UNIVERSITY OF IOANNINA

Department of Physics

Master Thesis

Heterotic string theories with reduced–rank non–abelian gauge symmetry.

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September 25, 2026

Abstract

The aim of this thesis is to study heterotic string theory vacua characterized a rank reduction of the non-abelian gauge symmetry. The study of these theories reduces to the analysis of two-dimensional conformal field theories—involving bosonic and fermionic fields—defined on Riemann surfaces with torus topology. Specifically, the work focuses on vacuum constructions using real fermions on the worldsheet in cases where the charge lattice construction, commonly found in the literature, is not applicable. Indeed, in standard vacuum constructions within the heterotic string framework, free fermionic fields can be paired into complex fields, directly leading to a representation of current operators from which the current algebra and corresponding charges can be calculated. This particularly elegant realization of operator product expansions (OPEs) and the current algebra breaks down if the boundary conditions of the free fermions do not respect the complex structure defined by the choice of currents.

This work begins with a review of two-dimensional Conformal Field Theory (CFT) on the torus, focusing on the construction of partition functions as traces over the theory's individual sectors (Hilbert spaces). Subsequently, the construction and analysis of simple free bosonic theories on the torus are presented, encompassing both non–compact and compact internal dimensions. The discussion extends to fermionic theories involving complex fermions, focusing on periodic and more general spin structures components essential for constructing string theory solutions. This formalism is applied to ten-dimensional superstring theories, where modular invariance and GSO projections ensure the consistency of the spectrum. Finally, we examine the construction of string models with rank reduction, investigating the equivalence of this construction to formulations using real fermions and to orbifold constructions.

Περίληψη

Στόχος της εργασίας είναι η μελέτη κενών της ετεροτικής Θεωρίας Χορδών με τη χαρακτηριστική ιδιότητα μείωσης τάξης (rank reduction) της μη αβελιανής συμμετρίας βαθμίδας. Η μελέτη των θεωριών αυτών ανάγεται στην ανάλυση δισδιάστατων συμμόρφων θεωριών μποζονικών και φερμιονικών πεδίων που ζουν σε επιφάνειες Riemann με την τοπολογία του τόρου. Συγκεκριμένα, η εργασία επικεντριώνεται σε κατασκευές κενών με χρήση μεθόδων πραγματικών φερμιονίων επί της κοσμικής επιφάνειας σε περιπτώσεις όπου η μελέτη μέσω πλεγμάτων φορτίου (charge lattice construction), που συναντάται συνήθως στη βιβλιογραφία, δεν είναι εφαρμόσιμη. Πράγματι, στις συνήθεις κατασκευές κενών στο πλαίσιο των ετεροτικών χορδών, τα ελεύθερα φερμιονικά πεδία μπορούν να συνδυασθούν ανά ζεύγη σε μιγαδικά πεδία, πράγμα που οδηγεί άμεσα σε μια αναπαράσταση των τελεστών ρεύματος, εκ των οποίων δύνανται να υπολογισθεί η άλγεβρα ρευμάτων καθώς και τα αντίστοιχα φορτία. Η ιδιαιτέρως κομψή αυτή υλοποίηση των αναπτυγμάτων γινομένου τελεστών (OPEs), και της άλγεβρας ρευμάτων καταρρέει στην περίπτωση που οι συνοριακές συνθήκες των ελευθέρων φερμιονίων δεν σέβονται τη μιγαδική δομή που ορίζει η επιλογή των ρευμάτων.

Η εργασία αυτή ξεκινά με μια ανασκόπηση της δισδιάστατης Σύμμορφης Θεωρίας Πεδίου (CFT) στον τόρο, με έμφαση στην κατασκευή των συναρτήσεων επιμερισμού ως ίχνη στους επιμέρους τομείς της θεωρίας (χώρους Hilbert). Στη συνέχεια, γίνεται κατασκευή και ανάλυση απλών ελευθέρων μποζονικών θεωριών στον τόρο, συμπεριλαμβανομένων μη συμπαγών και συμπαγών εσωτερικών διαστάσεων. Η συζήτηση επεκτείνεται στις φερμιονικές θεωρίες με μιγαδικά φερμιόνια, εστιάζοντας σε περιοδικές και πιο γενικές δομές σπιν, συστατικά τα οποία είναι απαραίτητα για την κατασκευή λύσεων της Θεωρίας Χορδών. Ο φορμαλισμός εφαρμόζεται στις δεκαδιάστατες θεωρίες υπερχορδών, όπου η modular invariance και οι προβολές GSO εξασφαλίζουν τη συνέπεια του φάσματος. Τέλος, εξετάζουμε την κατασκευή μοντέλων χορδών με rank reduction, διερευνώντας την ισοδυναμία της κατασκευής με πραγματικά φερμιόνια καθώς και ως orbifold.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my supervisor, Ioannis Florakis, for accepting me as his student and for giving me the opportunity to work on this topic. I am sincerely grateful for the time, patience, and guidance you have given me throughout the course of this thesis, and for the many hours spent discussing both technical details and broader ideas. I greatly appreciate everything you have taught me, not only about the subject itself, but also about how to approach problems, structure ideas, and communicate them clearly. I am also particularly thankful for your openness to exploring topics beyond the immediate scope of the thesis, which made this experience especially enriching.

I also want to thank my friends Maria, Ermioni and Anastasia, who have somehow managed to put up with me since the very beginning of undergrad. So much has changed since those first years, but somehow we are still here, with the same ability to turn absolutely nothing into a three-hour conversation and to make even the most ordinary moments memorable. Thank you for all the laughs, the countless stories, the stupid moments, the serious ones, and everything in between.

Finally, I want to thank my family, and especially my parents and my siblings, for being there for me through all of this and, really, through everything that came before it. To my sister Dimitra, for all the advice I never asked for but usually needed; to my brother Antonis, for turning sibling rivalry into a lifelong competition; and to my other brother Dimitris, for providing enough chaos over the years to give me a practical introduction to the subject. Lastly, to my parents, thank you for literally everything. I would not be where I am today without you.

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Chapter 1

Introduction

We begin this introductory chapter with a brief review of the bosonic string, starting from the relativistic point particle and proceeding through the classical formulation of the string to its quantization. The main focus of this thesis is on closed, orientable strings.

1.1 Relativistic Point Particle

A useful starting point for the formulation of string theory is the relativistic point particle. Consider a particle of mass m propagating in a D-dimensional Minkowski spacetime with metric $\eta_{\mu\nu} = \text{diag}(-,+,\ldots,+)$. The trajectory of the particle is described by $x^\mu(\tau)$, where τ is an arbitrary parameter on the worldline. The action is proportional to the proper length of the worldline and is therefore given by

$$S = -m\int_{\tau_1}^{\tau_2} d\tau\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \tag{1.1}$$

where a dot denotes differentiation with respect to τ. The minus sign is chosen such that the correct sign of the kinetic term is recovered in the non-relativistic limit. Since τ is arbitrary, the action is invariant under reparameterizations of the worldline. Under a transformation $\tau \to \widetilde{\tau}(\tau)$, the Jacobian arising from the integration measure is cancelled by the corresponding transformation of the square-root factor. Although the square-root form has a direct geometrical interpretation, it is inconvenient for calculations and becomes trivial in the massless limit. An equivalent formulation is obtained by introducing an auxiliary field $e(\tau)$, known as the einbein

$$\widehat{S} = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau\left(e^{-1}\dot{x}^2 - em^2\right) \tag{1.2}$$

The einbein does not represent an additional physical degree of freedom since its value is determined by its equation of motion. Varying the action with respect to $e(\tau)$ gives

$$\frac{\delta\widehat{S}}{\delta e} = 0 \quad \Longrightarrow \quad -\frac{1}{2}e^{-2}\dot{x}^2 - \frac{1}{2}m^2 = 0 \tag{1.3}$$

or equivalently

$$\dot{x}^2 + e^2m^2 = 0 \tag{1.4}$$

Solving this equation for e and substituting the result into the action reproduces the square-root formulation. The two actions are therefore classically equivalent. The auxiliary-field formulation is nevertheless more convenient because it remains meaningful for $m = 0$, eliminates the square root and has a structure

that will be directly analogous to the Polyakov action for the string. The reparameterization invariance of the particle action is reflected in the Hamiltonian formulation by the mass-shell constraint

$$\phi = p^2 + m^2 = 0 \tag{1.5}$$

This is a first-class constraint since it generates the gauge symmetry associated with reparameterizations. Infinitesimally, under $\tau \to \tau + \xi(\tau)$, the fields transform as

$$\delta x^\mu = -\xi \partial_\tau x^\mu \tag{1.6a}$$
$$\delta e = -\partial_\tau(\xi e) \tag{1.6b}$$

The connection between reparameterization invariance and first-class constraints will have a direct analogue in the string theory.

1.2 Classical Bosonic String

The relativistic string is obtained by generalizing the point particle to an extended one-dimensional object. While a point particle traces out a one-dimensional worldline as it propagates through spacetime, a string sweeps out a two-dimensional surface known as the worldsheet. We introduce the worldsheet coordinates $\sigma^a = (\tau, \sigma)$ with $0 \leq \sigma \leq \pi$ and describe the embedding of the worldsheet into the D-dimensional target spacetime by the functions $X^\mu(\tau, \sigma)$. The direct generalization of the relativistic particle action is the Nambu–Goto action, which is proportional to the proper area of the worldsheet

$$S_{\text{NG}} = -T \int d^2\sigma \sqrt{-\det \gamma_{ab}} \tag{1.7}$$

where T is the string tension and γ_{ab} is the induced metric on the worldsheet

$$\gamma_{ab} = \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{1.8}$$

For the choice $\sigma \in [0, \pi]$, the action can be written explicitly as

$$S_{\text{NG}} = -T \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \sqrt{-(\dot{X})^2 (X')^2 + (\dot{X} \cdot X')^2} \tag{1.9}$$

The square root is real when the induced worldsheet metric has Lorentzian signature. This requirement has a direct physical interpretation since the worldsheet must contain both timelike and spacelike directions. At a point on the worldsheet, consider the tangent vector

$$U^\mu = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma} \tag{1.10}$$

Its norm is

$$U^2 = \lambda^2 (X')^2 + 2\lambda (\dot{X} \cdot X') + \dot{X}^2 \tag{1.11}$$

For the tangent space to contain both timelike and spacelike directions, this quadratic expression must have real roots. Its discriminant therefore satisfies

$$(\dot{X} \cdot X')^2 - \dot{X}^2 (X')^2 \geq 0 \tag{1.12}$$

which is precisely the condition required for the determinant of the induced metric to have the appropriate sign

The Nambu–Goto action is invariant under global Poincaré transformations of the target spacetime

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu \tag{1.13}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and a^μ is a constant translation. In addition, the action is invariant under arbitrary reparameterizations of the worldsheet coordinates

$$\sigma^a \to \tilde{\sigma}^a(\sigma) \tag{1.14}$$

This local symmetry corresponds to worldsheet diffeomorphism invariance and represents a gauge redundancy of the theory. The equations of motion follow from varying the action with respect to X^μ. Introducing the canonical worldsheet momentum densities

$$\Pi^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \qquad \Pi^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} \tag{1.15}$$

the variation can be written as

$$\delta S = -T \int d^2\sigma \left[\partial_\tau \left(\delta X^\mu \Pi^\tau_\mu \right) + \partial_\sigma \left(\delta X^\mu \Pi^\sigma_\mu \right) - \delta X^\mu \left(\partial_\tau \Pi^\tau_\mu + \partial_\sigma \Pi^\sigma_\mu \right) \right] \tag{1.16}$$

The first two terms are boundary contributions and vanish when the appropriate boundary conditions are imposed. Since δX^μ is otherwise arbitrary in the interior of the worldsheet, the equation of motion becomes

$$\partial_\tau \Pi^\tau_\mu + \partial_\sigma \Pi^\sigma_\mu = 0 \tag{1.17}$$

A more convenient formulation of the string action is obtained in analogy with the auxiliary-field formulation of the relativistic particle. We introduce an independent worldsheet metric $g_{ab}(\tau, \sigma)$ and write the Polyakov action as

$$S_{\rm P} = -\frac{T}{2} \int d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu \tag{1.18}$$

The metric g_{ab} is initially independent of the induced metric γ_{ab} and acts as an auxiliary field. Varying the Polyakov action with respect to X^μ gives

$$\partial_a \left(\sqrt{-g} g^{ab} \partial_b X^\mu \right) = 0 \tag{1.19}$$

while variation with respect to g^{ab} gives

$$\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X_\mu = 0 \tag{1.20}$$

These equations can be expressed in terms of the worldsheet energy-momentum tensor

$$T_{ab} = -\frac{4\pi}{\sqrt{-g}} \frac{\delta S_{\rm P}}{\delta g^{ab}} \tag{1.21}$$

so that the metric equation of motion becomes

$$T_{ab} = 0 \tag{1.22}$$

The vanishing of the energy-momentum tensor therefore represents the set of constraints that accompany the dynamical equations for X^μ. The metric equation of motion implies that the intrinsic metric g_{ab} and the induced metric γ_{ab} differ only by a conformal factor

$$g_{ab} = f(\tau, \sigma) \gamma_{ab} \tag{1.23}$$

Substitution of this relation into the Polyakov action removes the conformal factor and reproduces the Nambu–Goto action. The two formulations are consequently classically equivalent. The Polyakov action has an additional local symmetry known as Weyl invariance. Under a Weyl transformation the embedding coordinates remain unchanged while the worldsheet metric transforms according to

$$X^\mu \to X^\mu \tag{1.24}$$

$$g_{ab} \to \Omega^2(\tau,\sigma) g_{ab} \tag{1.25}$$

Infinitesimally, with $\Omega^2 = e^{2\phi}$, this becomes

$$\delta g_{ab} = 2\phi(\tau,\sigma) g_{ab} \tag{1.26}$$

The combination of diffeomorphism and Weyl invariance allows the worldsheet metric to be greatly simplified. In two dimensions, a symmetric metric contains three independent components. Two can be removed using diffeomorphism invariance and the remaining conformal factor can be eliminated using Weyl invariance. The metric can therefore be brought to conformal gauge

$$g_{ab} = e^{2\phi}\eta_{ab} \tag{1.27}$$

and subsequently to

$$g_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.28}$$

This special role of two dimensions is essential to string theory since the available local symmetries are precisely sufficient to remove all local degrees of freedom of the worldsheet metric. The boundary conditions determine which classical configurations are allowed. For a closed string there are no endpoints and the embedding coordinates must be periodic

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi) \tag{1.29}$$

For an open string the endpoints are located at $\sigma = 0$ and $\sigma = \pi$. The vanishing of the boundary contribution requires either Dirichlet or Neumann boundary conditions. Dirichlet conditions fix the position of an endpoint in a given spacetime direction

$$\delta X^\mu(\tau,0) = \delta X^\mu(\tau,\pi) = 0 \tag{1.30}$$

whereas Neumann conditions allow the endpoint to move freely and require the momentum flux through the endpoint to vanish

$$\Pi^\sigma_\mu(\tau,0) = \Pi^\sigma_\mu(\tau,\pi) = 0 \tag{1.31}$$

The choice between these conditions can be made independently for each spacetime direction. Consequently, an open string may satisfy Neumann conditions in some directions and Dirichlet conditions in others. The hypersurface on which the Dirichlet directions are fixed is interpreted as a D-brane. To obtain the classical solutions in a convenient form, we introduce lightcone coordinates on the worldsheet

$$\sigma^\pm = \tau \pm \sigma \tag{1.32}$$

In conformal gauge, the equations of motion become

$$\partial_+\partial_- X^\mu = 0 \tag{1.33}$$

with general solution

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \tag{1.34}$$

The functions X_L^μ and X_R^μ describe left- and right-moving excitations respectively. The constraints $T_{ab} = 0$ reduce to

$$T_{++} = -2\pi T \partial_+ X \cdot \partial_+ X = 0 \tag{1.35}$$

and

$$T_{--} = -2\pi T \partial_- X \cdot \partial_- X = 0 \tag{1.36}$$

For the convention

$$T = \frac{1}{2\pi\alpha'} \tag{1.37}$$

the closed-string mode expansion can be written as

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(a_n^\mu e^{-in\sigma^+} + \widetilde{a}_n^\mu e^{-in\sigma^-} \right) \tag{1.38}$$

where x^μ and p^μ denote the center-of-mass position and momentum. Reality of X^μ requires

$$(a_n^\mu)^* = a_{-n}^\mu \qquad (\widetilde{a}_n^\mu)^* = \widetilde{a}_{-n}^\mu \tag{1.39}$$

The constraints can be expanded in Fourier modes and expressed in terms of the Virasoro generators

$$L_n = \frac{1}{2} \sum_m a_{n-m} \cdot a_m \tag{1.40}$$

and

$$\widetilde{L}_n = \frac{1}{2} \sum_m \widetilde{a}_{n-m} \cdot \widetilde{a}_m \tag{1.41}$$

The classical constraints are therefore

$$L_n = \widetilde{L}_n = 0 \tag{1.42}$$

with the zero modes satisfying

$$a_0^\mu = \widetilde{a}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \tag{1.43}$$

The equality of the left- and right-moving excitation levels gives the level-matching condition

$$N = \widetilde{N} \tag{1.44}$$

The classical mass-shell relation follows from

$$M^2 = -p^\mu p_\mu \tag{1.45}$$

and takes the form

$$M^2 = \frac{4}{\alpha'} N = \frac{4}{\alpha'} \widetilde{N} \tag{1.46}$$

For an open string with Neumann boundary conditions there is only one independent oscillator sector and the corresponding classical relation is

$$M^2 = \frac{1}{\alpha'} N \tag{1.47}$$

1.3 Quantum Bosonic String

Quantization is obtained by promoting the classical Poisson brackets to commutators. With the convention

$$\{A, B\}_{\text{P.B.}} \to -i[A, B] \tag{1.48}$$

the oscillator algebra becomes

$$[a_m^\mu, a_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu} \tag{1.49}$$

and

$$[\widetilde{a}_m^\mu, \widetilde{a}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu} \tag{1.50}$$

while the left- and right-moving sectors commute

$$[a_m^\mu, \widetilde{a}_n^\nu] = 0 \tag{1.51}$$

The oscillators with $n > 0$ are interpreted as annihilation operators while those with $n < 0$ act as creation operators. The vacuum is therefore defined by

$$a_n^\mu|0; p\rangle = 0 \qquad n > 0 \tag{1.52}$$

In the covariant formulation the Virasoro generators must be normal ordered because the oscillator operators do not commute. They are defined by

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} : a_{n-m} \cdot a_m : \tag{1.53}$$

Normal ordering produces a constant contribution to L_0, which is denoted by a. Physical states are required to satisfy

$$L_n|\phi\rangle = 0 \qquad n > 0 \tag{1.54}$$

together with

$$(L_0 - a)|\phi\rangle = 0 \tag{1.55}$$

The latter equation is the quantum mass-shell condition. Covariant quantization therefore retains Lorentz covariance but requires the Virasoro constraints to remove the unphysical degrees of freedom. In particular, the Lorentzian target-space metric allows negative-norm states to appear in the unconstrained Fock space. These states are eliminated by imposing the physical-state conditions An alternative procedure is lightcone quantization, in which the constraints are solved before quantization. We introduce target-space lightcone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right) \tag{1.56}$$

so that

$$ds^2 = -2dX^+dX^- + dX^i dX^i \qquad i = 1, \ldots, D-2 \tag{1.57}$$

A residual gauge symmetry can be used to fix the coordinate X^+ to

$$X^+ = x^+ + \alpha' p^+ \tau \tag{1.58}$$

The constraints then determine X^- in terms of the transverse coordinates. For example,

$$\partial_+ X \cdot \partial_+ X = 0 \tag{1.59}$$

gives

$$\partial_+ X^- = \frac{1}{\alpha' p^+} \partial_+ X^i \partial_+ X^i \tag{1.60}$$

and an analogous expression holds for the right-moving sector. Thus, neither X^+ nor X^- represents an independent physical oscillator. The physical degrees of freedom are contained entirely in the $D-2$ transverse coordinates X^i. The lightcone formulation therefore makes the physical content of the string particularly transparent, although Lorentz invariance is no longer manifest. The quantum mass spectrum of the closed string is

$$M^2 = \frac{4}{\alpha'} (N - a) = \frac{4}{\alpha'} \left(\widetilde{N} - \widetilde{a} \right) \tag{1.61}$$

with

$$N = \sum_{i=1}^{D-2} \sum_{m>0} a^i_{-m} a^i_m \tag{1.62}$$

and

$$\widetilde{N} = \sum_{i=1}^{D-2} \sum_{m>0} \widetilde{a}^i_{-m} \widetilde{a}^i_m \tag{1.63}$$

For the open string the corresponding expression is

$$M^2 = \frac{1}{\alpha'} (N - a) \tag{1.64}$$

The normal-ordering constant is determined by the number of transverse oscillators and is

$$a = \widetilde{a} = \frac{D-2}{24} \tag{1.65}$$

Although the lightcone formulation simplifies the physical spectrum, the choice of lightcone coordinates breaks manifest Lorentz covariance. Requiring the quantum Lorentz generators to satisfy the Lorentz algebra without anomalies determines the critical dimension. The algebra closes consistently only when $D = 26$, which, together with $a = \widetilde{a} = 1$ gives the consistent quantum bosonic string. The same result can be obtained from the covariant formulation by demanding the absence of negative-norm physical states. The critical dimension is therefore not an assumption but a consequence of quantum consistency. The resulting spectrum can now be interpreted in terms of spacetime particles. For the closed string, the ground state satisfies $N = \widetilde{N} = 0$ and therefore has

$$M^2 = -\frac{4}{\alpha'} \tag{1.66}$$

The ground state is consequently tachyonic. At the first excited level, $N = \widetilde{N} = 1$, the state is massless and is obtained by acting with one left-moving and one right-moving creation operator

$$|\Omega^{ij}\rangle = a^i_{-1} \widetilde{a}^j_{-1} |0; p\rangle \tag{1.67}$$

In the critical dimension the indices i, j each take 24 values, giving 24^2 states before decomposition into irreducible representations. These states separate into a symmetric traceless component, an antisymmetric component and a trace component. The symmetric traceless part is identified with the graviton, the antisymmetric part with the two-form field $B_{\mu\nu}$ and the trace with the dilaton. Higher closed-string states are obtained by acting on the vacuum with increasingly excited combinations of left- and right-moving oscillators. They are massive and organize into representations of the corresponding massive little group. For the open string, the ground state is again tachyonic, while the first excited state is massless and transforms as a vector. Higher oscillator levels produce massive states with increasingly rich spin content. If Dirichlet boundary conditions are imposed in some spacetime directions, additional scalar states appear, corresponding to fluctuations of the string endpoints in directions transverse to the D-brane.

Chapter 2

Conformal Field Theory and String Quantization

The discussion of conformal field theory (CFT) is broad and encompasses many interesting classical and quantum aspects of two–dimensional worldsheet theories. Keeping the discussion concise, we focus on those topics that are essential for the subsequent study of CFT on the torus. We firstly summarize some of the basic aspects of classical and quantum CFT in two dimensions, before introducing the spin-field formalism and discussing how it arises.

2.1 CFT Basics

A conformal field theory (CFT) is a two-dimensional quantum field theory invariant under transformations that preserve the metric up to a local scale factor. Under a Weyl transformation, the metric transforms as

$$g_{ab}(\sigma) \to e^{2\omega(\sigma)} g_{ab}(\sigma) \tag{2.1}$$

while an infinitesimal conformal transformation of the coordinates is written as

$$x^\mu \to x^\mu + \epsilon^\mu(x) \tag{2.2}$$

The requirement that the metric be preserved up to a Weyl rescaling gives the conformal Killing equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{2.3}$$

In two dimensions this equation becomes particularly restrictive on the complex coordinates

$$z = x^1 + ix^2, \qquad \bar{z} = x^1 - ix^2 \tag{2.4}$$

The conformal transformations then separate into holomorphic and antiholomorphic transformations,

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}) \tag{2.5}$$

The corresponding infinitesimal generators are

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{\,n+1}\partial_{\bar{z}} \tag{2.6}$$

which satisfy

$$[l_m, l_n] = (m-n)l_{m+n}, \tag{2.7}$$

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \qquad [l_m, \bar{l}_n] = 0 \tag{2.8}$$

The generators l_{-1}, l_0 and l_1, together with their antiholomorphic counterparts, form the globally defined conformal transformations on the sphere. The remaining generators describe local conformal transformations and are essential in two-dimensional CFT. The central object in a CFT is the stress-energy tensor. It is defined by varying the action with respect to the worldsheet metric,

$$T_{ab} = -\frac{4\pi}{\sqrt{-g}} \frac{\delta S}{\delta g^{ab}} \tag{2.9}$$

Classical Weyl invariance implies

$$T^a{}_a = 0 \tag{2.10}$$

In complex coordinates, conservation and tracelessness imply that the stress tensor separates into holomorphic and antiholomorphic components,

$$T_{zz} \equiv T(z), \qquad T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}) \tag{2.11}$$

The transformation properties of local operators are encoded in their operator product expansions. A primary operator $\mathcal{O}$ with conformal weights $(h, \bar{h})$ satisfies

$$T(z)\mathcal{O}(w,\bar{w}) \sim \frac{h}{(z-w)^2}\mathcal{O}(w,\bar{w}) + \frac{1}{z-w}\partial_w \mathcal{O}(w,\bar{w}) \tag{2.12}$$

and

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\mathcal{O}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}} \mathcal{O}(w,\bar{w}) \tag{2.13}$$

The stress tensor itself is not primary. Its OPE with itself is

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{2.14}$$

where c is the central charge of the holomorphic sector. The antiholomorphic sector is characterized independently by $\bar{c}$. Expanding the stress tensor in Laurent modes,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z) \tag{2.15}$$

and similarly for $\bar{T}$, gives the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \tag{2.16}$$

with an analogous antiholomorphic algebra. At the quantum level the trace of the stress tensor can acquire an anomalous contribution. On a curved two-dimensional background it is proportional to the scalar curvature,

$$\langle T^a{}_a \rangle = -\frac{c}{12} R \tag{2.17}$$

For string theory, Weyl invariance therefore requires cancellation of the total conformal anomaly between matter and ghost sectors. The reparametrization ghosts form an anticommuting (b,c) system with conformal weights $(2,-1)$ and central charge $c_{bc} = -26$. The corresponding cancellation condition will be used below when discussing string quantization.

2.2 Spin Field Formalism

The superstring extends the bosonic worldsheet theory by introducing fermionic fields $\psi^\mu(\sigma,\tau)$, $\widetilde{\psi}^\mu(\sigma,\tau)$. The action can be written schematically as

$$S = S_X + S_\psi, \tag{2.18}$$

where

$$S_\psi = \frac{i}{4\pi}\int d^2\sigma \left(\psi^\mu \rho^a \partial_a \psi_\mu + \widetilde{\psi}^\mu \rho^a \partial_a \widetilde{\psi}_\mu\right). \tag{2.19}$$

The fermionic equations of motion separate into left- and right-moving components. The resulting worldsheet theory possesses local supersymmetry, and after gauge fixing the superconformal ghosts (β,γ) appear in addition to the reparametrization ghosts. For a first-order system with conformal weights

$$h_b = \lambda, \qquad h_c = 1-\lambda \tag{2.20}$$

the reparametrization ghosts have $(h_b,h_c)=(2,-1)$, whereas the superconformal ghosts have $(h_\beta,h_\gamma) = \left(\frac{3}{2},-\frac{1}{2}\right)$. The latter are commuting fields even though they arise from fixing a fermionic gauge symmetry. The worldsheet fermions admit two consistent boundary conditions. In the Neveu–Schwarz sector they are antiperiodic,

$$\psi^\mu(\sigma+2\pi) = -\psi^\mu(\sigma) \tag{2.21}$$

while in the Ramond sector they are periodic,

$$\psi^\mu(\sigma+2\pi) = +\psi^\mu(\sigma) \tag{2.22}$$

Consequently, the fermionic modes are half-integer moded in the NS sector, $r\in\mathbb{Z}+\frac{1}{2}$, and integer moded in the Ramond sector, $r\in\mathbb{Z}$. Their oscillator algebra is

$$\{\psi^\mu_r,\psi^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{2.23}$$

The crucial difference between the two sectors is the presence of the fermionic zero modes in the Ramond sector. Setting $r=s=0$ gives

$$\{\psi^\mu_0,\psi^\nu_0\} = \eta^{\mu\nu} \tag{2.24}$$

Defining

$$\Gamma^\mu = \sqrt{2}\,\psi^\mu_0 \tag{2.25}$$

we obtain

$$\{\Gamma^\mu,\Gamma^\nu\} = 2\{\psi^\mu_0,\psi^\nu_0\} = 2\eta^{\mu\nu} \tag{2.26}$$

Hence, the Ramond zero modes generate the target-space Clifford algebra,

$$\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu} \tag{2.27}$$

Let the Ramond ground states be denoted by $|A\rangle_R$. The action of the zero modes is then

$$\psi^\mu_0|A\rangle_R = \frac{1}{\sqrt{2}}(\Gamma^\mu)_A{}^B|B\rangle_R \tag{2.28}$$

Thus the Ramond ground states form a representation of the Clifford algebra and consequently transform as spacetime spinors. The spinorial nature of the Ramond sector is therefore a direct consequence of the

fermionic zero modes. The same structure has a direct interpretation in the two-dimensional CFT through the operator-state correspondence. A state is associated with a local operator inserted at the origin. In the Ramond sector the corresponding operator is a spin field,

$$|A\rangle_R = \lim_{z\to 0} S_A(z)|0\rangle_{\rm NS} \tag{2.29}$$

The spin field changes the boundary conditions of the fermions and therefore creates the Ramond ground state from the NS vacuum. To construct these operators explicitly, it is convenient to bosonize the worldsheet fermions. Consider four real fermions, $\psi^i\ , i = 1, ..., 4$. They can be combined into two complex fermions,

$$\Psi^1 = \frac{1}{\sqrt{2}}\left(\psi^1 + i\psi^2\right), \qquad \Psi^2 = \frac{1}{\sqrt{2}}\left(\psi^3 + i\psi^4\right) \tag{2.30}$$

with conjugates

$$\overline{\Psi}^1 = \frac{1}{\sqrt{2}}\left(\psi^1 - i\psi^2\right), \qquad \overline{\Psi}^2 = \frac{1}{\sqrt{2}}\left(\psi^3 - i\psi^4\right) \tag{2.31}$$

Their OPE is

$$\Psi^i(z)\overline{\Psi}^j(w) \sim \frac{\delta^{ij}}{z-w} \tag{2.32}$$

Each complex fermion can be represented in terms of a free boson,

$$\Psi^i(z) \sim e^{+iH_i(z)}, \qquad \overline{\Psi}^i(z) \sim e^{-iH_i(z)} \tag{2.33}$$

where the bosons satisfy

$$H_i(z)H_j(w) \sim -\delta_{ij}\ln(z-w) \tag{2.34}$$

The corresponding Cartan currents are

$$J_i(z) =: \overline{\Psi}^i\Psi^i :\sim i\partial H_i(z) \tag{2.35}$$

The charge carried by the exponential operators follows directly from the bosonic OPE. From

$$H(z)H(w) \sim -\ln(z-w) \tag{2.36}$$

one obtains

$$\partial H(z)e^{iqH(w)} \sim -\frac{iq}{z-w}e^{iqH(w)} \tag{2.37}$$

Consequently,

$$i\partial H(z)e^{iqH(w)} \sim \frac{q}{z-w}e^{iqH(w)} \tag{2.38}$$

and therefore

$$J(z)e^{iqH(w)} \sim \frac{q}{z-w}e^{iqH(w)} \tag{2.39}$$

The operators e^{+iH} and e^{-iH} thus have charges $+1$ and -1, respectively. The conformal dimension of these operators is obtained from the stress-energy tensor of the free boson,

$$T(z) = -\frac{1}{2} : \partial H \partial H : \tag{2.40}$$

Its OPE with an exponential operator is

$$T(z)e^{iqH(w)} \sim \frac{q^2/2}{(z-w)^2}e^{iqH(w)} + \frac{1}{z-w}\partial e^{iqH(w)} \tag{2.41}$$

Comparison with the defining OPE of a primary field gives

$$h[e^{iqH}] = \frac{q^2}{2}. \tag{2.42}$$

For $q = \pm 1$, this reproduces the conformal weight $h = 1/2$ of a complex fermion. The Ramond spin fields correspond instead to half-integer bosonic charges. For the two bosons introduced above, they are

$$S_{s_1 s_2}(z) = \exp\left[\frac{i}{2}\left(s_1 H_1(z) + s_2 H_2(z)\right)\right], \qquad s_1, s_2 = \pm 1 \tag{2.43}$$

The four possible spin fields are

$$S_{++} = e^{\frac{i}{2}(H_1+H_2)}, \qquad S_{+-} = e^{\frac{i}{2}(H_1-H_2)} \tag{2.44}$$

$$S_{-+} = e^{\frac{i}{2}(-H_1+H_2)}, \qquad S_{--} = e^{-\frac{i}{2}(H_1+H_2)} \tag{2.45}$$

Using the conformal dimension of an exponential operator, the dimension of a spin field is

$$h_S = \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] = \frac{1}{4} \tag{2.46}$$

More generally, for D real fermions there are $D/2$ bosons and

$$S_{\mathbf{s}} = \exp\left[\frac{i}{2}\sum_{I=1}^{D/2} s_I H_I\right], \qquad s_I = \pm 1 \tag{2.47}$$

The conformal weight is then

$$h_S = \frac{1}{2}\sum_{I=1}^{D/2}\left(\frac{s_I}{2}\right)^2 = \frac{D}{16} \tag{2.48}$$

The spinorial interpretation is particularly transparent from the action of the Cartan currents. For

$$S_{\mathbf{s}} = e^{\frac{i}{2}\sum_I s_I H_I} \tag{2.49}$$

the current OPE gives

$$J_I(z) S_{\mathbf{s}}(w) \sim \frac{s_I/2}{z-w} S_{\mathbf{s}}(w) \tag{2.50}$$

Thus the spin field carries the charge vector

$$\boldsymbol{\mu} = \left(\frac{s_1}{2}, \frac{s_2}{2}, \ldots\right) \tag{2.51}$$

For the four-dimensional example the four charge vectors are

$$S_{++}: \quad \left(\frac{1}{2}, \frac{1}{2}\right), \qquad S_{+-}: \quad \left(\frac{1}{2}, -\frac{1}{2}\right) \tag{2.52}$$

$$S_{-+}: \quad \left(-\frac{1}{2}, \frac{1}{2}\right), \qquad S_{--}: \quad \left(-\frac{1}{2}, -\frac{1}{2}\right) \tag{2.53}$$

These are precisely the weight vectors of the spinor representation.

The four spin fields can be separated into two sets of opposite chirality. For one conventional assignment,

$$S_\alpha = (S_{++}, S_{--})\,, \tag{2.54}$$

and

$$C_{\dot\alpha} = (S_{+-}, S_{-+}) \tag{2.55}$$

The precise labels depend on the convention used for chirality, but the physical statement is independent of this choice: the Ramond ground-state representation decomposes into two Weyl representations of opposite chirality.

The equivalence between the oscillator description and the CFT description becomes explicit in the fermion–spin-field OPE. The action of a fermionic field on a spin field changes its spinor weight by one of the fermionic weights and produces a square-root branch cut,

$$\psi^\mu(z) S_A(w) \sim \frac{1}{\sqrt{2}} \frac{(\Gamma^\mu)_A{}^B S_B(w)}{(z-w)^{1/2}} \tag{2.56}$$

The gamma matrices appearing in this OPE are the CFT realization of the Clifford algebra generated by the Ramond zero modes.
The two descriptions can therefore be summarized as

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu} \quad \Longleftrightarrow \quad \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{2.57}$$

while the corresponding state-operator relation is

$$|A\rangle_R \quad \Longleftrightarrow \quad S_A(z) \tag{2.58}$$

Thus the spacetime spinor representation of the Ramond sector is encoded directly in the operator algebra of the two-dimensional CFT.

For the superstring, the NS and R sectors must subsequently be subjected to the GSO projection. In the NS sector the projection removes the tachyonic vacuum and retains the appropriate fermion-number parity, while in the Ramond sector it selects a definite chirality. The resulting left- and right-moving sectors can then be combined into the Type IIA and Type IIB theories according to their relative Ramond chiralities.

The same worldsheet fermions also provide the starting point for the heterotic construction. In the heterotic string, the supersymmetric sector is combined with a bosonic sector whose additional degrees of freedom can be represented by internal fermions. Bosonization converts these fermionic degrees of freedom into a lattice of charges, making the connection between worldsheet CFT, spin fields and the internal charge lattice explicit. This provides the bridge to the gauge-group and charge lattice construction developed in the following sections.

Chapter 3

CFT on the Torus

3.1 Torus Basics

Before discussing the supersting in ten dimensions through its charge lattice, which can be extracted directly from the structure of the partition function, let us first examine the CFT on the torus in some more detail. The reason is that the perturbative formulation of string theory is ultimately organized as an expansion over Riemann surfaces. Even though the focus of this thesis is on closed, orientable strings, and therefore closed orientable worldsheet topologies, it should be noted that boundaries and crosscaps must be taken into account when considering more general classes of worldsheet surfaces. Hence, open or non–orientable strings have a perturbative expansion that consists of different Riemann surfaces. Although the modularity constraints associated with higher–genus surfaces will also play a role, it is therefore useful to discuss the torus in some detail, as the simplest genus–one surface, characterized by a vanishing Euler characteristic.

In this first section, we first briefly compare the sphere and the torus and highlight the origin of the distinction between them and, in particular, the emergence of a non trivial complex structure in the case of the torus. Combined with the zero Euler characteristic, this introduces two equivalent coordinate choices where the dependence on the complex structure is manifest either on the definition of the coordinates or the metric components. We then explain how the Teichmüller and moduli spaces are recovered starting from the Polyakov path integral. Before analyzing the path integral and the gauge redundancies that should eventually be modded out, let us first try to understand, even before studying the metric variation, what distinguishes the sphere from the torus. Recall that the sphere is a genus-zero surface with Euler characteristic two. The Gauss–Bonnet theorem relates the Euler characteristic to the total curvature of the surface. More precisely, for a Riemann surface Σ, we have

$$\chi = 2 - 2g = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{g} R \tag{3.1}$$

This means that the sphere has positive curvature while the torus, which has $\chi = 0$, admits a flat metric. This is not the end of the story, however. Flatness is a local statement, and even though in two dimensions this means that locally the metric can be written as $g_{ab} = \delta_{ab}$, the way in which these locally flat coordinates are globally identified makes all the difference between the two surfaces. The torus is indeed locally indistinguishable from the Euclidean plane, but the nontrivial information lies in the global identifications required to make the plane compact.

The first homotopy group of the sphere is $\pi_1(S^2) = 0$, which means that every closed curve on the sphere can be continuously contracted to a point. It is simply connected, and therefore there are no

non–contractible cycles whose global identifications can carry additional information. Once the local conformal redundancy is removed, there is no remaining non–contractible cycle that can encode further global data.

The torus, on the other hand, has $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, meaning that it possesses two independent non–contractible cycles. The universal covering space of the flat torus is the complex plane $\mathbb{C}$, and the torus may be obtained by quotienting this plane by two independent translations,

$$T^2 = \mathbb{C}/\Lambda, \qquad \Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\} \tag{3.2}$$

The local metric is still the same Euclidean metric for every choice of ω_1 and ω_2; what changes is the global identification of points. This is precisely the information that cannot be removed by local coordinate transformations or Weyl rescalings. Although all these metrics are locally flat, their complex structures need not be equivalent. Thus, in contrast to the sphere, the torus provides the first example where gauge fixing the worldsheet and following the Faddeev–Popov procedure does not leave a unique representative.

Let us try to understand this in more detail by explicitly examining the terms that contribute to the variation of the two-dimensional metric. As mentioned above, the Polyakov path integral expression for the bosonic closed string is

$$Z = \int \mathcal{D}g\mathcal{D}X e^{-S[g,X]} \tag{3.3}$$

However, performing this integral directly, as expected, gives an infinite result. This comes as no surprise since there is a gauge redundancy arising from the Weyl and diffeomorphism symmetries of the worldsheet. This is already apparent in the simplest Gaussian integral example, where we have

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-(x-y)^2} = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du e^{-u^2} = V \int_{-\infty}^{\infty} du e^{-u^2} \tag{3.4}$$

where $u = x - y$ and $v = x + y$, up to an irrelevant normalization of the Jacobian. The divergence arises because the integrand is independent of v. Indeed, the transformation

$$x \to x + a, \qquad y \to y + a, \qquad \forall, a \in \mathbb{R} \tag{3.5}$$

leaves the integrand invariant. Thus, the integral includes an infinite volume corresponding to this redundant direction.

To get rid of this extra redundancy and ensure that the path integral passes exactly once through each gauge orbit, we have to mod out this redundant freedom. In the simple Gaussian example, this means dividing by the volume associated with the non-physical variable v. In the Polyakov bosonic path integral, the analogous redundancy is generated by Weyl transformations and diffeomorphisms.

Let us now commence with the variation of the metric. We have a symmetric 2×2 matrix, which means that its variation can be decomposed into a trace and a traceless part. More precisely,

$$\delta g_{ab} = \frac{1}{2} g_{ab} \delta g + \delta g_{ab}^{\text{traceless}} \tag{3.6}$$

where $\delta g = g^{ab}\delta g_{ab}$. Out of the three independent components of the metric, one is determined by the trace part and is associated with Weyl rescaling, while the other two are contained in the traceless part of the variation. This is not quite enough, however, since some additional detail is needed: the traceless part contains not only the contribution coming from diffeomorphisms, but also the remaining conformal

deformations associated with the construction of Teichmüller space. First of all, it is straightforward to see that the first term is precisely associated with the Weyl rescaling freedom, since

$$g_{ab}(\sigma) \to g'_{ab}(\sigma) = e^{2\omega(\sigma)} g_{ab}(\sigma) \tag{3.7}$$

and by expanding $e^{2\omega} = 1 + 2\omega + \mathcal{O}(\omega^2)$ gives

$$\delta_\omega g_{ab} = 2\omega g_{ab} \tag{3.8}$$

Choosing $\omega = \frac{1}{4}\delta g$ therefore precisely reproduces the first term in the decomposition above. Now consider an infinitesimal coordinate transformation

$$\sigma^a \to \sigma^a + \xi^a(\sigma) \tag{3.9}$$

Taking into account that an infinitesimal diffeomorphism generated by a vector field $\xi^a(\sigma)$ acts on the metric according to $\delta_\xi g_{ab} = \mathcal{L}_\xi g_{ab}$, and using metric compatibility in terms of the covariant derivative, we obtain

$$\delta_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \tag{3.10}$$

Taking the trace gives $g^{ab}\delta_\xi g_{ab} = 2\nabla_a \xi^a$. Consequently, the pure-trace part of this diffeomorphism–induced variation is

$$\frac{1}{2} g_{ab} \left(g^{cd} \delta_\xi g_{cd} \right) = g_{ab} \nabla_c \xi^c \tag{3.11}$$

Subtracting this from the complete diffeomorphism variation gives its traceless part:

$$\delta_\xi g_{ab}^{\text{traceless}} = \delta_\xi g_{ab} - \frac{1}{2} g_{ab} g^{cd} \delta_\xi g_{cd} = \nabla_a \xi_b + \nabla_b \xi_a - g_{ab} \nabla_c \xi^c \tag{3.12}$$

Or following the notation discussed in [**kiritsis2019string**]

$$(P_1 \xi) * ab \equiv \nabla_a \xi_b + \nabla_b \xi_a - g_{ab} \nabla_c \xi^c \tag{3.13}$$

At this point, the traceless metric variations split into two qualitatively different classes. Some are of the form $P_1\xi$ and are therefore generated by diffeomorphisms. The remainder cannot be produced by an infinitesimal diffeomorphism and instead represents a genuine change of conformal structure. To identify this remainder, we need an inner product on the space of traceless symmetric tensors. Define

$$\langle h, k \rangle = \int_\Sigma d^2\sigma \sqrt{g} h_{ab} k^{ab} \tag{3.14}$$

We can now determine the adjoint $P_1^\dagger$. Take a traceless symmetric tensor h_{ab}. Then

$$\langle h, P_1 \xi \rangle = \int_\Sigma d^2\sigma, \sqrt{g} h^{ab} \left(\nabla_a \xi_b + \nabla_b \xi_a - g_{ab} \nabla_c \xi^c \right) \tag{3.15}$$

Because h^{ab} is traceless,$h^{ab} g_{ab} = 0$,the third term vanishes:

$$\langle h, P_1 \xi \rangle = \int_\Sigma d^2\sigma \sqrt{g} h^{ab} \left(\nabla_a \xi_b + \nabla_b \xi_a \right) \tag{3.16}$$

Since $h^{ab} = h^{ba}$, the two terms are identical, and therefore

$$\langle h, P_1 \xi \rangle = 2 \int_\Sigma d^2\sigma \sqrt{g} h^{ab} \nabla_a \xi_b \tag{3.17}$$

Now using the product rule

$$\nabla_a(h^{ab}\xi_b) = (\nabla_a h^{ab})\xi_b + h^{ab}\nabla_a\xi_b \tag{3.18}$$

we eventually recover

$$\langle h, P_1\xi\rangle = 2\int_\Sigma d^2\sigma\sqrt{g}\nabla_a\left(h^{ab}\xi_b\right) - 2\int_\Sigma d^2\sigma\sqrt{g}\left(\nabla_a h^{ab}\right)\xi_b \tag{3.19}$$

Because Σ is closed, it has no boundary and the first integral vanishes by Stokes' theorem, so

$$\langle h, P_1\xi\rangle = -2\int_\Sigma d^2\sigma\sqrt{g}\left(\nabla_a h^{ab}\right)\xi_b \tag{3.20}$$

Comparing this with the definition $\langle P_1^\dagger h, \xi\rangle = \langle h, P_1\xi\rangle$, we obtain

$$(P_1^\dagger h)^b = -2\nabla_a h^{ab} \tag{3.21}$$

Consequently, a traceless tensor h_{ab} is orthogonal to every diffeomorphism direction precisely when $P_1^\dagger h = 0$. Using the explicit expression for the adjoint, this condition becomes $\nabla_a h^{ab} = 0$. Thus, the non-gauge directions are characterized by the two conditions:

$$g^{ab}h_{ab} = 0 \tag{3.22a}$$

$$\nabla_a h^{ab} = 0 \tag{3.22b}$$

These are the transverse-traceless metric deformations. The explicit solution of these equations for the sphere and the torus can be found on appendix B. The complete infinitesimal decomposition can therefore be written as

$$\delta g_{ab} = 2\omega g_{ab} + (P_1\xi)_{ab} + \sum_I \delta m_I h^I_{ab} \tag{3.23}$$

The parameters m_I are coordinates on Teichmüller space. This also explains precisely where the Faddeev–Popov determinant enters. When the functional integral over metrics is rewritten in terms of the variables ω, ξ, and m_I, the change of variables has a nontrivial Jacobian. The operator controlling the diffeomorphism part of this Jacobian is precisely P_1. Schematically,

$$\mathcal{D}g = \mathcal{D}\omega\mathcal{D}\xi\prod_I dm_I \det{}' P_1 \tag{3.24}$$

The prime is essential. If $P_1\xi = 0$, then the corresponding vector field produces no traceless metric deformation. Such vector fields are conformal Killing vectors and therefore represent residual gauge transformations rather than Teichmüller directions. Meanwhile, the physical infinitesimal deformations are associated with $\ker P_1^\dagger$, so the two spaces have completely different meanings. For the sphere, $\ker P_1^\dagger = 0$, so $\mathcal{T}(S^2) =$ point. There are nevertheless conformal Killing vectors on the sphere, $\ker P_1 \neq 0$ that generate the residual conformal transformations, whose global group is $PSL(2,\mathbb{C})$. They are gauge zero modes and must not be confused with moduli. For the torus, in contrast, $\dim_{\mathbb{C}}\mathcal{T}(T^2) = 1$. We therefore introduce a single complex coordinate,

$$\tau = \tau_1 + i\tau_2, \qquad \tau_2 \geq 0 \tag{3.25}$$

The Teichmüller space of the torus is consequently the upper half-plane, $\mathcal{T}(T^2) = \mathbb{H}$. At this stage, however, we have not yet obtained the physical moduli space. The reason is that the Faddeev–Popov

analysis above removes diffeomorphisms connected *continuously* to the identity. The torus possesses additional diffeomorphisms that are not continuously connected to the identity. These are the so–called large diffeomorphisms. They act by changing the choice of basis of the two independent homology cycles. If A and B are the two cycles, a large diffeomorphism can send them to

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{3.26}$$

with $a, b, c, d \in \mathbb{Z},\ ad - bc = 1$. Thus, the group of orientation-preserving large diffeomorphisms acts as $SL(2, \mathbb{Z})$. Its action on the modulus is

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \tag{3.27}$$

In particular, the two generators can be taken to be

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \tau \longrightarrow \tau + 1 \tag{3.28a}$$

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \longrightarrow -\frac{1}{\tau} \tag{3.28b}$$

But there is one final subtlety. The matrices M and $-M$ act identically on τ, since

$$\frac{(-a)\tau + (-b)}{(-c)\tau + (-d)} = \frac{a\tau + b}{c\tau + d} \tag{3.29}$$

Therefore, the effective group acting on Teichmüller space is not $SL(2, \mathbb{Z})$ itself, but $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\pm I$. Consequently, the moduli space for the torus is

$$\mathcal{M}(T^2) = \frac{\mathcal{T}(T^2)}{PSL(2, \mathbb{Z})} = \frac{\mathbb{H}}{PSL(2, \mathbb{Z})} \tag{3.30}$$

The important conceptual distinction is now precise: Weyl transformations and diffeomorphisms connected to the identity appear directly in the infinitesimal decomposition of the metric and therefore in the Faddeev–Popov procedure. The modular group does not appear as another infinitesimal term in this decomposition. It appears afterwards, because gauge fixing leaves a discrete ambiguity in the choice of global coordinates, or equivalently in the choice of basis for the non-contractible cycles of the torus. This is also the origin of modular invariance in the torus partition function. Since points in $\mathbb{H}$ related by a modular transformation describe the same conformal geometry, the partition function must be invariant under

$$Z(\tau, \bar{\tau}) = Z\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) \tag{3.31}$$

Modular invariance is therefore not an additional arbitrary requirement, but a consequence of the fact that the worldsheet path integral should assign the same physical amplitude to different choices of lattice basis describing the same torus.

