcal{X}'=\big\{(X,Y,0,g)\mid g\colon G(Y)\longrightarrow X \text{ is an epimorphism}\big\}\\
\mathcal{Y}'=\big\{(X,0,0,0)\mid X\in\mathscr{A}\big\}=\mathsf{Im}\,\mathsf{Z}_{\mathscr{A}}\\
\mathcal{Z}'=\big\{(X,Y,f,0)\mid \pi(f)\colon X\longrightarrow F'(Y) \text{ is a monomorphism}\big\}
\end{cases}$$

We will show that the above subcategories have a special structure in $\mathcal{M}(0,0)$. In section 1.2 we discussed the notion of torsion pairs in abelian categories. Let us briefly recall this notion.

Let $\mathscr{B}$ be an abelian category. Then a **torsion pair** in $\mathscr{B}$ is a pair $(\mathcal{U},\mathcal{V})$ of strict full subcategories of $\mathscr{B}$ satisfying the following conditions:

(i) $\mathsf{Hom}_{\mathscr{B}}(\mathcal{U},\mathcal{V})=0$, i.e. $\mathsf{Hom}_{\mathscr{B}}(U,V)=0$ for all $U\in\mathcal{U}$ and $V\in\mathcal{V}$.

(ii) For every object $B\in\mathscr{B}$ there exists a short exact sequence $0\longrightarrow U_B\xrightarrow{f_B} B\xrightarrow{g^B} V^B\longrightarrow 0$ in $\mathscr{B}$ such that $U_B\in\mathcal{U}$ and $V^B\in\mathcal{V}$.

In that case, the torsion class $\mathcal{U}$ is is closed under factors, extensions and coproducts and the torsion-free class $\mathcal{V}$ is closed under subobjects, extensions and products. Moreover we have $\mathcal{U} = {}^{\perp}\mathcal{V} = \{B \in \mathscr{B} \mid \mathsf{Hom}_{\mathscr{B}}(B, \mathcal{V}) = 0\}$ and $\mathcal{V} = \mathcal{U}^{\perp} = \{B \in \mathscr{B} \mid \mathsf{Hom}_{\mathscr{B}}(\mathcal{U}, B) = 0\}$. A triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ of strict full subcategories of $\mathscr{B}$ is called torsion, torsion-free triple or TTF-triple for short, if $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ are torsion pairs.

Definition 3.4.1. [15] A full subcategory $\mathscr{C}$ is called contravariantly finite in $\mathscr{B}$ if for any object $B$ in $\mathscr{B}$ there exists a map $f_B : C_B \longrightarrow B$, where $C_B$ lies in $\mathscr{C}$, such that the induced map $\mathsf{Hom}_{\mathscr{B}}(\mathscr{C}, f_B) : \mathsf{Hom}_{\mathscr{B}}(\mathscr{C}, C_B) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(\mathscr{C}, B)$ is surjective. In this case the map $f_B$ is called a right $\mathscr{C}$-approximation of $B$. Dually we have the notions of covariant finiteness and left approximations. Then $\mathscr{C}$ is called functorially finite if it is both contravariantly and covariantly finite.

Example 3.4.2. Let $\Lambda$ be an Artin algebra. Then the full subcategory $\mathsf{Proj}\,\Lambda$ of projective $\Lambda$-modules and the full subcategory $\mathsf{Inj}\,\Lambda$ of injective $\Lambda$-modules are functorially finite in $\mathsf{Mod}\text{-}\Lambda$. Moreover, the finitely generated projective and injective $\Lambda$-modules, $\mathsf{proj}\,\Lambda$ and $\mathsf{inj}\,\Lambda$ respectively, are functorially finite in $\mathsf{mod}\text{-}\Lambda$. This follows from [30, Chapter X] since the categories of modules $\mathsf{Mod}\text{-}\Lambda$ and $\mathsf{mod}\text{-}\Lambda$ are Nakayama categories (i.e. there is an adjoint pair of functors inducing an equivalence between the projectives and injectives).

On the other hand since $\Lambda$ is an Artin algebra, it is left and right Artin ring. Thus $\Lambda$ is left and right Noetherian, but moreover is right perfect and left coherent. Then from [24, Theorems 5.3, 5.7] we infer that the full subcategories $\mathsf{Proj}\,\Lambda$ and $\mathsf{Inj}\,\Lambda$ are functorially finite.

In what follows we shall need the following easy and well known observation.

Lemma 3.4.3. *[17] Let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in $\mathscr{B}$. Then $\mathcal{U}$ is contravariantly finite in $\mathscr{B}$ and $\mathcal{V}$ is covariantly finite in $\mathscr{B}$.*

Proof. Let $B$ be an object of $\mathscr{B}$. Since $(\mathcal{U}, \mathcal{V})$ is a torsion pair in $\mathscr{B}$ we have the exact sequence $0 \longrightarrow U_B \longrightarrow B \longrightarrow V^B \longrightarrow 0$ with $U_B \in \mathcal{U}$ and $V^B \in \mathcal{V}$. Let $h : U' \longrightarrow B$ be a morphism in $\mathscr{B}$ with $U \in \mathcal{U}$. Then the composition $h \circ g^B = 0$ since $\mathsf{Hom}_{\mathscr{B}}(\mathcal{U}, \mathcal{V}) = 0$, and therefore there exists a morphism $u : U' \longrightarrow U_B$ such that $u \circ f_B = h$. This implies that $f_B : U_B \longrightarrow B$ is a right $\mathcal{U}$-approximation. Hence $\mathcal{U}$ is contravariantly finite and similarly we show that $\mathcal{V}$ is covariantly finite. $\square$

The following result gives the precise structure of the subcategories $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{X}', \mathcal{Y}'$, $\mathcal{Z}'$ in the morita category $\mathcal{M}(0, 0)$.

Proposition 3.4.4. *Let $\mathcal{M}(0, 0)$ be a Morita category. Then the triples $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ and $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$ are TTF-triples in $\mathcal{M}(0, 0)$.*

Proof. Let $(X, Y, f, g)$ be an object of $\mathcal{M}(0, 0)$. We write $f = \kappa \circ \lambda$ for the factorization of $f$ through the $\mathsf{Im}\, f$. Then from the counit of the adjoint pair $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}})$ we have the following exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & (0, \mathsf{Ker}\, f, 0, 0) \arrow[r] & \mathsf{T}_{\mathscr{A}}(X) \arrow[r, "(\mathrm{Id}_X, f)"] \arrow[d] & (X, Y, f, g) \arrow[r] & (0, \mathsf{Coker}\, f, 0, 0) \arrow[r] & 0 \\
 & & \mathsf{Im}\,(\mathrm{Id}_X, f) \arrow[ur] & & &
\end{tikzcd}$$

where $\mathsf{Im}\,(\mathrm{Id}_X, f) = (X, \mathsf{Im}\, f, \kappa, G(\lambda)\circ g)$. But from the following commutative diagram:

$$\begin{tikzcd}
GF(X) \arrow[r, "G(f)"] \arrow[d, "G(\kappa)"'] \arrow[rr, bend left, "\Psi_X=0"] & G(Y) \arrow[r, "g"] & X \\
G(\mathsf{Im}\, f) \arrow[ur, "G(\lambda)"'] & &
\end{tikzcd}$$

it follows that the map $G(\lambda)\circ g$ is zero since the map $G(\kappa)$ is an epimorphism. Therefore we have the exact sequence

$$0 \longrightarrow (X, \mathsf{Im}\, f, \kappa, 0) \longrightarrow (X, Y, f, g) \longrightarrow (0, \mathsf{Coker}\, f, 0, 0) \longrightarrow 0$$

with $(X, \mathsf{Im}\, f, \kappa, 0) \in \mathcal{X}$ and $(0, \mathsf{Coker}\, f, 0, 0) \in \mathcal{Y}$. Let $(X, Y, f, 0) \in \mathcal{X}$ and $(0, Y', 0, 0) \in \mathcal{Y}$. Then $\mathsf{Hom}_{\mathcal{M}(0,0)}((X, Y, f, 0), (0, Y', 0, 0)) = 0$ since the map $f\colon F(X) \longrightarrow Y$ is an epimorphism. Hence $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{M}(0, 0)$. We continue now in order to show that $(\mathcal{Y}, \mathcal{Z})$ is also a torsion pair. Let $(X, Y, f, g)$ be an object of $\mathcal{M}(0, 0)$. From the unit of the adjoint pair $(\mathsf{U}_{\mathscr{A}}, \mathsf{H}_{\mathscr{A}})$ we have the following exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & (0, \mathsf{Ker}\, \rho(g), 0, 0) \arrow[r] & (X, Y, f, g) \arrow[r, "(\mathrm{Id}_X, \rho(g))"] \arrow[d] & \mathsf{H}_{\mathscr{A}}(X) \arrow[r] & (0, \mathsf{Coker}\, \rho(g), 0, 0) \arrow[r] & 0 \\
& & \mathsf{Im}\,(\mathrm{Id}_X, \rho(g)) \arrow[ur] & & &
\end{tikzcd}$$

where $\rho(g) = \delta'_Y \circ G'(g)$. We write $\rho(g) = \gamma \circ n$ for the factorization through the $\mathsf{Im}\, \rho(g)$. The image of the morphism $(\mathrm{Id}_X, \rho(g))$ is the object

$$\mathsf{Im}\,(\mathrm{Id}_X, \rho(g)) = (X, \mathsf{Im}\, \rho(g), m, \rho^{-1}(n))$$

where $m = f \circ \gamma$ and the map $\rho(\rho^{-1}(n)) = n\colon \mathsf{Im}\, \rho(g) \longrightarrow G'(X)$ is a monomorphism. But then the map $m = f \circ \gamma = \pi^{-1}(\pi(f) \circ F'(\gamma) = 0$ since the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "\pi(f)"] \arrow[rr, bend left, "\Psi'_X=0"] & F'(Y) \arrow[r, "F'(\rho(g))"] \arrow[d, "F'(\gamma)"'] & F'G'(X) \\
& F'(\mathsf{Im}\, \rho(g)) \arrow[ur, "F'(n)"'] &
\end{tikzcd}$$

Thus we obtain the exact sequence

$$0 \longrightarrow (0, \mathsf{Ker}\, \rho(g), 0, 0) \longrightarrow (X, Y, f, g) \longrightarrow (X, \mathsf{Im}\, \rho(g), 0, \rho^{-1}(n)) \longrightarrow 0$$

in $\mathcal{M}(0, 0)$ with $(0, \mathsf{Ker}\, \rho(g), 0, 0) \in \mathcal{Y}$ and $(X, \mathsf{Im}\, \rho(g), 0, \rho^{-1}(n)) \in \mathcal{Z}$. Let $(0, b)\colon (0, Y', 0, 0) \longrightarrow (X, Y, 0, g)$ be a morphism in $\mathcal{M}(0, 0)$ with $(0, Y', 0, 0) \in \mathcal{Y}$ and $(X, Y, 0, g) \in \mathcal{Z}$. Then from the following commutative diagrams:

$$\begin{tikzcd}
G(Y') \arrow[r] \arrow[d, "G(b)"'] & 0 \arrow[d] \\
G(Y) \arrow[r, "g"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[r] \arrow[d, "b"'] & 0 \arrow[d] \\
Y \arrow[r, "\rho(g)"] & G'(X)
\end{tikzcd}$$

it follows that $b = 0$ since the map $\rho(g)$ is a monomorphism. Thus $\mathsf{Hom}_{\mathcal{M}(0,0)}(\mathcal{Y}, \mathcal{Z}) = 0$ and therefore we conclude that $(\mathcal{Y}, \mathcal{Z})$ is a torsion pair in $\mathcal{M}(0,0)$. Similarly we prove that $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$ is a $\mathsf{TTF}$-triple in $\mathcal{M}(0,0)$. $\square$

Remark 3.4.5. Let $(X, Y, f, g)$ be an object of the morita category $\mathcal{M}(0,0)$ with $f\colon F(X) \longrightarrow Y$ an epimorphism. Then the morphism $g\colon G(Y) \longrightarrow X$ is zero since the composition $GF(X) \longrightarrow G(Y) \longrightarrow X$ is zero and the map $G(f)$ is an epimorphism. We used this argument in the above proof and note that the same holds for the objects of $\mathcal{Z}$, $\mathcal{X}'$ and $\mathcal{Z}'$ as well.

As a consequence of Proposition 3.4.4 and Lemma 3.4.3 we have the following.

Corollary 3.4.6. *Let $\mathcal{M}(0,0)$ be a Morita category.*

(i) *The full subcategory*
$$\mathcal{X} = \{(X, Y, f, 0) \mid f\colon F(X) \longrightarrow Y \text{ is an epimorphism}\}$$
*is contravariantly finite in $\mathcal{M}(0,0)$, closed under extensions, quotients and coproducts, and $\mathsf{T}_{\mathscr{A}}(\mathscr{A}) \subseteq \mathcal{X}$.*

(ii) *The full subcategory*
$$\mathcal{X}' = \{(X, Y, 0, g) \mid g\colon G(Y) \longrightarrow X \text{ is an epimorphism}\}$$
*is contravariantly finite in $\mathcal{M}(0,0)$, closed under extensions, quotients and coproducts, and $\mathsf{T}_{\mathscr{B}}(\mathscr{B}) \subseteq \mathcal{X}'$.*

(iii) *The full subcategory*
$$\mathcal{Z} = \{(X, Y, 0, g) \mid \rho(g)\colon Y \longrightarrow G'(X) \text{ is a monomorphism}\}$$
*is covariantly finite in $\mathcal{M}(0,0)$, closed under extensions, subobjects and products, and $\mathsf{H}_{\mathscr{A}}(\mathscr{A}) \subseteq \mathcal{Z}$.*

(iv) *The full subcategory*
$$\mathcal{Z}' = \{(X, Y, f, 0) \mid \pi(f)\colon X \longrightarrow F'(Y) \text{ is a monomorphism}\}$$
*is covariantly finite in $\mathcal{M}(0,0)$, closed under extensions, subobjects and products, and $\mathsf{H}_{\mathscr{B}}(\mathscr{B}) \subseteq \mathcal{Z}'$.*

The next result describes the categories $\mathscr{A}$ and $\mathscr{B}$ via the subcategories $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{X}'$, $\mathcal{Z}'$.

Corollary 3.4.7. *Let $\mathcal{M}(0,0)$ be a Morita category. Then there are equivalences*
$$\mathscr{A} \xrightarrow{\simeq} \mathcal{X} \cap \mathcal{Z} \quad \text{and} \quad \mathscr{B} \xrightarrow{\simeq} \mathcal{X}' \cap \mathcal{Z}'$$

Proof. From Proposition 4.1.4 we have the recollement $(\mathscr{B}, \mathcal{M}(0,0), \mathscr{A})$ and so the quotient category $\mathcal{M}(0,0)/\mathscr{B}$ is equivalent to $\mathscr{A}$. From Proposition 3.4.4 we have the $\mathsf{TTF}$-triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in $\mathcal{M}(0,0)$. Then from [30, Proposition 1.3] it follows that $\mathcal{M}(0,0)/\mathcal{Y} \simeq \mathcal{X} \cap \mathcal{Z}$. But since we can identify $\mathcal{Y}$ with $\mathscr{B}$ we infer that $\mathscr{A}$ is equivalent with $\mathcal{X} \cap \mathcal{Z}$. Similarly using the $\mathsf{TTF}$-triple $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$ and the recollement $(\mathscr{A}, \mathcal{M}(0,0), \mathscr{B})$ we infer that $\mathscr{B}$ is equivalent to $\mathcal{X}' \cap \mathcal{Z}'$. $\square$

We denote by $\mathcal{X}_0$ the full subcategory of $\mathcal{M}(0,0)$ whose objects are the tuples $(X, Y, f, g)$ such that there is an exact sequence $0 \longrightarrow K_0 \longrightarrow \mathsf{T}_{\mathscr{A}}(P_0) \longrightarrow (X, Y, f, g) \longrightarrow 0$ with $P_0 \in \mathsf{Proj}\, A$. Similarly we define the subcategories $\mathcal{Y}_0 = \{(X, Y, f, g) \in \mathcal{M}(0,0) \mid \exists\ 0 \longrightarrow (X, Y, f, g) \longrightarrow \mathsf{H}_{\mathscr{A}}(I_0) \longrightarrow L_0 \longrightarrow 0 \text{ exact with } I_0 \in \mathsf{Inj}\, A\}$, $\mathcal{X}'_0 = \{(X, Y, f, g) \in \mathcal{M}(0,0) \mid \exists\ 0 \longrightarrow K_0 \longrightarrow \mathsf{T}_{\mathscr{B}}(Q_0) \longrightarrow (X, Y, f, g) \longrightarrow 0 \text{ exact with } Q_0 \in \mathsf{Proj}\, B\}$

and $\mathcal{Y}_0' = \{(X, Y, f, g) \in \mathcal{M}(0,0) \mid \exists\ 0 \longrightarrow (X, Y, f, g) \longrightarrow \mathsf{H}_{\mathscr{B}}(J_0) \longrightarrow L_0 \longrightarrow 0$ exact with $J_0 \in \mathsf{Inj}\, B\}$.

The reason for defining the above subcategories is the following result which provides another description of the subcategories $\mathcal{X}, \mathcal{Z}, \mathcal{X}'$ and $\mathcal{Z}'$.

PROPOSITION 3.4.8. *Let $\mathcal{M}(0,0)$ be a Morita category. Then: $\mathcal{X} = \mathcal{X}_0$, $\mathcal{Z} = \mathcal{Y}_0$, $\mathcal{X}' = \mathcal{X}_0'$ and $\mathcal{Z}' = \mathcal{Y}_0'$.*

PROOF. Let $(X, Y, f, 0) \in \mathcal{X}$ and let $a\colon P \longrightarrow X$ be an epimorphism with $P \in \mathsf{Proj}\, \mathscr{A}$. Then we have the morphism $(a, F(a) \circ f)\colon \mathsf{T}_{\mathscr{A}}(P) \longrightarrow (X, Y, f, 0)$ in $\mathcal{M}(0,0)$ which is an epimorphism since $f\colon F(X) \longrightarrow Y$ is an epimorphism. Hence $(X, Y, f, 0) \in \mathcal{X}_0$. Conversely, let $(X, Y, f, g) \in \mathcal{X}_0$. Since there exists an epimorphism $(a, b)\colon \mathsf{T}_{\mathscr{A}}(P) \to (X, Y, f, g)$ it follows that $b = F(a) \circ f$ and $b$ is an epimorphism. Hence the map $f$ is an epimorphism and from Remark 3.4.5 it follows that the map $g = 0$. Thus the object $(X, Y, f, 0) \in \mathcal{X}$. Therefore we have $\mathcal{X} = \mathcal{X}_0$ and similarly we prove the other statements. $\square$

For the next result we need to recall the notion of bireflective subcategories. A full subcategory $\mathscr{C}$ of an abelian category $\mathscr{B}$ is said to be **reflective** if the inclusion functor $\mathsf{i}\colon \mathscr{C} \longrightarrow \mathscr{B}$ has a left adjoint. Dually the subcategory $\mathscr{C}$ is called **coreflective** if $\mathsf{i}\colon \mathscr{C} \longrightarrow \mathscr{B}$ has a right adjoint. Then $\mathscr{C}$ is **bireflective** if it is both reflective and coreflective. We refer to [54], [55] and [56] for more information about bireflective subcategories.

The following gives the exact properties of the natural subcategories of $\mathcal{M}(0,0)$.

THEOREM 3.4.9. *Let $\mathcal{M}(0,0)$ be a Morita category. Then the full subcategories*

$$\mathscr{A},\ \mathscr{B},\ (F \downarrow \mathrm{Id}),\ (\mathrm{Id} \downarrow G)$$

*are bireflective in $\mathcal{M}(0,0)$. In particular the above subcategories are functorially finite in $\mathcal{M}(0,0)$, closed under isomorphic images, direct sums, direct products, kernels and cokernels.*

PROOF. The categories $\mathscr{A}$ and $\mathscr{B}$ are bireflective subcategories of $\mathcal{M}(0,0)$ since from Corollary 3.3.2 we have the recollements $(\mathscr{A}, \mathcal{M}(0,0), \mathscr{B})$ and $(\mathscr{B}, \mathcal{M}(0,0), \mathscr{A})$. Let $\mathcal{F}\colon (F \downarrow \mathrm{Id}) \longrightarrow \mathcal{M}(0,0)$ be the functor defined on the objects $(X, Y, f) \in (F \downarrow \mathrm{Id})$ by $\mathcal{F}(X, Y, f) = (X, Y, f, 0)$ and given a morphism $(a, b)\colon (X, Y, f) \longrightarrow (X', Y', f')$ in $(F \downarrow \mathrm{Id})$ then $\mathcal{F}(a, b) = (a, b)\colon (X, Y, f, 0) \longrightarrow (X', Y', f', 0)$ is a morphism in $\mathcal{M}(0,0)$. Clearly the functor $\mathcal{F}$ is fully faithful. We will show that $\mathcal{F}$ has a left and a right adjoint. Let $(X, Y, f, g)$ be an object of $\mathcal{M}(0,0)$. Since $\phi_Y = 0$ we have the following commutative diagram:

$$\begin{tikzcd}
FG(Y) \arrow[r, "F(g)"] \arrow[dr, "0=\phi_Y"'] & F(X) \arrow[r, "F(\pi_X)"] \arrow[d, "f"] & F(\mathsf{Coker}\, g) \arrow[r] \arrow[dl, dashed, "h"] & 0 \\
& Y & &
\end{tikzcd}$$

Then define the functor $\mathcal{H}\colon \mathcal{M}(0,0) \longrightarrow (F \downarrow \mathrm{Id})$ by $\mathcal{H}(X, Y, f, g) = (\mathsf{Coker}\, g, Y, h)$ on objects $(X, Y, f, g) \in \mathcal{M}(0,0)$ and if $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ is a morphism in $\mathcal{M}(0,0)$ then $\mathcal{H}(a, b) = (\xi, b)$ where $\xi\colon \mathsf{Coker}\, g \longrightarrow \mathsf{Coker}\, g'$ is the unique morphism which makes the following diagram commutative:

$$\begin{tikzcd} G(Y) \arrow[r, "g"] \arrow[d, "G(b)"'] & X \arrow[r, two heads, "\pi_X"] \arrow[d, "a"'] & \mathsf{Coker}\, g \arrow[d, "\xi"] \\ G(Y') \arrow[r, "g'"] & X' \arrow[r, two heads, "\pi_{X'}"] & \mathsf{Coker}\, g' \end{tikzcd}$$

Note that $\mathcal{H}(a,b) = (\xi, b)$ is a morphism in $(F \downarrow \mathrm{Id})$ since $F(\pi_X) \circ h \circ b = f \circ b = F(a) \circ f' = F(a) \circ F(\pi_{X'}) \circ h' = F(\pi_X) \circ F(\xi) \circ h'$ and $F(\pi_X)$ is an epimorphism. We claim that $(\mathcal{H}, \mathcal{F})$ is an adjoint pair of functors. Let $(a,b)\colon \mathcal{H}(X,Y,f,g) \longrightarrow (X',Y',f')$ be a morphism in $(F \downarrow \mathrm{Id})$. We define the map

$$\mathcal{S}\colon \mathsf{Hom}_{(F\downarrow\mathrm{Id})}(\mathcal{H}(X,Y,f,g),(X',Y',f')) \longrightarrow \mathsf{Hom}_{\mathcal{M}(0,0)}((X,Y,f,g),(X',Y',f',0))$$

by $\mathcal{S}((a,b)) = (\pi_X \circ a, b)$. It is easy to verify that $\mathcal{S}$ is a natural isomorphism of abelian groups. Thus $(\mathcal{H}, \mathcal{F})$ is an adjoint pair of functors. Finally the assignment $(X,Y,f,g) \mapsto (X, \mathsf{Ker}\,\rho(g), j)$ induces a well defined functor $\mathcal{K}\colon \mathcal{M}(0,0) \longrightarrow (F \downarrow \mathrm{Id})$ which is the right adjoint of $F$. Note that since $f \circ \rho(g) = 0$ we have the map $j\colon F(X) \longrightarrow \mathsf{Ker}\,\rho(g)$. The details are left to the reader. Then we have the adjoint triple $(\mathcal{H}, \mathcal{F}, \mathcal{S})$ and therefore we conclude that $(F \downarrow \mathrm{Id})$ is a bireflective subcategory of $\mathcal{M}(0,0)$. Similarly we show that $(\mathrm{Id} \downarrow G)$ is bireflective. Then from [54], [56] it follows that the full subcategories $\mathscr{A}, \mathscr{B}, (F \downarrow \mathrm{Id}), (\mathrm{Id} \downarrow G)$ are functorially finite in $\mathcal{M}(0,0)$, closed under isomorphic images, direct sums, direct products, kernels and cokernels. □

REMARK 3.4.10. (i) By Proposition 3.4.4 the categories $\mathscr{A}$ and $\mathscr{B}$ are TTF-classes. This implies also that they are bireflective in $\mathcal{M}(0,0)$.

(ii) Let $(X,Y,f,g)$ be an object of $\mathcal{M}(0,0)$. Then the map $(\pi_X, 0)\colon (X,Y,f,g) \longrightarrow (\mathsf{Coker}\, g, 0, 0, 0)$ is the unique left $\mathsf{Im}\,\mathsf{Z}_{\mathscr{A}}$-approximation. Also the morphism $(\pi_X, \mathrm{Id}_Y)\colon (X,Y,f,g) \longrightarrow (\mathsf{Coker}\, g, Y, h, 0)$ is the left $\mathsf{Im}\,\mathcal{F}$-approximation. Similarly we obtain the descriptions of the left approximations from $\mathscr{B}$ and $(\mathrm{Id} \downarrow G)$, and dually we derive the right approximations.

We continue with the following result on finiteness of subcategories.

THEOREM 3.4.11. *Let $\mathcal{M}(0,0)$ be a Morita category.*

(i) *Let $\mathcal{U}$ be a covariantly finite subcategory of $\mathscr{A}$ such that $\mathcal{U} \subseteq \mathsf{Ker}\, G'$ and $\mathcal{V}$ a covariantly finite subcategory of $\mathscr{B}$ such that $\mathcal{V} \subseteq \mathsf{Ker}\, G$. Then the full subcategory*

$$\mathcal{W} = \big\{(X,Y,f,g) \in \mathcal{M}(0,0) \mid X \in \mathcal{U} \ \ \textit{and} \ \ Y \in \mathcal{V}\big\}$$

*is covariantly finite in $\mathcal{M}(0,0)$.*

(ii) *Let $\mathcal{U}$ be a contravariantly finite subcategory of $\mathscr{A}$ such that $\mathcal{U} \subseteq \mathsf{Ker}\, G'$ and $\mathcal{V}$ a contravariantly finite subcategory of $\mathscr{B}$ such that $\mathcal{V} \subseteq \mathsf{Ker}\, G$. Then the full subcategory*

$$\mathcal{W} = \big\{(X,Y,f,g) \in \mathcal{M}(0,0) \mid X \in \mathcal{U} \ \ \textit{and} \ \ Y \in \mathcal{V}\big\}$$

*is contravariantly finite in $\mathcal{M}(0,0)$.*

(iii) *Let $\mathcal{U}$ be a functorially finite subcategory of $\mathscr{A}$ such that $\mathcal{U} \subseteq \mathsf{Ker}\, G'$ and $\mathcal{V}$ a functorially finite subcategory of $\mathscr{B}$ such that $\mathcal{V} \subseteq \mathsf{Ker}\, G$. Then the full subcategory*

$$\mathcal{W} = \big\{(X,Y,f,g) \in \mathcal{M}(0,0) \mid X \in \mathcal{U} \ \ \textit{and} \ \ Y \in \mathcal{V}\big\}$$

*is functorially finite in $\mathcal{M}(0,0)$.*

PROOF. (i) Let $(A_1, B_1, f_1, g_1)$ be an object of $\mathcal{M}(0,0)$ and let $m\colon A_1 \longrightarrow X_1$ be a left $\mathcal{U}$-approximation. From the morphisms $F(m)$ and $f_1$ we have the following pushout diagram:

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(m)"] \arrow[d, "f_1"'] & F(X_1) \arrow[d, "\rho"] \\
B_1 \arrow[r, "\theta"] & I
\end{tikzcd}$$

and let $n\colon I \longrightarrow Y_1$ be a left $\mathcal{V}$-approximation. Then we claim that the map

$$(A_1, B_1, f_1, g_1) \xrightarrow{(m, \theta \circ n)} (X_1, Y_1, \rho \circ n, 0)$$

is a left $\mathcal{W}$-approximation. First the object $(X_1, Y_1, \rho \circ n, 0) \in \mathcal{W}$ since it is an object of $\mathcal{M}(0,0)$ with $X_1 \in \mathcal{U}$ and $Y_1 \in \mathcal{V}$. Also from the above pushout diagram and since $\mathsf{Hom}_{\mathscr{A}}(G(B_1), X_1) \simeq \mathsf{Hom}_{\mathscr{B}}(B_1, G'(X_1)) = 0$ it follows that the following diagrams are commutative:

$$\begin{tikzcd}
F(A_1) \arrow[r, "f_1"] \arrow[d, "F(m)"'] & B_1 \arrow[d, "\theta\circ n"] \\
F(X_1) \arrow[r, "\rho\circ n"] & Y_1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G(B_1) \arrow[r, "g_1"] \arrow[d, "G(\theta\circ n)"'] & A_1 \arrow[d, "m"] \\
G(Y_1) \arrow[r, "0"] & X_1
\end{tikzcd}$$

Thus the map $(m, \theta \circ n)$ is a morphism in $\mathcal{M}(0,0)$. Let $(\alpha, \beta)\colon (A_1, B_1, f_1, g_1) \longrightarrow (X, Y, f, g)$ be a morphism in $\mathcal{M}(0,0)$ with $(X, Y, f, g) \in \mathcal{W}$. Since $m\colon A_1 \longrightarrow X_1$ is a left $\mathcal{U}$-approximation and $X \in \mathcal{U}$ there exists a map $\gamma\colon X_1 \longrightarrow X$ such that $m \circ \gamma = \alpha$. Morover since $f_1 \circ \beta = F(a) \circ f$ there exists a map $\mu\colon I \longrightarrow Y$ such that the following diagram:

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(m)"] \arrow[d, "f_1"'] & F(X_1) \arrow[d, "\rho"] \arrow[ddr, bend left, "F(\gamma)\circ f"] & \\
B_1 \arrow[r, "\theta"] \arrow[drr, bend right, "\beta"'] & I \arrow[dr, dashed, "\mu"] & \\
& & Y
\end{tikzcd}$$

is commutative. Then since the map $n\colon I \longrightarrow Y_1$ is a left $\mathcal{V}$-approximation there exists a morphism $\delta\colon Y_1 \longrightarrow Y$ such that $n \circ \delta = \mu$. This implies that the following diagram is commutative:

$$\begin{tikzcd}
(A_1, B_1, f_1, g_1) \arrow[r, "(m,\theta\circ n)"] \arrow[d, "(\alpha,\beta)"'] & (X_1, Y_1, \rho \circ n, 0) \arrow[dl, "(\gamma,\delta)"] \\
(X, Y, f, g) &
\end{tikzcd}$$

Since $\rho \circ n \circ \delta = \rho \circ \mu = F(\gamma) \circ f$ and $G(Y_1) = 0$ we infer that $(\gamma, \delta)$ is a morphism in $\mathcal{M}(0,0)$ and therefore we have proved our claim. Part (ii) is dual to (i) and (iii) follows from (i) and (ii). □

REMARK 3.4.12. Note that the converse of Theorem 3.4.11 holds, i.e. if $\mathcal{W}$ is contravariantly (resp. covariantly) finite in $\mathcal{M}(0,0)$ then $\mathcal{U}$ is contravariantly (resp. covariantly) finite in $\mathscr{A}$ and $\mathcal{V}$ is contravariantly (resp. covariantly) finite in $\mathscr{B}$. Let $X$ be an object of $\mathscr{A}$. Then the object $\mathsf{Z}_{\mathscr{A}}(X)$ lies in $\mathcal{M}(0,0)$ and since $\mathcal{W}$ is contravariantly finite there exists a morphism $(a,0)\colon (X',Y',f',g') \longrightarrow (X,0,0,0)$ with $(X',Y',f',g') \in \mathcal{W}$. We claim that the morphism $a\colon X' \longrightarrow X$ is a right $\mathcal{U}$-approximation of $X$ in $\mathscr{A}$. First $X' \in \mathcal{U}$ and let $k\colon X'' \longrightarrow X$ be a morphism in $\mathscr{A}$ with $X'' \in \mathcal{U}$. Then since $(a,0)$ is a right $\mathcal{W}$-approximation and $(X'',0,0,0) \in \mathcal{W}$ we have the following commutative diagram:

$$\begin{tikzcd} & (X'',0,0,0) \arrow[dl, "(\lambda,0)"'] \arrow[d, "(k,0)"] \\ (X',Y',f',g') \arrow[r, "(a,0)"] & (X,0,0,0) \end{tikzcd}$$

This implies that $\lambda \circ a = k$ and then our claim follows. Hence $\mathcal{U}$ is contravariantly finite in $\mathscr{A}$ and similarly we show that $\mathcal{V}$ is contravariantly finite in $\mathscr{B}$.

In section 4.3 of Chapter 4 we will discuss applications of the above results to Morita rings.

## 3.5. Projective, Injective and Simple Objects

Let $\mathcal{M}(\phi,\psi)$ be a Morita category of the abelian categories $\mathscr{A}$ and $\mathscr{B}$ by the natural transformations $\phi$ and $\psi$. We always assume that the functors $F$ and $G$ are right exact in order that $\mathcal{M}(\phi,\psi)$ is abelian. Our aim in this section is to determine the projective, injective and simple objects of $\mathcal{M}(\phi,\psi)$.

**3.5.1. Projective and Injective Objects.** We begin with the following result which characterizes the projective objects of $\mathcal{M}(\phi,\psi)$.

Proposition 3.5.1. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category.*

(i) *Let $P$ be a projective object of $\mathscr{A}$. Then $\mathsf{T}_{\mathscr{A}}(P)$ is a projective object of $\mathcal{M}(\phi,\psi)$.*
(ii) *Let $Q$ be a projective object of $\mathscr{B}$. Then $\mathsf{T}_{\mathscr{B}}(Q)$ is a projective object of $\mathcal{M}(\phi,\psi)$.*
(iii) *Let $(X,Y,f,g)$ be a projective object of $\mathcal{M}(\phi,\psi)$. Then $\mathsf{C}_{\mathscr{A}}(X,Y,f,g) \in \mathsf{Proj}\,(\mathsf{Ker}\,\psi)$ and $\mathsf{C}_{\mathscr{B}}(X,Y,f,g) \in \mathsf{Proj}\,(\mathsf{Ker}\,\phi)$.*
(iv) *Let $(X,Y,f,g)$ be an object of $\mathcal{M}(\phi,\psi)$. Then $(X,Y,f,g) \in \mathsf{Proj}\,(\mathcal{M}(\phi,\psi))$ if and only if $(X,Y,f,g)$ is a direct summand of $\mathsf{T}_{\mathscr{A}}(P) \oplus \mathsf{T}_{\mathscr{B}}(Q)$ where $P$ is a projective object in $\mathscr{A}$ and $Q$ is a projective object in $\mathscr{B}$.*

Proof. The statements (i), (ii) and (iii) follow from Proposition 3.1.11 since we have the adjoint pairs $(\mathsf{T}_{\mathscr{A}},\mathsf{U}_{\mathscr{A}})$, $(\mathsf{T}_{\mathscr{B}},\mathsf{U}_{\mathscr{B}})$, $(\mathsf{C}_{\mathscr{A}},\mathsf{Z}_{\mathscr{A}})$, $(\mathsf{C}_{\mathscr{B}},\mathsf{Z}_{\mathscr{B}})$ and the functors $\mathsf{U}_{\mathscr{A}}$, $\mathsf{U}_{\mathscr{B}}$, $\mathsf{Z}_{\mathscr{A}}$, $\mathsf{Z}_{\mathscr{B}}$ are exact.

(iv) Let $(X,Y,f,g)$ be an object of $\mathcal{M}(\phi,\psi)$. Let $a\colon P \longrightarrow X$ be an epimorphism in $\mathscr{A}$ with $P \in \mathsf{Proj}\,\mathscr{A}$ and $b\colon Q \longrightarrow Y$ be an epimorphism in $\mathscr{B}$ with $Q \in \mathsf{Proj}\,\mathscr{B}$. Then the map $(a,F(a))\colon \mathsf{T}_{\mathscr{A}}(P) \longrightarrow \mathsf{T}_{\mathscr{A}}(X)$ is an epimorphism with $\mathsf{T}_{\mathscr{A}}(P) \in \mathsf{Proj}\,(\mathcal{M}(\phi,\psi))$, and the map $(G(b),b)\colon \mathsf{T}_{\mathscr{B}}(Q) \longrightarrow \mathsf{T}_{\mathscr{B}}(Y)$ is an epimorphism with $\mathsf{T}_{\mathscr{B}}(Q) \in \mathsf{Proj}\,(\mathcal{M}(\phi,\psi))$. Then we have the following commutative diagram:

$$\begin{tikzcd}[column sep=huge] \mathsf{T}_{\mathscr{A}}(P) \oplus \mathsf{T}_{\mathscr{B}}(Q) \arrow[r, two heads, "\left(\begin{smallmatrix} (a,F(a)) & 0 \\ 0 & (G(b),b) \end{smallmatrix}\right)"] \arrow[dr, dashed] & \mathsf{T}_{\mathscr{A}}(X) \oplus \mathsf{T}_{\mathscr{B}}(Y) \arrow[d, "\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)"] \\ & (X,Y,f,g) \end{tikzcd}$$

where $\mathsf{T}_{\mathscr{A}}(X)\oplus\mathsf{T}_{\mathscr{B}}(Y)\simeq\big(X\oplus G(Y),F(X)\oplus Y,\left(\begin{smallmatrix}\mathrm{Id}_{F(X)} & 0\\ 0 & \phi_Y\end{smallmatrix}\right),\left(\begin{smallmatrix}\psi_X & 0\\ 0 & \mathrm{Id}_{G(Y)}\end{smallmatrix}\right)\big)$ and the map ${}^t(c,d)\colon \mathsf{T}_{\mathscr{A}}(X)\oplus\mathsf{T}_{\mathscr{B}}(Y)\longrightarrow(X,Y,f,g)$ is an epimorphism since $c={}^t(\mathrm{Id}_X,g)\colon X\oplus G(Y)\longrightarrow X$ and $d={}^t(f,\mathrm{Id}_Y)\colon F(X)\oplus Y\longrightarrow Y$ are epimorphisms. Hence there is an epimorphism $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)\longrightarrow(X,Y,f,g)$ given by the composition $\left(\begin{smallmatrix}(a,F(a)) & 0\\ 0 & (G(b),b)\end{smallmatrix}\right)\circ\left(\begin{smallmatrix}c\\ d\end{smallmatrix}\right)$ and $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)$ is projective in $\mathcal{M}(\phi,\psi)$. Thus if $(X,Y,f,g)$ is projective then the map $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)\longrightarrow(X,Y,f,g)$ splits and so $(X,Y,f,g)$ is a direct summand of $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)$. Conversely if $(X,Y,f,g)$ is a direct summand of $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)$, where $P\in\mathsf{Proj}\,\mathscr{A}$ and $Q\in\mathsf{Proj}\,\mathscr{B}$, then $(X,Y,f,g)$ is a projective object of $\mathcal{M}(\phi,\psi)$ since $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)\in\mathsf{Proj}\,(\mathcal{M}(\phi,\psi))$. □

As a consequence we have the following.

Corollary 3.5.2. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category. If $\mathscr{A}$ and $\mathscr{B}$ have enough projective objects then $\mathcal{M}(\phi,\psi)$ has enough projectives.*

We state without proofs the dual results of Proposition 3.5.1 and Corollary 3.5.2. We use the notation introduced in section 3.2.

Proposition 3.5.3. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category such that the functors $F$ and $G$ have right adjoints.*

(i) *Let $I$ be an injective object of $\mathscr{A}$. Then $\mathsf{H}_{\mathscr{A}}(I)$ is an injective object of $\mathcal{M}(\phi,\psi)$.*
(ii) *Let $J$ be a injective object of $\mathscr{B}$. Then $\mathsf{H}_{\mathscr{B}}(J)$ is an injective object of $\mathcal{M}(\phi,\psi)$.*
(iii) *Let $(X,Y,f,g)$ be an injective object of $\mathcal{M}(\phi,\psi)$. Then $\mathsf{C}'_{\mathscr{A}}(X,Y,f,g)\in\mathsf{Inj}\,(\mathsf{Ker}\,\psi)$ and $\mathsf{C}'_{\mathscr{B}}(X,Y,f,g)\in\mathsf{Inj}\,(\mathsf{Ker}\,\phi)$.*
(iv) *Let $(X,Y,f,g)$ be an object of $\mathcal{M}(\phi,\psi)$. Then $(X,Y,f,g)\in\mathsf{Inj}\,(\mathcal{M}(\phi,\psi))$ if and only if $(X,Y,f,g)$ is a direct summand of $\mathsf{H}_{\mathscr{A}}(I)\oplus\mathsf{H}_{\mathscr{B}}(J)$ where $I$ is an injective object of $\mathscr{A}$ and $J$ is an injective object of $\mathscr{B}$.*

Corollary 3.5.4. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category such that the functors $F$ and $G$ have right adjoints. If $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects then $\mathcal{M}(\phi,\psi)$ has enough injectives.*

**3.5.2. Simple Objects.** In this subsection we determine the simple objects of $\mathcal{M}(\phi,\psi)$. Recall that an object $(X,Y,f,g)$ of $\mathcal{M}(\phi,\psi)$ is simple if it has no proper nonzero subobject. Note that the description of the simple objects follows from the recollement diagram of Proposition 3.3.1 and is due to Kuhn, see [81,82] for a general result about simple objects in recollements of abelian categories. For completeness we sketch the proof in our case.

Let $X$ be an object of $\mathscr{A}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathsf{T}_{\mathscr{A}}(X) \arrow[rr, "{(\mathrm{Id}_X,\mu_X)}"] \arrow[dr, two heads] & & \mathsf{H}_{\mathscr{A}}(X) \\
& \mathsf{C}_{\mathscr{A}}(X) \arrow[ur, hook] &
\end{tikzcd}$$

where $\mathsf{C}_{\mathscr{A}}(X)=\mathsf{Im}\,(\mathrm{Id}_X,\mu_X)=(X,\mathsf{Im}\,\mu_X,\kappa,\lambda)$ and $\mu_X=\delta'_{F(X)}\circ G'(\psi_X)$. Then the assigment $X\mapsto\mathsf{Im}\,(\mathrm{Id}_X,\mu_X)$ defines a functor $\mathsf{C}_{\mathscr{A}}\colon\mathscr{A}\longrightarrow\mathcal{M}(\phi,\psi)$ on objects and given a morphism $a\colon X\longrightarrow X'$ in $\mathscr{A}$ then $\mathsf{C}_{\mathscr{A}}(a)=(a,\theta)$, where $\theta\colon\mathsf{Im}\,\mu_X\longrightarrow\mathsf{Im}\,\mu_{X'}$ is the unique morphism which makes the following diagram commutative:

$$\begin{tikzcd}
F(X) \arrow[rr, "\mu_X"] \arrow[dd, "F(a)"'] \arrow[dr, two heads] & & G'(X) \arrow[dd, "G'(a)"] \\
& \mathsf{Im}\,\mu_X \arrow[ur, tail] \arrow[dd, "\theta"] & \\
F(X') \arrow[rr, "\mu_{X'}"] \arrow[dr, two heads] & & G'(X') \\
& \mathsf{Im}\,\mu_{X'} \arrow[ur, tail] &
\end{tikzcd}$$

From the above diagram it follows that the functor $\mathsf{C}_{\mathscr{A}} \colon \mathscr{A} \longrightarrow \mathcal{M}(\phi,\psi)$ preserves epimorphisms and monomorphisms since $F$ is right exact and $G'$ is left exact.

We have the following result which shows that the functor $\mathsf{C}_{\mathscr{A}}$ lifts simple modules.

Lemma 3.5.5. *Let $S$ be a simple object of $\mathscr{A}$. Then $\mathsf{C}_{\mathscr{A}}(S)$ is a simple object of $\mathcal{M}(\phi,\psi)$.*

Proof. Consider the following exact commutative diagram:

$$\begin{tikzcd}
& & \mathsf{T}_{\mathscr{A}}(S) \arrow[r] \arrow[d, two heads] & \mathsf{H}_{\mathscr{A}}(S) & \\
0 \arrow[r] & (X,Y,f,g) \arrow[r] & \mathsf{C}_{\mathscr{A}}(S) \arrow[r] \arrow[ur, tail] & (X',Y',f',g') \arrow[r] & 0
\end{tikzcd}$$

We claim that either $(X,Y,f,g) = 0$ or $(X',Y',f',g') = 0$. If we apply the exact functor $\mathsf{U}_{\mathscr{A}}$ we get the short exact sequence $0 \longrightarrow X \longrightarrow S \longrightarrow X' \longrightarrow 0$ and so $X = 0$ or $X' = 0$ since $S$ is simple. Hence $(X,Y,f,g) \in \mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$ or $(X',Y',f',g') \in \mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}$. Consider the subobject $(0,Y,0,0)$ of $\mathsf{C}_{\mathscr{A}}(S)$. Then the object $(0,Y,0,0)$ is also a subobject of $\mathsf{H}_{\mathscr{A}}(S)$ and $\mathsf{Hom}_{\mathcal{M}(\phi,\psi)}((0,Y,0,0),\mathsf{H}_{\mathscr{A}}(S)) \simeq \mathsf{Hom}_{\mathscr{A}}(\mathsf{U}_{\mathscr{A}}(0,Y,0,0),S) = 0$. This implies that $\mathsf{C}_{\mathscr{A}}(S)$ has no nonzero subobjects. Similarly if $(0,Y',0,0)$ is the quotient object of $\mathsf{C}_{\mathscr{A}}(S)$ then $(0,Y',0,0)$ is also a quotient object of $\mathsf{T}_{\mathscr{A}}(S)$. But $\mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(S),(0,Y',0,0)) \simeq \mathsf{Hom}_{\mathscr{A}}(S,\mathsf{U}_{\mathscr{A}}(0,Y',0,0)) = 0$ and then $\mathsf{C}_{\mathscr{A}}(S)$ has no nonzero quotient object. We infer that $(X,Y,f,g) = 0$ or $(X',Y',f',g') = 0$ and therefore the object $\mathsf{C}_{\mathscr{A}}(S)$ is simple. □

Lemma 3.5.6. *Let $(X,Y,f,g)$ be a simple object of $\mathcal{M}(\phi,\psi)$ such that $\mathsf{U}_{\mathscr{A}}(X,Y,f,g) = X \neq 0$. Then $X$ is a simple object of $\mathscr{A}$ and $\mathsf{C}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}}(X,Y,f,g) \simeq (X,Y,f,g)$.*

Proof. From the following commutative diagram:

$$\begin{tikzcd}
\mathsf{T}_{\mathscr{A}}(X) \arrow[r, "(\mathrm{Id}_X, f)"] \arrow[d, two heads] \arrow[dr] & (X,Y,f,g) \arrow[d, "(\mathrm{Id}_X, \delta'_Y \circ G'(g))"] \\
\mathsf{C}_A\mathsf{U}_{\mathscr{A}}(X,Y,f,g) \arrow[r, tail] & \mathsf{H}_{\mathscr{A}}(X)
\end{tikzcd}$$

we deduce that the map $\mathsf{C}_{\mathscr{A}}(X) \longrightarrow \mathsf{Im}\,(\mathrm{Id}_X, \delta'_Y \circ G'(g))$ is a monomorphism. But then since $(X,Y,f,g)$ is simple and $\mathsf{C}_{\mathscr{A}}(X) \neq 0$ it follows that

$$\mathsf{C}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}}(X,Y,f,g) \simeq (X,Y,f,g)$$

Let $X'$ be a subobject of $X$ and let $i \colon X' \longrightarrow X$ be the inclusion map. Then since the functor $\mathsf{C}_{\mathscr{A}}$ preserves monomorphisms we have the monomorphism $\mathsf{C}_{\mathscr{A}}(i) \colon \mathsf{C}_{\mathscr{A}}(X') \longrightarrow$

$\mathsf{C}_{\mathscr{A}}\mathsf{U}_{\mathscr{A}}(X,Y,f,g)$ and therefore $\mathsf{C}_{\mathscr{A}}(X')=0$ or $\mathsf{C}_{\mathscr{A}}(X')\simeq(X,Y,f,g)$. Thus if we apply the functor $\mathsf{U}_{\mathscr{A}}$ we get $X'=0$ or $X'\simeq X$ and therefore we conclude that the object $\mathsf{U}_{\mathscr{A}}(X,Y,f,g)=X$ is simple. $\square$

The following result describes the simple objects of $\mathcal{M}(\phi,\psi)$ and follows easily from Lemma 3.5.5 and Lemma 3.5.6.

