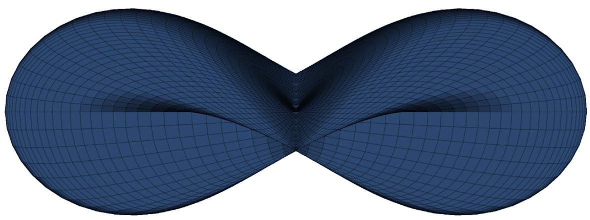
UNIVERSITY OF IOANNINA
DEPARTMENT OF MATHEMATICS

Entzi Zotai

Equivariant Lagrangian Mean Curvature Flow

Master's Thesis



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The present dissertation was carried out under the postgraduate program of the Department of Mathematics of the University of Ioannina in order to obtain the Master's Degree.

Accepted by the three members of the evaluation committee on 17-06-2026:

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The acceptance of this dissertation by the Department of Mathematics of the University of Ioannina does not imply the approval of the opinions of the author (section 202 par. 2 Law 5343/1932 and section 50 par. 8 Law 1268/1982).

Statutory Declaration

I lawfully declare with statutority that the present dissertation was carried out under the international ethical and academic rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis.

Dedicated to my respected parents and to my sister,
whom I deeply appreciate.

With love and respect.

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First and foremost, my deepest gratitude goes to my parents. Beyond their love and support, they have always treated me with honesty and respect. They encouraged me to face difficulties directly, to question my assumptions, and never to settle for comforting illusions instead of truth. From them, I learned the value of integrity, self-reflection, and perseverance.

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Abstract

An important problem arising in mathematical physics, particularly in mirror symmetry, concerns the existence of minimal Lagrangian tori in compact Calabi-Yau manifolds. A natural approach to constructing such objects is provided by the ***mean curvature flow*** (MCF). More precisely, let $F_0 : L \to N$ be a compact Lagrangian immersion, and consider the evolution governed by

$$\begin{cases} \dfrac{\partial F}{\partial t}(x,t) = H(x,t), \\ F(x,0) = F_0(x), \end{cases} \tag{MCF}$$

for $(x,t) \in L \times [0,T)$, where $H(x,t)$ denotes the mean curvature vector of the immersion $F_t : L \to N$, defined by

$$F_t(x) = F(x,t).$$

If L is compact, the initial value problem (MCF) admits a smooth unique solution (up to reparametrizations) on a maximal time interval $[0,T)$, where $0 < T \leq \infty$. A fundamental result due to Smoczyk [49] shows that the Lagrangian property is preserved along the flow when the ambient manifold is Calabi-Yau. However, geometric flows typically develop singularities in finite time, and understanding their formation and structure is a central problem in Geometric Analysis.

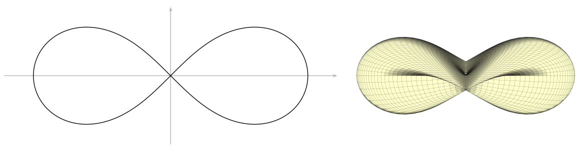
Building on foundational work of Hamilton [20] and Huisken [25], singularities can be analyzed via suitable rescalings of the flow. Depending on the blow-up rate of the squared norm of the second fundamental form, they are classified as ***Type I*** or ***Type II***. Roughly speaking, Type I singularities form more slowly. These singularities are modeled as submanifolds in Euclidean space.

This thesis investigates the evolution of ***equivariant Lagrangian submanifolds*** in $\mathbb{R}^{2m} = \mathbb{R}^m \otimes \mathbb{R}^m$ under (MCF). We focus on submanifolds invariant under the full action of $\mathbb{O}(m) \times \mathbb{O}(m)$. Such submanifolds admit a parametrization of the form

$$f(x,s) = (u(s)x, v(s)x),$$

where $x \in \mathbb{S}^{m-1}$ and $s \in I \subset \mathbb{R}$. The associated planar curve $z = (u,v)$ will be referred to as the ***profile curve***. As we will see below (see Theorem A), the most symmetric example in this class is the ***Whitney sphere***, generated by

$$z(s) = \frac{(-\sin(s), \sin(s)\cos(s))}{1+\cos^2(s)}.$$



The equivariance condition is preserved along the flow, and the evolution reduces to a curvature-driven equation with an additional forcing term. More precisely, the profile curve evolves according to

$$\frac{\partial z}{\partial t} = \kappa - (m-1)\frac{z^\perp}{|z|^2},$$

where $z : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ is a family of curves, κ denotes the curvature vector, and $z^\perp$ is the normal component of z. The main objective is to describe the long-time behavior of these submanifolds and to analyze the formation of singularities. This work is inspired by the results of:

1. K. Groh, M. Schwarz, K. Smoczyk, and K. Zehmisch, *Mean curvature flow of monotone Lagrangian submanifolds*, Math. Z. **257** (2007), 295-327.

2. K. Groh, *Singular behavior of equivariant Lagrangian mean curvature flow*, Hannover: Univ. Hannover, Fakultät für Mathematik und Physik (Diss.), 2007.

3. A. Savas-Halilaj and K. Smoczyk, *Lagrangian mean curvature flow of Whitney spheres*, Geom. Topol. **23** (2019), 1057-1084.

4. A. Ros and F. Urbano, *Lagrangian submanifolds of* $\mathbb{C}^n$ *with conformal Maslov form and the Whitney sphere*, J. Math. Soc. Japan **50** (1998), 203-226.

5. V. Borrelli, B. Chen, and J. Morvan, *A geometric characterisation of the Whitney sphere*, C. R. Acad. Sci., Paris, Sér. I **321** (1995), 1485-1490.

The main results presented in this thesis are the following:

Theorem A: (Ros-Urbano & Borrelli-Chen-Morvan) *Let* $F\colon L \to \mathbb{R}^{2m}$ *be an immersed Lagrangian submanifold satisfying*

$$\|A\|^2 = \frac{3}{m+2}\,\|H\|^2.$$

Then $F(L)$ *is either a flat subspace or a Whitney sphere.*

This result shows that the Whitney sphere is the Lagrangian analogue of a round sphere in the Euclidean space.

Theorem B: (Groh) *Let* $F_0\colon \mathbb{S}^1 \times \mathbb{S}^{m-1} \to \mathbb{R}^{2m}$ *be an equivariant Lagrangian immersion whose profile curve is closed, embedded, tamed, and does not contain the origin. Then the evolving submanifolds under the equivariant Lagrangian mean curvature flow (ELMCF) converge to an* $(m-1)$*-dimensional sphere as* $t \to T$*. Moreover, the maximal time is given by*

$$T = A_0/2\pi,$$

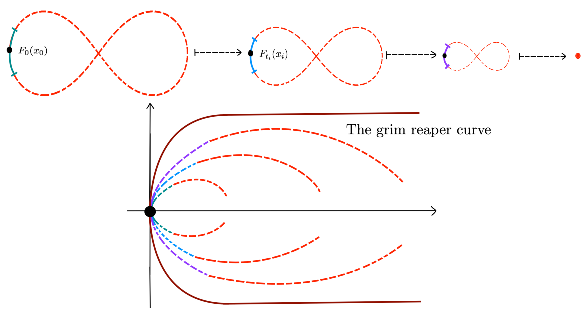
where A_0 *denotes the area enclosed by the initial curve.*

Theorem C: (Groh-Schwarz-Smoczyk-Zehmisch) *Let* $F\colon \mathbb{S}^1 \times \mathbb{S}^{m-1} \times [0,T) \to \mathbb{R}^{2m}$*,* $m \geq 2$*, be a solution of the equivariant Lagrangian mean curvature flow (ELMCF), with profile curves* $z\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2 - \{0\}$*. Suppose the initial curve is star-shaped. Then,*

1. *the profile curves remain star-shaped as long as a smooth solution exists,*
2. *the first singularity occurs at the origin, i.e.* $\liminf_{t\to T} \|F\| = 0$*,*
3. *depending on the initial data, both Type I and Type II singularities may occur.*

The behavior of the Whitney sphere under the mean curvature flow (MCF) differs significantly from that of a round sphere in Euclidean space. As a matter of fact, the submanifolds that evolve from the Whitney sphere cannot be Whitney spheres anymore; they do not simply scale by a time dependent dilation, like round spheres in the Euclidean space.

Theorem D: (Savas-Halilaj-Smoczyk) *Let $F_0 : \mathbb{S}^m \to \mathbb{R}^{2m}$, $m > 1$, be an equivariant Lagrangian immersion satisfying* $\mathrm{Ric}(g_0) \geq c\,\|F_0\|^2 g_0$, *for some constant $c > 0$. Then along the flow one has* $\mathrm{Ric}(g_t) \geq \varepsilon \|F_t\|^2 g_t$ *for a uniform constant $\varepsilon > 0$. Moreover, the flow develops a Type II singularity whose blow-up limit is given by the product of a grim reaper curve and a flat $(m-1)$-dimensional Lagrangian subspace in $\mathbb{R}^{2m-2}$.*



The thesis is organized into 3 main parts. The first part develops the differential geometric framework for Lagrangian and equivariant Lagrangian submanifolds in Euclidean space. It includes a characterization of the Whitney sphere, obtained independently by Ros-Urbano and Borrelli-Chen-Morvan, as well as a study of equivariant Lagrangian submanifolds with non-zero mean curvature, referred to as ***tamed equivariant Lagrangians***.

The second part establishes the analytical foundations of mean curvature flow, including short-time existence and regularity. Key tools such as the parabolic maximum principle are introduced, and the evolution equations for the relevant geometric quantities are derived. This framework is then used to analyze the formation of singularities, including the characterization of the maximal time of existence and techniques for modeling singular behavior.

The final part focuses on the singular behavior of the equivariant Lagrangian mean curvature flow, culminating in the proofs of Theorems B, C, and D.

Περιληψη

Ένα σημαντικό πρόβλημα που προκύπτει στη μαθηματική φυσική, αφορά την ύπαρξη ελαχιστικών Λαγκρανζιανών τόρων σε συμπαγή πολυπτύγματα Calabi-Yau. Μία προσέγγιση για την κατασκευή τέτοιων αντικειμένων παρέχεται από τη ***ροή μέσης καμπυλότητας*** (MCF). Συγκεκριμένα, δοθείσης μίας Λαγκρανζιανής εμβάπτισης $F_0 : L \to N$ θεωρούμε την εξέλιξη που διέπεται από το σύστημα διαφορικών εξισώσεων

$$\begin{cases} \dfrac{\partial F}{\partial t}(x,t) = H(x,t), \\ F(x,0) = F_0(x), \end{cases} \tag{MCF}$$

για $(x,t) \in L \times [0,T)$, όπου $H(x,t)$ το διάνυσμα μέσης καμπυλότητας της εμβάπτισης $F_t : L \to N$, που ορίζεται από τον τύπο

$$F_t(x) = F(x,t).$$

Αν το πολύπτυγμα L είναι συμπαγές, τότε το πρόβλημα αρχικών τιμών (MCF) διαθέτει ομαλή και μοναδική (ως προς αναπαραμετρήσεις) λύση σε ένα μεγιστοτικό χρονικό διάστημα $[0,T)$, $0 < T \leq \infty$. Σύμφωνα με ένα θεμελιώδες αποτέλεσμα του Smoczyk [49] η Λαγκρανζιανή ιδιότητα διατηρείται από τη ροή όταν ο περιβάλλοντας χώρος είναι Calabi-Yau. Ωστόσο, οι γεωμετρικές ροές συνήθως αναπτύσσουν ιδιομορφίες, και η κατανόηση του σχηματισμού και της δομής αυτών αποτελεί κεντρικό πρόβλημα στη Γεωμετρική Ανάλυση.

Βασιζόμενοι στις θεμελιώδεις εργασίες των Hamilton [20] και Huisken [25], οι ιδιομορφίες μπορούν να αναλυθούν μέσω κατάλληλων μεγεθύνσεων της ροής μέσης καμπυλότητας. Ανάλογα με τον ρυθμό έκρηξης της νόρμας της δεύτερης θεμελιώδους μορφής, ταξινομούνται ως ιδιομορφίες ***Τύπου I*** ή ***Τύπου II***. Σε κάθε περίπτωση, οι ιδιομορφίες αυτές μοντελοποιούνται από υποπολυπτύγματα του Ευκλειδείου χώρου.

Η παρούσα διατριβή μελετά την εξέλιξη μέσω ροής μέσης καμπυλότητας ***ισο-αναλλοίωτων Λαγκρανζιανών υποπολυπτυγμάτων*** στον $\mathbb{R}^{2m} = \mathbb{R}^m \otimes \mathbb{R}^m$, δηλαδή την εξέλιξη υποπολυπτυγμάτων που είναι αναλλοίωτα από την πλήρη δράση της ομάδας $\mathbb{O}(m) \times \mathbb{O}(m)$. Τέτοια υποπολυπτύγματα περιγράφονται από παραμετρήσεις της μορφής

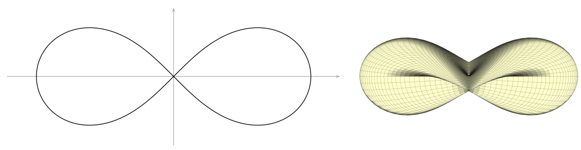
$$f(x, s) = (u(s)x, v(s)x),$$

όπου $x \in \mathbb{S}^{m-1}$ και $s \in I \subseteq \mathbb{R}$. Η αντίστοιχη επίπεδη καμπύλη

$$z = (u, v)$$

θα αναφέρεται ως ***γενέτειρα*** του ισοαναλλοίωτου υποπολυπτύγματος. Όπως θα δούμε παρακάτω (βλ. Θεώρημα A), το πλέον συμμετρικό παράδειγμα σε αυτή την κλάση είναι η ***σφαίρα του Whitney***, η οποία παράγεται από την καμπύλη

$$z(s) = \frac{(-\sin(s), \sin(s)\cos(s))}{1 + \cos^2(s)}.$$



Η ισοαναλλοιωτική ιδιότητα διατηρείται από τη ροή, και η εξέλιξη ανάγεται στην παρακάτω εξελικτική διαφορική εξίσωση

$$\frac{\partial z}{\partial t} = \kappa - (m-1)\frac{z^{\perp}}{|z|^2},$$

όπου

$$z : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$$

είναι μία οικογένεια καμπυλών, κ το διάνυσμα καμπυλότητας, και $z^{\perp}$ η κάθετη συνιστώσα του z.

Κύριος στόχος της διατριβής είναι η μελέτη και η περιγραφή της συμπεριφοράς των ισοαναλλοίωτων υποπολυπτυγμάτων κάτω από τη ροή μέσης καμπυλότητας και η ανάλυση των ιδιομορφιών. Στην εργασία αναλύονται αποτελέσματα των παρακάτω άρθρων:

1. K. Groh, M. Schwarz, K. Smoczyk, and K. Zehmisch, *Mean curvature flow of monotone Lagrangian submanifolds*, Math. Z. **257** (2007), 295-327.

2. K. Groh, *Singular behavior of equivariant Lagrangian mean curvature flow*, Hannover: Univ. Hannover, Fakultät für Mathematik und Physik (Diss.), 2007.

3. A. Savas-Halilaj and K. Smoczyk, *Lagrangian mean curvature flow of Whitney spheres*, Geom. Topol. **23** (2019), 1057-1084.

4. A. Ros and F. Urbano, *Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and the Whitney sphere*, J. Math. Soc. Japan **50** (1998), 203-226.

5. V. Borrelli, B. Chen, and J. Morvan, *A geometric characterisation of the Whitney sphere*, C. R. Acad. Sci., Paris, Sér. I **321** (1995), 1485-1490.

Τα κύρια θεωρήματα που παρουσιάζονται στη διατριβή είναι τα ακόλουθα:

Θεώρημα A: (Ros-Urbano & Borrelli-Chen-Morvan) *Έστω $F\colon L \to \mathbb{R}^{2m}$ ένα εμβαπτισμένο Λαγκρανζιανό υποπολύπτυγμα που ικανοποιεί*

$$\|A\|^2 = \frac{3}{m+2}\,\|H\|^2.$$

Τότε η εικόνα $F(L)$ είναι είτε ισόπεδο είτε μια σφαίρα του Whitney.

Το αποτέλεσμα αυτό δείχνει ότι η σφαίρα του Whitney αποτελεί το Λαγκρανζιανό ανάλογο της σφαίρας στον Ευκλείδειο χώρο.

Θεώρημα B: (Groh) *Έστω $F_0\colon \mathbb{S}^1 \times \mathbb{S}^{m-1} \to \mathbb{R}^{2m}$ μία λεία ισοαναλλοίωτη Λαγκρανζιανή εμβάπτιση με μη-μηδενικό διάνυσμα μέσης καμπυλότητας της οποίας η γενέτειρα είναι κλειστή, εμφυτευμένη καμπύλη που δεν περικλείει την αρχή των αξόνων. Τότε τα εξελισσόμενα υποπολυπτύγματα από τη ροή μέσης καμπυλότητας (ELMCF) συρρικνώνονται σε μία σφαίρα διάστασης $(m-1)$ καθώς $t \to T$. Επιπλέον, ο μεγιστοτικός χρόνος ύπαρξης της ροής δίνεται από τη σχέση*

$$T = \frac{A_0}{2\pi},$$

όπου A_0 είναι το εμβαδόν που περικλείεται από την αρχική καμπύλη.

Θεώρημα C: (Groh-Schwarz-Smoczyk-Zehmisch) *Έστω*

$$F \colon \mathbb{S}^1 \times \mathbb{S}^{m-1} \times [0, T) \to \mathbb{R}^{2m}, \quad m \geq 2,$$

λύση της ισοαναλλοίωτης ροής μέσης καμπυλότητας (ELMCF), με εξελισσόμενες γενέτειρες τις καμπύλες

$$z \colon \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2 - \{0\}.$$

Υποθέτουμε ότι η αρχική γενέτειρα είναι αστρόμορφη καμπύλη. Τότε,

1. *οι εξελισσόμενες γενέτειρες καμπύλες παραμένουν αστρόμορφες,*
2. *η πρώτη ιδιομορφία εμφανίζεται στην αρχή των αξόνων, δηλαδή όταν*

$$\liminf_{t \to T} \|F\| = 0,$$

3. *ανάλογα με τα αρχικά δεδομένα, μπορούν να εμφανιστούν τόσο ιδιομορφίες Τύπου I όσο και Τύπου II.*

Η συμπεριφορά της σφαίρας του Whitney υπό τη (MCF) διαφέρει σημαντικά από εκείνη μιας Ευκλείδειας σφαίρας στον $\mathbb{R}^{2m}$. Πράγματι, τα εξελισσόμενα υποπολυπτύγματα που προκύπτουν από τη σφαίρα του Whitney δεν παραμένουν πλέον σφαίρες του Whitney, δηλαδή δεν εξελίσσονται μέσω ομοιοθεσιών.

Θεώρημα D: (Savas-Halilaj–Smoczyk) *Έστω*

$$F_0 : \mathbb{S}^m \to \mathbb{R}^{2m}, \qquad m > 1,$$

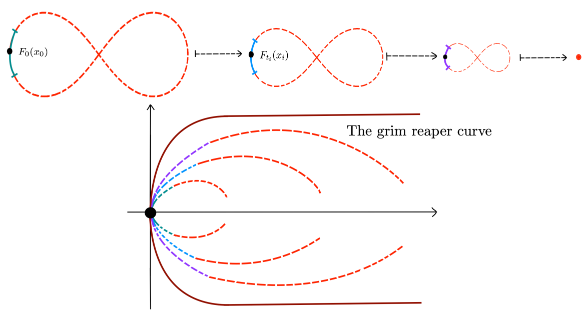
μία ισοαναλλοίωτη Λαγκρανζιανή εμβάπτιση που ικανοποιεί τη σχέση

$$\mathrm{Ric} \geq c \, \|F\|^2 g,$$

για κάποια σταθερά $c > 0$. Τότε υπάρχει μια ομοιόμορφη σταθερά $\varepsilon > 0$ ώστε

$$\mathrm{Ric} \geq \varepsilon \|F\|^2 g$$

όσο η ροή υπάρχει. Επιπλέον, η ροή αναπτύσσει κυλινδρική ιδιομορφία Τύπου II.



Η διατριβή αποτελείται από τρία κύρια μέρη. Το πρώτο μέρος αναπτύσσει το απαραίτητο διαφορο-γεωμετρικό πλαίσιο για τα ισοαναλλοίωτα Λαγκρανζιανά υποπολυπτύγματα στον Ευκλείδειο χώρο. Στη συνέχεια, αποδεικνύεται ένας χαρακτηρισμός της σφαίρας του Whitney, που οφείλεται στους Ros-Urbano και Borrelli-Chen-Morvan.

Στο δεύτερο μέρος εισάγεται η ροή μέσης καμπυλότητας και παρουσιάζονται τα βασικά εργαλεία, όπως η παραβολική αρχή μεγίστου και οι εξισώσεις εξέλιξης βασικών γεωμετρικών μεγεθών. Το πλαίσιο αυτό χρησιμοποιείται στη συνέχεια για την ανάλυση του σχηματισμού ιδιομορφιών, συμπεριλαμβανομένου και του χαρακτηρισμού του μέγιστοτικού χρόνου ύπαρξης της ροής και των τεχνικών μοντελοποίησης των ιδιομορφιών.

Το τελικό μέρος της μεταπτυχιακής διατριβής επικεντρώνεται στην ταξινόμηση ιδιομορφιών που δημιουργούνται εξελίσσοντας ισοαναλλοίωτα Λαγκρανζιανά υποπολυπτύγματα μέσω της ροής μέσης καμπυλότητας, καταλήγοντας στις αποδείξεις των Θεωρημάτων B, C και D.

Contents

CHAPTER 1

Riemannian Geometry

In this section we set up the notation and quickly review some basic facts from Riemannian geometry. We closely follow the exposition in [3, 10–14, 27, 30, 32, 38, 50].

1.1 Preliminaries

In this chapter, we will deal with a very important area of differential geometry, namely that of Riemannian submanifolds and we shall briefly review some basic facts about convergence of Riemannian manifolds.

1.1.1 Tensors and curvature

Throughout this thesis, M will denote a smooth manifold of dimension m. We will denote by $C^\infty(M)$ the space of smooth functions and by $\mathfrak{X}(M)$ the space of smooth vector fields on M. Riemannian metrics on M will be denoted by g_M, and their Levi-Civita connections by ∇^M. Quite often, we will simply write $g_M = \langle \cdot\,,\cdot\rangle$ and ∇.

Suppose from now on that M is equipped with a Riemannian metric. Then the ***Riemann curvature operator*** is given by

$$R(X,Y)Z \doteq \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Using the metric we can form the ***Riemann curvature tensor***

$$R(X,Y,Z,W) \doteq -\langle R(X,Y)Z, W\rangle.$$

The Riemann curvature tensor possesses several symmetries. One of the most important is the so-called ***first Bianchi identity***

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0. \tag{1.1}$$

There are several important geometric quantities associated with the Riemann curvature tensor. In particular:

1. For any pair of linearly independent vector fields X, Y, the quantity

$$K(X \wedge Y) \doteq \frac{R(X, Y, X, Y)}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}$$

   is called the ***sectional curvature*** of the plane spanned by X and Y.

2. The symmetric bilinear form

$$\mathrm{Ric}(X, Y) \doteq \operatorname{tr} R(X, \cdot, Y, \cdot)$$

   is called the ***Ricci curvature***.

3. The function

$$\mathrm{Sc} \doteq \operatorname{tr} \mathrm{Ric}$$

   is called the ***scalar curvature***.

1.1.2 Mappings, Hessian and Laplacian

Let (M, g_M, ∇^M) and (N, g_N, ∇^N) be Riemannian manifolds, and $f\colon M \to N$ be a smooth map. A ***vector field along*** f is a map

$$Z\colon M \to TN$$

such that

$$\pi_N \circ Z = f,$$

where $\pi_N : TN \to N$ is the natural projection map. Equivalently, a vector field Z along f assigns to each point $x \in M$ a vector

$$Z(x) \in T_{f(x)}N.$$

Fix a point $x_0 \in M$ and an orthonormal frame $\{E_1, \ldots, E_n\}$ defined on an open neighborhood $V \subset N$ of $f(x_0)$. Then, for any $x \in U = f^{-1}(V)$, we may write

$$Z(x) = \sum_{\alpha=1}^{n} z_\alpha(x)\, E_\alpha(f(x)),$$

where $z_1, \ldots, z_n$ are real-valued functions, called the ***components*** of Z with respect to the given frame. The vector field Z is called ***smooth*** if its components are smooth functions.

Denote by $\mathfrak{X}(f)$ the space of smooth vector fields along the map f. We would like to differentiate vector fields along f, namely, to define a natural connection ∇^f on the ***pullback bundle***

$$f^*TN \doteq \{(x, v) \in M \times TN : v \in T_{f(x)}N\}.$$

If a smooth vector field Z along f can be locally extended to a smooth vector field on a neighborhood of $f(M)$ in N, we may define

$$\nabla^f_X Z = \nabla^N_{df(X)} Z.$$

If f is an immersion, such an extension always exists. However, in general, such an extension is not always available. For example, let us consider the curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(x) = (x^2, x^3)$. Then

$$Z(x) = f'(x) = (2x, 3x^2)$$

cannot be extended to a smooth vector field in a neighborhood of $f(0) = (0, 0)$.

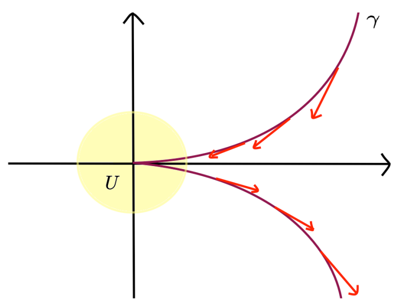


Figure 1.1: Neil's parabola.

To overcome this difficulty, we define the induced connection locally by

$$\nabla^f_X Z\big|_x = \sum_{\alpha=1}^{n} \left(X_x(z_\alpha)\, E_\alpha(f(x)) + z_\alpha(x)\, \nabla^N_{df_x(X)} E_\alpha\right). \tag{1.2}$$

The induced connection satisfies the properties of a linear connection and moreover

$$X g_N(Z_1, Z_2) = g_N(\nabla^f_X Z_1, Z_2) + g_N(Z_1, \nabla^f_X Z_2),$$

and

$$\nabla^f_{df(X)} df(Y) - \nabla^f_{df(Y)} df(X) = df([X,Y]),$$

for all $X, Y \in \mathfrak{X}(M)$ and $Z_1, Z_2 \in \mathfrak{X}(f)$.

With may define now the ***Hessian*** and the ***Laplacian*** of f as

$$\nabla^2 f(X,Y) \doteq \nabla^f_X df(Y) - df(\nabla^M_X Y), \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

and

$$\Delta f \doteq \mathrm{tr}_{g_M}(\nabla^2 f),$$

respectively. It turns out that the Hessian is symmetric. In the case where

$$\Delta f = 0,$$

we say that f is a ***harmonic map***.

1.1.3 Riemannian submanifolds

Suppose that M and N are two differentiable manifolds and $f : M \to N$ is an ***immersion***, i.e., a smooth map whose differential

$$df_x : T_x M \to T_{f(x)} N$$

is injective for every $x \in M$. Assume additionally that N is equipped with a Riemannian metric g_N. Then f induces a Riemannian metric $f^* g_N$ on M given by

$$(f^* g_N)(X,Y) \doteq g_N(df(X), df(Y)), \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

If M is already equipped with a Riemannian metric g_M, then map f is called an ***isometric immersion*** if

$$g_M = f^* g_N.$$

Assume in the sequel that $f : (M, g_M) \to (N, g_N)$ is an isometric immersion. To simplify the notation, we often use the convention

$$g_M = \langle \cdot\,, \cdot \rangle = g_N.$$

For each $x \in M$, the tangent space $T_{f(x)}N$ decomposes uniquely as

$$T_{f(x)}N = df_x(T_xM) \oplus N_{f(x)}M, \tag{1.3}$$

where $N_{f(x)}M$ denotes the orthogonal complement of $df_x(T_xM)$ with respect to the Riemannian metric on N. The vector space $N_{f(x)}M$ is called the ***normal space*** of f at x, and the set

$$N_fM = \cup_{x\in M} N_{f(x)}M$$

is called the ***normal bundle*** of f.

Definition 1.1.1. *A vector field $\xi \in \mathfrak{X}(f)$ along the map f is called* **normal** *if*

$$\xi(x) \in N_{f(x)}M,$$

for all $x \in M$.

Due to (1.3), each $V \in \mathfrak{X}(f)$ can be uniquely written as

$$V = V^\top + V^\perp,$$

where $V^\top \in df(TM)$ denotes the ***tangential*** and $V^\perp \in N_fM$ the ***normal component*** of V. Let ∇^M be the Levi-Civita connection of g_M and ∇^N is the Levi-Civita connection of g_N. Then, for any $X, Y \in \mathfrak{X}(M)$, we have

$$\nabla^N_X df(Y) = (\nabla^N_X df(Y))^\top + (\nabla^N_X df(Y))^\perp.$$

Definition 1.1.2. *The $C^\infty(M)$-bilinear symmetric tensor*

$$A \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to N_fM$$

defined by

$$A(X,Y) \doteq (\nabla^f_X df(Y))^\perp = \nabla^f_X df(Y) - df(\nabla^M_X Y).$$

is called the **second fundamental form** *of the immersion f. The trace H of A is called the mean curvature vector of f. The isometric immersion f is called* **totally geodesic** *if $A = 0$ and* **minimal** *if $H = 0$.*

Let ξ be a normal vector along f. The $C^\infty(M)$-bilinear symmetric form

$$a_\xi : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(\mathbb{R})$$

defined by

$$a_\xi(X, Y) \doteq \langle A(X, Y), \xi \rangle, \quad \text{for all } X, Y \in \mathfrak{X}(M)$$

is called the ***second fundamental form*** of f in the direction ξ. Since for a given $\xi \in N_f M$ the bilinear form a_ξ is symmetric, there exists a unique self-adjoint linear transformation

$$A_\xi : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that

$$a_\xi(X, Y) \doteq \langle A_\xi X, Y \rangle, \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

The tensor A_ξ is called the ***Weingarten operator*** of f in the direction ξ. The eigenvalues $\lambda_i(\xi)$, $i \in \{1, \ldots, n\}$, arranged in increasing order, of A_ξ are called the ***principal curvatures*** of f with respect to the direction ξ. The corresponding eigenvectors are called the ***principal directions*** of f with respect to the direction ξ. The principal curvatures are continuous and almost everywhere smooth according to the following general lemma.

Lemma 1.1.3 ([41, 46])**.** *Let S be an (arbitrary) symmetric $(1,1)$-tensor field on a m-dimensional Riemannian manifold. Then there exist m continuous functions*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$$

such that for every point $x \in M^m$ the numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ are the eigenvalues of the tensor S at the point x. In particular, the eigenvalues are almost everywhere smooth.

For a given $X \in \mathfrak{X}(M)$ the covariant derivative of A_ξ in the direction X is given by

$$(\nabla_X A_\xi)Y = \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y).$$

It turns out that the linear transformation $\nabla_X A_\xi$ is self-adjoint, i.e.,

$$\langle (\nabla_X A_\xi)Y, Z \rangle = \langle Y, (\nabla_X A_\xi)Z \rangle,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

One can easily verify that the Weingarten operator A_ξ of f satisfies the so-called ***Weingarten formula***

$$\nabla^f_X \xi = -df(A_\xi X) + (\nabla^f_X \xi)^\perp, \quad \text{for all } X \in \mathfrak{X}(M). \tag{1.4}$$

In the case of hypersurfaces the last term vanishes and the formula becomes

$$\nabla^f_X \xi = -df(A_\xi X), \quad \text{for all } X \in \mathfrak{X}(M). \tag{1.5}$$

In the following chapters we will need to consider compositions of immersions. Namely, let $f\colon M \to N$ and $g\colon N \to P$ be isometric immersions between Riemannian manifolds and let $F = g \circ f : M \to P$ the composition. Then by a direct computation we obtain

$$\begin{aligned} A_F(X,Y) &= \nabla^F_X dF(Y) - dF(\nabla^M_X Y) \\ &= \nabla^{g\circ f}_X dg(df(Y)) - dg(df(\nabla^M_X Y)) \\ &= \nabla^P_{dg(df(X))} dg(df(Y)) - dg(df(\nabla^M_X Y)), \end{aligned}$$

from where we deduce that

$$A_F(X,Y) = A_g(df(X), df(Y)) + dg(A_f(X,Y)), \tag{1.6}$$

for all $X, Y \in \mathfrak{X}(M)$, where A_F is the second fundamental form of F, A_g the second fundamental form of g and A_f the second fundamental form of f.

Let us conclude this subsection with an example.

Example 1.1.4 (Spheres)**.** Let $\mathbb{S}^n(R)$ be the sphere of radius R centered at the origin in $\mathbb{R}^{n+1}$. Consider the inclusion $f : \mathbb{S}^n(R) \to \mathbb{R}^{n+1}$ given by $f(x) = x$. We choose as unit normal the vector field

$$\xi(x) = \frac{x}{R}.$$

Using the Weingarten formula (1.5), we obtain

$$A_\xi(X) = -\frac{1}{R}X.$$

Therefore, the second fundamental form of the sphere is given by

$$A(X,Y) = -\frac{\langle X, Y\rangle}{R}\xi = -\frac{\langle X, Y\rangle}{R}\frac{x}{R}, \tag{1.7}$$

for all $X, Y \in \mathfrak{X}(\mathbb{S}^n)$. Moreover,

$$H = -\frac{n}{R}\xi.$$

The normal bundle of the submanifold admits a natural connection which we denote by $\nabla^{\perp}$, i.e.,

$$\nabla_X \xi \doteq (\nabla^f_X \xi)^{\perp}, \quad \text{for all } X \in \mathfrak{X}(M) \ \& \ \xi \in N_f M.$$

The $C^{\infty}(M)$-trilinear tensor

$$R^{\perp} \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times N_f M \to N_f M$$

defined by

$$R^{\perp}(X,Y)\xi \doteq \nabla^{\perp}_X \nabla^{\perp}_Y \xi - \nabla^{\perp}_Y \nabla^{\perp}_X \xi - \nabla^{\perp}_{[X,Y]}\xi,$$

is called the ***normal curvature tensor***.

Let us deal now with the following question. Consider an isometric immersion $f : M \to N$, and let R^M, R^N denote the curvature tensors of the metrics g_M and g_N, respectively. *Is it possible to express the second fundamental form A in terms of the curvature tensors R^M, R^N and $R^{\perp}$?* Recall that there are two ways to differentiate the second fundamental form. The first is by viewing A as a map with values in the induced bundle f^*TN, and the second by viewing A as taking values in the normal bundle. More precisely,

$$(\nabla^f_X A)(Y,Z) \doteq \nabla^f_X (A(Y,Z)) - A(\nabla^M_X Y, Z) - A(Y, \nabla^M_X Z),$$

and

$$(\nabla^{\perp}_X A)(Y,Z) \doteq \nabla^{\perp}_X (A(Y,Z)) - A(\nabla^M_X Y, Z) - A(Y, \nabla^M_X Z),$$

for $X, Y, Z \in \mathfrak{X}(M)$. It does not make much difference whether we use ∇^f or $\nabla^{\perp}$; however, we prefer $\nabla^{\perp}$, since A takes values in the normal bundle.

Lemma 1.1.5. *The identities*

$$(\nabla^f_Z A)(X,Y) = (\nabla^{\perp}_Z A)(X,Y) - \sum_{l=1}^{m} \langle A(X,Y), A(Z,e_l)\rangle df(e_l),$$

and

$$\nabla^f_X H = \nabla^{\perp}_X H - \sum_{l=1}^{m} \langle H, A(X,e_l)\rangle df(e_l),$$

hold for $X, Y, Z \in \mathfrak{X}(M)$, where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame.

Proof. Fix a point $x_0 \in M$ and consider a local orthonormal frame $\{e_1, \dots, e_m\}$ defined in an open neighborhood of x_0 and being normal at x_0. It suffices to compute the tangent part of $\nabla^f A$. Differentiating and evaluating at x_0 we find

$$\begin{aligned}(\nabla^{\top}_{e_k} A)(e_i, e_j) &= \sum_{i=1}^{m} \langle (\nabla^f_{e_k} A)(e_i, e_j), df(e_l)\rangle df(e_l) \\ &= \sum_{l=1}^{m} \langle \nabla^f_{e_k} A(e_i, e_j), df(e_l)\rangle df(e_l) \\ &= -\sum_{l=1}^{m} \langle A(e_i, e_j), \nabla^f_{e_k} df(e_l)\rangle df(e_l) \\ &= -\sum_{l=1}^{m} \langle A(e_i, e_j), A(e_k, e_l)\rangle df(e_l).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(\nabla^{\top}_{e_k} H) &= \sum_{i,l=1}^{m} \langle (\nabla^f_{e_k} A(e_i, e_i), df(e_l)\rangle df(e_l) \\ &= -\sum_{i,l=1}^{m} \langle A(e_i, e_i), \nabla^f_{e_k} df(e_l)\rangle df(e_l) \\ &= -\sum_{i,l=1}^{m} \langle A(e_i, e_i), A(e_k, e_l)\rangle df(e_l) \\ &= -\sum_{l=1}^{m} \langle H, A(e_k, e_l)\rangle df(e_l).\end{aligned}$$

This completes the proof. $\square$

The tensors R^M, R^N and $R^{\perp}$ are related with A through the ***Gauss-Codazzi-Ricci equations***. Namely,

Gauss equation: For every $X, Y, Z, W \in \mathfrak{X}(M)$, it holds

$$\begin{aligned}\langle R^M(X,Y)Z, W\rangle &= \langle R^N(df(X), df(Y))df(Z), df(W)\rangle \\ &\quad -\langle A(X,Z), A(Y,W)\rangle + \langle A(Y,Z), A(X,W)\rangle.\end{aligned} \tag{1.8}$$

Codazzi equation: For $X, Y, Z, W \in \mathfrak{X}(M)$, we have

$$\{R^N(df(X), df(Y))df(Z)\}^\perp = (\nabla^\perp_X A)(Y, Z) - (\nabla^\perp_Y A)(X, Z). \tag{1.9}$$

Ricci equation: For $X, Y \in \mathfrak{X}(M)$ and $\xi \in N_f M$, it holds

$$\{R^N(df(X), df(Y))\xi\}^\perp = R^\perp(X, Y)\xi - A(X, A_\xi Y) + A(Y, A_\xi X). \tag{1.10}$$

Remark 1.1.1. From the Gauss equation we immediately obtain the following:

1. The sectional curvatures K^M of M and K^N of N are related by

   $$K^N(df(X) \wedge df(Y)) = K^M(X \wedge Y) + \|A(X, Y)\|^2 - \langle A(X, X), A(Y, Y)\rangle,$$

   for any orthonormal pair $\{X, Y\} \subset TM$.

2. Suppose that M is a hypersurface of N. Let ξ be a unit normal vector field along f, A_ξ the associated shape operator, and $\{e_1, \dots, e_m\}$ a local orthonormal frame of principal directions, that is,

   $$A_\xi e_i = \lambda_i e_i, \quad \text{for all } i \in \{1, \dots, m\},$$

   where λ_i are the principal curvatures. Then

   $$K^N(df(e_i) \wedge df(e_j)) = K^M(e_i \wedge e_j) - \lambda_i \lambda_j,$$

   for all $i \neq j \in \{1, \dots, m\}$.

1.1.4 Local expressions

Let $f\colon M \to N$ be a smooth map between Riemannian manifolds. Choose a chart (U, φ) around a point $x \in M$ and a chart (V, ψ) around $f(x) \in N$. Assume that $\varphi : U \to \mathbb{R}^m$ is represented as $\varphi = (x^1, \dots, x^k)$ and suppose that $\psi : V \to \mathbb{R}^m$ is represented as $\psi = (y^1, \dots, y^m)$. Unless otherwise stated, we will use Latin indices to describe quantities on M and Greek indices for quantities on N. From the charts φ and ψ, we obtain for f the local expression

$$\psi \circ f \circ \varphi^{-1} = (f^1, \dots, f^m),$$

where

$$f^\alpha = y^\alpha \circ f \circ \varphi^{-1}.$$

Denote the basic vector fields associated with (U, φ) and (V, ψ) by $\{\partial_i\}$ and $\{\partial_\alpha\}$ and their corresponding dual 1-forms by $\{dx^i\}$ and $\{dy^\alpha\}$, respectively. With respect to these conventions, the metrics g_M and g_N can be written in the form

$$g_M = g_{ij} dx^i \otimes dx^j \quad \text{and} \quad g_N = g_{\alpha\beta} dy^\alpha \otimes dy^\beta,$$

where we have adopted Einstein's summation convention. The differential of the map f and the pull-back via f of the metric g are given by

$$df = f_i^\alpha \partial_\alpha \otimes dx^i \quad \text{and} \quad f^* g_N = g_{\alpha\beta} f_i^\alpha f_j^\beta dx^i \otimes dx^j,$$

where

$$f_i^\alpha \doteq (y^\alpha \circ f \circ \varphi^{-1})_{x^i}.$$

By a straightforward computation, we see that the Hessian and the Laplacian of f can be represented in the form

$$(\nabla^2 f)_{ij} \doteq A_{ij}^\alpha \partial_\alpha = \big(f_{ij}^\alpha - \Gamma_{ij}^s f_s^\alpha + \Gamma_{\beta\gamma}^\alpha f_i^\beta f_j^\gamma\big)\partial_\alpha$$

and

$$\Delta f = g^{ij} (\nabla^2 f)_{ij} = g^{ij} \big(f_{ij}^\alpha - \Gamma_{ij}^s f_s^\alpha + \Gamma_{\beta\gamma}^\alpha f_i^\beta f_j^\gamma\big)\partial_\alpha,$$

where

$$f_{ij}^\alpha = (y^\alpha \circ f \circ \varphi^{-1})_{x^i x^j}$$

are the second derivatives of f^α, Γ_{ij}^s are the Christoffel symbols of the Levi-Civita connection on M and $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of the Levi-Civita connection on N.

Let us restrict in the case where f is isometric immersion. Occasionally we will use local orthonormal frames instead of local coordinates. Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame on M and let $\{\xi_{m+1}, \dots, \xi_{m+n}\}$ be a local orthonormal frame of the normal bundle $N_f M$. We use Latin indices to denote components of tensors on the tangent bundle TM and Greek indices for components on the normal bundle of N. Then the set

$$\{e_1, \dots, e_m; \xi_{m+1}, \dots, \xi_m\}$$

is called ***adapted orthonormal frame*** along the (immersed) submanifold $f(M)$. The 1-forms ω_{ij} and $\omega_{\alpha\beta}$ given by

$$\omega_{ij}(X) \doteq g_M(\nabla_X^M e_i, e_j) \quad \text{and} \quad \omega_{\alpha\beta}(X) \doteq g_N(\nabla_X^\perp \xi_\alpha, \xi_\beta)$$

are called ***connections forms of the tangent*** and ***normal bundle***, respectively.

Following the aforementioned notation we have:

$$A^\alpha_{ij} = \langle A(e_i, e_j), \xi_\alpha \rangle = \langle A_{ij}, \xi_\alpha \rangle \quad \text{and} \quad H^\alpha = \langle H, \xi_\alpha \rangle = A^\alpha_{ii}.$$

Moreover, we set

$$R^N_{ijkl} \doteq (f^* R^N)(e_i, e_j, e_k, e_l) \quad \text{and} \quad R^N_{ij\alpha\beta} = R^N(df(e_i), df(e_j), \xi_\alpha, \xi_\beta).$$

Now the Gauss-Codazzi-Ricci fundamental equations of the submanifold $f(M)$ can be equivalently written in the form:

(a) **Gauss equation:**

$$R^M_{ijkl} = R^N_{ijkl} + \sum_{\alpha,k} \big(A^\alpha_{ik} A^\alpha_{jl} - A^\alpha_{jk} A^\alpha_{il} \big). \tag{1.11}$$

(b) **Codazzi equation:**

$$\big(\nabla^\perp_{e_i} A\big)^\alpha_{jk} - \big(\nabla^\perp_{e_j} A\big)^\alpha_{ik} = -R^N_{ijk\alpha}. \tag{1.12}$$

(c) **Ricci equation:**

$$R^\perp_{ij\alpha\beta} = R^N_{ij\alpha\beta} + \sum_{k} \big(A^\alpha_{ik} A^\beta_{jk} - A^\beta_{ik} A^\alpha_{jk} \big). \tag{1.13}$$

Since we will constantly deal with tensorial quantities, let us recall some basic linear algebraic definitions concerning tensors and norms of tensors induced by inner products. Let (V, g_V) and (W, g_W) be finite-dimensional inner product spaces of dimensions m and n, respectively, and let

$$T : \underbrace{V \times \cdots \times V}_{r\text{-times}} \longrightarrow \underbrace{W \times \cdots \times W}_{s\text{-times}}$$

be an (r, s)-tensor, i.e., a multilinear map. Let $\{v_1, \dots, v_m\}$ be an arbitrary basis of V and $\{w_1, \dots, w_n\}$ an arbitrary basis of W. Following the above mentioned conventions, we use Latin indices for quantities defined on V and Greek indices for quantities on W, and write

$$g_{ij} = g_V(v_i, v_j) \quad \text{and} \quad g_{\alpha\beta} = g_W(w_\alpha, w_\beta).$$

Moreover, denote by g^{ij} and $g^{\alpha\beta}$ the components of the inverse matrices. The ***components*** of T with respect to the chosen bases are defined by

$$T(v_{i_1}, \dots, v_{i_r}) \doteq \sum_{\alpha_1,\dots,\alpha_s} T_{i_1\cdots i_r}^{\alpha_1\cdots\alpha_s}\, w_{\alpha_1} \otimes \cdots \otimes w_{\alpha_s}.$$

The ***norm*** of T induced by the inner products g_V and g_W is then defined by

$$\|T\|^2_{(g_V,g_W)} \doteq \sum_{i_1,j_1,\dots,i_r,j_r} \sum_{\alpha_1,\beta_1,\dots,\alpha_s,\beta_s} g^{i_1j_1}\cdots g^{i_rj_r} g_{\alpha_1\beta_1}\cdots g_{\alpha_s\beta_s}\, T_{i_1\cdots i_r}^{\alpha_1\cdots\alpha_s}\, T_{j_1\cdots j_r}^{\beta_1\cdots\beta_s}.$$

If the chosen bases $\{v_1, \dots, v_m\}$ of V and $\{w_1, \dots, w_n\}$ of W are orthonormal with respect to g_V and g_W, respectively, then

$$g_{ij} = \delta_{ij}, \quad g^{ij} = \delta^{ij}, \quad g_{\alpha\beta} = \delta_{\alpha\beta}, \quad \text{and} \quad g^{\alpha\beta} = \delta^{\alpha\beta},$$

where δ denotes the Kronecker delta. Consequently, the expression for the norm simplifies to

$$\|T\|^2_{(g_V,g_W)} = \sum_{i_1,\dots,i_r} \sum_{\alpha_1,\dots,\alpha_s} \left(T_{i_1\cdots i_r}^{\alpha_1\cdots\alpha_s}\right)^2.$$

Thus, with respect to orthonormal bases, the norm of a (r, s)-tensor is simply the Euclidean norm of its components. When the underlying Riemannian metrics are fixed, we simply denote the norm of a tensor T by $\|T\|$.

The following lemma compares the tensor norms induced by two equivalent inner products.

Lemma 1.1.6. *Let g_1 and g_2 be inner products on the same finite-dimensional vector space V, and let (W, g) be another finite-dimensional inner product space. Assume that there exist constants $c_1, c_2 > 0$ such that*

$$c_1 g_1 \le g_2 \le c_2 g_1.$$

Let

$$T : \underbrace{V \times \cdots \times V}_{r\text{-times}} \longrightarrow \underbrace{W \times \cdots \times W}_{s\text{-times}}$$

be an (r, s)-tensor. Then

$$\|T\|^2_{(g_2,g)} \le c_1^{-r}\, \|T\|^2_{(g_1,g)} \quad \textit{and} \quad \|T\|^2_{(g_1,g)} \le c_2^{r}\, \|T\|^2_{(g_2,g)}.$$

Proof. Choose a g_1-orthonormal basis $\{e_1, \dots, e_m\}$ of the vector space V. In this basis, the matrix of g_1 is the identity, while the matrix of g_2 is a symmetric positive definite matrix. Suppose now that $\{e_1, \dots, e_m\}$ is a g_1-orthonormal basis which diagonalizes g_2. The assumption

$$c_1 g_1 \leq g_2 \leq c_2 g_1$$

implies that the eigenvalues of g_2 lie between c_1 and c_2. Consequently, with respect to this eigenbasis, we have

$$c_1 \delta_{ij} \leq (g_2)_{ij} = \lambda_i \delta_{ij} \leq c_2 \delta_{ij},$$

and

$$c_2^{-1} \delta^{ij} \leq (g_2)^{ij} = \lambda_i^{-1} \delta^{ij} \leq c_1^{-1} \delta^{ij},$$

respectively. Suppose further that $\{w_1, \dots, w_n\}$ is a g-orthonormal basis of W. Then

$$T(e_{i_1}, \dots, e_{i_r}) = \sum_{\alpha_1, \dots, \alpha_s} T^{\alpha_1 \cdots \alpha_s}_{i_1 \cdots i_r} \, w_{\alpha_1} \otimes \cdots \otimes w_{\alpha_s}.$$

and

$$\begin{aligned}
\|T\|^2_{(g_2,g)} &= \sum_{i_1, j_1, \dots, i_r, j_r} \sum_{\alpha_1, \dots, \alpha_s} (g_2)^{i_1 j_1} \cdots (g_2)^{i_r j_r} \, T^{\alpha_1 \cdots \alpha_s}_{i_1 \cdots i_r} T^{\alpha_1 \cdots \alpha_s}_{j_1 \cdots j_r} \\
&= \sum_{i_1, \dots, i_r} \sum_{\alpha_1, \dots, \alpha_s} \lambda_{i_1}^{-1} \cdots \lambda_{i_r}^{-1} \left(T^{\alpha_1 \cdots \alpha_s}_{i_1 \cdots i_r} \right)^2 \\
&\leq c_1^{-r} \sum_{i_1, \dots, i_r} \sum_{\alpha_1, \dots, \alpha_s} \left(T^{\alpha_1 \cdots \alpha_s}_{i_1 \cdots i_r} \right)^2 \\
&= c_1^{-r} \|T\|^2_{(g_1,g)}.
\end{aligned}$$

Therefore,

$$\|T\|^2_{(g_2,g)} \leq c_1^{-r} \|T\|^2_{(g_1,g)}.$$

Similarly, we prove that

$$\|T\|^2_{(g_1,g)} \leq c_2^{r} \, \|T\|^2_{(g_2,g)}.$$

This proves the claim. □

Let us illustrate with an important example.

Example 1.1.7 (Graphs). Let $u\colon U \subseteq \mathbb{R}^m \to \mathbb{R}$ be a smooth function and let

$$\Gamma \doteq \{(x, u(x)) \in \mathbb{R}^{m+1} : x \in U\}.$$

be its ***graph.*** The hypersurface $\Gamma \subset \mathbb{R}^{m+1}$ is the image of the embedding $F : U \to \mathbb{R}^{m+1}$ given by $F(x) = (x, u(x))$, for any $x \in U$. One can readily check that the components of the induced metric g on the graph are

$$g_{ij} = \delta_{ij} + u_i u_j, \quad \text{for } i, j \in \{1, \dots, m\},$$

where according the notation introduced in this section $u_i = u_{x^i}$. Following [37, page 1101], let us denote the gradient Du of u by the $(1 \times m)$-matrix

$$p \doteq (u_1, \dots, u_m).$$

Then, $\mathrm{g} = (g_{ij})$ may be written as

$$\mathrm{g} = I + p^T p, \tag{1.14}$$

where I is the identity matrix and q^T is the transpose of p. Note that

$$pp^T = \|p\|^2 = \|Du\|^2.$$

Now observe that

$$\mathrm{g}(p^T) = (I + p^T p)p^T = p^T + \|p\|^2 p^T = (1 + \|p\|^2)p^T.$$

Moreover, if v is a vector perpendicular to p, we have that

$$\mathrm{g}(v^T) = (I + p^T p)v^T = v^T + \langle p, v\rangle p^T = v^T.$$

Consequently, g has eigenvalues 1 with multiplicity $m - 1$ and $1 + \|p\|^2$ with multiplicity 1. This means that (g_{ij}) is invertible and

$$\det(g_{ij}) = 1 + \|p\|^2 = 1 + \|Du\|^2.$$

Let c be a constant to be determined. Then,

$$(I - cp^T p)(I + p^T p) = I + (1 - c - c\|p\|^2)p^T p.$$

We make the right side reduce to the identity by choosing

$$c \doteq \frac{1}{1 + \|p\|^2}.$$

Thus the components g^{ij} are given by

$$g^{ij} = \delta_{ij} - \frac{u_i u_j}{1 + \sum_{j=1}^m u_j^2} = \delta_{ij} - \frac{u_i u_j}{\det(g_{ij})}, \tag{1.15}$$

for $i, j \in \{1, \ldots, m\}$. Now the unit normal ν is given by

$$\nu = \frac{(-u_1, \ldots, -u_m, 1)}{\sqrt{1 + |Du|^2}} = \frac{(-u_1, \ldots, -u_m, 1)}{\sqrt{\det(g_{ij})}}. \tag{1.16}$$

The components of the second fundamental form of the graph are given by

$$A_{ij}^\nu = \langle F_{ij}, \nu \rangle = \frac{u_{ij}}{\sqrt{1 + |Du|^2}},$$

for any $i, j \in \{1, \ldots, m\}$. Therefore, the (scalar) mean curvature of the graph (in the direction ν) is given by

$$\begin{aligned} h = \langle H, \nu \rangle &= \sum_{i,j} g^{ij} A_{ij}^\nu \\ &= \frac{1}{\sqrt{1 + \|Du\|^2}} \sum_{i,j} u_{ij} \Big(\delta_{ij} - \frac{u_i u_j}{1 + \sum_{j=1}^m u_j^2} \Big). \end{aligned}$$

Therefore

$$h = \operatorname{div} \left(\frac{Du}{\sqrt{1 + \|Du\|^2}} \right). \tag{1.17}$$

We conclude with a result that we will use in the next chapters. As a matter of fact, we present a lemma which expresses the Laplacian of the squares norm $\|B\|^2$ of a tensor B with values in the normal bundle of a submanifolds in terms of the normal Laplacian and the covariant derivative of B with respect to the normal connection.

Lemma 1.1.8. *The following holds true*

$$\Delta \|B\|^2 = 2 \langle \Delta^\perp B, B \rangle + 2 \|\nabla^\perp B\|^2.$$

Proof. Since the formula is of tensorial nature, let us fix a point x_0 and a local orthonormal frame $\{e_1, \ldots, e_m\}$ in a neighborhood around x_0 such that

$$\nabla_{e_i} e_j \big|_{x_0} = 0, \quad \text{for all } i, j \in \{1, \ldots, m\}.$$

Differentiating and then evaluating at x_0, we find

$$\begin{aligned}\Delta\|B\|^2 &= \sum_i \sum_{i_1\dots i_r} e_i e_i \langle B(e_{i_1},\dots,e_{i_r}), B(e_{i_1},\dots,e_{i_r})\rangle \\ &= 2\sum_i \sum_{i_1\dots i_r} e_i \langle \nabla^\perp_{e_i} B(e_{i_1},\dots,e_{i_r}), B(e_{i_1},\dots,e_{i_r})\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\Delta\|B\|^2 \;=\; & 2\sum_i \sum_{i_1\dots i_r} e_i \langle (\nabla^\perp_{e_i} B)(e_{i_1},\dots,e_{i_r}), B(e_{i_1},\dots,e_{i_r})\rangle \\ & +2\sum_i \sum_{i_1\dots i_r} \sum_a e_i \langle B(e_{i_1},\dots,\nabla_{e_i} e_{i_a},\dots,e_{i_r}), B(e_{i_1},\dots,e_{i_r})\rangle \\ =\; & 2\sum_i \sum_{i_1\dots i_r} \langle (\nabla^\perp_{e_i}\nabla^\perp_{e_i} B)(e_{i_1},\dots,e_{i_r}), B(e_{i_1},\dots,e_{i_r})\rangle \\ & +2\sum_i \sum_{i_1\dots i_r} \langle (\nabla^\perp_{e_i} B)(e_{i_1},\dots,e_{i_r}), (\nabla^\perp_{e_i} B)(e_{i_1},\dots,e_{i_r})\rangle \\ & +2\sum_i \sum_{i_1\dots i_r} \sum_a \langle B(e_{i_1},\dots,\nabla_{e_i}\nabla_{e_i} e_{i_a},\dots,e_{i_r}), B(e_{i_1},\dots,e_{i_r})\rangle,\end{aligned}$$

where the sum with respect to a runs over the tensor slots, i.e., the index $a \in \{1,\dots,r\}$ indicates the position at which the connection term acts. Now note that at x_0 we have

$$\nabla_{e_i}\nabla_{e_i} e_{i_a} = \sum_j e_i(\omega_{i_a j}(e_i)) e_j.$$

Hence, the last term of the above equality becomes

$$\begin{aligned}\mathcal{B} &= 2\sum_i \sum_{i_1\dots i_r} \sum_a \langle B(e_{i_1},\dots,\nabla_{e_i}\nabla_{e_i} e_{i_a},\dots,e_{i_r}), B(e_{i_1},\dots,e_{i_r})\rangle \\ &= 2\sum_{i,j} \sum_{i_1\dots i_r} \sum_a \underbrace{e_i(\omega_{i_a j}(e_i))}_{\text{anti-symmetric}} \underbrace{\langle B_{i_1\dots i_{a-1} j i_{a+1}\dots i_r}, B_{i_1\dots i_r}\rangle}_{\text{symmetric}} \\ &= 0.\end{aligned}$$

Consequently,

$$\Delta\|B\|^2 = 2\langle \Delta^\perp B, B\rangle + 2\|\nabla^\perp B\|^2,$$

and this completes the proof. □

1.2 Convergence of Riemannian manifolds

In this chapter, we will introduce the concepts of limits of manifolds and submanifolds.

1.2.1 Ascoli-Arzelà theorem

Let us briefly recall in this subsection the Ascoli-Arzelà theorem for sequences of maps between Riemannian manifolds. Let M and N be smooth manifolds and let $k \in \mathbb{N} \cup \{0\}$. We define

$$C^k(M, N) \doteq \{f \colon M \to N : f \text{ is } k\text{-times continuously differentiable}\}.$$

Suppose now that M compact, let $k \in \mathbb{N} \cup \{0\}$ and $f \in C^k(M, N)$. Consider the sequence $\{f_i\}_{i \in \mathbb{N}} \subset C^k(M, N)$. We say that

$$f_i \mapsto f \quad \text{in } C^k(M, N)$$

if the following holds: For every coordinate chart (U, φ) on M and for every coordinate chart (V, ψ) on N with $f(U) \subset V$ and $f_i(U) \subset V$ for all sufficiently large i, the coordinate representations

$$\psi \circ f_i \circ \varphi^{-1},\ \psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \to \mathbb{R}^m$$

satisfy

$$\psi \circ f_i \circ \varphi^{-1} \mapsto \psi \circ f \circ \varphi^{-1} \quad \text{in } C^m(\varphi(U), \mathbb{R}^m),$$

i.e. all partial derivatives up to order k converge uniformly on compact subsets of $\varphi(U)$.

Theorem 1.2.1. *Let (M, g_M) and (N, g_N) be complete Riemannian manifolds, with M compact. Let $k \geq 0$, and let $\{f_i \colon M \to N\}_{i \in \mathbb{N}}$ be a sequence of C^{k+1} maps. Assume that:*

1. *There exists $x_0 \in M$ such that $\{f_i(x_0)\}_{i \in \mathbb{N}}$ has compact closure in N.*
2. *For each integer $j \in \{1, \ldots, k+1\}$, there exists a constant $c_j > 0$ such that*
$$\|\nabla^j f_i\| \leq c_j, \quad \textit{for all } i \in \mathbb{N}.$$

Then there exists a subsequence $\{f_{i_l} \colon M \to N\}_{l \in \mathbb{N}}$ converging in $C^k(M, N)$ to a map $f : M \to N$.

1.2.2 Cheeger-Gromov convergence

We begin with the notion of convergence of tensors on Riemannian manifolds.

Definition 1.2.2. *Let (M, g) be a Riemanian manifold and denote by ∇ the associated Levi-Civita connection. Suppose that $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{T}^{(r,s)}(M)$ is a sequence of (r, s)- tensors, with $s \in \{0, 1\}$. Let Ω be an open subset of M with compact closure $\overline{\Omega}$. Fix a natural number $k \in \mathbb{N}$. We say that the sequence $\{B_i\}_{i\in\mathbb{N}}$ converges in C^k-**topology** to $B_\infty \in \mathcal{T}^{(r,s)}(\overline{\Omega})$, if for every $\varepsilon > 0$, there exists $i_0 = i_0(\varepsilon) \in \mathbb{N}$, such that*

$$\max_{0\leq\alpha\leq k}\left\{\sup_{x\in\overline{\Omega}}\|\nabla^{(\alpha)}(B_i - B_\infty)\|\right\} < \varepsilon, \quad \textit{for all } i \geq i_0.$$

*We say that the sequence $\{B_i\}_{i\in\mathbb{N}}$ converges in C^∞-**topology** to $B_\infty \in \mathcal{T}^{(r,s)}(\overline{\Omega})$, if it converges in C^k to B_∞, for every $k \in \mathbb{N}$.*

Note that since we are working on compact sets, the choice of the metric g does not affect the notion of convergence. Moreover, we can use the above definition to describe the notion of convergence on compact sets.