We are now ready to introduce the notation that will be used throughout and to describe the two equivalent ways of writing the torus metric. We begin with real coordinates (σ^1, σ^2), each taken to have unit periodicity,

$$\sigma^1 \sim \sigma^1 + 1 \tag{3.32a}$$

$$\sigma^2 \sim \sigma^2 + 1 \tag{3.32b}$$

The complex structure is then encoded in the metric. After fixing diffeomorphism and Weyl invariance, a convenient representative of the metric is

$$g_{ab} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \tag{3.33}$$

The corresponding line element is

$$ds^2 = \frac{1}{\tau_2}\left[(d\sigma^1)^2 + 2\tau_1\, d\sigma^1 d\sigma^2 + |\tau|^2 (d\sigma^2)^2\right] \tag{3.34}$$

The dependence of the metric on the two real parameters τ_1 and τ_2 makes explicit that the remaining information after gauge fixing is precisely the complex structure of the torus. The overall scale of the metric has been fixed by Weyl invariance, leaving only the two real parameters encoded in τ. An equivalent description is obtained by introducing a complex coordinate,

$$z = \sigma^1 + \tau\sigma^2, \;\; \bar{z} = \sigma^1 + \bar{\tau}\sigma^2 \tag{3.35}$$

If we require the determinant of the metric to be normalized to one, the metric can be rewritten as

$$ds^2 = \frac{1}{\tau_2}\, dz\, d\bar{z} \tag{3.36}$$

The metric is therefore manifestly conformally flat in complex coordinates, while the dependence on τ has been transferred from the metric components to the definition of the coordinate itself. The global identifications are particularly simple in these coordinates:

$$z \sim z + 1, \qquad z \sim z + \tau \tag{3.37}$$

Thus, the torus is obtained by quotienting the complex plane by the lattice generated by 1 and τ, $T^2 = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$. This makes the geometric meaning of τ particularly transparent: it specifies the shape of the fundamental parallelogram of the lattice and therefore determines the complex structure of the torus. The two descriptions of the metric are therefore completely equivalent. In the real-coordinate description, the coordinates have fixed periodicities and the complex structure appears explicitly in the metric components. The first description therefore makes the modulus explicit in the metric, whereas the second makes the complex structure manifest through the global identification of the complex plane. The geometric picture of the torus as a quotient of the complex plane by a lattice can be visualized in Figure 1. The fundamental parallelogram is spanned by the two lattice vectors 1 and τ, with opposite edges identified to form the torus. Translating this parallelogram by integer combinations of 1 and τ tiles the complex plane, with all points related by such lattice translations representing the same point on the torus.

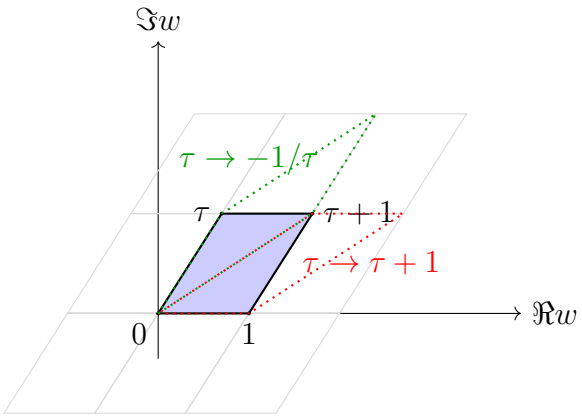


Figure 3.1: Lattice representation of the torus with fundamental and invariant parallelograms.

The fundamental domain $\mathcal{F}$ of the modular group, illustrated in Figure 2, provides a canonical representation of all inequivalent torus geometries. Each point $\tau \in \mathcal{F}$ corresponds to a distinct complex structure, and the metric on the torus can be expressed explicitly in terms of τ_1 and τ_2.

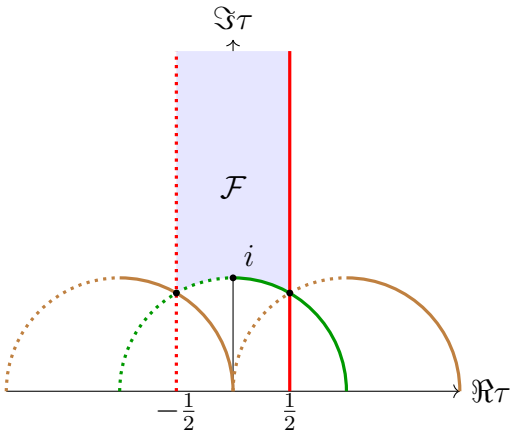


Figure 3.2: Fundamental domain $\mathcal{F}$ of T^2. Vertical red lines indicate the T identification, the green semicircle $|\tau| = 1$ shows the S identification, while brown semicircles illustrate additional modular images. Dotted and solid line segments indicate identified boundaries. The point i is the canonical modular fixed point.

3.2 Bosons on the Torus

3.2.1 Non-compact Periodic Boson

Trace method To compute the partition function we quantize the theory along the spatial cycle. The Hilbert space $\mathcal{H}$ consists of all states created by acting with oscillators on the vacuum and carrying continuous momentum in the non–compact target space directions. The torus partition function is defined as the trace over this Hilbert space

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\, e^{-2\pi\tau_2 H + 2\pi i \tau_1 P} \tag{3.38}$$

where H is the Hamiltonian generating Euclidean time translations and P is the momentum operator generating spatial translations on the worldsheet cylinder.

Equivalently, using the left and right moving Virasoro generators,

$$H = L_0 + \bar{L}_0 - \frac{c}{12}, \qquad P = L_0 - \bar{L}_0 \tag{3.39}$$

and introducing

$$q = e^{2\pi i \tau}, \;\; \bar{q} = e^{-2\pi i \bar{\tau}} \tag{3.40}$$

the partition function becomes

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right) \tag{3.41}$$

The trace runs over the Hilbert space of the theory defined on the spatial circle. The vacuum state corresponds to the lowest energy configuration and all other states are obtained by acting with oscillator creation operators. In the trace we sum over all these states weighted by their energy eigenvalues.

For a free boson the Hilbert space factorizes into a momentum sector and an oscillator sector. The states therefore take the schematic form

$$|p; \{N_n\}, \{\tilde{N}_n\}\rangle \tag{3.42}$$

Here p is the target–space momentum and $N_n, \tilde{N}_n$ are occupation numbers of left and right moving oscillators.

Scalar field spectrum For a non–compact free boson X^μ the Virasoro generators take the form

$$L_0 = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} nN_n, \qquad \bar{L}_0 = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} n\tilde{N}_n \tag{3.43}$$

The central charge for d free bosons is $c = d$. Substituting into the trace gives

$$Z(\tau, \bar{\tau}) = (q\bar{q})^{-d/24}\mathrm{Tr}\, q^{\frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} nN_n}\, \bar{q}^{\frac{\alpha'}{4}p^2 + \sum_{m=1}^{\infty} m\tilde{N}_m} \tag{3.44}$$

Explicitly writing the dependence on oscillators and momentum,

$$Z(\tau, \bar{\tau}) = (q\bar{q})^{-d/24}\mathrm{Tr}\, q^{\frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty}\sum_{\mu=1}^{d} nN_n^\mu}\, \bar{q}^{\frac{\alpha'}{4}p^2 + \sum_{m=1}^{\infty}\sum_{\mu=1}^{d} m\tilde{N}_m^\mu} \tag{3.45}$$

The trace now factorizes into a momentum contribution and an oscillator contribution.

Oscillator contribution For a single oscillator mode with occupation number $N_n = 0, 1, 2, \ldots$ we have

$$\sum_{N_n=0}^{\infty} q^{nN_n} = \frac{1}{1-q^n} \tag{3.46}$$

Taking the product over all modes and over the d bosonic directions gives

$$Z_{\mathrm{osc}} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^d(1-\bar{q}^n)^d} \tag{3.47}$$

Including the vacuum factor $(q\bar{q})^{-d/24}$ we obtain

$$Z_{\mathrm{osc}} = \frac{1}{|\eta(\tau)|^{2d}}, \tag{3.48}$$

where the Dedekind eta function is defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{3.49}$$

In the derivation above we have taken into account the theorem for the generating function of partitions introduced by Euler. A proof can be found in [1].

Theorem 1 (Euler's Partition Generating Function)**.** *Let $p(n)$ denote the number of partitions of the integer $n \geq 0$. Then, for all complex numbers x with $|x| < 1$ (i.e., inside the unit disk), the generating function of $p(n)$ is*

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}\frac{1}{1-x^k}$$

The infinite product converges absolutely for $|x| < 1$, and equality with the power series holds term by term.

Momentum integral Since the boson is non–compact, the momentum p^μ is continuous. The trace over the momentum sector becomes an integral

$$\int \frac{d^d p}{(2\pi)^d} q^{\frac{\alpha'}{4}p^2} \bar{q}^{\frac{\alpha'}{4}p^2} \tag{3.50}$$

where using

$$q^{\frac{\alpha'}{4}p^2} \bar{q}^{\frac{\alpha'}{4}p^2} = e^{-\pi\alpha'\tau_2 p^2}, \tag{3.51}$$

the Gaussian integral gives

$$\int d^d p\, e^{-\pi\alpha'\tau_2 p^2} = \frac{1}{(\alpha'\tau_2)^{d/2}} \tag{3.52}$$

Combining the oscillator and momentum contributions we finally obtain the torus partition function for d non–compact free bosons

$$Z(\tau,\bar{\tau}) = \frac{1}{(4\pi^2\alpha'\tau_2)^{d/2}} \frac{1}{|\eta(\tau)|^{2d}} \tag{3.53}$$

This expression exhibits the characteristic dependence on the modular parameter: the factor $\tau_2^{-d/2}$ originates from the Gaussian integration over continuous momenta, while the infinite product encoded in the Dedekind η function arises from summing over the oscillator excitations of the scalar field.

Path Integral method

Consider a free scalar field X defined on a Euclidean torus

$$\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),$$

The action of the free boson is then

$$S[X] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2 z\, \partial X \bar{\partial} X \tag{3.54}$$

and the partition function is formally defined by

$$Z = \int \mathcal{D}X\, e^{-S[X]} \tag{3.55}$$

The equation of motion obtained from the action is

$$\partial\bar{\partial} X = 0 \tag{3.56}$$

which is equivalent to the Laplace equation $\Delta X = 0$.

Classical and Quantum Decomposition To evaluate the path integral, we decompose the field into a classical solution and quantum fluctuations

$$X = X_{\text{cl}} + X_q. \tag{3.57}$$

On the torus the only harmonic functions satisfying periodic boundary conditions are constants. Hence

$$X_{\text{cl}} = x_0. \tag{3.58}$$

The path integral measure factorizes as

$$\int \mathcal{D}X = \int dx_0 \int \mathcal{D}X_q \tag{3.59}$$

The constant mode x_0 represents the center-of-mass position of the non–compact boson in target space. Because the target space is non–compact, the integral $\int dx_0$, diverges and produces a factor proportional to the target-space volume. This divergence reflects the translational invariance of the theory.

Eigenmodes of the Laplacian The quantum fluctuations X_q can be expanded in eigenfunctions of the Laplacian on the torus. Writing the Laplacian in (σ^1, σ^2) coordinates,

$$\Delta = \frac{1}{\tau_2^2}\left(|\tau|^2\partial_1^2 - 2\tau_1\partial_1\partial_2 + \partial_2^2\right) \tag{3.60}$$

The eigenfunctions compatible with torus periodicities are Fourier modes

$$\phi_{m,n}(\sigma^1, \sigma^2) = e^{2\pi i(m\sigma^1 + n\sigma^2)},\ (m,n) \in \mathbb{Z}^2 \tag{3.61}$$

Acting with the Laplacian yields

$$\Delta\phi_{m,n} = \lambda_{m,n}\phi_{m,n} \tag{3.62}$$

with eigenvalues

$$\lambda_{m,n} = \frac{4\pi^2}{\tau_2^2}|m + n\tau|^2 \tag{3.63}$$

The mode $(m,n) = (0,0)$ corresponds to the constant function and therefore has eigenvalue $\lambda_{0,0} = 0$. This is the zero mode of the Laplacian. We then expand the fluctuation field as

$$X_q = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n}\phi_{m,n} \tag{3.64}$$

Separating the zero mode,

$$X_q = X_0 + \tilde{X}, \tag{3.65}$$

where

$$X_0 = a_{0,0}, \qquad \tilde{X} = \sum_{(m,n)\neq(0,0)} a_{m,n}\phi_{m,n}$$

The action becomes

$$S[\tilde{X}] = \frac{1}{4\pi\alpha'}\sum_{(m,n)\neq(0,0)} \lambda_{m,n}|a_{m,n}|^2 \tag{3.66}$$

The path integral over oscillatory modes therefore produces

$$Z_{\text{osc}} = \prod_{(m,n)\neq(0,0)}\left(\frac{1}{\lambda_{m,n}}\right)^{1/2} = (\det{}'\Delta)^{-1/2} \tag{3.67}$$

The prime indicates that the zero eigenvalue has been removed.

Zeta Function Regularization The determinant of the Laplacian is formally divergent because it contains an infinite product over eigenvalues. A natural way to define it is through the spectral zeta function

$$\zeta_\Delta(s) = \sum_{(m,n)\neq(0,0)} \lambda_{m,n}^{-s} \tag{3.68}$$

The regularized determinant is defined as

$$\det{}'\Delta = \exp\left[-\zeta_\Delta'(0)\right] \tag{3.69}$$

Substituting the eigenvalues gives

$$\zeta_\Delta(s) = \left(\frac{\tau_2^2}{4\pi^2}\right)^s \sum_{(m,n)\neq(0,0)} \frac{1}{|m + n\tau|^{2s}} \tag{3.70}$$

The lattice sum appearing here is the non-holomorphic Eisenstein series. Using its analytic continuation to $s = 0$ one finds

$$\det{}'\Delta = \tau_2^2 \, |\eta(\tau)|^4 \tag{3.71}$$

where $\eta(\tau)$ is the Dedekind eta function. Thus, the oscillator contribution to the partition function becomes

$$Z_{\text{osc}} = \frac{1}{\sqrt{\det{}'\Delta}} = \frac{1}{\tau_2 |\eta(\tau)|^2} \tag{3.72}$$

Zero Mode Divergence The constant mode contributes $\int dx_0$. Since the boson is non-compact, this integral diverges and is usually interpreted as the volume of the target space V_d. This factor is removed by dividing by the target-space volume or by considering normalized correlation functions. The remaining finite contribution comes entirely from the oscillatory determinant computed above.

To conclude, the partition function factorizes as

$$Z = V \, (\det{}'\Delta)^{-1/2} \tag{3.73}$$

After zeta-function regularization of the determinant the dependence on the modular parameter of the torus is governed by the Dedekind eta function, which encodes the contribution of the infinite set of oscillatory modes. We can thus write the factorized result

$$Z(\tau, \bar{\tau}) = Z_{\text{zero}} \, Z_{\text{osc}} \tag{3.74}$$

The oscillator contribution formally reads

$$Z_{\text{osc}} = \prod_{(m,n)\neq(0,0)} \frac{1}{|m + n\tau|^2} \tag{3.75}$$

This infinite product requires regularization.

1. Zeta Function Regularization The determinant of the Laplacian on the torus can be written as

$$\det{}'\Delta = \prod_{(m,n)\neq(0,0)} \left(\frac{4\pi^2}{\tau_2^2} |m + n\tau|^2 \right) \tag{3.76}$$

Define the spectral zeta function

$$\zeta_\Delta(s) = \sum_{(m,n)\neq(0,0)} \left(\frac{4\pi^2}{\tau_2^2} |m + n\tau|^2 \right)^{-s} \tag{3.77}$$

The determinant is defined via

$$\log \det{}'\Delta = - \frac{d}{ds} \zeta_\Delta(s) \Big|_{s=0} \tag{3.78}$$

Using modular invariance properties of the lattice sum one obtains

$$\det{}'\Delta = \tau_2 |\eta(\tau)|^4 \tag{3.79}$$

Therefore

$$Z_{\text{osc}} = \frac{1}{\sqrt{\tau_2} \, |\eta(\tau)|^2} \tag{3.80}$$

The factor $q^{1/24}$ arises from the zeta regularization identity

$$\sum_{n=1}^{\infty} n = -\frac{1}{12} \tag{3.81}$$

2. Eisenstein Series Regularization An alternative derivation starts from the regulated lattice sum

$$E(s,\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}} \tag{3.82}$$

which defines the non–holomorphic Eisenstein series. The determinant of the Laplacian on the torus can be related to this object through

$$\log \det{}' \Delta = -\left.\frac{d}{ds} E(s,\tau)\right|_{s=0} \tag{3.83}$$

To make progress we first rewrite the lattice norm explicitly. Using

$$|m+n\tau|^2 = (m+n\tau_1)^2 + (n\tau_2)^2 \tag{3.84}$$

the Eisenstein series becomes

$$E(s,\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{[(m+n\tau_1)^2 + (n\tau_2)^2]^s} \tag{3.85}$$

It is convenient to separate the $n = 0$ contribution. This gives

$$E(s,\tau) = \sum_{m\neq 0} \frac{\tau_2^s}{|m|^{2s}} + \sum_{n\neq 0} \sum_{m=-\infty}^{\infty} \frac{\tau_2^s}{[(m+n\tau_1)^2 + (n\tau_2)^2]^s} \tag{3.86}$$

The first term is immediately expressed in terms of the Riemann zeta function,

$$\sum_{m\neq 0} \frac{1}{|m|^{2s}} = 2\zeta(2s) \tag{3.87}$$

so that

$$E(s,\tau) = 2\zeta(2s)\tau_2^s + \sum_{n\neq 0} \sum_{m=-\infty}^{\infty} \frac{\tau_2^s}{[(m+n\tau_1)^2 + (n\tau_2)^2]^s} \tag{3.88}$$

To evaluate the second term we use the integral representation

$$\frac{1}{A^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-At} \tag{3.89}$$

Applying this with $A = (m+n\tau_1)^2 + (n\tau_2)^2$ gives

$$E(s,\tau) = 2\zeta(2s)\tau_2^s + \frac{\tau_2^s}{\Gamma(s)} \sum_{n\neq 0} \sum_{m=-\infty}^{\infty} \int_0^\infty dt\, t^{s-1} e^{-t[(m+n\tau_1)^2 + (n\tau_2)^2]} \tag{3.90}$$

The sum over m is now Gaussian and can be evaluated using the Poisson resummation formula

$$\sum_{m=-\infty}^{\infty} e^{-\pi a(m+b)^2} = \frac{1}{\sqrt{a}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/a} e^{2\pi i k b} \tag{3.91}$$

After rescaling the integration variable so that the exponent matches the Gaussian form, the Poisson resummation produces

$$\sum_{m=-\infty}^{\infty} e^{-t(m+n\tau_1)^2} = \sqrt{\frac{\pi}{t}} \sum_{k=-\infty}^{\infty} e^{-\pi^2 k^2/t} e^{2\pi i k n \tau_1} \tag{3.92}$$

Substituting this back into the expression for $E(s,\tau)$ yields

$$E(s,\tau) = 2\zeta(2s)\tau_2^s + \frac{\sqrt{\pi}\tau_2^s}{\Gamma(s)} \sum_{n\neq 0} \sum_{k=-\infty}^{\infty} e^{2\pi i k n \tau_1} \int_0^\infty dt\, t^{s-\frac{3}{2}} e^{-t(n\tau_2)^2 - \pi^2 k^2/t} \tag{3.93}$$

The remaining integral is a standard representation of the modified Bessel function,

$$\int_0^\infty dt\, t^{\nu-1} e^{-\beta t - \gamma/t} = 2\left(\frac{\gamma}{\beta}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}) \tag{3.94}$$

Using this identity we finally obtain

$$E(s,\tau) = 2\zeta(2s)\tau_2^s + \frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)\tau_2^{1-s} + \frac{4\pi^s}{\Gamma(s)} \sum_{n\neq 0} |n|^{s-\frac{1}{2}} \sigma_{1-2s}(n) K_{s-\frac{1}{2}}(2\pi|n|\tau_2) e^{2\pi i n \tau_1} \tag{3.95}$$

where

$$\sigma_k(n) = \sum_{d|n} d^k \tag{3.96}$$

and K_ν is the modified Bessel function. The analytic continuation to $s = 0$ gives the determinant of the Laplacian on the torus,

$$\det{}' \Delta = \tau_2 |\eta(\tau)|^4 \tag{3.97}$$

Substituting this result into the path–integral expression for the oscillator partition function gives

$$Z_{\rm osc} = \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2} \tag{3.98}$$

Analytic continuation and Kronecker limit formula To extract the determinant we need the analytic continuation of the Eisenstein series around $s = 0$. From the previous expression we isolate the terms that are singular when $s \to 0$. Expanding the Riemann zeta function and the Gamma functions around the origin,

$$\zeta(2s) = \frac{1}{2s} + \mathcal{O}(1), \qquad \zeta(2s-1) = -\frac{1}{12} + \mathcal{O}(s) \tag{3.99}$$

and

$$\Gamma(s) = \frac{1}{s} - \gamma + \mathcal{O}(s) \tag{3.100}$$

where γ is the Euler constant. Substituting these expansions into the representation of $E(s,\tau)$ shows that the Eisenstein series has a simple pole at $s = 1$ and is finite at $s = 0$. The small–s expansion takes the schematic form

$$E(s,\tau) = \frac{\pi}{s} + 2\pi\left(\gamma - \log(2) - \log(\sqrt{\tau_2}\,|\eta(\tau)|^2)\right) + \mathcal{O}(s) \tag{3.101}$$

More precisely, one obtains the classical *Kronecker limit formula*,

$$E(s,\tau) = \frac{\pi}{s} + 2\pi\left(\gamma - \log(2\sqrt{\tau_2}\,|\eta(\tau)|^2)\right) + \mathcal{O}(s) \tag{3.102}$$

This formula provides the analytic relation between the Eisenstein series and the Dedekind η function.

Extraction of the Laplacian determinant Using the relation between the determinant and the Eisenstein series,

$$\log \det{}' \Delta = -\left.\frac{d}{ds} E(s,\tau)\right|_{s=0} \tag{3.103}$$

we differentiate the expansion of $E(s,\tau)$ around $s=0$. The pole term does not contribute after differentiation, and we obtain

$$\log \det{}' \Delta = \log \tau_2 + 4 \log|\eta(\tau)| \tag{3.104}$$

Exponentiating gives the determinant of the scalar Laplacian on the torus,

$$\det{}' \Delta = \tau_2 |\eta(\tau)|^4 \tag{3.105}$$

Oscillator partition function In the worldsheet path integral of a free scalar field the oscillator contribution is proportional to the inverse square root of the Laplacian determinant,

$$Z_{\text{osc}} = (\det{}' \Delta)^{-1/2} \tag{3.106}$$

Substituting the expression for the determinant yields

$$Z_{\text{osc}} = \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2} \tag{3.107}$$

This is the same result obtained from the canonical trace derivation. The factor $\tau_2^{-1/2}$ originates from the zero–mode normalization of the Gaussian path integral, while the Dedekind η function encodes the infinite tower of oscillator modes of the scalar field. [1]

3.2.2 Compact Periodic Boson

Trace method

We now consider the case in which the scalar field is compact. The target space coordinate is identified as

$$X \sim X + 2\pi R \tag{3.108}$$

so that the field takes values on a circle of radius R. This compactification changes the structure of the Hilbert space. In particular, momentum along the compact direction becomes quantized and new topological sectors corresponding to winding configurations appear.

[1] For a detailed explanation of the Kronecker limit formula derivation of the torus boson determinant, see: D. Zagier, "Eisenstein Series and the Dedekind Eta Function," in *The 1-2-3 of Modular Forms*

Because the worldsheet spatial direction is itself a circle, the field may wind around the target space circle. The boundary condition along the spatial cycle therefore becomes

$$X(\sigma + 2\pi, \tau) = X(\sigma, \tau) + 2\pi R\, w \tag{3.109}$$

where $w \in \mathbb{Z}$ is the winding number. At the same time, momentum along the compact direction is quantized,

$$p = \frac{n}{R}, \qquad n \in \mathbb{Z} \tag{3.110}$$

The Hilbert space therefore decomposes into sectors labeled by the pair (n, w) together with oscillator excitations,

$$|n, w; \{N_m\}, \{\tilde{N}_m\}\rangle \tag{3.111}$$

For a compact boson it is convenient to introduce left and right moving momenta

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'}, \; p_R = \frac{n}{R} - \frac{wR}{\alpha'} \tag{3.112}$$

In terms of these quantities the Virasoro generators take the form

$$L_0 = \frac{\alpha'}{4} p_L^2 + \sum_{m=1}^{\infty} m N_m \tag{3.113}$$

$$\bar{L}_0 = \frac{\alpha'}{4} p_R^2 + \sum_{m=1}^{\infty} m \tilde{N}_m \tag{3.114}$$

The torus partition function is again defined as the trace

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}\left(q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}} \right) \tag{3.115}$$

Substituting the explicit expressions for L_0 and $\bar{L}_0$ we obtain

$$Z(\tau, \bar{\tau}) = (q\bar{q})^{-1/24} \sum_{n,w \in \mathbb{Z}} \mathrm{Tr}_{\mathrm{osc}}\, q^{\frac{\alpha'}{4} p_L^2 + \sum_{m=1}^{\infty} m N_m}\, \bar{q}^{\frac{\alpha'}{4} p_R^2 + \sum_{m=1}^{\infty} m \tilde{N}_m} \tag{3.116}$$

As in the non–compact case, the trace factorizes into a momentum part and an oscillator part. The oscillator trace is identical to the non–compact case, since the oscillator algebra is unaffected by the compactification. Therefore

$$Z_{\mathrm{osc}} = \frac{1}{|\eta(\tau)|^2} \tag{3.117}$$

The essential difference with the non–compact case arises from the zero modes. Instead of integrating over a continuous momentum, we now sum over the discrete momentum and winding sectors. The zero mode contribution becomes

$$Z_0 = \sum_{n,w \in \mathbb{Z}} q^{\frac{\alpha'}{4} p_L^2} \bar{q}^{\frac{\alpha'}{4} p_R^2} \tag{3.118}$$

Substituting the expressions for p_L and p_R we obtain the lattice sum

$$Z_0 = \sum_{n,w \in \mathbb{Z}} q^{\frac{\alpha'}{4}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)^2}. \tag{3.119}$$

Combining the oscillator and zero mode contributions gives

$$Z(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{n,w \in \mathbb{Z}} q^{\frac{\alpha'}{4}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)^2} \tag{3.120}$$

The structure of the derivation closely parallels the non–compact case, but the zero mode sector is fundamentally different. For the non–compact boson the momentum is continuous and the trace produces a Gaussian integral, leading to the factor $\tau_2^{-1/2}$.
For the compact boson, momentum becomes quantized and additional winding sectors appear. As a result, the zero mode contribution is not an integral but a discrete double sum over (n, w), forming a two–dimensional lattice in momentum space. The partition function therefore takes the form of a lattice sum multiplied by the universal oscillator factor $|\eta(\tau)|^{-2}$. This lattice structure is responsible for properties such as modular invariance and T-duality, under which the spectrum is invariant under the transformation

$$R \longrightarrow \frac{\alpha'}{R} \tag{3.121}$$

which exchanges momentum and winding numbers,

$$n \leftrightarrow w \tag{3.122}$$

Path integral derivation

We now consider the case in which the bosonic field takes values on a circle of radius R,

$$X \sim X + 2\pi R \tag{3.123}$$

The action remains

$$S[X] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2z\, \partial X \bar{\partial} X \tag{3.124}$$

but the compactness of the target space modifies the allowed field configurations. In particular, fields on the torus can now wind around the target-space circle.

Modified Boundary Conditions Because of the periodic identification of X, the field is allowed to change by integer multiples of $2\pi R$ when going around the cycles of the torus. Denoting the fundamental cycles by $(\sigma^1, \sigma^2) \to (\sigma^1 + 1, \sigma^2)$ and $(\sigma^1, \sigma^2) \to (\sigma^1, \sigma^2 + 1)$, we impose

$$X(\sigma^1 + 1, \sigma^2) = X(\sigma^1, \sigma^2) + 2\pi R\, m \tag{3.125}$$

$$X(\sigma^1, \sigma^2 + 1) = X(\sigma^1, \sigma^2) + 2\pi R\, n \tag{3.126}$$

where $(m, n) \in \mathbb{Z}^2$ label the winding numbers around the two independent cycles of the torus. Thus the path integral must be decomposed into topological sectors,

$$Z = \sum_{m,n\in\mathbb{Z}} \int_{(m,n)} \mathcal{D}X\, e^{-S[X]} \tag{3.127}$$

Classical Solutions In contrast with the non–compact case, the classical solutions are not constants. Instead they must reproduce the winding boundary conditions. Solving the Laplace equation

$$\partial\bar{\partial} X_{\rm cl} = 0$$

with the above periodicities gives a linear function,

$$X_{\rm cl}(\sigma^1, \sigma^2) = 2\pi R(m\sigma^1 + n\sigma^2) \tag{3.128}$$

Substituting this configuration into the action gives the classical contribution

$$S_{\rm cl}(m, n) = \frac{\pi R^2}{\alpha'\tau_2} |m + n\tau|^2 \tag{3.129}$$

The partition function therefore becomes

$$Z = \sum_{m,n\in\mathbb{Z}} e^{-S_{\rm cl}(m,n)} \int \mathcal{D}X_q \, e^{-S[X_q]} \tag{3.130}$$

Quantum Fluctuations The quantum fluctuations are defined as

$$X = X_{\rm cl} + X_q \tag{3.131}$$

Importantly, the fluctuations satisfy *periodic* boundary conditions,

$$X_q(\sigma^1+1,\sigma^2) = X_q(\sigma^1,\sigma^2), \qquad X_q(\sigma^1,\sigma^2+1) = X_q(\sigma^1,\sigma^2) \tag{3.132}$$

Therefore the analysis of the oscillatory sector proceeds exactly as in the non–compact case. Expanding in Laplacian eigenmodes,

$$X_q = \sum_{(k,l)\in\mathbb{Z}^2} a_{k,l} e^{2\pi i(k\sigma^1 + l\sigma^2)} \tag{3.133}$$

the eigenvalues are again

$$\lambda_{k,l} = \frac{4\pi^2}{\tau_2^2}|k+l\tau|^2 \tag{3.134}$$

Separating the zero mode,

$$X_q = X_0 + \tilde{X}$$

the oscillatory contribution gives the same functional determinant

$$Z_{\rm osc} = (\det{}'\Delta)^{-1/2} = \frac{1}{\tau_2|\eta(\tau)|^2} \tag{3.135}$$

Again, a crucial difference from the non–compact case concerns the treatment of the constant mode. Because the field is compact,

$$X_0 \sim X_0 + 2\pi R$$

the integration over the zero mode becomes

$$\int dX_0 = 2\pi R \tag{3.136}$$

Thus the divergence present in the non–compact case is removed and replaced by a finite factor proportional to the circumference of the target circle.

Combining all contributions we obtain

$$Z = (2\pi R)\frac{1}{\tau_2|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} \exp\left[-\frac{\pi R^2}{\alpha'\tau_2}|m+n\tau|^2\right] \tag{3.137}$$

3.2.3 Multiple Compact Periodic Bosons

A next generalization of the problem in view of the orbifold constructions that will occur let us now consider the case of d compact periodic scalars. The most general metric takes the form

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \left(G_{IJ}\sqrt{g}g^{\alpha\beta}\partial_a X^I \partial_\beta X^j + iB_{IJ}\,\epsilon^{\alpha\beta}\partial_\alpha X^I \partial_\beta X^J\right) \tag{3.138}$$

Again, we are always working with the gauge-fixed metric (??), while the convention for the Levi-Civita symbol is $\epsilon^{12} = 1$. In this setup, G_{IJ} is a constant symmetric metric on the target torus and B_{IJ} is a constant antisymmetric two-form.The boundary conditions in the (σ^1, σ^2) coordinate system are

$$X^I(\sigma^1 + 1, \sigma^2) = X^I(\sigma^1, \sigma^2) + 2\pi n^I \ , n^I \in \mathbb{Z} \tag{3.139a}$$
$$X^I(\sigma^1, \sigma^2 + 1) = X^I(\sigma^1, \sigma^2) + 2\pi m^I \ , m^I \in \mathbb{Z} \tag{3.139b}$$

We will distinguish again the the compact fields into a classical and a quantum part. The quantum solutions will again as expected provide the characteristic Dedekind expression for the oscillatory part of the partition function. As we expected the contribution of the topological factor B_{IJ} will affect only the classical part. To derive an expression of the classical solutions in terms of n^I, m^I the (later to be interpreted) as winding modes and Kaluza-Klein momenta respectively, it is convenient to transition to the z-plane. For a reminder, we have

$$z = \sigma^2 - \tau\sigma^1 \tag{3.140}$$

or inversely

$$\sigma^1 = \frac{i}{2\tau_2}(\bar{\tau}z - \bar{z}\tau) \quad , \quad \sigma^2 = -\frac{i}{\tau_2}(z - \bar{z}) \tag{3.141}$$

The classical solutions are separated between a left-moving and a right-moving part, where in this context is equivalent to saying a holomorphic and an anti-holomorphic part. The equivalence relations are translated from $\sigma^1 \to \sigma^1 + 1$ and $\sigma^2 \to \sigma^2 + 1$ to $z \to z + 1$ and $z \to z + \tau$. In this context the boundary conditions are transformed to

$$X^I(\sigma^1 + 1, \sigma^2) = X^I_L(z + 1) + X^I_R(\bar{z} + 1) = X^I_L(z) + X^I_R(\bar{z}) + 2\pi n^I \tag{3.142a}$$
$$X^I(\sigma^1 + 1, \sigma^2) = X^I_L(z + 1) + X^I_R(\bar{z} + 1) = X^I_L(z) + X^I_R(\bar{z}) + 2\pi m^I \tag{3.142b}$$

Demanding the aforementioned boundary conditions to be true means that X^I_L, X^I_R are at most linear functions of their arguments. This means that

$$X^I_{cl} = a^I z + \bar{a}^I \bar{z} \tag{3.143}$$

and the two system of equations from the boundary conditions gives

$$a^I + \bar{a}^I = 2\pi n^I \tag{3.144a}$$
$$a^I\tau + \bar{a}^I\bar{\tau} = 2\pi m^I \tag{3.144b}$$

from which we recover

$$X^I_{cl} = 2\pi i \left[\frac{(m - \bar{\tau}n)^I}{2\tau_2} z - \frac{(m - \tau n)^I}{2\tau_2} \bar{z} \right] \tag{3.145}$$

or

$$X^I_{cl} = \frac{2\pi i}{\tau_2} \left[\frac{2i\tau_2}{2} n\sigma^1 + \frac{2i\tau_2}{2} m\sigma^2 \right] = 2\pi(n\sigma^1 + m\sigma^2) \tag{3.146}$$

We can then recover from the above action in Minkowski space, the Euclidean classical action contribution, which turns out to be

$$S_{class} = \frac{2\pi}{2\tau_2}\frac{(G+B)_{IJ}}{\alpha'}(m-\bar{\tau}n)^I(m-\tau n)^J \tag{3.147}$$

And the path integral calculation gives

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}X e^{-S} = \sum_{m^I,n^I\in\mathbb{Z}} e^{-\frac{2\pi(G-iB)_{IJ}}{2\tau_2\alpha'}(m-\bar{\tau}n)^I(m-\tau n)^J}\int \mathcal{D}X_q e^{-S_q} \\ &= \sum_{m^I,n^I\in\mathbb{Z}} e^{-\frac{\pi G_{IJ}}{\tau_2\alpha'}(m+\bar{\tau}n)^I(m+\tau n)^J+\frac{i\pi B_{IJ}}{\tau_2\alpha'}(\tau m^I n^J+\bar{\tau}m^J n^I)}\int \mathcal{D}X_q e^{-S_q} \\ &= \sum_{m^I,n^I\in\mathbb{Z}} e^{-\frac{\pi G_{IJ}}{\tau_2\alpha'}(m+\bar{\tau}n)^I(m+\tau n)^J+\frac{i\pi B_{IJ}}{\tau_2\alpha'}(2i\tau_2)m^I n^J}\int \mathcal{D}X_q e^{-S_q}\end{aligned} \tag{3.148}$$

The B-term contributes only through the winding numbers which reflects its topological character: it depends solely on the homology class of the map from the worldsheet torus into the target torus. On the other hand, the quantum fluctuations satisfy periodic boundary conditions and may be expanded in the same Laplacian eigenbasis as before,

$$X_q^I = \sum_{(m,n)\in\mathbb{Z}^2\neq(0,0)} a^I_{m,n} e^{2\pi i(m\sigma^1+n\sigma^2)} \tag{3.149}$$

The Laplacian eigenvalues remain

$$\lambda_{m,n} = \frac{4\pi^2}{\tau_2^2}|m+n\tau|^2 \tag{3.150}$$

Since there are now d scalar fields, the oscillator determinant is simply the d–th power of the single boson result,

$$Z_{\text{osc}} = \sqrt{\det G}\,(\det{}'\Delta)^{-d/2} = \frac{\sqrt{\det G}}{(\tau_2^{d/2})|\eta(\tau)|^{2d}} \tag{3.151}$$

Integration over zero modes gives the volume of the target torus,

$$\int d^d X_0 = (2\pi)^d \tag{3.152}$$

Unlike the non-compact case, this contribution is finite because the target space is compact.

Combining the classical sectors with the quantum determinant yields

$$Z = (2\pi)^d \frac{\sqrt{\det G}}{\tau_2^{d/2}|\eta(\tau)|^{2d}} \sum_{m^I,n^I\in\mathbb{Z}^d} \exp\left[-\frac{\pi}{\alpha'\tau_2}G_{IJ}(m+\tau n)^I(m+\bar{\tau}n)^J + \frac{i\pi}{\alpha'\tau_2}B_{IJ}m^I n^J\right] \tag{3.153}$$

Note that in this case The classical action depends on the target-space metric G_{ij}, while the antisymmetric tensor B_{ij} introduces a purely topological phase depending only on the winding numbers. The oscillator determinant is raised to the d–th power since there are d independent scalar fields. The lattice sum appearing above defines a quadratic form on the integer vectors (m,n) and can be rewritten in terms of left- and right-moving momenta. This structure leads to the Narain lattice description of toroidal compactifications.

This expression arises naturally in the path integral formulation and is often called the *Lagrangian representation* of the toroidal partition function. It is written as a sum over winding numbers around the two cycles of the worldsheet torus. However, it is often more useful to express the result in terms of left- and right-moving momenta. This leads to the *Hamiltonian representation* and to the appearance of the Narain lattice. The choice of which formulation to considered depends on the nature of the problem under consideration. Hence, in cases where decompactification and modular properties of the partition function are considered, the Lagrangian formalism is best fitted, while the Hamiltonian one is usually most useful when T-dualitites and the spectrum of the theory are investigated.

To obtain the Hamiltonian form we perform a Poisson resummation over one set of integers, say n^I. Using the multidimensional identity

$$\sum_{w\in\mathbb{Z}^d} f(w) = \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} d^d x\, f(x)\, e^{2\pi i n\cdot x} \tag{3.154}$$

the partition function can be rewritten as a sum over integer vectors $m_I, \quad n^I \in \mathbb{Z}^d$ where m_I are interpreted as quantized momenta along the compact directions. After performing the Gaussian integral one finds that the exponent can be written in terms of left- and right-moving momenta

$$p_{L,I} = \sqrt{\frac{\alpha'}{2}}\left[m_I + (B+G)_{IJ} n^J\right] \tag{3.155}$$

$$p_{R,I} = \sqrt{\frac{\alpha'}{2}}\left[m_I + (B-G)_{IJ} n^J\right] \tag{3.156}$$

Introducing the modular parameter $q = e^{2\pi i\tau}$ the partition function can now be written as

$$Z = \frac{1}{|\eta(\tau)|^{2d}} \sum_{n,w\in\mathbb{Z}^d} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \tag{3.157}$$

This expression has the interpretation of a trace over the Hilbert space

$$Z = \mathrm{Tr}\left(q^{L_0}\bar{q}^{\bar{L}_0}\right) \tag{3.158}$$

where

$$L_0 = \frac{1}{2}p_L^2 + N, \qquad \bar{L}_0 = \frac{1}{2}p_R^2 + \bar{N} \tag{3.159}$$

include contributions from both momenta and oscillator excitations. The set of vectors (p_L, p_R) generated by integer pairs (m, n) forms a lattice in $\mathbb{R}^{d,d}$. This lattice is called the *Narain lattice* and is denoted $\Gamma_{d,d}$.

It has the following properties:

- It is an even lattice:
$$p_L^2 - p_R^2 \in 2\mathbb{Z}$$
- It is self-dual with respect to the metric of signature (d, d).
- Its structure depends on the background fields G_{IJ} and B_{IJ}.

Duality Symmetry: The Narain lattice is invariant under the discrete group $O(d,d;\mathbb{Z})$ which acts linearly on the vector (m,n). This symmetry mixes momentum and winding numbers and corresponds to the T–duality symmetry of toroidal compactifications. Different choices of background fields (G,B) that are related by $O(d,d;\mathbb{Z})$ transformations lead to the same Narain lattice and therefore define equivalent conformal field theories.