PROPOSITION 3.5.7. *There is the following bijections*:

$$\{\textit{simple objects of } \mathsf{Ker}\,\mathsf{U}_{\mathscr{A}}\} \underset{\mathsf{i}_{\mathscr{A}}}{\overset{\mathrm{Id}}{\leftrightarrows}} \{\textit{simple objects } (X,Y,f,g)\in\mathcal{M}(\phi,\psi)\colon\ X=0\}$$

$$\{\textit{simple objects of } \mathscr{A}\} \underset{\mathsf{C}_{\mathscr{A}}}{\overset{\mathsf{U}_{\mathscr{A}}}{\leftrightarrows}} \{\textit{simple objects } (X,Y,f,g)\in\mathcal{M}(\phi,\psi)\colon\ X\neq0\}$$

$$\{\textit{simple objects of } \mathsf{Ker}\,\mathsf{U}_{\mathscr{B}}\} \underset{\mathsf{i}_{\mathscr{B}}}{\overset{\mathrm{Id}}{\leftrightarrows}} \{\textit{simple objects } (X,Y,f,g)\in\mathcal{M}(\phi,\psi)\colon\ Y=0\}$$

$$\{\textit{simple objects of } \mathscr{B}\} \underset{\mathsf{C}_{\mathscr{B}}}{\overset{\mathsf{U}_{\mathscr{B}}}{\leftrightarrows}} \{\textit{simple objects } (X,Y,f,g)\in\mathcal{M}(\phi,\psi)\colon\ Y\neq0\}$$

REMARK 3.5.8. Let $(0,Y,0,0)$ be a simple object of $\mathcal{M}(\phi,\psi)$. Thus $Y\in\mathscr{B}$ with $\phi_Y=0$ and we claim that $\mathsf{U}_{\mathscr{B}}(0,Y,0,0)=Y$ is a simple object of $\mathscr{B}$. Let $Y'$ be a non-zero subobject of $Y$. Then we have the inclusion $i\colon Y'\hookrightarrow Y$ and from the commutative diagram

$$\begin{tikzcd} FG(Y') \arrow[r, "FG(i)"] \arrow[d, "\phi_{Y'}"'] & FG(Y) \arrow[d, "\phi_Y=0"] \\ Y' \arrow[r, tail, "i"] & Y \end{tikzcd}$$

we obtain that $\phi_{Y'}=0$. Hence we have the monomorphism $(0,i)\colon(0,Y',0,0)\longrightarrow(0,Y,0,0)$ where $(0,Y',0,0)\neq0$ and $(0,Y,0,0)$ is simple. Then $(0,Y',0,0)\simeq(0,Y,0,0)$ and so $Y'\simeq Y$, i.e. $Y$ is a simple object of $\mathscr{B}$. Conversely if we start with a simple object $Y\in\mathscr{B}$ with $\phi_Y=0$ then it follows easily that the object $\mathsf{Z}_{\mathscr{B}}(Y)=(0,Y,0,0)$ is simple. Hence we deduce a bijection between simple objects $Y$ of $\mathscr{B}$ with $\phi_Y=0$ and simple objects $(X,Y,f,g)$ of $\mathcal{M}(\phi,\psi)$ such that $\mathsf{U}_{\mathscr{A}}(X,Y,f,g)=0$. Similarly we derive a bijection between simple objects $X$ of $\mathscr{A}$ with $\psi_X=0$ and simple objects $(X,Y,f,g)$ of $\mathcal{M}(\phi,\psi)$ such that $\mathsf{U}_{\mathscr{B}}(X,Y,f,g)=0$.

## 3.6. Constructing Resolutions

Let $(X,Y,f,g)$ be an arbitrary object of $\mathcal{M}(\phi,\psi)$. Our aim in this section is to construct a projective resolution of $(X,Y,f,g)$ in terms of a projective resolution of $X$ in $\mathscr{A}$ and a projective resolution of $Y$ in $\mathscr{B}$. In this section we assume that the abelian categories $\mathscr{A}$ and $\mathscr{B}$ have enough projectives.

CONSTRUCTION 3.6.1. *Let $(X,Y,f_0,g_0)$ be an object of $\mathcal{M}(\phi,\psi)$. Let $a_0\colon P_0\longrightarrow X$ be an epimorphism with $P_0$ projective in $\mathscr{A}$ and $b_0\colon Q_0\longrightarrow Y$ an epimorphism with $Q_0$*

*projective in $\mathscr{B}$. Then we have the following pullback diagrams:*

$$\begin{tikzcd}
K_0 \arrow[r, "\nu_0"] \arrow[d, "\mu_0"'] & G(Q_0) \arrow[d, "G(b_0)\circ g_0"] \\
P_0 \arrow[r, "a_0"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
L_0 \arrow[r, "\xi_0"] \arrow[d, "\zeta_0"'] & Q_0 \arrow[d, "b_0"] \\
F(P_0) \arrow[r, "F(a_0)\circ f_0"] & Y
\end{tikzcd}$$

*and the following exact sequences:*

$$0 \longrightarrow K_0 \xrightarrow{\begin{pmatrix}\mu_0 & \nu_0\end{pmatrix}} P_0 \oplus G(Q_0) \xrightarrow{\begin{pmatrix} a_0 \\ G(b_0)\circ g_0\end{pmatrix}} X \longrightarrow 0$$

$$0 \longrightarrow L_0 \xrightarrow{\begin{pmatrix}\zeta_0 & \xi_0\end{pmatrix}} F(P_0) \oplus Q_0 \xrightarrow{\begin{pmatrix} F(a_0)\circ f_0 \\ b_0\end{pmatrix}} Y \longrightarrow 0$$

*Since* $G(\xi_0)\circ G(b_0)\circ g_0 = G(\zeta_0)\circ GF(a_0)\circ G(f_0)\circ g_0 = G(\zeta_0)\circ GF(a_0)\circ\psi_X = G(\zeta_0)\circ\psi_{P_0}\circ a_0$ *there exist a unique morphism* $g_1\colon G(L_0) \longrightarrow K_0$ *such that* $G(\zeta_0)\circ\psi_{P_0} = g_1\circ\mu_0$ *and* $g_1\circ\nu_0 = G(\xi_0)$, *that is the following diagram is commutative:*

$$\begin{tikzcd}
G(L_0) \arrow[dr, dotted, "g_1"] \arrow[drr, bend left, "G(\xi_0)"] \arrow[ddr, bend right, "G(\zeta_0)\circ\psi_{P_0}"'] & & \\
& K_0 \arrow[r, "\nu_0"] \arrow[d, "\mu_0"] & G(Q_0) \arrow[d, "G(b_0)\circ g_0"] \\
& P_0 \arrow[r, "a_0"] & X
\end{tikzcd}$$

*Also since* $F(\mu_0)\circ F(a_0)\circ f_0 = F(\nu_0)\circ FG(b_0)\circ F(g_0)\circ f_0 = F(\nu_0)\circ FG(b_0)\circ\phi_Y = F(\nu_0)\circ \phi_{Q_0}\circ b_0$ *there exist a unique morphism* $f_1\colon F(K_0) \longrightarrow L_0$ *such that* $F(\nu_0)\circ\phi_{Q_0} = f_1\circ\xi_0$ *and* $f_1\circ\zeta_0 = F(\mu_0)$, *that is the following diagram is commutative:*

$$\begin{tikzcd}
F(K_0) \arrow[dr, dotted, "f_1"] \arrow[drr, bend left, "F(\nu_0)\circ\phi_{Q_0}"] \arrow[ddr, bend right, "F(\mu_0)"'] & & \\
& L_0 \arrow[r, "\xi_0"] \arrow[d, "\zeta_0"] & Q_0 \arrow[d, "b_0"] \\
& F(P_0) \arrow[r, "F(a_0)\circ f_0"] & X
\end{tikzcd}$$

*Using the above relations we will show that* $(K_0, L_0, f_1, g_1)$ *is an object of* $\mathcal{M}(\phi,\psi)$. *This means that we have to prove that the following diagrams are commutative:*

$$\begin{tikzcd}
GF(K_0) \arrow[r, "G(f_1)"] \arrow[d, "\psi_{K_0}"'] & G(L_0) \arrow[dl, "g_1"] \\
K_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
FG(L_0) \arrow[r, "F(g_1)"] \arrow[d, "\phi_{L_0}"'] & F(K_0) \arrow[dl, "f_1"] \\
L_0 &
\end{tikzcd}$$

*We have*

$$\begin{aligned}
G(f_1)\circ g_1\circ(\mu_0,\nu_0) &= (G(f_1)\circ g_1\circ\mu_0, G(f_1)\circ g_1\circ\nu_0)\\
&= (G(f_1)\circ G(\zeta_0)\circ\psi_{P_0}, G(f_1)\circ G(\xi_0))\\
&= (GF(\mu_0)\circ\psi_{P_0}, GF(\nu_0)\circ G(\phi_{Q_0}))\\
&= (\psi_{K_0}\circ\mu_0, \psi_{K_0}\circ\nu_0)\\
&= \psi_{K_0}\circ(\mu_0,\nu_0)
\end{aligned}$$

*and so* $G(f_1)\circ g_1=\psi_{K_0}$ *since* $(\mu_0,\nu_0)$ *is a monomorphism. Similarly we have*

$$\begin{aligned}
F(g_1)\circ f_1\circ(\zeta_0,\xi_0) &= (F(g_1)\circ f_1\circ\zeta_0, F(g_1)\circ f_1\circ\xi_0)\\
&= (F(g_1)\circ F(\mu_0), F(g_1)\circ F(\nu_0)\circ\phi_{Q_0})\\
&= (FG(\zeta_0)\circ F(\psi_{P_0}), FG(\xi_0)\circ\phi_{Q_0})\\
&= (\phi_{L_0}\circ\zeta_0, \phi_{L_0}\circ\xi_0)\\
&= \phi_{L_0}\circ(\zeta_0,\xi_0)
\end{aligned}$$

*and therefore* $F(g_1)\circ f_1=\phi_{L_0}$ *since* $(\zeta_0,\xi_0)$ *is a monomorphism. Hence the tuple* $(K_0,L_0,f_1,g_1)$ *lies in* $\mathcal{M}(\phi,\psi)$*. Then we have the exact sequence*:

$$0\longrightarrow(K_0,L_0,f_1,g_1)\xrightarrow{\iota_0}\mathsf{T}_{\mathscr{A}}(P_0)\oplus\mathsf{T}_{\mathscr{B}}(Q_0)\xrightarrow{\alpha_0}(X,Y,f,g)\longrightarrow 0 \tag{3.6.1}$$

*where the morphisms are*

$$\iota_0=\left(\begin{pmatrix}\mu_0 & \nu_0\end{pmatrix},\begin{pmatrix}\zeta_0 & \xi_0\end{pmatrix}\right)\qquad \alpha_0=\left(\begin{pmatrix}a_0\\ G(b_0)\circ g_0\end{pmatrix},\begin{pmatrix}F(a_0)\circ f_0\\ b_0\end{pmatrix}\right)$$

*and we view the direct sum* $\mathsf{T}_{\mathscr{A}}(P_0)\oplus\mathsf{T}_{\mathscr{B}}(Q_0)$ *as the object*:

$$\mathsf{T}_{\mathscr{A}}(P_0)\oplus\mathsf{T}_{\mathscr{B}}(Q_0)\simeq\left(P_0\oplus G(Q_0), F(P_0)\oplus Q_0,\begin{pmatrix}\mathrm{Id}_{F(P_0)} & 0\\ 0 & \phi_{Q_0}\end{pmatrix},\begin{pmatrix}\psi_{P_0} & 0\\ 0 & \mathrm{Id}_{G(Q_0)}\end{pmatrix}\right)$$

*Now we continue the construction for the object* $(K_0,L_0,f_1,g_1)$*. Let* $a_1\colon P_1\longrightarrow K_0$ *be an epimorphism with* $P_1$ *projective in* $\mathscr{A}$ *and* $b_1\colon Q_1\longrightarrow L_0$ *be an epimorphism with* $Q_1$ *projective in* $\mathscr{B}$*. Then we have the following pullback diagrams*:

$$\begin{tikzcd}
K_1 \arrow[r, "\nu_1"] \arrow[d, "\mu_1"'] & G(Q_1) \arrow[d, "G(b_1)\circ g_1"] & \\
P_1 \arrow[r, "a_1"] & K_0 \arrow[r, "\nu_0"] \arrow[d, "\mu_0"'] & G(Q_0) \arrow[d, "G(b_0)\circ g_0"] \\
& P_0 \arrow[r, "a_0"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
L_1 \arrow[r, "\xi_1"] \arrow[d, "\zeta_1"'] & Q_1 \arrow[d, "b_1"] & \\
F(P_1) \arrow[r, "F(a_1)\circ f_1"] & L_0 \arrow[r, "\xi_0"] \arrow[d, "\zeta_0"'] & Q_0 \arrow[d, "b_0"] \\
& F(P_0) \arrow[r, "F(a_0)\circ f_0"] & Y
\end{tikzcd}$$

*and the exact sequences*:

$$0\longrightarrow K_1\xrightarrow{\begin{pmatrix}\mu_1 & \nu_1\end{pmatrix}}P_1\oplus G(Q_1)\xrightarrow{\begin{pmatrix}a_1\\ G(b_1)\circ g_1\end{pmatrix}}K_0\longrightarrow 0$$

$$0\longrightarrow L_1\xrightarrow{\begin{pmatrix}\zeta_1 & \xi_1\end{pmatrix}}F(P_1)\oplus Q_1\xrightarrow{\begin{pmatrix}F(a_1)\circ f_1\\ b_1\end{pmatrix}}L_0\longrightarrow 0$$

*Then as above there exist unique morphisms* $g_2\colon G(L_1)\longrightarrow K_1$ *and* $f_2\colon F(K_1)\longrightarrow L_1$ *which satisfy analogue relations from the pullback diagrams and then we get that the object* $(K_1,L_1,f_2,g_2)\in\mathcal{M}(\phi,\psi)$*. Hence we have the exact sequence*:

$$0\longrightarrow(K_1,L_1,f_2,g_2)\xrightarrow{\iota_1}\mathsf{T}_{\mathscr{A}}(P_1)\oplus\mathsf{T}_{\mathscr{B}}(Q_1)\xrightarrow{\kappa_1}(K_0,L_0,f_1,g_1)\longrightarrow 0 \tag{3.6.2}$$

*where the morphisms are*

$$\iota_1 = \Big( \begin{pmatrix} \mu_1 & \nu_1 \end{pmatrix}, \begin{pmatrix} \zeta_1 & \xi_1 \end{pmatrix} \Big) \qquad \kappa_1 = \Big( \begin{pmatrix} a_1 \\ G(b_1)\circ g_1 \end{pmatrix}, \begin{pmatrix} F(a_1)\circ f_1 \\ b_1 \end{pmatrix} \Big)$$

*and we view the direct sum* $\mathsf{T}_{\mathscr{A}}(P_1) \oplus \mathsf{T}_{\mathscr{B}}(Q_1)$ *as the object*:

$$\mathsf{T}_{\mathscr{A}}(P_1) \oplus \mathsf{T}_{\mathscr{B}}(Q_1) \simeq \Big(P_1 \oplus G(Q_1), F(P_1) \oplus Q_1, \begin{pmatrix} \mathrm{Id}_{F(P_1)} & 0 \\ 0 & \phi_{Q_1} \end{pmatrix}, \begin{pmatrix} \psi_{P_1} & 0 \\ 0 & \mathrm{Id}_{G(Q_1)} \end{pmatrix}\Big)$$

*Then from the exact sequences* (3.6.1) *and* (3.6.2) *we obtain the exact sequence*:

$$0 \longrightarrow (K_1, L_1, f_2, g_2) \xrightarrow{\iota_1} \mathsf{T}_{\mathscr{A}}(P_1) \oplus \mathsf{T}_{\mathscr{B}}(Q_1) \xrightarrow{\alpha_1} \mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0) \xrightarrow{\alpha_0} (X, Y, f, g) \longrightarrow 0$$

*where the morphism* $\alpha_1$ *is described as follows*:

$$\begin{aligned}
\alpha_1 &= \kappa_1 \circ \iota_0 \\
&= \Big( \begin{pmatrix} a_1 \\ G(b_1)\circ g_1 \end{pmatrix}, \begin{pmatrix} F(a_1)\circ f_1 \\ b_1 \end{pmatrix} \Big) \circ \Big( \begin{pmatrix} \mu_0 & \nu_0 \end{pmatrix}, \begin{pmatrix} \zeta_0 & \xi_0 \end{pmatrix} \Big) \\
&= \Big( \begin{pmatrix} a_1\circ\mu_0 & a_1\circ\nu_0 \\ G(b_1)\circ g_1\circ\mu_0 & G(b_1)\circ g_1\circ\nu_0 \end{pmatrix}, \begin{pmatrix} F(a_1)\circ f_1\circ\zeta_0 & F(a_1)\circ f_1\circ\xi_0 \\ b_1\circ\zeta_0 & b_1\circ\xi_0 \end{pmatrix} \Big) \\
&= \Big( \begin{pmatrix} a_1\circ\mu_0 & a_1\circ\nu_0 \\ G(b_1)\circ G(\zeta_0)\circ\psi_{P_0} & G(b_1)\circ G(\xi_0) \end{pmatrix}, \begin{pmatrix} F(a_1)\circ F(\mu_0) & F(a_1)\circ F(\nu_0)\circ\phi_{Q_0} \\ b_1\circ\zeta_0 & b_1\circ\xi_0 \end{pmatrix} \Big)
\end{aligned}$$

*Therefore from the following sequence of pullback diagrams*:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & & & \\
\cdots \arrow[r] & K_i \arrow[r, "\mu_i"] \arrow[d, "\nu_i"'] & G(Q_i) \arrow[d, "G(b_i)\circ g_i"] & & & & \\
 & P_i \arrow[r, "a_i"] & K_{i-1} \arrow[r] \arrow[d] & \cdots & & & \\
 & & \vdots & \ddots & \vdots \arrow[d] & & \\
 & & & \cdots \arrow[r] & K_1 \arrow[r, "\nu_1"] \arrow[d, "\mu_1"'] & G(Q_1) \arrow[d, "G(b_1)\circ g_1"] & \\
 & & & & P_1 \arrow[r, "a_1"] & K_0 \arrow[r, "\nu_0"] \arrow[d, "\mu_0"'] & G(Q_0) \arrow[d, "G(b_0)\circ g_0"] \\
 & & & & & P_0 \arrow[r, "a_0"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & & & \\
\cdots \arrow[r] & L_i \arrow[r, "\xi_i"] \arrow[d, "\zeta_i"'] & Q_i \arrow[d, "b_i"] & & & & \\
 & F(P_i) \arrow[r, "F(a_i)\circ f_i"] & L_{i-1} \arrow[r] \arrow[d] & \cdots & & & \\
 & & \vdots & \ddots & \vdots \arrow[d] & & \\
 & & & \cdots \arrow[r] & L_1 \arrow[r, "\xi_1"] \arrow[d, "\zeta_1"'] & Q_1 \arrow[d, "b_1"] & \\
 & & & & F(P_1) \arrow[r, "F(a_1)\circ f_1"] & L_0 \arrow[r, "\xi_0"] \arrow[d, "\zeta_0"'] & Q_0 \arrow[d, "b_0"] \\
 & & & & & F(P_0) \arrow[r, "F(a_0)\circ f_0"] & Y
\end{tikzcd}$$

*we derive the following projective resolution of* $(X, Y, f, g)$:

$$\cdots \twoheadrightarrow \mathsf{T}_{\mathscr{A}}(P_i) \oplus \mathsf{T}_{\mathscr{B}}(Q_i) \xrightarrow{\alpha_i} \mathsf{T}_{\mathscr{A}}(P_{i-1}) \oplus \mathsf{T}_{\mathscr{B}}(Q_{i-1}) \twoheadrightarrow \cdots \twoheadrightarrow \mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0) \xrightarrow{\alpha_0} (X, Y, f, g) \twoheadrightarrow 0$$

*where*

$$\alpha_0 = \big( \big( \begin{smallmatrix} a_0 \\ G(b_0)\circ g_0 \end{smallmatrix} \big), \big( \begin{smallmatrix} F(a_0)\circ f_0 \\ b_0 \end{smallmatrix} \big) \big)$$

*and*

$$\alpha_i = \big( \big( \begin{smallmatrix} a_i\circ\mu_{i-1} & a_1\circ\nu_{i-1} \\ G(b_i)\circ G(\zeta_{i-1})\circ\psi_{P_{i-1}} & G(b_i)\circ G(\xi_{i-1}) \end{smallmatrix} \big), \big( \begin{smallmatrix} F(a_i)\circ F(\mu_{i-1}) & F(a_i)\circ F(\nu_{i-1})\circ\phi_{Q_{i-1}} \\ b_i\circ\zeta_{i-1} & b_i\circ\xi_{i-1} \end{smallmatrix} \big) \big)$$

*for every* $i \geq 1$.

We continue with another construction of a projective resolution for an object $(X, Y, f, g)$ of $\mathcal{M}(\phi, \psi)$. We remark that the following method was used in Proposition 3.5.1 and that the resulting resolution is the same with that of the above construction.

Construction 3.6.2. *Let* $(X, Y, f, g)$ *be an object of* $\mathcal{M}(\phi, \psi)$. *Let* $a_0 \colon P_0 \longrightarrow X$ *be an epimorphism with* $P_0$ *projective in* $\mathscr{A}$ *and* $b_0 \colon Q_0 \longrightarrow Y$ *be an epimorphism with* $Q_0$ *projective in* $\mathscr{B}$. *Then we have the following commutative diagram*:

$$\begin{tikzcd}
\mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0) \arrow[r, two heads, "\kappa_0"] \arrow[dr, dashed, "\alpha_0"'] & \mathsf{T}_{\mathscr{A}}(X) \oplus \mathsf{T}_{\mathscr{B}}(Y) \arrow[d, two heads, "\lambda_0"] \\
 & (X, Y, f, g)
\end{tikzcd}$$

*where the objects* $\mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0)$ *and* $\mathsf{T}_{\mathscr{A}}(X) \oplus \mathsf{T}_{\mathscr{B}}(Y)$ *as tuples are*:

$$\mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0) \simeq \big(P_0 \oplus G(Q_0), F(P_0) \oplus Q_0, \big( \begin{smallmatrix} \mathrm{Id}_{F(P_0)} & 0 \\ 0 & \phi_{Q_0} \end{smallmatrix} \big), \big( \begin{smallmatrix} \psi_{P_0} & 0 \\ 0 & \mathrm{Id}_{G(Q_0)} \end{smallmatrix} \big) \big)$$

$$\mathsf{T}_{\mathscr{A}}(X) \oplus \mathsf{T}_{\mathscr{B}}(Y) \simeq \big(X \oplus G(Y), F(X) \oplus Y, \big( \begin{smallmatrix} \mathrm{Id}_{F(X)} & 0 \\ 0 & \phi_Y \end{smallmatrix} \big), \big( \begin{smallmatrix} \psi_X & 0 \\ 0 & \mathrm{Id}_{G(Y)} \end{smallmatrix} \big) \big)$$

*and the morphisms are the following*:

$$\kappa_0 = \big( \big( \begin{smallmatrix} a_0 & 0 \\ 0 & G(b_0) \end{smallmatrix} \big), \big( \begin{smallmatrix} F(a_0) & 0 \\ 0 & b_0 \end{smallmatrix} \big) \big), \ \lambda_0 = \big( \big( \begin{smallmatrix} \mathrm{Id}_X \\ g \end{smallmatrix} \big), \big( \begin{smallmatrix} f \\ \mathrm{Id}_Y \end{smallmatrix} \big) \big), \ \alpha_0 = \kappa_0 \circ \lambda_0 = \big( \big( \begin{smallmatrix} a_0 \\ G(b_0)\circ g \end{smallmatrix} \big), \big( \begin{smallmatrix} F(a_0)\circ f \\ b_0 \end{smallmatrix} \big) \big)$$

*The kernel of the morphism* $\alpha_0$ *is the object* $(K_0, L_0, f_1, g_1)$ *where* $K_0 = \mathsf{Ker}\,({}^t(a_0 \quad G(b_0)\circ g))$ *and* $L_0 = \mathsf{Ker}\,({}^t(F(a_0)\circ f \quad b_0))$. *Also from the exact sequences*:

$$0 \longrightarrow K_0 \xrightarrow{\begin{pmatrix} \mu_0 & \nu_0 \end{pmatrix}} P_0 \oplus G(Q_0) \xrightarrow{\begin{pmatrix} a_0 \\ G(b_0)\circ g_0 \end{pmatrix}} X \longrightarrow 0$$

$$0 \longrightarrow L_0 \xrightarrow{\begin{pmatrix} \zeta_0 & \xi_0 \end{pmatrix}} F(P_0) \oplus Q_0 \xrightarrow{\begin{pmatrix} F(a_0)\circ f_0 \\ b_0 \end{pmatrix}} Y \longrightarrow 0$$

*we get the morphism* $\iota_0 = ((\mu_0, \nu_0), (\zeta_0, \xi_0))$ *from* $(K_0, L_0, f_1, g_1)$ *to* $\mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0)$. *Thus so far we have the exact sequence*:

$$0 \longrightarrow (K_0, L_0, f_1, g_1) \xrightarrow{\iota_0} \mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0) \xrightarrow{\alpha_0} (X, Y, f, g) \longrightarrow 0 \tag{3.6.3}$$

*Now we continue the above procedure for the object* $(K_0, L_0, f_1, g_1)$. *Let* $a_1 \colon P_1 \longrightarrow K_0$ *be an epimorphism with* $P_1$ *projective in* $\mathscr{A}$ *and* $b_1 \colon Q_1 \longrightarrow L_0$ *be an epimorphism with* $Q_1$ *projective in* $\mathscr{B}$. *Then we have the commutative diagram*

$$\begin{tikzcd}
\mathsf{T}_{\mathscr{A}}(P_1) \oplus \mathsf{T}_{\mathscr{B}}(Q_1) \arrow[r, two heads, "\kappa_1"] \arrow[dr, dashed, "\kappa_1\circ\lambda_1"'] & \mathsf{T}_{\mathscr{A}}(K_0) \oplus \mathsf{T}_{\mathscr{B}}(L_0) \arrow[d, two heads, "\lambda_1"] \\
 & (K_0, L_0, f_1, g_1)
\end{tikzcd}$$

*where the objects* $\mathsf{T}_{\mathscr{A}}(P_1) \oplus \mathsf{T}_{\mathscr{B}}(Q_1)$ *and* $\mathsf{T}_{\mathscr{A}}(K_0) \oplus \mathsf{T}_{\mathscr{B}}(L_0)$ *as tuples are*:

$$\mathsf{T}_{\mathscr{A}}(P_1) \oplus \mathsf{T}_{\mathscr{B}}(Q_1) \simeq \Big(P_1 \oplus G(Q_1), F(P_1) \oplus Q_1, \begin{pmatrix} \mathrm{Id}_{F(P_1)} & 0 \\ 0 & \phi_{Q_1} \end{pmatrix}, \begin{pmatrix} \psi_{P_1} & 0 \\ 0 & \mathrm{Id}_{G(Q_1)} \end{pmatrix}\Big)$$

$$\mathsf{T}_{\mathscr{A}}(K_0) \oplus \mathsf{T}_{\mathscr{B}}(L_0) \simeq \Big(K_0 \oplus G(L_0), F(K_0) \oplus L_0, \begin{pmatrix} \mathrm{Id}_{F(K_0)} & 0 \\ 0 & \phi_{L_0} \end{pmatrix}, \begin{pmatrix} \psi_{K_0} & 0 \\ 0 & \mathrm{Id}_{G(L_0)} \end{pmatrix}\Big)$$

*and the morphisms are the following*:

$$\kappa_1 = \Big(\begin{pmatrix} a_1 & 0 \\ 0 & G(b_1) \end{pmatrix}, \begin{pmatrix} F(a_1) & 0 \\ 0 & b_1 \end{pmatrix}\Big), \ \lambda_1 = \Big(\begin{pmatrix} \mathrm{Id}_{K_0} \\ g_1 \end{pmatrix}, \begin{pmatrix} f_1 \\ \mathrm{Id}_{L_0} \end{pmatrix}\Big), \ \kappa_1 \circ \lambda_1 = \Big(\begin{pmatrix} a_1 \\ G(b_1)\circ g_1 \end{pmatrix}, \begin{pmatrix} F(a_1)\circ f_1 \\ b_1 \end{pmatrix}\Big)$$

*The kernel of the epimorphism* $\kappa_1 \circ \lambda_1$ *is the object* $(K_1, L_1, f_1, g_1)$ *where*

$$K_1 = \mathsf{Ker}\,({}^t(a_1 \quad G(b_1)\circ g_1))$$

*and*

$$L_1 = \mathsf{Ker}\,({}^t(F(a_1)\circ f_1 \quad b_1))$$

*Also from the following exact sequences*:

$$0 \longrightarrow K_1 \xrightarrow{\begin{pmatrix} \mu_1 & \nu_1 \end{pmatrix}} P_1 \oplus G(Q_1) \xrightarrow{\begin{pmatrix} a_1 \\ G(b_1)\circ g_1 \end{pmatrix}} K_0 \longrightarrow 0$$

$$0 \longrightarrow L_1 \xrightarrow{\begin{pmatrix} \zeta_1 & \xi_1 \end{pmatrix}} F(P_1) \oplus Q_1 \xrightarrow{\begin{pmatrix} F(a_1)\circ f_1 \\ b_1 \end{pmatrix}} L_0 \longrightarrow 0$$

*we get the morphism* $\iota_1 = ((\mu_1, \nu_1), (\zeta_1, \xi_1))$ *from* $(K_1, L_1, f_2, g_2)$ *to* $\mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0)$. *Hence we have the exact sequence*:

$$0 \longrightarrow (K_1, L_1, f_2, g_2) \xrightarrow{\iota_1} \mathsf{T}_{\mathscr{A}}(P_1) \oplus \mathsf{T}_{\mathscr{B}}(Q_1) \xrightarrow{\kappa_1\circ\lambda_1} (K_0, L_0, f_1, g_1) \longrightarrow 0 \tag{3.6.4}$$

*Therefore the Yoneda composition of the short exact sequences* (3.6.3) *and* (3.6.4) *gives the following exact sequence*:

$$0 \longrightarrow (K_1, L_1, f_2, g_2) \xrightarrow{\iota_1} \mathsf{T}_{\mathscr{A}}(P_1) \oplus \mathsf{T}_{\mathscr{B}}(Q_1) \xrightarrow{\alpha_1} \mathsf{T}_{\mathscr{A}}(P_0) \oplus \mathsf{T}_{\mathscr{B}}(Q_0) \xrightarrow{\alpha_0} (X, Y, f, g) \longrightarrow 0$$

*where the morphism* $\alpha_1\colon \mathsf{T}_{\mathscr{A}}(P_1)\oplus\mathsf{T}_{\mathscr{B}}(Q_1)\longrightarrow\mathsf{T}_{\mathscr{A}}(P_0)\oplus\mathsf{T}_{\mathscr{B}}(Q_0)$ *is the following composition*:

$$\begin{aligned}\alpha_1 &= \kappa_1\circ\lambda_1\circ\iota_0 = \Big(\big(\begin{smallmatrix} a_1 \\ G(b_1)\circ g_1\end{smallmatrix}\big),\big(\begin{smallmatrix} F(a_1)\circ f_1 \\ b_1\end{smallmatrix}\big)\Big)\circ\big((\mu_0,\nu_0),(\zeta_0,\xi_0)\big)\\ &= \Big(\big(\begin{smallmatrix} a_1\circ\mu_0 & a_1\circ\nu_0 \\ G(b_1)\circ g_1\circ\mu_0 & G(b_1)\circ g_1\circ\nu_0\end{smallmatrix}\big),\big(\begin{smallmatrix} F(a_1)\circ f_1\circ\zeta_0 & F(a_1)\circ f_1\circ\xi_0 \\ b_1\circ\zeta_0 & b_1\circ\xi_0\end{smallmatrix}\big)\Big)\end{aligned}$$

*From the following commutative diagrams*:

$$\begin{tikzcd}
F(K_0) \arrow[r, "f_1"] \arrow[d, "(F(\mu_0),F(\nu_0))"'] & L_0 \arrow[d, "(\zeta_0,\xi_0)"] \\
F(P_0)\oplus FG(Q_0) \arrow[r, "\left(\begin{smallmatrix}\mathrm{Id}_{F(P_0)} & 0 \\ 0 & \phi_{Q_0}\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}F(a_0) \\ FG(b_0)\circ F(g)\end{smallmatrix}\right)"'] & F(P_0)\oplus Q_0 \arrow[d, "\left(\begin{smallmatrix}F(a_0)\circ f \\ b_0\end{smallmatrix}\right)"] \\
F(X) \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
G(L_0) \arrow[r, "g_1"] \arrow[d, "(G(\zeta_0),G(\xi_0))"'] & K_0 \arrow[d, "(\mu_0,\nu_0)"] \\
GF(P_0)\oplus G(Q_0) \arrow[r, "\left(\begin{smallmatrix}\psi_{P_0} & 0 \\ 0 & \mathrm{Id}_{G(Q_0)}\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}GF(a_0)\circ G(f) \\ G(b_0)\end{smallmatrix}\right)"'] & P_0\oplus G(Q_0) \arrow[d, "\left(\begin{smallmatrix}a_0 \\ G(b_0)\circ g\end{smallmatrix}\right)"] \\
G(Y) \arrow[r, "g"] & X
\end{tikzcd}$$

*we have* $f_1\circ\zeta_0 = F(\mu_0)$, $f_1\circ\xi_0 = F(\nu_0)\circ\phi_{Q_0}$, $g_1\circ\mu_0 = G(\zeta_0)\circ\psi_{P_0}$ *and* $g_1\circ\nu_0 = G(\xi_0)$. *Thus the morphism* $\alpha_1$ *is the following*:

$$\alpha_1 = \Big(\big(\begin{smallmatrix} a_1\circ\mu_0 & a_1\circ\nu_0 \\ G(b_1)\circ G(\zeta_0)\circ\psi_{P_0} & G(b_1)\circ G(\xi_0)\end{smallmatrix}\big),\big(\begin{smallmatrix} F(a_1)\circ F(\mu_0) & F(a_1)\circ F(\nu_0)\circ\phi_{Q_0} \\ b_1\circ\zeta_0 & b_1\circ\xi_0\end{smallmatrix}\big)\Big)$$

*and we see immediately that it is the same morphism with the one obtained in the Construction 3.6.1. If we repeat the same procedure for the object* $(K_1,L_1,f_2,g_2)$ *and continue in this way we finally deduce a projective resolution for* $(X,Y,f,g)$ *which is the same with that of Construction 3.6.1.*

## 3.7. Left Derived Functors and Extensions

In this section we describe the left derived functors of $\mathsf{T}_{\mathscr{A}}\colon\mathscr{A}\longrightarrow\mathcal{M}(\phi,\psi)$ and $\mathsf{T}_{\mathscr{B}}\colon\mathscr{B}\longrightarrow\mathcal{M}(\phi,\psi)$ and we derive also some formulas for $\mathsf{Ext}$ homology groups in $\mathcal{M}(\phi,\psi)$. In what follows we assume that $\mathscr{A}$ and $\mathscr{B}$ have enough projective objects.

Proposition 3.7.1. *Let* $\mathcal{M}(\phi,\psi)$ *be a Morita category.*

(i) *The left derived functor of* $\mathsf{T}_{\mathscr{A}}\colon\mathscr{A}\longrightarrow\mathcal{M}(\phi,\psi)$ *is the functor*

$$\mathsf{L}_n\mathsf{T}_{\mathscr{A}}\colon\mathscr{A}\longrightarrow\mathcal{M}(\phi,\psi),\ \ X\mapsto\mathsf{L}_n\mathsf{T}_{\mathscr{A}}(X) = (0,\mathsf{L}_nF(X),0,0),\ \ \forall n\geq 1$$

*where* $\mathsf{L}_nF(X)$ *is the left derived functor of* $F$.

(ii) *The left derived functor of* $\mathsf{T}_{\mathscr{B}}\colon\mathscr{B}\longrightarrow\mathcal{M}(\phi,\psi)$ *is the functor*

$$\mathsf{L}_n\mathsf{T}_{\mathscr{B}}\colon\mathscr{B}\longrightarrow\mathcal{M}(\phi,\psi),\ \ Y\mapsto\mathsf{L}_n\mathsf{T}_{\mathscr{B}}(Y) = (\mathsf{L}_nG(Y),0,0,0),\ \ \forall n\geq 1$$

*where* $\mathsf{L}_nG(Y)$ *is the left derived functor of* $G$.

(iii) *For every* $n\geq 1$ *we have the following natural isomorphism*:

$$\mathsf{U}_{\mathscr{B}}(\mathsf{L}_n\mathsf{T}_{\mathscr{A}}(-)) \xrightarrow{\ \simeq\ } \mathsf{L}_nF(-)$$

*and*

$$\mathsf{U}_{\mathscr{A}}(\mathsf{L}_n\mathsf{T}_{\mathscr{A}}(-)) = 0$$

(iv) *For every $n \geq 1$ we have the following natural isomorphism:*

$$\mathsf{U}_{\mathscr{A}}(\mathsf{L}_n\mathsf{T}_{\mathscr{B}}(-)) \xrightarrow{\simeq} \mathsf{L}_nG(-)$$

*and*

$$\mathsf{U}_{\mathscr{B}}(\mathsf{L}_n\mathsf{T}_{\mathscr{B}}(-)) = 0$$

(v) *Let $X$ be an object of $\mathscr{A}$. Then there exists the following exact sequences:*

$$0 \longrightarrow \mathsf{L}_1\mathsf{T}_{\mathscr{A}}(X) \longrightarrow \mathsf{T}_{\mathscr{A}}(\Omega_{\mathscr{A}}(X)) \longrightarrow \Omega_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X)) \longrightarrow 0$$

*and*

$$0 \longrightarrow \mathsf{L}_{i+1}\mathsf{T}_{\mathscr{A}}(X) \longrightarrow \mathsf{T}_{\mathscr{A}}(\Omega^{i+1}_{\mathscr{A}}(X)) \longrightarrow \Omega_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(\Omega^{i}_{\mathscr{A}}(X))) \longrightarrow 0$$

(vi) *Let $Y$ be an object of $\mathscr{B}$. Then there exists the following exact sequences:*

$$0 \longrightarrow \mathsf{L}_1\mathsf{T}_{\mathscr{B}}(Y) \longrightarrow \mathsf{T}_{\mathscr{B}}(\Omega_{B}(Y)) \longrightarrow \Omega_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(Y)) \longrightarrow 0$$

*and*

$$0 \longrightarrow \mathsf{L}_{i+1}\mathsf{T}_{\mathscr{B}}(Y) \longrightarrow \mathsf{T}_{\mathscr{B}}(\Omega^{i+1}_{\mathscr{B}}(Y)) \longrightarrow \Omega_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(\Omega^{i}_{\mathscr{B}}(Y))) \longrightarrow 0$$

(vii) *Let $X$ and $X'$ be objects of $\mathscr{A}$ such that $\psi_{X'} = 0$. Then we have the following natural isomorphism:*

$$\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X')) \xrightarrow{\simeq} \mathsf{Ext}^1_{\mathscr{A}}(X, X')$$

(viii) *Let $Y$ and $Y'$ are objects of $\mathscr{B}$ such that $\phi_{Y'} = 0$. Then we have the following natural isomorphism:*

$$\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(Y), \mathsf{Z}_{\mathscr{B}}(Y')) \xrightarrow{\simeq} \mathsf{Ext}^1_{\mathscr{B}}(Y, Y')$$

(ix) *Let $X$ and $X'$ be objects of $\mathscr{A}$ such that $\psi_X = 0 = \psi_{X'}$. Then the extension group $\mathsf{Ext}^n_{\mathscr{A}}(X, X')$ is a direct summand of $\mathsf{Ext}^n_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X'))$ for all $n \geq 1$. Furthermore there is a natural isomorphism:*

$$\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X')) \xrightarrow{\simeq} \mathsf{Ext}^1_{\mathscr{A}}(X, X') \oplus G$$

*where $G$ is the abelian group of all equivalence classes of short exact sequences in $\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X'))$ which split as short exact sequences in $\mathscr{A}$.*

(x) *If $Y$ and $Y'$ are objects of $\mathscr{B}$ such that $\phi_Y = 0 = \phi_{Y'}$ then the extension group $\mathsf{Ext}^n_{\mathscr{B}}(Y, Y')$ is a direct summand of $\mathsf{Ext}^n_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{B}}(Y), \mathsf{Z}_{\mathscr{B}}(Y'))$ for all $n \geq 1$. Furthermore there is a natural isomorphism:*

$$\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{B}}(Y), \mathsf{Z}_{\mathscr{B}}(Y')) \xrightarrow{\simeq} \mathsf{Ext}^1_{\mathscr{B}}(Y, Y') \oplus G'$$

*where $G'$ is the abelian group consisting of all equivalence classes of short exact sequences in $\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{B}}(Y), \mathsf{Z}_{\mathscr{B}}(Y'))$ which split as short exact sequences in $\mathscr{B}$.*

(xi) *If the functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ is exact then*

$$\mathsf{Ext}^n_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X), (X', Y', f', g')) \xrightarrow{\simeq} \mathsf{Ext}^n_{\mathscr{A}}(X, X')$$

*for all $(X', Y', f', g') \in \mathcal{M}(\phi,\psi)$ and $n \geq 0$.*

(xii) *If the functor* $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ *is exact then*

$$\mathsf{Ext}^n_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(Y),(X',Y',f',g')) \xrightarrow{\simeq} \mathsf{Ext}^n_{\mathscr{A}}(Y,Y')$$

*for all* $(X',Y',f',g') \in \mathcal{M}(\phi,\psi)$ *and* $n \geq 0$.

(xiii) *If* $\mathsf{L}_iF(X) = 0$, $\forall\ 1 \leq i \leq n$, *then we have the isomorphism*:

$$\mathsf{Ext}^i_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X),(X',Y',f',g')) \xrightarrow{\simeq} \mathsf{Ext}^i_{\mathscr{A}}(X,X')$$

*for every* $1 \leq i \leq n$ *and* $(X',Y',f',g') \in \mathcal{M}(\phi,\psi)$.

(xiv) *If* $\mathsf{L}_iG(Y) = 0$, $\forall\ 1 \leq i \leq n$, *then we have the isomorphism*:

$$\mathsf{Ext}^i_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(Y),(X',Y',f',g')) \xrightarrow{\simeq} \mathsf{Ext}^i_{\mathscr{B}}(Y,Y')$$

*for every* $1 \leq i \leq n$ *and* $(X',Y',f',g') \in \mathcal{M}(\phi,\psi)$.

Proof. (i) Let $X$ be an object of $\mathscr{A}$ and suppose that

$$\begin{tikzcd} \cdots \arrow[r] & P_3 \arrow[r, "a_3"] \arrow[d, "j_2"] & P_2 \arrow[r, "a_2"] \arrow[d, "j_1"] & P_1 \arrow[r, "a_1"] \arrow[d, "j_0"] & P_0 \arrow[r, "a_0"] & X \arrow[r] & 0 \\ & K_2 \arrow[ur, "i_2"'] & K_1 \arrow[ur, "i_1"'] & K_0 \arrow[ur, "i_0"'] & & & \end{tikzcd}$$

is a projective resolution of $X$ where $K_n = \mathsf{Ker}\, a_n$, $\forall\ n \geq 0$. From the short exact sequence $0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ we obtain the exact sequence:

$$0 \longrightarrow \mathsf{L}_1\mathsf{T}_{\mathscr{A}}(X) \longrightarrow \mathsf{T}_{\mathscr{A}}(K_0) \xrightarrow{\mathsf{T}_{\mathscr{A}}(i_0)} \mathsf{T}_{\mathscr{A}}(P_0) \xrightarrow{\mathsf{T}_{\mathscr{A}}(a_0)} \mathsf{T}_{\mathscr{A}}(X) \longrightarrow 0 \quad (3.7.1)$$

since $\mathsf{L}_1\mathsf{T}_{\mathscr{A}}(P_0) = 0$. Then we have

$$\mathsf{L}_1\mathsf{T}_{\mathscr{A}}(X) = \mathsf{Ker}\,\mathsf{T}_{\mathscr{A}}(i_0) = (0, \mathsf{Ker}\,F(i_0), 0, 0) = (0, \mathsf{L}_1F(X), 0, 0)$$

From the exact sequence (3.7.1) we have $\mathsf{L}_2\mathsf{T}_{\mathscr{A}}(X) \simeq \mathsf{L}_1\mathsf{T}_{\mathscr{A}}(K_0)$ and then we get that $\mathsf{L}_2\mathsf{T}_{\mathscr{A}}(X) \simeq (0, \mathsf{L}_2F(X), 0, 0)$ since

$$\mathsf{L}_1\mathsf{T}_{\mathscr{A}}(K_0) \simeq \mathsf{Ker}\,\mathsf{T}_{\mathscr{A}}(i_1) = (0, \mathsf{Ker}\,F(i_1), 0, 0) = (0, \mathsf{L}_1F(K_0), 0, 0) = (0, \mathsf{L}_2F(X), 0, 0)$$

Continuing as above we infer that $\mathsf{L}_n\mathsf{T}_{\mathscr{A}}(X) = (0, \mathsf{L}_nF(X), 0, 0)$ for every $n \geq 1$. Similarly we show (ii) and then (iii) and (iv) follow immediately from (i) and (ii).

(v) Let $0 \longrightarrow \Omega_{\mathscr{A}}(X) \longrightarrow P \longrightarrow X \longrightarrow 0$ be a short exact sequence with $P$ projective object of $\mathscr{A}$. Then from the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{L}_1\mathsf{T}_{\mathscr{A}}(X) \arrow[r] & \mathsf{T}_{\mathscr{A}}(\Omega_{\mathscr{A}}(X)) \arrow[r] \arrow[d] & \mathsf{T}_{\mathscr{A}}(P) \arrow[r] \arrow[d, equal] & \mathsf{T}_{\mathscr{A}}(X) \arrow[r] \arrow[d, equal] & 0 \\ & 0 \arrow[r] & \Omega_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X)) \arrow[r] & \mathsf{T}_{\mathscr{A}}(P) \arrow[r] & \mathsf{T}_{\mathscr{A}}(X) \arrow[r] & 0 \end{tikzcd}$$

we obtain the exact sequence:

$$0 \longrightarrow \mathsf{L}_1\mathsf{T}_{\mathscr{A}}(X) \longrightarrow \mathsf{T}_{\mathscr{A}}(\Omega_{\mathscr{A}}(X)) \longrightarrow \Omega_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X)) \longrightarrow 0$$

If we replace $X$ in the above exact sequence with $\Omega^i_{\mathscr{A}}(X)$ then we get the second exact sequence since we have the isomorphism $\mathsf{L}_{i+1}\mathsf{T}_{\mathscr{A}}(X) \simeq \mathsf{L}_1\mathsf{T}_{\mathscr{A}}(\Omega^i_{\mathscr{A}}(X))$ and $\Omega_{\mathscr{A}}(\Omega^i_{\mathscr{A}}(X)) = \Omega^{i+1}_{\mathscr{A}}(X)$. Dually we show (vi).