Definition 1.2.3. *Let (M, g) be a Riemannian manifold and $\{U_j\}_{j\in\mathbb{N}}$ be an exhaustion of M. Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of (r, s)-tensors, $s \in \{0, 1\}$, where each B_i is defined on a corresponding open set A_i. We say that $\{B_i\}_{i\in\mathbb{N}}$ **converges smoothly on compact sets** to $B_\infty \in \mathcal{T}^{(r,s)}(M)$ if:*

1. *For each $j \in \mathbb{N}$, there exists i_0 such that $\overline{U_j} \subset A_{i_0}$, for all $i \geq i_0$.*
2. *The sequence $\{B_i|_{U_j}\}_{j\geq j_0}$ converges smoothly to $B_\infty|_{U_j}$.*

Similarly we may define the concept of ***smooth convergence*** of a sequence of smooth maps $\{F_i\colon (M, g) \to (N, h)\}_{i\in\mathbb{N}}$. As a matter of fact, we require C^0-convergence of $\{F_i\}_{i\in\mathbb{N}}$ to a smooth map $F_\infty\colon M \to N$ (using the metrics of M and N) and smooth convergence of $\{F_i^* h\}_{i\in\mathbb{N}}$ to $F_\infty^* h$ on compact sets.

In the sequel, we will recall the notion of smooth convergence of sequences of Riemannian manifolds following ideas of Cheeger and Gromov.

Definition 1.2.4. *A **pointed Riemannian manifold** (M, g, p) is a Riemannian manifold (M, g) with a choice of **origin** (or a **base point**) $p \in M$. If g is complete we say that (M, g, p) is a **complete pointed Riemannian manifold**.*

Definition 1.2.5 (Cheeger-Gromov convergence). *A sequence $\{(M_i, g_i, p_i)\}_{i\in\mathbb{N}}$ of complete pointed Riemannian manifolds* ***smoothly converges in the sense of Cheeger-Gromov*** *to a complete pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$, if there exists:*

1. *An exhaustion $\{U_i\}_{i\in\mathbb{N}}$ of M_∞ with $p_\infty \in U_i$, for all $i \in \mathbb{N}$.*

2. *A sequence $\{\phi_i \colon U_i \to \phi_i(U_i) \subset M_i\}_{i\in\mathbb{N}}$ of diffeomorphisms with $\phi_i(p_\infty) = p_i$ such that $\{\phi_i^* g_i\}_{i\in\mathbb{N}}$ smoothly converges to g_∞ on compact sets in M_∞.*

In this case we write,

$$(M_i, g_i, p_i) \xrightarrow{C^\infty} (M_\infty, g_\infty, p_\infty).$$

The family $\{(U_i, \phi_i)\}_{i\in\mathbb{N}}$ is called ***family of convergence pairs****.*

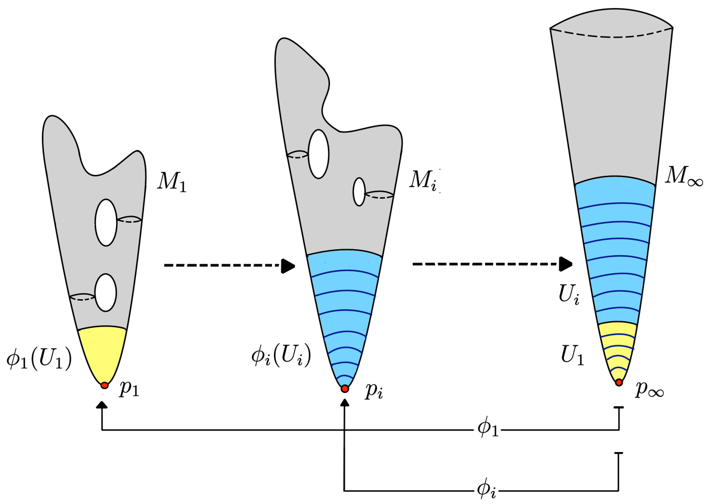


Figure 1.2: Convergence of pointed Riemannian manifolds.

Theorem 1.2.6. *The limit of a sequence $\{(M_i, g_i, p_i)\}_{i\in\mathbb{N}}$ of complete pointed Riemannian manifolds is unique in the following sense: If $(M_\infty, g_\infty, p_\infty)$ and $(N_\infty, h_\infty, q_\infty)$ are limits, then there exists a diffeomorphism $f\colon M_\infty \to N_\infty$ such that*

$$f^* h_\infty = g_\infty \quad \textit{and} \quad f(p_\infty) = q_\infty.$$

In the sequel, we will state an Ascoli-Arzelá type theorem for sequences of pointed Riemannian manifolds. To this end, we need to recall two more definitions from Riemannian geometry. Suppose that (M, g) is a complete Riemannian manifold. Recall that for $p \in M$ the ***injectivity radius at*** p is defined by

$$\text{inj}_g(p) = \sup\{r > 0 : \exp_p : B(0, r) \to M \text{ is diffeomorphism}\}.$$

Moreover, the ***injectivity radius of*** M is defined by

$$\text{inj}_p(M) = \inf_{p \in M} \text{inj}_g(p).$$

If M is compact, then $\text{inj}_p(M) > 0$, see [38]. However, in the complete case, this is not necessarily true; see the figure below.

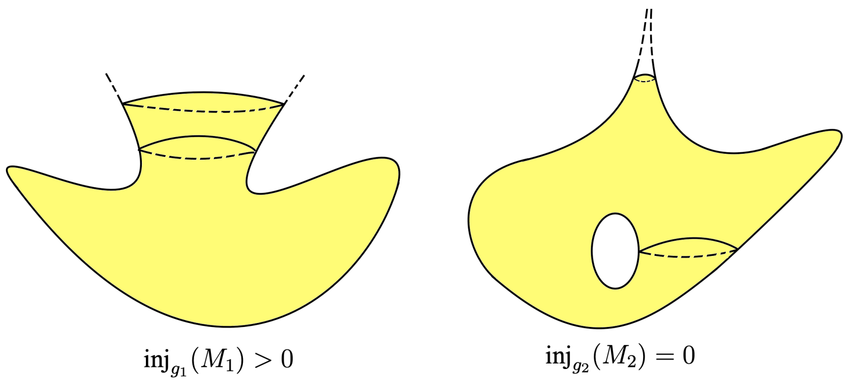


Figure 1.3: Injectivity radius of Riemannian manifolds.

Definition 1.2.7. *A complete Riemannian manifold has* ***bounded geometry*** *if the following conditions are satisfied:*

1. *For all* $k \in \mathbb{N}$, *there exists* $C_k > 0$ *such that* $\|\nabla^{(k)} R\| \leq C_k$.
2. $\text{inj}_g(M) > 0$.

For later use we will need the following refined version of the above definition.

Definition 1.2.8. *We say that the sequence of complete Riemannian manifolds* $\{(M_i, g_i, p_i)\}_{i \in \mathbb{N}}$ *has* ***uniformly bounded geometry*** *if the following conditions are satisfied:*

1. *For any $j \in \mathbb{N} \cup \{0\}$, there exists a uniform constant C_j such that, for each $i \in \mathbb{N}$, it holds*
$$\|\nabla^{(j)} R_{g_i}\| \leq C_j.$$
2. *There exists a uniform constant $r_0 > 0$ such that*
$$\mathrm{inj}_{g_i}(M_i) > 0,$$
for all $i \in \mathbb{N}$.

Theorem 1.2.9 (Cheeger-Gromov compactness)**.** *Suppose $\{(M_i, g_i, p_i)\}_{i\in\mathbb{N}}$ is a sequence of complete pointed Riemannian manifolds with uniformly bounded geometry. Then, the sequence $\{(M_i, g_i, p_i)\}_{i\in\mathbb{N}}$ subconverges smoothly to a complete pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$, i.e. there exists a subsequence $\{i_k\}_{k\in\mathbb{N}}$ such that*
$$(M_{i_k}, g_{i_k}, p_{i_k}) \xrightarrow{C^\infty} (M_\infty, g_\infty, p_\infty)$$
in the Cheeger-Gromov sense.

The above compactness theorem still holds under the weaker assumption that the injectivity radius is uniformly bounded from below by a positive constant only along the base points $\{p_i\}_{i\in\mathbb{N}}$, thereby avoiding the assumption of the uniform lower bound for $\mathrm{inj}_{g_i}(M_i)$; see [18].

The following theorem will be of crucial importance in the next subsection.

Theorem 1.2.10. *Suppose that (M, g) is a complete, Riemannian manifold of dimension m, with bounded geometry, $\{a_i\}_{i\in\mathbb{N}}$ be an increasing sequence of positive numbers tending to ∞ and $\{p_i\}_{i\in\mathbb{N}}$ be a sequence of points on M^m. Then, $\{(M, a_i^2 g, p_i)\}_{i\in\mathbb{N}}$ smoothly subconverges to the Euclidean space $(\mathbb{R}^m, g_{euc}, 0)$.*

1.2.3 Convergence of immersions

Let us begin our exposition with the definition of the limit of a sequence of isometric immersions.

Definition 1.2.11. *Let $\{(N_i, h_i, y_i)\}_{i\in\mathbb{N}}$ be a sequence of pointed Riemannian manifolds and let $\{F_i\colon (M_i, g_i) \to (N_i, h_i)\}_{i\in\mathbb{N}}$ be a sequence of isometric immersions with*
$$F_i(x_i) = y_i.$$

We say that $\{F_i\}_{i\in\mathbb{N}}$ ***converges smoothly*** *to an isometric immersion*

$$F_\infty\colon (M_\infty, g_\infty, x_\infty) \to (N_\infty, h_\infty, y_\infty)$$

if the following conditions are satisfied:

1. $(M_i, g_i, x_i) \xrightarrow{C^\infty} (M_\infty, g_\infty, x_\infty)$ *and* $(N_i, h_i, y_i) \xrightarrow{C^\infty} (N_\infty, h_\infty, y_\infty)$.
2. *If* $\{(U_i, \phi_i)\}_{i\in\mathbb{N}}$ *and* $\{(W_i, \psi_i)\}_{i\in\mathbb{N}}$ *are, respectively, convergence pairs of the corresponding sequences, then:*
   - $(F_i \circ \phi)(U_i) \subseteq \psi_i(W_i)$
   - $\{\psi_i^{-1} \circ F_i \circ \phi_i\colon U_i \subseteq M_\infty \to N_\infty\}_{i\in\mathbb{N}}$ *smoothly converges to* F_∞ *on compact sets.*

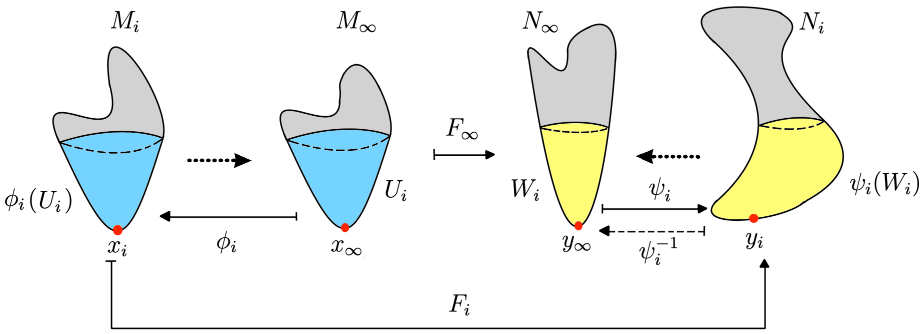


Figure 1.4: Convergence of isometric immersions.

We would like to have an Ascoli-Arzelà type theorem for sequences of isometric immersions. We would prefer the assumptions to be stated in terms of the second fundamental form. Before stating the result, let us mention the following lemma, which tells us that the injectivity radius can be controlled by the maximum of the norm of the second fundamental form.

Lemma 1.2.12. *Let* (N, h) *be a complete Riemannian manifold with bounded geometry. Fix* $C > 0$ *and denote by:*

$$\mathcal{M}_C = \left\{ \begin{matrix} (M, g)\colon g \text{ is complete and there exists isometric immersion} \\ F\colon (M, g) \to (N, h) \text{ with } \|A_F\| \leq C \end{matrix} \right\}.$$

Then, there exists $r > 0$ *(depending on* C*) such that*

$$\operatorname{inj}_g(M) > r > 0, \quad \textit{for all } (M, g) \in \mathcal{M}_C.$$

Now, we are ready to state the Ascoli-Arzelà theorem for isometric immersions.

Theorem 1.2.13 (Compactness for immersions)**.** *Consider* $\{(M_i, g_i, x_i)\}_{i\in\mathbb{N}}$ *and* $\{(N_i, h_i, y_i)\}_{i\in\mathbb{N}}$ *two sequences of complete pointed Riemannian manifolds. Let*

$$F_i\colon (M_i, g_i, x_i) \to (N_i, h_i, y_i), \quad i \in \mathbb{N},$$

be isometric immersions with

$$F_i(x_i) = y_i.$$

Assume that:

1. M_i *is compact, for each* $i \in \mathbb{N}$.
2. (N_i, h_i, y_i) *has uniformly bounded geometry, for each* $i \in \mathbb{N}$.
3. *For each* $k \in \mathbb{N}$*, there exists* $C_k > 0$ *such that*

$$\|(\nabla^{F_i})^{(k)} A_{F_i}\| \leq C_k, \quad \textit{for all } i \in \mathbb{N}.$$

Then, $\{F_i\}_{i\in\mathbb{N}}$ *smoothly subconverges to a complete isometric immersion*

$$F_\infty\colon (M_\infty, g_\infty, x_\infty) \to (N_\infty, h_\infty, y_\infty).$$

CHAPTER 2

Lagrangian Submanifolds

In this chapter, we introduce the main class of submanifolds that will be studied throughout this thesis. The material presented here follows [24, 33].

2.1 Kähler manifolds

Let M be a $2m$-dimensional smooth manifold equipped with a metric $g = \langle \cdot\,, \cdot \rangle$ and associated connection ∇. An ***almost complex structure*** on M is a $(1, 1)$-tensor J satisfying

$$J^2 = J \circ J = -\mathrm{Id}.$$

The pair (M, J) is called an ***almost complex manifold***. Additionally, if J satisfies

$$\langle JX, JY \rangle = \langle X, Y \rangle \quad \text{and} \quad (\nabla_X J)(Y) = 0, \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

then the triple (M, g, J) is called ***Kähler manifold***. A standard linear algebra argument shows that each tangent space of an almost complex manifold has an orthonormal basis of the form $\{e_1, Je_1, \dots, e_m, Je_m\}$. In $\mathbb{R}^{2m}$, $m \geq 1$, there are infinitely many almost complex structures. For example, let us identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$. Then, the multiplication with the complex number i, gives rise to an almost complex structure. Indeed, consider

$$(x^1 + iy^1, \dots, x^m + iy^m) \;\; \mapsto \;\; i \cdot (x^1 + iy^1, \dots, x^m + iy^m).$$

Thus,

$$J(x^1, y^1, \dots, x^m, y^m) = (-y^1, x^1, \dots, -y^m, x^m),$$

gives rise to a complex structure on $\mathbb{R}^{2m}$. Note that, up to sign, in $\mathbb{R}^2$ there exists only one complex structure. In particular, as the following lemma suggests, in any 2-dimensional Riemannian manifold locally there exists a complex structure.

Lemma 2.1.1. *Let M be a 2-dimensional Riemannian manifold. Then around each point x there exists an open neighborhood U which supports a Kähler structure J. Additionally, if M is oriented then J can be globally defined.*

Proof. Fix a point $x \in M$ and consider an orthonormal frame $\{e_1, e_2\}$ defined in a neighborhood U of x. Define on U the tensor $J\colon \mathfrak{X}(U) \to \mathfrak{X}(U)$ given by

$$Je_1 = e_2 \quad \text{and} \quad Je_2 = -e_1.$$

Clearly, J is an isometry. It remains to show that J is parallel. We compute

$$\begin{aligned}(\nabla_{e_1} J)(e_1) &= \nabla_{e_1}(Je_1) - J(\nabla_{e_1} e_1)\\ &= \nabla_{e_1} e_2 - J(\langle \nabla_{e_1} e_1, e_2\rangle e_2)\\ &= \nabla_{e_1} e_2 - \langle \nabla_{e_1} e_1, e_2\rangle Je_2\\ &= \langle \nabla_{e_1} e_2, e_1\rangle e_1 + \langle \nabla_{e_1} e_1, e_2\rangle e_1\\ &= 0.\end{aligned}$$

Similarly we have

$$(\nabla_{e_1} J)(e_2) = 0, \quad (\nabla_{e_2} J)(e_1) = 0 \quad \text{and} \quad (\nabla_{e_2} J)(e_2) = 0.$$

Assume that M is oriented. Let $x_0 \in M$, and choose any positively oriented orthonormal basis $\{e_1, e_2\}$ of $T_{x_0}M$. As above, define

$$Je_1 = e_2 \quad \text{and} \quad Je_2 = -e_1.$$

We must check that this definition does not depend on the chosen frame. Let $\{\varepsilon_1, \varepsilon_2\}$ be another positively oriented orthonormal frame around x_0. Since both bases are positively oriented and orthonormal, there exists an angle function θ such that

$$\varepsilon_1 = \cos(\theta)e_1 + \sin(\theta)e_2 \quad \text{and} \quad \varepsilon_2 = -\sin(\theta)e_1 + \cos(\theta)e_2.$$

Then

$$J\varepsilon_1 = \cos(\theta)Je_1 + \sin(\theta)Je_2 = \cos(\theta)e_2 - \sin(\theta)e_1 = \varepsilon_2.$$

Similarly,

$$J\varepsilon_2 = -\sin(\theta)Je_1 + \cos(\theta)Je_2 = -\sin(\theta)e_2 - \cos(\theta)e_1 = -\varepsilon_1.$$

Hence the definition of J is independent of the positively oriented orthonormal frame and extends to a global smooth 1-1 tensor J with all the required properties. This completes the proof. $\square$

Definition 2.1.2. *Let (M, g, J) be a Kähler manifold. Then, the 2-form Ω given by*

$$\Omega(X, Y) \doteq \langle JX, Y\rangle, \quad \textit{for all } X, Y \in \mathfrak{X}(M),$$

is called the associated ***Kähler form****.*

Theorem 2.1.3. *The Kähler form Ω is closed.*

Proof. Fix a point $x_0 \in M$ and local orthonormal frame $\{e_1, \dots, e_{2m}\}$ such that

$$\nabla_{e_j} e_i|_{x_0} = 0, \quad \text{for all} \quad i, j \in \{1, \dots, 2m\}.$$

Because our considerations are of tensorial nature it suffices to prove that

$$d\Omega(e_i, e_j, e_k)|_{x_0} = 0, \quad \text{for all} \quad i, j, k \in \{1, \dots, 2m\}.$$

Using the formula

$$\begin{aligned} d\Omega(X, Y, Z) \;&=\; X\Omega(Y, Z) - Y\Omega(X, Z) + Z\Omega(X, Y) \\ &\quad -\Omega([X, Y], Z) - \Omega([Y, Z], X) - \Omega([Z, X], Y), \end{aligned}$$

and evaluating at the point x_0, we get

$$\begin{aligned} d\Omega(e_i, e_j, e_k) &= e_i(\Omega(e_j, e_k)) - e_j(\Omega(e_i, e_k)) + e_k(\Omega(e_i, e_j)) \\ &= e_i\langle Je_j, e_k\rangle - e_j\langle Je_i, e_k\rangle + e_k\langle Je_i, e_j\rangle \\ &= \langle \nabla_{e_i} Je_j, e_k\rangle - \langle \nabla_{e_j} Je_i, e_k\rangle + \langle \nabla_{e_k} Je_i, e_j\rangle \\ &= \langle J(\nabla_{e_i} e_j), e_k\rangle - \langle J(\nabla_{e_j} e_i), e_k\rangle + \langle J(\nabla_{e_k} e_i), e_j\rangle \\ &= 0. \end{aligned}$$

This completes the proof. □

Theorem 2.1.4 (Kähler identities)**.** *Let (M, g, J) be a Kähler manifold. Then*

1. $R(X, Y)JZ = JR(X, Y)Z$*,*
2. $R(JX, JY)Z = R(X, Y)Z$*,*
3. $\mathrm{Ric}(X, Y) = \mathrm{Ric}(JX, JY) = -\frac{1}{2}\sum_{i=1}^{2m} R(JX, Y, e_i, Je_i)$*,*

where $X, Y, Z \in \mathfrak{X}(M)$ and $\{e_1, \dots, e_{2m}\}$ is orthonormal frame.

Proof. 1. We compute

$$\begin{aligned} R(X,Y)JZ &= \nabla_X\nabla_Y JZ - \nabla_Y\nabla_X JZ - \nabla_{[X,Y]}JZ \\ &= J(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) \\ &= JR(X,Y)Z. \end{aligned}$$

2. First note that

$$\begin{aligned} R(JX,JY,Z,W) &= R(Z,W,JX,JY) = -\langle R(Z,W)JX, JY\rangle \\ &= -\langle JR(Z,W)X, JY\rangle = -\langle R(Z,W)X, Y\rangle \\ &= R(X,Y,Z,W). \end{aligned}$$

Therefore,

$$R(JX,JY)Z = -\sum_{i=1}^{2m} R(JX,JY,Z,e_i)e_i = -\sum_{i=1}^{2m} R(X,Y,Z,e_i)e_i.$$

3. Consider an orthonormal frame $\{e_1, Je_1, \ldots, e_m, Je_m\}$. We compute

$$\begin{aligned} \mathrm{Ric}(JX,JY) &= \sum_{i=1}^{m} R(JX,e_i,JY,e_i) + \sum_{i=1}^{m} R(JX,Je_i,JY,Je_i) \\ &= \sum_{i=1}^{m} R(JJX,Je_i,JY,e_i) + \sum_{i=1}^{m} R(X,e_i,JY,Je_i) \\ &= -\sum_{i=1}^{m} R(X,Je_i,JY,e_i) + \sum_{i=1}^{m} R(JY,Je_i,X,e_i) \\ &= -\sum_{i=1}^{m} R(X,Je_i,JY,e_i) + \sum_{i=1}^{m} R(Y,e_i,X,e_i) \\ &= -\sum_{i=1}^{m} R(JY,e_i,X,Je_i) + \sum_{i=1}^{m} R(Y,e_i,X,e_i) \\ &= -\sum_{i=1}^{m} R(JJY,Je_i,X,Je_i) + \sum_{i=1}^{m} R(Y,e_i,X,e_i) \\ &= \sum_{i=1}^{m} R(Y,Je_i,X,Je_i) + \sum_{i=1}^{m} R(Y,e_i,X,e_i). \end{aligned}$$

Consequently,

$$\mathrm{Ric}(JX, JY) = \mathrm{Ric}(X, Y).$$

For the other identity, we compute

$$\begin{aligned}
\mathrm{Ric}(JX, JY) &= \sum_{i=1}^{m} R(JX, e_i, JY, e_i) + \sum_{i=1}^{m} R(JX, Je_i, JY, Je_i) \\
&= \sum_{i=1}^{m} R(JX, e_i, JJY, Je_i) + \sum_{i=1}^{m} R(JX, Je_i, Y, e_i) \\
&= -\sum_{i=1}^{m} R(JX, e_i, Y, Je_i) + \sum_{i=1}^{m} R(X, e_i, Y, e_i) \\
&= \sum_{i=1}^{m} R(Y, JX, e_i, Je_i) = \sum_{i=1}^{m} R(JY, JJX, e_i, Je_i) \\
&= -\sum_{i=1}^{m} R(JY, X, e_i, Je_i) = \sum_{i=1}^{m} R(X, JY, e_i, Je_i) \\
&= -\sum_{i=1}^{m} R(JX, Y, e_i, Je_i). \qquad (2.1)
\end{aligned}$$

Now observe that for

$$\{e_1, \ldots, e_m, e_{m+1} = Je_1, \ldots, e_{2m} = Je_m\}$$

we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2m} R(JX, Y, e_i, Je_i) \\
&\quad = \frac{1}{2}\sum_{i=1}^{m} R(JX, Y, e_i, Je_i) + \frac{1}{2}\sum_{i=1}^{m} R(JX, Y, Je_i, JJe_i) \\
&\quad = \frac{1}{2}\sum_{i=1}^{m} R(JX, Y, e_i, Je_i) - \frac{1}{2}\sum_{i=1}^{m} R(JX, Y, Je_i, e_i) \\
&\quad = \sum_{i=1}^{m} R(JX, Y, e_i, Je_i). \qquad (2.2)
\end{aligned}$$

From (2.1) and (2.2) we get the desired result.

This completes the proof. □

Definition 2.1.5. *The* ***Ricci form*** *of a Kähler manifold* (M, g, J) *is defined by*

$$\mathcal{R}(X, Y) \doteq \mathrm{Ric}(JX, Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

Lemma 2.1.6. *The following formula holds*

$$\mathcal{R}(X, Y) = \mathrm{Ric}(JX, Y) = \frac{1}{2}\sum_{i=1}^{2m} R(X, Y, e_i, Je_i),$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. We compute

$$\begin{aligned}
\mathcal{R}(X, Y) &= \sum_{i=1}^{m} R(JX, e_i, Y, e_i) + \sum_{i=1}^{m} R(JX, Je_i, Y, Je_i) \\
&= \sum_{i=1}^{m} R(X, e_i, Y, Je_i) - \sum_{i=1}^{m} R(X, Je_i, Y, e_i) \\
&= \sum_{i=1}^{m} R(Je_i, Y, X, e_i) + \sum_{i=1}^{m} R(Y, X, Je_i, e_i) + \sum_{i=1}^{m} R(X, e_i, Y, Je_i) \\
&= \sum_{i=1}^{m} R(X, Y, e_i, Je_i) + \sum_{i=1}^{m} R(X, e_i, Y, Je_i) - \sum_{i=1}^{m} R(X, e_i, Y, Je_i) \\
&= \sum_{i=1}^{m} R(X, Y, e_i, Je_i). \qquad (2.3)
\end{aligned}$$

Now observe that for $\{e_1, \ldots, e_m, e_{m+1} = Je_1, \ldots, e_{2m} = Je_m\}$ we have

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{2m} R(X, Y, e_i, Je_i) &= \frac{1}{2}\sum_{i=1}^{m} R(X, Y, e_i, Je_i) + \frac{1}{2}\sum_{i=1}^{m} R(X, Y, Je_i, J^2 e_i) \\
&= \frac{1}{2}\sum_{i=1}^{m} R(X, Y, e_i, Je_i) + \frac{1}{2}\sum_{i=1}^{m} R(X, Y, e_i, Je_i) \\
&= \sum_{i=1}^{m} R(X, Y, e_i, Je_i). \qquad (2.4)
\end{aligned}$$

From (2.3) and (2.4) we get the desired result. This completes the proof. □

Definition 2.1.7. *Suppose that M is a Kähler manifold equipped with complex structure J.*

1. *A 2-plane Π is called* ***complex*** *if it is invariant by J, i.e., $J(\Pi) = \Pi$.*
2. *A 2-plane Π is called* ***Lagrangian*** *if $J(\Pi)$ is perpendicular to Π.*
3. *Let Π be a complex 2-plane generated by the orthonormal frame $\{e_1, Je_1\}$. The* ***holomorphic sectional curvature*** *of Π is denoted by $\mathrm{Hol}(\Pi)$ and it is given by*
$$\mathrm{Hol}(\Pi) = \sec(\Pi) = \sec(e_1 \wedge Je_1) = R(e_1, Je_1, e_1, Je_1).$$
4. *We say that M has* ***constant holomorphic sectional curvature*** *if*
$$\mathrm{Hol}(\Pi) = \sigma \in \mathbb{R}$$
for every complex 2-plane Π.

We conclude this section with the following characterization.

Theorem 2.1.8. *Let (M, g, Ω) be a Kähler manifold with constant holomorphic sectional curvature σ. Then, for any $X, Y, Z, W \in \mathfrak{X}(M)$, it holds*

$$\begin{aligned} R(X,Y,Z,W) \;=\; & \frac{\sigma}{4}\{\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle \\ & +\Omega(X,Z)\Omega(Y,W) - \Omega(X,W)\Omega(Y,Z) \\ & +2\Omega(X,Y)\Omega(Z,W)\}. \end{aligned}$$

2.2 Complex and Lagrangian submanifolds

We present now an interesting category of Kähler submanifolds.

Definition 2.2.1. *A map $f\colon (M^{2m}, J_M) \to (N^{2n}, J_N)$ between two Kähler manifolds is called* ***holomorphic*** *if*

$$df \circ J_M = J_N \circ df.$$

If the map is a holomorphic isometric immersion, then $f(M)$ will be called an (immersed) ***complex submanifold*** *of N.*

Theorem 2.2.2. *Let $f\colon (M^{2m}, g_M, J_M) \to (N^{2n}, g_N, J_N)$ be a holomorphic isometric immersion. Then, the second fundamental form A satisfies*

$$A(J_M X, Y) = J_N A(X, Y),$$

for all $X, Y \in \mathfrak{X}(M)$. In particular, f is minimal.

Proof. We compute

$$\begin{aligned} A(J_M X, Y) &= \nabla^f_Y df(J_M X) - df(\nabla^M_Y(J_M X)) \\ &= \nabla^N_{df(Y)} df(J_M X) - df(\nabla^M_Y(J_M X)) \\ &= \nabla^N_{df(Y)} J_N(df(X)) - df(J_M(\nabla^M_Y X)) \\ &= J_N(\nabla^N_{df(Y)} df(X)) - J_N(df(\nabla^M_Y X)) \\ &= J_N A(X, Y). \end{aligned}$$

In particular,

$$\begin{aligned} A(J_M X, J_M Y) &= J_N A(X, J_M Y) = J_N A(J_M Y, X) = J_N^2 A(Y, X) \\ &= -A(X, Y). \end{aligned}$$

Consider an orthonormal frame of the form $\{e_1, J_M e_1, \ldots, e_m, J_M e_m\}$. Then,

$$\operatorname{tr} A = \sum_{i=1}^m A(e_i, e_i) + \sum_{i=1}^m A(J_M e_i, J_M e_i) = 0.$$

This completes the proof. $\square$

Definition 2.2.3. *Let (M^m, g_M) be a Riemannian manifold, $(N^{2m}, g_N, J_N, \Omega_N)$ a Kähler manifold, and $f\colon M^m \to N^{2m}$ an isometric immersion. Then f is called **Lagrangian** if*

$$f^*\Omega_N = 0,$$

or equivalently,

$$\langle df(X), J_N df(Y)\rangle = 0,$$

for every $X, Y \in \mathfrak{X}(M)$. In this case, the trilinear form C defined by

$$C(X, Y, Z) \doteq \langle A(X, Y), J_N df(Z)\rangle,$$

*for every $X, Y, Z \in \mathfrak{X}(M)$, is called the **fundamental cubic** of the Lagrangian submanifold.*

Clearly every curve in $\mathbb{R}^2$ is a Lagrangian submanifold.

Lemma 2.2.4. *The fundamental cubic C is fully symmetric. If additionally the Lagrangian is minimal, then C is traceless.*

Proof. The symmetry in the first two entries is obvious. So let us prove the symmetry in the last two entries. For every $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned} C(X,Y,Z) &= \langle A(X,Y), J_N df(Z)\rangle = \langle \nabla^N_{df(X)} df(Y), J_N df(Z)\rangle \\ &= -\langle df(Y), \nabla^N_{df(X)} J_N df(Z)\rangle = -\langle df(Y), J_N(\nabla^N_{df(X)} df(Z))\rangle \\ &= -\langle df(Y), J_N A(X,Z)\rangle = \langle J_N df(Y), A(X,Z)\rangle \\ &= \langle A(X,Z), J_N df(Y)\rangle \\ &= C(X,Z,Y). \end{aligned}$$

That minimality implies traceless C is trivial. This completes the proof. $\square$

Lemma 2.2.5. *The fundamental cubic C satisfies the identity*

$$\begin{aligned} (\nabla_X C)(Y,Z,W) &- (\nabla_Y C)(X,Z,W) \\ &= -R^N(df(X), df(Y), df(Z), J_N df(W)), \end{aligned}$$

for every $X, Y, Z, W \in \mathfrak{X}(M)$.

Proof. Because the formula is of tensorial nature, without loss of generality we may assume the $\{X, Y, Z\}$ is part of a normal frame at a fixed point $x_0 \in M$. At point x_0 we have,

$$\begin{aligned} (\nabla_X C)(Y,Z,W) &= XC(Y,Z,W) = X\langle A(Y,Z), J_N df(W)\rangle \\ &= \langle \nabla^\perp_X A(Y,Z), J_N df(W)\rangle + \langle A(Y,Z), \nabla^N_X J_N df(W)\rangle \\ &= \langle (\nabla^\perp_X A)(Y,Z), J_N df(W)\rangle + \langle A(Y,Z), J_N \nabla^N_X df(W)\rangle \\ &= \langle (\nabla^\perp_X A)(Y,Z), J_N df(W)\rangle + \langle A(Y,Z), J_N A(X,W)\rangle \\ &= \langle (\nabla^\perp_X A)(Y,Z), J_N df(W)\rangle. \end{aligned}$$

From the Codazzi equation (1.9), we have

$$\begin{aligned} (\nabla_X C)(Y,Z,W) \;=\; &\langle (\nabla^\perp_Y A)(X,Z), J_N df(W)\rangle \\ &- R^N(df(X), df(Y), df(Z), J_N df(W)). \end{aligned}$$

Taking the difference we immediately arrive at the desired conclusion. $\square$

Definition 2.2.6. *Let $f\colon M^m \to N^{2m}$ be a Lagrangian immersion. The form θ given by*

$$\theta(X) \doteq \langle J_N H, df(X)\rangle_N = -\operatorname{tr} C(X, \cdot, \cdot)$$

is called the ***Maslov form*** *of the Lagrangian.*

Lemma 2.2.7. *The Maslov form of a Lagrangian immersion $f\colon M^m \to N^{2m}$ in a Kähler-Einstein manifold is a closed form.*

Proof. Fix a point $x_0 \in M$ and let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame defined around x_0. Without loss of generality, we may assume that this frame is normal at x_0. Moreover, for simplicity, we denote the Einstein constant by σ. Recall that

$$d\theta(e_i, e_j) = e_i(\theta(e_j)) - e_j(\theta(e_i)) - \theta([e_i, e_j]),$$

for all $i, j \in \{1, \ldots, m\}$.

Differentiating and then evaluating at x_0, we get

$$\begin{aligned}
d\theta(e_i, e_j) &= e_i(\theta(e_j)) - e_j(\theta(e_i)) = e_i\langle J_N H, df(e_j)\rangle - e_j\langle J_N H, df(e_i)\rangle \\
&= \sum_{k=1}^m e_i \langle J_N A(e_k, e_k), df(e_j)\rangle - \sum_{k=1}^m e_j \langle J_N A(e_k, e_k), df(e_i)\rangle \\
&= -\sum_{k=1}^m e_i \langle A(e_k, e_k), J_N df(e_j)\rangle + \sum_{k=1}^m e_j \langle A(e_k, e_k), J_N df(e_i)\rangle \\
&= -\sum_{k=1}^m e_i C(e_k, e_k, e_j) + \sum_{k=1}^m e_j C(e_k, e_k, e_i) \\
&= -\sum_{k=1}^m (\nabla_{e_i} C)(e_j, e_k, e_k) + \sum_{k=1}^m (\nabla_{e_j} C)(e_i, e_k, e_k).
\end{aligned}$$

Hence,

$$d\theta(e_i, e_j) = -\sum_{k=1}^m R^N(df(e_i), df(e_j), df(e_k), J_N df(e_k)),$$

from where we see that

$$d\theta(e_i, e_j) = -\operatorname{Ric}^N(J_N df(e_i), df(e_j)) = -\sigma\langle J_N df(e_i), df(e_j)\rangle = 0.$$

This completes the proof. $\square$

2.3 Examples of Lagrangians

In this section, we investigate graphical Lagrangian submanifolds in $\mathbb{C}^m$. Here we regard $\mathbb{C}^m = \mathbb{R}^m \times \mathbb{R}^m$ as a Kähler manifold equipped with the complex structure

$$J(V, W) = (-W, V),$$

for every pair of vectors $V, W \in \mathbb{R}^m$. Let $F\colon \mathbb{R}^m \to \mathbb{R}^m$ be a smooth map, and let $G\colon \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ be the graphical immersion defined by

$$G(x) = (x, F(x)).$$

In the following lemma, we determine the conditions under which the graph G is Lagrangian.

Theorem 2.3.1. *The graph G is a Lagrangian immersion if and only if the map $F = (F_1, \dots, F_m)$ is the gradient of a function $f : \mathbb{R}^m \to \mathbb{R}$.*

Proof. Let $X \in \mathfrak{X}(\mathbb{R}^m)$. Then,

$$dG(X) = (D_X x, dF(X)) = (X, dF(X)).$$

The immersion G is Lagrangian if, for $X, Y \in \mathfrak{X}(\mathbb{R}^m)$ we have

$$\begin{aligned} 0 &= \langle dG(X), JdG(Y)\rangle = \langle (X, dF(X)), (-dF(Y), Y)\rangle \\ &= -\langle X, dF(Y)\rangle + \langle dF(X), Y\rangle, \end{aligned}$$

which equivalent can be written in the form

$$\langle X, dF(Y)\rangle = \langle dF(X), Y\rangle,$$

for all $X, Y \in \mathfrak{X}(\mathbb{R}^m)$. This means that dF is symmetric. Let $\{\partial_1, \dots, \partial_m\}$ denote the standard basis of $\mathbb{R}^m$. Then

$$\langle \partial_i, dF(\partial_j)\rangle = \langle dF(\partial_i), \partial_j\rangle .$$

Since

$$dF(\partial_j) = \sum_{k=1}^{m} \frac{\partial F_k}{\partial x_j} \partial_k,$$

it follows that

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}, \quad \text{for all } i, j \in \{1, \dots, m\}. \tag{2.5}$$

Now consider the 1-form α given by

$$\alpha = \sum_{i=1}^{m} F_i \, dx_i.$$

Then, using the symmetry relation (2.5), we conclude that

$$d\alpha = \sum_{i,j} \frac{\partial F_i}{\partial x_j} dx_j \wedge dx_i = \sum_{i<j} \left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} \right) dx_j \wedge dx_i = 0.$$

Hence the form α is closed. Since $\mathbb{R}^m$ is simply connected, the Poincaré Lemma [14] implies that every closed 1-form is exact. Therefore, there exists a smooth function $f \colon \mathbb{R}^m \to \mathbb{R}$ such that $df = \alpha$. Consequently, $F = Df$ and this completes the proof. ☐

We conclude this section by computing the second fundamental form and the mean curvature of a Lagrangian graph $G = (I, F)$.

Induced metric: We compute

$$\begin{aligned} g_G(X, Y) &= \langle dG(X), dG(Y) \rangle \\ &= \langle (X, dF(X)), (Y, dF(Y)) \rangle = \langle X, Y \rangle + \langle dF(X), dF(Y) \rangle \\ &= \langle X, Y \rangle + \langle X, dF^T dF(Y) \rangle. \end{aligned}$$

Hence,

$$\boxed{g_G = I + dF^T dF.}$$

Because, $F = Df$ it follows that

$$dF(X) = D_X Df \doteq S(X). \tag{2.6}$$

Consequently, since S is symmetric, we deduce that

$$g_G = \mathrm{Id} + S^2.$$

Second fundamental form: Let Z be a non-zero vector field on $\mathbb{R}^m$. Then,

$$\boxed{\xi = -J dG(Z) = (dF(Z), -Z) = (D_Z Df, -Z),}$$

is normal along the submanifold. We have

$$\begin{aligned} A^{\xi}(X,Y) &= \langle A(X,Y),\xi\rangle = \langle D_X dG(Y),\xi\rangle \\ &= \langle D_X(Y,dF(Y)),(D_Z Df,-Z)\rangle \\ &= \langle (D_X Y, D_X dF(Y)),(D_Z Df,-Z)\rangle \\ &= \langle (D_X Y, D_X D_Y Df),(D_Z Df,-Z)\rangle \\ &= \langle D_X Y, D_Z Df\rangle - \langle D_X D_Y Df, Z\rangle. \end{aligned} \tag{2.7}$$

Recall that

$$D^2 f(X,Y) = \langle D_X Df, Y\rangle. \tag{2.8}$$

From (2.7) and (2.8), we see that

$$\begin{aligned} A^{\xi}(X,Y) &= D^2 f(D_X Y, Z) - X\langle D_Y Df, Z\rangle + \langle D_Y Df, D_X Z\rangle \\ &= -X\{D^2 f(Y,Z)\} + D^2 f(D_X Y, Z) + D^2 f(Y, D_X Z) \\ &= -D^3 f(Y,Z,X). \end{aligned}$$

Consequently,

$$\boxed{A^{\xi}(X,Y) = -D^3 f(Y,Z,X) = -D^3 f(X,Y,Z).}$$

Mean curvature: Because the Hessian

$$S(X) = D_X Df$$

is symmetric, there exists an orthonormal frame $\{e_1,\dots,e_m\}$, with respect to the Euclidean metric, such that

$$S(e_i) = \lambda_i e_i, \quad \text{for all } i \in \{1,\dots,m\}.$$

As we already have seen in Lemma 1.1.3, the eigenvalues of the tensor S are almost everywhere smooth. Now note that,

$$g_G(e_i,e_j) = \langle e_i,e_j\rangle + \langle S(e_i),S(e_j)\rangle = (1+\lambda_i^2)\delta_{ij},$$

for all $i,j \in \{1,\dots,m\}$. Hence, the basis $\{v_1,\dots,v_m\}$ with

$$v_i = \frac{e_i}{\sqrt{1+\lambda_i^2}}, \quad \text{for } i \in \{1,\dots,m\}.$$

forms an orthonormal frame with respect to g_G. Moreover, the frame $\{\xi_1, \ldots, \xi_m\}$, defined by

$$\xi_i = -JdG(v_i) = -J(v_i, D_{v_i}Df) = -(-D_{v_i}Df, v_i) = (D_{v_i}Df, -v_i),$$

forms an orthonormal frame of the normal bundle of the submanifold. We now compute the trace of the shape operator in the direction of ξ_i. We have

$$\begin{aligned} \mathrm{tr}A^{\xi_i} &= \sum_{k=1}^{m} A^{\xi_i}(v_k, v_k) = -\sum_{k=1}^{m} D^3 f(v_k, v_k, v_i) \\ &= -\frac{1}{\sqrt{1+\lambda_i^2}} \sum_{k=1}^{m} \frac{1}{\sqrt{1+\lambda_k^2}} D^3 f(e_k, e_k, e_i). \end{aligned} \tag{2.9}$$

Without loss of generality, we can assume that at a fixed point x_0 the eigenvalues of S are smooth and

$$D_{e_i} e_j|_{x_0} = 0.$$

Therefore, at x_0 we have

$$D^3 f(e_k, e_k, e_i) = e_i(D^2 f(e_k, e_k)) = e_i(\lambda_k). \tag{2.10}$$

From (2.9) and (2.10) we get

$$\mathrm{tr}A^{\xi_i} = \frac{1}{\sqrt{1+\lambda_i^2}} \sum_{k=1}^{m} \frac{1}{1+\lambda_k^2} e_i(\lambda_k) = \frac{1}{\sqrt{1+\lambda_i^2}} \sum_{k=1}^{m} e_i(\arctan(\lambda_k)).$$

Therefore,

$$\boxed{H = -\sum_{i,k=1}^{m} \frac{1}{\sqrt{1+\lambda_i^2}}\, e_i(\arctan(\lambda_k))\, \xi_i.}$$

Hence, the Lagrangian graph is minimal if and only if there exists $\theta \in \mathbb{R}$ such that

$$\sum_{k=1}^{m} \arctan(\lambda_k) = \theta.$$

For $m = 2$, the above equation becomes

$$\cos(\theta)\Delta f = \sin(\theta)\big(1 - \det(\mathrm{Hess}\, f)\big).$$

For $m = 3$, it takes the form

$$\cos(\theta)\big(\Delta f - \det(\mathrm{Hess}\, f)\big) = \sin(\theta)\big(1 - \sigma_2(\mathrm{Hess}\, f)\big),$$

where σ_2 denotes the second elementary symmetric polynomial of Hess f.

CHAPTER 3

Equivariant Lagrangians

In this section, we investigate a special class of Lagrangian submanifolds in $\mathbb{C}^m$, namely those possessing a high degree of symmetry.

3.1 Symmetric Lagrangian submanifolds

We regard $\mathbb{C}^m$ as the product $\mathbb{R}^m \times \mathbb{R}^m$ equipped with the complex structure J given by

$$J(p, q) \doteq (-q, p),$$

for any $p, q \in \mathbb{R}^m$. As we have already seen in Chapter 2, in complex notation J is simply multiplication by i; that is, J is the map

$$p + iq \mapsto i \cdot (p + iq) = -q + ip.$$

In $\mathbb{C}^m$ we consider the following group action: Given $A \in \mathbb{O}(m)$, we define

$$G_A \colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$$

by

$$G_A(p, q) = (A(p), A(q)).$$

A submanifold $L^m \subset \mathbb{C}^m$ is called ***equivariant*** if and only if L^m is invariant under the group action of $\mathbb{O}(m)$, i.e., if and only if

$$G_A(L^m) = L^m, \quad \text{for all } A \in \mathbb{O}(m).$$

If L^m is equivariant then it must have reflective symmetry through the origin. Indeed, choosing

$$A = -I,$$

we see that if $(p, q) \in L^m$, then $(-p, -q)$ must be again a point in L^m.

Theorem 3.1.1. *An Lagrangian equivariant submanifold $L^m \subset \mathbb{C}^m$ can be parametrized by a map $F\colon (\varepsilon, \varepsilon) \times \mathbb{S}^{m-1} \to \mathbb{C}^m$ given by*

$$F(s,x) = (u(s)f(x), v(s)f(x)),$$

where $z = (u, v)$ is a planar curve and f the standard embedding of the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$.

Proof. Let

$$P = (p,q) \in L^m \subset \mathbb{C}^m,$$

and consider its $\mathbb{O}(m)$-orbit

$$\mathcal{O}_P \doteq \{(A(p), A(q)) : A \in \mathbb{O}(m)\}.$$

The orbit is a smooth submanifold of L^m; see [30]. Since L^m is Lagrangian, it follows that $\mathcal{O}_P$ is isotropic with respect to the symplectic form ω in $\mathbb{C}^m$ given by

$$\omega((X,Y),(Z,W)) = \langle J(X,Y),(Z,W)\rangle = \langle X,W\rangle - \langle Y,Z\rangle.$$

We distinguish two cases:

Case 1: Assume that $m \geq 3$. We will show that the points p and q are collinear. Suppose to the contrary that this is not the case. Denote by $\{e_1, \dots, e_m\}$ the standard basis of $\mathbb{R}^m$. Without loss of generality, we may assume

$$p = e_1 \quad \text{and} \quad q = ae_1 + be_2, \quad \text{with } b \neq 0.$$

Consider the following two one-parameter rotations: First, rotate in the (e_1, e_3)-plane. That is consider

$$\begin{cases} R_\theta(e_1) &= \quad \cos(\theta)e_1 + \sin(\theta)e_3, \\ R_\theta(e_3) &= -\sin(\theta)e_1 + \cos(\theta)e_3, \\ R_\theta(e_i) &= \quad e_i, \ \text{ for all other basis vectors,} \end{cases} \qquad \theta \in [-\pi, \pi).$$

Then the orbit curve at P is

$$\theta \mapsto \gamma(\theta) = (R_\theta(p), R_\theta(q)).$$

The corresponding tangent vector to the orbit at P is

$$\gamma'(0) = (e_3, ae_3).$$

Next, rotate in the (e_2, e_3)-plane. That is consider

$$\begin{cases} S_\theta(e_2) &= \quad \cos(\theta)e_2 + \sin(\theta)e_3, \\ S_\theta(e_3) &= -\sin(\theta)e_3 + \cos(\theta)e_2, \\ S_\theta(e_i) &= \quad e_i, \text{ for all other basis vectors,} \end{cases} \qquad \theta \in [-\pi, \pi).$$

Then the orbit curve at P is

$$\theta \mapsto \sigma(\theta) = (S_\theta(p), S_\theta(q)).$$

Similarly, the corresponding tangent vector to the orbit at P is

$$\sigma'(0) = (0, be_3).$$

From the Lagrangian property we have that

$$0 = \omega(\gamma'(0), \sigma'(0)) = \langle e_3, be_3 \rangle - \langle ae_3, 0 \rangle = b,$$

which gives a contradiction. Hence $b = 0$, and $q = ap$. Consequently, the point P can be written in the form $P = (u + iv)\, x$, where $x \in \mathbb{S}^{m-1}$. This implies that the submanifold L^m contains the whole sphere $(u + iv)\, \mathbb{S}^{m-1}$ sitting inside the m-plane

$$E \doteq (u + iv)\, \mathbb{R}^m \subset \mathbb{C}^m.$$

Take now the 2-plane

$$\mathit{\Pi} \doteq \mathbb{C}\, x = \{(\alpha + i\beta)x : \alpha, \beta \in \mathbb{R}\}.$$

The plane $\mathit{\Pi}$ is perpendicular to E and so also to the sphere at

$$P = (u + iv)x.$$

Because L^m has dimension m and the orbit sphere has dimension $m - 1$, the transverse direction inside L^m is 1-dimensional. Hence the intersection $L^m \cap \mathit{\Pi}$ is (locally) a smooth curve. Parametrize this curve in $E \cap L^m$ by

$$z(s) = u(s) + iv(s), \quad \text{for } s \in I \subset \mathbb{R}.$$

Then rotating the point $z(s)x$ by all $A \in \mathbb{O}(m)$, we get

$$L^m = \{z(s)\, x : x \in \mathbb{S}^{m-1}\}.$$

Therefore, the submanifold can by parametrized in the desired form.

Case 2: Let $L^2 \subset \mathbb{C}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ be a connected Lagrangian submanifold which is invariant under the action of $\mathbb{O}(2)$. We will prove that every point $(p, q) \in L$ satisfies p and q linearly dependent. Following the considerations made in part (1) we arrive at the conclusion that any smooth $\mathbb{SO}(2)$-equivariant surface in $\mathbb{R}^2 \times \mathbb{R}^2$ can be parametrized in the form

$$F(s, \theta) = (r(s)R_\theta(e_1),\ a(s)R_\theta(e_1) + b(s)R_\theta(e_2)),$$

where

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Then

$$F_s = (r' R_\theta(e_1), a' R_\theta(e_1) + b' R_\theta(e_2)) \text{ and } F_\theta = (rR_\theta(e_2), aR_\theta(e_2) - bR_\theta(e_1)).$$

On the other hand from the Lagrangian condition, we deduce that

$$0 = \omega(F_s, F_\theta) = -(rb)'.$$

Hence, rb is locally constant along the profile curve. Now observe that

$$rb = \det(p, q),$$

where $\det(p, q)$ denotes the determinant of the (2×2)-matrix with columns the vectors p and q. Indeed, in the above parametrization, $p = rR_\theta(e_1)$ and $q = aR_\theta(e_1) + bR_\theta(e_2)$, and therefore

$$\det(p, q) = \det(rR_\theta(e_1), aR_\theta(e_1) + bR_\theta(e_2)) = rb.$$

Thus the function $D : L \to \mathbb{R}$, given by $D(p, q) \doteq \det(p, q)$, is locally constant. Since L is connected, it follows that D is constant on L. Now we use the full $\mathbb{O}(2)$-invariance. Let $S \in \mathbb{O}(2)$ be any reflection, so that $\det S = -1$. Since L is $\mathbb{O}(2)$-invariant, if $(p, q) \in L$, then $(Sp, Sq) \in L$. But

$$D(Sp, Sq) = \det(Sp, Sq) = \det(S)\det(p, q) = -D(p, q).$$

Since D is constant on L, we also have

$$D(p, q) = D(Sp, Sq) = -D(p, q).$$

Thus $\det(p, q) = 0$, for all $(p, q) \in L$. Hence p and q are linearly dependent at every point of L. Thus, for every $(p, q) \in L$, there exist real numbers u, v and a unit vector $x \in \mathbb{S}^1$ such that $(p, q) = (ux, vx)$. Choosing a local smooth curve $s \mapsto (u(s), v(s))$ in the quotient, and rotating it by all $x \in \mathbb{S}^1$, we obtain the desired parametrization. □

From the above theorem it follows that the geometry of L^m is determined by the behaviour of a point symmetric curve

$$z \doteq L^m \cap \Pi,$$

coming from the intersection of L^m with an appropriate 2-plane. The curve z will be referred in the sequel as the ***profile curve*** of L^m; see Fig 3.1. Note that

$$z = L^m/\mathbb{O}(m), \tag{3.1}$$

where here the quotient is taken with respect to the natural equivalence relation induced by the group action.

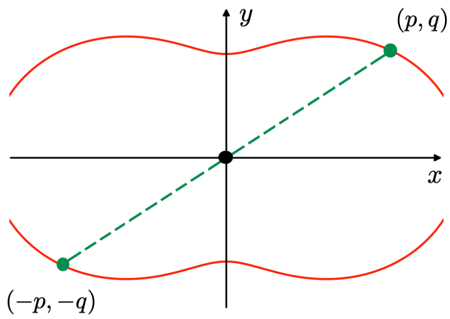


Figure 3.1: Point symmetric profile curve

In the sequel we will explore the geometry of the immersion F. Consider a local coordinate system $(s; x^1, \ldots, x^{m-1})$ on $(-\varepsilon, \varepsilon) \times \mathbb{S}^{m-1}$. Here Latin indices i, j, k are used in the range $\{1, \ldots, m-1\}$ and 0 for quantities depending on s.

Induced metric: We compute

$$dF(\partial_s) = F_s = (u' f, v' f) \quad \text{and} \quad dF(\partial_i) = F_{x^i} = (u\, f_{x^i}, v\, f_{x^i}),$$

where, throughout this chapter, $\{\cdot\}'$ denotes differentiation with respect to the parameter s. We keep in mind that f_{x^i} are smooth vector fields that do not depend on the parameter s. We have

$$\begin{cases} g_{00} = \langle F_s, F_s \rangle = \|F_s\|^2 = (u')^2 + (v')^2 = \|z'\|^2, \\[2ex] g_{0i} = \langle F_s, F_{x^i} \rangle = 0, \\[2ex] g_{ij} = \langle F_{x^i}, F_{x^j} \rangle = (u^2 + v^2)(g_S)_{ij}. \end{cases}$$

Consequently,

$$g \sim \begin{pmatrix} \|z'\|^2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \|z\|^2 g_S & \\ 0 & & & \end{pmatrix},$$

where g_S is the standard metric $\mathbb{S}^{m-1}$. Then, the induced by F metric on M^m is

$$g = \|z\|^2 g_S + \|z'\|^2 ds^2.$$

Hence, F is an immersion at points where $\|z\| > 0$. As we shall see later, there are cases where we can extend the submanifold even at points where $\|z\| = 0$.

Orthonormal frames: Let $\{e_1, \ldots, e_{m-1}\}$ be a local orthonormal frame field of $\mathbb{S}^{m-1}$. Then,

$$\{e_0 = \partial_s, e_1, \ldots, e_{m-1}\}$$

form a basis of TL. Moreover, the vector fields

$$E_0 = \frac{\partial_s}{\|z'\|},\ E_1 = \frac{e_1}{\|z\|},\ \ldots, E_{m-1} = \frac{e_{m-1}}{\|z\|},$$

constitutes a local orthonormal frame of L^m with respect to the induced metric g. Consider now the vector fields

$$\begin{cases} dF(E_0) = \|z'\|^{-1} F_s = \|z'\|^{-1}(u'f, v'f), \\ \\ dF(E_i) = \|z\|^{-1}(u\, df(e_i), v\, df(e_i)), \\ \\ \nu_0 = JdF(E_0) = \|z'\|^{-1}(-v'f, u'f), \\ \\ \nu_i = JdF(E_i) = \|z'\|^{-1}(-v\, df(e_i), u\, df(e_i)). \end{cases}$$

A direct computation shows that $\{\nu_0, \nu_1, \ldots, \nu_{m-1}\}$ form a basis of the normal bundle. Observe that E_0 corresponds to the tangent

$$\tau = \frac{z'}{\|z'\|}$$

of the profile curve and ν_0 corresponds to the inward pointing unit normal

$$\xi \doteq J\left(\frac{z'}{\|z'\|}\right) = \frac{(-v', u')}{\|z'\|},$$

of the profile curve z. We will denote the curvature of z by k.

Shape operators: Using the fact that

$$\langle f, f\rangle = 1,$$

we get

$$\begin{aligned} A^{\nu_0}(E_0, E_0) &= \frac{1}{\|z'\|^2}\langle A(\partial_s, \partial_s), \nu\rangle = \frac{1}{\|z'\|^2}\langle \nabla^F_{\partial_s} dF(\partial_s) - dF(\nabla_{\partial_s}\partial_s), \nu\rangle \\ &= \frac{1}{\|z'\|^2}\langle F_{ss}, \nu\rangle = \frac{1}{\|z'\|^3}\langle (u'' f, v'' f), (-v' f, u' f)\rangle \\ &= \frac{u'v'' - u''v'}{\{(u')^2 + (v')^2\}^{\frac{3}{2}}}. \end{aligned}$$

Hence,

$$A^{\nu_0}(E_0, E_0) = k.$$

Similarly, we find

$$A^{\nu_0}(E_0, E_i) = 0.$$

Furthermore, in view of Example 1.1.4, we have

$$\begin{aligned} A^{\nu_0}(E_i, E_j) &= \langle A(E_i, E_j), \nu_0\rangle = \frac{1}{\|z\|^2}\langle A(e_i, e_j), \nu_0\rangle = \frac{1}{\|z\|^2}\langle \nabla^F_{e_i} dF(e_j), \nu_0\rangle \\ &= \frac{1}{\|z\|^2}\langle \nabla^F_{e_i}(u\, df(e_j), v\, df(e_j)), \nu_0\rangle \\ &= \frac{1}{\|z\|^2}\langle (u\nabla^f_{e_i} df(e_j), v\nabla^f_{e_i} df(e_j)), \nu_0\rangle \\ &= \frac{1}{\|z'\|\,\|z\|^2}\langle (uA_f(e_i, e_j), vA_f(e_i, e_j)), (-v' f, u' f)\rangle \\ &= \frac{1}{\|z'\|\,\|z\|^2}\langle (-u\langle e_i, e_j\rangle f, -v\langle e_i, e_j\rangle f), (-v' f, u' f)\rangle \\ &= \frac{1}{\|z'\|\,\|z\|^2}(uv' - u'v)\delta_{ij}. \end{aligned}$$

Hence,

$$A^{\nu_0}(E_i, E_j) = -\frac{\langle z, \xi\rangle}{\|z\|^2}\delta_{ij}.$$

Define the function p given by

$$p = \frac{\langle z, \xi\rangle}{\|z\|^2}.$$

Then,

$$A^{\nu_0} \sim \begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & -p & 0 & \cdots & 0 \\ 0 & 0 & -p & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -p \end{pmatrix}.$$

Moreover, by straightforward computation we have

$$A^{\nu_i}(E_0, E_0) = 0.$$

On the other hand,

$$\begin{aligned} A^{\nu_i}(E_0, E_i) &= \frac{1}{\|z\|\,\|z'\|}\langle A(\partial_s, e_i), \nu_i\rangle = \frac{1}{\|z\|\,\|z'\|}\langle \nabla^F_{\partial_s} dF(e_i), \nu_i\rangle \\ &= \frac{1}{\|z\|\,\|z'\|}\langle \nabla^F_{\partial_s}(u\, df(e_i), v\, df(e_i)), \nu_i\rangle \\ &= \frac{1}{\|z\|\,\|z'\|}\langle \big(u' df(e_i), v' df(e_i)\big), (-v\, df(e_i), u\, df(e_i))\rangle \\ &= \frac{uv' - u'v}{\|z\|^2\,\|z'\|} = -\frac{\langle z, \xi\rangle}{\|z\|^2} \\ &= -p. \end{aligned}$$

For $i \neq j$, we obtain

$$\begin{aligned} A^{\nu_i}(E_0, E_j) &= \frac{1}{\|z\|\,\|z'\|}\langle A(\partial_s, e_j), \nu_i\rangle \\ &= \frac{1}{\|z\|\,\|z'\|}\langle \big(u'\, df(e_j), v'\, df(e_j)\big), (-v\, df(e_i), u\, df(e_i))\rangle. \end{aligned}$$

Because

$$\langle df(e_i), df(e_j)\rangle = 0,$$

we get that

$$A^{\nu_i}(E_0, E_j) = 0.$$

Hence,

$$A^{\nu_i}(E_0, E_j) = -p\,\delta_{ij}.$$

In addition, for $i \neq j$, we have

$$\begin{aligned}
A^{\nu_i}(E_k, E_j) &= \frac{1}{\|z\|^2}\langle A(e_k, e_j), \nu_i\rangle = \frac{1}{\|z\|^2}\langle \nabla^F_{e_k} dF(e_j), \nu_i\rangle \\
&= \frac{1}{\|z\|^3}\langle (u\,\nabla^f_{e_k} df(e_j), v\,\nabla^f_{e_k} df(e_j)), (v\,df(e_i), -u\,df(e_i))\rangle \\
&= \frac{1}{\|z\|^3}\{uv\langle \nabla^f_{e_k} df(e_j), df(e_i)\rangle - uv\langle \nabla^f_{e_k} df(e_j), df(e_i)\rangle\} \\
&= 0.
\end{aligned}$$

Summarizing, the shape operators with respect to the previous frames are

$$A^{\nu_0} \sim \begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & -p & 0 & \cdots & 0 \\ 0 & 0 & -p & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -p \end{pmatrix} \; \& \; A^{\nu_i} \sim \begin{pmatrix} 0 & 0 & \cdots & -p & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ -p & \vdots & & 0 & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Lemma 3.1.2. *The mean curvature vector of F is given by*

$$H = (k - (m-1)p)\,\nu_0.$$

and the squared norm of the second fundamental form by

$$\|A\|^2 = k^2 + 3(m-1)p^2.$$

Moreover, the eigenvalues of the Ricci tensor are given by

$$-(m-1)p(k+p) \quad \textit{and} \quad -p(k+p-(m-2)p)$$

the latter with multiplicity $m-1$.

