



**UNIVERSITY OF IOANNINA**  
Department of Mathematics



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TILTING THEORY, QUASI-ABELIAN CATEGORIES, AND STABILITY  
CONDITIONS

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MASTER THESIS

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*Dedicated to my family and friends.*



The present dissertation was carried out under the M.Sc program of the Department of Mathematics of the University of Ioannina in order to obtain the degree of Master of Science.

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Evangelos Gkotsis



# ABSTRACT

Tom Bridgeland in 2007, see the paper [2], motivated by papers of Michael Douglas in Theoretical Physics, see for instance the paper [4], regarding Dirichlet branes or D-branes in combination with the Homological Mirror Symmetry conjecture of Kontsevich in String Theory, introduced the notion of the *space of stability conditions*  $\text{Stab}(\mathcal{T})$  of a triangulated category  $\mathcal{T}$ , and proved that the space  $\text{Stab}(\mathcal{T})$  has the structure of a complex manifold.

Since then, the space of stability conditions  $\text{Stab}(\mathcal{T})$  of a triangulated category  $\mathcal{T}$ , plays an important role in Algebraic Geometry, where for instance  $\mathcal{T}$  is the derived category  $D^b(\text{coh}X)$  of coherent sheaves of a smooth projective variety  $X$ , in Representation Theory, where  $\mathcal{T}$  is the derived category  $D^b(\mathcal{A})$  of an abelian category  $\mathcal{A}$  of modules over an algebra, etc. The main reason is that the space of stability conditions provide us with finer moduli spaces in the study of a variety or an algebra.

The main purpose of this Master Thesis is to present, taking the paper [2] as a primary guide, an introduction in the theory of stability conditions of a triangulated category, mentioning some of its applications. In this direction, the theory of  $t$ -structures, see [1], tilting theory, see [7], and the theory of quasi-abelian categories, see [17], play an important role in analyzing the structure of stability condition spaces, the fundamental aspects of which will be examined in this Thesis.



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# INTRODUCTION

In modern mathematics, category theory has played an important role in unifying and advancing many areas. The main theory was developed by Eilenberg and MacLane in their famous series of 15 papers published between 1941 and 1954. From its inception, the impact of category theory in Algebra and Geometry has been profound, providing a common framework for concepts that had previously been regarded as unrelated.

Category theory provides the natural framework for a relatively new discipline in mathematics: homological algebra. Originally developed to uncover deep connections between algebra and topology, homological algebra found important applications, with the work of Grothendieck in derived categories and its student Verdier in triangulated categories, in many branches of mathematics. The derived category of an abelian category has a profound role in algebra, particularly in the study of modules over a ring, and in algebraic geometry, in the study of smooth projective varieties. The abelian category in the first case is the category of (left) modules over a ring and in the second case is the category of (quasi) coherent sheaves respectively. However, abelian categories fail to capture the structure of the derived category of an abelian category. This limitation led Verdier to introduce triangulated categories, which provide the appropriate framework for describing an important part of the internal structure of derived categories.

Derived categories and triangulated categories in general, as we have stated, have an influential role in algebraic geometry. Through the study of the bounded derived category of coherent sheaves of a smooth projective variety, one can uncover subtle geometric information about the variety. More recently, in the context of string theory and mirror symmetry, triangulated categories have taken on an even deeper role: they provide the natural home for Bridgeland's stability conditions. Motivated by ideas of Douglas on  $\Pi$ -stability in string theory, see [4], Bridgeland introduced in [2] the notion of a stability condition in a triangulated category. In reality, a stability condition consists of a pair of data: a group homomorphism and a slicing. slicings were also introduced by Bridgeland and are generalizations of  $t$ -structures. In particular

**Definition .** Let  $\mathcal{T}$  be a triangulated category. A **slicing**  $\mathcal{P}$  of  $\mathcal{T}$  is a collection of full additive subcategories  $\mathcal{P}(\phi)$  of  $\mathcal{T}$ , for every  $\phi \in \mathbb{R}$ , such that:

(s1)  $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ , for every  $\phi \in \mathbb{R}$ ,

(s2)  $\text{Hom}_{\mathcal{T}}(A_1, A_2) = 0$ , for ever  $\phi_1 > \phi_2$  and  $A_j \in \mathcal{P}(\phi_j)$ ,

(s3) for every non zero object  $E$  of  $\mathcal{T}$ , there exists a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of distinguished triangles

$$\begin{array}{ccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\ & & \swarrow [1] & & \swarrow [1] & & & & \swarrow [1] & & \swarrow [1] \\ & & A_1 & & A_2 & & & & A_n & & \end{array}$$

where  $A_j \in \mathcal{P}(\phi_j)$ , for all  $j$ .

In general,  $t$ -structures provide a way to break up objects of a triangulated category into pieces (cohomology objects) indexed by the integers. Slicings allow us to break up objects of the category into finer pieces indexed by the integers. Now, the definition of a stability condition is the following:

**Definition .** Let  $\mathcal{T}$  be a triangulated category. A **stability condition** on  $\mathcal{T}$  is a pair  $\sigma = (Z, \mathcal{P})$ , where

$$Z : K_0(\mathcal{T}) \rightarrow \mathbb{C}$$

is a group homomorphism and  $\mathcal{P}$  is a slicing of  $\mathcal{T}$ , such that for every nonzero object  $E$  of  $\mathcal{T}$  then

$$Z(E) = m(E)e^{i\pi\phi}$$

for some  $m(E) \in \mathbb{R}_+$ .

The group homomorphism  $Z$  is called the **central charge** of the stability condition.

However, in order to avoid degenerate examples and to build a richer theory, Bridgeland imposes an extra conditon on stability conditions, that of local- finiteness

**Definition .** A slicing  $\mathcal{P}$  of a triangulated category  $\mathcal{T}$  is **locally-finite** if there exists a real number  $\eta > 0$  such that for all  $t \in \mathbb{R}$  the quasi-abelian category  $\mathcal{P}((t - \eta, t + \eta)) \subseteq \mathcal{T}$  is of finite length. A stability condition  $(Z, \mathcal{P})$  is locally-finite if the slicing  $\mathcal{P}$  is.

By considering the space of all locally-finite stability conditions of a triangulated category  $\mathcal{T}$ , which he denotes by  $\text{Stab}(\mathcal{T})$ , he proves that there exists a natural topology on this set, and proves the following theorem

**Theorem .** *Let  $\mathcal{T}$  be a triangulated category. For each connected component  $\Sigma \subset \text{Stab}(\mathcal{T})$  there are a linear subspace  $V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C})$ , with a well-defined linear topology, and a local homeomorphism  $\mathcal{Z}: \Sigma \rightarrow V(\Sigma)$  which maps a stability condition  $(Z, \mathcal{P})$  to its central charge  $Z$ .*

From this it follows that each component  $\Sigma \subset \text{Stab}(\mathcal{T})$  is a manifold, locally modelled on the topological vector space  $V(\Sigma)$ . In order to give a complex structure to the stability manifold, Bridgeland assumes that  $\mathcal{T}$  is linear over a field  $\mathbb{K}$ , that is, the morphisms of  $\mathcal{T}$  have the structure of a vector space over  $\mathbb{K}$ , with respect to which the composition is bilinear. Furthermore, the category  $\mathcal{T}$  is assumed to be of finite type. This means that for every pair of objects  $E, F \in \mathcal{T}$  the vector space  $\bigoplus_i \text{Hom}_{\mathcal{T}}(E, F[i])$  is finite-dimensional. Hence we can define a free abelian group  $\mathcal{N}(\mathcal{T}) = K_0(\mathcal{T})/K_0^+(\mathcal{T})$ , called the numerical Grothendieck group of  $\mathcal{T}$ , and  $\mathcal{T}$  is called numerically finite if  $\mathcal{N}(\mathcal{T})$  has finite rank. If  $\mathcal{T}$  is of finite type over a field, a stability condition is called numerical if the central charge  $Z: K_0\mathcal{T} \rightarrow \mathbb{C}$  factors through  $\mathcal{N}(\mathcal{T})$ . By denoting  $\text{Stab}_{\mathcal{N}}(\mathcal{T})$  the subspace of  $\text{Stab}(\mathcal{T})$  which consists of the numerical finite stability conditions we have the following Corollary

**Theorem .** *Suppose that  $\mathcal{T}$  is numerically finite. For each connected component  $\Sigma \subset \text{Stab}_{\mathcal{N}}(\mathcal{T})$  there are a subspace  $V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C})$  and a local homeomorphism  $\mathcal{Z}: \Sigma \rightarrow V(\Sigma)$  which maps a stability condition to its central charge  $Z$ . In particular  $\Sigma$  is a finite-dimensional complex manifold.*

The introduction of stability conditions on triangulated categories has had a substantial impact on several areas of mathematics. By combining homological methods with geometric structures on the space of stability conditions, Bridgeland's theory offers a powerful framework for studying derived categories and their moduli, with applications ranging from algebraic geometry to representation theory and mathematical physics.

The aim of this thesis is to provide a systematic introduction to Bridgeland's theory of stability conditions on triangulated categories. In particular, it focuses on developing the categorical tools necessary to understand the construction and properties of stability conditions, such as  $t$ -structures, tilting theory, and quasi-abelian categories. Using these tools, the thesis presents the main results of Bridgeland's work and illustrates their significance in algebraic geometry and representation theory. The thesis is organized into four chapters. The first chapter is an introduction to the basic elements of the theory of abelian and triangulated categories, and the theory of localization in these contexts. The second chapter develops the necessary tools required for understanding Bridgeland's work and the third chapter is devoted to analyzing Bridgeland's original paper on stability conditions on triangulated categories. The fourth and final chapter discusses applications of Bridgeland's theory and outlines possible directions for further study for readers who have gained a solid understanding of the material.

Chapter 1: In the first chapter we give a brief overview of categories and functors. We also give the definition of abelian categories and present some basic results. Then we move on to the localization of categories. We present the general theory of localization of an arbitrary category and then turn our focus the localization theory of abelian categories. Next we give an introduction to the theory of triangulated categories and their localizations. We conclude by presenting the construction of the derived category of an abelian category.

Chapter 2: In the second chapter, we provide the basic tools which play an important role in Bridgeland's paper. We begin with the theory of  $t$ -structures, which allow us to recover abelian categories inside triangulated categories. Next we move on to the Happel-Reiten-Smalø (HRS) tilting theory, which produces new  $t$ -structures and abelian hearts. Finally, we discuss quasi-abelian categories, a generalization of abelian categories which arise naturally in Bridgeland's stability conditions.

Chapter 3: In the third chapter we focus on Bridgeland's breakthrough paper "Stability conditions on triangulated categories". We introduce the necessary notions that come up in his paper and we give the proof of the main result, Bridgeland's deformation theorem, which endows the space of stability conditions with the structure of a complex manifold.

Chapter 4: In the final chapter of the thesis we discuss applications of Bridgeland's theory and indicate briefly future directions of research in various fields, and ways of applications of stability conditions in Algebraic Geometry and Representation Theory.

# ABELIAN CATEGORIES, TRIANGULATED CATEGORIES, AND LOCALIZATION

In this chapter, we begin by giving the definition of a category. Then we proceed by defining functors between categories and introduce the notion of a natural transformation between functors. Then we shift our focus to abelian categories which is the right context in which we can do homological algebra, and we present some fundamental results about them. We then introduce the notion of a localization of an arbitrary category and then we concentrate on the localization of abelian and triangulated categories. Next we focus on triangulated categories which are of great importance in modern mathematics and play an important role in several contexts. We discuss some of their essential properties and then we present the construction of one of the most basic examples of a triangulated category, that is, the derived category of an abelian category. For this chapter we closely follow [9], [10], [12], [13], [14], [18] and [20], and we refer to them for more details.

## 1.1 Categories and Functors

Often in mathematics we want to study sets that have structure, so that we can understand connections between them and build a rich theory. Categories contain objects that already have a structure and provide us with a way to relate those objects without losing their structure. Categories and functors were introduced by Eilenberg and MacLane in their paper [5]. In this section we rely on [9], [10], [13] and [18].

We begin this section by introducing some fundamental definitions in Category Theory.

**Definition 1.1.1.** A *category*  $\mathcal{C}$  consists of the following data:

- (1) A class of objects, denoted by  $\text{Ob}(\mathcal{C})$ ,
- (2) A class of morphisms, denoted by  $\text{Hom}_{\mathcal{C}}$ , which satisfy the following properties:

- (i) For every two objects  $X, Y \in \text{Ob}(\mathcal{C})$ , there exists a set, denoted by  $\text{Hom}_{\mathcal{C}}(X, Y)$ , which is called the **set of morphisms from object X to object Y**. A morphism  $f$  from object  $X$  to object  $Y$  is represented by  $f : X \rightarrow Y$ .
- (ii) For every  $X, Y, Z \in \text{Ob}(\mathcal{C})$ , there exists a map, called the **composition** of morphisms

$$\circ : \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z), (f, g) \mapsto g \circ f$$

such that:

- (a) If  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  and  $h : Z \rightarrow W$  are morphisms, then the composition is associative, i.e.

$$(h \circ g) \circ f = h \circ (g \circ f)$$

- (b) For every object  $X \in \mathcal{C}$ , there exists a morphism  $1_X \in \text{Hom}_{\mathcal{C}}(X, X)$ , such that for arbitrary objects  $Y, Z \in \text{Ob}(\mathcal{C})$  and morphisms  $f : Y \rightarrow X$ ,  $g : X \rightarrow Z$  we have that:

$$f \circ 1_X = f \quad \text{and} \quad 1_X \circ g = g$$

**Remark 1.1.2.** We will write  $X \in \mathcal{C}$  instead of  $X \in \text{Ob}(\mathcal{C})$ , to denote that  $X$  is an object of the category  $\mathcal{C}$ .

- Example 1.1.3.** 1. The category of sets **Set**. The class of objects is the class of all sets, and the class of morphisms is the class of all maps between sets.
2. The category of abelian groups **Ab**. The class of objects is the class of all abelian groups, and the class of morphisms is the class of all homomorphisms between abelian groups.
3. The category of left R-modules **R-Mod**. The class of objects is the class of all left R-modules, and the class of morphisms is the class of left R-module homomorphisms.

As with many algebraic structures, it is natural to give a definition of a subcategory of a given category.

**Definition 1.1.4.** A category  $\mathcal{D}$  is called a **subcategory** of a category  $\mathcal{C}$  if:

- (1) Every object of  $\mathcal{C}$  is an object of  $\mathcal{D}$ , i.e.  $\text{Ob}(\mathcal{D}) \subseteq \text{Ob}(\mathcal{C})$
- (2) For every two objects  $X, Y \in \mathcal{D}$  we have that:

$$\text{Hom}_{\mathcal{D}}(X, Y) \subseteq \text{Hom}_{\mathcal{C}}(X, Y)$$

- (3) The composition of morphisms between objects of  $\mathcal{D}$ , when it is defined, is the same as the composition of morphisms in  $\mathcal{C}$
- (4) The identity map  $1_X$  of an object  $X \in \mathcal{D}$  coincides with the identity map of the object  $X$  in  $\mathcal{C}$ .

**Definition 1.1.5.** A subcategory  $\mathcal{D}$  of a category  $\mathcal{C}$  is called **full** if:

$$\forall X, Y \in \mathcal{D} : \text{Hom}_{\mathcal{D}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$$

**Definition 1.1.6.** Let  $\mathcal{C}$  be a category. The **opposite** category of  $\mathcal{C}$ , denoted by  $\mathcal{C}^{op}$ , is the category whose objects are the same as the objects of the category  $\mathcal{C}$ , i.e  $\text{Ob}(\mathcal{C}^{op}) = \text{Ob}(\mathcal{C})$ , and the class of morphisms consists of all the reversed arrows in  $\mathcal{C}$ , i.e

$$\text{Hom}_{\mathcal{C}^{op}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$$

The composition  $\circ^{op}$  of morphisms in  $\mathcal{C}^{op}$  is defined by  $f \circ^{op} g = g \circ f$ .

**Remark 1.1.7.** The opposite category  $\mathcal{C}^{op}$  of a given category  $\mathcal{C}$  is of great importance since it spares us unnecessary work in proving dual statements.

We proceed by giving some important classes of objects which appear frequently in category theory.

**Definition 1.1.8.** Let  $\mathcal{C}$  be a category and  $X$  an object of  $\mathcal{C}$ .

- (1) The object  $X$  is called an **initial object**, if for every object  $Y$  of  $\mathcal{C}$  the set  $\text{Hom}_{\mathcal{C}}(X, Y)$  is a singleton.
- (2) The object  $X$  is called a **terminal object**, if for every object  $Y$  of  $\mathcal{C}$  the set  $\text{Hom}_{\mathcal{C}}(Y, X)$  is a singleton.
- (3) The object  $X$  is called a **zero object**, if it is an initial and a terminal object.

Just like with objects, there also exist important classes of morphisms.

**Definition 1.1.9.** Let  $\mathcal{C}$  be a category,  $X, Y \in \mathcal{C}$  and  $f : X \rightarrow Y$  a morphism in  $\mathcal{C}$ .

- (1) The morphism  $f$  is called a **monomorphism**, if for every pair of morphisms  $g_1 : Z \rightarrow X$  and  $g_2 : Z \rightarrow X$  the following is true:

$$Z \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} X \xrightarrow{f} Y$$

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

- (2) The morphism  $f$  is called an **epimorphism**, if for every pair of morphisms  $g_1 : Y \rightarrow Z$  and  $g_2 : Y \rightarrow Z$  the following is true:

$$X \xrightarrow{f} Y \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} Z$$

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

- (3) The morphism  $f$  is called a **bimorphism**, if it is a monomorphism and an epimorphism.
- (4) The morphism  $f$  is called an **isomorphism**, if there exists a morphism  $g : Y \rightarrow X$  such that  $f \circ g = 1_Y$  and  $g \circ f = 1_X$ . If such morphism exists, we call the objects  $X$  and  $Y$  **isomorphic**, and we write  $X \cong Y$ .

**Example 1.1.10.** (1) In the category **Set** of sets, the monomorphisms are the functions which are injective, the epimorphisms are the functions which are surjective and the isomorphisms are functions which are injective and surjective.

- (2) In the category **Ab** of abelian groups, the monomorphisms are group homomorphisms between abelian groups which are injective, the epimorphisms are group homomorphisms between abelian groups which are surjective and the isomorphisms are group homomorphisms which are injective and surjective.

**Remark 1.1.11.** 1. The zero object of a category, if it exists, is unique up to a unique isomorphism. Therefore we can denote it by 0.

- 2. There are examples of bimorphisms which are not isomorphisms.

**Definition 1.1.12.** (1) Let  $\mathcal{C}$  be a category and let  $f : Y \rightarrow X, g : Z \rightarrow X$  be monomorphisms in  $\mathcal{C}$ . We call  $f$  and  $g$  **isomorphic monomorphisms** if there exists an isomorphism  $h : Y \xrightarrow{\cong} Z$  such that  $f = g \circ h$ . That is, the following diagram is commutative

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \exists h \downarrow \cong & \nearrow g & \\ Z & & \end{array}$$

- (2) Let  $\mathcal{C}$  be a category and let  $f : X \rightarrow Y, g : X \rightarrow Z$  be epimorphisms in  $\mathcal{C}$ . We call  $f$  and  $g$  **isomorphic epimorphisms** if there exists an isomorphism  $h : Y \xrightarrow{\cong} Z$  such that  $g = h \circ f$ . That is, the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \downarrow \cong \exists h \\ & & Z \end{array}$$





is surjective.

(3)  $F$  is called **essentially surjective** if is surjective onto equivalence classes of isomorphic objects, i.e.

$$\forall D \in \mathcal{D}, \exists C \in \mathcal{C} : F(C) \cong D$$

It is possible to compare two functors. The way to do this is to introduce the notion of a natural transformation between two functors. This was the idea that led Eilenberg and Mac Lane to create category theory. Via natural transformations, we may define when two categories are pretty much the same, i.e. when they are isomorphic.

**Definition 1.1.19.** A **natural transformation**  $\alpha : F \rightarrow G$ , where  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  are two functors, is a family of morphisms

$$\alpha = \{\alpha_X : F(X) \rightarrow G(X) \mid X \in \mathcal{C}\}$$

such that for every morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , we have  $G(f) \circ \alpha_X = \alpha_Y \circ F(f)$ , that is, the following diagram is commutative

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

The morphism  $\alpha_X : F(X) \rightarrow G(X)$  is called the **component** of  $\alpha$  at  $X$ .

**Definition 1.1.20.** Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be two functors between categories and let  $\alpha : F \rightarrow G$  be a natural transformation. We call  $\alpha$  a **natural isomorphism**, if for every object  $X \in \mathcal{C}$  the morphism  $\alpha_X : F(X) \rightarrow G(x)$  is an isomorphism. We write  $\alpha : F \xrightarrow{\cong} G$  and we call the functors  $F$  and  $G$  naturally isomorphic.

**Definition 1.1.21.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$  a functor. Then  $F$  is called an **isomorphism** if there exists a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $G \circ F = 1_{\mathcal{C}}$  and  $F \circ G = 1_{\mathcal{D}}$ . Then the categories  $\mathcal{C}$  and  $\mathcal{D}$  are called **isomorphic** and we write  $\mathcal{C} \cong \mathcal{D}$ .

For two categories to be isomorphic is a very strong condition. In reality, there exist categories with very similar properties which are not isomorphic. Hence we arrive at the following definition.

**Definition 1.1.22.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called an **equivalence of categories** if there exists a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $G \circ F \cong 1_{\mathcal{C}}$  and  $F \circ G \cong 1_{\mathcal{D}}$ . Then the categories  $\mathcal{C}$  and  $\mathcal{D}$  are called **equivalent** and we write  $\mathcal{C} \approx \mathcal{D}$ .

It would be convenient if the class of objects of the categories we are interested in were a set. These categories are called small categories.

**Definition 1.1.23.** A category  $\mathcal{C}$  is called **small** if the class of objects  $\text{Ob}(\mathcal{C})$  is a set.

**Definition 1.1.24.** A category  $\mathcal{C}$  is called **skeletally small** if it is equivalent to a small category, that is the class of isomorphism classes of its objects is a set.

The following Theorem gives a characterization of when a functor is an equivalence of categories.

**Theorem 1.1.25.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. The following are equivalent:

- (1)  $F$  is an equivalence of categories.
- (2)  $F$  is full, faithful and essentially surjective.

*Proof.* (1)  $\Rightarrow$  (2) : Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an equivalence of categories. Then, there exist a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  and natural isomorphisms  $\alpha : F \circ G \rightarrow 1_{\mathcal{D}}$ ,  $\beta : G \circ F \rightarrow 1_{\mathcal{C}}$ .

(a)  $F$  is faithful : Let  $f, g : X \rightarrow Y$  be two morphisms in  $\mathcal{C}$  such that  $F(f) = F(g)$ . Then  $(G \circ F)(f) = (G \circ F)(g)$ . In addition, we have the commutative diagrams

$$\begin{array}{ccc} (G \circ F)(X) & \xrightarrow{\beta_X} & X \\ (G \circ F)(f) \downarrow & & \downarrow f \\ (G \circ F)(Y) & \xrightarrow{\beta_Y} & Y \end{array} \qquad \begin{array}{ccc} (G \circ F)(X) & \xrightarrow{\beta_X} & X \\ (G \circ F)(g) \downarrow & & \downarrow g \\ (G \circ F)(Y) & \xrightarrow{\beta_Y} & Y \end{array}$$

By the commutativity of the above diagrams we have that  $f \circ \beta_X = \beta_Y \circ (G \circ F)(f)$  and  $g \circ \beta_X = \beta_Y \circ (G \circ F)(g)$ , thus  $f \circ \beta_X = g \circ \beta_X$ , and because  $\beta_X$  is an isomorphism,  $f = g$ .

(b)  $F$  is full : Let  $g : F(X) \rightarrow F(Y)$  be a morphism in  $\mathcal{D}$ . Thus, we have a morphism  $G(g) : (G \circ F)(X) \rightarrow (G \circ F)(Y)$  in  $\mathcal{C}$ . We define a morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  as follows

$$f = \alpha_Y \circ G(g) \circ \alpha_X^{-1} : X \xrightarrow{\beta_X^{-1}} (G \circ F)(X) \xrightarrow{G(g)} (G \circ F)(Y) \xrightarrow{\beta_Y} Y$$

Therefore, we have the commutative diagram

$$\begin{array}{ccc} (G \circ F)(X) & \xrightarrow[\cong]{\beta_X} & X \\ G(g) \downarrow & & \downarrow f \\ (G \circ F)(Y) & \xrightarrow[\beta_Y]{\cong} & Y \end{array}$$

so,  $G \circ \beta_{(G \circ F)(X)} = \beta_{(G \circ F)(Y)} \circ (G \circ F \circ G)(g)$ . We consider the morphism

$$F(f) = F(\beta_Y \circ G(g) \circ \beta_X^{-1}) = F(\beta_Y) \circ (F \circ G)(g) \circ F(\beta_X^{-1}) : F(X) \rightarrow F(Y)$$

We will show that  $F(f) = g$ , that is  $F(\beta_Y) \circ F(G(g)) \circ F(\beta_X^{-1}) = g$ . Since  $G$  is faithful, it suffices to show that

$$(G \circ F)(f) = (G \circ F)(\beta_Y) \circ (G \circ F \circ G)(g) \circ (G \circ F)(\beta_X^{-1}) = G(g)$$

However,  $\beta_Y$  is an isomorphism, therefore it is enough to show that

$$\beta_Y \circ (G \circ F)(\beta_Y) \circ (G \circ F \circ G)(g) \circ (G \circ F)(\beta_X^{-1}) = \beta_Y \circ G(g)$$

To prove that, we calculate the compositions of the left side of the above equation. We have the commutative diagram

$$\begin{array}{ccc} (G \circ F \circ G \circ F)(X) & \xrightarrow{\beta_{(G \circ F)(X)}} & (G \circ F)(X) \\ (G \circ F)(\beta_Y) \downarrow & & \downarrow G(g) \\ (G \circ F \circ G \circ F)(Y) & \xrightarrow{\beta_{G(F(Y))}} & G(F(X)) \end{array}$$

so,  $\beta_Y \circ (G \circ F)(\beta_Y) = \beta_Y \circ \beta_{(G \circ F)(Y)}$ . Thus, the left side of the equation becomes  $\beta_Y \circ \beta_{(G \circ F)(Y)} \circ (G \circ F \circ G)(g) \circ (G \circ F)(\beta_X^{-1})$ . For the composition  $\beta_{(G \circ F)(Y)} \circ (G \circ F \circ G)(g)$ , we have the commutative diagram

$$\begin{array}{ccc} (G \circ F \circ G \circ F)(X) & \xrightarrow{\beta_{(G \circ F)(X)}} & (G \circ F)(X) \\ \beta_{(G \circ F \circ G)(g)} \downarrow & & \downarrow G(g) \\ (G \circ F \circ G \circ F)(Y) & \xrightarrow{\beta_{(G \circ F)(Y)}} & (G \circ F)(Y) \end{array}$$

so,  $G \circ \beta_{(G \circ F)(X)} = \beta_{(G \circ F)(Y)} \circ (G \circ F \circ G)(g)$ , and the left side of the equation becomes  $\beta_Y \circ G(g) \circ \beta_{(G \circ F)(Y)} \circ (G \circ F)(\beta_X^{-1})$ . For the composition  $\beta_{(G \circ F)(X)} \circ (G \circ F)(\beta_X^{-1})$ , we have the commutative diagram

$$\begin{array}{ccc} (G \circ F)(X) & \xrightarrow{\beta_X} & X \\ \beta_{(G \circ F)(\beta_X^{-1})} \downarrow & & \downarrow \beta_X^{-1} \\ (G \circ F \circ G \circ F)(X) & \xrightarrow{\beta_{(G \circ F)(X)}} & (G \circ F)(X) \end{array}$$

so,  $\beta_{(G \circ F)(X)} \circ (G \circ F)(\beta_X^{-1}) = \beta_X^{-1} \circ \beta_Y = 1_{(G \circ F)(X)}$ , and then  $\beta_Y \circ G(g) \circ 1_{(G \circ F)(X)} = \beta_Y \circ G(g)$ , which proves that  $F$  is full.

3) Let  $D$  be an object in  $\mathcal{D}$ . Because  $\beta_D : (F \circ G)(D) \rightarrow D$  is an isomorphism, we have that  $F(G(D)) \cong D$ , so  $F$  is essentially surjective. ■

Before we move on to abelian categories, we define the notion of adjoint functors between two categories

**Definition 1.1.26.** Let  $\mathcal{C}, \mathcal{C}'$  be two categories and  $L: \mathcal{C} \rightarrow \mathcal{C}', R: \mathcal{C}' \rightarrow \mathcal{C}$  be two functors. The pair  $(L, R)$  is a **pair of adjoint functors** if there exists a natural isomorphism  $\eta: \mathcal{C}^{op} \times \mathcal{C}' \rightarrow \text{Ab}$  defined as follows:

$$\eta: \text{Hom}_{\mathcal{C}}(-, R(-)) \rightarrow \text{Hom}_{\mathcal{C}'}(L(-), -)$$

This means that for every pair of objects  $C \in \mathcal{C}, D \in \mathcal{C}'$  there exists an isomorphism

$$\eta_{C,D}: \text{Hom}_{\mathcal{C}}(C, R(D)) \rightarrow \text{Hom}_{\mathcal{C}'}(L(C), D)$$

such that for every morphism  $f: C \rightarrow C'$  in  $\mathcal{C}$  and every morphism  $g: D \rightarrow D'$  in  $\mathcal{C}'$ , the following diagrams are commutative

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(C', R(D)) & \xrightarrow{\eta_{C',D}} & \text{Hom}_{\mathcal{D}}(L(C'), D) \\ \text{Hom}_{\mathcal{C}}(f, R(D)) \downarrow & & \downarrow \text{Hom}_{\mathcal{D}}(L(f), D) \\ \text{Hom}_{\mathcal{C}}(C, R(D)) & \xrightarrow{\eta_{C,D}} & \text{Hom}_{\mathcal{D}}(L(C), D) \\ \\ \text{Hom}_{\mathcal{C}}(C, R(D)) & \xrightarrow{\eta_{C,D}} & \text{Hom}_{\mathcal{D}}(L(C), D) \\ \text{Hom}_{\mathcal{C}}(C, R(g)) \downarrow & & \downarrow \text{Hom}_{\mathcal{D}}(L(C), g) \\ \text{Hom}_{\mathcal{C}}(C, R(D')) & \xrightarrow{\eta_{C,D'}} & \text{Hom}_{\mathcal{D}}(L(C), D') \end{array}$$

## 1.2 Abelian categories

The notion of a category is general enough to have a rich theory on its own. However, in what concerns us, we would like to enrich our categories by giving an additional structure. Abelian categories provide the appropriate framework for doing homological algebra. Abelian categories come up often in algebra and geometry, an important example being the category of left  $R$ -modules of a ring  $R$ . We start giving the definition of preadditive categories, leading to the definition of additive categories. Once we have defined abelian categories, we will present, without proof, since this goes beyond the scope of this thesis, some fundamental lemmas and theorems about them. In this section we follow [9], [10], [13], [14], [18] and [20].

**Definition 1.2.1.** A category  $\mathcal{C}$  is called **preadditive** (or **Ab-category**) if for every two objects  $X, Y \in \mathcal{C}$  the set  $\text{Hom}_{\mathcal{C}}(X, Y)$  is an abelian group where the operation is the addition:

$$+ : \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Y), (f, g) \mapsto f + g$$

and the composition of morphisms in  $\mathcal{C}$  is bilinear, that is

$$\forall f, f_1, f_2 \in \text{Hom}_{\mathcal{C}}(X, Y), \forall g, g_1, g_2 \in \text{Hom}_{\mathcal{C}}(Y, Z) : \\ g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2 \quad \text{and} \quad f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$$

It is natural, since we have introduced preadditive categories, to define a specific type of functor, the so called additive functor, between them.

**Definition 1.2.2.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two preadditive categories.

(1) A covariant functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called an **additive functor** if for every two objects  $X, Y \in \mathcal{C}$  the map

$$F_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a group homomorphism, that is:

$$\forall f, g \in \text{Hom}_{\mathcal{C}}(X, Y) : F(f + g) = F(f) + F(g)$$

(2) A contravariant functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called an **additive functor** if for every two objects  $X, Y \in \mathcal{C}$  the map

$$F_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(Y), F(X))$$

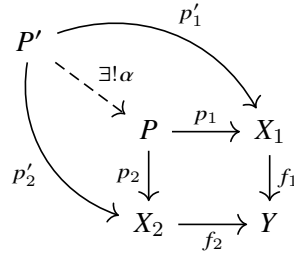
is a group homomorphism, that is:

$$\forall f, g \in \text{Hom}_{\mathcal{C}}(X, Y) : F(f + g) = F(f) + F(g)$$

**Definition 1.2.3.** Let  $\mathcal{C}$  be an abelian category and  $f_1 : X_1 \rightarrow Y, f_2 : X_2 \rightarrow Y$  be two morphisms. The **pullback** of  $f_1$  and  $f_2$  consists of an object  $P$  together with two morphisms  $p_1 : P \rightarrow X_1$  and  $p_2 : P \rightarrow X_2$ , such that  $f_1 \circ p_1 = f_2 \circ p_2$ , that is, the diagram

$$\begin{array}{ccc} P & \xrightarrow{p_1} & X_1 \\ p_2 \downarrow & & \downarrow f_1 \\ X_2 & \xrightarrow{f_2} & Y \end{array}$$

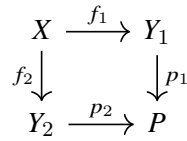
is commutative and the following universal property holds: for every other object  $P'$  and morphisms  $p'_1 : P' \rightarrow X_1$  and  $p'_2 : P' \rightarrow X_2$ , where  $f_1 \circ p'_1 = f_2 \circ p'_2$ , there exists a unique morphism  $\alpha : P' \rightarrow P$  such that  $p'_1 = p_1 \circ \alpha$  and  $p'_2 = p_2 \circ \alpha$ , that is, the diagram



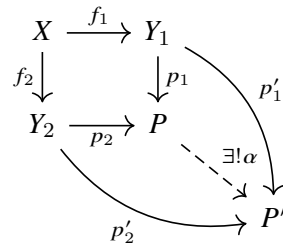
is commutative.

Dually, we have the notion of a pushout of two morphisms.

**Definition 1.2.4.** Let  $\mathcal{A}$  be an abelian category and  $f_1 : X \rightarrow Y_1$  and  $f_2 : X \rightarrow Y_2$  be two morphisms. The **pushout** of  $f_1$  and  $f_2$  consists of an object  $P$  together with two morphisms  $p_1 : Y_1 \rightarrow P$  and  $p_2 : Y_2 \rightarrow P$ , such that  $p_1 \circ f_1 = p_2 \circ f_2$ , that is, the diagram



is commutative and the following universal property holds: for every other object  $P'$  and morphisms  $p'_1 : Y_1 \rightarrow P'$  and  $p'_2 : Y_2 \rightarrow P'$ , where  $p'_1 \circ f_1 = p'_2 \circ f_2$ , there exists a unique morphism  $\alpha : P \rightarrow P'$  such that  $p'_1 = \alpha \circ p_1$  and  $p'_2 = \alpha \circ p_2$ , that is, the diagram



is commutative.

**Definition 1.2.5.** Let  $\mathcal{C}$  be a category and  $\{X_j\}_{j \in J}$  a family of objects in  $\mathcal{C}$ . The **coproduct** of the family  $\{X_j\}_{j \in J}$  is a pair  $(X, i_j)_{j \in J}$  where:

(1)  $X \in \mathcal{C}$

(2)  $i_j : X_j \rightarrow X, j \in J$ , is a family of morphisms in  $\mathcal{C}$  such that the following universal property holds: for every pair  $(Y, f_j)_{j \in J}$ , where  $Y \in \mathcal{C}$  and  $f_j : X_j \rightarrow Y$  is a family of morphisms, there exists a unique morphism  $f : X \rightarrow Y$  such that  $f \circ i_j = f_j, \forall j \in J$ , i.e. the following diagram is commutative

$$\begin{array}{ccc} X_j & \xrightarrow{i_j} & X \\ f_j \downarrow & \swarrow \exists! f & \\ Y & & \end{array}$$

Dually, we have the notion of the coproduct.

**Definition 1.2.6.** Let  $\mathcal{C}$  be a category and  $\{X_j\}_{j \in J}$  in  $\mathcal{C}$ . The **product** of the family  $\{X_j\}_{j \in J}$  is a pair  $(X, p_j)_{j \in J}$  where:

(1)  $X \in \mathcal{C}$

(2)  $p_j : X \rightarrow X_j, j \in J$ , is a family of morphisms in  $\mathcal{C}$  such that the following universal property holds: for every pair  $(Y, f_j)_{j \in J}$ , where  $Y \in \mathcal{C}$  and  $f_j : Y \rightarrow X_j$  is a family of morphisms, there exists a unique morphism  $f : Y \rightarrow X$  such that  $p_j \circ f = f_j, \forall j \in J$ , i.e. the following diagram is commutative

$$\begin{array}{ccc} Y & \xrightarrow{\exists! f} & X \\ f_j \searrow & & \downarrow p_j \\ & & X_j \end{array}$$

The object  $X$  is denoted by  $\prod_{j \in J} X_j$ .

**Remark 1.2.7.** (1) The product and coproduct, if they exist, are unique up to a unique isomorphism because they are solutions to a universal problem. Hence, for a family of objects  $\{X_j\}_{j \in J}$  we denote the product by  $\prod_{j \in J} X_j$ , and the coproduct by  $\coprod_{j \in J} X_j$ . Their existence depends on the category.

(2) The pull-back and push-out, if they exist, are unique up to a unique isomorphism because they are solutions to a universal problem.

**Example 1.2.8.** (1) In the category of sets the coproduct is the disjoint union and the product is the cartesian product.

- (2) In the category of groups the coproduct is the free product of groups and the product is the direct product of groups.
- (3) In the category of left  $R$ -modules over a ring  $R$ , the coproduct coincides with the product of left  $R$ -modules, if the family of modules is finite.

**Definition 1.2.9.** A category  $\mathcal{C}$  is called **additive** if:

- (1) it is preadditive,
- (2) there exists the zero object,
- (3) for every two objects  $X$  and  $Y$  in  $\mathcal{C}$  their coproduct  $X \oplus Y$  exists.

**Proposition 1.2.10.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an additive functor between additive categories. Then:

- (1)  $F$  preserves monomorphisms, epimorphisms and isomorphisms,
- (2)  $F$  preserves the zero object,
- (3)  $F$  preserves coproducts of finitely many objects.

Before we proceed with the definition of an abelian category, it is necessary to introduce kernels and cokernels.

**Definition 1.2.11.** Let  $\mathcal{C}$  be a category and  $f : X \rightarrow Y$  a morphism in  $\mathcal{C}$ .

- (1) The **kernel** of the morphism  $f$  is pair  $(K, k)$ , where  $K$  is an object of  $\mathcal{C}$  and  $k : K \rightarrow X$  is morphism in  $\mathcal{C}$  such that  $f \circ k = 0$  and the following universal property holds: for every pair  $(K', k')$  where  $K'$  is an object of  $\mathcal{C}$  and  $k' : K' \rightarrow X$  is a morphism of  $\mathcal{C}$  such that  $f \circ k' = 0$  there exists a unique morphism  $g : K' \rightarrow K$  such that  $k' = k \circ g$ , i.e. the following diagram is commutative

$$\begin{array}{ccccc}
 K & \xrightarrow{k} & X & \xrightarrow{f} & Y \\
 \uparrow & & \nearrow & & \\
 \exists! \downarrow g & & k' & & \\
 K' & & & & 
 \end{array}$$

- (2) The **cokernel** of the morphism  $f$  is a pair  $(C, c)$ , where  $C$  is an object in  $\mathcal{C}$  and  $c : Y \rightarrow C$  is a morphism in  $\mathcal{C}$  such that  $c \circ f = 0$  and the following universal property holds: for every pair  $(C', c')$  where  $C'$  is an object of  $\mathcal{C}$  and  $c' : Y \rightarrow C'$  is a morphism of  $\mathcal{C}$  such that  $c' \circ f = 0$  there exists a unique morphism  $g : C \rightarrow C'$  such that  $c' = g \circ c$ , i.e. the following diagram is commutative

$$\begin{array}{ccccc}
 & & & & C' \\
 & & & \nearrow c' & \uparrow \exists! g \\
 X & \xrightarrow{f} & Y & \xrightarrow{c} & C \\
 & & & & \downarrow \\
 & & & & C
 \end{array}$$

**Definition 1.2.12.** A category  $\mathcal{C}$  is called **pre-abelian** if it is additive with kernels and cokernels.