(vii) Let $\cdots \longrightarrow P_2 \xrightarrow{a_2} P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} X \longrightarrow 0$ be the start of a projective resolution of $X$ in $\mathscr{A}$. Then we have the following exact sequence in $\mathcal{M}(\phi,\psi)$:

$$0 \longrightarrow \mathsf{Ker}\,\mathsf{T}_{\mathscr{A}}(a_1) \longrightarrow \mathsf{T}_{\mathscr{A}}(P_1) \xrightarrow{\mathsf{T}_{\mathscr{A}}(a_1)} \mathsf{T}_{\mathscr{A}}(P_0) \xrightarrow{\mathsf{T}_{\mathscr{A}}(a_0)} \mathsf{T}_{\mathscr{A}}(X) \longrightarrow 0$$

where $\mathsf{Ker}\,\mathsf{T}_{\mathscr{A}}(a_1) = (\mathsf{Ker}\,a_1, \mathsf{Ker}\,F(a_1), m, n)$. Let $P_2 \longrightarrow \mathsf{Ker}\,a_1$ be the epimorphism from the projective resolution of $X$ and let $Q_2 \longrightarrow \mathsf{Ker}\,F(a_1)$ be an epimorphism with $Q_2$ projective object of $\mathscr{B}$. Then from the construction 3.6.1 we have the following projective resolution of $\mathsf{T}_{\mathscr{A}}(X)$:

$$\cdots \longrightarrow \mathsf{T}_{\mathscr{A}}(P_2) \oplus \mathsf{T}_{\mathscr{B}}(Q_2) \longrightarrow \mathsf{T}_{\mathscr{A}}(P_1) \longrightarrow \mathsf{T}_{\mathscr{A}}(P_0) \longrightarrow \mathsf{T}_{\mathscr{A}}(X) \longrightarrow 0$$

Since $\mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(Q_2), \mathsf{Z}_{\mathscr{A}}(X')) = 0$ and using the adjoint pair $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}})$ we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (\mathsf{T}_{\mathscr{A}}(X), \mathsf{Z}(X')) \arrow[r] \arrow[d, "\simeq"'] & (\mathsf{T}_{\mathscr{A}}(P_0), \mathsf{Z}(X')) \arrow[r] \arrow[d, "\simeq"'] & (\mathsf{T}_{\mathscr{A}}(P_1), \mathsf{Z}(X')) \arrow[r] \arrow[d, "\simeq"'] & (\mathsf{T}_{\mathscr{A}}(P_2), \mathsf{Z}(X')) \arrow[r] \arrow[d, "\simeq"'] & \cdots \\
0 \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(X, X') \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(P_0, X') \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(P_1, X') \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(P_2, X') \arrow[r] & \cdots
\end{tikzcd}$$

We infer that $\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X')) \simeq \mathsf{Ext}^1_{\mathscr{A}}(X, X')$. Similarly we get (viii).

(ix) Since $\mathsf{U}_{\mathscr{A}}$ and $\mathsf{Z}_{\mathscr{A}}$ are exact functors we have the following homomorphisms between the $\mathsf{Ext}$-groups:

$$\mathsf{U}^n_{\mathscr{A}} \colon \mathsf{Ext}^n_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X')) \longrightarrow \mathsf{Ext}^n_{\mathscr{A}}(X, X')$$

and

$$\mathsf{Z}^n_{\mathscr{A}} \colon \mathsf{Ext}^n_{\mathscr{A}}(X, X') \longrightarrow \mathsf{Ext}^n_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X'))$$

Since $\mathsf{U}_{\mathscr{A}} \circ \mathsf{Z}_{\mathscr{A}} = \mathrm{Id}_{\mathscr{A}}$ it follows that $\mathsf{U}^n_{\mathscr{A}} \circ \mathsf{Z}^n_{\mathscr{A}} = \mathrm{Id}_{\mathsf{Ext}^n_{\mathscr{A}}(X,X')}$. Hence the extension group $\mathsf{Ext}^n_{\mathscr{A}}(X, X')$ is a direct summand of $\mathsf{Ext}^n_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X'))$ for all $n \geq 1$. Then we have the following split exact sequence:

$$0 \longrightarrow \mathsf{Ker}\,\mathsf{U}^1_{\mathscr{A}} \longrightarrow \mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X')) \xrightarrow{\mathsf{U}^1_{\mathscr{A}}} \mathsf{Ext}^1_{\mathscr{A}}(X, X') \longrightarrow 0$$

where $\mathsf{Ker}\,\mathsf{U}^1_{\mathscr{A}}$ consists of all equivalence classes of short exact sequences in the group $\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X'))$ which split as short exact sequences in $\mathscr{A}$. Hence we have the isomorphism

$$\mathsf{Ext}^1_{\mathcal{M}(\phi,\psi)}(\mathsf{Z}_{\mathscr{A}}(X), \mathsf{Z}_{\mathscr{A}}(X')) \simeq \mathsf{Ext}^1_{\mathscr{A}}(X, X') \oplus G$$

where $G = \mathsf{Ker}\,\mathsf{U}^1_{\mathscr{A}}$. Dually we obtain (x).

(xi) Let $X$ be an object of $\mathscr{A}$ and let $\cdots \longrightarrow P_2 \xrightarrow{a_2} P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} X \longrightarrow 0$ be a projective resolution of $X$. Since $F$ is exact it follows that the functor $\mathsf{T}_{\mathscr{A}}(X)$ is exact and therefore the following exact sequence

$$\cdots \longrightarrow \mathsf{T}_{\mathscr{A}}(P_2) \xrightarrow{\mathsf{T}_{\mathscr{A}}(a_2)} \mathsf{T}_{\mathscr{A}}(P_1) \xrightarrow{\mathsf{T}_{\mathscr{A}}(a_1)} \mathsf{T}_{\mathscr{A}}(P_0) \xrightarrow{\mathsf{T}_{\mathscr{A}}(a_0)} \mathsf{T}_{\mathscr{A}}(X) \longrightarrow 0$$

is a projective resolution of $\mathsf{T}_{\mathscr{A}}(X)$. From the adjoint pair $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}})$ we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (\mathsf{T}_{\mathscr{A}}(X),(X',Y',f',g')) \arrow[r] \arrow[d, "\simeq"'] & (\mathsf{T}_{\mathscr{A}}(P_0),(X',Y',f',g')) \arrow[r] \arrow[d, "\simeq"'] & (\mathsf{T}_{\mathscr{A}}(P_1),(X',Y',f',g')) \arrow[r] \arrow[d, "\simeq"'] & \cdots \\
0 \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(X,X') \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(P_0,X') \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(P_1,X') \arrow[r] & \cdots
\end{tikzcd}$$

We conclude that $\mathsf{Ext}^n_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X),(X',Y',f',g')) \simeq \mathsf{Ext}^n_{\mathscr{A}}(X,X')$ for every $n \geq 0$. In the same way we obtain (xii).

(xiii) Let $X$ be an object of $\mathscr{A}$ and let $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be a projective resolution of $X$. Since $\mathsf{L}_iF(X) = 0$, for $1 \leq i \leq n$, it follows from (i) that $\mathsf{L}_i\mathsf{T}_{\mathscr{A}}(X) = 0$ for every $1 \leq i \leq n$. This implies that the following sequence:

$$\cdots \longrightarrow \mathsf{T}_{\mathscr{A}}(P_{n+1}) \longrightarrow \mathsf{T}_{\mathscr{A}}(P_n) \longrightarrow \cdots \longrightarrow \mathsf{T}_{\mathscr{A}}(P_0) \longrightarrow \mathsf{T}_{\mathscr{A}}(X) \longrightarrow 0$$

is part of a projective resolution of $\mathsf{T}_{\mathscr{A}}(X)$. Let $(X',Y',f',g')$ be an arbitrary object of $\mathcal{M}(\phi,\psi)$. Then using the adjoint pair $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}})$ we have the following commutative diagram:

$$\begin{tikzcd}
(\mathsf{T}_{\mathscr{A}}(X),(X',Y',f',g')) \arrow[r, tail] \arrow[d, "\simeq"'] & (\mathsf{T}_{\mathscr{A}}(P_0),(X',Y',f',g')) \arrow[r] \arrow[d, "\simeq"'] & (\mathsf{T}_{\mathscr{A}}(P_1),(X',Y',f',g')) \arrow[r] \arrow[d, "\simeq"'] & \cdots \\
\mathsf{Hom}_{\mathscr{A}}(X,X') \arrow[r, tail] & \mathsf{Hom}_{\mathscr{A}}(P_0,X') \arrow[r] & \mathsf{Hom}_{\mathscr{A}}(P_1,X') \arrow[r] & \cdots
\end{tikzcd}$$

Hence we have the isomorphism $\mathsf{Ext}^i_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X),(X',Y',f',g')) \simeq \mathsf{Ext}^i_{\mathscr{A}}(X,X')$ for every $1 \leq i \leq n$. Similarly using (ii) we get the desired isomorphism of (xiv). □

## 3.8. The Morita Category of Module Categories

In this section we give the intepretation of the Morita category of module categories over rings. This will lead us to the class of Morita rings that we are going to discuss thoroughly in Chapter 4. Moreover we examine when a Morita category is a module category.

**3.8.1. Morita rings.** Let $A$ and $B$ be rings and $M$ a $B$-$A$-bimodule and $N$ an $A$-$B$-bimodule. Let $\phi\colon M \otimes_A N \longrightarrow B$ be a $B$-$B$-bimodule homomorphism and let $\psi\colon N \otimes_B M \longrightarrow A$ be an $A$-$A$-bimodule homomorphism. Then we define the **Morita ring**:

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$$

where the addition of elements of $\Lambda_{(\phi,\psi)}$ is componentwise and multiplication is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \cdot \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(n \otimes m') & an' + nb' \\ ma' + bm' & bb' + \phi(m \otimes n') \end{pmatrix}$$

If the following diagrams are commutaitve:

$$\begin{tikzcd}
N \otimes_B M \otimes_A N \arrow[r, "\psi\otimes 1_N"] \arrow[d, "1_N\otimes\phi"'] & A \otimes_A N \arrow[d, "\simeq"] \\
N \otimes_B B \arrow[r, "\simeq"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes_A N \otimes_B M \arrow[r, "\phi\otimes 1_M"] \arrow[d, "1_M\otimes\psi"'] & B \otimes_B M \arrow[d, "\simeq"] \\
M \otimes_A A \arrow[r, "\simeq"] & M
\end{tikzcd}$$

that is

$$\phi(m\otimes n)m' = m\psi(n\otimes m') \quad \text{and} \quad n\phi(m\otimes n') = \psi(n\otimes m)m' \tag{3.8.1}$$

for every $m, m \in M$ and $n, n' \in N$, then the above multiplication defines an associative ring structure on the Morita ring $\Lambda_{(\phi,\psi)}$.

We will describe now the category of modules over $\Lambda_{(\phi,\psi)}$. For this reason we introduce a category which we will show that it is equivalent to $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$. Let $\mathcal{M}(\phi,\psi)$ be the category whose objects are tuples $(X,Y,f,g)$ where $X \in \mathsf{Mod}\text{-}A$, $Y \in \mathsf{Mod}\text{-}B$, $f\colon M\otimes_A X \longrightarrow Y$ and $g\colon N\otimes_B Y \longrightarrow X$ such that the following diagrams are commutative:

$$\begin{tikzcd}
N\otimes_B M\otimes_A X \arrow[r, "1_N\otimes f"] \arrow[d, "\psi\otimes 1_X"'] & N\otimes_B Y \arrow[d, "g"] \\
A\otimes_A X \arrow[r, "\simeq"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
M\otimes_A N\otimes_B Y \arrow[r, "1_M\otimes g"] \arrow[d, "\phi\otimes 1_Y"'] & M\otimes_A X \arrow[d, "f"] \\
B\otimes_B Y \arrow[r, "\simeq"] & Y
\end{tikzcd}$$

If $(X,Y,f,g)$ and $(X',Y',f',g')$ are objects of $\mathcal{M}(\phi,\psi)$, then a morphism $(X,Y,f,g) \longrightarrow (X',Y',f',g')$ in $\mathcal{M}(\phi,\psi)$ is a pair of homomorphisms $(a,b)$ where $a\colon X \longrightarrow X'$ is a morphism in $\mathsf{Mod}\text{-}A$ and $b\colon Y \longrightarrow Y'$ is a morphism in $\mathsf{Mod}\text{-}B$ such that the following diagrams are commutative:

$$\begin{tikzcd}
M\otimes_A X \arrow[r, "f"] \arrow[d, "1_M\otimes a"'] & Y \arrow[d, "b"] \\
M\otimes_A X' \arrow[r, "f'"] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
N\otimes_B Y \arrow[r, "g"] \arrow[d, "1_N\otimes b"'] & X \arrow[d, "a"] \\
N\otimes_B Y' \arrow[r, "g'"] & X'
\end{tikzcd}$$

Then it is well known that the categories $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ and $\mathcal{M}(\phi,\psi)$ are equivalent, see [59, Theorem 1.5]. For completeness we prove this equivalence giving some details. The relationship between $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ and $\mathcal{M}(\phi,\psi)$ is given via the functor $F\colon \mathcal{M}(\phi,\psi) \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ which is defined as follows. If $(X,Y,f,g) \in \mathcal{M}(\phi,\psi)$ then $F(X,Y,f,g) = X\oplus Y$ as abelian groups, and the $\Lambda_{(\phi,\psi)}$-module structure is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}(x,y) = (ax + g(n\otimes y), by + f(m\otimes x)) \tag{$*$}$$

for all $a \in A, b \in B, n \in N, m \in M, x \in X$ and $y \in Y$. One can verify easily that $F(X,Y,f,g)$ is a $\Lambda_{(\phi,\psi)}$-module. If $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ is a morphism in $\mathcal{M}(\phi,\psi)$ then $F(a,b) = a\oplus b\colon X\oplus Y \longrightarrow X'\oplus Y'$.

Proposition 3.8.1. *[59] [105] Let $\Lambda_{(\phi,\psi)} = \left(\begin{smallmatrix} A & {}_A N_B \\ {}_B M_A & B \end{smallmatrix}\right)$ be a Morita ring. Then the categories $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ and $\mathcal{M}(\phi,\psi)$ are equivalent.*

Proof. From the description of the functor $F$ on morphisms it follows that $F$ is faithful. Let $C$ be a $\Lambda_{(\phi,\psi)}$-module. Then $C \simeq e_1C \oplus e_2C$ as abelian groups where $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are idempotents elements of $\Lambda_{(\phi,\psi)}$. Observe that $e_1C$ is an $A$-module, $e_2C$ is a $B$-module and there exists a $B$-morphsim $f\colon M\otimes_A e_1C \longrightarrow e_2C$ and an $A$-morphism $g\colon N\otimes_B e_2C \longrightarrow e_1C$. Then from the relations (3.8.1) we obtain that the tuple $(e_1C, e_2C, f, g)$ is an object of $\mathcal{M}(\phi,\psi)$ and $F(e_1C,e_2C,f,g) \simeq C$. Hence $F$ is surjective on objects. Let $(X,Y,f,g)$ and $(X',Y',f',g')$ two objects of $\mathcal{M}(\phi,\psi)$ and let $\alpha\colon F(X,Y,f,g) \longrightarrow F(X',Y',f',g')$ be a morphism in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$. Then we have a map $\alpha\colon X\oplus Y \longrightarrow X'\oplus Y'$ and suppose that $\alpha(x,y) = (x',y')$ for $x \in X$ and $y \in Y$. Since $\alpha(x,0) = (x',0)$ and $\alpha(o,y) = (0,y')$ we obtain an $A$-morphism $\beta\colon X \longrightarrow X'$,

$\beta(x) = x'$ and a $B$-morphism $\gamma\colon Y \longrightarrow Y'$, $\gamma(y) = y'$. For $m \in M$ we write $m' = \left(\begin{smallmatrix} 0 & 0 \\ m & 0 \end{smallmatrix}\right)$ and for $n \in N$ we write $n' = \left(\begin{smallmatrix} 0 & n \\ 0 & 0 \end{smallmatrix}\right)$. Then we have

$$\begin{aligned}(0, \gamma(f(m \otimes x))) &= \alpha(0, f(m \otimes x)) = \alpha(m'(x, 0)) = m'\alpha(x, 0) \\ &= m'(\beta(x), 0) = (0, f'(m \otimes \beta(x)))\end{aligned}$$

and

$$(\beta(g(n \otimes y)), 0) = \alpha(g(n \otimes y), 0) = \alpha(n'(0, y)) = n'\alpha(0, y) = n'(0, \gamma(y)) = (g'(n \otimes \gamma(y)), 0)$$

This implies that the following diagrams are commutative:

$$\begin{tikzcd} M \otimes_A X \arrow[r, "f"] \arrow[d, "1_M \otimes \beta"'] & Y \arrow[d, "\gamma"] \\ M \otimes_A X' \arrow[r, "f'"] & Y' \end{tikzcd} \qquad \begin{tikzcd} N \otimes_B Y \arrow[r, "g"] \arrow[d, "1_N \otimes \gamma"'] & X \arrow[d, "\beta"] \\ N \otimes_B Y' \arrow[r, "g'"] & X' \end{tikzcd}$$

Thus the map $(\beta, \gamma)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ is a morphism in $\mathcal{M}(\phi, \psi)$ and $F(\beta, \gamma) = \alpha$. Hence the functor $F$ is full and we infer that $F$ is an equivalence of categories. □

In Chapter 4 we are going to work with the above description of $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$.

**3.8.2. Morita Categories of Module Categories.** Let

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$$

be a Morita ring as in the previous section. Consider the functors $F = M \otimes_A -\colon \mathsf{Mod}\text{-}A \to \mathsf{Mod}\text{-}B$ and $G = N \otimes_B -\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}A$. Then the $A$-$A$-bimodule homomorphism $\psi\colon N \otimes_B M \longrightarrow A$ defines a natural transformation $\Psi\colon GF \longrightarrow \mathrm{Id}_{\mathsf{Mod}\text{-}A}$, where $\Psi_X = \psi \otimes 1_X$, and the $B$-$B$-bimodule homomorphism $\phi\colon M \otimes_A N \longrightarrow B$ defines a natural transformation $\Phi\colon FG \longrightarrow \mathrm{Id}_{\mathsf{Mod}\text{-}B}$, where $\Phi_Y = \phi \otimes 1_Y$. Then

$$\mathcal{M}(\phi, \psi) = (\mathsf{Mod}\text{-}A, \mathsf{Mod}\text{-}B, M \otimes_A -, N \otimes_B -, \Phi, \Psi)$$

is a Morita category of $\mathsf{Mod}\text{-}A$ and $\mathsf{Mod}\text{-}B$ by the natural transformations $\Phi$ and $\Psi$.

PROPOSITION 3.8.2. *Let $\mathcal{M}(\phi, \psi)$ be a Morita category of the abelian categories $\mathscr{A}$ and $\mathscr{B}$ by the natural transformations $\phi\colon FG \longrightarrow \mathrm{Id}_{\mathscr{B}}$ and $\psi\colon GF \longrightarrow \mathrm{Id}_{\mathscr{A}}$. Let $X \in \mathscr{A}$, $Y \in \mathscr{B}$, $A = \mathsf{End}_{\mathscr{A}}(X)$, $B = \mathsf{End}_{\mathscr{B}}(Y)$ and $\Lambda = \mathsf{End}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X) \oplus \mathsf{T}_{\mathscr{B}}(Y))$. Then $\Lambda$ is a Morita ring, that is there exists a $A$-$B$-bimodule $N$ and a $B$-$A$-bimodule $M$, a $B$-$B$-bimodule morphism $\phi\colon M \otimes_A N \longrightarrow B$, a $A$-$A$-bimodule morphism $\psi\colon N \otimes_B M \longrightarrow A$, and an isomorphism of rings*

$$\Lambda \simeq \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$$

*Moreover, if $F = M \otimes_A -\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}B$, $G = N \otimes_B -\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}A$, then defining $\psi'_X\colon GF(X) \longrightarrow X$ by $\psi'_X = \psi \otimes 1_X$ and $\phi'_Y\colon FG(Y) \longrightarrow Y$ by $\phi'_Y = \phi \otimes 1_Y$ we get an equivalence of categories:*

$$\mathsf{Mod}\text{-}\Lambda \simeq \mathsf{Mod}\text{-}\begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix} \simeq \mathcal{M}(\phi', \psi') = (\mathsf{Mod}\text{-}A, \mathsf{Mod}\text{-}B, M \otimes_A -, N \otimes_B -, \phi', \psi')$$

PROOF. First it is easy to verify that any element of $\Lambda = \mathsf{End}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X)\oplus\mathsf{T}_{\mathscr{B}}(Y))$ is of the form:

$$[\begin{pmatrix} a & n \\ G(m)\circ\psi_X & G(b)\end{pmatrix}, \begin{pmatrix} F(a) & F(n)\circ\phi_Y \\ m & b\end{pmatrix}]$$

where $a \in A$, $b \in B$, $n \in \mathsf{Hom}_{\mathscr{A}}(X, G(Y))$ and $m \in \mathsf{Hom}_{\mathscr{B}}(Y, F(X))$. Using the adjoint pairs $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}})$ and $(\mathsf{T}_{\mathscr{B}}, \mathsf{U}_{\mathscr{B}})$, and the natural isomorphisms $\mathsf{U}_{\mathscr{A}}\mathsf{T}_{\mathscr{A}} = \mathrm{Id}_{\mathscr{A}}$ and $\mathsf{U}_{\mathscr{B}}\mathsf{T}_{\mathscr{B}} = \mathrm{Id}_{\mathscr{B}}$, we have the following isomorphism of abelian groups:

$$\begin{aligned}
\Lambda &= \mathsf{End}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X)\oplus\mathsf{T}_{\mathscr{B}}(Y)) \\
&\simeq \mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X), \mathsf{T}_{\mathscr{A}}(X)) \oplus \mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X), \mathsf{T}_{\mathscr{B}}(Y)) \\
&\oplus \mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(Y), \mathsf{T}_{\mathscr{A}}(X)) \oplus \mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(Y), \mathsf{T}_{\mathscr{B}}(Y)) \\
&\simeq \mathsf{Hom}_{\mathscr{A}}(X, \mathsf{U}_{\mathscr{A}}\mathsf{T}_{\mathscr{A}}(X)) \oplus \mathsf{Hom}_{\mathscr{A}}(X, \mathsf{U}_{\mathscr{A}}\mathsf{T}_{\mathscr{B}}(Y)) \\
&\oplus \mathsf{Hom}_{\mathscr{B}}(Y, \mathsf{U}_{\mathscr{B}}\mathsf{T}_{\mathscr{A}}(X)) \oplus \mathsf{Hom}_{\mathscr{B}}(Y, \mathsf{U}_{\mathscr{B}}\mathsf{T}_{\mathscr{B}}(Y)) \\
&\simeq A \oplus \mathsf{Hom}_{\mathscr{A}}(X, G(Y)) \oplus \mathsf{Hom}_{\mathscr{B}}(Y, F(X)) \oplus B
\end{aligned}$$

Set

$$N := \mathsf{Hom}_{\mathscr{A}}(X, G(Y)) \quad \text{and} \quad M := \mathsf{Hom}_{\mathscr{B}}(Y, F(X))$$

and define actions:

$$(a,n) \mapsto a\circ n,\ (n,b)\mapsto n\circ G(b) \quad \text{and} \quad (b,m)\mapsto b\circ m,\ (m,a)\mapsto m\circ F(a)$$

for every $a \in A$, $b \in B$, $n \in N$, $m \in M$. Then $N$ becomes an $A$-$B$-bimodule and $M$ becomes a $B$-$A$-bimodule. We define the following morphisms:

$$\phi\colon M\otimes_A N \longrightarrow B,\ \phi(m\otimes n) = m\circ F(n)\circ\phi_Y$$

$$\psi\colon N\otimes_B M \longrightarrow A,\ \psi(n\otimes m) = n\circ G(m)\circ\psi_X$$

Then it is easy to check that $\phi$ is a $B$-$B$-bimodule morphism and $\psi$ is an $A$-$A$-bimodule morphism. Let $(a,n,m,b)$, $(a',n',m',b')$ be two elements of the abelian group $A\oplus N\oplus M\oplus B$. We define multiplication by

$$(a,n,m,b)\cdot(a',n',m',b') = (aa'+\psi(n\otimes m'), an'+n\circ G(b'), m\circ F(a')+bm', \phi(m\otimes n')+bb')$$

Then it follows that $A\oplus N\oplus M\oplus B$ is a ring, and moreover it is an associative ring if

$$\psi(n\otimes m)n' = n\phi(m\otimes n') \quad \text{and} \quad \phi(m\otimes n)m' = m\psi(n\otimes m')$$

for every $n, n' \in N$ and $m, m' \in M$. We define the following map:

$$\alpha\colon A\oplus N\oplus M\oplus B \longrightarrow \Lambda = \mathsf{End}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X)\oplus\mathsf{T}_{\mathscr{B}}(Y))$$

$$(a,n,m,b) \mapsto [\begin{pmatrix} a & n \\ G(m)\circ\psi_X & G(b)\end{pmatrix}, \begin{pmatrix} F(a) & Fn\circ\phi_Y \\ m & s\end{pmatrix}]$$

Then $\alpha$ is an isomorphism of rings and if we consider the Morita ring $\left(\begin{smallmatrix} A & N \\ M & B\end{smallmatrix}\right)$ then the map $\omega\colon \left(\begin{smallmatrix} A & N \\ M & B\end{smallmatrix}\right) \longrightarrow \mathsf{End}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(X)\oplus\mathsf{T}_{\mathscr{B}}(Y))$ given by $\omega(\left(\begin{smallmatrix} a & n \\ m & b\end{smallmatrix}\right)) = (a,n,m,b)$ is an isomorphism of rings and we are done. Finally the last assertion follows from the equivalence of categories between $\mathsf{Mod}\text{-}\Lambda$ and $\mathcal{M}(\phi',\psi')$. $\square$

The following result describes explicitly the coproduct in $\mathcal{M}(\phi,\psi)$.

LEMMA 3.8.3. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category. If $\mathscr{A}$ and $\mathscr{B}$ are cocomplete and the functors $F$, $G$ preserve coproducts then $\mathcal{M}(\phi,\psi)$ is cocomplete.*

Proof. Suppose that $(X_i, Y_i, f_i, g_i)_{i\in I}$ is a family of objects in $\mathcal{M}(\phi,\psi)$. We will show that the coproduct of this family of objects is the tuple $\oplus_{i\in I}(X_i, Y_i, f_i, g_i) = (\oplus X_i, \oplus Y_i, \oplus f_i, \oplus g_i) \in \mathcal{M}(\phi,\psi)$, where $\oplus_{i\in I}X_i$ is the coproduct of $(X_i)_{i\in I}$ in $\mathscr{A}$ with injections $\kappa_i\colon X_i \longrightarrow \oplus_{i\in I}X_i$ and $\oplus_{i\in I}Y_i$ is the coproduct of $(Y_i)_{i\in I}$ in $\mathscr{B}$ with injections $\lambda_i\colon Y_i \longrightarrow \oplus_{i\in I}Y_i$. Since $F$, $G$ preserve coproducts and each $(X_i, Y_i, f_i, g_i) \in \mathcal{M}(\phi,\psi)$ we deduce that the object $(\oplus X_i, \oplus Y_i, \oplus f_i, \oplus g_i)$ lies in $\mathcal{M}(\phi,\psi)$. Also since the following diagrams are commutative:

$$\begin{tikzcd} F(X_i) \arrow[r, "f_i"] \arrow[d, "F(\kappa_i)"'] & Y_i \arrow[d, "\lambda_i"] \\ \oplus_{i\in I}F(X_i) \arrow[r, "\oplus f_i"] & \oplus_{i\in I}Y_i \end{tikzcd} \qquad \begin{tikzcd} G(Y_i) \arrow[r, "g_i"] \arrow[d, "G(\lambda_i)"'] & X_i \arrow[d, "\kappa_i"] \\ \oplus_{i\in I}G(Y_i) \arrow[r, "\oplus g_i"] & \oplus_{i\in I}X_i \end{tikzcd}$$

we infer that $(\kappa_i, \lambda_i)\colon (X_i, Y_i, f_i, g_i) \longrightarrow (\oplus X_i, \oplus Y_i, \oplus f_i, \oplus g_i)$ is a morphism in $\mathcal{M}(\phi,\psi)$. Let $(a_i, b_i)\colon (X_i, Y_i, f_i, g_i) \longrightarrow (X, Y, f, g)$ be a morphism in $\mathcal{M}(\phi,\psi)$. Then we have the following commutative diagrams:

$$\begin{tikzcd} X_i \arrow[r, "\kappa_i"] \arrow[d, "a_i"'] & \oplus_{i\in I}X_i \arrow[dl, "\eta"] \\ X & \end{tikzcd} \qquad \begin{tikzcd} Y_i \arrow[r, "\lambda_i"] \arrow[d, "b_i"'] & \oplus_{i\in I}Y_i \arrow[dl, "\theta"] \\ Y & \end{tikzcd}$$

and therefore the following diagram is commutative:

$$\begin{tikzcd} (X_i, Y_i, f_i, g_i) \arrow[r, "(\kappa_i,\lambda_i)"] \arrow[d, "(a_i,b_i)"'] & (\oplus X_i, \oplus Y_i, \oplus f_i, \oplus g_i) \arrow[dl, "(\eta,\theta)"] \\ (X, Y, f, g) & \end{tikzcd}$$

Note that $(\eta, \theta)$ is the unique morphism making the above diagram commutative. It remains to show that $(\eta, \theta)$ is a morphism in $\mathcal{M}(\phi,\psi)$, i.e. that the following diagrams are commutative:

$$\begin{tikzcd} \oplus_{i\in I}F(X_i) \arrow[r, "\oplus f_i"] \arrow[d, "F(\eta)"'] & \oplus Y_i \arrow[d, "\theta"] \\ F(X) \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} \oplus_{i\in I}G(Y_i) \arrow[r, "\oplus g_i"] \arrow[d, "G(\theta)"'] & \oplus_{i\in I}X_i \arrow[d, "\eta"] \\ G(Y) \arrow[r, "g"] & X \end{tikzcd}$$

Since $\kappa_i \circ \eta = a_i$ we have $F(\kappa_i) \circ F(\eta) \circ f = F(a_i) \circ f$ and also since $(\kappa_i, \lambda_i)$ and $(a_i, b_i)$ are morphisms in $\mathcal{M}(\phi,\psi)$ we have $F(\kappa_i) \circ \oplus f_i \circ \theta = f_i \circ \lambda_i \circ \theta = f_i \circ b_i = F(a_i) \circ f$. Then from the universal property of coproducts:

$$\begin{tikzcd} F(X_i) \arrow[r, "F(\kappa_i)"] \arrow[d, "F(a_i)\circ f"'] & \oplus_{i\in I}F(X_i) \arrow[dl, "F(\eta)\circ f"] \arrow[dl, bend left=60, "\oplus f_i\circ\theta"] \\ Y & \end{tikzcd}$$

it follows that $F(\eta) \circ f = \oplus f_i \circ \theta$. Similarly we show that $G(\theta) \circ g = \oplus g_i \circ \eta$. $\square$

Before we proceed we recall the following characterization for an abelian category to be a module category.

THEOREM 3.8.4. *[21] [51] [94] [106] A cocomplete abelian category $\mathscr{A}$ is equivalent with a module category if and only if $\mathscr{A}$ contains a compact (i.e. $\mathsf{Hom}_{\mathscr{A}}(P,-)$ preserves coproducts) projective generator $P$. In this case $\mathscr{A}$ is equivalent with $\mathsf{Mod}\text{-}\mathsf{End}_{\mathscr{A}}(P)$.*

We close this section with the next result where we characterize when a Morita category is a module category.

PROPOSITION 3.8.5. *Let $\mathcal{M}(\phi,\psi)$ be a Morita category of the abelian categories $\mathscr{A}$ and $\mathscr{B}$. Suppose that $\mathscr{A}$ and $\mathscr{B}$ have coproducts and enough projectives, and the functors $F$, $G$ preserves coproducts. Then the following are equivalent:*

(i) *$\mathcal{M}(\phi,\psi)$ is a module category.*
(ii) *$\mathscr{A}$ and $\mathscr{B}$ are module categories.*

*Moreover if one of the above equivalent statements hold and $\mathscr{A} = \mathsf{Mod}\text{-}A$, $\mathscr{B} = \mathsf{Mod}\text{-}B$, then $\mathcal{M}(\phi,\psi) \simeq \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$, where $\Lambda_{(\phi,\psi)}$ is the Morita ring of $A$ and $B$.*

PROOF. (ii) $\Rightarrow$ (i) If $\mathscr{A}$ and $\mathscr{B}$ are module categories then $\mathscr{A}$, resp. $\mathscr{B}$, admits a compact projective generator $P$, resp. $Q$. From Proposition 3.5.1 it follows that the object $\mathsf{T}_{\mathscr{A}}(P) \oplus \mathsf{T}_{\mathscr{B}}(Q)$ is projective in $\mathcal{M}(\phi,\psi)$. Let $(X_i,Y_i,f_i,g_i)_{i\in I}$ be a family of objects in $\mathcal{M}(\phi,\psi)$. Note that from Lemma 3.8.3 the functors $\mathsf{U}_{\mathscr{A}}$ and $\mathsf{U}_{\mathscr{B}}$ preserve coproducts. Then using the adjoint pairs $(\mathsf{T}_{\mathscr{A}},\mathsf{U}_{\mathscr{A}})$ and $(\mathsf{T}_{\mathscr{B}},\mathsf{U}_{\mathscr{B}})$ we have:

$$\begin{aligned}
\big(\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q),\oplus_{i\in I}(X_i,Y_i,f_i,g_i)\big) &\simeq \mathsf{Hom}_{\mathscr{A}}\big(P,\mathsf{U}_{\mathscr{A}}(\oplus_{i\in I}(X_i,Y_i,f_i,g_i))\big)\\
&\oplus \mathsf{Hom}_{\mathscr{B}}\big(Q,\mathsf{U}_{\mathscr{B}}(\oplus_{i\in I}(X_i,Y_i,f_i,g_i))\big)\\
&\simeq \mathsf{Hom}_{\mathscr{A}}\big(P,\oplus_{i\in I}\mathsf{U}_{\mathscr{A}}(X_i,Y_i,f_i,g_i)\big)\\
&\oplus \mathsf{Hom}_{\mathscr{B}}\big(Q,\oplus_{i\in I}\mathsf{U}_{\mathscr{B}}(X_i,Y_i,f_i,g_i)\big)\\
&\simeq \mathsf{Hom}_{\mathscr{A}}(P,\oplus X_i)\oplus\mathsf{Hom}_{\mathscr{B}}(Q,\oplus Y_i)\\
&= \big(\oplus\,\mathsf{Hom}_{\mathscr{A}}(P,X_i)\big)\oplus\big(\oplus\,\mathsf{Hom}_{\mathscr{B}}(Q,Y_i)\big)\\
&= \big(\oplus\,\mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(P),(X_i,Y_i,f_i,g_i))\big)\\
&\oplus \big(\oplus\,\mathsf{Hom}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{B}}(Q),(X_i,Y_i,f_i,g_i))\big)\\
&= \oplus_{i\in I}\mathsf{Hom}_{\mathcal{M}(\phi,\psi}(\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q),(X_i,Y_i,f_i,g_i))
\end{aligned}$$

Hence $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)$ is a compact object of $\mathcal{M}(\phi,\psi)$. Also $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)$ is a generator since $\mathscr{A}$ and $\mathscr{B}$ have enough projectives and so there exists a non-zero morphism $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q) \longrightarrow (X,Y,f,g)$ for every $(X,Y,f,g)\in\mathcal{M}(\phi,\psi)$, see the proof of Proposition 3.5.1. Finally, since $\mathscr{A}$ and $\mathscr{B}$ are cocomplete and $F$, $G$ preserves coproducts then from Lemma 3.8.3 it follows that $\mathcal{M}(\phi,\psi)$ is cocomplete. We infer that $\mathcal{M}(\phi,\psi)$ is the module category $\mathsf{Mod}\text{-}\mathsf{End}_{\mathcal{M}(\phi,\psi)}(\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q))$.

(i) $\Rightarrow$ (ii) Suppose that $\mathcal{M}(\phi,\psi)$ is a module category. Since the categories $\mathscr{A}$ and $\mathscr{B}$ are coreflective (see section 3.4 for the definition) it follows from [121, Chapter X] that they are cocomplete. Also from Proposition 3.5.1 we assume that $\mathsf{T}_{\mathscr{A}}(P)\oplus\mathsf{T}_{\mathscr{B}}(Q)$ is a projective compact generator of $\mathcal{M}(\phi,\psi)$. Then it is easy to see that $P$, resp. $Q$, is a projective generator for $\mathscr{A}$, resp. $\mathscr{B}$. Let $(X_i)_{i\in I}$ be a family of objects in $\mathscr{A}$. Since the object $\mathsf{T}_{\mathscr{A}}(P)$ is compact in $\mathcal{M}(\phi,\psi)$ and using the adjoint pair $(\mathsf{T}_{\mathscr{A}},\mathsf{U}_{\mathscr{A}})$ we have:

$$\begin{aligned}
\mathsf{Hom}_{\mathscr{A}}\big(P,\oplus_{i\in I}X_i) &\simeq \mathsf{Hom}_{\mathscr{A}}(P,\mathsf{U}_{\mathscr{A}}(\oplus X_i,\oplus FX_i,\mathrm{Id}_{\oplus FX_i},\oplus\psi_{X_i})\big)\\
&\simeq \mathsf{Hom}_{\mathcal{M}(\phi,\psi)}\big(\mathsf{T}_{\mathscr{A}}(P),(\oplus X_i,\oplus F(X_i),\mathrm{Id}_{\oplus F(X_i)},\oplus\psi_{X_i})\big)\\
&\simeq \mathsf{Hom}_{\mathcal{M}(\phi,\psi)}\big(\mathsf{T}_{\mathscr{A}}(P),\oplus(X_i,F(X_i),\mathrm{Id}_{F(X_i)},\psi_{X_i})\big)\\
&\simeq \oplus_{i\in I}\mathsf{Hom}_{\mathcal{M}(\phi,\psi)}\big(\mathsf{T}_{\mathscr{A}}(P),(X_i,F(X_i),\mathrm{Id}_{F(X_i)},\psi_{X_i})\big)\\
&\simeq \oplus_{i\in I}\mathsf{Hom}_{\mathscr{A}}\big(P,\mathsf{U}_{\mathscr{A}}(X_i,F(X_i),\mathrm{Id}_{F(X_i)},\psi_{X_i})\big)\\
&\simeq \oplus_{i\in I}\mathsf{Hom}_{\mathscr{A}}(P,X_i)
\end{aligned}$$

Thus the object $P$ is compact in $\mathscr{A}$. Similarly using the adjoint pair $(\mathsf{T}_{\mathscr{B}},\mathsf{U}_{\mathscr{B}})$ and that the object $\mathsf{T}_{\mathscr{B}}(Q)$ is compact it follows that $Q$ is a compact object of $\mathscr{B}$. Hence $\mathscr{A}$ is the module category $\mathsf{Mod}\text{-}\mathsf{End}\,P$ and $\mathscr{B}$ is the module category $\mathsf{Mod}\text{-}\mathsf{End}\,Q$. $\square$

CHAPTER 4

# Artin Algebras Arising from Morita Contexts

In this Chapter we study Morita rings

$$\Lambda_{(\phi,\psi)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$$

in the context of Artin algebras, concentrating mainly at representation-theoretic and homological aspects. First we investigate covariantly finite, contravariantly finite, and functorially finite subcategories of the module category of a Morita ring when the bimodule homomorphisms $\phi$ and $\psi$ are zero. Further, under some assumpions, we give bounds for the global dimension of a Morita ring $\Lambda_{(0,0)}$, as an Artin algebra, in terms of the global dimensions of $A$ and $B$ in the case when both $\phi$ and $\psi$ are zero. We illustrate our bounds with some examples. Finally we investigate when a Morita ring is a Gorenstein Artin algebra. In particular we show under a condition that $\Lambda_{(\phi,\psi)}$ is Gorenstein and as an application we determine the Gorenstein-projective modules over the Morita ring $\Lambda_{(\phi,\psi)}$ in case $A = N = M = B$ and $A$ an Artin algebra. The results of this Chapter are included in the paper entitled: On Artin algebras arising from Morita Contexts [60], which is joint work with Edward L. Green.

## 4.1. Preliminaries on Morita Rings

Let $A$ and $B$ be rings, ${}_AN_B$ an $A$-$B$-bimodule, ${}_BM_A$ a $B$-$A$-bimodule, and $\phi\colon M \otimes_A N \longrightarrow B$ a $B$-$B$-bimodule homomorphism, and $\psi\colon N \otimes_B M \longrightarrow A$ an $A$-$A$-bimodule homomorphism. Recall that from the Morita context $\mathcal{M} = (A, N, M, B, \phi, \psi)$ we defined before the Morita ring:

$$\Lambda_{(\phi,\psi)}(\mathcal{M}) = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$$

where the addition of elements of $\Lambda_{(\phi,\psi)}$ is componentwise and multiplication is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \cdot \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(n \otimes m') & an' + nb' \\ ma' + bm' & bb' + \phi(m \otimes n') \end{pmatrix}$$

We assume that $\phi(m \otimes n)m' = m\psi(n \otimes m')$ and $n\phi(m \otimes n') = \psi(n \otimes m)n'$ for all $m, m' \in M$ and $n, n' \in N$. This condition ensures that $\Lambda_{(\phi,\psi)}(\mathcal{M})$ is an associative ring. From now on we will write for simplicity $\Lambda_{(\phi,\psi)}$ instead of $\Lambda_{(\phi,\psi)}(\mathcal{M})$.

Remark 4.1.1. Morita rings have appeared in the literature under various names, for instance: the ring of the Morita context [92] and generalized matrix rings [59], [105].

Since we are interested in Artin algebras, the following easy result characterizes when a Morita ring is an Artin algebra. Recall that an Artin algebra $\Lambda$ is an $R$-algebra which is finitely generated as an $R$-module, where $R$ is a commutative artinian ring [18].

Proposition 4.1.2. *Let $\Lambda_{(\phi,\psi)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring. Then $\Lambda_{(\phi,\psi)}$ is an Artin algebra if and only if there is a commutative Artin ring $R$ such that $A$ and $B$ are*

*Artin $R$-algebras and $M$ and $N$ are finitely generated over $R$ which acts centrally on $M$ and $N$.*

PROOF. Suppose that there is a commutative Artin ring $R$ such that $A$ and $B$ are Artin $R$-algebras and $M$ and $N$ are finitely generated over $R$ which acts centrally on $M$ and $N$. Then there is a ring morphism $f_A\colon R \longrightarrow A$ whose image is in the center of $A$ and there is a ring morphism $f_B\colon R \longrightarrow B$ whose image is in the center of $B$. We define the ring morphism

$$f\colon R \longrightarrow \Lambda_{(\phi,\psi)},\ r \mapsto f(r) = \left(\begin{smallmatrix} f_A(r) & 0 \\ 0 & f_B(r) \end{smallmatrix}\right)$$

We claim that $\mathsf{Im}\, f \subseteq \mathsf{Z}(\Lambda_{(\phi,\psi)})$, where $\mathsf{Z}(\Lambda_{(\phi,\psi)})$ is the center of $\Lambda_{(\phi,\psi)}$. Indeed since $\mathsf{Im}\, f_A \subseteq \mathsf{Z}(A)$, $\mathsf{Im}\, f_B \subseteq \mathsf{Z}(B)$ and $R$ acts centrally on $M$ and $N$ we have:

$$\begin{aligned} \begin{pmatrix} f_A(r) & 0 \\ 0 & f_B(r) \end{pmatrix} \cdot \begin{pmatrix} a & m \\ n & b \end{pmatrix} &= \begin{pmatrix} f_A(r)\cdot a & f_A(r)\cdot m \\ f_B(r)\cdot n & f_B(r)\cdot b \end{pmatrix} \\ &= \begin{pmatrix} a\cdot f_A(r) & m\cdot f_A(r) \\ n\cdot f_B(r) & b\cdot f_B(r) \end{pmatrix} \\ &= \begin{pmatrix} a & m \\ n & b \end{pmatrix} \cdot \begin{pmatrix} f_A(r) & 0 \\ 0 & f_B(r) \end{pmatrix} \end{aligned}$$

and so our claim holds. Since $A$, $B$, $M$, $N$ are finitely generated over $R$ it follows that $\Lambda_{(\phi,\psi)}$ is also finitely generated over $R$. We infer that $\Lambda_{(\phi,\psi)}$ is an Artin algebra.