Proof. By straightforward computations, we have

$$H = \mathrm{tr}(A^{\nu_0})\nu_0 + \sum_{i=1}^{m-1} \mathrm{tr}(A^{\nu_i})\nu_i = (k - (m-1)p)\,\nu_0.$$

and

$$\begin{aligned}
\|A\|^2 &= \|A^{\nu_0}\|^2 + \sum_{i=1}^{m-1} \|A^{\nu_i}\|^2 \\
&= k^2 + (m-1)p^2 + \underbrace{2p^2 + \cdots + 2p^2}_{2(m-2)-\text{times}} \\
&= k^2 + 3(m-1)p^2
\end{aligned}$$

From the Gauss equation (1.8), we have

$$\begin{aligned}
\mathrm{Ric}(X,Y) = \langle A(X,Y), H\rangle - \langle A(X,E_0), A(Y,E_0)\rangle \\
- \sum_{l=1}^{m-1} \langle A(X,E_l), A(Y,E_l)\rangle.
\end{aligned}$$

Therefore,

$$\mathrm{Ric}(E_0,E_0) = \langle A(E_0,E_0), H\rangle - \|A(E_0,E_0)\|^2 - \sum_{l=1}^{m-1} \|A(E_0,E_l)\|^2,$$

from where it follows that

$$\begin{aligned}
\mathrm{Ric}(E_0,E_0) &= \langle k\nu_0, H\rangle - k^2 - (m-1)p^2 \\
&= k(k-(m-1)p) - k^2 - (m-1)p^2 \\
&= -(m-1)p(k+p).
\end{aligned}$$

Thus,

$$\mathrm{Ric}(E_0,E_0) = -(m-1)p(k+p).$$

Moreover,

$$\begin{aligned}
\mathrm{Ric}(E_0,E_i) &= \langle A(E_0,E_i), H\rangle - \langle A(E_0,E_0), A(E_i,E_0)\rangle \\
&\quad - \sum_{l=1}^{m-1} \langle A(E_0,E_l), A(E_i,E_l)\rangle \\
&= 0.
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
\mathrm{Ric}(E_i, E_j) &= \langle A(E_i, E_j), H\rangle - \langle A(E_0, E_i), A(E_0, E_j)\rangle \\
&\quad - \sum_{l=1}^{m-1} \langle A(E_i, E_l), A(E_j, E_l)\rangle \\
&= \langle A(E_i, E_j), H\rangle - \sum_{k=1}^{m-1} A^{\nu_k}(E_0, E_i) A^{\nu_k}(E_0, E_j) \\
&\quad - \sum_{k=1}^{m-1} A^{\nu_0}(E_i, E_l) A^{\nu_0}(E_j, E_l) \\
&= -p(k-(m-1)p)\,\delta_{ij} - \sum_{k=1}^{m-1} p^2 \delta_{ik}\delta_{jk} - \sum_{l=1}^{m-1} p^2 \delta_{il}\delta_{jl}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{Ric}(E_i, E_j) &= -p(k-(m-1)p)\delta_{ij} - 2p^2\delta_{ij} \\
&= -p(k+p-(m-2)p)\delta_{ij}.
\end{aligned}
$$

This completes the proof. □

Corollary 3.1.3. *The following formula holds*

$$
\|A\|^2 - \frac{3}{m+2}\|H\|^2 = \frac{m-1}{m-2}(k+3p)^2.
$$

Proof. By straight-forward computations we get

$$
\begin{aligned}
&\|A\|^2 - \frac{3}{m+2}\|H\|^2 \\
&= k^2 + 3(m-1)p^2 - \frac{3}{m+2}(k-(m-1)p)^2 \\
&= \frac{m-1}{m+2}(k^2 + 9p^2 + 6kp) \\
&= \frac{m-1}{m+2}(k+3p)^2.
\end{aligned}
$$

This completes the proof. □

3.2 Equivariant Lagrangian spheres

Let us examine the case in which the profile curve $z = (u, v)$ passes through the origin; that is, $z(0) = 0$ and $z(s) \neq 0$ for all $s \in (-\varepsilon, 0) \cup (0, \varepsilon)$. In the sequel, we always assume that z is real analytic.

Lemma 3.2.1. *The following facts hold true.*

1. *The function*
$$p = \frac{\langle z, \xi \rangle}{\|z\|^2}$$
can be analytically extended through the origin. In particular,
$$\lim_{s \to 0} p(s) = -\frac{1}{2} k(0).$$

2. *The image* $F((-\varepsilon, \varepsilon) \times \mathbb{S}^{m-1})$ *gives rise to Lagrangian submanifold of the Euclidean space* $\mathbb{C}^m = \mathbb{R}^m \times \mathbb{R}^m$.

Proof. Without loss of generality, we may assume that the x-axis is tangent to the curve. Moreover, the curve can be locally parametrized as $z(s) = (s, y(s))$, where $y\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ is a real analytic function satisfying
$$y(0) = y'(0) = 0.$$
Since y is real analytic, it follows that
$$y(s) = s^2 h(s), \quad \text{for } s \in (-\varepsilon, \varepsilon),$$
where h is a real analytic function.

1. For $s \neq 0$, we have
$$\begin{aligned} p(s) &= \frac{\langle z(s), \xi(s) \rangle}{\|z(s)\|^2} = \frac{\langle (s, y(s)), (-y'(s), 1) \rangle}{(s^2 + y^2(s))(1 + (y'(s))^2)^{1/2}} \\ &= \frac{y(s) - s\, y'(s)}{(s^2 + y^2(s))(1 + (y'(s))^2)^{1/2}} = \frac{-h(s) - s\, h'(s)}{(1 + s^2 h^2(s))(1 + (y'(s))^2)^{1/2}}. \end{aligned}$$
Therefore,
$$\lim_{s \to 0} p(s) = -h(0) = -\frac{y''(0)}{2} = -\frac{y''(0)}{2(1 + (y'(0))^2)^{3/2}} = -\frac{k(0)}{2}.$$

2. We claim that the tangent space of $F\colon (0,\varepsilon)\times\mathbb{S}^{m-1}\to\mathbb{C}^m$ extends smoothly over the set
$$\{0\}\times\mathbb{S}^{m-1}.$$
Again, without loss of generality, assume that $z\colon(0,\varepsilon)\to\mathbb{C}$ is parametrized as the graph
$$z(s)=(u(s),v(s))=(s,s\varphi(s)),$$
where φ is a real analytic function defined on the interval $(0,\varepsilon)$. Choose a local orthonormal frame $\{e_1,\dots,e_{m-1}\}$ on $\mathbb{S}^{m-1}$ and, for simplicity, set
$$\vec{e}_i \doteq df(e_i).$$
Then, a direct computation shows
$$\begin{aligned}
T_{(s,x)} &= dF(E_0)\wedge dF(E_1)\wedge\cdots\wedge dF(E_{m-1})\\
&=\frac{1}{\|z'\|\,\|z\|^{m-1}}dF(\partial_s)^\wedge dF(e_1)\wedge\cdots\wedge dF(e_{m-1})\\
&=\frac{1}{\|z'\|\,\|z\|^{m-1}}(u'f,v'f)\wedge(u\,\vec{e}_1,v\,\vec{e}_1))\wedge\cdots\wedge(u\,\vec{e}_{m-1},v\,\vec{e}_{m-1})\\
&=\frac{1}{\|z'\|\,\|z\|^{m-1}}(f,(s\varphi)'f)\wedge(s\vec{e}_1,s\,\varphi\,\vec{e}_1)\wedge\cdots\wedge(s\,\vec{e}_{m-1},s\,\varphi\,\vec{e}_{m-1})\\
&=\frac{s^{m-1}}{\|z'\|(s^2+s^2\varphi^2)^{\frac{m-1}{2}}}(f,(s\varphi)'f)\wedge(\vec{e}_1,y\vec{e}_1)\wedge\cdots\wedge(\vec{e}_{m-1},\varphi\,\vec{e}_{m-1})\\
&=\frac{1}{\|z'\|\,(1+y^2)^{\frac{m-1}{2}}}(f,(sy)'f)\wedge(\vec{e}_1,\varphi\,\vec{e}_1)\wedge\cdots\wedge(\vec{e}_{m-1},\varphi\,\vec{e}_{m-1}).
\end{aligned}$$
Clearly, the tangent space extends smoothly to $\{0\}\times\mathbb{S}^{m-1}$.

This completes the proof. □

The following theorem will be useful in the last chapter of the thesis.

Theorem 3.2.2. *Let $F:\mathbb{S}^m\to\mathbb{C}^m$ be an equivariant Lagrangian sphere, where $m>1$. Then its profile curve passes through the origin exactly once.*

Proof. Recall that an equivariant Lagrangian submanifold is invariant under the diagonal action of $\mathbb{O}(m)$. Away from the origin, every orbit is an $(m-1)$-sphere. Therefore, the orbit space $L/\mathbb{O}(m)$ is a one-dimensional manifold; the profile curve.

1. We first prove that the profile curve passes through the origin at least once.

Let us suppose, to the contrary, that the profile curve does not pass through the origin. Then all $\mathbb{O}(m)$-orbits are diffeomorphic to $\mathbb{S}^{m-1}$. Consequently, L is an $\mathbb{S}^{m-1}$-bundle over a closed one-dimensional manifold. Since L is compact and connected, the base manifold is diffeomorphic to $\mathbb{S}^1$. If $m = 2$, then L is an $\mathbb{S}^1$-bundle over $\mathbb{S}^1$, and hence it is either a torus or a Klein bottle. If $m > 2$, then the long exact sequence of homotopy groups yields

$$0 \cong \pi_1(\mathbb{S}^m) = \pi_1(L) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}.$$

In either case, L cannot be diffeomorphic to $\mathbb{S}^m$. This contradiction shows that the profile curve must pass through the origin.

2. We now prove uniqueness.

Set $\mathcal{F}_0 \doteq F^{-1}(0)$. Since F is an immersion, the set $\mathcal{F}_0$ is discrete. Because $\mathbb{S}^m$ is compact, it follows that $\mathcal{F}_0$ is finite. Assume, by contradiction, that the profile curve passes through the origin at least twice. Then $\mathcal{F}_0$ contains at least two points. Because $m > 1$, removing finitely many points from $\mathbb{S}^m$ leaves a connected manifold. Therefore, $\mathbb{S}^m - \mathcal{F}_0$ is connected. Since F is continuous, it follows that $L - \{0\} = F(\mathbb{S}^m - \mathcal{F}_0)$ is connected. On the other hand, removing the origin from the profile curve disconnects it into as many connected components as the number of times it passes through the origin. Thus, if the profile curve passed through the origin at least twice, then $(L - \{0\})/\mathbb{O}(m)$ would be disconnected. This is impossible, because the quotient of a connected space by a continuous group action is connected. Therefore, the profile curve can pass through the origin at most once. Combining the existence and uniqueness parts, we conclude that the profile curve passes through the origin exactly once. This completes the proof. ☐

We now study the local behavior of the profile curve at the origin. Suppose that $z\colon \mathbb{S}^1 \to \mathbb{C}$ is a real analytic curve that is centrally symmetric with respect to the origin. Thus, we may identify $\mathbb{S}^1$ with the interval $[-\pi, \pi]$ and assume that $z\colon [-\pi, \pi] \to \mathbb{C}$ is a closed real analytic curve satisfying

$$z(s) = -z(-s), \text{ for } s \in [-\pi, \pi], \quad \text{and} \quad z^{-1}(0) = \{-\pi, 0, \pi\}.$$

For such curves, we may introduce the signed distance function

$$r(s) \doteq \operatorname{sign}(s)\|z(s)\|.$$

Lemma 3.2.3. *The following identities holds true.*

1. $\|\nabla r\|^2 = 1 - r^2 p^2$,
2. $\Delta r = rp(k + p)$.

Proof. We following the notation introduced in the previous sections.

1. We have

$$\begin{aligned}\nabla r &= E_0(r)E_0 + \sum_{i=1}^{m-1} E_i(r)E_i = \frac{1}{\|z'\|^2}\partial_s(r)\partial_s + \frac{1}{r^2}\sum_{i=1}^{m-1} e_i(r)e_i \\ &= \frac{1}{\|z'\|^2}\partial_s(\langle z, z\rangle^{\frac{1}{2}})\partial_s = \frac{1}{\|z'\|^2}\frac{\langle z', z\rangle}{r}\partial_s \\ &= \frac{\langle \tau, z\rangle}{r}E_0.\end{aligned}$$

Therefore, $r^2\|\nabla r\|^2 = \langle \tau, z\rangle^2$. Now, note that

$$\frac{z}{r} = \frac{\langle z, \tau\rangle}{r}\tau + \frac{\langle z, \xi\rangle}{r}\xi.$$

Taking norms, we immediately see that

$$\frac{\langle z, \tau\rangle^2}{r^2} = 1 - \frac{\langle z, \xi\rangle^2}{r^2} = 1 - \frac{r^4p^2}{r^2} = 1 - r^2p^2.$$

This completes the proof of part (1).

2. Without loss of generality, we may assume that our curve is parametrized by arc-length. Then the Laplacian along the curve is given by $\Delta r = r_{ss}$. Recall from part (1) that,

$$rr_s = r\langle \nabla r, E_0\rangle = \langle \tau, z\rangle.$$

Differentiating the above identity with respect to s and using Serret-Frenet formulas, we have

$$r_{ss} = \frac{\langle \tau', z\rangle}{r} + \frac{1}{r} - \frac{\langle \tau, z\rangle r_s}{r^2} = \frac{k\langle z, \xi\rangle}{r} + \frac{1}{r} - \frac{\langle \tau, z\rangle^2}{r^3}.$$

Using the formula we proved in the first part and the definition of p we obtain the desired result.

This completes the proof. $\square$

Lemma 3.2.4. *The following facts hold true.*

1. *There exist a constant* $r_0 > 0$ *and a local point-symmetric real analytic parametrization of the curve, which for simplicity we again denote by* z*, defined on the interval* $(-r_0, r_0)$*, such that for all* $r \in (-r_0, r_0)$*,*
$$\|z(r)\| = |r|$$
and
$$z(r) = \sum_{n=0}^{\infty} z^{(2n+1)}(0)\frac{r^{2n+1}}{(2n+1)!}.$$
In particular,
$$\|z'(0)\| = 1.$$

2. *The functions* p/r *and* $(k+3p)/r^3$ *are real analytic. In particular,*
$$\lim_{r\to 0} p = 0 \quad \textit{and} \quad \lim_{r\to 0}\frac{k+3p}{r^2} = 0.$$

3. *If the curve* z *is not flat, then the function* k/p *is well defined and real analytic in a neighborhood of* $r = 0$ *and*
$$\lim_{r\to 0}\frac{k}{p} = -(2l+3),$$
where $2l+1$ *is the order of the zero of* p *at* $r = 0$*.*

Proof. Since $\|\nabla r\|^2 = 1 - r^2p^2$ and $\lim_{s\to 0} p = 0$, the inverse function theorem implies that the distance function r can be used as a local variable. Moreover, since z is point-symmetric with respect to the origin, then
$$z(r) = -z(-r), \quad \text{for } r \in (-r_0, r_0).$$
Consequently, by differentiating we see that
$$z^{(2k)}(0) = 0, \quad \text{for all } k \in \mathbb{N}.$$
Hence, its Taylor expansion takes the form
$$z(r) = \sum_{n=0}^{\infty} z^{(2n+1)}(0)\frac{r^{2n+1}}{(2n+1)!}.$$

Moreover, using the facts that $\|z(r)\| = r$ and $z(0) = 0$, we obtain

$$\|z'(0)\| = \lim_{r\to 0}\left|\frac{z(r)-z(0)}{r-0}\right| = \lim_{r\to 0}\left|\frac{z(r)}{r}\right| = 1.$$

From the Taylor expansion, we have

$$\begin{aligned} z'(r) &= \sum_{n=0}^{\infty}(2n+1)z^{(2n+1)}(0)\frac{r^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} z^{(2n+1)}(0)\frac{r^{2n}}{(2n)!} \\ &= z'(0) + r^2\underbrace{\left\{\frac{z''(0)}{2} + \cdots\right\}}_{\sigma(r)} \\ &= z'(0) + r^2\sigma(r). \end{aligned}$$

On the other hand, there exists a function h such that,

$$z(r) = z'(0) + r^3 h(r).$$

Consequently,

$$\begin{aligned} \langle z, Jz'\rangle &= \langle z'(0)r + r^3h(r), J(z'(0) + r^2\sigma(r))\rangle \\ &= r^3\langle z'(0), J\sigma(r)\rangle + r^3\langle h(r), Jz'(0)\rangle + r^5\langle h(r), \sigma(r)\rangle \\ &= r^3\varphi(r), \end{aligned}$$

for some analytic function ϕ. Thus,

$$\langle z, \xi\rangle = \frac{\langle z, Jz'\rangle}{\|z'\|} = \sum_{n=1}^{\infty} b_{2n+1}r^{2n+1},$$

from where it follows that

$$p = \frac{\langle z, \xi\rangle}{r^2} = r\, b(r),$$

where $b(r)$ is a real analytic function. Therefore, the function p/r is real analytic and

$$\lim_{r\to 0} p = 0.$$

Claim: *The following formula holds*

$$r\,\nabla p = -(k + 2p)\nabla r.$$

Proof of the claim: Without loss of generality, we may assume that s is the arc-length parameter of z. Using the Frenet-Serret formulas, we obtain

$$p' = \frac{\langle \tau, \xi \rangle + \langle z, \xi' \rangle}{r^2} - 2\frac{\langle z, \xi \rangle}{r^3} r' = \frac{-k\langle z, \tau \rangle}{r^2} - 2\,\frac{p}{r} r' = -\frac{k+2p}{r} r'.$$

Therefore,

$$r\,\nabla p = r\,p'\partial_s = -(k+2p)r'\partial_s = -(k+2p)\,\nabla r,$$

which is the desired formula. This completes the proof of the claim. ⊛

From the result of the Claim, it follows that

$$-rp_r = k + 2p.$$

Consequently,

$$k + 3p = -rp_r + p = r^2\,\frac{-p_r r + p}{r^2} = -r^2\left(\frac{p}{r}\right)'. \tag{3.2}$$

But since p/r is even, its derivative must have a zero at $r = 0$ and hence the function $k+3p$ must have a zero at $r = 0$ of order at least 3. Indeed! The Taylor expansion of p is

$$p(r) = \sum_{n=l\geq 0}^{\infty} c_{2n+1} r^{2n+1}.$$

Equivalently,

$$\frac{p}{r} = \sum_{k=l\geq 0}^{\infty} c_{2n+1} r^{2n}.$$

Hence,

$$\left(\frac{p}{r}\right)(-r) = \left(\frac{p}{r}\right)(r).$$

Differentiating with respect to r, we get

$$-\left(\frac{p}{r}\right)'(-r) = \left(\frac{p}{r}\right)'(r),$$

from where it follows that

$$\left(\frac{p}{r}\right)'(0) = 0.$$

Because $(p/r)'$ is analytic, we get

$$\left(\frac{p}{r}\right)' = r\,\mu(r). \tag{3.3}$$

Hence, from the identities (3.2) and (3.3) it follows that

$$k + 3p = -r^2\left(\frac{p}{r}\right)' = -r^3\,\mu(r),$$

where μ is a real-analytic function. Consequently, we see that

$$\frac{k+3p}{r^2} = -r\,\mu(r) \xrightarrow{r\to 0} 0$$

and

$$\frac{k+3p}{r^3} = -\mu(r) \quad \text{is real analytic.}$$

It remain to prove that

$$\lim_{r\to 0}\frac{k}{p} = -2l-3.$$

From $-rp_r = k + 2p$, we obtain

$$\frac{k}{p} = -r\frac{p_r}{p} - 2.$$

Since p is real analytic, it admits a Taylor expansion in a neighborhood of $r = 0$. Let $2l+1$, where $l \in \mathbb{N}\cup\{0\}$, denote the order of vanishing of p at $r = 0$. Recall that

$$p(r) = \sum_{n=l}^{\infty} c_{2n+1} r^{2n+1}.$$

Then

$$p_r(r) = \sum_{n=l}^{\infty}(2n+1)c_{2n+1} r^{2n},$$

and therefore

$$\lim_{r\to 0}\frac{rp_r}{p} = \frac{(2l+1)c_{2l+1}}{c_{2l+1}} = 2l+1.$$

Hence,

$$\lim_{r\to 0}\frac{k}{p} = -\lim_{r\to 0}\frac{rp_r}{p} - 2 = -(2l+1) - 2 = -(2l+3).$$

This completes the proof. $\square$

Lemma 3.2.5. *The following conditions are equivalent.*

1. *There exists* $\varepsilon > 0$ *such that* $\mathrm{Ric} \geq \varepsilon r^2 g$.
2. *There exist* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *such that the estimates*

$$-p \geq \varepsilon_1 r \quad \textit{and} \quad k + p \geq \varepsilon_1 r$$

hold on the profile curve.

Proof. By Lemma 3.1.2, the eigenvalues of the Ricci tensor are given by

$$-(m-1)p(k+p) \quad \text{and} \quad -p(k+p) + (m-2)p^2,$$

where the second eigenvalue has multiplicity $m-1$. Observe that 2. $\Rightarrow$ 1. is immediate. For the converse, note that

$$-(m-1)p(k+p) \geq \varepsilon\, r^2$$

implies

$$-\frac{p}{r}\,\frac{k+p}{r} \geq \frac{\varepsilon}{m-1} > 0. \tag{3.4}$$

From Lemma 3.2.4 we have

$$\frac{\varepsilon}{m-1} \leq \lim_{r\to 0}\Big(-\frac{p}{r}\,\frac{k+p}{r}\Big) = \lim_{r\to 0}\Big(-\frac{p}{r}\,\frac{k+3p-2p}{r}\Big) = 2\lim_{r\to 0}\frac{p^2}{r^2}. \tag{3.5}$$

Since the profile curve is closed and passes through the origin only once, there always exists a point at which $-p > 0$ (see Remark 3.2.1 below). Therefore, by (3.4) and (3.5), it follows that the functions $-p/r$ and $(k+p)/r$ are nonzero and positive. Consequently, there exist constants ε_1 and ε_2 as required. This completes the proof. □

Remark 3.2.1. Let z be a regular closed curve in $\mathbb{R}^2$. Suppose that s_0 is a point such that

$$r^2(s_0) = \max\{r^2(s) : s \in \mathbb{S}^1\} > 0.$$

Consider the function $f \doteq r^2$. Since s_0 is a maximum point of f, we have

$$0 = f'(s_0) = 2\langle z'(s_0), z(s_0)\rangle.$$

Therefore, $z'(s_0)$ is perpendicular to $z(s_0)$. Consequently, $\xi(s_0)$ is parallel to $z(s_0)$. Because ξ is inward pointing, it follows that

$$p(s_0) = \frac{\langle z(s_0), \xi(s_0)\rangle}{r^2(s_0)} < 0.$$

3.3 A characterization of the Whitney sphere

In this section we give a characterization of the Whitney sphere.

Lemma 3.3.1. *The following estimate holds*

$$\|A\|^2 - \frac{3}{m+2}\|H\|^2 \geq 0.$$

Proof. Denote by $F\colon L^m \to \mathbb{C}^m$ the immersion. Consider a local orthonormal frame $\{e_1, \ldots, e_m\}$ along L^m. Set

$$H_i \doteq \langle H, JdF(e_i)\rangle.$$

Consider also the functions

$$E_{ijk} = \frac{1}{m+2}(H_i\delta_{jk} + H_j\delta_{ik} + H_k\delta_{ij}).$$

We compute

$$\begin{aligned} E_{ijk}^2 \;=\; & \frac{1}{(m+2)^2}(H_i^2\delta_{jk} + H_j^2\delta_{ik} + H_k^2\delta_{ij} \\ & +2H_iH_j\delta_{jk}\delta_{ik} + 2H_iH_k\delta_{jk}\delta_{ij} + 2H_jH_k\delta_{ik}\delta_{ij}). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i,j,k=1}^m E_{ijk}^2 \;=\; & \frac{1}{(m+2)^2}\Big\{m\sum_{i=1}^m H_i^2 + m\sum_{j=1}^m H_j^2 + m\sum_{k=1}^m H_k^2 \\ & +2\sum_{i=1}^m H_i^2 + 2\sum_{j=1}^m H_j^2 + 2\sum_{k=1}^m H_k^2\Big\}. \end{aligned}$$

Therefore,

$$\sum_{i,j,k=1}^m E_{ijk}^2 = \frac{3}{m+2}\,\|H\|^2.$$

Recall from the previous chapter that the second fundamental form is completely determined by the fundamental cubic

$$C(X,Y,Z) = \langle A(X,Y), JdF(Z)\rangle.$$

Since C is fully symmetric, we get

$$\sum_{i=1}^{m} C(e_i, e_i, e_k) = \sum_{i=1}^{m} \langle A(e_i, e_i), JdF(e_k)\rangle = \langle H, JdF(e_k)\rangle = H_k.$$

Moreover,

$$\begin{aligned}
\sum_{i,j,k=1}^{m} E_{ijk}C_{ijk} &= \frac{1}{m+2}\sum_{i,j,k=1}^{m} (H_i\delta_{jk} + H_j\delta_{ik} + H_k\delta_{ij})C_{ijk} \\
&= \frac{1}{m+2}\Big\{\sum_{i,j=1}^{m} H_iC_{ijj} + \sum_{i,j=1}^{m} H_jC_{iij} + \sum_{i,k=1}^{m} H_kC_{iik}\Big\} \\
&= \frac{1}{m+2}\Big\{\sum_{i=1}^{m} H_i^2 + \sum_{j=1}^{m} H_j^2 + \sum_{k=1}^{m} H_k^2\Big\}.
\end{aligned}$$

Consequently,

$$\sum_{i,j,k=1}^{m} E_{ijk}C_{ijk} = \frac{3}{m+2}\,\|H\|^2.$$

Thus,

$$\sum_{i,j,k=1}^{m} E_{ijk}(C_{ijk} - E_{ijk}) = 0.$$

Now, we have

$$\begin{aligned}
\|A\|^2 &= \sum_{i,j,k=1}^{m} C_{ijk}^2 \\
&= \sum_{i,j,k=1}^{m} (E_{ijk} + C_{ijk} - E_{ijk})^2 \\
&= \sum_{i,j,k=1}^{m} E_{ijk}^2 + \sum_{i,j,k=1}^{m} (C_{ijk} - E_{ijk})^2 \geq \sum_{i,j,k=1}^{m} E_{ijk}^2 \qquad (3.6) \\
&= \frac{3}{m+2}\,\|H\|^2.
\end{aligned}$$

This completes the proof. $\square$

We shall describe an example which satisfy

$$\|A\|^2 = \frac{3}{m+2}\|H\|^2.$$

Example 3.3.2 (Whitney Sphere)**.** Recall that according to Corollary 3.1.3, for any equivariant Lagrangian submanifold we have

$$\|A\|^2 - \frac{3}{m+2}\|H\|^2 = \frac{m-1}{m+2}(k+3p)^2.$$

It suffices to find a curve satisfying

$$k + 3p = 0.$$

Consider the equivariant Lagrangian which is generated by the profile curve $z\colon \mathbb{S}^1 \to \mathbb{C}$ given by

$$z(s) = \left(\frac{-\sin(s)}{1+\cos^2(s)}, \frac{\sin(s)\cos(s)}{1+\cos^2(s)}\right).$$

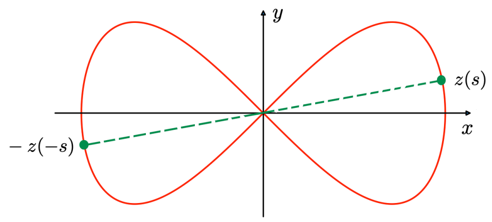


Figure 3.2: Profile curve of z.

A direct computation shows that

$$z'(s) = \left(\frac{\cos(s)(\sin^2(s)-2)}{(1+\cos^2(s))^2}, \frac{3\cos^2(s)-1}{(1+\cos^2(s))^2}\right)$$

and

$$\|z'\| = \frac{1}{\sqrt{1+\cos^2(s)}}.$$

Moreover,

$$p = \frac{-\sin(s)}{\sqrt{1+\cos^2(s)}} = -\mathrm{sign}(\sin(s))\|z\|.$$

Furthermore,

$$z''(s) = \left(\frac{\sin(s)(\cos^4(s) - 5\cos^2(s) + 2)}{(1+\cos^2 s)^3}, \frac{2\cos(s)\sin(s)(\cos^2(s) - 5)}{(1+\cos^2(s))^3} \right)$$

from where it follows that

$$k(s) = \frac{3\sin(s)}{\sqrt{1+\cos^2(s)}} = -3p(s).$$

This completes the claims of the example.

The Lagrangian submanifold generated by the profile curve z in Example 3.3.2 is known in the literature as the ***Whitney sphere***.

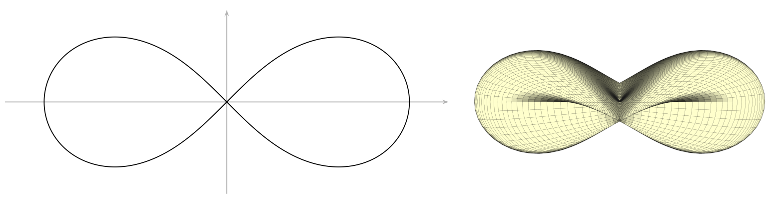


Figure 3.3: The Whitney sphere (figure adapted from [43]).

In the following theorem, we present a characterization of this submanifold, proved independently by Borelli, Chen, and Morvan [6], and by Ros and Urbano [40].

Theorem 3.3.3. *Let $L^m \subset \mathbb{C}^m$ be an immersed real analytic Lagrangian submanifold satisfying*

$$\|A\|^2 = \frac{3}{m+2}\|H\|^2.$$

Then, either L^m is a flat subspace or a Whitney shpere.

Proof of the theorem: Without loss of generality, we may assume that $H = 0$ only on a subset of measure zero; otherwise, by real analyticity, we would have $H = 0$ everywhere. In the latter case, the submanifold is flat, and there is nothing to prove. Therefore, we work on the set

$$U = \{x \in L \colon H(x) \neq 0\}.$$

We follow the notation and setup introduced in Lemma 3.3.1. Furthermore, throughout the proof, capital Latin indices range from 0 to $m-1$, while lower-case Latin indices range from 1 to $m-1$. Choose on U a local orthonormal frame

$$\{e_0, \ldots, e_{m-1}; \nu_0 = JdF(e_0), \ldots, \nu_{m-1} = JdF(e_{m-1})\}$$

such that

$$H \doteq h\,\nu_0.$$

With respect to this frame, we have

$$H_0 = h \quad \text{and} \quad H_1 = H_2 = \cdots = H_{m-1} = 0. \tag{3.7}$$

From our pinching assumption and (3.6), it follows that

$$C_{ijk} = E_{ijk} = \frac{1}{m+2}(H_i\delta_{jk} + H_j\delta_{ik} + H_k\delta_{ij}). \tag{3.8}$$

Using (3.7), (3.8), and the general identity

$$\begin{aligned} A(e_A, e_B) &= \langle A(e_A, e_B), \nu_0\rangle\nu_0 + \sum_{i=1}^{m-1}\langle A(e_A, e_B), \nu_i\rangle\nu_i \\ &= C_{AB0}\,\nu_0 + \sum_{i=1}^{m-1} C_{ABi}\,\nu_i, \end{aligned}$$

we find

$$\begin{cases} A(e_0, e_0) = C_{000}\nu_0 + \displaystyle\sum_{i=1}^{m-1} C_{00i}\nu_i = \frac{3h}{m+2}\nu_0, \\ A(e_0, e_i) = C_{0i0}\nu_0 + \displaystyle\sum_{j=1}^{m-1} C_{0ij}\nu_j = \frac{h}{m+2}\nu_i, \\ A(e_i, e_i) = C_{ii0}\nu_0 + \displaystyle\sum_{j=1}^{m-1} C_{iij}\nu_j = \frac{h}{m+2}\nu_0, \\ A(e_i, e_j) = C_{ij0}\nu_0 + \displaystyle\sum_{k=1}^{m-1} C_{ijk}\nu_k = 0, \quad \text{if } i \neq j. \end{cases}$$

In particular,

$$\begin{cases} A_{\nu_0}(e_0) = 3\lambda e_0, \\ A_{\nu_0}(e_i) = \lambda e_i, \\ A_{\nu_i}(e_0) = \lambda e_i, \\ A_{\nu_i}(e_j) = \lambda \delta_{ij} e_0, \end{cases} \tag{3.9}$$

where

$$\lambda \doteq \frac{h}{m+2}.$$

Claim I: *The connection forms of the normal bundle satisfy:*

$$\langle \nabla^{\perp}_{e_A} \nu_B, \nu_C \rangle = \langle \nabla_{e_A} e_B, e_C \rangle.$$

Proof of claim I. We have

$$\begin{aligned} \langle \nabla^{\perp}_{e_A} \nu_B, \nu_C \rangle &= \langle \nabla_{e_A} JdF(e_B), JdF(e_C) \rangle = \langle J\nabla_{e_A} dF(e_B), JdF(e_C) \rangle \\ &= \langle \nabla_{e_A} dF(e_B), dF(e_C) \rangle = \langle dF(\nabla_{e_A} e_B), dF(e_C) \rangle \\ &= \langle \nabla_{e_A} e_B, e_C \rangle. \end{aligned}$$

This completes the proof of Claim I. □

Claim II: *The following facts hold true:*

1. *The distributions*

$$D = \text{span}\{e_0\} \quad \text{and} \quad D^{\perp} = \text{span}\{e_1, \ldots, e_{m-1}\}$$

are integrable.

2. *The function h depends only on the e_0-direction.*

Proof of claim II. Let us assume first that $m > 2$. D is integrable because it is 1-dimensional. From the Codazzi equation (1.9), we have

$$(\nabla_X A_\xi)(Y) - A_{\nabla^{\perp}_X \xi}(Y) = (\nabla_Y A_\xi)(X) - A_{\nabla^{\perp}_Y \xi}(X), \quad \text{for } X, Y \in \mathfrak{X}(L).$$

Evaluating in special vectors we obtain

$$(\nabla_{e_i} A_{\nu_0})(e_j) - A_{\nabla^\perp_{e_i}\nu_0}(e_j) = (\nabla_{e_j} A_{\nu_0})(e_i) - A_{\nabla^\perp_{e_j}\nu_0}(e_i)$$

from where it follows that

$$\begin{aligned}
&e_i(\lambda)e_j + \lambda\nabla_{e_i}e_j - \langle\nabla_{e_i}e_j, e_0\rangle A_{\nu_0}(e_0)\\
&\qquad - \sum_{k=1}^{m-1}\langle\nabla_{e_i}e_j, e_k\rangle A_{\nu_0}(e_k) - \sum_{k=1}^{m-1}\langle\nabla^\perp_{e_i}\nu_0, \nu_k\rangle A_{\nu_k}(e_j)\\
&\quad = e_j(\lambda)e_i + \lambda\nabla_{e_j}e_i - \langle\nabla_{e_j}e_i, e_0\rangle A_{\nu_0}(e_0)\\
&\qquad - \sum_{k=1}^{m-1}\langle\nabla_{e_j}e_i, e_k\rangle A_{\nu_0}(e_k) - \sum_{k=1}^{m-1}\langle\nabla^\perp_{e_j}\nu_0, \nu_k\rangle A_{\nu_k}(e_i).
\end{aligned}$$

From equation (3.9), we arrive at

$$\begin{aligned}
&e_i(\lambda)e_j + \lambda\nabla_{e_i}e_j - 3\lambda\langle\nabla_{e_i}e_j, e_0\rangle e_0\\
&\qquad -\lambda\sum_{k=1}^{m-1}\langle\nabla_{e_i}e_j, e_k\rangle e_k - \lambda\langle\nabla_{e_i}e_0, e_j\rangle e_0\\
&\quad = e_j(\lambda)e_i + \lambda\nabla_{e_j}e_i - 3\lambda\langle\nabla_{e_j}e_i, e_0\rangle e_0\\
&\qquad -\lambda\sum_{k=1}^{m-1}\langle\nabla_{e_j}e_i, e_k\rangle e_k - \lambda\langle\nabla_{e_j}e_0, e_i\rangle e_0.
\end{aligned}$$

Collecting the terms, it follows that

$$\begin{aligned}
&e_i(\lambda)e_j - 2\lambda\langle\nabla_{e_i}e_j, e_0\rangle e_0 + \lambda\langle\nabla_{e_i}e_j, e_0\rangle e_0\\
&\quad = e_j(\lambda)e_i - 2\lambda\langle\nabla_{e_j}e_i, e_0\rangle e_0 + \lambda\langle\nabla_{e_j}e_i, e_0\rangle e_0
\end{aligned}$$

and so

$$e_i(\lambda)e_j - e_j(\lambda)e_i = \lambda\langle[e_i, e_j], e_0\rangle e_0. \tag{3.10}$$

Multiplying (3.10) with e_0, we obtain $\langle[e_i, e_j], e_0\rangle = 0$. Thus,

$$[e_i, e_j] \in \text{span}\{e_1, \ldots, e_{m-1}\}.$$

Fix i in (3.10). Multiplying with e_j, $j \neq i$, we arrive at $e_i(\lambda) = 0$. Therefore

$$e_i(\lambda) = 0, \quad \text{for all } \ i \in \{1, \ldots, m-1\}.$$

Let us examine now the case $m = 2$. In this situation we have a local orthonormal frame

$$\{e_0, e_1; \nu_0 = JdF(e_0), \nu_1 = JdF(e_1)\}.$$

In this situation, the second fundamental form is given by

$$A(e_0, e_0) = 3\lambda\nu_0, \;\; A(e_0, e_1) = \lambda\nu_1 \;\text{ and }\; A(e_1, e_1) = \lambda\nu_0.$$

Since we are working on the set

$$U = \{x \in L : H(x) \neq 0\},$$

we have

$$\lambda \neq 0.$$

For simplicity, let us write the connection coefficients as

$$\nabla_{e_0} e_0 = ae_1, \;\; \nabla_{e_0} e_1 = -ae_0, \;\; \nabla_{e_1} e_0 = be_1 \;\text{ and }\; \nabla_{e_1} e_1 = -be_0.$$

Then,

$$\nabla_{e_0} \nu_0 = a\nu_1, \;\; \nabla_{e_0} \nu_1 = -a\nu_0, \;\; \nabla_{e_1} \nu_0 = b\nu_1 \;\text{ and }\; \nabla_{e_1} \nu_1 = -b\nu_0.$$

From the Codazzi equation

$$(\nabla^{\perp}_{e_0} A)(e_1, e_1) = (\nabla^{\perp}_{e_1} A)(e_0, e_1),$$

we get

$$e_0(\lambda) = b\lambda \quad \text{and} \quad e_1(\lambda) = 3a\lambda. \tag{3.11}$$

Similarly, from

$$(\nabla^{\perp}_{e_0} A)(e_1, e_0) = (\nabla^{\perp}_{e_1} A)(e_0, e_0),$$

we obtain

$$e_0(\lambda) = b\lambda \quad \text{and} \quad a\lambda = 3e_1(\lambda). \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$e_1(\lambda) = 3a\lambda \quad \text{and} \quad a\lambda = 3e_1(\lambda),$$

from where it follows that

$$e_1(\lambda) = 9e_1(\lambda).$$

Hence

$$8e_1(\lambda) = 0,$$

which proves claim II in the case $m = 2$.

This completes the proof of claim II. $\square$

Claim III: *The following facts hold true:*

1. *The integral curves of the vector field* e_0 *are geodesics, that is,*

$$\nabla_{e_0} e_0 = 0.$$

2. *For every* $i \in \{1, \dots, m-1\}$*, we have*

$$\nabla_{e_i} e_0 = e_0(\log \lambda)\, e_i.$$

Proof of claim III. From the Codazzi equation (1.9), we have

$$(\nabla_{e_0} A_{\nu_0})(e_i) - A_{\nabla^{\perp}_{e_0}\nu_0}(e_i) = (\nabla_{e_i} A_{\nu_0})(e_0) - A_{\nabla^{\perp}_{e_i}\nu_0}(e_0),$$

from where it follows that

$$\begin{aligned} e_0(\lambda)e_i &+ \lambda\nabla_{e_0}e_i - \langle\nabla_{e_0}e_i, e_0\rangle A_{\nu_0}(e_0) \\ &- \sum_{k=1}^{m-1}\langle\nabla_{e_0}e_i, e_k\rangle A_{\nu_0}(e_k) - \sum_{k=1}^{m-1}\langle\nabla^{\perp}_{e_0}\nu_0, \nu_k\rangle A_{\nu_k}(e_i) \\ =\ & 3e_i(\lambda)e_0 + 3\lambda\nabla_{e_i}e_0 - \langle\nabla_{e_i}e_0, e_0\rangle A_{\nu_0}(e_0) \\ &- \sum_{k=1}^{m-1}\langle\nabla_{e_i}e_0, e_k\rangle A_{\nu_0}(e_k) - \sum_{k=1}^{m-1}\langle\nabla^{\perp}_{e_i}\nu_0, \nu_k\rangle A_{\nu_k}(e_0). \end{aligned}$$

Using again (3.9) we arrive at

$$\begin{aligned} e_0(\lambda)e_i + \lambda\nabla_{e_0}e_i - 3\lambda\langle\nabla_{e_0}e_i, e_0\rangle e_0 - \lambda\sum_{k=1}^{m-1}\langle\nabla_{e_0}e_i, e_k\rangle e_k - \lambda\langle\nabla_{e_0}e_0, e_i\rangle e_0 \\ = 3e_i(\lambda)e_0 + 3\lambda\nabla_{e_i}e_0 - \lambda\sum_{k=1}^{m-1}\langle\nabla_{e_i}e_0, e_k\rangle e_k - \lambda\sum_{k=1}^{m-1}\langle\nabla_{e_i}e_0, e_k\rangle e_k. \end{aligned}$$

Consequently,

$$e_0(\lambda)e_i + \lambda\langle\nabla_{e_0}e_0, e_i\rangle e_0 = \lambda\nabla_{e_i}e_0.$$

Hence,

$$\nabla_{e_0}e_0 = 0 \quad \text{and} \quad \nabla_{e_i}e_0 = e_0(\log\lambda)e_i.$$

This completes the proof. □

Claim IV: *The following facts hold true:*

1. The function λ solves the ODE
$$e_0e_0(\lambda) + 2\lambda^3 = 0.$$

2. *There exists a constant α such that*
$$\lambda^4 + \big(e_0(\lambda)\big)^2 = \alpha^4.$$

Proof of the claim IV. From the Gauss equation (1.8), we have

$$\begin{aligned} R(e_0, e_i, e_0, e_i) &= R_{0i0i} = -\langle A(e_i, e_0), A(e_0, e_i)\rangle + \langle A(e_0, e_0), A(e_i, e_i)\rangle \\ &= -\langle \lambda\nu_i, \lambda\nu_i\rangle + \langle 3\lambda\nu_0, \lambda\nu_0\rangle \\ &= 2\lambda^2. \end{aligned} \tag{3.13}$$

On the other hand, using the fact that the integral curves of e_0 are geodesics, we get

$$\begin{aligned} -R(e_0, e_i, e_0, e_i) &= \langle R(e_0, e_i)e_0, e_i\rangle = \langle \nabla_{e_0}\nabla_{e_i}e_0 - \nabla_{\nabla_{e_0}e_i - \nabla_{e_i}e_0}e_0, e_i\rangle \\ &= \underbrace{\langle \nabla_{e_0}\nabla_{e_i}e_0, e_i\rangle}_{\mathcal{A}} \underbrace{- \nabla_{\nabla_{e_0}e_i}e_0, e_i\rangle}_{\mathcal{B}} + \underbrace{\langle \nabla_{\nabla_{e_i}e_0}e_0, e_i\rangle}_{\mathcal{C}}. \end{aligned}$$

Let us compute each term separately. We have

$$\begin{aligned} \mathcal{A} &= \langle \nabla_{e_0}\nabla_{e_i}e_0, e_i\rangle = \langle \nabla_{e_0}(e_0(\log(\lambda)e_i), e_i\rangle = e_0e_0(\log(\lambda) \\ &= \lambda^{-1}e_0e_0(\lambda) - \lambda^{-2}(e_0(\lambda))^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{B} &= -\langle \nabla_{\nabla_{e_0}e_i}e_0, e_i\rangle = -\sum_{j=1}^{m} \omega_{ij}(e_0)\langle \nabla_{e_j}e_0, e_i\rangle = -\sum_{j=1}^{m} \omega_{ij}(e_0)e_0(\log\lambda)\delta_{ij} \\ &= 0 \end{aligned}$$

and

$$\mathcal{C} = \langle \nabla_{\nabla_{e_i}e_0}e_0, e_i\rangle = e_0(\log\lambda)\langle \nabla_{e_i}e_0, e_i\rangle = (e_0(\log\lambda))^2 = \lambda^2(e_0(\lambda))^2.$$

Adding $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we get

$$-R(e_0, e_i, e_0, e_i) = \lambda^{-1}e_0e_0(\lambda). \tag{3.14}$$

Combing (3.13) and (3.14), we obtain

$$e_0e_0(H) = -\frac{2H^3}{(m+2)^2}.$$

Define the function

$$f = \lambda^4 + (e_0(\lambda))^2.$$

Since λ depends only on the e_0-direction, the function f also depends only on the e_0-direction. Differentiating with respect to e_0 we have

$$\begin{aligned} e_0(f) &= e_0(\lambda^4) + e_0((e_0(\lambda))^2) \\ &= 4\lambda^3 e_0(\lambda) + 2e_0(\lambda)e_0e_0(\lambda) \\ &= 2e_0(\lambda)(2\lambda^3 + e_0e_0(\lambda)) \\ &= 0. \end{aligned}$$

Hence,

$$e_0(f) = 0$$

and f is constant. Therefore, there exists a constant α such that

$$\lambda^4 + (e_0(\lambda))^2 = \alpha^4,$$

and this completes the proof. □

Remark 3.3.1. If the integral curves of e_0 are closed, then from the relation

$$\alpha^4 = \lambda^4 + (e_0(\lambda))^2$$

in Claim IV, it follows that either λ is constant or it must change sign.

Claim V: *Let Σ be a leaf of $D^\perp$. Then the following facts hold: Σ is totally umbilical, and its image under F, namely $F(\Sigma)$, is a round sphere of radius*

$$R = \frac{\|\lambda\|}{\alpha^2} = \frac{\|H\|}{\alpha^2(m+2)},$$

where α is the constant given in Claim IV.

Proof of claim V. We already proved that the distribution $D^\perp$ is integrable and e_0 is perpendicular along the integral hypersurfaces of $D^\perp$. Moreover, we proved that λ depends only in the e_0-direction and

$$\nabla_{e_i}e_0 = e_0(\log\lambda)e_i, \quad i \in \{1,\ldots,m-1\}.$$

The latter is just minus the shape operator of a leaf Σ of $D^{\perp}$ with respect to the normal direction of e_0. As a matter of fact the second fundamental form is given by

$$A_{\Sigma\subset L^m}(e_i, e_j) = -e_0(\log\lambda)e_0\delta_{ij}.$$

Let us detect the shape of $F(\Sigma)$. By the composition formula (1.6) the second fundamental form of $F(\Sigma)\subset\mathbb{C}^m$ is given by

$$\begin{aligned} A_{F(\Sigma)\subset\mathbb{C}^m}(e_i, e_j) &= A_{L\subset\mathbb{C}^m}(e_i, e_j) + A_{\Sigma\subset\mathbb{C}^m}(e_i, e_j) \\ &= \big(\lambda\nu_0 - e_0(\log\lambda\big)dF(e_0))\delta_{ij}. \end{aligned}$$

Note that Σ is regarded as a submanifold of L^m and trivial identifications are used. Because the shape operator of $F(\Sigma)$ is diagonal, from a classical result of submanifold theory we deduce that $F(\Sigma)$ is a sphere. The mean curvature vector of $F(\Sigma)$ is

$$H_\Sigma = (m-1)\big(\lambda\nu_0 - e_0(\log\lambda)dF(e_0)\big).$$

In particular, $F(\Sigma)$ belongs to the hyperplane spanned by the vectors

$$\{dF(e_1), dF(e_2), \dots, dF(e_{m-1}), H_\Sigma\}.$$

Moreover,

$$\|H_\Sigma\|^2 = (m-1)^2\Big(\lambda^2 + \frac{(e_0(\lambda))^2}{\lambda^2}\Big) = (m-1)^2\frac{\alpha^4}{\lambda^2}.$$

Thus, $F(\Sigma)$ is a sphere of radius

$$R = \frac{\|\lambda\|}{\alpha^2} = \frac{\|\,H\|}{\alpha^2(m+2)}.$$

This completes the proof. $\square$

Claim VI: *The image of L^m is a Whitney sphere.*

Proof of claim VI. We will use the Gauss formula to recover the immersion. Let us consider a spherical coordinate system $\{s, x^1, \dots, x^{m-1}\}$ on $\mathbb{R}\times\mathbb{S}^{m-1}$. From the previous computations, we have

$$\nabla_{\partial_s}\partial_s = 0 \quad \text{and} \quad \nabla_{\partial_j}\partial_s = e_0(\log(\lambda))\partial_j = (\lambda'/\lambda)\partial_{j.}$$

Since our submanifold is within $\mathbb{C}^m$, we may regard the complex structure of $\mathbb{C}^m$ as multiplication with i. Note that

$$\begin{aligned} F_{ss} &= D_{\partial_s} dF(\partial_s) = A(\partial_s, \partial_s) + dF(\nabla_{\partial_s}\partial_s) \\ &= A(e_0, e_0) = 3\lambda\nu_0 = 3\lambda J dF(e_0) = 3\lambda J F_s \\ &= 3\lambda i F_s. \end{aligned} \tag{3.15}$$

By integration, we deduce that

$$F(s, x^1, \ldots, x^{m-1}) = A(x^1, \ldots, x^{m-1}) \int_0^s e^{3i\int_0^t \lambda(x)\,dx}\,dt + B(x^1, \ldots, x^{m-1})$$

where A and B are $\mathbb{C}^m$-valued smooth functions that we have to determine. Now observe that

$$\begin{aligned} F_{x^j s} &= D_{\partial_j} dF(\partial_s) = A(\partial_j, \partial_s) + dF(\nabla_{\partial_j}\partial_s) \\ &= \lambda J dF(\partial_j) + e_0(\log\lambda) dF(\partial_j) \\ &= (i\lambda + \mu) F_{x^j}. \end{aligned} \tag{3.16}$$

Setting

$$\theta(s) = \int_0^s e^{3i\int_0^t \lambda(x)dx} dt$$

and combining (3.15) and (3.16), we get

$$(A\theta')_{x^j} = (i\lambda + \mu)\big(A_{x^j}\theta + B_{x^j}\big)$$

and so

$$A_{x^j}\theta' = (i\lambda + \mu)\big(A_{x^j}\theta + B_{x^j}\big).$$

Hence,

$$(\theta' - (i\lambda + \mu)\theta) A_{x^j} = (i\lambda + \mu) B_{x^j}.$$

Consequently,

$$B_{x_j} = \frac{\theta' - (i\lambda + \mu)\theta}{i\lambda + \mu} A_{x_j}. \tag{3.17}$$

Since A and B are independent of s, we deduce that

$$\frac{\theta' - (i\lambda + \mu)\theta}{i\lambda + \mu} = c_1 = \text{constant}.$$

By integrating (3.17) we get that there is a constant vector $C \in \mathbb{C}^m$ such that

$$B = c_1 A + C. \tag{3.18}$$

Therefore,

$$F(s, x^1, \ldots, x^{m-1}) = (c_1 + \theta(s))A(x^1, \ldots, x^{m-1}). \tag{3.19}$$

From Claim V, for $s = s_0 =$ constant, the image $F(\{s_0\} \times \mathbb{S}^{m-1})$ is a sphere of radius $R(s_0)$. It follows that the vector-valued function A has constant norm equal to 1. In fact, A represents the position vector of $\mathbb{S}^{m-1}$. In particular, the immersion F can be written in the form

$$\begin{aligned} F(s,x) &= z(s)f(x) = (u(x) + iv(s))f(x) \\ &= u(s)f(x) + iv(s)f(x). \end{aligned} \tag{3.20}$$

where F is the standard embedding of the sphere $\mathbb{S}^{m-1}$. Recall from the previous section that an immersion of the form (3.20) satisfies our pinching condition only if the curvature satisfy

$$k = -3p. \tag{3.21}$$

Planar curves satisfying (3.21) are uniquely determined up to rigid motions. In particular, a non-trivial curve

$$z = u + iv$$

satisfying (3.21), must coincide with the profile curve of the Whitney sphere. Indeed, let $z : I \subset \mathbb{R} \to \mathbb{C}$ be a connected branch of the profile curve away from the origin, parametrized by arc-length. Let us represent the curve in polar coordinates as $z = re^{i\theta}$. Moreover, let α be the tangent angle of the curve, that is, the function such that $z_s = e^{i\alpha}$. With our convention $\xi = J(z_s)$, we have

$$p = -\theta_s \quad \text{and} \quad k = \alpha_s.$$

Therefore the equation $k + 3p = 0$ becomes $\alpha_s - 3\theta_s = 0$. Thus

$$\alpha = 3\theta + \theta_0$$

for some constant $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$. Since

$$z_s = e^{i\alpha} \quad \text{and} \quad z_s = (r_s + ir\theta_s)e^{i\theta},$$

we obtain

$$r_s + ir\theta_s = e^{i(\alpha - \theta)} = e^{i(2\theta + \theta_0)}.$$

Taking real and imaginary parts gives

$$r_s = \cos(2\theta + \theta_0) \quad \text{and} \quad r\theta_s = \sin(2\theta + \theta_0).$$

On any interval where $\theta_s \neq 0$, we therefore have

$$\frac{dr}{d\theta} = \frac{r_s}{\theta_s} = r\cot(2\theta + \theta_0),$$

from where we obtain that

$$\frac{1}{r}dr = \cot(2\theta + \theta_0)\, d\theta.$$

Integrating gives

$$\log r = \frac{1}{2}\log|\sin(2\theta + \theta_0)| + \log C,$$

where $c > 0$ is a constant. Therefore

$$r^2 = c^2|\sin(2\theta + \theta_0)|.$$

Note that on a connected branch away from the origin, the function $\sin(2\theta + \theta_0)$ has constant sign. Replacing θ_0 by $\theta_0 + \pi$, if necessary, we may write

$$r^2 = c^2\sin(2\theta + \theta_0).$$

Finally, rotating the plane by an angle φ replaces θ by $\theta+\varphi$, and therefore replaces θ_0 by $\theta_0 - 2\varphi$. Choosing φ appropriately, we may assume $\theta_0 = \pi/2$. Thus, up to rotation and homothety, we get $r^2 = \cos(2\theta)$. In Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$x^2 + y^2 = r^2 \quad \text{and} \quad x^2 - y^2 = r^2\cos(2\theta).$$

Since $r^2 = \cos(2\theta)$, it follows that

$$(x^2 + y^2)^2 = x^2 - y^2.$$

This is the Bernoulli lemniscate. Consequently, up to rotations and homotheties of the plane, the profile curve is the Whitney profile curve. By analyticity, F must therefore be the Whitney sphere. $\square$

3.4 Catenoids and tamed Lagrangians

This section is devoted to the classification of minimal equivariant Lagrangians, as well as to the classification of solutions with positive mean curvature vector.

3.4.1 Minimal Lagrangian catenoids

In this paragraph, we classify all minimal equivariant Lagrangian submanifolds. The following result is due to Lawson; see for example [24].

Theorem 3.4.1. *Let $L^m \subset \mathbb{C}^m$ be an equivariant Lagrangian submanifold generated by the profile curve $z = (u, v)$. Then L^m is minimal if and only if the profile curve, in polar coordinates, has the form*

$$r^m(\theta)\cos(c_1 + m\theta) = c_2,$$

where c_1 and c_2 are real constants. In particular, after a rigid motion, the curve z satisfies

$$\mathrm{Im}(z^m) = c,$$

where c is a constant.

Proof. We represent the curve in polar coordinates as

$$z(\theta) = r(\theta)(\cos(\theta), \sin(\theta)), \quad \text{where } \theta \in (-\varepsilon, \varepsilon).$$

From elementary computations, we obtain

$$k = \frac{-r\, r_{\theta\theta} + 2r_\theta^2 + r^2}{(r^2 + r_\theta^2)^{3/2}} \quad \text{and} \quad p = -\frac{1}{\sqrt{r^2 + r_\theta^2}}.$$

Therefore, L^m is minimal if and only if the function r satisfies the ODE

$$r\, r_{\theta\theta} - (m+1)r_\theta^2 - mr^2 = 0. \tag{3.22}$$

Set $r_\theta = w(r)$. where w is a function to be determined later. Substituting in (3.22), we get

$$rw\frac{dw}{dr} - (m+1)w^2 - mr^2 = 0. \tag{3.23}$$

Equivalently, we may write (3.23) in the form

$$\frac{dw}{dr} = \frac{(m+1)w^2 + mr^2}{rw} = \frac{(m+1)(w/r)^2 + m}{(w/r)}.$$

Setting $w(r) = r\theta(r)$ we find that θ satisfies the first order ODE

$$\frac{\theta d\theta}{\theta^2 + 1} = \frac{m}{r} dr.$$

By integrating we deduce $\theta^2 + 1 = cr^{2m}$ where c is a positive constant. Therefore, the distance function r satisfies

$$r_\theta = r\sqrt{cr^{2m} - 1}.$$

Integrating once more, we obtain the desired result. Variations of c_1 correspond to a rotation of the curve. By choosing c_1 appropriately, we may write the solution in the form $\mathrm{Im}(z^m) = c$. This completes the proof. $\square$

Definition 3.4.2. *The minimal Lagrangians obtained in Theorem 3.4.1 are called* ***Lagrangian catenoids****.*

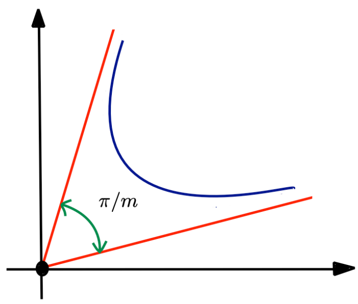


Figure 3.4: Lagrangian catenoid.

3.4.2 Convexity and tamed Lagrangians

Let $F\colon \mathbb{S}^1 \times \mathbb{S}^{m-1} \to \mathbb{C}^m - \{0\}$ is an equivariant Lagrangian submanifold generated by a closed profile curve $z\colon \mathbb{S}^1 \to \mathbb{C} - \{0\}$. Recall that the mean curvature of F is determined by the behavior of the function

$$h = k - (m-1)p.$$

Definition 3.4.3. *We say that the profile curve z is* ***tamed*** *if $h > 0$.*

If the curve z is tamed, then the equivariant Lagrangian submanifold generated by z has nowhere vanishing mean curvature vector. In the next lemma, we describe how to construct such Lagrangian submanifolds.

Lemma 3.4.4. *Let $\gamma\colon \mathbb{S}^1 \to \mathbb{C} - \{0\}$ be a regular curve with curvature k_γ. Consider the map $w\colon \mathbb{C} \to \mathbb{C}$ defined by*

$$w(z) = z^m,$$

and denote by $\widetilde{\gamma}$ the image of γ under w. Then:

1. *The support function β of the curve $\widetilde{\gamma}$ is given by*

$$\beta = -r^m \varrho.$$

2. *The curvature $k_{\widetilde{\gamma}}$ of the curve $\widetilde{\gamma}$ is given by*

$$k_{\widetilde{\gamma}} = \frac{k_\gamma - (m-1)p}{m\|\gamma\|^{m-1}} = \frac{h}{m\|\gamma\|^{m-1}}.$$

Proof. Since $\gamma \subset \mathbb{C} - \{0\}$ is a smooth regular curve with curvature k_γ, we may write γ in polar coordinates as

$$\gamma(\theta) = r(\theta)e^{i\theta} = r(\theta)(\cos\theta + i\sin\theta).$$

Applying the holomorphic map w, we obtain

$$\widetilde{\gamma}(\theta) = u(\theta) + iv(\theta) = r^m(\theta)\{\cos(m\theta) + i\sin(m\theta)\}.$$

1. We compute

$$\begin{aligned} u_\theta &= mr^{m-1}r_\theta\cos(m\theta) - mr^m\sin(m\theta),\\ v_\theta &= mr^{m-1}r_\theta\sin(m\theta) + mr^m\cos(m\theta). \end{aligned}$$

Hence,

$$u_\theta^2 + v_\theta^2 = m^2r^{2m-2}(r_\theta^2 + r^2).$$

Consequently, the inward-pointing unit normal of $\widetilde{\gamma}$ is given by

$$\xi_{\widetilde{\gamma}} = \frac{(-v_\theta, u_\theta)}{mr^{m-1}\sqrt{r_\theta^2 + r^2}}.$$

It follows that

$$\beta = \langle \widetilde{\gamma}, \xi_{\widetilde{\gamma}} \rangle = \frac{-r^{m+1}}{\sqrt{r_\theta^2 + r^2}}.$$

Hence,

$$\beta = -r^{m+1}p = -r^m \varrho.$$

2. Let us now derive the curvature of $\widetilde{\gamma}$. We compute

$$\begin{aligned} u_{\theta\theta} &= mr^{m-2}(rr_{\theta\theta} + (m-1)r_\theta^2 - mr^2)\cos(m\theta) - 2m^2 r^{m-1} r_\theta \sin(m\theta), \\ v_{\theta\theta} &= 2m^2 r^{m-1} r_\theta \cos(m\theta) + mr^{m-2}(rr_{\theta\theta} + (m-1)r_\theta^2 - m^2 r^2)\sin(m\theta). \end{aligned}$$

Therefore,

$$u_\theta v_{\theta\theta} - v_\theta u_{\theta\theta} = m^2 r^{2m-2}(-rr_{\theta\theta} + 2r_\theta^2 + r^2 + (m-1)(r_\theta^2 + r^2)).$$

and

$$\begin{aligned} k_{\widetilde{\gamma}} &= \frac{u_\theta v_{\theta\theta} - u_{\theta\theta} v_\theta}{(u_\theta^2 + v_\theta^2)^{\frac{3}{2}}} \\ &= \frac{-rr_{\theta\theta} + 2r_\theta^2 + r^2 + (m-1)(r_\theta^2 + r^2)}{mr^{m-1}(r_\theta^2 + r^2)^{\frac{3}{2}}} \\ &= \frac{k_\gamma - (m-1)p}{mr^{m-1}}, \end{aligned}$$

which completes the proof. □

Remark 3.4.1. We make a few remarks about the previous theorem.