Geometric Structure of the Narain Lattice: The Narain lattice $\Gamma_{d,d}$ can be viewed as a lattice embedded in $\mathbb{R}^{d,d}$ equipped with the metric of signature (d,d)

$$\eta = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \tag{3.160}$$

Introduce the $2d$–dimensional vector

$$\mathbf{k} = \begin{pmatrix} m_I \\ n^I \end{pmatrix}, \qquad m_I, n^I \in \mathbb{Z} \tag{3.161}$$

The left- and right-moving momenta can be combined into a vector

$$\mathbf{P} = \begin{pmatrix} p_L \\ p_R \end{pmatrix} \tag{3.162}$$

These are related by a background-dependent linear transformation

$$\mathbf{P} = E\,\mathbf{k}, \tag{3.163}$$

where the matrix E depends on the metric G and antisymmetric tensor B. A convenient parametrization is

$$E = \frac{1}{\sqrt{2\alpha'}} \begin{pmatrix} e^{-T} & e^{-T}(B+G) \\ e^{-T} & e^{-T}(B-G) \end{pmatrix} \tag{3.164}$$

where $G = e^T e$. The Narain lattice is therefore

$$\Gamma_{d,d} = \left\{ \mathbf{P} = E\,\mathbf{k} \mid \mathbf{k} \in \mathbb{Z}^{2d} \right\} \subset \mathbb{R}^{d,d} \tag{3.165}$$

Even Self–Dual Property: Using the (d,d) metric η, the lattice satisfies

$$\mathbf{P}^T \eta \mathbf{P} = p_L^2 - p_R^2 = 2\,m_I n^I \tag{3.166}$$

Hence

$$p_L^2 - p_R^2 \in 2\mathbb{Z} \tag{3.167}$$

which shows that the lattice is *even*. Moreover it is self–dual with respect to η, implying

$$\Gamma_{d,d} = \Gamma_{d,d}^* \tag{3.168}$$

Thus $\Gamma_{d,d}$ is an even self-dual lattice of signature (d,d).

Moduli Space of Toroidal Compactifications: Different choices of background fields (G, B) correspond to different embeddings of $\Gamma_{d,d}$ in $\mathbb{R}^{d,d}$. The continuous deformations preserving the lattice structure are described by

$$O(d, d; \mathbb{R}). \tag{3.169}$$

However, two embeddings related by an integer change of basis

$$\mathbf{k} \to M\mathbf{k}, \qquad M \in O(d, d; \mathbb{Z}),$$

define the same lattice. The moduli space of inequivalent toroidal conformal field theories is therefore

$$\mathcal{M}_d = O(d, d; \mathbb{Z}) \backslash \frac{O(d, d; \mathbb{R})}{O(d) \times O(d)} \tag{3.170}$$

This means that in this setup

$$O(d) \times O(d) \quad \text{rotates left and right movers,} \tag{3.171}$$
$$O(d, d; \mathbb{R}) \quad \text{acts on the background } (G, B), \tag{3.172}$$
$$O(d, d; \mathbb{Z}) \quad \text{implements T–duality} \tag{3.173}$$

The discrete group $O(d, d; \mathbb{Z})$ acts linearly on the vector $\mathbf{k} = (m, n)$, mixing momentum and winding numbers. This leaves the lattice $\Gamma_{d,d}$ invariant and therefore preserves the partition function. For $d = 1$ this symmetry reduces to the familiar transformation

$$R \longrightarrow \frac{\alpha'}{R} \tag{3.174}$$

which exchanges momentum and winding modes.

A convenient way to expose the geometric structure of toroidal compactifications is to combine the background fields G_{ij} and B_{ij} into the *generalized metric*

$$\mathcal{H}(G, B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{3.175}$$

This matrix acts on the $2d$–dimensional vector

$$\mathbf{k} = \begin{pmatrix} m_I \\ n^I \end{pmatrix}, \; m_I, n^I \in \mathbb{Z} \tag{3.176}$$

The left- and right-moving momenta can be expressed through the quadratic form

$$p_L^2 + p_R^2 = \frac{1}{\alpha'} \mathbf{k}^T \mathcal{H}(G, B) \mathbf{k} \tag{3.177}$$

The difference between left and right norms is determined by the $O(d, d)$ invariant metric

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{3.178}$$

so that

$$p_L^2 - p_R^2 = \frac{1}{\alpha'} \mathbf{k}^T \eta \, \mathbf{k} = \frac{2}{\alpha'} m_I n^I \tag{3.179}$$

The matrix $\mathcal{H}$ satisfies the defining relation

$$\mathcal{H}^T \eta \mathcal{H} = \eta \tag{3.180}$$

which shows that $\mathcal{H}$ is an element of $O(d,d;\mathbb{R})$. A duality transformation

$$M \in O(d,d;\mathbb{Z}) \tag{3.181}$$

acts on the lattice vector as

$$\mathbf{k} \to M\mathbf{k}. \tag{3.182}$$

To keep the spectrum invariant the generalized metric must transform as

$$\mathcal{H} \to M^T \mathcal{H} M \tag{3.183}$$

This transformation mixes the background fields G and B while preserving the Narain lattice and therefore the torus partition function. Using the generalized metric the Narain partition function can be written compactly as

$$Z(\tau) = \frac{1}{|\eta(\tau)|^{2d}} \sum_{\mathbf{k}\in\mathbb{Z}^{2d}} \exp\Big[i\pi\tau \, \frac{1}{2}(p_L^2 + p_R^2) + i\pi\bar{\tau} \, \frac{1}{2}(p_L^2 - p_R^2) \Big] \tag{3.184}$$

where the quadratic forms are determined by $\mathcal{H}$ and η.

3.3 Bosons General Boundary Conditions

3.3.1 Path Integral

In view of the analysis of orbifolds in the next section, we examine a complex boson $Z(\sigma^1, \sigma^2)$ with general boundary conditions on the torus

$$Z(\sigma^1 + 2\pi, \sigma^2) = e^{2\pi i h/N} Z(\sigma^1, \sigma^2) \tag{3.185a}$$

$$Z(\sigma^1, \sigma^2 + 2\pi) = e^{-2\pi i g/N} Z(\sigma^1, \sigma^2) \tag{3.185b}$$

Since the symmetry group under consideration is $\mathbb{Z}_N$, the indices h and g take integer values in $\{0, \ldots, N-1\}$. The value 0 corresponds to the untwisted sector, while the remaining values label the twisted sectors of the theory, which are required in the closed-string formulation to ensure modular invariance of the partition function. The standard choice of phases is adopted in order to simplify the characteristics of the associated theta functions. From an orbifold perspective, the value of h labels the (un)twisted sectors while g the corresponding $\mathbb{Z}_N$ projection.

To proceed in the analysis of the path integral derivation for the partition function of a compactified complex boson, we gauge fix the metric as

$$g_{ij} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \tag{3.186}$$

or for its contravariant components

$$g^{ij} = \tau_2 \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix} \tag{3.187}$$

where we have introduced a Weyl rescaling such that the volume of the torus T^2 is normalized to unity.

We work on a Euclidean worldsheet. The action for non-interacting massless bosons propagating on a toroidal target space is

$$S_E \;=\; \frac{1}{4\pi\alpha'}\int d^2w\,\Big[\sqrt{g}\,g^{ij}\,G_{IJ}\,\partial_i X^I\partial_j X^J \;+\; \epsilon^{ij}\,B_{IJ}\,\partial_i X^I\partial_j X^J\Big]\,. \tag{3.188}$$

Here g_{ij} is the two-dimensional worldsheet metric, while G_{IJ} and B_{IJ} are respectively the metric and antisymmetric Kalb–Ramond field on the target-space torus. We adopt the convention $\epsilon^{12}=1$. This action is identical to that of N periodic real compact bosons.

To simplify the analysis and prepare for orbifold constructions, we combine the two real bosons into a single complex boson. We define

$$Z = X^1 + UX^2, \qquad \bar{Z} = X^1 + \bar{U}X^2 \tag{3.189}$$

where $U = U_1 + iU_2$ is the complex structure modulus of the torus. The torus metric and antisymmetric tensor are

$$G_{IJ} = \frac{T_2}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, \qquad B_{IJ} = T_1\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.190}$$

where $T_2 = \Im T$ and $T_1 = \Re T$.

This allows us to rewrite the derivatives:

$$\partial_i X^1 = \frac{1}{2}\Big(\partial_i Z + \partial_i\bar{Z} - U\partial_i X^2 - \bar{U}\partial_i X^2\Big) \tag{3.191}$$

$$\partial_i X^2 = \frac{1}{2iU_2}\Big(\partial_i Z - \partial_i\bar{Z}\Big) \tag{3.192}$$

Substituting these expressions into the original action, we find that the kinetic term becomes

$$G_{IJ}\,\partial_i X^I\partial_j X^J = G_{Z\bar{Z}}\,\partial_i Z\partial_j\bar{Z} + G_{\bar{Z}Z}\,\partial_i\bar{Z}\partial_j Z, \tag{3.193}$$

where $G_{Z\bar{Z}}$ is a combination of the original G_{IJ} components, specifically

$$G_{Z\bar{Z}} = G_{\bar{Z}Z} = \frac{G_{11}G_{22} - G_{12}^2}{U_2}. \tag{3.194}$$

Similarly, the antisymmetric term transforms as

$$B_{IJ}\,\partial_i X^I\partial_j X^J = B_{Z\bar{Z}}\,\partial_i Z\partial_j\bar{Z} + B_{\bar{Z}Z}\,\partial_i\bar{Z}\partial_j Z, \tag{3.195}$$

so that the action can finally be written in the compact form

$$S = \frac{1}{2\pi\alpha'}\int d^2w\Big(G_{Z\bar{Z}}\,\partial Z\bar{\partial}\bar{Z} + G_{\bar{Z}Z}\,\partial\bar{Z}\bar{\partial}Z\Big) \tag{3.196}$$

where all metric and antisymmetric contributions are absorbed into $G_{Z\bar{Z}}$ and $B_{Z\bar{Z}}$.

Next, we consider orbifolds defined by a discrete rotation

$$Z \longrightarrow e^{2\pi i/N}Z \tag{3.197}$$

For this to be a symmetry of the torus lattice, the rotation must map lattice vectors to lattice vectors. That is, for all $n^1, n^2 \in \mathbb{Z}$,

$$e^{2\pi i/N}(n^1 + Un^2) = m^1 + Um^2, \quad m^1, m^2 \in \mathbb{Z} \tag{3.198}$$

This condition is only satisfied for $N = 2, 3, 4, 6$, as required by the crystallographic restriction theorem: in two dimensions, only rotations by angles $\pi, 2\pi/3, \pi/2, \pi/3$ preserve a lattice. For $N = 3, 4, 6$ the complex structure U is fixed to special values ($U = e^{2\pi i/3}, i, e^{\pi i/3}$), while for $N = 2$, the structure constant U can remain arbitrary. The Kähler modulus, however, is not constrained by the rotation and remains a free parameter.

The *Kähler modulus* T is defined in terms of the torus metric and antisymmetric field:

$$T = B_{12} + i\sqrt{\det G}, \qquad \det G = G_{11}G_{22} - G_{12}^2 \tag{3.199}$$

Here, $\mathrm{Re}\, T = B_{12}$ measures the antisymmetric background, while $\mathrm{Im}\, T = \sqrt{\det G}$ encodes the torus area. It controls the size and shape of the torus in a way that is independent of the complex structure, and survives orbifold projections as a continuous parameter.

In the same way as with fermions, we can then construct the path integral of the complex boson and subsequently express it as a determinant, with the difference that now the degrees of freedom are bosonic. Explicitly, we have

$$\mathcal{Z} = \int \mathcal{D}Z D\bar{Z}\; e^{-S_E} = \int \mathcal{D}Z D\bar{Z}\; \exp\left[-\frac{1}{2}\int Z \left(\frac{G_{Z\bar{Z}}}{2\pi\alpha'}\nabla^2\right) Z \right] = Z_{\text{classical}} \times \left[\det\left(\frac{G_{Z\bar{Z}}}{2\pi\alpha'}\nabla^2\right)\right]^{-1} \tag{3.200}$$

where the Laplacian acting on the complex boson is

$$\begin{aligned} \nabla^2 Z = g^{ij}\partial_i\partial_j Z &= \partial_1\left(\frac{|\tau|^2}{\tau_2}\partial_1 Z\right) + \partial_2\left(\frac{1}{\tau_2}\partial_2 Z\right) - 2\frac{\tau_1}{\tau_2}\partial_1\partial_2 Z \\ &= \frac{1}{\tau_2}\left[|\tau|^2\,\partial_1^2 Z + \partial_2^2 Z - 2\tau_1\,\partial_1\partial_2 Z\right] = \frac{1}{\tau_2}\left|\partial_2 - \tau\partial_1\right|^2 Z \end{aligned} \tag{3.201}$$

The most general solution of this equation naturally separates into a classical part, which satisfies the boundary conditions, and an oscillatory part, corresponding to the quantum fluctuations. For untwisted sectors, the torus identifications and orbifold constraints impose additional restrictions on the classical part. More precisely, the most general solution can be written as

$$Z(\sigma^1, \sigma^2) = Z_0 + A\sigma^1 + B\sigma^2 + \sum_{m,n\in\mathbb{Z}} Z_{n,m}\, e^{2\pi i\left(\left(n+\frac{h}{2N}\right)\sigma^1 + \left(m-\frac{g}{2N}\right)\sigma^2\right)} \tag{3.202}$$

where Z_0 is the constant zero mode, A and B encode the winding contributions along the two cycles of the torus, and $Z_{n,m}$ are the oscillator modes. The shifts by $\frac{h}{2N}$ and $\frac{g}{2N}$ implement the twisted boundary conditions characteristic of the $\mathbb{Z}_N$ orbifold sectors, ensuring invariance under the discrete rotation $Z \to e^{2\pi i/N}Z$.

The classical contributions Z_0, A, B are further constrained by the torus lattice identifications and by the orbifold rotation symmetry. For example, for a $\mathbb{Z}_N$ rotation, the allowed winding numbers must satisfy

$$A + UB \in \Lambda, \qquad \Lambda = \{n^1 + Un^2 \,|\, n^1, n^2 \in \mathbb{Z}\} \tag{3.203}$$

so that the rotated configuration remains on the lattice. This is precisely the restriction that only allows $N = 2, 3, 4, 6$ for a consistent orbifold, in accordance with the crystallographic restriction theorem. Finally, the Kähler modulus of the torus, defined as

$$T = B_{12} + i\sqrt{\det G}, \qquad \det G = G_{11}G_{22} - G_{12}^2 \tag{3.204}$$

controls the overall size of the torus and the background antisymmetric field. In the path integral, the classical action depends explicitly on T through the kinetic term and the winding contributions. Thus, the Kähler modulus appears as a continuous geometric parameter which survives the orbifold projection and hence plays a role in determining the spectrum of classical solutions.

Since the contribution of the classical and the oscillatory part is factorized in the partition function, we focus at first in the path integral derivation of the oscillatory one. The eigenmode $e^{2\pi i\left[\left(n+\frac{h}{2N}\right)\sigma^1+\left(m-\frac{g}{2N}\right)\sigma^2\right]}$ corresponds to eigenvalues

$$\lambda_{n,m} = -\frac{(2\pi)^2}{\tau_2}\left|m + \frac{g}{N} + \tau\left(n + \frac{h}{N}\right)\right|^2 \tag{3.205}$$

The path integral can then be rewritten as

$$\mathcal{Z} = Z_{classical} \times \left[\prod_{m,n\in\mathbb{Z}} -\frac{(2\pi)^2}{\tau_2}\left|m + \frac{g}{2N} + \tau\left(n + \frac{h}{2N}\right)\right|^2\right]^{-1/2} \tag{3.206}$$

This expression involves the same product over (m, n) modes as in the fermionic case. Consequently, the oscillatory contribution to the partition function is given by

$$\mathcal{Z}_{osc} = \left|\frac{\eta}{\theta\begin{bmatrix}1-\frac{2h}{N}\\ 1-\frac{2g}{N}\end{bmatrix}}\right|^2 \tag{3.207}$$

For the untwisted sector of the theory, where $(h, g) = (0, 0)$, recognizing the integral representation of the generalized zeta function in the heat kernel method, we precisely recover the Eisenstein series regularization technique for a periodic complex boson reviewed in a previous section.

3.3.2 Crystallographic Theorem

Theorem 2 (Crystallographic Restriction Theorem)**.** *In two dimensions, a lattice invariant under a rotation can only have rotational symmetries of order*

$$n = 1, 2, 3, 4, 6$$

Proof: Consider a 2D lattice Λ generated by basis vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\Lambda = \{m\mathbf{a} + n\mathbf{b} \,|\, m, n \in \mathbb{Z}\}$$

Let R_θ be a rotation by angle θ that leaves the lattice invariant:

$$R_\theta(\Lambda) = \Lambda$$

Then R_θ maps the lattice basis to integer linear combinations of itself:

$$R_\theta \mathbf{a} = m\mathbf{a} + n\mathbf{b}, \quad R_\theta \mathbf{b} = p\mathbf{a} + q\mathbf{b}$$

where $m, n, p, q \in \mathbb{Z}$.

In the $\{\mathbf{a}, \mathbf{b}\}$ basis, R_θ has the matrix:

$$R_\theta = \begin{pmatrix} m & p \\ n & q \end{pmatrix} \in GL(2, \mathbb{Z})$$

with determinant $\det(R_\theta) = \pm 1$ (area-preserving).
The trace of a rotation matrix is related to the rotation angle:

$$\mathrm{Tr}(R_\theta) = m + q = 2\cos\theta$$

Since $m + q$ is an integer, we require

$$2\cos\theta \in \mathbb{Z}$$

The only integers between -2 and 2 (the range of $2\cos\theta$) are

$$-2, -1, 0, 1, 2$$

Therefore, the allowed rotation angles θ satisfy:

$$\cos\theta = 1 \implies \theta = 0 \quad (n = 1)$$
$$\cos\theta = 0 \implies \theta = \frac{\pi}{2} \quad (n = 4)$$
$$\cos\theta = -1 \implies \theta = \pi \quad (n = 2)$$
$$\cos\theta = \frac{1}{2} \implies \theta = \frac{\pi}{3} \quad (n = 6)$$
$$\cos\theta = -\frac{1}{2} \implies \theta = \frac{2\pi}{3} \quad (n = 3)$$

Hence, the possible rotational symmetries are of order $n = 1, 2, 3, 4, 6$. □

3.3.3 The multiplicity factor $\Lambda(k, k')$

We now determine the factor $\Lambda(k, k')$ which accompanies the contribution of a sector twisted by g^k with an insertion of $g^{k'}$[2]. It is useful to distinguish this quantity from the geometrical fixed-point multiplicity of the orbifold action. The latter is obtained from the determinant of the lattice automorphism and counts solutions of a fixed-point equation on the torus. The factor $\Lambda(k, k')$ has a different origin. It measures the multiplicity associated with the common identifications generated by the two cyclic twists.

Let g be a generator of $\mathbb{Z}_N$. The two elements appearing in a sector with twist g^k and insertion $g^{k'}$ generate a subgroup of $\mathbb{Z}_N$. The structure of this subgroup is determined entirely by the integers k, k' and N. To make this explicit, consider the integer labels before imposing the cyclic identification. The original orbifold identifies labels which differ by N, while the two twists generate further identifications by k and k'. Thus the complete set of shifts which acts trivially on the labels is the integer subgroup

$$k\mathbb{Z} + k'\mathbb{Z} + N\mathbb{Z} = \{ak + bk' + cN \mid a, b, c \in \mathbb{Z}\} \tag{3.208}$$

[2]This subsection is based on related discussion appearing in [2]

Every element of this subgroup is divisible by $d = \gcd(k, k', N)$ and, by Bezout's identity, d itself can be expressed as an integer linear combination of k, k' and N. Hence, the subgroup generated by the three integers is precisely

$$k\mathbb{Z} + k'\mathbb{Z} + N\mathbb{Z} = d\mathbb{Z} \qquad d = \gcd(k, k', N) \tag{3.209}$$

The inequivalent labels are consequently elements of the quotient

$$\frac{\mathbb{Z}}{k\mathbb{Z} + k'\mathbb{Z} + N\mathbb{Z}} = \frac{\mathbb{Z}}{d\mathbb{Z}} \tag{3.210}$$

which contains d distinct classes. This gives the general multiplicity

$$\Lambda(k, k') = \gcd(k, k', N) \tag{3.211}$$

for the twisted contributions, with the completely untwisted contribution treated according to the normalization convention adopted for the partition function. The same conclusion follows directly from the group-theoretic description. Since $\mathbb{Z}_N$ is cyclic, every subgroup is cyclic and is generated by a suitable power of g. Let $d = \gcd(k, k', N)$. Since d divides both k and k', there exist integers u and v such that

$$k = du \qquad k' = dv \tag{3.212}$$

and therefore

$$g^k = (g^d)^u \qquad g^{k'} = (g^d)^v \tag{3.213}$$

which shows that $\langle g^k, g^{k'} \rangle \subseteq \langle g^d \rangle$.

The reverse inclusion follows from Bezout's identity. Since $d = \gcd(k, k', N)$, there exist integers a, b and c satisfying

$$ak + bk' + cN = d \tag{3.214}$$

and hence, inside $\mathbb{Z}_N$,

$$g^d = g^{ak+bk'+cN} = (g^k)^a (g^{k'})^b \tag{3.215}$$

so that g^d itself belongs to $\langle g^k, g^{k'} \rangle$. Consequently, $\langle g^k, g^{k'} \rangle = \langle g^d \rangle$.

The element g^d has order N/d. Thus the subgroup generated by the two twists contains N/d elements, leaving a quotient with d elements. This is precisely the multiplicity found from the integer lattice description. There is also a useful integer–matrix formulation of the same argument. Consider the 1×3 matrix

$$A_{k,k'} = \begin{pmatrix} k & k' & N \end{pmatrix} \tag{3.216}$$

Integer changes of basis bring this matrix to Smith normal form. Since there is only one row, its nonzero Smith invariant is simply the greatest common divisor of its entries,

$$\mathrm{SNF}(A_{k,k'}) = \begin{pmatrix} d & 0 & 0 \end{pmatrix} \qquad d = \gcd(k, k', N) \tag{3.217}$$

The Smith invariant therefore measures the index of the subgroup generated by the shifts k, k' and N. In this form the origin of the factor $\Lambda(k, k')$ is particularly transparent: it is an integer-lattice index and not an additional numerical factor inserted by hand.

The $\mathbb{Z}_6$ case: We can now specialize the general argument to the case $N = 6$. The twists are labelled by

$$k, k' \in \{0, 1, 2, 3, 4, 5\} \tag{3.218}$$

and the multiplicity is determined by

$$\Lambda(k, k') = \gcd(k, k', 6) \tag{3.219}$$

The structure can be understood without calculating the individual entries one by one. The integer 6 has only the prime factors 2 and 3, so a nontrivial common divisor can only arise from a common factor of 2 or a common factor of 3. Consider first a pair for which one of the twists is relatively prime to 6. The elements 1 and 5 generate $\mathbb{Z}_6$, and any pair containing either of them therefore generates the full cyclic group. Equivalently,

$$\gcd(1, k', 6) = \gcd(5, k', 6) = 1 \tag{3.220}$$

for every k'. The corresponding multiplicity is consequently one. The next possibility is a common factor of 2. The even elements of $\mathbb{Z}_6$ are 0, 2 and 4. Whenever the two twists are both even, their generated subgroup is contained in $\langle g^2 \rangle$. For nontrivial even twists this is the order–three subgroup

$$\langle g^2 \rangle = \{1, g^2, g^4\} \tag{3.221}$$

and the quotient by this subgroup contains two classes. Thus the corresponding multiplicity is two. In particular, $\langle g^2, g^4 \rangle = \langle g^2 \rangle$, so that $\Lambda(2, 4) = 2$, while the same subgroup argument applies to the other pairs of even twists. The other nontrivial possibility is a common factor of 3. The only nonzero element divisible by 3 is 3 itself, and therefore

$$\langle g^3 \rangle = \{1, g^3\} \tag{3.222}$$

is the order-two subgroup. The corresponding quotient contains three classes, giving

$$\Lambda(3, 3) = 3 \tag{3.223}$$

and, with the same convention for an untwisted entry paired with the g^3 insertion,

$$\Lambda(0, 3) = \Lambda(3, 0) = 3 \tag{3.224}$$

Finally, if one twist is even and the other is divisible by 3, the two twists have no nontrivial common divisor with 6. For example, the pair $(2, 3)$ satisfies $\gcd(2, 3, 6) = 1$, and Bezout's identity gives $2(-1) + 3(1) = 1$,so that g itself is generated by g^2 and g^3. Hence,$\langle g^2, g^3 \rangle = \mathbb{Z}_6$ and the corresponding multiplicity is one. Collecting these possibilities gives the complete $\mathbb{Z}_6$ result. With the rows and columns ordered as $0, 1, 2, 3, 4, 5$, one obtains

$$\Lambda(k, k') = \begin{pmatrix} 1 & 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{3.225}$$

where the completely untwisted entry has been normalized to one in accordance with the convention for the partition function. The symmetry under interchange of the twist and insertion is now immediate. Interchanging k and k' does not change the subgroup which they generate, $\langle g^k, g^{k'} \rangle = \langle g^{k'}, g^k \rangle$ and therefore $\Lambda(k, k') = \Lambda(k', k)$ which is also manifest from the arithmetic expression $\gcd(k, k', N)$.

The geometrical fixed-point multiplicity: The above counting should now be contrasted with the geometrical fixed-point problem. Let $T^2 = \mathbb{R}^2/\Lambda$, where Λ is a two-dimensional lattice, and let Q be an automorphism of Λ. Since a point of the torus is represented by $x \in \mathbb{R}^2$, with

$$x \sim x + \lambda \qquad \lambda \in \Lambda \tag{3.226}$$

Let a point be fixed by Q^k on the torus when $Q^k x$ represents the same torus point as x. The fixed–point condition is therefore

$$Q^k x = x + \lambda \qquad \lambda \in \Lambda \tag{3.227}$$

or, equivalently,

$$(1 - Q^k)x = \lambda \qquad \lambda \in \Lambda \tag{3.228}$$

This condition already shows why a determinant enters the calculation. Suppose that $1 - Q^k$ is nonsingular. The image $(1 - Q^k)\Lambda$ is then a full-rank sublattice of Λ. The fixed points are classified by the possible lattice vectors on the right-hand side of the fixed-point equation, modulo those vectors which give equivalent points on the torus. Thus the set of inequivalent fixed points is described by the finite quotient $\frac{\Lambda}{(1-Q^k)\Lambda}$. Consequently, the number of fixed points is the index

$$N_k = \Big[\Lambda : (1 - Q^k)\Lambda\Big] \tag{3.229}$$

The index can be evaluated directly by choosing a basis $\{e_1, e_2\}$ of the lattice. Since Q preserves Λ, its matrix in this basis has integer entries, and so does $1 - Q^k$. Write

$$1 - Q^k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.230}$$

The image lattice is generated by

$$(1 - Q^k)e_1 = ae_1 + ce_2 \qquad (1 - Q^k)e_2 = be_1 + de_2 \tag{3.231}$$

Thus, its fundamental parallelogram is obtained from a fundamental parallelogram of Λ by the linear transformation $1 - Q^k$. A linear transformation in two dimensions multiplies oriented areas by its determinant. Hence, the area of the fundamental cell of the image lattice is

$$\Big|\det\Big((1 - Q^k)e_1, (1 - Q^k)e_2\Big)\Big| = \Big|\det(1 - Q^k)\Big|\,|\det(e_1, e_2)| \tag{3.232}$$

The index of a full-rank sublattice is the ratio of the volume of its fundamental cell to the volume of the fundamental cell of the original lattice. Therefore,

$$\Big[\Lambda : (1 - Q^k)\Lambda\Big] = \Big|\det(1 - Q^k)\Big| \tag{3.233}$$

and the fixed–point number is

$$N_k = \Big|\det(1 - Q^k)\Big| \tag{3.234}$$

The determinant has therefore arisen directly from the lattice index: the fixed-point equation defines a finite quotient of the original lattice by the image of $1 - Q^k$, and the determinant measures precisely the index of that image lattice. For the standard order-six lattice automorphism

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \qquad Q^6 = 1 \tag{3.235}$$

the relevant powers are

$$Q^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \qquad Q^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Q^4 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad Q^5 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.236}$$

and hence,

$$1 - Q = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \qquad 1 - Q^2 = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \qquad 1 - Q^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{3.237}$$

The corresponding determinants are

$$|\det(1 - Q)| = 1 \qquad \left|\det(1 - Q^2)\right| = 3 \qquad \left|\det(1 - Q^3)\right| = 4 \tag{3.238}$$

and the inverse powers give the same values for $k = 5$ and $k = 4$. Thus,

$$N_k = \begin{cases} 1 & k = 1, 5 \\ 3 & k = 2, 4 \\ 4 & k = 3 \end{cases} \tag{3.239}$$

These numbers are geometrical fixed-point multiplicities and are therefore distinct from the factor $\Lambda(k, k')$. The distinction can be stated directly in terms of the two lattice quotients involved. For the fixed-point problem, one considers $\frac{\Lambda}{(1-Q^k)\Lambda}$, whose cardinality is determined by the determinant of $1 - Q^k$. For the multiplicity factor, one instead considers the quotient generated by the cyclic identifications, $\frac{\mathbb{Z}}{k\mathbb{Z}+k'\mathbb{Z}+N\mathbb{Z}}$ whose cardinality is determined by $\gcd(k, k', N)$. The two constructions therefore count different objects. In particular, the values 1, 2 and 3 appearing in the $\Lambda(k, k')$ table for the $\mathbb{Z}_6$ orbifold cannot be obtained by simply applying the geometrical determinant formula to Q^k. The determinant counts fixed points of a single geometrical action on the torus, whereas $\Lambda(k, k')$ records the multiplicity associated with the subgroup generated simultaneously by the twist and the insertion. Keeping these two lattice-index calculations separate is essential for the correct organization of the orbifold partition function.

3.4 Propagators on the Torus

We start again from our familiar Gaussian partition function,

$$Z[J] = \int \mathcal{D}X \, \exp\left[-\frac{1}{2}\int d^2\sigma \, X\Box X + \int d^2\sigma \, JX\right] \tag{3.240}$$

The purpose of this section is to determine the propagator explicitly on the torus, paying particular attention to the zero mode of the Laplacian. We begin by expanding the field in a complete orthonormal basis of eigenfunctions of $\Box$,

$$X(\sigma) = \sum_I X_I \psi_I(\sigma) = X_0\psi_0(\sigma) + \sum_{I\neq 0} X_I \psi_I(\sigma) \tag{3.241}$$

where

$$\Box\psi_I = \lambda_I \psi_I \tag{3.242a}$$

$$\Box\psi_0 = 0 \tag{3.242b}$$

and the eigenfunctions satisfy the orthonormality condition

$$(\psi_I, \psi_J) = \int d^2\sigma \, \sqrt{g} \, \psi_I^*(\sigma)\psi_J(\sigma) = \delta_{IJ} \tag{3.243}$$

The distinction between the zero mode and the non-zero modes will be essential. Since $\lambda_0 = 0$, the Laplacian cannot be inverted on the full space of functions. The inverse will therefore always be understood as acting on the subspace orthogonal to the zero mode.
We now calculate the two-point function directly from the generating functional. By definition,

$$\langle X(\sigma)X(\sigma')\rangle = \frac{1}{Z[0]}\,\frac{\delta}{\delta J(\sigma)}\frac{\delta}{\delta J(\sigma')}Z[J]\bigg|_{J=0} \tag{3.244}$$

Let us first rewrite the action in terms of the eigenmodes. Substituting

$$X = X_0\psi_0 + \sum_{I\neq 0} X_I\psi_I \tag{3.245}$$

into the kinetic term gives

$$-\frac{1}{2}(X,\Box X) = -\frac{1}{2}\sum_{I,J} X_I X_J(\psi_I,\Box\psi_J) \tag{3.246}$$

Using

$$\Box\psi_J = \lambda_J\psi_J \tag{3.247}$$

and the orthonormality relation,

$$(\psi_I,\Box\psi_J) = \lambda_J(\psi_I,\psi_J) = \lambda_J\delta_{IJ} \tag{3.248}$$

we obtain

$$-\frac{1}{2}(X,\Box X) = -\frac{1}{2}\sum_{I}\lambda_I X_I^2 \tag{3.249}$$

Since $\lambda_0 = 0$, the zero mode does not appear in the kinetic term:

$$-\frac{1}{2}(X,\Box X) = -\frac{1}{2}\sum_{I\neq 0}\lambda_I X_I^2 \tag{3.250}$$

Similarly, we expand the source as

$$J(\sigma) = \sum_I J_I\psi_I(\sigma) \tag{3.251}$$

so that

$$(J,X) = \sum_{I,J} J_I X_J(\psi_I,\psi_J) = \sum_I J_I X_I \tag{3.252}$$

Therefore,

$$(J,X) = J_0X_0 + \sum_{I\neq 0} J_I X_I \tag{3.253}$$

The functional measure decomposes correspondingly as

$$\mathcal{D}X = dX_0\prod_{I\neq 0} dX_I \tag{3.254}$$

The generating functional consequently becomes

$$Z[J] = \int dX_0\prod_{I\neq 0}\int_{-\infty}^{\infty} dX_I\,\exp\left[-\frac{1}{2}\lambda_I X_I^2 + J_I X_I\right] e^{J_0X_0} \tag{3.255}$$

The integral over the zero mode is simply

$$\int dX_0\, e^{J_0X_0} \tag{3.256}$$

which $J_0 = 0$, gives an infinite volume, $\int dX_0 = V$ that is the usual zero–mode volume. For the non–zero modes we use the standard Gaussian integral

$$\int_{-\infty}^{\infty} dX_I \, e^{-\frac{1}{2}\lambda_I X_I^2 + J_I X_I} = \sqrt{\frac{2\pi}{\lambda_I}} \, e^{\frac{J_I^2}{2\lambda_I}} \tag{3.257}$$

Hence

$$Z[J] = V \prod_{I\neq 0} \sqrt{\frac{2\pi}{\lambda_I}} \, \exp\left[\frac{1}{2}\sum_{I\neq 0}\frac{J_I^2}{\lambda_I}\right] \tag{3.258}$$

Up to the usual normalization of the functional measure, the product over eigenvalues is represented by the determinant of the Laplacian with its zero mode removed,

$$\det{}'\Box = \prod_{I\neq 0} \lambda_I, \tag{3.259}$$

and therefore

$$Z[J] = V(\det{}'\Box)^{-1/2} \exp\left[\frac{1}{2}\sum_{I\neq 0}\frac{J_I^2}{\lambda_I}\right] \tag{3.260}$$

Introducing the Green function

$$G'(\sigma,\sigma') = \langle\sigma|\Box^{-1}|\sigma'\rangle \tag{3.261}$$

where the inverse is restricted to the non-zero-mode subspace, we can write

$$\sum_{I\neq 0}\frac{J_I^2}{\lambda_I} = \iint d^2\sigma \, d^2\sigma' \, J(\sigma)G'(\sigma,\sigma')J(\sigma') \tag{3.262}$$

Thus

$$Z[J] = V(\det{}'\Box)^{-1/2} \exp\left[\frac{1}{2}\iint d^2\sigma \, d^2\sigma' \, J(\sigma)G'(\sigma,\sigma')J(\sigma')\right] \tag{3.263}$$

We can now explicitly differentiate with respect to the sources. The first functional derivative gives

$$\frac{\delta Z[J]}{\delta J(\sigma)} = V(\det{}'\Box)^{-1/2}\left[\int d^2\sigma' \, G'(\sigma,\sigma')J(\sigma')\right]\exp\left[\frac{1}{2}\iint JG'J\right] \tag{3.264}$$

Taking the second derivative gives

$$\frac{\delta^2 Z[J]}{\delta J(\sigma)\delta J(\sigma')} = V(\det{}'\Box)^{-1/2}\Bigg[G'(\sigma,\sigma') \tag{3.265}$$

$$+ \left(\int d^2\sigma'' \, G'(\sigma,\sigma'')J(\sigma'')\right)\left(\int d^2\sigma''' \, G'(\sigma',\sigma''')J(\sigma''')\right)\Bigg] e^{\frac{1}{2}\iint JG'J} \tag{3.266}$$

Setting $J = 0$ removes the second term, and therefore

$$\left.\frac{\delta^2 Z[J]}{\delta J(\sigma)\delta J(\sigma')}\right|_{J=0} = V(\det{}'\Box)^{-1/2} G'(\sigma,\sigma') \tag{3.267}$$

Since

$$Z[0] = V(\det{}'\Box)^{-1/2} \tag{3.268}$$

we finally obtain

$$\langle X(\sigma)X(\sigma')\rangle = G'(\sigma,\sigma') \tag{3.269}$$

We now construct this Green function explicitly. The eigenvalue equation may be written in operator notation as

$$\Box|\psi_I\rangle = \lambda_I|\psi_I\rangle \tag{3.270}$$

Taking the inner product with $\langle\psi_J|$ gives

$$\langle\psi_J|\Box|\psi_I\rangle = \lambda_I\delta_{JI} \tag{3.271}$$

Completeness on the full space reads

$$\sum_I |\psi_I\rangle\langle\psi_I| = \hat{1} \tag{3.272}$$

Since the zero mode cannot be inverted, however, the operator can be decomposed as

$$\Box = \sum_{I\neq 0} \lambda_I|\psi_I\rangle\langle\psi_I| \tag{3.273}$$

The inverse on the non-zero-mode subspace is therefore

$$\Box^{-1} = \sum_{I\neq 0} \frac{1}{\lambda_I}|\psi_I\rangle\langle\psi_I| \tag{3.274}$$

Consequently,

$$\Box^{-1}\Box = \sum_{I\neq 0} |\psi_I\rangle\langle\psi_I| = \hat{1} - |\psi_0\rangle\langle\psi_0| \tag{3.275}$$

Thus the Green function is not the inverse of $\Box$ on the complete space. Instead, it satisfies

$$\Box G'(\sigma,\sigma') = \delta^{(2)}(\sigma-\sigma') - \psi_0(\sigma)\psi_0^*(\sigma') \tag{3.276}$$

In position space,

$$G'(\sigma,\sigma') = \langle\sigma|\Box^{-1}|\sigma'\rangle = \sum_{I\neq 0}\frac{1}{\lambda_I}\langle\sigma|\psi_I\rangle\langle\psi_I|\sigma'\rangle = \sum_{I\neq 0}\frac{1}{\lambda_I}\psi_I(\sigma)\psi_I^*(\sigma') \tag{3.277}$$

We now specialize to the torus. Using the real coordinates (σ^1,σ^2), with both coordinates periodically identified, the eigenfunctions can be chosen as

$$\psi_{m,n}(\sigma) = e^{2\pi i(n\sigma^1+m\sigma^2)},\ m,n\in\mathbb{Z} \tag{3.278}$$

For the gauge-fixed torus metric

$$g_{ab} = \frac{1}{\tau_2}\begin{pmatrix}1 & \tau_1\\ \tau_1 & |\tau|^2\end{pmatrix} \tag{3.279}$$

the inverse metric is

$$g^{ab} = \frac{1}{\tau_2}\begin{pmatrix}|\tau|^2 & -\tau_1\\ -\tau_1 & 1\end{pmatrix} \tag{3.280}$$

Since $\det g = 1$, the scalar Laplacian is

$$\Box = g^{ab}\partial_a\partial_b \tag{3.281}$$

Therefore

$$\Box = \frac{1}{\tau_2}\left(|\tau|^2\partial_1^2 - 2\tau_1\partial_1\partial_2 + \partial_2^2\right) \tag{3.282}$$

Acting on the eigenfunction gives

$$\Box\psi_{m,n} = \frac{1}{\tau_2}\left[|\tau|^2(2\pi in)^2 - 2\tau_1(2\pi in)(2\pi im) + (2\pi im)^2\right]\psi_{m,n} \tag{3.283}$$

$$= -\frac{4\pi^2}{\tau_2}\left(|\tau|^2 n^2 - 2\tau_1 mn + m^2\right)\psi_{m,n} \tag{3.284}$$

The expression in parentheses can be written as

$$|\tau|^2 n^2 - 2\tau_1 mn + m^2 = |n\tau - m|^2 \tag{3.285}$$

Thus,

$$\lambda_{m,n} = -\frac{4\pi^2}{\tau_2}|n\tau - m|^2 \tag{3.286}$$

If instead one defines the positive Laplacian as $-\Box$, the eigenvalues are positive. In what follows, we use the positive convention $\Delta \equiv -\Box$, so that

$$\Delta\psi_{m,n} = \frac{4\pi^2}{\tau_2}|n\tau - m|^2\psi_{m,n} \tag{3.287}$$

The Green function is therefore

$$G'(\sigma,\sigma') = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{4\pi^2|n\tau - m|^2} e^{2\pi i[n(\sigma^1-\sigma'^1)+m(\sigma^2-\sigma'^2)]} \tag{3.288}$$

Setting $\sigma' = 0$ gives

$$G'(\sigma,0) = \frac{\tau_2}{4\pi^2}\sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(n\sigma^1+m\sigma^2)}}{|n\tau - m|^2} \tag{3.289}$$

We now evaluate this sum explicitly. The condition $(m,n) \neq (0,0)$ allows us to separate the $m = 0$ contribution from the terms with $m \neq 0$:

$$G'(\sigma,0) = \frac{\tau_2}{4\pi^2}\sum_{n\neq 0}\frac{e^{2\pi in\sigma^1}}{|n\tau|^2} + \frac{\tau_2}{4\pi^2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{e^{2\pi i(n\sigma^1+m\sigma^2)}}{|n\tau - m|^2} \tag{3.290}$$

Since

$$|n\tau|^2 = n^2\tau_2^2, \tag{3.291}$$

the first term becomes

$$\frac{\tau_2}{4\pi^2}\sum_{n\neq 0}\frac{e^{2\pi in\sigma^1}}{n^2\tau_2^2} = \frac{1}{4\pi^2\tau_2}\sum_{n\neq 0}\frac{e^{2\pi in\sigma^1}}{n^2} \tag{3.292}$$

Using

$$\sum_{n\neq 0}\frac{e^{2\pi inx}}{n^2} = 2\pi^2 B_2(x) \tag{3.293}$$

where $B_2(x) = x^2 - x + \frac{1}{6}$, and obtain

$$\frac{1}{4\pi^2\tau_2}\sum_{n\neq 0}\frac{e^{2\pi in\sigma^1}}{n^2} = \frac{1}{2\tau_2}B_2(\sigma^1) \tag{3.294}$$

We now turn to the double sum. Since $|n\tau - m|^2 = (n\tau_1 - m)^2 + n^2\tau_2^2$, we can write

$$\sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{e^{2\pi i(n\sigma^1+m\sigma^2)}}{(n\tau_1-m)^2+n^2\tau_2^2} \tag{3.295}$$

The useful step is to rewrite the denominator through a Fourier integral. For fixed $m \neq 0$ and $n \neq 0$, introduce

$$t = \frac{x - n\tau_1}{|n|\tau_2}, \tag{3.296}$$

so that $x = n\tau_1 + |n|\tau_2 t$. Consequently,

$$(x - n\tau_1)^2 + n^2\tau_2^2 = n^2\tau_2^2(t^2+1). \tag{3.297}$$

The Fourier representation that we need is

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i\alpha t}}{1+t^2}\, dt = \pi e^{-2\pi|\alpha|} \tag{3.298}$$

Taking $\alpha = n(\sigma^1 - m)$ and treating separately the cases $\alpha > 0$ and $\alpha < 0$ gives the same result through the absolute value,

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i n(\sigma^1-m)t}}{1+t^2}\, dt = \pi e^{-2\pi|n||\sigma^1-m|} \tag{3.299}$$

Therefore the non-zero m contribution can be expressed as

$$G'(\sigma,0) = \frac{1}{2\tau_2}B_2(\sigma^1) + \frac{1}{4\pi}\sum_{m\neq 0}\sum_{n\neq 0}\frac{1}{|n|}e^{2\pi i m\sigma^2}e^{2\pi i n\sigma^1}e^{-2\pi i mn\tau_1}e^{-2\pi|n|\tau_2|\sigma^1-m|} \tag{3.300}$$

At this stage it is useful to separate the positive and negative values of n. For $n > 0$ we obtain

$$\frac{1}{n}e^{2\pi i m\sigma^2}e^{2\pi i n\sigma^1}e^{-2\pi i mn\tau_1}e^{-2\pi n\tau_2|\sigma^1-m|} = \frac{1}{n}\exp\Big[2\pi i n\Big(\sigma^2 + \tau_1(\sigma^1-m) + i\tau_2|\sigma^1-m|\Big)\Big] \tag{3.301}$$

Similarly, for $n < 0$, writing $n = -k$ with $k > 0$ gives

$$\frac{1}{k}e^{2\pi i m\sigma^2}e^{-2\pi i k\sigma^1}e^{2\pi i km\tau_1}e^{-2\pi k\tau_2|\sigma^1-m|} = \frac{1}{k}\exp\Big[-2\pi i k\Big(\sigma^2 + \tau_1(\sigma^1-m) - i\tau_2|\sigma^1-m|\Big)\Big] \tag{3.302}$$

Renaming k back to n, we therefore have

$$G'(\sigma,0) = \frac{1}{2\tau_2}B_2(\sigma^1) + \frac{1}{4\pi}\sum_{m\neq 0}\sum_{n>0}\frac{1}{n}e^{2\pi i n[\sigma^2+\tau_1(\sigma^1-m)+i\tau_2|\sigma^1-m|]} + \frac{1}{4\pi}\sum_{m\neq 0}\sum_{n>0}\frac{1}{n}e^{-2\pi i n[\sigma^2+\tau_1(\sigma^1-m)-i\tau_2|\sigma^1-m|]} \tag{3.303}$$

We can now use the elementary series

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = -\log(1-x), \qquad |x| < 1 \tag{3.304}$$

Thus,

$$G'(\sigma,0) = \frac{1}{2\tau_2}B_2(\sigma^1) - \frac{1}{4\pi}\sum_{m\neq 0}\log\Big[1 - e^{2\pi i[\sigma^2+\tau_1(\sigma^1-m)+i\tau_2|\sigma^1-m|]}\Big] - \frac{1}{4\pi}\sum_{m\neq 0}\log\Big[1 - e^{-2\pi i[\sigma^2+\tau_1(\sigma^1-m)-i\tau_2|\sigma^1-m|]}\Big] \tag{3.305}$$

Combining the two logarithms gives

$$G'(\sigma,0)=\frac{1}{2\tau_2}B_2(\sigma^1)-\frac{1}{4\pi}\log\prod_{m\neq 0}\left[1-e^{2\pi i[\sigma^2+\tau_1(\sigma^1-m)+i\tau_2|\sigma^1-m|]}\right]\times\left[1-e^{-2\pi i[\sigma^2+\tau_1(\sigma^1-m)-i\tau_2|\sigma^1-m|]}\right] \tag{3.306}$$

After separating the $m>0$ and $m\leq 0$ contributions according to the absolute value $|\sigma^1-m|$, the product takes the form

$$\begin{aligned}G'(\sigma,0)=\frac{1}{2\tau_2}B_2(\sigma^1)-\frac{1}{4\pi}\log\Bigg\{&\prod_{m=1}^{\infty}\left(1-e^{2\pi iz}q^m\right)\left(1-e^{-2\pi iz}q^m\right)\\&\times\prod_{m=1}^{\infty}\left(1-e^{2\pi i\bar{z}}\bar{q}^m\right)\left(1-e^{-2\pi i\bar{z}}\bar{q}^m\right)\times\left(1-e^{2\pi iz}\right)\left(1-e^{-2\pi i\bar{z}}\right)\Bigg\}\end{aligned} \tag{3.307}$$

The two factors without powers of q can be combined using

$$\left(1-e^{2\pi iz}\right)\left(1-e^{-2\pi i\bar{z}}\right)=|2\sin(\pi z)|^2\,e^{i\pi(z-\bar{z})} \tag{3.308}$$

Hence,

$$G'(\sigma,0)=\frac{1}{2\tau_2}B_2(\sigma^1)-\frac{1}{4\pi}\log\left[|2\sin(\pi z)|^2\,e^{i\pi(z-\bar{z})}\left|\prod_{m=1}^{\infty}\left(1-e^{2\pi iz}q^m\right)\left(1-e^{-2\pi iz}q^m\right)\right|^2\right] \tag{3.309}$$

We now use the Jacobi product representation of the odd theta function,

$$\theta_1(z|\tau)=2q^{1/8}\sin(\pi z)\prod_{m=1}^{\infty}(1-q^m)(1-e^{2\pi iz}q^m)(1-e^{-2\pi iz}q^m) \tag{3.310}$$

Therefore,

$$|2\sin(\pi z)|^2\left|\prod_{m=1}^{\infty}(1-e^{2\pi iz}q^m)(1-e^{-2\pi iz}q^m)\right|^2=\frac{|\theta_1(z|\tau)|^2}{|q|^{1/4}\left|\prod_{m=1}^{\infty}(1-q^m)\right|^2} \tag{3.311}$$

Using the Dedekind eta function, we have

$$|2\sin(\pi z)|^2\left|\prod_{m=1}^{\infty}(1-e^{2\pi iz}q^m)(1-e^{-2\pi iz}q^m)\right|^2=\left|\frac{\theta_1(z|\tau)}{\eta(\tau)}\right|^2|q|^{-1/4+1/12} \tag{3.312}$$

which simplifies to

$$|2\sin(\pi z)|^2\left|\prod_{m=1}^{\infty}(1-e^{2\pi iz}q^m)(1-e^{-2\pi iz}q^m)\right|^2=\left|\frac{\theta_1(z|\tau)}{\eta(\tau)}\right|^2|q|^{-1/6}. \tag{3.313}$$

Since $|q|=e^{-2\pi\tau_2}$, we have $|q|^{-1/6}=e^{\pi\tau_2/3}$. Thus,

$$G'(\sigma,0)=\frac{1}{2\tau_2}B_2(\sigma^1)-\frac{1}{4\pi}\log\left[e^{i\pi(z-\bar{z})}e^{\pi\tau_2/3}\left|\frac{\theta_1(z|\tau)}{\eta(\tau)}\right|^2\right] \tag{3.314}$$

Using

$$B_2(\sigma^1) = (\sigma^1)^2 - \sigma^1 + \frac{1}{6}, \tag{3.315}$$

the polynomial contribution can be combined with the exponential factor. In particular, $e^{i\pi(z-\bar{z})} = e^{-2\pi\,\mathrm{Im}(z)}$ and, since $\mathrm{Im}(z) = \tau_2\sigma^1$, we recover $e^{i\pi(z-\bar{z})} = e^{-2\pi\tau_2\sigma^1}$. Combining this with the Bernoulli-polynomial contribution gives

$$G'(\sigma, 0) = -\frac{1}{4\pi} \log \left[e^{-2\pi \frac{[\mathrm{Im}(z)]^2}{\tau_2}} \left| \frac{\theta_1(z|\tau)}{\eta(\tau)} \right|^2 \right] \tag{3.316}$$

Thus, up to the overall normalization convention associated with the definition of the Laplacian, the torus Green function takes the standard form

$$G'(\sigma, 0) = -\frac{1}{4\pi} \log \left[e^{-2\pi \frac{[\mathrm{Im}(z)]^2}{\tau_2}} \left| \frac{\theta_1(z|\tau)}{\eta(\tau)} \right|^2 \right] \tag{3.317}$$

The appearance of the theta function is therefore not imposed from the beginning. It emerges naturally from the explicit evaluation of the non-zero-mode sum over the torus eigenfunctions. The Gaussian zero mode is excluded throughout, and the resulting Green function is the inverse of the Laplacian on the subspace orthogonal to the constant mode.