Suppose conversely that $\Lambda_{(\phi,\psi)}$ is an Artin algebra, i.e. there is a commutative Artin ring $R$ and a ring homomorphism $f\colon R \longrightarrow \Lambda_{(\phi,\psi)}$ with $\mathsf{Im}\, f \subseteq \mathsf{Z}(\Lambda_{(\phi,\psi)})$, and $\Lambda_{(\phi,\psi)}$ is a finitely generated $R$-module. From the following compositions:

$$R \longrightarrow \Lambda_{(\phi,\psi)} \longrightarrow A \quad \text{and} \quad R \longrightarrow \Lambda_{(\phi,\psi)} \longrightarrow B$$

it follows that $A$ and $B$ are finitely generated $R$-modules. Since $f(r) \in \mathsf{Z}(\Lambda_{(\phi,\psi)})$, i.e.

$$f(r) \cdot \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \cdot f(r) \tag{$*$}$$

we have $\mathsf{Im}\, f \subseteq \mathsf{Z}(A)$ and $\mathsf{Im}\, f \subseteq \mathsf{Z}(B)$. Hence $A$ and $B$ are Artin $R$-algebras. It follows also from the relation $(*)$ that the ring $R$ acts centrally on $M$ and $N$. $\square$

The description of the modules over a Morita ring $\Lambda_{(\phi,\psi)}$ is well known [59], and was defined before in section 3.8. But for completeness and due to our needs we include it also here. Let $\mathcal{M}(\Lambda)$ be the category whose objects are tuples $(X, Y, f, g)$ where $X \in \mathsf{Mod}\text{-}A$, $Y \in \mathsf{Mod}\text{-}B$, $f \in \mathsf{Hom}_B(M \otimes_A X, Y)$ and $g \in \mathsf{Hom}_A(N \otimes_B Y, X)$ such that the following diagrams are commutative:

$$\begin{tikzcd} N \otimes_B M \otimes_A X \arrow[r, "N\otimes f"] \arrow[d, "\psi\otimes \mathrm{Id}_X"'] & N \otimes_B Y \arrow[d, "g"] \\ A \otimes_A X \arrow[r, "\simeq"] & X \end{tikzcd} \qquad \begin{tikzcd} M \otimes_A N \otimes_B Y \arrow[r, "M\otimes g"] \arrow[d, "\phi\otimes \mathrm{Id}_Y"'] & M \otimes_A X \arrow[d, "f"] \\ B \otimes_B Y \arrow[r, "\simeq"] & Y \end{tikzcd}$$

We denote by $\Psi_X$ and $\Phi_Y$ the following compositions:

$$\begin{tikzcd} N\otimes_B M\otimes_A X \arrow[r, "\psi\otimes\mathrm{Id}_X"] \arrow[rr, bend left, "\Psi_X"] & A \otimes_A X \arrow[r, "\simeq"] & X \end{tikzcd} \qquad \begin{tikzcd} M \otimes_A N \otimes_B Y \arrow[r, "\phi\otimes\mathrm{Id}_Y"] \arrow[rr, bend left, "\Phi_Y"] & B \otimes_B Y \arrow[r, "\simeq"] & Y \end{tikzcd}$$

Let $(X, Y, f, g)$ and $(X', Y', f', g')$ be objects of $\mathcal{M}(\Lambda)$. Then a morphism $(X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathcal{M}(\Lambda)$ is a pair of homomorphisms $(a, b)$ where $a\colon X \longrightarrow X'$ is an $A$-morphism and $b\colon Y \longrightarrow Y'$ is a $B$-morphism such that the following diagrams are commutative:

$$\begin{tikzcd} M \otimes_A X \arrow[r, "f"] \arrow[d, "M\otimes a"'] & Y \arrow[d, "b"] \\ M \otimes_A X' \arrow[r, "f'"] & Y' \end{tikzcd} \qquad \begin{tikzcd} N \otimes_B Y \arrow[r, "g"] \arrow[d, "N\otimes b"'] & X \arrow[d, "a"] \\ N \otimes_B Y' \arrow[r, "g'"] & X' \end{tikzcd}$$

Dually since the functors $M\otimes_A -\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}B$ and $N\otimes_B -\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}A$ have right adjoints we can define the category $\widetilde{\mathcal{M}}(\Lambda)$. We denote by

$$\pi : \mathsf{Hom}_B(M \otimes_A X, Y) \xrightarrow{\simeq} \mathsf{Hom}_A(X, \mathsf{Hom}_B(M, Y)) \tag{4.1.1}$$

and

$$\rho : \mathsf{Hom}_A(N \otimes_B Y, X) \xrightarrow{\simeq} \mathsf{Hom}_B(Y, \mathsf{Hom}_A(N, X)) \tag{4.1.2}$$

the adjoint isomorphims and let $\epsilon\colon M \otimes_A \mathsf{Hom}_B(M, -) \longrightarrow \mathrm{Id}_{\mathsf{Mod}\text{-}B}$, resp. $\epsilon'\colon N \otimes_B \mathsf{Hom}_A(N, -) \longrightarrow \mathrm{Id}_{\mathsf{Mod}\text{-}A}$, and $\delta\colon \mathrm{Id}_{\mathsf{Mod}\text{-}A} \longrightarrow \mathsf{Hom}_B(M, M\otimes_A -)$, resp. $\delta'\colon \mathrm{Id}_{\mathsf{Mod}\text{-}B} \longrightarrow \mathsf{Hom}_A(N, N\otimes_B -)$, be the counit and the unit of the adjoint pair $(M\otimes_A -, \mathsf{Hom}_A(M, -))$, resp. $(N \otimes_B -, \mathsf{Hom}_A(N, -))$. The objects of $\widetilde{\mathcal{M}}$ are tuples $(X, Y, \kappa, \lambda)$ where $X \in \mathsf{Mod}\text{-}A$, $Y \in \mathsf{Mod}\text{-}B$, $\kappa\colon X \longrightarrow \mathsf{Hom}_B(M, Y)$ and $\lambda\colon Y \longrightarrow \mathsf{Hom}_A(N, X)$ such that the following diagrams are commutative:

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "\kappa"'] & \mathsf{Hom}_A(A, X) \arrow[d] \\ \mathsf{Hom}_B(M, Y) \arrow[r, "{(M,\lambda)}"] & \mathsf{Hom}_B(M, \mathsf{Hom}_A(N, X)) \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\simeq"] \arrow[d, "\lambda"'] & \mathsf{Hom}_B(B, Y) \arrow[d] \\ \mathsf{Hom}_A(N, X) \arrow[r, "{(N,\kappa)}"] & \mathsf{Hom}_A(N, \mathsf{Hom}_B(M, Y)) \end{tikzcd}$$

Let $(X, Y, \kappa, \lambda)$ and $(X', Y', \kappa', \lambda')$ be objects in $\widetilde{\mathcal{M}}(\Lambda)$. Then a morphism $(X, Y, \kappa, \lambda) \longrightarrow (X', Y', \kappa', \lambda')$ in $\widetilde{\mathcal{M}}(\Lambda)$ is a pair of homomorphisms $(c, d)$ where $c\colon X \longrightarrow X'$ is an $A$-morphism and $d\colon Y \longrightarrow Y'$ is a $B$-morphism such that the following diagrams are commutative:

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d, "c"'] & \mathsf{Hom}_B(M, Y) \arrow[d, "{\mathsf{Hom}_B(M,d)}"] \\ X' \arrow[r, "\kappa'"] & \mathsf{Hom}_B(M, Y') \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\lambda"] \arrow[d, "d"'] & \mathsf{Hom}_A(N, X) \arrow[d, "{\mathsf{Hom}_A(N,c)}"] \\ Y' \arrow[r, "\lambda'"] & \mathsf{Hom}_A(N, X') \end{tikzcd}$$

We define the functor $\mathsf{F}\colon \mathcal{M}(\Lambda) \longrightarrow \widetilde{\mathcal{M}}(\Lambda)$ by $\mathsf{F}(X, Y, f, g) = (X, Y, \pi(f), \rho(g))$ on objects and $\mathsf{F}(a, b) = (a, b)$ on morphisms. Then it is straightforward that $\mathsf{F}\colon \mathcal{M}(\Lambda) \longrightarrow \widetilde{\mathcal{M}}(\Lambda)$ is an isomorphism of categories with inverse $\mathsf{G}(X, Y, \kappa, \lambda) = (X, Y, \pi^{-1}(\kappa), \rho^{-1}(\lambda))$ on objects and $\mathsf{G}(a, b) = (a, b)$ on morphisms. The relationship between $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ and $\mathcal{M}(\Lambda)$ is given via the functor $\mathsf{F}'\colon \mathcal{M}(\Lambda) \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ which is defined as follows. If $(X, Y, f, g)$ is an oblect of $\mathcal{M}(\Lambda)$, then we define $\mathsf{F}'(X, Y, f, g) = X \oplus Y$ as abelian groups, and the $\Lambda_{(\phi,\psi)}$-module structure is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} (x, y) = (ax + g(n \otimes y), by + f(m \otimes x))$$

for all $a \in A, b \in B, n \in N, m \in M, x \in X$ and $y \in Y$. One can verify easily that the object $\mathsf{F}'(X,Y,f,g)$ is a $\Lambda_{(\phi,\psi)}$-module. If $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ is a morphism in $\mathcal{M}$ then $\mathsf{F}'(a,b) = a \oplus b\colon X \oplus Y \longrightarrow X' \oplus Y'$.

PROPOSITION 4.1.3. *Let $\Lambda_{(\phi,\psi)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring. Then the categories $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$, $\mathcal{M}(\Lambda)$ and $\widetilde{\mathcal{M}}(\Lambda)$ are equivalent.*

PROOF. See [59, Theorem 1.5] and Propostion 3.8.1. □

From now on we will identify the modules over $\Lambda_{(\phi,\psi)}$ with the objects of $\mathcal{M}(\Lambda)$. We define the following functors:

(i) The functor $\mathsf{T}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is defined by
$$\mathsf{T}_A(X) = (X, M \otimes_A X, \mathrm{Id}_{M\otimes X}, \Psi_X)$$
on the objects $X \in \mathsf{Mod}\text{-}A$ and given an $A$-morphism $a\colon X \longrightarrow X'$ then $\mathsf{T}_A(a) = (a, M \otimes a)$ is a morphism in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$.

(ii) The functor $\mathsf{U}_A\colon \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \longrightarrow \mathsf{Mod}\text{-}A$ is defined on the objects $(X,Y,f,g)$ of $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ by
$$\mathsf{U}_A(X,Y,f,g) = X$$
and given a morphism $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ then $\mathsf{U}_A(a,b) = a$.

(iii) The functor $\mathsf{T}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is defined by
$$\mathsf{T}_B(Y) = (N \otimes_B Y, Y, \Phi_Y, \mathrm{Id}_{N\otimes Y})$$
on the objects $Y \in \mathsf{Mod}\text{-}B$ and given a $B$-morphism $b\colon Y \longrightarrow Y'$ then $\mathsf{T}_B(b) = (N \otimes b, b)$ is a morphism in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$.

(iv) The functor $\mathsf{U}_B\colon \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \longrightarrow \mathsf{Mod}\text{-}B$ is defined on the objects $(X,Y,f,g)$ of $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ by
$$\mathsf{U}_B(X,Y,f,g) = Y$$
and given a morphism $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ then $\mathsf{U}_B(a,b) = b$.

(v) The functor $\mathsf{H}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is defined by
$$\mathsf{H}_A(X) = (X, \mathsf{Hom}_A(N,X), \delta'_{M\otimes X} \circ \mathsf{Hom}_A(N,\Psi_X), \epsilon'_X)$$
on the objects $X \in \mathsf{Mod}\text{-}A$ and given an $A$-morphism $a\colon X \longrightarrow X'$ then $\mathsf{H}_A(a) = (a, \mathsf{Hom}_A(N,a))$ is a morphism in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$.

(vi) The functor $\mathsf{H}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is defined by
$$\mathsf{H}_B(Y) = (\mathsf{Hom}_B(M,Y), Y, \epsilon_Y, \delta_{N\otimes Y} \circ \mathsf{Hom}_B(M,\Phi_Y))$$
on the objects $Y \in \mathsf{Mod}\text{-}B$ and given a $B$-morphism $b\colon Y \longrightarrow Y'$ then $\mathsf{H}_B(b) = (\mathsf{Hom}_B(M,b), b)$ is a morphism in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$.

(vii) Suppose that $\phi = 0 = \psi$. Then we define the functor $\mathsf{Z}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ by $\mathsf{Z}_A(X) = (X,0,0,0)$ on the objects $X \in \mathsf{Mod}\text{-}A$ and if $a\colon X \longrightarrow X'$ is an $A$-morphism then $\mathsf{Z}_A(a) = (a,0)$. Dually we define the functor $\mathsf{Z}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$.

The following result collects and recalls some useful properties about these functors and the module category over Morita rings.

PROPOSITION 4.1.4. (i) *The functors $\mathsf{T}_A$, $\mathsf{T}_B$ and $\mathsf{H}_A, \mathsf{H}_B$ are fully faithful.*

(ii) *The pairs* $(\mathsf{T}_A, \mathsf{U}_A)$, $(\mathsf{T}_B, \mathsf{U}_B)$ *and* $(\mathsf{U}_A, \mathsf{H}_A)$, $(\mathsf{U}_B, \mathsf{H}_B)$ *are adjoint pairs of functors.*

(iii) *The functors* $\mathsf{U}_A$ *and* $\mathsf{U}_B$ *are exact.*

(iv) *We have* $\operatorname{Ker}\mathsf{U}_A = \mathsf{Mod}\text{-}\Lambda/\Lambda e_1\Lambda \simeq \mathsf{Mod}\text{-}B/\operatorname{Im}\phi$, $\operatorname{Ker}\mathsf{U}_B = \mathsf{Mod}\text{-}\Lambda/\Lambda e_2\Lambda \simeq \mathsf{Mod}\text{-}A/\operatorname{Im}\psi$, $\mathsf{Mod}\text{-}e_1\Lambda e_1 = \mathsf{Mod}\text{-}A$ *and* $\mathsf{Mod}\text{-}e_2\Lambda e_2 = \mathsf{Mod}\text{-}B$, *where* $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and* $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *are idempotent elements of* $\Lambda_{(\phi,\psi)}$.

(v) *The following diagrams*:

$$\begin{tikzcd}
\mathsf{Mod}\text{-}B/\operatorname{Im}\phi \arrow[r, "\mathsf{inc}"] & \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{U}_A"] & \mathsf{Mod}\text{-}A \arrow[l, bend right, "\mathsf{T}_A"'] \arrow[l, bend left, "\mathsf{H}_A"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Mod}\text{-}A/\operatorname{Im}\psi \arrow[r, "\mathsf{inc}"] & \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{U}_B"] & \mathsf{Mod}\text{-}B \arrow[l, bend right, "\mathsf{T}_B"'] \arrow[l, bend left, "\mathsf{H}_B"]
\end{tikzcd}$$

*are recollements of abelian categories. If* $\phi = 0 = \psi$ *then we have the following recollements*:

$$\begin{tikzcd}
\mathsf{Mod}\text{-}B \arrow[r, "\mathsf{Z}_B"] & \mathsf{Mod}\text{-}\Lambda_{(0,0)} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{U}_A"] & \mathsf{Mod}\text{-}A \arrow[l, bend right, "\mathsf{T}_A"'] \arrow[l, bend left, "\mathsf{H}_A"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Mod}\text{-}A \arrow[r, "\mathsf{Z}_A"] & \mathsf{Mod}\text{-}\Lambda_{(0,0)} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{U}_B"] & \mathsf{Mod}\text{-}B \arrow[l, bend right, "\mathsf{T}_B"'] \arrow[l, bend left, "\mathsf{H}_B"]
\end{tikzcd}$$

PROOF. This result is a special case of Proposition 3.3.1 and Corollary 3.3.2. □

The next result describes the Morita rings $\Lambda_{(\phi,\psi)}$ with $\phi = 0 = \psi$. Recall from [18], [49] that if $R$ is a ring and ${}_RM_R$ a $R$-bimodule then the trivial extension $R \ltimes M$ is the ring with elements pairs $(r, m)$ with $r \in R$ and $m \in M$, addition is componentwise and multiplication is given by $(r, m)(r', m') = (rr', rm' + mr')$. For example let $\left(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\right)$ be a triangular matrix ring. Set $R = A \times B$ and consider $M$ as a $R$-bimodule by $(a, b)m = bm$ and $m(a, b) = ma$ for all $a \in A$, $b \in B$ and $m \in M$. Then the rings $\left(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\right)$ and $(A \times B) \ltimes M$ are isomorphic. For more details on trivial extension of rings we refer to Fossum-Griffith-Reiten [49].

PROPOSITION 4.1.5. *[49] Let* $\Lambda_{(0,0)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ *be a Morita ring where the bimodule morphisms* $\phi$ *and* $\psi$ *are zero. Then we have an isomorphism of rings*:

$$\Lambda_{(0,0)} \xrightarrow{\simeq} (A \times B) \ltimes M \oplus N$$

*where* $(A \times B) \ltimes M \oplus N$ *is the trivial extension ring of* $A \times B$ *by the* $(A \times B)$-$(A \times B)$-*bimodule* $M \oplus N$.

PROOF. First we have to show that the abelian group $M \oplus N$ is an $(A \times B)$-$(A \times B)$-bimodule. We define the morphisms:

$$(A \times B) \times M \oplus N \longrightarrow M \oplus N, \ [(a,b),(m,n)] \mapsto (bm, an)$$

and

$$M \oplus N \times (A \times B) \longrightarrow M \oplus N, \ [(m,n),(a,b)] \mapsto (ma, nb)$$

Then it is easy to establish that $M \oplus N$ is a left $A \times B$-module and a right $A \times B$-module. Also since ${}_A N_B$ and ${}_B M_A$ are bimodules it follows that

$$\begin{aligned}
(a,b)[(m,n)(a',b')] &= (a,b)(ma',nb') \\
&= [b(ma'),a(nb')] \\
&= [(bm)a'),(an)b'] \\
&= (bm,an)(a',b') \\
&= [(a,b)(m,n)](a',b')
\end{aligned}$$

and this shows that $M \oplus N$ is an $(A \times B)$-$(A \times B)$-bimodule. Hence we can define the trivial extension $(A \times B) \ltimes M \oplus N$ with elements $[(a,b),(m,n)]$ where $(a,b) \in A \times B$, $(m,n) \in M \oplus N$, the addition is componentwise and the multiplication is given by

$$\begin{aligned}
[(a_1,b_1),(m_1,n_1)] \cdot [(a_2,b_2),(m_2,n_2)] &= [(a_1,b_1) \cdot (a_2,b_2),(a_1,b_1) \cdot (m_2,n_2) \\
&+ (m_1,n_1) \cdot (a_2,b_2)] \\
&= [(a_1a_2,b_1b_2),(b_1m_2,a_1n_2) + (m_1a_2,n_1b_2)] \\
&= [(a_1a_2,b_1b_2),(b_1m_2 + m_1a_2,a_1n_2 + n_1b_2)]
\end{aligned}$$

Then it is straightforward that the morphism $\Lambda_{(0,0)} \longrightarrow (A \times B) \ltimes M \oplus N$, $\left(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\right) \mapsto [(a,b),(m,n)]$ is an isomorphism of rings. $\square$

**4.1.1. Examples of Morita Rings.** We continue by giving a variety of examples of Morita rings which will be used throughout this chapter.

The first example shows that any ring with an idempotent element is a Morita ring.

EXAMPLE 4.1.6. Let $R$ be a ring with an idempotent element $e$. Then from the Pierce decomposition of $R$ with respect to the idempotents $e, f = 1_R - e$ it follows that $R$ is the Morita ring with $A = eRe$, $B = (1-e)R(1-e)$, $N = eR(1-e)$, $M = (1-e)Re$ and the bimodule homomorphisms $\phi$, $\psi$ are induced by the multiplication in $R$.

The following example is well known from Morita equivalence.

EXAMPLE 4.1.7. Let $A$ be a ring and $P$ be an $A$-module. Then we have the following Morita ring:

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & \mathsf{Hom}_A(P,A) \\ P & \mathsf{End}_A(P) \end{pmatrix}$$

with bimodule homomorphisms $\phi\colon P \otimes_A \mathsf{Hom}_A(P,A) \longrightarrow \mathsf{End}_A(P)$, $p \otimes f \mapsto \phi(p \otimes f)(p') = pf(p')$ and $\psi\colon \mathsf{Hom}_A(P,A) \otimes_{\mathsf{End}_A(P)} P \longrightarrow A$, $f \otimes p \mapsto \psi(f \otimes p) = f(p)$. Hence any pair $(A, P_A)$, where $A$ is a ring and $P_A$ is a right $A$-module induces a Morita ring. Also it is well known that if the $A$-module $P$ is progenerator, then the rings $A$ and $\mathsf{End}_A(P)$ are Morita equivalent.

The next example shows that Morita rings are special cases of semi-trivial extensions introduced by Palmer and Roos, see [105].

Example 4.1.8. Let $R$ be a ring, $M$ a $R$-$R$-bimodule and $\theta\colon M \otimes_R M \longrightarrow M$ a $R$-$R$-bimodule homomorphism. Then on the Cartesian product $R \times M$ we define multiplication by $(r, m) \cdot (r', m') = (rr' + \theta(m, m'), rm' + mr')$ such that $\theta(m \otimes m')m'' = m\theta(m' \otimes m'')$, for every $r, r' \in R$ and $m, m', m'' \in M$. Then this data defines a ring structure with unit element on the Cartesian product set $R \times M$. This ring is denoted by $R \ltimes_\theta M$ and is called the semi-trivial extension of $R$ by $M$ and $\theta$. We refer to Palmer [105] for more details.

Let $\Lambda_{(\phi,\psi)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring. Set $R = A \times B$ and consider the $R$-$R$-bimodule $\widetilde{M} = N \times M$. Then the map $\theta\colon \widetilde{M} \otimes_R \widetilde{M} \longrightarrow R$ is a $R$-$R$-bimodule homomorphism, where $\widetilde{M} \otimes_R \widetilde{M} = N \otimes_B M \times N \otimes_B M$, and we have the following ring isomorphism: $R \ltimes_\theta M \simeq \Lambda_{(\phi,\psi)}$.

The following is the motivated example of this work.

Example 4.1.9. Let $\Lambda$ be an Artin algebra and $U, V$ two finitely generated $\Lambda$-modules. Consider the endomorphism Artin algebra $\mathsf{End}_\Lambda(U \oplus V)$. Then we have the Artin algebra:

$$\Lambda_{(\phi,\psi)} = \mathsf{End}_\Lambda(U \oplus V) \simeq \begin{pmatrix} \mathsf{End}_\Lambda(U) & \mathsf{Hom}_\Lambda(U, V) \\ \mathsf{Hom}_\Lambda(V, U) & \mathsf{End}_\Lambda(V) \end{pmatrix}$$

where the bimodule homomorphisms $\phi\colon \mathsf{Hom}_\Lambda(V, U) \otimes \mathsf{Hom}_\Lambda(U, V) \longrightarrow \mathsf{End}_\Lambda(V)$ and $\psi\colon \mathsf{Hom}_\Lambda(U, V) \otimes \mathsf{Hom}_\Lambda(V, U) \longrightarrow \mathsf{End}_\Lambda(U)$ are given by composition.

Example 4.1.10. Let $\Lambda$ be an Artin algebra and let $\{\epsilon_1, \cdots, \epsilon_n\}$ be a full set of primitive orthogonal idempotents. Suppose $n \geq 2$ and $1 \leq r < n$. Set $e_1 = \sum_{i=1}^r \epsilon_i$ and $e_2 = \sum_{i=r+1}^n \epsilon_i$. Then

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} e_1\Lambda e_1 & e_1\Lambda e_2 \\ e_2\Lambda e_1 & e_2\Lambda e_2 \end{pmatrix}$$

is a Morita ring having the structure of an Artin algebra and the bimodule homomorphisms $\phi\colon e_2\Lambda e_1 \otimes_{e_1\Lambda e_1} e_1\Lambda e_2 \longrightarrow e_2\Lambda e_2$ and $\psi\colon e_1\Lambda e_2 \otimes_{e_2\Lambda e_2} e_2\Lambda e_1 \longrightarrow e_1\Lambda e_1$ are given by multiplication.

Example 4.1.11. Let $A$ be a ring and let $I$ and $J$ be ideals in $A$. Then we have the Morita ring

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & J \\ I & A \end{pmatrix}$$

where $\phi\colon J \otimes_A I \longrightarrow A$ and $\psi\colon I \otimes_A J \longrightarrow A$ are multiplication maps. Interesting special cases are when $I = J$, $A$ is an Artin algebra and $I$ and $J$ are contained in the Jacobson radical of $A$, or $I = J = A$.

Later in this Chapter we will study the special case of Morita rings with $\phi = \psi = 0$. More generally, the case where $\phi = \psi$ is of interest. Towards this end, we have the following result and its corollary.

Lemma 4.1.12. *Suppose that*

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & N \\ A & A \end{pmatrix}$$

*is a Morita ring and that* $\alpha\colon N \longrightarrow A \otimes_A N$ *and* $\beta\colon N \longrightarrow N \otimes_A A$ *are given by* $\alpha(n) = 1 \otimes n$ *and* $\beta(n) = n \otimes 1$. *Then* $\phi \circ \alpha = \psi \circ \beta$.

PROOF. We note that $\phi(a \otimes n)a' = a\psi(n \otimes a')$ for all $a, a' \in A$ and $n \in N$. Taking $a = a' = 1_A$, we see that $\phi(\alpha(n)) = \phi(1 \otimes n) = \psi(n \otimes 1) = \psi(\beta(n))$ and the result follows. □

As a consequence, we have the following result.

COROLLARY 4.1.13. *Suppose that*

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

*is a Morita ring. Then* $\phi = \psi$.

PROOF. Noting that $A \otimes_A A$ is generated by $1 \otimes 1$, we need only to show that $\phi(1 \otimes 1) = \psi(1 \otimes 1)$. But, keeping the notation of the Lemma 4.1.12, with $N = A$, $\phi(1 \otimes 1) = \phi(\alpha(1)) = \psi(\beta(1)) = \psi(1 \otimes 1)$ and we are done. □

## 4.2. Projective, Injective and Simple Modules

In this section we describe the projective, injective and simple modules over $\Lambda_{(\phi,\psi)}$ as an Artin algebra and we examine also when $\Lambda_{(\phi,\psi)}$ is selfinjective. Throughout the section we work in the setting of finitely generated modules over the Artin algebra $\Lambda_{(\phi,\psi)}$.

**4.2.1. Projective and Injective $\Lambda_{(\phi,\psi)}$-modules.** We start with the next result which gives a description of the indecomposable projective $\Lambda_{(\phi,\psi)}$-modules.

PROPOSITION 4.2.1. *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring regarded as an Artin algebra. Then the indecomposable projective $\Lambda_{(\phi,\psi)}$-modules are objects of the form:*

$$\begin{cases} \mathsf{T}_A(P) = (P, M \otimes_A P, \mathrm{Id}_{M \otimes_A P}, \Psi_P) \\ \\ \mathsf{T}_B(Q) = (N \otimes_B Q, Q, \Phi_Q, \mathrm{Id}_{N \otimes_B Q}) \end{cases}$$

*where $P$ is an indecomposable projective $A$-module and $Q$ is an indecomposable projective $B$-module.*

PROOF. If $P = \left(\begin{smallmatrix} A & 0 \\ M & 0 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} 0 & N \\ 0 & B \end{smallmatrix}\right)$ then $\Lambda_{(\phi,\psi)} \simeq P \oplus Q$ as left $\Lambda_{(\phi,\psi)}$-modules. From Proposition 4.1.3 it follows that the object of $\mathcal{M}(\Lambda)$ which corresponds to the $\Lambda_{(\phi,\psi)}$-module $P$ is $(A, M \otimes_A A, \mathrm{Id}_{M \otimes A}, \Psi_A)$. Also the tuple $(N \otimes B, B, \Phi_B, \mathrm{Id}_{N \otimes B})$ is the object of $\mathcal{M}(\Lambda)$ corresponding to $Q$. Let $A = P_1 \oplus \cdots \oplus P_n$ be the decomposition of $A$ into a direct sum of indecomposable projective $A$-modules. Then we have the following decomposition of $(A, M \otimes_A A, \mathrm{Id}_{M \otimes A}, \Psi_A)$:

$$(A, M \otimes_A A, \mathrm{Id}_{M \otimes A}, \Psi_A) \simeq \mathsf{T}_A(P_1) \oplus \cdots \oplus \mathsf{T}_A(P_n)$$

where $\mathsf{T}_A(P_i) = (P_i, M \otimes_A P_i, \mathrm{Id}_{M \otimes_A P_i}, \Psi_{P_i})$, and for every $1 \leq i \leq n$ we have the isomorphism $\mathsf{End}_{\Lambda_{(\phi,\psi)}}(P_i, M \otimes_A P_i, \mathrm{Id}_{M \otimes_A P_i}, \Psi_{P_i}) \simeq \mathsf{End}_A(P_i)$ because $\mathsf{T}_A$ is fully faithful. Since the algebra $\mathsf{End}_A(P_i)$ is local it follows that $(P_i, M \otimes_A P_i, \mathrm{Id}_{M \otimes_A P_i}, \Psi_{P_i})$ is an indecomposable projective $\Lambda_{(\phi,\psi)}$-module, for all $1 \leq i \leq n$. Similarly if $B = Q_1 \oplus \cdots \oplus Q_m$ is the decomposition of $B$ into the direct sum of indecomposable projective $B$-modules, then we infer that $(N \otimes_B Q_i, Q_i, \Phi_{Q_i}, \mathrm{Id}_{N \otimes_B Q_i})$ is an indecomposable projective $\Lambda_{(\phi,\psi)}$-module for all $1 \leq i \leq m$. In this way we get all indecomposable projective $\Lambda_{(\phi,\psi)}$-modules up to isomorphism. □

Our aim now is to describe the injective $\Lambda_{(\phi,\psi)}$-modules. In order to do this we describe how the duality acts on the objects of $\widetilde{\mathcal{M}}(\Lambda)$ using the equivalence of categories: $\widetilde{\mathcal{M}}(\Lambda) \xrightarrow{\simeq} \mathcal{M}(\Lambda) \xrightarrow{\simeq} \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$. As usual we denote by $\mathsf{D}$ the duality functor of an Artin algebra, see [18, Section 3, Chapter 2] for more informations.

First we identify the opposite algebra $\Lambda^{\mathsf{op}}_{(\phi,\psi)}$ with the Morita ring

$$\Lambda^{\mathsf{op}}_{(\phi,\psi)} \simeq \begin{pmatrix} B^{\mathsf{op}} & {}_{B^{op}}N_{A^{\mathsf{op}}} \\ {}_{A^{\mathsf{op}}}M_{B^{op}} & A^{\mathsf{op}} \end{pmatrix}, \ \begin{pmatrix} a & n \\ m & b \end{pmatrix}^{\mathsf{op}} \mapsto \begin{pmatrix} b^{\mathsf{op}} & n \\ m & a^{\mathsf{op}} \end{pmatrix}$$

Let $(X, Y, f, g) \in \mathcal{M}(\Lambda)$ and recall from (4.1.1) and (4.1.2) the adjoint isomorphisms $\pi$ and $\rho$. Then the object $(X, Y, \pi(f), \rho(g))$ lies in $\widetilde{\mathcal{M}}(\Lambda)$ and applying the duality we obtain the morphisms $\mathsf{D}(\pi(f)) \in \mathsf{Hom}_{A^{\mathsf{op}}}(\mathsf{D}\,\mathsf{Hom}_B(M, Y), \mathsf{D}\,X)$ and $\mathsf{D}(\rho(g)) \in \mathsf{Hom}_{B^{\mathsf{op}}}(\mathsf{D}\,\mathsf{Hom}_A(N, X), \mathsf{D}\,Y)$. Let $0 \longrightarrow Y \longrightarrow I_0 \longrightarrow I_1$ be an injective coresolution of $Y$. Since the functors $\mathsf{D}\,\mathsf{Hom}_B(M, -)$, $M \otimes_{B^{\mathsf{op}}} \mathsf{D}(-)\colon \mathsf{mod}\text{-}B \longrightarrow \mathsf{mod}\text{-}A^{\mathsf{op}}$ are right exact we have the following exact commutative diagram:

$$\begin{tikzcd}
\mathsf{D}\,\mathsf{Hom}_B(M, I_1) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{D}\,\mathsf{Hom}_B(M, I_0) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{D}\,\mathsf{Hom}_B(M, Y) \arrow[r] \arrow[d, dashed, "\sigma"'] & 0 \\
M \otimes_{B^{\mathsf{op}}} \mathsf{D}\,I_1 \arrow[r] & M \otimes_{B^{\mathsf{op}}} \mathsf{D}\,I_0 \arrow[r] & M \otimes_{B^{\mathsf{op}}} \mathsf{D}\,Y \arrow[r] & 0
\end{tikzcd}$$

Hence the morphism $\sigma\colon M \otimes_{B^{\mathsf{op}}} \mathsf{D}\,Y \xrightarrow{\simeq} \mathsf{D}\,\mathsf{Hom}_B(M, Y)$ is an $A^{\mathsf{op}}$-isomorphism which is functorial in $Y$. Similarly we obtain a $B^{\mathsf{op}}$-isomorphism $\tau\colon N \otimes_{A^{\mathsf{op}}} \mathsf{D}\,X \xrightarrow{\simeq} \mathsf{D}\,\mathsf{Hom}_A(N, X)$ which is functorial in $X$. Then we have the object $(\mathsf{D}\,Y, \mathsf{D}\,X, \sigma \circ \mathsf{D}(\pi(f)), \tau \circ \mathsf{D}(\rho(g))) \in \mathcal{M}(\Lambda^{\mathsf{op}})$ where the morphisms are the following compositions:

$$M \otimes_{B^{\mathsf{op}}} \mathsf{D}\,Y \xrightarrow[\simeq]{\sigma} \mathsf{D}\,\mathsf{Hom}_B(M, Y) \xrightarrow{\mathsf{D}(\pi(f))} \mathsf{D}\,X$$

$$N \otimes_{A^{\mathsf{op}}} \mathsf{D}\,X \xrightarrow[\simeq]{\tau} \mathsf{D}\,\mathsf{Hom}_A(N, X) \xrightarrow{(\mathsf{D}\,\rho(g))} \mathsf{D}\,Y$$

If $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ is a morphism in $\mathcal{M}(\Lambda)$ then it is easy to check that

$$(\mathsf{D}(b), \mathsf{D}(a))\colon (\mathsf{D}\,Y', \mathsf{D}\,X', \sigma' \circ \mathsf{D}(\pi(f')), \tau' \circ \mathsf{D}(\rho(g'))) \longrightarrow (\mathsf{D}\,Y, \mathsf{D}\,X, \sigma \circ \mathsf{D}(\pi(f)), \tau \circ \mathsf{D}(\rho(g)))$$

is a morphism in $\mathcal{M}(\Lambda^{\mathsf{op}})$ and in this way we obtain a contravariant functor $\mathsf{D}\colon \mathcal{M}(\Lambda) \longrightarrow \mathcal{M}(\Lambda^{\mathsf{op}})$. Then from the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}(\Lambda) \arrow[r, "\simeq"] \arrow[d, "\mathsf{D}"'] & \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)} \arrow[d, "\mathsf{D}"] \\
\mathcal{M}(\Lambda^{\mathsf{op}}) \arrow[r, "\simeq"] & \mathsf{mod}\text{-}\Lambda^{\mathsf{op}}_{(\phi,\psi)}
\end{tikzcd}$$

we infer that the functor $\mathsf{D}\colon \mathcal{M}(\Lambda) \longrightarrow \mathcal{M}(\Lambda^{\mathsf{op}})$ is a duality, where $\mathsf{D}\colon \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)} \longrightarrow \mathsf{mod}\text{-}\Lambda^{op}_{(\phi,\psi)}$ is the usual duality of Artin algebras. We are now ready to describe the injective $\Lambda_{(\phi,\psi)}$-modules.

PROPOSITION 4.2.2. *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring regarded as an Artin algebra. Then the indecomposable injective $\Lambda_{(\phi,\psi)}$-modules are of the form:*

$$\begin{cases} \mathsf{H}_A(I) = (I, \mathsf{Hom}_A(N, I), \delta'_{M\otimes I} \circ \mathsf{Hom}_A(N, \Psi_I), \epsilon'_I) \\ \\ \mathsf{H}_B(J) = (\mathsf{Hom}_B(M, J), J, \epsilon_J, \delta_{N\otimes J} \circ \mathsf{Hom}_B(M, \Phi_J)) \end{cases}$$

*where $I$ is an indecomposable injective $A$-module and $J$ is an indecomposable injective $B$-module.*

PROOF. This follows from the description of the duality $\mathsf{D}\colon \mathcal{M}(\Lambda) \longrightarrow \mathcal{M}(\Lambda^{\mathsf{op}})$ and Proposition 4.2.1. □

**4.2.2. Simple $\Lambda_{(\phi,\psi)}$-modules.** In this subsection we determine the simple $\Lambda_{(\phi,\psi)}$-modules. Note that the description of the simple $\Lambda_{(\phi,\psi)}$-modules follows from Proposition 3.5.7. For completeness we state the result in our case.

Recall that for an $A$-module $X$ we have the following commutative diagram:

$$\begin{tikzcd} \mathsf{T}_A(X) \arrow[rr, "(\mathrm{Id}_X{,}\mu_X)"] \arrow[dr] & & \mathsf{H}_A(X) \\ & \mathsf{C}_A(X) \arrow[ur, hook] & \end{tikzcd}$$

where $\mathsf{C}_A(X) = \mathsf{Im}\,(\mathrm{Id}_X, \mu_X) = (X, \mathsf{Im}\,\mu_X, \kappa, \lambda)$ and $\mu_X = \delta'_{M\otimes X} \circ \mathsf{Hom}_A(N, \Psi_X)$. Then the assigment $X \mapsto \mathsf{Im}\,(\mathrm{Id}_X, \mu_X)$ gives a well defined functor $\mathsf{C}_A\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$. One crucial property of $\mathsf{C}_A$ is that it lifts simple modules, see Lemma 3.5.5.

The following result describes the simple $\Lambda_{(\phi,\psi)}$-modules in terms of simple $A$-modules, simple $B$-modules, simple $B/\,\mathsf{Im}\,\phi$-modules and simple $A/\,\mathsf{Im}\,\psi$-modules, and follows immediately from Proposition 3.5.7.

PROPOSITION 4.2.3. *There is the following bijections:*

$$\{\textit{simple } B/\,\mathsf{Im}\,\phi\textit{-modules}\} \underset{\mathsf{Z}_B}{\overset{\mathsf{U}_B}{\leftrightarrows}} \{\textit{simple } \Lambda_{(\phi,\psi)}\textit{-modules such that } X = 0\}$$

$$\{\textit{simple } A\textit{-modules}\} \underset{\mathsf{C}_A}{\overset{\mathsf{U}_A}{\leftrightarrows}} \{\textit{simple } \Lambda_{(\phi,\psi)}\textit{-modules such that } X \neq 0\}$$

$$\{\textit{simple } A/\,\mathsf{Im}\,\psi\textit{-modules}\} \underset{\mathsf{Z}_A}{\overset{\mathsf{U}_A}{\leftrightarrows}} \{\textit{simple } \Lambda_{(\phi,\psi)}\textit{-modules such that } Y = 0\}$$

$$\{\textit{simple } B\textit{-modules}\} \underset{\mathsf{C}_B}{\overset{\mathsf{U}_B}{\leftrightarrows}} \{\textit{simple } \Lambda_{(\phi,\psi)}\textit{-modules such that } Y \neq 0\}$$

REMARK 4.2.4. Let $\{S_1, \cdots, S_n\}$, resp. $\{S'_1, \cdots, S'_m\}$, be a complete set of simple $A$-modules, resp. $B$-modules. The simple $B/\,\mathsf{Im}\,\phi$-modules are the simple $B$-modules with the additional property that $\Phi_{S'_1} = \cdots = \Phi_{S'_m} = 0$. Then from the first two bijections of Proposition 4.2.3 the simple $\Lambda_{(\phi,\psi)}$-modules are of the form: $\mathsf{C}_A(S_1), \cdots, \mathsf{C}_A(S_n)$, $\mathsf{Z}_B(S'_1), \cdots, \mathsf{Z}_B(S'_m)$. Similarly the simple $A/\,\mathsf{Im}\,\psi$-modules are the simple $A$-modules such that $\Psi_{S_1} = \cdots = \Psi_{S_n} = 0$, and using the last two bijections of Proposition 4.2.3 we get that the simple $\Lambda_{(\phi,\psi)}$-modules are of the form: $\mathsf{C}_B(S'_1), \cdots, \mathsf{C}_B(S'_m), \mathsf{Z}_A(S_1), \cdots, \mathsf{Z}_A(S_n)$. It is easy to check that these two descriptions essentially coincide.

**4.2.3. Selfinjective Artin Algebras.** After the complete description of the indecomposable projective and injective $\Lambda_{(\phi,\psi)}$-modules we are interested to find conditions for the Artin algebra $\Lambda_{(\phi,\psi)}$ to be selfinjective. Recall that an Artin algebra $A$ is selfinjective if $A$ is an injective $A$-module.

The following result gives a sufficient condition for a Morita ring to be selfinjective.

Proposition 4.2.5. *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring which is an Artin algebra. Assume that the adjoint pair of functors $(M\otimes_A -, \mathsf{Hom}_B(M,-))$ induces an equivalence*

$$M\otimes_A -\colon \ \mathsf{proj}\,A) \ \underset{\longrightarrow}{\overset{\simeq}{\longleftarrow}} \ \mathsf{inj}\,B \ : \ \mathsf{Hom}_B(M,-)$$

*and the adjoint pair of functors $(N\otimes_B -, \mathsf{Hom}_A(N,-))$ induces an equivalence*

$$N\otimes_B -\colon \ \mathsf{proj}\,B \ \underset{\longrightarrow}{\overset{\simeq}{\longleftarrow}} \ \mathsf{inj}\,A \ : \ \mathsf{Hom}_A(N,-)$$

*Then the algebra $\Lambda_{(\phi,\psi)}$ is selfinjective.*

Proof. Let $\mathsf{T}_A(P)=(P, M\otimes_A P, \mathrm{Id}_{M\otimes_A P}, \Psi_P)$ be an indecomposable projective $\Lambda_{(\phi,\psi)}$-module, where $P$ is an indecomposable projective $A$-module. Since the categories $\mathsf{proj}\,A$ and $\mathsf{inj}\,B$ are equivalent we have isomorphisms $a\colon M\otimes_A P \xrightarrow{\simeq} J$ and $b^{-1}\colon \mathsf{Hom}_B(M,J)\xrightarrow{\simeq} P$ for some injective $B$-module $J$, where $b=\delta_P\circ\mathsf{Hom}_B(M,a)$. Then we have the injective module $\mathsf{H}_B(J)=(\mathsf{Hom}_B(M,J), J, \epsilon_J, \delta_{N\otimes_B J}\circ\mathsf{Hom}_B(M,\Phi_J))$ and we claim that the map $(b,a)\colon \mathsf{T}_A(P)\longrightarrow \mathsf{H}_B(J)$ is an isomorphism of $\Lambda_{(\phi,\psi)}$-modules. It is straightforward that the map $(b,a)$ is an isomorphism since both the maps $b$ and $a$ are isomorphims. Hence we have only to prove that $(b,a)$ is a morphism in $\mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$, i.e. we have to show that the following diagrams are commutative:

$$\begin{tikzcd}
M\otimes_A P \arrow[r, "\mathrm{Id}_{M\otimes P}"] \arrow[d, "M\otimes b"', "\simeq"] & M\otimes_A P \arrow[d, "a", "\simeq"'] \\
M\otimes_A \mathsf{Hom}_B(M,J) \arrow[r, "\epsilon_J", "\simeq"'] & J
\end{tikzcd}
\qquad
\begin{tikzcd}
N\otimes_B M\otimes_A P \arrow[r, "\Psi_P"] \arrow[d, "N\otimes a"', "\simeq"] & P \arrow[d, "b", "\simeq"'] \\
N\otimes_B J \arrow[r, "\delta_{N\otimes J}\circ\mathsf{Hom}_B(M,\Phi_J)"] & \mathsf{Hom}_B(M,J)
\end{tikzcd}$$

Clearly the first diagram is commutative and for the second we have

$$\begin{aligned}
(N\otimes_B a)\circ\delta_{N\otimes J}\circ\mathsf{Hom}_B(M,\Phi_J) &= \delta_{N\otimes M\otimes P}\circ\mathsf{Hom}_B(M, M\otimes_A N\otimes_B a)\circ\mathsf{Hom}_B(M,\Phi_J)\\
&= \delta_{N\otimes M\otimes P}\circ\mathsf{Hom}_B(M,\Phi_{M\otimes P})\circ\mathsf{Hom}_B(M,a)\\
&= \delta_{N\otimes M\otimes P}\circ\mathsf{Hom}_B(M, M\otimes\Psi_P)\circ\mathsf{Hom}_B(M,a)\\
&= \Psi_P\circ\delta_P\circ\mathsf{Hom}_B(M,a)\\
&= \Psi_P\circ b
\end{aligned}$$

Thus $\mathsf{T}_A(P)\simeq\mathsf{H}_B(J)$ and so the indecomposable projective $\Lambda_{(\phi,\psi)}$-module $\mathsf{T}_A(P)$ is injective. Moreover from Proposition 4.2.1 we have also the indecomposable projective $\Lambda_{(\phi,\psi)}$-module $\mathsf{T}_B(Q)=(N\otimes_B Q, Q, \phi\otimes\mathrm{Id}_Q, \mathrm{Id}_{N\otimes_B Q})$ for some indecomposable projective $B$-module $Q$. Then using the equivalence between $\mathsf{proj}\,B$ and $\mathsf{inj}\,A$ it follows as above that $\mathsf{T}_B(Q)\simeq\mathsf{H}_A(I)$ for some injective $A$-module $I$ and so $\mathsf{T}_B(Q)$ is an injective $\Lambda_{(\phi,\psi)}$-module. Since every indecomposable projective $\Lambda_{(\phi,\psi)}$-module is injective we infer that the Artin algebra $\Lambda_{(\phi,\psi)}$ is selfinjective. □

Example 4.2.6. Let $\Lambda$ be a selfinjective Artin algebra. Then from Proposition 4.2.5 we get that $\left(\begin{smallmatrix}\Lambda & \Lambda\\ \Lambda & \Lambda\end{smallmatrix}\right)_{(\phi,\phi)}$ is a selfinjective Artin algebra.

The following example shows that the converse of Proposition 4.2.5 is not true in general.

EXAMPLE 4.2.7. Let $\mathbb{K}$ be a field and $\mathbb{K}\mathcal{Q}$ be the path algebra where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd} \overset{v_1}{\circ} \arrow[rr, "a"] & & \overset{v_2}{\circ} \arrow[dl, "b"] \\ & \overset{v_3}{\circ} \arrow[ul, "c"] & \end{tikzcd}$$

Let $I$ be the ideal generated by $ba, cb$, and $ac$ and $\Lambda = \mathbb{K}\mathcal{Q}/I$. Then $\Lambda$ is a selfinfective finite dimensional $\mathbb{K}$-algebra. Setting $e = v_1$ and $e' = v_2 + v_3$, we view $\Lambda$ as a Morita ring via

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} e\Lambda e & e\Lambda e' \\ e'\Lambda e & e'\Lambda e' \end{pmatrix}$$

Note that, in this case, $\phi = \psi = 0$. Since $e\Lambda e$ has one indecomposable projective-injective module up to isomorphism and $e'\Lambda e'$ has two nonisomorphic indecomposable projective-injective modules, the converse to Proposition 4.2.5 fails.

For an Artin algebra $\Lambda$ we denote by $\ell\ell(\Lambda)$ the Loewy length of $\Lambda$, i.e. the smallest integer $n$ such that $\mathfrak{r}^n = 0$ where $\mathfrak{r}$ is the Jacobson radical of $\Lambda$. Recall that the **representation dimension** $\mathsf{rep.\,dim}\,\Lambda$ of $\Lambda$ in the sense of Auslander [10] is defined by

$$\mathsf{rep.\,dim}\,\Lambda = \min\{\mathsf{gl.\,dim}\,\mathsf{End}_\Lambda(X) \mid X\colon \text{ generator and cogenerator of } \mathsf{mod}\text{-}\Lambda\}$$

We recall also the following result for selfinjective algebras. We refer to [10] and [111] for old and recent developments respectively, on this important dimension of Auslander. See also section 5.2 of Chapter 5 for more applications of our results on representation dimension.

PROPOSITION 4.2.8. *[10] Let $\Lambda$ be a selfinjective Artin algebra. Then*

$$\mathsf{rep.\,dim}\,\Lambda \ \leq \ \ell\ell(\Lambda)$$

As an easy consequence of the above we have the following result.

COROLLARY 4.2.9. *Let $\Lambda$ be a selfinjective Artin algebra. Then*:

$$\mathsf{rep.\,dim} \begin{pmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{pmatrix}_{(0,0)} \ \leq \ 2\,\ell\ell(\Lambda)$$

PROOF. Since $\Lambda$ is selfinjective we have from Example 4.2.6 that the matrix Artin algebra $\left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)_{(0,0)}$ is selfinjective. Then the result follows from Proposition 4.2.8 since $\ell\ell\left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)_{(0,0)} = 2\,\ell\ell(\Lambda)$. □

## 4.3. Functorially Finite Subcategories

Our purpose in this section is to apply the results of section 3.4 in order to study finiteness conditions on subcategories of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$. The reason for restricting to the case where $\phi = \psi = 0$ is that we have full embeddings from the module categories $\mathsf{Mod}\text{-}A$, $\mathsf{Mod}\text{-}B$, $\mathsf{Mod}\text{-}\left(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\right)$ and $\mathsf{Mod}\text{-}\left(\begin{smallmatrix} A & {}_AN_B \\ 0 & B \end{smallmatrix}\right)$ to $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$. In particular we show that the above $\mathsf{natural}$ subcategories of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$ are bireflective and therefore functorially finite.

We start by defining the following full subcategories of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$:

$$\begin{cases} \mathcal{X} = \big\{(X,Y,f,0) \mid f\colon M\otimes_A X \longrightarrow Y \text{ is an epimorphism}\big\} \\ \mathcal{Y} = \big\{(0,Y,0,0) \mid Y \in \mathsf{Mod}\text{-}B\big\} = \mathsf{Im}\,\mathsf{Z}_B \\ \mathcal{Z} = \big\{(X,Y,0,g) \mid \rho(g)\colon Y \longrightarrow \mathsf{Hom}_A(N,X) \text{ is a monomorphism}\big\} \\ \mathcal{X}' = \big\{(X,Y,0,g) \mid g\colon N\otimes_B Y \longrightarrow X \text{ is an epimorphism}\big\} \\ \mathcal{Y}' = \big\{(X,0,0,0) \mid X \in \mathsf{Mod}\text{-}A\big\} = \mathsf{Im}\,\mathsf{Z}_A \\ \mathcal{Z}' = \big\{(X,Y,f,0) \mid \pi(f)\colon X \longrightarrow \mathsf{Hom}_B(M,Y) \text{ is a monomorphism}\big\} \end{cases}$$

The following resul describes the structure of the above subcategories in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$. The proof follows immediately from Proposition 3.4.4 and Corollary 3.4.6. For the notion of torsion pairs in abelian categories and functorially finiteness of subcategories see sections 1.2 and 3.4.

PROPOSITION 4.3.1. *Let $\Lambda_{(0,0)}$ be a Morita ring. Then the triples $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ and $(\mathcal{X}',\mathcal{Y}',\mathcal{Z}')$ are TTF-triples in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$. In particular the following hold.*

(i) *The full subcategory $\mathcal{X}$ is contravariantly finite in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$, closed under extensions, quotients and coproducts, and $\mathsf{T}_A(\mathsf{Mod}\text{-}A) \subseteq \mathcal{X}$.*
(ii) *The full subcategory $\mathcal{X}'$ is contravariantly finite in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$, closed under extensions, quotients and coproducts, and $\mathsf{T}_B(\mathsf{Mod}\text{-}B) \subseteq \mathcal{X}'$.*
(iii) *The full subcategory $\mathcal{Z}$ is covariantly finite in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$, closed under extensions, subobjects and products, and $\mathsf{H}_A(\mathsf{Mod}\text{-}A) \subseteq \mathcal{Z}$.*
(iv) *The full subcategory $\mathcal{Z}'$ is covariantly finite in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$, closed under extensions, subobjects and products, and $\mathsf{H}_B(\mathsf{Mod}\text{-}B) \subseteq \mathcal{Z}'$.*

The next consequence of Corollary 3.4.7 describes the categories of modules over $A$ and $B$ via the subcategories $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{X}'$, $\mathcal{Z}'$.