1. From Lemma 3.4.4 we see that there is a duality between tamed and convex curves.

2. Tamed curves can be viewed as convex curves in $\mathbb{C} - \{0\}$ when the latter is equipped with the conical metric

$$g = \|z\|^{2m-2}\|dz\|^2.$$

3. There is a beautiful formula in classical differential geometry which states the following. Let $w\colon \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ be a holomorphic function with $w_z \neq 0$, and let $\gamma\colon \mathbb{S}^1 \to \mathbb{C} - \{0\}$ be a regular curve. Then

$$k_{w(\gamma)} = \frac{1}{\|w_z(\gamma)\|}\left\{k_\gamma + \operatorname{Im}\left(\frac{w_{zz}(\gamma)}{w_z(\gamma)}\gamma'\right)\right\}. \tag{3.24}$$

The next lemma will be of crucial importance.

Lemma 3.4.5. *Let $\gamma\colon \mathbb{S}^1 \to \mathbb{C} - \{0\}$ be a regular, embedded tamed curve, not containing the origin. Then its image lies within the profile curve of a Lagrangian catenoid.*

Proof. From Lemma 3.4.4 the holomorphic function $w(z) = z^m$ will map γ in a closed convex curve $\widetilde{\gamma}$. The function w is 1-1 on sectors of $\mathbb{C}$ centered at the origin 0 and opening angle $2\pi/m$. As a matter of fact, w is 1-1 on any sector of the form

$$S_\alpha = \big\{z \in \mathbb{C} - \{0\} \colon \alpha < \arg(z) < \alpha + 2\pi/m\big\}.$$

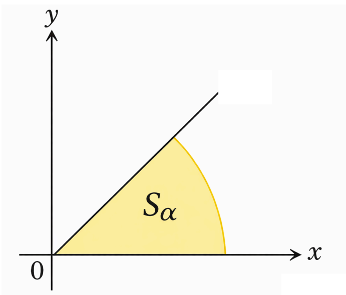


Figure 3.5: Sector S_α.

A priori, the curve $\widetilde{\gamma}$ may have self-intersections. We will exclude this possibility. We give a proof in three steps:

Step 1: At first we show that the winding number is zero. We have

$$\begin{aligned}
\mathrm{wind}_0(\widetilde{\gamma}) &= \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\widetilde{\gamma}'(t)}{\widetilde{\gamma}(t)}\,dt \\
&= \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{m\gamma^{m-1}(t)\gamma'(t)}{\gamma^m(t)}\,dt \\
&= \frac{m}{2\pi i}\int_{\mathbb{S}^1} \frac{\gamma'(t)}{\gamma(t)}\,dt.
\end{aligned}$$

Hence,

$$\mathrm{wind}_0(\widetilde{\gamma}) = m\,\mathrm{wind}_0(\gamma).$$

Because γ is embedded and does not contain the origin in its interior, we get that

$$\mathrm{wind}_0(\widetilde{\gamma}) = 0.$$

Step 2: We claim that
$$\mathrm{rot}(\widetilde{\gamma}) = \pm 1.$$
Parametrize γ by arc-length $s \in [0, L]$. Then, we may write
$$\gamma_s = e^{i\alpha},$$
where α is a continuous choice of tangent angle. The total change of the tangent angle is denoted by
$$\delta\alpha \doteq \alpha(L) - \alpha(0).$$
Hence
$$\mathrm{rot}(\gamma) \doteq \frac{\delta\alpha}{2\pi}.$$
Since the curve γ is closed and embedded, the turning number theorem gives
$$\mathrm{rot}(\gamma) = \pm 1.$$
Thus
$$\delta\alpha = \pm 2\pi.$$
Moreover, because $0 \notin \gamma$, we may write
$$\gamma = re^{i\theta},$$
where here θ is a continuous choice of argument of the position vector γ. Thus, by assumption,
$$0 = \mathrm{wind}_0(\gamma) = \frac{\delta\theta}{2\pi} = \frac{\theta(L) - \theta(0)}{2\pi},$$
Differentiating we get that
$$\widetilde{\gamma}_s = mr^{m-1}e^{i(m-1)\theta}e^{i\alpha} = mr^{m-1}e^{i\{\alpha+(m-1)\theta\}}.$$
Therefore a tangent angle $\widetilde{\alpha}$ of $\widetilde{\gamma}$ is given by
$$\widetilde{\alpha} = \alpha + (m-1)\theta,$$
from where it follows that
$$\delta\widetilde{\alpha} = \delta\alpha + (m-1)\delta\theta = \pm 2\pi.$$
Consequently,
$$\mathrm{rot}(\gamma^m) = \pm 1.$$

Step 3: We will prove now that the curve γ^m is embedded. Assume now that s is the arc-length parameter of $\widetilde{\gamma}$. Since

$$\widetilde{\alpha}_s = k_{\widetilde{\gamma}} > 0,$$

it follows that the tangent angle α is strictly increasing. Without loss of generality let us assume that

$$\text{rot}(\widetilde{\gamma}) = 1,$$

or, equivalently, that

$$\widetilde{\alpha}(L) - \widetilde{\alpha}(0) = 2\pi.$$

Since the function α is strictly increasing and increases by exactly 2π, every tangent direction occurs exactly once. Suppose to the contrary that the curve γ^m has a self-intersection. Then there exist $0 \leq s_1 < s_2 < L$ such that

$$\widetilde{\gamma}(s_1) = \widetilde{\gamma}(s_2).$$

The restriction of $\widetilde{\gamma}$ to $[s_1, s_2]$ is a closed immersed loop. Since $k_{\widetilde{\gamma}} > 0$, its tangent angle is strictly increasing along this loop. Hence the total turning of this loop is a positive integer multiple of 2π. In particular,

$$\alpha(s_2) - \alpha(s_1) \geq 2\pi.$$

But since α is strictly increasing on $[0, L]$ and

$$\alpha(L) - \alpha(0) = 2\pi,$$

we must have

$$0 < \alpha(s_2) - \alpha(s_1) < 2\pi,$$

a contradiction. Therefore $\widetilde{\gamma}$ has no self-intersections. Hence $\widetilde{\gamma}$ is embedded.

Due to the fact that $k_{\widetilde{\gamma}} > 0$, the tangent angle turns monotonically. Hence the image is the boundary of a strictly convex domain. Consequently, there exists a half-space $\mathcal{H}$ such that $0 \notin \mathcal{H}$ and $\widetilde{\gamma}(\mathbb{S}^1) \subset \mathcal{H}$. Denote by $\ell = \partial\mathcal{H}$. We now consider the inverse transformation

$$w^{-1}(z) = z^{1/m}.$$

Since the domain $\mathcal{H}$ is simply connected, we can choose a branch $\text{Log}_{\widetilde{\ell}}$ of the complex logarithm on $\mathcal{H}$, defined by removing a ray $\widetilde{\ell}$ parallel to the boundary ℓ of $\mathcal{H}$. We then define

$$w^{-1}(z) = \exp\left(\frac{1}{m}\,\text{Log}_{\widetilde{\ell}}(z)\right).$$

If we write ℓ in the form

$$\ell = \{z \in \mathbb{C} \colon a \ \mathrm{Re}(z) + b \ \mathrm{Im}(z) = c\},$$

with $(a, b) \neq (0, 0)$ and $c \in \mathbb{R}$, we easily see that

$$L = w^{-1}(\ell) = \{z \in \mathbb{C} \colon a \ \mathrm{Re}(z^m) + b \ \mathrm{Im}(z^m) = c\}.$$

By Theorem 3.4.1, L is a Lagrangian catenoid. The curve L separates the plane

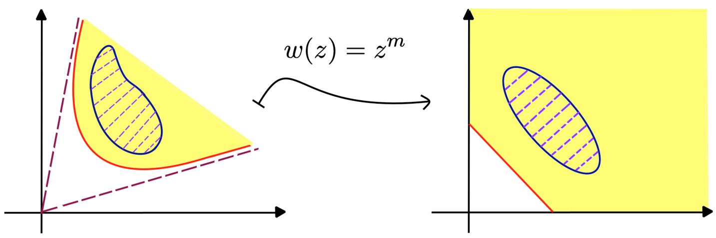


Figure 3.6: Tamed vs convex.

into two regions: let A be the region containing 0, and let B be its complement. By continuity, we have $w^{-1}(\mathcal{H}) \subset B$. Consequently,

$$w^{-1}(\tilde{\gamma}(\mathbb{S}^1)) = \gamma(\mathbb{S}^1) \subset B,$$

which completes the proof. □

Remark 3.4.2. Recall that the profile curve of a Lagrangian catenoid lies in a sector of opening angle π/m. In this sector, the map $w(z) = z^m$ is 1-1.

Remark 3.4.3. Consider the circle

$$C = \{z \in \mathbb{C} \colon \|z - z_0\|^2 = R^2\}, \quad \text{with} \quad \|z_0\|^2 > R^2.$$

Then the circle does not contain the origin in its interior. In particular, it lies in a half-space that does not contain the origin. Consider the curve $\gamma = w(C)$, where

$$w(z) = z^{1/m}.$$

We parametrize C by

$$z(\theta) = z_0 + Re^{i\theta}.$$

According to the formula in Remark 3.4.1, we have

$$
\begin{aligned}
k_\gamma &= \frac{\|z\|^{\frac{m-1}{m}}}{R}\left\{1+(m-1)\,\mathrm{Re}\left(\frac{z_0}{z}\right)\right\}\\
&= \frac{\|z\|^{\frac{m-1}{m}}}{R}\left\{1+(m-1)\,\mathrm{Re}\left(\frac{z_0\bar{z}}{\|z\|^2}\right)\right\}\\
&= \frac{\|z\|^{\frac{m-1}{m}}}{R}\left\{1+(m-1)\frac{\|z_0\|(\|z_0\|-R)}{\|z\|^2}\right\}\\
&> 0.
\end{aligned}
$$

Hence, γ is convex and tamed.

CHAPTER 4

Mean Curvature Flow

This chapter is devoted to the analytical foundations of mean curvature flow, including short-time existence and regularity. Moreover, we present key tools such as the parabolic maximum principle and derive the evolution equations for the relevant geometric quantities. This framework will be used to analyze the formation of singularities, including the characterization of the maximal time of existence and techniques for modeling singular behavior. The material presented here is largely based on the manuscripts [3, 5, 15, 42, 44, 48].

4.1 Short-time existence and regularity

4.1.1 Basic facts

Suppose that M is a smooth manifold of dimension m, let $T > 0$, and let $F\colon M\times[0,T)\to N$ be a smooth one-parameter family of immersions of M into a Riemannian manifold (N, g_N) of dimension n. We say that the one-parameter family of immersions $F\colon M\times[0,T)\to N$ evolves by ***mean curvature flow*** (MCF) with initial data the immersion $F_0\colon M\to N$ if it satisfies the initial value problem

$$\begin{cases} dF_{(x,t)}(\partial_t) = H_{F(x,t)}, & (x,t)\in M\times[0,T),\\ F(x,0) = F_0(x), & x\in M, \end{cases} \tag{MCF}$$

where $H_{F(x,t)}$ denotes the mean curvature vector of the immersion

$$F_t = F(\,\cdot\,,t)\colon M\to N, \quad x\mapsto F_t(x) = F(x,t),$$

at the point x.

Following Hadamard, given an initial immersion $F_0\colon M \to N$, one is interested in the following problems.

Problem I: *Does there exist a number $T > 0$ and a family of immersions $F\colon M \times [0,T) \to N$ solving the initial value problem* (MCF) *on $M \times [0,T)$?*

Problem II: *Is the solution to the initial value problem* (MCF) *unique?*

Problem III: *If the initial data $F_0\colon M \to N$ is smooth, is the corresponding solution F smooth in space-time? Moreover, is the solution real-analytic?*

4.1.2 First properties of MCF

Lemma 4.1.1. *Let $F\colon M \times [0,T) \to N$ be a smooth solution to* (MCF)*, and let Φ be an isometry of the ambient manifold (N, g_N). Then the family of immersions*

$$\widetilde{F}\colon M \times [0,T) \to N$$

defined by

$$\widetilde{F} = \Phi \circ F$$

is also a solution to (MCF).

Proof. Since Φ is an isometry, the metric induced by $\widetilde{F}_t$ coincides with the metric induced by F_t. Indeed,

$$\begin{aligned}(\widetilde{F}_t^* g_N)(X,Y) &= g_N(d\widetilde{F}_t(X), d\widetilde{F}_t(Y)) \\ &= g_N(d\Phi(dF_t(X)), d\Phi(dF_t(Y))) \\ &= g_N(dF_t(X), dF_t(Y)) \\ &= (F_t^* g_N)(X,Y),\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Let us compute the second fundamental form of $\widetilde{F}$. For every $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}A_{\widetilde{F}_t}(X,Y) &= \nabla^N_{d\widetilde{F}_t(X)} d\widetilde{F}_t(Y) - d\widetilde{F}_t(\nabla_X Y) \\ &= \nabla^N_{d\Phi(dF_t(X))} d\Phi(dF_t(Y)) - d\Phi(dF_t(\nabla_X Y)) \\ &= d\Phi(\nabla^N_{dF_t(X)} dF_t(Y) - dF_t(\nabla_X Y)) \\ &= d\Phi(A_{F_t}(X,Y)).\end{aligned}$$

Taking the trace, we obtain

$$H_{\widetilde{F}_t} = \operatorname{tr} A_{\widetilde{F}_t} = \operatorname{tr}(d\Phi \circ A_{F_t}) = d\Phi(H_{F_t}).$$

On the other hand,

$$d\widetilde{F}(\partial_t) = d\Phi(dF(\partial_t)) = d\Phi(H_{F_t}) = H_{\widetilde{F}_t}.$$

This completes the proof. □

Lemma 4.1.2 (Geometric invariance under tangential perturbations)**.** *Suppose*

$$F\colon M \times [0,T) \to N$$

is a smooth family of immersions, where M is compact. Assume that the map F satisfies

$$\begin{cases} dF(\partial_t) = H_F + dF(X), \\ F(\,\cdot\,,0) = F_0, \end{cases} \tag{4.1}$$

where X is a time-dependent vector field on $M \times [0,T)$. Then there exists a smooth time-dependent family of reparametrizations

$$\widehat{F}\colon M \times [0,T) \to N$$

of F such that

$$\begin{cases} d\widehat{F}(\partial_t) = H_{\widehat{F}}, \\ \widehat{F}(\,\cdot\,,0) = F_0. \end{cases} \tag{4.2}$$

In particular, the systems (4.1) *and* (4.2) *are equivalent up to time-dependent reparametrization.*

Proof. Let

$$\psi\colon M \times [0,T) \to M, \qquad \psi(x,t) = \psi_t(x),$$

be a one-parameter family of diffeomorphisms, to be determined later. Consider the family of immersions $\widehat{F}\colon M \times [0,T) \to N$ defined by

$$\widehat{F}(x,t) = F(\psi(x,t),t),$$

where F is a solution of (4.1). We will choose ψ suitably so that $\widehat{F}$ becomes a solution of (4.2).

Fix a point (x,t) in space time. From the chain rule, we have

$$\begin{aligned} d\widehat{F}_{(x,t)}(\partial_t) &= dF_{(\psi(x,t),t)}(d\psi_{(x,t)}(\partial_t)) + dF_{(\psi(x,t),t)}(\partial_t) \\ &= dF_{(\psi(x,t),t)}(d\psi_{(x,t)}(\partial_t)) + H_{F(\psi(x,t),t)} + dF(X_{(\psi(x,t),t)}). \end{aligned}$$

Therefore,

$$d\widehat{F}_{(x,t)}(\partial_t) = H_{F(\psi(x,t),t)} + dF_{(\psi(x,t),t)}(d\psi_{(x,t)}(\partial_t) + X_{(\psi(x,t),t)}). \tag{4.3}$$

Claim I: *The mean curvature vectors of the immersions F and $\widehat{F}$ are related by the formula*

$$H_{F(\psi(x,t),t)} = H_{\widehat{F}(x,t)}.$$

Proof of the Claim I. First, note that for any $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned} g_{\widehat{F}_t}(X,Y) &= g_N(d\widehat{F}_t(X), d\widehat{F}_t(Y)) \\ &= g_N(dF_t(d\psi_t(X)), dF_t(d\psi_t(Y))) \\ &= g_{F_t}(d\psi_t(X), d\psi_t(Y)) \\ &= (\psi_t^* g_{F_t})(X,Y). \end{aligned}$$

Therefore,

$$\psi_t \colon (M, g_{\widehat{F}_t}) \to (M, g_{F_t})$$

is an isometry. Consequently, the connections of the metrics $g_{\widehat{F}_t}$ and g_{F_t} are related by the formula

$$d\psi_t(\widehat{\nabla}_X Y) = \nabla_{d\psi_t(X)} d\psi_t(Y).$$

Now,

$$\begin{aligned} A_{\widehat{F}_t}(X,Y) &= \nabla^N_{d\widehat{F}_t(X)} d\widehat{F}_t(Y) - d\widehat{F}_t(\widehat{\nabla}_X Y) \\ &= \nabla^N_{dF_t(d\psi_t(X))} dF_t(d\psi_t(Y)) - dF_t(d\psi_t(\widehat{\nabla}_X Y)) \\ &= \nabla^N_{dF_t(d\psi_t(X))} dF_t(d\psi_t(Y)) - dF_t(\nabla_{d\psi_t(X)} d\psi_t(Y)) \\ &= A_{F_t}(d\psi_t(X), d\psi_t(Y)). \end{aligned}$$

Since the map ψ_t is an isometry, taking the trace yields the desired identity. This completes the proof of the Claim I. $\square$

Since M is compact and the vector field X is smooth, the classical Picard-Lindelöf theorem [26] guarantees the existence and uniqueness of a family of diffeomorphisms satisfying

$$\begin{cases} d\psi_{(x,t)}(\partial_t) = -X_{(\psi(x,t),t)}, \\ \psi(x,0) = x, \end{cases}$$

for all $(x,t) \in M \times [0,T)$. Combining this fact with Claim I and (4.3), we obtain the desired conclusion. □

Corollary 4.1.3. *Let M be a compact manifold, and let $F\colon M \times [0,T) \to N$ be a solution of*

$$\begin{cases} \{dF(\partial_t)\}^{\perp} = H_F, \\ F(\,\cdot\,,0) = F_0, \end{cases} \tag{MCF1}$$

where $\{\cdot\}^{\perp}$ denotes the orthogonal projection onto the normal bundle of the immersion F_t. Then F can be reparametrized into a solution of the standard mean curvature flow equation (MCF)*. More precisely, there exists a family of diffeomorphisms $\psi_t\colon M \to M$, $t \in [0,T)$, with $\psi_0 = \mathrm{Id}$, such that the family of immersions*

$$\widehat{F}\colon M \times [0,T) \to N, \qquad \widehat{F}(x,t) = F(\psi_t(x),t),$$

satisfies

$$\begin{cases} d\widehat{F}(\partial_t) = H_{\widehat{F}}, \\ \widehat{F}(\,\cdot\,,0) = F_0, \end{cases} \tag{4.4}$$

Proof. Equation (MCF1) can be rewritten as

$$H_F = \{dF(\partial_t)\}^{\perp} = dF(\partial_t) - \{dF(\partial_t)\}^{\top}. \tag{4.5}$$

Hence, there exists a time-dependent vector field X on M such that

$$\{dF(\partial_t)\}^{\top} = dF(X_{(x,t)}). \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$dF(\partial_t) = H_F + dF(X).$$

The conclusion now follows directly from Lemma 4.1.2. □

4.1.3 Examples of MCF

Example 4.1.4 (Spheres)**.** Let $f\colon \mathbb{S}^m \to \mathbb{R}^{m+1}$ denote the standard immersion of the sphere of radius $R_0 = 1$. We look for a solution of (MCF) of the form

$$F(x,t) = F_t(x) = R(t)f(x), \qquad t \in [0,T),$$

where $R(0) = R_0 = 1 > 0$. We compute

$$dF(\partial_t) = R' f. \tag{4.7}$$

On the other hand, the mean curvature vector is given by

$$H_F = -\frac{m}{R} f. \tag{4.8}$$

Combining (4.7) and (4.8), equation (MCF) gives

$$R' f = -\frac{m}{R} f.$$

Hence,

$$R' = -\frac{m}{R}.$$

Solving this ordinary differential equation, we obtain

$$R(t) = (R_0^2 - 2mt)^{1/2}, \qquad t \in [0, R_0^2/2m).$$

Consequently,

$$F_t = (R_0^2 - 2mt)^{1/2} f.$$

Thus, under mean curvature flow, round spheres contract homothetically to a point in finite time.

Example 4.1.5 (Cylinders)**.** Consider the cylinder $\mathbb{S}^m(R_0) \times \mathbb{R}^k \subset \mathbb{R}^{m+1} \times \mathbb{R}^k$. Denote by $\{e_1, \dots, e_{m+1}; v_1, \dots, v_k\}$ be the standard orthonormal basis of $\mathbb{R}^{m+1} \times \mathbb{R}^k$. The cylinder can be parametrized by the immersion

$$f(x, s_1, \dots, s_k) = R_0 x + \sum_{i=1}^{k} s_i v_i,$$

where here x denotes the position vector of the unit sphere in $\mathbb{R}^{m+1}$. We look for a solution of (MCF) of the form

$$F(x, s_1, \dots, s_k, t) = R(t)x + \sum_{i=1}^{k} s_i v_i.$$

We compute

$$dF(\partial_t) = R'x. \tag{4.9}$$

On the other hand, the mean curvature vector is

$$H_F = -\frac{m}{R}x. \tag{4.10}$$

Therefore, equation (MCF) is equivalent to the ordinary differential equation

$$R' = -\frac{m}{R}.$$

Solving this equation, we obtain

$$R(t) = \left(R_0^2 - 2mt\right)^{1/2}, \quad t \in [0, R_0^2/2m).$$

Hence, the solution is given by

$$F(x, s_1, \ldots, s_k, t) = \left(R_0^2 - 2mt\right)^{1/2} x + \sum_{i=1}^{k} s_i v_i.$$

Thus, under mean curvature flow, the spherical factor contracts homothetically to a point in finite time, while the Euclidean factor remains fixed; consequently, the cylinder collapses to the flat space $\mathbb{R}^k$.

Example 4.1.6 (Shrinkers-Expanders)**.** Suppose that $f\colon M^m \to \mathbb{R}^{m+k}$ is an immersion. Consider the family of immersions

$$F\colon M \times I \to \mathbb{R}^{m+k},$$

where $I \subseteq \mathbb{R}$ is an interval, defined by

$$F(x,t) = \lambda(t) f(x).$$

The question is to determine for which functions λ and immersions f the family F satisfies

$$\{dF(\partial_t)\}^{\perp} = H_F.$$

We have

$$dF(\partial_t) = \lambda' f. \tag{4.11}$$

Let us compute the second fundamental form of the immersion F_t. We begin with the computation of the induced metric. For any $X, Y \in \mathfrak{X}(M)$, we have

$$g_{F_t}(X,Y) = \langle dF_t(X), dF_t(Y)\rangle = \langle \lambda df(X), \lambda df(Y)\rangle = \lambda^2 g_f(X,Y).$$

Thus,

$$g_{F_t} = \lambda^2 g_f,$$

and therefore we must have

$$\lambda^2 > 0.$$

Let $\{e_1, \dots, e_m\}$ be an orthonormal frame with respect to g_f. Then

$$\left\{\widetilde{e}_1 = \frac{e_1}{\lambda}, \dots, \widetilde{e}_m = \frac{e_m}{\lambda}\right\}$$

is an orthonormal frame with respect to g_{F_t}. Moreover, if $\{\nu_1, \dots, \nu_k\}$ is an orthonormal frame of the normal bundle of f, then it is also an orthonormal frame of the normal bundle of F_t. We compute

$$\begin{aligned}
A_{F_t}(\widetilde{e}_i, \widetilde{e}_j) &= \sum_{l=1}^{k} \langle A_{F_t}(\widetilde{e}_i, \widetilde{e}_j), \nu_l\rangle \nu_l = \sum_{l=1}^{k} \langle D^{F_t}_{\widetilde{e}_i} dF_t(\widetilde{e}_j), \nu_l\rangle \nu_l \\
&= \lambda \sum_{l=1}^{k} \langle D^{f}_{\widetilde{e}_i} df(\widetilde{e}_j), \nu_l\rangle \nu_l = \lambda \sum_{l=1}^{k} \langle A_f(\widetilde{e}_i, \widetilde{e}_j), \nu_l\rangle \nu_l = \lambda A_f(\widetilde{e}_i, \widetilde{e}_j) \\
&= \frac{1}{\lambda} A_f(e_i, e_j).
\end{aligned}$$

Taking the trace, we obtain

$$H_{F_t} = \sum_{i=1}^{m} A_{F_t}(\widetilde{e}_i, \widetilde{e}_i) = \frac{1}{\lambda} \sum_{i=1}^{m} A_f(e_i, e_i) = \frac{1}{\lambda} H_f.$$

Consequently, F is a solution of

$$\{dF(\partial_t)\}^{\perp} = H_F$$

if and only if

$$\lambda' f^{\perp} = \frac{1}{\lambda} H_f. \tag{4.12}$$

If $f^{\perp}$ vanishes on an open neighbourhood, then $f(M)$ is flat and $H_f = 0$. This case is not interesting. Hence, we restrict ourselves to the case where

$$\|f^{\perp}\| \neq 0.$$

From (4.12) we obtain

$$\lambda(t)\lambda'(t) = \frac{\langle H_{f(x)}, f^{\perp}(x)\rangle}{\|f^{\perp}(x)\|^2} = \frac{\langle H_{f(x)}, f(x)\rangle}{\|f^{\perp}(x)\|^2}, \qquad (x,t) \in M \times I.$$

Therefore, there exists a constant $c \in \mathbb{R}$ such that

$$\lambda\lambda' \equiv c \quad \text{and} \quad H_f = cf^{\perp}.$$

Thus,

$$\lambda^2(t) = \lambda_0^2 + 2ct,$$

and the solution F_t is given by

$$F_t = \pm(\lambda_0^2 + 2ct)^{1/2} f, \tag{4.13}$$

while

$$H_f = cf^{\perp}. \tag{4.14}$$

Observe that:

- If $c > 0$, then for each fixed x we have

$$\lim_{t\to\infty} F_t(x) = \infty.$$

- If $c = 0$, then the solution F_t is minimal.
- If $c < 0$, then the maximal positive time of existence is

$$T = -\frac{\lambda_0^2}{2c}.$$

  Moreover, for each fixed x, we have

$$\lim_{t\to T} F_t(x) = 0.$$

Motivated by the above example, we give the following definition.

Definition 4.1.7. *Immersions $f\colon M^m \to \mathbb{R}^{m+k}$ satisfying*

$$H_f = cf^{\perp}$$

*where c is a non-zero constant, are called **self-similar solutions** of the* (MCF). *In particular,*

- *If $c < 0$ then f is called a **shrinker**.*
- *If $c > 0$ then f is called an **expander**.*

Example 4.1.8 (Graphs)**.** Let $u\colon \mathbb{R}^n \times [0,T) \to \mathbb{R}$ be a smooth function, and consider the family of graphs

$$\Gamma_t = \left\{(x, u(x,t)) \in \mathbb{R}^{n+1} \colon x \in \mathbb{R}^n\right\},$$

evolving by (MCF1). We derive the equation satisfied by u. The family of graphs Γ_t can be parametrized by the family of immersions

$$F\colon \mathbb{R}^n \times [0,T) \to \mathbb{R}^{n+1},$$

defined by

$$F(x,t) = (x, u(x,t)).$$

From Example 1.1.7, the scalar mean curvature is given by

$$h_F = \operatorname{div}\left(\frac{Du}{\sqrt{1+\|Du\|^2}}\right). \tag{4.15}$$

On the other hand,

$$dF(\partial_t) = (0, u_t).$$

The unit normal vector field along the graph F_t is

$$\xi = \frac{(-Du, 1)}{\sqrt{1+\|Du\|^2}}.$$

Therefore,

$$\{dF(\partial_t)\}^\perp = \langle dF(\partial_t), \xi\rangle \xi = \frac{u_t}{\sqrt{1+\|Du\|^2}}\xi.$$

Moreover,

$$H_F = h_F \xi = \operatorname{div}\left(\frac{Du}{\sqrt{1+\|Du\|^2}}\right)\xi.$$

Hence, u satisfies the equation

$$\frac{\partial u}{\partial t} = \sqrt{1+\|Du\|^2}\operatorname{div}\left(\frac{Du}{\sqrt{1+\|Du\|^2}}\right).$$

Let us examine this equation in the case $n = 1$. In this situation,

$$\frac{\partial u}{\partial t} = \frac{u_{xx}}{1+(u_x)^2} = (\arctan u_x)_x. \tag{4.16}$$

This is a nonlinear partial differential equation and, in general, it cannot be solved explicitly. Let us seek special solutions of the form

$$u(x,t) = v(x) + ct,$$

where c is a positive constant. Then equation (4.16) becomes

$$v_{xx} = c\big(1 + (v_x)^2\big),$$

or equivalently,

$$(v_x)_x = c\big(1 + (v_x)^2\big).$$

Integrating, we obtain

$$v_x = \tan(cx + c_1).$$

Consequently,

$$\frac{1}{c}\left(-c_1 - \frac{\pi}{2}\right) < x < \frac{1}{c}\left(\frac{\pi}{2} - c_1\right).$$

Integrating once more yields

$$v(x) = -\frac{1}{c}\log(\cos(cx + c_1)) + c_2,$$

where c_1 and c_2 are constants. Without loss of generality, we may assume that $c_1 = c_2 = 0$. Hence,

$$v(x) = -\frac{1}{c}\log(\cos(cx)).$$

Observe that, in this case, the graphs evolve purely by translation. The graph of v is known in the literature as the ***grim reaper curve***.

Example 4.1.9 (Translators). Let $f\colon M^m \to \mathbb{R}^{m+k}$ be an immersion, and let v be a unit vector in $\mathbb{R}^{m+k}$. A direct computation shows that the family

$$F\colon M^m \times \mathbb{R} \to \mathbb{R}^{m+k}, \qquad F(x,t) = f(x) + tv,$$

is a solution of (MCF1) if and only if the immersion f satisfies

$$H_f = v^\perp,$$

where $v^\perp$ denotes the normal projection of v onto the normal bundle of f. In this case, the family F evolves by translation in the direction of v.

Motivated by the above example we give the following definition.

Definition 4.1.10. *An immersion $f\colon M^m \to \mathbb{R}^{m+k}$ satisfying*

$$H_f = v^\perp$$

is called a ***translating soliton (translator)*** *of the mean curvature flow.*

4.1.4 Quasilinear parabolic system of PDEs

Let $\Omega \subseteq \mathbb{R}^m$ be an open bounded set, and let $T > 0$. Suppose that

$$u = (u_1, \dots, u_n) \colon \Omega \times [0, T) \to \mathbb{R}^n$$

is a smooth time-dependent map. We use throughout this subsection Latin indices i, j, k to denote quantities associated with $\Omega \subseteq \mathbb{R}^m$, and Greek indices α, β, γ to denote quantities associated with the target space $\mathbb{R}^n$. We say that the map u satisfies a ***quasilinear system*** of partial differential equations if each component u_α satisfies

$$\frac{\partial u^\alpha}{\partial t} - \sum_{\beta,i,j} A^{ij}_{\alpha\beta}(x, t, u, Du) u^\beta_{x_i x_j} = h^\alpha(x, t, u, Du), \tag{4.17}$$

where

$$A^{ij}_{\alpha\beta}, h^\alpha \in C^\infty(\Omega \times [0, T) \times \mathbb{R}^n \times \mathbb{R}^{nm}).$$

In geometric problems, the coefficients $A^{ij}_{\alpha\beta}$ and h^α are usually smooth. However, depending on the problem under consideration, these coefficients may only be continuous or even merely bounded.

We say that (4.17) is ***parabolic*** if, for every $\xi = (\xi_1, \dots, \xi_m)$ and $\eta = (\eta^1, \dots, \eta^n)$, the inequality

$$\lambda \|\xi\|^2 \|\eta\|^2 \leq \sum_{\alpha,\beta,i,j} A^{ij}_{\alpha\beta}(x, t, z, p)\, \xi_i \xi_j \eta^\alpha \eta^\beta \leq \Lambda \|\xi\|^2 \|\eta\|^2 \tag{4.18}$$

holds, where $\Lambda, \lambda \in \mathbb{R}$ satisfy

$$\Lambda \geq \lambda > 0.$$

Suppose now that $(M, g(t))$ is a Riemannian manifold endowed with a family of time-dependent Riemannian metrics, and let (N, g_N) be another Riemannian manifold. We say that a smooth map

$$u \colon M \times [0, T) \to N$$

satisfies a ***quasilinear system*** of partial differential equations if, in some local coordinate system (and hence in every local coordinate system), it satisfies a system of the form (4.17), whose coefficients satisfy the parabolicity condition (4.18).

According to a classical result in the theory of partial differential equations, the following theorem holds; for more details we refer to [2,5,21,28]

Theorem 4.1.11. *Suppose that* $(M, g(t))$ *is a compact manifold equipped with a family of Riemannian metrics, and let* (N, g_N) *be a complete Riemannian manifold. Given a smooth map*

$$u_0 \colon (M, g(0)) \to (N, g_N),$$

the quasilinear parabolic system

$$\begin{cases} \dfrac{\partial u^\alpha}{\partial t} - \displaystyle\sum_{\beta,i,j} A^{ij}_{\alpha\beta}(x,t,u,Du)\, u^\beta_{x_i x_j} = h^\alpha(x,t,u,Du), \\ u^\alpha(x,0) = u^\alpha_0, \end{cases} \tag{4.19}$$

where $\alpha \in \{1, \dots, n\}$*, admits a smooth solution on a maximal time interval* $[0, T)$*. Moreover, if the metrics* $g(t)$ *and* g_N *are real analytic, then the solution is real analytic on* $M \times (0, T)$*.*

4.1.5 Short time existence of MCF

Let us express (MCF) in local coordinates. Suppose that $F \colon M^m \times [0, T) \to N^n$ is a solution of (MCF). With respect to local charts ϕ and ψ, the map F admits the representation

$$\begin{array}{ccc} M \times [0,T] & \xrightarrow{F} & N \\ \big\downarrow \phi & & \big\downarrow \psi \\ \mathbb{R}^m \times [0,T] & \xrightarrow{\widetilde{F}} & \mathbb{R}^n \end{array}$$

The map

$$\widetilde{F} = \psi \circ F \circ \phi^{-1}$$

can be written in the form

$$\widetilde{F}(x^1, \dots, x^m, t) = \big(F^1(x^1, \dots, x^m, t), \dots, F^n(x^1, \dots, x^m, t)\big).$$

As usual, we use Latin indices for quantities on M and Greek indices for quantities on N. Denote by g_{ij} and Γ^k_{ij} the components of the induced metric and the Christoffel symbols on M, respectively. Observe that these quantities depend on the immersion F. Likewise, denote by $g_{\alpha\beta}$ and $\Gamma^\gamma_{\alpha\beta}$ the components of the metric and the Christoffel symbols on N, respectively. Let us compute:

- **Induced metric:**

$$\begin{aligned} g_{ij} &= g_N(d\widetilde{F}(\partial_i), d\widetilde{F}(\partial_j)) = g_N(\widetilde{F}_{x^i}, \widetilde{F}_{x^j}) \\ &= g_N\Big(\sum_{\alpha} \widetilde{F}^{\alpha}_{x^i}\partial_\alpha, \sum_{\beta} \widetilde{F}^{\beta}_{x^j}\partial_\beta\Big) \\ &= \sum_{\alpha,\beta} \widetilde{F}^{\alpha}_{x^i}\widetilde{F}^{\beta}_{x^j} g_N(\partial_\alpha, \partial_\beta). \end{aligned}$$

Therefore,

$$g_{ij} = \sum_{\alpha,\beta} g_{\alpha\beta}\widetilde{F}^{\alpha}_{x^i}\widetilde{F}^{\beta}_{x^j}.$$

- **Christoffel symbols:**

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

By computing each term seperately, we get

$$\Gamma^k_{ij} = \sum_{\alpha,\beta,l} g_{\alpha\beta}g^{kl}\widetilde{F}^{\alpha}_{x^l}\widetilde{F}^{\beta}_{x^ix^j} + \text{ lower order terms (l.o.t).}$$

- **Evolution equation for F^α:**

$$\begin{aligned} \frac{\partial \widetilde{F}^\gamma}{\partial t} &= \sum_{i,j} g^{ij}\Big(\widetilde{F}^{\gamma}_{x_ix_j} - \sum_k \Gamma^k_{ij}\widetilde{F}^{\gamma}_{x_k} + \sum_{\alpha,\beta}\Gamma^{\gamma}_{\alpha\beta}\widetilde{F}^{\beta}_{x_i}\widetilde{F}^{\alpha}_{x_j}\Big) \\ &= \sum_{i,j} g^{ij}\Big(\widetilde{F}^{\gamma}_{x_ix_j} - \sum_k \Gamma^k_{ij}\widetilde{F}^{\gamma}_{x_k}\Big) + \text{l.o.t.} \\ &= \sum_{i,j} g^{ij}\Big(\widetilde{F}^{\gamma}_{x_ix_j} - \sum_{\alpha,\beta,k,l} g_{\alpha\beta}g^{kl}\widetilde{F}^{\alpha}_{x_l}\widetilde{F}^{\beta}_{x_ix_j}\widetilde{F}^{\gamma}_{x_k}\Big) + \text{l.o.t.} \\ &= \sum_{i,j} g^{ij}\Big(\sum_{\alpha}\delta_{\alpha\gamma}\widetilde{F}^{\alpha}_{x_ix_j} - \sum_{\alpha,\beta,k,l} g_{\alpha\beta}g^{kl}\widetilde{F}^{\beta}_{x_l}\widetilde{F}^{\gamma}_{x_k}\widetilde{F}^{\alpha}_{x_ix_j}\Big) + \text{l.o.t.} \\ &= \sum_{\alpha,i,j} g^{ij}\Big(\delta_{\alpha\gamma} - \sum_{\beta,k,l} g_{\alpha\beta}g^{kl}\widetilde{F}^{\beta}_{x_l}\widetilde{F}^{\gamma}_{x_k}\Big)\widetilde{F}^{\alpha}_{x_ix_j} + \text{l.o.t.} \end{aligned}$$

Interchanging the indices α and γ, the equation becomes

$$\frac{\partial \widetilde{F}^\alpha}{\partial t} = \sum_{\gamma,i,j} g^{ij}\Big(\delta_{\alpha\gamma} - \sum_{\beta,k,l} g_{\beta\gamma}g^{kl}\widetilde{F}^{\beta}_{x_l}\widetilde{F}^{\alpha}_{x_k}\Big)\widetilde{F}^{\gamma}_{x_ix_j} + \text{l.o.t.}$$

Finally, interchanging the indices β and γ, we obtain

$$\frac{\partial \widetilde{F}^{\alpha}}{\partial t} = \sum_{\beta,i,j} g^{ij}\Big(\delta_{\alpha\beta} - \sum_{\gamma,k,l} g_{\beta\gamma} g^{kl} \widetilde{F}^{\gamma}_{x_l} \widetilde{F}^{\alpha}_{x_k}\Big) \widetilde{F}^{\beta}_{x_i x_j} + \text{l.o.t.} \tag{4.20}$$

In the special case $M = \mathbb{S}^1$ and $N = \mathbb{R}^2$, the system (4.20) can be written in the form

$$\begin{pmatrix} \dfrac{\partial F^1}{\partial t} \\ \dfrac{\partial F^2}{\partial t} \end{pmatrix} = \frac{1}{\|F_x\|^4} \begin{pmatrix} (F^2_x)^2 & -F^1_x F^2_x \\ -F^1_x F^2_x & (F^1_x)^2 \end{pmatrix} \begin{pmatrix} F^1_{xx} \\ F^2_{xx} \end{pmatrix}.$$

From (4.20), we see that (MCF) is a second-order quasilinear system of partial differential equations, with coefficients

$$A^{ij}_{\alpha\beta} = g^{ij}\Big(\delta_{\alpha\beta} - \sum_{\gamma,k,l} g_{\beta\gamma} g^{kl} \widetilde{F}^{\gamma}_{x_l} \widetilde{F}^{\alpha}_{x_k}\Big).$$

The nature of the tensor $A^{ij}_{\alpha\beta}$ depends on the behaviour of the tensor $T = (T_{\alpha\beta})$, where

$$T_{\alpha\beta} \doteq \delta_{\alpha\beta} - \sum_{\gamma,k,l} g_{\beta\gamma} g^{kl} \widetilde{F}^{\gamma}_{x_l} \widetilde{F}^{\alpha}_{x_k}.$$

In the next lemma, we show that the tensor A does not satisfy the parabolicity condition (4.18).

Lemma 4.1.12. *Let $Z \in F^*TN$. Then the following identity holds:*

$$T(Z) = Z^{\perp}.$$

In particular,

$$T(Z^{\top}) = 0,$$

which shows that T vanishes on the tangent bundle of the immersion F. Moreover, the kernel of T is m-dimensional.

Proof. Fix $t \in [0, T)$ and let $Z \in F_t^*TN$. We work in the local coordinates introduced above. Decompose

$$Z = Z^{\top} + Z^{\perp}.$$

We begin by computing the tangential component $Z^\top$ of Z. We may write

$$Z^\top = \sum_l \lambda_l \widetilde{F}_{x^l}.$$

Consequently,

$$g_N(Z^\top, \widetilde{F}_{x^k}) = \sum_l \lambda_l g_N(\widetilde{F}_{x^l}, \widetilde{F}_{x^k}),$$

from which it follows that

$$g_N(Z, \widetilde{F}_{x^k}) = \sum_l \lambda_l g_{kl}.$$

Multiplying both sides by g^{ks}, we obtain

$$\lambda_s = \sum_k g^{ks} g_N(Z, \widetilde{F}_{x^k}).$$

Renaming the index s as l, we get

$$\begin{aligned}
\lambda_l &= \sum_k g^{kl} g_N(Z, \widetilde{F}_{x^k}) \\
&= \sum_k g^{kl} g_N\big(\sum_\beta Z^\beta\, \partial_\beta, \sum_\gamma \widetilde{F}^\gamma_{x^k}\, \partial_\gamma\big) \\
&= \sum_{k,\beta,\gamma} Z^\beta g^{kl} g_{\beta\gamma} \widetilde{F}^\gamma_{x^k}.
\end{aligned}$$

Hence,

$$\lambda_l = \sum_{k,\beta,\gamma} Z^\beta g^{kl} g_{\beta\gamma} \widetilde{F}^\gamma_{x^k}$$

and so

$$\begin{aligned}
Z^\top &= \sum_l \lambda_l\, \widetilde{F}_{x^l} = \sum_{\beta,\gamma,k,l} Z^\beta g^{kl} g_{\beta\gamma} \widetilde{F}^\gamma_{x^k} \widetilde{F}_{x^l} \\
&= \sum_{\alpha,\beta,\gamma,k,l} Z^\beta g^{kl} g_{\beta\gamma} \widetilde{F}^\gamma_{x^k} \widetilde{F}^\alpha_{x^l}\, \partial_\alpha.
\end{aligned}$$

Therefore,

$$Z^\perp = Z - Z^\top = \sum_{\alpha,\beta} Z^\beta \big(\delta_{\alpha\beta} - \sum_{\gamma,k,l} g^{kl} g_{\beta\gamma} \widetilde{F}^\gamma_{x_k} \widetilde{F}^\alpha_{x_l}\big)\partial_\alpha = T(Z),$$

which completes the proof. □

According to the previous lemma, the mean curvature flow (MCF) is not parabolic, and therefore Theorem 4.1.11 cannot be applied directly. We will prove short-time existence, up to reparametrization, as well as uniqueness for (MCF), by using an argument analogous to the DeTurck trick for the Ricci flow.

Theorem 4.1.13 (Short-time existence). *Let M be a compact Riemannian manifold, and let $F_0\colon M \to N$ be an immersion into a Riemannian manifold N. Then the mean curvature flow equation* (MCF)*, with initial data F_0, admits a smooth solution on a maximal time interval $[0,T)$.*

Remark 4.1.1. The ***maximal time of existence*** T of the solution is defined by

$$T \doteq \sup\left\{T_0 > 0\colon F_t \text{ is an immersion solving (MCF) for all } t \in [0,T_0)\right\}.$$

Proof. The idea is to modify (MCF) by adding suitable tangential terms in order to obtain a parabolic system. According to the computations carried out at the beginning of this section, we have

$$\begin{aligned}\frac{\partial \widetilde{F}}{\partial t} &= \sum_{\alpha} \frac{\partial \widetilde{F}^{\alpha}}{\partial t}\,\partial_\alpha = \sum_{\alpha,i,j} g^{ij}\Big(\widetilde{F}^{\alpha}_{x_i x_j} - \sum_k \Gamma^k_{ij}\widetilde{F}^{\alpha}_{x_k} + \sum_{\beta,\gamma}\Gamma^{\alpha}_{\beta\gamma}\widetilde{F}^{\beta}_{x_i}\widetilde{F}^{\gamma}_{x_j}\Big)\partial_\alpha \\ &= \sum_{\alpha,i,j} g^{ij}\widetilde{F}^{\alpha}_{x_i x_j}\,\partial_\alpha + \sum_{\alpha,\beta,\gamma,i,j} g^{ij}\Gamma^{\alpha}_{\beta\gamma}\widetilde{F}^{\beta}_{x_i}\widetilde{F}^{\gamma}_{x_j}\,\partial_\alpha - \sum_{\alpha,i,j,k} g^{ij}\Gamma^k_{ij}\widetilde{F}^{\alpha}_{x_k}\,\partial_\alpha.\end{aligned}$$

The last term is responsible for the failure of parabolicity. We now modify this term. Fix a Riemannian metric $\widehat{g}$ on M, and denote by $\widehat{\Gamma}^k_{ij}$ its Christoffel symbols. Then

$$\begin{aligned}\sum_{\alpha,i,j,k} g^{ij}\Gamma^k_{ij}\widetilde{F}^{\alpha}_{x_k}\,\partial_\alpha &= \sum_{i,j,k} g^{ij}\Gamma^k_{ij} d\widetilde{F}(\partial_k) = d\widetilde{F}\Big(\sum_{i,j,k} g^{ij}\Gamma^k_{ij}\,\partial_k\Big) \\ &= d\widetilde{F}\Big(\sum_{i,j,k} g^{ij}(\Gamma^k_{ij} - \hat{\Gamma}^k_{ij} + \hat{\Gamma}^k_{ij})\,\partial_k\Big) \\ &= d\widetilde{F}\Big(\sum_{i,j,k} g^{ij}(\Gamma^k_{ij} - \widehat{\Gamma}^k_{ij})\partial_k\Big) + d\widetilde{F}\Big(\sum_{i,j,k} g^{ij}\widehat{\Gamma}^k_{ij}\,\partial_k\Big).\end{aligned}$$

Observe that the second term in the last equality is a lower-order term. The first term is the image under dF of the globally defined vector field

$$X_{DT} \doteq \mathrm{tr}_g B$$

where B is the symmetric tensor given by

$$B(X,Y) \doteq \nabla_X Y - \widehat{\nabla}_X Y.$$

From the above observation, it follows that the modified flow

$$dF(\partial_t) = H_F + dF(X_{DT})$$

is uniformly parabolic. Therefore, by Theorem 4.1.11, it admits a smooth solution. Applying Lemma 4.1.2, we conclude that the original mean curvature flow equation (MCF) also admits a smooth solution. □

Definition 4.1.14. *Let $F\colon M \times [0,T) \to N$ be a solution of* (MCF). *Fix a Riemannian metric $\widehat{g}$ on M, and consider the vector field*

$$X_{DT} \doteq \mathrm{tr}_g(\nabla - \widehat{\nabla}).$$

Then the modified flow

$$\frac{\partial F}{\partial t} = H_F + dF(X_{DT}) \tag{DMCF}$$

is called the ***DeTurck mean curvature flow****.*

Lemma 4.1.15. *The vector field X_{DT} is minus the Laplacian of the identity map*

$$I\colon (M,g) \to (M,\widehat{g}).$$

Proof. The second fundamental form of I is given by

$$\begin{aligned} A(X,Y) &= \nabla^I_X dI(Y) - dI(\nabla_X Y) \\ &= \widehat{\nabla}_{dI(X)} dI(Y) - dI(\nabla_X Y) \\ &= \widehat{\nabla}_X Y - \nabla_X Y \\ &= -B(X,Y). \end{aligned}$$

Therefore,

$$\Delta_{g,\widehat{g}} I = \mathrm{tr}_g A = -\mathrm{tr}_g B = -X_{DT}.$$

This completes the proof. □

Theorem 4.1.16 (Uniqueness)**.** *Let M be a compact Riemannian manifold and $F_0\colon M \to N$ an immersion into a Riemannian manifold N. Then the solution of* (MCF)*, with initial data F_0, is unique up to reparametrizations.*

Proof. Let $F_1, F_2\colon M \times [0,T) \to N$ two solutions of (MCF) with initial data the immersion F_0, i.e.,

$$\begin{cases} dF_1(\partial_t) = H_{F_1}, \\ F_1(\,\cdot\,,0) = F_0, \end{cases} \qquad \text{and} \qquad \begin{cases} dF_2(\partial_t) = H_{F_2}, \\ F_2(\,\cdot\,,0) = F_0. \end{cases}$$

Fix a Riemannian metric $\widehat{g}$ on M, and let g_1 and g_2 denote the metrics induced on M by F_1 and F_2, respectively. Define the families of diffeomorphisms

$$\psi_1, \psi_2\colon M \times [0,T) \to M$$

as the solutions of

$$\begin{cases} d\psi_1(\partial_t) = \Delta_{g_1,\widehat{g}}\psi_1, \\ \psi_1(\,\cdot\,,0) = \mathrm{Id}, \end{cases} \qquad \text{and} \qquad \begin{cases} d\psi_2(\partial_t) = \Delta_{g_2,\widehat{g}}\psi_2, \\ \psi_2(\,\cdot\,,0) = \mathrm{Id}, \end{cases} \tag{4.21}$$

where

$$\Delta_{g_k,\widehat{g}}\psi_k = \sum_{i,j} g^{ij}\big(\widehat{\nabla}_{d\psi_k(\partial_i)}d\psi_k(\partial_j) - d\psi_k(\nabla_{\partial_i}\partial_j)\big), \quad \text{for } k = 1,2.$$

The systems (4.21) are parabolic and therefore admit unique solutions. Consider now the immersions $\widetilde{F}_1$ and $\widetilde{F}_2$ defined by

$$\begin{cases} \widetilde{F}_1(x,t) = F_1(\psi_1(x,t),t), \\ \widetilde{F}_2(x,t) = F_2(\psi_2(x,t),t). \end{cases} \qquad (x,t) \in M \times [0,T).$$

The induced metrics on M by $\widetilde{F}_1$ and $\widetilde{F}_2$ are

$$\begin{cases} \widetilde{g}_1 = \psi_1^* g_1, \\ \widetilde{g}_2 = \psi_2^* g_2. \end{cases}$$

From the computations carried out in Lemma 4.1.2, we conclude that $\widetilde{F}_1$ and $\widetilde{F}_2$ satisfy

$$\begin{aligned} \frac{\partial \widetilde{F}_1}{\partial t}(x,t) &= H_{\widetilde{F}_1(x,t)} + dF_1|_{(\psi_1(x,t),t)}\left(d\psi_1|_{(x,t)}(\partial_t)\right) \\ &= H_{\widetilde{F}_1(x,t)} + dF_1|_{(\psi_1(x,t),t)}\left((\Delta_{g_1,\hat{g}}\psi_1)(x,t)\right). \end{aligned}$$

From the composition formula (1.6) it follows that

$$(\Delta_{g_1,\widehat{g}}\psi_1)(x,t) = d\psi_1|_{(x,t)}(X_{DT}|_{(x,t)}), \quad \text{for } (x,t) \in M \times [0,T).$$

Therefore,

$$\frac{\partial \widetilde{F}_1}{\partial t}(x,t) = H_{\widetilde{F}_1(x,t)} + d\widetilde{F}_1(X_{DT}|_{(x,t)}).$$

Similarly,

$$\frac{\partial \widetilde{F}_2}{\partial t}(x,t) = H_{\widetilde{F}_2(x,t)} + d\widetilde{F}_2(X_{DT}|_{(x,t)}).$$

Hence, $\widetilde{F}_1$ and $\widetilde{F}_2$ satisfy the DeTurck mean curvature flow DMCF. Moreover,

$$\begin{cases} \widetilde{F}_1(x,0) = F_1(\psi_1(x,0),0) = F_1(x,0) = F_0(x), \\ \\ \widetilde{F}_2(x,0) = F_2(\psi_2(x,0),0) = F_2(x,0) = F_0(x). \end{cases}$$

Since (DMCF) is parabolic, it follows that

$$\widetilde{F}_1 \equiv \widetilde{F}_2.$$

Hence,

$$F_1(\psi_1(x,t),t) = F_2(\psi_2(x,t),t).$$

Consequently, the maps F_1 and F_2 differ only by a family of diffeomorphisms. This completes the proof. □

Let us conclude this chapter with an application of Theorem 4.1.16.

Theorem 4.1.17. *Let $F_0\colon M \to \mathbb{R}^n$ be an immersion. Then* (MCF)*, with initial data F_0, preserves all ambient isometries leaving $M_0 = F_0(M)$ invariant.*

Proof. Assume that $M_0 = F_0(M)$ is invariant under the action of an isometry $\Psi\colon \mathbb{R}^n \to \mathbb{R}^n$. That is, $M_0 = \Psi(M_0)$. Let F be a solution of (MCF) with initial data F_0, and consider the family $\widetilde{F} = \Psi \circ F$. By Lemma 4.1.1, the family $\widetilde{F}$ is again a solution of (MCF). Moreover,

$$\widetilde{F}_0(M) = (\Psi \circ F_0)(M) = \Psi(F_0(M)) = \Psi(M_0) = M_0 = F_0(M).$$

Therefore, $\widetilde{F}$ and F have the same initial data. By the uniqueness statement in Theorem 4.1.16, we conclude that

$$\widetilde{F}_t(M) = F_t(M), \quad \text{for all } t \in [0,T).$$

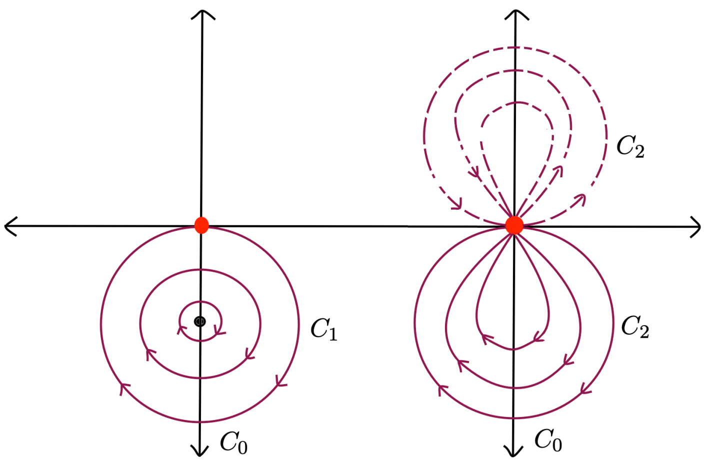
Hence, the symmetry generated by Ψ is preserved along the flow. □

Remark 4.1.2. Let us make some comments on the mean curvature flow of curves in $\mathbb{R}^2$. We may initiate the mean curvature flow from Lipschitz curves; see for example [15] and [4, Theorem B]. In this case, the family of solutions $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ is continuous on $\mathbb{S}^1 \times [0, T)$ and analytic on $\mathbb{S}^1 \times (0, T)$. According to a beautiful result of Angenent [4, Theorem 3.1 & Theorem 8.1], if $F_0 : \mathbb{S}^1 \to \mathbb{R}^2$ is a regular curve with Hölder continuous curvature, or with bounded L_p-norm of the curvature for some suitable $p > 1$, then (MCF) admits a unique solution with initial data F_0. Consequently, in Theorem 4.1.17 one might also weaken appropriately the regularity assumptions on the initial condition. For topics concerning weak solutions and uniqueness of the mean curvature flow of curves, we refer the reader to the monograph [23].

Remark 4.1.3. The mean curvature flow of a non-closed curve may fail to be unique. Let us consider the following example. Define the curve

$$C_0 = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 1)^2 = 1 \ \& \ y < 0\}.$$

We claim that there exist infinitely many solutions of the mean curvature flow with initial data the curve C_0. Indeed, consider the following extensions of C_0: (a) the extension C_1 obtained by taking the entire circle centered at $(0, -1)$ with radius 1, and (b) the extension C_2 obtained by reflecting C_0 through the origin. The curve C_1 is smooth, whereas C_2 is Lipschitz and has bounded L_p-norm of the curvature. The curve shortening flow deforms C_1 through concentric circles shrinking to the point $(0, -1)$, while the evolution of C_2 produces curves which are symmetric with respect to the origin $(0, 0)$. Consequently, the evolving curves arising from C_2 pass through the origin; see also the figure below.



4.2 Evolution equations

In this section we will derive the evolution under the (MCF) of various geometric quantities.

4.2.1 Time dependent bundle

At first we need to fix the notation and the set up the general frame work. Since we work on the product $M \times [0, T)$, $T > 0$, the first question is:

What is the most natural connection to differentiate time dependent quantities?

Denote by $\pi_1 : M \times [0, T) \to M$ and $\pi_2 : M \times [0, T) \to [0, T)$ the natural projections of $M \times [0, T)$, i.e.,

$$\pi_1(p, t) = p \quad \text{and} \quad \pi_2(p, t) = t.$$

$$\begin{tikzcd} & M \times [0,T) \arrow[dl, mapsto, "\pi_1"'] \arrow[dr, mapsto, "\pi_2"] & \\ M & & {[0,T)} \end{tikzcd}$$

Figure 4.1: The natural projections.

We can split the tangent bundle of $M \times [0, T)$ as

$$T(M \times [0, T)) = \mathcal{H} \oplus \mathbb{R}\, \partial_t,$$

where

$$\mathcal{H} = \{V \in \mathfrak{X}(M \times [0, T)) \colon d\pi_2(V) = 0\}.$$

In other words, we decompose the tangent bundle of $M \times [0, T)$ as the sum of a space parallel to ∂_t and a complementary space, which we call $\mathcal{H}$. The vector bundle $\mathcal{H}$ is called the ***spatial tangent bundle***, and sections of $\mathcal{H}$ are called ***spatial vector fields***.

Suppose that $\{g = g(t)\}_{t\in[0,T)}$ is a time-dependent smooth family of Riemannian metrics on $M \times [0,T)$. This means that each slice $M \times \{t\}$ is equipped with the smooth Riemannian metric $g(t)$. Denote by ∇^t the Levi-Civita connection associated with $g(t)$. We can use these structures to induce a natural connection on the manifold $M \times [0,T)$. The idea is as follows: consider a local coordinate system $(x = (x^1, \dots, x^m); t)$ defined on an open subset W of $M \times [0,T)$. If $V \in \mathfrak{X}(W)$, then with respect to the local coordinate system (x,t), we have the representation

$$V_{(x,t)} = \sum_{i=1}^{m} a_i(x,t)\partial_i + b(x,t)\partial_t,$$

where a_i, b_i, $i \in \{1, \dots, m\}$, are smooth time-dependent functions. Define now on W the connection ∇ by requiring

$$\nabla_{\partial_t}\partial_i = 0, \quad \nabla_{\partial_t}\partial_t = 0, \quad \nabla_{\partial_i}\partial_t = 0, \quad \nabla_{\partial_i}\partial_j = \nabla^t_{\partial_i}\partial_j. \tag{4.22}$$

Because,

$$[\partial_t, \partial_i] = 0 = [\partial_i, \partial_j],$$

we can express the identities (4.22), globally as

$$\nabla_X Y = \nabla^t_X Y, \;\; \nabla_{\partial_t}\partial_t = 0, \;\; \nabla_{\partial_X}\partial_t = 0, \;\; \nabla_{\partial_t} X = [\partial_t, X], \tag{4.23}$$

for any $X, Y \in \mathcal{H}$. We can easily check the following properties:

1. For every $X, Y, Z \in \mathcal{H}$, we have

$$Xg(Z,Y) = g(\nabla_X Z, Y) + g(Z, \nabla_X Y).$$

2. For every $X, Y \in \mathcal{H}$, it holds

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

3. If $F\colon M \times [0,T) \to N$ is a smooth map, then

$$\nabla^F_{\partial_t} dF(X) - \nabla^F_X dF(\partial_t) = dF([\partial_t, X]) = dF(\nabla_{\partial_t} X),$$

for every $X \in \mathcal{H}$.

Let $F\colon M \times [0,T) \to N$ be a 1-parameter family of isometric immersions, i.e., the maps $F_t\colon M \to N$, $t \in [0,T)$, given by

$$F_t(x) = F(x,t), \quad (x,t) \in M \times [0,T),$$

are smooth immersions. Define the ***extended second fundamental form*** $\widetilde{A}$ by

$$\begin{cases} \widetilde{A}(X,Y) = A(X,Y), \\ \widetilde{A}(\partial_t, X) = \nabla^F_{\partial_t} dF(X) - dF(\nabla_{\partial_t} X), \\ \widetilde{A}(X,\partial_t) = \nabla^F_X dF(\partial_t) - dF(\nabla_X \partial_t) = \nabla^F_X dF(\partial_t), \\ \widetilde{A}(\partial_t,\partial_t) = \nabla^F_{\partial_t} dF(\partial_t) - dF(\nabla_{\partial_t}\partial_t) = \nabla^F_{\partial_t} dF(\partial_t), \end{cases}$$

for all $X, Y \in \mathcal{H}$. Clearly, $\widetilde{A}$ is symmetric. We denote the extended curvature and normal curvature operators by $\widetilde{R}$ and $\widetilde{R}^\perp$, respectively, i.e. define

$$\widetilde{R}(\partial_t, Y)Z = \nabla_{\partial_t}\nabla_Y Z - \nabla_Y \nabla_{\partial_t} Z - \nabla_{[\partial_t, Y]} Z$$

and

$$\widetilde{R}^\perp(\partial_t, Y)\xi = \nabla^\perp_{\partial_t}\nabla^\perp_Y \xi - \nabla^\perp_Y \nabla^\perp_{\partial_t} \xi - \nabla^\perp_{[\partial_t, Y]} \xi,$$

for any $X, Y \in \mathcal{H}$ and $\xi \in \mathcal{N}_{F_t} M$. Recall that, if $X \in \mathcal{H}$ then $\nabla_{\partial_t} X \in \mathcal{H}$.