In what follows, we fix an additive category  $\mathcal{C}$ .

**Remark 1.2.13.** (1) The kernel and cokernel of a morphism don't always exist. Their existence depends on the category. However, since they are defined as solutions of a universal problem, they are unique up to isomorphism.

(2) If the kernel and the cokernel of a morphism  $f : X \rightarrow Y$  exist, then  $\ker(f)$  is a monomorphism and  $\operatorname{coker}(f)$  is an epimorphism.

From the above Remark 1.2.13 we are led to the following definition.

**Definition 1.2.14.** Let  $\mathcal{C}$  be a pre-abelian category. The objects  $K, C$  and the morphisms  $k, c$  on the pairs  $(K, k)$  and  $(C, c)$ , will be denoted by  $\operatorname{Ker}(f), \operatorname{Coker}(f)$  and  $\ker(f), \operatorname{coker}(f)$  respectively.

**Remark 1.2.15.** Let  $f : X \rightarrow Y$  be a morphism with a kernel and a cokernel. We present a construction of an important morphism.

Suppose that  $\ker(f)$  has a cokernel and  $\operatorname{coker}(f)$  has a kernel. We denote the cokernel of  $\ker(f)$  by  $(I, i)$ , where  $I = \operatorname{Coker}(k) = \operatorname{Coker}(\operatorname{Ker}(f))$  and  $i = \operatorname{Coker}(k) = \operatorname{Coker}(\operatorname{Ker}(f))$ , and the kernel of  $\operatorname{Coker}(f)$  by  $(J, j)$ , where  $J = \operatorname{Ker}(c) = \operatorname{Ker}(\operatorname{Coker}(f))$  and  $j = \operatorname{Ker}(c) = \operatorname{Ker}(\operatorname{Coker}(f))$ . Since  $f \circ k = 0$ ,  $f$  factors through the  $I = \operatorname{Coker}(k)$ , that is, there exists a unique morphism  $\alpha : I \rightarrow Y$  such that  $f = \alpha \circ i$ . However,  $0 = c \circ f = c \circ \alpha \circ i$ . Since  $i$  is an epimorphism, we have that  $c \circ \alpha = 0$ , which means that  $\alpha$  is factored through the kernel of  $c$ , that is, there exists a unique morphism  $\bar{f} : I \rightarrow J$ , or  $\bar{f} : \operatorname{Coker}(k) \rightarrow \operatorname{Ker}(c)$ , or  $\bar{f} : \operatorname{Coker}(\operatorname{Ker}(f)) \rightarrow \operatorname{Ker}(\operatorname{Coker}(f))$ , such that  $\alpha = j \circ \bar{f}$ . Therefore, the morphism  $f$  can be expressed as the composition

$$f = \alpha \circ i = j \circ \bar{f} \circ i$$

The above morphisms fit into the following commutative diagram

$$\begin{array}{ccccccc}
 K & \xrightarrow{k} & X & \xrightarrow{f} & Y & \xrightarrow{c} & C \\
 & & \downarrow i & \nearrow \exists! \alpha & \uparrow j & & \\
 & & \operatorname{Coker}(k) & \xrightarrow{\exists! \bar{f}} & \operatorname{Ker}(c) & & 
 \end{array} \tag{1.1}$$

The morphism  $\bar{f} : \operatorname{Coker}(k) \rightarrow \operatorname{Ker}(c)$  is called the **induced (parallel) morphism** of  $f$ .

**Definition 1.2.16.** A morphism  $f : X \rightarrow Y$  is called **strict** if its induced (parallel) morphism  $\bar{f} : \text{Coim}(f) \rightarrow \text{Im}(f)$  is an isomorphism.

We are now in position to give the definition of an abelian category.

**Definition 1.2.17.** A category  $\mathcal{A}$  is called **abelian** if:

- (1) it is additive
- (2) every morphism has a kernel and a cokernel,
- (3) every monomorphism is the kernel of its cokernel,
- (4) every epimorphism is the cokernel of its kernel.

**Remark 1.2.18.** (1) There is an equivalent way to define abelian categories. One can replace (2) and (3) of Definition 1.2.17 by demanding that every morphism is strict. The following proposition essentially proves one direction.

(2) Every abelian, and in general every pre-abelian category, has pull-backs and push-outs.

**Proposition 1.2.19.** Let  $\mathcal{A}$  be an abelian category. Then every morphism is strict.

*Proof.* Let  $f : X \rightarrow Y$  be a morphism in an abelian category  $\mathcal{A}$ . By the construction in Remark 1.2.15 we have the commutative diagram

$$\begin{array}{ccccccc}
 K & \xrightarrow{k} & X & \xrightarrow{f} & Y & \xrightarrow{c} & C \\
 & & \downarrow i & & \uparrow j & & \\
 & & \text{Coker}(k) & \xrightarrow{\bar{f}} & \text{Ker}(c) & & 
 \end{array}$$

$\exists! \alpha$  (dashed arrow from  $\text{Coker}(k)$  to  $Y$ )  
 $\exists! \tau$  (dashed arrow from  $Y$  to  $\text{Ker}(c)$ )

First we prove that  $\alpha$  is a monomorphism. Let  $\phi_1, \phi_2 : Z \rightarrow \text{Coker}(\text{Ker}(f))$  be two morphisms such that  $\alpha \circ \phi_1 = \alpha \circ \phi_2$ . We will show that  $\phi_1 = \phi_2$ . We set  $\sigma := \phi_1 - \phi_2$  and then  $\alpha \circ \sigma = 0$ . We consider the pullback of  $i$  and  $\sigma$ . Then there exist an object  $W$ , a morphism  $\tau : W \rightarrow Z$  and a morphism  $\pi : W \rightarrow X$  such that the following diagram is commutative

$$\begin{array}{ccc}
 W & \xrightarrow{\tau} & Z \\
 \pi \downarrow & & \downarrow \sigma \\
 X & \xrightarrow{i} & \text{Coker}(\text{Ker}(f))
 \end{array}$$

We compute,

$$f \circ \pi = \alpha \circ i \circ \pi = \alpha \circ \sigma \circ \tau = 0 \circ \tau = 0$$

Therefore, from the universal property of the cokernel, there exists a unique morphism  $w : W \rightarrow K$  such that  $\pi = k \circ w$ , i.e, the following diagram is commutative

$$\begin{array}{ccccc}
 & & W & \xrightarrow{\tau} & Z \\
 & \swarrow \exists! & \downarrow \pi & & \downarrow \sigma \\
 K & \xrightarrow{k} & X & \xrightarrow{i} & \text{Coker}(\text{Ker}(f))
 \end{array}$$

Since  $i$  is the cokernel of  $k$ , we have that  $i \circ k = 0$ . But

$$\sigma \circ \tau = i \circ \pi = i \circ k \circ w = 0 \circ w = 0$$

so  $\sigma \circ \tau = 0$ . Since  $i$  is a monomorphism, then  $\tau$  is an epimorphism, thus  $\sigma = 0$ , which means that  $\phi_1 = \phi_2$ . This implies that  $\alpha$  is a monomorphism. Since  $i$  is an epimorphism, we have that  $\text{Coker}(\alpha) = \text{Coker}(f)$ . The morphism  $\alpha$  is a monomorphism, thus, from the third axiom of an abelian category, it is the kernel of its cokernel. This means that  $\alpha$  is the kernel of the cokernel of  $f$ . In addition,  $j$  is also the kernel of the cokernel of  $f$ . Consequently, since the kernel is unique up to isomorphism, we have that  $\bar{f}$  is an isomorphism. ■

By Proposition 1.2.19 we have the following useful characterizations of monomorphisms, epimorphisms, and isomorphisms.

**Proposition 1.2.20.** *Let  $\mathcal{A}$  be an abelian category and  $f : X \rightarrow Y$  a morphism in  $\mathcal{A}$ . Then:*

- (1)  $f$  is a monomorphism  $\iff \text{Ker}(f) = 0$
- (2)  $f$  is an epimorphism  $\iff \text{Coker}(f) = 0$
- (3)  $f$  is an isomorphism  $\iff f$  is a monomorphism and an epimorphism

**Remark 1.2.21.** The third part of the above Proposition 1.2.20, is not necessarily true if  $\mathcal{A}$  is not abelian. For instance, in a quasi-abelian category, (3) does not hold, despite the fact that they have kernels and cokernels and some extra properties.

**Definition 1.2.22.** *Let  $\mathcal{A}$  be an abelian category and  $f : X \rightarrow Y$  be a morphism in  $\mathcal{A}$ .*

- (1) We call **image** of the morphism  $f$  the pair  $(\text{Im}(f), \text{im}(f))$ , where  $\text{Im}(f) = \text{Ker}(\text{Coker}(f))$  and  $\text{im}(f) : \text{Im}(f) \rightarrow Y$ , where  $\text{im} = j$  in Remark 1.2.15, in diagram (1.1).

(2) We call **coimage** of the morphism  $f$  the pair  $(\text{Coim}(f), \text{coim}(f))$ , where  $\text{Coim}(f) = \text{Coker}(\text{Ker}(f))$  and  $\text{coim}(f) : X \rightarrow \text{Coim}(f)$ , where  $\text{coim}(f) = i$  in Remark 1.2.15, in diagram (1.1).

**Remark 1.2.23.** Let  $\mathcal{A}$  be an abelian category and  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  two morphisms in  $\mathcal{A}$ , such that  $g \circ f = 0$ . Consider the following diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

We want to define the notion of **exactness** at the "position"  $Y$ . Since we work in an abelian category, the induced morphism of  $f$  is an isomorphism, hence we have the factorization  $f = \text{coim}(f) \circ \bar{f} \circ \text{im}(f)$ , where  $\text{coim}(f) : X \rightarrow \text{Im}(f)$  and  $\text{im}(f) : \text{Im}(f) \rightarrow Y$  are defined up to isomorphism, hence we may define  $\text{coim}(f)$  as  $i$  in Definition 1.2.22. Therefore we can factor  $f$  through its image as such

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \xrightarrow{g} Z \\ & \searrow^{i=\text{coim}(f)} & \nearrow_{\alpha=\text{im}(f)} \\ & & \text{Im}(f) \end{array}$$

We have  $g \circ f = g \circ \alpha \circ i = 0$ , and because  $i$  is an epimorphism we get  $g \circ \alpha = 0$ . As a result  $\alpha$  can be factored through the kernel of  $g$ , that is, there exists a unique morphism  $h : \text{Im}(f) \rightarrow \text{Ker}(g)$  such that  $\alpha = \text{Ker}(g) \circ h$ . Thus we have the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & \searrow^{i} & \nearrow_{\alpha} & & \\ & & \text{Im}(f) & \xrightarrow[\text{h}]{\exists!} & \text{Ker}(g) \end{array}$$

and since  $\alpha = \text{im}(f)$  is a monomorphism, it follows that  $h$  is a monomorphism i.e.  $\text{Im}(f)$  is a subobject of  $\text{Ker}(g)$ . The morphism  $h$  is called the **canonical map** from  $\text{Im}(f)$  to  $\text{Ker}(g)$ .

**Definition 1.2.24.** Let  $\mathcal{A}$  be an abelian category. We say that the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is **exact** at  $Y$ , if the canonical map  $h : \text{Im}(f) \rightarrow \text{Ker}(g)$  is an isomorphism. If we have a sequence

$$\dots \longrightarrow X^{n-1} \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1} \longrightarrow \dots$$

we say that it is **exact**, if it is exact at every  $X^n$ ,  $n \in \mathbb{Z}$ .

**Notation 1.2.25.** If  $\mathcal{A}$  is an abelian category then:

(1) if we have a sequence of the form

$$0 \longrightarrow X \xrightarrow{f} Y$$

then the morphism  $f$  is a monomorphism, i.e. the sequence is exact at  $X$ ,

(2) if we have a sequence of the form

$$Y \xrightarrow{f} Z \longrightarrow 0$$

then the morphism  $f$  is an epimorphism, i.e. the sequence is exact at  $Z$ .

**Definition 1.2.26.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two abelian categories and  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an covariant additive functor. Then  $F$  is called **exact** if for every short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in  $\mathcal{A}$  we have that

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is a short exact sequence in  $\mathcal{B}$ . Respectively, we have the definition of exactness for a contravariant additive functor.

**Remark 1.2.27.** The following are some important results in abelian categories. We present them without giving a proof.

**Theorem 1.2.28** (Mitchell Embedding Theorem). Let  $\mathcal{A}$  be a small abelian category. Then there exist a unitary ring  $R$  and a fully faithful exact functor  $F: \mathcal{A} \rightarrow R\text{-Mod}$ .

*Proof.* For the proof, we refer the reader to [13, Chapter IV, Section 7]. ■

**Lemma 1.2.29** (5-Lemma). Let  $\mathcal{A}$  be an abelian category. Consider the following exact commutative diagram in  $\mathcal{A}$

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_2 & \longrightarrow & X_4 & \longrightarrow & X_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & Y_4 & \longrightarrow & Y_5 \end{array}$$

If the morphisms  $f_2, f_4$  are isomorphisms, the morphism  $f_1$  is an epimorphism and the morphism  $f_5$  is a monomorphism, then the morphism  $f_3$  is an isomorphism.

*Proof.* For the proof, we refer the reader to [10, Chapter XII, Section 3]. ■

**Lemma 1.2.30** (Snake Lemma). *Let  $\mathcal{A}$  be an abelian category. Consider the following exact commutative diagram in  $\mathcal{A}$ .*

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ 0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 \end{array}$$

*There exists a morphism  $\partial : \text{Ker}(f_3) \rightarrow \text{Coker}(f_1)$  which makes the following sequence exact*

$$\text{Ker}(f_1) \longrightarrow \text{Ker}(f_2) \longrightarrow \text{Ker}(f_3) \xrightarrow{\partial} \text{Coker}(f_1) \longrightarrow \text{Coker}(f_2) \longrightarrow \text{Coker}(f_3)$$

*In addition, if the morphism  $X_1 \rightarrow X_2$  is a monomorphism, the morphism  $\text{Ker}(f_1) \rightarrow \text{Ker}(f_2)$  is also a monomorphism, and if the morphism  $Y_2 \rightarrow Y_3$  is an epimorphism, then the morphism  $\text{Coker}(f_2) \rightarrow \text{Coker}(f_3)$  is also an epimorphism. By these assumptions the initial diagram becomes*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ 0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 \longrightarrow 0 \end{array}$$

*and the above exact sequence can be extended to the exact sequence*

$$0 \longrightarrow \text{Ker}(f_1) \longrightarrow \text{Ker}(f_2) \longrightarrow \text{Ker}(f_3) \xrightarrow{\partial} \text{Coker}(f_1) \longrightarrow \text{Coker}(f_2) \longrightarrow \text{Coker}(f_3) \longrightarrow 0$$

*The morphism  $\partial$  is called a **connecting morphism**.*

*Proof.* For the proof, we refer the reader to [20, Chapter 1, Section 1.3]. ■

## 1.3 Localization

In this section, we discuss the notion of the localization of category  $\mathcal{C}$ . On one hand, the localization of an arbitrary category is itself important. On the other hand, in its full generality, we can run into some set theoretic problems. Therefore it is reasonable to focus on localization of categories such as abelian and triangulated categories, imposing several conditions ensuring that the localization exists and has an accessible description. In this section we closely follow [12].

### 1.3.1 Localization of categories

The main idea for the localization of an arbitrary category  $\mathcal{C}$ , is that we want to "invert" a class of morphisms  $S$ , that is, creating a new category  $\mathcal{C}[S^{-1}]$  in which these inverted morphisms are isomorphisms. We present only the uniqueness, up to isomorphism, of this new category. We refer the reader to [12] for the construction in general.

**Definition 1.3.1.** Let  $\mathcal{C}$  be a category and  $S$  a class of morphisms of  $\mathcal{C}$ . The **localization** of  $\mathcal{C}$  with respect to  $S$  is category denoted by  $\mathcal{C}[S^{-1}]$  along with a functor  $Q: \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  such that the following are true:

- (1) the morphism  $Q(s)$  is an isomorphism for every  $s \in S$ ,
- (2) the following universal property holds: for every category  $\mathcal{D}$  and functor  $F: \mathcal{C} \rightarrow \mathcal{D}$ , such that  $F(s)$  is an isomorphism for every  $s \in S$ , there exists a unique functor  $G: \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$  which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ Q \downarrow & \nearrow G & \\ \mathcal{C}[S^{-1}] & & \end{array}$$

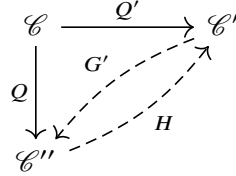
**Theorem 1.3.2.** Let  $\mathcal{C}$  be a category and  $S$  a subclass of  $\text{Hom}_{\mathcal{C}}$ . Then there exists a category  $\mathcal{C}[S^{-1}]$  and a functor  $Q: \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  such that:

- (1)  $Q(s)$  is an isomorphism for every  $s \in S$ ,
- (2) for every category  $\mathcal{D}$  and functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  such that  $F(s)$  is an isomorphism for every  $s \in S$ , there exists a unique functor  $G: \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$  such that  $F = G \circ Q$ , that is, the following diagram is commutative

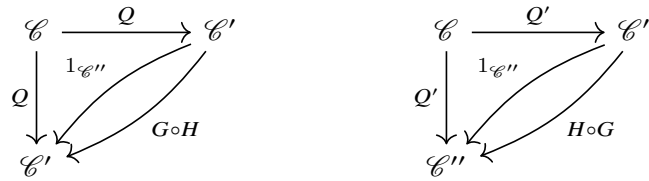
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ Q \downarrow & \xrightarrow{\exists!} & \mathcal{D} \\ \mathcal{C}[S^{-1}] & \xrightarrow{G} & \mathcal{D} \end{array}$$

The category  $\mathcal{C}[S^{-1}]$  is unique up to equivalence of categories.

*Proof.* For the construction of the category  $\mathcal{C}[S^{-1}]$  we refer the reader to [12]. Here we will prove the uniqueness. Let  $(\mathcal{C}', Q)$  and  $(\mathcal{C}'', Q')$  be two pairs which satisfy the universal property (2). Then there exist unique functors  $G: \mathcal{C}' \rightarrow \mathcal{C}''$  and  $H: \mathcal{C}'' \rightarrow \mathcal{C}'$  such that  $Q' = G \circ Q$  and  $Q = H \circ Q'$  respectively. Therefore we have the commutative diagram



By combining the above equalities we have  $Q' = G \circ Q = G \circ H \circ Q' = (G \circ H) \circ Q'$  and  $Q = H \circ Q' = H \circ G \circ Q = (H \circ G) \circ Q$ . So we get the following commutative diagrams



From the universal property we have that  $H \circ G = 1_{\mathcal{C}'}$  and  $G \circ H = 1_{\mathcal{C}''}$ . Thus we have that  $H$  and  $G$  are isomorphic. ■

### 1.3.2 Localization classes of morphisms

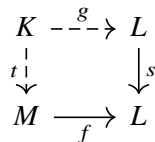
As we have stated before, the localization of a category  $\mathcal{C}$  with respect to an arbitrary subclass  $S$  of  $\text{Hom}_{\mathcal{C}}$ , always exists. However, set theoretic problems can arise. In addition, in most cases, we can give little to none information about the localization  $\mathcal{C}[S^{-1}]$  in the completely general case. In order to build a rich theory, we introduce the notion of a localizing class of a category, which permits a better description of the morphisms of  $\mathcal{C}[S^{-1}]$ .

**Definition 1.3.3.** Let  $\mathcal{C}$  be a category and  $S$  a class of morphisms in  $\mathcal{C}$ . The class  $S$  is called a **localizing class** if:

(LC1) For every  $X$  in  $\mathcal{C}$ , the morphism  $1_X$  is in  $S$ ,

(LC2) If  $s, t \in S$  then  $s \circ t \in S$ ,

(LC3.a) For every pair of morphisms  $f \in \text{Hom}_{\mathcal{C}}$  and  $s \in S$ , there exist morphisms  $g \in \text{Hom}_{\mathcal{C}}$  and  $t \in S$  such that  $s \circ g = f \circ t$ , i.e. the following square is commutative



(LC3.b) For every pair of morphisms  $f \in \text{Hom}_{\mathcal{C}}$  and  $s \in S$ , there exist morphisms  $g \in \text{Hom}_{\mathcal{C}}$  and  $t \in S$  such that  $s \circ g = f \circ t$ , i.e. the following square is commutative

$$\begin{array}{ccc} K & \xleftarrow{g} & L \\ \uparrow t & & \uparrow s \\ M & \xleftarrow{f} & N \end{array}$$

(LC4) If  $f, g : M \rightarrow N$  are morphisms, then there exists a morphism  $s \in S$  such that  $s \circ f = s \circ g$  if and only if there exists a morphism  $t \in S$  such that  $f \circ t = g \circ t$ .

**Remark 1.3.4.** If  $S$  is a localizing class in  $\mathcal{C}$  then it is also a localizing class in  $\mathcal{C}^{op}$ .

The following proposition explains why it is convenient to work with localizing classes. It provides a way to give a better description of morphisms in the localization  $\mathcal{C}[S^{-1}]$ . For the proof, we refer the reader to [12].

**Proposition 1.3.5.** Let  $\mathcal{C}$  be a category,  $S$  a localizing class of  $\mathcal{C}$  and  $Q : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  the localization functor. Then every morphism in  $\mathcal{C}[S^{-1}]$  can be written as  $Q(f) \circ Q(s)^{-1}$ , where  $f \in \text{Hom}_{\mathcal{C}}$  and  $s \in S$ , or  $Q(t)^{-1} \circ Q(g)$ , where  $g \in \text{Hom}_{\mathcal{C}}$  and  $t \in S$ .

Another advantage of working with localizing classes is that it helps to visualize the morphisms in the localization with diagrams. Therefore we have the following definition.

**Definition 1.3.6.** Let  $\mathcal{C}$  be a category and  $S$  a localizing class of  $\mathcal{C}$ .

(1) A diagram of the form

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & \sim & N \end{array}$$

where  $s \in S$ , is called a **left roof** between the objects  $M$  and  $N$ , and is denoted by  $(s, f)$  or  $s/f$ .

(2) A diagram of the form

$$\begin{array}{ccc} & L & \\ g \nearrow & & \nwarrow t \\ M & \sim & N \end{array}$$

where  $t \in S$ , is called a **right roof** between the objects  $M$  and  $N$ , and is denoted by  $(g, t)$  or  $g/t$ .

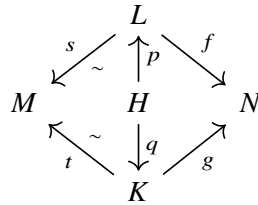
We use the symbol  $\sim$  to indicate that this morphism is in  $S$ .

**Remark 1.3.7.** By switching from  $\mathcal{C}$  to  $\mathcal{C}^{\text{op}}$ , we go from left roofs to right roofs and vice versa. So it is enough to study the left roofs alone.

**Definition 1.3.8.** Let



be two left roofs. They are called **equivalent** if there exist an object  $H$  and morphisms  $p : H \rightarrow L$  and  $q : H \rightarrow K$  such that  $s \circ p = t \circ q \in S$ , i.e. the following diagram is commutative

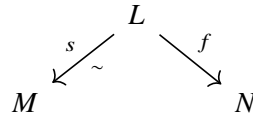


**Remark 1.3.9.** According to the above Definition 1.3.8, if two left roofs are equivalent, then the morphism  $Q(p \circ s) = Q(p) \circ Q(s)$  is an isomorphism in  $\mathcal{C}[S^{-1}]$ . Since  $Q(s)$  is an isomorphism, then  $Q(p)$  is also an isomorphism. Furthermore  $Q(q)$  is also an isomorphism. Therefore we have

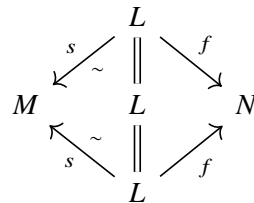
$$\begin{aligned}
 Q(f) \circ Q(s)^{-1} &= Q(f) \circ Q(p) \circ Q(p)^{-1} \circ Q(s)^{-1} = Q(f \circ p) \circ Q(s \circ p)^{-1} = \\
 Q(g \circ q) \circ Q(t \circ q)^{-1} &= Q(g) \circ Q(q) \circ Q(q)^{-1} \circ Q(t)^{-1} = Q(g) \circ Q(t)^{-1} =
 \end{aligned}$$

**Lemma 1.3.10.** The relation in Definition 1.3.8 of left roofs is an equivalence relation.

*Proof.* Reflexive: Let



be a left roof. From the following diagram

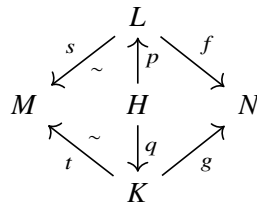


it follows that every left roof is equivalent to itself.

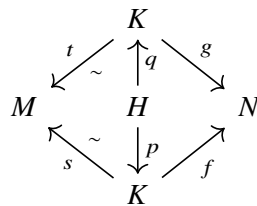
Symmetric: Let



be two left roofs which are equivalent. Therefore there exist an object  $H$  and morphisms  $p : H \rightarrow L$  and  $q : H \rightarrow K$ , such that  $f \circ p = g \circ q$  and  $s \circ p = t \circ q$ , i.e. the following diagram is commutative

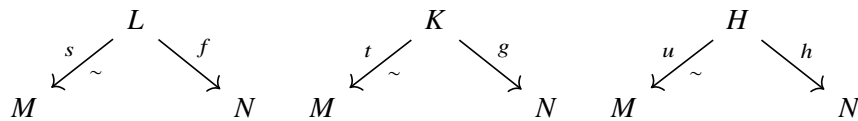


Clearly, from the commutative diagram



we have that the relation is symmetric.

Transitive: Let



be three left roofs and



be two equivalences of left roofs, where  $s \circ p = t \circ q \in S$  and  $u \circ v = t \circ r \in S$ . Consider the morphisms  $s \circ p : H \rightarrow M$  and  $t \circ r : Q \rightarrow M$ . Since  $t \circ r \in S$ , from (LC3.a), there exist an object  $R$  and morphisms  $z : R \rightarrow H$  and  $a : R \rightarrow Q$  such that  $z \in S$  and the following diagram is commutative

$$\begin{array}{ccc} R & \xrightarrow{a} & Q \\ z \downarrow \sim & & \sim \downarrow t \circ r \\ H & \xrightarrow{s \circ p} & M \end{array}$$

Now, consider the morphisms  $b := q \circ z : R \rightarrow K$  and  $c := r \circ a : R \rightarrow K$ . Then we have that

$$t \circ b = t \circ q \circ z = s \circ p \circ z = t \circ r \circ a = t \circ c$$

Therefore, from (LC4), we have that there exists an object  $T$  and a morphism  $w : T \rightarrow R$  such that  $b \circ w = c \circ w$ . We define  $x := p \circ z \circ w$  and  $y := v \circ a \circ w$ . Then

$$s \circ x = s \circ p \circ z \circ w = t \circ q \circ z \circ w = t \circ b \circ w = t \circ c \circ w = t \circ r \circ a \circ w = u \circ v \circ a \circ w = u \circ y$$

Moreover, since  $s \circ p, z, w \in S$ , we get  $s \circ x = u \circ y \in S$ . In addition

$$h \circ y = h \circ v \circ a \circ w = g \circ r \circ a \circ w = g \circ c \circ w = g \circ b \circ w = g \circ q \circ z \circ w = f \circ p \circ z \circ w = f \circ x$$

i.e. the following diagram is commutative

$$\begin{array}{ccccc} & & L & & \\ & s \swarrow & \uparrow x & \searrow f & \\ M & & T & & N \\ & u \swarrow & \downarrow y & \searrow h & \\ & & H & & \end{array}$$

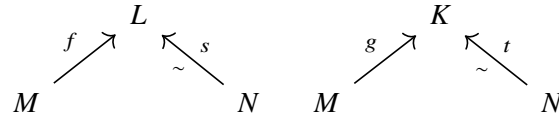
which implies that the left roofs

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \quad \begin{array}{ccc} & H & \\ u \swarrow & & \searrow h \\ M & & N \end{array}$$

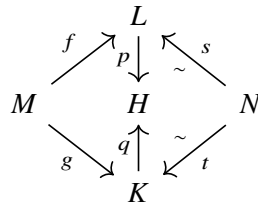
are equivalent. ■

Dually, we can define the equivalence of two right roofs.

**Definition 1.3.11.** Let



be two right roofs. They are called **equivalent** if there exist an object  $H$  and morphisms  $p : L \rightarrow H$  and  $q : K \rightarrow H$  such that  $p \circ s = q \circ t \in S$ , i.e. the following diagram is commutative



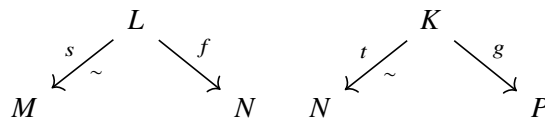
**Remark 1.3.12.** By switching from  $\mathcal{C}$  to  $\mathcal{C}^{\text{op}}$ , if we have two left roofs in  $\mathcal{C}$  which are equivalent, we obtain two right roofs in  $\mathcal{C}^{\text{op}}$  which are equivalent.

**Lemma 1.3.13.** The above relation of right roofs is an equivalence relation.

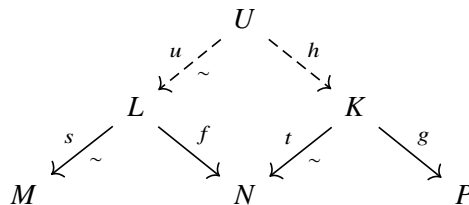
*Proof.* The proof is dual to the proof using left roofs. ■

**Notation 1.3.14.** If  $(s, f)$  is a left roof, then we denote its equivalence class by  $[s, f]$ . Respectively, if  $(g, t)$  is a right roof, we denote its equivalence class by  $[g, t]$ .

**Remark 1.3.15.** Let

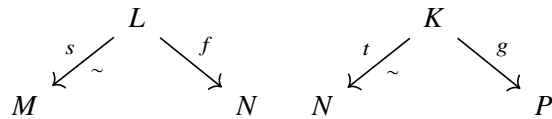


From (LC3.a) there exists an object  $U$  and morphisms  $u : U \rightarrow L$  in  $S$  and  $h : U \rightarrow K$  such that  $f \circ u = t \circ h$ , i.e. the following diagram is commutative

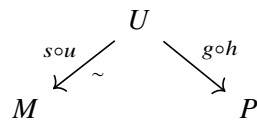


Hence we arrive at the following definition of the composition of two left roofs.

**Definition 1.3.16.** *Let*

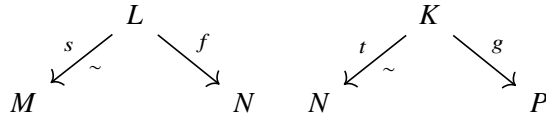


We define the **composition** of the above left roofs to be the left roof

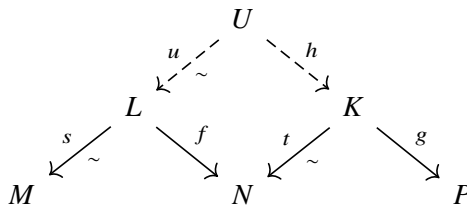


**Proposition 1.3.17.** *The equivalence class of the composition of two left roofs is independent of the choice of  $U$ ,  $u$  and  $h$  from the previous Definition 1.3.16.*

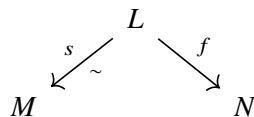
*Proof.* Let



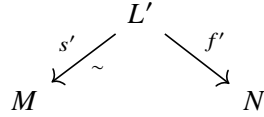
be two left roofs and



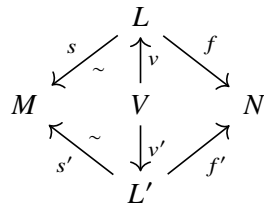
be their composition. For a start, we will prove it for the class of the left roof



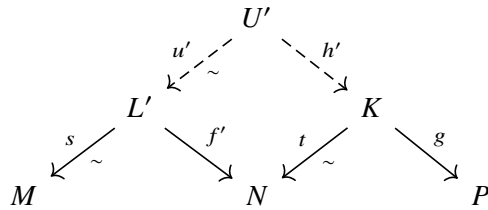
Let



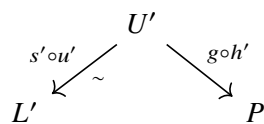
be another left roof which is equivalent to the first one, i.e. there exist an object  $V$  and morphisms  $v : V \rightarrow L$ ,  $v' : V \rightarrow L'$ , such that  $s \circ v = s' \circ v' \in S$  and the diagram



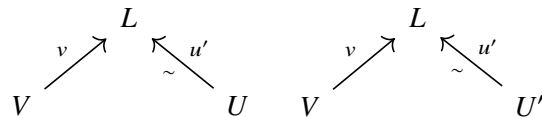
is commutative. Then there exist an object  $U'$  and morphisms  $v' : U' \rightarrow L$ ,  $h' : U' \rightarrow K$  such that  $f' \circ u' = h' \circ t$ , i.e. the diagram



is commutative. Hence we have the composition which is the left roof



By applying (LC3.a) to the right roofs



it follows that there exist objects  $W, W'$ , and morphisms  $w : W \rightarrow V \in S$ ,  $a : W \rightarrow U$ ,  $w' : W' \rightarrow V \in S$  and  $a' : W \rightarrow U'$  such that the following diagrams are commutative

$$\begin{array}{ccc}
 W & \xrightarrow{a} & U \\
 \downarrow w \sim & & \downarrow u \\
 V & \xrightarrow{v} & L
 \end{array}
 \quad
 \begin{array}{ccc}
 W' & \xrightarrow{a'} & U' \\
 \downarrow w' \sim & & \downarrow u' \\
 V & \xrightarrow{v'} & L'
 \end{array}$$

Now, by applying (LC3.a) to the left roof

$$\begin{array}{ccc}
 & & V \\
 & \nearrow w' & \\
 W' & & \\
 & \nwarrow w & \\
 & & W
 \end{array}$$

there exist an object  $R$  and morphisms  $r : R \rightarrow W$  and  $r' : R \rightarrow W' \in S$ , such that the following square is commutative

$$\begin{array}{ccc}
 R & \xrightarrow{r} & W \\
 \downarrow r' \sim & & \downarrow w \\
 W' & \xrightarrow{w'} & V
 \end{array}$$

Notice that

$$s \circ u \circ a \circ r = s \circ v \circ w \circ r = s \circ v \circ w' \circ r' = s' \circ v' \circ w' \circ r' = s' \circ u' \circ a' \circ r' \in S$$

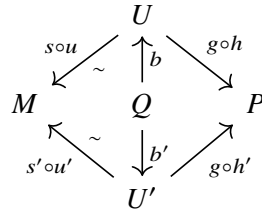
because  $s \circ v, w, r \in S$ . In addition,

$$t \circ h \circ a \circ r = f \circ u \circ a \circ r = f \circ v \circ w \circ r = f' \circ u' \circ w' \circ r' = f' \circ u' \circ a' \circ r' = t \circ h' \circ a' \circ r'$$

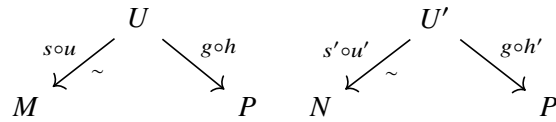
Therefore from (LC4), there exists an object  $Q$  and morphisms  $q : Q \rightarrow R$ , with  $q \in S$ , such that  $h \circ a \circ r \circ q = h' \circ a' \circ r' \circ q$ . We consider now the morphisms  $b := a \circ r \circ q : Q \rightarrow U$  and  $b' := a' \circ r' \circ q : Q \rightarrow U'$ , and then we notice that

- $s \circ u \circ b = s \circ u \circ a \circ r \circ q = s' \circ u' \circ a' \circ r' \circ q = s' \circ u' \circ b' \in S$
- $g \circ h \circ b = g \circ h' \circ b'$

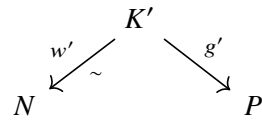
Consequently we have the commutative diagram



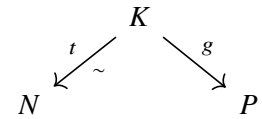
from which it follows that the right roofs



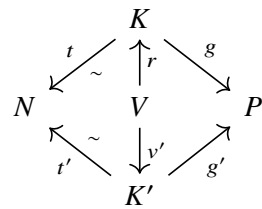
are equivalent. It remains to prove the statement for the class of the left roof on the right. In this direction, consider the left roof



which is equivalent to the initial left roof on the right



By the definition of equivalence, there exist an object  $V$  and morphisms  $v : V \rightarrow K$  and  $v' : V \rightarrow K'$ , such that  $t \circ v = t' \circ v' \in S$ , i.e. the following diagram is commutative



By applying (LC3.a) to the diagram

$$\begin{array}{ccc}
 & V & \\
 & \sim \downarrow t \circ v & \\
 L & \xrightarrow{f} & N
 \end{array}$$

there exist an object  $U$  and morphisms  $u : U \rightarrow L \in S$  and  $a : U \rightarrow V$ , such that the diagram

$$\begin{array}{ccc}
 U & \xrightarrow{a} & V \\
 u \downarrow \sim & & \sim \downarrow t \circ v \\
 L & \xrightarrow{f} & N
 \end{array}$$

is commutative. Hence we have the diagram

$$\begin{array}{ccccc}
 & & U & & \\
 & & \swarrow & \searrow & \\
 & & u & r \circ a & \\
 & & \downarrow \sim & & \\
 & & L & & K \\
 & \swarrow & & \searrow & \\
 & s & & t & g \\
 & \downarrow \sim & & \downarrow \sim & \\
 M & & N & & P
 \end{array}$$

which implies that the composition of the left roofs

$$\begin{array}{ccc}
 & L & \\
 & \swarrow & \searrow \\
 & s & f \\
 & \downarrow \sim & \\
 M & & N
 \end{array}
 \quad
 \begin{array}{ccc}
 & K & \\
 & \swarrow & \searrow \\
 & t & g \\
 & \downarrow \sim & \\
 N & & P
 \end{array}$$

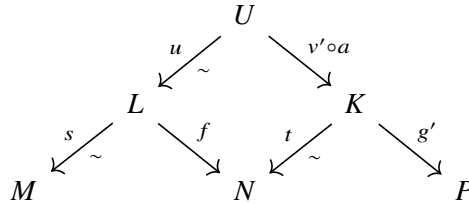
is the left roof

$$\begin{array}{ccc}
 & U & \\
 & \swarrow & \searrow \\
 & s \circ u & g \circ v \circ a \\
 & \downarrow \sim & \\
 M & & P
 \end{array}$$

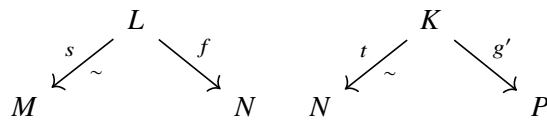
Similarly, by applying (LC3.a) to the diagram

$$\begin{array}{ccc}
 & V & \\
 & \downarrow v' \circ t' & \\
 L & \xrightarrow{f} & K'
 \end{array}$$

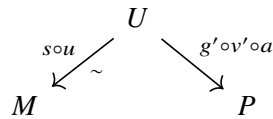
we have the following commutative diagram



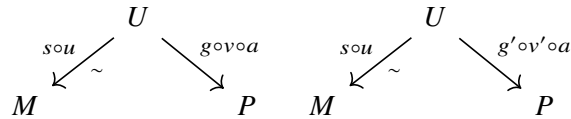
Hence the composition of the left roofs



is the left roof



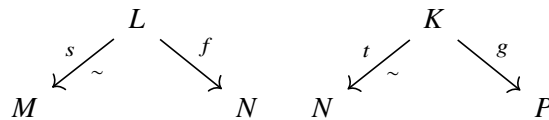
Since  $g \circ v \circ a = g' \circ v' \circ a$ , the above left roofs, i.e. the compositions



are the same, which completes the proof. ■

Dually, we have the composition of two right roofs, and it can be proved that it is well defined.