COROLLARY 4.3.2. *Let $\Lambda_{(0,0)}$ be a Morita ring. Then there is an equivalence*

$$\mathsf{Mod}\text{-}A \xrightarrow{\simeq} \mathcal{X}\cap\mathcal{Z} \quad \textit{and} \quad \mathsf{Mod}\text{-}B \xrightarrow{\simeq} \mathcal{X}'\cap\mathcal{Z}'$$

We denote by $\mathcal{X}_0$ the full subcategory of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$ whose objects are the tuples $(X,Y,f,g)$ such that there is an exact sequence $0 \longrightarrow K_0 \longrightarrow \mathsf{T}_A(P_0) \longrightarrow (X,Y,f,g) \longrightarrow 0$ with $P_0 \in \mathsf{Proj}\,A$. Similarly we define the subcategories $\mathcal{Y}_0 = \{(X,Y,f,g) \in \mathsf{Mod}\text{-}\Lambda_{(0,0)} \mid \exists\; 0 \longrightarrow (X,Y,f,g) \longrightarrow \mathsf{H}_A(I_0) \longrightarrow L_0 \longrightarrow 0 \text{ exact with } I_0 \in \mathsf{Inj}\,A\}$, $\mathcal{X}'_0 = \{(X,Y,f,g) \in \mathsf{Mod}\text{-}\Lambda_{(0,0)} \mid \exists\; 0 \longrightarrow K_0 \longrightarrow \mathsf{T}_B(Q_0) \longrightarrow (X,Y,f,g) \longrightarrow 0 \text{ exact with } Q_0 \in \mathsf{Proj}\,B\}$ and $\mathcal{Y}'_0 = \{(X,Y,f,g) \in \mathsf{Mod}\text{-}\Lambda_{(0,0)} \mid \exists\; 0 \longrightarrow (X,Y,f,g) \longrightarrow \mathsf{H}_B(J_0) \longrightarrow L_0 \longrightarrow 0 \text{ exact with } J_0 \in \mathsf{Inj}\,B\}$.

From Proposition 3.4.8 we have another description of the subcategories $\mathcal{X},\mathcal{Z},\mathcal{X}',\mathcal{Z}'$.

COROLLARY 4.3.3. *Let $\Lambda_{(0,0)}$ be a Morita ring. Then: $\mathcal{X} = \mathcal{X}_0$, $\mathcal{Z} = \mathcal{Y}_0$, $\mathcal{X}' = \mathcal{X}'_0$ and $\mathcal{Z}' = \mathcal{Y}'_0$.*

The following main result of this section gives the exact properties of the natural subcategories of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$. For the notion of bireflective subcategory see section 3.4. Although the next result follows from Theorem 3.4.9 we give an alternative proof.

THEOREM 4.3.4. *Let* $\Lambda_{(0,0)}$ *be a Morita ring. Then the full subcategories*

$$\mathsf{Mod}\text{-}A,\ \mathsf{Mod}\text{-}B,\ \mathsf{Mod}\text{-}\big(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\big),\ \mathsf{Mod}\text{-}\big(\begin{smallmatrix} A & {}_AN_B \\ 0 & B \end{smallmatrix}\big)$$

*are bireflective in* $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$*. In particular the above subcategories are functorially finite in* $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$*, closed under isomorphic images, direct sums, direct products, kernels and cokernels.*

PROOF. Since $\phi = \psi = 0$ it follows that the following maps:

$$\begin{aligned}
\theta_1\colon \Lambda_{(0,0)} &\longrightarrow A,\ \theta_1\big(\big(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\big)\big) = a \\
\theta_2\colon \Lambda_{(0,0)} &\longrightarrow B,\ \theta_2\big(\big(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\big)\big) = b \\
\theta_3\colon \Lambda_{(0,0)} &\longrightarrow \big(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\big),\ \theta_3\big(\big(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\big)\big) = \big(\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\big) \\
\theta_4\colon \Lambda_{(0,0)} &\longrightarrow \big(\begin{smallmatrix} A & {}_AN_B \\ 0 & B \end{smallmatrix}\big),\ \theta_4\big(\big(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\big)\big) = \big(\begin{smallmatrix} a & n \\ 0 & b \end{smallmatrix}\big)
\end{aligned}$$

are surjective ring homomorphisms. Therefore the above maps are ring epimorphisms, i.e. epimorphisms in the category of rings. Hence from [54], [56], see also Theorem 1.3.1, we infer that the essential images of the restriction functors $\mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$, $\mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$, $\mathsf{Mod}\text{-}\big(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\big) \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ and $\mathsf{Mod}\text{-}\big(\begin{smallmatrix} A & {}_AN_B \\ 0 & B \end{smallmatrix}\big) \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ are bireflective subcategories. Since bireflective subcategories are functorially finite, it follows that the full subcategories $\mathsf{Mod}\text{-}A, \mathsf{Mod}\text{-}B, \mathsf{Mod}\text{-}\big(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\big), \mathsf{Mod}\text{-}\big(\begin{smallmatrix} A & {}_AN_B \\ 0 & B \end{smallmatrix}\big)$ are functorially finite in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$, closed under isomorphic images, direct sums, direct products, kernels and cokernels. □

REMARK 4.3.5. From Theorem 4.3.4 the full embedding

$$\mathcal{F}\colon\ \mathsf{Mod}\text{-}\big(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\big) \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$$

has a left and right adjoint. But it is easy to observe that $\mathsf{Mod}\text{-}\big(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\big)$ is not closed under extensions and therefore it is not a Serre subcategory of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$. Hence from the full embedding $\mathcal{F}$ we cannot derive a recollement of module categories.

We continue with the following result on finiteness of subcategories which is a consequence of Theorem 3.4.11.

THEOREM 4.3.6. *Let* $\Lambda_{(0,0)}$ *be a Morita ring.*

(i) *Let* $\mathcal{U}$ *be a covariantly finite subcategory of* $\mathsf{Mod}\text{-}A$ *such that* $\mathcal{U} \subseteq \mathsf{Ker}\,\mathsf{Hom}_A(N,-)$ *and* $\mathcal{V}$ *a covariantly finite subcategory of* $\mathsf{Mod}\text{-}B$ *such that* $\mathcal{V} \subseteq \mathsf{Ker}\,N \otimes_B -$. *Then the full subcategory*

$$\mathcal{W} = \big\{(X, Y, f, g) \in \mathsf{Mod}\text{-}\Lambda_{(0,0)} \mid X \in \mathcal{U}\ \ \textit{and}\ \ Y \in \mathcal{V}\big\}$$

*is covariantly finite in* $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$.

(ii) *Let* $\mathcal{U}$ *be a contravariantly finite subcategory of* $\mathsf{Mod}\text{-}A$ *such that* $\mathcal{U} \subseteq \mathsf{Ker}\,\mathsf{Hom}_A(N,-)$ *and* $\mathcal{V}$ *a contravariantly finite subcategory of* $\mathsf{Mod}\text{-}B$ *such that* $\mathcal{V} \subseteq \mathsf{Ker}\,N \otimes_B -$. *Then the full subcategory*

$$\mathcal{W} = \big\{(X, Y, f, g) \in \mathsf{Mod}\text{-}\Lambda_{(0,0)} \mid X \in \mathcal{U}\ \ \textit{and}\ \ Y \in \mathcal{V}\big\}$$

*is contravariantly finite in* $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$.

(iii) *Let* $\mathcal{U}$ *be a functorially finite subcategory of* $\mathsf{Mod}\text{-}A$ *such that* $\mathcal{U} \subseteq \mathsf{Ker}\,\mathsf{Hom}_A(N,-)$ *and* $\mathcal{V}$ *a functorially finite subcategory of* $\mathsf{Mod}\text{-}B$ *such that* $\mathcal{V} \subseteq \mathsf{Ker}\,N \otimes_B -$. *Then the full subcategory*

$$\mathcal{W} = \big\{(X, Y, f, g) \in \mathsf{Mod}\text{-}\Lambda_{(0,0)} \mid X \in \mathcal{U}\ \ \textit{and}\ \ Y \in \mathcal{V}\big\}$$

*is functorially finite in* $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$.

REMARK 4.3.7. Note that the converse of Theorem 4.3.6 holds, i.e. if $\mathcal{W}$ is contravariantly (resp. covariantly) finite in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$ then $\mathcal{U}$ is contravariantly (resp. covariantly) finite in $\mathsf{Mod}\text{-}A$ and $\mathcal{V}$ is contravariantly (resp. covariantly) finite in $\mathsf{Mod}\text{-}B$. For more details see Remark 3.4.12.

If the bimodule $N = 0$ then from Theorem 4.3.6 and Remark 4.3.7 we have the following well known result due to Smalø.

COROLLARY 4.3.8. *[120, Theorem 2.1] Let $\Lambda = \left(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\right)$ be a triangular matrix ring, $\mathcal{U}$ a full subcategory of $\mathsf{Mod}\text{-}A$, $\mathcal{V}$ a full subcategory of $\mathsf{Mod}\text{-}B$ and let $\mathcal{W} = \big\{(X, Y, f) \in \mathsf{Mod}\text{-}\Lambda \mid X \in \mathcal{U} \;\text{ and }\; Y \in \mathcal{V}\big\}$.*

(i) *The subcategory $\mathcal{W}$ is covariantly finite in $\mathsf{Mod}\text{-}\Lambda$ if and only if $\mathcal{U}$ is covariantly finite in $\mathsf{Mod}\text{-}A$ and $\mathcal{V}$ is covariantly finite in $\mathsf{Mod}\text{-}B$.*
(ii) *The subcategory $\mathcal{W}$ is contravariantly finite in $\mathsf{Mod}\text{-}\Lambda$ if and only if $\mathcal{U}$ is contravariantly finite in $\mathsf{Mod}\text{-}A$ and $\mathcal{V}$ is contravariantly finite in $\mathsf{Mod}\text{-}B$.*
(iii) *The subcategory $\mathcal{W}$ is functorially finite in $\mathsf{Mod}\text{-}\Lambda$ if and only if $\mathcal{U}$ is functorially finite in $\mathsf{Mod}\text{-}A$ and $\mathcal{V}$ is functorially finite in $\mathsf{Mod}\text{-}B$.*

We continue with the following applications for Artin algebras. For the notion of Auslander-Reiten sequences we refer to [18].

COROLLARY 4.3.9. *Let $\Lambda_{(0,0)}$ be a Morita ring regarded as an Artin algebra. Then the full subcategories $\mathsf{mod}\text{-}A$ and $\mathsf{mod}\text{-}B$ of $\mathsf{mod}\text{-}\Lambda_{(0,0)}$ have relative Auslander-Reiten sequences in $\mathsf{mod}\text{-}\Lambda_{(0,0)}$.*

PROOF. It is well known that functorially finite subcategories which are closed under extensions have Auslander-Reiten sequences, see [19]. Then the result follows from Theorem 4.3.4. □

COROLLARY 4.3.10. *Let $\Lambda_{(0,0)}$ be a Morita ring which is as an Artin algerba. Let $\mathcal{U}$ be an extension closed functorially finite subcategory of $\mathsf{mod}\text{-}A$ such that $\mathcal{U} \subseteq \mathsf{Ker}\,\mathsf{Hom}_A(N, -)$ and $\mathcal{V}$ an extension closed functorially finite subcategory of $\mathsf{mod}\text{-}B$ such that $\mathcal{V} \subseteq \mathsf{Ker}\, N \otimes_B -$. Then the full subcategory $\mathcal{W} = \{(X, Y, f, g) \in \mathsf{mod}\text{-}\Lambda_{(0,0)} \mid X \in \mathcal{U} \;\text{ and }\; Y \in \mathcal{V}\}$ has Auslander-Reiten sequences in $\mathsf{mod}\text{-}\Lambda_{(0,0)}$.*

PROOF. Since $\mathcal{U}$ and $\mathcal{V}$ are closed under extensions it follows that $\mathcal{W}$ is also closed under extensions. Then the result follows from Theorem 4.3.6 and [19]. □

The last part in the paper of Smalø [120] deals with the full subcategory of modules of finite projective dimension. In particular, if $\Lambda = \left(\begin{smallmatrix} A & 0 \\ {}_BM_A & B \end{smallmatrix}\right)$ is a triangular matrix Artin algebra, then the category of $\Lambda$-modules of finite projective dimension is contravariantly finite in $\mathsf{mod}\text{-}\Lambda$ if and only if the category of $A$-modules of finite projective dimension is contravariantly finite in $\mathsf{mod}\text{-}A$ and the category of $B$-modules of finite projective dimension is contravariantly finite in $\mathsf{mod}\text{-}B$, see [120, Proposition 2.3]. This result follows from Corollary 4.3.8 and the description of the subcategory of $\Lambda$-modules of finite projective dimension. Recall that a $\Lambda$-module $(X, Y, f)$ is of finite projective dimension if and only if the projective dimensions of ${}_AX$ and ${}_BY$ are finite.

We close this section with the next example which shows that the subcategory of $\Lambda_{(0,0)}$-modules of finite projective dimension cannot be described as in the lower triangular case. This distinguishes our situation from the lower triangular situation.

EXAMPLE 4.3.11. Let $\mathbb{K}$ be a field and $\mathbb{K}\mathcal{Q}$ be the path algebra where $\mathcal{Q}$ is the quiver

$$\mathcal{Q} = \begin{tikzcd} \overset{v_1}{\circ} \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \overset{v_2}{\circ} \arrow[l, bend left, "c"] \arrow[loop right, "d"] \end{tikzcd}$$

Let $I$ be the ideal generated by $a^2, bc, cb, d^2, ba - db$, and $cd - ac$ and let $\Lambda = \mathbb{K}\mathcal{Q}/I$. It is not hard to show that $\Lambda$ is a selfinjective finite dimensional $\mathbb{K}$-algebra. The structure of the indecomposable projective-injective modules look like:

$$\begin{tikzcd}[column sep=small] & v_1 \arrow[dl, dash, "a"'] \arrow[dr, dash, "b"] & \\ v_1 \arrow[dr, dash, "b"'] & & v_2 \arrow[dl, dash, "d"] \\ & v_2 & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & v_2 \arrow[dl, dash, "d"'] \arrow[dr, dash, "c"] & \\ v_2 \arrow[dr, dash, "c"'] & & v_1 \arrow[dl, dash, "a"] \\ & v_1 & \end{tikzcd}$$

Setting $e = v_1$ and $e' = v_2$, we view $\Lambda$ as the Morita ring via

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} e\Lambda e & e\Lambda e' \\ e'\Lambda e & e'\Lambda e' \end{pmatrix}$$

Note that, in this case, $\phi = \psi = 0$. One sees that $\Lambda$ is a selfinjective finite dimensional biserial algebra. Consider the string module of the form

$$D = \begin{tikzcd}[column sep=small] & v_1 \arrow[dl, dash] \arrow[dr, dash] & & v_2 \arrow[dl, dash] \\ v_1 & & v_2 & \end{tikzcd}$$

Viewing $D$ as module over the algebra $\Lambda_{(\phi,\psi)}$, $D = (X, Y, f, g)$, we see that $X$ is isomorphic to $e\Lambda e$ as a left $e\Lambda e$-module, $Y$ is isomorphic to $e'\Lambda e'$ as a left $e'\Lambda e'$-module, and $g = 0$. Thus, we have that $\mathsf{pd}_{e\Lambda e} X < \infty$, $\mathsf{pd}_{e'\Lambda e'} Y < \infty$, but $\mathsf{pd}_\Lambda D = \infty$ (since $D$ is not a projective $\Lambda$-module and $\Lambda$ is selfinjective). Finally, letting $R = \mathbb{K}[x]/(x^2)$, then it is easy to see that $\Lambda_{(0,0)} \cong \left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right)$.

## 4.4. Bounds on the Global Dimension

Let $\Lambda_{(0,0)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring which is as an Artin algebra with $\phi = \psi = 0$. In this section we show that, under certain restictions on either $M$ or $N$, there is a bound on the global dimension of $\Lambda_{(0,0)}$ in terms of the global dimensions of $A$ and $B$. This is achieved via the notion of tight projective module and tight projective resolution that we introduce in the first subsection.

**4.4.1. Tight Resolutions and Upper Bounds.** Before we begin with some preliminary definitions and results, we give an example which shows that we will need some restrictions to get a bound on the global dimension of $\Lambda_{(0,0)}$ in terms of the global dimensions of $A$ and $B$.

EXAMPLE 4.4.1. Let $\mathbb{K}$ be a field and $\mathcal{Q}$ be the quiver

$$\begin{tikzcd} \overset{v}{\circ} \arrow[r, shift left, "a"] & \overset{w}{\circ} \arrow[l, shift left, "b"] \end{tikzcd}$$

Let $\Lambda = \mathbb{K}\mathcal{Q}/\langle ab, ba\rangle$. Let $P$ (respectively $Q$) be the projective $\Lambda$-cover of the simple module having $\mathbb{K}$ at vertex $v$ (resp. $w$) and $0$ at vertex $w$ (resp. $v$). Then $\Lambda = P \oplus Q$.

Hence $\Lambda$ is isomorphic to $\mathsf{Hom}_\Lambda(P \oplus Q, P \oplus Q)^{\mathsf{op}}$, which in turn is isomorphic to the matrix algebra

$$\begin{pmatrix} \mathsf{End}_\Lambda(P)^{\mathsf{op}} & \mathsf{Hom}_\Lambda(P,Q) \\ \mathsf{Hom}_\Lambda(Q,P) & \mathsf{End}_\Lambda(Q)^{\mathsf{op}} \end{pmatrix}$$

Each entry in this $2 \times 2$-matrix is $\mathbb{K}$ but the multiplication of two elements, one of the form $\left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right)$ and the other of the form $\left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right)$, in any order, is 0. Thus, as a Morita ring $\left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$, $A = B = M = N = \mathbb{K}$ and $\phi = \psi = 0$. Hence $A$ and $B$ have global dimension 0 and $M$ and $N$ have projective dimension 0 over both $A$ and $B$. But $\Lambda$ has infinite global dimension. $\square$

We introduce the following notion which is crucial for our results of this section.

Definition 4.4.2. If $P = (P_A, 0, 0, 0)$ is a projective $\Lambda_{(0,0)}$-module for some left $A$-module $P_A$, then $P$ is called an $A$-**tight projective** $\Lambda_{(0,0)}$-module. We say that a left $\Lambda_{(0,0)}$-module $(X, 0, 0, 0)$ has an $A$-**tight projective** $\Lambda_{(0,0)}$-**resolution** if $(X, 0, 0, 0)$ has a projective $\Lambda_{(0,0)}$-resolution in which each projective $\Lambda_{(0,0)}$-module is $A$-tight.

Note that if $(P_A, 0, 0, 0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module then $P_A$ is a projective $A$-module and $M \otimes_A P_A = 0$. Conversely, if $P_A$ is a projective $A$-module and $M \otimes_A P_A = 0$, then $(P_A, 0, 0, 0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module. It is easy to see the following.

- (i) A direct sum of modules having $A$-tight projective $\Lambda_{(0,0)}$-resolutions also has an $A$-tight projective $\Lambda_{(0,0)}$-resolution.
- (ii) A summand of an $A$-tight projective $\Lambda_{(0,0)}$-module is again an $A$-tight projective $\Lambda_{(0,0)}$-module.
- (iii) If $X$ is an $A$-module such that $(X, 0, 0, 0)$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution, then $\mathsf{pd}_A X = \mathsf{pd}_{\Lambda_{(0,0)}}(X, 0, 0, 0)$.

The next result classifies $\Lambda_{(0,0)}$-modules having $A$-tight projective $\Lambda_{(0,0)}$-resolutions.

Proposition 4.4.3. *A $\Lambda_{(0,0)}$-module of the form $(X, 0, 0, 0)$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution if and only if $M \otimes_A P = 0$, where $P$ is the direct sum of projective $A$-modules in a minimal projective $A$-resolution of $X$.*

Proof. Suppose that $\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow X \longrightarrow 0$ is a minimal projective $A$-resolution of $X$. Set $P = \oplus_{n \geq 0} P^n$. If $M \otimes_A P = 0$ then $M \otimes_A P^n = 0$, for all $n \geq 0$. It follows that $(X, 0, 0, 0)$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution. On the other hand, if $M \otimes_A P \neq 0$, then there is a smallest $n \geq 0$, such that $M \otimes_A P^n \neq 0$. It follows that there is a minimal projective $\Lambda_{(0,0)}$-resolution of $(X, 0, 0, 0)$ that starts

$$(P^n, M \otimes_A P^n, \mathrm{Id}_{M \otimes P^n}, 0) \longrightarrow (P^{n-1}, 0, 0, 0) \longrightarrow \cdots \longrightarrow (P^0, 0, 0, 0) \longrightarrow (X, 0, 0, 0) \longrightarrow 0$$

Hence the $n+1$-st syzygy is of the form $(\Omega_A^{n+1}(X), M \otimes_A P^n, \mathrm{Id}_M \otimes \kappa, 0)$, where $\kappa$ is the monomorphism $\kappa \colon \Omega_A^{n+1}(X) \longrightarrow P^n$. It follows that the next projective in the above minimal $\Lambda_{(0,0)}$-resolution of $(X, 0, 0, 0)$ is not $A$-tight and the result follows. $\square$

We use the next result a number of times in what follows.

Lemma 4.4.4. *If $P$ is a projective $A$-module such that $(P, 0, 0, 0)$ is not an $A$-tight projective $\Lambda_{(0,0)}$-module, then*:

$$\mathsf{pd}_{\Lambda_{(0,0)}}(P, 0, 0, 0) = 1 + \mathsf{pd}_{\Lambda_{(0,0)}}(0, M \otimes_A P, 0, 0)$$

*If $Q$ is a projective $B$-module such that $(0, Q, 0, 0)$ is not a $B$-tight projective $\Lambda_{(0,0)}$-module, then*:

$$\mathsf{pd}_{\Lambda_{(0,0)}}(0, Q, 0, 0) = 1 + \mathsf{pd}_{\Lambda_{(0,0)}}(N \otimes_B Q, 0, 0, 0)$$

PROOF. The first statement follows from the following short exact sequence of $\Lambda_{(0,0)}$-modules

$$0 \longrightarrow (0, M \otimes_A P, 0, 0) \longrightarrow (P, M \otimes_A P, \mathrm{Id}_{M\otimes_A P}, 0) \longrightarrow (P, 0, 0, 0) \longrightarrow 0$$

and the proof of the second statement is similar. □

We define $B$-tight projective $\Lambda_{(0,0)}$-modules $(0, Q, 0, 0)$ in a similar fashion as $A$-tight projective $\Lambda_{(0,0)}$-modules and also $\Lambda_{(0,0)}$-modules $(0, Y, 0, 0)$ having $B$-tight projective $\Lambda_{(0,0)}$-resolutions.

We also have the following result.

LEMMA 4.4.5. *Let $X$ be an $A$-$B$-bimodule such that $(X, 0, 0, 0)$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution. If $Q$ is a projective $B$-module, then $(X \otimes_B Q, 0, 0, 0)$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution.*

PROOF. Since direct sums of modules that have $A$-tight projective $\Lambda_{(0,0)}$-resolutions are modules having $A$-tight projective $\Lambda_{(0,0)}$-resolutions, we may assume that $Q = Be$, for some primitive idempotent $e$ in $B$. Since $X \otimes_B Be \simeq Xe$ is a summand of $X$, the result follows. □

The next lemma is a useful tool in what follows.

LEMMA 4.4.6. *Let $X$ be an $A$-module and $Y$ be a $B$-module. Then*:

(i) $\mathsf{pd}_{\Lambda_{(0,0)}}(X, 0, 0, 0) \ \leq \ 1 + \max\{\mathsf{pd}_{\Lambda_{(0,0)}}(\Omega^1_A(X), 0, 0, 0), \mathsf{pd}_{\Lambda_{(0,0)}}(0, M, 0, 0)\}$

(ii) $\mathsf{pd}_{\Lambda_{(0,0)}}(0, Y, 0, 0) \ \leq \ 1 + \max\{\mathsf{pd}_{\Lambda_{(0,0)}}(0, \Omega^1_B(Y), 0, 0), \mathsf{pd}_{\Lambda_{(0,0)}}(N, 0, 0, 0)\}$

PROOF. We only prove (i) since the proof of (ii) is similar. Let $\alpha\colon P \longrightarrow X$ be a projective $A$-cover of $X$ with kernel $\Omega^1_A(X)$. Then we have a short exact sequence

$$0 \longrightarrow (\Omega^1_A(X), M \otimes_A P, M \otimes k, 0) \xrightarrow{(k, \mathrm{Id}_{M\otimes P})} (P, M \otimes_A P, \mathrm{Id}_{M\otimes P}, 0) \xrightarrow{(\alpha,0)} (X, 0, 0, 0) \longrightarrow 0$$

in which the middle term is a projective $\Lambda_{(0,0)}$-module. Therefore it follows that $\mathsf{pd}_{\Lambda_{(0,0)}}(X, 0, 0, 0) \ \leq \ 1 + \mathsf{pd}_{\Lambda_{(0,0)}}(\Omega^1_A(X), M \otimes_A P, M \otimes k, 0)$. Next we note that we have a short exact sequence

$$0 \longrightarrow (0, M \otimes_A P, 0, 0) \longrightarrow (\Omega^1_A(X), M \otimes_A P, M \otimes k, 0) \longrightarrow (\Omega^1_A(X), 0, 0, 0) \longrightarrow 0$$

We observe that $M \otimes_A P$ is direct sum of summands of $M$. Hence we infer that $\mathsf{pd}_{\Lambda_{(0,0)}}(0, M \otimes_A P, 0, 0) \leq \mathsf{pd}_{\Lambda_{(0,0)}}(0, M, 0, 0)$ and the result now follows. □

We get an immediate consequence of the previous lemma.

COROLLARY 4.4.7. *Let $X$ be an $A$-module and $Y$ be a $B$-module. Then*:

(i) $\mathsf{pd}_{\Lambda_{(0,0)}}(X, 0, 0, 0) \ \leq \ \mathsf{pd}_A X + 1 + \mathsf{pd}_{\Lambda_{(0,0)}}(0, M, 0, 0)$

(ii) $\mathsf{pd}_{\Lambda_{(0,0)}}(0, Y, 0, 0) \ \leq \ \mathsf{pd}_B Y + 1 + \mathsf{pd}_{\Lambda_{(0,0)}}(N, 0, 0, 0)$

Proof. We only prove (i). If $\mathsf{pd}_A X$ is not finite, then the result follows. Assume that $\mathsf{pd}_A X = n < \infty$. If we apply Lemma 4.4.6 first to $(X,0,0,0)$ and then to $(\Omega^1_A(X),0,0,0)$, we get

$$\mathsf{pd}_{\Lambda_{(0,0)}}(X,0,0,0) \leq 2 + \max\{\mathsf{pd}_{\Lambda_{(0,0)}}(\Omega^2_A(X),0,0,0), \mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)\}$$

Continuing in this fashion, we get

$$\mathsf{pd}_{\Lambda_{(0,0)}}(X,0,0,0) \leq n + \max\{\mathsf{pd}_{\Lambda_{(0,0)}}(\Omega^n_A(X),0,0,0), \mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)\}$$

By assumption the nth syzygy $\Omega^n_A(X)$ is a projective $A$-module. Applying Lemma 4.4.6 to $(\Omega^n_A(X),0,0,0)$, we obtain the desired result. $\square$

We are now in a position to state our first set of results. For simplicity we write that a left $A$-module $X$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution meaning that the object $(X,0,0,0)$, as a left $\Lambda_{(0,0)}$-module, has an $A$-tight projective $\Lambda_{(0,0)}$-resolution. We make the same agreement for left $B$-modules having $B$-tight projective $\Lambda_{(0,0)}$-resolution.

Proposition 4.4.8. *Let $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ be a Morita ring which is an Artin algebra and let $X$ be an $A$-module and $Y$ be a $B$-module. If $M$ has a $B$-tight projective $\Lambda_{(0,0)}$-resolution, then*

$$\mathsf{pd}_{\Lambda_{(0,0)}}(X,0,0,0) \ \leq \ \mathsf{pd}_A X + 1 + \mathsf{pd}_B M$$

*If $N$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution, then*

$$\mathsf{pd}_{\Lambda_{(0,0)}}(0,Y,0,0) \ \leq \ \mathsf{pd}_B Y + 1 + \mathsf{pd}_A N$$

Proof. The result follows from Corollary 4.4.7 and the fact that $(0,M,0,0)$ having a $B$-tight projective resolution implies that $\mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0) = \mathsf{pd}_B M$. Similarly we obtain the second inequality. $\square$

Theorem 4.4.9. *Let $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ be a Morita ring which is an Artin algebra and suppose that $M$ has a $B$-tight projective $\Lambda_{(0,0)}$-resolution and $N$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution. Then:*

$$\mathsf{gl.\,dim}\,\Lambda_{(0,0)} \ \leq \ \mathsf{gl.\,dim}\,A + \mathsf{gl.\,dim}\,B + 1$$

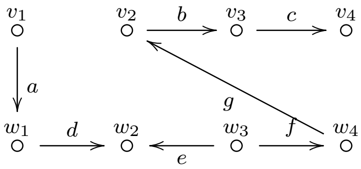
Proof. Since $\phi = \psi = 0$ it follows from Proposition 4.2.3 that the simple $\Lambda_{(0,0)}$-modules are of the form $(S,0,0,0)$, where $S$ is a simple $A$-module or of the form $(0,T,0,0)$, where $T$ is a simple $B$-module. Now

$$\mathsf{gl.\,dim}\,\Lambda_{(0,0)} \leq \max\{\mathsf{pd}_{\Lambda_{(0,0)}}(S,0,0,0), \mathsf{pd}_{\Lambda_{(0,0)}}(0,T,0,0) \mid S\colon \text{simple } A\text{-module}, \quad T\colon \text{simple } B\text{-module}\}$$

By Proposition 4.4.8 we have $\mathsf{pd}_{\Lambda_{(0,0)}}(S,0,0,0) \leq \mathsf{pd}_A S + \mathsf{pd}_B M + 1$. Thus, $\mathsf{pd}_{\Lambda_{(0,0)}}(S,0,0,0) \leq \mathsf{gl.\,dim}\,A + \mathsf{gl.\,dim}\,B + 1$. Similarly, we infer that $\mathsf{pd}_{\Lambda_{(0,0)}}(0,T,0,0) \leq \mathsf{gl.\,dim}\,A + \mathsf{gl.\,dim}\,B + 1$ and then the result follows. $\square$

We provide two examples, the first of which shows that the inequality of Theorem 4.4.9 is sharp and the second shows that the inequality can be proper.

EXAMPLE 4.4.10. Let $\mathbb{K}$ be a field and $\mathcal{Q}$ the quiver




Let $I$ be the ideal in $\mathbb{K}\mathcal{Q}$ generated by all paths of length 2 and let $\Lambda = \mathbb{K}\mathcal{Q}/I$. We see the global dimension of $\Lambda$ is 4. Now set $\epsilon_1 = v_1 + v_2 + v_3 + v_4$ and $\epsilon_2 = w_1 + w_2 + w_3 + w_4$. View $\Lambda$ as the Morita ring

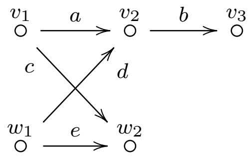
$$\begin{pmatrix} \epsilon_1\Lambda\epsilon_1 & \epsilon_1\Lambda\epsilon_2 \\ \epsilon_2\Lambda\epsilon_1 & \epsilon_2\Lambda\epsilon_2 \end{pmatrix}$$

The global dimension of $\epsilon_1\Lambda\epsilon_1$ is 2 and the global dimension of $\epsilon_2\Lambda\epsilon_2$ is 1. Thus

$$\mathsf{gl.dim}\,\Lambda = \mathsf{gl.dim}\,\epsilon_1\Lambda\epsilon_1 + \mathsf{gl.dim}\,\epsilon_2\Lambda\epsilon_2 + 1$$

Now $M = \epsilon_2\Lambda\epsilon_1$, which, as a left $\epsilon_2\Lambda\epsilon_2$-module is isomorphic to the simple module at $w_1$. We see that $N = \epsilon_1\Lambda\epsilon_2$, which, as a left $\epsilon_1\Lambda\epsilon_1$-module is isomorphic to the simple module at $v_2$. The reader may check that $(0, M, 0, 0)$ and $(N, 0, 0, 0)$ have tight projective $\Lambda$-resolutions. We note that $\phi$ and $\psi$ are both 0 for this example. $\square$

EXAMPLE 4.4.11. Let $\mathbb{K}$ be a field and $\mathcal{Q}$ the quiver




We again take $I$ to be the ideal generated by all paths of length 2 and set $\Lambda = \mathbb{K}\mathcal{Q}/I$. Now set $\epsilon_1 = v_1 + v_2 + v_3$ and $\epsilon_2 = w_1 + w_2$. View $\Lambda$ as the Morita ring

$$\begin{pmatrix} \epsilon_1\Lambda\epsilon_1 & \epsilon_1\Lambda\epsilon_2 \\ \epsilon_2\Lambda\epsilon_1 & \epsilon_2\Lambda\epsilon_2 \end{pmatrix}$$

The reader may check that the hypotheses of Theorem 4.4.9 are satisfied. But the global dimension of $\Lambda$ is 2 while the global dimension of $\epsilon_1\Lambda\epsilon_1$ is 2 and the global dimension of $\epsilon_2\Lambda\epsilon_2$ is 1. $\square$

We now turn to the case where either $(0, M, 0, 0)$ or $(N, 0, 0, 0)$ does not have a tight projective $\Lambda_{(0,0)}$-resolution. If $M$ is not a projective $B$-module then a projective cover of $(0, M, 0, 0)$ is of the form $(0, \beta)\colon (N \otimes_B Q, Q, 0, \mathrm{Id}_{N \otimes Q}) \longrightarrow (0, M, 0, 0)$, where $\beta\colon Q \longrightarrow M$ is projective $B$-cover of $M$. In particular, $N \otimes_B Q$ is a sum of summands of $N$ over which we have little control. The next bound results will require that $M$, as a left $B$-module, is projective and $N$, as a left $A$-module is projective.

We state a preliminary lemma.

LEMMA 4.4.12. *Suppose $M$ is a $B$-$A$-bimodule which is projective as a left $B$-module and $N$ is an $A$-$B$-bimodule which is projective as a left $A$-module. Then*

(i) $(M \otimes_A N)^{\overset{t}{\otimes}_B}$ *is a projective left $A$-module, for all $t \geq 1$.*
(ii) $N \otimes_B (M \otimes_A N)^{\overset{t}{\otimes}_B}$ *is a projective left $A$-module, for all $t \geq 0$.*
(iii) $(N \otimes_B M)^{\overset{s}{\otimes}_A}$ *is a projective left $B$-module, for all $s \geq 1$.*
(iv) $M \otimes_A (N \otimes_B M)^{\overset{s}{\otimes}_A}$ *is a projective left $B$-module, for all $s \geq 0$.*

Proof. (i) Suppose $P$ is a projective left $A$-module. Then $P$ is isomorphic to a direct sum of indecomposable projective $A$-modules of the form $Ae$, where $e$ is a primitive idempotent in $A$. It follows that $M \otimes_A P$ is a direct sum of modules of the form $Me$. Since $M$ is assumed to be a projective left $B$-module, $Me$ is a projective left $B$-module and, hence, $M \otimes_A P$ is a projective left $B$-module. The result now follows by induction on $t$.

(iii) Similarly if $Q$ is a projective left $B$-module then $N \otimes_B Q$ is also a projective left $A$-module and then our statement follows. In the same way we get (ii) and (iv). □

The next result concerns tight projective $\Lambda_{(0,0)}$-modules.

Lemma 4.4.13. *Suppose $M$ is a $B$-$A$-bimodule which is projective as a left $B$-module and $N$ is an $A$-$B$-bimodule which is projective as a left $A$-module.*

(i) *If $(0, M \otimes_A (N \otimes_B M)^{\overset{s}{\otimes}_A}, 0, 0)$ is a $B$-tight projective $\Lambda_{(0,0)}$-module, for some $s \geq 0$, then $(0, (M \otimes_A N)^{\overset{s+1}{\otimes}_B}, 0, 0)$ also is a $B$-tight projective $\Lambda_{(0,0)}$-module.*

(ii) *If $((N \otimes_B M)^{\overset{s}{\otimes}_A}, 0, 0, 0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module, for some $s \geq 0$, then $(N \otimes_B (M \otimes_A N)^{\overset{s}{\otimes}_B}, 0, 0, 0)$ also is an $A$-tight projective $\Lambda_{(0,0)}$-module.*

(iii) *If $((N \otimes_B M)^{\overset{s}{\otimes}_A}, 0, 0, 0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module, for some $s \geq 0$, then $(N \otimes_B (M \otimes_A N)^{\overset{s+1}{\otimes}_B}, 0, 0, 0)$ also is an $A$-tight projective $\Lambda_{(0,0)}$-module.*

(iv) *If $(0, (M \otimes_A N)^{\overset{s}{\otimes}_B}, 0, 0)$ is a $B$-tight projective $\Lambda_{(0,0)}$-module, for some $s \geq 0$, then $(0, M \otimes_A (N \otimes_B M)^{\overset{s}{\otimes}_A}, 0, 0)$ also is a $B$-tight projective $\Lambda_{(0,0)}$-module.*

Proof. We only prove part (i) since the proofs of the other parts are similar. Assume that $(0, M \otimes_A (N \otimes_B M)^{\overset{s}{\otimes}_A}, 0, 0)$ is a $B$-tight projective $\Lambda_{(0,0)}$-module, for some $s \geq 0$. Then tensoring $M \otimes_A (N \otimes_B M)^{\overset{s}{\otimes}_A}$ on the right by $\otimes_A N$, we obtain $(M \otimes_A N)^{\overset{s+1}{\otimes}_B}$. The assumption that $N$ is a projective left $A$-module implies that $(M \otimes_A N)^{\overset{s+1}{\otimes}_B}$ is a direct sum of summands of $M \otimes_A (N \otimes_B M)^{\overset{s}{\otimes}_A}$. The result now follows. □

We are now in a position to state the second result on bounding the global dimension of $\Lambda_{(0,0)}$.

Theorem 4.4.14. *Let $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ be a Morita ring which is an Artin algebra. Suppose that the global dimensions of $A$ and $B$ are finite, $M$ is a projective left $B$-module, and $N$ is a projective left $A$-module.*

(i) *If $((N \otimes_B M)^{\overset{s}{\otimes}_A}, 0, 0, 0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module, for some $s \geq 1$, then:*

$$\mathsf{gl.dim}\, \Lambda_{(0,0)} \;\leq\; \max\{\mathsf{gl.dim}\, A + 2s, \mathsf{gl.dim}\, B + 2s + 1\}$$

(ii) *If $(0, M \otimes_A (N \otimes_B M)^{\overset{s}{\otimes}_A}, 0, 0)$ is a $B$-tight projective $\Lambda_{(0,0)}$-module, for some $s \geq 0$, then:*

$$\mathsf{gl.dim}\, \Lambda_{(0,0)} \;\leq\; \max\{\mathsf{gl.dim}\, A + 2s + 1, \mathsf{gl.dim}\, B + 2(s + 1)\}$$

(iii) *If $(N \otimes_B (M \otimes_A N)^{\overset{s}{\otimes}_B}, 0, 0, 0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module, for some $s \geq 0$, then:*

$$\mathsf{gl.dim}\, \Lambda_{(0,0)} \;\leq\; \max\{\mathsf{gl.dim}\, A + 2(s + 1), \mathsf{gl.dim}\, B + 2s + 1\}$$

(iv) *If* $(0,(M\otimes_A N)^{\overset{s}{\otimes}_B},0,0)$ *is a* $B$*-tight projective* $\Lambda_{(0,0)}$*-module, for some* $s\geq 1$*, then:*

$$\mathsf{gl.\,dim}\,\Lambda_{(0,0)} \;\leq\; \max\{\mathsf{gl.\,dim}\,A+2s+1,\mathsf{gl.\,dim}\,B+2s\}$$

(v) *If* $((N\otimes_B M)^{\overset{s}{\otimes}_A},0,0,0)$ *is an* $A$*-tight projective* $\Lambda_{(0,0)}$*-module and if* $(0,(M\otimes_A N)^{\overset{s}{\otimes}_B},0,0)$ *is an* $B$*-tight projective* $\Lambda_{(0,0)}$*-module, for some* $s\geq 1$*, then:*

$$\mathsf{gl.\,dim}\,\Lambda_{(0,0)} \;\leq\; \max\{\mathsf{gl.\,dim}\,A+2s,\mathsf{gl.\,dim}\,B+2s\}$$

(vi) *If* $(N\otimes_B (M\otimes_A N)^{\overset{s}{\otimes}_B},0,0,0)$ *is an* $A$*-tight projective* $\Lambda_{(0,0)}$*-module, and if* $(0,M\otimes_A (N\otimes_B M)^{\overset{s}{\otimes}_A},0,0)$ *is a* $B$*-tight projective* $\Lambda_{(0,0)}$*-module, for some* $s\geq 0$*, then:*

$$\mathsf{gl.\,dim}\,\Lambda_{(0,0)} \;\leq\; \max\{\mathsf{gl.\,dim}\,A+2s+1,\mathsf{gl.\,dim}\,B+2s+1\}$$

Proof. Let $\mathsf{gl.\,dim}\,A=d<\infty$ and $\mathsf{gl.\,dim}\,B=e<\infty$. We only prove part (i) with the remaining parts having similar proofs. We assume that $((N\otimes_B M)^{\overset{s}{\otimes}_A},0,0,0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module, for some $s\geq 1$. First let $S$ be a simple $A$-module. By Lemma 4.4.6 we have $\mathsf{pd}_{\Lambda_{(0,0)}}(S,0,0,0)\leq 1+\max\{\mathsf{pd}_{\Lambda_{(0,0)}}(\Omega^1_A(S),0,0,0),\mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)\}$. By applying Lemma 4.4.6 again, this time to $(\Omega^1_A(S),0,0,0)$, we get

$$\mathsf{pd}_{\Lambda_{(0,0)}}(S,0,0,0)\leq 2+\max\{\mathsf{pd}_{\Lambda_{(0,0)}}(\Omega^2_A(S),0,0,0),\mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)\}$$

Continuing in this fashion, we get

$$\mathsf{pd}_{\Lambda_{(0,0)}}(S,0,0,0)\leq d+\max\{\mathsf{pd}_{\Lambda_{(0,0)}}(\Omega^d_A(S),0,0,0),\mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)\}$$

Now $\Omega^d_A(S)$ is a projective $A$-module, so the next time we apply Lemma 4.4.6, we obtain

$$\mathsf{pd}_{\Lambda_{(0,0)}}(S,0,0,0)\leq d+1+\mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)$$

If $(0,M,0,0)$ is a $B$-tight projective $\Lambda$-module, then we are done. Suppose that $(0,M,0,0)$ is not a $B$-tight projective module. Since $M$ is a projective $B$-module, by Lemma 4.4.4 it follows that $\mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)\leq 1+\mathsf{pd}_{\Lambda_{(0,0)}}(N\otimes_B M,0,0,0)$. If $(N\otimes_B M,0,0,0)$ is an $A$-tight projective $\Lambda$-module, we are done. Suppose that $(N\otimes_B M,0,0,0)$ is not an $A$-tight projective $\Lambda$-module. Since $N\otimes_B M$ is a projective $A$-module by Lemma 4.4.12, we see again by Lemma 4.4.4 that $\mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)\leq 2+\mathsf{pd}_{\Lambda_{(0,0)}}(0,M\otimes_A N\otimes_B M,0,0)$. Continuing in this fashion, we obtain

$$\mathsf{pd}_{\Lambda_{(0,0)}}(0,M,0,0)\leq 2s-1+\mathsf{pd}_{\Lambda_{(0,0)}}((N\otimes_B M)^{\overset{s}{\otimes}_A},0,0,0)$$

By assumption, $((N\otimes_B M)^{\overset{s}{\otimes}_A},0,0,0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module. Hence, we see that

$$\mathsf{pd}_{\Lambda_{(0,0)}}(S,0,0,0)\leq d+2s=\mathsf{gl.\,dim}\,A+2s \qquad (*)$$

Now let $T$ be a simple $B$-module. By Lemma 4.4.6 we have

$$\mathsf{pd}_{\Lambda_{(0,0)}}(0,T,0,0)\leq 1+\max\{\mathsf{pd}_{\Lambda_{(0,0)}}(0,\Omega^1_B(T),0,0),\mathsf{pd}_{\Lambda_{(0,0)}}(N,0,0,0)\}$$

Continuing in a similar fashion to the first part of the proof, we obtain

$$\mathsf{pd}_{\Lambda_{(0,0)}}(T,0,0,0)\leq e+\max\{\mathsf{pd}_{\Lambda_{(0,0)}}(0,\Omega^e_B(T),0,0),\mathsf{pd}_{\Lambda_{(0,0)}}(N,0,0,0)\}$$

Now $N$ is a projective $A$-module, and we see by Lemma 4.4.4 that $\mathsf{pd}_{\Lambda_{(0,0)}}(N,0,0,0) \leq 1+\mathsf{pd}_{\Lambda_{(0,0)}}(0, M\otimes_A N, 0,0)$. Again, following similar arguments to the first part of the proof, we obtain

$$\mathsf{pd}_{\Lambda_{(0,0)}}(N,0,0,0) \leq 2s + \mathsf{pd}_{\Lambda_{(0,0)}}(N\otimes_B (M\otimes_A N)^{\overset{s}{\otimes}_B}, 0,0,0)$$

By Lemma 4.4.13, $(N\otimes_B (M\otimes_A N)^{\overset{s}{\otimes}_B}, 0,0,0)$ is an $A$-tight projective $\Lambda_{(0,0)}$-module, and we have

$$\mathsf{pd}_{\Lambda_{(0,0)}}(0,T,0,0) \leq e+2s+1 = \mathsf{gl.dim}\, B + 2s + 1 \tag{$**$}$$

Since $\Lambda_{(0,0)}$ is an Artin algebra and since from Proposition 4.2.3 a simple $\Lambda_{(0,0)}$-module is isomorphic to either a module of the form $(S,0,0,0)$ or $(0,T,0,0)$, for some simple $A$-module $S$ or some simple $B$-module $T$, part (1) follows from $(*)$ and $(**)$. $\square$

We conclude this section with an example showing that the bounds in the above theorem are sharp.