Lemma 4.2.1 (Time-like Gauss and Ricci equations)**.** *Let $F\colon M \times [0,T) \to N$ be a solution of the* (MCF)*. Then:*

1. *For any $Y, Z, W \in \mathcal{H}$, we have*

$$\begin{aligned} \widetilde{R}(\partial_t, Y, Z, W) \;=\; & R^N(H, dF(Y), dF(Z), dF(W)) \\ & -\langle \nabla^\perp_W H, A(Y,Z)\rangle + \langle \nabla^\perp_Z H, A(Y,W)\rangle \\ & +\langle \nabla^\perp_Y H, A(Z,W)\rangle + \langle H, (\nabla^\perp_Y A)(Z,W)\rangle. \end{aligned}$$

2. *For any $\xi, \eta \in \mathcal{N}_{F_t} M$, we have*

$$\widetilde{R}^\perp(\partial_t, Y, \xi, \eta) = R^N(H, dF(Y), \xi, \eta) + \langle \xi, \nabla^\perp_{A_\eta Y} H\rangle - \langle \eta, \nabla^\perp_{A_\xi Y} H\rangle.$$

Proof. Since F is evolving by mean curvature flow, we have that

$$dF(\partial_t) = H.$$

We compute,

$$\begin{aligned}
&\langle R^N(dF(\partial_t), dF(Y))dF(Z), dF(W)\rangle - \langle \widetilde{R}(\partial_t, Y)Z, W\rangle \\
&= \underbrace{\langle \nabla^F_{\partial_t}\nabla^F_Y dF(Z), dF(W)\rangle - \langle \nabla_{\partial_t}\nabla_Y Z, W\rangle}_{\mathcal{A}} \\
&\quad - \underbrace{\langle \nabla^F_Y\nabla^F_{\partial_t} dF(Z), dF(W)\rangle + \langle \nabla_Y\nabla_{\partial_t} Z, W\rangle}_{\mathcal{B}} \\
&\quad - \underbrace{\langle \nabla^F_{[\partial_t, Y]} dF(Z), dF(W)\rangle + \langle \nabla_{[\partial_t, Y]} Z, W\rangle}_{\mathcal{C}}.
\end{aligned}$$

Let us compute each term $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ separately. Recall that

$$\nabla_{\partial_t} Z \in \mathcal{H} \quad \text{and} \quad \widetilde{A}(X, Y) = A(X, Y)$$

for all $X, Y, Z \in \mathcal{H}$. Moreover, observe that for all $\xi \in \mathcal{N}_{F_t}M$, it holds

$$\langle \nabla_{\partial_t}\xi, dF(X)\rangle = -\langle \xi, \nabla^F_{\partial_t} dF(X)\rangle = -\langle \xi, \nabla^F_X dF(\partial_t)\rangle = -\langle \xi, \nabla^\perp_X H\rangle.$$

Keeping in mind the above identities, we have

$$\begin{aligned}
\mathcal{C} &= \langle \nabla_{\nabla_{\partial_t} Y} Z, W\rangle - \langle \nabla^F_{\nabla_{\partial_t} Y} dF(Z), dF(W)\rangle \\
&= \langle dF(\nabla_{\nabla_{\partial_t} Y} Z), dF(W)\rangle - \langle \nabla^F_{\nabla_{\partial_t} Y} dF(Z), dF(W)\rangle \\
&= \langle dF(\nabla_{\nabla_{\partial_t} Y} Z) - \nabla^F_{\nabla_{\partial_t} Y} dF(Z), dF(W)\rangle \\
&= -\langle A(\nabla_{\partial_t} Y, Z), dF(W)\rangle \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{A} &= \langle \nabla^F_{\partial_t}\nabla^F_Y dF(Z), dF(W)\rangle - \langle \nabla_{\partial_t}\nabla_Y Z, W\rangle \\
&= \langle \nabla^F_{\partial_t}\{A(Y, Z) + dF(\nabla_Y Z)\}, dF(W)\rangle - \langle dF(\nabla_{\partial_t}\nabla_Y Z), dF(W)\rangle \\
&= -\langle A(Y, Z), \nabla^F_{\partial_t} dF(W)\rangle + \langle \widetilde{A}(\partial_t, \nabla_Y Z), dF(W)\rangle \\
&= -\langle A(Y, Z), \nabla^\perp_W H\rangle + \langle \nabla^F_{\nabla_Y Z} H, dF(W)\rangle.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\mathcal{B} &= -\langle \nabla^F_Y \nabla^F_{\partial_t} dF(Z), dF(W)\rangle + \langle \nabla_Y \nabla_{\partial_t} Z, W\rangle \\
&= -\langle \nabla^F_Y \{\nabla^F_Z H + dF([\partial_t, Z])\}, dF(W)\rangle + \langle \nabla_Y \nabla_{\partial_t} Z, W\rangle\rangle \\
&= -\langle \nabla^F_Y \nabla^F_Z H, dF(W)\rangle - \langle A(Y, \nabla_{\partial_t} Z), dF(W)\rangle \\
&= -\langle \nabla^F_Y \nabla^F_Z H, dF(W)\rangle.
\end{aligned}$$

Therefore,

$$\mathcal{A} + \mathcal{B} + \mathcal{C} = -\langle A(Y, Z), \nabla^\perp_W H\rangle - \langle \nabla^2_{Y,Z} H, dF(W)\rangle.$$

It remains to compute the tangential part of the Hessian of H. For simplicity, we consider a local orthonormal frame $\{e_1, \ldots, e_m\}$ that is normal at a fixed point $x_0 \in M$. We have

$$(\nabla^F_{e_i} H)^\top = \sum_k \langle \nabla^F_{e_i} H, dF(e_k)\rangle dF(e_k) = -\sum_k \langle H, A(e_i, e_k)\rangle dF(e_k). \tag{4.24}$$

On the other hand, at the point x_0, we have

$$\begin{aligned}
\langle \nabla^F_{e_i} \nabla^F_{e_j} H, dF(e_l)\rangle &= \langle \nabla^F_{e_i} \{(\nabla^F_{e_j} H)^\top + \nabla^\perp_{e_j} H\}, dF(e_l)\rangle \\
&= \langle \nabla^F_{e_i} (\nabla^F_{e_j} H)^\top, dF(e_l)\rangle + \langle \nabla^F_{e_i} (\nabla^\perp_{e_j} H), dF(e_l)\rangle \\
&= \langle \nabla^F_{e_i} (\nabla^F_{e_j} H)^\top, dF(e_l)\rangle - \langle \nabla^\perp_{e_j} H, A(e_i, e_l)\rangle.
\end{aligned} \tag{4.25}$$

Using (4.24), we find that at x_0 it holds

$$\begin{aligned}
\langle \nabla^F_{e_i} (\nabla^F_{e_j} H)^\top, dF(e_l)\rangle &= -\sum_k \langle \nabla^F_{e_i} \left(\langle H, A(e_j, e_k)\rangle dF(e_k)\right), dF(e_l)\rangle \\
&= -\sum_k \langle e_i \langle H, A(e_j, e_k)\rangle dF(e_k), dF(e_l)\rangle \\
&\quad -\sum_k \langle H, A(e_j, e_k)\rangle \langle \nabla^F_{e_i} dF(e_k), dF(e_l)\rangle \\
&= -\sum_k e_i \langle H, A(e_j, e_k)\rangle \delta_{kl},
\end{aligned}$$

from where we obtain

$$\langle \nabla^F_{e_i} (\nabla^F_{e_j} H)^\top, dF(e_l)\rangle = -\langle \nabla^\perp_{e_i} H, A(e_j, e_l)\rangle - \langle H, \nabla^\perp_{e_i} A(e_j, e_l)\rangle. \tag{4.26}$$

Combining (4.24), (4.25) and (4.26), we deduce

$$\begin{aligned}\mathcal{A}+\mathcal{B}+\mathcal{C} &= \langle \nabla^{\perp}_{Z}H, A(Y,W)\rangle + \langle H, (\nabla^{\perp}_{Y}A)(Z,W)\rangle \\ &\quad -\langle \nabla^{\perp}_{W}H, A(Y,Z)\rangle + \langle \nabla^{\perp}_{Y}H, A(Z,W)\rangle.\end{aligned}$$

Putting everything together, we get the desired formula.

To prove the second property, first note that if $\xi \in \mathcal{N}_{F_t}M$ then

$$\begin{aligned}\nabla^{F}_{\partial_t}\xi &= \nabla^{\perp}_{\partial_t}\xi + \sum_k \langle \nabla^{F}_{\partial_t}\xi, dF(e_k)\rangle dF(e_k) \\ &= \nabla^{\perp}_{\partial_t}\xi - \sum_k \langle \xi, \nabla^{F}_{\partial_t}dF(e_k)\rangle dF(e_k) \\ &= \nabla^{\perp}_{\partial_t}\xi - \sum_k \langle \xi, \nabla^{F}_{e_k}dF(\partial_t)\rangle dF(e_k) \\ &= \nabla^{\perp}_{\partial_t}\xi - \sum_k \langle \xi, \nabla^{\perp}_{e_k}H\rangle dF(e_k).\end{aligned}$$

Hence,

$$\nabla^{F}_{\partial_t}\xi = \nabla^{\perp}_{\partial_t}\xi - \sum_k \langle \xi, \nabla^{\perp}_{e_k}H\rangle dF(e_k). \tag{4.27}$$

Using (4.27) and Weingarten formula (1.4), we get

$$\begin{aligned}R^{N}(dF(\partial_t), dF(Y))\xi &= \nabla^{F}_{\partial_t}\nabla^{F}_{Y}\xi - \nabla^{F}_{Y}\nabla^{F}_{\partial_t}\xi - \nabla^{F}_{\nabla_{\partial_t}Y}\xi \\ &= \nabla^{F}_{\partial_t}\{\nabla^{\perp}_{Y}\xi - dF(A_{\xi}Y)\} - \nabla^{\perp}_{\nabla_{\partial_t}Y}\xi + dF(A_{\xi}\nabla_{\partial_t}Y) \\ &\quad -\nabla^{F}_{Y}\Big\{\nabla^{\perp}_{\partial_t}\xi - \sum_k \langle \xi, \nabla^{\perp}_{e_k}H\rangle dF(e_k)\Big\} \\ &= \nabla^{F}_{\partial_t}\nabla^{\perp}_{Y}\xi - \nabla^{F}_{\partial_t}dF(A_{\xi}Y) - \nabla^{\perp}_{\nabla_{\partial_t}Y}\xi + dF(A_{\xi}\nabla_{\partial_t}Y) \\ &\quad -\nabla^{F}_{Y}\nabla^{\perp}_{\partial_t}\xi + \sum_k \nabla^{F}_{Y}\left\{\langle \xi, \nabla^{\perp}_{e_k}H\rangle dF(e_k)\right\}.\end{aligned}$$

Consequently, using (4.27), we obtain

$$\begin{aligned}
&-R^N(H, dF(Y), \xi, \eta) \\
&\quad = -\widetilde{R}^{\perp}(\partial_t, Y, \xi, \eta) - \langle \nabla^F_{\partial_t} dF(A_\xi Y), \eta\rangle + \sum_k \langle \xi, \nabla^{\perp}_{e_k} H\rangle \langle \nabla^F_Y dF(e_k), \eta\rangle \\
&\quad = -\widetilde{R}^{\perp}(\partial_t, Y, \xi, \eta) + \langle dF(A_\xi Y), \nabla^F_{\partial_t}\eta\rangle + \sum_k \langle \xi, \nabla^{\perp}_{e_k} H\rangle \langle A(Y, e_k), \eta\rangle \\
&\quad = -\widetilde{R}^{\perp}(\partial_t, Y, \xi, \eta) - \sum_k \langle \eta, \nabla^{\perp}_{e_k} H\rangle \langle A_\xi Y, e_k\rangle + \sum_k \langle A_\eta Y, e_k\rangle \langle \xi, \nabla^{\perp}_{e_k} H\rangle \\
&\quad = -\widetilde{R}^{\perp}(\partial_t, Y, \xi, \eta) - \sum_k \langle \eta, \nabla^{\perp}_{\langle A_\xi Y, e_k\rangle e_k} H\rangle + \sum_k \langle \xi, \nabla^{\perp}_{\langle A_\eta Y, e_k\rangle e_k} H\rangle.
\end{aligned}$$

Therefore,

$$-R^N(H, dF(Y), \xi, \eta) = -\widetilde{R}^{\perp}(\partial_t, Y, \xi, \eta) - \langle \eta, \nabla^{\perp}_{A_\xi Y} H\rangle + \langle \xi, \nabla^{\perp}_{A_\eta Y} H\rangle.$$

This completes the proof. $\square$

We conclude with the following lemmas which will be useful in the characterization of the maximal time of existence of (MCF).

Lemma 4.2.2 (Ricci Identity)**.** *Let $F_t\colon M \to N$ be a solution of* (MCF) *and*

$$T\colon \underbrace{\mathfrak{X}(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r\text{-times}} \to \mathcal{N}_{F_t} M$$

be a tensor with values in the normal bundle of the evolved submanifolds. Then, for any $X_1, \dots, X_r, Y \in \mathcal{H}$ it holds:

$$\begin{aligned}
&(\nabla^{\perp}_{\partial_t} \nabla^{\perp} T)(X_1, \dots, X_r, Y) \\
&\qquad = (\nabla^{\perp} \nabla^{\perp}_{\partial_t} T)(X_1, \dots, X_r, Y) + \widetilde{R}^{\perp}(\partial_t, Y) T(X_1, \dots, X_r) \\
&\qquad\quad - T\big(\widetilde{R}(\partial_t, Y) X_1, \dots, X_r\big) - \cdots - T\big(X_1, \dots, (\widetilde{R}^{\perp}(\partial_t, Y) X_r\big).
\end{aligned}$$

Proof. Consider a local coordinate chart $(x = (x_1, \dots, x_m); t)$ which is normal at $(x_0, t_0) \in M \times [0, T)$. On the domain of the coordinate chart, we have

$$\nabla_{\partial_t} \partial_i = 0 = \nabla_{\partial_i} \partial_t, \quad i \in \{1, \dots, m\}.$$

Differentiating and evaluating at (x_0, t_0), we obtain

$$\begin{aligned}\mathcal{A} &= (\nabla^{\perp}_{\partial_t}\nabla^{\perp}T)(\partial_1,\dots,\partial_r,\partial_s)\\ &= \nabla^{\perp}_{\partial_t}\{(\nabla^{\perp}T)(\partial_1,\dots,\partial_r,\partial_s)\} = \nabla^{\perp}_{\partial_t}\big\{(\nabla^{\perp}_{\partial_s}T)(\partial_1,\dots,\partial_r)\big\}\\ &= \nabla^{\perp}_{\partial_t}\big\{\nabla^{\perp}_{\partial_s}T(\partial_1,\dots,\partial_r) - T(\nabla_{\partial_s}\partial_1,\dots,\partial_r) - \cdots - T(\partial_1,\dots,\nabla_{\partial_s}\partial_r)\big\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{A} &= \nabla^{\perp}_{\partial_t}\nabla^{\perp}_{\partial_s}T(\partial_1,\dots,\partial_r)\\ &\quad -\nabla^{\perp}_{\partial_t}T(\nabla_{\partial_s}\partial_1,\dots,\partial_r) - \cdots - \nabla^{\perp}_{\partial_t}T(\partial_1,\dots,\nabla_{\partial_s}\partial_r)\\ &= \nabla^{\perp}_{\partial_t}\nabla^{\perp}_{\partial_s}T(\partial_1,\dots,\partial_r)\\ &\quad -(\nabla^{\perp}_{\partial_t}T)(\nabla_{\partial_s}\partial_1,\dots,\partial_r)\\ &\qquad -T(\nabla_{\partial_t}\nabla_{\partial_s}\partial_1,\dots,\partial_r) - \cdots - T(\nabla_{\partial_t}\partial_1,\dots,\nabla_{\partial_s}\partial_r)\\ &\quad\ \vdots \qquad \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots\\ &\quad -(\nabla^{\perp}_{\partial_t}T)(\partial_1,\dots,\nabla_{\partial_s}\partial_r)\\ &\qquad -T(\nabla_{\partial_s}\partial_1,\dots,\nabla_{\partial_t}\partial_r) - \cdots - T(\partial_1,\dots,\nabla_{\partial_t}\nabla_{\partial_s}\partial_r)\\ &= \nabla^{\perp}_{\partial_t}\nabla^{\perp}_{\partial_s}T(\partial_1,\dots,\partial_r)\\ &\quad -T(\nabla_{\partial_t}\nabla_{\partial_s}\partial_1,\dots,\partial_r)\cdots - T(\partial_1,\dots,\nabla_{\partial_t}\nabla_{\partial_s}\partial_r).\end{aligned}$$

Similarly, at the point (x_0, t_0), we compute

$$\begin{aligned}\mathcal{B} &= (\nabla^{\perp}\nabla^{\perp}_{\partial_t}T)(\partial_1,\dots,\partial_r,\partial_s) = (\nabla^{\perp}_{\partial_s}\nabla^{\perp}_{\partial_t}T)(\partial_1,\dots,\partial_r)\\ &= \nabla^{\perp}_{\partial_s}\big\{(\nabla^{\perp}_{\partial_t}T)(\partial_1,\dots,\partial_r)\big\}\\ &\quad -(\nabla^{\perp}_{\partial_t}T)(\nabla_{\partial_s}\partial_1,\dots,\partial_r) - \cdots - (\nabla^{\perp}_{\partial_t}T)(\partial_1,\dots,\nabla_{\partial_s}\partial_r)\\ &= \nabla^{\perp}_{\partial_s}\big\{(\nabla^{\perp}_{\partial_t}T)(\partial_1,\dots,\partial_r)\big\}\\ &= \nabla^{\perp}_{\partial_s}\big\{\nabla^{\perp}_{\partial_t}T(\partial_1,\dots,\partial_r) - T(\nabla_{\partial_t}\partial_1,\dots,\partial_r) - \cdots - T(\partial_1,\dots,\nabla_{\partial_t}\partial_r)\big\}\\ &= \nabla^{\perp}_{\partial_s}\nabla^{\perp}_{\partial_t}T(\partial_1,\dots,\partial_r).\end{aligned}$$

By taking the difference between $\mathcal{A}$ and $\mathcal{B}$, we obtain

$$\begin{aligned}\mathcal{A}-\mathcal{B} &= \nabla^{\perp}_{\partial_t}\nabla^{\perp}_{\partial_s}T(\partial_1,\dots,\partial_r) - \nabla^{\perp}_{\partial_s}\nabla^{\perp}_{\partial_t}T(\partial_1,\dots,\partial_r)\\ &\quad -T(\nabla_{\partial_t}\nabla_{\partial_s}\partial_1,\dots,\partial_r) - \cdots - T(\partial_1,\dots,\nabla_{\partial_t}\nabla_{\partial_s}\partial_r)\\ &= \widetilde{R}^{\perp}(\partial_t,\partial_s)T(\partial_1,\dots,\partial_r)\\ &\quad -T(\widetilde{R}(\partial_t,\partial_s)\partial_1,\dots,\partial_r) - \cdots - T(\partial_1,\dots,\widetilde{R}(\partial_t,\partial_s)\partial_r).\end{aligned}$$

This completes the proof. $\square$

The following lemma gives a Bochner-type identity describing the commutation between $\Delta^\perp$ and $\nabla^\perp$ acting on normal-valued tensors.

Lemma 4.2.3. *Let $F\colon M \to N$ be an isometric immersion and*

$$T\colon \underbrace{\mathfrak{X}(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r\text{-times}} \to \mathcal{N}_F M$$

be a normal valued tensor as in the Lemma 4.2.2. Then for all

$$X_1, \ldots, X_r, X \in \mathcal{H}$$

it holds

$$\begin{aligned}
&(\Delta^\perp \nabla^\perp T - \nabla^\perp \Delta^\perp T)(X_1, \ldots, X_r, X) \\
&\qquad = -2 \sum_{i=1}^{m} \sum_{a=1}^{r} (\nabla^\perp_{e_i} T)(X_1, \ldots, R(e_i, X)X_a, \ldots, X_r) \\
&\qquad\quad - \sum_{i=1}^{m} \sum_{a=1}^{r} T(X_1, \ldots, (\nabla_{e_i} R)(e_i, X, X_a), \ldots, X_r) \\
&\qquad\quad - \sum_{i=1}^{m} \left(\nabla^\perp_{R(e_i,X)e_i} T\right)(X_1, \ldots, X_r) \\
&\qquad\quad + \sum_{i=1}^{m} \left(\nabla^\perp_{e_i} R^\perp\right)(e_i, X, T(X_1, \ldots, X_r)) \\
&\qquad\quad + 2 \sum_{i=1}^{m} R^\perp(e_i, X)(\nabla^\perp_{e_i} T)(X_1, \ldots, X_r),
\end{aligned}$$

where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame, and the sum with respect to a runs over the tensor slots, i.e., the index

$$a \in \{1, \ldots, r\}$$

indicates the position at which the curvature term acts.

Proof. Let us first compute the Laplacian $\Delta^\perp T$. For simplicity, we perform our computations in the case where T is a 2-tensor. Fix a point $x_0 \in M$ and consider a local orthonormal frame defined on an open neighbourhood U which is normal at the point x_0. Let us begin with a general computation of the Laplacian of the tensor T.

On the neighbourhood U, we have

$$\begin{aligned}(\Delta^\perp T)(e_1,e_2) &= \sum_i (\nabla^\perp_{e_i}\nabla^\perp_{e_i}T)(e_1,e_2) - \sum_i (\nabla^\perp_{\nabla_{e_i}e_i}T)(e_1,e_2)\\ &= \sum_i \nabla^\perp_{e_i}\big\{(\nabla^\perp_{e_i}T)(e_1,e_2)\big\} - \sum_i (\nabla^\perp_{e_i}T)(\nabla_{e_i}e_1,e_2)\\ &\quad - \sum_i (\nabla^\perp_{e_i}T)(e_1,\nabla_{e_i}e_2) - \sum_i \nabla^\perp_{\nabla_{e_i}e_i}T(e_1,e_2)\\ &\quad + \sum_i T(\nabla_{\nabla_{e_i}e_i}e_1,e_2) + \sum_i T(e_1,\nabla_{\nabla_{e_i}e_i}e_2).\end{aligned}$$

Consequently,

$$\begin{aligned}&(\Delta^\perp T)(e_1,e_2)\\ &\quad = \sum_i \nabla^\perp_{e_i}\left\{\nabla^\perp_{e_i}T(e_1,e_2) - T(\nabla_{e_i}e_1,e_2) - T(e_1,\nabla_{e_i}e_2)\right\}\\ &\qquad - \sum_i \nabla^\perp_{e_i}T(\nabla_{e_i}e_1,e_2) + \sum_i T(\nabla_{e_i}\nabla_{e_i}e_1,e_2) + \sum_i T(\nabla_{e_i}e_1,\nabla_{e_i}e_2)\\ &\qquad - \sum_i \nabla^\perp_{e_i}T(e_1,\nabla_{e_i}e_2) + \sum_i T(\nabla_{e_i}e_1,\nabla_{e_i}e_2) + \sum_i T(e_1,\nabla_{e_i}\nabla_{e_i}e_2)\\ &\qquad - \sum_i \nabla^\perp_{\nabla_{e_i}e_i}T(e_1,e_2) + \sum_i T(\nabla_{\nabla_{e_i}e_i}e_1,e_2) + \sum_i T(e_1,\nabla_{\nabla_{e_i}e_i}e_2),\end{aligned}$$

from where it follows that

$$\begin{aligned}&(\Delta^\perp T)(e_1,e_2)\\ &\quad = \sum_i \nabla^\perp_{e_i}\nabla^\perp_{e_i}T(e_1,e_2) - 2\sum_i \nabla^\perp_{e_i}T(\nabla_{e_i}e_1,e_2) - 2\sum_i \nabla^\perp_{e_i}T(e_1,\nabla_{e_i}e_2)\\ &\qquad +2\sum_i T(\nabla_{e_i}e_1,\nabla_{e_i}e_2) + \sum_i T(\nabla_{e_i}\nabla_{e_i}e_1,e_2) + \sum_i T(e_1,\nabla_{e_i}\nabla_{e_i}e_2)\\ &\qquad - \sum_i \nabla^\perp_{\nabla_{e_i}e_i}T(e_1,e_2) + \sum_i T(\nabla_{\nabla_{e_i}e_i}e_1,e_2) + \sum_i T(e_1,\nabla_{\nabla_{e_i}e_i}e_2).\end{aligned}$$

Differentiating and estimating at the point x_0, we obtain

$$(\nabla^\perp\Delta^\perp T)(e_1,e_2,X) = (\nabla^\perp_X\Delta^\perp T)(e_1,e_2) = \nabla^\perp_X\{(\Delta^\perp T)(e_1,e_2)\}.$$

Therefore,

$$
\begin{aligned}
(\nabla^\perp\Delta^\perp T)(e_1,e_2,X) &= \sum_i \nabla^\perp_X\nabla^\perp_{e_i}\nabla^\perp_{e_i}T(e_1,e_2)\\
&\quad -2\sum_i \underbrace{\nabla^\perp_X\nabla^\perp_{e_i}T(\nabla_{e_i}e_1,e_2)}_{A_1} -2\sum_i \underbrace{\nabla^\perp_X\nabla^\perp_{e_i}T(e_1,\nabla_{e_i}e_2)}_{A_2}\\
&\quad +\sum_i \underbrace{(\nabla^\perp_X T)(\nabla_{e_i}\nabla_{e_i}e_1,e_2)}_{A_1} +\sum_i \underbrace{T(\nabla_X\nabla_{e_i}\nabla_{e_i}e_1,e_2)}_{A_1}\\
&\quad +\sum_i \underbrace{(\nabla^\perp_X T)(e_1,\nabla_{e_i}\nabla_{e_i}e_2)}_{A_2} +\sum_i \underbrace{T(e_1,\nabla_X\nabla_{e_i}\nabla_{e_i}e_2)}_{A_2}\\
&\quad -\sum_i \underbrace{\nabla^\perp_X\nabla^\perp_{\nabla_{e_i}e_i}T(e_1,e_2)}_{A_3} +\sum_i T(\nabla_X\nabla_{\nabla_{e_i}e_i}e_1,e_2)\\
&\quad +\sum_i T(e_1,\nabla_X\nabla_{\nabla_{e_i}e_i}e_2).
\end{aligned}
$$

Let us compute each term separately. We have

$$
\begin{aligned}
A_1 &= -\sum_i \nabla^\perp_X\nabla^\perp_{e_i}T(\nabla_{e_i}e_1,e_2) - \sum_i \nabla^\perp_X\nabla^\perp_{e_i}T(\nabla_{e_i}e_1,e_2)\\
&\quad +\sum_i(\nabla^\perp_X T)(\nabla_{e_i}\nabla_{e_i}e_1,e_2) + \sum_{i=1}^m T(\nabla_X\nabla_{e_i}\nabla_{e_i}e_1,e_2).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
A_1 &= -\sum_i \nabla^\perp_X\nabla^\perp_{e_i}T(\nabla_{e_i}e_1,e_2) + \sum_i(\nabla^\perp_X T)(\nabla_{e_i}\nabla_{e_i}e_1,e_2)\\
&\quad +\sum_i T(\nabla_X\nabla_{e_i}\nabla_{e_i}e_1,e_2) - \sum_i \nabla^\perp_X(\nabla^\perp_{e_i}T)(\nabla_{e_i}e_1,e_2)\\
&\quad -\sum_i \nabla^\perp_X T(\nabla_{e_i}\nabla_{e_i}e_1,e_2) - \sum_i \nabla^\perp_X T(\nabla_{e_i}e_1,\nabla_{e_i}e_2).
\end{aligned}
$$

Hence,

$$\begin{aligned}
A_1 \;=\; & -\sum_i \nabla^\perp_X \nabla^\perp_{e_i} T(\nabla_{e_i} e_1, e_2) + \sum_i (\nabla^\perp_X T)(\nabla_{e_i}\nabla_{e_i} e_1, e_2) \\
& + \sum_i T(\nabla_X \nabla_{e_i}\nabla_{e_i} e_1, e_2) - \sum_i (\nabla^\perp_{e_i} T)(\nabla_X \nabla_{e_i} e_1, e_2) \\
& - \sum_i (\nabla^\perp_X T)(\nabla_{e_i}\nabla_{e_i} e_1, e_2) - \sum_i T(\nabla_X \nabla_{e_i}\nabla_{e_i} e_1, e_2) \\
=\; & -\sum_i \nabla^\perp_X \nabla^\perp_{e_i} T(\nabla_{e_i} e_1, e_2) - \sum_i (\nabla^\perp_{e_i} T)(\nabla_X \nabla_{e_i} e_1, e_2).
\end{aligned}$$

Similarly, we have

$$A_2 = -\sum_i \nabla^\perp_X \nabla^\perp_{e_i} T(e_1, \nabla_{e_i} e_2) - \sum_i (\nabla^\perp_{e_i} T)(e_1, \nabla_X \nabla_{e_i} e_2).$$

Moreover,

$$\begin{aligned}
A_3 \;=\; & -\sum_i \nabla^\perp_X \nabla^\perp_{\nabla_{e_i} e_i} T(e_1, e_2) = -\sum_i R(X, \nabla_{e_i} e_i) T(e_1, e_2) \\
& -\sum_i \nabla^\perp_{\nabla_{e_i} e_i} \nabla^\perp_X T(e_1, e_2) - \sum_i \nabla^\perp_{[X, \nabla_{e_i} e_i]} T(e_1, e_2) \\
=\; & -\sum_i \nabla^\perp_{\nabla_X \nabla_{e_i} e_i} T(e_1, e_2) + \sum_i \nabla^\perp_{\nabla_{\nabla_{e_i} e_i} X} T(e_1, e_2) \\
=\; & -\sum_i \nabla^\perp_{\nabla_X \nabla_{e_i} e_i} T(e_1, e_2) = -\sum_i (\nabla^\perp_{\nabla_X \nabla_{e_i} e_i} T)(e_1, e_2) \\
& -\sum_i T(\nabla_{\nabla_X \nabla_{e_i} e_i} e_1, e_2) - \sum_i T(e_1, \nabla_{\nabla_X \nabla_{e_i} e_i} e_2) \\
=\; & -\sum_i (\nabla^\perp_{\nabla_X \nabla_{e_i} e_i} T)(e_1, e_2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& (\nabla^\perp \Delta^\perp T)(e_1, e_2, X) \qquad\qquad (4.28) \\
& \quad = \sum_i \nabla^\perp_X \nabla^\perp_{e_i} \nabla^\perp_{e_i} T(e_1, e_2) - \sum_i \nabla^\perp_X \nabla^\perp_{e_i} T(\nabla_{e_i} e_1, e_2) \\
& \qquad - \sum_i (\nabla^\perp_{e_i} T)(\nabla_X \nabla_{e_i} e_1, e_2) - \sum_i \nabla^\perp_X \nabla^\perp_{e_i} T(e_1, \nabla_{e_i} e_2) \\
& \qquad - \sum_i (\nabla^\perp_{e_i} T)(e_1, \nabla_X \nabla_{e_i} e_2) - \sum_i (\nabla^\perp_{\nabla_X \nabla_{e_i} e_i} T)(e_1, e_2).
\end{aligned}$$

The next step is to compute the Laplacian of the covariant derivative of the normal bundle valued tensor T. We always keep in mind that our choice of frames is such that

$$\nabla_{e_j} e_i = 0$$

for all indices

$$i, j \in \{1, \ldots, m\}.$$

Differentiating and evaluating at the point x_0 we have

$$\begin{aligned}
&(\Delta^\perp \nabla^\perp T)(e_1, e_2, X) \\
&\quad = \sum_i (\nabla^\perp_{e_i} \nabla^\perp_{e_i} \nabla^\perp T - \nabla^\perp_{\nabla_{e_i} e_i} \nabla^\perp T)(e_1, e_2, X) \\
&\quad = \sum_i (\nabla^\perp_{e_i} \nabla^\perp_{e_i} \nabla^\perp T)(e_1, e_2, X) \\
&\qquad - \sum_i (\nabla^\perp_{\nabla_{e_i} e_i} \nabla^\perp T)(e_1, e_2, X) \\
&\quad = \sum_i \nabla^\perp_{e_i} \left\{ (\nabla^\perp_{e_i} \nabla^\perp T)(e_1, e_2, X) \right\} \\
&\quad = \sum_i \nabla^\perp_{e_i} \big\{ \nabla^\perp_{e_i} [(\nabla^\perp T)(e_1, e_2, X)] - (\nabla^\perp T)(\nabla_{e_i} e_1, e_2, X) \\
&\qquad\qquad - (\nabla^\perp T)(e_1, \nabla_{e_i} e_2, X) - (\nabla^\perp T)(e_1, e_2, \nabla_{e_i} X) \big\} .
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\Delta^\perp \nabla^\perp T)(e_1, e_2, X) \\
&\quad = \sum_i \nabla^\perp_{e_i} \{ \nabla^\perp_{e_i} [\nabla^\perp_X T(e_1, e_2) - T(\nabla_X e_1, e_2) - T(e_1, \nabla_X e_2)] \} \\
&\qquad - \sum_i \nabla^\perp_{e_i} \{ \nabla^\perp_X T(\nabla_{e_i} e_1, e_2) - T(\nabla_X \nabla_{e_i} e_1, e_2) - T(\nabla_{e_i} e_1, \nabla_X e_2) \} \\
&\qquad - \sum_i \nabla^\perp_{e_i} \{ \nabla^\perp_X T(e_1, \nabla_{e_i} e_2) - T(\nabla_X e_1, \nabla_{e_i} e_2) - T(e_1, \nabla_X \nabla_{e_i} e_2) \} \\
&\qquad - \sum_i \nabla^\perp_{e_i} \{ \nabla^\perp_{\nabla_{e_i} X} T(e_1, e_2) - T(\nabla_{\nabla_{e_i} X} e_1, e_2) - T(e_1, \nabla_{\nabla_{e_i} X} e_2) \}.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&(\Delta^\perp \nabla^\perp T)(e_1, e_2, X)\\
&= \sum_i \underbrace{\nabla^\perp_{e_i}\nabla^\perp_{e_i}\nabla^\perp_X T(e_1,e_2)}_{\mathcal{A}} - \sum_i \underbrace{\nabla^\perp_{e_i}\nabla^\perp_{e_i} T(\nabla_X e_1, e_2)}_{\mathcal{B}}\\
&\quad - \sum_i \underbrace{\nabla^\perp_{e_i}\nabla^\perp_{e_i} T(e_1, \nabla_X e_2)}_{\mathcal{B}} - \sum_i \underbrace{\nabla^\perp_{e_i}\nabla^\perp_X T(\nabla_{e_i} e_1, e_2)}_{\mathcal{C}}\\
&\quad - \sum_i \underbrace{\nabla^\perp_{e_i}\nabla^\perp_X T(e_1, \nabla_{e_i} e_2)}_{\mathcal{C}} - \sum_i \underbrace{\nabla^\perp_{e_i}\nabla^\perp_{\nabla_{e_i}X} T(e_1, e_2)}_{\mathcal{D}}\\
&\quad + \sum_i \underbrace{\nabla^\perp_{e_i} T(\nabla_X \nabla_{e_i} e_1, e_2)}_{\mathcal{E}} + \sum_i \underbrace{\nabla^\perp_{e_i} T(\nabla_{e_i} e_1, \nabla_X e_2)}_{\mathcal{E}}\\
&\quad + \sum_i \underbrace{\nabla^\perp_{e_i} T(\nabla_X e_1, \nabla_{e_i} e_2)}_{\mathcal{E}} + \sum_i \underbrace{\nabla^\perp_{e_i} T(e_1, \nabla_X \nabla_{e_i} e_2)}_{\mathcal{E}}\\
&\quad + \sum_i \underbrace{\nabla^\perp_{e_i} T(\nabla_{\nabla_{e_i}X} e_1, e_2)}_{\mathcal{E}} + \sum_i \underbrace{\nabla^\perp_{e_i} T(e_1, \nabla_{\nabla_{e_i}X} e_2)}_{\mathcal{E}}.
\end{aligned}
$$

Let us compute separately each term. We have

$$
\begin{aligned}
\mathcal{A} &= \nabla^\perp_{e_i}\nabla^\perp_{e_i}\nabla^\perp_X T(e_1,e_2)\\
&= \nabla^\perp_{e_i}\big(R^\perp(e_i,X)T(e_1,e_2) + \nabla^\perp_X\nabla^\perp_{e_i}T(e_1,e_2) + \nabla^\perp_{[e_i,X]}T(e_1,e_2)\big)\\
&= \nabla^\perp_{e_i}R^\perp(e_i,X)T(e_1,e_2) + \nabla^\perp_{e_i}\nabla^\perp_X\nabla^\perp_{e_i}T(e_1,e_2)\\
&\quad +\nabla^\perp_{e_i}\nabla^\perp_{\nabla_{e_i}X}T(e_1,e_2) - \nabla^\perp_{e_i}\nabla^\perp_{\nabla_X e_i}T(e_1,e_2)\\
&= (\nabla^\perp_{e_i}R^\perp)(e_i,X,T(e_1,e_2)) + R^\perp(e_i,X)(\nabla^\perp_{e_i}T)(e_1,e_2)\\
&\quad +R^\perp(e_i,X)(\nabla^\perp_{e_i}T)(e_1,e_2) + \nabla^\perp_X\nabla^\perp_{e_i}\nabla^\perp_{e_i}T(e_1,e_2)\\
&\quad +(\nabla^\perp_{\nabla_{e_i}\nabla_{e_i}X}T)(e_1,e_2) - (\nabla^\perp_{\nabla_{e_i}\nabla_X e_i}T)(e_1,e_2).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\mathcal{B} &= -\nabla^\perp_{e_i}\nabla^\perp_{e_i}T(\nabla_X e_1,e_2) - \nabla^\perp_{e_i}\nabla^\perp_{e_i}T(e_1,\nabla_X e_2)\\
&= -\nabla^\perp_{e_i}\big((\nabla^\perp_{e_i}T)(\nabla_X e_1,e_2) + T(\nabla_{e_i}\nabla_X e_1,e_2) + T(\nabla_X e_1,\nabla_{e_i}e_2)\big)\\
&\quad -\nabla^\perp_{e_i}\big((\nabla^\perp_{e_i}T)(e_1,\nabla_X e_2) + T(\nabla_{e_i}e_1,\nabla_X e_2) + T(e_1,\nabla_{e_i}\nabla_X e_2)\big)\\
&= -(\nabla^\perp_{e_i}T)(\nabla_{e_i}\nabla_X e_1,e_2) - (\nabla^\perp_{e_i}T)(\nabla_{e_i}\nabla_X e_1,e_2)\\
&\quad -T(\nabla_{e_i}\nabla_{e_i}\nabla_X e_1,e_2) - (\nabla^\perp_{e_i}T)(e_1,\nabla_{e_i}\nabla_X e_2)\\
&\quad -(\nabla^\perp_{e_i}T)(e_1,\nabla_{e_i}\nabla_X e_2) - T(e_1,\nabla_{e_i}\nabla_{e_i}\nabla_X e_2).
\end{aligned}
$$

We leave the term $\mathcal{C}$ as it is because it will be combined with corresponding terms from (4.28). For the term $\mathcal{D}$ we have

$$\begin{aligned}\mathcal{D} &= -\nabla^{\perp}_{e_i}\nabla^{\perp}_{\nabla_{e_i}X}T(e_1,e_2)\\ &= -R^{\perp}(e_i,\nabla_{e_i}X)T(e_1,e_2)\\ &\quad -\nabla^{\perp}_{\nabla_{e_i}X}\nabla^{\perp}_{e_i}T(e_1,e_2) - \nabla^{\perp}_{[e_i,\nabla_{e_i}X]}T(e_1,e_2)\\ &= -\nabla^{\perp}_{[e_i,\nabla_{e_i}X]}T(e_1,e_2) = -\nabla^{\perp}_{\nabla_{e_i}\nabla_{e_i}X}T(e_1,e_2)\\ &= -(\nabla^{\perp}_{\nabla_{e_i}\nabla_{e_i}X}T)(e_1,e_2).\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{E} &= (\nabla^{\perp}_{e_i}T)(\nabla_X\nabla_{e_i}e_1,e_2) + T(\nabla_{e_i}\nabla_X\nabla_{e_i}e_1,e_2)\\ &\quad +(\nabla^{\perp}_{e_i}T)(e_1,\nabla_X\nabla_{e_i}e_2) + T(e_1,\nabla_{e_i}\nabla_X\nabla_{e_i}e_2)\\ &\quad +T(\nabla_{e_i}\nabla_{\nabla_{e_i}X}e_1,e_2) + T(e_1,\nabla_{e_i}\nabla_{\nabla_{e_i}X}e_2).\end{aligned}$$

Therefore,

$$\begin{aligned}&(\Delta^{\perp}\nabla^{\perp}T)(e_1,e_2,X) \qquad (4.29)\\ &= \sum_i \nabla^{\perp}_X\nabla^{\perp}_{e_i}\nabla^{\perp}_{e_i}T(e_1,e_2) - \sum_i \nabla^{\perp}_{e_i}\nabla^{\perp}_X T(e_1,\nabla_{e_i}e_2)\\ &\quad -\sum_i \nabla^{\perp}_{e_i}\nabla^{\perp}_X T(\nabla_{e_i}e_1,e_2) - \sum_i(\nabla^{\perp}_{e_i}T)(R(e_i,X)e_1,e_2)\\ &\quad -\sum_i(\nabla^{\perp}_{e_i}T)(e_1,R(e_i,X)e_2) - \sum_i T(e_1,(\nabla_{e_i}R)(e_i,X,e_2))\\ &\quad -\sum_i T((\nabla_{e_i}R)(e_i,X,e_1),e_2) - \sum_i(\nabla^{\perp}_{e_i}T)(\nabla_{e_i}\nabla_X e_1,e_2)\\ &\quad -\sum_i(\nabla^{\perp}_{e_i}T)(e_1,\nabla_{e_i}\nabla_X e_2) + \sum_i(\nabla^{\perp}_{e_i}R^{\perp})(e_i,X,T(e_1,e_2))\\ &\quad -\sum_i(\nabla^{\perp}_{\nabla_{e_i}\nabla_X e_i}T)(e_1,e_2) + 2\sum_i R^{\perp}(e_i,X)(\nabla^{\perp}_{e_i}T)(e_1,e_2).\end{aligned}$$

Taking the difference between (4.28) and (4.29), we get the following

$$
\begin{aligned}
&(\Delta^{\perp}\nabla^{\perp}T - \nabla^{\perp}\Delta^{\perp}T)(e_1, e_2, X)\\
&\qquad = -\sum_i R^{\perp}(e_i, X)T(\nabla_{e_i}e_1, e_2) - \sum_i R^{\perp}(e_i, X)T(e_1, \nabla_{e_i}e_2)\\
&\qquad\quad -2\sum_i(\nabla^{\perp}_{e_i}T)(e_1, R(e_i, X)e_2) - 2\sum_i(\nabla^{\perp}_{e_i}T)(R(e_i, X)e_1, e_2)\\
&\qquad\quad -\sum_i T(e_1, (\nabla_{e_i}R)(e_i, X, e_2)) - \sum_i T((\nabla_{e_i}R)(e_i, X, e_1), e_2)\\
&\qquad\quad -\sum_i \left(\nabla^{\perp}_{R(e_i, X)e_i}T\right)(e_1, e_2) + \sum_i(\nabla^{\perp}_{e_i}R^{\perp})(e_i, X, T(e_1, e_2))\\
&\qquad\quad +2\sum_i R^{\perp}(e_i, X)(\nabla^{\perp}_{e_i}T)(e_1, e_2).
\end{aligned}
$$

This completes the proof. ☐

4.2.2 Evolution of the metric

Suppose that $F\colon M \times [0, T) \to N$ is a solution of the (MCF).

Lemma 4.2.4. *The following facts are true*

1. *The induced metrics g evolve in time under the equation*
$$(\nabla_{\partial_t}g)(X, Y) = -2g(A(X, Y), H) = -2A^H(X, Y),$$
for every $X, Y \in \mathcal{H}$.
2. *Given* $(x_0, t_0) \in M \times (0, T)$*, there exists an open neighbourhood* U *around* (x_0, t_0) *and a smooth time-dependent orthonormal field* $\{e_1, \ldots, e_m\}$ *on* U*. Moreover, there exists a corresponding time dependent orthonormal frame field along the normal bundles of the evolving submanifolds.*
3. *The induced volume form* $d\mu$ *on* (M, g) *evolves according to the equation*
$$\nabla_{\partial_t}d\mu = -\|H\|^2 d\mu.$$

Proof. Recall that for any $X, Y \in \mathcal{H}$, it holds
$$g(X, Y) = g_N(dF(X), dF(Y)).$$

1. Fix a local coordinate system $\{\partial_1, \ldots, \partial_m; \partial_t\}$. Differentiating with respect to ∂_t, we deduce that

$$\begin{aligned}
(\nabla_{\partial_t} g)(\partial_i, \partial_j) &= \partial_t(g(\partial_i, \partial_j)) - g(\nabla_{\partial_t}\partial_i, \partial_j) - g(\partial_i, \nabla_{\partial_t}\partial_j) \\
&= \partial_t(g_N(dF(\partial_i), dF(\partial_j))) \\
&= (\nabla^N_{dF(\partial_t)} g_N)(dF(\partial_i), dF(\partial_j)) \\
&\quad + g_N(\nabla^F_{\partial_t} dF(\partial_i), dF(\partial_j)) + g_N(dF(\partial_i), \nabla^F_{\partial_t} dF(\partial_j)) \\
&= g_N(\nabla^F_{\partial_i} dF(\partial_t), dF(\partial_j)) + g_N(dF(\partial_i), \nabla^F_{\partial_j} dF(\partial_t)) \\
&= g_N(\nabla^F_{\partial_i} H, dF(\partial_j)) + g_N(dF(\partial_i), \nabla^F_{\partial_j} H) \\
&= -g_N(H, \nabla^F_{\partial_i} dF(\partial_j)) - g_N(\nabla^F_{\partial_j} dF(\partial_i), H) \\
&= -2g_N(A(\partial_i, \partial_j), H) \\
&= -2A^H(\partial_i, \partial_j).
\end{aligned}$$

2. Consider the $(1,1)$-tensor $P\colon \mathcal{H} \to \mathcal{H}$ given by,

$$A^H(X, Y) = g(PX, Y).$$

Let $\{v_1, \ldots, v_m\}$ be an orthonormal frame at $t = 0$. Define a time-dependent orthonormal frame

$$\{e_1 = e_1(t), \ldots, e_m = e_m(t)\}$$

as the unique solution to the system

$$\begin{cases} \nabla_{\partial_t} e_i = Pe_i \\ \\ e_i(0) \ = v_i, \end{cases} , \quad \text{for any } i \in \{1, \ldots, m\}$$

We claim that $\{e_1, \ldots, e_m\}$ is orthonormal for every t. Indeed

$$\begin{aligned}
\partial_t\{g(e_i, e_j)\} &= (\nabla_{\partial_t} g)(e_i, e_j) + g(\nabla_{\partial_t} e_i, e_j) + g(e_i, \nabla_{\partial_t} e_j) \\
&= -2A^H(e_i, e_j) + g(Pe_i, e_j) + g(e_i, Pe_j) \\
&= -2g(Pe_i, e_j) + g(Pe_i, e_j) + g(e_i, Pe_j). \\
&= 0.
\end{aligned}$$

Hence, the functions $\partial_t g(e_i, e_j)$ are independent of time and therefore vanish. Consequently, $\{e_1, \ldots, e_m\}$ is a local time-dependent orthonormal frame. By standard linear algebra, one obtains a local orthonormal frame for the normal bundle of the evolving submanifolds.

3. Denote by $\{\omega_1, \ldots, \omega_m\}$ the corresponding dual forms of $\{e_1, \ldots, e_m\}$ that we constructed above, i.e.,
$$\omega_i(e_j) = \delta_{ij},$$
for all $i, j \in \{1, \ldots, m\}$. Then,
$$d\mu = \omega_1 \wedge \cdots \wedge \omega_m.$$
Observe that
$$\begin{aligned}(\nabla_{\partial_t}\omega_i)(e_j) &= \partial_t(\omega_i(e_j)) - \omega_i(\nabla_{\partial_t} e_j) = -\omega_i(Pe_j) \\ &= -\sum_k \omega_i(g(Pe_j, e_k)e_k) = -\sum_k \omega_i(A^H_{jk} e_k),\end{aligned}$$
from where it follows
$$(\nabla_{\partial_t}\omega_i)(e_j) = -A^H_{ij}.$$
On the other hand
$$\begin{aligned}\nabla_{\partial_t} d\mu &= \nabla_{\partial_t}(\omega_1 \wedge \cdots \wedge \omega_m) \\ &= (\nabla_{\partial_t}\omega_1) \wedge \cdots \wedge \omega_m + \cdots + \omega_1 \wedge \cdots \wedge (\nabla_{\partial_t}\omega_m) \\ &= -\sum_k (A^H_{1k}\omega_k) \wedge \cdots \wedge \omega_m - \cdots - \omega_1 \wedge \cdots \wedge \sum_k (A^H_{mk}\omega_k).\end{aligned}$$
Consequently,
$$\begin{aligned}\nabla_{\partial_t} d\mu &= -(A^H_{11} + A^H_{22} + \cdots + A^H_{mm})\omega_1 \wedge \cdots \wedge \omega_m \\ &= -(A^H_{11} + A^H_{22} + \cdots + A^H_{mm})d\mu.\end{aligned}$$
Hence,
$$\nabla_{\partial_t} d\mu = -\|H\|^2 d\mu.$$

This completes the proof. □

4.2.3 Evolution of the second fundamental form

In this section we will compute the evolution of the second fundamental form of the evolved submanifolds.

Lemma 4.2.5. *The time-derivative of the second fundamental form is given by*

$$(\nabla^{\perp}_{\partial_t} A)^{\alpha}_{ij} = (\nabla^{2\perp} H)^{\alpha}_{ij} - \sum_{\beta,k} H^{\beta} A^{\beta}_{jk} A^{\alpha}_{ik} - \sum_{\beta} H^{\beta} R^{N}_{\beta ij\alpha}$$

where all indices are with respect to a local adapted orthonormal frame.

Proof. Let $\{e_1, \ldots, e_m; \xi_{m+1}, \ldots, \xi_{m+n}\}$ is a local adapted orthonormal frame field around at a fixed point (x_0, t_0). Without loss of generality, we may assume that at (x_0, t_0) it holds

$$\nabla_{e_i} e_j = 0, \quad \text{for all} \quad i, j \in \{1, \ldots, m\}$$

Recall that

$$\nabla_{\partial_t} e_i = [\partial_t, e_i] = Pe_i = \sum_{j,\beta} H^{\beta} A^{\beta}_{ij} e_j.$$

Differentiating and estimating at (x_0, t_0), we have

$$\begin{aligned} (\nabla^F_{\partial_t} A)(e_i, e_j) &= \nabla^F_{\partial_t} A(e_i, e_j) - A\left(\nabla_{\partial_t} e_i, e_j\right) - A\left(e_i, \nabla_{\partial_t} e_j\right) \\ &= \nabla^F_{\partial_t} \nabla^F_{e_i} dF(e_j) - \nabla^F_{\partial_t} dF(\nabla_{e_i} e_j) \\ &\quad - \sum_{k,\beta} H^{\beta} A^{\beta}_{ik} A(e_k, e_j) - \sum_{k,\beta} H^{\beta} A^{\beta}_{jk} A(e_i, e_k). \end{aligned} \tag{4.30}$$

Note that at (x_0, t_0) we have

$$\nabla^F_{\partial_t} dF(\nabla_{e_i} e_j) = \nabla^F_{\nabla_{e_i} e_j} dF(\partial_t) + dF([\partial_t, \nabla_{e_i} e_j]) = dF(\nabla_{\partial_t} \nabla_{e_i} e_j) \tag{4.31}$$

and

$$\begin{aligned} & R^N(dF(\partial_t)), dF(e_i), dF(e_j)) \\ &\quad = \nabla^F_{\partial_t} \nabla^F_{e_i} dF(e_j) - \nabla^F_{e_i} \nabla^F_{\partial_t} dF(e_j) - \nabla^F_{[\partial_t, e_i]} dF(e_j) \\ &\quad = \nabla^F_{\partial_t} \nabla^F_{e_i} dF(e_j) - \nabla^F_{e_i} \nabla^F_{\partial_t} dF(e_j) - \nabla^F_{\nabla_{\partial_t} e_i} dF(e_j) \\ &\quad = \nabla^F_{\partial_t} \nabla^F_{e_i} dF(e_j) - \nabla^F_{e_i} \nabla^F_{\partial_t} dF(e_j) - A(\nabla_{\partial_t} e_i, e_j). \end{aligned} \tag{4.32}$$

Combining (4.31), (4.32) with (4.30), we get

$$\begin{aligned}
(\nabla^F_{\partial_t} A)_{ij} &= \nabla^F_{e_i}\nabla^F_{\partial_t} dF(e_j) + R^N(H, dF(e_i), dF(e_j)) \\
&\quad - \sum_{k,\beta} H^\beta A^\beta_{jk} A_{ik} - dF(\nabla_{\partial_t}\nabla_{e_i} e_j) \\
&= \nabla^F_{e_i}\big(\nabla^F_{e_j} dF(\partial_t) + dF([\partial_t, e_j])\big) - dF(\nabla_{\partial_t}\nabla_{e_i} e_j) \\
&\quad + \sum_{\beta} H^\beta R^N_{\beta ij} - \sum_{k,\beta} H^\beta A^\beta_{jk} A_{ik} \\
&= \nabla^F_{e_i}\nabla^F_{e_j} H + \nabla^F_{e_i} dF(\nabla_{\partial_t} e_j) - dF(\nabla_{\partial_t}\nabla_{e_i} e_j) \\
&\quad + \sum_{\beta} H^\beta R^N_{\beta ij} - \sum_{k,\beta} H^\beta A^\beta_{jk} A_{ik} \\
&= \nabla^F_{e_i}\nabla^F_{e_j} H + A(e_i, \nabla_{\partial_t} e_j) + dF(\nabla_{e_i}\nabla_{\partial_t} e_j - \nabla_{\partial_t}\nabla_{e_i} e_j) \\
&\quad + \sum_{\beta} H^\beta R^N_{\beta ij} - \sum_{k,\beta} H^\beta A^\beta_{jk} A_{ik}.
\end{aligned}$$

Multiplying with ξ_α, we have

$$\begin{aligned}
(\nabla^\perp_{\partial_t} A)^\alpha_{ij} &= g_N(\nabla^F_{e_i}\nabla^F_{e_j} H, \xi_\alpha) + g_N(A(e_i, \nabla_{\partial_t} e_j), \xi_\alpha) \\
&\quad - \sum_{\beta} H^\beta R^N_{\beta ij\alpha} - \sum_{k,\beta} H^\beta A^\beta_{jk} A^\alpha_{ik}.
\end{aligned}$$

From the last equality we deduce that

$$(\nabla^\perp_{\partial_t} A)^\alpha_{ij} = g_N(\nabla^F_{e_i}\nabla^F_{e_j} H, \xi_\alpha) - \sum_{\beta} H^\beta R^N_{\beta ij\alpha}.$$

Let us compute each term of the last equality seperately. We have

$$\begin{aligned}
g_N(\nabla^F_{e_i}\nabla^F_{e_j}H,\xi_a) &= g_N(\nabla^\perp_{e_i}(\nabla^F_{e_j}H),\xi_\alpha)\\
&= g_N(\nabla^\perp_{e_i}\nabla^\perp_{e_i}H,\xi_\alpha) + \sum_k g_N(\nabla^\perp_{e_i}(g_N(\nabla^F_{e_j}H,dF(e_k))dF(e_k)),\xi_\alpha)\\
&= (\nabla^{2\perp}H)^\alpha_{ij} + \sum_k g_N(\nabla^F_{e_j}H,dF(e_k))g_N(\nabla^\perp_{e_i}dF(e_k),\xi_\alpha)\\
&= (\nabla^{2\perp}H)^\alpha_{ij} - \sum_k g_N(H,\nabla^F_{e_j}dF(e_k))g_N(\nabla^F_{e_i}dF(e_k),\xi_\alpha)\\
&= (\nabla^{2\perp}H)^\alpha_{ij} - \sum_k g_N\left(H,A_{jk}\right)g_N\left(A_{ik},\xi_a\right)\\
&= (\nabla^{2\perp}H)^\alpha_{ij} - \sum_{k,\beta} H^\beta A^\beta_{jk}A^\alpha_{ik}.
\end{aligned}$$

Hence, putting everything together we get the desired result. $\square$

Lemma 4.2.6. *The following hold:*

1. *The mean curvature evolves in time under the equation*

$$(\nabla^\perp_{\partial_t}H)^\alpha = (\Delta^\perp H)^\alpha - \sum_{i,\beta} H^\beta R^N_{\beta ii\alpha} + \sum_{i,j,\beta} H^\beta A^\beta_{ij}A^\alpha_{ij},$$

where all indices are with respect to a local adapted orthonormal frame.

2. *The squared norm of the squared mean curvature evolve under the equation*

$$\partial_t\|H\|^2 - \Delta\|H\|^2 = -2\|\nabla^\perp H\|^2 + 2\|A^H\|^2 - \sum_{i,\alpha,\beta} H^\alpha H^\beta R^N_{\alpha ii\beta},$$

where all indices are with respect to a local adapted orthonormal frame on our manifold.

Proof. Consider a fixed point (x_0,t_0) and frames as in the Lemma 4.2.5. Then, we have

$$\begin{aligned}
(\nabla^{\perp}_{\partial_t} H)^{\alpha} &= \sum_i (\nabla^{\perp}_{\partial_t} A(e_i, e_i))^{\alpha} = \sum_i ((\nabla^{\perp}_{\partial_t} A)(e_i, e_i) + 2A(\nabla_{\partial_t} e_i, e_i))^{\alpha} \\
&= \sum_i (\nabla^{\perp}_{\partial_t} A)^{\alpha}_{ii} + 2 \sum_{i,k,\beta} H^{\beta} A^{\beta}_{ik} A^{\alpha}_{ik} \\
&= \sum_i (\nabla^{2\perp} H)^{\alpha}_{ii} - \sum_{i,\beta} H^{\beta} R^{N}_{\beta i i \alpha} - \sum_{i,k,\beta} H^{\beta} A^{\beta}_{ik} A^{\alpha}_{ik} \\
&\quad + 2 \sum_{i,k,\beta} H^{\beta} A^{\beta}_{ik} A^{\alpha}_{ik} \\
&= (\Delta^{\perp} H)^{\alpha} - \sum_{i,\beta} H^{\beta} R^{N}_{\beta i i \alpha} + \sum_{i,k,\beta} H^{\beta} A^{\beta}_{ik} A^{\alpha}_{ik}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_t \|H\|^2 &= 2\langle \nabla^{\perp}_{\partial_t} H, H \rangle \\
&= 2 \sum_{\alpha} H^{\alpha} \langle \nabla^{\perp}_{\partial_t} H, \xi_{\alpha} \rangle = 2 \sum_{\alpha} H^{\alpha} (\nabla^{\perp}_{\partial_t} H)^{\alpha} \\
&= 2 \sum_{\alpha} H^{\alpha} (\Delta^{\perp} H)^{\alpha} - 2 \sum_{i,\alpha,\beta} H^{\alpha} H^{\beta} R^{N}_{\beta i i \alpha} + 2 \sum_{i,k,\alpha,\beta} H^{\alpha} H^{\beta} A^{\beta}_{ik} A^{\alpha}_{ik}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta \|H\|^2 &= \sum_i e_i e_i (\|H\|^2) = 2 \sum_{i=1}^{m} e_i \langle \nabla^{\perp}_{e_i} H, H \rangle \\
&= 2 \sum_{i=1}^{m} \langle \nabla^{\perp}_{e_i} \nabla^{\perp}_{e_i} H, H \rangle + 2 \|\nabla^{\perp} H\|^2 \\
&= 2 \sum_{\alpha} H^{\alpha} \langle \Delta^{\perp} H, \xi_{\alpha} \rangle + 2 \|\nabla^{\perp} H\|^2 \\
&= 2 \sum_{\alpha} H^{\alpha} (\Delta^{\perp} H)^{\alpha} + 2 \|\nabla^{\perp} H\|^2.
\end{aligned}$$

Thus,

$$\partial_t \|H\|^2 - \Delta \|H\|^2 = -2 \|\nabla^{\perp} H\|^2 + 2 \|A^H\|^2 - \sum_{i,\alpha,\beta} H^{\alpha} H^{\beta} R^{N}_{\beta i i \alpha}.$$

This completes the proof. □

4.2.4 Simons' formula

In this subsection, we derive the evolution equation for the squared norm $\|A\|^2$ of the second fundamental form. The preceding subsection established the time derivative of A; it remains to determine its Laplacian. This identity is due to Simons [47], in his seminal work on minimal submanifolds in the sphere. As the arguments are local, we fix a point (x_0, t_0) and select an orthonormal frame $\{e_1, \dots, e_m\}$ in a neighborhood of (x_0, t_0) satisfying

$$\nabla_{e_i} e_j = 0, \quad \text{for all } i, j \in \{1, \dots, m\}.$$

From the Codazzi equation (1.9), we have

$$\begin{aligned}(\nabla^{\perp}_{e_k} A)(e_i, e_j) - (\nabla^{\perp}_{e_i} A)(e_k, e_j) &= \big\{R^N(dF(e_k), dF(e_i))dF(e_j)\big\}^{\perp} \\ &= -\sum_{\alpha} R^N(dF(e_k), dF(e_i), dF(e_j), \xi_\alpha)\xi_\alpha.\end{aligned}$$

Differentiating the preceding identity once more and evaluating at the point (x_0, t_0), we obtain

$$\begin{aligned}(\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k} A)(e_i, e_j) &= \nabla^{\perp}_{e_k}\{(\nabla^{\perp}_{e_k} A)(e_i, e_j)\} = \nabla^{\perp}_{e_k}\{(\nabla^{\perp}_{e_i} A)(e_k, e_j)\} \\ &\quad - \sum_{\alpha} \nabla^{\perp}_{e_k}\big\{R^N\left(dF(e_k), dF(e_i), dF(e_j), \xi_\alpha\right)\xi_\alpha\big\} \\ &= \nabla^{\perp}_{e_k}\big\{\nabla^{\perp}_{e_i} A(e_k, e_j) - A(\nabla_{e_i} e_k, e_j) - A(e_k, \nabla_{e_i} e_j)\big\} \\ &\quad - \sum_{\alpha} e_k\big(R^N(dF(e_k), dF(e_i), dF(e_j), \xi_a)\big)\xi_a \\ &\quad - \sum_{\alpha} R^N\left(dF(e_k), dF(e_i), dF(e_j), \xi_a\right)\nabla^{\perp}_{e_k}\xi_a \\ &= \nabla^{\perp}_{e_k}\nabla^{\perp}_{e_i} A(e_k, e_j) - \nabla^{\perp}_{e_k} A(\nabla_{e_i} e_k, e_j) - \nabla^{\perp}_{e_k} A(e_k, \nabla_{e_i} e_j) \\ &\quad - \sum_{\alpha} e_k\big(R^N(dF(e_k), dF(e_i), dF(e_j), \xi_\alpha)\big)\xi_\alpha \\ &\quad - \sum_{\alpha} R^N\left(dF(e_k), dF(e_i), dF(e_j), \xi_\alpha\right)\nabla^{\perp}_{e_k}\xi_\alpha.\end{aligned}$$

Denote by $\mathcal{A}$ the first three terms in the above equality, which involve the second fundamental form, and by $\mathcal{C}$ the remaining curvature terms. We will compute each of these separately.