**Definition 1.3.18.** *Let*

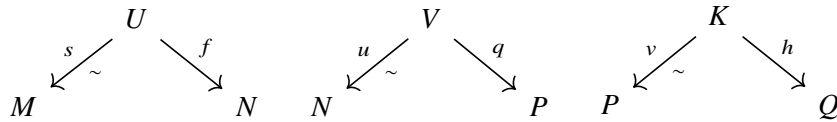


*be two left roofs. We define the composition of the above left roofs, to be the map from the cartesian product of the two equivalence classes of the left roofs, to the equivalence class of the composition.*

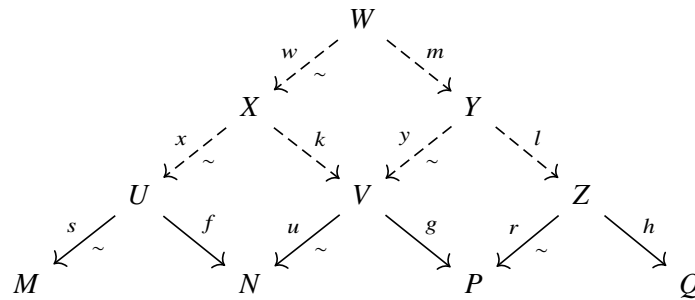
**Remark 1.3.19.** *From the Proposition 1.3.17, the composition which is defined in the above Definition 1.3.18 is well defined.*

**Proposition 1.3.20.** *The composition of the equivalence classes of left roofs is associative.*

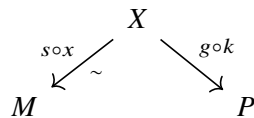
*Proof.* Let



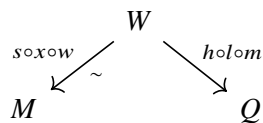
be representatives of left roofs. By applying (LC3.a) to the above left roofs, we have the commutative diagram



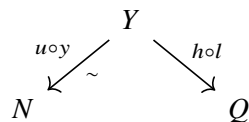
The composition of the first and the second left roof is the left roof



and then its composition with the third left roof is the left roof



On the other hand, the composition of the second and the third left roof is the left roof



and then its composition with the first left roof is the left roof

$$\begin{array}{ccc}
 & W & \\
 \text{soxow} \swarrow & & \searrow \text{holom} \\
 M & \sim & Q
 \end{array}$$

This completes the proof of the proposition, since the two compositions produce the same left roofs. ■

**Definition 1.3.21.** We define the **identity morphism** of an object  $M \in \mathcal{C}$ , the equivalence class of the left roof

$$\begin{array}{ccc}
 & M & \\
 \parallel \swarrow & & \searrow \parallel \\
 M & \sim & M
 \end{array}$$

and the above left roof is denoted by  $1_M$ .

**Definition 1.3.22.** We define the category whose objects are the objects of  $\mathcal{A}$ , which we denote by  $\mathcal{C}_S^l$  and the set of morphisms between two objects  $M$  and  $N$  is the equivalence class of all the left roofs between  $M$  and  $N$ . That is

- $\text{Ob}(\mathcal{C}_S^l) = \text{Ob}(\mathcal{C})$ ,
- $\text{Hom}_{\mathcal{C}_S^l}(M, N) = \{[s, f] \mid \text{where } (s, f) \text{ is a left roof between } X \text{ and } Y \}$ ,
- $1_M$  in  $\mathcal{C}_S^l$  is defined as  $1_M := [1_M/1_M]$ .

We define the category  $\mathcal{C}_S^r$  dually.

From the above discussion the following Lemma is straightforward.

**Lemma 1.3.23.** The categories  $\mathcal{C}_S^l$  and  $\mathcal{C}_S^r$  are isomorphic.

*Proof.* The proof follows from the above analysis. ■

**Remark 1.3.24.** From the above Lemma 1.3.23, we will use the symbol  $\mathcal{C}_S$  to denote either of the above two categories. An isomorphism of the categories  $\mathcal{C}_S^l$  and  $\mathcal{C}_S^r$  can be defined as follows:

- $\text{Ob}(\mathcal{C}_S^l) \rightarrow \text{Ob}(\mathcal{C}_S^r), M \mapsto M$ ,
- $\text{Hom}_{\mathcal{C}_S^l}(M, N) \rightarrow \text{Hom}_{\mathcal{C}_S^r}(M, N), [s, f] \mapsto [f, s]$ .

We proceed by defining a functor from  $\mathcal{C}$  to  $\mathcal{C}_S$  as such:

$$(1) \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C}_S), M \mapsto M$$

$$(2) \text{Hom}_{\mathcal{C}}(M, N) \rightarrow \text{Hom}_{\mathcal{C}_S}(M, N), f : M \rightarrow N \mapsto [1_M, f]$$

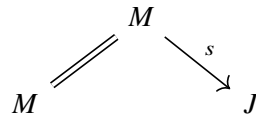
**Lemma 1.3.25.** *The map  $Q : \mathcal{C} \rightarrow \mathcal{C}_S$  which is defined above, is indeed a functor.*

*Proof.* For the proof we refer the reader to [12, Chapter 1, Section 1, Paragraph 3, p. 18]. ■

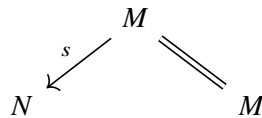
This functor  $Q$  will serve as the localization functor of  $\mathcal{C}_S$ ,  $Q$  of  $\mathcal{C}$ , which is what we are aiming to prove.

**Lemma 1.3.26.** *Let  $s : M \rightarrow N \in S$ . Then  $Q(s)$  is an isomorphism in  $\mathcal{C}_S$ .*

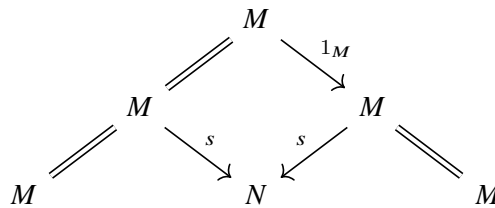
*Proof.* We compose the representative of  $Q(s)$



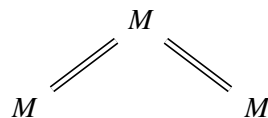
from the right, with the equivalence class of the left roof



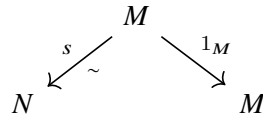
and we end up with the following commutative diagram



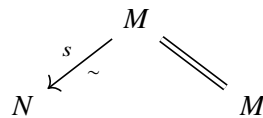
Therefore the composition is given by the equivalence class of the left roof



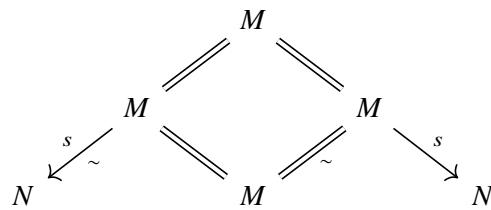
Hence, the equivalence class of the left roof



is a right inverse of  $Q(s)$ . On the other hand, by composing from the right with the equivalence class of the left roof



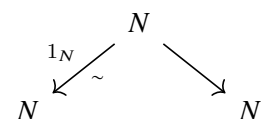
we obtain the commutative diagram



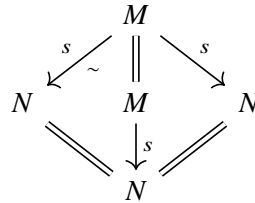
Therefore the composition is given by the equivalence class of the left roof



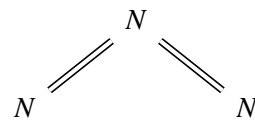
However, the last left roof is equivalent to the left roof



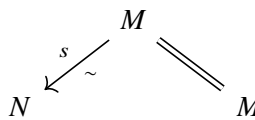
from the commutative diagram



This implies that the composition is given by the equivalence class of the left roof



hence the equivalence class of the left roof



is a right inverse of  $Q(s)$ . ■

We are now in a position to construct the localization  $\mathcal{C}[S^{-1}]$  of a category  $\mathcal{C}$  with respect to a localizing class  $S$ . Essentially, we will show that the category  $\mathcal{C}_S$  along with the functor  $Q : \mathcal{C} \rightarrow \mathcal{C}_S$  satisfy the axioms of Definition 1.3.1. Before we proceed with prove an auxiliary lemma.

**Lemma 1.3.27.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $S$  a localizing class in  $\mathcal{C}$  such that  $F(s)$  is an isomorphism in  $\mathcal{D}$ , for every  $s \in S$ . Then the map*

$$\left[ \begin{array}{ccc} & L & \\ \swarrow s & & \searrow f \\ M & & N \end{array} \right] \mapsto F([s, f]) = F(f) \circ F(s)^{-1} : F(M) \rightarrow F(N)$$

is constant on equivalence classes.

*Proof.* Let



be two left roofs which are equivalent. Then there exist  $U \in \mathcal{C}$  and morphisms  $u : U \rightarrow L$  and  $v : U \rightarrow K$ , where  $s \circ u = t \circ v \in S$ , such that the following diagram is commutative

$$\begin{array}{ccccc}
 & & L & & \\
 & s \swarrow & \uparrow u & \searrow f & \\
 M & & U & & N \\
 & t \swarrow & \downarrow v & \searrow g & \\
 & & K & & 
 \end{array}$$

Since  $s \circ u \in S$ , then  $F(s \circ u)$  is an isomorphism, and because  $F(s)$  is an isomorphism, we also have that  $F(u)$  is an isomorphism. Similarly,  $F(v)$  is also an isomorphism. Therefore we have that

$$s \circ u = t \circ v \implies F(s \circ u) = F(t \circ v) \implies F(s) \circ F(u) = F(t) \circ F(v)$$

and

$$f \circ u = g \circ v \implies F(f \circ u) = F(g \circ v) \implies F(f) \circ F(u) = F(g) \circ F(v)$$

However, we have that  $(F(s) \circ F(u))^{-1} = (F(t) \circ F(v))^{-1} \implies F(u)^{-1} \circ F(s)^{-1} = F(v)^{-1} \circ F(t)^{-1}$ . Hence we have

$$F(f) \circ F(s)^{-1} = F(f) \circ F(f) \circ F(u)^{-1} \circ F(s)^{-1} = F(g) \circ F(v) \circ F(v)^{-1} \circ F(t)^{-1} = F(g) \circ F(t)^{-1}$$

or

$$F(f) \circ F(s)^{-1} = F(g) \circ F(t)^{-1}$$

which proves the Lemma. ■

We are now able to prove the main theorem of this section

**Theorem 1.3.28.** *The pair  $(\mathcal{C}_S, Q)$ , just like it was defined in this section, is the localization of the category  $\mathcal{C}$  with respect to the localizing class  $S$ .*

*Proof.* For a start, we will prove that the diagram

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
 Q \downarrow & \nearrow G & \\
 \mathcal{C}_S & & 
 \end{array}$$

is commutative and that the pair  $(\mathcal{C}_S, Q)$  satisfies the universal property of the localization. Let  $M \in \mathcal{C}_S$ . Then  $(G \circ Q)(M) = G(Q(M)) = G(M) = F(M)$ . Let now  $f : M \rightarrow N$  be a morphism in  $\mathcal{C}$ .

Then we have  $G(Q(f)) = G \left( \left[ \begin{array}{ccc} & M & \\ 1_M \swarrow \sim & & \searrow f \\ M & & N \end{array} \right] \right) = F(f) \circ F(1_M)^{-1} = F(f) \circ 1_{F(M)} = F(f)$ .

Hence  $G \circ Q = F$ . It is left to prove that the pair  $(\mathcal{C}_S, Q)$  satisfies the universal property of the localization. Let  $H : \mathcal{C}_S \rightarrow \mathcal{D}$  be a functor such that  $H \circ Q = F$ . Then  $H(M) = F(M) = G(M)$ , for every  $M \in \mathcal{C}_S$ . Now, let  $\phi : M \rightarrow N$  be a morphism in  $\mathcal{C}_S$  which is represented by the left roof

$$\begin{array}{ccc} & L & \\ s \swarrow \sim & & \searrow f \\ M & & N \end{array}$$

Then we have that  $\phi = Q(f) \circ Q(s)^{-1}$  and we compute

$$H(\phi) = H(Q(f) \circ Q(s)^{-1}) = H(Q(f)) \circ H(Q(s)^{-1}) = H(Q(f)) \circ H(Q(s))^{-1} = F(f) \circ F(s)^{-1} = G(Q(f)) \circ G(Q(s))^{-1} = G(\phi)$$

Therefore  $G = H$ , which proves the theorem. ■

With a slight change of notation, from the last Theorem 1.3.28 we prove that  $\mathcal{C}_S = \mathcal{C}[S^{-1}]$ .

**Lemma 1.3.29.** *Let*

$$\begin{array}{ccc} & L_i & \\ s_i \swarrow \sim & & \searrow f_i \\ M & & N \end{array}$$

*be left roofs which represent the morphisms  $\phi_i : M \rightarrow N$ , for  $1 \leq i \leq n$  in  $\mathcal{C}[S^{-1}]$ . Then there exist an object  $L \in \mathcal{C}$ , a morphism  $s \in S$  and morphisms  $g_i : L \rightarrow N$  in  $\mathcal{C}$ , such that the left roofs*

$$\begin{array}{ccc} & L & \\ s \swarrow \sim & & \searrow g_i \\ M & & N \end{array}$$

*represent the morphisms  $\phi_i$ , for every  $1 \leq i \leq n$ .*

*Proof.* We will prove this Lemma by induction on  $n$ , which is the number of the left roofs.

$n = 1$ : Trivial.

Induction hypothesis: Suppose that  $n > 1$  and that there exist an object  $K \in \mathcal{C}$ , a morphism  $s \in S$  and morphisms  $h_i : K \rightarrow N$ , where  $1 \leq i \leq n - 1$ , such that

$$\begin{array}{ccc} & K & \\ t \swarrow & & \searrow h_i \\ M & & N \end{array}$$

$\sim$

which represent the morphisms  $\phi_i$ , for every  $1 \leq i \leq n - 1$ .

Inductive step: From (LC3.a) there exist an object  $U$  and morphisms  $u : U \rightarrow K \in S$  and  $u' : U \rightarrow L_u$  such that the following diagram is commutative

$$\begin{array}{ccc} U & \xrightarrow{u'} & L_u \\ u \downarrow \sim & & \sim \downarrow s_u \\ K & \xrightarrow{t} & M \end{array}$$

Let  $s := t \circ u = s_u \circ u' : U \rightarrow M \in S$ . Then the diagram

$$\begin{array}{ccc} & K & \\ t \swarrow & \uparrow u & \searrow h_i \\ M & U & N \\ s \swarrow & \parallel & \nearrow h_i \circ u \\ & U & \end{array}$$

is commutative, hence the left roofs

$$\begin{array}{ccc} & K & \\ t \swarrow & & \searrow h_i \\ M & & N \end{array}$$

$\sim$

are represented by the left roofs

$$\begin{array}{ccc} & U & \\ s \swarrow & & \searrow h_i \circ u \\ M & & N \end{array}$$

$\sim$

that is, from the morphisms  $\phi_i$ , for every  $1 \leq i \leq n - 1$ . In addition, from the commutative diagram

$$\begin{array}{ccccc}
 & & L_u & & \\
 & s_n \swarrow & \uparrow u' & \searrow h_n & \\
 M & & U & & N \\
 & \swarrow \sim & \parallel & \searrow f_n \circ u' & \\
 & s & U & & 
 \end{array}$$

it follows that the left roof

$$\begin{array}{ccc}
 & U & \\
 s \swarrow & & \searrow f_n \circ u' \\
 M & & N
 \end{array}$$

is represented by the morphism  $\phi_n$ . Therefore by setting  $L := U$ , and  $g_i := h_i$ , for  $1 \leq i \leq n - 1$  and  $g_n := f_n \circ u'$  we have proved the claim. ■

**Remark 1.3.30.** One can think the above Lemma 1.3.29 as being the reduction of fractions to a common denominator but with left roofs. From the duality of left and right roofs, the above Lemma is also true for right roofs.

### 1.3.3 Localization of subcategories

One might naturally wonder if there exists a relation between the localization of a subcategory  $\mathcal{D}$  of a given category  $\mathcal{C}$ , with the localization of the category  $\mathcal{C}$ . In this direction, let  $\mathcal{C}$  be a category and  $\mathcal{D}$  a full subcategory of  $\mathcal{C}$ . Let also  $S$  be a localizing class of  $\mathcal{C}$  and we suppose that  $S_{\mathcal{D}} := S \cap \text{Hom}_{\mathcal{D}}$  is a localizing class of  $\mathcal{D}$ . Then, we can define a functor  $F : \mathcal{D}[S_{\mathcal{D}}^{-1}] \rightarrow \mathcal{C}[S^{-1}]$  as follows:

- $\text{Ob}(\mathcal{D}[S_{\mathcal{D}}^{-1}]) \rightarrow \text{Ob}(\mathcal{C}[S^{-1}])$ , where  $M \mapsto M$

- $\text{Hom}_{\mathcal{D}[S_{\mathcal{D}}^{-1}]}(M, N) \rightarrow \text{Hom}_{\mathcal{C}[S^{-1}]}(M, N)$ , where  $\left[ \begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \right] \mapsto \left[ \begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \right]$

**Proposition 1.3.31.** Let  $\mathcal{C}$  be a category and  $\mathcal{D}$  a full subcategory of  $\mathcal{C}$ . Let also  $S$  be a localizing class of  $\mathcal{C}$  and suppose that:

- (1)  $S_{\mathcal{D}} = S \cap \text{Hom}_{\mathcal{D}}$  is a localizing class of  $\mathcal{D}$

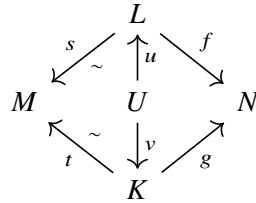
(2) for every morphism  $s : N \rightarrow M$ , with  $s \in S$  and  $M \in \mathcal{D}$ , there exist a morphism  $u : P \rightarrow N$ , such that  $s \circ u \in S$  and  $P \in \mathcal{D}$

Then the functor  $F : \mathcal{D}[S_{\mathcal{D}}^{-1}] \rightarrow \mathcal{C}[S^{-1}]$  is fully faithful.

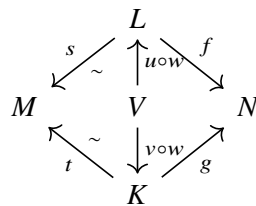
*Proof.* Let  $M, N \in \mathcal{D}$ . We need to prove that the map  $\text{Hom}_{\mathcal{D}[S_{\mathcal{D}}^{-1}]}(M, N) \rightarrow \text{Hom}_{\mathcal{C}[S^{-1}]}(M, N)$  is a bijection. We first prove that is one to one. Let



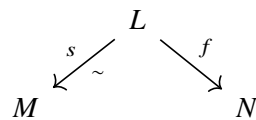
be two left roofs that represent two morphisms in  $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ . From the above functor, they also represent two right roofs in  $\mathcal{C}[S^{-1}]$ . Then, the two right roofs are equivalent, hence there exist  $U \in \mathcal{C}$  and morphisms  $u : U \rightarrow L$  and  $v : U \rightarrow K$ , such that  $s \circ u = t \circ v \in S$  and the diagram



is commutative. From the second part of our hypothesis, there exist  $V \in \mathcal{D}$  and morphism  $w : V \rightarrow U$  such that  $s \circ u \circ w = t \circ v \circ w \in S$ . Therefore we have the commutative diagram



From the last diagram it follows that the left roofs that we started with are equivalent, hence they represent the same morphism in  $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ . To prove that it is onto, let



be a left roof which represents the morphism  $\phi$  in  $\text{Hom}_{\mathcal{C}[S^{-1}]}(M, N)$ . From the second part of our assumption, there exist object  $U \in \mathcal{D}$  and morphism  $u : U \rightarrow L \in S$  such that  $s \circ u \in S$ . Therefore we have the commutative diagram

$$\begin{array}{ccccc}
 & & L & & \\
 & s \swarrow & \uparrow u & \searrow f & \\
 M & & U & & N \\
 & \swarrow \sim & \parallel & \nearrow f \circ u & \\
 & sou \swarrow & U & \nearrow f \circ u & \\
 & & & & 
 \end{array}$$

Hence, the two left roofs

$$\begin{array}{ccc}
 & L & \\
 s \swarrow & & \searrow f \\
 M & & N
 \end{array}
 \quad
 \begin{array}{ccc}
 & U & \\
 sou \swarrow & & \searrow f \circ u \\
 M & & N
 \end{array}$$

are equivalent, which means that they represent the same morphism  $\phi$ . In addition the second left roof determines a morphism from  $M$  to  $N$  in  $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ , whose image from the map  $\text{Hom}_{\mathcal{D}[S_{\mathcal{D}}^{-1}]}(M, N) \rightarrow \text{Hom}_{\mathcal{C}[S^{-1}]}(M, N)$  is  $\phi$ . This means that this map is onto. ■

**Remark 1.3.32.** Under the assumptions of the Proposition 1.3.31, we may regard  $\mathcal{D}[S_{\mathcal{D}}^{-1}]$  as a full subcategory of  $\mathcal{C}[S^{-1}]$ .

### 1.3.4 Localization of abelian categories

It is natural to turn our interest in the localization of additive and abelian categories. Our main focus is to prove that the localization of an additive category  $\mathcal{A}$  with respect to a localizing class  $S$  has an additive, respectively abelian, structure and the localization functor is additive, respectively exact. We fix an additive category  $\mathcal{A}$  and a localizing class  $S$  in  $\mathcal{A}$ . Note that (LC4) can be replaced with the following:

(LC4') Let  $f : M \rightarrow N$  be a morphism in  $\mathcal{A}$ . Then there exists  $s \in S$  such that  $s \circ f = 0$  if and only if there exists  $t \in S$  such that  $f \circ t = 0$ .

**Lemma 1.3.33.** Let  $M, N \in \mathcal{A}$  and  $\phi, \psi \in \text{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ . Then, there exists an object  $L$  and morphisms  $s \in S$  and  $f, g : L \rightarrow M$ , such that these morphisms are represented by the left roofs

$$\begin{array}{ccc}
 & L & \\
 s \swarrow & & \searrow f \\
 M & & N
 \end{array}
 \quad
 \begin{array}{ccc}
 & L & \\
 s \swarrow & & \searrow g \\
 M & & N
 \end{array}$$

Then the morphism  $M \rightarrow N$ , which is determined by the left roof

$$\begin{array}{ccc}
 & L & \\
 s \swarrow & & \searrow f+g \\
 M & \underset{\sim}{\downarrow} & N
 \end{array}$$

depends only from the morphisms  $\phi$  and  $\psi$ , i.e., it is independent of  $L$ , and their representatives  $s$ ,  $f$  and  $g$ .

*Proof.* For the proof we refer to [12, Lemma 2.1.1]. ■

The above Lemma 1.3.33 implies that we have a well defined binary operation in  $\mathcal{A}[S^{-1}]$  which allows us to add two morphisms. In particular, let  $M, N \in \mathcal{A}$  and  $\phi, \psi : M \rightarrow N$  be two morphisms in  $\mathcal{A}[S^{-1}]$ . From the above Lemma there exist  $L \in \mathcal{A}$  and morphisms  $f, g : L \rightarrow N$  and  $s : L \rightarrow M$ , where  $s \in S$ , such that the morphisms  $\phi$  and  $\psi$  are represented by the left roofs

$$\begin{array}{ccc}
 & L & \\
 s \swarrow & & \searrow f \\
 M & \underset{\sim}{\downarrow} & N
 \end{array}
 \quad
 \begin{array}{ccc}
 & L & \\
 s \swarrow & & \searrow g \\
 M & \underset{\sim}{\downarrow} & N
 \end{array}$$

Then we define the bilinear operation in the equivalence class of left roofs as follows:

$$\begin{aligned}
 \text{Hom}_{\mathcal{A}[S^{-1}]}(M, N) \times \text{Hom}_{\mathcal{A}[S^{-1}]}(M, N) &\rightarrow \text{Hom}_{\mathcal{A}[S^{-1}]}(M, N) \\
 (\phi, \psi) &\mapsto \phi + \psi
 \end{aligned}$$

where  $\phi + \psi$  is the morphism represented by the left roof

$$\begin{array}{ccc}
 & L & \\
 s \swarrow & & \searrow f+g \\
 M & \underset{\sim}{\downarrow} & N
 \end{array}$$

Note that if we denote by 0 the morphism in  $\mathcal{A}[S^{-1}]$  represented by the left roof

$$\begin{array}{ccc}
 & M & \\
 \parallel \swarrow & & \searrow 0 \\
 M & \underset{\sim}{\downarrow} & N
 \end{array}$$

then 0 is the identity in  $\text{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ . In addition if  $\phi$  is a morphism in  $\text{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$  represented by the left roof

$$\begin{array}{ccc} & L & \\ \swarrow & & \searrow \\ M & & N \end{array}$$

(Note: The arrow from L to M is labeled with a tilde ~, and the arrow from L to N is labeled with f.)

then the left roof

$$\begin{array}{ccc} & L & \\ \swarrow & & \searrow \\ M & & N \end{array}$$

(Note: The arrow from L to M is labeled with a tilde ~, and the arrow from L to N is labeled with -f.)

is the opposite of  $\phi$ . Therefore the set  $\mathcal{A}[S^{-1}](M, N)$  has the structure of an abelian group. The zero object of  $\mathcal{A}$  coincides with the zero object of  $\mathcal{A}[S^{-1}]$ . Lastly by defining the direct sum of two objects  $M, N \in \mathcal{A}[S^{-1}]$  to be the direct sum  $M \oplus N$  in  $\mathcal{A}$ , we have the following Proposition.

**Proposition 1.3.34.** *Let  $\mathcal{A}$  be an additive category and  $S$  a localizing class in  $\mathcal{A}$ . Then the localization  $\mathcal{A}[S^{-1}]$  is additive.*

*Proof.* The sketch of the proof is given from the above discussion. For more details we refer to [12, pp. 28-34] ■

Therefore, from the above, we have the following Theorem.

**Theorem 1.3.35.** *Let  $\mathcal{A}$  be an additive category and  $S$  a localizing class in  $\mathcal{A}$ . There exist an additive category  $\mathcal{A}[S^{-1}]$  and an additive functor  $Q: \mathcal{A} \rightarrow \mathcal{A}[S^{-1}]$  such that:*

- (1)  $Q(s)$  is an isomorphism for every  $s \in S$ .
- (2) For any additive category  $\mathcal{B}$  and additive functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  such that  $F(s)$  is an isomorphism for every  $s \in S$ , there exists a unique additive functor  $G: \mathcal{A}[S^{-1}] \rightarrow \mathcal{B}$  such that  $F = G \circ Q$ , that is, the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ Q \downarrow & \nearrow G & \\ \mathcal{A}[S^{-1}] & & \end{array}$$

In addition, the category  $\mathcal{A}[S^{-1}]$  is unique up to isomorphism.

*Proof.* From Proposition 1.3.34 and the discussion before that, it is enough to show the existence of the functor  $Q: \mathcal{A} \rightarrow \mathcal{A}[S^{-1}]$  and the universal property holds. Let  $f, g: M \rightarrow N$  be two morphisms

in  $\mathcal{A}$ . Then the morphisms  $Q(f), Q(g)$  in  $\mathcal{A}[S^{-1}]$  are represented by the left roofs

$$\begin{array}{ccc} & M & \\ // & \searrow f & \\ M & & N \end{array} \quad \begin{array}{ccc} & M & \\ // & \searrow g & \\ M & & N \end{array}$$

Hence the morphism  $Q(f) + Q(g)$  is represented by the left roof

$$\begin{array}{ccc} & M & \\ // & \searrow f+g & \\ M & & N \end{array}$$

This implies that  $Q(f) + Q(g) = Q(f + g)$ , which means that the localization functor  $Q : \mathcal{A} \rightarrow \mathcal{A}[S^{-1}]$  is additive. For the universal property, let  $\mathcal{B}$  be an abelian category and  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor such that  $F(s)$  is an isomorphism for every  $s \in S$ . Then from Theorem 1.3.2 there exists a unique functor  $G : \mathcal{A}[S^{-1}] \rightarrow \mathcal{B}$  such that  $F = G \circ Q$ . We have to prove that it is additive. For every  $M \in \mathcal{A}$  we have that  $G(M) = F(M)$ . In addition, for every morphism  $\phi : M \rightarrow N$  which is represented by the left roof

$$\begin{array}{ccc} & L & \\ & \swarrow s \searrow f & \\ M & & N \end{array}$$

we have that  $G(\phi) = F(f) \circ F(s)^{-1}$ . Now, let  $\phi, \psi : M \rightarrow N$  be two morphisms in  $\mathcal{A}[S^{-1}]$ . From Lemma 1.3.29 they are represented by the left roofs

$$\begin{array}{ccc} & L & \\ & \swarrow s \searrow f & \\ M & & N \end{array} \quad \begin{array}{ccc} & L & \\ & \swarrow s \searrow f & \\ M & & N \end{array}$$

Then the morphism  $\phi + \psi$  is represented by the left roof

$$\begin{array}{ccc} & L & \\ & \swarrow s \searrow f+g & \\ M & & N \end{array}$$

Therefore we have that

$$G(\phi + \psi) = F(f + g) \circ F(s)^{-1} = F(f) \circ F(s)^{-1} + F(g) \circ F(s)^{-1} = G(\phi) + G(\psi)$$

which means that  $G$  is additive. The proof of the uniqueness is identical to that of Theorem 1.3.2, therefore it is omitted. ■

The following two Lemmas provide a characterization of zero morphisms in localizations.

**Lemma 1.3.36.** *Let  $\phi : M \rightarrow N$  be a morphism in  $\mathcal{A}[S^{-1}]$  represented by the left roof*

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array}$$

$\sim$

Then the following are equivalent:

- (1)  $\phi = 0$ .
- (2) There exists  $t \in S$  such that  $f \circ t = 0$ .
- (3) There exists  $t \in S$  such that  $t \circ f = 0$ .

*Proof.* We will only show that (1)  $\iff$  (2) since (1)  $\iff$  (3) is dual.

(1)  $\implies$  (2): Suppose that  $\phi = 0$ . Then  $0 = Q(f) \circ Q(s)^{-1}$  and  $Q(f) = 0$ . Therefore the left roof

$$\begin{array}{ccc} & L & \\ 1_L \swarrow & & \searrow f \\ L & & N \end{array}$$

$\sim$

represents the zero morphism in  $\text{Hom}_{\mathcal{A}[S^{-1}]}(L, N)$ . The zero morphism between  $L$  and  $N$  is represented by the left roof

$$\begin{array}{ccc} & L & \\ \parallel \swarrow & & \searrow 0 \\ L & & N \end{array}$$

$\sim$

Hence, the above two roofs are equivalent, which means that there exist an object  $U \in \mathcal{A}$  and a morphism  $t : U \rightarrow L$  such that the diagram

$$\begin{array}{ccccc} & & L & & \\ & & \uparrow & \searrow f & \\ & & U & & N \\ & & \downarrow t & \nearrow 0 & \\ L & & L & & \end{array}$$

$\sim$

is commutative. This implies that  $f \circ t = 0$ .

(2)  $\implies$  (1): Suppose that there exists  $t \in S$  such that  $t \circ f = 0$ . Then we have that  $Q(f) \circ Q(t) = 0$ . This implies that  $Q(f) = 0$  and  $\phi = Q(f) \circ Q(s)^{-1} = 0$ .  $\blacksquare$

We focus now on abelian categories, so let  $\mathcal{A}$  be an abelian category and  $S$  be a localizing class of  $\mathcal{A}$ . In the rest of this section, we will show that the localization  $\mathcal{A}[S^{-1}]$  is also an abelian category. We first prove that any morphism in  $\mathcal{A}[S^{-1}]$  has kernel and cokernel.

**Lemma 1.3.37.** *Let  $\phi : M \rightarrow N$  be a morphism in  $\mathcal{A}[S^{-1}]$ . Then  $\phi$  has a kernel and a cokernel.*

*Proof.* Let

$$\begin{array}{ccc} & L & \\ f \nearrow & & \nwarrow s \\ M & & N \end{array}$$

be a representative of  $\phi$ . Then we have  $\phi = Q(s)^{-1} \circ Q(f)$ . Since  $Q(s)^{-1}$  is an isomorphism, then  $x : K \rightarrow M$  is a kernel in  $\mathcal{A}[S^{-1}]$  if and only if  $x$  is a kernel of  $Q(f)$  in  $\mathcal{A}[S^{-1}]$ . Since  $\mathcal{A}$  is abelian, let  $k : K \rightarrow M$  be the kernel of  $f$ . We claim that  $x := Q(k) : K \rightarrow M$  is a kernel of  $Q(f)$ . Let  $\psi : P \rightarrow M$  be a morphism in  $\mathcal{A}[S^{-1}]$  such that  $Q(f) \circ \psi = 0$ . If

$$\begin{array}{ccc} & U & \\ t \swarrow & & \searrow g \\ P & & M \end{array}$$

is a representative of  $\psi$ , then we have that  $\psi = Q(g) \circ Q(t)^{-1}$ . We compute

$$0 = Q(f) \circ \psi = Q(f) \circ Q(g) \circ Q(t)^{-1} = Q(f \circ g) \circ Q(t)^{-1}$$

and since  $Q(t)$  is an isomorphism, it follows that  $Q(f \circ g) = 0$ . Hence, from Lemma 1.3.29, there exists a morphism  $v : V \rightarrow U \in S$ , such that  $f \circ g \circ v = 0$ . This means that  $g \circ v$  factors uniquely through the kernel  $k : K \rightarrow M$  of  $f$ , i.e. there exists  $w : W \rightarrow K$  such that  $k \circ w = g \circ v$ . Therefore

$$Q(k \circ w) = Q(g \circ v) \implies Q(k) \circ Q(w) = Q(g) \circ Q(v) \implies Q(g) = Q(k) \circ Q(w) \circ Q(v)^{-1}$$

Consequently we have that

$$\psi = Q(g) \circ Q(t)^{-1} = Q(k) \circ Q(w) \circ Q(v)^{-1} \circ Q(t)^{-1} = x \circ Q(w) \circ Q(v)^{-1} \circ Q(t)^{-1}$$

which implies that  $\psi$  factors through  $K$ . It remains to show that this factorization is unique. Let  $\psi = x \circ \alpha = x \circ \beta$  be two factorizations of  $\psi$  through  $K$ . Then  $x \circ (\alpha - \beta) = 0$ . Since  $k$  is a monomorphism, then  $x$  is also a monomorphism since  $Q$  is additive, thus  $\alpha - \beta = 0 \Rightarrow \alpha = \beta$ . Dually, in the opposite category of  $\mathcal{A}[S^{-1}]$ , we can prove that  $\psi$  has a cokernel. ■

From Remark 1.2.15, for the morphism  $\phi$  we have the commutative diagram

$$\begin{array}{ccccccc}
 \text{Ker}(\phi) & \xrightarrow{x} & M & \xrightarrow{\phi} & N & \xrightarrow{\rho} & \text{Coker}(\phi) \\
 & & \alpha \downarrow & \nearrow \psi & \uparrow \beta & & \\
 & & \text{Coim}(\phi) & \xrightarrow{\bar{\phi}} & \text{Im}(\phi) & & 
 \end{array}$$

To prove that  $\mathcal{A}[S^{-1}]$  is abelian, it remains to show that  $\bar{\phi}$  is an isomorphism. This is proved in the following Lemma.

**Lemma 1.3.38.** *The morphism  $\bar{\phi}$  is an isomorphism.*

*Proof.* Let

$$\begin{array}{ccc}
 & L & \\
 s \swarrow & & \searrow f \\
 P & \xrightarrow{\sim} & M
 \end{array}$$

be a left roof which is a representative of  $\phi$ . Therefore we have that  $\phi = Q(f) \circ Q(s)^{-1}$ . Since  $\mathcal{A}$  is abelian, from Remark 1.2.15 we have the commutative diagram

$$\begin{array}{ccccccc}
 \text{Ker}(f) & \xrightarrow{Q(k)} & L & \xrightarrow{Q(f)} & M & \xrightarrow{Q(c)} & \text{Coker}(f) \\
 & & Q(\alpha) \downarrow & & \uparrow Q(b) & & \\
 & & \text{Coim}(f) & \xrightarrow{Q(\bar{f})} & \text{Im}(f) & & 
 \end{array}$$

From Lemma 1.3.37 we have that  $Q(k) : \text{Ker}(f) \rightarrow L$  is a kernel of  $Q(f)$ , and  $Q(c) : C \rightarrow \text{Coker}(f)$  is a cokernel of  $Q(f)$ . Therefore  $Q(a) : L \rightarrow \text{Coim}(f)$  is a coimage and  $Q(b) : \text{Im}(f) \rightarrow N$  is an image of  $Q(f)$ . Since  $\phi = Q(f) \circ Q(s)^{-1}$ , we can assume that  $Q(c) : N \rightarrow \text{Coker}(f)$  is a cokernel of  $\phi : M \rightarrow N$ , that is, we can let  $\text{Coker}(\phi) = \text{Coker}(f)$  and  $\rho = Q(c)$ . By using the same argument, we have that  $Q(b) : \text{Im}(f) \rightarrow N$  is a kernel of  $\rho : N \rightarrow \text{Coker}(\phi)$ , that is, we can set  $\text{Im}(\phi) = \text{Im}(f)$  and  $\beta = (Q(b))$ . Finally, since  $\phi = Q(f) \circ Q(s)^{-1}$ , the morphism  $Q(s) \circ Q(k) : \text{Ker}(f) \rightarrow M$  is a kernel of  $\phi$ , thus we can let  $\text{Ker}(\phi) = \text{Ker}(f)$  and  $x = Q(s) \circ Q(k)$ . Analogously, we have that  $Q(\alpha) \circ$

$Q(s)^{-1} : M \rightarrow \text{Coim}(f)$  is a cokernel of  $x$ , thus we can let  $\text{Coim}(\phi) = \text{Coim}(f)$  and  $a = Q(\alpha) \circ Q(s)^{-1}$ . Then we have that

$$\begin{aligned} Q(f) \circ Q(s)^{-1} &= \phi = \beta \circ \bar{\phi} \circ \alpha = Q(b) \circ \bar{\phi} \circ Q(a) \circ Q(s)^{-1} \\ Q(b) \circ \bar{\phi} \circ Q(a) &= Q(f) = Q(b) \circ \bar{f} \circ Q(a) \end{aligned}$$

Because  $Q(b)$  is a monomorphism, we have that  $\bar{\phi} \circ Q(a) = Q(\bar{f}) \circ Q(a)$ . However, since  $Q(a)$  is an epimorphism, it follows that  $\bar{\phi} = Q(\bar{f})$ , which completes the proof. ■

Using the last two Lemmas, Lemma 1.3.37 and Lemma 1.3.38 we may prove one of our main results which is the following

**Theorem 1.3.39.** *Let  $\mathcal{A}$  be an abelian category and  $S$  a localizing class in  $\mathcal{A}$ . Then the localization  $\mathcal{A}[S^{-1}]$  is also an abelian category. In addition, the localization functor  $Q : \mathcal{A} \rightarrow \mathcal{A}[S^{-1}]$  is exact.*

*Proof.* We only need to prove that  $Q$  is exact, since the first part is proved in the previous two Lemmas, Lemma 1.3.37 and Lemma 1.3.38. Let

$$M \xrightarrow{f} N \xrightarrow{g} P$$

be an exact sequence in  $\mathcal{A}$ . We need to show that the sequence

$$M \xrightarrow{Q(f)} N \xrightarrow{Q(g)} P$$

is exact in  $\mathcal{A}[S^{-1}]$ . Clearly, we have that  $Q(g) \circ Q(f) = 0$  since  $Q$  is additive. From the proof of the previous two Lemmas, if  $i : \text{Im}(f) \rightarrow N$  is the image of  $f$ , then  $Q(i) : \text{Im}(f) \rightarrow N$  is the image of  $Q(f)$ . Also, if  $k : \text{Ker}(g) \rightarrow N$  is the kernel of  $g$ , then  $Q(k) : \text{Ker}(g) \rightarrow N$  is the kernel of  $Q(g)$ . Therefore, the exactness of the second sequence comes from the exactness of the first one. ■

In the context of abelian categories, there exists an important class of subcategories of  $\mathcal{A}$  called thick subcategories.