3.5 Fermions on the Torus

We start by formulating the action for a free massless fermion in $1+1$ dimensions in the Minkowski spacetime. The metric signature is $(\sigma^0, \sigma^1) = (-, +)$ which introduces an extra overall minus sign in the Dirac action

$$S = -\frac{1}{4\pi} \int d^2\sigma \; \chi^\dagger i\gamma^1\gamma^\alpha \partial_\alpha \chi \tag{3.318}$$

Here, χ denotes a two-component Dirac spinor, and γ^α are the Dirac matrices satisfying the Clifford algebra. In two dimensions, we make the convenient choice

$$\gamma^0 = -i\sigma^2 \; , \gamma^1 = \sigma^1 \tag{3.319}$$

where σ^i are the standard Pauli matrices. This representation indeed satisfies the canonical anti-commutation relations

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta} \tag{3.320}$$

Generally, a Dirac spinor can be decomposed into two Weyl spinors, corresponding to left- and right-moving components. In two dimensions, it is possible to impose both the Weyl (chirality) and Majorana (reality) conditions simultaneously. This restriction results in real, one-component fermions, which greatly simplifies the formulation of free fermion theories and the analysis of their holomorphic and anti-holomorphic sectors.

To transition to Euclidean spacetime, we perform a Wick rotation on the zeroth components of the coordinates and fields, $\sigma^0 \to -i\sigma^0_E$. The resulting Euclidean action is

$$S_E \equiv -iS = \frac{1}{2\pi} \int d^2w \left(\psi\bar{\partial}\psi - \tilde{\psi}\partial\tilde{\psi} \right) \tag{3.321}$$

where $w = \sigma^1 + i\sigma^0_E$, $\partial \equiv \partial_w$ and $\bar{\partial} \equiv \partial_{\bar{w}}$. The corresponding equations of motion for the left- and right-moving components are

$$\bar{\partial}\psi = 0 \ , \partial\tilde{\psi} = 0 \tag{3.322}$$

indicating that ψ is holomorphic and $\tilde{\psi}$ is anti-holomorphic.

In the free theory, the singular part of the operator product expansions (OPEs) is given by

$$\psi(z)\psi(w) \sim \frac{1}{z-w} \ , \ \tilde{\psi}(\bar{z})\tilde{\psi}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}} \tag{3.323}$$

The OPE between a left-moving and a right-moving fermion is purely regular. These OPEs form the basis for constructing correlation functions and defining the operator algebra in the two-dimensional fermionic conformal field theory. The holomorphic and anti-holomorphic components of the stress-energy tensor, which generate the Virasoro algebra, are expressed as

$$T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) : \tag{3.324a}$$

$$\bar{T}(\bar{z}) = -\frac{1}{2} : \tilde{\psi}(\bar{z})\bar{\partial}\tilde{\psi}(\bar{z}) : \tag{3.324b}$$

Here, normal ordering is denoted by colons and ensures that singular contributions from coincident points are removed. These operators encapsulate the local energy and momentum densities of the theory and act as the generators of the Virasoro algebra. In the free fermion conformal field theory, this decomposition into holomorphic and anti-holomorphic sectors allows one to treat left- and right-moving degrees of freedom independently. Moreover, the explicit form of $T(z)$ and $\bar{T}(\bar{z})$ allows us to compute their operator product expansions, and hence the determination of the central charges, which for a single real fermion equals $c = \bar{c} = 1/2$.

We are now interested in further studying a complex chiral free fermion. To describe it, one can combine two left-moving real (Majorana-Weyl) fermions, ψ_1 and ψ_2, into a single complex field. Each real chiral fermion carries only one degree of freedom, corresponding to its left-moving component, and satisfies the holomorphic equation of motion $\bar{\partial}\psi_i = 0$. By forming a complex linear combination,

$$\Psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \qquad \bar{\Psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \tag{3.325}$$

we obtain a complex field that still preserves chirality, meaning that Ψ remains purely left-moving.

The Euclidean action for the complex chiral fermion can be written directly in terms of Ψ and $\bar{\Psi}$ as

$$S_E[\Psi] = \frac{1}{2\pi}\int d^2w \, \bar{\Psi} \, \bar{\partial}\Psi \tag{3.326}$$

Here, $\bar{\partial}$ acts on the holomorphic coordinate w, and the action describes the dynamics of a single left-moving complex fermionic degree of freedom. If one wishes to construct a right-moving complex fermion instead, one simply combines two right-moving real chiral fermions and replaces $\bar{\partial}$ by ∂ in the action. In both cases, the resulting field continues to satisfy the appropriate chiral equation of motion and exhibits the expected holomorphic or anti-holomorphic behavior. Besides the global $O(N)$ symmetry under which the action remains invariant, there exists also a $\mathbb{Z}_2$ symmetry. The two symmetries do commute.

The OPEs of the complex chiral fermion reflect the independence of the underlying real chiral components. Explicitly, we have

$$\Psi(z)\bar{\Psi}(w) \sim \frac{1}{z-w}, \qquad \Psi(z)\Psi(w) \sim 0, \qquad \bar{\Psi}(z)\bar{\Psi}(w) \sim 0 \tag{3.327}$$

Only the mixed combination $\Psi(z)\bar{\Psi}(w)$ exhibits singular behavior, while the OPEs of Ψ with itself or $\bar{\Psi}$ with itself are regular, which encodes the fact that the two real chiral fermions contributing to Ψ are independent.

Finally, the stress-energy tensor for the left-moving complex fermion is

$$T(z) = - : \bar{\Psi}\partial\Psi : \tag{3.328}$$

where again colons denote normal ordering to remove singular contributions at coincident points. Since the complex chiral fermion is composed of two left-moving real chiral fermions, its central charge is $c = 1$.

In two dimensions, an interesting feature of fermionic theories is the possibility of *bosonization*, which provides an exact equivalence between fermions and compactified bosons (on a circle of a specified radius R). This correspondence arises because in 2D the algebra of holomorphic (or anti-holomorphic) fermions can be represented in terms of the derivatives of a scalar field. Intuitively, the one-dimensional nature of left- or right-moving excitations allows fermionic operators to be expressed as exponentials of a chiral boson field, reproducing the correct operator product expansions and correlation functions. We will return to this perspective in the next section to illustrate how bosonization can be used as a complementary method for evaluating the partition function of a complex chiral free fermion on a torus with arbitrary γ, δ boundary conditions.

Let us now elaborate more on the structure of a complex free chiral fermion defined on the cylinder. Although our primary objective is the computation of its partition function on the torus, it is conceptually clarifying to first dissect the theory on the simpler cylindrical geometry. Indeed, the dynamics along the spacetime direction σ^1 is treated in the same manner on both surfaces, and the transition from the cylinder to the torus introduces only a controlled modification: the fermion acquires a non-trivial monodromy parameter δ along the Euclidean time circle. Once the additional monodromy parameter δ associated with temporal winding on the torus is introduced, the trace should be modified only by a phase insertion.

We parametrize the cylinder by complex coordinates $w = \sigma^1 + i\sigma^2$, where the spatial coordinate is compactified as $\sigma^1 \sim \sigma^1 + 2\pi$. The boundary conditions imposed on the fermion field are specified by a discrete parameter $\gamma \in \{0, 1\}$. The case $\gamma = 1$ corresponds to periodic (Ramond) boundary conditions, whereas $\gamma = 0$ enforces anti-periodicity (Neveu–Schwarz). Explicitly,

$$\Psi(\sigma^1 + 2\pi, \sigma^2) \;=\; -e^{-i\pi\gamma}\Psi(\sigma^1, \sigma^2) \tag{3.329}$$

and

$$\bar{\Psi}(\sigma^1 + 2\pi, \sigma^2) \;=\; -e^{i\pi\gamma}\bar{\Psi}(\sigma^1, \sigma^2) \tag{3.330}$$

These spatial boundary conditions fix the fermionic mode quantization. One finds that the admissible frequencies take the general form

$$\nu \;=\; n+\frac{1-\gamma}{2}\,, \qquad n\in\mathbb{Z} \tag{3.331}$$

so that the NS sector corresponds to half-integer moding while the R sector corresponds to integer one.

In terms of these frequencies, the holomorphic and anti-holomorphic components of the fermion field decompose as

$$\Psi(w) \;=\; \sum_{n\in\mathbb{Z}} \Psi_{\,n+\frac{1-\gamma}{2}}\, \exp\!\Big[i\left(n+\frac{1-\gamma}{2}\right)w\Big]\,, \qquad \bar\Psi(\bar w) \;=\; \sum_{n\in\mathbb{Z}} \bar\Psi_{\,-n-\frac{1-\gamma}{2}}\, \exp\!\Big[-i\left(n+\frac{1-\gamma}{2}\right)\bar w\Big] \tag{3.332}$$

The associated mode operators satisfy the canonical anti-commutation relations

$$\{\Psi_\nu,\bar\Psi_\mu\} \;=\; \delta_{\nu+\mu,\,0}\,, \qquad \{\Psi_\nu,\Psi_\mu\}=\{\bar\Psi_\nu,\bar\Psi_\mu\}=0 \tag{3.333}$$

ensuring a consistent fermionic Fock space. The vacuum is required to be annihilated by all positive-frequency modes, with the notion of positivity determined by the shifted integer $n+\frac{1-\gamma}{2}$. In the Ramond sector, special attention must be paid to the existence of fermionic zero modes and the corresponding two-dimensional Clifford algebra, whose representation space produces a ground-state degeneracy. In contrast, the NS sector admits a unique vacuum.

To construct the physical operators of interest—namely, the $U(1)$ charge and the Hamiltonian—we introduce the fermionic occupation numbers through standard normal-ordered products. The holomorphic oscillator number operator is

$$N \;=\; \sum_{n>0} :\Psi_{-n+\frac{1-\gamma}{2}}\,\bar\Psi_{\,n-\frac{1-\gamma}{2}}: \;\equiv\; \sum_{n>0} N_{\,n-\frac{1-\gamma}{2}} \tag{3.334}$$

while its anti-holomorphic counterpart reads

$$\bar N \;=\; \sum_{n\geq 0} :\bar\Psi_{-n-\frac{1-\gamma}{2}}\,\Psi_{\,n+\frac{1-\gamma}{2}}: \;\equiv\; \sum_{n\geq 0} \bar N_{\,n+\frac{1-\gamma}{2}} \tag{3.335}$$

Upon substituting these expressions into the Noether current associated with the global $U(1)$ symmetry, and employing a regularization scheme (by introducing for instance an exponential cutoff in the frequency) to treat the formally divergent sums, one arrives at the cylinder charge operator

$$(J_0)_{\rm cyl} \;=\; N-\bar N+\frac{\gamma}{2} \tag{3.336}$$

The vacuum shift $\frac{\gamma}{2}$ encodes the difference between the Ramond and Neveu–Schwarz sectors; it arises from the normal-ordering of the fermionic bilinear and reflects the distinct ground-state structures in the two spin sectors.

The Hamiltonian on the cylinder, expressed in terms of the Virasoro zero mode L_0 of the energy–momentum tensor, takes the form

$$(L_0)_{\rm cyl} \;=\; \sum_{n>0}\left(n-\frac{1-\gamma}{2}\right) N_{\,n-\frac{1-\gamma}{2}} + \sum_{n\geq 0}\left(n+\frac{1-\gamma}{2}\right)\bar N_{\,n+\frac{1-\gamma}{2}} \tag{3.337}$$

Expressed in terms of the Euclidean action, the path integral has the form

$$Z=\int \mathcal{D}\Psi\mathcal{D}\bar\Psi\mathcal{D}\tilde\Psi\tilde{\bar\Psi}e^{-S_E} \tag{3.338}$$

We are now ready to discuss the complex fermion CFT on the torus, meaning that we should impose the boundary conditions

$$\Psi(\sigma^1+2\pi,\sigma^2)=-e^{-i\pi\gamma}\Psi(\sigma^1,\sigma^2)=e^{i\pi(1-\gamma)}\Psi(\sigma^1,\sigma^2) \tag{3.339a}$$

$$\Psi(\sigma^1,\sigma^2+2\pi)=-e^{i\pi\delta}\Psi(\sigma^1,\sigma^2)=e^{i\pi(\delta-1)}\Psi(\sigma^1,\sigma^2) \tag{3.339b}$$

or if we wish to transition to the z-plane, where $z=e^{-iw}$,

$$\Psi(e^{-2\pi i}z)=e^{-\pi i\gamma}\Psi(z)\ ,\ \bar{\Psi}(e^{-2\pi i}z)=e^{\pi i\gamma}\bar{\Psi}(z) \tag{3.340}$$

The parameters $\gamma,\delta\in(-1,1]$ and for the values $0,1$ they correspond to (anti-) periodic boundary conditions to the two cycles of the torus. As expected the spatial part remains the same as in the cylinder. Solving the equations of motion we obtain the mode expansions

$$\Psi(\sigma^1,\sigma^2)=\sum_{n,m\in\mathbb{Z}}\Psi_{n,m}e^{2\pi i\left[\left(n+\frac{1-\gamma}{2}\right)\sigma^1+\left(m+\frac{\delta-1}{2}\right)\sigma^2\right]} \tag{3.341}$$

$$\bar{\Psi}(\sigma^1,\sigma^2)=\sum_{n,m\in\mathbb{Z}}\bar{\Psi}_{n,m}e^{-2\pi i\left[\left(n+\frac{1-\gamma}{2}\right)\sigma^1+\left(m+\frac{\delta-1}{2}\right)\sigma^2\right]} \tag{3.342}$$

As implied by the equations of motion satisfied for the complex fermion, it suffices to focus on the left moving part of the path integral and then generalize the acquired result. Hence, the left-moving action in the Euclidean space is

$$\begin{aligned}S_{E(L)}&=\frac{1}{2\pi}\int d^2w\ \Psi\bar{\partial}\bar{\Psi}\\&=\frac{1}{2\pi}\int d^2w\sum_{n,m\in\mathbb{Z}}\sum_{\bar{n},\bar{m}\in\mathbb{Z}}\Psi_{n,m}\bar{\Psi}_{\bar{n},\bar{m}}e^{2\pi i\left[(n-\bar{n})\sigma^1+(m-\bar{m})\sigma^2\right]}\left(-\frac{i}{2\tau_2}\right)2\pi i\left[\tau\left(\bar{n}+\frac{1-\gamma}{2}\right)-\left(\bar{m}+\frac{\delta-1}{2}\right)\right]\\&=\frac{1}{2\tau_2}\sum_{n,m\in\mathbb{Z}}\Psi_{n,m}\bar{\Psi}_{n,m}\left[\tau\left(n+\frac{1-\gamma}{2}\right)-\left(m+\frac{\delta-1}{2}\right)\right]\end{aligned} \tag{3.343}$$

which leads to the calculation of the determinant

$$det(\bar{\partial})=\prod_{n,m\in\mathbb{Z}}\frac{1}{2\tau_2}\left[\tau\left(n+\frac{1-\gamma}{2}\right)-\left(m+\frac{\delta-1}{2}\right)\right] \tag{3.344}$$

However, there exists a subtlety. The difficulty in regularizing the determinant of the chiral Dirac operator $\bar{\partial}$ on a Euclidean manifold originates from the fact that its spectrum is not confined to a single ray in the complex plane. Since $\bar{\partial}$ is not a Hermitian operator, its eigenvalues are generically complex. As a consequence, the zeta-function definition of $\log\left(\det\bar{\partial}\right)$ requires a choice of branch cut for the complex logarithm, but no canonical choice exists. The resulting determinant is therefore intrinsically ambiguous up to a complex phase, reflecting the familiar chiral anomaly. This ambiguity disappears if instead one considers the positive elliptic operator obtained by pairing ∂ with its adjoint $\bar{\partial}$, since this product has real eigenvalues.

Trace: The trace representation of the chiral complex fermion partition function on the torus can be obtained by comparing it with the corresponding construction on the cylinder. In the cylindrical geometry the fermion modes are defined with a fixed choice of spin structure, encoded in the parameter γ. When

the theory is placed on a torus, however, the fermionic fields acquire an additional nontrivial monodromy along the time cycle, parametrized by δ, such that transporting a fermion once around the cycle yields a phase $e^{\pi i\delta}$.

To implement this modified periodicity condition within the operator formalism, it is convenient to insert into the trace a phase operator that produces exactly this twist. The natural choice is the fermion number (or charge-parity) operator J_0, introduced in (3.336), whose eigenvalues shift the phase appropriately for the δ-dependent monodromy. Hence, the holomorphic part of torus partition function takes the form

$$Z\begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \mathrm{Tr}_{\mathcal{H}_\gamma}\left[e^{\pi i\delta J_0}\, q^{L_0-1/24}\right] \tag{3.345}$$

The factor $q^{L_0-1/24}$ arises from spatial translations around the Euclidean cycle, with the $1/24$ shift accounting for the Casimir energy of a chiral fermion. Since the theory is fermionic, all occupation numbers $M_n, \bar{M}_n$ obey Fermi statistics and therefore take values in $\{0,1\}$. A general state for such fermion takes the form

$$|\{M,\bar{M}\};\gamma\rangle = \prod_{n>0}\left(\Psi_{-n+\frac{1-\gamma}{2}}\right)^{M_{n-\frac{1-\gamma}{2}}} \prod_{n\geq 0}\left(\bar{\Psi}_{-n-\frac{1-\gamma}{2}}\right)^{\bar{M}_{n+\frac{1-\gamma}{2}}} |0_\gamma\rangle \tag{3.346}$$

Evaluating the trace, one may separate the zero-mode (when present) and oscillator contributions. The zero-mode yields the phase

$$e^{i\pi\gamma\delta/2}\, q^{\gamma^2/8-1/24} \tag{3.347}$$

The oscillators factorize into independent contributions from each Fourier mode, and after a convenient shift of the summation indices the final product representation becomes

$$\begin{aligned} Z\begin{bmatrix}\gamma\\ \delta\end{bmatrix} &= e^{i\pi\gamma\delta/2}\, q^{\gamma^2/8-1/24} \prod_{n>0}\sum_{M_n=0,1} e^{i\pi\delta\, M_n}\, q^{(n-\frac{1-\gamma}{2})\, M_n} \prod_{\bar{n}\geq 0}\sum_{\bar{M}_n=0,1} e^{-i\pi\delta\,\bar{M}_n} q^{\left(\bar{n}+\frac{1-\gamma}{2}\right)\bar{M}_n} \\ &= e^{i\pi\gamma\delta/2}\, q^{\gamma^2/8-1/24} \prod_{n>0}\left(1+q^{n+\frac{\gamma}{2}-\frac{1}{2}}e^{\pi i\delta}\right)\left(1+q^{n-\frac{\gamma}{2}-\frac{1}{2}}e^{-\pi i\delta}\right) \end{aligned} \tag{3.348}$$

This representation makes explicit the decomposition into zero-mode and oscillator contributions and demonstrates the dependence on the spin structure through the parameters γ and δ, providing the operator-formalism derivation of the chiral complex fermion partition function on the torus.

Path Integral (Formal): To evaluate the partition function of the chiral contribution via the path integral formalism, we define

$$\det(\partial\bar{\partial}) = \exp\left[-\,\partial_s\big|_{s=0}\sum_{m,n}|\lambda_{m,n}|^{-2s}\right] = \exp\left[-\,\partial_s\big|_{s=0}\,\mathrm{Tr}(\partial\bar{\partial})^{-s}\right] \tag{3.349}$$

Using the Mellin transform representation of the spectral zeta function,

$$x^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{\,s-1}e^{-xt}, \tag{3.350}$$

we rewrite the logarithm of the determinant in heat–kernel form:

$$\log\det(\partial\bar{\partial}) = -\,\partial_s|_{s=0}\left[\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\sum_{m,n\in\mathbb{Z}} e^{-|\lambda_{m,n}|^2 t}\right] \tag{3.351}$$

The integrand diverges for $t \to 0$ when $s = 0$, so interchanging the integral with the double sum over (m, n) is not permissible. To proceed, we initially Poisson resum over the integer m, which appears linearly in the eigenvalue expression. This yields

$$\begin{aligned}\sum_{m\in\mathbb{Z}} e^{-|\lambda_{m,n}|^2 t} &= \sum_{m\in\mathbb{Z}} e^{-\frac{t}{(2\tau_2)^2}\left[m+\frac{1-\delta}{2}+\tau_1\left(n+\frac{1-\gamma}{2}\right)\right]^2-\frac{t}{4}(n+\frac{1-\gamma}{2})^2} \\ &= 2\tau_2\sqrt{\frac{\pi}{t}}\sum_{m\in\mathbb{Z}} e^{-\frac{(2\pi m\tau_2)^2}{t}-\left(\frac{n}{2}+\frac{1-\gamma}{2}\right)^2 t} e^{-2\pi i m\left[\frac{1-\delta}{2}+\tau_1(n+\frac{1-\gamma}{2})\right]}\end{aligned} \tag{3.352}$$

At this stage it is useful to separate the contributions of the zero mode ($m = 0$) and the nonzero modes ($m \neq 0$). The $m \neq 0$ sector is finite at $s = 0$ and can be evaluated directly. Using the integral representation of the modified Bessel function K_ν, the result takes the form

$$\log\left|\prod_{n>0}\left(1+e^{i\pi\delta}\,q^{n+\frac{\gamma}{2}-\frac{1}{2}}\right)\left(1+e^{-i\pi\delta}\,q^{n-\frac{\gamma}{2}-\frac{1}{2}}\right)\right|^2. \tag{3.353}$$

Remark. The analytic structure of this contribution closely parallels the one encountered in the path–integral computation of a compact scalar CFT on the torus. In both theories, Poisson resummation isolates an infinite tower of winding or momentum modes, and the resulting product formula reorganizes naturally into a modified Bessel integral of the second kind.

The $m = 0$ term contains the ultraviolet divergence responsible for the obstruction at $s = 0$. To regularize it, one must use Hurwitz zeta regularization before taking the limit $s \to 0$. After analytic continuation, the zero–mode contribution evaluates to

$$-4\pi\tau_2\left(\frac{\gamma^2}{8}-\frac{1}{24}\right) \tag{3.354}$$

Finally, combining the nonzero–mode sector with the regularized zero–mode term, we obtain the determinant of the chiral Dirac operator:

$$det(\partial\bar{\partial}) = \left|q^{\frac{\gamma^2}{8}-\frac{1}{24}}\frac{\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0|\tau)}{\eta(\tau)}\right|^2 \tag{3.355}$$

which agrees with the trace representation outcome up to a phase ambiguity.

Path Integral (Informal): There exists an alternative in the treatment of the path integral which however is not that rigorous since it splits divergent products in two independent infinite products, without a guarantee that this is correct. We begin from the expression

$$det(\partial\bar{\partial}) = \prod_{n,m\in\mathbb{Z}}\frac{1}{(2\tau_2)^2}\left|\tau\left(n+\frac{1-\gamma}{2}\right)-\left(m+\frac{\delta-1}{2}\right)\right|^2 \tag{3.356}$$

To make sense of this product in terms of zeta regularization techniques, we commence considering the infinite product with respect to m, i.e. the one not multiplied by the modulus τ

$$\begin{aligned} det(\partial\bar{\partial}) &= \prod_{n\in\mathbb{Z}} \frac{1}{(2\tau_2)^2} \left|\tau\left(n+\frac{1-\gamma}{2}\right)+\frac{1-\delta}{2}\right|^2 \prod_{m>0} \left|\left(\tau\left(n+\frac{1-\gamma}{2}\right)+\frac{1-\delta}{2}\right)^2-m^2\right|^2 \\ &= \prod_{n\in\mathbb{Z}} \left|\tau\left(n+\frac{1-\gamma}{2}\right)+\frac{1-\delta}{2}\right|^2 \prod_{m>0} (-m)^4 \left|1-\frac{\left(\tau\left(n+\frac{1-\gamma}{2}\right)+\frac{1-\delta}{2}\right)^2}{m^2}\right|^2 \end{aligned} \tag{3.357}$$

In (3.357) we recognize Weierstrass factorization formula for $\sin\pi z$

$$\sin\pi z = \pi z \prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}} = \pi z \prod_{n=1}^{\infty}\left[1-\left(\frac{z}{n}\right)^2\right] \tag{3.358}$$

The above formula can be derived if we consider the exponential expression of the Mittag-Leffler theorem for the sine function. To get rid of the constant terms arising in the process, we are adopting the regularization schemes discussed in Ginsparg's lecture notes, namely that for a constant $a\in\mathbb{C}$

$$\prod_{n=1}^{\infty} a = a^{-1/2}\ ,\ \prod_{n=-\infty}^{\infty} a = 1 \tag{3.359}$$

If we expand the sine function in terms of exponentials, we end up with

$$det(\partial\bar{\partial}) = \prod_{n\in\mathbb{Z}} \left|e^{i\pi\frac{1-\delta}{2}} q^{\frac{1}{2}\left(n+\frac{1-\gamma}{2}\right)} - e^{-i\pi\frac{1-\delta}{2}} q^{-\frac{1}{2}\left(n+\frac{1-\gamma}{2}\right)}\right|^2 \tag{3.360}$$

We now decompose the sum in two separate sums of negative and positive n integer values and make the change $n\to -n$ to the negative integers sum

$$det(\partial\bar{\partial}) = \prod_{n\geq 0} \left|q^{-\frac{1}{2}\left(n+\frac{1-\gamma}{2}\right)}\right|^2 \left|1-q^{n-\frac{\gamma}{2}+\frac{1}{2}} e^{2\pi i\frac{1-\delta}{2}}\right|^2 \prod_{n>0} \left|q^{-\frac{1}{2}\left(n-\frac{1-\gamma}{2}\right)}\right|^2 \left|1-q^{n+\frac{\gamma}{2}-\frac{1}{2}} e^{-2\pi i\frac{1-\delta}{2}}\right|^2 \tag{3.361}$$

Both these products are a multiplication of a convergent times a divergent term, meaning we can extract the divergent parentheses out of the product and obtain using the Hurwitz zeta function regularization scheme

$$det(\partial\bar{\partial}) = \left|q^{\frac{\gamma^2}{8}-\frac{1}{24}} \prod_{n>0}\left(1+q^{n-\frac{\gamma}{2}-\frac{1}{2}}e^{-\pi i\delta}\right)\left(1+q^{n+\frac{\gamma}{2}-\frac{1}{2}}e^{\pi i\delta}\right)\right|^2 = \left|q^{\frac{\gamma^2}{8}-\frac{1}{24}} \frac{\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0|\tau)}{\eta(\tau)}\right|^2 \tag{3.362}$$

where we have used the Kiritsis's notation for the definition of the theta and the Dedekind eta function.

This specifies $det(\bar{\partial})$ up to a phase ambiguity A, which is obtained by matching the solution with the trace representation. Here $A = e^{i\pi\gamma\delta/2}$ and this phase ambiguity is related to the global gravitational anomaly.

Bosonization: As a last method of obtaining the partition function, we may exploit the property of bosonization in two dimensions. We start with the identifications

$$J = \Psi\bar{\Psi} = i\partial H(z) \tag{3.363}$$

$$\Psi = e^{iH(z)} \ , \ \bar{\Psi} = e^{-iH(z)} \tag{3.364}$$

Expanding the current in modes on the complex plane,

$$J(z) = \sum_{n\in\mathbb{Z}} \frac{J_n}{z^{n+1}}, \qquad J_n = \oint_{C_0} \frac{dz}{2\pi i}\, z^n J(z) \tag{3.365}$$

we find that the J_n satisfy the $U(1)$ Kac-Moody algebra $[J_m, J_n] = m\,\delta_{m+n,0}$. In particular, the zero mode J_0 represents the fermion number operator, $Q = J_0$, and its eigenvalues correspond to the quantized $U(1)$ charge of the fermionic states.

The bosonic field $H(z)$ is compactified on a circle of radius R, such that the vertex operators $e^{\pm iH(z)}$ are single-valued around the origin. This requirement enforces the canonical normalization of the $U(1)$ charge, yielding

$$R = \sqrt{\frac{\alpha'}{2}} \equiv 1 \tag{3.366}$$

At this radius, the vacuum $|0\rangle$ is annihilated by all positive modes of the current,

$$J_{n>0}|0\rangle = 0, \tag{3.367}$$

ensuring that it is a lowest-weight state of the $U(1)$ algebra and that the Fock space is built consistently by acting with negative modes.

In this formulation a general state can be represented as

$$|\{M\}; Q\rangle = \prod_{n>0} (J_{-n})^{M_n}|Q\rangle \ , \ Q \in \frac{\gamma}{2} + \mathbb{Z} \tag{3.368}$$

where the occupation numbers $M_n = 0, 1, 2, \ldots$ are now bosonic. Hence,

$$Z\begin{bmatrix}\gamma\\ \delta\end{bmatrix} = q^{-\frac{1}{24}} \prod_{n>0} \sum_{M_n=0}^{\infty} q^{nM_n} \sum_{Q\in\frac{\gamma}{2}+\mathbb{Z}} q^{\frac{Q^2}{2}} e^{\pi i\delta Q} = \frac{\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0|\tau)}{\eta(\tau)} \tag{3.369}$$

The sum representation of Jacobi's theta functions is generally (following Kiritsis' notation)

$$\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{\gamma}{2})^2} e^{2\pi i(z-\frac{\delta}{2})(n-\frac{\gamma}{2})} \tag{3.370}$$

If we insert the above definition to the full partition function involving both left and right moving parts, we obtain

$$Z_T = \frac{\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\bar{\theta}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}}{\eta\bar{\eta}} = \frac{1}{\eta\bar{\eta}} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}(m-\frac{\gamma}{2})^2} \bar{q}^{\frac{1}{2}(n-\frac{\gamma}{2})^2} e^{-i\pi\delta(m-n)} \tag{3.371}$$

Shifting the summation variable of the anti-holomorphic part to $n \to m - n$ we end up with

$$Z_T = \frac{1}{\eta\bar{\eta}} \sum_{m,n\in\mathbb{Z}} q^{\frac{\alpha'}{4}\left(\frac{m-n/2-\gamma/2}{\sqrt{\alpha'/2}}+\frac{n}{\sqrt{2\alpha'}}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{m-n/2-\gamma/2}{\sqrt{\alpha'/2}}-\frac{n}{\sqrt{2\alpha'}}\right)^2} e^{-i\pi\delta n} \tag{3.372}$$

In the above expression we can identify the sum with the $(1,1)$ Narain lattice with a fermion radius $R = \sqrt{\frac{\alpha'}{2}}$ which coincides with the aforementioned bosonization radius.

3.6 Orbifolds

We close this chapter by a short review of orbifolds. Let a conformal field theory have a finite symmetry generated by an element g satisfying $g^N = 1$. The orbifold theory is obtained by identifying configurations related by this symmetry, $\mathcal{C} \sim g\mathcal{C}$, and therefore by the cyclic group

$$\mathbb{Z}_N = \{1, g, g^2, \ldots, g^{N-1}\} \tag{3.373}$$

There are two logically distinct operations involved in the orbifold construction. The first is the projection onto states invariant under the action of g. The second is the introduction of twisted sectors in which closed strings are allowed to close only up to the action of an element of the orbifold group. The first operation produces the group average $\frac{1}{N}\sum_{a=0}^{N-1} g^a$, while the second produces the twisted sectors. When the theory is defined on a torus, both operations appear simultaneously and lead to the two-index objects $Z_{h,a}$. The purpose of the construction is therefore not merely to define a quotient of the original conformal field theory. The resulting string theory must also be consistent with the different choices of cycles used to describe the worldsheet torus. This requirement is expressed through modular invariance. The relation between the group action, the twisted sectors, the torus amplitudes and modular transformations will be developed below.

3.6.1 The $\mathbb{Z}_N$ representation and the orbifold projector

There is an important point concerning the representation used in the orbifold construction. The irreducible representations of the abelian group $\mathbb{Z}_N$ are one-dimensional. The natural N-dimensional representation is instead the regular representation, which can also be understood as the direct sum of the N one-dimensional characters of $\mathbb{Z}_N$. Introducing $\omega = e^{2\pi i/N}$ and states $|k'\rangle$, $\quad k' = 0, 1, \ldots, N-1$, such that

$$g|k'\rangle = \omega^{k'}|k'\rangle \tag{3.374}$$

We see that

$$g^a|k'\rangle = e^{2\pi i a k'/N}|k'\rangle \tag{3.375}$$

One may instead choose the conjugate convention

$$g^a|k'\rangle = e^{-2\pi i a k'/N}|k'\rangle \tag{3.376}$$

This latter convention will be used below since it is the convention that produces the phase appearing in the orbifold trace. The main reference for conventions is again [3]. A physical state of the quotient theory must be invariant under the group action. Consider a general state

$$|\psi\rangle = \sum_{k'=0}^{N-1} c_{k'}|k'\rangle \tag{3.377}$$

The invariance condition is $g|\psi\rangle = |\psi\rangle$. Using the representation of g, we see that

$$g|\psi\rangle = \sum_{k'} c_{k'} e^{2\pi i k'/N} |k'\rangle \tag{3.378}$$

Consequently, invariance requires $c_{k'} e^{2\pi i k'/N} = c_{k'}$, and therefore Hence, only the zero-charge state survives the projection. This condition can be implemented directly by introducing the projection operator

$$P = \frac{1}{N} \sum_{a=0}^{N-1} g^a \tag{3.379}$$

The projection can be inserted into the partition function of the original untwisted Hilbert space $\mathcal{H}_0$. This gives

$$Z_{\text{projected}} = \text{Tr}_{\mathcal{H}_0} \left(P q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right) \tag{3.380}$$

Substituting the explicit form of the projector gives

$$Z_{\text{projected}} = \frac{1}{N} \sum_{a=0}^{N-1} \text{Tr}_{\mathcal{H}_0} \left(g^a q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right) \tag{3.381}$$

This expression describes the projected untwisted sector. It is not yet the complete orbifold theory because the quotient permits additional closed-string boundary conditions.

3.6.2 Twisted sectors and torus amplitudes

After orbifolding, the string may instead close only up to the action of an element of the orbifold group. The corresponding boundary condition is

$$X(\sigma + 2\pi, \tau) = g^h X(\sigma, \tau) \tag{3.382}$$

where $h = 0, 1, \ldots, N-1$. Since $g^N = 1$, the values h and $h+N$ define the same boundary condition. The two indices have distinct physical meanings. The index h specifies the twisted Hilbert space obtained by quantizing along the spatial cycle, while a specifies the group element inserted when propagating around the second cycle. The torus therefore naturally requires two group labels rather than only the single label that was needed for the twisted boundary condition of a closed string.

The geometric quotient is consequently $T^d/\mathbb{Z}_N$ and the action of the orbifold group is directly connected to the action of the corresponding matrix on the compactification lattice. The additional basis vectors used in the construction impose further conditions on the allowed states and consequently modify the gauge structure. The successive stages of this construction are illustrated in Figure 3.3. The figure shows how the introduction of the additional basis vectors changes the lattice description and reduces the set of degrees of freedom that remain compatible with all the imposed identifications. The successive rows should therefore be read as successive stages of the same construction rather than as independent cases.

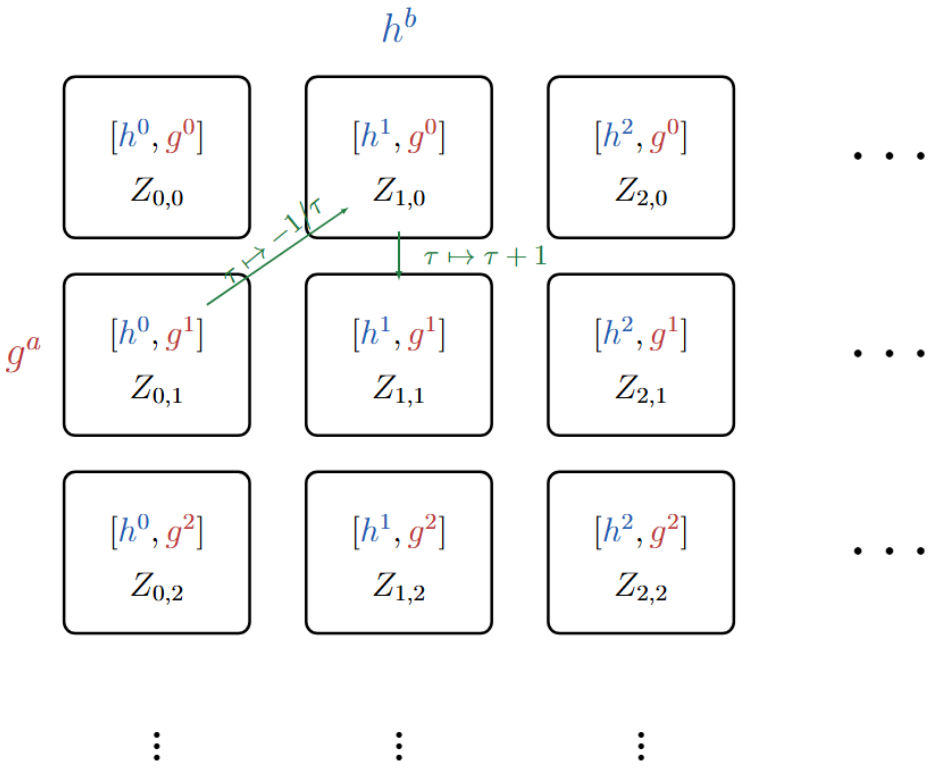


Figure 3.3: Successive stages of the orbifold construction showing the effect of the additional basis vectors on the allowed lattice structure and the corresponding reduction of the independent degrees of freedom

The gauge-theoretic consequence of the same construction is displayed in Figure 3.4. Each row corresponds to the addition of a new basis vector together with all the vectors introduced in the preceding rows. The resulting sequence shows how the allowed gauge degrees of freedom are progressively restricted. In particular, the rank reduction and the surviving gauge group can be followed directly as the successive basis vectors are included. The figure therefore provides the connection between the lattice or basis-vector construction and the gauge group obtained after the orbifold projection

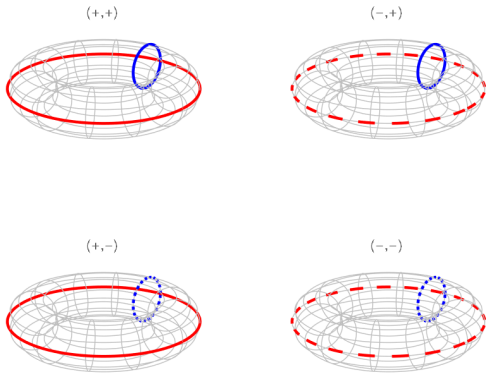


Figure 3.4: Visualization of the four different possibilities for periodic and antiperiodic boundary conditions in the two non–contractible cycles of the torus.

The geometric and worldsheet descriptions are therefore two descriptions of the same quotient construction. The geometric condition $\theta\Lambda = \Lambda$ ensures that the discrete action is compatible with the compactification lattice, while the worldsheet description expresses the same action through twisted boundary conditions. The basis-vector construction determines which states remain compatible with the discrete identifications and consequently determines the surviving gauge symmetry

3.6.3 Modular transformations and consistency

The torus contains two independent cycles and the modular group acts by changing the basis chosen for these cycles. Since the orbifold boundary conditions are specified by their group holonomies, modular transformations necessarily act on the pair (h, a). The torus has fundamental cycles

$$\alpha : z \longrightarrow z + 1 \tag{3.383a}$$
$$\beta : z \longrightarrow z + \tau \tag{3.383b}$$

and assign the holonomies

$$U_\alpha = g^h, \quad U_\beta = g^a \tag{3.384}$$

The complete information about the orbifold boundary conditions on the torus is therefore contained in the pair of holonomies $(U_\alpha, U_\beta) = (g^h, g^a)$. Under a T modular transformation, the labels transform as

$$T : (h, a) \longmapsto (h, a + h) \tag{3.385}$$

and the corresponding amplitudes satisfy

$$Z_{h,a}(\tau + 1) = Z_{h,a+h}(\tau) \tag{3.386}$$

The transformation can also be understood directly from the path integral. The T transformation changes the choice of temporal cycle by adding one copy of the spatial cycle. Since the spatial cycle carries the holonomy g^h, this additional cycle contributes precisely the factor g^h to the temporal holonomy. The resulting temporal insertion is therefore g^{a+h}.