We start by computing the term $\mathcal{A}$ at the point (x_0, t_0). We have

$$\begin{aligned}
\mathcal{A} &= \nabla^\perp_{e_k}\nabla^\perp_{e_i}A(e_k, e_j) - A(\nabla_{e_k}\nabla_{e_i}e_k, e_j) - A(e_k, \nabla_{e_k}\nabla_{e_i}e_j) \\
&= \nabla^\perp_{e_i}\nabla^\perp_{e_k}A(e_k, e_j) + R^\perp(e_k, e_i)A(e_k, e_j) \\
&\quad -A(\nabla_{e_k}\nabla_{e_i}e_k, e_j) - A(e_k, \nabla_{e_k}\nabla_{e_i}e_j) \\
&= \nabla^\perp_{e_i}\{\nabla^\perp_{e_k}A(e_j, e_k)\} + R^\perp(e_k, e_i)A(e_k, e_j) \\
&\quad -A(\nabla_{e_k}\nabla_{e_i}e_k, e_j) - A(e_k, \nabla_{e_k}\nabla_{e_i}e_j) \\
&= \nabla^\perp_{e_i}\{(\nabla^\perp_{e_k}A)(e_j, e_k) + A(\nabla_{e_k}e_j, e_k) + A(e_j, \nabla_{e_k}e_k)\} \\
&\quad +R^\perp(e_k, e_i)A(e_k, e_j) - A(\nabla_{e_k}\nabla_{e_i}e_k, e_j) - A(e_k, \nabla_{e_k}\nabla_{e_i}e_j) \\
&= \nabla^\perp_{e_i}\{(\nabla^\perp_{e_k}A)(e_j, e_k)\} + A(\nabla_{e_i}\nabla_{e_k}e_j, e_k) + A(e_j, \nabla_{e_i}\nabla_{e_k}e_k) \\
&\quad -A(\nabla_{e_k}\nabla_{e_i}e_k, e_j) - A(e_k, \nabla_{e_k}\nabla_{e_i}e_j) + R^\perp(e_k, e_i)A(e_k, e_j).
\end{aligned}$$

Thus, by the Codazzi equation (1.9) and the fact that at (x_0, t_0) one has

$$[e_i, e_j] = 0, \qquad \text{for all } i, j \in \{1, \ldots, m\},$$

it follows that

$$\begin{aligned}
\mathcal{A} &= \nabla^\perp_{e_i}\{(\nabla^\perp_{e_j}A)(e_k, e_k)\} - \nabla^\perp_{e_i}\big\{R^N(dF(e_k), dF(e_j), dF(e_k), \xi_\alpha)\big\}^\perp \\
&\quad +R^\perp(e_k, e_i)A(e_k, e_j) + A\left(R(e_i, e_k)e_j, e_k\right) + A\left(e_j, R(e_i, e_k)e_k\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{A} &= \nabla^\perp_{e_i}\big\{\nabla^\perp_{e_j}A(e_k, e_k) - 2A(\nabla_{e_j}e_k, e_k)\big\} \\
&\quad -\sum_\alpha e_i\big\{R^N(dF(e_k), dF(e_j), dF(e_k), \xi_\alpha)\big\}\xi_\alpha \\
&\quad -\sum_\alpha R^N(dF(e_k), dF(e_j), dF(e_k), \xi_\alpha)\nabla^\perp_{e_i}\xi_\alpha \\
&\quad -\sum_l R(e_i, e_k, e_j, e_l)A(e_l, e_k) - \sum_l R(e_i, e_k, e_k, e_l)A(e_j, e_l) \\
&\quad +\sum_\alpha A^\alpha(e_k, e_j)R^\perp(e_k, e_i)\xi_\alpha.
\end{aligned}$$

Expanding the expressions and consistently evaluating at (x_0, t_0), we obtain

$$\begin{aligned}
\mathcal{A} \,=\, & \nabla^\perp_{e_i}\nabla^\perp_{e_j}A(e_k,e_k) - 2A(\nabla_{e_i}\nabla_{e_j}e_k,e_k) \\
& - \sum_\alpha (\nabla^N_{dF(e_i)}R^N)(dF(e_k),dF(e_j),dF(e_k),\xi_\alpha)\xi_\alpha \\
& - \sum_\alpha R^N(A(e_i,e_k),dF(e_j),dF(e_k),\xi_\alpha)\xi_\alpha \\
& - \sum_\alpha R^N(dF(e_k),A(e_i,e_j),dF(e_k),\xi_\alpha)\xi_\alpha \\
& - \sum_\alpha R^N(dF(e_k),dF(e_j),A(e_i,e_k),\xi_\alpha)\xi_\alpha \\
& - \sum_\alpha R^N(dF(e_k),dF(e_j),dF(e_k),\nabla^N_{dF(e_i)}\xi_\alpha)\xi_\alpha \\
& - \sum_\alpha R^N(dF(e_k),dF(e_j),dF(e_k),\xi_\alpha)\nabla^\perp_{e_i}\xi_\alpha \\
& - \sum_l R(e_i,e_k,e_j,e_l)A(e_l,e_k) - \sum_l R(e_i,e_k,e_k,e_l)A(e_j,e_l) \\
& + \sum_\alpha A^\alpha(e_k,e_j)R^\perp(e_k,e_i)\xi_\alpha.
\end{aligned} \tag{4.33}$$

In what follows, we estimate the contributions involving covariant derivatives of ξ_α. First observe that

$$\begin{aligned}
\nabla^N_{dF(e_i)}\xi_\alpha &= (\nabla^N_{dF(e_i)}\xi_\alpha)^\perp + (\nabla^N_{dF(e_i)}\xi_\alpha)^\top \\
&= \nabla^\perp_{e_i}\xi_\alpha + \sum_k \langle \nabla^N_{dF(e_i)}\xi_\alpha, dF(e_k)\rangle dF(e_k) \\
&= \nabla^\perp_{e_i}\xi_\alpha - \sum_k \langle \xi_\alpha, \nabla^N_{dF(e_i)}dF(e_k)\rangle dF(e_k) \\
&= \nabla^\perp_{e_i}\xi_\alpha - \sum_k \langle \xi_\alpha, A(e_i,e_k)\rangle dF(e_k) \\
&= \nabla^\perp_{e_i}\xi_\alpha - \sum_k A^\alpha(e_i,e_k)dF(e_k).
\end{aligned}$$

On the other hand, using the connection forms of the normal bundle, we obtain

$$\nabla^N_{dF(e_i)}\xi_\alpha = -\sum_k A^\alpha_{ik}dF(e_k) + \sum_\beta \omega_{\alpha\beta}(e_i)\xi_\beta.$$

Therefore,

$$
\begin{aligned}
&-\sum_{\alpha} R^N(dF(e_k), dF(e_j), dF(e_k), \nabla^N_{dF(e_i)}\xi_\alpha)\xi_\alpha \\
&-\sum_{\alpha} R^N(dF(e_k), dF(e_j), dF(e_k), \xi_\alpha)\nabla^\perp_{e_i}\xi_\alpha \\
&\quad = \sum_{\alpha,l} A^\alpha_{il} R^N_{kjkl}\xi_\alpha - \sum_{\alpha,\beta} \omega_{\alpha\beta}(e_i) R^N_{kjk\beta}\xi_\alpha - \sum_{\alpha,\beta} \omega_{\alpha\beta}(e_i) R^N_{kjk\alpha}\xi_\beta \\
&\quad = \sum_{\alpha,l} A^\alpha_{il} R^N_{kjkl}\xi_\alpha - \sum_{\alpha,\beta} \omega_{\alpha\beta}(e_i) R^N_{kjk\beta}\xi_\alpha - \sum_{\alpha,\beta} \omega_{\beta\alpha}(e_i) R^N_{kjk\beta}\xi_\alpha \\
&\quad = \sum_{\alpha,l} A^\alpha_{il} R^N_{kjkl}\xi_\alpha.
\end{aligned}
$$

from where it follows that

$$
\begin{aligned}
\mathcal{A} \;=\; & \nabla^\perp_{e_i}\nabla^\perp_{e_j} A_{kk} - 2A(\nabla_{e_i}\nabla_{e_j} e_k, e_k) \\
& - \sum_{\alpha} (\nabla^N_{dF(e_i)} R^N)_{kjk\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^\beta_{ik} R^N_{\beta jk\alpha}\xi_\alpha \\
& - \sum_{\alpha,\beta} A^\beta_{ij} R^N_{k\beta k\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^\beta_{ik} R^N_{kj\beta\alpha}\xi_\alpha \\
& + \sum_{\alpha,l} A^\alpha_{il} R^N_{kjkl}\xi_\alpha - \sum_{\alpha,l} R_{ikjl} A^\alpha_{lk}\xi_\alpha \\
& - \sum_{\alpha,l} R_{ikkl} A^\alpha_{jl}\xi_\alpha + \sum_{\beta} A^\beta_{kj} R^\perp(e_k, e_i)\xi_\beta.
\end{aligned}
$$

From the Gauss equation (1.8), one has

$$
R_{ijkl} = R^N_{ijkl} + \sum_\alpha A^\alpha_{ik} A^\alpha_{jl} - \sum_\alpha A^\alpha_{il} A^\alpha_{jk}.
$$

Moreover, the Ricci equation (1.10) yields

$$
R^\perp(e_i, e_j)\xi_\beta = -\sum_\alpha R^N_{ij\beta\alpha}\,\xi_\alpha + \sum_{\alpha,l} A^\beta_{jl} A^\alpha_{il}\,\xi_\alpha - \sum_{\alpha,l} A^\alpha_{jl} A^\beta_{il}\,\xi_\alpha.
$$

Taking into account the Gauss and Ricci equations, we obtain

$$\begin{aligned}
\mathcal{A} \;=\; & \nabla^{\perp}_{e_i}\nabla^{\perp}_{e_j}A_{kk} - 2A(\nabla_{e_i}\nabla_{e_j}e_k, e_k) - \sum_{\alpha}(\nabla^N_{dF(e_i)}R^N)_{kjk\alpha}\xi_\alpha \\
& - \sum_{\alpha,\beta} A^{\beta}_{ik}R^N_{\beta jk\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^{\beta}_{ij}R^N_{k\beta k\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^{\beta}_{ik}R^N_{kj\beta\alpha}\xi_\alpha \\
& + \sum_{\alpha,l} A^{\alpha}_{il}R^N_{kjkl}\xi_\alpha - \sum_{\alpha,l} A^{\alpha}_{lk}R^N_{ikjl}\xi_\alpha - \sum_{\alpha,\beta,l} A^{\alpha}_{lk}A^{\beta}_{ij}A^{\beta}_{kl}\xi_\alpha \\
& + \sum_{\alpha,\beta,l} A^{\alpha}_{jl}A^{\beta}_{il}A^{\beta}_{kk}\xi_\alpha + \sum_{\alpha,\beta,l} A^{\alpha}_{lk}A^{\beta}_{il}A^{\beta}_{jk}\xi_\alpha - \sum_{a,l} A^{\alpha}_{jl}R^N_{ikkl}\xi_\alpha \\
& - \sum_{\alpha,\beta,l} A^{\alpha}_{jl}A^{\beta}_{ik}A^{\beta}_{kl}\xi_\alpha - \sum_{\alpha,\beta} A^{\beta}_{kj}R^N_{ki\beta\alpha}\xi_\alpha + \sum_{\alpha,\beta,l} A^{\beta}_{kj}A^{\beta}_{il}A^{\alpha}_{kl}\xi_\alpha \\
& - \sum_{\alpha,\beta,l} A^{\beta}_{kj}A^{\alpha}_{il}A^{\beta}_{kl}\xi_\alpha.
\end{aligned}$$

We now compute the term $\mathcal{C}$. We have

$$\begin{aligned}
\mathcal{C} \;=\; & -\sum_{\alpha} e_k\big(R^N(dF(e_k), dF(e_i), dF(e_j), \xi_\alpha)\big)\xi_\alpha \\
& -\sum_{\alpha} R^N(dF(e_k), dF(e_i), dF(e_j), \xi_\alpha)\nabla^{\perp}_{e_k}\xi_\alpha \\
=\; & -\sum_{\alpha}(\nabla^N_{dF(e_k)}R^N)(e_k, e_i, e_j, \xi_\alpha)\xi_\alpha \\
& -\sum_{\alpha} R^N(A(e_k, e_k), dF(e_i), dF(e_j), \xi_\alpha)\xi_\alpha \\
& -\sum_{\alpha} R^N(dF(e_k), A(e_k, e_i), dF(e_j), \xi_\alpha)\xi_\alpha \\
& -\sum_{\alpha} R^N(dF(e_k), dF(e_i), A(e_k, e_j), \xi_\alpha)\xi_\alpha \\
& -\sum_{\alpha} R^N(dF(e_k), dF(e_i), dF(e_j), \nabla^N_{dF(e_k)}\xi_\alpha)\xi_\alpha \\
& -\sum_{\alpha} R^N(dF(e_k), dF(e_i), dF(e_j), \xi_\alpha)\nabla^{\perp}_{e_k}\xi_\alpha.
\end{aligned}$$

As before, the final two terms in the last identity combine, yielding

$$\begin{aligned}\mathcal{C} \;=\; &-\sum_{\alpha}\left(\nabla^N_{dF(e_k)}R^N\right)_{kija}\xi_\alpha-\sum_{\alpha,\beta}A^\beta_{kk}R^N_{\beta ij\alpha}\xi_\alpha\\ &-\sum_{\alpha,\beta}A^\beta_{ki}R^N_{k\beta j\alpha}\xi_\alpha-\sum_{\alpha,\beta}A^\beta_{kj}R^N_{ki\beta\alpha}\xi_\alpha+\sum_{\alpha,l}A^\alpha_{kl}R^N_{kijl}\xi_\alpha.\end{aligned}$$

Before adding $\mathcal{A}+\mathcal{C}$, lets us also observe that

$$\begin{aligned}\sum_k A\left(\nabla_{e_i}\nabla_{e_j}e_k,e_k\right) &= \sum_{k,l}A\left(\nabla_{e_i}(\omega_{kl}(e_j)e_l),e_k\right)\\ &= \sum_{k,l}A\left(e_i(\omega_{kl}(e_j))e_l+\omega_{kl}(e_j)\nabla_{e_i}e_l,e_k\right)\\ &= \sum_{k,l}\underbrace{e_i(\omega_{kl}(e_j))}_{\text{antisymmetric}}\;\underbrace{A_{lk}}_{\text{symmetric}}\\ &= 0.\end{aligned}$$

At the point (x_0,t_0), we have

$$(\Delta^\perp A)^\alpha_{ij}=\sum_k\left(\nabla^\perp_{e_k}\nabla^\perp_{e_k}A-\nabla^\perp_{\nabla_{e_k}e_k}A\right)^\alpha_{ij}=\sum_k\left(\nabla^\perp_{e_k}\nabla^\perp_{e_k}A\right)^\alpha_{ij}.$$

Taking $\mathcal{A}+\mathcal{C}$ and summing over the index k, we obtain the Laplacian of the second fundamental form. In particular,

$$\begin{aligned}(\Delta^\perp A)^\alpha_{ij} \;=\; &(\nabla^{2\perp}H)^\alpha_{ij}-\sum_k\left(\nabla^N_{dF(e_i)}R^N\right)_{kjk\alpha}-\sum_{\beta,k}A^\beta_{ik}R^N_{\beta jk\alpha}\\ &-\sum_{\beta,k}A^\beta_{ij}R^N_{k\beta k\alpha}-\sum_{\beta,k}A^\beta_{ik}R^N_{kj\beta\alpha}+\sum_{k,l}A^\alpha_{il}R^N_{kjkl}\\ &-\sum_{k,l}A^\alpha_{lk}R^N_{ikjl}-\sum_{\beta,k,l}A^\alpha_{lk}A^\beta_{ij}A^\beta_{kl}+\sum_{\beta,k,l}A^\alpha_{lk}A^\beta_{il}A^\beta_{jk}\\ &-\sum_{k,l}A^\alpha_{jl}R^N_{ikkl}-\sum_{\beta,k,l}A^\alpha_{jl}A^\beta_{ik}A^\beta_{kl}-\sum_{\beta,k}A^\beta_{kj}R^N_{ki\beta\alpha}\\ &+\sum_{\beta,k,l}A^\beta_{kj}A^\beta_{il}A^\alpha_{kl}-\sum_{\beta,k,l}A^\beta_{kj}A^\alpha_{il}A^\beta_{kl}-\sum_k\left(\nabla^N_{dF(e_k)}R^N\right)_{kij\alpha}\\ &-\sum_{\beta,k}A^\beta_{kk}R^N_{\beta ij\alpha}-\sum_{\beta,k}A^\beta_{ki}R^N_{k\beta j\alpha}-\sum_{\beta,k}A^\beta_{kj}R^N_{ki\beta\alpha}\\ &+\sum_{k,l}A^\alpha_{kl}R^N_{kijl}+\sum_{\beta,k,l}A^\beta_{kk}A^\beta_{il}A^\alpha_{lj}.\end{aligned}$$

Using the first Bianchi identity (1.1), we obtain

$$R^N_{\beta jk\alpha} + R^N_{kj\beta\alpha} + R^N_{k\beta j\alpha} = -R^N_{jk\beta\alpha} + R^N_{kj\beta\alpha} = 2R^N_{kj\beta\alpha}.$$

Combining these identities, we deduce

$$\begin{aligned}
(\Delta^{\perp} A)^{\alpha}_{ij} \;=\; & \big(\nabla^{2\perp} H\big)^{\alpha}_{ij} - \sum_k \big(\nabla^N_i R^N\big)_{kjk\alpha} - \sum_k \big(\nabla^N_k R^N\big)_{kij\alpha} \\
& - \sum_{\beta} H^{\beta} R^N_{\beta ij\alpha} - \sum_{\beta,k} A^{\beta}_{ij} R^N_{k\beta k\alpha} + \sum_{k,l} A^{\alpha}_{il} R^N_{kjkl} \\
& + 2\sum_{k,l} A^{\alpha}_{kl} R^N_{kijl} + \sum_{k,l} A^{\alpha}_{jl} R^N_{kikl} - 2\sum_{\beta,k} A^{\beta}_{kj} R^N_{ki\beta\alpha} \\
& - 2\sum_{\beta,k} A^{\beta}_{ik} R^N_{kj\beta\alpha} + \sum_{\beta,k,l} A^{\alpha}_{kl}\big\{A^{\beta}_{il}A^{\beta}_{kj} - A^{\beta}_{ij}A^{\beta}_{kl}\big\} \\
& + \sum_{\beta,k,l} A^{\alpha}_{lj}\big\{A^{\beta}_{kk}A^{\beta}_{il} - A^{\beta}_{ik}A^{\beta}_{kl}\big\} + \sum_{\beta,k,l} A^{\beta}_{kj}\big\{A^{\beta}_{il}A^{\alpha}_{kl} - A^{\alpha}_{il}A^{\beta}_{kl}\big\}.
\end{aligned}$$

Recall from Lemma 4.2.5 that the second fundamental form satisfies the following evolution equation

$$(\nabla^{\perp}_{\partial_t} A)^{a}_{ij} = \big(\nabla^{2\perp} H\big)^{\alpha}_{ij} - \sum_{\beta,k} H^{\beta} A^{\beta}_{jk} A^{a}_{ik} - \sum_{\beta} H^{\beta} R^N_{\beta ija}.$$

Combining the preceding identities, we obtain the evolution under (MCF) of the second fundamental form. In particular,

$$\begin{aligned}
\big(\nabla^{\perp}_{\partial_t} A\big)^{\alpha}_{ij} - \big(\Delta^{\perp} A\big)^{\alpha}_{ij} = & \sum_k \big(\nabla^N_i R^N\big)_{kjk\alpha} + \sum_k \big(\nabla^N_k R^N\big)_{kij\alpha} \\
& + \sum_{\beta,k} A^{\beta}_{ij} R^N_{k\beta k\alpha} - \sum_{k,l} A^{\alpha}_{il} R^N_{kjkl} - 2\sum_{k,l} A^{\alpha}_{kl} R^N_{kijl} \\
& - \sum_{k,l} A^{\alpha}_{jl} R^N_{kikl} + 2\sum_{\beta,k} A^{\beta}_{kj} R^N_{ki\beta\alpha} + 2\sum_{\beta,k} A^{\beta}_{ik} R^N_{kj\beta\alpha} \\
& - \sum_{\beta,k,l} A^{\alpha}_{kl}\big\{A^{\beta}_{il}A^{\beta}_{kj} - A^{\beta}_{ij}A^{\beta}_{kl}\big\} - \sum_{\beta,k,l} A^{\alpha}_{lj}\big\{A^{\beta}_{kk}A^{\beta}_{il} - A^{\beta}_{ik}A^{\beta}_{kl}\big\} \\
& - \sum_{\beta,k,l} A^{\beta}_{kj}\big\{A^{\beta}_{il}A^{\alpha}_{kl} - A^{\alpha}_{il}A^{\beta}_{kl}\big\} - \sum_{\beta} H^{\beta} A^{\beta}_{jk} A^{\alpha}_{ik}.
\end{aligned}$$

We now compute the evolution of $\|A\|^2$ under the mean curvature flow (MCF). We begin with its time derivative. We have

$$
\begin{aligned}
\partial_t\|A\|^2 &= 2\sum_{i,j}\langle\nabla^\perp_{\partial_t}A(e_i,e_j),A(e_i,e_j)\rangle \\
&= \sum_{i,j}\langle(\nabla^\perp_{\partial_t}A)(e_i,e_j),A(e_i,e_j)\rangle \\
&\quad +2\sum_{i,j}\langle A(\nabla_{\partial_t}e_i,e_j),A(e_i,e_j)\rangle+2\sum_{i,j}\langle A(e_i,\nabla_{\partial_t}e_j),A(e_i,e_j)\rangle.
\end{aligned}
$$

Using that our orthonormal frames are chosen so that

$$\nabla_{\partial_t}e_i=\sum_{j,\beta}H^\beta A^\beta_{ij}e_j$$

we deduce that

$$\partial_t\|A\|^2=2\sum_{i,j}\langle(\nabla^\perp_{\partial_t}A)_{ij},A_{ij}\rangle+2\sum_{i,j,k,\alpha,\beta}H^\beta A^\beta_{ik}A^\alpha_{kj}A^\alpha_{ij}+2\sum_{i,j,k,\alpha,\beta}H^\beta A^\beta_{jk}A^\alpha_{ik}A^\alpha_{ij}.$$

It remains to compute the Laplacian of $\|A\|^2$. As the calculations are pointwise, we may assume that at a fixed point $(x_0,t_0)\in M\times[0,T)$ the frame satisfies

$$\nabla_{e_i}e_j=0,\quad \text{for all } i,j\in\{1,\dots,m\}.$$

At the point (x_0,t_0), from Lemma 1.1.8 we have

$$\Delta\|A\|^2=2\sum_{i,j}\langle\left(\Delta^\perp A\right)(e_i,e_j),A(e_i,e_j)\rangle+2\|\nabla^\perp A\|^2.$$

Hence,

$$
\begin{aligned}
\partial_t\|A\|^2-\Delta\|A\|^2 &= 2\sum_{i,j,\alpha}\left(\nabla^\perp_{\partial_t}A-\Delta^\perp A\right)^\alpha_{ij}A^\alpha_{ij}-2\|\nabla^\perp A\|^2 \\
&\quad +2\sum_{i,j,k,\alpha,\beta}H^\beta A^\beta_{ik}A^\alpha_{kj}A^\alpha_{ij}+2\sum_{i,j,k,\alpha,\beta}H^\beta A^\beta_{jk}A^\alpha_{ik}A^\alpha_{ij}.
\end{aligned}
$$

Using the result of the previous lemma, we get

$$
\begin{aligned}
\partial_t\|A\|^2 \; - \; \Delta\|A\|^2 = -2\|\nabla^{\perp}A\|^2 &+ 2\sum_{i,j,k,\alpha}\left(\nabla_i^N R^N\right)_{kjk\alpha} A_{ij}^{\alpha} \\
&+2\sum_{i,j,k,\alpha}\left(\nabla_k^N R^N\right)_{kij\alpha} A_{ij}^{\alpha} + \underbrace{2\sum_{i,j,k,\alpha,\beta} A_{ij}^{\beta}A_{ij}^{\alpha}R_{k\beta k\alpha}^N}_{A_1} \\
&\underbrace{-2\sum_{i,j,k,l,\alpha} A_{il}^{\alpha}A_{ij}^{\alpha}R_{kjkl}^N}_{B_1}\underbrace{-2\sum_{i,j,k,l,\alpha} A_{lj}^{\alpha}A_{ij}^{\alpha}R_{kikl}^N}_{B_1} \\
&\underbrace{+4\sum_{i,j,k,\alpha,\beta} A_{kj}^{\beta}A_{ij}^{\alpha}R_{ki\beta\alpha}^N}_{A_1}\underbrace{-4\sum_{i,j,k,l,\alpha} A_{kl}^{\alpha}A_{ij}^{\alpha}R_{kijl}^N}_{B_1} \\
&\underbrace{+4\sum_{i,j,k,\alpha,\beta} A_{ik}^{\beta}A_{ij}^{\alpha}R_{kj\beta\alpha}^N}_{A_1}\underbrace{-2\sum_{i,j,k,l,\alpha,\beta} A_{kl}^{\alpha}A_{ij}^{\alpha}A_{kj}^{\beta}A_{il}^{\beta}}_{\Gamma_1} \\
&\underbrace{+2\sum_{i,j,k,l,\alpha,\beta} A_{kl}^{\alpha}A_{ij}^{\alpha}A_{ij}^{\beta}A_{kl}^{\beta}}_{\Delta_1}\underbrace{+2\sum_{i,j,k,l,\alpha,\beta} A_{lj}^{\alpha}A_{ij}^{\alpha}A_{ik}^{\beta}A_{kl}^{\beta}}_{\Delta_1} \\
&\underbrace{-2\sum_{i,j,k,l,\alpha,\beta} A_{kj}^{\beta}A_{ij}^{\alpha}A_{il}^{\beta}A_{kl}^{\alpha}}_{\Gamma_1}\underbrace{+2\sum_{i,j,k,l,\alpha,\beta} A_{kj}^{\beta}A_{ij}^{\alpha}A_{kl}^{\beta}A_{il}^{\alpha}}_{\Delta_1}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
A_1 &= 2\sum_{i,j,k,\alpha,\beta} A_{ij}^{\beta}A_{ij}^{\alpha}R_{k\beta k\alpha}^N + 8\sum_{i,j,k,\alpha,\beta} A_{jk}^{\beta}A_{ij}^{\alpha}R_{ki\beta\alpha}^N \\
&= 2\sum_{i,j,k,\alpha,\beta} A_{ij}^{\beta}A_{ij}^{\alpha}R_{k\beta k\alpha}^N + 8\sum_{i,j,k,\alpha,\beta} A_{jk}^{\alpha}A_{ik}^{\beta}R_{ji\alpha\beta}^N \\
&= 2\sum_{i,j,k,\alpha,\beta} A_{ij}^{\beta}A_{ij}^{\alpha}R_{k\beta k\alpha}^N + 8\sum_{i,j,k,\alpha,\beta} A_{jk}^{\alpha}A_{ik}^{\beta}R_{\alpha\beta ji}^N \\
&= 2\sum_{i,j,k,\alpha,\beta}\left(4A_{jk}^{\alpha}A_{ik}^{\beta}R_{\alpha\beta ji}^N + A_{jk}^{\alpha}A_{jk}^{\beta}R_{i\beta i\alpha}^N\right).
\end{aligned}
$$

Hence, we obtain

$$A_1 = 2 \sum_{i,j,k,\alpha,\beta} \left(4A^{\alpha}_{jk} A^{\beta}_{ik} R^N_{\alpha\beta ji} + A^{\alpha}_{jk} A^{\beta}_{jk} R^N_{\alpha i\beta i} \right).$$

On the other hand note that

$$\begin{aligned}
B_1 &= -2 \sum_{i,j,k,l,\alpha} A^{\alpha}_{il} A^{\alpha}_{ij} R^N_{kjkl} - 2 \sum_{i,j,k,l,\alpha} A^{\alpha}_{lj} A^{\alpha}_{ij} R^N_{kikl} - 4 \sum_{i,j,k,l,\alpha} A^{\alpha}_{kl} A^{\alpha}_{ij} R^N_{kijl} \\
&= -4 \sum_{i,j,k,l,\alpha} A^{\alpha}_{il} A^{\alpha}_{ij} R^N_{kjkl} - 4 \sum_{i,j,k,l,\alpha} A^{\alpha}_{kl} A^{\alpha}_{ij} R^N_{kijl} \\
&= -4 \sum_{i,j,k,l,a} A^{\alpha}_{il} A^{\alpha}_{lj} R^N_{kjki} - 4 \sum_{i,j,k,l,\alpha} A^{\alpha}_{kl} A^{\alpha}_{ij} R^N_{kijl} \\
&= -4 \sum_{i,j,k,\alpha} \Big(\sum_{p} A^{\alpha}_{ip} A^{\alpha}_{jp} \Big) R^N_{kjki} - 4 \sum_{i,j,k,l,\alpha=1}^{m} A^{\alpha}_{kl} A^{\alpha}_{ij} R^N_{kijl} \\
&= -4 \sum_{i,j,k,l,\alpha} \delta_{kl} \Big(\sum_{p} A^{\alpha}_{ip} A^{\alpha}_{jp} R^N_{kjli} \Big) - 4 \sum_{i,j,k,l,\alpha} A^{\alpha}_{kl} A^{\alpha}_{ij} R^N_{kijl} \\
&= -4 \sum_{i,j,k,l,\alpha} \delta_{kl} \Big(\sum_{p} A^{\alpha}_{ip} A^{\alpha}_{jp} R^N_{kilj} \Big) - 4 \sum_{i,j,k,l,\alpha} A^{\alpha}_{kl} A^{\alpha}_{ij} R^N_{kilj}.
\end{aligned}$$

Therefore,

$$B_1 = 4 \sum_{i,j,k,l,\alpha} \Big\{ A^{\alpha}_{kl} A^{\alpha}_{ij} - \delta_{kl} \sum_{p} A^{\alpha}_{ip} A^{\alpha}_{jp} \Big\} R^N_{kilj}.$$

Additionally,

$$\begin{aligned}
\Gamma_1 &= -2 \sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{kl} A^{\alpha}_{ij} A^{\beta}_{kj} A^{\beta}_{il} - 2 \sum_{i,j,k,l,\alpha,\beta} A^{\beta}_{kj} A^{\alpha}_{ij} A^{\beta}_{il} A^{\alpha}_{kl} \\
&= -4 \sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{kl} A^{\beta}_{kj} A^{\alpha}_{ij} A^{\beta}_{il} \\
&= -4 \sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ik} A^{\beta}_{kj} A^{\alpha}_{lj} A^{\beta}_{li}.
\end{aligned}$$

Hence,

$$\Gamma_1 = -4 \sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ik} A^{\beta}_{kj} A^{\alpha}_{lj} A^{\beta}_{li}.$$

Finally,

$$
\begin{aligned}
\Delta_1 &= 2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{kl}A^{\alpha}_{ij}A^{\beta}_{ij}A^{\beta}_{kl} + 2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{lj}A^{\alpha}_{ij}A^{\beta}_{ik}A^{\beta}_{kl} \\
&\quad +2\sum_{i,j,k,l,\alpha,\beta} A^{\beta}_{kj}A^{\alpha}_{ij}A^{\beta}_{kl}A^{\alpha}_{il} \\
&= 2\sum_{i,j,k,l}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2 + 2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ij}A^{\alpha}_{jl}A^{\beta}_{ik}A^{\beta}_{kl} \\
&\quad +2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ij}A^{\beta}_{jk}A^{\alpha}_{il}A^{\beta}_{lk} \\
&= 2\sum_{i,j,k,l}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2 + 2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ji}A^{\beta}_{ik}A^{\alpha}_{jl}A^{\beta}_{lk} \\
&\quad +2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ij}A^{\beta}_{jk}A^{\alpha}_{il}A^{\beta}_{lk}.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\Delta_1 &= 2\sum_{i,j,k,l}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2 + 2\sum_{j,k,\alpha,\beta}\Big(\sum_{l} A^{\alpha}_{jl}A^{\beta}_{lk}\Big)^2 \\
&\quad +2\sum_{i,k,\alpha,\beta}\Big(\sum_{l} A^{\alpha}_{il}A^{\beta}_{lk}\Big)^2 \\
&= 2\sum_{i,j,k,l=1}^{m}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big(\sum_{l} A^{\alpha}_{jl}A^{\beta}_{li}\Big)^2 \\
&\quad +2\sum_{i,j,\alpha,\beta}\Big(\sum_{l} A^{\alpha}_{il}A^{\beta}_{lj}\Big)^2 \\
&= 2\sum_{i,j,k,l=1}^{m}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big(\sum_{k} A^{\alpha}_{jk}A^{\beta}_{ki}\Big)^2 \\
&\quad +2\sum_{i,j,\alpha,\beta}\Big(\sum_{k} A^{\alpha}_{ik}A^{\beta}_{kj}\Big)^2.
\end{aligned}
$$

Therefore,

$$
\Delta_1 = 2\sum_{i,j,k,l}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big(\sum_{k} A^{\alpha}_{ik}A^{\beta}_{kj}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big(\sum_{k} A^{\alpha}_{jk}A^{\beta}_{ki}\Big)^2.
$$

Now, adding Γ_1 and Δ_1 we have

$$\begin{aligned}\Gamma_1 + \Delta_1 &= 2\sum_{i,j,k,l}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big(\sum_{k} A^{\alpha}_{ik}A^{\beta}_{kj}\Big)^2 \\ &\quad +2\sum_{i,j,\alpha,\beta}\Big(A^{\beta}_{ik}A^{\alpha}_{kj}\Big)^2 - 4\sum_{i,j,\alpha,\beta}\Big(\sum_{k,l} A^{\alpha}_{ik}A^{\beta}_{kj}A^{\beta}_{il}A^{\alpha}_{lj}\Big) \\ &= 2\sum_{i,j,\alpha,\beta}\Big\{\sum_{k}\Big(A^{\alpha}_{ik}A^{\beta}_{kj} - A^{\beta}_{ik}A^{\alpha}_{kj}\Big)\Big\}^2 + 2\sum_{i,j,k,l}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2.\end{aligned}$$

Collecting the above identities, we arrive at the following result.

Theorem 4.2.7 (Simons' formula)**.** *Let $F : M \times [0,T) \to N$ be a solution of* (MCF)*. The squared norm $\|A\|^2$ of the second fundamental form A evolves in time under the equation*

$$\begin{aligned}\partial_t\|A\|^2 - \Delta\|A\|^2 &= -2\|\nabla^{\perp}A\|^2 + 2\|A\circ A\|^2 + 2\|\sigma^{\perp}\|^2 \\ &+ 4\sum_{i,j,k,l,\alpha}\{A^{\alpha}_{kl}A^{\alpha}_{ij} - \delta_{kl}\sum_{p}A^{\alpha}_{ip}A^{\alpha}_{pj}\}R^N_{kilj} \\ &+ 2\sum_{i,j,k,\alpha,\beta}\{4A^{\alpha}_{jk}A^{\beta}_{ik}R^N_{\alpha\beta ji} + A^{\alpha}_{jk}A^{\beta}_{jk}R^N_{\alpha i\beta i}\} \\ &+ 2\sum_{i,j,k,\alpha}A^{\alpha}_{ij}\{(\nabla^N_i R^N)_{kjk\alpha} + (\nabla^N_k R^N)_{kij\alpha}\},\end{aligned}$$

where $A\circ A$ is the $(4,0)$-tensor given by

$$(A\circ A)(X,Y,Z,W) = \langle A(X,Y), A(Z,W)\rangle,$$

and

$$\|\sigma^{\perp}\|^2 = \sum_{i,j,\alpha,\beta}\{\sum_{k}(A^{\alpha}_{ik}A^{\beta}_{kj} - A^{\beta}_{ik}A^{\alpha}_{kj})\}^2.$$

We now examine how this formula simplifies in the case of space forms. Recall that in this case

$$R^N_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

Theorem 4.2.8. *In a space of constant sectional curvature $c \in \mathbb{R}$, we have*

$$\begin{aligned}\partial_t\|A\|^2 &= \Delta\|A\|^2 - 2\|\nabla^{\perp}A\|^2 \\ &\quad +2\|A\circ A\|^2 + 2\|\sigma^{\perp}\|^2 + 4c\|H\|^2 - 2mc\|A\|^2.\end{aligned}$$

In codimension one, the above formula becomes

$$\partial_t \|A\|^2 = \Delta \|A\|^2 - 2\|\nabla A\|^2 + 2\|A\|^2(\|A\|^2 - mc) + 4c\|H\|^2.$$

In particular, for codimension one minimal submanifolds we get

$$\Delta \|A\|^2 = 2\|\nabla A\|^2 + 2\|A\|^2(mc - \|A\|^2).$$

Proof. We have

$$\begin{aligned}
\mathcal{A}_1 &= 4 \sum_{i,j,k,l,\alpha} \big\{A^\alpha_{kl} A^\alpha_{ij} - \delta_{kl} \sum_p A^\alpha_{ip} A^\alpha_{pj}\big\} R^N_{kilj} \\
&= 4c \sum_{i,j,k,l,\alpha} A^\alpha_{kl} A^\alpha_{ij} (\delta_{kl}\delta_{ij} - \delta_{il}\delta_{kj}) - 4c \sum_{i,j,k,l,p,\alpha} \delta_{kl} A^\alpha_{ip} A^\alpha_{pj} (\delta_{kl}\delta_{ij} - \delta_{il}\delta_{kj}) \\
&= 4c \sum_{i,k,\alpha} A^\alpha_{kk} A^\alpha_{ii} - 4c \sum_{i,j,\alpha} (A^\alpha_{ij})^2 - 4c \sum_{i,j,k,p,\alpha} A^\alpha_{ip} A^\alpha_{pj} (\delta_{ij} - \delta_{ik}\delta_{kj}).
\end{aligned}$$

Hence,

$$\mathcal{A}_1 = 4c\|H\|^2 - 4mc\|A\|^2.$$

Moreover,

$$\begin{aligned}
\mathcal{A}_2 &= 2 \sum_{i,j,k,\alpha,\beta} (4A^\alpha_{jk} A^\beta_{ik} R^N_{\alpha\beta ji} + A^\alpha_{jk} A^\beta_{jk} R^N_{\alpha i\beta i}) \\
&= 2c \sum_{i,j,k,\alpha,\beta} A^\alpha_{jk} A^\beta_{jk} (\delta_{\alpha\beta}\delta_{ii} - \delta_{i\beta}\delta_{\alpha i}) \\
&= 2c \sum_{i,j,k,\alpha} (A^\alpha_{jk})^2,
\end{aligned}$$

from where we find that

$$\mathcal{A}_2 = 2mc\|A\|^2.$$

Thus,

$$\mathcal{A}_1 + \mathcal{A}_2 = 4c\|H\|^2 - 2mc\|A\|^2.$$

Combining these expressions, we obtain the evolution equation of $\|A\|^2$ in space forms. $\square$

4.2.5 Gap theorems for minimal hypersurfaces

There are many applications of Simons' formula. Let us present only one here.

Theorem 4.2.9. *Let $f\colon M^m \to \mathbb{S}^{m+1}$ be a compact minimal hypersurface with $\|A\|^2 < m$. Then, $f(M^m) \equiv \mathbb{S}^m$.*

Proof. From our assumption, there exists $\varepsilon > 0$ such that $m - \|A\|^2 > \varepsilon$. On the other hand, from Theorem 4.2.8, we have

$$\begin{aligned}\Delta\|A\|^2 &= 2\|\nabla A\|^2 + 2\|A\|^2(m - \|A\|^2)\\ &\geq 2\|A\|^2(m - \|A\|^2)\\ &\geq 2\,\varepsilon\,\|A\|^2. \end{aligned} \tag{4.34}$$

Applying Stokes' theorem, we obtain

$$0 = \int \Delta\|A\|^2\, d\mu \geq 2\,\varepsilon \int \|A\|^2\, d\mu,$$

which implies that

$$\|A\|^2 = 0.$$

Consequently, $f(M^m)$ is totally geodesic, and hence $f(M^m) \equiv \mathbb{S}^m$, completing the proof. ☐

Remark 4.2.1. A stronger result than the previous theorem holds. In particular, if we assume that

$$\|A\|^2 \leq m,$$

then we obtain the estimate

$$\Delta\|A\|^2 \geq 2\|A\|^2(m - \|A\|^2) \geq 0.$$

By the Hopf maximum principle [39], we deduce that $\|A\|^2$ must be constant. Returning to (4.34), it follows that

$$0 \geq \|\nabla A\|^2 + \|A\|^2(m - \|A\|^2) \geq 0.$$

Hence,

$$\|A\|^2 = 0 \quad \text{or} \quad \|A\|^2 = m.$$

In the first case, $f(M^m) = \mathbb{S}^m$. In the second case, we have $\|A\|^2 = m$ and $\|\nabla A\|^2 = 0$. By a theorem of Lawson [29], it follows that $f(M^m)$ is a Clifford torus. Similar results also hold for higher codimensional compact or complete minimal submanifolds in spheres; see [31].

4.3 Parabolic comparison principles

In this chapter we will present the main tool to investigate the behaviour of (MCF) as we approach the maximal time of existence.

4.3.1 The scalar maximum principle

Throughout this section M will be a compact manifold equipped with a time dependent family of Riemannian metrics $g = g(t),\ \ t \in [0,T)$. We remind that the Laplacian Δ of g is the operator which in local coordinates has the form

$$\Delta u = \sum_{i,j} g^{ij}\big(u_{x_i x_j} - \sum_k \Gamma_{ij}^k u_{x_k}\big).$$

Theorem 4.3.1. *Let M be a compact manifold equipped with a family of time-dependent Riemannian metrics $g = g(t)$. Let $u \in C^2(M \times [0,T))$ be a solution of*

$$\frac{\partial u}{\partial t} - \Delta u - g(X, \nabla u) \geq 0,$$

where X is a bounded time-dependent vector field. If $u(x,0) \geq c$ for all $x \in M$, then

$$u(x,t) \geq c,$$

for all $(x,t) \in M \times [0,T)$.

Proof. The most natural idea is to examine the function $f\colon [0,T) \to \mathbb{R}$ given by

$$f(t) = \min_{x \in M} u(x,t).$$

However, this function is only Lipschitz continuous (not differentiable everywhere). To avoid such problems we follow another idea. Let $\varepsilon > 0$ and define $u^\varepsilon \in C^2(M \times [0,T))$ by

$$u^\varepsilon(x,t) = u(x,t) + \varepsilon(1+t).$$

Then

$$\frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon - g(X, \nabla u^\varepsilon) = \frac{\partial u}{\partial t} + \varepsilon - \Delta u - g(X, \nabla u) \geq \varepsilon > 0. \tag{4.35}$$

Claim: *For all* $(x,t) \in M \times [0,T)$ *it holds*

$$u^\varepsilon(x,t) > c.$$

Proof of the claim: Suppose to the contrary that our assertion is false. Note that

$$u^\varepsilon(x,0) = u(x,0) + \varepsilon \geq c + \varepsilon > c, \quad \text{for all} \quad x \in M.$$

By compactness, continuity, and our assumption, it follows that there exist a first time $0 < t_0 < T$ and a point $x_0 \in M$ with the following properties:

- $u^\varepsilon(x_0,t_0) = c$,
- $u^\varepsilon(x,t) \quad > c, \quad \text{for all} \quad (x,t) \in M \times [0,t_0)$,
- $u^\varepsilon(x,t) \quad \geq c, \quad \text{for all} \quad (x,t) \in M \times [0,t_0]$,

Then, at (x_0,t_0) we have:

- $u^\varepsilon(x_0,t_0) \quad = c$, (minimum point),
- $\nabla u^\varepsilon(x_0,t_0) = 0$,
- $\Delta u^\varepsilon(x_0,t_0) \geq 0$,
- $\dfrac{\partial u^\varepsilon}{\partial t}(x_0,t_0) \leq 0$, ($t_0$ is a boundary point).

From the inequality (4.35), we get

$$0 \geq \frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon - g(X, \nabla u^\varepsilon) \geq \varepsilon > 0.$$

which leads to a contradiction. This completes the proof of the claim. ⊛

By taking the limit when $\varepsilon \to 0$ we get

$$c \leq \lim_{\varepsilon \to 0} u^\varepsilon(x,t) = u(x,t), \quad \text{for all} \quad (x,t) \in M \times [0,T)$$

and the proof of the theorem is concluded. □

Following exactly the same lines, we can prove the following theorems.

Theorem 4.3.2. *Let M be a compact manifold equipped with a time-dependent family of Riemannian metrics $g = g(t)$. Let $u \in C^2(M \times [0,T))$ be a solution of*

$$\frac{\partial u}{\partial t} - \Delta u - g(X, \nabla u) \leq 0,$$

where X is a bounded time-dependent vector field. If $u(x,0) \leq c$ for all $x \in M$, then

$$u(x,t) \leq c,$$

for all $(x,t) \in M \times [0,T)$.

Theorem 4.3.3. *Let M be a compact manifold equipped with a family of time-dependent Riemannian metrics $g = g(t)$. Suppose $u, v \in C^2(M \times [0,T))$ such that*

$$\frac{\partial u}{\partial t} - \Delta u - g(X, \nabla u) - \psi(u,t) \geq \frac{\partial v}{\partial t} - \Delta v - g(X, \nabla v) - \psi(v,t)$$

where X is a bounded time-dependent vector field and $\psi \colon \mathbb{R} \times [0,T) \to \mathbb{R}$ is a continuous function which is Lipschitz in the first factor. If

$$u(x,0) \geq v(x,0), \quad \textit{for all} \quad x \in M,$$

then

$$u(x,t) \geq v(x,t)$$

for all $(x,t) \in M \times [0,T)$.

Proof. Consider the function $w = u - v$. We compute

$$\frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}, \quad \nabla w = \nabla u - \nabla v \quad \text{and} \quad \Delta w = \Delta u - \Delta v.$$

Therefore,

$$\frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} \geq \Delta w + g(X, \nabla w) + \psi(u,t) - \psi(v,t). \tag{4.36}$$

Because ψ is Lipschitz continuous in the first factor, we get

$$\|\psi(u,t) - \psi(v,t)\| \leq c(t)\|u - v\| \leq k\|u - v\|.$$

Therefore,

$$-k\|u - v\| \leq \psi(u,t) - \psi(v,t) \leq k\|u - v\|.$$

Hence, (4.36) becomes

$$\frac{\partial w}{\partial t} \geq \Delta w + g(X, \nabla w) - k\|w\|. \tag{4.37}$$

Let us introduce the function w^ε given by

$$w^\varepsilon(x,t) \doteq w(x,t) + \varepsilon e^{2kt}, \quad \text{for all } (x,t) \in M \times [0,T).$$

Then,

- $\dfrac{\partial w^\varepsilon}{\partial t} = \frac{\partial w}{\partial t} + 2k\varepsilon e^{2kt}$,
- $\nabla w^\varepsilon = \nabla w$,
- $\Delta w^\varepsilon = \Delta w$,

from where we deduce that

$$\begin{aligned}\frac{\partial w^\varepsilon}{\partial t} &- \Delta w^\varepsilon - g(X, \nabla w^\varepsilon) \\ &= \frac{\partial w}{\partial t} + 2k\varepsilon e^{2kt} - \Delta w - g(X, \nabla w) \\ &\geq -k\|w\| + 2k\varepsilon e^{2kt}.\end{aligned} \tag{4.38}$$

Claim: *For all $(x,t) \in M \times [0,T)$, it holds $w^\varepsilon(x,t) > 0$.*

Proof of the claim: Suppose, to the contrary, that the assertion of our claim is false. By compactness, continuity, and our assumption, it follows that there exist a first time $0 < t_0 < T$ and a point $x_0 \in M$ such that:

- $w^\varepsilon(x_0, t_0) = 0$,
- $w^\varepsilon(x,t) \quad > 0, \quad \text{for all} \quad (x,t) \in M \times [0, t_0)$,
- $w^\varepsilon(x,t) \quad \geq 0, \quad \text{for all} \quad (x,t) \in M \times [0, t_0]$.

Therefore at the point (x_0, t_0) we have

- $w(x_0, t_0) = -\varepsilon e^{2kt_0}$,

- $\dfrac{\partial w^\varepsilon}{\partial t}(x_0,t_0) \le 0$,
- $\nabla w^\varepsilon(x_0,t_0) = 0$,
- $\Delta w^\varepsilon(x_0,t_0) \ge 0$.

Hence from (4.38) we get

$$\begin{aligned}0 &\ge \frac{\partial w^\varepsilon}{\partial t}(x_0,t_0) - \Delta w^\varepsilon(x_0,t_0) - g(X,\nabla w^\varepsilon)(x_0,t_0)\\ &\ge k\varepsilon e^{2kt_0}\\ &> 0.\end{aligned}$$

which leads to a contradiction and proves the claim. ⊛

Letting $\varepsilon \to 0$, we conclude the proof of the theorem. □

Theorem 4.3.4 (Comparison maximum principle)**.** *Suppose M is a compact manifold equipped with a family of Riemannian metrics $g = g(t)$. Let $u \in C^2(M \times [0,T))$ is a solution of*

$$\frac{\partial u}{\partial t} \ge \Delta u + g(X,\nabla u) + \psi(u,t),$$

where X is a bounded time-dependent vector field and ψ a continuous function which is Lipschitz in the first factor. Consider, the associated ODE

$$\begin{cases} y'(t) = \psi(y(t),t),\\ y(0) = \min_{x\in M} u(x,0), \end{cases}$$

Then,

$$u(x,t) \ge y(t), \quad \textit{for all} \quad (x,t) \in M \times [0,T).$$

The same conclusion holds if we reverse all the inequalities and instead of minimum we take the maximum.

Proof. Consider the function $v(x,t) = y(t)$. Then, for every $x \in M$, it holds

$$u(x,0) \ge \min_{x\in M} u(x,0) = y(0) = v(x,0).$$

Moreover,

$$\frac{\partial v}{\partial t} - \Delta v - g(X, \nabla v) - \psi(v, t) = y' - \psi(y, t) = 0.$$

Therefore,

$$\frac{\partial u}{\partial t} - \Delta u - g(X, \nabla u) - \psi(u, t) \geq 0 = \frac{\partial v}{\partial t} - \Delta v - g(X, \nabla v) - \psi(v, t).$$

and the result follows from Theorem 4.3.3. □

4.3.2 Hamilton's second derivative test

Let (M, g) be a Riemannian manifold and B a symmetric 2-tensor. Denote by $A\colon TM \to TM$ the associated to B self-adjoint operator, i.e., the $(1,1)$-tensor given by

$$B(X, Y) = g(AX, Y), \quad \text{for all} \quad X, Y \in \mathfrak{X}(M).$$

The eigenvalues of A, are denoted by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m,$$

and with the above ordering they give rise to Lipschitz continuous functions. Moreover, for given $x \in M$ we denote by

$$\mathrm{Eig}_x(\lambda) \doteq \{v_x \in T_xM \colon \, Av_x = \lambda v_x\}.$$

Multiplying with w_x we get

$$\mathrm{Eig}_x(\lambda) = \{v_x \in T_xM \colon \, B(v_x, w_x) = \lambda g(v_x, w_x), \text{ for all } w_x \in T_xM\}.$$

Theorem 4.3.5 (Hamilton [22]). *Suppose that (M, g) is a Riemannian manifold and B is a symmetric 2-tensor. If the biggest eigenvalue λ_1 of B attains a (local) maximum λ at an (interior) point $x_0 \in M$, then*

$$\begin{cases} (\nabla_X B)(V, V) = 0, \\ (\Delta B)(V, V) \quad \leq 0, \end{cases}$$

for each $X \in T_{x_0}M$ and for every $V \in \mathrm{Eig}_{x_0}(\lambda)$. Replacing B with $-B$, we get the corresponding result for the smallest eigenvalue λ_m of B.

Proof. Let $v \in \mathrm{Eig}_{x_0}(\lambda)$ be a unit vector. Consider a local smooth extension V of v, on an open neighborhood U, such that

$$V_{x_0} = v, \ \ \|V\| = 1 \ \text{ and } \ \nabla_X V|_{x_0} = 0, \quad \text{for all } X \in \mathfrak{X}(M).$$

Define the symmetric 2-tensor S given by

$$S(X,Y) \doteq B(X,Y) - \lambda g(X,Y), \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

If $\{e_1, \dots, e_m\}$ is an orthonormal frame with respect to g diagonalizing B then

$$\begin{aligned} S(e_i, e_j) &= B(e_i, e_j) - \lambda g(e_i, e_j) = g(Ae_i, e_j) - \lambda g(e_i, e_j) \\ &= \mu_i \delta_{ij} = (\lambda_i - \lambda)\delta_{ij}. \end{aligned}$$

Therefore, the biggest eigenvalue of S is

$$\mu_1 = \lambda_1 - \lambda \leq 0.$$

Additionally,

$$\mu_1(x_0) = \lambda_1(x_0) - \lambda = 0.$$

Hence, for every $w \in T_{x_0}M$, we have

$$\begin{aligned} S(v,w) &= B(v,w) - \lambda g(v,w) \\ &= \lambda_1(x_0) g(v,w) - \lambda_1(x_0) g(v,w) \\ &= 0. \end{aligned}$$

Consequently,

$$S(v,w) = 0, \quad \text{for all } w \in T_{x_0}M.$$

Define the smooth function $f \colon U \to \mathbb{R}$ given by

$$f(x) = S(V_x, V_x) = B(V_x, V_x) - \lambda g(V_x, V_x).$$

Therefore, f can be viewed as a smooth approximation of $\lambda_1 - \lambda$. Observe that

- For every $x \in U$ it holds,

$$f(x) = S(V_x, V_x) \leq \mu_1(x)\|V_x\|^2 \leq \mu_1(x_0) = 0. \tag{4.39}$$

- At the point x_0, we have

$$f(x_0) = S(V_{x_0}, V_{x_0}) = 0.$$

For all $X \in \mathfrak{X}(U)$, we have

$$X(f) = X\{S(V,V)\} = (\nabla_X S)(V,V) + 2S(\nabla_X V, V).$$

Differentiating once more with respect to X, we get

$$\begin{aligned} XX(f) \;&=\; (\nabla_X \nabla_X S)(V,V) + 4(\nabla_X S)(\nabla_X V, V) \\ &\quad +2S(\nabla_X \nabla_X V, V) + 2S(\nabla_X V, \nabla_X V). \end{aligned}$$

If $\{e_1, \ldots, e_m\}$ is an arbitrary local orthonormal frame on U, then

$$\begin{aligned} \Delta f \;&=\; \sum_{i=1}^{m} \big(e_i e_i(f) - \nabla_{e_i} e_i(f)\big) \\ &=\; (\Delta S)(V,V) + 2S(\Delta V, V) \\ &\quad +2\sum_{i=1}^{m} \big(2(\nabla_{e_i} S)(\nabla_{e_i} V, V) + S(\nabla_{e_i} V, \nabla_{e_i} V)\big). \end{aligned}$$

Estimating at x_0 we get

$$(\nabla_{X_{x_0}} S)(V,V) = 0,$$

which proves our first claim, and

$$0 \geq \Delta f(x_0) = (\Delta S)(V,V),$$

which proves the second claim. This completes the proof of the theorem. □

CHAPTER 5

Singularity Formation

In this chapter, we present a detailed proof of the characterization of the maximal time of existence for the mean curvature flow. This result plays a central role in understanding the long-term behavior and singularity formation of evolving submanifolds. In the case of codimension one submanifolds in Euclidean space, a complete proof can be found in the book of Ecker [15]. For the case of higher codimension submanifolds in Euclidean space, we refer the reader to [5].

5.1 The maximal time of existence

The purpose of this section is to characterize the maximal time of existence and particularly to prove the following result.

Theorem 5.1.1. *Let $F\colon M \times [0, T) \to N$ be solution of* (MCF) *defined on a finite maximal time interval, where M is compact and N is complete. Then,*

$$\limsup_{t \to T} \left(\max_{M \times [0,t]} \|A\| \right) = \infty.$$

Proof of the theorem: The proof proceeds by contradiction. Suppose that

$$\|A_F\| \leq c,$$

where $c > 0$ is a uniform constant. We will show that the flow can be smoothly extended beyond T, contradicting the maximality of T. This will be achieved using the Ascoli-Arzelà theorem.

The proof of this theorem is divided into four steps.

Step 1: Estimates for the image of F.

First, note that the evolving submanifolds lie in a compact region of N. Indeed, fix $x_0 \in M$ and consider the curve $\gamma\colon [0,T) \to N$ defined by $\gamma(t) = F(x_0,t)$. Then,

$$\begin{aligned} L(\gamma)\big|_0^T &= \lim_{s\to T}\int_0^s \|\gamma'(t)\|dt = \lim_{s\to T}\int_0^s \big\|dF_{(x_0,t)}(\partial_t)\big\|dt \\ &= \lim_{s\to T}\int_0^s \|H(x_0,t)\|dt \le \lim_{s\to T}\int_0^s \sqrt{m}\,c\,dt \\ &= \sqrt{m}\,c\,T. \end{aligned}$$

Therefore,

$$\operatorname{dist}(F(x_0,0),F(x_0,t)) \le L(\gamma)\big|_0^t \le \sqrt{m}\,c\,T. \tag{5.1}$$

Lemma 5.1.2. *Fix a smooth, time-independent Riemannian metric $\widetilde{g}$ on M. There exists a constant $C>0$ such that*

$$C^{-1}\widetilde{g} \le g \le C\,\widetilde{g}, \quad \text{on } M\times[0,T).$$

Moreover, there exists a constant c_0, such that

$$\operatorname{diam}(M,g_t) \le e^{c_0 t/2}\operatorname{diam}(M,g_0),$$

for any $t\in[0,T)$.