**Definition 1.3.40.** *Let  $\mathcal{A}$  be an abelian category. A non trivial full subcategory  $\mathcal{B}$  of  $\mathcal{A}$  is called **thick**, if for every short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in  $\mathcal{A}$ , we have that  $M \in \mathcal{B}$  if and only if  $M', M'' \in \mathcal{B}$ .

**Remark 1.3.41.** A thick subcategory of an abelian category always contains the zero object due to the following short exact sequence

$$0 \longrightarrow N \xrightarrow{\cong} N \longrightarrow 0 \longrightarrow 0$$

where  $N \in \mathcal{B}$ .

**Lemma 1.3.42.** *Let  $\mathcal{A}$  be an abelian category and  $\mathcal{B}$  a thick subcategory of  $\mathcal{A}$ . Then:*

- (1)  $\mathcal{B}$  is a strictly full subcategory of  $\mathcal{A}$ ,
- (2)  $\mathcal{B}$  is an abelian category,
- (3) any subobject and every quotient-object  $M \in \mathcal{B}$  is in  $\mathcal{B}$ ,
- (4) any extension of any two objects in  $\mathcal{B}$  is in  $\mathcal{B}$ .

*Proof.* (1) Let  $M \in \mathcal{B}$  and  $i : N \rightarrow M$  be an isomorphism. Then

$$0 \longrightarrow N \xrightarrow{\cong} M \longrightarrow 0 \longrightarrow 0$$

is a short exact sequence, hence  $N \in \mathcal{B}$ .

- (2) Let  $M, N \in \mathcal{B}$ . Then we have the short exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0$$

in  $\mathcal{A}$ . Therefore  $M \oplus N$  is in  $\mathcal{B}$ , hence  $\mathcal{B}$  is additive. Let now  $f : M \rightarrow N$  be a morphism in  $\mathcal{B}$ . Then  $f$  is also a morphism in  $\mathcal{A}$ , therefore its kernel, cokernel, image and coimage are in  $\mathcal{A}$ . Since  $\mathcal{B}$  is thick, all of the above objects are also in  $\mathcal{B}$ . In addition, they represent the kernel, cokernel, image, and coimage of  $f$  in  $\mathcal{B}$ . This means that the canonical map of  $f$  in  $\mathcal{A}$  is also the canonical map of  $f$  in  $\mathcal{B}$ , and this proves the second part of the lemma.

- (3) Let  $M \in \mathcal{B}$  and  $M'$  be a subobject of  $M$  in  $\mathcal{A}$ . Then we have the short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in  $\mathcal{A}$ . Since  $\mathcal{B}$  is thick, it follows that  $M'$  and  $M''$  are also in  $\mathcal{B}$ .

- (4) Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence in  $\mathcal{A}$ , where  $M', M'' \in \mathcal{B}$ . Then  $M$  is an extension of  $M'$  by  $M''$ . ■

The following Lemma gives a characterization of thick subcategories.

**Lemma 1.3.43.** *Let  $\mathcal{A}$  be an abelian category and  $S$  a localizing class of  $\mathcal{A}$ . Let also  $\mathcal{B}$  be the full subcategory of  $\mathcal{A}$  whose objects are isomorphic to the zero object in  $\mathcal{A}[S^{-1}]$ , i.e.  $\mathcal{B} = \text{Ker}(Q) = \{B \in \mathcal{A} \mid Q(B) = 0\}$ . Then  $\mathcal{B}$  is thick.*

*Proof.* Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence in  $\mathcal{A}$ . Since the localization functor  $Q : \mathcal{A} \rightarrow \mathcal{A}[S^{-1}]$  is exact, then

$$0 \longrightarrow Q(M') \longrightarrow Q(M) \longrightarrow Q(M'') \longrightarrow 0$$

is a short exact sequence in  $\mathcal{A}[S^{-1}]$ . If now  $M \in \mathcal{B}$ , then  $Q(M) = 0$ . Due to exactness  $M', M'' \in \mathcal{B}$ , hence  $Q(M') = Q(M'') = 0$ . Conversely, if  $M', M'' \in \mathcal{B}$ , we have that  $Q(M') = Q(M'') = 0$ , and again from exactness we have  $Q(M) = 0$ . Therefore  $M \in \mathcal{B}$ , which means that  $\mathcal{B}$  is thick. ■

**Definition 1.3.44.** *Let  $\mathcal{A}$  be an abelian category and  $\mathcal{B}$  a thick subcategory of  $\mathcal{A}$ . We define  $S_{\mathcal{B}}$  to be the class of all morphisms  $f : M \rightarrow N$  in  $\mathcal{A}$  such that  $\text{Ker}(f)$  and  $\text{Coker}(f)$  are in  $\mathcal{B}$ .*

**Lemma 1.3.45.** *The class  $S_{\mathcal{B}}$  defined above, is a localizing class in  $\mathcal{A}$ .*

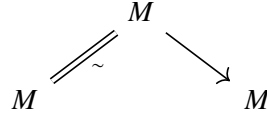
*Proof.* We refer the reader to [12, Lemma 2.2.5] for the proof, since it is long but not that difficult. ■

**Lemma 1.3.46.** *Let  $\mathcal{A}$  be an abelian category and  $\mathcal{B}$  a thick subcategory of  $\mathcal{A}$ . Let also  $S_{\mathcal{B}}$  be the localizing class defined in 1.3.44. Then  $\mathcal{B}$  is the thick subcategory whose objects are the objects of  $\mathcal{A}$  which are isomorphic to the zero object in  $\mathcal{A}[S_{\mathcal{B}}^{-1}]$ .*

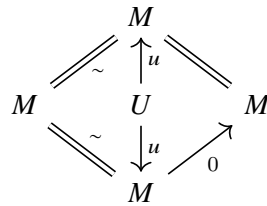
*Proof.* Let  $M \in \mathcal{B}$ . The kernel and cokernel of the morphism  $M \rightarrow 0$ , is  $M$  and  $0$  respectively. Therefore the morphism  $M \rightarrow 0$  is in  $S_{\mathcal{B}}$ , and  $M \cong 0$  in  $\mathcal{A}[S_{\mathcal{B}}^{-1}]$ . On the other hand, suppose that  $M \cong 0$  in  $\mathcal{A}[S_{\mathcal{B}}^{-1}]$ . Then the identity morphism of  $M$ , which has the left roof

$$\begin{array}{ccc} & M & \\ \cong & & \cong \\ M & & M \end{array}$$

has to be isomorphic to the zero morphism, whose representative is the left roof



This means that there exists a morphism  $u : U \rightarrow M$  such that the diagram



is commutative and  $u \in S_{\mathcal{B}}$ . Therefore it follows that  $u$  is the zero morphism, which means that  $\text{Coker}(u) = M$ , hence  $M \in \mathcal{B}$ . ■

**Definition 1.3.47.** We denote the localization  $\mathcal{A}[S_{\mathcal{B}}^{-1}]$  with  $\mathcal{A}/\mathcal{B}$  and we call it the **quotient category** of  $\mathcal{A}$  with respect to the thick subcategory  $\mathcal{B}$ .

By Theorem 1.3.39, the quotient category  $\mathcal{A}/\mathcal{B}$  is abelian and the quotient functor  $Q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$  is exact and is universal for exact functors  $\mathcal{A} \xrightarrow{F} \mathcal{C}$ , where  $\mathcal{C}$  is abelian such that  $F(\mathcal{B}) = 0$ .

**Proposition 1.3.48.** Let  $\mathcal{A}$  be an abelian category and  $\mathcal{B}, \mathcal{C}$  two thick subcategories of  $\mathcal{A}$ . Then:

- (1) the full subcategory  $\mathcal{B} \cap \mathcal{C}$  is a thick subcategory of  $\mathcal{A}$ .
- (2) the natural functor  $\mathcal{B}/\mathcal{B} \cap \mathcal{C} \rightarrow \mathcal{A}/\mathcal{C}$  is fully faithful.

*Proof.* For the proof we refer the reader to [12, Proposition 2.2.6]. ■

## 1.4 Triangulated Categories

Triangulated categories play an important role in modern mathematics. The notion of a triangulated category was introduced independently from Puppe, in [15], and Verdier, in [19]. The first wanted to study the stable homotopy category, and the second the derived category of an abelian category. Just as Verdier showed, a basic example of a triangulated category is the derived category  $D(\mathcal{A})$  of an abelian category  $\mathcal{A}$ . By definition a triangulated category is not necessarily abelian, therefore we can not have a notion of exactness, since the existence of kernels and cokernels can not be guaranteed. Instead of s.e.s, in triangulated categories we have the class of distinguished triangles

which play the role of s.e.s in abelian categories. One can divide this section in two parts. In the first part, we give the definition of a triangulated category and we present some basic results concerning them. Next we present the localization of triangulated categories, where we prove that the localization of a triangulated category with respect to a localizing class, with some extra properties, is also a triangulated category. We conclude the first part by discussing triangulated subcategories of triangulated categories. The second part is about the derived category of an abelian category. We start by giving the definition of the category of complexes and then passing on to the homotopic category of complexes of an abelian category. We close the second part with the definition of the derived category and present some basic results about them. We will closely follow Milicic notes [12].

### 1.4.1 Definition of triangulated categories

In this subsection we begin by giving the definition of a triangulated category. We then present functors between triangulated categories that preserve their triangulated structure, called triangulated functors. We introduce the notion of cohomology in triangulated categories through cohomological functors, thereby generalizing the classical notion of cohomology in abelian categories.

**Definition 1.4.1.** Let  $\mathcal{T}$  be an additive category and  $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$  an additive functor which is an automorphism. The functor  $\Sigma$  is called the **suspension** (translation, shift) **functor** of  $\mathcal{T}$ .

**Notation 1.4.2.** If  $X \in \mathcal{T}$ , we also write  $X[n]$  instead of  $\Sigma^n(X)$ , for  $n \in \mathbb{Z}$ .

**Definition 1.4.3.** Let  $\mathcal{T}$  be an additive category.

(1) A **triangle** in  $\mathcal{T}$  is a diagram of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

A triangle can also be represented by the diagram

$$\begin{array}{ccc} & Z & \\ & \swarrow & \searrow \\ X & & Y \\ & \xrightarrow{\quad} & \end{array}$$

[1]

justifying the name "triangle" for the above diagram. We write [1] to indicate that the morphism is of the form  $Z \rightarrow \Sigma(X)$ .

(2) A **morphism of triangles** is a commutative diagram of the form

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
\downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma(u) \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X')
\end{array}$$

A morphism of triangles is called an **isomorphism of triangles** if the vertical morphisms are isomorphisms.

**Definition 1.4.4.** A category  $\mathcal{T}$  is called a **triangulated category** if it is additive and it is equipped with a suspension functor and a class of triangles called **distinguished triangles** which satisfy the following axioms

(TR1.a) Every triangle isomorphic to a distinguished triangle is a distinguished triangle.

(TR1.b) For every object  $X \in \mathcal{T}$  the triangle

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \Sigma(X)$$

is a distinguished triangle.

(TR1.c) Every morphism  $f : X \rightarrow Y$  in  $\mathcal{T}$  can be embedded in a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

(TR2) The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

is distinguished if and only if the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma(f)} \Sigma(Y)$$

is distinguished.

(TR3) Let

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
u \downarrow & & \downarrow v & & & & \downarrow \Sigma(u) \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X')
\end{array}$$

be a diagram in  $\mathcal{T}$  where the rows are distinguished triangles and the first square is commutative. Then there exists a morphism  $w : Z \rightarrow Z'$  such that the following commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ u \downarrow & & \downarrow v & & \downarrow w & & \downarrow \Sigma(u) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \end{array}$$

is a morphism of triangles.

(TR4) Let  $f : X \rightarrow Y, g : Y \rightarrow Z$  and  $h : X \rightarrow Z$  be morphisms in  $\mathcal{T}$  such that  $h = g \circ f$ . Let also

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\alpha} & Z' & \longrightarrow & \Sigma(X) \\ \parallel & & \downarrow g & & & & \parallel \\ X & \xrightarrow{h} & Z & \xrightarrow{b} & Y' & \longrightarrow & \Sigma(X) \\ \downarrow f & & \parallel & & & & \downarrow \\ Y & \xrightarrow{g} & Z & \xrightarrow{c} & X' & \longrightarrow & \Sigma(Y) \end{array}$$

be a diagram in  $\mathcal{T}$  where the first column is commutative. Then the above diagram can be completed to the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\alpha} & Z' & \longrightarrow & \Sigma(X) \\ \parallel & & \downarrow g & & \downarrow u & & \parallel \\ X & \xrightarrow{h} & Z & \xrightarrow{b} & Y' & \longrightarrow & \Sigma(X) \\ \downarrow f & & \parallel & & \downarrow v & & \downarrow \\ Y & \xrightarrow{g} & Z & \xrightarrow{c} & X' & \longrightarrow & \Sigma(Y) \\ \downarrow \alpha & & \downarrow b & & \parallel & & \downarrow \Sigma(a) \\ Z' & \xrightarrow{u} & Y' & \xrightarrow{v} & X' & \xrightarrow{w} & \Sigma(Z') \end{array}$$

where the cone triangle

$$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} \Sigma(Z')$$

is distinguished

**Remark 1.4.5.** (1) The axiom (TR2) is called the **rotation of triangles** axiom. Because it is an "if and only if" statement, that means we can also rotate the triangle "counterclockwise". Therefore we also have an equivalent of the (TR2) axiom which is the following:  
The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

is distinguished if and only if the triangle

$$\Sigma^{-1}(Z) \xrightarrow{-\Sigma^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z$$

is distinguished.

(2) The axiom (TR4) is also called the **octahedral** axiom. This name comes from the fact that the involved triangles in the diagram can be arranged to form an octahedron in three dimensions.

**Definition 1.4.6.** Let  $\mathcal{T}, \mathcal{T}'$  be two triangulated categories and  $F : \mathcal{T} \rightarrow \mathcal{T}'$  be an additive functor. The functor  $F$  is called **graded** if the functor  $F \circ \Sigma$  is naturally isomorphic to the functor  $\Sigma \circ F$ .

Let  $F : \mathcal{T} \rightarrow \mathcal{T}'$  be a graded functor between triangulated categories and  $\eta : F \circ \Sigma \rightarrow \Sigma \circ F$  be a natural isomorphism between  $F \circ \Sigma$  and  $\Sigma \circ F$ . If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \tag{1.2}$$

is a distinguished triangle in  $\mathcal{T}$ , by applying  $F$  we get the diagram

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(\Sigma(X)) \xrightarrow{\eta_X} \Sigma(F(X)) \tag{1.3}$$

Consequently, we get the triangle

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\eta_{X \circ F(h)}} \Sigma(F(X))$$

Let

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ u \downarrow & & \downarrow v & & \downarrow w & & \downarrow \Sigma(u) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \end{array}$$

be a morphism of triangles in  $\mathcal{T}$ . By applying  $F$  we get the commutative diagram

$$\begin{array}{ccccccc} F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(\Sigma(X)) \xrightarrow{\eta_X} \Sigma(F(X)) \\ F(u) \downarrow & & \downarrow F(v) & & \downarrow F(w) & & \downarrow F(\Sigma(u)) \quad \downarrow \Sigma(F(u)) \\ F(X') & \longrightarrow & F(Y') & \longrightarrow & F(Z') & \longrightarrow & F(\Sigma(X')) \xrightarrow{\eta_{X'}} \Sigma(F(X')) \end{array}$$

and by combining the last two squares we get the morphism of triangles

$$\begin{array}{ccccccc} F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & \Sigma(F(X)) \\ F(u)\downarrow & & \downarrow F(v) & & \downarrow F(w) & & \downarrow \Sigma(F(u)) \\ F(X') & \longrightarrow & F(Y') & \longrightarrow & F(Z') & \longrightarrow & \Sigma(F(X)) \end{array}$$

If the initial morphism of triangles is an isomorphism, then clearly the one induced from  $F$  is also an isomorphism since  $F$  is additive.

**Definition 1.4.7.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two triangulated categories and  $F : \mathcal{T} \rightarrow \mathcal{T}'$  be a graded functor. The functor  $F$  is called **exact**, or in more modern terminology **triangulated functor**, if it maps distinguished triangles to distinguished triangles. Explicitly, if for any distinguished triangle (1.2), the triangle (1.3) is distinguished.

**Definition 1.4.8.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two triangulated categories and  $F, G : \mathcal{T} \rightarrow \mathcal{T}'$  be two exact functors. A natural transformation  $\omega : F \rightarrow G$  of functors is called a **graded morphism** if the diagram

$$\begin{array}{ccc} F(\Sigma(X)) & \xrightarrow{\eta_{F,X}} & \Sigma(F(X)) \\ \omega_{\Sigma(X)}\downarrow & & \downarrow \Sigma(\omega_X) \\ G(\Sigma(X)) & \xrightarrow{\eta_{G,X}} & \Sigma(G(X)) \end{array}$$

is commutative for all  $X \in \mathcal{T}$ .

**Remark 1.4.9.** If we have a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

then by the above Definition 1.4.8, we get the following commutative diagram

$$\begin{array}{ccccccccc} F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(\Sigma(X)) & \longrightarrow & \Sigma(F(X)) \\ \omega_X\downarrow & & \downarrow \omega_Y & & \downarrow \omega_Z & & \downarrow \omega_{\Sigma(X)} & & \downarrow \Sigma(\omega_X) \\ G(X) & \longrightarrow & G(Y) & \longrightarrow & G(Z) & \longrightarrow & G(\Sigma(X)) & \longrightarrow & \Sigma(G(X)) \end{array}$$

By combining the last two squares we get the following commutative diagram

$$\begin{array}{ccccccc} F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & \Sigma(F(X)) \\ \omega_X\downarrow & & \downarrow \omega_Y & & \downarrow \omega_Z & & \downarrow \Sigma(\omega_X) \\ G(X) & \longrightarrow & G(Y) & \longrightarrow & G(Z) & \longrightarrow & \Sigma(G(X)) \end{array}$$

Since  $F$  and  $G$  are exact functors, the last morphism is a morphism of distinguished triangles.

**Remark 1.4.10.** If  $\mathcal{T}$  is a triangulated category, then its opposite category  $\mathcal{T}^{\text{op}}$  can be equipped with a natural triangulated structure. We define the shift functor on  $\mathcal{T}^{\text{op}}$  to be inverse of the shift functor  $\Sigma$  of  $\mathcal{T}$ . Also, if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

is a distinguished triangle in  $\mathcal{T}$ , we declare the triangle

$$Z \xrightarrow{g} Y \xrightarrow{f} X \xrightarrow{-\Sigma^{-1}(h)} \Sigma(Z)$$

to be a distinguished triangle in  $\mathcal{T}^{\text{op}}$ .

**Proposition 1.4.11.** *The category  $\mathcal{T}^{\text{op}}$  with the above shift functor and class of distinguished triangles, is a triangulated category.*

*Proof.* For the proof we refer to [12, Proposition 1.2.1]. ■

**Remark 1.4.12.** If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

is a distinguished triangle, then we have the following diagram

$$\dots \xrightarrow{\Sigma^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{\Sigma(f)} \dots$$

**Lemma 1.4.13.** *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

*be a distinguished triangle. Then any two consecutive compositions of the triangle is 0, i.e.*

$$g \circ f = h \circ f = f \circ h = 0$$

*Proof.* From (TR2) it is enough to prove that  $g \circ f = 0$ . Consider the diagram

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma(X) \\ \parallel & & \downarrow f & & & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \end{array}$$

From (TR1) the lines are distinguished triangles, therefore from (TR3) there exists a morphism  $u : 0 \rightarrow Z$  such that the following diagram is commutative

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma(X) \\ \parallel & & \downarrow f & & \downarrow u & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \end{array}$$

Since the middle square is commutative, we have that  $g \circ f = u \circ (X \rightarrow 0)$ . However  $u$  is the zero morphism, hence  $g \circ f = 0$ .  $\blacksquare$

**Remark 1.4.14.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{A}$  an abelian category. Let also  $F : \mathcal{T} \rightarrow \mathcal{A}$  be an additive functor. For every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

from the previous Lemma 1.4.13, we have that  $g \circ f = 0 \Rightarrow F(g \circ f) = 0 \Rightarrow F(g) \circ F(f) = 0$ . Also, from the diagram

$$\dots \xrightarrow{\Sigma^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{\Sigma(f)} \dots$$

we get the diagram

$$\dots \xrightarrow{F(\Sigma^{-1}(h))} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(\Sigma(X)) \xrightarrow{F(\Sigma(f))} \dots$$

of objects is  $\mathcal{A}$ .

**Definition 1.4.15.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{A}$  be an abelian category. An additive functor  $F : \mathcal{T} \rightarrow \mathcal{A}$  is called a **cohomological functor** if for every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

is exact in  $\mathcal{A}$ . In particular, the sequence

$$\cdots \xrightarrow{F(\Sigma^{-1}(h))} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(\Sigma(X)) \xrightarrow{F(\Sigma(f))} \cdots$$

is exact in  $\mathcal{A}$ .

**Remark 1.4.16.** Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{T}$ . Then, for every object  $U \in \mathcal{T}$ , there exists an induced morphism  $f_* : \text{Hom}_{\mathcal{T}}(U, X) \rightarrow \text{Hom}_{\mathcal{T}}(U, Y)$ , which is defined as  $f_*(\phi) = f \circ \phi$ , where  $\phi$  is a morphism of the form  $\phi : U \rightarrow X$ , and  $f^* : \text{Hom}_{\mathcal{T}}(Y, U) \rightarrow \text{Hom}_{\mathcal{T}}(X, U)$ , which is defined as  $f^*(\psi) = \psi \circ f$ , where  $\psi$  is a morphism of the form  $\psi : Y \rightarrow U$ . Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

be a distinguished triangle and  $U$  an object of  $\mathcal{T}$ . Then  $f, g$  and  $h$  induce morphisms, particularly group homomorphisms, of abelian groups and thus we get the following sequences

$$\begin{aligned} \cdots \longrightarrow \text{Hom}_{\mathcal{T}}(U, X) \xrightarrow{f_*} \text{Hom}_{\mathcal{T}}(U, Y) \xrightarrow{g_*} \text{Hom}_{\mathcal{T}}(U, Z) \xrightarrow{h_*} \text{Hom}_{\mathcal{T}}(U, \Sigma(X)) \xrightarrow{\Sigma(f_*)} \cdots \\ \cdots \xrightarrow{\Sigma(f^*)} \text{Hom}_{\mathcal{T}}(\Sigma(X), U) \xrightarrow{h^*} \text{Hom}_{\mathcal{T}}(Z, U) \xrightarrow{g^*} \text{Hom}_{\mathcal{T}}(Y, U) \xrightarrow{f^*} \text{Hom}_{\mathcal{T}}(X, U) \longrightarrow \cdots \end{aligned}$$

The following proposition proves that the above sequences are exact sequences of abelian groups.

**Proposition 1.4.17.** Let  $U \in \mathcal{T}$ . Then:

- (1) The functor  $\text{Hom}_{\mathcal{T}}(U, -) : \mathcal{T} \rightarrow \mathbf{Ab}$  is a homological functor,
- (2) The functor  $\text{Hom}_{\mathcal{T}}(-, U) : \mathcal{T}^{op} \rightarrow \mathbf{Ab}$  is a cohomological functor.

*Proof.* We will prove the first part since the proof of the second part is dual using the rotation axiom. It is enough to prove that  $\text{Im}(f_*) = \text{Ker}(g_*)$ . Since  $\text{Hom}_{\mathcal{T}}(U, -)$  is additive functor we have that  $\text{Im}(f_*) \subseteq \text{Ker}(g_*)$ . For the other inclusion, let  $u : U \rightarrow Y$  be a morphism in  $\text{Ker}(g_*)$ , i.e.  $g_*(u) = 0 \Rightarrow g \circ u = 0$ . Consider the diagram

$$\begin{array}{ccccccc} U & \xlongequal{\quad} & U & \longrightarrow & 0 & \longrightarrow & \Sigma(U) \\ & & \downarrow u & & \downarrow & & \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma(X) \end{array}$$

where the middle square is commutative and the rows are distinguished triangles. By rotating the triangles we get the diagram

$$\begin{array}{ccccccc}
U & \longrightarrow & 0 & \longrightarrow & \Sigma(U) & \xrightarrow{-1_U} & \Sigma(U) \\
\downarrow u & & \downarrow & & & & \\
Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma(X) & \xrightarrow{-\Sigma(f)} & \Sigma(Y)
\end{array}$$

From (TR3) we can complete it to the following commutative diagram

$$\begin{array}{ccccccc}
U & \longrightarrow & 0 & \longrightarrow & \Sigma(U) & \xrightarrow{-1_U} & \Sigma(U) \\
\downarrow u & & \downarrow & & \downarrow \Sigma(v) & & \\
Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma(X) & \xrightarrow{-\Sigma(f)} & \Sigma(Y)
\end{array}$$

and by rotating the triangles we have the following morphism of triangles

$$\begin{array}{ccccccc}
U & \xlongequal{\quad} & U & \longrightarrow & 0 & \longrightarrow & \Sigma(U) \\
\downarrow v & & \downarrow u & & \downarrow & & \downarrow \Sigma(v) \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma(X)
\end{array}$$

Therefore we have constructed a morphism  $v : U \rightarrow X$  such that  $u = f \circ v = f_*(v)$ , which means that  $u \in \text{Im}(f_*)$ .  $\blacksquare$

The following Lemma is the analogue of the 5-Lemma 1.2.29 in abelian categories.

**Lemma 1.4.18** (Five lemma for triangulated categories). *Let*

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
\downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma(u) \\
X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & \Sigma(X')
\end{array}$$

*be a morphism of distinguished triangles where two of the morphisms  $u, v$  and  $w$  are isomorphisms. Then the third one is also an isomorphism.*

*Proof.* By rotating the triangles we can assume that  $u$  and  $v$  are isomorphisms, and by applying the the functor  $\text{Hom}_{\mathcal{T}}(Z', -)$  we get the following commutative diagram

$$\begin{array}{ccccccccc}
\text{Hom}_{\mathcal{T}}(Z', X) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Z', Y) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Z', Z) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Z', \Sigma(X)) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Z', \Sigma(Y)) \\
\downarrow u_* & & \downarrow v_* & & \downarrow w_* & & \downarrow \Sigma(u_*) & & \downarrow \Sigma(v_*) \\
\text{Hom}_{\mathcal{T}}(Z', X') & \longrightarrow & \text{Hom}_{\mathcal{T}}(Z', Y') & \longrightarrow & \text{Hom}_{\mathcal{T}}(Z', Z) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Z', \Sigma(X)) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Z', \Sigma(Y))
\end{array}$$

where the lines are exact and all maybe except the morphism  $w_*$ , are isomorphisms. From the five lemma for abelian categories we have that  $w_*$  is an isomorphism. Therefore there exists a morphism  $\alpha : Z' \rightarrow Z$  such that  $w_*(\alpha) = w \circ \alpha = 1_{Z'}$ . Now by applying the functor  $\text{Hom}_{\mathcal{D}}(-, Z)$  to the original diagram, we the following commutative diagram

$$\begin{array}{ccccccccc} \text{Hom}_{\mathcal{D}}(\Sigma(Y'), Z) & \longrightarrow & \text{Hom}_{\mathcal{D}}(\Sigma(X'), Z) & \longrightarrow & \text{Hom}_{\mathcal{D}}(Z', Z) & \longrightarrow & \text{Hom}_{\mathcal{D}}(Y', Z) & \longrightarrow & \text{Hom}_{\mathcal{D}}(X', Z) \\ \downarrow \Sigma(v^*) & & \downarrow \Sigma(u^*) & & \downarrow w^* & & \downarrow v^* & & \downarrow u^* \\ \text{Hom}_{\mathcal{D}}(\Sigma(Y), Z) & \longrightarrow & \text{Hom}_{\mathcal{D}}(\Sigma(X), Z) & \longrightarrow & \text{Hom}_{\mathcal{D}}(Z, Z) & \longrightarrow & \text{Hom}_{\mathcal{D}}(Y, Z) & \longrightarrow & \text{Hom}_{\mathcal{D}}(X, Z) \end{array}$$

where the lines are exact and all maybe except the morphism  $w^*$ , are isomorphisms. Again, from the five lemma for abelian categories we have that there exists a morphism  $\beta : Z' \rightarrow Z$  such that  $w^*(\beta) = \beta \circ w = 1_Z$ . Hence we have that  $\beta = \beta \circ (w \circ \alpha) = (\beta \circ w) \circ \alpha = \alpha$ , which means that  $w$  is an isomorphism. ■

**Remark 1.4.19.** Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{T}$  and consider the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \parallel & & \parallel & & & & \parallel \\ X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \end{array}$$

where the first square is commutative. From (TR3) we can complete the above diagram in the following morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \parallel & & \parallel & & \downarrow w & & \parallel \\ X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \end{array}$$

From the five lemma for triangulated categories we have that  $w$  is an isomorphism, hence the object  $Z$  is unique up to isomorphism but not up to unique isomoprhism, since in general there are many isomorphisms  $w$  making the above diagram commutative.

**Definition 1.4.20.** The object  $Z$  in the above Remark 1.4.19 is called the **cone** of the morphism  $f$ .

**Lemma 1.4.21.** Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

be a triangle in  $\mathcal{T}$ . If two of its vertices are isomorphic to the zero object, then its third is also isomorphic to the zero object.

*Proof.* The proof is straightforward and is left to the reader. ■

**Lemma 1.4.22.** *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

be a distinguished triangle. The following are equivalent:

- (1)  $f$  is an isomorphism,
- (2)  $Z = 0$ .

*Proof.* Consider the following commutative diagram

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma(X) \\ \parallel & & \downarrow f & & \downarrow & & \downarrow 1_{\Sigma(X)} \\ X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \end{array}$$

(1)  $\Rightarrow$  (2) : Suppose that  $f$  is an isomorphism. Then the first and second vertical morphisms are isomorphisms and from the five lemma for triangulated categories 1.4.18 the morphism  $0 \rightarrow Z$  is also an isomorphism, hence  $Z = 0$ .

(2)  $\Rightarrow$  (1) : Suppose that  $Z = 0$ . Then the first and third vertical morphisms are isomorphisms and from the five lemma for triangulated categories 1.4.18  $f : X \rightarrow Y$  is also an isomorphism. ■

The following Proposition is an improvement of the axiom (TR3).

**Proposition 1.4.23.** *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \quad \text{and} \quad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma(X')$$

be two distinguished triangles and  $v : Y \rightarrow Y'$  be a morphism. Then we get the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma(u) \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \xrightarrow{h} & \Sigma(X') \end{array}$$

and the following are equivalent:

- (1)  $g' \circ v \circ f = 0$ ,
- (2) there exists a morphism  $u : X \rightarrow X'$  such that the first square is commutative, i.e.  $v \circ f = f \circ u$ ,
- (3) there exists a morphism  $w : Z \rightarrow Z'$  such that the second square is commutative, i.e.  $w \circ g = g' \circ v$ ,
- (4) there exist a morphism  $u : X \rightarrow X'$  and a morphism  $w : Z \rightarrow Z'$  such that the diagram is commutative, i.e.  $v \circ f = f \circ u$  and  $w \circ g = g' \circ v$ .

Furthermore, if the above properties are satisfied and also  $\text{Hom}_{\mathcal{T}}(X, Z'[-1]) = 0$ , then the morphisms  $u$  and  $w$  are unique.

*Proof.* (1)  $\Rightarrow$  (2): From Proposition 1.4.17 we have the exact sequence

$$\text{Hom}_{\mathcal{T}}(X, Z'[-1]) \longrightarrow \text{Hom}_{\mathcal{T}}(X, X') \xrightarrow{f'_*} \text{Hom}_{\mathcal{T}}(X, Y') \xrightarrow{g'_*} \text{Hom}_{\mathcal{T}}(X, Z')$$

Then, if  $g'_*(v \circ f) = g' \circ v \circ f = 0$ , we have that  $v \circ f \in \text{Ker}(g'_*)$  and because the above sequence is exact we have that  $v \circ f \in \text{Im}(f'_*)$ . Hence there exists a morphism  $u : X \rightarrow X'$  such that  $v \circ f = f'_*(u) = f' \circ u$ . In addition if  $\text{Hom}_{\mathcal{T}}(X, Z'[-1]) = 0$ , then the morphism  $u$  is unique because  $f'_*$  is a group monomorphism.

(2)  $\Rightarrow$  (1): If there exists such morphism  $u$  then we have  $g' \circ v \circ f = g' \circ f' \circ u = 0 \circ u = 0$ .

(1)  $\Rightarrow$  (3): Suppose that  $g' \circ v \circ f = 0$ . Then again from Proposition 1.4.17 we have the exact sequence

$$\text{Hom}_{\mathcal{T}}(X[1], Z') \longrightarrow \text{Hom}_{\mathcal{T}}(Z, Z') \xrightarrow{g^*} \text{Hom}_{\mathcal{T}}(Y, Z') \xrightarrow{f^*} \text{Hom}_{\mathcal{T}}(X', Z')$$

Therefore, since  $f^*(g' \circ v) = g' \circ v \circ f = 0$ , and because the above sequence is exact, therefore also exact at  $\text{Hom}_{\mathcal{T}}(Y, Z')$ , there exists a morphism  $w : Z \rightarrow Z'$  such that  $g^*(w) = w \circ g = g' \circ v$ . Furthermore, if  $\text{Hom}_{\mathcal{T}}(X, Z'[-1]) = 0$  then the morphism  $w$  is unique since then  $g^*$  is a group monomorphism.

(3)  $\Rightarrow$  (1): If there exists such morphism  $w$ , then we have that  $g' \circ v \circ f = w \circ g \circ f = w \circ 0 = 0$ .

(2)  $\Rightarrow$  (4): Immediate from (TR3).

(4)  $\Rightarrow$  (2): Immediate. ■

**Corollary 1.4.24.** *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

*be a distinguished triangle such that  $\text{Hom}(X, Z[-1]) = 0$ . Then:*

(1) *If*

$$X \xrightarrow{f} Y \xrightarrow{g'} Z \xrightarrow{h'} \Sigma(X)$$

*is another distinguished triangle based on  $f : X \rightarrow Y$ , there exists a unique isomorphism  $u : Z \rightarrow Z'$  such that the diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{g} & \Sigma(X) \\ \parallel & & \parallel & & \downarrow u & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X) \end{array}$$

*is an isomorphism of triangles.*

(2) *If*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h'} \Sigma(X)$$

*is another distinguished triangle, then  $h' = h$ .*

*Proof.* For the proof, we refer the reader to [12, Lemma 1.4.6] ■

### 1.4.2 Localization of triangulated categories

Previously in this chapter we studied the localization of a category with respect to a localizing class. We then considered the localization of additive and abelian categories, and under certain assumptions we showed that their localizations are additive and abelian respectively. In the same spirit, here now show that the localization of a triangulated category with respect to a localizing class which "respects" the triangulated structure of the category, is a triangulated category.

**Definition 1.4.25.** *Let  $\mathcal{T}$  be a triangulated category. A localizing class  $S$  of  $\mathcal{T}$  is called **compatible with the triangulated structure** if:*

(1) *For every morphism  $s$  in  $\mathcal{T}$  we have that  $s \in S \Leftrightarrow \Sigma(s) \in S$ ,*

(2) *The diagram*

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
\downarrow s & & \downarrow t & & & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X')
\end{array}$$

where the lines are distinguished triangles, the first square is commutative and  $s \in S$ , can be completed to the morphism of triangles

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
\downarrow s & & \downarrow t & & \downarrow p & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X')
\end{array}$$

where  $p \in S$ .

**Remark 1.4.26.** Let  $\mathcal{T}$  be a triangulated category, and let  $S$  be a localizing class compatible with the triangulation, and let  $Q : \mathcal{T} \rightarrow \mathcal{T}[S^{-1}]$  be the quotient functor. Then for every  $s \in S$  we have that  $(Q \circ \Sigma)(s) = Q(\Sigma(s))$  is an isomorphism. Therefore, from the universal property of the localization, the functor  $Q \circ \Sigma$  factors uniquely through  $\mathcal{T}[S^{-1}]$ , that is, there exists a unique additive functor  $\Sigma_S : \mathcal{T}[S^{-1}] \rightarrow \mathcal{T}[S^{-1}]$  which makes the following diagram commutative

$$\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\Sigma} & \mathcal{T} \\
Q \downarrow & & \downarrow Q \\
\mathcal{T}[S^{-1}] & \xrightarrow{\Sigma_S} & \mathcal{T}[S^{-1}]
\end{array}$$

Clearly the functor  $\Sigma_S$  is an automorphism of the localization  $\mathcal{T}[S^{-1}]$ . In the following, by abuse of notation, we will also use the symbol  $\Sigma$  for the functor  $\Sigma_S$ .

In order to have a triangulated structure in the localization  $\mathcal{T}[S^{-1}]$  of a triangulated category  $\mathcal{T}$  with respect to a localizing class  $S$  compatible with the triangulated structure, we need to define the distinguished triangles, in  $\mathcal{T}[S^{-1}]$

**Definition 1.4.27.** A triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

in  $\mathcal{T}[S^{-1}]$  is distinguished if there exists a distinguished triangle

$$U \longrightarrow V \longrightarrow W \longrightarrow \Sigma(U)$$

in  $\mathcal{T}$  and an isomorphism of triangles

$$\begin{array}{ccccccc} U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & \Sigma(U) \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma(a) \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(x) \end{array}$$

in  $\mathcal{T}[S^{-1}]$ .

**Theorem 1.4.28.** *Let  $\mathcal{T}$  be a triangulated category and  $S$  a localizing class compatible with the triangulated structure. The category  $\mathcal{T}[S^{-1}]$  is triangulated and the localization functor  $Q : \mathcal{T} \rightarrow \mathcal{T}[S^{-1}]$  is exact.*

*Proof.* We will first prove that the localization  $\mathcal{T}[S^{-1}]$  is a triangulated category.

- (TR1) Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{T}[S^{-1}]$ . Then  $f$  is represented by the left roof

$$\begin{array}{ccc} & U & \\ s \swarrow & & \searrow g \\ X & & Y \end{array}$$

where  $s \in S$ . Since  $\mathcal{T}$  is triangulated, there exists a distinguished triangle

$$U \xrightarrow{g} Y \xrightarrow{v} V \xrightarrow{w} \Sigma(U)$$

with base  $g : U \rightarrow Y$ . Consider the diagram

$$\begin{array}{ccccccc} U & \xrightarrow{Q(g)} & Y & \xrightarrow{Q(v)} & V & \xrightarrow{Q(w)} & \Sigma(U) \\ Q(s) \downarrow & & \parallel & & \parallel & & \downarrow \Sigma(Q(s)) \\ X & \xrightarrow{Q(s)^{-1} \circ Q(g)} & Y & \xrightarrow{Q(v)} & V & \xrightarrow{\Sigma(Q(s)) \circ Q(w)} & \Sigma(X) \end{array}$$

which is an isomorphism of triangles in  $\mathcal{T}[S^{-1}]$ . Therefore the triangle

$$X \xrightarrow{f} Y \longrightarrow V \xrightarrow{\Sigma(Q(s)) \circ Q(w)} \Sigma(X)$$

is a distinguished triangle in  $\mathcal{T}[S^{-1}]$  with base  $f : X \rightarrow Y$ .