On the other hand, the S transformation exchanges the two fundamental cycles. The new spatial cycle is the old temporal cycle with reversed orientation. Consequently the labels transform according to

$$S : (h, a) \longmapsto (a, -h) \tag{3.387}$$

and the corresponding amplitudes satisfy

$$Z_{h,a}\left(-\frac{1}{\tau}\right) = Z_{a,-h}(\tau) \tag{3.388}$$

These transformations show that the individual functions $Z_{h,a}$ do not have to be modular invariant. Instead, modular transformations permute the collection of twisted sectors and group insertions. The complete partition function is constructed precisely so that it is invariant under this permutation. Under the T transformation,

$$Z_{\text{orb}}(\tau + 1) = \frac{1}{N}\sum_{h,a} Z_{h,a}(\tau + 1) = \frac{1}{N}\sum_{h,a} Z_{h,a+h}(\tau) \tag{3.389}$$

Introduce $a' = a + h \pmod N$. For fixed h, as a runs over all elements of $\mathbb{Z}_N$, the shifted variable a' also runs over all elements of $\mathbb{Z}_N$.Therefore,

$$Z_{\text{orb}}(\tau + 1) = \frac{1}{N} \sum_{h,a'} Z_{h,a'}(\tau) = Z_{\text{orb}}(\tau) \tag{3.390}$$

The same argument can be applied to the S transformation. Starting from the definition of the full partition function gives

$$Z_{\text{orb}}\left(-\frac{1}{\tau}\right) = \frac{1}{N} \sum_{h,a} Z_{h,a}\left(-\frac{1}{\tau}\right) = \frac{1}{N} \sum_{h,a} Z_{a,-h}(\tau) \tag{3.391}$$

Define $h' = a$ and $a' = -h$. Then, the map $(h, a) \longrightarrow (h', a')$ is a bijection of $\mathbb{Z}_N \times \mathbb{Z}_N$. Hence

$$Z_{\text{orb}}\left(-\frac{1}{\tau}\right) = \frac{1}{N} \sum_{h',a'} Z_{h',a'}(\tau) = Z_{\text{orb}}(\tau) \tag{3.392}$$

Since S and T generate the modular group $SL(2, \mathbb{Z})$, invariance under these two transformations is sufficient to establish modular invariance of the complete orbifold partition function. The distinction between the individual blocks and their complete sum is important. A particular $Z_{h,a}$ represents one choice of worldsheet holonomies and therefore need not be invariant under a change of basis of the torus cycles. Modular transformations map this choice into another allowed pair of holonomies. The complete sum over all (h, a) is invariant because these transformations merely permute the sectors included in the sum.

The orbifold construction therefore combines three structures. The finite group determines the quotient and its representations. The worldsheet boundary conditions determine the twisted sectors and the corresponding Hilbert spaces. The modular group acts on the two torus cycles and consequently on the twisted-sector labels. The consistency of the resulting string theory requires these three structures to fit together, with the twisted sectors completing the projected theory and the full collection of amplitudes forming a modular-invariant torus partition function

Chapter 4

The Superstring

The bosonic string provides a consistent framework for describing a relativistic string, but its spectrum contains a tachyon and does not contain spacetime fermions. A natural way to overcome these limitations is to enlarge the worldsheet theory by introducing fermionic degrees of freedom. In the Ramond–Neveu–Schwarz (RNS) formulation, the worldsheet fermions are supersymmetric partners of the bosonic coordinates X^μ. Worldsheet supersymmetry then leads to a theory whose quantum consistency requires a ten-dimensional spacetime and whose spectrum, after the GSO projection, contains equal numbers of bosonic and fermionic degrees of freedom.

4.1 Classical Superstring

The RNS formulation is obtained by extending the Polyakov action with worldsheet fermions ψ^μ. In conformal gauge, the action can be written as

$$S = -\frac{1}{2\pi}\int d^2\sigma \left(\partial_a X_\mu \partial^a X^\mu + \bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right) \tag{4.1}$$

where ρ^a are the two-dimensional Dirac matrices and ψ^μ is a two-dimensional Majorana spinor carrying a spacetime index μ. The fermionic fields are Grassmann-valued and therefore anticommute. Decomposing the Majorana spinor into its two worldsheet chiral components gives

$$S = \frac{1}{\pi}\int d^2\sigma \left(2\partial_+ X \partial_- X + i\psi_- \partial_+ \psi_- + i\psi_+ \partial_- \psi_+\right) \tag{4.2}$$

The equations of motion show that the two fermionic components are independent left- and right-moving fields,

$$\partial_+ \psi_-^\mu = 0, \qquad \partial_- \psi_+^\mu = 0 \tag{4.3}$$

while the Majorana condition ensures that the fermions can be chosen to be real in an appropriate representation,

$$\psi_+{}^* = \psi_+, \qquad \psi_-{}^* = \psi_- \tag{4.4}$$

The addition of the fermions is not arbitrary. The action possesses a local supersymmetry on the worldsheet, which relates the bosonic coordinates to their fermionic partners. The corresponding supersymmetry transformations take the form

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \tag{4.5}$$

and

$$\delta \psi^\mu = \rho^a \partial_a X^\mu \epsilon \tag{4.6}$$

Thus the bosonic coordinates and worldsheet fermions form supersymmetric multiplets. This worldsheet supersymmetry is the starting point for the construction of spacetime supersymmetric string theories. For closed strings, the left- and right-moving fermions can independently be either periodic or antiperiodic around the spatial direction of the worldsheet. Periodic fermions define the Ramond (R) sector, while antiperiodic fermions define the Neveu–Schwarz (NS) sector. Consequently, a closed superstring has four possible sectors,

$$\mathrm{NS}-\mathrm{NS}, \qquad \mathrm{NS}-\mathrm{R}, \qquad \mathrm{R}-\mathrm{NS}, \qquad \mathrm{R}-\mathrm{R}$$

The different periodicities lead directly to different oscillator modings. In the Ramond sector the fermions have integer modes,

$$\psi_+^\mu = \sqrt{\frac{1}{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^+}, \qquad \psi_-^\mu = \sqrt{\frac{1}{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^-} \tag{4.7}$$

whereas in the NS sector the modes are half-integer,

$$\psi_+^\mu = \sqrt{\frac{1}{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma^+}, \qquad \psi_-^\mu = \sqrt{\frac{1}{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma^-} \tag{4.8}$$

The different modings are particularly important after quantization, since they determine both the structure of the ground states and the normal-ordering constants.

Worldsheet supersymmetry also modifies the constraints of the theory. In addition to the stress-energy tensor, the theory possesses a conserved supercurrent. In conformal gauge, the physical configurations must satisfy the vanishing of both the energy-momentum tensor and the supercurrent,

$$T_{++} = T_{--} = 0 \tag{4.9}$$

and

$$J_+ = J_- = 0 \tag{4.10}$$

These constraints are the classical counterparts of the Virasoro and super-Virasoro constraints that appear after quantization.

4.2 Quantum Superstring

Upon quantization, the bosonic oscillators obey the usual commutation relations, while the fermionic oscillators obey anticommutation relations. In the NS sector one has

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{4.11}$$

and in the Ramond sector

$$\{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{n+m,0} \tag{4.12}$$

The ground states are defined by the annihilation conditions

$$a_m^\mu|0\rangle_{\mathrm{NS}} = b_r^\mu|0\rangle_{\mathrm{NS}} = 0 \qquad m, r > 0 \tag{4.13}$$

in the NS sector and

$$a_m^\mu|0\rangle_{\mathrm{R}} = d_m^\mu|0\rangle_{\mathrm{R}} = 0 \qquad m > 0 \tag{4.14}$$

in the Ramond sector.

A crucial difference between the two sectors comes from the Ramond zero modes. Since the fermionic modes are integer-moded in the Ramond sector, the zero modes satisfy a Clifford algebra and therefore act on the ground state as gamma matrices. The Ramond ground state consequently forms a spinor representation of the spacetime Clifford algebra. The NS ground state, in contrast, is a spacetime scalar. More generally, states obtained by acting with an even number of fermionic creation operators are spacetime bosons, whereas states in the Ramond sector are spacetime fermions.

Quantum consistency imposes a much stronger restriction on the theory. The Virasoro and super-Virasoro constraints remove unphysical states, but negative-norm states can otherwise remain in the spectrum. Requiring these states to decouple, or equivalently requiring Lorentz invariance of the light-cone quantized theory, fixes the critical spacetime dimension to ten, $D = 10$. The same result follows from the cancellation of the worldsheet superconformal anomaly. The matter sector contains ten bosonic coordinates, contributing $c_X = 10$ and ten Majorana fermions, contributing $c_\psi = 10 \times \frac{1}{2} = 5$, so that the total matter central charge is

$$c_{\text{matter}} = 15 \tag{4.15}$$

The gauge-fixed theory contains the usual (b, c) ghosts with $c = -26$ together with the superconformal (β, γ) ghosts with $c = 11$. Hence

$$c_{\text{ghost}} = -26 + 11 = -15 \tag{4.16}$$

and therefore

$$c_{\text{matter}} + c_{\text{ghost}} = 0 \tag{4.17}$$

This cancellation is the quantum statement of superconformal consistency and provides an independent derivation of the critical dimension.

4.3 Superstring Spectrum and GSO Projection

The physical spectrum can be determined by imposing the mass-shell conditions in the Ramond and NS sectors. In light-cone gauge there are $D - 2 = 8$ physical transverse directions. In the Ramond sector the mass-shell condition takes the form

$$\alpha' M^2 = \sum_{n=1}^{\infty} a^i_{-n} a^i_n + \sum_{n=1}^{\infty} n\, d^i_{-n} d^i_n \tag{4.18}$$

where the normal-ordering constant is $a_R = 0$. Consequently, the Ramond ground state is massless,

$$\alpha' M^2 = 0 \tag{4.19}$$

and, because of the fermionic zero modes, it is a spacetime spinor. The physical state conditions reduce the number of independent components, resulting in a Majorana–Weyl fermion in ten dimensions. All excited Ramond states are massive spacetime fermions. The NS sector differs because its fermions are half-integer moded. Its mass-shell condition is

$$\alpha' M^2 = \sum_{n=1}^{\infty} a^i_{-n} a^i_n + \sum_{r=\frac{1}{2}}^{\infty} r\, b^i_{-r} b^i_r - \frac{1}{2} \tag{4.20}$$

The normal-ordering constant is therefore $a_{\text{NS}} = \frac{1}{2}$ and the NS vacuum satisfies

$$\alpha' M^2 = -\frac{1}{2} \tag{4.21}$$

so that it is tachyonic. The first fermionic excitation has the smallest possible mode number, $r = \frac{1}{2}$, and is obtained by acting with $b^i_{-1/2}$ on the vacuum. Its mass is

$$\alpha' M^2 = -\frac{1}{2} + \frac{1}{2} = 0 \tag{4.22}$$

and hence $b^i_{-1/2}|0\rangle_{\rm NS}$ is a massless spacetime boson transforming as the vector representation of the transverse little group $SO(8)$. Higher excitations are obtained by acting with arbitrary combinations of the bosonic and fermionic creation operators and give massive spacetime bosons. The tachyonic NS ground state is incompatible with the desired supersymmetric spectrum. It is removed by the GSO projection, which simultaneously imposes the condition required for modular invariance and selects a consistent set of physical states. In the NS sector the projection can be expressed through the fermion-number operator as

$$(-1)^F = (-1)^{\sum_{r=\frac{1}{2}}^{\infty} b^i_{-r} b^i_r} = -1 \tag{4.23}$$

Thus states with an odd number of NS fermionic excitations are retained, while the NS vacuum is projected out. The first surviving state is therefore the massless vector $b^i_{-1/2}|0\rangle_{\rm NS}$.
In the Ramond sector the projection acts on the spinor ground state through its chirality. In ten dimensions the chirality operator is Γ_{11}, and the corresponding projection operators are

$$P_\pm = \frac{1}{2}(1 \pm \Gamma_{11}) \tag{4.24}$$

The GSO projection therefore selects one chirality of the Ramond ground state. The resulting spectrum contains a massless spacetime boson from the NS sector and a massless spacetime fermion from the Ramond sector, together with their massive excitations. The equality between bosonic and fermionic degrees of freedom is then extended throughout the complete supersymmetric spectrum.
For closed strings, the left- and right-moving sectors are independent and their states are combined by tensor products. The four possible sectors are therefore

$$\text{NS}-\text{NS}, \qquad \text{NS}-\text{R}, \qquad \text{R}-\text{NS}, \qquad \text{R}-\text{R} \tag{4.25}$$

The NS–NS and R–R sectors contain spacetime bosons, whereas the mixed NS–R and R–NS sectors contain spacetime fermions. In particular, the NS–NS sector contains the tensor product of two massless transverse vectors,

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \tag{4.26}$$

which correspond respectively to the dilaton, the antisymmetric two-form $B_{\mu\nu}$, and the graviton $g_{\mu\nu}$. The remaining sectors provide their supersymmetric partners.
The relative chirality chosen for the two Ramond sectors determines the two Type II theories. If the left- and right-moving Ramond ground states have the same chirality, the resulting theory is Type IIB. If their chiralities are opposite, the theory is Type IIA. Both theories are therefore obtained from the same worldsheet construction, with the difference arising from the choice of GSO projection.
The structure of the closed superstring also provides the starting point for the heterotic construction. The essential observation is that the left- and right-moving sectors of a closed string can be treated independently. Instead of choosing the same worldsheet theory in both sectors, one can combine a supersymmetric ten-dimensional sector on one side with a bosonic sector on the other. This asymmetric construction is the heterotic string.

4.4 From the Superstring to the Heterotic Construction

The heterotic construction exploits the independence of the two chiral sectors while preserving the same ten-dimensional spacetime interpretation on both sides. One sector is supersymmetric and contains the ten-dimensional fields X^μ together with their fermionic partners ψ^μ. The other sector is bosonic and must have the critical central charge of the bosonic string. Since ten spacetime bosons contribute only $c = 10$, an additional internal conformal field theory with central charge

$$c_{\text{int}} = 16 \tag{4.27}$$

is required.
The internal degrees of freedom can be represented either by sixteen compact bosons or, after fermionization, by thirty-two real right-moving fermions. The equivalence follows immediately from their central charges. Each compact boson contributes one unit of central charge, while each real fermion contributes one half, giving

$$16 \times 1 = 32 \times \frac{1}{2} = 16 \tag{4.28}$$

Thus the right-moving sector contains the ten spacetime bosons together with an internal conformal field theory of central charge sixteen, giving

$$c_R = 10 + 16 = 26 \tag{4.29}$$

as required for cancellation against the (b, c) ghost contribution. The fermionic realization is particularly useful because the internal fermions naturally generate current algebras and therefore gauge symmetries. This construction leads to the two consistent heterotic theories with gauge groups $SO(32)$ and $E_8 \times E_8$, which will be constructed in the following section.
In the fermionic formulation, the heterotic action can be written schematically as

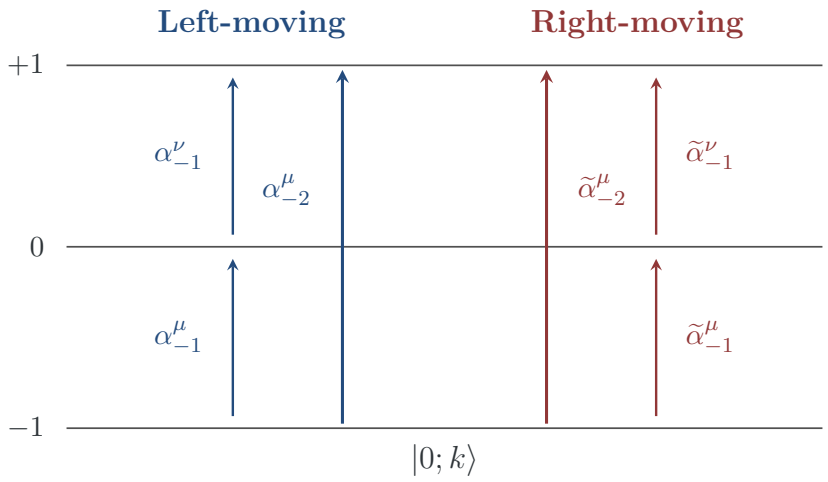
$$S = \frac{1}{\pi} \int d^2\sigma \left(2\partial_+ X_\mu \partial_- X^\mu + i\psi^\mu \partial_+ \psi_\mu + i \sum_{A=1}^{32} \lambda^A \partial_- \lambda^A \right) \tag{4.30}$$

where ψ^μ belongs to the supersymmetric sector and the λ^A are internal right-moving fermions. The index A is not a spacetime index but labels the internal degrees of freedom. Different consistent boundary conditions and projections on these internal fermions lead to the two allowed gauge structures. The resulting charge lattice construction will therefore determine the gauge symmetry of the heterotic string.

4.5 Charge Lattice Construction

4.5.1 The Bosonic String

The bosonic string provides the simplest setting in which the left- and right-moving sectors of a closed string can be studied. The oscillator structure of the closed bosonic string is illustrated in the following diagram



The diagram shows how the bosonic creation operators raise the oscillator level of the string. The vacuum is located at level -1 in each chiral sector. This negative value is the normal-ordering contribution to the Virasoro generator and is responsible for the tachyonic ground state of the bosonic string. Acting with an oscillator α^i_{-n} increases the left-moving level by n, while $\widetilde{\alpha}^i_{-n}$ increases the right-moving level by the same amount. Thus, a single α^i_{-1} excitation raises the left-moving sector from -1 to 0, while an α^i_{-2} excitation raises it directly from -1 to $+1$. The same structure is present in the right-moving sector through the operators $\widetilde{\alpha}^i_{-1}$ and $\widetilde{\alpha}^i_{-2}$.

For a closed string, the left- and right-moving excitations are independent, but physical states must satisfy the level-matching condition. In terms of the oscillator numbers, this requires $N_L = N_R$. Consequently, although the partition function contains terms with different powers of q and $\bar{q}$, only the level-matched combinations correspond to physical closed-string states.

The spectrum can be organised using the character expansion of the partition function. Since there are 24 transverse bosons in each chiral sector, the left-moving oscillator contribution is

$$Z_L(q) = \frac{1}{\eta(q)^{24}} \tag{4.31}$$

and the corresponding right-moving contribution is

$$Z_R(\bar{q}) = \frac{1}{\eta(\bar{q})^{24}} \tag{4.32}$$

Using the expansion of the Dedekind eta function, this becomes

$$Z(q,\bar{q}) = \left(q^{-1} + 24 + 324q + 3200q^2 + \cdots\right)\left(\bar{q}^{-1} + 24 + 324\bar{q} + 3200\bar{q}^2 + \cdots\right) \tag{4.33}$$

where the coefficient of each power of q or $\bar{q}$ counts the corresponding transverse oscillator states in one chiral sector. More generally, the partition function can be written as

$$Z = \sum_{\Delta,\bar{\Delta}} d_{\Delta,\bar{\Delta}} q^{\Delta} \bar{q}^{\bar{\Delta}} \tag{4.34}$$

where $d_{\Delta,\bar{\Delta}}$ denotes the degeneracy of states with left- and right-moving conformal weights Δ and $\bar{\Delta}$. Expanding the product gives

$$\begin{aligned} Z = \frac{1}{q\bar{q}} + \frac{24}{q} + 324\frac{\bar{q}}{q} + 3200\frac{\bar{q}^2}{q} + 24^2 + \frac{24}{\bar{q}} + 24\cdot 324\,\bar{q} + 24\cdot 3200\,\bar{q}^2 \\ + 324\frac{q}{\bar{q}} + 324\cdot 24\,q + 324^2 q\bar{q} + 324\cdot 3200 q\bar{q}^2 + \cdots \end{aligned} \tag{4.35}$$

The leading contribution $1/(q\bar{q})$ corresponds to the bosonic-string vacuum. Since each chiral sector has a vacuum energy of -1, the corresponding state has $L_0^{\rm osc} = \tilde{L}_0^{\rm osc} = -1$ and is therefore tachyonic. The terms containing unequal powers of q and $\bar{q}$, such as $24/q$ or $324q/\bar{q}$, do not satisfy level matching and are therefore not physical closed-string states.

The first physical excited level occurs for $N_L = N_R = 1$, corresponding to the term 24^2. The associated states are

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;k\rangle$$

with $i, j = 1, \ldots, 24$. There are therefore $24 \times 24 = 576$ states in the transverse oscillator Fock space. Since the left- and right-moving indices transform independently, these states form the tensor product $\mathbf{24} \otimes \mathbf{24}$, which decomposes into symmetric and antisymmetric parts as

$$\mathbf{24} \otimes \mathbf{24} = \mathrm{Sym}^2(\mathbf{24}) \oplus \Lambda^2(\mathbf{24}) \tag{4.36}$$

The symmetric part has dimension

$$\dim \mathrm{Sym}^2(\mathbf{24}) = \frac{24(24+1)}{2} = 300 \tag{4.37}$$

while the antisymmetric part has dimension

$$\dim \Lambda^2(\mathbf{24}) = \frac{24(24-1)}{2} = 276 \tag{4.38}$$

The symmetric tensor can then be separated into its traceless component and its trace, giving $300 = 299 + 1$. Hence, the total number of massless states decomposes as

$$576 = 299 + 276 + 1 \tag{4.39}$$

These three representations correspond to the graviton, the antisymmetric Kalb–Ramond two-form, and the dilaton. The graviton originates from the symmetric traceless part of $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;k\rangle$, the antisymmetric two-form from the antisymmetric part, and the dilaton from the trace. We can now consider the next level in the spectrum, represented by the term $324^2 q\bar{q}$. Since the vacuum contributes q^{-1} and $\bar{q}^{-1}$, obtaining $q\bar{q}$ requires

$$N_L - 1 = 1, \qquad N_R - 1 = 1$$

and $N_L = N_R = 2$. We therefore need to count the oscillator states at level $N = 2$ in each chiral sector. There are two possible ways to obtain oscillator number two, namely $2 = 2$ and $2 = 1 + 1$. The corresponding left-moving states are $\alpha^i_{-2}|0\rangle$ and $\alpha^i_{-1}\alpha^j_{-1}|0\rangle$, where $i, j = 1, \ldots, 24$. The first class contains 24 states because the index i can take any of the 24 transverse values. For the second class, the oscillators commute,

$$[\alpha^i_{-1}, \alpha^j_{-1}] = 0 \tag{4.40}$$

so the two indices form a symmetric pair. The number of independent states is therefore

$$\dim \mathrm{Sym}^2(\mathbf{24}) = \frac{24(24+1)}{2} = 300 \tag{4.41}$$

Consequently, the total left–moving degeneracy at level two is $d_2 = 24 + 300 = 324$. The right–moving sector has precisely the same structure. Its level-2 states are $\tilde{\alpha}^i_{-2}|0\rangle$ and $\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle$ with 24 and 300 states, respectively. Thus the right-moving degeneracy is also $\tilde{d}_2 = 24 + 300 = 324$.

A closed-string state at level $(2,2)$ is obtained by independently choosing a level-2 state from each chiral sector. The possible combinations are

Left-moving excitation	Right-moving excitation	Degeneracy
α^i_{-2}	$\widetilde{\alpha}^j_{-2}$	24×24
α^i_{-2}	$\widetilde{\alpha}^j_{-1}\widetilde{\alpha}^k_{-1}$	24×300
$\alpha^i_{-1}\alpha^j_{-1}$	$\widetilde{\alpha}^k_{-2}$	300×24
$\alpha^i_{-1}\alpha^j_{-1}$	$\widetilde{\alpha}^k_{-1}\widetilde{\alpha}^\ell_{-1}$	300×300

The total degeneracy is therefore

$$d_{2,2} = 24^2 + 24 \times 300 + 300 \times 24 + 300^2 = 324^2 = 104976 \tag{4.42}$$

Thus, the coefficient of $q\bar{q}$ in the character expansion counts 104976 states in the transverse Fock space at oscillator levels $N_L = N_R = 2$. The factorisation $d_{2,2} = d_2\widetilde{d}_2 = 324 \times 324$, reflects the independent choice of a left-moving and a right-moving oscillator state.

It is important, however, to distinguish this Fock-space degeneracy from the number of independent physical polarizations. The character expansion counts states in the transverse oscillator construction, whereas the physical spectrum is obtained after imposing the appropriate Virasoro constraints and removing null states. Consequently, the coefficient 324^2 should be interpreted as the raw oscillator degeneracy at this level rather than directly as the number of physical massive degrees of freedom.

4.5.2 General Setup for the Superstring

We begin by considering the most general one-loop partition function of the superstring before specializing to the Type II or heterotic constructions. The left- and right-moving worldsheet fermions can each be taken in either the Neveu–Schwarz (NS) or Ramond (R) sector. We therefore introduce the spin-structure labels $a, b, \bar{a}, \bar{b} \in \{0, 1\}$,where $a, \bar{a} = 0$ correspond to the NS sector and $a, \bar{a} = 1$ to the Ramond sector. The labels $b, \bar{b}$ specify the corresponding spin structures around the other cycle of the torus. The contribution of the eight transverse bosonic degrees of freedom together with the four complex worldsheet fermions can be written as

$$Z = \frac{1}{\eta^8\bar{\eta}^8} \frac{\Theta^4 \begin{bmatrix} a \\ b \end{bmatrix}}{\eta^4} \frac{\Theta^4 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}}{\bar{\eta}^4} \tag{4.43}$$

Here, $\eta(\tau)$ is the Dedekind eta function, while $\Theta \begin{bmatrix} a \\ b \end{bmatrix}$ denotes the theta function associated with the corresponding spin structure. The modular consistency of the theory requires the partition function to be invariant under the modular group generated by

$$T : \tau \to \tau + 1, \qquad S : \tau \to -\frac{1}{\tau}$$

These transformations impose non-trivial constraints on the relative phases with which the different spin structures are combined. For the T-transformation, the theta functions transform according to

$$\Theta^4 \begin{bmatrix} a \\ b \end{bmatrix} (\tau + 1) = e^{-\frac{i\pi}{4}a(a-2)} \Theta^4 \begin{bmatrix} a \\ b + a - 1 \end{bmatrix} \tag{4.44}$$

It is useful to isolate the left-moving contribution,

$$Z_L\begin{bmatrix}a\\b\end{bmatrix}(\tau)=\frac{1}{\eta^8}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4} \tag{4.45}$$

Using the modular transformation of the eta function together with that of the theta function gives

$$Z_L\begin{bmatrix}a\\b\end{bmatrix}(\tau+1)=\frac{1}{\eta^8}\frac{\Theta^4\begin{bmatrix}a\\b+a-1\end{bmatrix}}{\eta^4}\frac{e^{-i\pi a(a-2)}}{e^{2\pi i 12/24}}=(-1)^{a+1}Z_L\begin{bmatrix}a\\b+a-1\end{bmatrix}(\tau) \tag{4.46}$$

Similarly, under the S-transformation, the left-moving contribution satisfies

$$Z_L\begin{bmatrix}a\\b\end{bmatrix}(-1/\tau)=(-i\tau)^{-4}\exp\left(\frac{i\pi ab}{2}\right)Z_L\begin{bmatrix}b\\-a\end{bmatrix}(\tau) \tag{4.47}$$

The full partition function is obtained by summing over the four spin-structure labels and introducing coefficients $C\begin{bmatrix}a,\bar{a}\\b,\bar{b}\end{bmatrix}$, which encode the possible relative phases between the left- and right-moving sectors. Modular invariance then constrains these coefficients. Under T, the transformed partition function can be written as

$$\frac{1}{4}\sum_{a,b,\bar{a},\bar{b}}(-1)^{a+\bar{a}}\frac{\Theta^4\begin{bmatrix}a\\b+a-1\end{bmatrix}\Theta^4\begin{bmatrix}\bar{a}\\\bar{b}+\bar{a}-1\end{bmatrix}}{\eta^4\bar{\eta}^4}\frac{1}{\eta^8\bar{\eta}^8}(-1)^{a+b}(-1)^{\bar{a}+\bar{b}}C\begin{bmatrix}a,\bar{a}\\b,\bar{b}\end{bmatrix}=$$

$$=\frac{1}{4}\sum_{a,b',\bar{a},\bar{b}'}\frac{\Theta^4\begin{bmatrix}a\\b'\end{bmatrix}\Theta^4\begin{bmatrix}\bar{a}\\\bar{b}'\end{bmatrix}}{\eta^4\bar{\eta}^4}\frac{1}{\eta^8\bar{\eta}^8}(-1)^{a+b'}(-1)^{\bar{a}+\bar{b}'}C\begin{bmatrix}a,\bar{a}\\b'-a+1,\bar{b}'-\bar{a}+1\end{bmatrix}$$

In the second line we have simply relabelled the summation variables according to

$$b'=b+a-1,\qquad \bar{b}'=\bar{b}+\bar{a}-1$$

Since the transformed partition function must coincide with the original one, we obtain the first modular constraint,

$$C\begin{bmatrix}a,\bar{a}\\b-a+1,\bar{b}-\bar{a}+1\end{bmatrix}=C\begin{bmatrix}a,\bar{a}\\b,\bar{b}\end{bmatrix} \tag{4.48}$$

The S-transformation gives a second independent constraint. Repeating the same argument using the transformation of the theta functions, one finds

$$C\begin{bmatrix}a,\bar{a}\\b,\bar{b}\end{bmatrix}=\exp\left(-\frac{i\pi ab}{2}\right)\exp\left(\frac{i\pi\bar{a}\bar{b}}{2}\right)C\begin{bmatrix}b,\bar{b}\\a,\bar{a}\end{bmatrix} \tag{4.49}$$

Equations (4.48) and (4.49) are the basic one-loop modular consistency conditions. They determine which relative phases between the different spin structures are allowed. A convenient way of parametrizing these phases is

$$C\begin{bmatrix}a,\bar{a}\\b,\bar{b}\end{bmatrix}=(-1)^{k_{11}ab+k_{22}\bar{a}\bar{b}+k_{12}(a\bar{b}+b\bar{a})} \tag{4.50}$$

where $k_{ij} \in \{0,1\}$. Substituting this expression into the T-invariance condition gives

$$(-1)^{k_{12}\left(a(\bar{b}-\bar{a}+1)+(b-a+1)\bar{a}\right)} = (-1)^{k_{12}(a\bar{b}+b\bar{a})}$$

and therefore

$$(-1)^{k_{12}(-a\bar{a}-\bar{a}a+a+\bar{a})} = 1$$

Since $2a\bar{a}$ is always even, this reduces to

$$(-1)^{k_{12}(a+\bar{a})} = 1 \qquad \Longrightarrow \qquad k_{12} = 0 \tag{4.51}$$

We can make the parametrization even more general by allowing phases linear in the spin-structure labels:

$$C\begin{bmatrix} a, \bar{a} \\ b, \bar{b} \end{bmatrix} = (-1)^{k_{11}ab+k_{22}\bar{a}\bar{b}+k_{12}(a\bar{b}+b\bar{a})+\lambda_1(a+b)+\lambda_2(\bar{a}+\bar{b})} \tag{4.52}$$

This parametrization is general enough to encode all possible phases compatible with the $\mathbb{Z}_2$ spin structures. Applying the first modular constraint now gives

$$(-1)^{k_{12}(a+\bar{a})}(-1)^{\lambda_1(a+b-a+1)+\lambda_2(\bar{a}+\bar{b}-\bar{a}+1)} = (-1)^{\lambda_1(a+b)+\lambda_2(\bar{a}+\bar{b})}$$

or equivalently

$$(-1)^{k_{12}(a+\bar{a})+\lambda_1 a+\lambda_2\bar{a}+\lambda_1+\lambda_2} = 1$$

Hence,

$$(-1)^{(k_{12}+\lambda_1)a+(k_{12}+\lambda_2)\bar{a}+\lambda_1+\lambda_2} = 1, \tag{4.53}$$

which requires

$$\begin{aligned} k_{12}+\lambda_1 &= 0 \pmod 1 \\ k_{12}+\lambda_2 &= 0 \pmod 1 \\ \lambda_1+\lambda_2 &= 0 \pmod 1 \end{aligned} \qquad \Longrightarrow \qquad \lambda_1 = \lambda_2 = k_{12} \tag{4.54}$$

Thus, the most general allowed coefficient can be written as

$$C\begin{bmatrix} a, \bar{a} \\ b, \bar{b} \end{bmatrix} = (-1)^{k_{11}ab+k_{22}\bar{a}\bar{b}+k_{12}(a\bar{b}+b\bar{a}+a+b+\bar{a}+\bar{b})} \tag{4.55}$$

The important point is that the remaining freedom is encoded in the three binary parameters $k_{11},\ k_{22},\ k_{12}$. The two qualitatively different possibilities for k_{12} lead to different ways of combining the left- and right-moving spin structures. We shall therefore consider the cases $k_{12} = 1,\ k_{12} = 0$ separately.

We begin with $k_{12} = 1$. In this case the partition function contains the phase coupling the left- and right-moving spin structures, $(-1)^{a\bar{b}+\bar{a}b}$. The corresponding sum can be written as

$$\frac{1}{2}\sum_{a,b}\frac{1}{2}\sum_{\bar{a},\bar{b}}(-1)^{a\bar{b}+\bar{a}b}\frac{\Theta^4\begin{bmatrix} a \\ b \end{bmatrix}\Theta^4\begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}}{\eta^4\bar{\eta}^4}\frac{1}{\eta^8\bar{\eta}^8}(-1)^{ab+\bar{a}\bar{b}}(-1)^{(k_{11}+1)ab+(k_{22}+1)\bar{a}\bar{b}}$$

Introducing the relative spin structures

$$\bar{a} = a + h, \qquad \bar{b} = b + g$$

the sum becomes

$$\frac{1}{2}\sum_{a,b}\frac{1}{2}\sum_{h,g}(-1)^{hg}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}\Theta^4\begin{bmatrix}a+h\\b+g\end{bmatrix}}{\eta^4\bar{\eta}^4}\frac{1}{\eta^8\bar{\eta}^8}$$

Using the Jacobi identity, the sum over the relative spin structures collapses, giving

$$\frac{1}{2}\sum_{a,b}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}\Theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4\bar{\eta}^4}\frac{1}{\eta^8\bar{\eta}^8}=\frac{1}{2}\sum_{a,b}\left|\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^{12}}\right|^2 \tag{4.56}$$

The identity used here is

$$\frac{1}{2}\left(\bar{\Theta}^4\begin{bmatrix}a\\b\end{bmatrix}+\bar{\Theta}^4\begin{bmatrix}a\\b+1\end{bmatrix}+\bar{\Theta}^4\begin{bmatrix}a+1\\b\end{bmatrix}+\bar{\Theta}^4\begin{bmatrix}a+1\\b+1\end{bmatrix}\right)=\bar{\Theta}^4\begin{bmatrix}a\\b\end{bmatrix}. \tag{4.57}$$

This is the partition function of the *Type 0* superstring. In this construction the spectrum contains no spacetime fermions.

We now turn to the relation between the worldsheet spin-structure sum and the charge lattice. This is the step that will eventually allow us to identify the relevant gauge and spacetime representations. In general, the theta function can be written as

$$\Theta\begin{bmatrix}a\\b\end{bmatrix}(\tau)=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n-\frac{a}{2}\right)^2}e^{-i\pi b\left(n-\frac{a}{2}\right)} \tag{4.58}$$

Consequently,

$$\frac{1}{2}\sum_b\Theta^4\begin{bmatrix}a\\b\end{bmatrix}(\tau)=\sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}}q^{\frac{1}{2}\left[\left(n_1-\frac{a}{2}\right)^2+\left(n_2-\frac{a}{2}\right)^2+\left(n_3-\frac{a}{2}\right)^2+\left(n_4-\frac{a}{2}\right)^2\right]}\frac{1}{2}\sum_b e^{-i\pi b\left(n_1+n_2+n_3+n_4-4\times\frac{a}{2}\right)} \tag{4.59}$$

The sum over b acts as a projection onto states satisfying the appropriate parity condition. In particular, $\frac{1}{2}\sum_b e^{-i\pi b\left(n_1+n_2+n_3+n_4-4\times\frac{a}{2}\right)}$ implements the corresponding $\mathbb{Z}_2$ constraint on the lattice charges. It is also useful to note the periodicity and reflection properties of the theta functions,

$$\Theta^4\begin{bmatrix}a\\b\end{bmatrix}=\Theta^4\begin{bmatrix}2a-a\\b\end{bmatrix}=\Theta^4\begin{bmatrix}-a\\b\end{bmatrix}=\Theta^4\begin{bmatrix}a\\2b-b\end{bmatrix}=\left(e^{i\pi ab}\Theta^4\begin{bmatrix}a\\-b\end{bmatrix}\right)=\Theta^4\begin{bmatrix}a\\-b\end{bmatrix} \tag{4.60}$$

For example, in the NS sector $a=0$, the spin-structure projection can be written as

$$\frac{1}{2}\sum_b(-1)^b\Theta^4\begin{bmatrix}0\\b\end{bmatrix}=\frac{1}{2}\sum_b(-1)^b\sum_{Q_i=n_i\in\mathbb{Z}}q^{\frac{1}{2}\left(Q_1^2+Q_2^2+Q_3^2+Q_4^2\right)}\frac{1}{2}\sum_b e^{-i\pi b(Q_1+Q_2+Q_3+Q_4)} \tag{4.61}$$

The sum over b is precisely the familiar Gliozzi–Scherk–Olive (GSO) projection. It removes states with the wrong worldsheet fermion-number parity and, in the bosonized description, becomes a condition on the allowed charge vectors. Explicitly,

$$\frac{1}{2}\sum_b(-1)^b\Theta^4\begin{bmatrix}0\\b\end{bmatrix}=\sum_{\substack{Q_i=n_i\in\mathbb{Z}\\ \sum_i Q_i=1 \mod 2}}q^{\frac{1}{2}\left(Q_1^2+Q_2^2+Q_3^2+Q_4^2\right)}\qquad\leftrightarrow\qquad e^{i\vec{Q}\cdot\vec{H}} \tag{4.62}$$

Thus the GSO projection translates directly into a restriction on the charge lattice. At the first excited level, the relevant conditions are

$$Q_i \in \mathbb{Z} \tag{4.63a}$$

$$\sum_i Q_i \in 2\mathbb{Z} + 1 \tag{4.63b}$$

$$\sum_i Q_i^2 = 1 \tag{4.63c}$$

These conditions select the eight vectors

$$(\pm 1, 0, 0, 0)$$

$$(0, \pm 1, 0, 0)$$

$$(0, 0, \pm 1, 0)$$

$$(0, 0, 0, \pm 1)$$

They are precisely the weights of the vector representation of $SO(8)$, written in Cartan–Weyl form. In the bosonized description, the corresponding currents are $J_i = \partial H_i$, and the charge vectors $\vec{Q}$ specify the momenta of the bosonized fields through $e^{i\vec{Q}\cdot\vec{H}}$.

The appearance of the $SO(8)$ charge lattice illustrates the general role of the spin-structure sum: modular invariance and the GSO projection do not merely determine relative signs in the partition function; they determine which charge vectors are actually present in the physical spectrum. The same mechanism generalizes immediately from a $\mathbb{Z}_2$ projection to a $\mathbb{Z}_N$ projection. The basic identity is

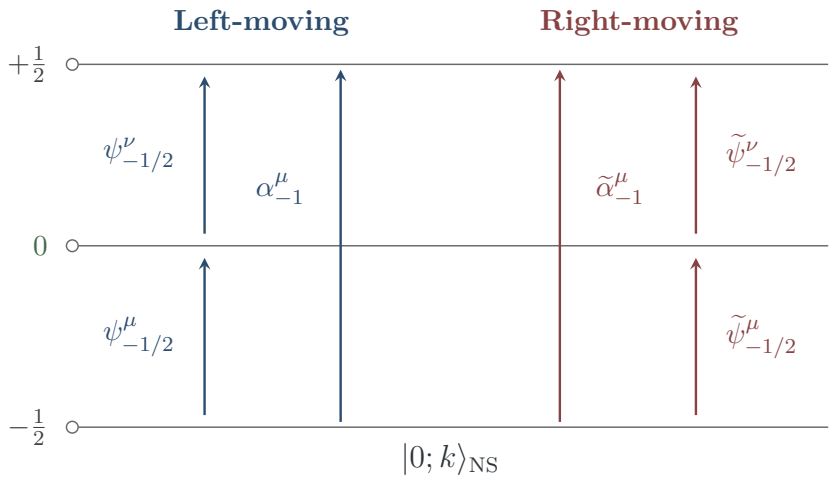
$$\frac{1}{N}\sum_{b\in\mathbb{Z}} e^{2\pi i \frac{b\cdot l}{N}} = \begin{cases} 1, & \text{iff } l = 0 \mod N, \\ 0, & \text{otherwise} \end{cases} \tag{4.64}$$

Thus the sum over the discrete spin-structure variable acts as a projector onto the allowed charge sublattice.

We are now in a position to distinguish the two remaining classes of superstring constructions. The choice of the phase structure encoded by C, and in particular the value of k_{12}, determines how the left- and right-moving charge lattices are correlated. The $k_{12} = 1$ branch led to the Type 0 construction discussed above. We now turn to the complementary case, $k_{12} = 0$ which leads to the Type II superstrings.

4.5.3 Type II String

In the same manner as the bosonic string, we an depict the oscillator structure of the Type II superstring as



It is useful to keep in mind the complete worldsheet field content of the theory. The left- and right-moving sectors each contain matter fields together with their corresponding ghost systems:

	Left-moving	Right-moving
Matter	$X^{\mu}_{L},\ \psi^{\mu}_{L}$	$X^{\mu}_{R},\ \psi^{\mu}_{R}$
Reparametrization ghosts	(b_L, c_L)	(b_R, c_R)
Superconformal ghosts	(β_L, γ_L)	(β_R, γ_R)

(4.65)

We now turn to the complementary possibility, $k_{12} = 0$. In this case the left- and right-moving spin structures are not coupled through the mixed phase $a\bar{b} + b\bar{a}$. The resulting partition function takes the form

$$Z_{II} = \frac{1}{2}\sum_{a,b}\frac{1}{2}\sum_{\bar{a},\bar{b}}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}\Theta^4\begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix}}{\eta^4\bar{\eta}^4}\frac{1}{\eta^8\bar{\eta}^8}(-1)^{a+b+ab}(-1)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} \tag{4.66}$$

Expanding the partition function in powers of q and $\bar{q}$, we may write schematically

$$Z_{II} = \sum_{\Delta,\bar{\Delta}} d_{\Delta,\bar{\Delta}} q^{\Delta}\bar{q}^{\bar{\Delta}} \tag{4.67}$$

where $d_{\Delta,\bar{\Delta}} = n_B - n_F$, is the net number of bosonic minus fermionic states at a given left- and right-moving conformal weight.

The two possible Type II theories differ in the relative chirality of their left- and right-moving Ramond sectors. This gives the two familiar possibilities: the chiral Type IIB theory and the non-chiral Type IIA theory. Their partition functions can be written as

$$Z_{IIB} = \frac{1}{2}\sum_{a,b}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}\frac{1}{\eta^8}(-1)^{a+b+ab}\frac{1}{2}\sum_{\bar{a},\bar{b}}\frac{\Theta^4\begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix}}{\bar{\eta}^4}\frac{1}{\bar{\eta}^8}(-1)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} \tag{4.68a}$$

$$Z_{IIA} = \frac{1}{2}\sum_{a,b}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}\frac{1}{\eta^8}(-1)^{a+b+ab}\frac{1}{2}\sum_{\bar{a},\bar{b}}\frac{\Theta^4\begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix}}{\bar{\eta}^4}\frac{1}{\bar{\eta}^8}(-1)^{\bar{a}+\bar{b}} \tag{4.68b}$$

The different sectors are obtained by choosing the values of a and $\bar{a}$. In particular, $a = \bar{a} = 0$ gives the NS–NS sector and therefore bosonic states. When $a = 0, \quad \bar{a} = 1$, we obtain the NS–R sector,

while $a = 1, \qquad \bar{a} = 0$ gives the R–NS sector. These two sectors contain spacetime fermions. Finally, $a = \bar{a} = 1$ gives the R–R sector, which again contains bosonic states.

Let us first examine the NS–NS sector of Type IIA. Setting $a = \bar{a} = 0$, we obtain

$$Z_{IIA}^{NS-NS} = \frac{1}{2}\sum_b \frac{\Theta^4\begin{bmatrix}0\\b\end{bmatrix}}{\eta^4}\frac{1}{\eta^8}(-1)^b \frac{1}{2}\sum_{\bar{b}} \frac{\Theta^4\begin{bmatrix}0\\\bar{b}\end{bmatrix}}{\bar{\eta}^4}\frac{1}{\bar{\eta}^8}(-1)^{\bar{b}} \tag{4.69}$$

Using the q-expansion of the theta and eta functions gives

$$Z_{IIA}^{NS-NS} = \frac{\left(1+8q^{1/2}+\cdots\right)\left(1+8\bar{q}^{1/2}+\cdots\right)}{q^{1/2}\prod_{n>0}(1-q^n)^{12}\,\bar{q}^{1/2}\prod_{\bar{n}>0}(1-\bar{q}^{\bar{n}})^{12}} \approx (8q^0)(8\bar{q}^0)+\cdots . \tag{4.70}$$

The massless contribution therefore contains $8 \times 8 = 64$ states. These are precisely the states obtained from $\psi^\mu_{-1/2}\tilde{\psi}^\nu_{-1/2}|0\rangle_{NS}$. They decompose into the graviton, the Kalb–Ramond two-form, and the dilaton, as expected for the NS–NS sector of a superstring.The potentially tachyonic terms that would have appeared before the GSO projection do not survive. In the present language, the modular consistency condition is reflected in the fact that only the appropriate integer powers of q and $\bar{q}$ remain in the physical spectrum.