Proof. Fix a nonzero, time-independent vector field X, and define the function $f\colon M\times[0,T)\to\mathbb{R}$ by

$$f(x,t) = \frac{g(X_x,X_x)}{\widetilde{g}(X_x,X_x)}.$$

We compute the time derivative of f as follows

$$\partial_t f = \frac{\partial_t g(X,X)}{\widetilde{g}(X,X)} = \frac{(\nabla_{\partial_t} g)(X,X)}{\widetilde{g}(X,X)} = \frac{-2A^H(X,X)}{\widetilde{g}(X,X)} = \frac{-2A^H(X,X)}{g(X,X)}\,f.$$

Hence,

$$|\partial_t f| = 2\left|\frac{A^H(X,X)}{g(X,X)}\right| f.$$

Since $\|A\|$ is uniformly bounded, we have

$$2\left|\frac{A^H(X,X)}{g(X,X)}\right| \leq C,$$

where C is a positive constant that does not depend on X or on time t. Thus,

$$|\partial_t f| \leq Cf.$$

Consequently,

$$f(0)\, e^{-Ct} \leq f(t) \leq f(0)e^{Ct}, \quad \text{for all } t \leq T.$$

Choose $\widetilde{g} = g_0$ and let $\gamma : [0,1] \to M$ be any smooth curve. Its length with respect to g_t is

$$L_{g_t}(\gamma) = \int_0^1 \sqrt{g_t(\gamma', \gamma')}\, d\mu.$$

From the first conclusion, there exists a $c_0 > 0$ such that

$$g_t \leq e^{c_0 t} g_0.$$

Hence,

$$\sqrt{g_t(\gamma',\gamma')} \leq \sqrt{e^{c_0 t} g_0(\gamma',\gamma')} = e^{c_0 t/2}\sqrt{g_0(\gamma',\gamma')}.$$

Therefore

$$L_{g_t}(\gamma) \leq e^{c_0 t/2} L_{g_0}(\gamma).$$

Taking the infimum over all curves joining two points $x, y \in M$, we obtain

$$d_{g_t}(x,y) \leq e^{c_0 t/2} d_{g_0}(x,y).$$

Consequently,

$$\text{diam}(M, g_t) \leq e^{c_0 t/2}\, \text{diam}(M, g_0), \quad \text{for any } t < T.$$

which proves our assertion. □

It remains to control the diameter of $F_t(M)$. Since M is compact, its diameter with respect to g_0 is finite. From Lemma 5.1.2, it follows that there exists a constant c_0 such that

$$\text{diam}(M, g_t) \leq e^{c_0 T/2}\, \text{diam}(M, g_0), \quad \text{for any } t < T.$$

From Lemma 5.1.2 and (5.1), for any $x \in M$, we have

$$\begin{aligned} \operatorname{dist}(F(x,t), F(x_0,0)) &\le \operatorname{dist}(F(x,t), F(x_0,t)) + \operatorname{dist}(F(x_0,t), F(x_0,0)) \\ &\le \operatorname{diam}(M, g_t) + \sqrt{m}\, c\, T \\ &\le e^{c_0 T/2} \operatorname{diam}(M, g_0) + \sqrt{m}\, c\, T. \end{aligned}$$

Therefore

$$F_t(M) \subset \overline{B}_N\big(F(x_0,0), e^{c_0 T/2} \operatorname{diam}(M, g(0)) + \sqrt{m}\, c\, T\big), \quad \text{for any } t < T,$$

where B_N is the ball of N with radius $R = e^{c_0 T/2} \operatorname{diam}(M, g(0)) + \sqrt{m}\, c\, T$ centered at $F(x_0, 0)$. Since N is complete, the Hopf-Rinow theorem implies that closed bounded balls in N are compact. Hence there exists a compact set $K \subset N$ such that

$$F_t(M) \subset K$$

for all $t \in [0,T)$. Since N is complete, the images $F_t(M)$, for $t \in [0,T)$, remain within a compact subset $K \subseteq N$. Consequently, we may assume that the curvature tensor R^N and all of its covariant derivatives are uniformly bounded on K.

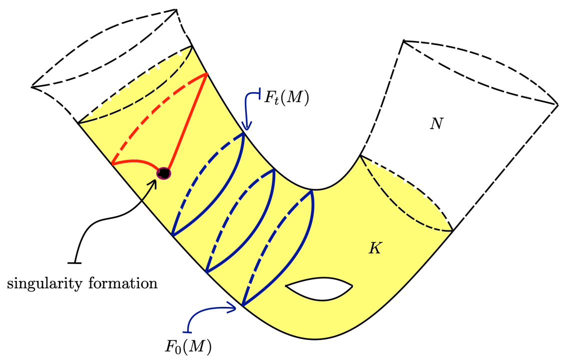


Figure 5.1: Singularity formation.

Before proceeding, we introduce Hamilton's $*$-notation:

Notation: *For tensors T_1 and T_2, the expression $T_1 * T_2$ denotes any linear combination of contractions (compositions and traces) of T_1 and T_2.*

Step 2: Higher-order derivative estimates for A.

Let us start with the following lemma.

Lemma 5.1.3. *Suppose B is an r-tensor taking values in the normal bundle of the evolving submanifolds under* (MCF). *Then:*

1. *The squared norm of B evolves in time under the equation,*

$$\partial_t \|B\|^2 = 2 \sum_{i_1 \cdots i_r} \langle (\nabla^\perp_{\partial_t} B)_{i_1 \cdots i_r}, B_{i_1 \cdots i_r} \rangle$$
$$+2 \sum_{i_1 \cdots i_r j} A^H_{i_1 j} \langle B_{j i_2 \cdots i_r}, B_{i_1 \cdots i_r} \rangle + \cdots + 2 \sum_{i_1 \cdots i_r j} A^H_{i_r j} \langle B_{i_1 \ldots i_{r-1} j}, B_{i_1 \cdots i_r} \rangle,$$

where the indices are with respect to a local time-dependent orthonormal frame.

2. *The Laplacian of B satisfies*

$$\Delta \|B\|^2 = 2 \sum_{i_1 \cdots i_r} \langle (\Delta^\perp B)_{i_1 \cdots i_r}, B_{i_1 \cdots i_r} \rangle + 2 \sum_{i_1 \cdots i_r j} \|(\nabla^\perp_j B)_{i_1 \cdots i_r}\|^2,$$

where the indices are with respect to a local time-dependent orthonormal frame.

Proof. We compute

$$\begin{aligned}
\partial_t \|B\|^2 \ =\ & 2 \sum_{i_1 \cdots i_r} \langle \nabla^\perp_{\partial_t} B(e_{i_1}, \ldots, e_{i_r}), B(e_{i_1}, \ldots, e_{i_r}) \rangle \\
& +2 \sum_{i_1 \cdots i_r j} \langle (\nabla^\perp_{\partial_t} B)(e_{i_1}, \ldots, e_{i_r}), B(e_{i_1}, \ldots, e_{i_r}) \rangle \\
& +2 \sum_{i_1 \cdots i_r} \sum_a \langle B(e_{i_1}, \ldots, \nabla_{\partial_t} e_{i_a}, \ldots, e_{i_r}), B(e_{i_1}, \ldots, e_{i_r}) \rangle \\
=\ & 2 \sum_{i_1 \cdots i_r} \langle (\nabla^\perp_{\partial_t} B)_{i_1 \cdots i_r}, B_{i_1 \cdots i_r} \rangle \\
& +2 \sum_{i_1 \cdots i_r} \sum_a A^H_{i_a j} \langle B_{i_1 \cdots i_{a-1} j i_{a+1} \cdots i_r}, B_{i_1 \cdots i_r} \rangle,
\end{aligned}$$

where the sum with respect to the index $a \in \{i, \ldots, r\}$ runs over the tensor slots. The second formula for the Laplacian of B follows from Lemma 1.1.8. This completes the proof. ∎

Lemma 5.1.4. *The evolution of the k-th derivative of A satisfies*

$$\begin{aligned}\nabla^{\perp}_{\partial_t}\big(\nabla^{(k)\perp}A\big)\; &-\; \Delta^{\perp}\big(\nabla^{(k)\perp}A\big)\\ &= \sum_{i+j+l=k}\nabla^{(i)\perp}A * \nabla^{(j)\perp}A * \nabla^{(l)\perp}A\\ &\quad+\sum_i\big(\nabla^{(i)N}R^N\big) * \big(\nabla^{(k-i)\perp}A\big)\\ &\quad+\sum_{i+j+l=k-1}\nabla^{(i)N}R^N * \nabla^{(j)\perp}A * \nabla^{(l)\perp}A\\ &\quad+\nabla^{(k+1)N}R^N * (\text{constants}),\end{aligned}$$

where all indices range from 0 *to* k*, and in the case* $k=0$ *the third term in the right hand side does not exist.*

Proof. We argue by induction on k.

The case $k=0$ follows from the evolution equation for the second fundamental form.

Suppose that the result holds up to order $k-1$. Differentiating the k-th covariant derivative of A with respect to time, and using the time-like Gauss and Ricci equations from Lemma 4.2.1, as well as the Ricci identity from Lemma 4.2.2, we obtain

$$\begin{aligned}&\nabla^{\perp}_{\partial_t}(\nabla^{(k)\perp}A) = \nabla^{\perp}_{\partial_t}\{\nabla^{\perp}(\nabla^{(k-1)\perp}A)\}\\ &\;= \nabla^{\perp}\{\nabla^{\perp}_{\partial_t}(\nabla^{(k-1)\perp}A)\} + A * \nabla^{\perp}A * \nabla^{(k-1)\perp}A + R^N * A * \nabla^{(k-1)\perp}A,\end{aligned}$$

from where we deduce that

$$\begin{aligned}\nabla^{\perp}_{\partial_t}(\nabla^{(k)\perp}A) &= \nabla^{\perp}\Big\{\Delta^{\perp}(\nabla^{(k-1)\perp}A) + \nabla^{(k)N}R^N * (\text{constants})\\ &\quad+\sum_{i+j+l=k-1}\nabla^{(i)\perp}A * \nabla^{(j)\perp}A * \nabla^{(l)\perp}A + \sum_{i=0}^{k-1}\nabla^{(i)N}R^N * \nabla^{(k-1-i)\perp}A\\ &\quad+\sum_{i+j+l=k-2}\nabla^{(i)N}R^N * \nabla^{(j)\perp}A * \nabla^{(l)\perp}A\Big\}\\ &\quad+A * \nabla^{\perp}A * \nabla^{(k-1)\perp}A + R^N * A * \nabla^{(k-1)\perp}A.\end{aligned}$$

Therefore,

$$\begin{aligned}\nabla^{\perp}_{\partial_t}(\nabla^{(k)\perp}A) &= \nabla^{\perp}\Delta^{\perp}\nabla^{(k-1)\perp}A + \nabla^{(k+1)N}R^N * (\text{constants}) \\ &+ \sum_{i+j+l=k} \nabla^{(i)\perp}A * \nabla^{(j)\perp}A * \nabla^{(l)\perp}A + \sum_{i=0}^{k} \nabla^{(i)N}R^N * \nabla^{(k-i)\perp}A \\ &+ \sum_{i+j+l=k-1} \nabla^{(i)N}R^N * \nabla^{(j)\perp}A * \nabla^{(l)\perp}A.\end{aligned}$$

Combining Lemma 4.2.3 with Lemma 4.2.1, we obtain the conclusion. □

Lemma 5.1.5. *The evolution of* $\|\nabla^{(k)\perp}A\|^2, k \geq 1$, *is of the form*

$$\begin{aligned}\partial_t\|\nabla^{(k)\perp}A\|^2 \; - \; & \Delta\|\nabla^{(k)\perp}A\|^2 \\ = \; & -2\|\nabla^{(k+1)\perp}A\|^2 + \nabla^{(k+1)N}R^N * \nabla^{(k)\perp}A \\ & + \sum_{i+j+l=k} \nabla^{(i)\perp}A * \nabla^{(j)\perp}A * \nabla^{(l)\perp}A * \nabla^{(k)\perp}A \\ & + \sum_{i+j+l=k-1} \nabla^{(i)N}R^N * \nabla^{(j)\perp}A * \nabla^{(l)\perp}A * \nabla^{(k)\perp}A \\ & + \sum_{i=0}^{k} \nabla^{(i)N}R^N * \nabla^{(k-i)\perp}A * \nabla^{(k)\perp}A.\end{aligned}$$

Proof. Applying Lemma 5.1.3 with $B = \nabla^{(k)\perp}A$, we obtain

$$\begin{aligned}\partial_t\|\nabla^{(k)\perp}A\|^2 \; - \; & \Delta\|\nabla^{(k)\perp}A\|^2 \\ = \; & 2\sum_{i_1\cdots i_r} \langle(\nabla^{\perp}_{\partial_t}\nabla^{(k)\perp}A - \Delta^{\perp}\nabla^{(k)\perp}A)_{i_1\cdots i_r}, (\nabla^{(k)\perp}A)_{i_1\cdots i_r}\rangle \\ & +2\sum_{i_1\cdots i_r j} A^H_{i_1 j}\langle(\nabla^{(k)\perp}A)_{j i_2\cdots i_r}, (\nabla^{(k)\perp}A)_{i_1\cdots i_r}\rangle \\ & \vdots \\ & +2\sum_{i_1\cdots i_r j} A^H_{i_r j}\langle(\nabla^{(k)\perp}A)_{i_1\cdots i_{r-1} j}, (\nabla^{(k)\perp}A)_{i_1\cdots i_r}\rangle \\ & -2\sum_{i_1\cdots i_r j} \|(\nabla^{\perp}_j\nabla^{(k)\perp}A)_{i_1\cdots i_r}\|^2,\end{aligned}$$

where all the indices are with respect to orthonormal frames. The conclusion follows from Lemma 5.1.5. □

In the following lemma, we will prove that the higher covariant derivatives of the second fundamental form A can be controlled by $\sup \|A\|$ and the geometry of the Riemannian manifolds M and N.

Lemma 5.1.6. *Suppose $F\colon M \times [0,\tau) \to N$ is a solution of (MCF) on a time interval $[0,\tau)$, with $\tau > 0$. Assume that there exists a constant $c_0 = c_0(\tau) > 0$ such that*

$$\sup_{M\times[0,\tau)} \|A\|^2 \leq c_0.$$

Then, for every $k \in \mathbb{N}$, there exists a constant $c_k > 0$ such that

$$\sup_{M\times[0,\tau)} \|\nabla^k A\|^2 \leq c_k,$$

where c_k depends only on k, c_0, the dimensions of M and N, and the geometry of the Riemannian manifold N.

Proof. Define the functions

$$\varPhi_k = \|\nabla^{(k)\perp} A\|^2, \quad k \in \mathbb{N}\cup\{0\}.$$

The proof proceeds by induction on $k \geq 0$. Assume that the result holds for $k-1$. Using the fact that the submanifolds lie in a subset of N with bounded geometry, together with Lemma 5.1.5 and the Cauchy-Schwarz inequality, we obtain the following inequality

$$\begin{aligned}
\partial_t \varPhi_k - \Delta\varPhi_k \;\leq\; & -2\|\nabla^{(k+1)\perp}A\|^2 \\
& + a_1 \sum_{i+j+l=k} \|\nabla^{(i)\perp}A\|\,\|\nabla^{(j)\perp}A\|\,\|\nabla^{(l)\perp}A\|\,\|\nabla^{(k)\perp}A\| \\
& + a_2 \sum_{j+l=k-1} \|\nabla^{(j)\perp}A\|\,\|\nabla^{(l)\perp}A\|\,\|\nabla^{(k)\perp}A\| \\
& + a_3 \sum_{i=0}^{k} \|\nabla^{(k-i)\perp}A\|\,\|\nabla^{(k)\perp}A\| \\
& + a_4 \|\nabla^{(k)\perp}A\|,
\end{aligned}$$

where a_1, a_2, a_3 and a_3 are positive constants. Using the inequality

$$a\,b \leq \frac{a^2+b^2}{2},$$

for all $a, b \in \mathbb{R}$, we arrive at

$$\partial_t \Phi_k - \Delta \Phi_k \leq c_1 \Phi_k + c_2$$

where c_1 and c_2 are positive constants. The solution of the associated ODE

$$\begin{cases} y'(t) &= c_1 y(t) + c_2, \quad t \in (0, \tau), \\ y(0) &= \max_{x \in M} \Phi_k(x, 0), \end{cases}$$

is

$$y(t) = \Big(y(0) + \frac{c_2}{c_1}\Big) e^{c_1 t} - \frac{c_2}{c_1}, \quad t < \tau < \infty.$$

Hence, from the maximum principle we get that

$$\Phi_k(x, t) \leq y(t) \leq y(\tau) \leq \infty,$$

for all $(x, t) \in M \times [0, \tau)$. This gives the desired bound on Φ_k and completes the proof. $\square$

Step 3: Higher-order derivative bounds for F.

The estimates obtained in Step 2 imply that all derivatives $\|\nabla^{(k)} F\|$ are bounded. However, we cannot apply the Ascoli-Arzelà theorem, since the connection ∇ is time-dependent. Therefore, we require estimates for the derivatives of F with respect to a fixed, time-independent Riemannian metric and connection on the base manifold M. Fix a smooth, time-independent Riemannian metric $\widetilde{g}$ on M, and denote by $\widetilde{\nabla}$ its Levi-Civita connection. We extend $\widetilde{g}$ to a Riemannian metric $\widetilde{g}_{\mathcal{H}}$ on $\mathcal{H}$ as follows: Let

$$(x, t) = (x_1, \ldots, x_m, t)$$

be a local coordinate system on $M \times [0, T)$, where $T > 0$. For $X, Y \in \mathcal{H}$, such that

$$X = \sum_i a_i \partial_i \quad \text{and} \quad Y = \sum_j b_j \partial_j,$$

where a_i, b_i, $i, j \in \{1, \ldots, m\}$, are time-dependent smooth functions, define

$$\widetilde{g}_{\mathcal{H}}(X, Y) = \sum_{i,j} \alpha_i(x, t) b_j(x, t) \widetilde{g}(\partial_i, \partial_j).$$

Clearly, the tensor $\widetilde{g}_{\mathcal{H}}$ is globally well-defined. Furthermore, we extend $\widetilde{\nabla}$ to a connection ${}^{\mathcal{H}}\widetilde{\nabla}$ on $M \times [0, T)$ as follows:

$$\begin{cases} {}^{\mathcal{H}}\widetilde{\nabla}_X Y = \widetilde{\nabla}_X Y, \\ {}^{\mathcal{H}}\widetilde{\nabla}_{\partial_t} X = [\partial_t, X], \\ {}^{\mathcal{H}}\widetilde{\nabla}_X \partial_t = 0 = {}^{\mathcal{H}}\widetilde{\nabla}_{\partial_t}\partial_t, \end{cases}$$

for all $X, Y \in \mathcal{H}$. As usual, we denote the extensions $\widetilde{g}_{\mathcal{H}}$ and ${}^{\mathcal{H}}\widetilde{\nabla}$ simply by $\widetilde{g}$ and $\widetilde{\nabla}$, respectively.

Let us introduce now the 2-tensor S given by

$$S = \nabla - \widetilde{\nabla}.$$

Consider a local coordinate system $(x_1, \ldots, x_m, t)$ on $M \times [0, T)$. Then

$$S(\partial_i, \partial_j) = \nabla_{\partial_i}\partial_j - \widetilde{\nabla}_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k - \sum_k \widetilde{\Gamma}_{ij}^k \partial_k.$$

Differentiating with respect to t, we find

$$(\widetilde{\nabla}_{\partial_t} S)(\partial_i, \partial_j) = \widetilde{\nabla}_{\partial_t} S(\partial_i, \partial_j) = \sum_k \partial_t(\Gamma_{ij}^k)\partial_k. \tag{5.2}$$

On the other hand,

$$R^{\nabla}(\partial_t, \partial_i)\partial_j = \nabla_{\partial_t}\nabla_{\partial_i}\partial_j - \nabla_{\partial_i}\nabla_{\partial_t}\partial_j - \nabla_{[\partial_t, \partial_i]}\partial_j = \sum_k \nabla_{\partial_t}(\Gamma_{ij}^k \partial_k).$$

Combining with (5.2), we find

$$R^{\nabla}(\partial_t, \partial_i)\partial_j = \sum_k \partial_t(\Gamma_{ij}^k)\partial_k = (\widetilde{\nabla}_{\partial_t} S)(\partial_i, \partial_j). \tag{5.3}$$

In view of the time-like Gauss equation in Lemma 4.2.1, we have

$$\widetilde{\nabla}_{\partial_t} S = R^{\nabla}(\partial_t, \cdot)\cdot = R^N * A + A * \nabla^{\perp} A.$$

Due to Lemma 5.1.2 and Lemma 1.1.6, we get that there exist constants a_1 and a_2 such that

$$\|\widetilde{\nabla}_{\partial_t} S\|_{\widetilde{g}} \leq a_1 \|\widetilde{\nabla}_{\partial_t} S\|_g \leq a_2(\|R^N\|_{g_N}\|A\|_g + \|A\|_g\|\nabla^{\perp} A\|_g).$$

We now show that the covariant derivatives of F with respect to the connection $\widetilde{\nabla}$ can be expressed in terms of A, S, and their derivatives. Observe that

$$\begin{aligned}(\widetilde{\nabla}^2 F)(X,Y) &= \nabla^F_X dF(Y) - dF(\widetilde{\nabla}_X Y)\\ &= A(X,Y) + dF(\nabla_X Y) - dF(\widetilde{\nabla}_X Y)\\ &= A(X,Y) + dF(S(X,Y)).\end{aligned}$$

Consequently,

$$\widetilde{\nabla}^2 F = A + dF \circ S. \tag{5.4}$$

From Lemma 5.1.2 and our assumptions, we see that

$$\|S\|_{\widetilde{g}} \leq c_0\,(1 + \|\widetilde{\nabla}^2 F\|_{\widetilde{g}}),$$

where c_0 is a positive constant. Moreover, in view of (5.4) we obtain

$$\begin{aligned}(\widetilde{\nabla}^3 F)(X,Y,Z) &= \{\widetilde{\nabla}(\widetilde{\nabla}^2 F)\}(X,Y,Z) = \{\widetilde{\nabla}_Z(\widetilde{\nabla}^2 F)\}(X,Y)\\ &= \nabla^F_Z\{(\widetilde{\nabla}^2 F)(X,Y)\}\\ &\quad -(\widetilde{\nabla}^2 F)(\widetilde{\nabla}_Z X, Y) - (\widetilde{\nabla}^2 F)(X, \widetilde{\nabla}_Z Y)\\ &= \nabla^F_Z\{A(X,Y) + dF(S(X,Y))\}\\ &\quad -A(\widetilde{\nabla}_Z X, Y) - dF(S(\widetilde{\nabla}_Z X, Y))\\ &\quad -A(X, \widetilde{\nabla}_Z Y) - dF(S(X, \widetilde{\nabla}_Z X))\\ &= (\nabla^F_Z A)(X,Y) + A(\nabla_Z X, Y) + A(X, \nabla_Z Y)\\ &\quad +(\widetilde{\nabla}^2 F)(Z, S(X,Y)) + dF(\widetilde{\nabla}_Z S(X,Y))\\ &\quad -A(\widetilde{\nabla}_Z X, Y) - dF(S(\widetilde{\nabla}_Z X, Y))\\ &\quad -A(X, \widetilde{\nabla}_Z Y) - dF(S(X, \widetilde{\nabla}_Z Y)).\end{aligned}$$

Therefore,

$$\begin{aligned}(\widetilde{\nabla}^3 F)(X,Y,Z) &= (\nabla^F_Z A)(X,Y)\\ &\quad +A(S(Z,X),Y) + A(X, S(Z,Y))\\ &\quad +(\widetilde{\nabla}^2 F)(Z, S(X,Y)) + dF((\widetilde{\nabla}_Z S)(X,Y)).\end{aligned} \tag{5.5}$$

As seen in Lemma 1.1.5, the tangential part of $\nabla^F A$ involves only terms depending on A. Hence, by Step 2, it follows that the norm

$$\|\nabla^F A\|_g$$

is also uniformly bounded. If $\|\widetilde{\nabla}^2 F\|_{\widetilde{g}}$ is uniformly bounded, from the identity (5.5) it follows that there exists a constant $c_1 > 0$ such that

$$\|\widetilde{\nabla} S\|_{\widetilde{g}} \leq c_1\big(1 + \|\widetilde{\nabla}^3 F\|_{\widetilde{g}}\big).$$

By induction on k, we obtain the following conclusion:

Lemma 5.1.7. *Suppose that the functions*

$$Q_i = \|\widetilde{\nabla}^{(i)} F\|_{\widetilde{g}}$$

are uniformly bounded for all indices $i \in \{1, \dots, k-1\}$. Then there exists a constant c_{k-2} such that

$$\|\widetilde{\nabla}^{(k-2)} S\|_{\widetilde{g}} \leq c_{k-2}\big(1 + \|\widetilde{\nabla}^{(k)} F\|_{\widetilde{g}}\big),$$

for all $k \geq 2$.

We are now in a position to prove the key estimate for the higher-order derivatives of F.

Lemma 5.1.8. *For every integer $k \geq 0$, the functions*

$$Q_k = \|\widetilde{\nabla}^{(k)} F\|_{\widetilde{g}} \quad \text{and} \quad Q_{k,1} = \|\widetilde{\nabla}_{\partial_t} \widetilde{\nabla}^{(k)} F\|_{\widetilde{g}}$$

are uniformly bounded in time.

Proof. We have shown that $F_t(M)$ remains in a compact region of N. Moreover, by Lemma 5.1.2, there exists a constant $\widehat{c}_1$ such that

$$\|dF\|_{\widetilde{g}} \leq \widehat{c}_1 \sqrt{m}.$$

Consider a local coordinate system $(x_1, \dots, x_m, t)$. Then,

$$\begin{aligned}
(\widetilde{\nabla}_{\partial_t} \widetilde{\nabla}^2 F)(\partial_i, \partial_j) &= \nabla^F_{\partial_t}\{(\widetilde{\nabla}^2 F)(\partial_i, \partial_j)\} \\
&= \nabla^F_{\partial_t}\{A(\partial_i, \partial_j) + dF(S(\partial_i, \partial_j))\} \\
&= (\nabla^F_{\partial_t} A)(\partial_i, \partial_j) + \nabla^F_{\partial_t} dF(S(\partial_i, \partial_j)) \\
&= (\nabla^F_{\partial_t} A)(\partial_i, \partial_j) + dF([\partial_t, S(\partial_i, \partial_j)]) + \nabla^F_{S(\partial_i, \partial_j)} H \\
&= (\nabla^F_{\partial_t} A)(\partial_i, \partial_j) + dF((\widetilde{\nabla}_{\partial_t} S)(\partial_i, \partial_j)) + \nabla^F_{S(\partial_i, \partial_j)} H.
\end{aligned}$$

From (5.3), it follows that the quantity $\nabla_{\partial_t} S(\partial_i, \partial_j)$ involves time derivatives of the Christoffel symbols, which, in view of the time-like Gauss equation in Lemma 4.2.1, can be expressed in terms of the second fundamental form A and its derivatives. Consequently,

$$\|\widetilde{\nabla}_{\partial_t}\widetilde{\nabla}^2 F\|_{\widetilde{g}} \leq \widetilde{c}\,(1 + \|S\|_{\widetilde{g}}) \leq \widehat{c}\,(1 + \|\widetilde{\nabla}^2 F\|_{\widetilde{g}}), \tag{5.6}$$

where $\widetilde{c}$ and $\widehat{c}$ are positive constants. By the Cauchy-Schwarz inequality,

$$\partial_t \|\widetilde{\nabla}^2 F\|_{\widetilde{g}} = \frac{\langle \widetilde{\nabla}_{\partial_t}\widetilde{\nabla}^2 F, \widetilde{\nabla}^2 F\rangle_{\widetilde{g}}}{\|\widetilde{\nabla}^2 F\|_{\widetilde{g}}} \leq \|\widetilde{\nabla}_{\partial_t}\widetilde{\nabla}^2 F\|_{\widetilde{g}} \leq \widehat{c}\,(1 + \|\widetilde{\nabla}^2 F\|_{\widetilde{g}}).$$

Solving the above differential inequality, we deduce that the function

$$Q_2 = \|\widetilde{\nabla}^2 F\|_{\widetilde{g}}$$

is bounded. Consequently, from (5.6) it follows that

$$Q_{2,1} = \|\widetilde{\nabla}_{\partial_t}\widetilde{\nabla}^2 F\|_{\widetilde{g}}$$

is uniformly bounded as well. By induction on k and using Lemma 5.1.7, we conclude that there exist constants $\widetilde{c}_k$ and $\widehat{c}_k$ such that

$$\|\widetilde{\nabla}_{\partial_t}\widetilde{\nabla}^{(k)} F\|_{\widetilde{g}} \leq \widetilde{c}_k \big(1 + \|\widetilde{\nabla}^{(k-2)} S\|_{\widetilde{g}}\big) \leq \widehat{c}_k \big(1 + \|\widetilde{\nabla}^{(k)} F\|_{\widetilde{g}}\big).$$

It follows that all derivatives of F are bounded.

Step 4: Arriving at a contradiction.

By the Ascoli-Arzelà Theorem 1.2.1, the family $\{F_t\}_{t\in[0,T)}$ converges in C^∞ to a limit map $F_T\colon M \to N$, which is an immersion. Applying the short-time existence result, we can extend the solution of (MCF) beyond T, contradicting the maximality of the time of existence. □

Remark 5.1.1. It is not possible to characterize the maximal time of existence of (MCF) in terms of curvature quantities when the ambient Riemannian manifold is not complete. For example, consider the unit circle centered at the origin in $\mathbb{R}^2$. As seen in Example 4.1.5, the maximal time of existence is

$$T_{\max} = 1/2.$$

Consider the ambient space given by the non-complete Riemannian manifold $N = \mathbb{R}^2 - \{(0, 1/4)\}$. In this case, the mean curvature flow terminates when the radius of the evolving circles approaches $1/4$. Note that the curvature and all its derivatives remain bounded.

5.2 Modeling the singularities

In this subsection, we describe what is meant by a singularity in mean curvature flow (MCF).

Theorem 5.2.1 (Singularity formation)**.** *Let M be a compact manifold and let $F\colon M\times[0,T)\to N$ be a solution of* (MCF)*, where (N,h) is a complete Riemannian manifold with bounded geometry and $T\leq\infty$ its maximal time of existence. Suppose that there exists a point $x_\infty\in M$ and a sequence $\{(x_i,t_i)\}_{i\in\mathbb{N}}$ in $M\times[0,T)$ with*

- $\lim_{i\to\infty} x_i = x_\infty$,
- $\lim_{i\to\infty} t_i = T$ *(t_i increasing),*
- $a_i = \|A_{F_{(x_i,t_i)}}\| = \max_{M\times[0,t_i]}\|A_{F(x,t)}\|\to\infty$.

Consider the new family of maps

$$F_i\colon M\times[L_i,R_i)\to(N,a_i^2h),\quad i\in\mathbb{N}$$

given by

$$F_i(x,s)=F_{i,s}(x)=F\left(x,\frac{s}{a_i^2}+t_i\right)$$

where

$$L_i=-a_i^2t_i\quad\textit{and}\quad R_i=\begin{cases}a_i^2(T-t_i), & T<\infty\\ \infty, & T=\infty.\end{cases}$$

Then, the following hold:

1. *For each fixed $i\in\mathbb{N}$, the maps $\{F_{i,s}\}_{s\in[L_i,R_i)}$ evolves by* (MCF) *in time s. The second fundamental forms A_{F_i} satisfy:*
   - $A_{F_i(x,s)} = A_{F(x,s/a_i^2+t_i)}$,
   - $\|A_{F_i(x,s)}\|_{g_{F_i}} = \frac{1}{a_i}\|A_{F(x,s/a_i^2+t_i)}\|_{g_F}$,

   where all the norms are with respect to the Riemannian metrics induced by the corresponding immersions. Hence, for any $s\leq 0$ and $i\in\mathbb{N}$, we have

   $$\|A_{F_i}\|_{g_{F_i}}\leq 1\quad\textit{and}\quad\|A_{F_i(x_i,0)}\|_{g_{F_i}}=1.$$

2. *For any fixed* $s \leq 0$, *the sequence* $\{(M, F_{i,s}^*(a_i^2 h), x_i)\}_{i\in\mathbb{N}}$ *of the pointed Riemannian manifolds smoothly subcoverges to a connected, complete, pointed manifold* $(M_\infty, g_\infty, x_\infty)$, *where* M_∞ *does not depend on* s. *Moreover,* $\{(N, a_i^2 h, F_i(x_i, t_i))\}_{i\in\mathbb{N}}$ *smoothly subconverges to* $(\mathbb{R}^n, g_{euc}, 0)$.

3. *There is an ancient solution* $F_\infty \colon M_\infty \times (-\infty, 0] \to \mathbb{R}^n$ *of* (MCF) *such that, for each fixed* $s \leq 0$, $\{F_{i,s}\}_{i\in\mathbb{N}}$ *smoothly subconverges to* $F_{\infty,s}$. *This convergence is uniform with respect to* s. *Additionally,*

$$\|A_{F_\infty}\| \leq 1 \quad \textit{and} \quad \|A_{F_{\infty(x_\infty,0)}}\| = 1.$$

4. *Suppose that* $R_i \to \infty$. *Then,* F_∞, *which we obtained in part 3, can be constructed on* $(-\infty, \infty)$ *and gives rise to an eternal solution of* (MCF).

Proof. That $\{(N, a_i^2 h, y_i)\}_{i\in\mathbb{N}}$ smoothly subconverges to the Euclidean space $(\mathbb{R}^n, g_{euc}, 0)$ is a consequence of Theorem 1.2.10. Now, let us compute the second fundamental form of the immersion F_i.

- **Induced metric:** We compute,

$$\begin{aligned} g_{F_i}(X,Y) &= a_i^2 h(dF_i(X), dF_i(Y)) = a_i^2 h(dF(X), dF(Y)) \\ &= a_i^2 (F^* h)(X,Y). \end{aligned}$$

  Therefore,

$$g_{F_i} = a_i^2 g_F.$$

  If $\{e_1, e_2, \ldots, e_m\}$ is an orthonormal frame with respect to g_F, then

$$\left\{ \tilde{e_1} = \frac{e_1}{a_i}, \tilde{e_2} = \frac{e_2}{a_i}, \ldots, \tilde{e_m} = \frac{e_m}{a_i} \right\}$$

  is orthonormal frame with respect to the metric g_{F_i}.

- **Second fundamental form**: We have,

$$\begin{aligned} A_{F_i}(X,Y) &= \nabla_X^{F_i} dF_i(Y) - dF_i(\nabla_X^{g_{F_i}} Y) \\ &= \nabla_{dF_i(X)}^N dF_i(Y) - dF_i(\nabla_X^{g_{F_i}} Y) \\ &= \nabla_{dF(X)}^N dF(Y) - dF(\nabla_X^{g_{F_i}} Y). \end{aligned} \tag{5.7}$$

Relating the connections of conformal metrics [38, Exercise 4.7.14.] we have

$$\begin{aligned}\nabla^{g_{F_i}}_X Y &= \nabla^{a_i^2 g_F}_X Y \\ &= \nabla^{g_F}_X Y + \frac{1}{a_i^2}\{X(a_i^2)Y + Y(a_i^2)X - g_F(X,Y)\nabla^{g_F} a_i^2\} \\ &= \nabla^{g_F}_X Y.\end{aligned}$$

Therefore (5.7) becomes

$$A_{F_i}(X,Y) = \nabla^N_{dF(X)} dF(Y) - dF(\nabla^{g_F}_X Y) = A_F(X,Y).$$

Hence,

$$A_{F_i}(X,Y) = A_F(X,Y).$$

Note that,

$$H_{F_i} = \mathrm{tr}_{g_{F_i}} A_{F_i} = \sum_i A_{F_i}(\tilde{e}_i, \tilde{e}_i) = \sum_i \frac{1}{a_i^2} A_F(e_i, e_i) = \frac{1}{a_i^2} H_F.$$

Consequently,

$$H_{F_i} = \frac{1}{a_i^2} H_F.$$

Moreover,

$$\begin{aligned}\|A_{F_i}\|^2_{g_{F_i}} &= \sum_{k,l} a_i^2 h(A_{F_i}(\tilde{e_k}, \tilde{e_l}), A_{F_i}(\tilde{e_k}, \tilde{e_l})) \\ &= a_i^2 \frac{1}{a_i^4} \sum_{k,l} h(A_F(e_k, e_l), A_F(e_k, e_l)) \\ &= \frac{1}{a_i^2} \|A_F\|^2_{g_F}. \end{aligned} \tag{5.8}$$

In particular, we have

$$\|A_{F_i(x,s)}\|^2_{g_{F_i}} = \frac{1}{a_i^2} \|A_{F(x, s/a_i^2 + t_i)}\|^2_{g_F} \leq 1$$

and

$$\|A_{F_i(x,0)}\|^2_{g_{F_i}} = \frac{1}{a_i^2} \|A_{F(x_i,t_i)}\|^2_{g_F} = 1.$$

- **Evolution of F_i:** We have,

$$\begin{aligned} dF_i|_{(x,s)}(\partial_s) &= \frac{\partial F_i}{\partial s}(x,s) = \frac{1}{a_i^2}\frac{\partial F}{\partial s}(x, s/a_i^2 + t_i) \\ &= \frac{1}{a_i^2} dF\big|_{(x,s/a_i^2+t_i)}(\partial_s) = \frac{1}{a_i^2} H_{F(x,s/a_i^2+t_i)} \\ &= H_{F_i(x,s)}. \end{aligned}$$

  Thus,

$$dF_i(\partial_s) = H_{F_i}.$$

  Therefore, F_i evolves under (MCF). Since

$$\|A_{F_i}\|^2_{g_{F_i}} \leq 1,$$

  according to Theorem 5.1.1, all the derivatives of A_{F_i} are bounded. Hence, due to Lemma 1.2.12 it follows that the sequence $\{(M_i, F_i^*(a_i^2 h), x_i)\}_{i\in\mathbb{N}}$ has injectivity radius positive and bounded away from zero. From Gauss formula we have

$$\begin{aligned} R_{g_i}(X,Y,Z,W) &= R^N(dF_i(X), dF_i(Y), dF_i(Z), dF_i(W)) \\ &\quad + \langle A_{F_i}(X,Z), A_{F_i}(Y,W)\rangle - \langle A_{F_i}(X,W), A_{F_i}(Y,Z)\rangle. \end{aligned}$$

  Differentiating the above equation and using the facts that the derivatives of A are bounded and that $(N, a_i^2 h)$ has bounded geometry, we conclude that R_{g_i} and all its derivatives are bounded. Thus, $\{(M_i, F_i^*(a_i^2 h), x_i)\}_{i\in\mathbb{N}}$ has uniformly bounded geometry and from Theorem 1.2.9 it subconverges smoothly to a Riemannian manifold $(M_\infty, g_\infty, x_\infty)$. That M_∞ does not depend of s we refer to [9, Theorem 2.3.1].

This completes the proof. ☐

Remark 5.2.1. Let us conclude this subsection with some comments.

1. The flow F_∞ which we obtained in Theorem 5.2.1 is called ***singularity***. In other words singularity of a (MCF) is another mean curvature flow.

2. If $\lim R_i \in (0,\infty)$ then the singularity is called ***Type I***. This happens if there exists a constant C such that

$$\|A_F\|^2 \leq \frac{C}{T-t}.$$

   A singularity is called of ***Type II*** if it is not of Type I.

3. In the case of a mean curvature flow $F\colon M \times [0,T) \to \mathbb{R}^n$ in the Euclidean space instead of rescaling the metric of the ambient space , equivalently we can scale the immersion itself. In particular, we can consider the following blow up sequence
$$F_i\colon M \times [L_i, R_i) \to \mathbb{R}^n$$
given by
$$F_i(x,s) = F_{i,s}(x) = \alpha_i\left(F\Big(x, \frac{s}{\alpha_i^2} + t_i\Big) - F\left(x_i, t_i\right)\right).$$
Note that,
$$F_i\left(x_i, 0\right) = F_{i,0}\left(x_i\right) = 0,$$
for all $i \in \mathbb{N}$.

4. Formation of the grim reaper curve

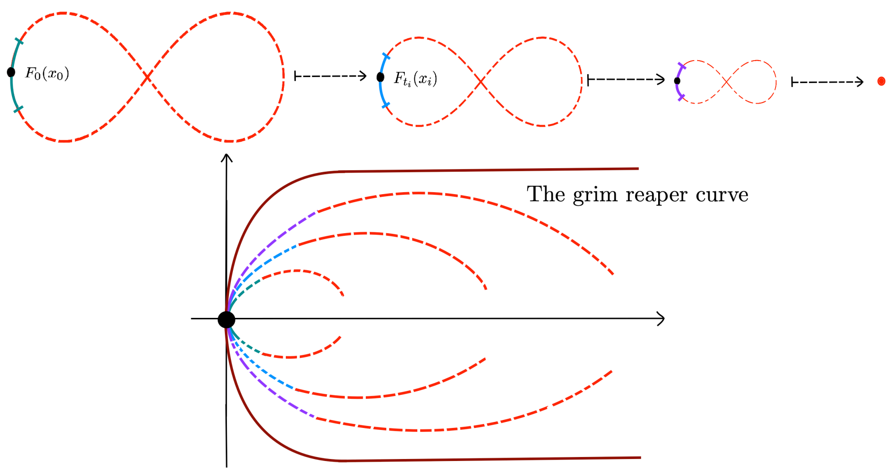


Figure 5.2: Formation of the grim reaper curve.

CHAPTER 6

Singular Behaviour of ELMCF

In this chapter we will explore the singularity formation along the equivariant mean curvature flow (ELMCF, for short).

6.1 Evolution equations of basic quantities

Let us follow the consideration of Chapter 3. Namely, let $z = (u, v)$ be a closed profile curve generating the equivariant Lagrangian submanifold $L_0 \subset \mathbb{C}^m$ parametrized by the map

$$F_0(s, x) = (u(s)f(x), v(s)f(x)),$$

where $f\colon \mathbb{S}^{m-1} \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is the standard embedding of the unit sphere. Since the mean curvature flow preserves the symmetries, the property of being equivariant is preserved along the flow. Therefore, the generated submanifolds L_t are equivariant and can be parametrized by the immersions given by

$$F(s, x; t) = (u(s, t)f(x), v(s, t)f(x)),$$

for every point $(s, x; t)$ in space-time. Let us begin with the following general observation.

Lemma 6.1.1. *Suppose that $F\colon M \times [0, T) \to \mathbb{R}^n$ is a solution of* (MCF) *(not necessarily Lagrangian), where M is a compact m-dimensional manifold and T the maximal time of existence. Then T is always finite.*

Proof. Consider the distance function $r\colon M \times [0, T) \to \mathbb{R}$ given by

$$r \doteq \|F\|^2 = \langle F, F\rangle.$$

Let us compute the evolution of r. We have,

$$r_t = 2\langle dF(\partial_t), F\rangle = 2\langle H, F\rangle.$$

For any $X \in \mathcal{H}$ we have

$$X(r) = X\langle F, F\rangle = 2\langle dF(X), F\rangle,$$

and

$$XX(r) = 2\langle A(X,X), F\rangle + 2\langle dF(X), dF(X)\rangle + (\nabla_X X)(r).$$

Therefore,

$$(\nabla^2 r)(X,X) = 2\langle A(X,X), F\rangle + 2\|dF(X)\|^2.$$

Hence,

$$\Delta r = 2\langle H, F\rangle + 2m$$

and so,

$$r_t - \Delta r = -2m.$$

Consider the ODE,

$$\begin{cases} y'(t) = -2m, \quad t \in [0,T), \\ y(0) = \max_{x\in M} r(x,0) = r_0. \end{cases}$$

Solving the ODE, we have

$$y(t) = -2mt + r_0, \quad t \in [0,T).$$

From the parabolic maximum principle we get

$$0 \le r(x,t) \le y(t) = -2mt + r_0, \quad \text{for all } (x,t) \in M \times [0,T).$$

Consequently,

$$t \le \frac{r_0}{2m}.$$

Taking the limit as t tends to T, we have

$$T \le \frac{r_0}{2m}.$$

This completes the proof. $\square$

From the above lemma we conclude that the evolution of a compact equivariant Lagrangian in $\mathbb{R}^{2m}$ always develops a finite-time singularity. In the next lemmas we describe how various quantities of the profile curve evolve in time.

Lemma 6.1.2. *The profile curve evolves under the*

$$\frac{\partial z}{\partial t} - \Delta z = -(m-1)p\,\xi.$$

Proof. Note that

$$dF(\partial_t) = H.$$

On the other hand, from the considerations in Chapter 3 and Lemma 3.1.2, we have

$$F(s,x;t) = (u(s,t)f(x), v(s,t)f(x)), \tag{6.1}$$

for every point $(s,x;t)$ in space-time and

$$H = (k-(m-1)p)\nu_0 \tag{6.2}$$

where

$$\nu_0 = \frac{(-v_s f, u_s f)}{\sqrt{u_s^2+v_s^2}}.$$

From the relations (6.1) and (6.2) we have

$$\left(\frac{\partial u}{\partial t}f, \frac{\partial v}{\partial t}f\right) = (k-(m-1)p)\frac{(-v_s f, u_s f)}{\sqrt{u_s^2+v_s^2}}.$$

Hence,

$$\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right) = (k-(m-1)p)\frac{(-v_s, u_s)}{\sqrt{u_s^2+v_s^2}},$$

from where we see that

$$\frac{\partial z}{\partial t} = (k-(m-1)p)\xi.$$

As we already have seen in Chapter 3, it holds

$$\Delta z = k\xi.$$

Taking the difference, we get

$$\frac{\partial z}{\partial t} - \Delta z = -(m-1)p\,\xi.$$

This completes the proof. □

According to the previous lemma, the evolution by mean curvature flow of an equivariant Lagrangian is completely determined by the curvature flow with an extra driving force

$$\boxed{\frac{\partial z}{\partial t} = \kappa - (m-1)\frac{z^{\perp}}{\|z\|^2},} \tag{ECSF}$$

where κ is the curvature vector of z and $\{\cdot\}^{\perp}$ stands for the projection onto the normal bundle of z. Hence, the speed is determined by the function

$$h \doteq k - (m-1)p.$$

In particular,

$$\frac{\partial z}{\partial t} = h\,\xi. \tag{6.3}$$

Note that for $m = 1$, equation (ECSF) is precisely the ***curve shortening flow.***

As in the standard curve shortening flow, we would like to perform computations with respect to a time-dependent arc-length parameter. Consider the function $\alpha\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}$ given by

$$\alpha(s,t) \doteq \|z_s(s,t)\| = \langle z_s(s,t), z_s(s,t)\rangle^{\frac{1}{2}}.$$

Then, the length element of the evolved curve is

$$d\mu = \alpha\, ds.$$

The arc-length parameter is given by

$$\sigma(s,t) = \int_0^s \alpha(y,t)\, dy.$$

Hence, we may introduce σ as a parameter. Note that

$$\partial_\sigma = \alpha^{-1}\partial_s.$$

Lemma 6.1.3. *The following formula holds true*

$$\nabla_{\partial_t} d\mu = -hk\, d\mu.$$

Proof. We compute

$$\nabla_{\partial_t} d\mu = \partial_t(\alpha) ds = \alpha_t\, ds.$$

On the other hand, from the Serret-Frenet equations and (6.3), we obtain

$$\begin{aligned}\alpha_t &= \frac{\langle z_{ts}, z_s\rangle}{\|z_s\|} = \frac{\langle (h\,\xi)_s, z_s\rangle}{\|z_s\|} \\ &= \frac{\langle h_s\xi, z_s\rangle + \langle h\xi_s, z_s\rangle}{\|z_s\|} = \frac{\langle h\xi_s, z_s\rangle}{\|z_s\|} \\ &= -hk\|z_s\|,\end{aligned}$$

from where we find

$$\alpha_t = -hk\alpha.$$

This completes the proof. □

Lemma 6.1.4. *The following formula holds*

$$\partial_t\,\partial_\sigma = \partial_\sigma\,\partial_t + kh\,\partial_\sigma.$$

Proof. We have

$$\begin{aligned}\partial_t\partial_\sigma &= \partial_t(\alpha^{-1}\partial_s) = \alpha^{-1}\partial_t\partial_s + \partial_t(\alpha^{-1})\partial_s \\ &= \alpha^{-1}\partial_s\partial_t - \alpha^{-2}\partial_t(\alpha)\partial_s \\ &= \partial_\sigma\partial_t + \alpha^{-1}kh\,\partial_s \\ &= \partial_\sigma\partial_t + kh\,\partial_\sigma.\end{aligned}$$

This completes the proof. □

Lemma 6.1.5. *The tangent and the normal evolve in time under the equation*

$$\nabla_{\partial_t}\tau = h_\sigma\xi \quad \textit{and} \quad \nabla_{\partial_t}\xi = -h_\sigma\tau.$$

Proof. From the Serret-Frenet equations and (6.3), we obtain

$$\begin{aligned}\partial_t(\tau) &= \partial_t(\alpha^{-1}z_s) = \partial_t(\alpha^{-1})z_s + \alpha^{-1}\partial_t\partial_s(z) \\ &= -\alpha^{-2}\partial_t(\alpha)z_s + \alpha^{-1}\partial_s\partial_t(z) \\ &= \alpha^{-1}khz_s + \alpha^{-1}\partial_s(h\,\xi) \\ &= \alpha^{-1}khz_s + \alpha^{-1}h_s\xi + h\alpha^{-1}\xi_s \\ &= khz_\sigma + h_\sigma\xi + h\xi_\sigma \\ &= h_\sigma\xi.\end{aligned}$$

Note that $\xi = J\tau$, where J is the complex structure on $\mathbb{R}^2$. Hence,

$$\begin{aligned}\nabla_{\partial_t}\xi &= \nabla_{\partial_t}(J\tau) = J(\nabla_{\partial_t}\tau)\\ &= J(h_\sigma\xi) = h_\sigma J\xi\\ &= -h_\sigma\tau.\end{aligned}$$

This completes the proof. □

Lemma 6.1.6. *The function $r = \|z\|$ satisfy the following equations:*

- $r_\sigma^2 \;= 1 - r^2p^2.$
- $r_{\sigma\sigma} = pr(k+p).$
- $r_t \;\; = r_{\sigma\sigma} - mrp^2.$

Proof. We compute

$$r_t = \partial_t(\|z\|) = \frac{\langle z_t, z\rangle}{\|z\|} = h\,\frac{\langle z,\xi\rangle}{\|z\|^2}\,\|z\| = hrp. \tag{6.4}$$

Moreover, the identities for r_σ and $r_{\sigma\sigma}$ are proved in Lemma 3.2.3. Hence, taking the difference, we get

$$r_t - r_{\sigma\sigma} = hrp - pr(p+k) = -mrp^2.$$

This completes the proof. □

Lemma 6.1.7. *The curvature k of z satisfies*

$$k_t = h_{\sigma\sigma} + hk^2.$$

Proof. Using Lemma 6.1.4 and Lemma 6.1.5, we obtain

$$\begin{aligned}k_t &= -\partial_t\langle\xi_\sigma,\tau\rangle = -\langle\partial_t\partial_\sigma(\xi),\tau\rangle - \langle\xi_\sigma, h_\sigma\,\xi\rangle = -\langle\partial_t\partial_\sigma(\xi),\tau\rangle\\ &= -\langle\partial_\sigma(\partial_t(\xi)),\tau\rangle - \langle h\,k\,\xi_\sigma,\tau\rangle = \langle\partial_\sigma(h_\sigma\tau),\tau\rangle + \langle h\,k^2\tau,\tau\rangle\\ &= \langle h_{\sigma\sigma}\tau + h_\sigma\tau_\sigma,\tau\rangle + hk^2 = h_{\sigma\sigma} + hk^2\\ &= h_{\sigma\sigma} + hk^2.\end{aligned}$$

This completes the proof. □

Let us consider the ***(normalized) support function*** of the evolved curve

$$\varrho \doteq \frac{\langle z, \xi\rangle}{r} = r\,p.$$

The function ϱ measures the angle between the position vector z and the unit normal vector ξ.

Lemma 6.1.8. *The support function ϱ satisfies the following identity*

$$\varrho_\sigma = -r_\sigma(k + \varrho/r).$$

Moreover, it evolves in time under the equation

$$\varrho_t - \varrho_{\sigma\sigma} = (m-1)\frac{\varrho_\sigma r_\sigma}{r} - 2m\frac{\varrho(1-\varrho^2)}{r^2} + \varrho\Big(k + \frac{\varrho}{r}\Big)^2.$$

Proof. By straight-forward computations and using the previous lemmas, we get

$$\begin{aligned}\varrho_t &= \partial_t(r^{-1}\langle z, \xi\rangle) = -r^{-2}r_t\langle z, \xi\rangle + r^{-1}\langle z_t, \xi\rangle + r^{-1}\langle z, \xi_t\rangle \\ &= -hr^{-1}p\langle z, \xi\rangle + r^{-1}h - r^{-1}h_\sigma\langle z, \tau\rangle \\ &= h(r^{-1} - r^{-1}\varrho^2) - r^{-1}h_\sigma\langle z, \tau\rangle. \end{aligned} \tag{6.5}$$

Using the identities

$$\langle z, \tau\rangle = r\,r_\sigma, \quad r_\sigma^2 = 1 - \varrho^2 \quad \text{and} \quad h = k - (m-1)r^{-1}\varrho, \tag{6.6}$$

we get

$$\begin{aligned}\varrho_t \;&=\; \Big(\frac{k}{r} - (m-1)\frac{\varrho}{r^2}\Big)(1-\varrho^2) \\ &\quad -k_\sigma\frac{\langle z, \tau\rangle}{r} + (m-1)\frac{\varrho_\sigma}{r^2}\,\langle z, \tau\rangle - (m-1)\frac{\varrho\, r_\sigma}{r^3}\,\langle z, \tau\rangle \\ &=\; \Big(\frac{k}{r} - (m-1)\frac{\varrho}{r^2}\Big)(1-\varrho^2) \\ &\quad -k_\sigma r_\sigma + (m-1)\frac{\varrho_\sigma r_\sigma}{r} - (m-1)\frac{\varrho\, r_\sigma^2}{r^2} \\ &=\; \Big(\frac{k}{r} - 2(m-1)\frac{\varrho}{r^2}\Big)(1-\varrho^2) - k_\sigma r_\sigma + (m-1)\frac{\varrho_\sigma r_\sigma}{r}.\end{aligned}$$

Hence,

$$\varrho_t = \Big(\frac{k}{r} - 2(m-1)\frac{\varrho}{r^2}\Big)(1-\varrho^2) - k_\sigma r_\sigma + (m-1)\frac{\varrho_\sigma r_\sigma}{r}.$$

On the other hand,

$$\begin{aligned}
\varrho_\sigma &= (r^{-1}\langle z,\xi\rangle)_\sigma = -r^{-2}r_\sigma\varrho\, r - r^{-1}\langle z, k\tau\rangle \\
&= -r^{-1}r_\sigma\varrho - kr^{-1}\langle z,\tau\rangle = -r^{-1}r_\sigma\varrho - kr_\sigma \\
&= -r_\sigma\Big(k+\frac{\varrho}{r}\Big).
\end{aligned}$$

Hence,

$$\varrho_\sigma = -r_\sigma\Big(k+\frac{\varrho}{r}\Big) \quad \text{and} \quad \varrho_\sigma r_\sigma = -\Big(k+\frac{\varrho}{r}\Big)(1-\varrho^2). \tag{6.7}$$

Differentiating once more, we get

$$\begin{aligned}
\varrho_{\sigma\sigma} &= -k_\sigma r_\sigma - kr_{\sigma\sigma} - \frac{\varrho_\sigma\, r_\sigma}{r} - \frac{\varrho\, r_{\sigma\sigma}}{r} + \frac{\varrho\, r_\sigma^2}{r^2} \\
&= -k_\sigma r_\sigma - \frac{\varrho_\sigma\, r_\sigma}{r} + \frac{\varrho\, r_\sigma^2}{r^2} - r_{\sigma\sigma}\Big(k+\frac{\varrho}{r}\Big) \\
&= -k_\sigma r_\sigma + \Big(k+\frac{\varrho}{r}\Big)\frac{1-\varrho^2}{r} + \frac{\varrho(1-\varrho^2)}{r^2} - \varrho\Big(k+\frac{\rho}{r}\Big)^2.
\end{aligned}$$

Hence,

$$\varrho_t = \Big(\frac{k}{r} + 2\frac{\varrho}{r^2} - 2m\frac{\varrho}{r^2}\Big)(1-\varrho^2) - k_\sigma r_\sigma + (m-1)\frac{\varrho_\sigma r_\sigma}{r}.$$

and

$$\varrho_{\sigma\sigma} = -k_\sigma r_\sigma - \varrho\Big(k+\frac{\varrho}{r}\Big)^2 + \Big(k+2\frac{\varrho}{r}\Big)\frac{1-\varrho^2}{r}.$$

Taking the difference we get the desired result. □

Lemma 6.1.9. *The function h evolves under the equation*

$$h_t - h_{\sigma\sigma} = (m-1)\frac{h_\sigma r_\sigma}{r} + h\Big(k^2 + \frac{m-1}{r^2}(2\varrho^2-1)\Big).$$

Proof. Again by straightforward computations and using the previous lemmas, we get

$$h_t = k_t - (m-1)\frac{\varrho_t}{r} + (m-1)\frac{\varrho\, r_t}{r^2}.$$

Hence,

$$
\begin{aligned}
h_t &= h_{\sigma\sigma} + hk^2 + (m-1)\frac{k_\sigma r_\sigma}{r} - \frac{(m-1)^2}{r}\frac{\rho_\sigma r_\sigma}{r} \\
&\quad - \frac{m-1}{r}\Big(\frac{h}{r} - (m-1)\frac{\varrho}{r^2}\Big)(1-\varrho^2) + (m-1)\frac{h\,\varrho^2}{r^2} \\
&= h_{\sigma\sigma} + h\Big(k^2 + \frac{m-1}{r^2}(2\varrho^2-1)\Big) \\
&\quad + \frac{m-1}{r}r_\sigma\Big(k_\sigma - \frac{m-1}{r}\varrho_\sigma + \frac{m-1}{r^2}\varrho\, r_\sigma\Big) \\
&= h_{\sigma\sigma} + h\Big(k^2 + \frac{m-1}{r^2}(2\varrho^2-1)\Big) + \frac{m-1}{r}r_\sigma h_\sigma .
\end{aligned}
$$

This completes the proof. □

6.2 Singularities of equivariant Lagrangian tori

Throughout this section we consider solutions of the equivariant mean curvature flow of the form $F\colon \mathbb{S}^1 \times \mathbb{S}^{m-1} \times [0,T) \to \mathbb{R}^{2m}$, generated by the family of profile curves

$$z = (u,v)\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2 - \{(0,0)\},$$

where T is the maximal time of existence of (MCF). Recall that while F evolves by the (MCF), the corresponding profile curve evolves by

$$\frac{\partial z}{\partial t} = (k-(m-1)p)\xi.$$

Perhaps the maximal time of existence of (ECSF) may be larger than the maximal time T of (MCF). Let us investigate this question in more detail.

Lemma 6.2.1. *Let $z\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2 - \{(0,0)\}$ be a closed curve evolving under* (ECSF)*, where T is the maximal time of existence of* (MCF)*. As t tends to T, we have*

$$\limsup_{t\to T} \|k\| = \infty \quad \textit{or} \quad \liminf_{t\to T} r = 0.$$

Moreover, if $m>1$ and there exists $T_{\text{origin}} \in (0,T]$ such that

$$\liminf_{t\to T_{\text{origin}}} r = 0,$$

then $T_{\text{origin}} = T$.

Proof. From Lemma 3.1.2 we have

$$\|A\|^2 = k^2 + 3(m-1)p^2.$$

Hence, we approach the maximal time of existence T if k^2 tends to infinity or if p^2 tends to infinity. We will show that p^2 tends to infinity if and only if r tends to zero. Suppose that

$$\liminf_{t \to T_{\text{origin}}} r = 0.$$

Consider the function $R\colon [0, T_{\text{origin}}) \to \mathbb{R}$ given by $R(t) \doteq \min_{x \in \mathbb{S}^1} r(x,t)$. Because r is a continuous function, for each t there exists $x_t \in \mathbb{S}^1$ such that $R(t) = r(x_t, t)$. By our assumption, there exists a sequence $\{(x_i, t_i)\}_{i \in \mathbb{N}}$ such that

$$R(t_i) = r(x_i, t_i) = \min_{x \in \mathbb{S}^1} r(x_i, t_i) \to 0.$$

Because x_i is a minimum point of $r(\,\cdot\,, t_i)$ we have

$$0 = r_\sigma^2(x_i, t_i) = 1 - \varrho^2(x_i, t_i)$$

from where it follows that $\varrho^2(x_i, t_i) = 1$. Therefore,

$$p^2(x_i, t_i) = \frac{\varrho^2(x_i, t_i)}{r^2(x_i, t_i)} = \frac{1}{R^2(t_i)} \to \infty.$$

Consequently,

$$\|A\|^2(x_i, t_i) \geq 3(m-1)p^2(x_i, t_i) \to \infty.$$

Thus, $T_{\text{origin}} = T$, and this completes the proof. $\square$

In view of the last lemma, we distinguish the following three cases:

C_1. $\limsup_{t\to T} \|k\| = \infty$ and $\liminf_{t\to T} r = 0$.

C_2. $\limsup_{t\to T} \|k\| = \infty$ and $\liminf_{t\to T} r > 0$.

C_3. $\limsup_{t\to T} \|k\| < \infty$ and $\liminf_{t\to T} r = 0$.

Singularities of type C_3 are called ***fake singularities*** because (ECSF) might still exist beyond time T. However, the existence of fake singularities is still unclear, since the curvature k may tend to infinity while r approaches zero.

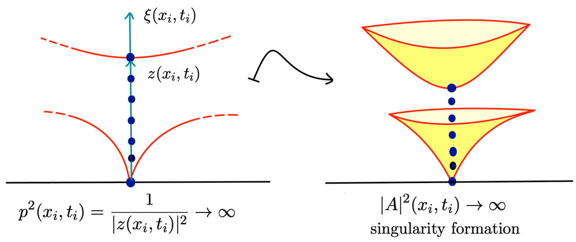


Figure 6.1: Conical singularity forming at the origin.

6.3 Star-shaped equivariant Lagrangians

In this section we investigate equivariant Lagrangians that are star-shaped with respect to the origin.

Theorem 6.3.1. *Let $F\colon \mathbb{S}^1 \times \mathbb{S}^{m-1} \times [0,T) \to \mathbb{R}^{2m}$, with $m \geq 2$, be a smooth solution of* (MCF)*, with corresponding profile curves*

$$z\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2 - \{(0,0)\},$$

where T is the maximal time of existence of (MCF)*. Suppose that the initial curve z_0 is star-shaped. Then:*

1. *The evolved profile curves remain star-shaped as long as a smooth solution of* (MCF) *exists in $\mathbb{R}^2 - \{(0,0)\}$.*
2. *The first singularity occurs at the origin, that is,*

$$\liminf_{t\to T} r = 0.$$

Proof. With the choice of ξ pointing toward the interior of the curve, the curve z is star-shaped with respect to the origin if and only if

$$\varrho = \frac{\langle z, \xi \rangle}{r} < 0.$$

1. We have to show that ρ stays negative for all times $t \in [0,T)$. In other words we will prove that, if $\varrho(x,0) < 0$, for all $x \in \mathbb{S}^1$, then

$$\varrho(x,t) < 0, \text{ for all } (x,t) \in \mathbb{S}^1 \times [0,T).$$

By compactness and continuity, there exists $\varepsilon > 0$ such that

$$\varrho(x,t) < 0, \text{ for all } (x,t) \in \mathbb{S}^1 \times [0,\varepsilon).$$

Suppose to the contrary that there exists a first time $t_0 \in [0,T)$ such that $\varrho(x,t) < 0$ for all $(x,t) \in \mathbb{S}^1 \times [0,t_0)$ and

$$\varrho(x_0,t_0) = 0, \quad \text{for some } x_0 \in \mathbb{S}^1.$$

Let us fix a positive time $0 < t_* < t_0$. Then on $\mathbb{S}^1 \times [0,t_*]$ we have:

- $-\dfrac{2m}{r^2}(1-\rho^2) + \Big(k + \dfrac{\varrho}{r}\Big)^2 \geq \lambda \in \mathbb{R}$,
- $\dfrac{r_\sigma}{r}$ is bounded,
- $\varrho\Big(-\dfrac{2m}{r^2}(1-\varrho^2) + \Big(k + \dfrac{\varrho}{r}\Big)^2\Big) \leq \varrho\,\lambda$.