- (TR2) Immediate from the definition of distinguished triangles in  $\mathcal{T}[S^{-1}]$ .

- (TR3) Without the loss of generality, we can assume that both distinguished triangles came from distinguished triangles in  $\mathcal{T}$ , i.e. in the following diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{Q(f)} & Y & \xrightarrow{Q(g)} & Z & \xrightarrow{Q(h)} & \Sigma(X) \\
 \downarrow \phi & & \downarrow \psi & & & & \downarrow \Sigma(\phi) \\
 X' & \xrightarrow{Q(f')} & Y' & \xrightarrow{Q(g')} & Z' & \xrightarrow{Q(h')} & \Sigma(X')
 \end{array} \tag{1.4}$$

the rows are distinguished triangles in  $\mathcal{T}[S^{-1}]$  and the first square is commutative. The morphisms  $\phi$  and  $\psi$  are represented by the left roofs

$$\begin{array}{ccc}
 & U & \\
 s \swarrow & & \searrow u \\
 X & & X'
 \end{array}
 \qquad
 \begin{array}{ccc}
 & V & \\
 t \swarrow & & \searrow v \\
 Y & & Y'
 \end{array}$$

thus we have the commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 s \uparrow \sim & & \sim \uparrow t \\
 U & & V \\
 u \downarrow & & \downarrow v \\
 X' & \xrightarrow{f'} & Y'
 \end{array}$$

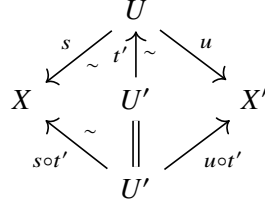
Consider now the morphisms  $f \circ s : U \rightarrow Y$  and  $t : V \rightarrow Y$ . Since  $S$  is a localizing class, the diagram

$$\begin{array}{ccc}
 & V & \\
 & t \downarrow \sim & \\
 U & \xrightarrow{f \circ s} & Y
 \end{array}$$

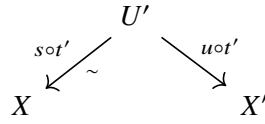
can be completed to the diagram

$$\begin{array}{ccc}
 U' & \xrightarrow{a} & V \\
 t' \downarrow \sim & & \sim \downarrow t \\
 U & \xrightarrow{f \circ s} & Y
 \end{array}$$

in  $\mathcal{T}$ . On the other hand, from the commutative diagram



we have that the top left roof is equivalent to the bottom left roof of the above diagram. Hence we can choose as a representative of  $\phi$  the left roof



and the above rectangular diagram becomes

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 s \circ t' \uparrow \sim & & \sim \uparrow t \\
 U' & \xrightarrow{a} & V \\
 u \circ t' \downarrow & & \downarrow v \\
 X' & \xrightarrow{f'} & Y'
 \end{array} \tag{1.5}$$

where  $f \circ s \circ t' = a \circ t$ , i.e. the top square is commutative in  $\mathcal{T}$ . By just changing the symbols for the objects and the morphisms, we can assume that our initial diagram (1.5) was

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 s \uparrow \sim & & \sim \uparrow t \\
 U & \xrightarrow{a} & V \\
 u \downarrow & & \downarrow v \\
 X' & \xrightarrow{f'} & Y
 \end{array} \tag{1.6}$$

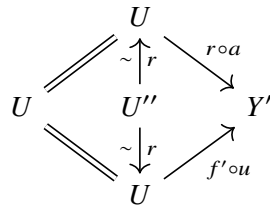
where the top square is commutative in  $\mathcal{T}$ . We have that  $\phi = Q(u) \circ Q(s)^{-1}$  and  $\psi = Q(v) \circ Q(t)^{-1}$ , therefore from the commutativity of the top square of the diagram (1.6) we get  $\psi \circ Q(f) = Q(f') \circ \phi$ , that is

$$Q(u) \circ Q(t)^{-1} \circ Q(f) = Q(f') \circ Q(u) \circ Q(s)^{-1}$$

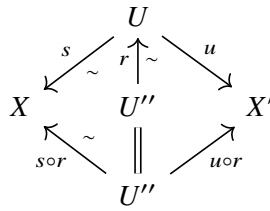
Therefore we have that

$$Q(v) \circ Q(t)^{-1} \circ Q(f) \circ Q(s) = Q(f') \circ Q(u)$$

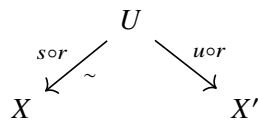
However, from the commutativity of the top square of the diagram (1.6), we get that  $Q(f) \circ Q(s) = Q(f \circ s) = Q(t \circ a) = Q(t) \circ Q(a)$ , hence  $Q(v) \circ Q(a) = Q(f') \circ Q(u)$ , i.e. the bottom square is commutative in  $\mathcal{T}[S^{-1}]$ . Hence there exists a morphism  $r : U'' \rightarrow U$ , where  $r \in S$ , such that the following square is commutative



This implies that the top left roof is equivalent to the bottom left roof. In particular we have that  $r \circ a \circ r = f' \circ u \circ r$ . Since the diagram



is commutative in  $\mathcal{T}$ , that is, one representative of  $\phi$  is the left roof



we can replace the original diagram (1.5) with the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 s \circ r \uparrow \sim & & \sim \uparrow t \\
 U'' & \xrightarrow{a \circ r} & V \\
 u \circ r \downarrow & & \downarrow v \\
 X' & \xrightarrow{f} & Y'
 \end{array} \tag{1.7}$$

where the top and bottom squares are commutative. By changing again the symbols for the objects and morphisms, without the loss of generality, we can assume that the diagram that we started with, the diagram (1.5), was

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 s \uparrow \sim & & \sim \uparrow t \\
 U & \xrightarrow{a} & V \\
 \downarrow u & & \downarrow v \\
 X' & \xrightarrow{f'} & Y'
 \end{array} \tag{1.8}$$

where the top and bottom squares are commutative in  $\mathcal{T}$ . Let

$$U \xrightarrow{a} V \longrightarrow W \longrightarrow \Sigma(U)$$

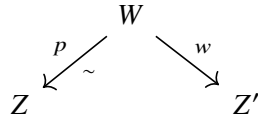
be a distinguished triangle in  $\mathcal{T}$  with base  $a : U \rightarrow V$ . Then the above rectangular diagram (1.8) can be considered to be a part of the following diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\
 s \uparrow \sim & & t \uparrow \sim & & & & \sim \downarrow \Sigma(s) \\
 U & \xrightarrow{a} & V & \longrightarrow & W & \longrightarrow & \Sigma(U) \\
 \downarrow u & & \downarrow v & & & & \Sigma(u) \downarrow \\
 X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X')
 \end{array} \tag{1.9}$$

where the lines are distinguished triangles in  $\mathcal{T}$ . From (LT2) there exists a morphism  $p : W \rightarrow Z$ , where  $p \in S$ , which completes the top part of the above diagram (1.9) to a morphism of triangles in  $\mathcal{T}$ . Now, from (TR3), there exists a morphism  $w : W \rightarrow Z'$  which completes the bottom part of the above diagram (1.9) to a morphism of triangles in  $\mathcal{T}$ . Hence we get the commutative diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\
 s \uparrow \sim & & t \uparrow \sim & & p \uparrow \sim & & \sim \downarrow \Sigma(s) \\
 U & \xrightarrow{a} & V & \longrightarrow & W & \longrightarrow & \Sigma(U) \\
 \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma(u) \\
 X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X')
 \end{array}$$

where all the squares are commutative. Let  $x : Z \rightarrow Z'$  be a morphism in  $\mathcal{T}[S^{-1}]$  which is represented by the left roof



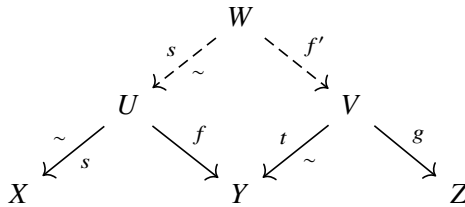
Then the above diagram (1.4) can be considered as a morphism of distinguished triangles in  $\mathcal{T}[S^{-1}]$  as such

$$\begin{array}{ccccccc}
 X & \xrightarrow{Q(f)} & Y & \xrightarrow{Q(g)} & Z & \xrightarrow{Q(h)} & \Sigma(X) \\
 \downarrow \phi & & \downarrow \psi & & \downarrow x & & \downarrow \Sigma(\phi) \\
 X' & \xrightarrow{Q(f')} & Y' & \xrightarrow{Q(g')} & Z' & \xrightarrow{Q(h')} & \Sigma(X')
 \end{array}$$

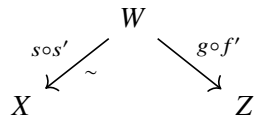
- (TR4) Let  $\phi : X \rightarrow Y$  and  $\psi : Y \rightarrow Z$  be two morphisms in  $\mathcal{T}[S^{-1}]$  with representatives the left roofs



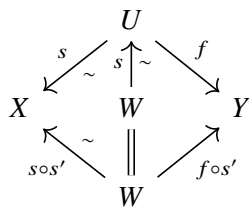
respectively. Their composition is the diagram



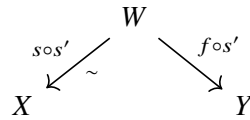
and is represented by the left roof



From the commutative diagram

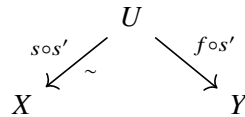


we have that the top left roof is equivalent to the bottom left roof. This means that  $\phi$  can also be represented by the left roof

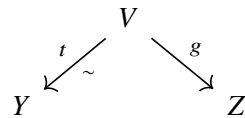


Again, just like we did for the proof of (TR3), by changing the symbols for the objects and morphisms, we can assume that:

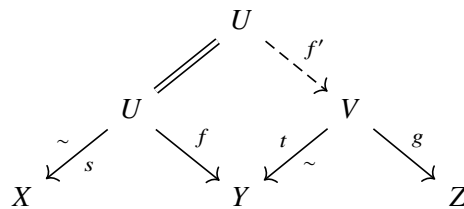
- (1) the morphism  $\phi : X \rightarrow Y$  can be represented by the left roof



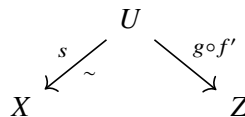
- (2) the morphism  $\psi : Y \rightarrow Z$  can be represented by the left roof



- (3) the composition  $x = \psi \circ \phi : X \rightarrow Z$  is given by the diagram



and therefore can be represented by the left roof



Let  $h := g \circ f'$  and  $W = Z$ . Since  $\mathcal{T}$  is triangulated we can construct through (TR4) an octahedral diagram, which is determined by the morphisms  $f', g$  and  $h$ , i.e. we have the commutative diagram

$$\begin{array}{ccccccc}
U & \xrightarrow{f'} & V & \longrightarrow & W' & \longrightarrow & \Sigma(U) \\
\parallel & & \downarrow g & & \downarrow u' & & \parallel \\
U & \xrightarrow{h} & W & \longrightarrow & V' & \longrightarrow & \Sigma(U) \\
\downarrow f' & & \parallel & & \downarrow v' & & \downarrow \Sigma(f') \\
V & \xrightarrow{g} & W & \longrightarrow & U' & \longrightarrow & \Sigma(V) \\
\downarrow & & \downarrow & & \parallel & & \downarrow \\
W' & \xrightarrow{u'} & V' & \xrightarrow{v} & U' & \xrightarrow{w} & \Sigma(W')
\end{array}$$

in  $\mathcal{T}$ . The image of the above diagram through the localization functor  $Q$  is a diagram of the same form. Consider the initial diagram for the assumptions of (TR4), that is, the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\phi} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \\
\parallel & & \downarrow \psi & & & & \parallel \\
X & \xrightarrow{x} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma(X) \\
\downarrow \phi & & \parallel & & & & \downarrow \Sigma(\phi) \\
Y & \xrightarrow{\psi} & Z & \longrightarrow & X' & \longrightarrow & \Sigma(Y)
\end{array}$$

in  $\mathcal{T}$ . The top part can be extended to the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\phi} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \\
\uparrow Q(s) & & \uparrow Q(t) & & & & \uparrow \Sigma(Q(s)) \\
U & \xrightarrow{Q(f')} & V & \longrightarrow & W' & \longrightarrow & \Sigma(U) \\
\downarrow & & \downarrow Q(g) & & \downarrow Q(u') & & \parallel \\
U & \longrightarrow & W & \longrightarrow & V' & \longrightarrow & \Sigma(U) \\
\downarrow Q(s) & & \parallel & & & & \downarrow \Sigma(Q(s)) \\
X & \xrightarrow{x} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma(X)
\end{array}$$

where the top and bottom squares in the first column are commutative in  $\mathcal{T}[S^{-1}]$  by construction. The middle row is the morphism of distinguished triangles from the octahedral axiom. Since we have already proved (TR3) for  $\mathcal{T}[S^{-1}]$ , we can complete the first and last row of the above diagram with the morphisms  $\alpha : W' \rightarrow Z'$  and  $\beta : V' \rightarrow Y'$  to the following diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{\phi} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \\
 \uparrow Q(s) & & \uparrow Q(t) & & & & \uparrow \Sigma(Q(s)) \\
 U & \xrightarrow{Q(f')} & V & \longrightarrow & W' & \xrightarrow{\alpha} & \Sigma(U) \\
 \downarrow & & \downarrow Q(g) & & \downarrow Q(u') & & \parallel \\
 U & \longrightarrow & W & \longrightarrow & V' & \longrightarrow & \Sigma(U) \\
 \downarrow Q(s) & & \parallel & & \downarrow \beta & & \downarrow \Sigma(Q(s)) \\
 X & \xrightarrow{x} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma(X)
 \end{array}$$

in  $\mathcal{T}[S^{-1}]$  where all the rows are morphisms of triangles. Since the five lemma for triangulated categories is independent of (TR4), we have that the morphisms  $\alpha$  and  $\beta$  are isomorphisms in  $\mathcal{T}[S^{-1}]$ . Doing the same thing to the middle row

$$\begin{array}{ccccccc}
 X & \xrightarrow{x} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma(X) \\
 \phi \downarrow & & \parallel & & & & \downarrow \Sigma(\phi) \\
 Y & \xrightarrow{\psi} & Z & \longrightarrow & X' & \longrightarrow & \Sigma(Y)
 \end{array}$$

we can complete the above diagram to the diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{x} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma(X) \\
 \uparrow Q(s) & & \parallel & & \uparrow \beta & & \uparrow \Sigma(Q(s)) \\
 U & \xrightarrow{Q(h)} & W & \longrightarrow & V' & \longrightarrow & \Sigma(U) \\
 \downarrow Q(f') & & \parallel & & \downarrow Q(v') & & \downarrow \Sigma(f') \\
 V & \xrightarrow{Q(g)} & W & \longrightarrow & U' & \longrightarrow & \Sigma(V) \\
 \downarrow Q(t) & & \parallel & & & & \downarrow \Sigma(Q(s)) \\
 Y & \xrightarrow{\psi} & Z & \longrightarrow & X' & \longrightarrow & \Sigma(Y)
 \end{array}$$

By using the same argument again, that is, because of the commutativity of the first square in the last row, and since we have already proved (TR3), there exists a morphism  $\gamma : X' \rightarrow U'$  which completes the previous diagram into the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{x} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma(X) \\
\uparrow Q(s) & & \parallel & & \uparrow \beta & & \uparrow \Sigma(Q(s)) \\
U & \xrightarrow{Q(h)} & W & \longrightarrow & V' & \longrightarrow & \Sigma(U) \\
\downarrow Q(f') & & \parallel & & \downarrow Q(v') & & \downarrow \Sigma(f') \\
V & \xrightarrow{Q(g)} & W & \longrightarrow & U' & \longrightarrow & \Sigma(V) \\
\downarrow Q(t) & & \parallel & & \downarrow \gamma & & \downarrow \Sigma(Q(s)) \\
Y & \xrightarrow{\psi} & Z & \longrightarrow & X' & \longrightarrow & \Sigma(Y)
\end{array}$$

Again, since the five lemma for triangulated categories is independent of (TR4),  $\gamma$  is an isomorphism in  $\mathcal{T}[S^{-1}]$ . Let  $u := \beta \circ Q(u') \circ \alpha$ ,  $v := \gamma \circ Q(v') \circ \beta^{-1}$  and  $w := \Sigma(\alpha) \circ Q(w') \circ \gamma^{-1}$ . Then, from the commutative diagram

$$\begin{array}{ccccccc}
W' & \xrightarrow{Q(u')} & V' & \xrightarrow{Q(v')} & U' & \xrightarrow{Q(w')} & \Sigma(W') \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \Sigma(\alpha) \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & \Sigma(Z')
\end{array}$$

we get that the second row is a distinguished triangle in  $\mathcal{T}[S^{-1}]$ . In conclusion, by combining all of the above, we get the octahedral diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\phi} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \\
\parallel & & \downarrow \psi & & \downarrow u & & \parallel \\
X & \xrightarrow{x} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma(X) \\
\phi \downarrow & & \parallel & & \downarrow v & & \downarrow \Sigma(\phi) \\
Y & \xrightarrow{\psi} & Z & \longrightarrow & X' & \longrightarrow & \Sigma(Y) \\
\downarrow & & \downarrow & & \parallel & & \downarrow \\
Z' & \xrightarrow{u} & Y' & \xrightarrow{v} & X' & \xrightarrow{w} & \Sigma(Z')
\end{array}$$

Therefore  $\mathcal{T}[S^{-1}]$  is a triangulated category.

For the exactness of the localization functor  $Q$ , if

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

is a distinguished triangle in  $\mathcal{T}$ , then immediately we have that the triangle

$$Q(X) \xrightarrow{Q(f)} Q(Y) \xrightarrow{Q(g)} Q(Z) \xrightarrow{Q(h)} \Sigma(Q(X))$$

is distinguished in  $\mathcal{T}[S^{-1}]$ . ■

**Theorem 1.4.29.** *Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be two triangulated categories and  $F : \mathcal{T}_1 \rightarrow \mathcal{T}_2$  an exact functor. Let also  $S$  be a localizing class in  $\mathcal{T}_1$  compatible with the triangulated structure such that, if  $s \in S$ , then  $F(s)$  is an isomorphism in  $\mathcal{T}_2$ . Then there exists a unique functor  $F_S : \mathcal{T}_1[S^{-1}] \rightarrow \mathcal{T}_2$  such that the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{T}_1 & & \\ \downarrow Q & \searrow F & \\ \mathcal{T}_1[S^{-1}] & \xrightarrow{\exists! F_S} & \mathcal{T}_2 \end{array}$$

*Proof.* The existence of an additive functor  $F_S$  such that  $F = F_S \circ Q$ , comes from Theorem ???. So, it is left to prove that  $F_S$  is exact. For a start, we have that  $\Sigma \circ F_S \circ Q = \Sigma \circ F$  and  $F_S \circ \Sigma \circ Q = F_S \circ Q \circ \Sigma = F \circ \Sigma$ , which implies that the functors  $\Sigma \circ F_S \circ Q$  and  $F_S \circ \Sigma \circ Q$  are naturally isomorphic. Let  $\eta$  be a natural isomorphism of the functors  $F \circ \Sigma$  and  $\Sigma \circ F$ . Then, for every object  $X \in \mathcal{T}_1$ , the morphism  $\eta_X : (F \circ \Sigma)(X) \rightarrow (\Sigma \circ F)(X)$  is an isomorphism. Also, for every morphism  $f : X \rightarrow Y$  in  $\mathcal{T}_1$  the following square is commutative

$$\begin{array}{ccc} (F \circ \Sigma)(X) & \xrightarrow{\eta_X} & (\Sigma \circ F)(X) \\ (F \circ \Sigma)(f) \downarrow & & \downarrow (\Sigma \circ F)(f) \\ (F \circ \Sigma)(Y) & \xrightarrow{\eta_Y} & (\Sigma \circ F)(Y) \end{array}$$

Since the objects of  $\mathcal{T}_1$  are the same with the objects of  $\mathcal{T}_1[S^{-1}]$ , we have the isomorphisms  $\eta_X : (F_S \circ \Sigma)(X) \rightarrow (\Sigma \circ F_S)(X)$ . In addition, if  $\phi : X \rightarrow Y$  is a morphism in  $\mathcal{T}_1[S^{-1}]$ , let

$$\begin{array}{ccc} & U & \\ s \swarrow & & \searrow g \\ X & \xrightarrow{\sim} & Y \end{array}$$

be a left roof that is a representative of  $\phi$ . Thus, we get the following commutative diagrams

$$\begin{array}{ccc}
(F \circ \Sigma)(U) & \xrightarrow{\eta_U} & (\Sigma \circ F)(U) \\
(F \circ \Sigma)(s) \downarrow & & \downarrow (\Sigma \circ F)(s) \\
(F \circ \Sigma)(X) & \xrightarrow{\eta_X} & (\Sigma \circ F)(X)
\end{array}
\qquad
\begin{array}{ccc}
(F \circ \Sigma)(U) & \xrightarrow{\eta_U} & (\Sigma \circ F)(U) \\
(F \circ \Sigma)(g) \downarrow & & \downarrow (\Sigma \circ F)(g) \\
(F \circ \Sigma)(Y) & \xrightarrow{\eta_Y} & (\Sigma \circ F)(Y)
\end{array}$$

and the equalities  $(\Sigma \circ F)(s) \circ \eta_U = \eta_X \circ (F \circ \Sigma)(s)$ ,  $(\Sigma \circ F)(g) \circ \eta_U = \eta_Y \circ (\Sigma \circ F)(g)$ . From the first equality we have that

$$\begin{aligned}
(\Sigma \circ F)(s) \circ \eta_U = \eta_X \circ (F \circ \Sigma)(s) &\Rightarrow (\Sigma \circ F_S \circ Q)(s) \circ \eta_U = \eta_X \circ (F_S \circ Q \circ \Sigma)(s) \Rightarrow \\
(\Sigma \circ F_S)(Q(s)) \circ \eta_U = \eta_X \circ (F_S \circ \Sigma \circ Q)(s) &\Rightarrow (\Sigma \circ F_S)(Q(s)) \circ \eta_U = \eta_X \circ (F_S \circ \Sigma)(Q(s)) \Rightarrow \\
\eta_U \circ (F_S \circ \Sigma)(Q(s))^{-1} &= (F_S \circ \Sigma)(Q(s))^{-1} \circ \eta_X
\end{aligned}$$

Therefore

$$\begin{aligned}
\eta_Y \circ (F_S \circ \Sigma)(\phi) &= \eta_Y \circ (F_S \circ \Sigma)(Q(g) \circ Q(s)^{-1}) = \eta_Y \circ (F_S \circ \Sigma)(Q(g)) \circ (F_S \circ \Sigma)(Q(s)^{-1}) = \\
&= (\Sigma \circ F_S)(Q(g)) \circ \eta_U \circ (F_S \circ \Sigma)(Q(s)^{-1}) = (\Sigma \circ F_S)(Q(g)) \circ (\Sigma \circ F_S)(Q(s)^{-1}) \circ \eta_X = \\
&= (\Sigma \circ F_S)(Q(g) \circ Q(s)^{-1}) \circ \eta_X = (\Sigma \circ F_S)(\phi) \circ \eta_X
\end{aligned}$$

which implies that the following diagram is commutative

$$\begin{array}{ccc}
(\Sigma \circ F_S)(X) & \xrightarrow{(F \circ \Sigma)(\phi)} & (\Sigma \circ F_S)(Y) \\
\eta_X \downarrow & & \downarrow \eta_Y \\
(F_S \circ \Sigma)(X) & \xrightarrow{(\Sigma \circ F_S)(\phi)} & (F_S \circ \Sigma)(Y)
\end{array}$$

Hence  $\Sigma \circ F_S \cong F_S \circ \Sigma$ , which implies that  $F_S$  is graded. It is left to prove that  $F_S$  sends distinguished triangles from  $\mathcal{T}_1[S^{-1}]$  to distinguished triangles in  $\mathcal{T}_2$ . Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

be a distinguished triangle in  $\mathcal{T}_1[S^{-1}]$ . By definition, there exists a distinguished triangle

$$U \longrightarrow V \longrightarrow W \longrightarrow \Sigma(U)$$

in  $\mathcal{T}_1$  and an isomorphism of triangles

$$\begin{array}{ccccccc}
U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & \Sigma(U) \\
\downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma(a) \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X)
\end{array}$$

in  $\mathcal{T}_1[S^{-1}]$ . Applying  $F_S$  to the above diagram and since  $F_S$  is graded, we obtain the commutative diagram

$$\begin{array}{ccccccccc} F_S(U) & \longrightarrow & F_S(V) & \longrightarrow & F_S(W) & \longrightarrow & \Sigma(U) & \xrightarrow{\eta_U} & \Sigma(F_S(U)) \\ \downarrow F_S(a) & & \downarrow F_S(b) & & \downarrow F_S(c) & & \downarrow F_S(\Sigma(a)) & & \downarrow \Sigma(F_S(a)) \\ F_S(X) & \longrightarrow & F_S(Y) & \longrightarrow & F_S(Z) & \longrightarrow & F_S(\Sigma(X)) & \xrightarrow{\eta_X} & \Sigma(F_S(X)) \end{array}$$

Combining the last two squares into one yields the isomorphism of triangles

$$\begin{array}{ccccccc} F(U) & \longrightarrow & V & \longrightarrow & F_S(W) & \longrightarrow & \Sigma(F(U)) \\ \downarrow F_S(a) & & \downarrow F_S(b) & & \downarrow F_S(c) & & \downarrow \Sigma(F_S(a)) \\ F_S(X) & \longrightarrow & F_S(Y) & \longrightarrow & F_S(Z) & \longrightarrow & F_S(\Sigma(X)) \end{array}$$

in  $\mathcal{T}_2$ . Since  $F$  is exact, the top triangle is a distinguished triangle in  $\mathcal{T}_2$ . Therefore the bottom triangle is also a distinguished triangle, hence  $F_S$  is exact. ■

**Remark 1.4.30.** Let  $\mathcal{T}^{\text{op}}$  be the opposite category of a category  $\mathcal{T}$  and  $S$  a localizing class of  $\mathcal{T}$ . We know that  $S$  is also a localizing class of  $\mathcal{T}^{\text{op}}$ . Also, there exists an isomorphism of categories  $a : \mathcal{T}^{\text{op}}[S^{-1}] \rightarrow \mathcal{T}[S^{-1}]^{\text{op}}$ . From its construction and from Theorem 1.4.29, we have that  $a$  is additive.

**Theorem 1.4.31.** *The functor  $a : \mathcal{T}^{\text{op}}[S^{-1}] \rightarrow \mathcal{D}[S^{-1}]^{\text{op}}$  is an isomorphism of triangulated categories.*

*Proof.* The proof follows from the above analysis. ■

**Proposition 1.4.32.** *Let  $\mathcal{T}$  be a triangulated category, let  $\mathcal{A}$  be an abelian category, and let  $F : \mathcal{D} \rightarrow \mathcal{A}$  be a cohomological functor. Let also  $S$  be a localizing class of  $\mathcal{T}$  compatible with the triangulated structure, such that if  $s \in S$ , then  $F(s)$  is an isomorphism. Then there exists a unique functor  $F_S : \mathcal{T}[S^{-1}] \rightarrow \mathcal{A}$  such that the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{T} & & \\ \downarrow Q & \searrow F & \\ \mathcal{T}[S^{-1}] & \xrightarrow{\exists! F_S} & \mathcal{A} \end{array}$$

Also the functor  $F_S$  is a cohomological functor.

*Proof.* The existence of an additive functor  $F_S : \mathcal{T}[S^{-1}] \rightarrow \mathcal{A}$  such that  $F = F_S \circ Q$ , comes from Theorem 1.3.35. So it is left to prove that  $F_S$  is a cohomological functor. Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

be a distinguished triangle in  $\mathcal{T}[S^{-1}]$ . By definition, there exists a distinguished triangle

$$U \longrightarrow V \longrightarrow W \longrightarrow \Sigma(U)$$

in  $\mathcal{T}$  and an isomorphism of triangles

$$\begin{array}{ccccccc} U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & \Sigma(U) \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma(a) \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \end{array}$$

in  $\mathcal{T}[S^{-1}]$ . By applying  $F_S$  to the above diagram we get the commutative diagram

$$\begin{array}{ccccccc} F(U) & \longrightarrow & F(V) & \longrightarrow & F(W) & & \\ \downarrow F_S(a) & & \downarrow F_S(b) & & \downarrow F_S(c) & & \\ F_S(X) & \longrightarrow & F_S(Y) & \longrightarrow & F_S(Z) & & \end{array}$$

Since  $F$  is a cohomological functor, we have that the top row is exact in  $\mathcal{A}$ . Therefore, immediately, the bottom row is also exact, hence  $F_S$  is a cohomological functor.  $\blacksquare$

### 1.4.3 Triangulated subcategories

This subsection presents the definitions of triangulated subcategories and their localization, which provide a framework for studying substructures and quotient constructions in triangulated categories.

Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{T}'$  a full subcategory of  $\mathcal{T}$  such that:

(TS1)  $0 \in \mathcal{T}'$ ,

(TS2) for every  $X, Y$  in  $\mathcal{T}'$ , we have that  $X \oplus Y$  is in  $\mathcal{T}'$ ,

(TS3) for every  $X, Y \in \mathcal{T}'$ , we have that  $X$  is in  $\mathcal{T}' \Leftrightarrow (\Sigma(X))$  is in  $\mathcal{T}'$

(TS4) for all  $X, Y$  in  $\mathcal{T}'$  and for every morphism  $f : X \rightarrow Y$ , there exists an object  $Z$  in  $\mathcal{T}'$  such that the triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

is a distinguished triangle in  $\mathcal{T}$ .

Immediately we see that  $\mathcal{T}'$  is additive. Furthermore, the triangles in  $\mathcal{T}'$  with vertices that are objects in  $\mathcal{T}$  define a triangulated structure in  $\mathcal{T}'$ , that is,  $\mathcal{T}'$  is a triangulated category.

**Definition 1.4.33.** *The above category  $\mathcal{T}'$  is called a **full triangulated subcategory** of  $\mathcal{T}$ .*

**Proposition 1.4.34.** *Let  $\mathcal{T}$  be a triangulated category, and let  $S$  a localizing class of  $\mathcal{T}$  compatible with its triangulated structure, and let  $\mathcal{T}'$  be a full triangulated subcategory of  $\mathcal{T}$ . Suppose that:*

- (1)  $S_{\mathcal{T}'} = S \cap \text{Mor}(\mathcal{T}')$  is a localizing class of  $\mathcal{T}'$ ,
- (2) for every morphism  $s : Y \rightarrow X$  in  $S$  and  $X$  in  $\mathcal{T}'$ , there exists a morphism  $u : Z \rightarrow Y$  such that  $s \circ u \in S$  and  $Z$  is in  $\mathcal{T}'$ .

Then  $S_{\mathcal{T}'}$  is compatible with the triangulated structure of  $\mathcal{T}'$ , and  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$  is a full triangulated subcategory of  $\mathcal{T}[S^{-1}]$ .

*Proof.* Clearly  $S_{\mathcal{T}'}$  is compatible with the triangulated structure of  $\mathcal{T}$ . Therefore the natural inclusion of  $\mathcal{T}'$  in  $\mathcal{T}$  induces an exact functor  $\iota : \mathcal{T}'[S_{\mathcal{T}'}^{-1}] \rightarrow \mathcal{T}[S^{-1}]$ . The functor  $\iota$  is the identity on the objects, and from Theorem 1.3.31 is fully faithful. Therefore we have that  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$  is a full additive subcategory of  $\mathcal{T}[S^{-1}]$ . It is left to prove that  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$  is a full triangulated subcategory of  $\mathcal{T}[S^{-1}]$ . Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

be a distinguished triangle in  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$ . Then it is a distinguished triangle in  $\mathcal{T}[S^{-1}]$ , since  $\iota$  is an exact functor. On the other hand, if

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

is a distinguished triangle in  $\mathcal{T}[S^{-1}]$  whose vertices are objects of  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$ , then there exists a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z' \longrightarrow \Sigma(X)$$

in  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$ . Hence we have the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \parallel & & \parallel & & & & \parallel \\ X & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \end{array}$$

whose first square is commutative. The above diagram can be completed to the isomorphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \parallel & & \parallel & & \downarrow & & \parallel \\ X & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \end{array}$$

in  $\mathcal{T}[S^{-1}]$ . Since  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$  is a full subcategory, the above is an isomorphism of triangles in  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$ . Therefore we have shown that the top triangle is a distinguished triangle in  $\mathcal{T}'[S_{\mathcal{T}'}^{-1}]$ . ■

#### 1.4.4 The category of complexes

Our goal for the rest of this chapter is to define the derived category of an abelian category. Derived categories provide one of the most typical examples of triangulated categories. We start by defining the category of complexes of an abelian category and then move on the homotopy category of complexes. The derived category is obtained from the localization of the homotopic category of complexes with respect to the class of quasi-isomorphisms in the abelian category.

**Definition 1.4.35.** Let  $\mathcal{A}$  be an abelian category. A **graded  $\mathcal{A}$ -object** is a family  $X^\bullet = \{X^n\}_{n \in \mathbb{Z}}$  of objects of  $\mathcal{A}$ . The object  $X^n$  is called the **homogenous component of degree  $n$**  of  $X^\bullet$ .

**Definition 1.4.36.** Let  $X^\bullet$  and  $Y^\bullet$  be two graded  $\mathcal{A}$ -objects and  $n \in \mathbb{Z}$ . With  $\text{Hom}^p(X^\bullet, Y^\bullet)$  we denote the set of all the **graded morphisms of degree  $p$** , i.e.

$$\text{Hom}^p(X^\bullet, Y^\bullet) = \{\{f^n\}_{n \in \mathbb{Z}} \mid f^n \in \text{Hom}(X^n, Y^{n+p})\}$$

**Definition 1.4.37.** Let  $\mathcal{A}$  be an additive category.

- (1) A **cochain complex** of  $\mathcal{A}$ -objects is a pair  $(X^\bullet, d_X)$ , which consists of the graded  $\mathcal{A}$ -object  $X^\bullet$  and the graded morphism  $d_X \in \text{Hom}^1(X^\bullet, X^\bullet)$  such that  $d_X^{n-1} \circ d_X^n = 0$ . The morphism  $d_X$  is called the **differential** of the cochain complex.
- (2) A **chain complex** of  $\mathcal{A}$ -objects is a pair  $(X^\bullet, d_X)$ , which consists of the the graded  $\mathcal{A}$ -object  $X^\bullet$  and the graded morphism  $d_X \in \text{Hom}^{-1}(X^\bullet, X^\bullet)$  such that  $d_X^n \circ d_X^{n-1} = 0$ . The morphism  $d_X$  is called the **differential** of the chain complex.

**Remark 1.4.38.** Chain and cochain complexes can be represented by the following diagrams respectively

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}^X} X_n \xrightarrow{d_n^X} X_{n-1} \longrightarrow \cdots$$

$$\dots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \dots$$

In what follows, the term complex is used to denote a cochain complex.

**Definition 1.4.39.** Let  $(X^\bullet, d_X)$  and  $(Y^\bullet, d_Y)$  be two complexes of  $\mathcal{A}$ -objects. A **chain map**  $f : (X^\bullet, d_X) \rightarrow (Y^\bullet, d_Y)$  is a graded morphism  $f \in \text{Hom}^0(X^\bullet, Y^\bullet)$  such that

$$f^n \circ d_X^{n-1} = d_Y^{n-1} \circ f^{n-1}, \forall n \in \mathbb{Z}$$

i.e. the following diagram is commutative

$$\begin{array}{ccccccc} \dots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \dots \\ & & \downarrow f^{n-1} & & \downarrow f^n & & \downarrow f^{n+1} & & \\ \dots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow & \dots \end{array}$$

**Remark 1.4.40.** The composition of two chain maps  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  and  $g^\bullet : Y^\bullet \rightarrow Z^\bullet$  is defined as  $g^\bullet \circ f^\bullet : X^\bullet \rightarrow Z^\bullet = \{g^p \circ f^p\}_{p \in \mathbb{Z}}$ .

We are now in a position to define the category of complexes of an additive category  $\mathcal{A}$ .

**Definition 1.4.41.** Let  $\mathcal{A}$  be an additive category. The category which has as objects complexes of  $\mathcal{A}$ -objects and as morphisms chain maps between complexes of  $\mathcal{A}$ -objects is called **the category of chain complexes** and is denoted by  $\mathcal{C}(\mathcal{A})$ .

**Definition 1.4.42.** Let  $X^\bullet$  and  $Y^\bullet$  be two complexes of  $\mathcal{A}$ -objects. We denote by  $\text{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet)$  the abelian group of all chain maps from  $X^\bullet$  to  $Y^\bullet$ .

**Definition 1.4.43.** Let  $\mathcal{A}$  be an additive category and  $\mathcal{C}(\mathcal{A})$  the category of complexes of  $\mathcal{A}$ . We define the functor  $\Sigma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$  which maps a complex  $X^\bullet$  to the complex  $\Sigma(X^\bullet)$ , where  $\Sigma(X^\bullet)^n = X^{n+1}$  and  $d_{\Sigma(X^\bullet)}^n = -d_X^{n+1}, \forall n \in \mathbb{Z}$ , and every chain map  $f : X^\bullet \rightarrow Y^\bullet$  to the chain map  $\Sigma(f) : \Sigma(X^\bullet) \rightarrow \Sigma(Y^\bullet)$ , where  $\Sigma(f)^n = f^{n+1}, \forall n \in \mathbb{Z}$ .

**Remark 1.4.44.** The functor  $\Sigma$  is an automorphism of the category of complexes  $\mathcal{C}(\mathcal{A})$ . This automorphism will play the role of the shift functor in the homotopy category of complexes, which will be defined later, and together with a class of distinguished triangles, will give the homotopic category its triangulated structure. Therefore, mimicking the notion of the shift functor of triangulated categories, most of the times we will write  $X^\bullet[p]$  instead of writing  $\Sigma^p(X^\bullet)$ , where  $X^\bullet$  is the complex  $X^\bullet$  shifted to the left  $p$  times.

**Remark 1.4.45.** The trivial complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is the zero object of  $C(\mathcal{A})$ .

**Definition 1.4.46.** Let  $X^\bullet$  and  $Y^\bullet$  be two complexes. We define the complex  $X^\bullet \oplus Y^\bullet$ , where  $(X^\bullet \oplus Y^\bullet)^p = X^p \oplus Y^p$  and  $d_{X^\bullet \oplus Y^\bullet}^p = d_X^p \oplus d_Y^p : X^p \oplus Y^p \rightarrow X^{p+1} \oplus Y^{p+1}$ ,  $\forall p \in \mathbb{Z}$ . We call the complex  $X^\bullet \oplus Y^\bullet$  the **direct sum** of the complexes  $X^\bullet$  and  $Y^\bullet$ .

**Remark 1.4.47.** Since in Definition 1.4.46 we have a direct sum, we have the following canonical morphisms

- (1)  $i_X : X^\bullet \rightarrow X^\bullet \oplus Y^\bullet$ ,
- (2)  $i_Y : Y^\bullet \rightarrow X^\bullet \oplus Y^\bullet$ ,
- (3)  $p_X : X^\bullet \oplus Y^\bullet \rightarrow X^\bullet$ ,
- (4)  $p_Y : X^\bullet \oplus Y^\bullet \rightarrow Y^\bullet$ ,

From Remarks 1.4.47 and 1.4.45, and from Definition 1.4.46, we have the following Lemma.

**Lemma 1.4.48.** The category  $C(\mathcal{A})$  is additive.

*Proof.* For the details of the proof, we refer the reader to [12, Lemma 1.1.1]. ■

Since  $\mathcal{A}$  and  $C(\mathcal{A})$  are additive categories we can define an additive functor between them.

**Definition 1.4.49.** Let  $\mathcal{A}$  be an additive category. We define the functor  $C : \mathcal{A} \rightarrow C(\mathcal{A})$  where

$$C(X)^n = \begin{cases} X, & n = 0 \\ 0, & n \neq 0 \end{cases} \quad \text{and} \quad d_{C(X)} = 0$$

for all  $X \in \mathcal{A}$ , and

$$C(f)^p = \begin{cases} f, & p = 0 \\ 0, & p \neq 0 \end{cases}$$

for every morphism  $f : X \rightarrow Y$  in  $\mathcal{A}$ .