EXAMPLE 4.4.15. Let $\mathbb{K}$ be a field and let $\mathcal{Q}$ be the quiver

$$\begin{tikzcd} \overset{v_1}{\circ} \arrow[d] & & \overset{v_3}{\circ} \arrow[d] & & \overset{v_5}{\circ} \\ \overset{v_2}{\circ} \arrow[urr] & & \overset{v_4}{\circ} \arrow[urr] & & \end{tikzcd}$$

Let $\Lambda$ be the quotient $\mathbb{K}\mathcal{Q}/I$, where $I$ is the ideal generated by all paths of length 2. Let $\epsilon_1 = v_1+v_3+v_5$ and $\epsilon_2 = v_2+v_4$. View $\Lambda$ as the Morita ring

$$\begin{pmatrix} \epsilon_1\Lambda\epsilon_1 & \epsilon_1\Lambda\epsilon_2 \\ \epsilon_2\Lambda\epsilon_1 & \epsilon_2\Lambda\epsilon_2 \end{pmatrix}_{(0,0)}$$

Using the notation ${}_i\mathbb{K}_j$ to denote the simple $\Lambda$-module, which on the left is isomorphic to the simple $\Lambda$-module at vertex $v_i$, and on the right is isomorphic to the simple $\Lambda$-module at vertex $v_j$, we see that

$$M = \epsilon_2\Lambda\epsilon_1 = {}_4\mathbb{K}_3 \oplus {}_2\mathbb{K}_1, \text{ and } N = \epsilon_1\Lambda\epsilon_2 = {}_5\mathbb{K}_4 \oplus {}_3\mathbb{K}_2.$$

Now the global dimensions of $A = \epsilon_1\Lambda\epsilon_1$ and $B = \epsilon_2\Lambda\epsilon_2$ are both 0. Clearly, $M$ is a projective left $B$-module and $N$ is a projective left $A$-module. We see that $N\otimes_B(M\otimes_A N)$ is isomorphic to ${}_5\mathbb{K}_2$. Moreover, $(N\otimes_B(M\otimes_A N),0,0,0)$ is an $A$-tight projective $\Lambda$-module. Thus, we can apply part (3) of Theorem 4.4.14 with $s=1$ to get $\mathsf{gl.dim}\,\Lambda \leq 4$. But the global dimension of $\Lambda$ is 4, and we have shown that the inequality in part (3) is sharp. This example can be adjusted to get that all the inequalities are sharp. $\square$

**4.4.2. Some Lower Bounds.** In this subsection we provide some lower bounds for the global dimension of a Morita ring.

LEMMA 4.4.16. *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring.*

(i) *If the bimodule ${}_BM_A$ is flat as a right $A$-module then $\mathsf{pd}_A X = \mathsf{pd}_{\Lambda_{(\phi,\psi)}} \mathsf{T}_A(X)$.*
(ii) *If the bimodule ${}_AN_B$ is flat as a right $B$-module then $\mathsf{pd}_B Y = \mathsf{pd}_{\Lambda_{(\phi,\psi)}} \mathsf{T}_B(Y)$.*

PROOF. Suppose that the bimodule ${}_BM_A$ is flat as a right $A$-module. Then the functor $M\otimes_A - \colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}B$ is exact and therefore the functor $\mathsf{T}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is exact. Let $X$ be a $A$-module with $\mathsf{pd}_A X = n$ and let $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be the projective resolution of $X$. Then if we apply the exact functor

$\mathsf{T}_A$ we get that $\mathsf{pd}_{\Lambda_{(\phi,\psi)}} \mathsf{T}_A(X) \leq n = \mathsf{pd}_A X$ since $\mathsf{T}_A$ preserves projectives. Conversely suppose that $\mathsf{pd}_{\Lambda_{(\phi,\psi)}} \mathsf{T}_A(X) = m < \infty$. Let $0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be an exact sequence with $P_0 \in \mathsf{Proj}\, A$ and $K_0 = \mathsf{Ker}\, a_0$, where $a_0 \colon P_0 \longrightarrow X$. Since $\mathsf{T}_A$ is exact the sequence $0 \longrightarrow \mathsf{T}_A(K_0) \longrightarrow \mathsf{T}_A(P_0) \longrightarrow \mathsf{T}_A(X) \longrightarrow 0$ is exact. Now we continue with the same procedure. This means that we take an epimorphism $a_1 \colon P_1 \longrightarrow K_0$ with $P_1$ a projective $A$-module, $K_1 = \mathsf{Ker}\, a_1$ and then we apply the functor $\mathsf{T}_A$. After $m$-steps we obtain the exact sequence: $0 \longrightarrow \mathsf{T}_A(K_{m-1}) \longrightarrow \mathsf{T}_A(P_{m-1}) \longrightarrow \cdots \longrightarrow \mathsf{T}_A(P_0) \longrightarrow \mathsf{T}_A(X) \longrightarrow 0$ where $\mathsf{T}_A(K_{m-1})$ is projective since $\mathsf{pd}_{\Lambda_{(\phi,\psi)}} \mathsf{T}_A(X) = m$. Then if we apply the functor $\mathsf{U}_A$ we get the exact sequence: $0 \longrightarrow K_{m-1} \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ and we claim that $\Omega^m(X) = K_{m-1}$ is projective in $\mathsf{Mod}\text{-}A$. But this is straightforward since $\mathsf{T}_A(K_{m-1})$ is projective. Thus we have $\mathsf{pd}_A X \leq m = \mathsf{pd}_{\Lambda_{(\phi,\psi)}} \mathsf{T}_A(X)$. We infer that $\mathsf{pd}_A X = \mathsf{pd}_{\Lambda_{(\phi,\psi)}} \mathsf{T}_A(X)$ and similarly we prove that $\mathsf{pd}_B Y = \mathsf{pd}_{\Lambda_{(\phi,\psi)}} \mathsf{T}_B(Y)$ when the functor $N \otimes_B - \colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}A$ is exact. $\square$

As a consequence of the above result we have the following lower bound.

PROPOSITION 4.4.17. *[88, Lemma 1.2] Let $\Lambda_{(\phi,\psi)}$ be a Morita ring and suppose that $M_A$ is a flat right $A$-module and $N_B$ is a flat right $B$-module. Then:*

$$\mathsf{gl.dim}\, \Lambda_{(\phi,\psi)} \;\geq\; \max\{\mathsf{gl.dim}\, A, \mathsf{gl.dim}\, B\}$$

**4.4.3. Comparing Tight Resolutions.** In this subsection we discuss the assumption of Theorem 4.4.9 about tight resolutions. Our aim is to compare our result with some well known bounds for the global dimension of trivial extensions rings.

Let $\Lambda_{(0,0)}$ be a Morita ring regarded as an Artin algebra. Then from Proposition 4.1.5 we have the isomorphism of rings $\Lambda_{(0,0)} \simeq (A\times B)\ltimes M\oplus N$, where $(A\times B)\ltimes M\oplus N$ is the trivial extension ring of $A\times B$ by the $(A\times B)$-$(A\times B)$-bimodule $M\oplus N$. Then the module category $\mathsf{mod}\text{-}\Lambda_{(0,0)}$ is equivalent to the trivial extension of abelian categories $(\mathsf{mod}\text{-}A \times \mathsf{mod}\text{-}B) \ltimes H$, see [49], where $H$ is the endofunctor

$$H\colon \mathsf{mod}\text{-}A \times \mathsf{mod}\text{-}B \longrightarrow \mathsf{mod}\text{-}A \times \mathsf{mod}\text{-}B,\;\; H(X,Y) = (N \otimes_B Y, M \otimes_A X)$$

Suppose that ${}_AN_B$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution and ${}_BM_A$ has a $B$-tight projective $\Lambda_{(0,0)}$-resolution. This implies that we have projective resolutions $\cdots \longrightarrow {}_AP_1 \longrightarrow {}_AP_0 \longrightarrow {}_AN \longrightarrow 0$ and $\cdots \longrightarrow {}_BQ_1 \longrightarrow {}_BQ_0 \longrightarrow {}_BM \longrightarrow 0$ such that $M \otimes_A P_i = 0$ and $N \otimes_B Q_i = 0$. If we aply the functor $M\otimes_A -$ to the projective resolution of $N$ we get that $M \otimes_A N = 0$. Similarly if we apply the functor $N \otimes_B -$ to the projective resolution of $M$ we obtain that $N \otimes_B M = 0$. Also we derive that $\mathsf{Tor}_i^A(M,N) = 0$ and $\mathsf{Tor}_i^B(N,M) = 0$ for every $i \geq 0$. Since $M \otimes_A N = 0$ if and only if $M \otimes_A N \otimes_B - = 0$ and $N \otimes_B M = 0$ if and only if $N \otimes_B M \otimes_A - = 0$ it follows that $H^2 = 0$. From Corollary 7.6 of [26] it follows that if the left derived functor $\mathsf{L}_iH^j(H(P,Q)) = 0$ for every $i,j \geq 1$ and $P \in \mathsf{proj}\, A, Q \in \mathsf{proj}\, B$, then

$$\mathsf{gl.dim}\, \Lambda_{(0,0)} \leq c(H) + 2 \cdot \max\{\mathsf{gl.dim}\, A, \mathsf{gl.dim}\, B\} \tag{$*$}$$

where $c(H) = \min\{\kappa \in \mathbb{N} : H^{\kappa+1} = 0\}$ is the nilpotency class of $H$. From the projective resolutions of $N$ and $M$ we have the following projective resolution of $H(A,B)$:

$$\cdots \longrightarrow ({}_AP_1, {}_BQ_1) \longrightarrow ({}_AP_0, {}_BQ_0) \longrightarrow H(A,B) \longrightarrow 0$$

in $\mathsf{mod}\text{-}A\times B$. Hence if we apply the functor $H$ to the above exact sequence we obtain the zero complex. We infer that $\mathsf{L}_iH^j(H(P,Q)) = 0$ for every $i,j \geq 1$ and $P \in \mathsf{proj}\, A, Q \in \mathsf{proj}\, B$. Hence the assumption of Corollary 7.6 of [26] is satisfied and so we have the bound of the relation $(*)$. In particular we obtain that $\operatorname{gl.dim}\Lambda_{(0,0} \leq 1 + 2\cdot\max\{\operatorname{gl.dim}A, \operatorname{gl.dim}B\}$ since $H^2 = 0$. But the bound of Theorem 4.4.9 is $\operatorname{gl.dim}\Lambda_{(0,0)} \leq \operatorname{gl.dim}A + \operatorname{gl.dim}B + 1$ which is better than the above bound but the assumption of Corollary 7.6 of [26] is weaker than the assumption of tight resolutions for $N$ and $M$. Note also that since $\mathsf{L}_iH(H(P,Q)) = 0$ for every $i \geq 0$ and $P \in \operatorname{proj}A, Q \in \operatorname{proj}B$, there exists an explicit formula for the global dimension of $\Lambda_{(0,0)}$, see Corollary 7.17 of [26].

**4.4.4. Trivial Extensions of Artin Algebras.** The main property of the assumption that $M$ has a $B$-tight projective $\Lambda_{(0,0)}$-resolution and $N$ has an $A$-tight projective $\Lambda_{(0,0)}$-resolution is that $\operatorname{pd}_{\Lambda_{(0,0)}}(0,M,0,0) = \operatorname{pd}_B M$ and $\operatorname{pd}_{\Lambda_{(0,0)}}(N,0,0,0) = \operatorname{pd}_A N$. Our aim in this subsection is to prove a version of Theorem 4.4.9 for a trivial extension of Artin algebras $\Lambda = A \ltimes N$. We start by recalling some basic facts for trivial extensions. We refer to [49] for more details.

Let $\Lambda = A \ltimes N$ be a trivial extension of rings which is an Artin algebra (see the discussion before Proposition 4.1.5 for the notion of trivial extension of rings). The objects of $\operatorname{mod-}\Lambda$ are pairs $(X,f)$ where $X \in \operatorname{mod-}A$ and $f\colon N\otimes_A X \longrightarrow X$ is an $A$-morphism such that $N\otimes_A f \circ f = 0$. A morphism $a\colon (X,f) \longrightarrow (Y,g)$ is an $A$-morphism $a\colon X \longrightarrow Y$ such that $f\circ a = N\otimes_A a \circ g$. We recall also the following functors. The functor $\mathsf{T}\colon \operatorname{mod-}A \longrightarrow \operatorname{mod-}\Lambda$ is defined by $\mathsf{T}(X) = (X\oplus(N\otimes_A X), t_X) \in \operatorname{mod-}\Lambda$ on the $A$-modules $X$, where $t_X = \left(\begin{smallmatrix} 0 & \mathrm{Id}_{N\otimes X} \\ 0 & 0 \end{smallmatrix}\right)\colon (N\otimes_A X)\oplus(N\otimes_A N\otimes_A X) \longrightarrow X\oplus(N\otimes_A X)$ and given an $A$-morphism $a\colon X \longrightarrow Y$ then $\mathsf{T}(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & F(a) \end{smallmatrix}\right)\colon \mathsf{T}(X) \longrightarrow \mathsf{T}(Y)$ is a $\Lambda$-morphism. The functor $\mathsf{Z}\colon \operatorname{mod-}A \longrightarrow \operatorname{mod-}\Lambda$ is defined by $\mathsf{Z}(X) = (X,0) \in \operatorname{mod-}\Lambda$ on the $A$-modules $X$ and if $a\colon X \longrightarrow Y$ is an $A$-morphism, then $\mathsf{Z}(a) = a$.

We need the following result which is the analogue of Lemma 4.4.6.

Lemma 4.4.18. *Let $\Lambda = A\ltimes N$ be a trivial extension of Artin algebras and let $X$ be an $A$-module. Then*:

$$\operatorname{pd}_\Lambda \mathsf{Z}(X) \leq 1 + \max\{\operatorname{pd}_\Lambda \mathsf{Z}(\Omega^1_A(X)), \operatorname{pd}_\Lambda \mathsf{Z}(N)\}$$

Proof. Let $\alpha\colon P \longrightarrow X$ be a projective $A$-cover of $X$ with kernel $\Omega^1_A(X)$. Then we have a short exact sequence of $\Lambda$-modules

$$\begin{tikzcd}
(N\otimes_A \Omega^1_A(X))\oplus(N\otimes_A N\otimes_A P) \arrow[r, "\left(\begin{smallmatrix} N\otimes k & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & N\otimes k \\ 0 & 0 \end{smallmatrix}\right)"] & (N\otimes_A P)\oplus(N\otimes_A N\otimes_A P) \arrow[r, two heads, "\left(\begin{smallmatrix} N\otimes\alpha \\ 0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] & N\otimes_A X \arrow[d, "0"] \\
\Omega^1_A(X)\oplus(N\otimes_A P) \arrow[r, tail, "\left(\begin{smallmatrix} k & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & P\oplus(N\otimes_A P) \arrow[r, two heads, "\left(\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right)"] & X
\end{tikzcd}$$

in which the middle term is a projective $\Lambda$-module. It follows that

$$\operatorname{pd}_\Lambda \mathsf{Z}(X) \leq 1 + \operatorname{pd}_\Lambda\big((N\otimes_A\Omega^1_A(X))\oplus(N\otimes_A N\otimes_A P) \longrightarrow \Omega^1_A(X)\oplus(N\otimes_A P)\big) \quad (1)$$

Next we note that we have the following exact commutative diagram

$$\begin{tikzcd}
N\otimes_A N\otimes_A P \arrow[r, tail, "\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "0"] & (N\otimes_A \Omega^1_A(X))\oplus(N\otimes_A N\otimes_A P) \arrow[r, two heads, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & N\otimes k \\ 0 & 0 \end{smallmatrix}\right)"] & N\otimes_A \Omega^1_A(X) \arrow[d, "0"] \\
N\otimes_A P \arrow[r, tail, "\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)"] & \Omega^1_A(X)\oplus(N\otimes_A P) \arrow[r, two heads, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \Omega^1_A(X)
\end{tikzcd}$$

and so we have

$$\mathsf{pd}_\Lambda\big((N\otimes_A\Omega^1_A(X))\oplus(N\otimes_A N\otimes_A P)\to\Omega^1_A(X)\oplus(N\otimes_A P)\big)\leq$$

$$\max\{\mathsf{pd}_\Lambda\mathsf{Z}(N\otimes_A P),\mathsf{pd}_\Lambda\mathsf{Z}(\Omega^1_A(X))\}\ (2)$$

Since we have that $N\otimes_A P$ is direct sum of summands of $N$ it follows that $\mathsf{pd}_\Lambda\mathsf{Z}(N\otimes_A P)\leq\mathsf{pd}_\Lambda\mathsf{Z}(N)$. Hence the result follows from the relations (1) and (2). □

We have the following result and its consequence.

PROPOSITION 4.4.19. *Let $\Lambda = A\ltimes N$ be a trivial extension of Artin algebras. Then*:

$$\mathsf{gl.dim}\,\Lambda\ \leq\ \mathsf{gl.dim}\,A+\mathsf{pd}_\Lambda\mathsf{Z}(N)+1$$

PROOF. Let $X$ be an $A$-module. We will first prove that

$$\mathsf{pd}_\Lambda\mathsf{Z}(X)\leq\mathsf{pd}_A X+\mathsf{pd}_\Lambda\mathsf{Z}(N)+1 \qquad (*)$$

If $\mathsf{pd}_A X$ is not finite, then the result follows. Assume that $\mathsf{pd}_A X = n$. If we apply Lemma 4.4.18 first to $\mathsf{Z}(X)$ and then to $\mathsf{Z}(\Omega^1_A(X))$, we get that

$$\mathsf{pd}_\Lambda\mathsf{Z}(X)\leq 2+\max\{\mathsf{pd}_\Lambda\mathsf{Z}(\Omega^2_A(X)),\mathsf{pd}_\Lambda\mathsf{Z}(N)\}$$

Continuing in this fashion, we obtain

$$\mathsf{pd}_\Lambda\mathsf{Z}(X)\leq n+\max\{\mathsf{pd}_\Lambda\mathsf{Z}(\Omega^n_A(X)),\mathsf{pd}_\Lambda\mathsf{Z}(N)\}$$

By assumption, $\Omega^n_A(X)$ is a projective $A$-module. Applying again Lemma 4.4.18 to $\mathsf{Z}(\Omega^n_A(X))$ we obtain the relation $(*)$. Recall from [49] that the simple $\Lambda$-modules are of the form $\mathsf{Z}(S)$, where $S$ is a simple $A$-module. Then from the relation $(*)$ we have $\mathsf{pd}_\Lambda\mathsf{Z}(S)\leq\mathsf{pd}_A S+\mathsf{pd}_\Lambda\mathsf{Z}(N)+1$. Thus $\mathsf{pd}_\Lambda\mathsf{Z}(S)\leq\mathsf{gl.dim}\,A+\mathsf{pd}_\Lambda\mathsf{Z}(N)+1$ and so the result follows. □

COROLLARY 4.4.20. *Let $\Lambda = A\ltimes N$ be a trivial extension of Artin algebras such that $\mathsf{pd}_A N=\mathsf{pd}_\Lambda\mathsf{Z}(N)$. Then*:

$$\mathsf{gl.dim}\,\Lambda\ \leq\ 2\cdot\mathsf{gl.dim}\,A+1$$

## 4.5. Gorenstein Artin Algebras and Cohen-Macaulay Modules

In this section we investigate when a Morita ring, which is an Artin algebra, is Gorenstein. Moreover we determine the Gorenstein-projective modules over the matrix algebra with $A = M = N = B = \Lambda$, where $\Lambda$ is an Artin algebra. We start by recalling the notion of Gorenstein algebras.

DEFINITION 4.5.1. [15, 16] An Artin algebra $\Lambda$ is called Gorenstein if $\mathsf{id}_\Lambda\Lambda<\infty$ and $\mathsf{id}\,\Lambda_\Lambda<\infty$

Equivalently, $\Lambda$ is Gorenstein if $(\mathsf{proj}\,\Lambda)^{<\infty}=(\mathsf{inj}\,\Lambda)^{<\infty}$, where $(\mathsf{proj}\,\Lambda)^{<\infty}$, resp. $(\mathsf{inj}\,\Lambda)^{<\infty}$, is the full subcategory of $\mathsf{mod}$-$\Lambda$ consisting of the $\Lambda$-modules of finite projective, resp. injective, dimension. Note that for a Gorenstein Artin algebra we have $\mathsf{id}_\Lambda\Lambda=\mathsf{id}\,\Lambda_\Lambda$ [16]. Important classes of Gorenstein algebras are the algebras of finite global dimension and the selfinjective algebras.

We start with the next result which describes the left derived functors of $\mathsf{T}_A$, $\mathsf{T}_B$ and gives also some useful isomorphisms for the extensions groups $\mathsf{Ext}$.

LEMMA 4.5.2. *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring.*

(i) *For every $n \geq 1$ we have the following natural isomorphisms*:

$$\mathsf{U}_B\mathsf{L}_n\mathsf{T}_A(-) \xrightarrow{\simeq} \mathsf{Tor}_n^A(M,-) \quad \textit{and} \quad \mathsf{U}_A\mathsf{L}_n\mathsf{T}_A(-) = 0$$

(ii) *For every $n \geq 1$ we have the following natural isomorphisms*:

$$\mathsf{U}_A\mathsf{L}_n\mathsf{T}_B(-) \xrightarrow{\simeq} \mathsf{Tor}_n^B(N,-) \quad \textit{and} \quad \mathsf{U}_B\mathsf{L}_n\mathsf{T}_B(-) = 0$$

(iii) *If $\mathsf{Tor}_i^A(M,X) = 0$, $\forall 1 \leq i \leq n$, then we have an isomorphism*:

$$\mathsf{Ext}^i_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(X),(X',Y',f',g')) \xrightarrow{\simeq} \mathsf{Ext}^i_A(X,X')$$

*for every $1 \leq i \leq n$ and $(X',Y',f',g') \in \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$.*

(iv) *If $\mathsf{Tor}_i^B(N,Y) = 0$, $\forall 1 \leq i \leq n$, then we have an isomorphism*:

$$\mathsf{Ext}^i_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_B(Y),(X',Y',f',g')) \xrightarrow{\simeq} \mathsf{Ext}^i_B(Y,Y')$$

*for every $1 \leq i \leq n$ and $(X',Y',f',g') \in \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$.*

Proof. This result was proved in the general framework of Morita categories, see Proposition 3.7.1 of Chapter 3. □

The following main result of this section gives a sufficient condition for a Morita ring to be Gorenstein.

Theorem 4.5.3. *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring which is an Artin algebra such that the adjoint pair of functors $(M \otimes_A -, \mathsf{Hom}_B(M,-))$ induces an equivalence*

$$M \otimes_A - \colon (\mathsf{proj}\, A)^{<\infty} \underset{\simeq}{\rightleftarrows} (\mathsf{inj}\, B)^{<\infty} \ \colon \ \mathsf{Hom}_B(M,-)$$

*and the adjoint pair of functors $(N \otimes_B -, \mathsf{Hom}_A(N,-))$ induces an equivalence*

$$N \otimes_B - \colon (\mathsf{proj}\, B)^{<\infty} \underset{\simeq}{\rightleftarrows} (\mathsf{inj}\, A)^{<\infty} \ \colon \ \mathsf{Hom}_A(N,-)$$

*Then the Morita ring $\Lambda_{(\phi,\psi)}$ is Gorenstein.*

Proof. We will show that $(\mathsf{proj}\,\Lambda_{(\phi,\psi)})^{<\infty} = (\mathsf{inj}\,\Lambda_{(\phi,\psi)})^{<\infty}$. In oder to prove our claim it suffices to show that any projective $\Lambda_{(\phi,\psi)}$-module has finite injective dimension and any injective $\Lambda_{(\phi,\psi)}$-module has finite projective dimension. Thus from Proposition 4.2.1 and Proposition 4.2.2 it suffices to show that $\mathsf{id}_{\Lambda_{(\phi,\psi)}}\mathsf{T}_A(P) < \infty$, $\mathsf{id}_{\Lambda_{(\phi,\psi)}}\mathsf{T}_B(Q) < \infty$, $\mathsf{pd}_{\Lambda_{(\phi,\psi)}}\mathsf{H}_A(I) < \infty$ and $\mathsf{pd}_{\Lambda_{(\phi,\psi)}}\mathsf{H}_B(J) < \infty$ for any $P \in \mathsf{proj}\,A$, $Q \in \mathsf{proj}\,B$, $I \in \mathsf{inj}\,A$ and $J \in \mathsf{inj}\,B$. Let $I \in \mathsf{inj}\,A$. By hypothesis the counit $\varepsilon'_I \colon N \otimes_B \mathsf{Hom}_A(N,I)) \longrightarrow I$ is an isomorphism and consider the $\Lambda_{(\phi,\psi)}$-modules $\mathsf{H}_A(I) = (I, \mathsf{Hom}_A(N,I), \delta'_{M\otimes I} \circ \mathsf{Hom}_A(N,\Psi_I), \epsilon'_I)$ and

$$\mathsf{T}_B(\mathsf{Hom}_A(N,I)) = (N \otimes_B \mathsf{Hom}_A(N,I), \mathsf{Hom}_A(N,I), \Phi_{\mathsf{Hom}_A(N,I)}, \mathrm{Id}_{N\otimes_B \mathsf{Hom}_A(N,I)})$$

Since $\mathsf{H}_A(I)$ is a $\Lambda_{(\phi,\psi)}$-module we have the following commutative diagram:

$$\begin{tikzcd}
M \otimes_A N \otimes_B \mathsf{Hom}_A(N,I) \arrow[r, "M\otimes\varepsilon'_I"] \arrow[d, "\phi\otimes\mathrm{Id}_{\mathsf{Hom}_A(N,I)}"'] & M \otimes_A I \arrow[d, "\delta'_{M\otimes I}\circ\mathsf{Hom}_A(N,\Psi_I)"] \\
B \otimes_B \mathsf{Hom}_A(N,I) \arrow[r, "\simeq"] & \mathsf{Hom}_A(N,I)
\end{tikzcd}$$

and therefore we have the following map:

$$(\varepsilon'_I, \mathrm{Id}_{\mathsf{Hom}_A(N,I)})\colon \mathsf{T}_B(\mathsf{Hom}_A(N,I)) \longrightarrow \mathsf{H}_A(I) \qquad (*)$$

which is an isomorphism of $\Lambda_{(\phi,\psi)}$-modules. Since $I$ is an injective $A$-module it follows that the $B$-module $\mathsf{Hom}_A(N,I)$ has finite projective dimension. Let

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathsf{Hom}_A(N,I) \longrightarrow 0$$

be a projective resolution of $\mathsf{Hom}_A(N,I)$ in $\mathsf{mod}\text{-}B$. Since the functor $N\otimes_B -$ is an equivalence restricted to the subcategory $(\mathsf{proj}\,B)^{<\infty}$ it follows that the complex

$$0 \longrightarrow N\otimes_B P_n \longrightarrow \cdots \longrightarrow N\otimes_B P_0 \longrightarrow N\otimes_B \mathsf{Hom}_A(N,I) \longrightarrow 0$$

is exact. This implies that $\mathsf{Tor}_n^B(N,\mathsf{Hom}_A(N,I)) = 0$, $\forall n\geq 1$, and then from Lemma 4.5.2 we have the following isomorphism:

$$\mathsf{Ext}^n_{\Lambda_{(\phi,\psi)}}\big(\mathsf{T}_B(\mathsf{Hom}_A(N,I)),(X,Y,f,g)\big) \xrightarrow{\ \simeq\ } \mathsf{Ext}^n_B\big(\mathsf{Hom}_A(N,I),Y\big)$$

for every $n\geq 1$ and $(X,Y,f,g)\in \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$. Since $\mathsf{pd}_B\,\mathsf{Hom}_A(N,I)<\infty$ it follows from the above isomorphism that $\mathsf{pd}_{\Lambda_{(\phi,\psi)}}\mathsf{T}_B(\mathsf{Hom}_A(N,I))<\infty$. Hence from the relation $(*)$ we infer that the projective dimension of $\mathsf{H}_A(I)$ is finite. Similarly we prove that $\mathsf{id}_{\Lambda_{(\phi,\psi)}}\mathsf{T}_A(P)<\infty$, $\mathsf{id}_{\Lambda_{(\phi,\psi)}}\mathsf{T}_B(Q)<\infty$ and $\mathsf{pd}_{\Lambda_{(\phi,\psi)}}\mathsf{H}_B(J)<\infty$. We infer that $(\mathsf{proj}\,\Lambda_{(\phi,\psi)})^{<\infty} = (\mathsf{inj}\,\Lambda_{(\phi,\psi)})^{<\infty}$ and therefore the Morita ring $\Lambda_{(\phi,\psi)}$ is Gorenstein. □

REMARK 4.5.4. (i) Let $\Lambda_{(\phi,\psi)}$ be a Morita ring and assume as above that the adjoint pair of functors $(M\otimes_A -,\mathsf{Hom}_B(M,-))$ induces quasi-inverse equivalences between $(\mathsf{proj}\,A)^{<\infty}$ and $(\mathsf{inj}\,B)^{<\infty}$, and the adjoint pair of functors $(N\otimes_B -,\mathsf{Hom}_A(N,-))$ induces equivalences between $(\mathsf{proj}\,B)^{<\infty}$ and $(\mathsf{inj}\,A)^{<\infty}$. Since $A\in(\mathsf{proj}\,A)^{<\infty}$ it follows that $\mathsf{id}\,_BM<\infty$ and $A\simeq \mathsf{End}_B(M)$, and similarly since $B\in(\mathsf{proj}\,B)^{<\infty}$ we get that $\mathsf{id}\,_AN<\infty$ and $B\simeq\mathsf{End}_A(N)$.

(ii) Since selfinjective algebras are Gorenstein, it follows from Example 4.2.7 that the converse of Theorem 4.5.3 is not true in general.

If $\Lambda_{(\phi,\psi)}$ is a Morita ring with $A=M=N=B$ then we know from Corollary 4.1.13 that the bimodule homomorphisms $\phi$ and $\psi$ are equal. From now on we denote the Morita ring with all entries a ring $\Lambda$ by $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix}\Lambda & \Lambda\\ \Lambda & \Lambda\end{smallmatrix}\right)$. It is known from Fossum-Griffith-Reiten [49], see also Happel [62], that if $\Lambda$ is a Gorenstein Artin algebra then the upper triangular matrix algebra $\left(\begin{smallmatrix}\Lambda & \Lambda\\ \Lambda & 0\end{smallmatrix}\right)$ is Gorenstein. In this connection we have the next result, which is a consequence of Theorem 4.5.3, and shows that $\Delta_{(\phi,\phi)}$ is Gorenstein when $\Lambda$ is as well.

COROLLARY 4.5.5. *Let $\Lambda$ be an Artin algebra. Then $\Lambda$ is Gorenstein if and only if the Morita ring $\Delta_{(\phi,\phi)}$ is Gorenstein Artin algebra.*

PROOF. Suppose that $\Lambda$ is Gorenstein. Then we have $(\mathsf{proj}\,\Lambda)^{<\infty} = (\mathsf{inj}\,\Lambda)^{<\infty}$ and so from Theorem 4.5.3 it follows that matrix algebra $\Delta_{(\phi,\phi)}$ is Gorenstein. Conversely assume that $\Delta_{(\phi,\phi)}$ is Gorenstein and let $I$ be an injective $\Lambda$-module. Then the injective $\Delta_{(\phi,\phi)}$-module $\mathsf{H}_\Lambda(I)$ has finite projective dimension since $(\mathsf{proj}\,\Delta_{(\phi,\phi)})^{<\infty} = (\mathsf{inj}\,\Delta_{(\phi,\phi)})^{<\infty}$. Consider the exact sequence

$$\cdots \longrightarrow \mathsf{T}_\Lambda(P_1)\oplus\mathsf{T}_\Lambda(Q_1) \longrightarrow \mathsf{T}_\Lambda(P_0)\oplus\mathsf{T}_\Lambda(Q_0) \longrightarrow \mathsf{H}_\Lambda(I) \longrightarrow 0$$

which is the start of a finite projective resolution of $\mathsf{H}_\Lambda(I)$. Note that such a resolution exists from Construction 3.6.1. Then applying the functor $\mathsf{U}_\Lambda\colon \mathsf{mod}\text{-}\Delta_{(\phi,\phi)} \longrightarrow \mathsf{mod}\text{-}\Lambda$ we obtain the exact sequence $\cdots \longrightarrow P_1\oplus Q_1 \longrightarrow P_0\oplus Q_0 \longrightarrow I \longrightarrow 0$ and this implies that $\mathsf{pd}_\Lambda I < \infty$. Similarly we show that $\mathsf{id}_\Lambda P < \infty$ for every $P \in \mathsf{proj}\,\Lambda$. We infer that $(\mathsf{proj}\,\Lambda)^{<\infty} = (\mathsf{inj}\,\Lambda)^{<\infty}$ and therefore the Artin algebra $\Lambda$ is Gorenstein. $\square$

We continue now in order to determine the Gorenstein-projective modules over $\Delta_{(\phi,\phi)}$. Before this we recall some basic results for Gorenstein-projective modules. An acyclic complex of projective $\Lambda$-modules $\mathsf{P}^\bullet\colon \cdots \longrightarrow P^{i-1} \longrightarrow P^i \longrightarrow P^{i+1} \longrightarrow \cdots$ is called **totally acyclic**, if the complex $\mathsf{Hom}_\Lambda(\mathsf{P}^\bullet, \Lambda)$ is acyclic.

Definition 4.5.6. [46, 47] A $\Lambda$-module $X$ is called **Gorenstein-projective** if it is of the form $X = \mathsf{Coker}\,(P^{-1} \longrightarrow P^0)$ for some totally acyclic complex $\mathsf{P}^\bullet$ of projective $\Lambda$-modules.

$$\begin{tikzcd} \mathsf{P}^\bullet\colon & \cdots \arrow[r] & P^{-1} \arrow[r] & P^0 \arrow[r] \arrow[d] & P^1 \arrow[r] & P^2 \arrow[r] & P^2 \arrow[r] & \cdots \\ & & & X \arrow[ur] & & & & \end{tikzcd}$$

For an Artin algebra $\Lambda$ we denote by $\mathsf{Gproj}\,\Lambda$ the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of the finitely generated Gorenstein-projective $\Lambda$-modules. Let $X$ be a left $\Lambda$-module. Then we have the contravariant functor $\mathsf{Hom}_\Lambda(-,\Lambda)\colon \mathsf{Mod}\text{-}\Lambda \longrightarrow \mathsf{Mod}\text{-}\Lambda^{\mathsf{op}}$ and the **evaluation map**:

$$\mathsf{ev}_X\colon X \longrightarrow \mathsf{Hom}_{\Lambda^{\mathsf{op}}}(\mathsf{Hom}_\Lambda(X,\Lambda),\Lambda),\ x \mapsto \mathsf{ev}_X(x)(f) = f(x)$$

A $\Lambda$-module $X$ is called **reflexive** if the $\Lambda$-morphism $\mathsf{ev}_X$ is an isomorphism. For example, if $P$ is a finitely generated projective $\Lambda$-module then $P$ is reflexive and this gives the well known equivalence between finitely generated projective left $\Lambda$-modules and finitely generated projective left $\Lambda^{\mathsf{op}}$-modules.

Lemma 4.5.7. *[47] [36] Let $X$ be a finitely generated $\Lambda$-module. Then the following are equivalent:*

(i) $X \in \mathsf{Gproj}\,\Lambda$.

(ii) *There is an exact sequence*

$$0 \longrightarrow X \longrightarrow P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} \cdots$$

*with $P^i \in \mathsf{proj}\,\Lambda$ and every cocycle $\mathsf{Ker}\,d^i \in {}^{1\perp\infty}\Lambda$.*

(iii) $X \in {}^{1\perp\infty}\Lambda$, $\mathsf{Hom}_\Lambda(X,\Lambda) \in {}^{1\perp\infty}(\Lambda_\Lambda)$ *and $X$ is reflexive.*

Proof. (i) $\Rightarrow$ (ii) Since $X$ is a Gorenstein-projective $\Lambda$-module it follows that every cocycle $\mathsf{Ker}\,d^i$ has a projective resolution of the form:

$$\cdots \longrightarrow P^{i-3} \longrightarrow P^{i-2} \longrightarrow P^{i-1} \longrightarrow \mathsf{Ker}\,d^i \longrightarrow 0$$

which remains exact after applying the functor $\mathsf{Hom}_\Lambda(-,\Lambda)$. Then $\mathsf{Ext}^n_\Lambda(\mathsf{Ker}\,d^i,\Lambda) = 0$ for every $n \geq 1$ and therefore every cocycle $\mathsf{Ker}\,d^i$ lies in ${}^{1\perp\infty}\Lambda$. The result now follows.

(ii) $\Rightarrow$ (i) Let $\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow X \longrightarrow 0$ be a projective resolution of $X$. Then from the hypothesis we have the following exact complex:

$$\begin{tikzcd} \mathsf{P}^\bullet\colon \cdots \arrow[r] & P^{-2} \arrow[r, "d^{-2}"] & P^{-1} \arrow[r, dashed, "d^{-1}"] \arrow[d] & P^0 \arrow[r, "d^0"] & P^1 \arrow[r, "d^1"] & P^2 \arrow[r] & \cdots \\ & & \mathsf{Ker}\,d^0 \simeq X \arrow[ur] & & & & \end{tikzcd}$$

and $\mathsf{Ker}\,d^n \in {}^{1\perp\infty}\Lambda$ for every $n \geq 0$. The module $X \in {}^{1\perp\infty}\Lambda$ and $P^{-1} \in {}^{1\perp\infty}\Lambda$ as well. Then $\mathsf{Ker}\,d^{-1} \in {}^{1\perp\infty}\Lambda$ since the full subcategory ${}^{1\perp\infty}\Lambda$ is closed under kernels of epimorphisms. Continuing in this way we infer that all the cocycles of $\mathsf{P}^\bullet$ belong to ${}^{1\perp\infty}\Lambda$, i.e $\mathsf{Ext}^n_\Lambda(\mathsf{Ker}\, d^i, \Lambda) = 0$ for every $n \geq 1$. This implies that the complex $\mathsf{Hom}_\Lambda(\mathsf{P}^\bullet, \Lambda)$ is exact and therefore $X \in \mathsf{Gproj}\,\Lambda$.

(i) $\Rightarrow$ (iii) Since $X$ is Gorenstein-projective it follows that $X \in {}^{1\perp\infty}\Lambda$. Also the complex $\mathsf{Hom}_\Lambda(\mathsf{P}^\bullet, \Lambda)$ is an exact complex of finitely generated projective left $\Lambda^{\mathsf{op}}$-modules and from the evaluation morphisms it follows that $\mathsf{Hom}_{\Lambda^{\mathsf{op}}}(\mathsf{Hom}_\Lambda(\mathsf{P}^\bullet, \Lambda), \Lambda) \simeq \mathsf{P}^\bullet$ $(*)$. Hence $\mathsf{Hom}_\Lambda(X, \Lambda)$ is Gorenstein-projective, and therefore $\mathsf{Hom}_\Lambda(X, \Lambda) \in {}^{1\perp\infty}(\Lambda_\Lambda)$, but moreover from the isomorphism $(*)$ we get that $X$ is reflexive.

(iii) $\Rightarrow$ (i) Let $\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow X \longrightarrow 0$ be a projective resolution of $X$. Then the complex $0 \longrightarrow \mathsf{Hom}_\Lambda(X, \Lambda) \longrightarrow \mathsf{Hom}_\Lambda(P^{-1}, \Lambda) \longrightarrow \cdots$ is exact since $X \in {}^{1\perp\infty}\Lambda$. Let $\cdots \longrightarrow Q^{-2} \longrightarrow Q^{-1} \longrightarrow \mathsf{Hom}_\Lambda(X, \Lambda) \longrightarrow 0$ be a projective resolution of $\mathsf{Hom}_\Lambda(X, \Lambda)$ where $Q^i \in \mathsf{proj}\,\Lambda^{\mathsf{op}}$. Then if we apply the functor $\mathsf{Hom}_{\Lambda^{\mathsf{op}}}(-, \Lambda)$ we get the complex

$$0 \longrightarrow \mathsf{Hom}_{\Lambda^{\mathsf{op}}}(\mathsf{Hom}_\Lambda(X, \Lambda), \Lambda) \longrightarrow \mathsf{Hom}_{\Lambda^{\mathsf{op}}}(Q^{-1}, \Lambda) \longrightarrow \mathsf{Hom}_{\Lambda^{\mathsf{op}}}(Q^{-2}, \Lambda) \longrightarrow \cdots$$

which is exact since $\mathsf{Hom}_\Lambda(X, \Lambda) \in {}^{1\perp\infty}(\Lambda_\Lambda)$. Then using that $X$ is reflexive we derive the following acyclic complex of projectives $\Lambda$-modules:

$$\begin{tikzcd}
\mathsf{P}^\bullet\colon \cdots \arrow[r] & P^{-2} \arrow[r, "d^{-2}"] & P^{-1} \arrow[rr, dashed, "d^{-1}"] \arrow[d] & & \mathsf{Hom}_{\Lambda^{\mathsf{op}}}(Q^{-1}, \Lambda) \arrow[r] & \mathsf{Hom}_{\Lambda^{\mathsf{op}}}(Q^{-2}, \Lambda) \arrow[r] & \cdots \\
& & X \simeq \mathsf{Hom}_{\Lambda^{\mathsf{op}}}(\mathsf{Hom}_\Lambda(X, \Lambda), \Lambda) \arrow[urr] & & & &
\end{tikzcd}$$

such that $\mathsf{Hom}_\Lambda(\mathsf{P}^\bullet, \Lambda)$ is exact. We infer that $X$ is Gorenstein-projective. $\square$

Next we also recall some well known descriptions of Gorenstein-projective modules.

PROPOSITION 4.5.8. *[47] [36] [27, Proposition 3.10] Let $\Lambda$ be an Artin algebra.*

(i) *We have*:
$$\mathsf{Gproj}\,\Lambda \subseteq {}^{1\perp\infty}\Lambda = \big\{X \mid \mathsf{Ext}^n_\Lambda(X, \Lambda) = 0, \ \forall n \geq 1\big\}$$

(ii) *If* $\mathsf{gl.dim}\,\Lambda < \infty$ *then*:
$$\mathsf{Gproj}\,\Lambda = \mathsf{proj}\,\Lambda$$

(iii) *If $\Lambda$ is selfinjective then*:
$$\mathsf{Gproj}\,\Lambda = \mathsf{mod}\text{-}\Lambda$$

(iv) *If $\Lambda$ is a Gorenstein Artin algebra then*:
$$\mathsf{Gproj}\,\Lambda = {}^{1\perp\infty}\Lambda$$

PROOF. (i) This follows immediately from the definition of Gorenstein-projectives. (ii) Let $P$ be a projective $\Lambda$-module. Then we have the complex

$$\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{\mathrm{Id}_P} P \longrightarrow 0 \longrightarrow \cdots$$

and this implies that $\mathsf{proj}\,\Lambda \subseteq \mathsf{Gproj}\,\Lambda$. Now let $X$ be a Gorenstein-projective $\Lambda$-module. If $\mathsf{pd}_\Lambda X = m < \infty$ then $\mathsf{Ext}^m_\Lambda(X, \Lambda) \neq 0$, but this is a contradiction from (i). Thus either $X \in \mathsf{proj}\,\Lambda$ or $\mathsf{pd}_\Lambda X = \infty$. Since the algebra $\Lambda$ has finite global dimension it follows that $X$ is a projective module and then our statement follows.

(iii) Let $X$ be a $\Lambda$-module and let $P^\bullet \longrightarrow X$, respectively $X \longrightarrow I^\bullet$, be a projective resolution, respectively an injective coresolution, of $X$. Then since the injectives are projectives and the functor $\mathsf{Hom}_\Lambda(-,\Lambda)$ is exact it follows that $X$ is Gorenstein-projective. Thus we have $\mathsf{Gproj}\,\Lambda = \mathsf{mod}\text{-}\Lambda$.

(iv) Suppose that $\mathsf{id}\,{}_\Lambda\Lambda \leq n$ and $\mathsf{id}\,\Lambda_\Lambda \leq n$, i.e. $\Lambda$ is Gorenstein. Let $X \in {}^{1\perp\infty}\Lambda$ and let $\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow X \longrightarrow 0$ be a projective resolution of $X$. Then the following complex:

$$\begin{tikzcd}[column sep=small]
\mathsf{Q}^\bullet\colon\ 0 \arrow[r] & \mathsf{Hom}_\Lambda(X,\Lambda) \arrow[r] & \mathsf{Hom}_\Lambda(P^{-1},\Lambda) \arrow[r] & \cdots \arrow[r] & \mathsf{Hom}_\Lambda(P^{-n},\Lambda) \arrow[r] \arrow[d] & \mathsf{Hom}_\Lambda(P^{-n-1},\Lambda) \arrow[r] & \cdots \\
 & & & & Y \arrow[ur, tail] & &
\end{tikzcd}$$

is exact. Since $\mathsf{Ext}^m_{\Lambda^{\mathsf{op}}}(\mathsf{Hom}_\Lambda(X,\Lambda),\Lambda) \simeq \mathsf{Ext}^{m+n}_{\Lambda^{\mathsf{op}}}(Y_\Lambda,\Lambda_\Lambda)$, $\forall m \geq 1$, and $\mathsf{id}\,\Lambda_\Lambda \leq n$ it follows that $\mathsf{Ext}^m_{\Lambda^{\mathsf{op}}}(\mathsf{Hom}_\Lambda(X,\Lambda),\Lambda) = 0$, $\forall m \geq 1$. Thus $\mathsf{Hom}_\Lambda(X,\Lambda) \in {}^{1\perp\infty}(\Lambda_\Lambda)$. Similarly we obtain that every cocycle of the exact sequence $\mathsf{Q}^\bullet$ belongs to ${}^{1\perp\infty}(\Lambda_\Lambda)$ and therefore the complex $\mathsf{Hom}_{\Lambda^{\mathsf{op}}}(\mathsf{Q}^\bullet,\Lambda)$ is exact. Since every projective $P^{-i}$ is reflexive we get that $\mathsf{Hom}_{\Lambda^{\mathsf{op}}}(\mathsf{Hom}_\Lambda(X,\Lambda),\Lambda) \simeq X$, that is $X$ is reflexive. Then from Lemma 4.5.7 we infer that $X \in \mathsf{Gproj}\,\Lambda$ and therefore from (i) we conclude that $\mathsf{Gproj}\,\Lambda = {}^{1\perp\infty}\Lambda$. $\square$

Recall from [27], [29] that an Artin algebra $\Lambda$ is said to be of **finite Cohen-Macaulay type** if the category $\mathsf{Gproj}\,\Lambda$ of finitely generated Gorenstein-projective $\Lambda$-modules is of finite representation type, i.e. the set of isomorphism classes of its indecomposable objects is finite. Recently Li and Zhang [87] determined the Gorenstein-projective modules over the triangular matrix algebra $\left(\begin{smallmatrix}\Lambda & \Lambda\\ 0 & \Lambda\end{smallmatrix}\right)$, when $\Lambda$ is a Gorenstein Artin algebra, and using this they obtained a criterion for the Cohen-Macaulay finiteness of $\left(\begin{smallmatrix}\Lambda & \Lambda\\ 0 & \Lambda\end{smallmatrix}\right)$ in case that $\Lambda$ is a Gorenstein Artin algebra of finite Cohen-Macaulay type.

Our aim now is to describe the Gorenstein-projective modules over the algebra $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix}\Lambda & \Lambda\\ \Lambda & \Lambda\end{smallmatrix}\right)$. For the ring $\Delta_{(\phi,\phi)}$ we denote by $\mathsf{T}'_\Lambda$, resp. $\mathsf{H}'_\Lambda$, the functor $\mathsf{T}_B$, resp. $\mathsf{H}_B$, where the algebra $B$ is now $\Lambda$.

We need the following observation.

Lemma 4.5.9. *Let $\Delta_{(\phi,\phi)}$ be a Morita ring. Then we have isomorphisms of functors:* $\mathsf{T}_\Lambda(-) \simeq \mathsf{H}'_\Lambda(-)$ *and* $\mathsf{T}'_\Lambda(-) \simeq \mathsf{H}_\Lambda(-)$.