Before moving to the mixed R–NS and NS–R sectors, it is useful to recall the Jacobi "abstruse" identity, which will be responsible for the crucial cancellations required by spacetime supersymmetry. In the notation used here,

$$\Theta^4\begin{bmatrix}a\\b\end{bmatrix} - \Theta^4\begin{bmatrix}a\\b+1\end{bmatrix} - \Theta^4\begin{bmatrix}a+1\\b\end{bmatrix} + \Theta^4\begin{bmatrix}a+1\\b+1\end{bmatrix} = \sum_{\gamma,\delta}\Theta^4\begin{bmatrix}a+\gamma\\b+\delta\end{bmatrix}(-1)^{\gamma+\delta} = (-1)^{a+b}\sum_{\gamma',\delta'}\Theta^4\begin{bmatrix}\gamma'\\\delta'\end{bmatrix}(-1)^{\gamma'+\delta'}$$

In the first line we have made the relabelling

$$\gamma' = a+\gamma, \qquad \delta' = \delta + b$$

Therefore,

$$2(-1)^{a+b}\frac{1}{2}\sum_{\gamma',\delta'}\Theta^4\begin{bmatrix}\gamma'\\\delta'\end{bmatrix}(-1)^{\gamma'+\delta'} = 0 \tag{4.71}$$

which is the familiar identity

$$\theta_3^4 - \theta_4^4 - \theta_2^4 = 0 \tag{4.72}$$

We use the conventions

$$\theta_3 = \Theta\begin{bmatrix}0\\0\end{bmatrix} \tag{4.73a}$$

$$\theta_4 = \Theta\begin{bmatrix}0\\1\end{bmatrix} \tag{4.73b}$$

$$\theta_2 = \Theta\begin{bmatrix}1\\0\end{bmatrix} \tag{4.73c}$$

$$\theta_1 = \Theta\begin{bmatrix}1\\1\end{bmatrix} \tag{4.73d}$$

We can now turn to the mixed sectors, which contain the spacetime fermionic degrees of freedom. For Type IIA, the R–NS sector is obtained by taking $a = 1, \qquad \bar{a} = 0$. The corresponding contribution is

$$Z_{IIA}^{R-NS} = \frac{1}{2}\sum_{a,b} \frac{\Theta^4 \begin{bmatrix} 1 \\ b \end{bmatrix}}{\eta^4} \frac{1}{\eta^8}(-1)^b \frac{1}{2}\sum_{\bar{a},\bar{b}} \frac{\Theta^4 \begin{bmatrix} 0 \\ \bar{b} \end{bmatrix}}{\bar{\eta}^4} \frac{1}{\bar{\eta}^8}(-1)^{\bar{b}} \tag{4.74}$$

The Ramond sector is described naturally in terms of bosonized fermions. The GSO projection becomes a condition on the allowed charge vectors,

$$\frac{1}{2}\sum_b \sum_{\substack{Q_i = n_i - \frac{1}{2} \\ n_i \in \mathbb{Z}}} q^{\frac{1}{2}\sum_i Q_i^2} e^{-i\pi b\left(\sum_i Q_i\right)} = \sum_{\substack{Q_i \in \mathbb{Z} - \frac{1}{2} \\ \sum_i Q_i \in 2\mathbb{Z}}} q^{\frac{1}{2}\sum_i Q_i^2} \tag{4.75}$$

Since $\left|n_i - \frac{1}{2}\right| \geq \frac{1}{2}$, we immediately obtain

$$\frac{1}{2}\sum_i Q_i^2 \geq \frac{1}{2} \times 4 \times \frac{1}{4} = \frac{1}{2} \tag{4.76}$$

Thus

$$\text{no tachyons} \qquad \Longrightarrow \qquad \text{the Ramond sector is at least massless} \tag{4.77}$$

To identify the massless states, we must saturate this bound:

$$\frac{1}{2}\sum_i Q_i^2 = \frac{1}{2} \qquad \Longrightarrow \qquad \sum_i Q_i^2 = 1 \tag{4.78}$$

Therefore $|Q_i| = \frac{1}{2} \qquad \forall i$ and the charge vectors are of the form

$$\vec{Q} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \tag{4.79}$$

The GSO projection further requires $\sum_i Q_i = 0 \mod 2$. This leaves $2^{8/2-1} = 8$ massless states. These eight states form one of the two Weyl spinor representations of $SO(8)$, which we denote by S. Explicitly, the allowed weights can be represented schematically as

$$\begin{pmatrix} ++++ & ---- \\ ++-- & --++ \\ +-+- & -+-+ \\ +--+ & -++- \end{pmatrix} \tag{4.80}$$

The conjugate Weyl spinor, which we denote by C, contains the remaining eight possible massless Ramond states. These are projected out by the GSO condition:

$$\begin{pmatrix} +++- & ---+ \\ ++-+ & --+- \\ +-++ & -+-- \\ +--- & -+++ \end{pmatrix} \tag{4.81}$$

The projection can be understood directly in terms of the inner products between the Ramond charge vectors and the vector weights. For example,

$$(++++) \cdot (1,0,0,0) \to \left(\frac{3}{2}, +, +, +\right) \qquad \checkmark$$

whereas

$$(++++)\cdot(-1,0,0,0)\to(-,+,+,+) \qquad \times$$

Similarly,

$$(++++)\cdot(0,1,0,0)\to\left(+,\frac{3}{2},+,+\right) \qquad \checkmark$$

while

$$(++++)\cdot(0,-1,0,0)\to(+,-,+,+) \qquad \times$$

The same pattern holds for the remaining two directions,

$$(++++)\cdot(0,0,1,0)\to\left(+,+,\frac{3}{2},+\right) \qquad \checkmark$$

$$(++++)\cdot(0,0,-1,0)\to(+,+,-,+) \qquad \times$$

and

$$(++++)\cdot(0,0,0,1)\to\left(+,+,+,\frac{3}{2}\right) \qquad \checkmark$$

$$(++++)\cdot(0,0,0,-1)\to(+,+,+,-) \qquad \times$$

Thus the GSO projection selects one chirality of the $SO(8)$ spinor representation. The massless R–NS states are therefore obtained schematically as $S\,\tilde{\psi}^{\mu}_{-1/2}|0\rangle_{NS}$. Since the Ramond ground state transforms as a spinor and the NS excitation as a vector, we have $8\otimes 8 = 56\oplus 8$, or, in representation-theoretic notation,

$$\frac{1}{2}\otimes 1 = \frac{3}{2}\oplus\frac{1}{2}$$

These states correspond to the gravitino together with the remaining Weyl-fermion degrees of freedom. For the other mixed sector, the distinction between Type IIA and Type IIB is precisely the relative chirality of the right-moving Ramond ground state. Thus,

$$\text{IIA}: \qquad \psi^{\mu}_{-1/2}\tilde{C}|0\rangle_{NS}$$

whereas

$$\text{IIB}: \qquad \psi^{\mu}_{-1/2}\tilde{S}|0\rangle_{NS}$$

Finally, we consider the R–R sector. The left- and right-moving Ramond ground states are both spinors of $SO(8)$. Their tensor product therefore determines the R–R spectrum.
For Type IIA the two chiralities are opposite,

$$\text{IIA}: \qquad S\cdot\tilde{C} = 8\times 8' = [1]+[3] = 8+56. \tag{4.82}$$

For Type IIB the two chiralities are the same,

$$\text{IIB}: \qquad S\cdot\tilde{S} = 8\times 8 = [0]+[2]+[4] = 1+28+35. \tag{4.83}$$

The antisymmetric tensor structure of these representations can be seen directly by adding the corresponding charge vectors. For example,

$$(++++)\cdot(++++)\to\Psi^1\Psi^2\Psi^3\Psi^4\to\Psi^{\mu}\Psi^{\nu}\Psi^{\rho}\Psi^{\sigma}\to[4]_{+} \tag{4.84}$$

whose number of independent components is $\binom{8}{4} = 70$ with the self-duality condition reducing this to 35. Likewise,

$$(++++)\cdot(----)\to(0,0,0,0) \tag{4.85}$$

which gives a singlet, $[0] = 1$. For a two-form,

$$(++++)\cdot(++--) \to (1,1,0,0) \to :\Psi^\mu\Psi^\nu: \to \text{rank-2 antisymmetric tensor} \tag{4.86}$$

whose dimension is $\frac{8\cdot 7}{2} = 28$. Similarly,

$$(++++)\cdot(+++-) \to (1,1,1,0) \to \Psi^\mu\Psi^\nu\Psi^\rho \to [3], \tag{4.87}$$

with $\binom{8}{3} = 56$,while

$$(++++)\cdot(+---) \to (1,0,0,0) \to \Psi^\mu \to [1] \tag{4.88}$$

giving 8 states. Thus the R–R sector naturally organizes itself in terms of antisymmetric tensor representations of $SO(8)$, with the precise collection of forms depending on whether the two Ramond sectors have equal or opposite chirality.

It is useful at this point to introduce the $SO(2n)$ characters, since they provide a compact way of organizing the different spin structures and their corresponding charge lattices.
In general, we define

$$\chi^\gamma_\lambda(\tau) = \frac{1}{2}\sum_\delta e^{i\pi\delta\lambda}e^{-i\pi n\delta\gamma/2}\frac{\theta^n\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0|\tau)}{\eta^n} \tag{4.89}$$

The four basic $SO(2n)$ characters are then

$$\begin{aligned}
O_{2n}(\tau) &= \frac{1}{2\,\eta(\tau)^n}\left(\theta_3^n + \theta_4^n\right) = \frac{1}{q^{n/24}}\left[1 + \frac{n(n-1)}{2} + \cdots\right]\\
V_{2n}(\tau) &= \frac{1}{2\,\eta(\tau)^n}\left(\theta_3^n - \theta_4^n\right) = \frac{1}{q^{n/24}}\left[2n\,q^{1/2} + \cdots\right]\\
S_{2n}(\tau) &= \frac{1}{2\,\eta(\tau)^n}\left(\theta_2^n + i^{-n}\theta_1^n\right) = \frac{1}{q^{n/24}}\left[2^{n-1}q^{n/8} + \cdots\right]\\
C_{2n}(\tau) &= \frac{1}{2\,\eta(\tau)^n}\left(\theta_2^n - i^{-n}\theta_1^n\right) = \frac{1}{q^{n/24}}\left[2^{n-1}q^{1/8} + \cdots\right]
\end{aligned} \tag{4.90}$$

Equivalently, in terms of the spin-structure labels, these characters can be written as

$$\begin{aligned}
\chi_0^0 &= O_{2n}(\tau) = \frac{1}{2}\sum_b \frac{\Theta^n\begin{bmatrix}0\\ b\end{bmatrix}}{\eta^n(\tau)}\\
\chi_1^0 &= V_{2n}(\tau) = \frac{1}{2}\sum_b (-1)^b\frac{\Theta^n\begin{bmatrix}0\\ b\end{bmatrix}}{\eta^n(\tau)}\\
\chi_0^1 &= S_{2n}(\tau) = \frac{1}{2}\sum_b \frac{\Theta^n\begin{bmatrix}1\\ b\end{bmatrix}}{\eta^n(\tau)}e^{-i\pi nb/2}\\
\chi_1^1 &= C_{2n}(\tau) = -\frac{1}{2}\sum_b \frac{\Theta^n\begin{bmatrix}0\\ b\end{bmatrix}}{\eta^n(\tau)}e^{-i\pi nb/2}
\end{aligned} \tag{4.91}$$

The relation between the theta-function description and the character description follows from the discrete Fourier transform over the spin-structure labels. In particular,

$$\frac{1}{2}\sum_{\lambda}(-1)^{b\lambda}\chi_\lambda^\gamma = \frac{1}{2}\sum_b \left(\frac{\Theta^n \begin{bmatrix} a \\ b \end{bmatrix}}{\eta(\tau)^n} e^{-i\pi abn/2} \right) \frac{1}{2}\sum_\lambda (-1)^{\lambda(b-b')} = \frac{1}{2}\frac{\Theta^n \begin{bmatrix} a \\ b' \end{bmatrix}}{\eta(\tau)^n} e^{-i\pi ab'n/2} \tag{4.92}$$

Therefore,

$$\frac{\Theta^n \begin{bmatrix} a \\ b' \end{bmatrix}}{\eta(\tau)^n} = e^{i\pi abn/2}\sum_\lambda (-1)^\lambda \chi_\lambda^a \tag{4.93}$$

For the superstring we have $n = 4$, so that the relevant transverse symmetry group is $SO(8)$. The one-loop spin-structure sum can then be expressed directly in terms of the $SO(8)$ characters:

$$\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab}\frac{\Theta^4 \begin{bmatrix} a \\ b \end{bmatrix}}{\eta(\tau)^4} = \frac{1}{2}\sum_{a,b}(-1)^{a+b+ab}\sum_l (-1)^{bl}\chi_l^a = \sum_a (-1)^a \sum_l \frac{1}{2}\sum_b (-1)^{b(1+a+l)}\chi_l^a$$

The sum over b projects onto

$$l = 1 + a \mod 2 \tag{4.94}$$

so that

$$\sum_a (-1)^a \chi^a_{(1+a) \mod 2} = \chi_1^0 - \chi_0^1 = V_8 - S_8 = 0 \tag{4.95}$$

The final equality is again the Jacobi identity. Its physical interpretation is particularly important: the complete cancellation is the character-level manifestation of spacetime supersymmetry. This is also where the special role of $SO(8)$ becomes manifest. The vector and the two inequivalent Weyl spinor representations have the same dimension, $\dim V_8 = \dim S_8 = \dim C_8 = 8$ and are related by the celebrated triality symmetry,

$$S_8 = V_8 = C_8 \tag{4.96}$$

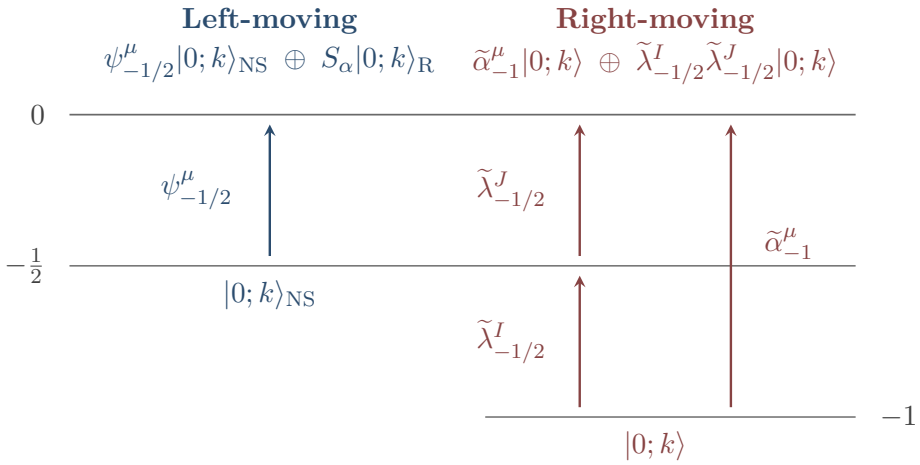
In ten-dimensional Type II superstring theory we therefore have $\mathcal{N} = 2$. The relevant $SO(8)$ representations appearing in the light-cone spectrum are identified from their q-expansions as

$$O_8 \longrightarrow 1, \text{ Adj} \tag{4.97a}$$
$$V_8 \longrightarrow \psi^\mu \tag{4.97b}$$
$$S_8 \longrightarrow S_\alpha(z) \tag{4.97c}$$
$$C_8 \longrightarrow C_{\dot\alpha}(z) \tag{4.97d}$$

Thus the entire Type II spectrum can be understood as arising from the interplay between the spin-structure sum, the GSO projection, the $SO(8)$ charge lattice, and the chirality choice in the left- and right-moving Ramond sectors. The two inequivalent choices of Ramond chirality give Type IIA and Type IIB, while the character identity $V_8 - S_8 = 0$ encodes the spacetime supersymmetry of the resulting ten-dimensional theories.

4.5.4 Heterotic String

For the heterotic superstring example, we have



We now turn to the heterotic string, which combines a supersymmetric left-moving sector with a purely bosonic right-moving sector. The resulting theory is therefore asymmetric between left- and right-movers. The one-loop partition function can be written as

$$Z_{\rm Heterotic} = \frac{1}{\eta^8\bar{\eta}^8}\left(\frac{1}{2}\sum_{a,b}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}(-1)^{a+b+ab}\right)\left(\frac{1}{2}\sum_{k,l}\frac{\bar{\Theta}^{16}\begin{bmatrix}k\\l\end{bmatrix}}{\bar{\eta}^{16}}C\begin{bmatrix}k, & a\\ l, & b\end{bmatrix}\right) \tag{4.98}$$

The asymmetry of the heterotic construction is reflected directly in its worldsheet field content. The left-moving sector contains the supersymmetric degrees of freedom, while the right-moving sector contains the additional internal conformal field theory required to reproduce the bosonic central charge. The field content is therefore

Left-moving sector	**Right-moving sector**
$X^\mu_L, \quad \mu = 0,\dots,9$	$X^\mu_R, \quad \mu = 0,\dots,9$
ψ^μ_L	—
(b_L, c_L)	(b_R, c_R)
(β_L, γ_L)	—
Internal right-moving CFT with $c_{\rm int} = 16$	
$\left\{M^I_R\right\}_{I=1}^{16} \iff \left\{\lambda^A_R\right\}_{A=1}^{32}$	
16 compact bosons	32 real fermions

Thus, the right-moving internal sector may equivalently be described in terms of 16 compact bosons or 32 real fermions. The corresponding internal conformal field theory has $c_{\rm int} = 16$.

The left-moving spin structures are labelled by a, b, while the right-moving internal sector is labelled by

k, l. We write the phase factor in the general form

$$C \longrightarrow (-1)^{\begin{pmatrix} k, & a \end{pmatrix} M \begin{pmatrix} l \\ b \end{pmatrix}}, \qquad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{4.99}$$

Under T the modular transformation, the spin structures transform according to

$$(k, l) \longrightarrow (k, l - k + 1), \qquad (a, b) \longrightarrow (a, b - a + 1)$$

Consequently, modular invariance requires

$$C \begin{bmatrix} k, & a \\ l - k + 1, & b - a + 1 \end{bmatrix} = C \begin{bmatrix} k, & a \\ l, & b \end{bmatrix} \tag{4.100}$$

Writing $X = \begin{pmatrix} k \\ a \end{pmatrix}$ and $Y = \begin{pmatrix} l \\ b \end{pmatrix}$, this condition becomes

$$X^T M(Y - X + \mathbb{I}) = X^T M Y \tag{4.101}$$

and hence

$$X^T M(X - \mathbb{I}) = 0 \mod 2 \tag{4.102}$$

We must also impose invariance under the second S modular transformation. In this case the two sets of spin structures are exchanged, giving

$$C \begin{bmatrix} k, & a \\ l, & b \end{bmatrix} = C \begin{bmatrix} l, & b \\ k, & a \end{bmatrix} \tag{4.103}$$

Therefore,

$$X^T M Y = Y^T M X \mod 2 \tag{4.104}$$

which implies

$$X^T (M - M^T) Y = 0 \mod 2 \tag{4.105}$$

Since this must hold for arbitrary X and Y, we obtain

$$M^T = M \mod 2 \tag{4.106}$$

Let us now impose the first modular constraint explicitly. For $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, we have

$$\begin{pmatrix} k, & a \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} k - 1 \\ a - 1 \end{pmatrix} = 0 \mod 2$$

Using $M_{21} = M_{12}$, this becomes

$$\begin{pmatrix} kM_{11} + aM_{12}, & kM_{12} + aM_{22} \end{pmatrix} \begin{pmatrix} k - 1 \\ a - 1 \end{pmatrix} = 0 \mod 2$$

$$\implies \quad k(k-1)M_{11} + a(k-1)M_{12} + k(a-1)M_{12} + a(a-1)M_{22} = 0 \mod 2$$

Since $k(k-1) = a(a-1) = 0 \mod 2$, the remaining condition is

$$aM_{12} + kM_{12} = 0 \mod 2$$

or equivalently

$$(a-k)M_{12} = 0 \mod 2 \qquad \forall\, a,k \tag{4.107}$$

This must hold for every choice of a and k, and therefore

$$M_{12} = 0 \mod 2 \tag{4.108}$$

The phase factor consequently reduces to

$$C = (-1)^{M_{11}kl + M_{22}ab} \tag{4.109}$$

The second factor can be removed by an appropriate choice of convention. Furthermore, chirality considerations for the heterotic string require $M_{11} = 0$. Thus, for the $SO(32)$ heterotic string, we arrive at the partition function

$$Z_{32} = \frac{1}{\eta^8\bar\eta^8}\left(\frac{1}{2}\sum_{a,b}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}(-1)^{a+b+ab}\right)\left(\frac{1}{2}\sum_{k,l}\frac{\bar\Theta^{16}\begin{bmatrix}k\\l\end{bmatrix}}{\bar\eta^{16}}\right) \tag{4.110}$$

The $SO(32)$ spectrum: Let us now examine the spectrum and, in particular, first ask whether the theory contains a tachyon.
A potential tachyon would arise in the bosonic sector, corresponding to $a = 0$. In the present construction, such a state would occur at conformal weight $-\frac{1}{2}$, and would schematically correspond to $\tilde\lambda^i_{-1/2}$. However, this state is projected out by the GSO projection. Hence, the $SO(32)$ heterotic string is tachyon-free.

We can therefore proceed to the massless spectrum. For $a = 0$, the massless states include

$$\psi^\mu_{-1/2}\tilde a^\nu_{-1}|0\rangle_{\rm NS} \longrightarrow \text{graviton},\ B_{\mu\nu},\ \text{dilaton} \tag{4.111a}$$

$$\psi^\mu_{-1/2} : \tilde\lambda^i_{-1/2}\tilde\lambda^j_{-1/2} : |0\rangle_{\rm NS} \longrightarrow \text{gauge bosons} \tag{4.111b}$$

Both classes of states are allowed by the GSO projection, namely by the b- and l-projections. The gauge bosons can be expressed in terms of the right-moving currents. Schematically,

$$A^{ij}_\mu\, \psi^\mu_{-1/2}(z)\tilde J^{ij}(\bar z)|0\rangle_{\rm NS} \tag{4.112}$$

generates the current-algebra sector. The corresponding Kac–Moody algebra, or current algebra, satisfies the operator product expansion

$$J^a(z)J^b(0) \sim \frac{k\,\delta^{ab}}{z^2} + i\frac{f^{ab}{}_c}{z}J^c(0) + \cdots \tag{4.113}$$

leading to the $\widehat{SO(32)}_{k=1}$ current algebra.
For the Ramond case, $a = 1$, we similarly obtain

$$S_\alpha(z)\,\tilde a^\nu_{-1}|0\rangle_{\rm NS} \longrightarrow \text{gravitino},\ \text{Weyl fermions} \tag{4.114a}$$

$$S_\alpha(z) : \tilde\lambda^i_{-1/2}\tilde\lambda^j_{-1/2} : |0\rangle_{\rm NS} \longrightarrow \text{Weyl fermions in the adjoint representation of } SO(32) \longrightarrow \text{gaugini} \tag{4.114b}$$

The fermionic spectrum therefore combines with the bosonic spectrum into $\mathcal{N} = 1$ supersymmetry in ten dimensions, corresponding to 16 supercharges.

The $E_8 \times E_8$ heterotic string: There is a second consistent heterotic construction in ten dimensions, obtained by splitting the internal right-moving degrees of freedom into two sets of eight. The corresponding partition function is

$$Z_{E_8\times E_8} = \frac{1}{\eta^8\bar{\eta}^8}\left(\frac{1}{2}\sum_{a,b}\frac{\Theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}(-1)^{a+b+ab}\right)\left(\frac{1}{2}\sum_{k,l}\frac{\bar{\Theta}^8\begin{bmatrix}k\\l\end{bmatrix}}{\bar{\eta}^8}\right)\left(\frac{1}{2}\sum_{\rho,\sigma}\frac{\bar{\Theta}^8\begin{bmatrix}\rho\\\sigma\end{bmatrix}}{\bar{\eta}^8}\right) \tag{4.115}$$

We can again examine the possibility of tachyonic states. Such states would correspond to $a = k = \rho = 0$ and schematically take the form

$$\mathbb{I}|0\rangle_L \otimes \tilde{\lambda}_{-1/2}|0\rangle_R \tag{4.116}$$

For this sector, the relevant contribution is

$$Z^{a=k=\rho=0}_{E_8\times E_8} = \frac{1}{\eta^8\bar{\eta}^8}\left(\frac{1}{2}\sum_{b}\frac{\Theta^4\begin{bmatrix}0\\b\end{bmatrix}}{\eta^4}(-1)^{b}\right)\left(\frac{1}{2}\sum_{l}\frac{\bar{\Theta}^8\begin{bmatrix}0\\l\end{bmatrix}}{\bar{\eta}^8}\right)\left(\frac{1}{2}\sum_{\sigma}\frac{\bar{\Theta}^8\begin{bmatrix}0\\\sigma\end{bmatrix}}{\bar{\eta}^8}\right)$$

To make the GSO projection explicit, consider the first factor. Introducing the corresponding charge variables, we may write

$$\frac{1}{2}\sum_b \Theta^4\begin{bmatrix}0\\b\end{bmatrix}(-1)^b = \sum_{Q_1,Q_2,Q_3,Q_4} q^{\frac{1}{2}\left[Q_1^2+Q_2^2+Q_3^2+Q_4^2\right]}\frac{1}{2}\sum_b e^{-i\pi b[Q_1+Q_2+Q_3+Q_4+1]} \tag{4.117}$$

The sum over b imposes the condition $Q_1 + Q_2 + Q_3 + Q_4 + 1 = 0 \mod 2$ or equivalently,

$$Q_1 + Q_2 + Q_3 + Q_4 = 1 \mod 2 \tag{4.118}$$

Thus the potentially tachyonic contribution is removed by the GSO projection, and the $E_8 \times E_8$ heterotic string is also tachyon-free.

The triality of $SO(8)$ can again be seen directly from the q-expansions of the characters. For $n = 4$, we have

$$\begin{aligned} V_8(\tau) &= \frac{1}{q^{n/24}}\left[2\times 4\, q^{1/2} + \cdots\right] \\ S_8(\tau) &= \frac{1}{q^{n/24}}\left[2^{4-1}q^{4/8} + \cdots\right] \\ C_8(\tau) &= \frac{1}{q^{n/24}}\left[2^{4-1}q^{4/8} + \cdots\right] \end{aligned} \tag{4.119}$$

Since $2\times 4 = 8$, and $2^{4-1} = 8$, the vector and the two Weyl-spinor representations all contain eight massless states, making the triality symmetry of $SO(8)$ manifest. Using the character definitions introduced previously, the $E_8 \times E_8$ partition function can then be expressed compactly as

$$Z = \frac{1}{\eta^8\bar{\eta}^8}\left(V_8 - S_8\right)\left(\bar{O}_{16} + \bar{S}_{16}\right)\left(\bar{O}_{16} + \bar{S}_{16}\right) \tag{4.120}$$

Expanding the product gives

$$Z = \frac{1}{\eta^8\bar{\eta}^8}\Big(V_8\bar{O}_{16}\bar{O}_{16} + V_8\bar{O}_{16}\bar{S}_{16} + V_8\bar{S}_{16}\bar{O}_{16} + V_8\bar{S}_{16}\bar{S}_{16} - S_8\bar{O}_{16}\bar{O}_{16} - S_8\bar{O}_{16}\bar{S}_{16} - S_8\bar{S}_{16}\bar{O}_{16} - S_8\bar{S}_{16}\bar{S}_{16}\Big) \tag{4.121}$$

Let us illustrate how the physical states are identified. Consider first $\frac{1}{\eta^8\bar{\eta}^8}V_8\bar{O}_{16}\bar{O}_{16}$. Taking the identity contributions from the relevant η-functions and the leading terms of the two $\bar{O}_{16}$ characters, the V_8 factor supplies the left-moving excitation $\psi^\mu_{-1/2}$, while the right-moving spacetime oscillator supplies $\tilde{a}^\nu_{-1}$. The resulting state carries two spacetime vector indices, (μ,ν). Its symmetric traceless part gives the graviton, $g_{\mu\nu}$, its antisymmetric part gives the Kalb–Ramond field, $B_{\mu\nu}$ and its trace gives the dilaton, ϕ.

The internal sectors similarly provide the gauge bosons. Schematically,

$$\psi^\mu_{-1/2}\tilde{J}^{ab}\cdot\hat{\mathbb{I}}' + \psi^\mu_{-1/2}\hat{\mathbb{I}}\cdot\tilde{J}'^{ab} \longrightarrow \text{gauge bosons in } SO(16)\times SO(16)'$$

The first set of currents belongs to the adjoint of one E_8, while the second belongs to the adjoint of the second, conventionally denoted E_8'. The resulting gauge group is therefore $E_8\times E_8$.

It is useful to recall the relation between the $SO(2n)$ characters and the theta-function representation. Starting from the character definition, one has

$$\sum_\lambda \chi^a_\lambda e^{-i\pi\beta'\lambda} = \frac{1}{2}\sum_{\beta,\lambda} e^{\pi i(\beta-\beta')\lambda}e^{-i\pi a\beta/2}\frac{\Theta^n\begin{bmatrix}a\\ \beta\end{bmatrix}}{\eta^n} = e^{-i\pi n a\beta'/2}\frac{\Theta^n\begin{bmatrix}a\\ \beta'\end{bmatrix}}{\eta^n} \tag{4.122}$$

Equivalently,

$$\frac{\Theta^n\begin{bmatrix}a\\ \beta\end{bmatrix}}{\eta^n} = e^{i\pi a\beta n/2}\sum_\lambda e^{-\pi i\beta\lambda}\chi^a_\lambda \tag{4.123}$$

For $n=4$, the left-moving spin-structure sum becomes

$$\text{RNS} = \frac{1}{2}\sum_{a,b}(-1)^{a+b+ab}\sum_\lambda e^{-i\pi b\lambda}\chi^a_{8,\lambda} = \sum_a(-1)^a\frac{1}{2}\sum_{b,\lambda}(-1)^{b(1+a+\lambda)}\chi^a_{8,\lambda}$$

The sum over b projects onto

$$\lambda = 1+a \mod 2 \tag{4.124}$$

and hence,

$$\text{RNS} = \sum_a(-1)^a\chi^a_{8,(1+a)\mod 2} = \chi^0_{8,1}-\chi^1_{8,0} = V_8 - S_8 \tag{4.125}$$

For the right-moving internal sector, taking $n=16$, we similarly find

$$\text{gauge} = \frac{1}{2}\sum_{k,l}\sum_\lambda(-1)^{\lambda l}\bar{\chi}^k_{16,\lambda} = \sum_k\bar{\chi}^k_{16,0} \tag{4.126}$$

Therefore,

$$\text{gauge} = \bar{\chi}^0_{16,0}+\bar{\chi}^1_{16,0} = \bar{O}_{16}+\bar{S}_{16} \tag{4.127}$$

This is the character-theoretic origin of the E_8 structure in the heterotic construction. Each $SO(16)$ factor is enlarged by the inclusion of its spinor conjugacy class, producing the exceptional group E_8. Thus, the two internal $SO(16)$ sectors combine into $E_8\times E_8$. The complete heterotic construction therefore exhibits the same essential ingredients encountered in the Type II analysis, but with an asymmetric left and right structure. The left-moving sector provides the spacetime supersymmetry, while the additional right-moving degrees of freedom generate the gauge symmetry. In ten dimensions this leads to the two consistent supersymmetric heterotic theories, $SO(32)$ and $E_8\times E_8$ both with $\mathcal{N}=1$ supersymmetry and 16 spacetime supercharges.

Chapter 5

Spin Field Formalism with Real Fermions

The present chapter is based primary on the constructions developed in [4], [5]. We follow the two papers chronologically and examine in detail two different examples where rank reduction occurs. Both examples involve periodic and antiperiodic free real fermions, rather than fermions that can be globally organized into independently diagonal complex pairs. In each case, we retain the notation and conventions of the original paper, which differ from those adopted in the previous chapter, particularly in the choice of boundary conditions along the two cycles for the fermions. The first model suggests a ten–dimensional example, while the second one is formulated in four dimensions, as already suggested from the additional internal degrees of freedom defining the basis vectors.

5.1 Setup and conventions

The main goal of this thesis is to understand in greater detail models whose set of basis vectors contain at least one vector W_i whose boundary conditions cannot be simultaneously diagonalized using a complex–fermion basis. In particular, we are interested in situations in which the boundary conditions of two real fermions are different for at least one basis vector. In such cases, the conventional complexification of the real fermions fails to diagonalize the corresponding monodromy. To illustrate the basic issue, consider two real fermionic modes d_n^1 and d_n^2, and define

$$b_n(\sigma^i) = \frac{1}{\sqrt{2}}\left(d_n^1 + i d_n^2\right) \tag{5.1a}$$

$$d_n(\sigma^i) = \frac{1}{\sqrt{2}}\left(d_n^1 - i d_n^2\right) \tag{5.1b}$$

with anticommutation relations

$$\{d_m^\dagger, d_n\} = \{d_m^\dagger, d_n\} = \delta_{mn} \tag{5.2a}$$

$$\{d_m^{\dagger\, i}, d_n^j\} = \delta_{mn}\delta^{ij} \tag{5.2b}$$

We see that, using the convention to express periodicities as $\sigma^i \sim \sigma^i + 2\pi$, if

$$d_n^1(\sigma^i + 2\pi) = d_n^1(\sigma^i) \tag{5.3a}$$

$$d_n^2(\sigma^i + 2\pi) = -d_n^2(\sigma^i) \tag{5.3b}$$

we eventually obtain

$$b_n(\sigma^i + 2\pi) = \frac{1}{\sqrt{2}}\left(d_n^1 - id_n^2\right) = d_n(\sigma^i) \tag{5.4}$$

and similarly $d_n(\sigma^i + 2\pi) = b_n(\sigma^i)$. Thus, in the complex basis, the monodromy exchanges the two complex–conjugate fermions. In matrix form,

$$\begin{pmatrix} b_n \\ d_n \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_n \\ d_n \end{pmatrix} \tag{5.5}$$

Thus, the complex basis does not diagonalize the monodromy when the two real fermions have different periodicities. The boundary conditions are diagonal in the real basis. The issue in the models considered here, however, is not resolved by choosing a different complexification. The relevant question is whether one can find a single complexification that diagonalizes the boundary conditions associated with every basis vector W_i in the set. For the models of interest, this is not possible. At least one basis vector remains non-diagonal in the complexified description. It is this type of vector that is referred to as a broken vector in [5]. Since the complexification and diagonalization of the fermions are essential for identifying the currents and determining the gauge group, the presence of such a vector requires the projection conditions to be reconsidered directly in terms of real fermions. This leads to the questions of how physical states are characterized, which states are projected out, and how the corresponding spin fields are constructed. It also raises the question of whether the massless vector bosons arising in these models necessarily furnish the adjoint representation of a Lie group.

Nevertheless, the first question to address is which constraints must be satisfied by a set of candidate basis vectors W_i in order to define a consistent model. The same question arises for the spin-structure coefficients k_{ij}. Although the k_{ij} are not uniquely defined, their possible values are restricted by a set of equations, subject to the corresponding ambiguities. Moreover, even when some of the k_{ij} may be chosen freely and the remaining coefficients determined from them, the particular choice of the independent coefficients matters a priori for the resulting model. This is illustrated by the ten-dimensional $E_8 \times E_8$ superstring, which, as discussed in [4, 5], is recovered for a suitable choice of the basis vectors W_i together with a specific choice of the k_{ij}, leading to the corresponding spectrum.

It is important at this point to make explicit the conventions used in the two papers. The boundary conditions of an RNS fermion around the two independent non–contractible cycles are written as

$$\lambda^j(\sigma_1 + 2\pi, \sigma_2) = e^{-2\pi i v}\lambda^j(\sigma_1, \sigma_2) \tag{5.6a}$$

$$\lambda^j(\sigma_1, \sigma_2 + 2\pi) = e^{-2\pi i u}\lambda^j(\sigma_1, \sigma_2) \tag{5.6b}$$

with $u, v \in \{0, \frac{1}{2}\}$. These are the familiar entries $\{0, 1\}$, but in a different convention. Note also that the presence of this extra factor 2 must be carried to the GSO–like phase constraint, and hence all the extra modularity constraints will be modulo one instead. Note also that because of the absence of the extra minus sign on the phase conventions, the antiperiodic or NS fermions are the ones having $\frac{1}{2}$ in the positions of the basis vectors and the 0 ones are the Ramond or the periodic ones. This means that in ten dimensions, the identity vector that should be included in any consistent candidate set is $W_0 = \left(\frac{1}{2}^8|\frac{1}{2}^{32}\right)$.

In a complex *diagonal* basis, the boundary conditions corresponding to a general rotation $SO(32)$, are for complex fermions Ψ^l

$$\Psi^l(\sigma_1+2\pi,\sigma_2)=e^{-2\pi i W^l}\Psi^l(\sigma_1,\sigma_2) \tag{5.7a}$$
$$\Psi^l(\sigma_1,\sigma_2+2\pi)=e^{-2\pi i W'^l}\Psi^l(\sigma_1,\sigma_2) \tag{5.7b}$$

For models of complex fermions the physical requirements that should be satisfied in general are Lorentz invariance, unitarity and also a consistent gravity framework. These translate to constraints imposed on the two–dimensional worldsheet and most of them are already implemented in the derivation of the ten–dimensional Lagrangian of the heterotic superstring. Here, the constraints that need to be carried out in detail are the ones associated with modularity. The initial condition is one–loop modularity along with a physically sensible projection of states. Again, the thought process is the same; starting from a sum of one–loop contributions with different boundary conditions, we explicitly write modular transformations and demand modular invariance of the partition function. It turns out that this leads to

$$k_{ij}+k_{ji}=W_i\cdot W_j \qquad \text{mod } 1 \tag{5.8}$$

$$m_j k_{ij}=0 \qquad \text{mod } 1 \tag{5.9}$$

$$k_{ii}+k_{i0}+s_i-\frac{1}{2}W_i\cdot W_i=0 \qquad \text{mod } 1 \tag{5.10}$$

For the case of complex free fermions the physical state projection condition is

$$\left[W_i\cdot N_\alpha-\sum_j k_{ij}a_j-s_i-k_{0i}+W_i\cdot\overline{\alpha V}\right]|\text{phys}\rangle_{\text{complex}}=0 \qquad \text{mod } 1 \tag{5.11}$$

These four equations turn out to be the necessary and sufficient set of conditions needed to imposed one–loop modularity and the GSO phase restriction. What is more, for models whose basis vectors can be simultaneously complexified, these conditions are in fact necessary and sufficient to also guarantee higher–loop modular invariance. For a more general real–fermion model though, and not this subset of models that at least for one complexification all vectors are diagonalized at the same time, the picture is a bit more subtle. First of all together with the set of equations (5.8)-(5.10), which turn out to remain the same, there is an additional cubic constraint, namely

$$4\sum_{\ell\in\text{real}}W_i^\ell W_j^\ell W_k^\ell=0 \qquad \text{mod } 1 \tag{5.12}$$

The origin of this extra equation comes from imposing

$$\overline{\alpha W}\cdot\overline{\alpha W}=0 \qquad \text{mod}1 \tag{5.13}$$

since linear combinations of the basis vector must also arise when summing all possible combinations in the partition function. The physical projection condition becomes

$$\left[W_i\cdot N'_\alpha+\frac{1}{4}(1-\Gamma_i^{\alpha W})-\sum_j k_{ij}a_j-s_i-k_{0i}+W_i\cdot\overline{\alpha W}\right]|\text{phys}\rangle_{\text{real}}=0 \qquad \text{mod } 1 \tag{5.14}$$

with $\Gamma_i^{\alpha W}\equiv g_i^0\prod_j\left(g_i^j\right)^{\alpha_j}$. As it can be seen, here we have separated the fermion number contribution coming from Ramond or NS in general and the contribution coming from the zero Ramond modes. This second terms containing the zero modes will give the nontrivial contribution as explicitly discussed in

the first example. In their most recent paper there is an altering of notations that should be highlighted, namely

$$\left[V_i \cdot N'_\alpha + \frac{1}{2}(1-\Gamma_i^{\alpha V}) - \sum_j k_{ij}a_j - s_i + V_i \cdot \overline{\alpha V}\right] |\text{phys}\rangle_{\text{real}} = 0 \qquad \text{mod } 1 \tag{5.15}$$

where $\Gamma_i^{\alpha V} \equiv \prod_j \left(g_i^j\right)^{\alpha_j}$ and $g_i^j = (i)^{n/2}\gamma^{l_1}...\gamma^{l_n}$. In the end, this difference in the definition of the Ramond zero modes can be absorbed in a redefinition of the convenient structure constants k_{ij}.

There is also the question of whether this is equivalent to multi–loop modularity. Here it turns out the equation is much less trivial. From discussing two–loop modular invariance, an additional quartic condition is acquired

$$8 \sum_{\ell \,\in\, \text{real}} W_i^\ell W_j^\ell W_k^\ell W_n^\ell = 0 \text{ mod } 1 \tag{5.16}$$

Since for higher loops however there exists a phase ambiguity also in Dihn twists, the end result is specified modulo phase ambiguities. As it is commented, even though the quartic condition is sufficient for two–loop considerations, it might be the case that the cubic constraint is necessary and sufficient for highr order Riemann surfaces' modular invariance. To determine this precisely the precise definitions of real fermions determinants must be determined for higher genus Riemann surfaces.

Before going to the first analytic example of the thesis lets us make some last comments around definitions and conventions. First pf all, the fermion occupation numbers are related via the fermion number as

$$N_{\overline{\alpha W}}^l = \sum_q \left(n_q^l - \bar{n}_q^l\right) \tag{5.17}$$

To find the spectrum, we must examine $\overline{\alpha W}$ for each value of α, remembering that $\alpha \in \{0, 1, ..., m_i\}$. In terms of the oscillatory modes, we can also write that

$$\hat{N}_\nu = \sum_{n=1}^{\infty} \left[b_{n+\nu-1}^\dagger b_{n+\nu-1} - d_{n-\nu}^\dagger d_{n-\nu}\right] \tag{5.18}$$

and the Hamiltonian

$$\hat{H}_\nu = \sum_{n=1}^{\infty} \left[(n+\nu-1)\, b_{n+\nu-1}^\dagger b_{n+\nu-1} - (n+\nu)\, d_{n-\nu}^\dagger d_{n-\nu}\right] + \frac{1}{2}\left(\nu^2 - \nu + \frac{1}{6}\right) \tag{5.19}$$

here for Ramond we have $\nu = 0$ and for NS we get $\nu = 1/2$.

5.2 E_8 model in 10D

We start working in the heterotic string case in ten-dimensions. This model turns out to be very useful since it displays three important properties; first of all, for a specific case of structure constants k_{ij}, we can recover the E_8 group. Hence, we recover a rank 8 model, instead of a rank 16 one which would normally be displayed in the case of a model where everything is consistently reorganized in terms of a single complexification. Secondly, the states and the currents that will constitute the generators for the gauge group of the model will not come from a single sector, but in fact 32 a priori. Lastly, it also displays the importance of applying the physical projection condition not only for each sector consistently, but also for *every* basis vector W_i of the model. This is happening because each such projection condition may either kill some of the already existing candidate states, give some additional , non–previously existing state (by creating new candidate generators), or give nothing new.

The set of basis vectors under discussion is

$$\begin{aligned}
W_0 &= \left(\frac{1}{2}^8 | \frac{1}{2}^{32}\right) \\
W_1 &= \left(0^8 | \frac{1}{2}^{16}, 0^{16}\right) \\
W_2 &= \left(0^8 | \frac{1}{2}^8, 0^8, \frac{1}{2}^8, 0^8\right) \\
W_3 &= \left(0^8 | \frac{1}{2}^4, 0^4, \frac{1}{2}^4, 0^4, \frac{1}{2}^4, 0^4, \frac{1}{2}^4, 0^4\right) \\
W_4 &= \left(0^8 | \frac{1}{2}^2, 0^2, \frac{1}{2}^2, 0^2, \frac{1}{2}^2, 0^2, \frac{1}{2}^2, 0^2, \frac{1}{2}^2, 0^2, \frac{1}{2}^2, 0^2, \frac{1}{2}^2, 0^2, \frac{1}{2}^2, 0^2\right) \\
W_5 &= \left(0^8 | \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0\right)
\end{aligned} \tag{5.20}$$

A sector $\overline{\alpha W}$ is defined from a linear combination of the given basis vectors. The overline symbol means exactly that we subtract integers from the components of the sum of vectors such that they lie in the range $[-\frac{1}{2}, \frac{1}{2})$. In our case of study we see that the components are either 0 or $\frac{1}{2}$. Since m_i is defined as the smallest integer such that $m_i W_i \in \mathbb{Z}$, $\forall i$, we obtain trivially that $m_i = 2$. This also gives from (5.9) that $k_{ij} \in \{0, \frac{1}{2}\}$. Keeping the overbar notation means that a basis vector either exists or not, hence signaling that the massless gauge bosons should be searched in $2^6 = 64$ sectors.