**Lemma 1.4.50.** The functor  $C : \mathcal{A} \rightarrow C(\mathcal{A})$  is fully faithful.

*Proof.* For the proof we refer the reader to [12][Lemma 1.1.2] ■

**Remark 1.4.51.** From Lemma 1.4.50 the additive category  $\mathcal{A}$  is isomorphic to the full subcategory of  $C(\mathcal{A})$  whose objects are the complexes  $X^\bullet$ , where  $X^p = 0$ , for  $p \neq 0$ .

We are interested in subcategories of  $C(\mathcal{A})$  which contain particular types of complexes. Therefore we have the following definition.

**Definition 1.4.52.** Let  $X^\bullet$  be a complex.

- (1) the complex  $X^\bullet$  is called **bounded from below**, if there exists  $n_0 \in \mathbb{Z}$  such that  $X^n = 0$ , for every  $n < n_0$ . We denote with  $C^-(\mathcal{A})$  the full subcategory of  $C(\mathcal{A})$  whose objects are the bounded from below complexes.
- (2) the complex  $X^\bullet$  is called **bounded from above**, if there exists  $n_0 \in \mathbb{Z}$  such that  $X^n = 0$ , for every  $n > n_0$ . We denote with  $C^+(\mathcal{A})$  the full subcategory of  $C(\mathcal{A})$  whose objects are the bounded from above complexes.
- (3) the complex  $X^\bullet$  is called **bounded**, if it is bounded from below and from above. We denote with  $C^b(\mathcal{A})$  the full subcategory of  $C(\mathcal{A})$  whose objects are the bounded complexes.

**Remark 1.4.53.** The categories  $C^-(\mathcal{A})$ ,  $C^+(\mathcal{A})$  and  $C^b(\mathcal{A})$  are invariant under for the action of the translation functor, and they are also additive.

**Notation 1.4.54.** In the following, we will denote any of the above three categories that we defined in Definition 1.4.52 by  $C^*(\mathcal{A})$ .

Let  $\mathcal{A}$  be an additive category. We will now give a relation between  $C(\mathcal{A})$  and  $C(\mathcal{A}^{\text{op}})$ .

**Definition 1.4.55.** We define the functor  $\iota : C(\mathcal{A})^{\text{op}} \rightarrow C(\mathcal{A}^{\text{op}})$  as follows:

- for every complex  $X^\bullet \in C(\mathcal{A})$ , we define the complex  $\iota(X^\bullet) \in C(\mathcal{A}^{\text{op}})$ , where:

$$\begin{aligned} \iota(X^\bullet)^p &= X^{-p}, \text{ for every } p \in \mathbb{Z} \\ d_{\iota(X)}^p : \iota(X^\bullet)^p &\rightarrow \iota(X^\bullet)^{p+1}, d_X^{-p-1} : X^{-p-1} \rightarrow X^{-p}, \text{ for every } p \in \mathbb{Z} \end{aligned}$$

- for every chain map  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  we define  $\iota(f^\bullet)$ :

$$\iota(f)^\bullet = f^{-\bullet} : Y^{-\bullet} \rightarrow X^{-\bullet}, \text{ for every } p \in \mathbb{Z}, \text{ in } \mathcal{A}^{\text{op}}$$

**Remark 1.4.56.** We compute

$$d_{\iota(X)}^p \circ \iota(f)^\bullet = f^{-p} \circ d_X^{-p-1} = d_Y^{-p-1} \circ f^{-p-1} = \iota(f)^{p+1} \circ d_{\iota(Y)}^p, \text{ for every } p \in \mathbb{Z}$$

Hence  $\iota(f)$  is a chain map from  $\iota(Y^\bullet)$  to  $\iota(X^\bullet)$  in  $C(\mathcal{A}^{\text{op}})$ . We can easily verify that  $\iota : C(\mathcal{A})^{\text{op}} \rightarrow C(\mathcal{A}^{\text{op}})$  is an isomorphism of categories. Therefore, via  $\iota$ , there are induced isomorphisms of categories  $C^+(\mathcal{A})^{\text{op}} \rightarrow C^-(\mathcal{A}^{\text{op}})$ ,  $C^-(\mathcal{A})^{\text{op}} \rightarrow C^+(\mathcal{A}^{\text{op}})$ , and  $C^b(\mathcal{A})^{\text{op}} \rightarrow C^b(\mathcal{A}^{\text{op}})$

**Notation 1.4.57.** We will denote with the same symbol  $\Sigma$  the shift functors of  $C(\mathcal{A})$  and  $C(\mathcal{A}^{\text{op}})$ .

We can see that:

$$\begin{aligned}\Sigma(\iota(X^\bullet)) &= \iota(\Sigma^{-1}(X^\bullet)), \text{ for every } X^\bullet \in C(\mathcal{A}) \\ \Sigma(\iota(f^\bullet)) &= \iota(\Sigma^{-1}(f^\bullet)), \text{ for every } f^\bullet : X^\bullet \rightarrow Y^\bullet\end{aligned}$$

i.e.,

$$\Sigma \circ \iota = \iota \circ \Sigma^{-1}$$

### 1.4.5 Homotopies

Homotopy plays a crucial role in Homological Algebra. In this section we introduce the notion of homotopy and we define the homotopic category of complexes of an additive category. For the rest of this section, let  $\mathcal{A}$  be an additive category.

**Definition 1.4.58.** Let  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map in  $C(\mathcal{A})$ . We say that the chain map  $f^\bullet$  is **homotopic to zero**, if there exists  $h^\bullet \in \text{Hom}^{-1}(X^\bullet, Y^\bullet)$  such that:

$$\begin{aligned}f^\bullet &= d_Y \circ h^\bullet + h^\bullet \circ d_X, \text{ or} \\ f^n &= d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n, \forall n \in \mathbb{Z}\end{aligned}$$

Diagrammatically:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\ & & \downarrow f^{n-1} & \swarrow h^n & \downarrow f^n & \swarrow h^{n+1} & \downarrow f^{n+1} & & \\ \cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow & \cdots \end{array}$$

The morphism  $h^\bullet$  is called a **homotopy**.

**Notation 1.4.59.** We will denote with  $\text{Ht}(X^\bullet, Y^\bullet)$  the set of all chain maps in  $\text{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$  which are homotopic to zero.

**Lemma 1.4.60.** The set  $\text{Ht}(X^\bullet, Y^\bullet)$  is a subgroup of the abelian group  $\text{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$

*Proof.* For a start, we have that  $0^\bullet \in \text{Ht}(X^\bullet, Y^\bullet)$  by choosing  $h^\bullet = 0^\bullet$ . Let now  $f^\bullet, g^\bullet \in \text{Ht}(X^\bullet, Y^\bullet)$ . Then there exist homotopies  $h^\bullet, k^\bullet$  such that  $f^\bullet = d_Y \circ h^\bullet + h^\bullet \circ d_X$  and  $g^\bullet = d_Y \circ k^\bullet + k^\bullet \circ d_X$ . By combining the above we have  $f^\bullet + g^\bullet = d_Y \circ h^\bullet + h^\bullet \circ d_X + d_Y \circ k^\bullet + k^\bullet \circ d_X = d_Y(h^\bullet + k^\bullet) +$

$(h^\bullet + k^\bullet) \circ d_X$ , that is,  $f^\bullet + g^\bullet$  is homotopic to zero. This means that  $\text{Ht}(X^\bullet, Y^\bullet)$  is closed under addition. Finally, if  $f^\bullet \in \text{Ht}(X^\bullet, Y^\bullet)$ , then  $-f^\bullet \in \text{Ht}(X^\bullet, Y^\bullet)$ , because  $-f^\bullet = d_Y \circ (-h^\bullet) + (-h^\bullet) \circ d_Y$ . This implies that  $-f^\bullet$  is homotopic to zero. Consequently  $\text{Ht}(X^\bullet, Y^\bullet)$  is a subgroup of  $\text{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet)$ . ■

Next we define when two chain maps are homotopic.

**Definition 1.4.61.** Two chain maps  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  and  $g^\bullet : X^\bullet \rightarrow Y^\bullet$  are called **homotopic**, if  $f^\bullet - g^\bullet \in \text{Ht}(X^\bullet, Y^\bullet)$ , that is, there exists a chain map  $h^\bullet$  such that

$$\begin{aligned} f^n - g^n &= d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n, \text{ or} \\ f^\bullet - g^\bullet &= d_Y \circ h^\bullet + h^\bullet \circ d_X \end{aligned}$$

Diagrammatically:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\ & & \downarrow g^{n-1} & \swarrow h^n & \downarrow g^n & \swarrow h^{n+1} & \downarrow g^{n+1} & & \\ & & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow & \cdots \end{array}$$

We denote that two chain maps  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  and  $g^\bullet : X^\bullet \rightarrow Y^\bullet$  are homotopic by  $f^\bullet \sim g^\bullet$ .

**Lemma 1.4.62.** Homotopy induces an equivalence relation in  $\text{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet)$ .

*Proof.* Reflexive: Let  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map. Obviously  $f^\bullet \sim f^\bullet$ , by choosing  $h^\bullet = 0^\bullet$ .

Symmetric: Let  $f^\bullet, g^\bullet : X^\bullet \rightarrow Y^\bullet$  be two chain maps such that  $f^\bullet \sim g^\bullet$ . Then, by definition, there exists  $h^\bullet \in \text{Hom}^{-1}(X^\bullet, Y^\bullet)$  such that  $f^\bullet - g^\bullet = d_Y \circ h^\bullet + h^\bullet \circ d_X$ . By choosing  $-h^\bullet$  we have that  $g^\bullet - f^\bullet = d_Y \circ (-h^\bullet) + (-h^\bullet) \circ d_X$ , which is equivalent to  $g^\bullet \sim f^\bullet$ .

Transitive: Let  $f^\bullet, g^\bullet, h^\bullet : X^\bullet \rightarrow Y^\bullet$  be chain maps such that  $f^\bullet \sim g^\bullet$  and  $g^\bullet \sim h^\bullet$ . Then there exist chain maps  $z^\bullet$  and  $w^\bullet$  such that  $f^\bullet - g^\bullet = d_Y \circ z^\bullet + z^\bullet \circ d_X$  and  $g^\bullet - h^\bullet = d_Y \circ w^\bullet + w^\bullet \circ d_X$ . By combining the above equations we have  $f^\bullet - h^\bullet = d_Y \circ (z^\bullet - w^\bullet) + (z^\bullet - w^\bullet) \circ d_X$ , which means that  $f^\bullet \sim h^\bullet$ . ■

**Lemma 1.4.63.** Let  $X^\bullet, Y^\bullet$  and  $Z^\bullet$  be complexes and  $f^\bullet : X^\bullet \rightarrow Y^\bullet, g^\bullet : Y^\bullet \rightarrow Z^\bullet$  two chain maps. If  $f^\bullet$  or  $g^\bullet$  is homotopic to zero, then so is  $g^\bullet \circ f^\bullet$ .

*Proof.* Suppose that  $f^\bullet$  is homotopic to zero. Then there exists a homotopy  $h^\bullet \in \text{Hom}^{-1}(X^\bullet, Y^\bullet)$  such that

$$f^\bullet = d_Y \circ h + h \circ d_X$$

Therefore  $g^\bullet \circ f^\bullet = g^\bullet \circ d_Y \circ h^\bullet + g^\bullet \circ h^\bullet \circ d_X = d_Z \circ g^\bullet \circ h^\bullet + g^\bullet \circ h^\bullet \circ d_X$ , where  $g^\bullet \circ h^\bullet \in \text{Hom}^{-1}(X^\bullet, Z^\bullet)$ . Hence  $g^\bullet \circ h^\bullet$  is a homotopy, so  $g^\bullet \circ f^\bullet \in \text{Ht}(X^\bullet, Z^\bullet)$ . We use the same argument if  $g^\bullet \in \text{Ht}(Y^\bullet, Z^\bullet)$ . ■

We now proceed to the definition of the homotopy category of complexes of an additive category  $\mathcal{A}$ .

**Definition 1.4.64.** Let  $\mathcal{A}$  be an additive category. We define the category  $\mathcal{K}(\mathcal{A})$  as such:

- Its objects are cochain complexes
- The set of morphisms between two complexes  $X^\bullet$  and  $Y^\bullet$  is defined to be:

$$\text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet) := \text{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet) / \text{Ht}(X^\bullet, Y^\bullet)$$

- The composition of two chain maps  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  and  $g^\bullet : Y^\bullet \rightarrow Z^\bullet$  is defined to be:

$$\begin{aligned} \circ : \text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet) \times \text{Hom}_{\mathcal{K}(\mathcal{A})}(Y^\bullet, Z^\bullet) &\rightarrow \text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Z^\bullet) \\ (f^\bullet, g^\bullet) &\mapsto g^\bullet \circ f^\bullet \end{aligned}$$

just as in  $\mathcal{C}(\mathcal{A})$ .

**Remark 1.4.65.** Let  $X^\bullet, Y^\bullet$  and  $Z^\bullet$  be complexes. From Lemma 1.4.63, the composition  $(g^\bullet, f^\bullet) \mapsto g^\bullet \circ f^\bullet$  from  $\text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet) \times \text{Hom}_{\mathcal{K}(\mathcal{A})}(Y^\bullet, Z^\bullet)$  to  $\text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Z^\bullet)$  defines a biadditive map

$$\text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet) \times \text{Hom}_{\mathcal{K}(\mathcal{A})}(Y^\bullet, Z^\bullet) \rightarrow \text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Z^\bullet)$$

such that the following diagram is commutative

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}(\mathcal{A})}(Y^\bullet, Z^\bullet) \times \text{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet) & \longrightarrow & \text{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Z^\bullet) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{C}(\mathcal{A})}(Y^\bullet, Z^\bullet) \times \text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet) & \longrightarrow & \text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Z^\bullet) \end{array}$$

This implies that the composition in the Definition 1.4.64 is well defined.

**Lemma 1.4.66.** For any additive category  $\mathcal{A}$ , the homotopy category  $\mathcal{K}(\mathcal{A})$  is additive.

*Proof.* For more details we refer the reader to [12, Lemma 1.3.3]. ■

The following is an important technical lemma that will be used in the following.

**Lemma 1.4.67.** *Let  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map. Then the following are equivalent:*

- (1)  $f^\bullet \sim 0^\bullet$ ,
- (2)  $\Sigma(f^\bullet) \sim 0^\bullet$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $f^\bullet \sim 0^\bullet$ . Then there exists a homotopy  $h \in \text{Hom}^{-1}(X^\bullet, Y^\bullet)$  such that  $f = d_Y \circ h + h \circ d_X$ . In particular,  $h$  can be considered as a family of morphisms  $h^n : X^n \rightarrow Y^{n-1}$ . Therefore, we can consider  $h$  as a morphism  $k \in \text{Hom}^{-1}(\Sigma(X^\bullet), \Sigma(Y^\bullet))$ , and then  $\Sigma(f)^n = f^{n+1} = d_Y^n \circ h^{n+1} + h^{n+2} \circ d_X^{n+1} = -d_{\Sigma(Y)}^{n-1} \circ k^n - k^{n+1} \circ d_{\Sigma(X)}^n$ , for every  $n \in \mathbb{Z}$ . This means that  $\Sigma(f^\bullet) \sim 0^\bullet$  with homotopy  $-k$ . Diagrammatically

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{f_X^n} & X^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow f^{n-1} & \swarrow h^n & \downarrow d_X^n & \swarrow h^{n+1} & \downarrow f^{n+1} & & \\
 \cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n+1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow & \cdots
 \end{array}$$

and

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & X^n & \xrightarrow{-d_X^n} & X^{n+1} & \xrightarrow{f_X^n} & X^{n+2} & \longrightarrow & \cdots \\
 & & \downarrow & \swarrow h^{n+1}=k^n & \downarrow & \swarrow h^{n+2}=k^{n+1} & \downarrow & & \\
 \cdots & \longrightarrow & Y^n & \xrightarrow{-d_Y^n} & Y^{n+1} & \xrightarrow{-d_Y^{n+1}} & Y^{n+2} & \longrightarrow & \cdots
 \end{array}$$

(2)  $\Rightarrow$  (1) : The proof is similar to that of (1)  $\Rightarrow$  (2). ■

**Remark 1.4.68.** From the above, the shift functor defines a group isomorphism  $\text{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet) \xrightarrow{\cong} \text{Hom}_{\mathcal{K}(\mathcal{A})}(\Sigma(X^\bullet), \Sigma(Y^\bullet))$ , and therefore an automorphism of  $\mathcal{K}(\mathcal{A})$ .

**Notation 1.4.69.** *By abuse of notation, the induced automorphism of  $\mathcal{K}(\mathcal{A})$  will be denoted again by  $\Sigma$ .*

**Definition 1.4.70.** *Let  $\mathcal{A}$  be an additive category.*

- (1) We define the **full subcategory of bounded from below chain complexes** of  $\mathcal{K}(\mathcal{A})$ , and we denote it by  $\mathcal{K}^-(\mathcal{A})$ ,

(2) We define the **full subcategory of bounded from above chain complexes** of  $K(\mathcal{A})$ , and we denote it by  $K^+(\mathcal{A})$ ,

(3) We define the **full subcategory of bounded chain complexes** of  $K(\mathcal{A})$ , and we denote it by  $K^b(\mathcal{A})$ .

**Notation 1.4.71.** As we did for the category of complexes, we will denote any of the above three categories by  $K^*(\mathcal{A})$ , when  $*$  =  $-$ ,  $+$ ,  $b$

**Remark 1.4.72.** Any of the above three categories are additive and invariant under the shift functor  $\Sigma$ .

**Definition 1.4.73.** Let  $\mathcal{A}$  be an additive category. We define the functor  $\Pi : C(\mathcal{A}) \rightarrow K(\mathcal{A})$  as:

- $\text{Ob}(K(\mathcal{A})) \rightarrow \text{Ob}(C(\mathcal{A})), X^\bullet \mapsto \Pi(X^\bullet) = X^\bullet$
- $\text{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet) \rightarrow \text{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet), f^\bullet \mapsto [f^\bullet]$

**Remark 1.4.74.** The functor  $\Pi$  is additive and commutes with direct sums.

**Definition 1.4.75.** From the functors  $C : \mathcal{A} \rightarrow C(\mathcal{A})$  and  $\Pi : C(\mathcal{A}) \rightarrow K(\mathcal{A})$ , we define the functor  $K = \Pi \circ C : \mathcal{A} \rightarrow K(\mathcal{A})$ .

**Lemma 1.4.76.** The functor  $K$  is fully faithful.

*Proof.* For the proof, we refer the reader to [12, Lemma 1.3.5]. ■

**Corollary 1.4.77.** From the above Lemma, there exists an isomorphism of categories from  $\mathcal{A}$  to the full subcategory of  $K(\mathcal{A})$ , whose objects are the complexes  $X^\bullet$  where  $X^n = 0$ , for every  $n \neq 0$ .

## 1.4.6 Cohomology

We are now ready to define the cohomological functors, which play a crucial role in the study of the homotopy category of complexes. In what follows, the category  $\mathcal{A}$  will be an abelian category.

**Definition 1.4.78.** Let  $\mathcal{A}$  be an abelian category and  $X^\bullet$  a complex in  $C(\mathcal{A})$ . For every  $n \in \mathbb{N}$ , we define the functor  $H : C(\mathcal{A}) \rightarrow \mathcal{A}$  as follows:

- $H^n(X^\bullet) = \text{Ker}(d_X^n) / \text{Im}(d_X^{n-1})$ , for every  $X^\bullet \in C(\mathcal{A})$ .
- Every chain map  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  in  $C(\mathcal{A})$ , where  $f^n(\text{Ker}(d_X^n)) \subseteq \text{Ker}(d_Y^n)$  and  $f^n(\text{Im}(d_X^{n-1})) \subseteq \text{Im}(d_Y^{n-1})$ , induces a morphism

$$H^n(X^\bullet) : H^n(X^\bullet) \rightarrow H^n(Y^\bullet)$$

For every  $n \in \mathbb{Z}$ , the functors  $H^n$  are called **cohomological functors**.

**Remark 1.4.79.** (1) The functors  $H^n$  are additive, for every  $n \in \mathbb{Z}$ ,

(2) We have that:

$$(i) \ H^n(\Sigma(X^\bullet)) = \text{Ker}(d_{\Sigma(X)}^n) / \text{Im}(d_{\Sigma(X)}^{n+1}) = \text{Ker}(d_X^{n+1}) / \text{Im}(d_X^n) = H^{n+1}(X^\bullet),$$

$$(ii) \ H^n(\Sigma(f)) = H^{n+1}(f).$$

Therefore from the above, we have that  $H^n = H^0 \circ \Sigma^n$ , for every  $n \in \mathbb{Z}$ . So it is enough to study the functor  $H^0 : C(\mathcal{A}) \rightarrow \mathcal{A}$ .

**Lemma 1.4.80.** Let  $f^\bullet, g^\bullet : X^\bullet \rightarrow Y^\bullet$  be two chain maps which are homotopic. Then  $H^n(f^\bullet) = H^n(g^\bullet)$ .

*Proof.* From Remark 1.4.79, it is enough to prove that  $H^0(f^\bullet) = H^0(g^\bullet)$ . Let  $h^\bullet$  be a homotopy of  $f^\bullet$  and  $g^\bullet$ . Then, by definition, we have that  $f^0 - g^0 = d_Y^{-1} \circ h^0 + h^1 \circ d_X^0$ . Hence the restriction of  $f^0 - g^0$  to the kernel of  $d_X^0$  is equal to the morphism  $d_Y^{-1} \circ h^0$ , i.e.  $(f^0 - g^0)|_{\text{Ker}(d_X^0)} = d_Y^{-1} \circ h^0$ . This means that  $\text{Im}(f^0 - g^0) \subseteq \text{Im}(d_Y^{-1})$ , which implies that  $f^0 - g^0$  induces a unique morphism from  $\text{Ker}(d_X^0)$  to  $H^0(Y^\bullet) = \text{Ker}(d_Y^0) / \text{Im}(d_Y^{-1})$ . This leads to the fact that  $H^0(f^\bullet) - H^0(g^\bullet) = H^0(f^\bullet - g^\bullet) : H^0(X^\bullet) \rightarrow H^0(Y^\bullet)$  is the zero morphism. ■

**Remark 1.4.81.** For every  $n \in \mathbb{Z}$ , the functor  $H^n : C(\mathcal{A}) \rightarrow \mathcal{A}$  induces a functor  $H^n : K(\mathcal{A}) \rightarrow \mathcal{A}$ . These functors are also additive and we have that  $H^n = H^0 \circ \Sigma^n$ , for every  $n \in \mathbb{Z}$ .

In our effort to define the homotopy category of an additive category, we introduce the notion of the cone of a chain map, which plays an important role in the development of the theory.

**Remark 1.4.82.** Let  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map in  $C(\mathcal{A})$ . We define the graded object  $C_f^\bullet$  as follows:

$$C_f^n := X^{n+1} \oplus Y^n, n \in \mathbb{Z}$$

Then we have that

$$d_{C_f}^{n+1} \circ d_{C_f}^n = \begin{pmatrix} -d_X^{n+2} & 0 \\ f^{n+2} & d_Y^{n+1} \end{pmatrix} \circ \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} = \begin{pmatrix} d_X^{n+2} \circ d_X^{n+1} & 0 \\ -f^{n+2} \circ d_X^{n+1} + d_Y^{n+1} \circ f^{n+1} & d_Y^{n+1} \circ d_Y^n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which implies that  $d_{C_f}$  is a differential and the pair  $(C_f^\bullet, d_{C_f})$  is a complex in  $C^*(\mathcal{A})$ .

**Definition 1.4.83.** The complex defined in Remark 1.4.82 is called the **cone** of the chain map  $f^\bullet$ .

(1) Diagrammatically, the complex  $(C_f^\bullet, d_{C_f})$  is expressed by the following diagram

$$\dots \longrightarrow X^n \oplus Y^{n-1} \xrightarrow{d_{C_f}^{n-1}} X^{n+1} \oplus Y^n \xrightarrow{d_{C_f}^n} X^{n+2} \oplus Y^{n+1} \longrightarrow \dots$$

(2) Consider now the graded morphism  $i_f^\bullet : Y^\bullet \rightarrow C_f^\bullet$ , where  $i_f^n := i_{Y^n} : Y^n \rightarrow C_f^n$ ,  $n \in \mathbb{Z}$ , i.e. the inclusion of  $Y^n$  in  $C_f^n = X^{n+1} \oplus Y^n$ . Then we have

$$d_{C_f}^n \circ i_f^n = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} \begin{pmatrix} 0 \\ 1_{Y^n} \end{pmatrix} = \begin{pmatrix} 0 \\ d_Y^n \end{pmatrix} = i_f^{n+1} \circ d_Y^n, \quad \forall n \in \mathbb{Z}$$

which means that  $i_f : Y^\bullet \rightarrow C_f^\bullet$  is a chain map in  $C^*(\mathcal{A})$ .

(3) Analogously, we define the graded morphism  $p_f : C_f^\bullet \rightarrow \Sigma(X^\bullet)$ , where  $p_f := p_{X^{n+1}} : C_f^n \rightarrow X^{n+1}$ ,  $n \in \mathbb{Z}$ , i.e. the projection of  $C_f^n = X^{n+1} \oplus Y^n$  to  $X^{n+1}$ . Then we have

$$p_f^{n+1} \circ d_{C_f}^n = (1_{X^{n+2}} \ 0) \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} = (-d_X^{n+1} \ 0) = d_{\Sigma(X)} \circ p_f^n, \quad \forall n \in \mathbb{Z}$$

that is,  $p_f : C_f^\bullet \rightarrow \Sigma(X)$  is a chain map in  $C^*(\mathcal{A})$ .

(4) In addition, we have that

$$p_f^{n+1} \circ i_f^n = (1_{X^{n+2}} \ 0) \begin{pmatrix} 0 \\ 1_{Y^n} \end{pmatrix} = 0$$

**Lemma 1.4.84.** *Let  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map in  $C^*(\mathcal{A})$ . Then the sequence*

$$H^0(Y^\bullet) \xrightarrow{H^0(i_f^\bullet)} H^0(C_f^\bullet) \xrightarrow{H^0(p_f^\bullet)} H^0(X^\bullet)$$

is exact.

*Proof.* Consider  $C_{f^\bullet}^0 = X^1 \oplus Y^0$ , the subcomplex  $\text{Im}(d_X^0) \oplus Y^0$ , and the morphism

$$\begin{pmatrix} d_X^0 & 0 \\ 0 & 1_{Y^0} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ f^0 & 1_{Y^0} \end{pmatrix} = \begin{pmatrix} d_X^0 & 0 \\ -f^0 & 0 \end{pmatrix} : X^0 \oplus Y^0 \rightarrow \text{Im}(d_X^0) \oplus Y^0$$

Clearly, the above morphism is equal to the composition  $-d_{C_f^\bullet}^0 \circ \begin{pmatrix} 1_{X^0} & 0 \\ 0 & 0 \end{pmatrix}$ . Therefore, it induces the zero morphism from  $X^0 \oplus Y^0$  to the quotient object  $\text{Im}(d_X^0) \oplus Y^0$  by  $\text{Im}(d_{C_f^\bullet}^{-1})$ . Hence the morphisms

$$\begin{pmatrix} d_X^0 & 0 \\ 0 & 1_{Y^0} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ f^0 & 1_{Y^0} \end{pmatrix}$$

induce the same morphism from  $X^0 \oplus Y^0$  to  $\text{Im}(d_X^0) \oplus Y^0 / \text{Im}(d_{C_f^\bullet}^{-1})$ . Thus we have  $(0 \oplus Y^0) \text{Im}(d_{C_f^\bullet}^{-1}) = \text{Im}(d_X^0) \oplus Y^0$ . We compute

$$\text{Ker}(d_{C_f^\bullet}^0) \cap (\text{Im}(d_X^0) \oplus Y^0) = \text{Ker}(d_{C_f^\bullet}^0) \cap ((0 \oplus Y^0) + \text{Im}(d_{C_f^\bullet}^0)) = (0 \oplus \text{Ker}(d_Y^0) + \text{Im}(d_{C_f^\bullet}^0)^{-1})$$

Hence  $\text{Ker}(H^0(p_f)) = \text{Im}(H^0(i_f))$ . ■

### 1.4.7 Standard triangles

To give a triangulated structure in the homotopy category of complexes, we need to define the distinguished triangles. The first step is the following definition.

**Definition 1.4.85.** Let  $\mathcal{A}$  be an additive category and  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map in  $C^*(\mathcal{A})$ . The diagram

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i_f^\bullet} C_f^\bullet \xrightarrow{p_f^\bullet} \Sigma(X^\bullet) \quad (1.10)$$

is called the **standard triangle** in  $C^*(\mathcal{A})$  attached to  $f^\bullet$ .

**Lemma 1.4.86.** Let

$$\begin{array}{ccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet \\ u^\bullet \downarrow & & \downarrow v^\bullet \\ X_1^\bullet & \xrightarrow{g^\bullet} & Y_1^\bullet \end{array} \quad (1.11)$$

be a diagram in  $C^*(\mathcal{A})$ , which is commutative up to homotopy, meaning that there is a homotopy  $v^\bullet \circ f^\bullet \sim g^\bullet \circ u^\bullet$ . Then there exists a chain map  $w^\bullet : C_f^\bullet \rightarrow C_g^\bullet$  such that the following diagram is commutative

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{i_f^\bullet} & C_f^\bullet & \xrightarrow{p_f^\bullet} & \Sigma(X^\bullet) \\ \downarrow u^\bullet & & \downarrow v^\bullet & & \downarrow w^\bullet & & \downarrow \Sigma(u^\bullet) \\ X_1^\bullet & \xrightarrow{g^\bullet} & Y_1^\bullet & \xrightarrow{i_g^\bullet} & C_g^\bullet & \xrightarrow{p_g^\bullet} & \Sigma(X_1^\bullet) \end{array} \quad (1.12)$$

*Proof.* From the hypothesis, we have that  $v^\bullet \circ f^\bullet \sim g^\bullet \circ u^\bullet$ , so there exists a homotopy  $h^\bullet : X^\bullet \rightarrow Y^\bullet$ , such that  $g^\bullet \circ u^\bullet + v^\bullet \circ f^\bullet = d_{Y_1}^\bullet \circ h^\bullet + h^\bullet \circ d_X^\bullet$ . We define a graded morphism  $w^\bullet : C_f^\bullet \rightarrow C_g^\bullet$  as  $w^n := \begin{pmatrix} u^{n+1} & 0 \\ h^{n+1} & v^n \end{pmatrix}$ ,  $\forall n \in \mathbb{Z}$ . We compute

$$\begin{aligned} d_{C_g}^n \circ w^n &= \begin{pmatrix} -d_{X_1}^{n+1} & 0 \\ g^{n+1} & d_{Y_1}^n \end{pmatrix} \begin{pmatrix} u^{n+1} & 0 \\ -h^{n+1} & v^n \end{pmatrix} = \begin{pmatrix} -d_{X_1}^{n+2} \circ u^{n+1} & 0 \\ g^{n+1} \circ u^{n+1} - d_{Y_1} \circ h^{n+1} & d_{Y_1} \circ v^n \end{pmatrix} \\ &= \begin{pmatrix} u^{n+2} & 0 \\ -h^{n+2} & v^{n+1} \end{pmatrix} \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} = w^{n+1} \circ d_{C_f}^n, \quad \forall n \in \mathbb{Z} \end{aligned}$$

Hence,  $d_{C_g} \circ w = w \circ d_{C_f}$ , which means that  $w$  is a chain map. In addition, we have

$$w^n \circ i_f^n = \begin{pmatrix} u^{n+1} & 0 \\ -h^{n+1} & v^n \end{pmatrix} \begin{pmatrix} 0 \\ 1_{Y^n} \end{pmatrix} = \begin{pmatrix} 0 \\ v^n \end{pmatrix} = i_g^n \circ v^n, \quad \forall n \in \mathbb{Z}$$

that is,  $u \circ p_f = p_g \circ w$ . This implies that the last square of the previous diagram is commutative. Finally, by choosing  $h = 0$ , the first square of the above diagram is commutative by hypothesis.  $\blacksquare$

We now make some observations that will be useful in what follows.

**Remark 1.4.87.** (1) Let  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map, and  $i_f^\bullet : Y^\bullet \rightarrow C_f^\bullet$  the canonical morphism. Then we can define the cone of  $i_f^\bullet$ , which we denote by  $D_f^\bullet$ , and then  $D_f^n = Y^{n+1} \oplus C_f^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n$ ,  $\forall n \in \mathbb{Z}$ , and the differentials are

$$d_{D_f^n} = \begin{pmatrix} -d_Y^{n+1} & 0 & 0 \\ i_f^{n+1} & d_{C_f^n} & 0 \\ 1_{Y^{n+1}} & f^{n+1} & d_Y^n \end{pmatrix}$$

Now, we define the graded morphism  $\alpha : \Sigma(X^\bullet) \rightarrow D_f^\bullet$  as  $\begin{pmatrix} -f^{n+1} \\ 1_{X^{n+1}} \\ 0 \end{pmatrix}$ ,  $\forall n \in \mathbb{Z}$ , and we compute

$$\begin{aligned} d_{D_f^n} \circ \alpha &= \begin{pmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ 1_{Y^{n+1}} & f^{n+1} & d_Y^n \end{pmatrix} \begin{pmatrix} -f^{n+1} \\ 1_{X^{n+1}} \\ 0 \end{pmatrix} \begin{pmatrix} d_Y^{n+1} \circ f^{n+1} \\ -d_X^{n+1} \\ 0 \end{pmatrix} \begin{pmatrix} f^{n+2} \circ d_X^{n-1} \\ -d_X^{n+1} \\ 0 \end{pmatrix} \\ &= - \begin{pmatrix} f^{n+2} \\ -1_{X^{n+2}} \end{pmatrix} \circ d_X^{n+1} \\ &= \alpha^{n+1} \circ d_{\Sigma(X^\bullet)}^n, \quad \forall n \in \mathbb{Z} \end{aligned}$$

This means that  $\alpha$  is a chain map.

(2) Analogously, we define  $\beta : D_f^\bullet \rightarrow \Sigma(X^\bullet)$  as  $\beta = \begin{pmatrix} 0 & 1_{X^{n+1}} & 0 \end{pmatrix}$ ,  $\forall n \in \mathbb{Z}$ , and we compute

$$d_{\Sigma(X^\bullet)}^n \circ \beta^n = \begin{pmatrix} 0 & -d_X^{n+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1_{X^{n+1}} & 0 \end{pmatrix} \begin{pmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ 1_{Y^{n+1}} & f^{n+1} & d_Y^n \end{pmatrix} = \beta^{n+1} \circ d_{D_f^n}^n, \quad \forall n \in \mathbb{Z}$$

This means that  $\beta$  is also a chain map.

**Lemma 1.4.88.** The chain map  $\alpha : \Sigma(X^\bullet) \rightarrow D_f$  defined in Remark 1.4.87 is an isomorphism in  $\mathcal{K}(\mathcal{A})$ .

*Proof.* For a start, we have

$$\beta^n \circ \alpha^n = \begin{pmatrix} 0 & 1_{X^{n+1}} & 0 \end{pmatrix} \begin{pmatrix} -f^{n+1} \\ 1_{X^{n+1}} \\ 0 \end{pmatrix} = 1_{X^{n+1}}, \quad \forall n \in \mathbb{Z}$$

hence  $\beta^\bullet \circ \alpha^\bullet = 1_{\Sigma(X^\bullet)}$ . On the other hand, we define the graded morphism  $h^\bullet : D_f^\bullet \rightarrow D_f^\bullet$ , which is of degree -1, as

$$h^n = \begin{pmatrix} 0 & 0 & 1_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We compute

$$\begin{aligned} & \begin{pmatrix} -d_Y^n & 0 & 0 \\ 0 & -d_X^n & 0 \\ 1_{Y^n} & f^n & d_Y^{n+1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1_{Y^{n+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ 1_{Y^{n+1}} & f^{n+1} & d_Y^n \end{pmatrix} = \\ & \begin{pmatrix} 0 & 0 & -d_Y^n \\ 0 & 0 & 0 \\ 0 & 0 & 1_Y^n \end{pmatrix} + \begin{pmatrix} 1_{Y^{n+1}} & f^{n+1} & d_Y^n \\ 0 & 1_{X^{n+1}} & 0 \\ 0 & 0 & 1_{Y^n} \end{pmatrix} = \begin{pmatrix} 1_{Y^{n+1}} & f^{n+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_{Y^n} \end{pmatrix} = \begin{pmatrix} 1_{Y^{n+1}} & 0 & 0 \\ 0 & 1_{X^{n+1}} & 0 \\ 0 & 0 & 1_{Y^n} \end{pmatrix} - \begin{pmatrix} 0 & f^{n+1} & 0 \\ 0 & 1_{X^{n+1}} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \\ & 1_{D_f^n} - \alpha^n \circ \beta^n, \forall n \in \mathbb{Z} \end{aligned}$$

This means that the composition  $\alpha^\bullet \circ \beta^\bullet : D_f^\bullet \rightarrow D_f^\bullet$  is homotopic to the identity chain map, which completes the proof.  $\blacksquare$

**Lemma 1.4.89.** *The following diagram is commutative up to homotopy.*

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{i_{f^\bullet}} & C_f^\bullet & \xrightarrow{p_{f^\bullet}} & \Sigma(X^\bullet) & \xrightarrow{-\Sigma(f^\bullet)} & \Sigma(Y^\bullet) \\ \parallel & & \parallel & & \downarrow & & \parallel \\ Y^\bullet & \xrightarrow{i_{f^\bullet}} & C_f^\bullet & \xrightarrow{i_{f^\bullet}} & D_f^\bullet & \xrightarrow{p_{i_{f^\bullet}}} & \Sigma(Y^\bullet) \end{array}$$

*Proof.* The proof is straightforward and left to the reader.  $\blacksquare$

### 1.4.8 The homotopy category of complexes

In this section, we prove that the homotopy category of complexes of an abelian category is a triangulated category. We begin by defining the distinguished triangles in  $K(\mathcal{A})$ .

**Definition 1.4.90.** *Let  $\mathcal{A}$  be an additive category, and let  $K^*(\mathcal{A})$  be the homotopy category of complexes, and let  $\Sigma$  be the shift functor of  $K^*(\mathcal{A})$ . A triangle*

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow \Sigma(X^\bullet)$$

*in  $K^*(\mathcal{A})$  is called **distinguished** if it is isomorphic to the image, under the projection functor  $C^*(\mathcal{A}) \rightarrow K(\mathcal{A})$  of a standard triangle in  $K^*(\mathcal{A})$ .*

The proof that  $K^*(\mathcal{A})$  is a triangulated category proceeds in several steps.