Proof. Let $X$ be a $\Lambda$-module and $f\colon \mathsf{Hom}_\Lambda(\Lambda,X) \xrightarrow{\simeq} X$, $g\colon \Lambda\otimes_\Lambda X \xrightarrow{\simeq} X$ the standard isomorphisms. Consider the maps $(f^{-1},g)\colon \mathsf{T}_\Lambda(X) = (X,\Lambda\otimes_\Lambda X,\mathrm{Id}_{\Lambda\otimes X},\Phi_X) \to \mathsf{H}'_\Lambda(X) = (\mathsf{Hom}_\Lambda(\Lambda,X),X,\epsilon_X,\delta_{\Lambda\otimes X}\circ\mathsf{Hom}_\Lambda(\Lambda,\Phi_X))$ and $(g,f^{-1})\colon \mathsf{T}'_\Lambda(X) = (\Lambda\otimes_\Lambda X,X,\Phi_X,\mathrm{Id}_{\Lambda\otimes X}) \to \mathsf{H}_\Lambda(X) = (X,\mathsf{Hom}_\Lambda(\Lambda,X),\delta'_{\Lambda\otimes X}\circ\mathsf{Hom}_\Lambda(\Lambda,\Phi_X),\epsilon'_X)$. We claim that $(f^{-1},g)$ and $(g,f^{-1})$ are isomorphisms of $\Delta_{(\phi,\phi)}$-modules. We prove that the map $(f^{-1},g)$ is an isomorphism. Since $f^{-1}$ and $g$ are isomorphisms we have to show that the following diagrams are commutative:

$$\begin{tikzcd}
\Lambda\otimes_\Lambda X \arrow[r, "\mathrm{Id}_{\Lambda\otimes X}"] \arrow[d, "1_\Lambda\otimes f^{-1}"'] & \Lambda\otimes_\Lambda X \arrow[d, "g"] \\
\Lambda\otimes_\Lambda \mathsf{Hom}_\Lambda(\Lambda,X) \arrow[r, "\epsilon_X"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda\otimes_\Lambda\Lambda\otimes_\Lambda X \arrow[r, "\Phi_X"] \arrow[d, "1_\Lambda\otimes g"'] & X \arrow[d, "f^{-1}"] \\
\Lambda\otimes_\Lambda X \arrow[r, "\delta_{\Lambda\otimes X}\circ\mathsf{Hom}(\Lambda,\Phi_X)"] & \mathsf{Hom}_\Lambda(\Lambda,X)
\end{tikzcd}$$

Let $\lambda\otimes x$ be an element of $\Lambda\otimes_\Lambda X$. The map $f^{-1}$ sends an element $x$ to $f_x\colon \Lambda \longrightarrow X$ defined by $f_x(\lambda) = \lambda x$. Also $\epsilon_X(\lambda\otimes f) = f(\lambda)$. Then $g(\lambda\otimes x) = \lambda x$ and $\epsilon_X(1_\Lambda\otimes f^{-1}(\lambda\otimes x)) = \epsilon_X(\lambda\otimes f_x) = f_x(\lambda) = \lambda x$. Hence the first diagram is commutative. For the second diagram let $\lambda\otimes\lambda'\otimes x$ be an element of $\Lambda\otimes_\Lambda\Lambda\otimes_\Lambda X$. Then $\Phi_X(\lambda\otimes\lambda'\otimes x) = g(\phi\otimes 1_X(\lambda\otimes\lambda'\otimes x)) = g(\phi(\lambda\otimes\lambda')\otimes x) = \phi(\lambda\otimes\lambda')x$ and

$$f^{-1}(\Phi_X(\lambda\otimes\lambda'\otimes x)) = f^{-1}(\phi(\lambda\otimes\lambda')x) = f_{\phi(\lambda\otimes\lambda')x}$$

where for $\lambda''\in\Lambda$ we have:

$$f_{\phi(\lambda\otimes\lambda')x}(\lambda'') = \lambda''\phi(\lambda\otimes\lambda')x \qquad (*)$$

On the other hand we have

$$\begin{array}{ccl} \Lambda\otimes_\Lambda X & \xrightarrow{\delta_{\Lambda\otimes X}} & \mathsf{Hom}_\Lambda(\Lambda,\Lambda\otimes_\Lambda\Lambda\otimes_\Lambda X) \xrightarrow{\mathsf{Hom}(\Lambda,\Phi_X)} \mathsf{Hom}_\Lambda(\Lambda,X) \\ \lambda\otimes x & \mapsto & g_{\lambda\otimes x}\colon \Lambda\longrightarrow\Lambda\otimes_\Lambda\Lambda\otimes_\Lambda X,\ g_{\lambda\otimes x}(\lambda') = \lambda'\otimes\lambda\otimes x \\ & \mapsto & h\colon\Lambda\longrightarrow X,\ h(\lambda') = \phi(\lambda'\otimes\lambda)x \end{array}$$

We compute

$$\begin{aligned} \mathsf{Hom}_\Lambda(\Lambda,\Phi_X)(\delta_{\Lambda\otimes X}(1_\Lambda\otimes g(\lambda\otimes\lambda'\otimes x)))(\lambda'') &= \mathsf{Hom}_\Lambda(\Lambda,\Phi_X)(\delta_{\Lambda\otimes X}(\lambda\otimes\lambda' x))(\lambda'') \\ &= \mathsf{Hom}_\Lambda(\Lambda,\Phi_X)(g_{\lambda\otimes\lambda' x}(\lambda'')) \\ &= \mathsf{Hom}_\Lambda(\Lambda,\Phi_X)(\lambda''\otimes\lambda\otimes\lambda' x) \\ &= \phi(\lambda''\otimes\lambda)\lambda' x \end{aligned}$$

Since $\phi$ is a $\Lambda$-$\Lambda$-bimodule homomorphism it follows from $(*)$ that the second diagram is commutative. Thus the map $(f^{-1},g)$ is an isomorphism. $\square$

The following result characterizes when a module over the algebra $\Delta_{(\phi,\phi)}$ is Gorenstein-projective.

Corollary 4.5.10. *Let $\Lambda$ be a Gorenstein Artin algebra. Then a $\Delta_{(\phi,\phi)}$-module $(X,Y,f,g)$ is Gorenstein-projective if and only if $X$ and $Y$ are Gorenstein-projective $\Lambda$-modules.*

Proof. Let $0\longrightarrow\Lambda\longrightarrow I_0\longrightarrow I_1\longrightarrow\cdots\longrightarrow I_n\longrightarrow 0$ be an injective coresolution of ${}_\Lambda\Lambda$. If we apply the functors $\mathsf{H}_\Lambda,\mathsf{H}'_\Lambda\colon \text{mod-}\Lambda\longrightarrow\text{mod-}\Delta_{(\phi,\phi)}$ we obtain the exact sequences $0\longrightarrow\mathsf{H}_\Lambda(\Lambda)\longrightarrow\mathsf{H}_\Lambda(I_0)\longrightarrow\cdots\longrightarrow\mathsf{H}_\Lambda(I_n)\longrightarrow 0$ and $0\longrightarrow\mathsf{H}'_\Lambda(\Lambda)\longrightarrow\mathsf{H}'_\Lambda(I_0)\longrightarrow\cdots\longrightarrow\mathsf{H}'_\Lambda(I_n)\longrightarrow 0$ which are injective coresolutions of $\mathsf{H}_\Lambda(\Lambda)$ and $\mathsf{H}'_\Lambda(\Lambda)$ respectively. From Lemma 4.5.9 the above resolutions can be regarded as injective coresolutions of $\mathsf{T}'_\Lambda(\Lambda)$ and $\mathsf{T}_\Lambda(\Lambda)$ respectively. Then using the adjoint pairs of functors $(\mathsf{U}'_\Lambda,\mathsf{H}'_\Lambda)$ and $(\mathsf{U}_\Lambda,\mathsf{H}_\Lambda)$ we have the following commutative diagrams:

$$\begin{array}{ccccccccc} 0\to & \big((X,Y,f,g),\mathsf{T}_\Lambda(\Lambda)\big) & \to & \big((X,Y,f,g),\mathsf{H}'_\Lambda(I_0)\big) & \to\cdots\to & \big((X,Y,f,g),\mathsf{H}'_\Lambda(I_n)\big) & \to 0 \\ & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & \\ 0\longrightarrow & \mathsf{Hom}_\Lambda(Y,\Lambda) & \longrightarrow & \mathsf{Hom}_\Lambda(Y,I_0) & \longrightarrow\cdots\longrightarrow & \mathsf{Hom}_\Lambda(Y,I_n) & \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0\to & \big((X,Y,f,g),\mathsf{T}'_\Lambda(\Lambda)\big) & \to & \big((X,Y,f,g),\mathsf{H}_\Lambda(I_0)\big) & \to\cdots\to & \big((X,Y,f,g),\mathsf{H}_\Lambda(I_n)\big) & \to 0 \\ & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & \\ 0\longrightarrow & \mathsf{Hom}_\Lambda(X,\Lambda) & \longrightarrow & \mathsf{Hom}_\Lambda(X,I_0) & \longrightarrow\cdots\longrightarrow & \mathsf{Hom}_\Lambda(X,I_n) & \longrightarrow 0 \end{array}$$

Then $\mathsf{Ext}^n_\Lambda(Y,\Lambda)=0$, $\forall n\geq 1$, if and only if $\mathsf{Ext}^n_{\Delta_{(\phi,\phi)}}((X,Y,f,g),\mathsf{T}_\Lambda(\Lambda))=0$, $\forall n\geq 1$, and $\mathsf{Ext}^n_\Lambda(X,\Lambda)=0$, $\forall n\geq 1$, if and only if $\mathsf{Ext}^n_{\Delta_{(\phi,\phi)}}((X,Y,f,g),\mathsf{T}'_\Lambda(\Lambda))=0$, $\forall n\geq 1$.

Since $\Delta_{(\phi,\phi)} \simeq \mathsf{T}_\Lambda(\Lambda) \oplus \mathsf{T}'_\Lambda(\Lambda)$ as $\Delta_{(\phi,\phi)}$-modules it follows from Corollary 4.5.5 and Proposition 4.5.8 (iv) that $(X, Y, f, g) \in \mathsf{Gproj}\,\Delta_{(\phi,\phi)}$ if and only if $X, Y \in \mathsf{Gproj}\,\Lambda$. $\square$

After the above description it would be interesting to find conditions such that the matrix algebra $\Delta_{(\phi,\phi)}$ is of finite Cohen-Macaulay type.

We close this section with the next result which gives the connection between the category of Gorenstein-projective modules over the matrix algebra $\Delta_{(\phi,\phi)}$ and the corresponding category of $\Lambda$.

Corollary 4.5.11. *Let $\Lambda$ be a Gorenstein Artin algebra. Then the recollement situation of* $\mathsf{mod}\text{-}\Delta_{(\phi,\phi)}$ *is restricted to the categories of Gorenstein-projective modules* $\mathsf{Gproj}\,\Delta_{(\phi,\phi)}$ *and* $\mathsf{Gproj}\,\Lambda$.

Proof. Let $X$ be a Gorenstein-projective $\Lambda$-module. Since $\Lambda$ is Gorenstein there exists an exact sequence $0 \longrightarrow X \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ where each $P^i$ is a projective $\Lambda$-module. If we apply the exact functors $\mathsf{T}_\Lambda$ and $\mathsf{T}'_\Lambda$ we get that $\mathsf{T}_\Lambda(X)$ and $\mathsf{T}'_\Lambda(X)$ are Gorenstein-projective $\Delta_{(\phi,\phi)}$-modules, since from Corollary 4.5.5 the Artin algebra $\Delta_{(\phi,\phi)}$ is Gorenstein. Also from Corollary 4.5.10 it follows that $\mathsf{U}_\Lambda(X, Y, f, g)$ is Gorenstein-projective for every $(X, Y, f, g) \in \mathsf{Gproj}\,\Delta_{(\phi,\phi)}$ and finally we have $\mathsf{Ker}\,\mathsf{U}_\Lambda = \{(0, Y, 0, 0) \in \mathsf{Gproj}\,\Delta_{(\phi,\phi)} \mid Y \in \mathsf{Gproj}\,\Lambda\}$. $\square$

CHAPTER 5

# Representation Dimension and Rouquier's Dimension

In this final Chapter we apply the abstract homological theory developed in Chapter 2 to the two main ingredients of this thesis, namely Auslander's representation dimension and Rouquier's dimension of triangulated categories. In the first section we investigate recollements of abelian categories $(\mathscr{A},\mathscr{B},\mathscr{C})$ which lift to triangulated recollements of the associated bounded derived categories $(\mathbf{D}^b(\mathscr{A}),\mathbf{D}^b(\mathscr{B}),\mathbf{D}^b(\mathscr{C}))$. Moreover we investigate the Rouquier dimension of triangulated categories. In particular we give upper and lower bounds for the dimension of a triangulated category $\mathcal{T}$ in a recollement situation $(\mathcal{U},\mathcal{T},\mathcal{V})$ of triangulated categories. As an application we derive bounds for the dimension of $\mathbf{D}^b(\mathscr{B})$ in terms of the dimensions of $\mathbf{D}^b(\mathscr{A})$ and $\mathbf{D}^b(\mathscr{C})$ for a recollement of abelian categories $(\mathscr{A},\mathscr{B},\mathscr{C})$. Finally we give bounds for the Rouquier dimension of the bounded derived category of rings with an idempotent element as well as to triangular matrix rings. In the last section of this Chapter we investigate the representation dimension and in particular how it behaves in a recollement situation $(\mathscr{A},\mathscr{B},\mathscr{C})$. As an application we generalize a classical result of Auslander relating the representation dimension of $\Lambda$ and $\mathsf{End}_\Lambda(P)$, where $\Lambda$ is an Artin algebra and $P$ a finitely generated projective $\Lambda$-module. We also give several interesting connections with finitistic dimension. In particular by applying the results of Chapter 2 about global and finitistic dimension we provide several applications for the representation dimension of Artin algebras and we present also an interesting interplay between representation and finitistic dimension, see Theorem 5.2.15 for more details. The results of this Chapter are included in the paper entitled: Homological Theory of Recollements of Abelian Categories [108].

## 5.1. Triangulated Categories and Rouquier's Dimension

In this section we investigate recollements of bounded derived categories arising from recollements of abelian categories and moreover we give bounds for the Rouquier dimension of a triangulated category in a recollement situation.

**5.1.1. Recollements of Bounded Derived Categories.** In this subsection we investigate recollements of abelian categories $(\mathscr{A},\mathscr{B},\mathscr{C})$ which lift to triangulated recollements of the associated bounded derived categories $(\mathbf{D}^b(\mathscr{A}),\mathbf{D}^b(\mathscr{B}),\mathbf{D}^b(\mathscr{C}))$. It turns out that the crucial conditions ensuring the existence of such a lifting are: $(\alpha)$ the functor $\mathsf{i}\colon \mathscr{A}\longrightarrow\mathscr{B}$ is a homological embedding, and $(\beta)$ certain homological finiteness conditions on the left and/or right adjoint of $\mathsf{i}$ and the quotient functor $\mathsf{e}\colon\mathscr{B}\longrightarrow\mathscr{C}$.

First we recall the notion of a recollement of triangulated categories, see [22] and [133] for more details.

Definition 5.1.1. A **recollement situation** between triangulated categories $\mathcal{U}, \mathcal{T}$ and $\mathcal{V}$ is a diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mathsf{i}"] & \mathcal{T} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathcal{V} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \qquad \mathsf{R_{tr}}(\mathcal{U}, \mathcal{T}, \mathcal{V})$$

of triangulated functors, henceforth denoted by $(\mathcal{U}, \mathcal{T}, \mathcal{V})$, satisfying the following conditions:

**1.** $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ is an adjoint triple.
**2.** $(\mathsf{q}, \mathsf{i}, \mathsf{p})$ is an adjoint triple.
**3.** The functors $\mathsf{i}$, $\mathsf{l}$, and $\mathsf{r}$ are fully faithful.
**4.** $\mathsf{Im}\,\mathsf{i} = \mathsf{Ker}\,\mathsf{e}$.

The next result explain us the above definition and in particular it shows how we get recollements of triangulated categories. Compare it with Remark 1.1.3 and Remark 1.1.4 for recollemenents of abelian categories.

Lemma 5.1.2. (i) *[38, Theorem 1.1] Suppose that we have the following diagram of triangulated categories:*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\mathsf{e}"] & \mathcal{V} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd} \qquad (*)$$

*such that*
($\alpha$) $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ *is an adjoint triple, and*
($\beta$) *the functor* $\mathsf{l}$ *(or* $\mathsf{r}$*) is fully faithful.*
*Then the diagram* $(*)$ *can be completed to the following recollement of triangulated categories:*

$$\begin{tikzcd} \mathsf{Ker}\,\mathsf{e} \arrow[r, "\mathsf{inc}"] & \mathcal{T} \arrow[l, bend right, dashed, "\mathsf{q}"'] \arrow[l, bend left, dashed, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathcal{V} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

(ii) *[39, Theorem 2.1] Suppose that we have the following diagram of triangulated categories:*

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mathsf{i}"] & \mathcal{T} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \end{tikzcd} \qquad (**)$$

*such that*
($\alpha$) $(\mathsf{q}, \mathsf{i}, \mathsf{p})$ *is an adjoint triple, and*
($\beta$) *the functor* $\mathsf{i}$ *is fully faithful.*

*Then the diagram* $(**)$ *can be completed to the following recollement of triangulated categories*:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mathsf{i}"] & \mathcal{T} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathcal{V} \arrow[l, bend right, dashed, "\mathsf{l}"'] \arrow[l, bend left, dashed, "\mathsf{r}"] \end{tikzcd}$$

*where* $\mathcal{V}$ *is the Verdier quotient* $\mathcal{T}/\operatorname{\mathsf{Im}}\mathsf{i}$.

REMARK 5.1.3. A recollement situation $(\mathcal{U}, \mathcal{T}, \mathcal{V})$ of triangulated categories produces some localization and colocalization functors for $\mathcal{T}$ and moreover these functors determines the recollement up to an equivalence. For more details we refer to [80].

The following result indicates a useful characterization for a functor to be a homological embedding, see Definition 2.1.6, and it will be used in our main result of this subsection. If $\mathscr{A}$ is a Serre subcategory of an abelian category $\mathscr{B}$ then we denote by $\mathbf{D}^b_{\mathscr{A}}(\mathscr{B})$ the full subcategory of $\mathbf{D}^b(\mathscr{B})$ consisting of all complexes whose homology lie in $\mathscr{A}$.

THEOREM 5.1.4. *[128, Theorem 2.1] Let* $\mathscr{B}$ *be an abelian category and* $\mathscr{A}$ *a Serre subcategory of* $\mathscr{B}$. *Then the functor* $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ *is a homological embedding if and only if the canonical triangulated functor* $\mathbf{D}^b(\mathsf{i})\colon \mathbf{D}^b(\mathscr{A}) \longrightarrow \mathbf{D}^b(\mathscr{B})$ *induces a triangle equivalence*:

$$\mathbf{D}^b(\mathscr{A}) \xrightarrow{\simeq} \mathbf{D}^b_{\mathscr{A}}(\mathscr{B})$$

We need the following notion.

DEFINITION 5.1.5. We define a functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ between abelian categories with enough injectives to be of locally finite cohomological dimension if for every $A \in \mathscr{A}$ there exists $k_A \geq 0$ such that the right derived functor $\mathsf{R}^n F(A) = 0$ for all $n > k_A$.

Dually we define a functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ between abelian categories with enough projectives to be of locally finite homological dimension.

The following first main result of this section gives necessary and suficcient conditions such that a recollement of abelian categories induces a recollement in the bounded derived categories. Note that a special case of the following result for finite dimensional algebras over a field was proved in [95], and for left noetherian or semiprimary rings was proved in [96].

THEOREM 5.1.6. *Let* $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ *be a recollement of abelian categories and assume that* $\mathscr{B}$ *and* $\mathscr{C}$ *have enough projective and injective objects.*

(i) *The following statements are equivalent*:
   (a) *The functor* $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ *is a homological embedding, and the functor* $\mathsf{q}\colon \mathscr{B} \longrightarrow \mathscr{A}$, *resp.* $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$, *is of locally finite homological, resp. cohomological, dimension.*
   (b) *There exists a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}^b(\mathscr{A}) \arrow[r, "\mathbf{D}^b(\mathsf{i})"] & \mathbf{D}^b(\mathscr{B}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{q}"'] \arrow[l, bend left, "\mathbf{R}^b\mathsf{p}"] \arrow[r, "\mathbf{D}^b(\mathsf{e})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right, "\mathsf{l}'"'] \arrow[l, bend left, "\mathsf{r}'"] \end{tikzcd}$$

(ii) *The following statements are equivalent*:
 (a) *The functor* $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ *is a homological embedding, and the functor* $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$*, resp.* $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$*, is of locally finite homological, resp. cohomological, dimension.*
 (b) *There exists a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}^b(\mathscr{A}) \arrow[r, "\mathbf{D}^b(\mathsf{i})"] & \mathbf{D}^b(\mathscr{B}) \arrow[l, bend right, "\mathsf{q}'"'] \arrow[l, bend left, "\mathsf{p}'"] \arrow[r, "\mathbf{D}^b(\mathsf{e})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{l}"'] \arrow[l, bend left, "\mathbf{R}^b\mathsf{r}"] \end{tikzcd}$$

(iii) *The following statements are equivalent*:
 (a) *The functor* $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ *is a homological embedding, and the functors* $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ *and* $\mathsf{q}\colon \mathscr{B} \longrightarrow \mathscr{A}$*, resp. the functors* $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ *and* $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$*, are of locally finite homological, resp. cohomological, dimension.*
 (b) *There exists a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}^b(\mathscr{A}) \arrow[r, "\mathbf{D}^b(\mathsf{i})"] & \mathbf{D}^b(\mathscr{B}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{q}"'] \arrow[l, bend left, "\mathbf{R}^b\mathsf{p}"] \arrow[r, "\mathbf{D}^b(\mathsf{e})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{l}"'] \arrow[l, bend left, "\mathbf{R}^b\mathsf{r}"] \end{tikzcd}$$

Proof. First note that the exact sequence of abelian categories $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$ lifts to an exact sequence $0 \longrightarrow \mathbf{D}^b_{\mathscr{A}}(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{C}) \longrightarrow 0$ of triangulated categories, see [73] and [95].

(i) (a) $\Rightarrow$ (b) Since the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding then it follows from Theorem 5.1.4 that there is an equivalence of triangulated categories $\mathbf{D}^b(\mathscr{A}) \simeq \mathbf{D}^b_{\mathscr{A}}(\mathscr{B})$. Then

$$0 \longrightarrow \mathbf{D}^b(\mathscr{A}) \longrightarrow \mathbf{D}^b(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{C}) \longrightarrow 0$$

is an exact sequence of triangulated categories. Let $B^\bullet \in \mathbf{D}^b(\mathscr{B})$. Suppose first that $B^\bullet$ is concentrated in degree zero, so we deal with an object $B \in \mathscr{B}$. Then there is a quasi-isomorphism $B^\bullet \longrightarrow I^\bullet$ where $I^\bullet \in \mathbf{K}^+(\mathsf{Inj}\,\mathscr{B})$ is an injective coresolution of $B$. Since the functor $\mathsf{p}$ is of locally finite cohomological dimension there exists $k_B \geq 0$ such that $\mathsf{R}^n\mathsf{p}(B) = 0$ for every $n > k_B$, so $\mathsf{p}(I^\bullet)$ has bounded cohomology. Hence the complex $\mathsf{p}(I^\bullet)$ is quasi-isomorphic to a bounded complex and so $\mathbf{R}^b\mathsf{p}(B^\bullet)$ lies in $\mathbf{D}^b(\mathscr{A})$. Suppose now that $B^\bullet$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow 0 \longrightarrow \cdots$$

Then we have the following triangle in $\mathbf{D}^b(\mathscr{B})$:

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[d] & 0 \arrow[d] & B_1 \arrow[d] & B_1 \arrow[d] \\ B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 & 0 & 0 & 0 \end{tikzcd}$$

$$B_1[0] \longrightarrow B_2[0] \longrightarrow B^\bullet \longrightarrow B_1[1]$$

Since the objects $\mathbf{R}^b\mathsf{p}(B_1)$, $\mathbf{R}^b\mathsf{p}(B_2)$ and $\mathbf{R}^b\mathsf{p}(B_1[1])$ lie in $\mathbf{D}^b(\mathscr{A})$ it follows from the triangle

$$\mathbf{R}^b\mathsf{p}(B_1) \longrightarrow \mathbf{R}^b\mathsf{p}(B_2) \longrightarrow \mathbf{R}^b\mathsf{p}(B^\bullet) \longrightarrow \mathbf{R}^b\mathsf{p}(B_1[1])$$

that $\mathbf{R}^b\mathsf{p}(B^\bullet)$ belongs also to $\mathbf{D}^b(\mathscr{A})$. Continuing inductively on the length of the complex $B^\bullet$ we infer that $\mathbf{R}^b\mathsf{p}(B^\bullet)$ lies in $\mathbf{D}^b(\mathscr{A})$. Hence $\mathbf{R}^b\mathsf{p}(\mathbf{D}^b(\mathscr{B})) \subseteq \mathbf{D}^b(\mathscr{A})$. Similarly we show that $\mathbf{L}^b\mathsf{q}(B^\bullet) \in \mathbf{D}^b(\mathscr{A})$ using that the functor $\mathsf{q}$ is of locally finite homological dimension.

By [74, Lemma 15.6] we have a natural isomorphism

$$\mathsf{Hom}_{\mathbf{D}(\mathscr{A})}(\mathbf{L}^b\mathsf{q}(B^\bullet), A^\bullet) \simeq \mathsf{Hom}_{\mathbf{D}(\mathscr{B})}(B^\bullet, \mathbf{D}^b(\mathsf{i})(A^\bullet))$$

for any complexes $B^\bullet \in \mathbf{D}^-(\mathscr{B})$ and $A^\bullet \in \mathbf{D}^+(\mathscr{A})$. Since for $B^\bullet \in \mathbf{D}^b(\mathscr{B})$ and $A^\bullet \in \mathbf{D}^b(\mathscr{A})$, we have $\mathbf{L}^b\mathsf{q}(B^\bullet) \in \mathbf{D}^b(\mathscr{A})$ and $\mathbf{D}^b(\mathsf{i})(A^\bullet) \in \mathbf{D}^b(\mathscr{B})$, it follows that $(\mathbf{L}^b\mathsf{q}, \mathbf{D}^b(\mathsf{i}))$ is an adjoint pair and similarly we prove that $(\mathbf{D}^b(\mathsf{i}), \mathbf{R}^b\mathsf{p})$ is an adjoint pair. On the other hand by Lemma 5.1.2(ii) it follows that the derived functor $\mathbf{D}^b(\mathsf{e})\colon \mathbf{D}^b(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{C})$ admits a left and right adjoint. We conclude that $(\mathbf{D}^b(\mathscr{A}), \mathbf{D}^b(\mathscr{B}), \mathbf{D}^b(\mathscr{C}))$ is a recollement of triangulated categories.

(b) $\Rightarrow$ (a) Let $B \in \mathscr{B}$. Then $\mathbf{L}^b\mathsf{q}(B) = \mathsf{q}(P^\bullet) \in \mathbf{D}^b(\mathscr{A})$ where $P^\bullet$ is a projective resolution of $B$. This means that the complex $\mathsf{q}(P^\bullet)$ has bounded cohomology and so there exists $m_B \geq 0$ such that $\mathsf{L}_n\mathsf{q}(B) = 0$ for every $n > m_B$. Hence the functor $\mathsf{q}$ is of locally finite homological dimension and similarly we prove that $\mathsf{p}$ is of locally finite cohomological dimension. Now the exact sequence of triangulated categories $0 \longrightarrow \mathbf{D}^b(\mathscr{A}) \longrightarrow \mathbf{D}^b(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{C}) \longrightarrow 0$ implies that the canonical functor $\mathbf{D}^b(\mathscr{A}) \longrightarrow \mathbf{D}^b_{\mathscr{A}}(\mathscr{B})$ is an equivalence. We infer from Theorem 5.1.4 that the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding.

The proof of part (ii) is similar to the proof of part (i), noting that in the setting of (ii) the left and right adjoint of the functor $\mathbf{D}^b(\mathsf{i})\colon \mathbf{D}^b(\mathscr{A}) \longrightarrow \mathbf{D}^b(\mathscr{B})$ follows from Lemma 5.1.2(i). The implication (b) $\Rightarrow$ (a) of (iii) follows from the corresponding implications of (i) and (ii). If (iii)(a) holds then as above we obtain the recollements of (i) and (ii). But then from the adjoint pairs $(\mathsf{l}', \mathbf{D}^b(\mathsf{e}))$ and $(\mathbf{L}^b\mathsf{l}, \mathbf{D}^b(\mathsf{e}))$ it follows that $\mathsf{l}' \simeq \mathbf{L}^b\mathsf{l}$ and similarly from the adjoint pairs $(\mathbf{D}^b(\mathsf{e}), \mathsf{r}')$ and $(\mathbf{D}^b(\mathsf{e}), \mathbf{R}^b\mathsf{r})$ we have $\mathsf{r}' \simeq \mathbf{R}^b\mathsf{r}$. This shows the implication (a) $\Rightarrow$ (b) of (iii). $\square$

**5.1.2. Rouquier Dimension.** In this subsection we study the connections between the dimension of $\mathbf{D}^b(\mathscr{B})$, in the sense of Rouquier [113], and the dimensions of $\mathbf{D}^b(\mathscr{A})$ and $\mathbf{D}^b(\mathscr{C})$ for a recollement of abelian categories $(\mathscr{A}, \mathscr{B}, \mathscr{C})$. Before this we show a general upper and lower bound for the dimension of a triangulated category in a recollement situation. We start by recalling the notion of dimension of a triangulated category due to Rouquier.

Let $\mathcal{T}$ be a triangulated category and let $\mathcal{U}$ and $\mathcal{V}$ be two subcategories of $\mathcal{T}$. We denote by $\mathcal{U} \star \mathcal{V}$ the full subcategory of $\mathcal{T}$ consisting of objects $A$ such that there exists a triangle $U \longrightarrow A \longrightarrow V \longrightarrow U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. We denote by $\langle \mathcal{U} \rangle$ the smallest full subcategory of $\mathcal{T}$ which contains $\mathcal{U}$ and is closed under finite direct sums, direct summands and shifts. Set $\mathcal{U} \diamond \mathcal{V} := \langle \mathcal{U} \star \mathcal{V} \rangle$ and define inductively $\langle \mathcal{U} \rangle_0 = 0$ and $\langle \mathcal{U} \rangle_n = \langle \mathcal{U} \rangle_{n-1} \diamond \langle \mathcal{U} \rangle$.

DEFINITION 5.1.7. [113] The dimension of a triangulated category $\mathcal{T}$ is defined by

$$\mathsf{dim}\, \mathcal{T} = \min\{n \geq 0 \mid \text{there exists } X \in \mathcal{T} \text{ such that } \langle X \rangle_{n+1} = \mathcal{T}\}$$

To proceed we need the following preliminary result.

LEMMA 5.1.8. (i) *Let $\mathcal{T}$ be a triangulated category and let $X, Y$ be objects of $\mathcal{T}$. Then*

$$\langle X \rangle_n \star \langle Y \rangle_m \ \subseteq \ \langle X \oplus Y \rangle_{n+m}$$

(ii) *Let $0 \longrightarrow \mathcal{U} \xrightarrow{\mathsf{i}} \mathcal{T} \xrightarrow{\mathsf{e}} \mathcal{V} \longrightarrow 0$ be an exact sequence of triangulated categories and assume that the functor $\mathsf{i}$ admits a left adjoint $\mathsf{q}$ and the functor $\mathsf{e}$ admits a left adjoint $\mathsf{l}$. Then for every $A \in \mathcal{T}$ there exists a triangle*

$$\mathsf{le}(A) \longrightarrow A \longrightarrow \mathsf{iq}(A) \longrightarrow \mathsf{le}(A)[1]$$

PROOF. (i) Let $U \longrightarrow A \longrightarrow V \longrightarrow U[1]$ be a triangle in $\mathcal{T}$ with $U \in \langle X \rangle_n$ and $V \in \langle Y \rangle_m$. Assume that $n = 1$. Then $U \in \langle X \rangle \subseteq \langle X \oplus Y \rangle$ and $V \in \langle Y \rangle_m \subseteq \langle X \oplus Y \rangle_m$. Hence $A \in \langle X \oplus Y \rangle_{m+1}$. Assume that $n = 2$. Since $U \in \langle X \oplus Y \rangle_2$ we have a triangle $U_1 \longrightarrow U \longrightarrow U_2 \longrightarrow U_1[1]$ with $U_1, U_2 \in \langle X \oplus Y \rangle$. Applying the octahedral axiom, see [25, Proposition 2.1], to the composition $U_1 \longrightarrow U \longrightarrow A$ we have the following commutative diagram:

$$\begin{tikzcd}
U_1 \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & U_2 \arrow[r] \arrow[d] & U_1[1] \arrow[d, equal] \\
U_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & U_1[1] \arrow[d] \\
U \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & U[1] \arrow[d] \\
0 \arrow[r] & U_2[1] \arrow[r, equal] & U_2[1] \arrow[r] & 0
\end{tikzcd}$$

Since $U_2 \in \langle X \oplus Y \rangle$ and $V \in \langle X \oplus Y \rangle_m$ it follows that $Y \in \langle X \oplus Y \rangle_{m+1}$. We infer that $A \in \langle X \oplus Y \rangle_{m+2}$ since $U_1 \in \langle X \oplus Y \rangle$. Then continuing inductively on $n$ the result follows.

(ii) From the counit of the adjoint pair $(\mathsf{l}, \mathsf{e})$ we have a triangle

$$\mathsf{le}(A) \longrightarrow A \longrightarrow A' \longrightarrow \mathsf{le}(A)[1]$$

in $\mathcal{T}$. Applying the functor $\mathsf{e}\colon \mathcal{T} \longrightarrow \mathcal{V}$ we deduce that $e(A') = 0$ and therefore $A' \simeq \mathsf{i}(U)$ for some $U \in \mathcal{U}$. Since $\mathsf{ql} = 0$ we have the isomorphism $\mathsf{qi}(U) \simeq \mathsf{q}(A)$ and then we get $\mathsf{i}(U) \simeq \mathsf{iqi}(U) \simeq \mathsf{iq}(A)$. This shows that $\mathsf{le}(A) \longrightarrow A \longrightarrow \mathsf{iq}(A) \longrightarrow \mathsf{le}(B)[1]$ is a triangle in $\mathcal{T}$. □

We need also the following easy observation.

Lemma 5.1.9. *[113, Lemma 3.4] Let $F\colon \mathcal{T} \longrightarrow \mathcal{T}'$ be a triangulated functor which is surjective on objects. Then*

$$\dim \mathcal{T}' \ \leq \ \dim \mathcal{T}$$

Proof. Let $\mathcal{T} = \langle X \rangle_n$. Since $F$ is surjective on objects then $\mathcal{T}' = \langle F(X) \rangle_n$ and so we are done. □

The following main result of this section gives bounds for the dimension of a triangulated category $\mathcal{T}$ in a recollement $\mathsf{R}_{\mathsf{tr}}(\mathcal{U}, \mathcal{T}, \mathcal{V})$.

Theorem 5.1.10. *Let $(\mathcal{U}, \mathcal{T}, \mathcal{V})$ be a recollement of triangulated categories. Then:*

$$\max\{\dim \mathcal{U}, \dim \mathcal{V}\} \ \leq \ \dim \mathcal{T} \ \leq \ \dim \mathcal{U} + \dim \mathcal{V} + 1$$

Proof. The functors $\mathsf{e}$ and $\mathsf{q}$ are essentially surjective since $\mathsf{e} \circ \mathsf{l} = \mathrm{Id}_{\mathcal{V}}$ and $\mathsf{q} \circ \mathsf{i} = \mathrm{Id}_{\mathcal{U}}$. Then Lemma 5.1.9 implies that $\dim \mathcal{T} \geq \dim \mathcal{V}$ and $\dim \mathcal{T} \geq \dim \mathcal{U}$, hence

$$\dim \mathcal{T} \geq \max\{\dim \mathcal{U}, \dim \mathcal{V}\}$$

Assume that $\dim \mathcal{U} = n < \infty$ and $\dim \mathcal{V} = m < \infty$, so there exist objects $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $\mathcal{U} = \langle U \rangle_{n+1}$ and $\mathcal{V} = \langle V \rangle_{m+1}$. Set $X := \mathsf{l}(V)$ and $Y := \mathsf{i}(U)$. Let $B \in \mathcal{T}$. Then from Lemma 5.1.8 we have the following triangle in $\mathcal{T}$:

$$\mathsf{le}(B) \longrightarrow B \longrightarrow \mathsf{iq}(B) \longrightarrow \mathsf{le}(B)[1] \tag{5.1.1}$$

Clearly $\mathsf{l}(\langle V \rangle_i) \subseteq \langle \mathsf{l}(V) \rangle_i$ and $\mathsf{i}(\langle U \rangle_j) \subseteq \langle \mathsf{i}(U) \rangle_j$ for every $0 \leq i \leq m+1$ and $0 \leq j \leq n+1$ respectively. Thus $\mathsf{le}(B)$ lies in $\langle X \rangle_{m+1}$ and $\mathsf{iq}(B)$ lies in $\langle Y \rangle_{n+1}$, so by applying Lemma 5.1.8 to the triangle (5.1.1) we infer that

$$\mathcal{T} = \langle \langle X \rangle_{m+1} \star \langle Y \rangle_{n+1} \rangle = \langle X \oplus Y \rangle_{n+m+2}$$

Therefore $\dim \mathcal{T} \leq n + m + 1 = \dim \mathcal{U} + \dim \mathcal{V} + 1$. □

Remark 5.1.11. The proof of Theorem 5.1.10 uses only the exact sequence of triangulated categories $0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{T} \longrightarrow \mathcal{V} \longrightarrow 0$ and the adjoint pairs $(\mathsf{q}, \mathsf{i})$, $(\mathsf{l}, \mathsf{e})$. Since the existence of a left/right adjoint either of $\mathsf{i}$ or $\mathsf{e}$ induces a left/right adjoint to the other functor, the above result holds for any exact sequence of triangulated categories such that one of the involved functors has a left or right adjoint.

The following result provides bounds for the dimension of the triangulated category $\mathbf{D}^b(\mathscr{B})$ in terms of the dimensions of $\mathbf{D}^b(\mathscr{A})$ and $\mathbf{D}^b(\mathscr{C})$.

Theorem 5.1.12. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Then:*

$$\dim \mathbf{D}^b(\mathscr{B}) \ \geq \ \dim \mathbf{D}^b(\mathscr{C})$$

*Suppose that the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding, and one of the following conditions hold.*

(i) *The functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is of locally finite homological dimension.*
(ii) *The functor $\mathsf{r}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is of locally finite cohomological dimension.*

(iii) *The functor* $\mathsf{q}\colon \mathscr{B} \longrightarrow \mathscr{A}$ *is of locally finite homological dimension.*
(iv) *The functor* $\mathsf{p}\colon \mathscr{B} \longrightarrow \mathscr{A}$ *is of locally finite cohomological dimension.*

*Then*:

$$\max\{\dim \mathbf{D}^b(\mathscr{A}), \dim \mathbf{D}^b(\mathscr{C})\} \ \leq\ \dim \mathbf{D}^b(\mathscr{B}) \ \leq\ \dim \mathbf{D}^b(\mathscr{A}) + \dim \mathbf{D}^b(\mathscr{C}) + 1$$

Proof. Since the quotient functor $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ is exact, the derived functor $\mathbf{D}^b(\mathsf{e})\colon \mathbf{D}^b(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{C})$ exists. Let $C^\bullet\colon 0 \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$ be an object of $\mathbf{D}^b(\mathscr{C})$. Then each $C_i \in \mathscr{C}$ and $\mathsf{el}(C_i) \simeq C_i$ with $\mathsf{l}(C_i) \in \mathscr{B}$. The bounded complex $\mathsf{l}(C^\bullet)\colon 0 \longrightarrow \mathsf{l}(C_n) \longrightarrow \cdots \longrightarrow \mathsf{l}(C_0) \longrightarrow 0$ in $\mathbf{D}^b(\mathscr{B})$ is such that $\mathbf{D}^b(\mathsf{e})(\mathsf{l}(C^\bullet)) = C^\bullet$, so $\mathbf{D}^b(\mathsf{e})$ is essentially surjective and then $\dim \mathbf{D}^b(\mathscr{B}) \geq \dim \mathbf{D}^b(\mathscr{C})$ by Lemma 5.1.9.

(i) Assume that the functor $\mathsf{l}\colon \mathscr{C} \longrightarrow \mathscr{B}$ is of locally finite homological dimension. Since the functor $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding, then from Theorem 5.1.6 we have the following diagram

$$\begin{tikzcd}
\mathbf{D}^b(\mathscr{A}) \arrow[r, "\mathbf{D}^b(\mathsf{i})"] & \mathbf{D}^b(\mathscr{B}) \arrow[l, bend right, "\mathsf{q}'"'] \arrow[r, "\mathbf{D}^b(\mathsf{e})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{l}"']
\end{tikzcd}$$

where $0 \longrightarrow \mathbf{D}^b(\mathscr{A}) \longrightarrow \mathbf{D}^b(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{C}) \longrightarrow 0$ (1) is an exact sequence of triangulated categories and $(\mathsf{q}', \mathbf{D}^b(\mathsf{i}))$, $(\mathbf{L}^b\mathsf{l}, \mathbf{D}^b(\mathsf{e}))$ are adjoint pairs of functors. Therefore the result follows from Theorem 5.1.10 and Remark 5.1.11.

(iii) Suppose now that the functor $\mathsf{q}\colon \mathscr{B} \longrightarrow \mathscr{A}$ is of locally finite homological dimension. Then the derived functor $\mathbf{L}^b\mathsf{q}\colon \mathbf{D}^b(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{A})$ induces a left adjoint $\mathsf{l}'\colon \mathbf{D}^b(\mathscr{C}) \longrightarrow \mathbf{D}^b(\mathscr{B})$ of $\mathbf{D}^b(\mathsf{e})$ and thus we have the exact sequence of triangulated categories (1) and the adjoint pairs $(\mathbf{L}^b\mathsf{q}, \mathbf{D}^b(\mathsf{i}))$, $(\mathsf{l}', \mathbf{D}^b(\mathsf{e}))$. Hence our result follows as in the proof of Theorem 5.1.10.

The proof, using the assumptions for $\mathsf{r}$ and $\mathsf{p}$, is similar as above and is left to the reader. □

The following is a consequence of Theorem 5.1.6 and Theorem 5.1.12 for comma categories.

Corollary 5.1.13. *Let* $\mathscr{C} = (G, \mathscr{B}, \mathscr{A})$ *be a comma category.*

(i) *We have the following exact sequences of triangulated categories*:

$$0 \longrightarrow \mathbf{D}^b(\mathscr{A}) \longrightarrow \mathbf{D}^b(\mathscr{C}) \longrightarrow \mathbf{D}^b(\mathscr{B}) \longrightarrow 0$$

*and*

$$0 \longrightarrow \mathbf{D}^b(\mathscr{B}) \longrightarrow \mathbf{D}^b(\mathscr{C}) \longrightarrow \mathbf{D}^b(\mathscr{A}) \longrightarrow 0$$

(ii) *The functor* $\mathsf{q}\colon \mathscr{C} \longrightarrow \mathscr{A}$ *is of locally finite homological dimension if and only if there exists a recollement of triangulated categories*:

$$\begin{tikzcd}
\mathbf{D}^b(\mathscr{A}) \arrow[r, "\mathbf{D}^b(\mathsf{Z}_{\mathscr{A}})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{q}"'] \arrow[l, bend left, "\mathbf{D}^b(\mathsf{U}_{\mathscr{A}})"] \arrow[r, "\mathbf{D}^b(\mathsf{U}_{\mathscr{B}})"] & \mathbf{D}^b(\mathscr{B}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

(iii) *The functor $G \colon \mathscr{B} \longrightarrow \mathscr{A}$ is of locally finite homological dimension if and only if there exists a recollement of triangulated categories:*

$$\begin{tikzcd}
\mathbf{D}^b(\mathscr{A}) \arrow[r, "\mathbf{D}^b(\mathsf{Z}_{\mathscr{A}})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathbf{D}^b(\mathsf{U}_{\mathscr{B}})"] & \mathbf{D}^b(\mathscr{B}) \arrow[l, bend right, "\mathbf{L}^b\mathsf{T}_{\mathscr{B}}"'] \arrow[l, bend left, "\mathbf{D}^b(\mathsf{Z}_{\mathscr{B}})"]
\end{tikzcd}$$

(iv) *The functor $\mathsf{p} \colon \mathscr{C} \longrightarrow \mathscr{B}$ is of locally finite cohomological dimension if and only if there exists a recollement of triangulated categories:*

$$\begin{tikzcd}
\mathbf{D}^b(\mathscr{B}) \arrow[r, "\mathbf{D}^b(\mathsf{Z}_{\mathscr{B}})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right, "\mathbf{D}^b(\mathsf{U}_{\mathscr{B}})"'] \arrow[l, bend left, "\mathbf{R}^b\mathsf{p}"] \arrow[r, "\mathbf{D}^b(\mathsf{U}_{\mathscr{A}})"] & \mathbf{D}^b(\mathscr{A}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

(v) *The functor $G' \colon \mathscr{A} \longrightarrow \mathscr{B}$ is of locally finite cohomological dimension if and only if there exists a recollement of triangulated categories:*

$$\begin{tikzcd}
\mathbf{D}^b(\mathscr{B}) \arrow[r, "\mathbf{D}^b(\mathsf{Z}_{\mathscr{B}})"] & \mathbf{D}^b(\mathscr{C}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathbf{D}^b(\mathsf{U}_{\mathscr{A}})"] & \mathbf{D}^b(\mathscr{A}) \arrow[l, bend right, "\mathbf{D}^b(\mathsf{Z}_{\mathscr{A}})"'] \arrow[l, bend left, "\mathbf{R}^b\mathsf{H}_{\mathscr{A}}"]
\end{tikzcd}$$

(vi) *We have:*

$$\max\{\dim \mathbf{D}^b(\mathscr{A}), \dim \mathbf{D}^b(\mathscr{B})\} \ \leq \ \dim \mathbf{D}^b(\mathscr{C}) \ \leq \ \dim \mathbf{D}^b(\mathscr{A}) + \dim \mathbf{D}^b(\mathscr{B}) + 1$$

Proof. From Example 1.1.12 we have the recollements $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ and $(\mathscr{B}, \mathscr{C}, \mathscr{A})$. Since the functors $\mathsf{U}_{\mathscr{A}} \colon \mathscr{C} \longrightarrow \mathscr{A}$ and $\mathsf{U}_{\mathscr{B}} \colon \mathscr{C} \longrightarrow \mathscr{B}$ are exact it follows from Theorem 2.1.10 that the functors $\mathsf{Z}_{\mathscr{A}} \colon \mathscr{A} \longrightarrow \mathscr{C}$ and $\mathsf{Z}_{\mathscr{B}} \colon \mathscr{B} \longrightarrow \mathscr{C}$ are homological embeddings. Hence as in the first part of the proof of Theorem 5.1.6 we obtain the exact sequences of (i). The bound on the dimension $\dim \mathbf{D}^b(\mathscr{C})$ follows from Theorem 5.1.12. Finally, the remaining assertions follow from Theorem 5.1.6 using that $\mathsf{L}_n\mathsf{T}_{\mathscr{B}}(B) = (\mathsf{L}_nG(B), 0, 0)$ $\forall n \geq 1$ and $B \in \mathscr{B}$, and $\mathsf{R}^n\mathsf{H}_{\mathscr{A}}(A) = (0, \mathsf{R}^nG'(A), 0)$ $\forall n \geq 1$ and $A \in \mathscr{A}$. $\square$

**5.1.3. Recollements of Bounded Derived Categories of Rings.** For a ring $R$ we write $\mathbf{D}^b(R)$ for the bounded derived category of $\mathsf{Mod}\text{-}R$. The next consequence follows from Theorem 5.1.12.

Corollary 5.1.14. *Let $R$ be a ring and $e^2 = e \in R$. Then:*

$$\dim \mathbf{D}^b(R) \ \geq \ \dim \mathbf{D}^b(eRe)$$

*If the natural map $R \longrightarrow R/ReR$ is a homological epimorphism, i.e. $ReR \in \mathfrak{X}_\infty$, and one of the following holds:*

(i) $\mathsf{pd}_R\, R/ReR < \infty$
(ii) $\mathsf{fd}_R\, R/ReR < \infty$
(iii) $\mathsf{pd}_{eRe}\, eR < \infty$
(iv) $\mathsf{fd}_{eRe}\, Re < \infty$

*then*:

$$\max\{\dim \mathbf{D}^b(eRe), \dim \mathbf{D}^b(R/ReR)\} \ \le\ \dim \mathbf{D}^b(R) \ \le\ \dim \mathbf{D}^b(eRe)+\dim \mathbf{D}^b(R/ReR)+1$$

Recall that the module category over a triangular matrix ring is a comma category, see [18] and [49]. Then we have the following consequence of Corollary 5.1.13, parts (ii) and (iii) of which generalize Corollary 2.5 of [38], see also [5] for part (iii).

COROLLARY 5.1.15. *Let* $\Lambda = \left(\begin{smallmatrix} R & {}_RN_S \\ 0 & S \end{smallmatrix}\right)$ *be a triangular matrix ring and let* $e_1 = \left(\begin{smallmatrix} 1_R & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and* $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_S \end{smallmatrix}\right)$.