Then, from Lemma 6.1.8, we obtain

$$\varrho_t - \varrho_{\sigma\sigma} = (m-1)\frac{\varrho_\sigma\, r_\sigma}{r} + \lambda\,\varrho.$$

Consider the ODE,

$$\begin{cases} y'(t) = \lambda\, y(t) \\ y(0) = \max_{x\in\mathbb{S}^1} \varrho(x,0) < 0. \end{cases} \tag{6.8}$$

Solving the ODE, we have

$$y(t) = e^{\lambda t}\, y(0).$$

From the comparison maximum principle we get

$$\varrho(x,t) \leq y(t) = e^{\lambda t}\, y(0)$$

and so

$$\varrho(x,t_*) \leq e^{\lambda t_*}\, y(0).$$

Passing to the limit as t_* tends to t_0, we get

$$\varrho(x,t_0) \leq e^{\lambda t_0}\, y(0).$$

Evaluating at $x = x_0$, we find

$$\varrho(x_0,t_0) \leq e^{\lambda t_0}\, y(0) < 0,$$

which contradicts our assumption.

2. To show that $r \to 0$ as $t \to T$, we need a better estimate for ϱ.

Claim I: *For any $n \geq 2$, there exists a constant $\varepsilon_n > 0$ such that*

$$r^{-n}\,\varrho \leq -\varepsilon_n$$

holds as long as a smooth solution of (MCF) *exists in* $\mathbb{R}^2 - \{(0,0)\}$.

Proof of claim I: Consider the function

$$\beta \doteq r^{-n}\,\varrho.$$

Then

$$\beta_t = -nr^{-n-1}r_t\varrho + r^{-n}\varrho_t \quad \text{and} \quad \beta_\sigma = -nr^{-n-1}r_\sigma\varrho + r^{-n}\varrho_\sigma. \tag{6.9}$$

Differentiating once more we get

$$\beta_{\sigma\sigma} = n(n+1)r^{-n-2}r_\sigma^2\varrho - nr^{-n-1}r_{\sigma\sigma}\varrho + r^{-n}\varrho_{\sigma\sigma} - 2nr^{-n-1}r_\sigma\varrho_\sigma.$$

Therefore, taking the difference we get

$$\begin{aligned}\beta_t - \beta_{\sigma\sigma} \;&=\; -nr^{-n-1}\varrho(r_t - r_{\sigma\sigma}) + r^{-n}(\varrho_t - \varrho_{\sigma\sigma})\\ &\quad -n(n+1)r^{-n-2}r_\sigma^2\varrho + 2nr^{-n-1}r_\sigma\varrho_\sigma.\end{aligned}$$

Using the formulas for r_σ, ϱ_σ and the evolutions of r and ϱ, we get

$$\begin{aligned}\beta_t - \beta_{\sigma\sigma} \;&=\; (m-1+2n)\frac{\beta_\sigma r_\sigma}{r}\\ &\quad +\beta\Big\{(n^2-2n-2m)\frac{1-\varrho^2}{r^2} + \frac{nm}{r^2} + \Big(k+\frac{\varrho}{r}\Big)^2\Big\}.\end{aligned} \tag{6.10}$$

We will show that β cannot admit an increasing negative maximum. Note that the function

$$B(t) \doteq \max_{x\in\mathbb{S}^1}\beta(x,t) = \beta(x_t,t),$$

is Lipschitz and so almost everywhere smooth. At a (smooth) minimum point of β, we have

$$0 = \beta_\sigma = -nr^{-n-1}r_\sigma\varrho + r^{-n}\varrho_\sigma = -r^{-n}r_\sigma\Big\{k + (n+1)\frac{\varrho}{r}\Big\}.$$

Case 1: At the (smooth) maximum point of β we have $r_\sigma = 0$. In this case $\varrho = -1$. Thus,

$$\beta_t \leq \beta\left\{\frac{nm}{r^2} + \Big(k+\frac{1}{r}\Big)^2\right\} < 0.$$

Case 2: At the (smooth) minimum point of β we have

$$k = -(n+1)\frac{\varrho}{r}.$$

In this case we have

$$\begin{aligned}\beta_t &\leq \beta\Big\{(n^2-2n-2m)\frac{1-\varrho^2}{r^2}+\frac{nm}{r^2}+n^2\,\frac{\varrho^2}{r^2}\Big\}\\ &= \beta\left\{\frac{(n-2)(n+m)}{r^2}+2\frac{n+m}{r^2}\varrho^2\right\}\\ &< 0.\end{aligned}$$

In each case, the minimum of β is non-decreasing. This completes the proof of the Claim I. $\circledast$

Claim II: *The distance function r tends to zero as time t approaches T the maximal time of existence.*

Proof of claim II: We argue by contradiction. Assume to the contrary that

$$\liminf_{t\to T} r > 0,$$

and we will show that $\|k\|$ is uniformly bounded in time. This implies that $\|A\|^2$ is uniformly bounded in time, contradicting the definition of T. To achieve this, we proceed as follows. Let a be defined by

$$\cos(a) = \varrho = \frac{\langle z, \xi\rangle}{r}.$$

We seek a smooth negative function $f(a) < -\varepsilon$, where ε is a sufficiently small positive constant, such that

$$g = f(a)h$$

is uniformly bounded.

Step 1: *Evolution equation of the angle function* $a \doteq \arccos(\varrho)$.

We compute

$$a_t = -\frac{\varrho_t}{\sqrt{1-\varrho^2}} \quad \text{and} \quad a_\sigma = -\frac{\varrho_\sigma}{\sqrt{1-\varrho^2}}.$$

Differentiating once more we get

$$a_{\sigma\sigma} = -\frac{\varrho_{\sigma\sigma}}{\sqrt{1-\varrho^2}} - \frac{\varrho\varrho_\sigma^2}{(1-\varrho^2)^{\frac{3}{2}}}.$$

Subtracting, we get

$$a_t - a_{\sigma\sigma} = -\frac{\varrho_t - \varrho_{\sigma\sigma}}{(1-\varrho^2)^{\frac{1}{2}}} + \frac{\varrho\,\varrho_\sigma^2}{(1-\varrho^2)^{\frac{3}{2}}}.$$

Using the formulas for ϱ_σ^2, r_σ^2, and the evolution equation of ϱ from Lemma 6.1.8 we get

$$a_t - a_{\sigma\sigma} = \frac{m-1}{r}\Big(k + \frac{\varrho}{r}\Big)\sqrt{1-\varrho^2} + 2m\,\frac{\varrho}{r^2}\sqrt{1-\varrho^2}.$$

Step 2: *Evolution equation of the function $g = f(a)h$.*

We compute

$$g_t = f'a_t h + f h_t \quad \text{and} \quad g_\sigma = f'a_\sigma h + f h_\sigma. \tag{6.11}$$

Differentiating once more we get

$$g_{\sigma\sigma} = f'' a_\sigma^2 h + f' a_{\sigma\sigma} h + 2f' a_\sigma h_\sigma + f h_{\sigma\sigma}.$$

Therefore, by taking the difference, we get

$$g_t - g_{\sigma\sigma} = f'h(a_t - a_{\sigma\sigma}) + f(h_t - h_{\sigma\sigma}) - f'' a_\sigma^2 h - 2f' a_\sigma h_\sigma.$$

Replacing the evolution equations of a and h and using the identities satisfied by the gradients of these functions, we find

$$\begin{aligned} g_t - g_{\sigma\sigma} \;=\; & \frac{m-1}{r} f h_\sigma r_\sigma - 2f' a_\sigma h_\sigma \\ & + g\Big\{k^2 + \frac{m-1}{r^2}(2\varrho^2 - 1)\Big\} + g\frac{m-1}{r}\frac{f'}{f}\Big(k + \frac{\varrho}{r}\Big)\sqrt{1-\varrho^2} \\ & + g\frac{2m}{r^2}\frac{f'}{f}\varrho\sqrt{1-\varrho^2} - g\frac{f''}{f}\Big(k + \frac{\varrho}{r}\Big)^2. \end{aligned} \tag{6.12}$$

We have to explore the nature of the first two terms in the equality above. From (6.11) we get

$$\frac{m-1}{r} f h_\sigma r_\sigma = \frac{m-1}{r} g_\sigma r_\sigma - g\frac{m-1}{r}\frac{f'}{f}\Big(k + \frac{\varrho}{r}\Big)\sqrt{1-\varrho^2}$$

and

$$-2f' h_\sigma a_\sigma = -2\frac{f'}{f} g_\sigma a_\sigma + 2g\Big(\frac{f'}{f}\Big)^2\Big(k + \frac{\varrho}{r}\Big)^2.$$

Therefore,

$$\begin{aligned}\frac{m-1}{r} f h_\sigma r_\sigma - 2f' h_\sigma a_\sigma \;=\; & \frac{m-1}{r} g_\sigma r_\sigma - g\frac{m-1}{r}\frac{f'}{f}\Big(k+\frac{\varrho}{r}\Big)\sqrt{1-\varrho^2} \\ & -2\frac{f'}{f} g_\sigma a_\sigma + 2g\Big(\frac{f'}{f}\Big)^2\Big(k+\frac{\varrho}{r}\Big)^2.\end{aligned}$$

Consequently, (6.12) becomes

$$\begin{aligned} g_t - g_{\sigma\sigma} \;=\; & (m-1)\frac{g_\sigma r_\sigma}{r} - \frac{2f_\sigma g_\sigma}{f} \\ & + g\left\{k^2 + \frac{m-1}{r^2}(2\varrho^2-1) + \frac{2m}{r^2}\,\frac{f'}{f}\,\varrho\sqrt{1-\rho^2}\right. \\ & \left. + \Big(2\Big(\frac{f'}{f}\Big)^2 - \frac{f''}{f}\Big)\Big(k+\frac{\varrho}{r}\Big)^2\right\}. \end{aligned} \tag{6.13}$$

Step 3: *Arriving at a contradiction.*

Since we want to show that g is bounded, above and below, let us consider

$$\widetilde{g} = g^2.$$

Differentiating with respect to t and σ respectively, we have

$$\widetilde{g}_t = 2gg_t, \quad \widetilde{g}_\sigma = 2gg_\sigma \quad \text{and} \quad \widetilde{g}_{\sigma\sigma} = 2g_\sigma^2 + 2gg_{\sigma\sigma}.$$

Taking the difference, we have

$$g_t - g_{\sigma\sigma} = 2g(g_t - g_{\sigma\sigma}) - 2g_\sigma^2$$

from where we see that

$$\begin{aligned} \widetilde{g}_t - \widetilde{g}_{\sigma\sigma} \;\leq\; & \frac{m-1}{r}\widetilde{g}_\sigma r_\sigma - 2\,\frac{f_\sigma}{f}\,\widetilde{g}_\sigma \\ & + 2\widetilde{g}\left\{k^2 + \frac{m-1}{r^2}(2\varrho^2-1) + \frac{2m}{r^2}\,\frac{f'}{f}\,\varrho\sqrt{1-\rho^2}\right. \\ & \left. + \Big(2\Big(\frac{f'}{f}\Big)^2 - \frac{f''}{f}\Big)\Big(k+\frac{\varrho}{r}\Big)^2\right\}. \end{aligned}$$

Set

$$\begin{aligned} \mathcal{A} \;\doteq\; & k^2 + \frac{m-1}{r^2}(2\varrho^2-1) + \frac{2m}{r^2}\,\frac{f'}{f}\,\varrho\sqrt{1-\rho^2} \\ & + \Big(2\Big(\frac{f'}{f}\Big)^2 - \frac{f''}{f}\Big)\Big(k+\frac{\varrho}{r}\Big)^2. \end{aligned}$$

The goal is to estimate $\mathcal{A}$ from above, provided that r stays bounded away from zero. Since k^2 is nonnegative, the only way to achieve this is to choose f such that

$$2\left(\frac{f'}{f}\right)^2 - \frac{f''}{f} \leq -\mu^2,$$

with $\mu > 1$. The solution of the ODE

$$2\left(\frac{f'}{f}\right)^2 - \frac{f''}{f} = -\mu^2,$$

is

$$f(a) = \frac{1}{\cos(\mu a)}.$$

So a natural choice for f is

$$f(a) = \frac{1}{\cos(\mu a)},$$

for an appropriate $\mu > 1$. To satisfy our assumptions, we first show that there exists $\mu > 1$ such that

$$\cos(\mu a) < -\varepsilon < 0.$$

Indeed, since the evolving curves do not approach the origin, the continuous function r attains a positive minimum r_0. Using Claim I with $n = 2$, we obtain the existence of a constant $\varepsilon_2 > 0$ such that $\varrho \leq -\varepsilon_2 r^2$ from where we see that

$$\cos(a) \leq -\varepsilon_2 r^2 \leq -\varepsilon_2 r_0^2 \doteq -\delta.$$

Since $\varrho \leq -\delta < 0$, we have

$$a = \arccos(\varrho) \in [a_0, \pi],$$

where

$$a_0 = \arccos(-\delta) > \frac{\pi}{2}.$$

Choose $\mu > 1$ sufficiently close to 1 so that $\mu\pi < 3\pi/2$. Then

$$\mu a \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

for all $a \in [a_0, \pi]$. Consequently,

$$\cos(\mu a) \leq -\varepsilon < 0$$

for some constant $\varepsilon > 0$ which proves our assertion.

With this choice of f, the quantity $\mathcal{A}$ becomes

$$
\begin{aligned}
\mathcal{A} &= k^2 + \frac{m-1}{r^2}(2\rho^2 - 1) - \mu^2\Big(k + \frac{\varrho}{r}\Big)^2 \\
&\quad + \frac{2m\mu}{r^2}\frac{\sin(\mu a)}{\cos(\mu a)}\varrho\sqrt{1-\varrho^2} \\
&= k^2 - \mu^2 k^2 + 2\mu^2 k\frac{\varrho}{r} - \mu^2\frac{\varrho^2}{r^2} + \frac{m-1}{r^2}(2\varrho^2 - 1) \\
&\quad + \frac{2m\mu}{r^2}\frac{\sin(\mu a)}{\cos(\mu a)}\varrho\sqrt{1-\varrho^2}.
\end{aligned}
$$

In the sequel we will use repeatedly the Peter-Paul inequality,

$$2\alpha\beta \leq \frac{\alpha^2}{\varepsilon} + \varepsilon\,\beta^2.$$

Hence,

$$
\begin{aligned}
\mathcal{A} &\leq k^2 - \mu^2 k^2 + \varepsilon k^2 + \frac{\varrho^2}{r^2\varepsilon} - \mu^2\frac{\varrho^2}{r^2} + \frac{m-1}{r^2}(2\varrho^2 - 1) \\
&\quad + \frac{2m\mu}{r^2}\frac{\sin(\mu a)}{\cos(\mu a)}\varrho\sqrt{1-\varrho^2}.
\end{aligned}
$$

Choosing ε sufficiently small and using the facts that $r \geq r_0 > 0$ and $\cos(\mu a) \leq -\varepsilon_1 < 0$, we deduce that there exist $\mu_1, \mu_2 > 0$ such that

$$\mathcal{A} \leq (1 - \mu^2 + \varepsilon)k^2 + \mu_2 \leq -\mu_1 k^2 + \mu_2 \leq \mu_2.$$

Therefore,

$$\widetilde{g}_t - \widetilde{g}_{\sigma\sigma} \leq \frac{m-1}{r}\widetilde{g}_\sigma r_\sigma - 2\frac{f_\sigma}{f}\widetilde{g}_\sigma + 2\mu_2\widetilde{g}.$$

From the parabolic maximum principle, it follows that there exists a constant $c > 0$ such that $g^2 < c$. Hence,

$$h^2 \leq c\cos^2(\mu a) \leq c.$$

Since,

$$h = k - (m-1)\frac{\varrho}{r}.$$

and $r \geq r_0 > 0$ it follows that $|k|$ is uniformly bounded. But in this case $\|A\|^2$ does not explode which contradicts the definition of the maximal time of existence.

Consequently, r must tend to zero, and this completes the proof. $\square$

In the previous theorem we showed that a star-shaped curve with respect to the origin develops a singularity at the origin. In some cases it may develop a Type I singularity, while in other situations it forms a Type II singularity. Let us illustrate this with two examples.

Example 6.3.2. Consider the equivariant Lagrangian submanifold generated by the profile curve

$$z_0(s) = (\cos(s), \sin(s)).$$

The submanifold is parametrized by the embedding

$$F_0 \colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^4$$

given by

$$F_0(s,x) = (\cos(s)\cos(x), \cos(s)\sin(x), \sin(s)\cos(x), \sin(s)\sin(x)),$$

whose image is the Clifford torus. We expect that the evolution of z_0 under (ECSF) is of the form

$$z(s,t) = R(t)(\cos(s), \sin(s)),$$

where R is a function to be determined. Note that

$$\xi = -z_0, \qquad p = \frac{\langle z, \xi\rangle}{\|z\|^2} = -\frac{1}{R}, \qquad \text{and} \qquad k = \frac{1}{R}.$$

Then equation (ECSF) becomes

$$\frac{\partial z}{\partial t} = (k-p)\xi = -\frac{2}{R} z_0.$$

On the other hand,

$$\frac{\partial z}{\partial t} = R' z_0.$$

Hence, combining the two formulas above, we obtain

$$R(t) = \sqrt{1-4t}, \quad \text{for all } \ t \in (-\infty, 1/4).$$

Thus,

$$z(s,t) = \sqrt{1-4t}\,(\cos(s), \sin(s)), \qquad \text{for all } (s,t) \in \mathbb{S}^1 \times (-\infty, 1/4).$$

In this case, the evolved Lagrangian submanifolds are self-similar solutions of the mean curvature flow (MCF), namely self-shrinkers evolving by homotheties. Consequently, the singularity is of Type I.

Example 6.3.3. Consider the ellipse

$$C_{a,b} = \left\{ (x,y) \in \mathbb{R}^2 \colon \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}.$$

According to a result of Neves [34–36], for a sufficiently large and b sufficiently small, the equivariant Lagrangian generated by the curve $C_{a,b}$ develops a Type II singularity at the origin; see the figure below.

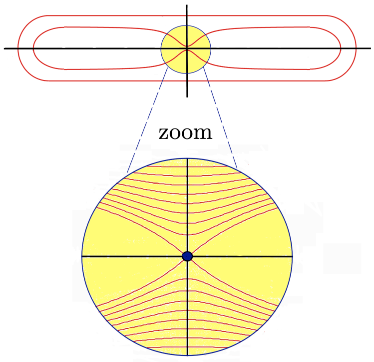


Figure 6.2: Type II singularity formation.

6.4 Evolution of tamed Lagrangians

In this section we deal with closed curves $z = u + iv \colon \mathbb{S}^1 \to \mathbb{C} - \{0\}$ with

$$h = k - (m-1)\frac{\varrho}{r} > 0.$$

Recall that such curves are called tamed curves. The goal is to give a proof of the following Theorem.

Theorem 6.4.1. *Let $F_0 \colon \mathbb{S}^1 \times \mathbb{S}^{m-1} \to \mathbb{C}^m$ be an equivariant Lagrangian. If the profile curve generating F_0 is closed, embedded, tamed, does not contain the origin, then the evolved submanifolds $F_t(\mathbb{S}^1 \times \mathbb{S}^{m-1})$ under the* (MCF) *converges to a sphere of dimension $m-1$ as time t goes to T. Moreover, the singular time is determined by*

$$T = \frac{A_0}{2\pi}$$

where A_0 is the area enclosed by the initial curve.

Proof. In Chapter 3, we proved that the initial curve is contained in a region R, enclosed by the profile curve of a Lagrangian catenoid that does not contain the origin. By the maximum principle, the evolved profile curves remain embedded and trapped inside the region R. In particular, the singularity forms away from the origin. We complete the proof in seven steps.

Step 1: *The enclosed areas of the evolved curves satisfy:*

$$\frac{dA}{dt} = -\int_{\mathbb{S}^1} h\, d\mu = -2\pi$$

and

$$A(t) = A(0) - 2\pi t.$$

Indeed, by Green's theorem, we have

$$A = -\frac{1}{2}\int_{\mathbb{S}^1} \langle z, \xi\rangle\, d\mu.$$

The minus sign in front of the integral appears because the normal ξ is chosen to point inward. Differentiating, we obtain

$$\begin{aligned} \frac{dA}{dt} &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}^1}\langle z,\xi\rangle d\mu = -\frac{1}{2}\int_{\mathbb{S}^1}\frac{d}{dt}(\langle z,\xi\rangle d\mu) \\ &= -\frac{1}{2}\int_{\mathbb{S}^1} h\, d\mu + \frac{1}{2}\int_{\mathbb{S}^1} h_\sigma\langle z,\tau\rangle d\mu + \frac{1}{2}\int_{\mathbb{S}^1} hk\langle z,\xi\rangle d\mu. \end{aligned} \tag{6.14}$$

Let us compute the second integral in the last equality. Integration by parts, yields

$$\frac{1}{2}\int_{\mathbb{S}^1} h_\sigma\langle z,\tau\rangle d\mu = -\frac{1}{2}\int_{\mathbb{S}^1} h\, d\mu - \frac{1}{2}\int_{\mathbb{S}^1} hk\langle z,\xi\rangle d\mu.$$

Hence,

$$\begin{aligned} \frac{dA}{dt} &= -\int_{\mathbb{S}^1} h\, d\mu = -\int_{\mathbb{S}^1} k\, d\mu + (m-1)\int_{\mathbb{S}^1}\frac{\langle z,\xi\rangle}{\|z\|^2}\, d\mu \\ &= -2\pi\,\mathrm{rot}(z) + 2\pi(m-1)\,\mathrm{wind}_0(z) \\ &= -2\pi. \end{aligned}$$

Integrating, we get

$$A(t) - A(0) = -2\pi t,$$

which proofs our assertion.

Step 2: *The positivity of h is preserved under the flow.*

Indeed, from Lemma 6.1.9 the function h evolves under (ECSF) by

$$h_t - h_{\sigma\sigma} = (m-1)\frac{h_\sigma r_\sigma}{r} + h\left\{k^2 + \frac{m-1}{r^2}(2\varrho^2 - 1)\right\}.$$

Since the evolved curves lie inside the region bounded by the profile curve of a Lagrangian catenoid, we have that

$$\min_{\mathbb{S}^1\times[0,T)} r \geq r_0 > 0.$$

We have to show that h stays positive for all times $t \in [0,T)$. By compactness and continuity, there exists $\varepsilon > 0$ such that

$$h(x,t) > 0, \quad \text{for all} \quad (x,t) \in \mathbb{S}^1 \times [0,\varepsilon).$$

Suppose to the contrary that there exists a first time $t_0 \in [0,T)$ such that

- $h(x,t) > 0, \qquad$ for all $\quad (x,t) \in \mathbb{S}^1 \times [0,t_0)$,
- $h(x_0,t_0) = 0, \quad$ for some $\quad x_0 \in \mathbb{S}^1$.

On the set $\mathbb{S}^1 \times [0,t_0]$ the functions $\|A\|$, ϱ and r are bounded. Consequently, k is also bounded. Hence, there exists a real number $\lambda \in \mathbb{R}$ such that

- $k^2 + \dfrac{m-1}{\varrho}(2\varrho^2 - 1) \geq \lambda,$
- $\dfrac{\nabla r}{r}$ is bounded,
- $h\Big\{k^2 + \dfrac{m-1}{\rho}(2\varrho^2 - 1)\Big\} \geq h\lambda.$

Then,

$$h_t - \Delta h \geq (m-1)\frac{h_\sigma r_\sigma}{r} + h\lambda.$$

Consider the ODE,

$$\begin{cases} \quad y' &= \lambda y, \\ y(0) &= \min_{x\in\mathbb{S}^1} h(x,0) > 0. \end{cases}$$

From the parabolic comparison principle, we get

$$h(x,t) \geq y(t) = y(0)e^{\lambda t}, \quad \text{for every} \quad (x,t) \in \mathbb{S}^1 \times [0,t_0].$$

Therefore,

$$0 = h(x_0,t_0) \geq y(0)e^{\lambda t_0} > 0,$$

which is a contradiction. Consequently, such a t_0 does not exist and the positivity of h is preserved under the flow on $t \in [0,T)$.

Step 3: *Derivation of a good test function.*

We proceed arguing by contradiction. Suppose that the area enclosed by the curves does not tend to zero as $t \to T$, i.e.,

$$\text{Area}(t) \not\to 0, \quad \text{as } t \to T.$$

Define the quantity

$$E \doteq \frac{h}{\beta - \hat{\beta}},$$

where $\beta \doteq -r^m \varrho$ and $\hat{\beta}$ is a time-independent smooth function to be specified shortly. Recall from the discussion in Chapter 3, the function β is minus the (unnormalized) support function of the curve

$$w \circ z = z^m.$$

Since z^m is strictly convex, it follows that β is positive. Because the area of the regions enclosed by the sequence $\{z_t^m\}_{t\in[0,T)}$ does not tend to zero, by Blaschke's selection theorem (see for example [45]), the convex curves $\{z_t^m\}_{t\in[0,T)}$ subconverge, in the Hausdorff sense, to a compact convex set K_∞ with non-empty interior. Therefore, there exist $R > 0$ and $y_0 \in \mathbb{R}^2$ such that

$$\overline{B_R(y_0)} \subset \text{int}(K_\infty).$$

Let $\widehat{\beta}$ denote the support function of the circle $\partial B_R(y_0)$. Since the inclusion is strict, there exists $\varepsilon > 0$ such that the support function β_∞ of K_∞ satisfies

$$\beta_\infty - \hat{\beta} \geq 2\varepsilon.$$

On the other hand, Hausdorff convergence of convex curves is equivalent to the uniform convergence of their support functions; see for example [45]. Hence

$$\beta(\cdot,t) \to \beta_\infty, \quad \text{uniformly as} \quad t \to T.$$

Therefore, for all t sufficiently close to T, we have that

$$|\beta(\cdot, t) - \beta_\infty| \leq \varepsilon.$$

Consequently,

$$\beta - \hat{\beta} \geq \beta_\infty - \hat{\beta} - \varepsilon \geq \varepsilon > 0.$$

Hence, after restricting to times sufficiently close to T, we obtain

$$\beta - \hat{\beta} \geq \varepsilon > 0.$$

As a matter of fact, we have

$$\hat{\beta}(x) = R + \alpha_1 \cos(x) + \alpha_2 \sin(x).$$

Let us reparametrize the above circle C using an appropriate time dependent function $\theta(\sigma, t)$. Then, it support function becomes

$$\hat{\beta}(\sigma, t) \doteq R + \alpha_1 \cos(\theta(\sigma, t)) + \alpha_2 \sin(\theta(\sigma, t)),$$

where θ is a smooth function to be determined later.

Differentiating E with respect to t and σ, respectively, we obtain

$$E_t = \frac{h_t}{\beta - \hat{\beta}} - \frac{h(\beta_t - \hat{\beta})}{(\beta - \hat{\beta})^2} \quad \text{and} \quad E_\sigma = \frac{h_\sigma}{\beta - \hat{\beta}} - \frac{h\,(\beta_\sigma - \hat{\beta}_\sigma)}{(\beta - \hat{\beta})^2}.$$

Differentiating once more with respect to σ, we get

$$E_{\sigma\sigma} = \frac{h_{\sigma\sigma}}{\beta - \hat{\beta}} - \frac{2h_\sigma(\beta_\sigma - \hat{\beta}_\sigma)}{(\beta - \hat{\beta})^2} - \frac{h(\beta_{\sigma\sigma} - \hat{\beta}_{\sigma\sigma})}{(\beta - \hat{\beta})^2} - \frac{2h(\beta_\sigma - \hat{\beta}_\sigma)^2}{(\beta - \hat{\beta})^3}.$$

Taking the difference, we have

$$\begin{aligned} E_t - E_{\sigma\sigma} \;=\; & \frac{h_t - h_{\sigma\sigma}}{\beta - \hat{\beta}} - h\frac{\beta_t - \beta_{\sigma\sigma}}{(\beta - \hat{\beta})^2} \\ & + \frac{2h_\sigma(\beta_\sigma - \hat{\beta}_\sigma) - h\hat{\beta}_{\sigma\sigma}}{(\beta - \hat{\beta})^2} + \frac{2h(\beta_\sigma - \hat{\beta}_\sigma)^2}{(\beta - \hat{\beta})^3}. \end{aligned}$$

From Lemma 6.1.9, we get

$$
\begin{aligned}
E_t - E_{\sigma\sigma} &= \left\{ \frac{m-1}{r} r_\sigma + 2\,\frac{\beta_\sigma - \hat{\beta}_\sigma}{\beta - \hat{\beta}} \right\} E_\sigma \\
&\quad + \left\{ \frac{1}{\beta - \hat{\beta}} + \frac{\hat{\beta}_t - \hat{\beta}_{\sigma\sigma}}{h^2(\beta - \hat{\beta})^2} - \frac{\beta}{(\beta - \hat{\beta})^2} \right\} h^3 \\
&\quad +2 \left\{ -(m-1)\frac{\varrho}{r} + m\frac{r^{m-1}}{\beta - \hat{\beta}} \right\} \frac{h^2}{\beta - \hat{\beta}} \\
&\quad + \left\{ (m^2-1)\frac{\varrho^2}{r^2} - \frac{m-1}{r^2} - \frac{m-1}{r}\frac{\hat{\beta}_\sigma\, r_\sigma}{\beta - \hat{\beta}} \right\} \frac{h}{\beta - \hat{\beta}}.
\end{aligned} \tag{6.15}
$$

Differentiating the function $\hat{\beta}$ with respect to t and σ, respectively, we have

$$\hat{\beta}_t = -\alpha_1 \sin(\theta)\,\theta_t + \alpha_2 \cos(\theta)\,\theta_t,$$

and

$$\hat{\beta}_\sigma = -\alpha_1 \sin(\theta)\,\theta_\sigma + \alpha_2 \cos(\theta)\,\theta_\sigma.$$

Differentiating once more with respect to σ, we obtain

$$\hat{\beta}_{\sigma\sigma} = -\alpha_1 \cos(\theta)\,\theta_\sigma^2 - \alpha_1 \sin(\theta)\,\theta_{\sigma\sigma} - \alpha_2 \sin(\theta)\,\theta_\sigma^2 + \alpha_2 \cos(\theta)\,\theta_{\sigma\sigma}.$$

Therefore, the support function $\hat{\beta}$ evolves in time under the equation

$$
\begin{aligned}
\hat{\beta}_t - \hat{\beta}_{\sigma\sigma} &= (-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))(\theta_t - \theta_{\sigma\sigma}) + \theta_\sigma^2(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta)) \\
&= (-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))(\theta_t - \theta_{\sigma\sigma}) + (\hat{\beta} - R)\,\theta_\sigma^2.
\end{aligned} \tag{6.16}
$$

Step 4: *Selection of a good function θ.*

In order to eliminate h^2 appearing in the denominator of the term

$$\hat{\beta}_t - \hat{\beta}_{\sigma\sigma}$$

in (6.15), it is natural to choose a function θ satisfying

$$\theta_\sigma = h.$$

Due to Lemma 2.2.7, at least locally, such a function exists. We now make this more precise. Let us write in polar coordinates

$$z = re^{i\phi} \quad \text{and} \quad z_\sigma = e^{i\alpha},$$

where

$$\phi = \arg(\gamma)$$

and α is the tangential angle of the curve. From classical differential geometry (see for example [14]), we know that

$$\alpha_\sigma = k.$$

Differentiating z, we obtain

$$z_\sigma = r_\sigma e^{i\phi} + i\phi_\sigma r e^{i\phi} = e^{i\phi}(r_\sigma + ir\phi_\sigma).$$

Hence,

$$\phi_\sigma = \frac{\operatorname{Im}(e^{-i\phi}\, z_\sigma)}{r} = \frac{\operatorname{Im}(\bar{z} z_\sigma)}{r^2} = -\frac{\varrho}{r}.$$

Consequently, the function

$$\theta \doteq \alpha + (m-1)\,\phi$$

has the property

$$\theta_\sigma = k - (m-1)\,\frac{\varrho}{r} = h. \tag{6.17}$$

The function θ is referred to the bibliography as the ***Lagrangian angle*** of the equivariant Lagrangian submanifolds.

Step 5: *Evolution equation of θ.*

It remains to compute the evolution of θ. But this is easy since we can follow the same lines as above to get

$$\phi_t = \frac{\operatorname{Im}(e^{-i\phi}\, z_t)}{r} = h\frac{\operatorname{Im}(e^{-i\phi}\xi)}{r} = h\frac{\operatorname{Im}(\bar{z}\,\xi)}{r^2} = h\frac{r_\sigma}{r}$$

and

$$\alpha_t = \operatorname{Im}(e^{-i\alpha}(z_\sigma)_t) = \operatorname{Im}(\overline{z_\sigma}\, h_\sigma\, \xi) = h_\sigma.$$

Therefore, using the above identities and (6.17), we obtain

$$\theta_t = h_\sigma + (m-1)\frac{r_\sigma h}{r} = \theta_{\sigma\sigma} + (m-1)\frac{r_\sigma h}{r},$$

or, equivalently,

$$\theta_t - \theta_{\sigma\sigma} = (m-1)\,\frac{r_\sigma h}{r}.$$

Step 6: *The area of the enclosed curves tends to zero as t tends to T.*

Plugging in (6.16), we get

$$\hat{\beta}_t - \hat{\beta}_{\sigma\sigma} = (\hat{\beta} - R)\, h^2 + (m-1)\,(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))\,\frac{r_\sigma h}{r}. \tag{6.18}$$

In view of (6.18), we can arrange the terms in (6.15) and rewrite the equation in the form

$$\begin{aligned} E_t - E_{\sigma\sigma} \;=\; & \left\{\frac{m-1}{r} r_\sigma + 2\,\frac{\beta_\sigma - \hat{\beta}_\sigma}{\beta - \hat{\beta}}\right\} E_\sigma \\ & + \left\{\frac{1}{\beta - \hat{\beta}} + \frac{\hat{\beta} - R}{(\beta - \hat{\beta})^2} - \frac{\beta}{(\beta - \hat{\beta})^2}\right\} h^3 \\ & + 2\left\{-(m-1)\frac{\varrho}{r} + m\frac{r^{m-1}}{\beta - \hat{\beta}}\right\} \frac{h^2}{\beta - \hat{\beta}} \\ & + \left\{(m^2-1)\frac{\varrho^2}{r^2} - \frac{m-1}{r^2}\right\} \frac{h}{\beta - \hat{\beta}}. \end{aligned} \tag{6.19}$$

Hence, (6.19) can be written in the form

$$\begin{aligned} E_t - E_{\sigma\sigma} \;=\; & \{\text{bounded terms}\}\, E_\sigma - \frac{R\,h^3}{(\beta - \hat{\beta})^2} \\ & + \{\text{bounded terms}\}\, h^2 + \{\text{bounded terms}\}\, h. \end{aligned} \tag{6.20}$$

We claim that E is bounded above. Suppose, to the contrary that $\max E \to \infty$. Since

$$\beta - \hat{\beta} \geq \varepsilon > 0,$$

it follows that $h \to \infty$. However, along such a sequence, (6.20) implies that $\max E$ is non-increasing, a contradiction. Hence, there exists a constant c such that

$$\frac{h}{\beta - \hat{\beta}} \leq c,$$

from where it follows that

$$h \leq c(\beta - \hat{\beta}) \ll \infty.$$

Therefore, h is bounded. This contradicts the definition of the maximal time of existence. Consequently, the enclosed area must tend to zero.

Step 7: *Final step.*

Now there are two options. Either $\{z_t^m\}_{t\in[0,T)}$ converges to a point or to a line. We claim that the second possibility cannot occur. Indeed, by Lemma 3.4.4, the curvature of z_t^m is given by

$$k_{z^m} = \frac{h}{m|z|^{m-1}}.$$

By Steps 2 and 6, there exists a constant $c > 0$ such that $h \geq c$ for all sufficiently large times. Moreover, since the profile curves remain trapped away from the origin, there exists $r_0 > 0$ such that $|z| \geq r_0$. Therefore,

$$k_{z^m} \geq \frac{c}{mr_0^{m-1}} > 0.$$

Consequently, the curvature of z_t^m is uniformly bounded from below by a positive constant. On the other hand, if z_t^m converged to a line segment, then its curvature would converge uniformly to zero. This contradicts the above estimate. Thus, z_t^m cannot converge to a line segment. Therefore, z_t^m must converge to a point. Since z_t^m converges to a point, the enclosed area tends to zero. Hence the original equivariant Lagrangians shrink to a point, which completes the proof. □

6.5 Singularities of Whitney spheres

In this section, we describe the main ideas of the proof of the following theorem, due to Savas-Halilaj and Smoczyk [43].

Theorem 6.5.1. *Let $F_0 : \mathbb{S}^m \to \mathbb{R}^{2m}$, $m > 1$, be an equivariant Lagrangian immersion satisfying*

$$\mathrm{Ric}(g_0) \geq c\,\|F_0\|^2 g_0,$$

for some constant $c > 0$. Then along the flow one has

$$\mathrm{Ric}(g_t) \geq \varepsilon\|F_t\|^2 g_t,$$

for a uniform constant $\varepsilon > 0$. Moreover, the flow develops a Type II singularity whose blow-up limit is given by the product of a grim reaper curve and a flat $(m-1)$-dimensional Lagrangian subspace in $\mathbb{R}^{2m-2}$.

Proof: Let us briefly discuss the main steps of the proof following [43]. That the singularity is of Type II follows by a deep result due to Neves [36].

Step 1: *Preservation of the pinching condition.*

According to Lemma 3.2.5 it sufficies to prove that the conditions

$$\max_{t=0} r^{-1}p < -\varepsilon_1 < 0 \quad \text{and} \quad \min_{t=0}(k+p)r^{-1} \geq \varepsilon_1 > 0,$$

are preserved under the evolution. To achieve this goal, for any $\alpha \in \mathbb{R}$, we set:

$$p_\alpha \doteq r^{-1-\alpha}p, \quad h_\alpha \doteq r^{-1-\alpha}h, \quad q_\alpha \doteq h_\alpha + mp_\alpha = (k+p)r^{-1-\alpha}.$$

Using Lemmas 6.1.6, 6.1.8, and 6.1.9, (6.7), (6.9) and (6.10), it follows that the functions p_α, h_α, and q_α satisfy the following evolution equations:

$$\begin{aligned}(p_\alpha)_t \;=\;& (p_\alpha)_{\sigma\sigma} + (2\alpha+m+1)(p_\alpha)_\sigma(\log r)_\sigma - 2(q_\alpha+2p_\alpha)(\log r)^2_\sigma \\ & +p_\alpha\big\{\alpha(\alpha+m)(\log r)^2_\sigma + m(\alpha+2)p^2 + (k+p)^2\big\},\end{aligned} \tag{6.21}$$

$$\begin{aligned}(h_\alpha)_t \;=\;& (h_\alpha)_{\sigma\sigma} + (2\alpha+m+1)(h_\alpha)_\sigma(\log r)_\sigma - 2phq_\alpha \\ & +h_\alpha\big\{\alpha(\alpha+m)(\log r)^2_\sigma + m(\alpha+2)p^2 + (k+p)^2\big\},\end{aligned} \tag{6.22}$$

$$\begin{aligned}(q_\alpha)_t \;=\;& (q_\alpha)_{\sigma\sigma} + (2\alpha+m+1)(q_\alpha)_\sigma(\log r)_\sigma \\ & +\big\{(\alpha(\alpha+m)-2m)q_\alpha - 4mp_\alpha\big\}(\log r)^2_\sigma \\ & +q_\alpha\big\{m(\alpha+2)p^2 + (k+p)^2 - 2ph\big\}.\end{aligned} \tag{6.23}$$

Lemma 6.5.2. *Assume that* $m \geq 2$.

1. *Suppose* $\sup_{t=0} r^{-1}p < 0$. *Then* $\sup_{t=t_0} p_\alpha$ *is attained for any* $\alpha > 0$ *and any* $t_0 \in [0,T)$. *Moreover,*

$$\max_{t=t_0} p_\alpha \leq \max_{t=0} p_\alpha.$$

2. *Suppose* $\inf_{t=0} r^{-1}h > 0$. *Then* $\inf_{t=t_0} h_\alpha$ *is attained for any* $\alpha > 0$ *and any* $t_0 \in [0,T)$. *Furthermore,*

$$\min_{t=t_0} h_\alpha \geq \min_{t=0} h_\alpha.$$

3. *Suppose* $\sup_{t=0} r^{-1}p < 0$ *and* $\inf_{t=0}(k+p)r^{-1} > 0$. *Then* $\sup_{t=t_0} p_\alpha$ *and* $\inf_{t=t_0} q_\alpha$ *are attained for any* $\alpha > 0$ *and any time* $t_0 \in [0,T)$. *Additionally,*

$$\min_{t=t_0} q_\alpha \geq \min\big\{\min_{t=0} q_\alpha, -2\max_{t=0} p_\alpha\big\}.$$

Proof. Since $m > 1$, from the conclusions of Chapter 3, it follows that for all $t \in [0, T)$ the profile curves are given by point-symmetric maps $z : [-\pi, \pi] \to \mathbb{C}$ such that

$$z^{-1}(0) = \{-\pi, 0, \pi\}.$$

In particular, locally around each of the zeros of z we can introduce a signed distance function r and consequently from the real analyticity of z we get a Taylor expansion of p, k, h in terms of r at these points.

1. By assumption

$$\sup_{t=0} r^{-1}p < 0.$$

Therefore, the first coefficient c_1 in the Taylor expansion

$$p(r) = \sum_{n=0}^{\infty} c_{2n+1} r^{2n+1}$$

of p is negative. Since the coefficients smoothly depend on t and $\alpha > 0$, the function p_α will tend to $-\infty$ at points on the curve with $r = 0$ on some maximal time interval $[0, \lambda)$ where $0 < \lambda \leq T$. Therefore, for any $t_0 \in [0, \lambda)$ the supremum $\sup_{t=t_0} p_\alpha$ will be attained at some point s_0 with $r(s_0) > 0$. Note that at such a point we get

$$0 = (p_\alpha)_\sigma(s_0) = -\big\{q_\alpha(s_0) + (2+\alpha)p_\alpha(s_0)\big\}(\log r)_\sigma(s_0).$$

Consequently at s_0 we have

$$0 = \big\{q_\alpha + (2+\alpha)p_\alpha\big\}(\log r)_\sigma^2 \quad \text{and} \quad (p_\alpha)_{\sigma\sigma}(s_0) \leq 0.$$

Therefore, from (6.21) that at s_0, it holds

$$(p_\alpha)_t \leq p_\alpha\big\{\alpha(\alpha+m+2)(\log r)_\sigma^2 + m(\alpha+2)p^2 + (k+p)^2\big\}.$$

Thus, the function p_α cannot admit a negative maximum that is increasing in time. Consequently,

$$\sup_{[0,\lambda)} p_\alpha \leq \max_{t=0} p_\alpha < 0.$$

Since this holds for any $\alpha > 0$, we may let α tend to zero and obtain that the first coefficient c_1 in the Taylor expansion of p is non-increasing in time and hence strictly negative for all t. In particular λ must coincide with T and our estimate holds for all $t_0 < T$.

2. Similarly as above, from the assumption $\alpha > 0$ and $\inf_{t=0} r^{-1}h > 0$, we conclude that the function h_α will tend to $+\infty$ at points on the curve with $r = 0$ on some maximal time interval $[0, \lambda)$ where $0 < \lambda \le T$. In addition for any $t_0 \in [0, \lambda)$ the infimum $\inf_{t=t_0} h_\alpha$ will be attained at some point s_0 with $r(s_0) > 0$. Observe that $2phq_\alpha = 2p(k+p)h_\alpha$ and
$$m(\alpha+2)p^2 + (k+p)^2 - 2p(k+p) = k^2 + \big(m(\alpha+2)-1\big)p^2.$$
Then from (6.22) we deduce that at a minimum s_0 of h_α we have
$$(h_\alpha)_t \ge h_\alpha\big\{\alpha(\alpha+m)(\log r)^2_\sigma + k^2 + \big(m(\alpha+2)-1\big)p^2\big\}.$$
Therefore h_α cannot attain a decreasing positive minimum at points where $r > 0$. The rest of the proof is the same as in the first part.

3. First note that
$$m(\alpha+2)p^2 + (k+p)^2 - 2ph = \big(m(\alpha+4)-1\big)p^2 + k^2 \ge 0.$$
Taking into account the conclusions of the first part, we get that
$$(\alpha(\alpha+m) - 2m)\min_{t=t_0} q_\alpha - 4mp_\alpha \ge -2m(\min_{t=t_0} q_\alpha + 2\max_{t=0} p_\alpha).$$
So, if $\min_{t=t_0} q_\alpha$ is positive but smaller than $-2\max_{t=0} p_\alpha$, then the minimum cannot be decreasing. Moreover, when $p < 0$, we have
$$k + p > k + 3p.$$
Consequently, we obtain the desired lower bound for the minimum of q_α.

This completes the proof. □

For $\alpha = 0$, Lemma 3.2.5 and Lemma 6.5.2 imply the following estimate on the Ricci curvature.

Lemma 6.5.3. *Let $F_0\colon \mathbb{S}^m \to \mathbb{C}^m$, $m > 1$, be an equivariant Lagrangian immersion such that the Ricci curvature satisfies* Ric $\ge cr^2 g$, *where c is a positive constant. Then for all $t_0 \in [0, T)$ we have*
$$\sup_{t=t_0} r^{-1}p \le \sup_{t=0} r^{-1}p < 0 \quad \textit{and} \quad \inf_{t=t_0} r^{-1}(k+p) \ge \inf_{t=0} r^{-1}(k+p) > 0.$$
In particular the Ricci curvature of the Lagrangian submanifolds evolving under the mean curvature flow satisfies Ric $\ge \varepsilon r^2 g$, *with a positive constant ε not depending on t.*

Step 2: *Behaviour of the angle functions.*

Recall that the profile curve $z\colon [-\pi, \pi] \to \mathbb{C}$ of a Lagrangian sphere is point symmetric with respect to the origin. So it can be divided into the union of the two arcs

$$\gamma_\ell \doteq z([0, \pi]) \quad \text{and} \quad -\gamma_\ell \doteq z([-\pi, 0]).$$

It is clear that it is sufficient to study one of these arcs under the equivariant curve shortening flow.

Let α_0 denote the angle between the (outer) unit normal ν of γ_ℓ at $s = 0$ and the unit vector $e_1 = (1, 0)$. Without loss of generality we may assume that $\alpha_0 \in (0, \pi/2)$. Additionally, after rotating the curve we can assume without loss of generality that $z(s) = (x(s), y(s))$ satisfies $x(s) < 0$ and $y(s) > 0$ for sufficiently small $s > 0$. Throughout this text we will always make this assumption. The arc γ_ℓ then starts and ends at the origin and is nonzero elsewhere. It does not have to be embedded and it might wind around the origin, e.g. the arc in Figure 6.3 generates a profile curve of an immersed Lagrangian sphere.

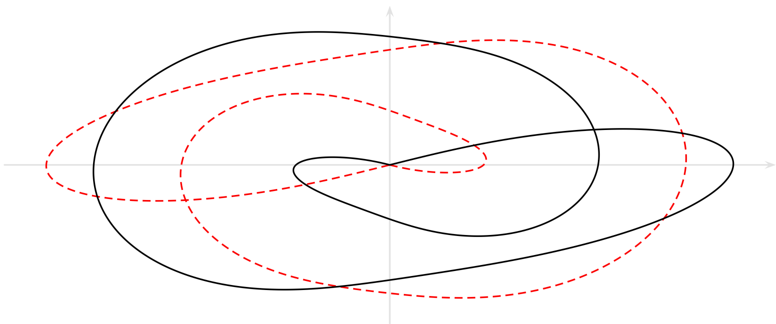


Figure 6.3: The arcs $\pm\gamma_\ell$. Figure taken from [43].

For any $s \in [0, \pi]$, define the ***Umlaufwinkel*** by

$$\alpha(s) \doteq \alpha_0 + \int_0^s k \, d\mu,$$

see Figure 6.4. Then for all $s \in [0, \pi]$ the function α measures the angle between the (outer) unit normal ν and e_1 modulo some integer multiple of 2π.

Similarly we define the ***polar angle*** by

$$\phi(s) \doteq \phi_0 - \int_0^s p\, d\mu, \tag{6.24}$$

where

$$\phi_0 \doteq \alpha_0 + \pi/2,$$

see Figure 6.4. For any $s \in [0, \pi]$, the number $\phi(s)$ is the angle between e_1 and the normalized position vector $z(s)/r(s)$, again up to some integer multiple of 2π. For $s \to 0$ the normalized position vector converges to the unit tangent vector τ at $s = 0$.

In addition we define

$$\beta \doteq \phi - \alpha.$$

Then up to multiples of 2π the angle β is the angle between the unit normal ν and the normalized position vector z/r, i.e., $\cos\beta = rp = \varrho$, see Figure 6.4. In particular

$$\beta_0 \doteq \beta(0) = \pi/2.$$

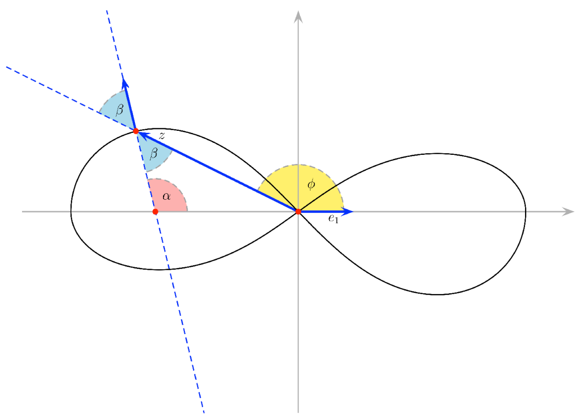


Figure 6.4: The angles α, β and ϕ. Figure taken from [43].

The condition $k > 0$ implies that the Umlaufwinkel α is a strictly increasing function in s. The polar angle ϕ is strictly increasing, if $p < 0$. In view of

$$\beta = \phi - \alpha, \quad d\alpha = k\, d\mu \quad \text{and} \quad d\phi = -p d\mu$$

we conclude that the condition $k + p > 0$ implies that β is strictly decreasing in s. From this, Lemma 3.2.5 and Lemma 6.5.2 we conclude the following result.

Lemma 6.5.4. *Let $F_0 : \mathbb{S}^m \to \mathbb{C}^m$, $m > 1$, be an equivariant Lagrangian immersion such that the Ricci curvature satisfies* $\mathrm{Ric} \geq cr^2 g$, *where c is a positive constant. Then, under the (ECSF), the angles α and ϕ remain strictly increasing functions on $[0, \pi]$ for all $t \in [0, T)$, whereas the angle β stays strictly decreasing on $[0, \pi]$.*

The following lemma plays a crucial role in the next step.

Lemma 6.5.5. *Let $z : [0, \infty) \to \mathbb{C}$ be a real analytic curve, parametrized by arc length σ such that*

$$z(0) = 0, \quad k \geq -p \geq 0 \quad \text{and} \quad \int_0^\infty p(\sigma)d\sigma > -\infty.$$

Then $p \equiv 0$ and z is a ray.

Proof. As in (6.24) let

$$\phi(\sigma) = \phi_0 - \int_0^\sigma p(\sigma)d\sigma$$

be the polar angle and define

$$\phi_\infty \doteq \phi_0 - \int_0^\infty p(\sigma)d\sigma.$$

Since $p \leq 0$, the function ϕ is an increasing function.

We will show $\phi \equiv \phi_0$, hence $p \equiv 0$. We distinguish two cases.

Case 1: Suppose there exists $\sigma_0 \in [0, \infty)$ such that $\phi(\sigma_0) = \phi_\infty$. Then from the monotonicity of ϕ we get $\phi(\sigma) = \phi_\infty$ for all $\sigma \geq \sigma_0$ and then the real analyticity implies $\phi(\sigma) = \phi_\infty$ for all $\sigma \in [0, \infty)$. But then $\phi_0 = \phi_\infty$ and $p \equiv 0$.

Case 2: In this case we have $\phi(\sigma) < \phi_\infty$ for every $\sigma \in [0, \infty)$. Since ϕ is monotone and bounded, there exists $\sigma_0 \in [0, \infty)$ such that the curve $z|_{[\sigma_0, \infty)}$ can be represented as a graph over the flat line passing through the origin in direction of $v = (\cos \phi_\infty, \sin \phi_\infty)$. After a rotation around the origin we may assume without loss of generality that $\phi_\infty = \pi$ such that $z|_{[\sigma_0, \infty)}$ can be represented as the graph of a real analytic function $y : (-\infty, s_0] \to \mathbb{R}$, $s_0 < 0$, where y is a strictly decreasing concave function, because $k \geq -p \geq 0$; see Figure 6.5.

On the interval $(-\infty, s_0]$ the function $\cos\beta = -rp$, is given by

$$\cos\beta = \frac{sy' - y}{\sqrt{(1 + (y')^2)(s^2 + y^2)}}.$$

Since $y'' \leq 0$, $s < 0$ and $(sy' - y)' = sy'' \geq 0$, by integrating we conclude that for $s < s_0$ we must have

$$y'(s) \geq \frac{y(s)}{s} + \frac{s_0 y'(s_0) - y(s_0)}{s}. \tag{6.25}$$

Since $y' \leq 0$ and $y'' \leq 0$ we observe that

$$a \doteq \lim_{s \to -\infty} y'(s)$$

exists and is non-positive. From inequality (6.25) we conclude that

$$a \geq \lim_{s \to -\infty} \frac{y(s)}{s}.$$

Now, for $\lambda < s_0$, let

$$\epsilon(s) = y'(\lambda)s + b(\lambda)$$

be the tangent line to the graph of the function y passing through the point $z(\lambda) = (\lambda, y(\lambda))$; see Figure 6.5.

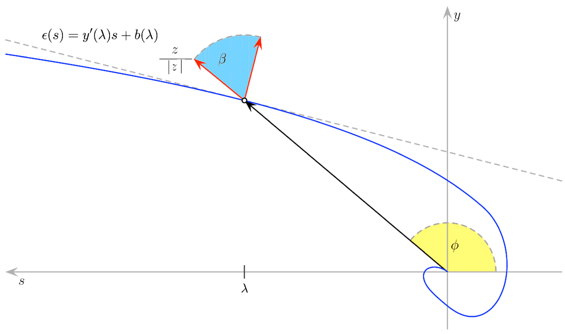


Figure 6.5: The angle β. Figure taken from [43].

Since y is concave we derive $y(s) \leq \epsilon(s)$, for all $s \leq s_0$. Therefore we obtain the estimate

$$\frac{y(s)}{s} \geq y'(\lambda) + \frac{b(\lambda)}{s}, \qquad \text{for all } s \leq \lambda. \tag{6.26}$$

Passing to the limit we obtain that

$$\lim_{s\to-\infty}\frac{y(s)}{s}\geq y'(\lambda).$$

Since the above inequality holds for any $\lambda<s_0$ we may let λ tend to $-\infty$ which implies

$$\lim_{s\to-\infty}\frac{y(s)}{s}\geq a.$$

Thus we have shown

$$\lim_{s\to-\infty}\left(y'(s)-y(s)/s\right)=0$$

and this yields

$$\lim_{s\to-\infty}\cos\beta(s)=0.$$

We will now see that this implies that $\cos\beta\equiv 0$ on the whole curve z. First note that the graphical property implies that for sufficiently large σ the distance function r is a strictly increasing function in terms of the arc length parameter σ, so that $r'=r_\sigma$. Since

$$(\cos\beta)'=-(rp)'=-\varrho'=(k+p)r'\geq 0$$

we conclude that the function $\cos\beta$ is increasing for sufficiently large σ. Because $\cos\beta=-rp\geq 0$ and $\lim_{\sigma\to\infty}\cos\beta(\sigma)=0$, this implies $\cos\beta(\sigma)=0$ for all sufficiently large σ and then the real analyticity also implies the same for all $\sigma\in[0,\infty)$. Hence $p\equiv 0$ and $\phi\equiv\phi_0$ as claimed. □

Step 3: *Rescaling the singularity.*

Set $a_j\doteq\max_{\mathbb{S}^m\times[0,t_j]}\|A\|$, where $\{t_j\}_{j\in\mathbb{N}}$ is an increasing sequence such that $t_j\to T$. Following Theorem 5.2.1 and Remark 5.2.1, take a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $\mathbb{S}^m$ such that

$$\|A\|(x_j,t_j)=a_j.$$

Let

$$F_j(x,\lambda)=a_j\big(F(x,\lambda/a_j^2+t_j)-F(x_j,t_j)\big)$$

be the rescaled Lagrangian spheres and denote the distance between the rescaled blow-up point (the origin) and the rescaled intersection point $-a_jF(x_j,t_j)$ by

$$r_j\doteq a_jr(x_j,t_j).$$

Claim: *It holds* $\limsup_{j\to\infty} r_j = \infty$.

Proof of claim: Suppose to the contrary that

$$\limsup_{j\to\infty} r_j = r_\infty < \infty.$$

Then, from Theorem 5.2.1, we obtain an eternal (ECSF) F_∞ generated by limit profile curves $z_\infty \colon M_\infty \to \mathbb{C}$, where here M_∞ is a 1-dimensional complete pointed Riemannian manifold. Because the map F_∞ is an eternal solution and eternal solutions in $\mathbb{R}^{2m}$ are known to be non-compact, M_∞ is diffeomorphic to $\mathbb{R}$. The limit profile curves z_∞ evolve again by an equivariant curve shortening flow with a different center, which without loss of generality can be assumed to be the origin. Because k and p have the same scaling behavior, the limit flow still satisfies

$$rk \geq -rp \geq 0.$$

For any fixed time $\lambda \in \mathbb{R}$ there exists a local parametrization of $z_\infty(M_\infty \times \{\lambda\})$ by a real analytic arc $z : [0, \infty) \to \mathbb{C}$, parametrized by arc length σ such that

$$z(0) = 0, \quad k \geq -p \geq 0 \quad \text{and} \quad k \not\equiv 0,$$

because

$$\|A\|^2 = k^2 + 3(m-1)p^2.$$

Moreover, since by Lemma 6.5.4 the opening angle is decreasing, we conclude that

$$\int_0^\infty p(\sigma) d\sigma > -\infty.$$

Applying Lemma 6.5.5 and using the point-symmetry we get that $z_\infty(M_\infty \times \{\lambda\})$ must be straight line, which is a contradiction. This proves the claim. $\circledast$

Instead of taking a blow-up sequence for F we may, equivalently, consider a blow-up sequence for the corresponding profile curves z. Therefore consider a sequence of points $\{s_j\}_{j\in\mathbb{N}}$ in $\mathbb{S}^1$ such that

$$\big(k^2 + (m-1)p^2\big)(s_j, t_j) = a_j^2.$$

Following the statements of Theorem 5.2.1, let us define the family of curves

$$z_j : \mathbb{S}^1 \times [L_j, R_j) \to \mathbb{C}, \quad j \in \mathbb{N},$$

given by
$$z_j(s,\lambda) \doteq z_{j,\lambda}(s) \doteq a_j\big(z(s,\lambda/a_j^2+t_j)-\gamma_j\big),$$
where
$$L_j=-a_j^2t_j, \quad R_j=a_j^2(T-t_j) \quad \text{and} \quad \gamma_j=z(s_j,t_j).$$
One can easily verify that the curves $\{z_j\}_{j\in\mathbb{N}}$ evolve by the equation
$$\frac{dz_j}{d\lambda}=\left(k_j-(m-1)\frac{\langle z_j+a_j\gamma_j,\xi_j\rangle}{|z_j+a_j\gamma_j|^2}\right)\xi_j,$$
where
$$k_j(s,\lambda)=a_j^{-1}k(s,\lambda/a_j^2+t_j)$$
is the signed curvature and
$$\xi_j(s,\lambda)=\xi(s,\lambda/a_j^2+t_j)$$
is the inner unit normal of the rescaled curve $z_{j,\lambda}$ at s. Note that with the notation from above we have
$$r_j=a_j|\gamma_j|=a_jr(s_j,t_j).$$
From Cauchy-Schwarz' inequality we obtain
$$\left|\frac{\langle z_j+a_j\gamma_j,\xi_j\rangle}{|z_j+a_j\gamma_j|^2}\right|\leq\frac{1}{|z_j+a_j\gamma_j|}.$$
For large r_j we can thus estimate
$$\left|\frac{\langle z_j+a_j\gamma_j,\xi_j\rangle}{|z_j+a_j\gamma_j|^2}\right|\leq\frac{1}{r_j-|z_j|}.$$
So for each $s\in\mathbb{S}^1$ where $z_j(s,\lambda)$ converges as $j\to\infty$ we conclude that
$$\lim_{j\to\infty}\frac{\langle z_j+a_j\gamma_j,\xi_j\rangle}{|z_j+a_j\gamma_j|^2}=0.$$
Since the intersection point after rescaling tends to infinity, this and the point-symmetry show that the limiting flow z_∞ in this case is an eternal and weakly convex non-flat solution of the curve shortening flow. It is well known by results of Altschuler [1] and Hamilton [19] that such a solution is a grim reaper. It is clear that the parabolic rescaling of the Lagrangian spheres then converge in the Cheeger-Gromov sense to the product of a grim-reaper with a flat Lagrangian space.

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