**Remark 1.4.91.** By definition, the axioms (TR1.a) and (TR1.c) are immediate. The following lemma proves (TR1.b).

**Lemma 1.4.92.** *Let  $X^\bullet$  be a complex. Then the cone  $C_{1_{X^\bullet}}^\bullet$  of the identity chain map of  $X^\bullet$  is isomorphic to the zero object in  $K^*(\mathcal{A})$ .*

*Proof.* Consider the complex  $C^\bullet = C_{1_{X^\bullet}}^\bullet = \Sigma(X^\bullet) \oplus X^\bullet$ , and the graded morphism of degree  $-1$   $h^\bullet: C_{1_{X^\bullet}}^\bullet \rightarrow C_{1_{X^\bullet}}^\bullet$ , defined as  $h^n = \begin{pmatrix} 0 & 1_{X^n} \\ 0 & 0 \end{pmatrix}$ ,  $\forall n \in \mathbb{Z}$ . We compute

$$\begin{aligned} d_{C^\bullet}^{n-1} \circ h^n + h^{n+1} \circ d_C^n &= \begin{pmatrix} -d_X^n & 0 \\ 1_{X^n} & d_X^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1_{X^n} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1_{X^{n+1}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_X^{n+1} & 0 \\ 1_{X^{n+1}} & d_X^n \end{pmatrix} = \\ &= \begin{pmatrix} 0 & -d_X^n \\ 0 & 1_{X^n} \end{pmatrix} + \begin{pmatrix} 1_{X^{n+1}} & 0 \\ 0 & 1_{X^{n+1}} \end{pmatrix} = 1_{C^\bullet}^\bullet, \quad \forall n \in \mathbb{Z} \end{aligned}$$

Therefore we proved that  $d_C \circ h + h \circ d_C = 1_{C^\bullet}$ , and since  $1_{C^\bullet}^\bullet \sim 0^\bullet$ , we have that  $C^\bullet = 0^\bullet$  in  $K^*(\mathcal{A})$ . This means that the diagram

$$\begin{array}{ccccccc} X^\bullet & \xlongequal{\quad} & X^\bullet & \longrightarrow & 0^\bullet & \longrightarrow & \Sigma(X^\bullet) \\ & & \parallel & & \downarrow 0 & & \parallel \\ X^\bullet & \xlongequal{\quad} & X^\bullet & \longrightarrow & C_{1_{X^\bullet}}^\bullet & \longrightarrow & \Sigma(X^\bullet) \end{array}$$

is commutative in  $K^*(\mathcal{A})$ , and the vertical arrows are isomorphisms. Because the bottom row is a standard triangle, then also the top row is a standard triangle.  $\blacksquare$

We proceed with the proof of the rest of the axioms.

**Lemma 1.4.93.** *Let  $\mathcal{A}$  be an additive category and  $K^*(\mathcal{A})$  be the homotopy category of complexes, together with the translation functor  $\Sigma$  defined in Notation 1.4.69 and the class of distinguished triangles in Definition 1.4.90. Then the axiom (TR2) of a triangulated category in Definition 1.4.3 is true for  $K^*(\mathcal{A})$ .*

*Proof.* We prove (TR2). Let

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet)$$

be a distinguished triangle in  $K^*(\mathcal{A})$ . By definition, there exists a standard triangle

$$U^\bullet \xrightarrow{\alpha^\bullet} V^\bullet \xrightarrow{i_\alpha^\bullet} C_\alpha^\bullet \xrightarrow{p_\alpha^\bullet} \Sigma(U^\bullet)$$

such that its image in  $K^*(\mathcal{A})$  is isomorphic to the previous triangle. This means that we have the commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \xrightarrow{h^\bullet} & \Sigma(X^\bullet) \\ \downarrow u^\bullet & & \downarrow v^\bullet & & \downarrow w^\bullet & & \downarrow \Sigma(u^\bullet) \\ U^\bullet & \xrightarrow{[\alpha^\bullet]} & V^\bullet & \xrightarrow{[i_\alpha]^\bullet} & C_\alpha^\bullet & \xrightarrow{[p_\alpha]^\bullet} & \Sigma(U^\bullet) \end{array}$$

in  $K^*(\mathcal{A})$ . From Lemma 1.4.88 and 1.4.89, the image of the triangle

$$V^\bullet \xrightarrow{i_\alpha^\bullet} C_\alpha^\bullet \xrightarrow{p_\alpha^\bullet} \Sigma(U^\bullet) \xrightarrow{-\Sigma(\alpha^\bullet)} \Sigma(V^\bullet)$$

is isomorphic to the image of a standard triangle in  $K^*(\mathcal{A})$ . Hence, it is a distinguished triangle in  $K^*(\mathcal{A})$  and the diagram

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \xrightarrow{h^\bullet} & \Sigma(X^\bullet) & \xrightarrow{-\Sigma(f^\bullet)} & \Sigma(Y^\bullet) \\ \downarrow v^\bullet & & \downarrow w^\bullet & & \downarrow \Sigma(u^\bullet) & & \downarrow \Sigma(v^\bullet) \\ V^\bullet & \xrightarrow{[i_\alpha]^\bullet} & C_\alpha^\bullet & \xrightarrow{p_\alpha^\bullet} & \Sigma(U^\bullet) & \xrightarrow{-\Sigma([\alpha]^\bullet)} & \Sigma(U^\bullet) \end{array}$$

is an isomorphism of triangles in  $K^*(\mathcal{A})$ . Since the bottom row is a distinguished triangle, the top row is also a distinguished triangle from (TR1.a), so the triangle

$$Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet) \xrightarrow{\Sigma(f^\bullet)} \Sigma(Y^\bullet)$$

is distinguished. Conversely, suppose that

$$Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet) \xrightarrow{\Sigma(f^\bullet)} \Sigma(Y^\bullet)$$

is a distinguished triangle in  $K^*(\mathcal{A})$ . By definition, there exists a standard triangle

$$U^\bullet \xrightarrow{\alpha^\bullet} V^\bullet \xrightarrow{i_\alpha^\bullet} C_\alpha^\bullet \xrightarrow{p_\alpha^\bullet} \Sigma(U^\bullet)$$

such that its image in  $K^*(\mathcal{A})$  is isomorphic with the above triangle, that is, there exists an isomorphism of triangles

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \xrightarrow{h^\bullet} & \Sigma(X^\bullet) & \xrightarrow{-\Sigma(f^\bullet)} & \Sigma(Y^\bullet) \\ \downarrow u & & \downarrow v^\bullet & & \downarrow w^\bullet & & \downarrow \Sigma(u^\bullet) \\ U^\bullet & \xrightarrow{[\alpha]^\bullet} & V^\bullet & \xrightarrow{[i_\alpha]^\bullet} & C_\alpha^\bullet & \xrightarrow{[p_\alpha]^\bullet} & \Sigma(U^\bullet) \end{array} \quad (1.13)$$

in  $K^*(\mathcal{A})$ . Consider now the chain map  $\Sigma^{-2}(\alpha^\bullet) : \Sigma^{-2}(U^\bullet) \rightarrow \Sigma^{-2}(V^\bullet)$ , and the standard triangle

$$\Sigma^{-2}(U^\bullet) \xrightarrow{\Sigma^{-2}(\alpha^\bullet)} \Sigma^{-2}(V^\bullet) \xrightarrow{i_{\Sigma^{-2}(\alpha^\bullet)}} C_{\Sigma^{-2}(\alpha^\bullet)}^\bullet \xrightarrow{p_{\Sigma^{-2}(\alpha^\bullet)}} \Sigma^{-1}(U^\bullet)$$

More specifically,

$$C_{\Sigma^{-2}(\alpha^\bullet)}^n = \Sigma^{-1}(U^\bullet)^n \oplus \Sigma^{-2}(V^\bullet)^n = U^{n-1} \oplus V^{n-2}, \quad \forall n \in \mathbb{Z}$$

and the differential is

$$d_{C_{\Sigma^{-2}(\alpha^\bullet)}^n}^n = \begin{pmatrix} -d_{\Sigma^{-2}(U^\bullet)}^{n+1} & 0 \\ \Sigma^{-2}(\alpha^\bullet)^{n+1} & d_{\Sigma^{-2}(V^\bullet)}^{n-2} \end{pmatrix} = \begin{pmatrix} -d_u^{n-1} & 0 \\ \alpha^{n-1} & d_v^{n-2} \end{pmatrix} = d_{\Sigma^{-2}(C_\alpha^\bullet)}, \quad \forall n \in \mathbb{Z}$$

Consequently  $C_{\alpha[2]}^\bullet = C_\alpha^\bullet[2]$  and

$$\Sigma^{-2}(U^\bullet) \xrightarrow{\Sigma^{-2}([\alpha^\bullet])} \Sigma^{-2}(V^\bullet) \xrightarrow{\Sigma^{-2}([i_{\alpha^\bullet}])} \Sigma^{-2}(C_\alpha^\bullet) \xrightarrow{\Sigma^{-2}([p_{\alpha^\bullet}])} \Sigma^{-2}(U^\bullet)$$

is a distinguished triangle in  $K^*(\mathcal{A})$ . By applying  $\Sigma^{-2}$  to the above isomorphism of triangles (1.13), we obtain the distinguished triangle

$$\Sigma^{-2}(Y^\bullet) \xrightarrow{\Sigma^{-2}(g^\bullet)} \Sigma^{-2}(Z^\bullet) \xrightarrow{\Sigma^{-2}(h^\bullet)} \Sigma^{-1}(X^\bullet) \xrightarrow{-\Sigma^{-1}(f^\bullet)} \Sigma^{-1}(Y^\bullet)$$

By applying the above argument multiple times we get that

$$\Sigma^{-1}(X^\bullet) \xrightarrow{-\Sigma^{-1}(f^\bullet)} \Sigma^{-1}(Y^\bullet) \xrightarrow{-\Sigma^{-1}(g^\bullet)} \Sigma^{-1}(Z^\bullet) \xrightarrow{-\Sigma^{-1}(h^\bullet)} X^\bullet$$

$$\Sigma^{-1}(Y^\bullet) \xrightarrow{-\Sigma^{-1}(g^\bullet)} \Sigma^{-1}(Z^\bullet) \xrightarrow{-\Sigma^{-1}(h^\bullet)} X^\bullet \xrightarrow{f^\bullet} Y^\bullet$$

$$\Sigma^{-1}(Z^\bullet) \xrightarrow{-\Sigma^{-1}(h^\bullet)} X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet$$

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet)$$

are all distinguished triangles in  $K^*(\mathcal{A})$ . Thus we have completed the proof of (TR2) ■

We proceed with the proof of (TR3).

**Lemma 1.4.94.** *Let  $\mathcal{A}$  be an additive category and  $K^*(\mathcal{A})$  be the homotopy category of complexes, together with the translation functor  $\Sigma$  defined in Notation 1.4.69 and the class of distinguished triangles in Definition 1.4.90. Then the axiom (TR3) of a triangulated category in Definition 1.4.3 is true for  $K^*(\mathcal{A})$ .*

*Proof.* Let

$$\begin{array}{ccccccc} X^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & \Sigma(X^\bullet) \\ \downarrow & & \downarrow & & & & \downarrow \\ X_1^\bullet & \longrightarrow & Y_1^\bullet & \longrightarrow & Z_1^\bullet & \longrightarrow & \Sigma(X_1^\bullet) \end{array}$$

be a diagram in  $K^*(\mathcal{A})$  where the lines are distinguished triangles and the first square is commutative. Then there exist standard triangles

$$U^\bullet \xrightarrow{a^\bullet} V^\bullet \longrightarrow C_\alpha^\bullet \longrightarrow \Sigma(U^\bullet)$$

and

$$U_1^\bullet \xrightarrow{b^\bullet} V_1^\bullet \longrightarrow C_\beta^\bullet \longrightarrow \Sigma(U_1^\bullet)$$

such that their images in  $K(\mathcal{A})$  are isomorphic to the above distinguished triangles. Therefore there exist chain maps  $u^\bullet: U^\bullet \rightarrow U_1^\bullet$  and  $v^\bullet: V^\bullet \rightarrow V_1^\bullet$  such that the image of the diagram

$$\begin{array}{ccccccc} U^\bullet & \longrightarrow & V^\bullet & \longrightarrow & C_\alpha^\bullet & \longrightarrow & \Sigma(U^\bullet) \\ u^\bullet \downarrow & & \downarrow v^\bullet & & & & \downarrow \Sigma(u^\bullet) \\ U_1^\bullet & \longrightarrow & V_1^\bullet & \longrightarrow & C_\beta^\bullet & \longrightarrow & \Sigma(U_1^\bullet) \end{array}$$

in  $K(\mathcal{A})$ , is isomorphic with the above diagram. In particular, the first square is commutative up to homotopy. From Lemma 1.4.86 there exists a chain map  $w^\bullet: C_\alpha^\bullet \rightarrow C_\beta^\bullet$  such that the diagram

$$\begin{array}{ccccccc} U^\bullet & \longrightarrow & V^\bullet & \longrightarrow & C_\alpha^\bullet & \longrightarrow & \Sigma(U^\bullet) \\ u^\bullet \downarrow & & \downarrow v^\bullet & & \downarrow w^\bullet & & \downarrow \Sigma(u^\bullet) \\ U_1^\bullet & \longrightarrow & V_1^\bullet & \longrightarrow & C_\beta^\bullet & \longrightarrow & \Sigma(U_1^\bullet) \end{array}$$

is commutative up to homotopy. This implies that there exists a chain map  $Z^\bullet \rightarrow Z_1^\bullet$  which completes the diagram at the start of the proof to the commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & \Sigma(X^\bullet) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_1^\bullet & \longrightarrow & Y_1^\bullet & \longrightarrow & Z_1^\bullet & \longrightarrow & \Sigma(X_1^\bullet) \end{array} \quad \blacksquare$$

It is left to prove (TR4). However, before we do that, we first prove an auxiliary lemma which provides a characterization of distinguished triangles in  $K^*(\mathcal{A})$ .

**Lemma 1.4.95.** *Let  $f^\bullet: X^\bullet \rightarrow Y^\bullet$  be a morphism in  $K^*(\mathcal{A})$  and  $\alpha^\bullet: X^\bullet \rightarrow Y^\bullet$  a chain map which represents  $f^\bullet$ . Then the following are equivalent:*

(1) *The triangle*

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \longrightarrow Z^\bullet \longrightarrow \Sigma(X^\bullet)$$

*is distinguished,*

(2) *There exists a morphism  $u^\bullet: Z^\bullet \rightarrow C_\alpha^\bullet$  such that the diagram*

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & \Sigma(X^\bullet) \\ \parallel & & \parallel & & \downarrow u & & \parallel \\ X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{[i_{\alpha^\bullet}]} & C_\alpha^\bullet & \xrightarrow{[p_{\alpha^\bullet}]} & \Sigma(X^\bullet) \end{array}$$

*is an isomorphism of triangles.*

*Proof.* (1)  $\Rightarrow$  (2) : The image of the standard triangle

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow C_\alpha^\bullet \longrightarrow \Sigma(X^\bullet)$$

is a distinguished triangle in  $K^*(\mathcal{A})$ . Therefore we have the diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & \Sigma(X^\bullet) \\ \parallel & & \parallel & & & & \parallel \\ X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{[i_{\alpha^\bullet}]} & C_\alpha^\bullet & \xrightarrow{[p_{\alpha^\bullet}]} & \Sigma(X^\bullet) \end{array}$$

where both rows are distinguished triangles and the first square is commutative in  $K^*(\mathcal{A})$ . Since we have already prove (TR3), the above diagram can be completed to the following commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & \Sigma(X^\bullet) \\ \parallel & & \parallel & & \downarrow & & \downarrow \\ X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{[i_{\alpha^\bullet}]} & C_\alpha^\bullet & \xrightarrow{[p_{\alpha^\bullet}]} & \Sigma(X^\bullet) \end{array}$$

In addition, from Lemma 1.4.18 this does not depend on the octahedral axiom, therefore this morphism is an isomorphism.

(2)  $\Rightarrow$  (1) : From the definition of the distinguished triangles in  $K^*(\mathcal{A})$ , it is immediate. ■

We proceed with the proof of (TR4).

**Lemma 1.4.96.** *Let  $\mathcal{A}$  be an additive category and  $K^*(\mathcal{A})$  be the homotopy category of complexes, together with the translation functor  $\Sigma$  defined in Notation 1.4.69 and the class of distinguished triangles in Definition 1.4.90. Then the axiom (TR4) of a triangulated category in Definition 1.4.3 is true for  $K^*(\mathcal{A})$ .*

*Proof.* Let  $f^\bullet: X^\bullet \rightarrow Y^\bullet$ ,  $g^\bullet: Y^\bullet \rightarrow Z^\bullet$  and  $h^\bullet: X^\bullet \rightarrow Z^\bullet$ , where  $h^\bullet = g^\bullet \circ f^\bullet$  be three morphisms  $K^*(\mathcal{A})$ . Consider the diagram

$$\begin{array}{ccccccc}
 X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z_1^\bullet & \longrightarrow & \Sigma(X^\bullet) \\
 \parallel & & \downarrow g^\bullet & & & & \parallel \\
 X^\bullet & \xrightarrow{h^\bullet} & Z^\bullet & \longrightarrow & Y_1^\bullet & \longrightarrow & \Sigma(X^\bullet) \\
 f^\bullet \downarrow & & \parallel & & & & \downarrow \Sigma(f^\bullet) \\
 Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \longrightarrow & X_1^\bullet & \longrightarrow & \Sigma(Y^\bullet)
 \end{array}$$

where the lines are distinguished triangles and the left squares are commutative. From Lemma 1.4.95, there exist chain maps  $a^\bullet: X^\bullet \rightarrow Y^\bullet$ ,  $b^\bullet: Y^\bullet \rightarrow Z^\bullet$  and  $c^\bullet: X^\bullet \rightarrow Z^\bullet$ , where  $c^\bullet = b^\bullet \circ a^\bullet$ , such that the triangles

$$\begin{array}{ccccccc}
 X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z_1^\bullet & \longrightarrow & \Sigma(X^\bullet) \\
 Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \longrightarrow & X_1^\bullet & \longrightarrow & \Sigma(Y^\bullet) \\
 X^\bullet & \xrightarrow{h^\bullet} & Z^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & \Sigma(X^\bullet)
 \end{array}$$

are isomorphic to the images of the standard triangles

$$\begin{array}{ccccccc}
 X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \longrightarrow & C_a^\bullet & \longrightarrow & \Sigma(X^\bullet) \\
 Y^\bullet & \xrightarrow{b^\bullet} & Z^\bullet & \longrightarrow & C_b^\bullet & \longrightarrow & \Sigma(Y^\bullet) \\
 X^\bullet & \xrightarrow{c^\bullet} & Z^\bullet & \longrightarrow & C_c^\bullet & \longrightarrow & \Sigma(X^\bullet)
 \end{array}$$

respectively, and the isomorphisms are induced by the chain maps  $1_{X^\bullet}$ ,  $1_{Y^\bullet}$  and  $1_{Z^\bullet}$ . Therefore the above diagram is isomorphic to the image of the diagram

$$\begin{array}{ccccccc}
 X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \xrightarrow{i_a^\bullet} & C_a^\bullet & \xrightarrow{p_a^\bullet} & \Sigma(X^\bullet) \\
 \parallel & & \downarrow b^\bullet & & & & \parallel \\
 X^\bullet & \longrightarrow & Z^\bullet & \xrightarrow{i_c^\bullet} & C_c^\bullet & \xrightarrow{p_c^\bullet} & \Sigma(X^\bullet) \\
 a^\bullet \downarrow & & \parallel & & & & \downarrow \Sigma(a^\bullet) \\
 Y^\bullet & \xrightarrow{b^\bullet} & Z^\bullet & \xrightarrow{i_b^\bullet} & C_b^\bullet & \xrightarrow{p_b^\bullet} & \Sigma(Y^\bullet)
 \end{array}$$

where the squares on the left are commutative. In the proof of Lemma 1.4.86, we have shown that the morphisms  $u^\bullet: C_a^\bullet \rightarrow C_c^\bullet$  and  $v^\bullet: C_c^\bullet \rightarrow C_b^\bullet$  are given by

$$u^n = \begin{pmatrix} 1_{X^{n+1}} & 0 \\ 0 & b^n \end{pmatrix} \quad \text{and} \quad v^n = \begin{pmatrix} a^{n+1} & 0 \\ 0 & 1_{Z^n} \end{pmatrix}, \quad \forall n \in \mathbb{Z}$$

and they complete the above diagram to the commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \xrightarrow{i_a^\bullet} & C_a^\bullet & \xrightarrow{p_a^\bullet} & \Sigma(X^\bullet) \\ \parallel & & \downarrow b^\bullet & & \downarrow u^\bullet & & \parallel \\ X^\bullet & \longrightarrow & Z^\bullet & \xrightarrow{i_c^\bullet} & C_c^\bullet & \xrightarrow{p_b^\bullet} & \Sigma(X^\bullet) \\ a^\bullet \downarrow & & \parallel & & \downarrow v^\bullet & & \downarrow \Sigma(a^\bullet) \\ Y^\bullet & \xrightarrow{b^\bullet} & Z^\bullet & \xrightarrow{i_b^\bullet} & C_b^\bullet & \xrightarrow{p_b^\bullet} & \Sigma(Y^\bullet) \end{array}$$

To prove (TR4), it is enough to prove the following claim:

Claim: The triangle

$$\begin{array}{ccccccc} C_a^\bullet & \xrightarrow{u^\bullet} & C_c^\bullet & \xrightarrow{v^\bullet} & C_b^\bullet & \xrightarrow{\Sigma(i_a^\bullet) \circ p_b^\bullet} & \Sigma(C_a^\bullet) \\ \parallel & & \parallel & & & & \parallel \\ C_A^\bullet & \xrightarrow{u^\bullet} & C_c^\bullet & \xrightarrow{i_a^\bullet} & C_u^\bullet & \xrightarrow{p_u^\bullet} & \Sigma(C_a^\bullet) \end{array}$$

Proof of Claim: It is enough to complete the diagram

$$\begin{array}{ccccccc} C_a^\bullet & \xrightarrow{u^\bullet} & C_c^\bullet & \xrightarrow{v^\bullet} & C_b^\bullet & \xrightarrow{\Sigma(i_a^\bullet) \circ p_b^\bullet} & \Sigma(C_a^\bullet) \\ \parallel & & \parallel & & & & \parallel \\ C_a^\bullet & \xrightarrow{u} & C_c^\bullet & \xrightarrow{i_u^\bullet} & C_u^\bullet & \xrightarrow{p_u^\bullet} & \Sigma(C_a^\bullet) \end{array}$$

to the diagram

$$\begin{array}{ccccccc} C_a^\bullet & \xrightarrow{u^\bullet} & C_c^\bullet & \xrightarrow{v^\bullet} & C_b^\bullet & \xrightarrow{\Sigma(i_a^\bullet) \circ p_b^\bullet} & \Sigma(C_a^\bullet) \\ \parallel & & \parallel & & \downarrow w^\bullet & & \parallel \\ C_a^\bullet & \xrightarrow{u} & C_c^\bullet & \xrightarrow{i_u^\bullet} & C_u^\bullet & \xrightarrow{p_u^\bullet} & \Sigma(C_a^\bullet) \end{array}$$

which is commutative up to homotopy, and the morphism  $w^\bullet$  induces a morphism in  $K^*(\mathcal{A})$ . Then the image of the first row in  $K^*(\mathcal{A})$  is a triangle which is isomorphic to the image of a standard triangle in  $K^*(\mathcal{A})$ , i.e. a distinguished triangle. As complexes,  $C_b^\bullet$  and  $C_c^\bullet$  are defined as

$$C_b^\bullet = \Sigma(Y^\bullet) \oplus Z^\bullet$$

$$C_u^\bullet = \Sigma(C_a^\bullet) \oplus C_a^\bullet = \Sigma^2(X^\bullet) \oplus \Sigma(Y^\bullet) \oplus \Sigma(X^\bullet) \oplus Z^\bullet$$

with differentials

$$d_{C_b}^n = \begin{pmatrix} -d_Y^{n+1} & 0 \\ b^{n+1} & d_Z^n \end{pmatrix}$$

$$d_{C_u}^n = \begin{pmatrix} -d_{C_a}^{n+1} & 0 \\ u^{n+1} & d_{C_c}^n \end{pmatrix} = \begin{pmatrix} d_X^{n+2} & 0 & 0 & 0 \\ -a^{n+2} & -d_Y^{n+2} & 0 & 0 \\ 1_{X^{n+2}} & 0 & -d_X^{n+1} & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{pmatrix}$$

From the above complexes, we can define a chain map  $w^\bullet : C_b^\bullet \rightarrow C_u^\bullet$ , which is of degree zero, as

$$w^n = \begin{pmatrix} 0 & 0 \\ 1_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & 1_Z^n \end{pmatrix}, \forall n \in \mathbb{Z}$$

and then we have

$$\begin{aligned} w^{n+1} \circ d_{C_b}^n &= \begin{pmatrix} 0 & 0 \\ 1_{Y^{n+2}} & 0 \\ 0 & 0 \\ 0 & 1_{Z^{n+1}} \end{pmatrix} \begin{pmatrix} -d_Y^{n+1} & 0 \\ b^{n+1} & d_Z^n \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -d_Y^{n+1} & 0 \\ 0 & 0 \\ b^{n+1} & d_Z^n \end{pmatrix} \\ &= \begin{pmatrix} d_X^{n+2} & 0 & 0 & 0 \\ -a^{n+2} & -d_Y^{n+1} & 0 & 0 \\ 1_{X^{n+2}} & 0 & -d_X^{n+1} & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & 1_Z^n \end{pmatrix} \\ &= d_{C_u}^n \circ w^n, \forall n \in \mathbb{Z} \end{aligned}$$

Now we have,  $\forall n \in \mathbb{Z}$

$$p_u^n \circ w^n = \begin{pmatrix} 1_{X^{n+2}} & 0 & 0 & 0 \\ 0 & 1_{Y^{n+1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1_{Y^{n+1}} & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1_{Y^{n+1}} \end{pmatrix} (1_{Y^{n+1}} \ 0) = \Sigma(i_a)^n \circ p_b^n$$

The last computation yields that the third square of the above diagram is commutative. We now show that the middle square is commutative up to homotopy. We compute

$$\begin{aligned}
w^n \circ v^n - i_u^n &= \begin{pmatrix} 0 & 0 \\ 1_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & 1_{Z^n} \end{pmatrix} \begin{pmatrix} a^{n+1} & 0 \\ 0 & 1_{Z^n} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1_{X^{n+1}} & 0 \\ 0 & 1_{Z^n} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ a^{n+1} & 0 \\ 0 & 0 \\ 0 & 1_{Z^n} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1_{X^{n+1}} & 0 \\ 0 & 1_{Z^n} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ a^{n+1} & 0 \\ -1_{X^{n+1}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall n \in \mathbb{Z}
\end{aligned}$$

Consider now the chain map  $h^\bullet : C_c^\bullet \rightarrow C_u^\bullet$  which is defined as

$$h^n = \begin{pmatrix} 1_{X^{n+1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall n \in \mathbb{Z}$$

Then we have

$$\begin{aligned}
d_{C_u}^{n+1} \circ h^n + h^{n+1} \circ d_{C_c}^n &= \begin{pmatrix} d_X^{n+1} & 0 & 0 & 0 \\ -a^{n+1} & -d_Y^n & 0 & 0 \\ 1_{X^{n+1}} & 0 & -d_X^n & 0 \\ 0 & b^n & c^n & d_Z^{n-1} \end{pmatrix} \begin{pmatrix} 1_{X^{n+1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1_{X^{n+2}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_X^{n+1} & 0 \\ c^{n+1} & d_Z^n \end{pmatrix} \\
&= \begin{pmatrix} d_X^{n+1} & 0 \\ -a^{n+1} & 0 \\ 1_{X^{n+1}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -d_X^{n+1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -a^{n+1} & 0 \\ 1_{X^{n+1}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall n \in \mathbb{Z}
\end{aligned}$$

Hence,

$$d_{C_u} \circ h + h \circ d_{C_c} = i_u - w \circ v$$

which means that  $w \circ v \sim i_u$ . It is left to show that  $w$  is an isomorphism in  $K^*(\mathcal{A})$ . We define the chain map  $\theta^\bullet : C_u^\bullet \rightarrow C_b^\bullet$  as follows:

$$\theta^n = \begin{pmatrix} 0 & 1_{Y^{n+1}} & a^{n+1} & 0 \\ 0 & 0 & 0 & 1_{Z^n} \end{pmatrix}, \forall n \in \mathbb{Z}$$

Then we compute

$$\begin{aligned} \theta^{n+1} \circ d_{C_b}^n &= \begin{pmatrix} 0 & 1_{Y^{n+2}} & a^{n+2} & 0 \\ 0 & 0 & 0 & 1_{Z^{n+1}} \end{pmatrix} \begin{pmatrix} d_X^{n+2} & 0 & 0 & 0 \\ -a^{n+2} & -d_Y^{n+1} & 0 & 0 \\ 1_{X^{n+2}} & 0 & -d_X^{n+1} & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{pmatrix} \\ &= \begin{pmatrix} 0 & -d_Y^{n+1} & -a^{n+2} \circ d_X^{n+1} & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{pmatrix} \\ &= \begin{pmatrix} 0 & -d_Y^{n+1} & -d_Y^{n+1} \circ a^{n+1} & 0 \\ 0 & b^{n+1} & b^{n+1} \circ a^{n+1} & d_Z^n \end{pmatrix} \\ &= \begin{pmatrix} -d_Y^{n+1} & 0 \\ b^{n+1} & d_Z^n \end{pmatrix} \begin{pmatrix} 0 & 1_{Y^{n+1}} & a^{n+1} & 0 \\ 0 & 0 & 0 & 1_{Z^n} \end{pmatrix} \\ &= d_{C_b}^n \circ \theta^n, \forall n \in \mathbb{Z} \end{aligned}$$

Thus we proved that  $\theta$  is a chain map. In addition we have

$$\theta^n \circ w^n = \begin{pmatrix} 0 & 1_{Y^{n+1}} & a^{n+1} & 0 \\ 0 & 0 & 0 & 1_{Z^n} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & 1_{Z^n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{Y^{n+1}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{Z^n} \end{pmatrix}, \forall n \in \mathbb{Z}$$

By considering the graded morphism  $x : C_u^\bullet \rightarrow C_u^\bullet$  of degree -1 as follows:

$$x^n = \begin{pmatrix} 0 & 0 & 1_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \forall n \in \mathbb{Z}$$

we have

$$\begin{aligned}
d_{C_u}^{n-1} \circ x^n + x^{n+1} \circ d_{C_u}^n &= \begin{pmatrix} d_X^{n+1} & 0 & 0 & 0 \\ -a^{n+1} & -d_Y^n & 0 & 0 \\ 1_{X^{n+1}} & 0 & -d_X^n & 0 \\ 0 & b^n & c^n & d_Z^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \\
\begin{pmatrix} 0 & 0 & 1_{X^{n+2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_X^{n+2} & 0 & 0 & 0 \\ -a^{n+2} & -d_Y^{n+1} & 0 & 0 \\ 1_{X^{n+2}} & 0 & -d_X^n & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{pmatrix} &= \begin{pmatrix} 0 & 0 & d_X^{n+1} & 0 \\ 0 & 0 & -a^{n+1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1_{X^{n-2}} & 0 & -d_X^{n+1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \\
&= \begin{pmatrix} 1_{X^{n+2}} & 0 & 0 & 0 \\ 0 & 0 & -a^{n+1} & 0 \\ 0 & 0 & 1_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \forall n \in \mathbb{Z}
\end{aligned}$$

Hence we have that  $d_{C_u} \circ x^\bullet + x^\bullet \circ d_{C_u} = 1_{C_u} - w \circ \theta^\bullet$ , which means that  $w^\bullet \circ \theta^\bullet \sim 1_{C_u}^\bullet$ . Thus we proved that  $w$  induces an isomorphism in  $K^*(\mathcal{A})$ , which completes the proof of the claim.  $\blacksquare$

We are now in a position to prove the following Theorem.

**Theorem 1.4.97.** *Let  $\mathcal{A}$  be an additive category. The additive category  $K(\mathcal{A})$ , equipped with the translation functor in Remark 1.4.69 and the distinguished triangles in Definition 1.4.90, is a triangulated category.*

*Proof.* By combining Remark 1.4.91 and Lemmas 1.4.92, 1.4.93, 1.4.94, 1.4.95 and 1.4.96 we have the result.  $\blacksquare$

### 1.4.9 The homotopy category of an abelian category

In this section, we shift our focus to abelian categories. Unless otherwise stated  $\mathcal{A}$  will be an abelian category.

**Theorem 1.4.98.** *The functor  $H^0 : K^*(\mathcal{A}) \rightarrow \mathcal{A}$  is a cohomological functor*

*Proof.* We have to prove that for any distinguished triangle

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{s^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet)$$

in  $K^*(\mathcal{A})$ , the sequence

$$H^0(Y^\bullet) \longrightarrow H^0(Z^\bullet) \longrightarrow H^0(\Sigma(X^\bullet))$$

is exact in  $\mathcal{A}$ . Let  $a^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map, which is a representative of  $f^\bullet$ , and

$$X^\bullet \xrightarrow{a^\bullet} Y^\bullet \xrightarrow{i_a^\bullet} C_a^\bullet \xrightarrow{p_a^\bullet} \Sigma(X^\bullet)$$

be the standard triangle of the chain map  $a^\bullet$ . Then we have the isomorphism of triangles

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \xrightarrow{h^\bullet} & \Sigma(X^\bullet) \\ \parallel & & \parallel & & \downarrow u & & \parallel \\ X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{[i_a^\bullet]} & C_a^\bullet & \xrightarrow{[p_a^\bullet]} & \Sigma(X^\bullet) \end{array}$$

in  $K^*(\mathcal{A})$ , where the bottom triangle is the image of the top triangle in  $K^*(\mathcal{A})$ . Thus we get the diagram

$$\begin{array}{ccccc} H^0(Y^\bullet) & \xrightarrow{H^0(g)} & H^0(Z^\bullet) & \xrightarrow{H^0(h)} & H^0(\Sigma(X^\bullet)) \\ \parallel & & \downarrow H^0(u) & & \parallel \\ H^0(Y^\bullet) & \xrightarrow{H^0(i_a^\bullet)} & H^0(C_a^\bullet) & \xrightarrow{H^0(p_a^\bullet)} & H^0(\Sigma(X^\bullet)) \end{array}$$

in  $\mathcal{A}$ , where the vertical arrows are isomorphisms. Therefore, it is enough to show that the bottom row is exact. However, this is proved in Lemma 1.4.84, and this completes the proof. ■

We now have the following Corollary.

**Corollary 1.4.99.** *Let*

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet)$$

*be a distinguished triangle in  $K^*(\mathcal{A})$ . Then the sequence*

$$\cdots \longrightarrow H^n(X^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(Y^\bullet) \xrightarrow{H^n(g^\bullet)} H^n(Z^\bullet) \xrightarrow{H^n(h^\bullet)} H^{n+1}(X^\bullet) \longrightarrow \cdots$$

*is exact in  $\mathcal{A}$ .*

*Proof.* The proof follows from the above Theorem. ■

**Definition 1.4.100.** *The above exact sequence is called the **long exact sequence of cohomology** associated to the triangle*

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet)$$

Let  $\mathcal{A}^{\text{op}}$  be the opposite category of  $\mathcal{A}$ . Then the functors  $H^n : K(\mathcal{A}) \rightarrow \mathcal{A}$ ,  $n \in \mathbb{Z}$  induce functors from  $K(\mathcal{A})^{\text{op}}$  to  $\mathcal{A}$ , which we denote again by  $H^n$ .

**Remark 1.4.101.** If

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet)$$

is a distinguished triangle in  $K(\mathcal{A})^{\text{op}}$ , then the triangle

$$Z^\bullet \xrightarrow{g^\bullet} Y^\bullet \xrightarrow{f^\bullet} X^\bullet \xrightarrow{\Sigma(h^\bullet)} \Sigma(Z^\bullet)$$

is a distinguished in  $K(\mathcal{A})$ . Therefore, we have the long exact sequence

$$\cdots \longrightarrow H^n(X^\bullet) \xrightarrow{H^n(g^\bullet)} H^n(Y^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(X^\bullet) \xrightarrow{H^{n+1}(h^\bullet)} H^{n+1}(Z^\bullet) \longrightarrow \cdots$$

If we see the above sequence as a sequence in  $\mathcal{A}^{\text{op}}$ , we obtain the exact sequence

$$\cdots \longrightarrow H^n(Z^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(Y^\bullet) \xrightarrow{H^n(g^\bullet)} H^n(Z^\bullet) \longrightarrow H^{n-1}(X^\bullet) \longrightarrow \cdots$$

Therefore we can consider  $H^0$  as a cohomological functor from  $K(\mathcal{A})^{\text{op}}$  to  $\mathcal{A}^{\text{op}}$ .

### 1.4.10 Quasi-isomorphisms

In this section, let  $\mathcal{A}$  be an abelian category and  $K^*(\mathcal{A})$  be the homotopy category of complexes with its triangulated structure. In order to construct the derived category we need a localizing class in  $K(\mathcal{A})$ . This localizing class consists of chain maps called quasi-isomorphism.

**Definition 1.4.102.** A chain map  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  in  $C^*(\mathcal{A})$  is called a **quasi-isomorphism**, if  $H^n(f^\bullet) : H^n(X^\bullet) \rightarrow H^n(Y^\bullet)$  is an isomorphism for any  $n \in \mathbb{Z}$ .

**Remark 1.4.103.** Let  $\mathcal{A}$  be an abelian category and  $f^\bullet$  a quasi-isomorphism, and  $g^\bullet$  be a chain map. If  $f^\bullet \simeq g^\bullet$ , then  $H^n(f^\bullet) = H^n(g^\bullet)$ ,  $\forall n \in \mathbb{Z}$ , hence  $g^\bullet$  is also a quasi-isomorphism. As a convention we will say that a chain map is a quasi-isomorphism in  $K^*(\mathcal{A})$ , if all of its representatives are quasi-isomorphisms.

**Notation 1.4.104.** We denote by  $S^*$  the class of all quasi-isomorphism in  $K^*(\mathcal{A})$ .

**Definition 1.4.105.** A chain complex  $X^\bullet$  in  $K^*(\mathcal{A})$  is called **acyclic** if  $H^n(X^\bullet) = 0$ ,  $\forall n \in \mathbb{Z}$ .

The following provides a criterion for when a chain map is a quasi-isomorphism.

**Lemma 1.4.106.** Let  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  be a chain map in  $K^*(\mathcal{A})$ . The following are equivalent:

- (1)  $f^\bullet$  is a quasi-isomorphism,  
 (2) the cone of  $f^\bullet$  is acyclic.

*Proof.* (1)  $\Rightarrow$  (2): Let

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \longrightarrow Z^\bullet \longrightarrow \Sigma(X^\bullet)$$

be a distinguished triangle with base  $f^\bullet$ . From Corollary 1.4.99, we obtain the long exact sequence

$$\cdots \longrightarrow H^n(X^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(Y^\bullet) \longrightarrow H^n(Z^\bullet) \longrightarrow H^{n+1}(X^\bullet) \xrightarrow{H^{n+1}(f^\bullet)} H^{n+1}(Y^\bullet) \longrightarrow \cdots$$

Since  $f^\bullet$  is a quasi-isomorphism, then  $H^n(f^\bullet)$  and  $H^{n+1}(f^\bullet)$  are isomorphisms, hence  $H^n(Z^\bullet) = 0$ ,  $\forall n \in \mathbb{Z}$ . This implies that  $Z^\bullet$  is acyclic.

(2)  $\Rightarrow$  (1): Suppose that  $Z^\bullet$  is acyclic. Again, we have the long exact sequence

$$\cdots \longrightarrow H^n(X^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(Y^\bullet) \longrightarrow H^n(Z^\bullet) \longrightarrow H^{n+1}(X^\bullet) \xrightarrow{H^{n+1}(f^\bullet)} H^{n+1}(Y^\bullet) \longrightarrow \cdots$$

We have that  $H^n(Z^\bullet) = H^{n+1}(Z^\bullet) = 0$ , hence  $H^n(f^\bullet)$  is an isomorphism for every  $n \in \mathbb{Z}$ . By definition, this implies that  $f^\bullet$  is a quasi-isomorphism.  $\blacksquare$

In our effort to define the derived category, we first prove that the class of quasi-isomorphisms  $S^*$ , is a localizing class of  $K^*(\mathcal{A})$ .

**Proposition 1.4.107.** *The class  $S^*$  of quasi-isomorphisms of  $K^*(\mathcal{A})$  is a localizing class compatible with the triangulated structure of  $K^*(\mathcal{A})$ .*

*Proof.* We will first show that  $S^*$  is a localizing class.

(LC1) Let  $s^\bullet, t^\bullet$  be quasi-isomorphisms. Then by definition  $H^n(s^\bullet)$  and  $H^n(t^\bullet)$  are isomorphisms for every  $n \in \mathbb{Z}$ . This implies that  $H^n(s^\bullet \circ t^\bullet) = H^n(s^\bullet) \circ H^n(t^\bullet)$  is an isomorphism for every  $n \in \mathbb{Z}$ , hence  $s^\bullet \circ t^\bullet$  is a quasi-isomorphism. Therefore  $s^\bullet \circ t^\bullet \in S^*$ .