(i) *We have the following exact sequences of triangulated categories*:

$$0 \longrightarrow \mathbf{D}^b(R) \longrightarrow \mathbf{D}^b(\Lambda) \longrightarrow \mathbf{D}^b(S) \longrightarrow 0$$

*and*

$$0 \longrightarrow \mathbf{D}^b(S) \longrightarrow \mathbf{D}^b(\Lambda) \longrightarrow \mathbf{D}^b(R) \longrightarrow 0$$

(ii) *If* $\mathsf{fd}_\Lambda \Lambda/\Lambda e_2\Lambda < \infty$ *or* $\mathsf{fd}_S N < \infty$, *then* $(\mathbf{D}^b(R), \mathbf{D}^b(\Lambda), \mathbf{D}^b(S))$ *is a triangulated recollement.*

(iii) *If* $\mathsf{pd}_\Lambda \Lambda/\Lambda e_1\Lambda < \infty$ *or* $\mathsf{pd}_R N < \infty$, *then* $(\mathbf{D}^b(S), \mathbf{D}^b(\Lambda), \mathbf{D}^b(R))$ *is a triangulated recollement.*

(iv) *We have*:

$$\max\{\dim \mathbf{D}^b(R), \dim \mathbf{D}^b(S)\} \ \le\ \dim \mathbf{D}^b(\Lambda) \ \le\ \dim \mathbf{D}^b(R) + \dim \mathbf{D}^b(S) + 1$$

REMARK 5.1.16. Let $A$ be a finite-dimensional algebra over a field such that there exist a recollement of triangulated categories $(\mathbf{D}^b(A'), \mathbf{D}^b(A), \mathbf{D}^b(A''))$ for some finite dimensional algebras $A'$ and $A''$. Then from [125], see also [63], it follows that $\mathsf{gl.dim}\, A < \infty$ if and only if $\mathsf{gl.dim}\, A' < \infty$ and $\mathsf{gl.dim}\, A'' < \infty$. If $e$ is an idempotent element of $A$ and $AeA \in \mathsf{Proj}\, A$ then from Corollary 2.4.5 it follows that $\mathsf{gl.dim}\, A < \infty$ if and only if $\mathsf{gl.dim}\, A/AeA < \infty$ and $\mathsf{gl.dim}\, eAe < \infty$. Suppose that the idempotent element $e$ of $\Lambda$ is primitive. Then from [96, Proposition 5] it follows that $(\mathbf{D}^b(A/AeA), \mathbf{D}^b(A), \mathbf{D}^b(eAe))$ is a recollement of triangulated categories if and only if $AeA$ is projective both as a left and right $A$-module. This shows that for finite dimensional algebras the assumption $AeA \in \mathsf{Proj}\, A$ is enough to ensure the finiteness of the global dimension of $A$ in connection with the finiteness of the global dimension of $A/AeA$ and $eAe$, without passing to the corresponding recollement in the bounded derived categories.

## 5.2. Representation Dimension

In this section we concentrate on the behavior of representation dimension of the abelian categories involved in a recollement situation and we give application to Artin algebras. The main consequences of 5.2.2 are on the relation of the representation dimension of $\Lambda$, where $\Lambda$ is an Artin algebra, with that of $e\Lambda e$, where $e$ is an idempotent element of $\Lambda$. Finally is subsection 5.2.3 we show an interesting interplay between representation and finitistic dimension of Artin algebras.

**5.2.1. Recollements and Representation Dimension.** We begin by reviewing Auslander's notion of representation dimension in the context of abelian categories, see [10]. Let $\mathcal{M}$ be an abelian Krull-Schmidt category, i.e. every object of $\mathcal{M}$ decomposes into a finite direct sum of objects having local endomorphism rings. An object $X$ of $\mathcal{M}$ is called $\mathsf{generator}$, resp. $\mathsf{cogenerator}$, if any object of $\mathcal{M}$ is a factor, resp. subobject, of a direct summand of a finite direct sum of copies of $X$. Recall that if $\mathcal{X}$ is a full subcategory of $\mathcal{M}$ then $\mathsf{add}\,\mathcal{X}$ denotes the full subcategory of $\mathcal{M}$ consisting of all direct summands of finite coproducts of objects of $\mathcal{X}$.

DEFINITION 5.2.1. The representation dimension of $\mathcal{M}$, denoted by $\mathsf{rep.dim}\,\mathcal{M}$, is defined as follows:

$$\mathsf{rep.dim}\,\mathcal{M} = \inf\{\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X) \mid X \text{ is a generator-cogenerator of } \mathcal{M} \\ \text{and } \mathsf{add}\,X \text{ is functorially finite in } \mathcal{M}\}$$

Note that contravariant finiteness of $\mathsf{add}\,X$ implies that $\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)$ is abelian and covariant finiteness of $\mathsf{add}\,X$ implies that $\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)^{\mathsf{op}}$ is abelian. In order to explain this we need to recall some basics from the theory of coherent functors [8].

An additive functor $F\colon \mathcal{M}^{\mathsf{op}} \longrightarrow \mathfrak{Ab}$ is called coherent if there exists an exact sequence of functors:

$$\mathsf{Hom}_{\mathcal{M}}(-, M^1) \longrightarrow \mathsf{Hom}_{\mathcal{M}}(-, M^0) \longrightarrow F \longrightarrow 0$$

We denote by $\mathsf{mod}\text{-}\mathcal{M}$ the category of coherent functors over $\mathcal{M}$. Given a morphism $f\colon Y \longrightarrow Z$ in $\mathcal{M}$, then a morphism $g\colon X \longrightarrow Y$ is a weak kernel for $f$ provided that the induced sequence of functors:

$$\mathsf{Hom}_{\mathcal{M}}(-, X) \longrightarrow \mathsf{Hom}_{\mathcal{M}}(-, Y) \longrightarrow \mathsf{Hom}_{\mathcal{M}}(-, Z)$$

is exact. Then a classical result of Freyd [52, Theorem 1.4], see also [10, Chapter 3, Section 2], asserts that the category of coherent functors $\mathsf{mod}\text{-}\mathcal{M}$ is abelian if and only if $\mathcal{M}$ has weak kernels. Consider now the category of coherent functors $\mathsf{mod}\text{-}\mathsf{add}\,X$ over $\mathsf{add}\,X$. Then from Auslander [10] there is an equivalence of categories:

$$\mathsf{mod}\text{-}\mathsf{add}\,X \xrightarrow{\simeq} \mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)$$

where $\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)$ is the category of finitely generated left $\mathsf{End}_{\mathcal{M}}(X)$-modules. Let $f\colon M_1 \longrightarrow M_2$ be a morphism in $\mathsf{add}\,X$ with kernel $\mathsf{Ker}\,f$. Since the subcategory $\mathsf{add}\,X$ is contravariantly finite in $\mathcal{M}$ we get the exact sequence:

$$0 \longrightarrow \mathsf{Hom}_{\mathcal{M}}(-, \mathsf{Ker}\,f)|_{\mathsf{add}\,X} \longrightarrow \mathsf{Hom}_{\mathcal{M}}(-, M_1)|_{\mathsf{add}\,X} \longrightarrow \mathsf{Hom}_{\mathcal{M}}(-, M_2)|_{\mathsf{add}\,X} \longrightarrow 0$$

and therefore $\mathsf{add}\,X$ has weak kernels. Summarizing the above discussion we have:

$$\begin{array}{lcl} \mathsf{add}\,X\colon \text{ contravariantly finite} & \Longrightarrow & \mathsf{add}\,X\colon \text{ has weak kernels} \\ & \stackrel{\text{Freyd}}{\Longrightarrow} & \mathsf{mod}\text{-}\mathsf{add}\,X\colon \text{ abelian category} \\ & \stackrel{\text{Auslander}}{\Longrightarrow} & \mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)\colon \text{ abelian category} \end{array}$$

Similarly, from the covariant finiteness of $\mathsf{add}\,X$, we derive that $\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)^{\mathsf{op}}$ is abelian. Moreover if $\mathsf{add}\,X$ is functorially finite, then it follows that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)$ $= \mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)^{\mathsf{op}}$, see [23], and this common value is the weak global dimension of $\mathsf{End}_{\mathcal{M}}(X)$.

Let $X$ be a generator-cogenerator of an abelian Krull-Schmidt category $\mathcal{M}$ such that $\mathsf{add}\,X$ is functorially finite in $\mathcal{M}$. Then it is well known, see [10] and [48], that the following statements are equivalent:

(i) $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X) \leq n$.

(ii) For every $A \in \mathcal{M}$ there exists an exact sequence

$$0 \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

with $X_i \in \mathsf{add}\, X$ such that the following induced sequence is exact:

$$0 \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X, X_{n-2}) \longrightarrow \cdots \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X, X_0) \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X, A) \longrightarrow 0$$

(iii) For every $A \in \mathcal{M}$ there exists an exact sequence

$$0 \longrightarrow A \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^{n-2} \longrightarrow 0$$

with $X_i \in \mathsf{add}\, X$ such that the following induced sequence is exact:

$$0 \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X^{n-2}, X) \longrightarrow \cdots \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X^0, X) \longrightarrow \mathsf{Hom}_{\mathcal{M}}(A, X) \longrightarrow 0$$

For completeness we give the proof.

Proof. (i) $\Rightarrow$ (ii) Let $A \in \mathcal{M}$ and $f \colon A \longrightarrow X^0$ be a monomorphism with $X^0 \in \mathsf{add}\, X$. Note that such a morphism exists since the object $X$ is a cogenerator of $\mathcal{M}$. For the cokernel of $f$ there is also a monomorphism for some object $X^1$ of $\mathsf{add}\, X$. Then we have the exact sequence:

$$0 \longrightarrow A \xrightarrow{f} X^0 \xrightarrow{g} X^1$$

with $X^0, X^1 \in \mathsf{add}\, X$. Applying the functor $\mathsf{Hom}_{\mathcal{M}}(X, -)$ we get the following exact sequence:

$$0 \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X, A) \xrightarrow{(X,f)} \mathsf{Hom}_{\mathcal{M}}(X, X^0) \xrightarrow{(X,g)} \mathsf{Hom}_{\mathcal{M}}(X, X^1) \longrightarrow \mathsf{Coker}\,(X, g) \longrightarrow 0$$

Note that the functor $\mathsf{Hom}_{\mathcal{M}}(X, -)$ induces the following equivalence of categories:

$$\mathsf{Hom}_{\mathcal{M}}(X, -) \colon \ \mathsf{add}\, X \ \xrightarrow{\simeq} \ \mathsf{proj}(\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X))$$

Thus, since $\mathsf{Hom}_{\mathcal{M}}(X, X^0), \mathsf{Hom}_{\mathcal{M}}(X, X^1) \in \mathsf{proj}\, \mathsf{End}_{\mathcal{M}}(X)$ and $\mathsf{gl.dim}\, \mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X) \leq n$ it follows that there is an exact sequence:

$$0 \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X, X_{n-2}) \longrightarrow \cdots \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X, X_0) \longrightarrow \mathsf{Hom}_{\mathcal{M}}(X, A) \longrightarrow 0$$

with $X_0, \dots, X_{n-2} \in \mathsf{add}\, X$. Then (ii) follows.

(ii) $\Rightarrow$ (i) Let $Y$ be an object of $\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X)$ and consider the exact sequence

$$\mathsf{Hom}_{\mathcal{M}}(X, X^0) \xrightarrow{(X,h)} \mathsf{Hom}_{\mathcal{M}}(X, X^1) \longrightarrow Y \longrightarrow 0$$

with $X^0, X^1 \in \mathsf{add}\, X$, i.e. $\mathsf{Hom}_{\mathcal{M}}(X, X^0)$ and $\mathsf{Hom}_{\mathcal{M}}(X, X^1)$ are projective $\mathsf{End}_{\mathcal{M}}(X)$-modules. Then for the object $\mathsf{Ker}\, h$ there exists an exact sequence

$$0 \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow \mathsf{Ker}\, h \longrightarrow 0$$

with $X_i \in \mathsf{add}\, X$ such that the following induced sequence is exact:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{M}}(X, X_{n-2}) \arrow[r, tail] & \cdots \arrow[r] & \mathsf{Hom}_{\mathcal{M}}(X, X_0) \arrow[r, dashed] \arrow[d] & \mathsf{Hom}_{\mathcal{M}}(X, X^0) \arrow[r] & \mathsf{Hom}_{\mathcal{M}}(X, X^1) \arrow[r, two heads] & Y \\
 & & \mathsf{Hom}_{\mathcal{M}}(X, \mathsf{Ker}\, h) \arrow[ur, tail] & & &
\end{tikzcd}$$

We infer that the projective dimension of $Y$ is at most $n$. Hence $\mathsf{gl.dim}\, \mathsf{mod}\text{-}\mathsf{End}_{\mathcal{M}}(X) \leq n$. Similarly we show that (i) is equivalent with (iii). $\square$

An object $X \in \mathcal{M}$ that realizes the minimal $n$ is called an Auslander generator of $\mathcal{M}$. If the sequence (ii) exists for an object $A \in \mathcal{M}$ then we say that $A$ has an $\mathsf{add}\, X$-resolution of length $\leq n-2$ and if the sequence (iii) exists then we say that $A$ has an $\mathsf{add}\, X$-coresolution of length $\leq n-2$. Hence the representation dimension of $\mathcal{M}$ is the smaller integer $n \geq 2$ or $\infty$ such that there exist a generator-cogenerator $X \in \mathcal{M}$ with the property that $\mathsf{add}\, X$ is functorially finite and every object $A \in \mathcal{M}$ has an $\mathsf{add}\, X$-resolution of length $n-2$ or equivalently an $\mathsf{add}\, X$-coresolution of length $n-2$. Note that $\mathsf{rep.dim}\, \mathcal{M} = 2$ if and only if $\mathcal{M}$ is of finite representation type, i.e $\mathcal{M}$ has only finitely many isoclasses of indecomposable objects.

Throughout this section we fix a recollement $\mathsf{R_{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$, where $\mathscr{B}$ is a Krull-Schmidt category, and, for simplicity, we assume that for any object $X \in \mathscr{B}$, the subcategory $\mathsf{add}\, X$ is covariantly finite in $\mathscr{B}$. Clearly then $\mathscr{A}$ and $\mathscr{C}$ are Krull-Schmidt categories. Note that if $\mathscr{B}$ is of finite representation type, then since $\mathscr{A}$ and $\mathscr{C}$ are fully embedded in $\mathscr{B}$, it follows that $\mathscr{A}$ and $\mathscr{C}$ are of finite representation type. Our first result in this section compares the representation dimension of $\mathscr{B}$ with the representation dimension of $\mathscr{A}$ and $\mathscr{C}$ when $\mathsf{rep.dim}\, \mathscr{B} \leq 3$.

THEOREM 5.2.2. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that* $\mathsf{rep.dim}\, \mathscr{B} \leq 3$.

(i) $\mathsf{rep.dim}\, \mathscr{C} \leq 3$.
(ii) *If the functor* $\mathsf{q} \colon \mathscr{B} \longrightarrow \mathscr{A}$ *is exact, then*: $\mathsf{rep.dim}\, \mathscr{A} \leq 3$.

PROOF. (i) Let $B$ be an Auslander generator of $\mathscr{B}$. Then $\mathsf{e}(B)$ is a generator-cogenerator of $\mathscr{C}$ and since the functor $\mathsf{e}$ has left and right adjoint it follows that $\mathsf{add}\, \mathsf{e}(B)$ is functorially finite in $\mathscr{C}$. We will show that any object $C$ of $\mathscr{C}$ has an $\mathsf{add}\, \mathsf{e}(B)$-resolution of length at most one. Let $C$ be an object of $\mathscr{C}$. Since the object $\mathsf{r}(C) \in \mathscr{B}$ and $\mathsf{rep.dim}\, \mathscr{B} \leq 3$ there exists an exact sequence

$$0 \longrightarrow B_1 \xrightarrow{b_1} B_0 \xrightarrow{b_0} \mathsf{r}(C) \longrightarrow 0 \tag{5.2.1}$$

with $B_1, B_0 \in \mathsf{add}\, B$ such that the sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, B_1) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, B_0) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, \mathsf{r}(C)) \longrightarrow 0 \tag{5.2.2}$$

is exact. Applying the functor $\mathsf{e} \colon \mathscr{B} \longrightarrow \mathscr{C}$ to (5.2.1) we obtain an exact sequence

$$0 \longrightarrow \mathsf{e}(B_1) \xrightarrow{\mathsf{e}(b_1)} \mathsf{e}(B_0) \xrightarrow{\mathsf{e}(b_0)} C \longrightarrow 0$$

with $\mathsf{e}(B_1), \mathsf{e}(B_2) \in \mathsf{add}\, \mathsf{e}(B)$ and we have to show that the induced sequence

$$0 \to \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), \mathsf{e}(B_1)) \to \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), \mathsf{e}(B_0)) \to \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), C) \to 0 \tag{5.2.3}$$

is exact, i.e. we have to prove that the morphism $\mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), \mathsf{e}(b_0)) \colon \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), \mathsf{e}(B_0)) \to \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), C)$ is an epimorphism. Let $c \colon \mathsf{e}(B) \longrightarrow C$ be a morphism in $\mathscr{C}$. Then the following composition of morphisms

$$B \xrightarrow{\nu_B} \mathsf{re}(B) \xrightarrow{\mathsf{r}(c)} \mathsf{r}(C)$$

belongs to $\mathsf{Hom}_{\mathscr{B}}(B, \mathsf{r}(C))$, where $\nu_B$ is the unit of the adjoint pair $(\mathsf{e}, \mathsf{r})$. Hence from the exact sequence (5.2.2) there exists a morphism $\kappa \in \mathsf{Hom}_{\mathscr{B}}(B, B_0)$ such that $\kappa \circ b_0 = \nu_B \circ \mathsf{r}(c)$. Then $\mathsf{e}(\kappa) \circ \mathsf{e}(b_0) = c$ and so the sequence (5.2.3) is exact. Therefore we deduce that $\mathsf{rep.dim}\, \mathscr{C} \leq 3$.

(ii) Let $B$ be an Auslander generator of $\mathscr{B}$. Since the functor $\mathsf{q}$ is exact and $\mathsf{qi} \simeq \mathrm{Id}_{\mathscr{A}}$ it follows that the object $\mathsf{q}(B)$ is a generator-cogenerator of $\mathscr{A}$. Further the category $\mathsf{add}\,\mathsf{q}(B)$ is functorially finite in $\mathscr{A}$. Let $A$ be an object of $\mathscr{A}$. Since $\mathsf{rep.dim}\,\mathscr{B} \leq 3$ there exists an exact sequence

$$0 \longrightarrow B_1 \xrightarrow{b_1} B_0 \xrightarrow{b_0} \mathsf{i}(A) \longrightarrow 0 \tag{5.2.4}$$

with $B_1, B_0 \in \mathsf{add}\,B$ such that the sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, B_1) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, B_0) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A)) \longrightarrow 0 \tag{5.2.5}$$

is exact. Applying the functor $\mathsf{q} \colon \mathscr{B} \longrightarrow \mathscr{A}$ to (5.2.4) we get an exact sequence

$$0 \longrightarrow \mathsf{q}(B_1) \xrightarrow{\mathsf{q}(b_1)} \mathsf{q}(B_0) \xrightarrow{\mathsf{q}(b_0)} A \longrightarrow 0$$

with $\mathsf{q}(B_1), \mathsf{q}(B_2) \in \mathsf{add}\,\mathsf{q}(B)$ and we claim that the following sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathscr{A}}(\mathsf{q}(B), \mathsf{q}(B_1)) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(\mathsf{q}(B), \mathsf{q}(B_0)) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(\mathsf{q}(B), A) \longrightarrow 0 \tag{5.2.6}$$

is exact. Let $a \colon \mathsf{q}(B) \longrightarrow A$ be a map in $\mathscr{A}$. Then the morphism

$$B \xrightarrow{\lambda_B} \mathsf{iq}(B) \xrightarrow{\mathsf{i}(a)} \mathsf{i}(A)$$

belongs to $\mathsf{Hom}_{\mathscr{B}}(B, \mathsf{i}(A))$, where $\lambda$ is the unit of the adjoint pair $(\mathsf{i}, \mathsf{q})$. Thus from the exact sequence (5.2.5) there exists a morphism $\kappa \colon B \longrightarrow B_0$ such that $\kappa \circ b_0 = \lambda_B \circ \mathsf{i}(a)$. Applying the functor $\mathsf{q}$ we have $\mathsf{q}(\kappa) \circ \mathsf{q}(b_0) = a$ and therefore the sequence (5.2.6) is exact. We infer that $\mathsf{rep.dim}\,\mathscr{A} \leq 3$. □

The following result shows that $\mathsf{rep.dim}\,\mathscr{C} \leq \mathsf{rep.dim}\,\mathscr{B}$ holds in general, provided that there exists an Auslander-generator of $\mathscr{B}$ enjoying a special property.

THEOREM 5.2.3. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. Let $B \in \mathscr{B}$ be an Auslander-generator such that $\mathsf{add}\,\mathsf{le}(B) \subseteq \mathsf{add}\,B$. Then:*

$$\mathsf{rep.dim}\,\mathscr{C} \;\leq\; \mathsf{rep.dim}\,\mathscr{B}$$

PROOF. Let $X$ be an arbitrary object of $\mathscr{B}$ and let $a \colon B_1 \longrightarrow X$ be a right $\mathsf{add}\,B$-approximation. Let $C \in \mathsf{add}\,\mathsf{e}(B)$. Then $\mathsf{e}(B)^n \simeq C \oplus C'$ for some $n$ and if we apply the functor $\mathsf{l} \colon \mathscr{C} \longrightarrow \mathscr{B}$ we deduce that $\mathsf{l}(C) \in \mathsf{add}\,B$ since $\mathsf{add}\,\mathsf{le}(B) \subseteq \mathsf{add}\,B$. But this implies that the sequence $\mathsf{Hom}_{\mathscr{C}}(C, \mathsf{e}(B_1)) \longrightarrow \mathsf{Hom}_{\mathscr{C}}(C, \mathsf{e}(X)) \longrightarrow 0$ is exact since the sequence $\mathsf{Hom}_{\mathscr{B}}(\mathsf{l}(C), B_1) \longrightarrow \mathsf{Hom}_{\mathscr{C}}(\mathsf{l}(C), X) \longrightarrow 0$ is exact. Hence we infer that that the morphism $\mathsf{e}(a) \colon \mathsf{e}(B_1) \longrightarrow \mathsf{e}(X)$ is a right $\mathsf{add}\,\mathsf{e}(B)$-approximation. Let $C$ be an object of $\mathscr{C}$ and assume that $\mathsf{rep.dim}\,\mathscr{B} = n$; since $\mathsf{l}(C) \in \mathscr{B}$ there exist an exact sequence

$$0 \longrightarrow B_{n-2} \xrightarrow{b_{n-2}} \cdots \xrightarrow{b_1} B_0 \xrightarrow{b_0} \mathsf{l}(C) \longrightarrow 0$$

with $B_i \in \mathsf{add}\,B$ such that the sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, B_{n-2}) \longrightarrow \cdots \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, B_0) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(B, \mathsf{l}(C)) \longrightarrow 0$$

is exact. Then we have the exact sequence

$$0 \longrightarrow \mathsf{e}(B_{n-2}) \xrightarrow{\mathsf{e}(b_{n-2})} \cdots \xrightarrow{\mathsf{e}(b_1)} \mathsf{e}(B_0) \xrightarrow{\mathsf{e}(b_0)} C \longrightarrow 0$$

with $\mathsf{e}(B_i) \in \mathsf{add}\,\mathsf{e}(B)$. Recall that $\mathsf{e}(B)$ is a generator-cogenerator of $\mathscr{C}$ and $\mathsf{add}\,\mathsf{e}(B)$ is functorially finite in $\mathscr{C}$. Since the morphisms $b_0 \colon B_0 \longrightarrow \mathsf{l}(C)$, $B_1 \longrightarrow \mathsf{Ker}\,b_0, \cdots$,

$B_{n-3} \longrightarrow \mathsf{Ker}\, b_{n-4}$ are right $\mathsf{add}\, B$-approximations it follows that the morphisms $\mathsf{e}(b_0)\colon \mathsf{e}(B_0) \longrightarrow C$, $\mathsf{e}(B_1) \longrightarrow \mathsf{e}(\mathsf{Ker}\, b_0)$, $\cdots$, $\mathsf{e}(B_{n-3}) \longrightarrow \mathsf{e}(\mathsf{Ker}\, b_{n-4})$ are right $\mathsf{add}\, \mathsf{e}(B)$-approximations. This implies that the sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), \mathsf{e}(B_{n-2})) \longrightarrow \cdots \longrightarrow \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), \mathsf{e}(B_0)) \longrightarrow \mathsf{Hom}_{\mathscr{C}}(\mathsf{e}(B), C) \longrightarrow 0$$

is exact. Hence any object $C \in \mathscr{C}$ admits an $\mathsf{add}\, \mathsf{e}(B)$-resolution of length at most $n-2$ and therefore $\mathsf{rep.dim}\, \mathscr{C} \leq n = \mathsf{rep.dim}\, \mathscr{B}$. □

**5.2.2. Representation Dimension of Artin Algebras.** In this subsection we work in the setting of finitely generated modules over an Artin algebra $\Lambda$. We recall the notion of representation dimension due to Auslander.

DEFINITION 5.2.4. [10, Auslander] The **representation dimension** $\mathsf{rep.dim}\, \Lambda$ of $\Lambda$ is defined by

$$\mathsf{rep.dim}\, \Lambda = \min\{\mathsf{gl.dim}\, \mathsf{End}_\Lambda(X) \mid X\colon \text{ generator and cogenerator of } \mathsf{mod}\text{-}\Lambda\}$$

Note that for $\mathcal{M} = \mathsf{mod}\text{-}\Lambda$, Definitions 5.2.1 and 5.2.4 coincide. We start with the following consequences of Theorems 5.2.2 and 5.2.3.

COROLLARY 5.2.5. *Let $\Lambda$ be an Artin algebra with $\mathsf{rep.dim}\, \Lambda \leq 3$ and $e$ an idempotent element of $R$. Then:*

(i) $\mathsf{rep.dim}\, e\Lambda e \leq 3$.
(ii) *If the $\Lambda$-module $\Lambda/\Lambda e\Lambda$ is projective, then:* $\mathsf{rep.dim}\, \Lambda/\Lambda e\Lambda \leq 3$.

COROLLARY 5.2.6. *Let $M$ be an Auslander-generator of $\Lambda$ and $e^2 = e$ an idempotent of $\Lambda$. If the multiplication map $\Lambda e \otimes_{e\Lambda e} eM \longrightarrow M$ is split monomorphism, then:*

$$\mathsf{rep.dim}\, e\Lambda e \leq \mathsf{rep.dim}\, \Lambda$$

PROOF. Since the map $\Lambda e \otimes_{e\Lambda e} eM \longrightarrow M$ is split monomorphism, it follows that $\mathsf{add}(\Lambda e \otimes_{e\Lambda e} eM) \subseteq \mathsf{add}\, M$ and therefore the result follows from Theorem 5.2.3. □

Let $P$ be a finitely generated projective $\Lambda$-module. Auslander posed the question of how the representation dimensions of $\Lambda$ and $\mathsf{End}_\Lambda(P)$ are related, see [10], where he proved that if $\Lambda$ is of finite representation type, then so is $\mathsf{End}_\Lambda(P)$. Now as an application of Theorem 5.2.2(i) we have the following result which provides a partial answer to the above question.

COROLLARY 5.2.7. *Let $\Lambda$ be an Artin algebra. Then:*

$$\mathsf{rep.dim}\, \Lambda \leq 3 \iff \mathsf{rep.dim}\, \mathsf{End}_\Lambda(P) \leq 3$$

*for any finitely generated projective $\Lambda$-module $P$.*

We continue with an application to the finitistic dimension. By a well known result of Igusa-Todorov, see [66], if $\Lambda$ is an Artin algebra of $\mathsf{rep.dim}\, \Lambda \leq 3$ then $\mathsf{fin.dim}\, \Lambda < \infty$. The following, which is the main result of A. Zhang and S. Zhang [132], is a direct consequence of Corollary 5.2.5(i) and Igusa-Todorov's result.

COROLLARY 5.2.8. *[132, Theorem 2.3] Let $\Lambda$ be an Artin algebra with $\mathsf{rep.dim}\, \Lambda \leq 3$ and $e^2 = e \in \Lambda$. Then:*

$$\mathsf{fin.dim}\, e\Lambda e < \infty$$

EXAMPLE 5.2.9. Many classes of algebras are known to have representation dimension at most three. For instance we mention:

(i) Algebras with radical square zero. Auslander proved in [10] that if $\Lambda$ is an Artin algebra with radical $\mathfrak{r}$, Loewy length $\ell\ell(\Lambda) = n$ and $\mathsf{rep.dim}\,\Lambda/\mathfrak{r}^{n-1} \leq 2$, then $\mathsf{rep.dim}\,\Lambda \leq 3$. This implies that every Artin algebra $\Lambda$ with radical square zero has $\mathsf{rep.dim}\,\Lambda \leq 3$.

(ii) Hereditary algebras. This is also a well known result of Auslander [10]. The Auslander generator in this case is the direct sum of all the non-isomorphic indecomposable projective modules with all the non-isomorphic indecomposable injective modules.

(iii) Stably hereditary algebras. Changchang Xi has shown that if an Artin algebra $\Lambda$ is stably hereditary then $\mathsf{rep.dim}\,\Lambda \leq 3$, see [127, Theorem 3.5]. Recall that an Artin algebra $\Lambda$ is called stably hereditary if $(\alpha)$ each indecomposable submodule of an indecomposable projective module is either projective or simple and $(\beta)$ each indecomposable factor module of an indecomposable injective module is either injective or simple. We refer to [127] for more details.

(iv) Special biserial algebras. Erdmann, Holm, Iyama and Schröer have shown that any special biserial algebra has representation dimension at most three, see [48, Corollary 1.3]. This is an application of their main result, which asserts that if there is a radical embedding $f\colon A \longrightarrow B$ and $\mathsf{rep.dim}\,B \leq 2$ then $\mathsf{rep.dim}\,A \leq 3$. See [48] for more details as well as for the definition of special biserial algebras.

(v) Tilted algebras. Let $\Lambda$ be an Artin algebra. Recall that a $\Lambda$-module $T$ is tilting if $\mathsf{pd}_\Lambda T \leq 1$, $\mathsf{Ext}^1_\Lambda(T,T) = 0$ and there exists a short exact sequence $0 \longrightarrow \Lambda \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0$ with $T^i \in \mathsf{add}\,T$. Then the endomorphism algebra $\mathsf{End}_\Lambda(T)$ of a tilting module $T$ over a hereditary algebra $\Lambda$ is called a tilted algebra. Assem, Platzeck and Todorov proved that the representation dimension of a tilted algebra is at most three. Moreover they showed that the class of strict laura algebras has also representation dimension bounded by three. For more details see [6].

(vi) Quasi-tilted algebras. Oppermann has shown that if $\Lambda$ is a quasi-tilted algebra over an algebraically closed field then $\mathsf{rep.dim}\,\Lambda \leq 3$. In particular, any quasi-tilted non-tilted algebra has representation dimension three. Recall that a finite dimensional $k$-algebra $\Lambda$ is called quasi-tilted if $\Lambda \simeq \mathsf{End}_{\mathscr{A}}(T)$ for some tilting object $T$ in a hereditary abelian category $\mathscr{A}$ (i.e. $\mathsf{Ext}^2_{\mathscr{A}}(X,Y) = 0$ for any $X, Y \in \mathscr{A}$). Hence if $\mathscr{A}$ is the module category of a hereditary algebra then $\Lambda$ is tilted. For more details we refer to [102].

(vii) Piecewise hereditary. An Artin algebra $\Lambda$ is said to be piecewise hereditary if the derived category $\mathbf{D}^b(\mathsf{mod}\text{-}\Lambda)$ is triangle equivalent with $\mathbf{D}^b(\mathscr{A})$ for some hereditary abelian category $\mathscr{A}$. For this important class of algebras, Happel and Unger have shown that the representation dimension is at most three, see [64, Corollary 3.2].

(viii) Torsionless-finite algebras. Let $\Lambda$ be an Artin algebra and denote by $\mathsf{Sub}\,\Lambda$ the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of the submodules of the projectives. The Artin algebra $\Lambda$ is said to be torsionless-finite if $\mathsf{Sub}\,\Lambda$ is of finite representation type, i.e. $\mathsf{Sub}\,\Lambda = \mathsf{add}\,X$ for some $\Lambda$-module $X$. Ringel [111], and later Beligiannis [28] gave a simpler proof, has shown that if $\Lambda$ is a torsionless-finite

Artin algebra then $\operatorname{rep.dim}\Lambda \leq 3$. This important class of algebras contains many other well known classes. For instance the algebras with radical square zero and the hereditary ones are torsionless finite. There are more classes of algebras that are torsionless finite and thus they have representation dimension at most 3 but not all Artin algebras with representation dimension at most 3 are torsionless finite. See [111] for more details on torsionless-finite algebras.

Therefore for any algebra $\Lambda$ in the above list and any idempotent element $e \in \Lambda$ we have $\operatorname{rep.dim} e\Lambda e \leq 3$ and $\operatorname{fin.dim} e\Lambda e < \infty$.

Auslander in [10] proved that if $\Lambda$ is a hereditary Artin algebra then $\operatorname{rep.dim}\Lambda \leq 3$. Then he asked if $\operatorname{rep.dim}\Lambda \leq 2 + \operatorname{gl.dim}\Lambda$ for all Artin algebras $\Lambda$, and so the general question is how we can relate representation dimension with global dimension. In this connection we have the following results.

Consider the Artin algebra $\Gamma = \mathsf{End}_\Lambda(\Lambda \oplus \mathsf{D}\,\Lambda)$, where $\mathsf{D}$ denotes the usual duality functor of Artin algebras, viewed as a Morita ring via the isomorphism $\mathsf{End}_\Lambda(\Lambda\oplus\mathsf{D}\,\Lambda) \simeq \left(\begin{smallmatrix} \Lambda & \mathsf{D}\,\Lambda \\ \mathsf{Hom}_\Lambda(\mathsf{D}\,\Lambda,\Lambda) & \Lambda \end{smallmatrix}\right)$. Recall that for the idempotent element $e_\Lambda = \left(\begin{smallmatrix} 1_\Lambda & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \Gamma$ we write $\operatorname{gl.dim}_{\Gamma/\Gamma e_\Lambda\Gamma}\Gamma = \sup\{\mathsf{pd}_\Gamma X \mid X \in \mathsf{mod}\text{-}\Gamma/\Gamma e_\Lambda \Gamma\}$.

Corollary 5.2.10. *Let $\Lambda$ be an Artin algebra and $\Gamma = \mathsf{End}_\Lambda(\Lambda \oplus \mathsf{D}\,\Lambda)$. Then:*

$$\operatorname{rep.dim}\Lambda \;\leq\; \operatorname{gl.dim}\Lambda + \operatorname{gl.dim}_{\Gamma/\Gamma e_\Lambda\Gamma}\Gamma + 1$$

*and*

$$\operatorname{gl.dim}_{\Gamma/\Gamma e_\Lambda\Gamma}\Gamma \;\leq\; \operatorname{gl.dim}\Gamma/\Gamma e_\Lambda\Gamma + \mathsf{pd}_\Gamma\,\Gamma/\Gamma e_\Lambda\Gamma$$

Proof. For an Artin algebra $\Lambda$ we always have $\operatorname{rep.dim}\Lambda \leq \operatorname{gl.dim}\mathsf{End}_\Lambda(\Lambda\oplus\mathsf{D}\,\Lambda)$ since $\Lambda\oplus\mathsf{D}\,\Lambda$ is a generator-cogenerator of $\mathsf{mod}\text{-}\Lambda$. By Example 1.1.8 we have the recollement $(\mathsf{mod}\text{-}\Gamma/\Gamma e_\Lambda\Gamma, \mathsf{mod}\text{-}\Gamma, \mathsf{mod}\text{-}e_\Lambda\Gamma e_\Lambda)$ of module categories and $\mathsf{mod}\text{-}\Lambda \simeq \mathsf{mod}\text{-}e_\Lambda\Gamma e_\Lambda$. Then the assertion follows from Proposition 2.2.5. □

The following consequence of Corollary 5.2.10, implied also by Proposition 2.2.5, is a result of Xi.

Corollary 5.2.11. *[126, Proposition 5.3] Let $\Lambda$ be an Artin algebra such that $\mathsf{Hom}_\Lambda(\mathsf{D}\,\Lambda,\Lambda) = 0$. Then:*

$$\operatorname{rep.dim}\Lambda \;\leq\; 2\operatorname{gl.dim}\Lambda + 1$$

Proof. Since $\mathsf{Hom}_\Lambda(\mathsf{D}\,\Lambda,\Lambda) = 0$ we have the recollement of module categories $(\mathsf{mod}\text{-}\Lambda, \mathsf{mod}\text{-}\left(\begin{smallmatrix} \Lambda & \mathsf{D}\,\Lambda \\ 0 & \Lambda \end{smallmatrix}\right), \mathsf{mod}\text{-}\Lambda)$. Since $\operatorname{gl.dim}\Gamma/\Gamma e_\Lambda\Gamma = \operatorname{gl.dim}\Lambda$ and $\mathsf{pd}_\Gamma\,\Gamma/\Gamma e_\Lambda\Gamma = 0$ then the result follows from Corollary 5.2.10. □

We have the following consequence of Corollaries 2.4.2 and 2.2.17 applied to recollements of module categories of the form $(\mathsf{mod}\text{-}\Gamma/\Gamma e_\Lambda\Gamma, \mathsf{mod}\text{-}\Gamma, \mathsf{mod}\text{-}\Lambda)$.

Corollary 5.2.12. *Let $\Lambda$ be an Artin algebra and $\Gamma = \mathsf{End}_\Lambda(\Lambda \oplus \mathsf{D}\,\Lambda)$.*

(i) *If the natural map $\Gamma \longrightarrow \Gamma/\Gamma e_\Lambda\Gamma$ is a homological epimorphism, i.e. $\Gamma e_\Lambda\Gamma \in \mathcal{X}_\infty$, then:*

$$\operatorname{rep.dim}\Lambda \;\leq\; \mathsf{pd}_\Gamma\,\Gamma/\Gamma e_\Lambda\Gamma + \max\{\mathsf{pd}_{\Gamma^{\mathsf{op}}}\,\Gamma/\Gamma e_\Lambda\Gamma + \operatorname{gl.dim}\Gamma/\Gamma e_\Lambda\Gamma, \operatorname{gl.dim}\Lambda\}$$

(ii) *If the idempotent ideal $\Gamma e_\Lambda\Gamma \in \mathsf{proj}\,\Gamma$ and $\Gamma e_\Lambda\Gamma \in \mathsf{proj}\,\Gamma^{\mathsf{op}}$, then:*

$$\operatorname{rep.dim}\Lambda \;\leq\; \max\{\operatorname{gl.dim}\Gamma/\Gamma e_\Lambda\Gamma + 2, \operatorname{gl.dim}\Lambda + 1\}$$

REMARK 5.2.13. In Corollaries 5.2.10 and 5.2.12 we used the recollement of module categories $(\mathsf{mod}\text{-}\Gamma/\Gamma e_\Lambda\Gamma, \mathsf{mod}\text{-}\Gamma, \mathsf{mod}\text{-}\Lambda)$ defined by the idempotent $e_\Lambda = \left(\begin{smallmatrix} 1_\Lambda & 0 \\ 0 & 0 \end{smallmatrix}\right)$ of $\Gamma$. Note that we have also the idempotent $e'_\Lambda = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_\Lambda \end{smallmatrix}\right)$ of $\Gamma$. Hence we have another recollement $(\mathsf{mod}\text{-}\Gamma/\Gamma e'_\Lambda\Gamma, \mathsf{mod}\text{-}\Gamma, \mathsf{mod}\text{-}\Lambda)$ which is different from the first one since the categories $\mathsf{mod}\text{-}\Gamma/\Gamma e'_\Lambda\Gamma$ and $\mathsf{mod}\text{-}\Gamma/\Gamma e_\Lambda\Gamma$ are not equivalent in general. Using the recollement defined by the idempotent element $e'_\Lambda$ we obtain analogous results; details are left to the reader.

COROLLARY 5.2.14. *Let $\Lambda$ be an Artin algebra with* $\mathsf{rep.dim}\,\Lambda = \mathsf{gl.dim}\,\mathsf{End}_\Lambda\,\Delta$, *where $\Delta = \mathsf{End}_\Lambda(\Lambda \oplus X)$ for an Auslander generator $\Lambda \oplus X$. Let $e_\Lambda = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_{\mathsf{End}_\Lambda(X)} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be the induced idempotents of $\Delta$.*

(i) *If the $\Delta$-module $\Delta e_\Lambda\Delta$ is projective, then:*

$$\mathsf{max}\{\mathsf{gl.dim}\,\Lambda, \mathsf{gl.dim}\,\Delta/\Delta e_\Lambda\Delta\} < \infty$$

(ii) *If the $\Delta$-module $\Delta e_{\mathsf{End}_\Lambda(X)}\Delta$ is projective, then:*

$$\mathsf{max}\{\mathsf{gl.dim}\,\mathsf{End}_\Lambda(X), \mathsf{gl.dim}\,\Delta/\Delta e_{\mathsf{End}_\Lambda(X)}\Delta\} < \infty$$

PROOF. As before we view the endomorphism Artin algebra $\mathsf{End}_\Lambda(\Lambda \oplus X)$ as the matrix algebra $\Delta = \left(\begin{smallmatrix} \Lambda & X \\ \mathsf{Hom}_\Lambda(X,\Lambda) & \mathsf{End}_\Lambda(X) \end{smallmatrix}\right)$. Then we have the recollements of module categories $(\mathsf{mod}\text{-}\Delta/\Delta e_\Lambda\Delta, \mathsf{mod}\text{-}\Delta, \mathsf{mod}\text{-}\Lambda)$ and $(\mathsf{mod}\text{-}\Delta/\Delta e_{\mathsf{End}_\Lambda(X)}\Delta, \mathsf{mod}\text{-}\Delta, \mathsf{mod}\text{-}\mathsf{End}_\Lambda(X))$. Note that we have the equivalences $\mathsf{mod}\text{-}e_{\mathsf{End}_\Lambda(X)}\Delta e_{\mathsf{End}_\Lambda(X)} \simeq \mathsf{mod}\text{-}\mathsf{End}_\Lambda(X)$ and $\mathsf{mod}\text{-}e_\Lambda\Delta e_\Lambda \simeq \mathsf{mod}\text{-}\Lambda$. Then the result follows from Corollary 2.4.5. $\square$

**5.2.3. Finitistic Versus Representation Dimension.** Let $\Lambda$ be an Artin algebra. We call a pair $(\Gamma, e)$, consisting of an Artin algebra $\Gamma$ and an idempotent $e^2 = e \in \Gamma$, an **Auslander pair** for $\Lambda$, if $\Gamma$ has finite global dimension and $\Lambda \simeq e\Gamma e$. By a basic result of Auslander, see [10], any Artin algebra admits an Auslander pair and Dlab-Ringel showed that $\Gamma$ can be chosen to be quasi-hereditary, see [44]. In fact one can take $\Gamma = \mathsf{End}_\Lambda(\oplus_{i=1}^n \Lambda/\mathfrak{r}^i)$, where $\mathfrak{r}$ is the Jacobson radical of $\Lambda$ and $n = \ell\ell(\Lambda)$ its Loewy length. In this case Auslander proved that $\mathsf{gl.dim}\,\Gamma \leq \ell\ell(\Lambda)$. We call $(\Gamma, e)$, where $\Gamma = \mathsf{End}_\Lambda(\oplus_{i=1}^n \Lambda/\mathfrak{r}^i)$, the **natural Auslander pair** of $\Lambda$.

The main result of this section gives an interesting interplay between representation dimension and finitistic dimension, and presents situations where the finitistic dimension of an Artin algebra is finite.

THEOREM 5.2.15. *Let $\Lambda$ be an Artin algebra.*

(i) *Let $\Gamma = \mathsf{End}_\Lambda(\Lambda \oplus \mathsf{D}\,\Lambda)$ and assume that the Nakayama functor $\nu = \mathsf{D}\,\Lambda \otimes_\Lambda -$ has locally finite homological dimension. Then:*

$$\mathsf{fin.dim}\,\Lambda \;\leq\; \mathsf{fin.dim}\,\Gamma + \mathsf{l.b.hom.dim}\,\nu$$

(ii) *Let $(\Gamma, e)$ be an Auslander pair for $\Lambda$. If the functor $\Gamma e \otimes_{e\Gamma e} -\colon$ $\mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}\Gamma$ has locally bounded homological dimension, then* $\mathsf{fin.dim}\,\Lambda < \infty$. *More precisely:*

$$\mathsf{fin.dim}\,\Lambda \;\leq\; \mathsf{gl.dim}\,\Gamma + \mathsf{l.b.hom.dim}\;\; \Gamma e \otimes_{e\Gamma e} - \;<\; \infty$$

*In particular let $(\mathsf{End}_\Lambda(\oplus_{i=1}^{\ell\ell(\Lambda)} \Lambda/\mathfrak{r}^i), e)$ be the natural Auslander pair of $\Lambda$. If the functor $\Gamma e \otimes_{e\Gamma e} -\colon$ $\mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}\Gamma$ has locally bounded homological dimension, then:*

$$\mathsf{fin.dim}\,\Lambda \;\leq\; \ell\ell(\Lambda) + \mathsf{l.b.hom.dim}\;\; \Gamma e \otimes_{e\Gamma e} - \;<\; \infty$$

(iii) *Let* $\Lambda \oplus X$ *be an Auslander generator of* $\Lambda$. *If the functor* $\mathsf{Hom}_\Lambda(X, \Lambda) \otimes_\Lambda -\colon \mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}\Gamma$ *has locally bounded homological dimension, then*:

$$\mathsf{fin.dim}\,\Lambda \;\leq\; \mathsf{rep.dim}\,\Lambda + \mathsf{l.b.hom.dim}\ \mathsf{Hom}_\Lambda(X, \Lambda) \otimes_\Lambda - \;<\; \infty$$

Proof. Part (i) follows from Corollary 2.4.10. Part (ii) follows from Theorem 2.3.2 applied to the recollement induced by an Auslander pair, see Example 1.1.11. For part (iii), let $\Delta = \mathsf{End}_\Lambda(\Lambda \oplus X)$. Since $\mathsf{gl.dim}\,\Delta := n < \infty$ it follows that $\mathsf{fin.dim}\,\Delta = \mathsf{rep.dim}\,\Lambda = \mathsf{gl.dim}\,\Delta$. Therefore if we view $\Delta$ as a Morita ring which is an Artin algebra, as in Corollary 5.2.14, then the result follows from Corollary 2.4.10. Hence if the locally bounded homological dimension of the functor $\mathsf{Hom}_\Lambda(X, \Lambda) \otimes_\Lambda -$ is $\mathsf{l.b.hom.dim}\ \mathsf{Hom}_\Lambda(X, \Lambda) \otimes_\Lambda - = m$, then we infer that $\mathsf{fin.dim}\,\Lambda \leq n + m < \infty$. □

It follows from Theorem 5.2.15 that an Artin algebra $\Lambda$ has $\mathsf{fin.dim}\,\Lambda < \infty$, provided that one of the following conditions hold:

(i) There exists a summand $X$ of an Auslander generator such that the functor $\mathsf{Hom}_\Lambda(X, \Lambda) \otimes_\Lambda -$ has locally bounded homological dimension.

(ii) The natural quasi-hereditary algebra $\Gamma$ and the idempotent $e \in \Gamma$ associated to $\Lambda$ has the property that the functor $\Gamma e \otimes_{e\Gamma e} -$ has locally bounded homological dimension.

**ABSTRACT**: In this thesis we investigate homological invariants arising in the representation theory of Artin algebras. The main focus of our study is on the representation and finitistic dimension of Artin algebras, the class of Cohen-Macaulay modules and the Rouquier dimension of triangulated categories. The proper conceptual framework, from our perspective, for this study is the general setting of recollements of abelian categories, a concept which is fundamental in algebra, geometry and topology, and the closely related omnipresent class of Morita rings. Our aim is to investigate homological aspects of recollements of abelian categories and to study Morita rings in the context of Artin algebras, concentrating mainly at representation-theoretic and homological aspects. Moreover we classify recollements of abelian categories whose terms are module categories, thus solving a conjecture by Kuhn. Our interest in recollements is motivated from questions and problems on representation and finitistic dimension of Artin algebras and the interrelation between them. On the other hand our interest in Morita rings is motivated by the frequent occurrence of this class of matrix rings in the representation theory of Artin algebras and elsewhere, and the interpretation of their module categories via suitable recollements.

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