A massless gauge boson is a vector in spacetime. Therefore, under the little group $SO(8)$, it must be in the vector representation $\mathbf{8}_\mathrm{v}$. So, in the right hand side we must have $b^i_{-1/2}|0\rangle$. Since only the identity basis vector W_0 has NS real fermions on the right hand side, gauge bosons come only from sectors that include W_0, meaning 32 sectors. The Lorentz product for these basis vectors are

$$W_0 \cdot W_0 = \frac{1}{8}(32 - 8) = 3 \tag{5.21}$$

$$W_0 \cdot W_i = \frac{1}{8}(32 - 16) = 2 \tag{5.22}$$

$$W_i \cdot W_i = \frac{1}{8}(16 - 0) = 2 \tag{5.23}$$

$$W_i \cdot W_j = \frac{1}{8}(8-0) = 1 \quad , i \neq j \tag{5.24}$$

Note that this is the usual Lorentz product with an extra factor $\frac{1}{2}$ for this real fermion basis, as imposed by the correct normalization of the physical phase conditions. Let us first see why only the 31 sectors turn out contributing. We explicitly discuss the W_0 sector. If we apply the physical projection ondition in the real fermion basis, we obtain

$$W_0 \cdot N_{W_0} = k_{00} + s_0 + k_{00} - W_0 \cdot W_0 = \frac{1}{2} \mod 1 \Rightarrow$$

$$\Rightarrow \frac{1}{2}\sum_{l=1}^{40}\sum_{q=1}^{\infty} n_q^l = \frac{1}{2} \mod 1 \tag{5.25}$$

for the identity basis vector, while for the rest

$$W_i \cdot N_{W_0} = k_{i0} + s_0 + k_{0i} - W_0 \cdot W_i \ = 0 \mod 1$$

or explicitly

$$\frac{1}{2}\sum_{l=9}^{24}\sum_{q=1}^{\infty} n_q^l = 0 \mod 1 \tag{5.26}$$

$$\frac{1}{2}\sum_{l\in\{9-16,25-32\}}\sum_{q=1}^{\infty} n_q^l = 0 \mod 1 \tag{5.27}$$

$$\frac{1}{2}\sum_{l\in\{9-12,17-20,25-28,33-36\}}\sum_{q=1}^{\infty} n_q^l = 0 \mod 1 \tag{5.28}$$

$$\frac{1}{2}\sum_{l\in\{9-10,13-14,17-18,21-22,25-26,29-3\}}\sum_{q=1}^{\infty} n_q^l = 0 \mod 1 \tag{5.29}$$

$$\frac{1}{2}\sum_{l\in\{9,11,13,15,17,19,21,23,25,27,29,31\}}\sum_{q=1}^{\infty} n_q^l = 0 \mod 1 \tag{5.30}$$

We can also calculate the left and right vacuum energies for this sector,

$$E_R^0 = 8\left(-\frac{1}{48}\right) - \frac{D-2}{24} = -\frac{1}{2} \tag{5.31a}$$

$$E_L^0 = 32\left(-\frac{1}{48}\right) - \frac{D-2}{24} = -1 \tag{5.31b}$$

This means that for massless states we should take only one right moving n_1^l, $(l = 1, .., 8)$. From simultaneously imposing the above constraints we see that such condition cannot hold at the same time for only one right-moving occupation number, meaning no massless state can be constructed in this sector. What is changing for the other sectors is the left-moving vacuum energy which equals

$$E_L^0 = 16\left(-\frac{1}{48}\right) + 16\left(\frac{1}{24}\right) - \frac{D-2}{24} = -0 \tag{5.32}$$

This occurs because there are always 16 NS and 16 Ramond real fermions, but in different positions. What this means is that in the "Kac–Moody" part of these sectors, the contribution necessarily only comes from the zero Ramond modes solely. Since all these sectors turn to contribute 8 gauge bosons each, we state them explicitly

$$\overline{W_0+W_1} = \left(\tfrac{1}{2}^8 \middle| 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \checkmark$$
$$\overline{W_0+W_2} = \left(\tfrac{1}{2}^8 \middle| 0,0,0,0,0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \checkmark$$
$$\overline{W_0+W_3} = \left(\tfrac{1}{2}^8 \middle| 0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \times$$
$$\overline{W_0+W_4} = \left(\tfrac{1}{2}^8 \middle| 0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \checkmark$$
$$\overline{W_0+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2}\right) \quad \times \ \checkmark$$
$$\overline{W_0+W_4+W_5} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2}\right) \quad \times \ \checkmark$$
$$\overline{W_0+W_3+W_5} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2}\right) \quad \times \ \times$$
$$\overline{W_0+W_3+W_4} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \times$$
$$\overline{W_0+W_2+W_5} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2}\right) \quad \times \ \checkmark$$
$$\overline{W_0+W_2+W_4} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \checkmark$$
$$\overline{W_0+W_1+W_2} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \checkmark$$
$$\overline{W_0+W_2+W_3} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \times$$
$$\overline{W_0+W_1+W_5} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0\right) \quad \times \ \checkmark$$
$$\overline{W_0+W_1+W_4} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \checkmark$$
$$\overline{W_0+W_1+W_3} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right) \quad \checkmark \ \times$$
$$\overline{W_0+W_1+W_2+W_4} = \left(\tfrac{1}{2}^8 \middle| 0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \checkmark$$
$$\overline{W_0+W_2+W_4+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2}\right) \quad \times \ \checkmark$$
$$\overline{W_0+W_3+W_4+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2}\right) \quad \times \ \times$$
$$\overline{W_0+W_2+W_3+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2}\right) \quad \times \ \times$$
$$\overline{W_0+W_2+W_3+W_4} = \left(\tfrac{1}{2}^8 \middle| 0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \times$$
$$\overline{W_0+W_1+W_4+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0\right) \quad \times \ \checkmark$$
$$\overline{W_0+W_1+W_3+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0\right) \quad \times \ \times$$
$$\overline{W_0+W_1+W_3+W_4} = \left(\tfrac{1}{2}^8 \middle| 0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \times$$
$$\overline{W_0+W_1+W_2+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2}\right) \quad \times \ \checkmark$$
$$\overline{W_0+W_1+W_2+W_3} = \left(\tfrac{1}{2}^8 \middle| 0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \times$$
$$\overline{W_0+W_1+W_2+W_4+W_5} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2}\right) \quad \times \ \checkmark$$
$$\overline{W_0+W_1+W_3+W_4+W_5} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2}\right) \quad \times \ \times$$
$$\overline{W_0+W_2+W_3+W_4+W_5} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2}\right) \quad \times \ \times$$
$$\overline{W_0+W_1+W_2+W_3+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2}\right) \quad \times \ \times$$
$$\overline{W_0+W_1+W_2+W_3+W_4} = \left(\tfrac{1}{2}^8 \middle| \tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,\tfrac{1}{2},\tfrac{1}{2}\right) \quad \checkmark \ \times$$
$$\overline{W_0+W_1+W_2+W_3+W_4+W_5} = \left(\tfrac{1}{2}^8 \middle| 0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,0,\tfrac{1}{2}\right) \times \times$$

From the conditions (5.8) - (5.10) and the Lorentzian products for the basis vectors, we have that for the spin–structure constants

$$k_{ij} = k_{ji} \qquad \text{mod } 1 \tag{5.33a}$$
$$k_{ii} = k_{i0} \qquad \text{mod } 1 \ , i \neq 0 \tag{5.33b}$$

As we can easily verify there is no single complexification that simultaneously diagonalizes all basis vectors, but of course given the complexification the non–diagonalizable vector changes. Consider two cases of complexification, namely

$$\begin{aligned}\text{Complexification A} : = \{&(1,2),(3,4),(5,6),(7,8),(9,10),(11,12),(13,14),(15,16),(17,18),(19,20),\\ &(21,22),(23,24),(25,26),(27,28),(29,30),(31,32)\}\end{aligned} \tag{5.34}$$

and

$$\begin{aligned}\text{Complexification B} : = \{&(1,5),(2,6),(3,7),(4,8),(9,13),(10,14),(11,15),(12,16),(17,21),(18,22),\\ &(19,23),(20,24),(25,29),(26,30),(27,31),(28,32)\}\end{aligned} \tag{5.35}$$

Here we consider only the 32 left–moving part real fermions and the numbers in the parentheses correspond to the fermions in these fixed positions that will be combined for the formation of a free complex fermion. As we see out of the 31 sectors, both choices can complexify 15, but different ones each time. Whether the sector under discussion is complexified by either of these cases is indicated with the related color. Note also that the so–called broken vector for A is W_5 and while B, it is W_3, where in this scenario the complexification imposes combining a periodic with an antiperiodic real fermion, creating the aforementioned antidiagonal picture. Note also that there are sectors where neither of these complexifications work. This is not a problem, however, since an alternative complexification will always exist. In fact, this is one of the most important takeaway messages of the analysis since there will always exist a consistent complexification for at least two of these sectors. Then, in this complexification picture, we may compute the OPEs and the commutation relations for the generators of each sector and derive the structure constants that will determine the gauge group.

Let's focus on the sector $W_{01} = \overline{W_0 + W_1}$, which is in fact an example diagonalized by both complexifications. We will do explicitly the analysis to see which are in the end the physical states that survive the GSO-like projection for all the basis vectors. We write the five projection relations for $i = 1, ..., 5$ since W_0 turns out to be trivially encoded in these ones. More precisely, we get that

$$W_1 \cdot N_{W_{01}} = \frac{1}{4}\left(1 + \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8\gamma^9\gamma^{10}\gamma^{11}\gamma^{12}\gamma^{13}\gamma^{14}\gamma^{15}\gamma^{16}\right) = \frac{1}{4}\left(1 + \Gamma_1\right) = k_{11} \quad \text{mod } 1 \tag{5.36a}$$
$$W_2 \cdot N_{W_{01}} = \frac{1}{4}\left(1 + \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8\right) = \frac{1}{4}\left(1 + \Gamma_2\right) = k_{21} \qquad \text{mod } 1 \tag{5.36b}$$
$$W_3 \cdot N_{W_{01}} = \frac{1}{4}\left(1 + \gamma^1\gamma^2\gamma^3\gamma^4\gamma^9\gamma^{10}\gamma^{11}\gamma^{12}\right) = \frac{1}{4}\left(1 + \Gamma_3\right) = k_{31} \qquad \text{mod } 1 \tag{5.36c}$$
$$W_4 \cdot N_{W_{01}} = \frac{1}{4}\left(1 + \gamma^1\gamma^2\gamma^5\gamma^6\gamma^9\gamma^{10}\gamma^{13}\gamma^{14}\right) = \frac{1}{4}\left(1 + \Gamma_4\right) = k_{41} \qquad \text{mod } 1 \tag{5.36d}$$
$$W_5 \cdot N_{W_{01}} = \frac{1}{4}\left(1 + \gamma^1\gamma^3\gamma^5\gamma^7\gamma^9\gamma^{11}\gamma^{13}\gamma^{15}\right) = \frac{1}{4}\left(1 + \Gamma_5\right) = k_{51} \qquad \text{mod } 1 \tag{5.36e}$$

Here the γ_i are the Ramond zero modes up to a square root, so they satisfy the anticommutation relations

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \quad , \gamma^i = \sqrt{2} b_0^i \tag{5.37}$$

Using the complexification B, we define the i–th Ramond charge component as

$$Q^i = \frac{i}{2}\gamma^i \gamma^{i+4} \tag{5.38}$$

From (5.37), we see that $(Q^i)^2 = \frac{1}{4}$, so for a state in each sector where the vacuum energy on the left hand side is zero and only the zero modes contribute, meaning for the eigenvalue problem we have

$$Q_i|q\rangle = q_i|q\rangle \quad , q_i = \pm\frac{1}{2} \tag{5.39}$$

For the entries in the left hand side that have NS real fermion the charge is always zero, since there is no contribution for the massless bosons. This means that for that sector

$$|q\rangle_{01} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0\right) \tag{5.40}$$

in the same way for the $W_{01} = \overline{W_0 + W_1}$ sector, we recover

$$|q\rangle_{04} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, 0, 0, \pm\frac{1}{2}, \pm\frac{1}{2}, 0, 0, \pm\frac{1}{2}, \pm\frac{1}{2}, 0, 0, \pm\frac{1}{2}, \pm\frac{1}{2}, 0, 0\right) \tag{5.41}$$

In the same way as a complex fermion problem, we define 16 $U(1)$ currents as

$$J_i(z) = i\partial H_i(z) \tag{5.42a}$$
$$J_{i,0} = Q_i \tag{5.42b}$$

meaning that

$$J_{i,0}|q\rangle_{\text{sector}} = q_i|q\rangle_{\text{sector}} \tag{5.43}$$

For an exponential operator we recover

$$V_\alpha(z) = :\exp\left(i\sum_m \alpha_m H_m(z)\right): \tag{5.44}$$

since for a free boson OPE

$$H_i(z)H_j(w) \sim \delta_{ij}\ln(z-w) \tag{5.45a}$$
$$\partial H_i(z)\partial H_j(w) \sim \delta_{ij}\frac{1}{z-w} \tag{5.45b}$$

meaning than in general

$$J_{i,0}V_\alpha(0)|0\rangle = \alpha_i V_\alpha(0)|0\rangle \tag{5.46}$$

as it will turn out though this is not the case for our system's spin fields. In each sector, we have for these 16 fields that

$$J_{i,0}V_\alpha(0)|0\rangle = \alpha_i V_{16+1-\alpha}(0)|0\rangle \tag{5.47a}$$
$$J_{i,0}V_{16+1-\alpha}(0)|0\rangle = \alpha_i V_\alpha(0)|0\rangle \tag{5.47b}$$

The true Cartan currents hence will be eight and will be either the sum of H_i and H_{16+1-i} or their difference (depending on what will be projected out and what not). In any spin field though, out of the 16 q_i eigenvalues, only 8 will correspond to Ramond *complex* fermions, so always 8 are nonzero. We can initially write

$$S_{q_i}(z) =: \exp\left[i\sum_{i=1}^{16} q_i H_i(z)\right] :=: \exp\left[\frac{i}{2}\sum_{i=1}^{16}\epsilon_i H_i(z)\right] : \tag{5.48}$$

There are initially $2^8 = 256$ candidate eigenstate spin fields in each sector. After the five boundary projections in each sector only eight will turn out to survive. To see which ones we have to study each case separately. We will consider the scenario where all the structure constants $k_{ij} = 0$ and then comment on what changes for alternate cases.

We can rewrite the Γ_1 projection in (5.74) as

$$\frac{1}{4}\left(1+\Gamma_1\right)|\text{phys}\rangle_{01} = \frac{1}{4}\left[1+\prod_{i=1}^{8}(-2iQ_i)\right]|\text{phys}\rangle_{01} = 0 \qquad \text{mod } 1 \tag{5.49}$$

The first condition then from the definition of the charges ends up giving the constraint

$$\prod_{i=1}^{8}\epsilon_i = -1 \tag{5.50}$$

with $\epsilon_i = \pm 1$. This means that there must be an odd number of minus signs. So, the multiplicity of the minus sign in this product of eight components must be $m = 1, 3, 5, 7$. More explicitly,

$$\binom{8}{1} + \binom{8}{3} + \binom{8}{5} + \binom{8}{7} = 8 + 56 + 56 + 8 = 128 \tag{5.51}$$

This means that after imposing the projection with one out of five physical boundary vectors W_1, half of the 256 initial states are projected out. Notice also that had we taken $k_{11} = 1$ the result would be again the projection of half of the initial states, but the ones whose product have multiplicity $m = 2, 4, 6, 8$.

Doing the exact same thing for the Γ_2 projection, we recover the constraint

$$\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1 \tag{5.52}$$

which combined with (5.50) gives

$$\epsilon_5\epsilon_6\epsilon_7\epsilon_8 = 1 \tag{5.53}$$

So, in the end, the survived states are

$$\left[\binom{4}{3} + \binom{4}{1}\right] + \left[\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right] = (4+4)\times(1+6+1) = 64 \tag{5.54}$$

As mentioned we see that W_3 is the basis vector not complexified with this complexification choice, so we treat it last. We continue with the Γ_4 projection which gives

$$\epsilon_1\epsilon_2\epsilon_5\epsilon_6 = -1 \tag{5.55}$$

Setting

$$\epsilon_1\epsilon_2 = a \tag{5.56a}$$
$$\epsilon_3\epsilon_4 = b \tag{5.56b}$$
$$\epsilon_5\epsilon_6 = c \tag{5.56c}$$
$$\epsilon_7\epsilon_8 = d \tag{5.56d}$$

and combining all three epsilon conditions, we obtain

$$ab = -1 \tag{5.57a}$$
$$cd = 1 \tag{5.57b}$$
$$ac = -1 \tag{5.57c}$$

which gives a and $b = c = d = -a$, so the surviving states are $2 \times 2^4 = 32$. The Γ_5 projection ends up giving

$$\epsilon_1\epsilon_3\epsilon_5\epsilon_7 = -1 \tag{5.58}$$

resulting in the survival of only 16 states out of the initial 256 ones. For the W_{01} sector these are explicitly

$$\Sigma_1^{01} = \left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_2^{01} = \left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_3^{01} = \left(+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_4^{01} = \left(+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_5^{01} = \left(+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_6^{01} = \left(+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_7^{01} = \left(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_8^{01} = \left(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_9^{01} = \left(-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_{10}^{01} = \left(-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$$
$$\Sigma_{11}^{01} = \left(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$$

$$\Sigma_{12}^{01} = \left(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$

$$\Sigma_{13}^{01} = \left(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$$

$$\Sigma_{14}^{01} = \left(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$

$$\Sigma_{15}^{01} = \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$

$$\Sigma_{16}^{01} = \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0\right)$$

Notice that this set naturally seems to be organized in different pairs for which $q_i \to -q_i$, such as $\Sigma_1^{01} \leftrightarrow \Sigma_{16}^{01}$.

The condition associated with the Γ_3 projection says that for the final physical surviving states we should additionally have

$$\gamma^1\gamma^2\gamma^3\gamma^4\gamma^9\gamma^{10}\gamma^{11}\gamma^{12}|\text{phys}\rangle_{01} = -|\text{phys}\rangle_{01} \tag{5.59}$$

We immediately see the problem, since in the complexification we have chosen this condition cannot be translated in terms of conditions of Ramond charge components Q_i. For instance $Q_1 = \frac{i}{2}\gamma_1\gamma_5$, but γ_1 and γ_5 must both be present. The question that should now be answered is how the single γ_i act on a state, or maybe even better, ask whether the previous states are still eigenstates of the single γ_i. So, before giving the eight final *physical* eigenstates let us first understand that.

We have

$$\psi_0^i = \frac{1}{2}\left(\gamma^i + i\gamma^{i+4}\right) \tag{5.60a}$$

$$\psi_0^{i,\dagger} = \frac{1}{2}\left(\gamma^i - i\gamma^{i+4}\right) \tag{5.60b}$$

$$\{\psi_0^i, \psi_0^{i,\dagger}\} = \delta_{ij}, i = 1, ..., 16 \tag{5.60c}$$

so

$$\gamma^i = \psi_0^i + \psi_0^{i,\dagger} \tag{5.61a}$$

$$\gamma^{i+4} = -i(\psi_0^i - \psi_0^{i,\dagger}) \tag{5.61b}$$

which allows us to rewrite $Q_i = \psi_0^{i,\dagger}\psi_0^i - \frac{1}{2}$. This means that the Ramond state has occupation number $n_i = q_i + \frac{1}{2}$, so for instance

$$|\Sigma_1^{01}\rangle = \left(11101001\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\right) \tag{5.62}$$

Here the $\tilde{0}$ means this position is occupied by an NS Ramond fermion. For a complex zero mode, we define the two states of the i—th fermionic zero mode as

$$|0\rangle_i \ , \ |1\rangle_i = \psi_0^{i,\dagger}|0\rangle_i \tag{5.63}$$

By definition $\psi_0^i|0\rangle_i = 0$, while we may also obtain that $\psi_0^{i,\dagger}|1\rangle_i = 0$ and $\psi_0^i|1\rangle_i = |0\rangle_i$. By the definition, this means that

$$\gamma_i : \ |0\rangle_i \leftrightarrow |1\rangle_i \tag{5.64}$$

or in matrix form in the basis $(|0\rangle_i, |1\rangle_i)$

$$\gamma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.65}$$

Now the full Ramond in the W_{01} sector is

$$\gamma_i|n_1 n_2 n_3 n_4 n_5 n_6 n_7 n_8 \tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\rangle_{01} = (-1)^{n_1+n_2+...+n_{i-1}}|n_1, ...(1-n_i)...\rangle_{01} \tag{5.66}$$

where we have used the convention

$$|n_1 n_2 n_3 n_4 n_5 n_6 n_7 n_8 \tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\rangle_{01} = (\psi_0^{1,\dagger})^{n_1}(\psi_0^{2,\dagger})^{n_2}(\psi_0^{3,\dagger})^{n_3}(\psi_0^{4,\dagger})^{n_4}(\psi_0^{5,\dagger})^{n_5}(\psi_0^{6,\dagger})^{n_6}(\psi_0^{7,\dagger})^{n_7}(\psi_0^{8,\dagger})^{n_8}|0\rangle_i \tag{5.67}$$

And the sign appears precisely because we have to pass through the occupied modes with index smaller than i. By following this procedure for all the eight γ_i appearing in the equation, we recover than

$$\Gamma_5|\Sigma_1^{01}\rangle = -|\Sigma_{16}^{01}\rangle \tag{5.68a}$$
$$\Gamma_5|\Sigma_{16}^{01}\rangle = -|\Sigma_1^{01}\rangle \tag{5.68b}$$

or equivalently

$$\Gamma_5 \begin{pmatrix} \Sigma_1^{01} \\ \Sigma_{16}^{01} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Sigma_1^{01} \\ \Sigma_{16}^{01} \end{pmatrix} \tag{5.69}$$

so the Γ_5 eigenstates are

$$\Psi_1^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_1^{01} + \Sigma_{16}^{01}\right) \tag{5.70a}$$
$$\Psi_1^{\prime 01} = \frac{1}{\sqrt{2}}\left(\Sigma_1^{01} - \Sigma_{16}^{01}\right) \tag{5.70b}$$

The projection though says that $\Gamma_5|\text{phys}\rangle_{01} = -|\text{phys}\rangle_{01}$, so the first *physical* state is

$$|\text{phys1}\rangle_{01} = \Psi_1^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_1^{01} + \Sigma_{16}^{01}\right) \tag{5.71}$$

In a similar manner, we identify the eight physical states of the W_{01} sector to be

$$|\text{phys1}\rangle_{01} = \Psi_1^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_1^{01} + \Sigma_{16}^{01}\right) \tag{5.72a}$$

$$|\text{phys2}\rangle_{01} = \Psi_2^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_2^{01} + \Sigma_{15}^{01}\right) \tag{5.72b}$$

$$|\text{phys3}\rangle_{01} = \Psi_3^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_3^{01} + \Sigma_{14}^{01}\right) \tag{5.72c}$$

$$|\text{phys4}\rangle_{01} = \Psi_4^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_4^{01} + \Sigma_{13}^{01}\right) \tag{5.72d}$$

$$|\text{phys5}\rangle_{01} = \Psi_5^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_5^{01} + \Sigma_{12}^{01}\right) \tag{5.72e}$$

$$|\text{phys6}\rangle_{01} = \Psi_6^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_6^{01} + \Sigma_{11}^{01}\right) \tag{5.72f}$$

$$|\text{phys7}\rangle_{01} = \Psi_7^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_7^{01} + \Sigma_{10}^{01}\right) \tag{5.72g}$$

$$|\text{phys8}\rangle_{01} = \Psi_8^{01} = \frac{1}{\sqrt{2}}\left(\Sigma_8^{01} + \Sigma_9^{01}\right) \tag{5.72h}$$

This also means that the spin fields must be expressed as sums of the related pair's spin fields. For instance,

$$S_1^{01} = \frac{1}{\sqrt{2}}\left(V_1^{01} + V_{16}^{01}\right) = \frac{1}{\sqrt{2}}\left[: e^{\frac{i}{2}(H_1+H_2+H_3-H_4+H_5-H_6-H_7+H_8)} : + : e^{\frac{i}{2}(-H_1-H_2-H_3+H_4-H_5+H_6+H_7-H_8)} :\right] =$$

$$= \sqrt{2}\cos\left(H_1 + H_2 + H_3 - H_4 + H_5 - H_6 - H_7 + H_8\right) = \sqrt{2}\cos\left(q_1^{01}\cdot H\right) \tag{5.73}$$

Notice here that in case we had selected $k_{51} = \frac{1}{2}$, the surviving states would be the subtraction of paired states and the spin field operator would be sine instead.

For the $W_{05} = \overline{W_0 + W_5}$ sector, which is a sector case not complexified by the first complexification, we get

$$W_1 \cdot N_{W_{05}} = \frac{1}{4}\left(1 + \gamma^1\gamma^3\gamma^5\gamma^7\gamma^9\gamma^1 1\gamma^1 3\gamma^1 5\right) = \frac{1}{4}\left(1 + \Gamma_1\right) = k_{11} \quad \text{mod } 1 \tag{5.74a}$$

$$W_2 \cdot N_{W_{05}} = \frac{1}{4}\left(1 + \gamma^1\gamma^3\gamma^5\gamma^7\gamma^{17}\gamma^{19}\gamma^{21}\gamma^{23}\right) = \frac{1}{4}\left(1 + \Gamma_2\right) = k_{25} \quad \text{mod } 1 \tag{5.74b}$$

$$W_3 \cdot N_{W_{05}} = \frac{1}{4}\left(1 + \gamma^1\gamma^3\gamma^9\gamma^{11}\gamma^{17}\gamma^{19}\gamma^{25}\gamma^{27}\right) = \frac{1}{4}\left(1 + \Gamma_3\right) = k_{35} \quad \text{mod } 1 \tag{5.74c}$$

$$W_4 \cdot N_{W_{05}} = \frac{1}{4}\left(1 + \gamma^1\gamma^5\gamma^9\gamma^{13}\gamma^{17}\gamma^{21}\gamma^{25}\gamma^{29}\right) = \frac{1}{4}\left(1 + \Gamma_4\right) = k_{45} \quad \text{mod } 1 \tag{5.74d}$$

$$W_5 \cdot N_{W_{05}} = \frac{1}{4}\left(1 + \gamma^1\gamma^3\gamma^5\gamma^7\gamma^9\gamma^{11}\gamma^{13}\gamma^{15}\gamma^{17}\gamma^{19}\gamma^{21}\gamma^{23}\gamma^{25}\gamma^{27}\gamma^{29}\gamma^{31}\right) = \frac{1}{4}\left(1 + \Gamma_5\right) = k_{55} \quad \text{mod } 1 \tag{5.74e}$$

From the $\Gamma_{1,2,4,5}$ projections, we similarly obtain sixteen states

$$\Sigma_1^{05} = \left(+\frac{1}{2}, 0, +\frac{1}{2}, 0, +\frac{1}{2}, 0, -\frac{1}{2}, 0, +\frac{1}{2}, 0, -\frac{1}{2}, 0, -\frac{1}{2}, 0, +\frac{1}{2}, 0\right)$$

$$\Sigma_{2}^{05} = \left(+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0\right)$$

$$\Sigma_{3}^{05} = \left(+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0\right)$$

$$\Sigma_{4}^{05} = \left(+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0\right)$$

$$\Sigma_{5}^{05} = \left(+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0\right)$$

$$\Sigma_{6}^{05} = \left(+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0\right)$$

$$\Sigma_{7}^{05} = \left(+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0\right)$$

$$\Sigma_{8}^{05} = \left(+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0\right)$$

$$\Sigma_{9}^{05} = \left(-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0\right)$$

$$\Sigma_{10}^{05} = \left(-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0\right)$$

$$\Sigma_{11}^{05} = \left(-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0\right)$$

$$\Sigma_{12}^{05} = \left(-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0\right)$$

$$\Sigma_{13}^{05} = \left(-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0\right)$$

$$\Sigma_{14}^{05} = \left(-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0\right)$$

$$\Sigma_{15}^{05} = \left(-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0\right)$$

$$\Sigma_{16}^{05} = \left(-\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0\right)$$

these simultaneously satisfy the four conditions obtained for the ϵ_i Ramond charge, namely

$$\epsilon_1\epsilon_3\epsilon_5\epsilon_7 = -1 \tag{5.75a}$$

$$\epsilon_9\epsilon_{11}\epsilon_{13}\epsilon_{15} = 1 \tag{5.75b}$$

$$\epsilon_1\epsilon_3\epsilon_9\epsilon_{11} = -1 \tag{5.75c}$$

$$\epsilon_1\epsilon_5\epsilon_9\epsilon_{13} = -1 \tag{5.75d}$$

After the Γ_3 one, supposing zero values for the spin structures, the physical states are again the sum of the paired states

$$|\text{phys1}\rangle_{05} = \Psi_1^{05} = \frac{1}{\sqrt{2}}\left(\Sigma_1^{05} + \Sigma_{16}^{05}\right) \tag{5.76a}$$

$$|\text{phys2}\rangle_{05} = \Psi_2^{05} = \frac{1}{\sqrt{2}}\left(\Sigma_2^{05} + \Sigma_{15}^{05}\right) \tag{5.76b}$$

$$|\text{phys3}\rangle_{05} = \Psi_3^{05} = \frac{1}{\sqrt{2}}\left(\Sigma_3^{05} + \Sigma_{14}^{05}\right) \tag{5.76c}$$

$$|\text{phys4}\rangle_{05} = \Psi_4^{05} = \frac{1}{\sqrt{2}}\left(\Sigma_4^{05} + \Sigma_{13}^{05}\right) \tag{5.76d}$$

$$|\text{phys5}\rangle_{05} = \Psi_5^{05} = \frac{1}{\sqrt{2}}\left(\Sigma_5^{05} + \Sigma_{12}^{05}\right) \tag{5.76e}$$

$$|\text{phys6}\rangle_{05} = \Psi_6^{05} = \frac{1}{\sqrt{2}}\left(\Sigma_6^{05} + \Sigma_{11}^{05}\right) \tag{5.76f}$$

$$|\text{phys7}\rangle_{05} = \Psi_7^{05} = \frac{1}{\sqrt{2}}\left(\Sigma_7^{05} + \Sigma_{10}^{05}\right) \tag{5.76g}$$

$$|\text{phys8}\rangle_{05} = \Psi_8^{05} = \frac{1}{\sqrt{2}}\left(\Sigma_8^{05} + \Sigma_9^{05}\right) \tag{5.76h}$$

In the same manner, an example of a spin field associated with the first physical state is

$$S_1^{05} = \frac{1}{\sqrt{2}}\left(V_1^{05} + V_{16}^{05}\right) = \frac{1}{\sqrt{2}}\left[: e^{\frac{i}{2}(H_1+H_3+H_5-H_7+H_9-H_{11}-H_{13}+H_{15})} : + : e^{\frac{i}{2}(-H_1-H_3-H_5+H_7-H_9+H_{11}+H_{13}-H_{15})} :\right] =$$

$$= \sqrt{2}\cos\left(H_1 + H_3 + H_5 - H_7 + H_9 - H_{11} - H_{13} + H_{15}\right) = \sqrt{2}\cos\left(q_1^{05}\cdot H\right) \tag{5.77}$$

We are now ready to determine the OPEs to identify the gauge group. Before that, it is important to make just a small comment regarding the algebra's generators. In general generators corresponding to states for the normal case scenario are

$$E_{\alpha_i} = \oint \frac{dz}{2\pi z} : e^{2i\sum_j \alpha_j X_j(z)} : \hat{C}(\alpha_i) \tag{5.78}$$

where $X_j(z)$ are the bosonized coordinates and $\hat{C}$ the cocycle operator accounting for the phase ambiguity. The commutation relations are then given after setting the Cartan exponential variables equal to zero, ie

$$[E_\alpha, E_\beta] = \epsilon(\alpha,\beta)E_{\alpha+\beta} \tag{5.79a}$$

$$[H_{\alpha_i=0}, E_\beta] = \beta_i\epsilon(\alpha_i = 0,\beta)E_\beta \tag{5.79b}$$

For each of our two sectors, we have that there are eight Ramond zero charge components whose square $q_i^2 = \frac{1}{4}$. So, in total for the eight physical states index j, $(q_j^{01})^2 = (q_j^{05})^2 = 2$. We have

$$S_1^{01}(z)\cdot S_1^{01}(w) = \frac{1}{2}\left[e^{iq_1^{01}\cdot H(z)}e^{iq_1^{01}\cdot H(w)} + e^{iq_1^{01}\cdot H(z)}e^{-iq_1^{01}\cdot H(w)} + e^{-iq_1^{01}\cdot H(z)}e^{iq_1^{01}\cdot H(w)} + e^{-iq_1^{01}\cdot H(z)}e^{-iq_1^{01}\cdot H(w)}\right] \tag{5.80}$$

The first and the last term are regular, so they have no contribution in the derivation of the structure constraint. If we expand

$$H^I(z) - H^I(z) = (z-w)\partial H^I(w) + \mathcal{O}(w^2) \tag{5.81}$$

where I are the nonzero eight Cartan–like currents in the exponential that are nonzero in each sector. This eventually gives

$$S_1^{01}(z) \cdot S_1^{01}(w) \sim \frac{1}{(z-w)^2} + \text{regular} \tag{5.82}$$

At first glance, this resembles the OPE form of holomorphic level $k = 1$ currents, since in general

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{f_c^{ab}J^c(w)}{z-w} + \text{regular} \tag{5.83}$$

Another example between spin fields of the same sectors is

$$S_1^{01}(z) \cdot S_2^{01}(w) = \sqrt{2}\cos\left(q_1^{01} \cdot H(z)\right)\sqrt{2}\cos\left(q_2^{01} \cdot H(w)\right) \sim 0 + \text{regular} \tag{5.84}$$

more generally

$$S_i^{01}(z) \cdot S_j^{01}(w) \sim \frac{\delta_{ij}}{(z-w)^2} + \text{regular} \tag{5.85}$$

For the OPE of two spin fields of different sectors we have for instance

$$S_1^{01}(z) \cdot S_1^{05}(w) \sim \frac{1}{2(z-w)} : \cos\left[\left(q_1^{01} - q_1^{05}\right) \cdot H\right] : + \text{regular} \tag{5.86}$$

We observe we obtain a value $\frac{1}{2}$ for this structure constant, while the Cartan q_k^{01} belonging to the W_{01} sector equal to zero, we see that the commutator result belongs to the same sector. If we repeat the same procedure for all the 248 *physical* generators, we recover the E_8 algebra commutation relations. Lastly, let us comment that had we chosen some structure constants to be $\frac{1}{2}$ instead, we would recover sine spin field generators. However, their OPE will then turn out to be a cosine operator, hence resulting in an operator that is not part of the original algebra.

The Lie algebra $\mathfrak{e}_8$ admits a decomposition with respect to its maximal subalgebra $\mathfrak{so}(16)$ as

$$\mathfrak{e}_8 \cong \mathfrak{so}(16) \oplus \mathbf{128} \tag{5.87}$$

where **128** denotes one of the two chiral spinor representations of $\mathfrak{so}(16)$. Since $\dim \mathfrak{so}(16) = \binom{16}{2} = 120$,, the adjoint representation of $\mathfrak{e}_8$ decomposes as
$\mathbf{248} \longrightarrow \mathbf{120} \oplus \mathbf{128}$.

Let $M_{ab} = -M_{ba}$, with $a, b = 1, \ldots, 16$, denote the generators of $\mathfrak{so}(16)$, and let Q_α, $\alpha = 1, \ldots, 128$, denote the generators transforming in the chiral spinor representation. The commutation relations of $\mathfrak{e}_8$ can then be organized according to the above decomposition. First, the generators M_{ab} close among themselves and satisfy the standard $\mathfrak{so}(16)$ commutation relations,

$$[M_{ab}, M_{cd}] = \delta_{bc}M_{ad} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac} + \delta_{ad}M_{bc} \tag{5.88}$$

Thus, $[\mathbf{120}, \mathbf{120}] \subseteq \mathbf{120}$. The action of $\mathfrak{so}(16)$ on the spinor generators is given by

$$[M_{ab}, Q_\alpha] = \frac{1}{2}(\Gamma_{ab})_\alpha{}^\beta Q_\beta \tag{5.89}$$

where $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$, are the generators of $\mathfrak{so}(16)$ in the spinor representation. Consequently, $[\mathbf{120}, \mathbf{128}] \subseteq \mathbf{128}$.
Finally, the commutator of two spinor generators closes back into the $\mathfrak{so}(16)$ sector. It can be written schematically as

$$[Q_\alpha, Q_\beta] = C_{\alpha\beta}{}^{ab} M_{ab} \tag{5.90}$$

where the coefficients $C_{\alpha\beta}{}^{ab}$ are determined by the $\mathfrak{so}(16)$ gamma matrices and the invariant bilinear form on the spinor representation. Hence, $[\mathbf{128}, \mathbf{128}] \subseteq \mathbf{120}$.

This happens here also since the generators simultaneously complexified with the B choice result in one generator belonging to these 15 sectors. On the other hand, the commutation relation of two generators from sectors that are not complexified by the B complexification choice, results in one of the 120 generators of the 15 sectors complexified by it.

Let us close this example paragraph regarding the needed complexifications. It is easy to check that taking two out of the thirty one sectors, we can always find a complexification where at least these two sectors are complexified and their commutation relation can be calculated to recover the corresponding structure constants. Say we have the W_{01} sector and we want to check the commutation relation of its generators, with the ones of a sector not complexified by B. To proceed we just mention for completion that instead of complexifying W_{01} by a different choice, we can alternatively define the transition maps that gives us the coefficients of the states of the sector from one to the other complexification choice, as analyzed in Appendix E of [5].

Mathematica Code for Pairings

```
case1 = {1, 1, 1, 1, 1, 1, 1, 1};
case2 = {0, 0, 0, 0, 0, 0, 0, 0};

pairings[list_] :=
  If[list === {}, {{}},
   Flatten[Table[
     Map[Prepend[#, {First[list], x}] &,
      pairings[DeleteCases[list, First[list] | x]]],
     {x, Rest[list]}],
     1]];

allPairings = pairings[Range[8]];

validQ[p_, c_] :=
  And @@ (c[[#[[1]]]] == c[[#[[2]]]] & /@ p);

common =
  Select[allPairings,
   validQ[#, case1] && validQ[#, case2] &];

{Length[common], common}
```

5.3 Real fermion models in 4D

We are now mentioning a model in four dimensions, as the increase of internal degrees of freedom suggests.

$$\begin{aligned}
V_0 &= (\frac{1}{2}^2 0|\frac{1}{2}^{44}) \\
V_1 &= (0^2, (0, \frac{1}{2}, \frac{1}{2})^6|\frac{1}{2}^{28}, 0^{16}) \\
V_2 &= (0^2, (\frac{1}{2}\frac{1}{2}, 0)^6|\frac{1}{2}^1 4, 0^{14}, \frac{1}{2}^6, 0^{10}) \\
V_3 &= (0^2, (0, \frac{1}{2}, \frac{1}{2})^2, (\frac{1}{2}, \frac{1}{2}, 0)^2, (\frac{1}{2}, 0, \frac{1}{2})^2|\frac{1}{2}^{44}) \\
V_4 &= (0^2, (0,0,0)^5, (\frac{1}{2}, 0, \frac{1}{2})|(\frac{1}{2}^6, 0^8), (\frac{1}{2}^3, 0^{11}), (0^3, \frac{1}{2}^3), (\frac{1}{2}^6, 0^4)) \\
V_5 &= (0^2, (0,0,0)^4, (\frac{1}{2}, 0, \frac{1}{2})|0^{14}, (0^3, \frac{1}{2}, 0^{10})(\frac{1}{2}, 0^5), 0^{10}) \\
V_6 &= (0^2, (0, \frac{1}{2}, \frac{1}{2}), (0,0,0)^5|(0^6, \frac{1}{2}, 0^7), (0^4, \frac{1}{2}, 0^9), 0^{16})
\end{aligned} \tag{5.91}$$

In fact the first four vectors of the above model defined also a consistent model referred to as M_3 in [4]. However, as can be shown by the simultaneous complexification of the basis vecto,r the rank of the gauge group is 22 for M_3. We want to actually see if viable models can be constructed with reduced rank. The role of V_3 is not to further reduce the gauge group, but to guarantee the $\mathcal{N} = 1$ supersymmetry of the model.

The simpler thing about this model is that, at least for a given choice of structure constants k_{ij}, the gauge bosons are coming solely from the V_0 sector. This means we can easily define bilinear currents that will eventually be the generators of the gauge group, but most importantly, "read" the gauge group directly from the positions that the Ramond and NS fermions fill in the left–hand side of the basis vectors. We commence from the consideration of only V_0 and V_1. Since the action of V_0 is the same for all 44 real fermions up to a phase, we obtain no nontrivial contribution. Of course, this does not mean it will not affect the matter states.

Gauge bosons will come from level one currents of the form

$$J^{fq}(z) =: \psi^f \psi^g : (z) \tag{5.92}$$

where f, g are real fermions and the gauge algebra $g = \text{span}\{J^{fg}\}$. For each vector V_i we can define a projector (which eventually leads to the aforementioned condition) as

$$P_i = \frac{1}{2}\left(1 + \exp\left(2\pi i (V_i \cdot N)\right)\right) \tag{5.93}$$

which for bilinear currents reduces to the simple formula that

$$V_i \cdot N = V_i\left(f\right) + V_i\left(g\right) \quad \mod 1 \tag{5.94}$$

A candidate current will survive only if it satisfies the condition $P_i J^{fg} = J^{fg}$, for all basis vectors. This means if we focus on the exponential part, that eventually

$$V_i(f) + V_i(g) = 0 \quad \mod 1 \tag{5.95}$$

We start with the V_1 sector, and $A = \{1, ..., 28\}$ and $B = \{29, ..., 44\}$, where $V_1 = \begin{cases} \frac{1}{2} & f \in A \\ 0 & B \in B \end{cases}$, we recover

$$\text{A} - \text{A current} : V_1(f) + V_1(g) = \frac{1}{2} + \frac{1}{2} = 0 \quad \text{mod } 1 \checkmark \tag{5.96a}$$

$$\text{B} - \text{B current} : V_1(f) + V_1(g) = 0 + 0 = 0 \quad \text{mod } 1 \checkmark \tag{5.96b}$$

$$\text{A} - \text{B current} : V_1(f) + V_1(g) = \frac{1}{2} + 0 = \frac{1}{2} \quad \text{mod } 1 \times \text{projected out} \tag{5.96c}$$

We easily verify that the same holds in the $V_0 + V_1$ sector. The currents then are

$$\text{A} - \text{A bilinears} : J^{ab}(z) =: \psi^a \psi^b : (z) \; a, b = 1, ..., 28 \tag{5.97a}$$

$$\text{B} - \text{B bilinears} : J^{ij}(z) =: \psi^i \psi^j : (z) \; i, j = 29, ..., 44 \tag{5.97b}$$

We now examine the OPEs of the surviving currents. Starting from the free–fermion OPE

$$\psi^I(z)\psi^J(w) \sim \frac{\delta^{IJ}}{z - w} \tag{5.98}$$

we first consider the $A - A$ OPEs

$$\begin{aligned} J^{ab}(z)J^{cd}(w) &=: \psi^a\psi^b : (z) : \psi^c\psi^d : (w) \\ &\sim \frac{\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}}{(z-w)^2} + \frac{\delta^{bc}J^{ad}(w) - \delta^{ac}J^{bd}(w) - \delta^{bd}J^{ac}(w) + \delta^{ad}J^{bc}(w)}{z - w} \end{aligned}$$

Since all indices a, b, c, d belong to A, all the currents appearing on the right–hand side are again of the form J^{ab}. Hence, the $A-A$ currents close among themselves and generate $\mathfrak{so}(28)$. Likewise, for the $B-B$ currents' OPEs, we find

$$\begin{aligned} J^{ij}(z)J^{kl}(w) &=: \psi^i\psi^j : (z) : \psi^k\psi^l : (w) \\ &\sim \frac{\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}}{(z-w)^2} + \frac{\delta^{jk}J^{il}(w) - \delta^{ik}J^{jl}(w) - \delta^{jl}J^{ik}(w) + \delta^{il}J^{jk}(w)}{z - w} \end{aligned}$$

Again, all indices on the right-hand side belong to B, so the $B - B$ currents close among themselves and generate $\mathfrak{so}(16)$. It remains to consider the OPE between the two different sets of surviving currents. Taking one current from the A sector and one from the B sector gives

$$\begin{aligned} J^{ab}(z)J^{ij}(w) &=: \psi^a\psi^b : (z) : \psi^i\psi^j : (w) \\ &\sim \frac{\delta^{ai}\delta^{bj} - \delta^{aj}\delta^{bi}}{(z-w)^2} + \frac{\delta^{bi}J^{aj}(w) - \delta^{ai}J^{bj}(w) - \delta^{bj}J^{ai}(w) + \delta^{aj}J^{bi}(w)}{z - w} \sim 0 \end{aligned} \tag{5.99}$$

Thus, there is no singular OPE between the $A-A$ and $B-B$ currents. In particular, no mixed generator is produced by the OPE of the two surviving sectors.

For completeness, the would-be mixed current is $J^{ai}(z) =: \psi^a\psi^i : (z), a \in A, i \in B$. However, this current is already removed by the V_1 projection, since

$$V_1(a) + V_1(i) = \frac{1}{2} + 0 = \frac{1}{2} \not\equiv 0 \quad \text{mod } 1 \tag{5.100}$$

Hence $J^{ai}(z) \longrightarrow 0$ in the projected spectrum. Therefore, the two sets of surviving currents form two independent current algebras, $\mathfrak{g} = \mathfrak{so}(28) \oplus \mathfrak{so}(16)$, and hence the gauge group is $G = SO(28) \times SO(16)$.

We can repeat the process to recover the gauge group by adding basis vectors and checking all the projection relations. The vector V_6 does not affect the $V_0 + V_1$ sector fermions as long as one chooses $k_{60}+k_{61} = 0$. All of these models have $\mathcal{N} = 1$ supersymmetry provided one chooses $k_{40}+k_{43} = k_{50}+k_{53} = k_{60} + k_{63} = 0$.

Rank		Gauge group
22	V_0	SO(44)
22	V_1	SO(28) × SO(16)
22	V_2	SO(14) × SO(14) × SO(6) × SO(10)
22	V_3	SO(14) × SO(14) × SO(6) × SO(10)
20	V_4	SO(6) × SO(8) × SU(2) $\overset{-1}{\times}$ SO(11) × SU(2) $\overset{-1}{\times}$ SU(2) × SU(4) × SU(2) × SU(2)
20	V_5	SO(6) × SO(8) × SU(2) × SO(10) × U(1) × SU(2) × SU(4) × $[\mathrm{SU}(2)]^2$
18	V_6	SO(6) $\overset{-1}{\times}$ SO(7) × SU(2) × $\overset{-1}{\mathrm{SO}}$(9) × U(1) × SU(2) × SU(4) × $[\mathrm{SU}(2)]^2$

Figure 5.1: Rank reduction and gauge group after adding each of the new basis vectors. Each case contains the vector of the same row and all the aforementioned ones.

This case should also be contrasted with the previous example, since the currents here are bilinears. In addition to that, since the real fermion organization in the left hand side of the basis vectors is very asymmetrical, the rank, up to the definition of a model with these seven basis vectors, has only been reduced from 22 to 18.

Chapter 6

Conclusions

The first goal of the present thesis was to understand how to formulate on the torus, a theory of free (non–)compact bosons and fermions with periodic or more general boundary conditions along the non–contractible cycles. We then introduced some basic notions of orbifolds and briefly reviewed the charge lattice construction for both bosonic and superstring theories, seeing how physical consistency conditions, in particular modular invariance, arise naturally from the q–expansion of the one–loop partition function. It was then possible to identify directly the different components of the spectrum at the various excitation levels.

The main focus of the thesis was to understand CFTs with real fermions. In particular, by adopting the spin-field formalism, in which the theory was formulated, we considered systems whose defining set of basis vectors cannot be simultaneously complexified within a single complexification choice of the fermions. Although this may initially seem problematic, it turns out to be precisely the feature that leads to the rank-reduced models encountered in the examples studied here.

This constitutes a very interesting setup for both phenomenological and more formal investigations. First of all, it is interesting to ask how far this setup can be pushed towards lower rank and what particle content can be obtained in the process. Another natural question is what the minimal rank that can be reached is, starting from rank 26 in 10D and rank 22 in 4D.

From a CFT viewpoint the connection between the spin-field formalism and the orbifold construction for such models seems also nontrivial. The appearance of linear combinations of states as the physical eigenstates after imposing the full set of projection conditions seems to suggest that some permutation-orbifold interpretation may be relevant. This can already be seen indirectly from the states

$$\Sigma_1^{01} = \left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},0,0,0,0,0,0,0,0\right)$$

and

$$\Sigma_1^{05} = \left(+\frac{1}{2},0,+\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0,-\frac{1}{2},0,-\frac{1}{2},0,+\frac{1}{2},0\right)$$

of the two different sectors of the first example explicitly discussed. Both contain eight Ramond zero modes, five of which have charge eigenvalue $+\frac{1}{2}$ and three of which have charge eigenvalue $-\frac{1}{2}$. What changes between the two sectors is which of the sixteen positions are Ramond and which are NS. The positions of the non-zero charges therefore matter. This is why the permutation structure matters: we do not, in general, have the freedom to interchange the positions without changing the physical content of the state.

Why was the rank eight in the first example? Did we not initially have sixteen Cartan generators? Here we were careful to refer to them as Cartan-like currents. As we saw, these are not necessarily eigenstates of the full set of projections, since we have relations of the form

$$V_i^{\text{sector}} \rightleftharpoons V_{\sigma(i)}^{\text{sector}}$$

This suggests that the physical Cartan directions are not simply the individual H_i, but rather the combinations

$$H_i^{\text{phys}} = H_i \pm H_{\sigma(i)} \tag{6.1}$$

where the sign is determined by which combination is retained by the projection and which one is projected out. The permutation $\sigma(i)$ specifies how the indices are paired. For example, in the first example considered in this thesis, we had $\sigma(i) = 16 + 1 - i$.

The appearance of these linear combinations also provides a useful way of thinking about the rank reduction observed in the examples. We start with sixteen Cartan-like directions associated with the internal degrees of freedom. The permutation pairs these directions, giving eight pairs of the form $(A_i, A_{\sigma(i)})$. Before imposing the projection, each pair can be represented by two independent linear combinations. The projection then distinguishes between these combinations, with one combination surviving in the physical spectrum and the other being projected out. Schematically,

$$\begin{array}{ccccc} A_1, A_2, \ldots, A_{16} & & A_1 \pm A_{\sigma(1)} & & A_1 + A_{\sigma(1)} \\ & & \vdots & & \vdots \\ \#\text{dof} = 16 & \longrightarrow & A_8 \pm A_{\sigma(8)} & \xrightarrow{\text{projection}} & A_8 + A_{\sigma(8)} \\ & & \#\text{dof} = 16 & & \#\text{dof} = 8 \end{array}$$

This is the structure that we observe in the examples considered here. Importantly, the schematic picture should not be understood as a proof of a general mechanism for rank reduction. Rather, it summarises the way in which the physical combinations arise in the examples studied. In particular, the fact that the physical states are linear combinations of permutation-related states suggests that the rank reduction may be related to the reorganisation of the original degrees of freedom under the permutation and to the subsequent projection. Whether this observation can be formulated as a general statement for the class of models considered here remains to be established.

The underlying idea has some useful analogies with familiar physical systems, although these should not be taken as direct analogies with the CFT construction. Consider, for example, two coupled degrees of freedom. Their natural eigenmodes are often the symmetric and antisymmetric combinations $\frac{A_1 \pm A_2}{\sqrt{2}}$, rather than the original variables A_1 and A_2. One of the two combinations may have a different energy from the other, and depending on the physical situation one may effectively observe only a particular mode. A similar situation occurs in solid-state physics, where the physically relevant excitations are collective normal modes, such as acoustic and optical phonons, rather than the original microscopic degrees of freedom.

Appendix A

Finite Determination of Modular Forms

A.1 Sturm Bound

Theorem 3 (Sturm)**.** *Let $f(\tau) = \sum_{n\geq 0} a_n q^n$ and $g(\tau) = \sum_{n\geq 0} b_n q^n$ be modular forms of weight k for a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$. If*

$$a_n = b_n \qquad \text{for all } n \leq N$$

where

$$N \geq \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma]$$

then $f = g$.

Proof. Let $h = f - g$. Then,

$$h(\tau) = \sum_{n>N} c_n q^n$$

so h vanishes at the cusp ∞ to order at least $N + 1$. By the valence formula for modular forms on Γ,

$$\sum_{z\in\Gamma\backslash\mathbb{H}} \operatorname{ord}_z(h) + \operatorname{ord}_\infty(h) = \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma]$$

If N satisfies the bound in the theorem then $\operatorname{ord}_\infty(h)$ already exceeds the total number of zeros allowed unless $h \equiv 0$. Hence $f = g$. □

A.2 Determination by Polar Part

Proposition 1. *Let $f(\tau)$ be a weakly holomorphic modular form of weight k for $SL_2(\mathbb{Z})$ with expansion*

$$f(\tau) = \sum_{n\geq -M} a_n q^n$$

Then f is uniquely determined by finitely many coefficients of its q–expansion, equivalently by its polar part together with a finite number of regular coefficients.

Proof. Let $\Delta(\tau)$ denote the modular discriminant, a cusp form of weight 12 with expansion

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

If f has poles of order at most M at the cusp then

$$g(\tau) = f(\tau)\Delta(\tau)^M$$

is a holomorphic modular form of weight $k + 12M$. The space $M_{k+12M}(SL_2(\mathbb{Z}))$ is finite dimensional and is generated by the Eisenstein series E_4 and E_6:

$$M_* = \mathbb{C}[E_4, E_6]$$

Thus g is a polynomial in E_4 and E_6, which is determined by finitely many Fourier coefficients. Consequently f is determined by finitely many coefficients of its q–expansion. □

Appendix B

Teichmüller spaces

B.1 General Solution

Following the analysis of Chapter (3), we now want to make the distinction between the sphere and the torus more explicit by solving the transverse–traceless conditions directly. We start from the same problem in both cases: we seek all symmetric tensors h_{ab} satisfying

$$g^{ab}h_{ab} = 0 \tag{B.1a}$$

$$\nabla_a h^{ab} = 0 \tag{B.1b}$$

The first condition imposes tracelessness, corresponding to the fact that the Weyl direction has already been removed. The second condition is precisely the condition $h \in \ker P_1^\dagger$, so that h_{ab} is orthogonal to the infinitesimal diffeomorphism directions. The important point is that these equations must ultimately be solved *globally* on the Riemann surface. Locally, the solution has the same form on the sphere and the torus. The difference between the two surfaces appears only after imposing their respective global regularity or periodicity conditions. Let us first work in complex coordinates $(z, \bar{z})$. A general symmetric rank-two tensor can be written as

$$h = h_{zz}dz^2 + 2h_{z\bar{z}}dzd\bar{z} + h_{\bar{z}\bar{z}}d\bar{z}^2 \tag{B.2}$$

For a two-dimensional conformal metric, we may write

$$ds^2 = 2g_{z\bar{z}}dzd\bar{z} \tag{B.3}$$

where

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, \qquad g_{z\bar{z}} = g_{\bar{z}z} \neq 0 \tag{B.4}$$

Consequently, the inverse metric also has only mixed components,

$$g^{zz} = g^{\bar{z}\bar{z}} = 0, \qquad g^{z\bar{z}} = g^{\bar{z}z} \neq 0 \tag{B.5}$$

Firstly, we start by imposing the tracelessness condition. Expanding the contraction gives

$$g^{zz}h_{zz} + g^{z\bar{z}}h_{z\bar{z}} + g^{\bar{z}z}h_{\bar{z}z} + g^{\bar{z}\bar{z}}h_{\bar{z}\bar{z}} = 0 \tag{B.6}$$

The first and last terms vanish because $g^{zz} = g^{\bar{z}\bar{z}} = 0$. Since h_{ab} is symmetric, $h_{z\bar{z}} = h_{\bar{z}z}$, and therefore

$$2g^{z\bar{z}}h_{z\bar{z}} = 0 \tag{B.7}$$

Since $g^{z\bar{z}} \neq 0$, we obtain $h_{z\bar{z}} = 0$. Thus every traceless symmetric tensor takes the form

$$h = h_{zz}dz^2 + h_{\bar{z}\bar{z}}d\bar{z}^2 \tag{B.8}$$

The two independent components of a traceless symmetric tensor are therefore h_{zz} and $h_{\bar{z}\bar{z}}$. Now consider the second condition. A conformally flat metric is written

$$ds^2 = e^{2\phi(z,\bar{z})}dzd\bar{z} \tag{B.9}$$

where the precise factor of 2 in the definition of $g_{z\bar{z}}$ depends on convention and will not affect the result. The non-vanishing Christoffel symbols are

$$\Gamma^z_{zz} = 2\partial_z\phi, \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\partial_{\bar{z}}\phi \tag{B.10}$$

while

$$\Gamma^z_{z\bar{z}} = \Gamma^{\bar{z}}_{z\bar{z}} = 0 \tag{B.11}$$

We first need to raise the indices on h_{ab}. Since the metric has only mixed components,

$$h^{zz} = g^{z\bar{z}}g^{z\bar{z}}h_{\bar{z}\bar{z}}, \quad h^{\bar{z}\bar{z}} = g^{\bar{z}z}g^{\bar{z}z}h_{zz} \tag{B.12}$$

The mixed components remain zero. We now consider the $\bar{z}$ component of the divergence equation,

$$\nabla_a h^{a\bar{z}} = 0 \tag{B.13}$$

Since $h^{z\bar{z}} = 0$, the only non-vanishing contribution comes from

$$\nabla_{\bar{z}} h^{\bar{z}\bar{z}} = 0 \tag{B.14}$$

For a rank-two contravariant tensor,

$$\nabla_{\bar{z}} h^{\bar{z}\bar{z}} = \partial_{\bar{z}} h^{\bar{z}\bar{z}} + \Gamma^{\bar{z}}_{\bar{z}c} h^{c\bar{z}} + \Gamma^{\bar{z}}_{\bar{z}c} h^{\bar{z}c} \tag{B.15}$$

only $c = \bar{z}$ contributes, giving

$$\nabla_{\bar{z}} h^{\bar{z}\bar{z}} = \partial_{\bar{z}} h^{\bar{z}\bar{z}} + 2\Gamma^{\bar{z}}_{\bar{z}\bar{z}} h^{\bar{z}\bar{z}} \tag{B.16}$$

Using the Christoffel definitions, we get

$$\nabla_{\bar{z}} h^{\bar{z}\bar{z}} = \partial_{\bar{z}} h^{\bar{z}\bar{z}} + 4(\partial_{\bar{z}}\phi) h^{\bar{z}\bar{z}} \tag{B.17}$$

Now,

$$h^{\bar{z}\bar{z}} = (g^{z\bar{z}})^2 h_{zz} \tag{B.18}$$

Since $g^{z\bar{z}} \propto e^{-2\phi}$, we have $h^{\bar{z}\bar{z}} \propto e^{-4\phi}h_{zz}$. Therefore,

$$\partial_{\bar{z}} h^{\bar{z}\bar{z}} \propto e^{-4\phi}\left[\partial_{\bar{z}} h_{zz} - 4(\partial_{\bar{z}}\phi) h_{zz}\right] \tag{B.19}$$

Substituting this into the divergence equation, the terms involving the conformal factor cancel, meaning

$$\nabla_a h^{a\bar{z}} = 0 \Rightarrow \partial_{\bar{z}} h_{zz} = 0 \tag{B.20}$$

Thus, $h_{zz} = q(z)$, where q is holomorphic. The other component of the divergence equation gives, similarly, $h_{\bar{z}\bar{z}} = \bar{q}(\bar{z})$. Thus the complete local solution of the transverse–traceless conditions is

$$h = q(z)dz^2 + \bar{q}(\bar{z})d\bar{z}^2 \tag{B.21}$$

B.2 Sphere

We start by solving the problem globally on S^2, the simplest scenario. We use the stereographic coordinate z, which covers the sphere except for one point, corresponding to $z = \infty$. Since q is holomorphic on the finite complex plane, it admits a power–series expansion,

$$q(z) = a_0 + a_1 z + a_2 z^2 + \mathcal{O}(z^3) \tag{B.22}$$

However, holomorphicity on the finite plane is not sufficient. The tensor h must also be smooth at the point $z = \infty$. To study this point, introduce a second coordinate $w = \frac{1}{z}$. Then

$$z = \frac{1}{w} \Rightarrow dz = -\frac{dw}{w^2} \tag{B.23}$$

and therefore

$$dz^2 = \frac{dw^2}{w^4} \tag{B.24}$$

The holomorphic part of the tensor becomes

$$q(z)dz^2 = q\left(\frac{1}{w}\right)\frac{dw^2}{w^4} \tag{B.25}$$

Thus, in the w coordinate, the coefficient of dw^2 is $q_w(w) = \frac{1}{w^4} q\left(\frac{1}{w}\right)$. Substituting the power–series expansion of q gives

$$q\left(\frac{1}{w}\right) = a_0 + a_1 w^{-1} + a_2 w^{-2} + \mathcal{O}(w^{-3}) \tag{B.26}$$

and hence

$$q_w(w) = a_0 w^{-4} + a_1 w^{-5} + a_2 w^{-6} + \mathcal{O}(w^{-7}) \tag{B.27}$$

For the tensor to be regular at $w = 0$, the coefficient $q_w(w)$ must be holomorphic there. Every non-zero coefficient a_n would produce a pole at $w = 0$. Therefore, all must vanish, which means that $q(z) = 0$. The antiholomorphic component vanishes by the same argument. We therefore find $h_{ab} = 0$. Thus, $\ker P_1^\dagger(S^2) = 0$ and therefore $\dim_{\mathbb{C}} \mathcal{T}(S^2) = 0$. The important point is that this conclusion did not follow from the local differential equations alone. Locally, there are many solutions $q(z)$. It is the global regularity condition at $z = \infty$ that eliminates all of them. The absence of Teichmüller moduli for the sphere is therefore a global statement.

B.3 Torus

We now solve exactly the same equations on the torus. The torus admits a globally defined flat metric, so we may choose

$$ds^2 = \frac{1}{\tau_2} dz d\bar{z} \tag{B.28}$$

In these coordinates the connection vanishes, and

$$\nabla_a h^{ab} = 0 \Rightarrow \partial_a h^{ab} = 0 \tag{B.29}$$

Since z and $z + 1$ represent the same point on the torus, the tensor must satisfy

$$h(z + 1) = h(z) \tag{B.30}$$

For the holomorphic part, since $d(z+1) = dz$, we obtain

$$q(z+1) = q(z) \tag{B.31}$$

Similarly, because $z \sim z + \tau$, we get $q(z+\tau) = q(z)$. Thus q is a holomorphic doubly periodic function:

$$q(z+1) = q(z), \qquad q(z+\tau) = q(z) \tag{B.32}$$

We can now determine the form of such a function without simply assuming that it is constant. Consider the fundamental parallelogram

$$F = \{x + y\tau \mid 0 \le x \le 1,\ 0 \le y \le 1\} \tag{B.33}$$

Since F is compact and q is continuous, $|q|$ attains a maximum on F. Thus there exists some finite M such that $|q(z)| \le M,\ z \in F$. Now take any $z \in \mathbb{C}$. Since the lattice translations tile the complex plane, there exist integers m, n such that $z - m - n\tau \in F$. Using also the periodicity of q, we have

$$q(z) = q(z - m - n\tau) \tag{B.34}$$

and

$$|q(z)| \le M \tag{B.35}$$

for every $z \in \mathbb{C}$. Hence q is an entire bounded function. Liouville's theorem now implies that q must be a complex constant. Therefore, the general transverse–traceless deformation on the torus is

$$h = Cdz^2 + \bar{C}d\bar{z}^2 \tag{B.36}$$

The complex constant C contains two real parameters, $\Re C$ and $\Im C$. Consequently, $\dim_{\mathbb{C}} \ker P_1^\dagger(T^2) = 1$.

Appendix C

Special Functions and Modular Properties

This appendix collects several standard definitions and identities that were used throughout the analysis, mainly considering the CFT discussions on the torus. We remind that the modular group $SL(2,\mathbb{Z})$ is generated by the two transformations

$$T : \tau \longmapsto \tau + 1, \qquad S : \tau \longmapsto -\frac{1}{\tau}$$

C.1 Dedekind Eta Function

The Dedekind eta function is defined on the upper half-plane by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right) \tag{C.1}$$

Since $\tau_2 > 0$, one has $|q| < 1$, and hence the infinite product converges absolutely.

The eta function is a modular form of weight $1/2$, up to the non–trivial multiplier system characteristic of half–integral weight modular forms. Its transformation under the generators T and S of $SL(2,\mathbb{Z})$ is

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau) \tag{C.2a}$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\, \eta(\tau) \tag{C.2b}$$

Here and below the square root is taken with the standard branch on $\mathbb{H}$. The first relation follows directly from the product definition. On the other hand, the S transformation is more substantial and can be derived using Poisson resummation or from the transformation properties of theta functions. In particular, it is exactly the transformation that implies that $\eta(\tau)$ has modular weight $1/2$. More generally, under

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

one may write

$$\eta(\gamma\tau) = \epsilon(a,b,c,d)\, (c\tau + d)^{1/2} \eta(\tau), \tag{C.3}$$

where $\epsilon(a,b,c,d)$ is a twelfth root of unity. Thus, while η is not itself invariant under $SL(2,\mathbb{Z})$, its modular behavior is completely controlled by its weight and multiplier system.

A particularly useful consequence of the product representation is the function $\eta^{24}(\tau)$ is a holomorphic modular form of integer weight 12 and, in fact, is the Ramanujan cusp form

$$\Delta(\tau) = \eta(\tau)^{24} \tag{C.4}$$

Its transformation properties are consequently simpler,

$$\Delta(\tau+1) = \Delta(\tau) \tag{C.5}$$

$$\Delta\left(-\frac{1}{\tau}\right) = \tau^{12}\Delta(\tau) \tag{C.6}$$

C.2 Jacobi Theta Functions and Poisson Resummation

The Jacobi theta functions arise naturally from Gaussian lattice sums[1]. A convenient definition of the general theta function with characteristics a and b is

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} \exp\Big[\pi i\tau(n+a)^2 + 2i(n+a)(z+\pi b)\Big] \qquad \tau\in\mathbb{H} \tag{C.7}$$

where $q = e^{2\pi i\tau}$. For τ in the upper half-plane, the Gaussian factor $e^{-\pi(n+a)^2\tau_2}$ guarantees convergence. The characteristics a and b determine the two quasi-periodicities

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z+\pi|\tau) = e^{2\pi i a}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) \tag{C.8a}$$

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z+\pi\tau|\tau) = e^{-i\pi\tau-2iz-2\pi ib}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) \tag{C.8b}$$

The four standard Jacobi theta functions can be defined through the series

$$\vartheta_1(z|\tau) = \sum_{n\in\mathbb{Z}}(-1)^{n-\frac{1}{2}}q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{(2n+1)iz} \tag{C.9a}$$

$$\vartheta_2(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{(2n+1)iz} \tag{C.9b}$$

$$\vartheta_3(z|\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}e^{2inz} \tag{C.9c}$$

$$\vartheta_4(z|\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2}e^{2inz} \tag{C.9d}$$

Equivalently, ϑ_1 can be written in the more familiar form

$$\vartheta_1(z|\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}\sin\Big[(2n+1)z\Big] \tag{C.10}$$

The Jacobi triple-product identity gives the corresponding infinite product representations:

[1] The conventions used in this section are the ones of [3].

$$\vartheta_1(z|\tau) = 2q^{1/8}\sin z\prod_{n=1}^{\infty}(1-q^n)(1-2q^n\cos(2z)+q^{2n}) \tag{C.11a}$$

$$\vartheta_2(z|\tau) = 2q^{1/8}\cos z\prod_{n=1}^{\infty}(1-q^n)(1+2q^n\cos(2z)+q^{2n}) \tag{C.11b}$$

$$\vartheta_3(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+2q^{n-\frac{1}{2}}\cos(2z)+q^{2n-1}) \tag{C.11c}$$

$$\vartheta_4(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)(1-2q^{n-\frac{1}{2}}\cos(2z)+q^{2n-1}) \tag{C.11d}$$

At $z = 0$, the first theta function vanishes,

$$\vartheta_1(0|\tau) = 0 \tag{C.12}$$

whereas the other three have the useful representations

$$\vartheta_2(0|\tau) = 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 \tag{C.13a}$$

$$\vartheta_3(0|\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2 \tag{C.13b}$$

$$\vartheta_4(0|\tau) = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2 \tag{C.13c}$$

The modular transformation T acts on the theta functions as

$$\vartheta_1(z|\tau+1) = e^{\pi i/4}\vartheta_1(z|\tau) \tag{C.14a}$$

$$\vartheta_2(z|\tau+1) = e^{\pi i/4}\vartheta_2(z|\tau) \tag{C.14b}$$

$$\vartheta_3(z|\tau+1) = \vartheta_4(z|\tau) \tag{C.14c}$$

$$\vartheta_4(z|\tau+1) = \vartheta_3(z|\tau) \tag{C.14d}$$

whilst the S one is

$$\vartheta_1\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = -i(-i\tau)^{1/2}e^{iz^2/(\pi\tau)}\vartheta_1(z|\tau) \tag{C.15a}$$

$$\vartheta_2\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = (-i\tau)^{1/2}e^{iz^2/(\pi\tau)}\vartheta_4(z|\tau) \tag{C.15b}$$

$$\vartheta_3\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = (-i\tau)^{1/2}e^{iz^2/(\pi\tau)}\vartheta_3(z|\tau) \tag{C.15c}$$

$$\vartheta_4\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = (-i\tau)^{1/2}e^{iz^2/(\pi\tau)}\vartheta_2(z|\tau) \tag{C.15d}$$

Thus $\vartheta_3(z|\tau)$ maps into itself under S, whereas $\vartheta_2(z|\tau)$ and $\vartheta_4(z|\tau)$ are exchanged. The theta functions consequently form a closed system under the modular group.

A fundamental tool in deriving these transformations is Poisson resummation. For a sufficiently well-behaved function $f(x)$, we can define its Fourier transform by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)e^{-2\pi ikx} \tag{C.16}$$

The Poisson summation formula states that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m) \tag{C.17}$$

For the Gaussian

$$f(x) = e^{-\pi a x^2}, \qquad \mathrm{Re}(a) > 0$$

its Fourier transform is

$$\widehat{f}(k) = a^{-1/2} e^{-\pi k^2/a} \tag{C.18}$$

Hence, its Poisson resummation gives the known identity

$$\sum_{n\in\mathbb{Z}} e^{-\pi a n^2} = \frac{1}{\sqrt{a}} \sum_{m\in\mathbb{Z}} e^{-\pi m^2/a} \tag{C.19}$$

Setting $a = \tau_2$ yields the basic relation between a lattice sum at τ_2 and its dual at $1/\tau_2$. More generally, including a phase gives

$$\sum_{n\in\mathbb{Z}} e^{-\pi a(n+b)^2 + 2\pi i n c} = \frac{1}{\sqrt{a}} \sum_{m\in\mathbb{Z}} e^{-\pi (m-c)^2/a + 2\pi i m b} \tag{C.20}$$

This form is particularly useful for converting the aforementioned momentum sums into winding ones.

C.3 Bessel Functions of the Second Kind

The Bessel equation of order ν is

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \tag{C.21}$$

Two linearly independent solutions are usually denoted by $J_\nu(x)$ and $Y_\nu(x)$, where J_ν is the Bessel function of the first kind and Y_ν is the Bessel function of the second kind. For non-integer ν, the latter can be defined in terms of the first kind functions as

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)} \tag{C.22}$$

The integer-order functions $Y_n(x)$ are obtained by taking the appropriate limit as $\nu \to n$. For small positive x, the behavior of $Y_\nu(x)$ for $\nu \notin \mathbb{Z}$ is dominated by

$$Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu, \qquad x \to 0^+, \qquad \nu > 0 \tag{C.23}$$

For integer order, logarithmic terms can appear. In particular,

$$Y_0(x) \sim \frac{2}{\pi} \left[\log\left(\frac{x}{2}\right) + \gamma\right], \qquad x \to 0^+ \tag{C.24}$$

where γ is the Euler–Mascheroni constant. For large positive x, the second-kind Bessel function has the oscillatory asymptotic expansion

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right), \qquad x \to \infty \tag{C.25}$$

Correspondingly,

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \tag{C.26}$$

The Bessel functions satisfy useful recurrence relations. For $Z_\nu = J_\nu$ or Y_ν,

$$Z_{\nu-1}(x) + Z_{\nu+1}(x) = \frac{2\nu}{x} Z_\nu(x) \tag{C.27a}$$

$$Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2Z'_\nu(x) \tag{C.27b}$$

Equivalently,

$$\frac{d}{dx}\left[x^\nu Z_\nu(x)\right] = x^\nu Z_{\nu-1}(x), \tag{C.28a}$$

$$\frac{d}{dx}\left[x^{-\nu} Z_\nu(x)\right] = -x^{-\nu} Z_{\nu+1}(x) \tag{C.28b}$$

The Wronskian of the two independent solutions J_ν and Y_ν is

$$W[J_\nu, Y_\nu] = J_\nu(x)Y'_\nu(x) - J'_\nu(x)Y_\nu(x) = \frac{2}{\pi x} \tag{C.29}$$

Thus any solution of the Bessel equation can be expressed as a linear combination

$$y(x) = AJ_\nu(x) + BY_\nu(x) \tag{C.30}$$

The modified Bessel functions I_ν and K_ν are obtained by continuation to imaginary arguments. This one was used via the Eisenstein series method for the formal path integral derivation of both bosons and fermions. Their large-x behavior is

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \tag{C.31a}$$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \tag{C.31b}$$

C.4 Divisor Sum Function

For a positive integer n and complex parameter s, the divisor sum function is defined by

$$\sigma_s(n) = \sum_{d|n} d^s \tag{C.32}$$

where the sum runs over all positive divisors d of n. The commonly used notation $\sigma(n) = \sigma_1(n)$ therefore corresponds to the sum of the positive divisors of n. For example,

$$\sigma_0(n) = \sum_{d|n} 1 = d(n) \tag{C.33a}$$

$$\sigma_1(n) = \sum_{d|n} d \tag{C.33b}$$

$$\sigma_2(n) = \sum_{d|n} d^2 \tag{C.33c}$$

Here $d(n)$ denotes the number of positive divisors of n. If n has prime factorization

$$n = \prod_{j=1}^{r} p_j^{a_j} \tag{C.34}$$

then $\sigma_s(n)$ is multiplicative and can be written as

$$\sigma_s(n) = \prod_{j=1}^{r} \left(1 + p_j^s + p_j^{2s} + \cdots + p_j^{a_j s}\right) \tag{C.35}$$

while for $s \neq 0$, this can equivalently be written as

$$\sigma_s(n) = \prod_{j=1}^{r} \frac{p_j^{(a_j+1)s} - 1}{p_j^s - 1} \tag{C.36}$$

For $s = 0$ one obtains

$$\sigma_0(n) = \prod_{j=1}^{r} (a_j + 1) \tag{C.37}$$

Additionally, the multiplicativity property states that if $\gcd(m, n) = 1$, then

$$\sigma_s(mn) = \sigma_s(m)\sigma_s(n) \tag{C.38}$$

The divisor functions also possess a useful Dirichlet–series representation. For $\mathrm{Re}(s)$ sufficiently large,

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \zeta(s)\zeta(s-a) \tag{C.39}$$

where $\zeta(s)$ is the Riemann zeta function. This follows by interchanging the order of summation:

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \sum_{n=1}^{\infty} \sum_{d|n} \frac{d^a}{n^s} = \sum_{d=1}^{\infty} d^{a-s} \sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s-a)\zeta(s) \tag{C.40}$$

Divisor sums occur naturally as Fourier coefficients of modular forms. For instance, the Eisenstein series of even weight $k > 2$ has the Fourier expansion

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{C.41}$$

where B_k denotes the k-th Bernoulli number. In particular,

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \tag{C.42}$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \tag{C.43}$$

Finally, the asymptotic size of $\sigma_s(n)$ can be estimated from standard divisor-sum bounds. For fixed $s > 0$, the function grows polynomially up to factors that depend mildly on the arithmetic structure of n. A simple bound following from the definition is

$$\sigma_s(n) \leq n^s d(n) \tag{C.44}$$

since every divisor d of n satisfies $d \leq n$. Such estimates are often sufficient when establishing convergence of Fourier expansions or bounding contributions from large Fourier modes.

Appendix D

Tree Level Amplitudes in the Bosonic String

Even though the goal of this thesis was not the study of scattering amplitudes, some basic notions are mentioned here. More specifically, we state the S–matrix equivalent in string theory and shortly review the steps that lead to the most famous string theory amplitude: the Veneziano amplitude.

D.1 Basic Formalism Setup

As in QFT, the starting point to calculate scattering amplitudes is the S-matrix[1]. The idea of the Feynman path integral generalizes and in principle what we will now have to consider is summing over all worldsheets that have the appropriate initial and final states as boundary conditions. Nonetheless, the analysis does not end here. As we can see from D.1 due to state–operator correspondence, in the end, the worldsheet topology is independent of the number of states. The only dependence regarding the number of states is encoded in the insertion of the vertex operators.

$$S_{j_1\cdots j_n}(k_1,\ldots,k_n) = \sum_{\text{compact topologies}} \int \frac{[dX\, dg]}{\text{Vol}(\text{Diff}\times\text{Weyl})} e^{(-S_X-\lambda\chi)} \prod_{i=1}^{n} \left(\int d^2\sigma_i \sqrt{g(\sigma_i)}\, V_{j_i}(k_i,\sigma_i) \right) \tag{D.1}$$

To carry out explicit calculations, it is essential to work with the gauge-fixed form of the S-matrix. Applying the Faddeev-–Popov procedure to the above expression leads to the considerably lengthy formula,

$$\begin{aligned} S_{j_1\cdots j_n}(k_1,\ldots,k_n) = \sum_{\text{compact topologies}} \int_{\mathcal{F}} \frac{d^\mu t}{n_R} \int [d\phi\, db\, dc]\, \exp\left(-S_m - S_g - \lambda\chi\right) \\ \prod_{(a,i)\notin f} \int d\sigma_i^a \left(\prod_{k=1}^{\mu} \frac{1}{4\pi} (b, \partial_k \hat{g}) \right) \left(\prod_{(a,i)\in f} c_i^a(\hat{\sigma}_i) \right) \prod_{i=1}^{n} \sqrt{\hat{g}(\sigma_i)}\, V_{j_i}(k_i,\sigma_i) \end{aligned} \tag{D.2}$$

D.2 Veneziano amplitude

This formula, while inherently intricate, admits significant simplification in the specific case under consideration—namely, the Veneziano amplitude. The Veneziano amplitude describes the tree–level scattering of four tachyons in oriented open string theory. At tree level, there are precisely three Riemann surfaces

[1]The conventions used in this section are the ones of Chapters 5,6 of [6], which is the main reference, with [7] and [3].

with positive Euler characteristic χ, among which the surface relevant for oriented open string scattering is the disk $\mathbb{D}^2$ which is conformally equivalent to the upper half-plane.

To proceed with our analysis, we now invoke the *Riemann–Roch theorem*[2], which establishes a relationship between the number of metric moduli μ, conformal Killing vectors κ, and the Euler characteristic χ via the relation:

$$\mu - \kappa = -3\chi \tag{D.3}$$

while simultaneously

$$\chi > 0 \Rightarrow \kappa = 3\chi \ , \quad \mu = 0 \tag{D.4}$$

$$\chi < 0 \Rightarrow \kappa = 0 \ , \quad \mu = -3\chi \tag{D.5}$$

Since the disk is a zero genus Riemann surface that has one boundary its Euler characteristic is $\chi_{\mathbb{D}^2} = 1$, meaning that $\mu = 0$ and $\kappa = 3$. This can also be verified by its conformal group $PSL(2, \mathbb{R})$, which will allow us to fix three out of the four insertions at fixed points.

Before presenting the simplified expression for the Veneziano amplitude, it is important to make some additional remarks. In the case of the oriented open string, the vertex operators must be inserted along the boundary of the worldsheet. When working in the upper half-plane formulation, this boundary corresponds to the real axis. Moreover, one must account for all inequivalent orderings of vertex operator insertions. For a four-point amplitude, there are six such inequivalent orderings. The amplitude is then given by

$$S_V(k_1, k_2, k_3, k_4) = \sum_{\pi \in S_4/\mathbb{Z}_4} \int \frac{[dX\, dg]}{\text{Vol(Diff} \times \text{Weyl)}} \exp\left(-S_X - \lambda\chi\right) \prod_{i=1}^{4} \left(\int d^2\sigma_i \sqrt{g(\sigma_i)}\, V_{j_i}(k_i, \sigma_i) \right) \tag{D.6}$$

where the sum is over inequivalent cyclic orderings of the vertex insertions.

At this point we will use the extra symmetry provided by the conformal group $PSL(2, \mathbb{R})$ to to fix the positions of three vertex operators without loss of generality. We choose:

$$y_1 = 0, \quad y_2 = 1, \quad y_3 \to \infty \tag{D.7}$$

This gauge fixing reduces the moduli of the configuration to a single real variable, corresponding to the position of the fourth vertex, $y_4 \in \mathbb{R}$

$$S_{\mathbb{D}^2}(k_1; k_2; k_3; k_4) = g_o e^{-\lambda} \int_{-\infty}^{\infty} dy_4 \langle \prod_{i=1}^{3} : c_i(y_i) e^{ik_i \cdot X(y_i)} :: e^{ik_4 \cdot X(y_4)} : \rangle + (k_2 \longleftrightarrow k_3) \tag{D.8}$$

The constant $g_o = e^{\lambda/2}$ is the *open string coupling*. The fourth power of g_o reflects the presence of four external tachyons, as each integrated vertex operator contributes a factor of g_o from its normalization. Since the ghost and matter sectors of the worldsheet CFT are decoupled we can factorize the path integral into separate correlators for the ghost and matter contributions. The ghost c is a primary field with conformal weight $h = -1$, so the three-point correlation function for $< c_1(y_1)c_2(y_2)c_3(y_3) >$ has the fixed form

[2] A complete proof and statement of the theorem can be found in [8].

$$\langle c_1(y_1)c_2(y_2)c_3(y_3)\rangle \propto \frac{1}{(y_1-y_2)^{-1}(y_2-y_3)^{-1}(y_1-y_3)^{-1}} \tag{D.9}$$

This follows from the general 3-point function of three primary fields $\phi_1(z_1),\ \phi_2(z_2),\ \phi_3(z_3)$ of conformal weights h_1, h_2h_3

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\, z_{13}^{h_1+h_3-h_2}\, z_{23}^{h_2+h_3-h_1}} \tag{D.10}$$

where the constant C_{123} depends on the nature of the problem.

In general, the correlation function of n tachyon vertex operators can be computed to be on the disk $\mathbb{D}^2$

$$\left\langle \prod_{i=1}^{n} : e^{ik_i\cdot X(z_i,\bar{z}_i)} : \right\rangle_{\mathbb{D}^2} = iC_X^{\mathbb{D}^2}(2\pi)^d\delta^{(d)}\left(\sum_i k_i\right)\prod_{i=1}^{n}|z_i-\bar{z}_i|^{\frac{\alpha' k_i^2}{2}}\prod_{\substack{i,j=1\\ i<j}}^{n}|z_i-z_j|^{\alpha' k_i\cdot k_j}|z_i-\bar{z}_j|^{\alpha' k_i\cdot k_j} \tag{D.11}$$

Inserting both expressions on (D.8) we can reach the form

$$S_{\mathbb{D}^2}(k_1;k_2;k_3;k_4) = i\, g_o^4\, C_{\mathbb{D}^2}\,(2\pi)^{26}\delta^{26}\left(\sum_i k_i\right)|y_{12}\, y_{13}\, y_{23}|\int_{-\infty}^{\infty} dy_4 \prod_{i<j}|y_{ij}|^{2\alpha' k_i\cdot k_j} + (k_2\leftrightarrow k_3) \tag{D.12}$$

where we have used the abbreviation $y_{ij} = y_i - y_j$. It is now time to change into Mandelstam variables as they were introduced in the introduction along with the fact that $s+t+u = \sum_i m_i = -\frac{4}{a'}$. More precisely,

$$2\alpha' k_i\cdot k_j = \alpha'\left[(k_i+k_j)^2 - k_i^2 - k_j^2\right] = -2 + \alpha'(k_i+k_j)^2 \tag{D.13}$$

To avoid potential confusion, we emphasize that in the present section k_i denote the full momentum vectors, not their magnitudes (in contrast to the notation used in the appendix beginning formula). By expressing the momentum dependence in terms of the Mandelstam variables and substituting into (D.12), we obtain an expression that closely resembles the integral representation of the Euler beta function. The primary difference lies in the limits of integration, which initially range over the entire real axis $y_4 \in (-\infty,\infty)$, rather than the interval $[0,1]$. However, there is another trick we can use which amounts to splitting the integral in three regions, namely

$$-\infty < y_4 \leq 0, \quad 0 < y_4 \leq 1, \quad 1 < y_4 < \infty$$

which gives equal contributions thanks to $PSL(2,\mathbb{R})$ transformations we can perform to map the first and third region in a form like the intermediate one. After these manipulations, we arrive at the well-known form of the Veneziano amplitude

$$S_{\mathbb{D}^2}(k_1;k_2;k_3;k_4) = 2i\, g_4^2\, C_{\mathbb{D}^2}\,(2\pi)^{26}\delta^{26}\left(\sum_{i=1}^{4} k_i\right)[I(s,t)+I(t,u)+I(u,s)] \tag{D.14}$$

where each term $I(s,t)$ corresponds to an integral of beta function type

$$I(s,t) = \int_0^1 dy\; y^{-\alpha' s-2}(1-y)^{-\alpha' t-2} = B(-\alpha' s-1, -\alpha' t-1) \tag{D.15}$$

Appendix E

Coset Theory

Let G be a simple Lie algebra of rank r_G, and let $\widehat{G}_k$ denote its affine extension at level k.[1] A useful way of understanding the corresponding conformal field theory is to separate the degrees of freedom associated with the Cartan subalgebra from those associated with the remaining root directions. The latter form the parafermionic sector of the theory.

The stress tensor of an affine algebra is obtained through the Sugawara construction. In the standard normalization, it is given by

$$T_G(z) = \frac{1}{2(k+h_G^\vee)} \sum_a : J^a J^a : (z) \tag{E.1}$$

where $h_G^\vee$ is the dual Coxeter number of G, and the index a runs over all generators of the Lie algebra. The corresponding central charge is

$$c_G = \frac{k, \dim, G}{k + h_G^\vee} \tag{E.2}$$

We now compare this central charge with that of the Cartan subalgebra. Since the rank of G is r_G, its Cartan subalgebra is r_G–dimensional and can be represented by r_G free bosons. In particular, the Cartan currents can be written as

$$H^i(z) = i\partial\phi^i(z), \qquad i = 1, \ldots, r_G \tag{E.3}$$

where the ϕ^i are free scalar fields. The corresponding free-boson stress tensor is

$$T_{U(1)^{r_G}}(z) = -\frac{1}{2} \sum_{i=1}^{r_G} : \partial\phi^i \partial\phi^i : (z) \tag{E.4}$$

and has central charge

$$c_{U(1)^{r_G}} = r_G \tag{E.5}$$

The simplest case is that of a simply-laced algebra at level $k = 1$. In the standard normalization, in which all roots have length squared $\alpha^2 = 2$, a simply-laced algebra satisfies

$$\dim G = r_G \, (h_G^\vee + 1) \tag{E.6}$$

Consequently, at level one,

$$c_{G_1} = \frac{\dim, G}{h_G^\vee + 1} = r_G \tag{E.7}$$

Thus, the central charge of the affine theory is precisely equal to that of r_G free bosons.

[1] This section follows the discussion of coset constructions in [9].

This equality provides the basis for the free–boson, or vertex–operator, construction of simply–laced affine algebras at level one. In addition to the Cartan currents in (E.3), the currents associated with the roots can be represented schematically as

$$E_\alpha(z) \sim : e^{i\alpha\cdot\phi(z)} : \tag{E.8}$$

With the normalization $\alpha^2 = 2$, the conformal weight of such a vertex operator is $h_\alpha = \frac{\alpha^2}{2} = 1$. The root vertex operators are therefore dimension–one operators and can be identified with the affine currents. Together with the Cartan currents, they reproduce the full affine algebra. Since there is no additional central charge beyond that of the Cartan bosons, one may write

$$T_{G_1}(z) = T_{U(1)^{r_G}}(z) \tag{E.9}$$

For $k > 1$, however, the situation changes. The affine theory has central charge

$$c_{G_k} = \frac{k \dim G}{k + h^\vee_G} \tag{E.10}$$

which in general is larger than r_G. The Cartan currents still provide a $U(1)^{r_G}$ sector with central charge r_G, but this sector no longer accounts for all the degrees of freedom of the affine theory. The remaining degrees of freedom define the coset theory $\frac{G_k}{U(1)^{r_G}}$, whose central charge is

$$c_{\text{para}} = c_{G_k} - r_G = \frac{k, \dim, G}{k + h^\vee_G} - r_G \tag{E.11}$$

At the level of the stress tensor, this decomposition can be expressed as

$$T_G(z) = T_{U(1)^{r_G}}(z) + T_{\frac{G_k}{U(1)^{r_G}}}(z) \tag{E.12}$$

The second contribution is referred to as the parafermionic or coset sector. This construction also makes clear why the free–boson description is special to simply–laced algebras at level one. For a non-simply–laced algebra, the relation (E.6) does not hold in the same way, and consequently the affine theory contains degrees of freedom that cannot be represented solely by the Cartan bosons, even at level $k = 1$. As a concrete example, consider $SO(5)$. Its Lie algebra has $r_{SO(5)} = 2$, $\dim SO(5) = 10$, $h^\vee_{SO(5)} = 3$. At level one, its affine central charge is therefore

$$c_{SO(5)_1} = \frac{10}{1+3} = \frac{5}{2} \tag{E.13}$$

The Cartan sector consists of two free bosons and hence has $c_{U(1)^2} = 2$. The remaining coset sector consequently has central charge

$$c_{\text{para}} = c_{SO(5)_1} - 2 = \frac{1}{2} \tag{E.14}$$

In fact, the corresponding coset is equivalent to the Ising conformal field theory,

$$\frac{SO(5)_1}{U(1)^2} \cong \text{Ising CFT} \tag{E.15}$$

Thus, the $SO(5)_1$ theory can be viewed as two free bosons together with an additional $c = \frac{1}{2}$ Majorana or Ising sector. This additional contribution is precisely what is absent in the simply–laced level–one construction.

A similar phenomenon occurs for simply–laced groups when the affine level is increased above one[2]. The simplest example is $SU(2)_k$. Since $\dim SU(2) = 3$, $h^\vee_{SU(2)} = 2$, $r_{SU(2)} = 1$, the affine central charge is

$$c_{SU(2)k} = \frac{3k}{k+2} \tag{E.16}$$

The Cartan subalgebra contributes one free boson with $c = 1$. Consequently, the parafermionic sector has

$$c_{\text{para}} = \frac{3k}{k+2} - 1 = \frac{2(k-1)}{k+2} \tag{E.17}$$

The resulting coset

$$\frac{SU(2)_k}{U(1)} \tag{E.18}$$

is the Z_k parafermion conformal field theory. Notice that at $k = 1$, the parafermionic central charge vanishes, recovering the purely bosonic description. For $k > 1$, however, a non-trivial parafermionic sector is required in addition to the Cartan boson.

More generally, the coset construction provides a natural decomposition of an affine theory into a bosonic Cartan sector and a non–abelian remainder. The latter contains the degrees of freedom associated with the root directions that cannot be captured by the $U(1)^{r_G}$ currents alone.

[2]For an in–depth description of related systems, see [10].

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