(LC2) Immediate since  $1_X^\bullet$  is a quasi-isomorphism.

(LC3.a) Consider the diagram

$$\begin{array}{ccc} & & Z^\bullet \\ & & \downarrow f^\bullet \\ X^\bullet & \xrightarrow{s^\bullet} & Y^\bullet \end{array}$$

We can embed  $s^\bullet$  to the distinguished triangle

$$X^\bullet \xrightarrow{s^\bullet} Y^\bullet \xrightarrow{i^\bullet} U^\bullet \xrightarrow{p^\bullet} \Sigma(X^\bullet)$$

From Lemma 1.4.106 we have that  $U^\bullet$  is acyclic. By rotating the above triangle, we obtain the distinguished triangle

$$Y^\bullet \xrightarrow{i^\bullet} U^\bullet \xrightarrow{p^\bullet} \Sigma(X^\bullet) \xrightarrow{-\Sigma(s^\bullet)} \Sigma(Y^\bullet)$$

In addition, we can embed the composition  $i^\bullet \circ f^\bullet$  in a distinguished triangle

$$Z^\bullet \xrightarrow{i^\bullet \circ f^\bullet} U^\bullet \longrightarrow V^\bullet \xrightarrow{u^\bullet} \Sigma(Z^\bullet)$$

therefore we obtain the commutative diagram

$$\begin{array}{ccccccc} Z^\bullet & \xrightarrow{i^\bullet \circ f^\bullet} & U^\bullet & \longrightarrow & V^\bullet & \xrightarrow{u^\bullet} & \Sigma(Z^\bullet) \\ f^\bullet \downarrow & & \parallel & & & & \downarrow \\ Y^\bullet & \xrightarrow{i^\bullet} & U^\bullet & \xrightarrow{p^\bullet} & \Sigma(X^\bullet) & \xrightarrow{-\Sigma(s^\bullet)} & \Sigma(Y^\bullet) \end{array}$$

From (TR3) the above diagram can be completed to the commutative diagram

$$\begin{array}{ccccccc} Z^\bullet & \xrightarrow{i^\bullet \circ f^\bullet} & U^\bullet & \longrightarrow & V^\bullet & \xrightarrow{u^\bullet} & \Sigma(Z^\bullet) \\ f^\bullet \downarrow & & \parallel & & \downarrow v^\bullet & & \downarrow \\ Y^\bullet & \xrightarrow{i^\bullet} & U^\bullet & \xrightarrow{p^\bullet} & \Sigma(X^\bullet) & \xrightarrow{-\Sigma(s^\bullet)} & \Sigma(Y^\bullet) \end{array}$$

where both rows are distinguished triangles. Since  $U^\bullet$  is acyclic, from Lemma 1.4.106, we have that  $u^\bullet$  is a quasi-isomorphism. Therefore, by applying the functor  $\Sigma^{-1}$  and by setting  $W^\bullet := V^\bullet[-1]$ ,  $t^\bullet := u^\bullet[-1]$  and  $g^\bullet := -v^\bullet[-1]$ , we obtain the commutative diagram

$$\begin{array}{ccc} W^\bullet & \xrightarrow{t^\bullet} & Z^\bullet \\ g^\bullet \downarrow & & \downarrow f^\bullet \\ X^\bullet & \xrightarrow{s^\bullet} & Y^\bullet \end{array}$$

where  $s^\bullet, t^\bullet \in S^*$ .

(LC3.b) Analogously, consider the diagram

$$\begin{array}{ccc} X^\bullet & \xrightarrow{s^\bullet} & Y^\bullet \\ f^\bullet \downarrow & & \\ Z^\bullet & & \end{array}$$

We can embed the morphism  $s^\bullet$  to a distinguished triangle

$$X^\bullet \xrightarrow{s^\bullet} Y^\bullet \xrightarrow{i^\bullet} U^\bullet \xrightarrow{p^\bullet} \Sigma(X^\bullet)$$

From Lemma 1.4.106, since  $s^\bullet$  quasi-isomorphism, we have that  $U^\bullet$  is acyclic. By rotating the above triangle, we obtain the distinguished triangle

$$U^\bullet[-1] \xrightarrow{-p^\bullet[-1]} X^\bullet \xrightarrow{s^\bullet} Y^\bullet \xrightarrow{i^\bullet} \Sigma(U^\bullet[-1])$$

In addition, we can embed the chain map  $-f^\bullet \circ p^\bullet[-1]: U^\bullet[-1] \rightarrow Z^\bullet$  to a distinguished triangle

$$U^\bullet[-1] \xrightarrow{-f^\bullet \circ p^\bullet[-1]} Z^\bullet \xrightarrow{t^\bullet} V^\bullet \xrightarrow{i^\bullet} \Sigma(U^\bullet[-1])$$

Therefore we have the commutative diagram

$$\begin{array}{ccccccc} U^\bullet[-1] & \xrightarrow{-p^\bullet[-1]} & X^\bullet & \xrightarrow{s^\bullet} & Y^\bullet & \xrightarrow{i^\bullet} & U^\bullet \\ \parallel & & \downarrow f^\bullet & & & & \parallel \\ U^\bullet[-1] & \xrightarrow{-f^\bullet \circ p^\bullet[-1]} & Z^\bullet & \xrightarrow{t^\bullet} & V^\bullet & \longrightarrow & U^\bullet \end{array}$$

From (TR3) the above diagram can be completed to the commutative diagram

$$\begin{array}{ccccccc} U^\bullet[-1] & \xrightarrow{-p^\bullet[-1]} & X^\bullet & \xrightarrow{s^\bullet} & Y^\bullet & \xrightarrow{i^\bullet} & U^\bullet \\ \parallel & & \downarrow f^\bullet & & \downarrow g^\bullet & & \parallel \\ U^\bullet[-1] & \xrightarrow{-f^\bullet \circ p^\bullet[-1]} & Z^\bullet & \xrightarrow{t^\bullet} & V^\bullet & \longrightarrow & U^\bullet \end{array}$$

From Lemma 1.4.106, since  $U^\bullet$  is acyclic, we have that  $t^\bullet$  is a quasi-isomorphism. Therefore the middle square completes the diagram at the start to the commutative diagram

$$\begin{array}{ccc} X^\bullet & \xrightarrow{s^\bullet} & Y^\bullet \\ f^\bullet \downarrow & & \downarrow g^\bullet \\ Z^\bullet & \xrightarrow{t^\bullet} & V^\bullet \end{array}$$

where  $s^\bullet, t^\bullet \in S^*$ .

(LC4) Let  $f^\bullet: X^\bullet \rightarrow Y^\bullet$ . We have to prove that  $s^\bullet \circ f^\bullet = 0$ , for some  $s^\bullet \in S^*$ , if and only if  $f^\bullet \circ t^\bullet = 0$ , for some  $t^\bullet \in S^*$ .

( $\Rightarrow$ ) : Suppose that there exists  $s^\bullet \in S^*$  such that  $s^\bullet \circ f^\bullet = 0$ . Consider the diagram

$$\begin{array}{ccccccc} X^\bullet & \longrightarrow & 0 & \longrightarrow & \Sigma(X^\bullet) & \xrightarrow{-1_{\Sigma(X^\bullet)}} & \Sigma(X^\bullet) \\ f^\bullet \downarrow & & \downarrow & & & & \downarrow \Sigma(f^\bullet) \\ Y^\bullet & \xrightarrow{s^\bullet} & Z^\bullet & \xrightarrow{i^\bullet} & U^\bullet & \xrightarrow{p^\bullet} & \Sigma(Y^\bullet) \end{array}$$

where the first triangle is obtained from the distinguished triangle with base  $1_{X^\bullet}$  after we have rotated it, and the second triangle is the distinguished triangle with base  $s^\bullet$ . From (TR3) the above diagram can be completed to the commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \longrightarrow & 0 & \longrightarrow & \Sigma(X^\bullet) & \xrightarrow{-1_{\Sigma(X^\bullet)}} & \Sigma(X^\bullet) \\ f^\bullet \downarrow & & \downarrow & & \downarrow_{-v^\bullet} & & \downarrow_{\Sigma(f^\bullet)} \\ Y^\bullet & \xrightarrow{s^\bullet} & Z^\bullet & \xrightarrow{i^\bullet} & U^\bullet & \xrightarrow{p^\bullet} & \Sigma(Y^\bullet) \end{array}$$

Therefore, from the last commutative square of the above diagram, we have that  $f^\bullet = p^\bullet[-1] \circ v^\bullet[-1]$ . Since  $s^\bullet$  is a quasi-isomorphism, we have that  $U^\bullet$  is acyclic. Hence, by considering the triangle

$$X^\bullet \xrightarrow{v^\bullet[-1]} U^\bullet[-1] \longrightarrow V^\bullet \xrightarrow{t^\bullet} \Sigma(X^\bullet)$$

with base  $v^\bullet[-1]$ , we have that  $t^\bullet$  is a quasi-isomorphism. In addition, from Remark 1.4.16, we have that  $v^\bullet[-1] \circ t^\bullet = 0$ . Thus we have  $f^\bullet \circ t^\bullet = p^\bullet[-1] \circ v^\bullet[-1] \circ t^\bullet = 0$ .

( $\Leftarrow$ ) : Suppose now that  $f^\bullet \circ t^\bullet = 0$  and consider the diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{t^\bullet} & Y^\bullet & \xrightarrow{u^\bullet} & V^\bullet & \longrightarrow & \Sigma(X^\bullet) \\ \downarrow & & \downarrow_{f^\bullet} & & \downarrow & & \downarrow \\ 0^\bullet & \longrightarrow & Z^\bullet & \xlongequal{\quad} & Z^\bullet & \longrightarrow & 0^\bullet \end{array}$$

where the first row is a distinguished triangle with base  $t^\bullet$ , and the second row is the distinguished triangle with base  $1_{Z^\bullet}$  after we have rotated it. From (TR3), the above triangle can be completed to the commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{t^\bullet} & Y^\bullet & \xrightarrow{u^\bullet} & V^\bullet & \longrightarrow & \Sigma(X^\bullet) \\ \downarrow & & \downarrow_{f^\bullet} & & \downarrow_{v^\bullet} & & \downarrow \\ 0^\bullet & \longrightarrow & Z^\bullet & \xlongequal{\quad} & Z^\bullet & \longrightarrow & 0^\bullet \end{array}$$

Hence we have that  $f^\bullet = v^\bullet \circ u^\bullet$ . Since  $t^\bullet$  is a quasi-isomorphism, the complex  $V^\bullet$  is acyclic. Consider now the triangle

$$V^\bullet \xrightarrow{v^\bullet} Z^\bullet \xrightarrow{s^\bullet} W^\bullet \longrightarrow \Sigma(V^\bullet)$$

with base  $v^\bullet$ . From Lemma 1.4.106 we have that  $s^\bullet$  is a quasi-isomorphism. In addition, again from Lemma 1.4.106 we have that  $s^\bullet \circ v^\bullet = 0$ , hence  $s^\bullet \circ f^\bullet = s^\bullet \circ v^\bullet \circ u^\bullet = 0$ .

It remains to prove that the localizing class  $S^*$  is compatible with the triangulated structure of  $K(\mathcal{A})$ .

(LT1) By definition, the localizing class  $S^*$  is invariant under the suspension functor  $\Sigma$ .

(LT2) Consider the morphism of triangles

$$\begin{array}{ccccccc} X^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & \Sigma(Z^\bullet) \\ s^\bullet \downarrow & & \downarrow t^\bullet & & \downarrow u^\bullet & & \downarrow \Sigma(s^\bullet) \\ X_1^\bullet & \longrightarrow & Y_1^\bullet & \longrightarrow & Z_1^\bullet & \longrightarrow & \Sigma(X_1^\bullet) \end{array}$$

For every  $n \in \mathbb{Z}$ , we obtain the commutative diagram

$$\begin{array}{ccccccccc} H^n(X^\bullet) & \longrightarrow & H^n(Y^\bullet) & \longrightarrow & H^n(Z^\bullet) & \longrightarrow & H^{n+1}(X^\bullet) & \longrightarrow & H^{n+1}(Y^\bullet) \\ H^n(s^\bullet) \downarrow & & \downarrow H^n(t^\bullet) & & \downarrow H^n(u^\bullet) & & \downarrow H^{n+1}(s^\bullet) & & \downarrow H^{n+1}(t^\bullet) \\ H^n(X_1^\bullet) & \longrightarrow & H^n(Y_1^\bullet) & \longrightarrow & H^n(Z_1^\bullet) & \longrightarrow & H^{n+1}(X_1^\bullet) & \longrightarrow & H^{n+1}(Y_1^\bullet) \end{array}$$

Since  $H^n(s^\bullet)$ ,  $H^n(t^\bullet)$ ,  $H^{n+1}(s^\bullet)$ ,  $H^{n+1}(t^\bullet)$  are isomorphisms, because  $s^\bullet$  and  $t^\bullet$  are quasi-isomorphisms, from the five lemma, we obtain that  $H^n(u^\bullet)$  is an isomorphism for every  $n \in \mathbb{Z}$ . This implies that  $u^\bullet$  is a quasi-isomorphism.  $\blacksquare$

We are now in a position to give the definition of the derived category of an abelian category. In what follows, let  $\mathcal{A}$  be an abelian category and  $C^*(\mathcal{A})$  be the category of complexes and  $K^*(\mathcal{A})$  be the homotopy category of complexes. From Theorem 1.4.97 the category  $K^*(\mathcal{A})$  is triangulated. Let also  $S^*$  be the class of all quasi-isomorphism of  $K^*(\mathcal{A})$ .

**Definition 1.4.108.** Let  $\mathcal{A}$  be an abelian category and  $K(\mathcal{A})$  the homotopy category of complexes. The localization of  $K(\mathcal{A})$  at the class of quasi-isomorphisms  $S^*$  is called the **derived category** of the abelian category  $\mathcal{A}$ , and is denoted by  $D(\mathcal{A})$ .

From the theory that we have developed so far, the following Theorem summarizes the triangulated structure of the derived category  $D(\mathcal{A})$  and the localizing functor  $Q: K(\mathcal{A}) \rightarrow D(\mathcal{A})$

**Theorem 1.4.109.** Let  $\mathcal{A}$  be an abelian category and  $D(\mathcal{A})$  its derived category. Then:

- (1)  $D(\mathcal{A})$  is a triangulated category.
- (2) The suspension functor is induced by the shift of complexes:

$$\Sigma : D(\mathcal{A}) \rightarrow D(\mathcal{A}), \quad X^\bullet \mapsto \Sigma(X^\bullet), \quad d_{X[1]}^n = -d_X^{n+1}$$

- (3) The class of distinguished triangles is the image under the localization functor of the distinguished triangles in  $K(\mathcal{A})$ .

(4) *The localization functor*

$$Q : D(\mathcal{A}) \rightarrow D(\mathcal{A})$$

sends complexes and chain maps to their images in  $D(\mathcal{A})$ , and inverts quasi-isomorphisms.

**Remark 1.4.110.** Let  $\mathcal{A}$  be an abelian category. Just like with  $C(\mathcal{A})$  and  $K(\mathcal{A})$ , we define the following subcategories of  $D(\mathcal{A})$ .

- (1) We define the **full subcategory of bounded from below complexes** of  $D^-(\mathcal{A})$ , and we denote it by  $D^-(\mathcal{A})$ ,
- (2) We define the **full subcategory of bounded from above complexes** of  $D(\mathcal{A})$ , and we denote it by  $D^+(\mathcal{A})$ ,
- (3) We define the **full subcategory of bounded complexes** of  $D(\mathcal{A})$ , and we denote it by  $D^b(\mathcal{A})$ .

**Remark 1.4.111.** By definition, the cohomological functor  $H^0 : K^*(\mathcal{A}) \rightarrow \mathcal{A}$  maps quasi-isomorphisms to isomorphisms. Therefore from Theorem 1.4.32, there exists an induced cohomological functor from  $D^*(\mathcal{A})$  to  $\mathcal{A}$ . By abuse of notation, we will denote this functor again by  $H^0$ . In particular, let

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet)$$

be a distinguished in  $D^*(\mathcal{A})$ . Then the long sequence

$$\dots \longrightarrow H^n(X^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(Y^\bullet) \xrightarrow{H^n(g^\bullet)} H^n(Z^\bullet) \longrightarrow H^{n+1}(X^\bullet) \longrightarrow \dots$$

is exact in  $\mathcal{A}$ . Just like in  $K^*(\mathcal{A})$ , the above sequence is called the **long exact sequence of cohomology** associated to the distinguished triangle

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} \Sigma(X^\bullet)$$

Therefore we have the canonical functors

$$C^*(\mathcal{A}) \rightarrow K^*(\mathcal{A}) \rightarrow D^*(\mathcal{A})$$

In addition, every morphism  $s^\bullet \in \tilde{S}^*$  induces an isomorphism in  $D^*(\mathcal{A})$ . From Definition 1.3.1, the above functor factors through the localization  $C^*(\mathcal{A})[\tilde{S}^{*-1}]$ , that is, we have the following commutative diagram

$$\begin{array}{ccc} C^*(\mathcal{A}) & \longrightarrow & K^*(\mathcal{A}) \\ \tilde{Q} \downarrow & & \downarrow Q \\ C^*(\mathcal{A})[\tilde{S}^{*-1}] & \xrightarrow{\iota} & D^*(\mathcal{A}) \end{array}$$

**Theorem 1.4.112.** *The functor  $\iota: C^*(\mathcal{A})[\tilde{S}^*] \rightarrow D^*(\mathcal{A})$  is an isomorphism of categories.*

*Proof.* The functor  $\iota$  acts as the identity on objects. Let now  $X^\bullet, Y^\bullet$  be two complexes and  $f^\bullet, g^\bullet: X^\bullet \rightarrow Y^\bullet$  two homotopic chain maps.

Claim:  $\tilde{Q}(f^\bullet) = \tilde{Q}(g^\bullet)$ .

Proof of the claim: Applying part of the proof from Lemma 1.4.86 to the diagram

$$\begin{array}{ccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet \\ \parallel & & \parallel \\ X^\bullet & \xrightarrow{g^\bullet} & Y^\bullet \end{array}$$

which is commutative up to homotopy, we have the existence of a chain map  $u^\bullet: C_f^\bullet \rightarrow C_g^\bullet$  such that the diagram

$$\begin{array}{ccc} Y^\bullet & \xrightarrow{i_b^\bullet} & C_f^\bullet \\ i_g^\bullet \downarrow & \swarrow u^\bullet & \\ C_g^\bullet & & \end{array}$$

is commutative in  $C^*(\mathcal{A})$ , and  $u^\bullet$  is defined as

$$u^n = \begin{pmatrix} 1_{X^{n+1}} & 0 \\ -h^{n+1} & 1_{Y^n} \end{pmatrix}, \quad \forall n \in \mathbb{Z}$$

where the chain maps  $i_b^\bullet: Y^\bullet \rightarrow C_f^\bullet$  and  $i_g^\bullet: Y^\bullet \rightarrow C_g^\bullet$  come from the diagram (1.4.86). Therefore, by applying again part of the proof of the Lemma 1.4.86 for the commutative diagram

$$\begin{array}{ccc} Y^\bullet & \xrightarrow{i_f^\bullet} & C_f^\bullet \\ \parallel & & \downarrow u^\bullet \\ Y^\bullet & \xrightarrow{i_d^\bullet} & C_g^\bullet \end{array}$$

we have the existence of a chain map  $v^\bullet: D_f^\bullet \rightarrow D_g^\bullet$  such that the diagram

$$\begin{array}{ccc} D_f^\bullet & \xrightarrow{p_{i_f^\bullet}^\bullet} & Y^\bullet \\ v^\bullet \downarrow & \swarrow p_{i_g^\bullet}^\bullet & \\ D_g^\bullet & & \end{array}$$

commutes in  $C^*(\mathcal{A})$ , and  $v^\bullet$  is defined as

$$v^n = \begin{pmatrix} 1_{Y^{n+1}} & 0 & 0 \\ 0 & 1_{X^{n+1}} & 0 \\ 0 & -h^{n+1} & 1_{Y^n} \end{pmatrix}, \forall n \in \mathbb{Z}$$

where the chain maps  $p_{i_f^\bullet}^\bullet: D_f^\bullet \rightarrow Y^\bullet$  and  $v^\bullet: D_f^\bullet \rightarrow D_g^\bullet$  again come from the diagram (1.4.86). We compute

$$\beta_g^n \circ v^n = (0 \ 1_{X^{n+1}} \ 0) \begin{pmatrix} 1_{Y^{n+1}} & 0 & 0 \\ 0 & 1_{X^{n+1}} & 0 \\ 0 & -h^{n+1} & 1_{Y^n} \end{pmatrix} = (0 \ 1_{X^{n+1}} \ 0) = \beta_f^n$$

that is,  $\beta_g^\bullet \circ v = \beta_f^\bullet$  in  $C^*(\mathcal{A})$ . Therefore  $\beta_g^\bullet[-1] \circ v^\bullet[-1] = \beta_f^\bullet[-1]$  and  $\tilde{Q}(\beta_g^\bullet[-1]) \circ \tilde{Q}(v^\bullet[-1]) = \tilde{Q}(\beta_f^\bullet[-1])$ . From Lemma 1.4.88, the chain maps  $\beta_f^\bullet[-1]: D_f^\bullet[-1] \rightarrow X^\bullet$  and  $\beta_g^\bullet[-1]: D_g^\bullet[-1] \rightarrow X^\bullet$  are isomorphisms in  $K(\mathcal{A})$ , hence also quasi-isomorphisms. This implies that  $\tilde{Q}(\beta_f^\bullet[-1])$  and  $\tilde{Q}(\beta_g^\bullet[-1])$  are isomorphisms in  $C^*(\mathcal{A})[S^{\sim-1}]$ . From the proof of the Lemma 1.4.88, we have that  $\beta_f^\bullet[-1] \circ \alpha_f^\bullet[-1] = 1_{X^\bullet}$  and  $\beta_g^\bullet[-1] \circ \alpha_g^\bullet[-1] = 1_{X^\bullet}$ . Therefore

$$\tilde{Q}(\beta_f^\bullet[-1]) \circ \tilde{Q}(\alpha_f^\bullet[-1]) = 1_{X^\bullet} \Rightarrow \tilde{Q}(\alpha_f^\bullet[-1]) = \tilde{Q}(\beta_f^\bullet[-1])^{-1}$$

and

$$\tilde{Q}(\beta_g^\bullet[-1]) \circ \tilde{Q}(\alpha_g^\bullet[-1]) = 1_{X^\bullet} \Rightarrow \tilde{Q}(\alpha_g^\bullet[-1]) = \tilde{Q}(\beta_g^\bullet[-1])^{-1}$$

By combining the above we have that  $\tilde{Q}(v^\bullet[-1]) \circ \tilde{Q}(\alpha_f^\bullet[-1]) = \tilde{Q}(\alpha_g^\bullet[-1])$ . From the proof of the Lemma 1.4.89, the diagram

$$\begin{array}{ccc} \Sigma(X^\bullet) & \xrightarrow{-\Sigma(f^\bullet)} & \Sigma(Y^\bullet) \\ \alpha_f^\bullet \downarrow & \nearrow p_{i_f^\bullet}^\bullet & \\ C_f^\bullet & & \end{array}$$

is commutative in  $C^*(\mathcal{A})$ . By applying the functor  $\Sigma^{-1}$  and changing the signs of the chain maps we obtain the diagram

$$\begin{array}{ccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet \\ \alpha_f^\bullet[-1] \downarrow & \nearrow -p_{i_f^\bullet[-1]} & \\ D_f^\bullet[-1] & & \end{array}$$

which is commutative in  $C^*(\mathcal{A})$ . Hence we have the factorization  $f^\bullet = -p_{i_f}^\bullet[-1] \circ \alpha_f^\bullet[-1]$ . Respectively we have  $g^\bullet = -p_{i_g}^\bullet[-1] \circ \alpha_g^\bullet[-1]$ . Thus we compute

$$\begin{aligned}\tilde{Q}(f^\bullet) &= -\tilde{Q}(p_{i_f}^\bullet[-1]) \circ \tilde{Q}(\alpha_f^\bullet[-1]) \\ &= -\tilde{Q}(p_{i_g}^\bullet[-1]) \circ \tilde{Q}(v^\bullet[-1]) \circ \tilde{Q}(\alpha_f^\bullet[-1]) \\ &= -\tilde{Q}(p_{i_g}^\bullet[-1]) \circ \tilde{Q}(\alpha_g^\bullet[-1]) \\ &= \tilde{Q}(g^\bullet)\end{aligned}$$

This means that the natural quotient functor  $\tilde{Q} : C^*(\mathcal{A}) \rightarrow C^*(\mathcal{A})[\widetilde{S^{*-1}}]$  factors through  $K^*(\mathcal{A})$ , i.e. the following diagram is commutative

$$\begin{array}{ccc} C^*(\mathcal{A}) & \longrightarrow & K^*(\mathcal{A}) \\ \tilde{Q} \downarrow & \searrow \phi & \downarrow \varrho \\ C^*(\mathcal{A})[\widetilde{S^{*-1}}] & \xrightarrow{\iota} & D^*(\mathcal{A}) \end{array}$$

where the square and left triangle are commutative. Since  $C^*(\mathcal{A}) \rightarrow K^*(\mathcal{A})$  is the identity on objects, and it is also onto on morphisms, the right triangle is also commutative. In addition,  $\phi$  maps quasi-isomorphisms to isomorphisms, so it also factors through  $D^*(\mathcal{A})$ , i.e. the following diagram is commutative

$$\begin{array}{ccc} K^*(\mathcal{A}) & \xrightarrow{\varrho} & D^*(\mathcal{A}) \\ \varrho \downarrow & \searrow \phi & \downarrow \iota \\ D^*(\mathcal{A}) & \xrightarrow{\psi} & C^*(\mathcal{A})[\widetilde{S^{*-1}}] \end{array}$$

From the universal property of the localization, we have that  $\iota \circ \psi = 1_{D^*(\mathcal{A})}$ . Now, by combining the above, we have the commutative diagram

$$\begin{array}{ccccc} C^*(\mathcal{A}) & \longrightarrow & K^*(\mathcal{A}) & \xrightarrow{\varrho} & D^*(\mathcal{A}) \\ \tilde{Q} \downarrow & \searrow \phi & \downarrow \varrho & \searrow \phi & \uparrow \iota \\ C^*(\mathcal{A})[\widetilde{S^{*-1}}] & \xrightarrow{\iota} & D^*(\mathcal{A}) & \xrightarrow{\psi} & C^*(\mathcal{A})[\widetilde{S^{*-1}}] \end{array}$$

Again, from the universal property of the localization, we have that  $\psi \circ \iota = 1_{D^*(\mathcal{A})}$ . ■

### 1.4.11 Truncation functors

Truncation functors in the derived category of an abelian category allow us to isolate portions of a complex above or below a given degree, producing new complexes that retain part of the original

structure. They play a fundamental role in the construction of  $t$ -structures that will be discussed in the next chapter, and in defining cohomology objects in derived categories.

**Definition 1.4.113.** Let  $\mathcal{A}$  be an abelian category and  $A^\bullet$  a complex of  $\mathcal{A}$ . We define the complex  $\tau_{\leq n}(A^\bullet)$  as follows:

$$\tau_{\leq n}(A^\bullet)^p = \begin{cases} A^p, & p < n \\ \text{Ker}(d_A^n), & p = n \\ 0, & p > n \end{cases}$$

Let  $i^\bullet : \tau_{\leq n}(A^\bullet) \rightarrow A^\bullet$  be the canonical injection. The following result is immediate from the definition.

**Lemma 1.4.114.** The morphism  $H^p(i) : H^p(\tau_{\leq n}(A^\bullet)) \rightarrow H^p(A^\bullet)$  is an isomorphism for  $p \leq n$ , and 0 for  $p > n$ .

*Proof.* The proof is immediate from the above Definition 1.4.113 and is left to the reader.  $\blacksquare$

Let  $B^\bullet$  be a complex and  $f^\bullet : A^\bullet \rightarrow B^\bullet$  a chain map. Then it is true that  $d^n \circ f^n = f^{n+1} \circ d^n$ , hence  $f^n(\text{Ker}(d^n)) \subseteq \text{Ker}(d^n)$ . This means that the chain map  $f^\bullet$  induces a chain map  $\tau_{\leq n}(f^\bullet) : \tau_{\leq n}(A^\bullet) \rightarrow \tau_{\leq n}(B^\bullet)$ . Then it is easy to see that the functor  $\tau_{\leq n} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$  is additive.

Suppose now that  $f^\bullet, g^\bullet : A^\bullet \rightarrow B^\bullet$  are two homotopic chain maps, that is,  $f - g = d \circ h + h \circ d$ . Then  $\tau_{\leq n}(f^\bullet)$  and  $\tau_{\leq n}(g^\bullet)$  are also homotopic via the restriction of the homotopy of  $f^\bullet$  and  $g^\bullet$ . This means that  $\tau_{\leq n}$  induces a functor  $\tau_{\leq n} : \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ . Obviously, we have

$$H^n(\tau_{\leq n}(f^\bullet)) = \begin{cases} H^n(f^\bullet), & p \leq n \\ 0, & p > n \end{cases}$$

By combining the above with Lemma 1.4.114 we have that if  $f^\bullet : A^\bullet \rightarrow B^\bullet$  is a quasi-isomorphism, then  $\tau_{\leq n}(f^\bullet)$  is also a quasi-isomorphism. This means that  $\tau_{\leq n}$  induces a functor  $\tau_{\leq n} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$ .

**Definition 1.4.115.** Let  $\mathcal{A}$  be an abelian category and  $A^\bullet$  a complex in  $\mathcal{A}$ . We define the complex  $\tau_{\geq n}(A^\bullet)$  as:

$$\tau_{\geq n}(A^\bullet)^p = \begin{cases} 0, & p < n \\ \text{Coker}(d^{n-1}), & p = n \\ A^n, & p > n \end{cases}$$

Let  $q^\bullet : A^\bullet \rightarrow \tau_{\geq n}(A^\bullet)$  be the canonical projection. Then we have the following result.

**Lemma 1.4.116.** *The morphism  $H^p(q^\bullet) : H^p(A^\bullet) \rightarrow H^p(\tau_{\geq n}(A^\bullet))$  is an isomorphism for  $p \geq n$ , and 0 for  $p < n$ .*

*Proof.* The proof is immediate from the above Definition 1.4.115 and is left to the reader. ■

**Remark 1.4.117.** Let  $B^\bullet$  be a complex and  $f^\bullet : A^\bullet \rightarrow B^\bullet$  a chain map. Then  $d^{n-1} \circ f^{n-1} = f^n \circ d^{n-1}$ , hence  $f^n(\text{Im}(d^{n-1})) \subseteq \text{Im}(d^{n-1})$ . This means that  $f^\bullet$  induces a chain map  $\tau_{\geq n}(f^\bullet) : C(\mathcal{A}) \rightarrow C(\mathcal{A})$ , and the functor  $\tau_{\geq n} : C(\mathcal{A}) \rightarrow C(\mathcal{A})$  is additive.

Suppose now that  $f^\bullet, g^\bullet : A^\bullet \rightarrow B^\bullet$  are two homotopic chain maps, i.e., there exists a graded morphism  $h^\bullet : A^\bullet \rightarrow B^\bullet$  of degree -1, such that  $f^\bullet - g^\bullet = d \circ h + h \circ d$ . Then,  $\tau_{\geq n}(f^\bullet)$  and  $\tau_{\geq n}(g^\bullet)$  are also homotopic via the induced homotopy between  $f^\bullet$  and  $g^\bullet$ . This means that  $\tau_{\geq n} : C(\mathcal{A}) \rightarrow C(\mathcal{A})$  induces a functor  $\tau_{\geq n} : K(\mathcal{A}) \rightarrow K(\mathcal{A})$ . Immediately we have

$$H^p(\tau_{\geq n}(f^\bullet)) = \begin{cases} H^p(f^\bullet), & p \geq n \\ 0, & p < n \end{cases}$$

So, by combining the above with Lemma 1.4.116, we have that if  $f^\bullet : A^\bullet \rightarrow B^\bullet$  is a quasi-isomorphism, then  $\tau_{\geq n}(f^\bullet)$  is also a quasi-isomorphism.

**Definition 1.4.118.** *The functors  $\tau_{\leq n}, \tau_{\geq n} : D(\mathcal{A}) \rightarrow D(\mathcal{A})$  are called the **truncation functors** of  $D(\mathcal{A})$ .*

**Remark 1.4.119.** The natural functor  $\overline{K}^-(\mathcal{A}) \rightarrow K(\mathcal{A})$  induces the functor  $D^-(\mathcal{A}) \rightarrow D(\mathcal{A})$ . In addition, the localizing class  $\overline{S}^{-1}$  consists of all the morphisms in  $S$  which lie in  $\overline{K}^-(\mathcal{A})$ . Let now  $X^\bullet, Y^\bullet$  be two chain complexes, where  $X^\bullet$  is bounded from above, and  $s^\bullet : Y^\bullet \rightarrow X^\bullet$  is a quasi-isomorphism. Since  $X^\bullet$  is bounded from above, there exists  $n \in \mathbb{Z}$  such that  $H^p(X^\bullet) = 0, \forall p > n$ . In addition, because  $s^\bullet$  is a quasi-isomorphism, we also have  $H^p(Y^\bullet) = 0, \forall p > n$ . So, from Lemma 1.4.114,  $i^\bullet : \tau_{\leq n}(Y^\bullet) \rightarrow Y^\bullet$  is a quasi-isomorphism, which makes  $s^\bullet \circ i^\bullet : \tau_{\leq n}(Y^\bullet) \rightarrow X^\bullet$  also an quasi-isomorphism. Hence we arrive to the following Proposition.

**Proposition 1.4.120.** *The natural functors  $D^-(\mathcal{A}) \rightarrow D(\mathcal{A})$  and  $D^+(\mathcal{A}) \rightarrow D(\mathcal{A})$ , are fully faithful.*

*Proof.* The proof follows from the above analysis and from Proposition 1.3.31 ■

**Remark 1.4.121.** Immediately, from the above proposition, we have that  $D^-(\mathcal{A})$  and  $D^+(\mathcal{A})$  are full subcategories of  $D(\mathcal{A})$ .

**Theorem 1.4.122.** *The isomorphism  $\iota : D(\mathcal{A})^{\text{op}} \rightarrow D(\mathcal{A}^{\text{op}})$  induces an isomorphism  $\iota : D^+(\mathcal{A})^{\text{op}} \rightarrow D^-(\mathcal{A}^{\text{op}})$ ,  $\iota : D^-(\mathcal{A})^{\text{op}} \rightarrow D^+(\mathcal{A}^{\text{op}})$  and  $\iota : D^{\text{op}}(\mathcal{A})^{\text{op}} \rightarrow D^{\text{b}}(\mathcal{A}^{\text{op}})$ .*

*Proof.* For the proof we refer the reader to [12][Theorem 3.4.6.] ■

**Remark 1.4.123.** The full subcategories  $D^b(\mathcal{A})$ ,  $D^+(\mathcal{A})$  and  $D^-(\mathcal{A})$  are invariant under the functors  $\tau_{\leq n}$  and  $\tau_{\geq n}$ . Therefore they induce respective truncation functors which we denote with the same symbol.

**Notation 1.4.124.** We denote by  $D^* : \mathcal{A} \rightarrow D^*(\mathcal{A})$  the composition of the functors  $C : \mathcal{A} \rightarrow K^*(\mathcal{A})$  and  $Q : K^*(\mathcal{A}) \rightarrow D^*(\mathcal{A})$ .

**Theorem 1.4.125.** The functor  $D^* : \mathcal{A} \rightarrow D^*(\mathcal{A})$  is fully faithful.

*Proof.* 1-1: Let  $X, Y \in \mathcal{A}$  and  $f : X \rightarrow Y$  be a morphism in  $\mathcal{A}$ . Then  $H^0(D(f)) = f$  and the map  $\text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{D(\mathcal{A})}(D(X), D(Y))$  is injective.

onto: Let  $\phi : D(M) \rightarrow D(N)$  be a morphism in  $D^*(\mathcal{A})$  and suppose that

$$\begin{array}{ccc} & X^\bullet & \\ s^\bullet \swarrow & & \searrow f^\bullet \\ D(M) & & D(N) \end{array}$$

(Note: The arrow from  $X^\bullet$  to  $D(M)$  is labeled with a tilde  $\sim$  below it.)

is a representative, where  $s^\bullet : X^\bullet \rightarrow D(M)$  is a quasi-isomorphism. Then  $H^p(X^\bullet) = 0$ ,  $\forall p \neq 0$ . Hence, from Lemma 1.4.114,  $i^\bullet : \tau_{\leq 0}(X^\bullet) \rightarrow X^\bullet$  is a quasi-isomorphism. By setting  $Y^\bullet = \tau_{\leq 0}(X^\bullet)$ , the diagram

$$\begin{array}{ccccc} & & X^\bullet & & \\ & s & \swarrow & & \searrow f^\bullet \\ & & D(M) & & D(N) \\ & & \uparrow \sim i^\bullet & & \\ & & Y^\bullet & & \\ & s^\bullet \circ i^\bullet & \swarrow & \parallel & \searrow f^\bullet \circ i^\bullet \\ & & Y^\bullet & & \end{array}$$

is commutative. This means that  $\phi$  has a representative a left roof with  $X^p = 0$ ,  $\forall p > 0$ . Thus we have the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow F^0 & & & & & & \\ \dots & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

for a representative  $F^0$  of the homotopic class of  $f^\bullet$ . Obviously, all the homotopies from  $X^\bullet$  to  $D(N)$  are 0, so  $F^0$  is unique. In addition,  $F^0$  is zero restricted to  $\text{Im}(d_X^{-1})$ , because  $F^0 \circ d_X^{-1} = 0$  and due to the fact that it factors through the zero morphism. This means that  $F^0$  factors through

$H^0(F) : H^0(X^\bullet) \rightarrow N$ , and  $H^0(F) = H^0(f^\bullet) = H^0(\phi) \circ H^0(s^\bullet)$ . From the above, the diagram

$$\begin{array}{ccccc}
 & & X^\bullet & & \\
 & \swarrow s^\bullet & \parallel & \searrow f^\bullet & \\
 D(M) & & X^\bullet & & D(N) \\
 & \searrow \cong & \downarrow \sim s^\bullet & \nearrow H^0(\phi) & \\
 & & D(M) & & 
 \end{array}$$

is commutative. So  $\phi = D(H^0(\phi))$ , which proves the statement.  $\blacksquare$

### 1.4.12 Short exact sequences and distinguished triangles

In this subsection we discuss the relationship between short exact sequences in an abelian category  $\mathcal{A}$  and the distinguished triangles in the homotopy category of complexes  $K(\mathcal{A})$ .

Let  $\mathcal{A}$  be an abelian category. Then the category  $C^*(\mathcal{A})$  is also abelian. Let

$$0 \longrightarrow X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \longrightarrow 0$$

be a short exact sequence in  $C^*(\mathcal{A})$  and

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i_f^\bullet} C_f^\bullet \xrightarrow{p_f^\bullet} \Sigma(X^\bullet)$$

a distinguished triangle with base  $f^\bullet$ . Let  $m^\bullet : \Sigma(X^\bullet) \oplus Y^\bullet \rightarrow Z^\bullet$  be the graded morphism which is the composition of the natural projection  $q : \Sigma(X^\bullet) \oplus Y^\bullet \rightarrow Y^\bullet$  with the chain map  $g^\bullet : Y^\bullet \rightarrow Z^\bullet$ . We compute  $\forall n \in \mathbb{Z}$

$$\begin{aligned}
 m^{n+1} \circ d_{C_f}^n &= (0 \quad g^{n+1}) \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^{n+1} \end{pmatrix} = (g^{n+1} \circ f^{n+1} \quad g^{n+1} \circ d_Y^n) = (0 \quad d_Z^n \circ g^n) = \\
 &= d_Z^n \circ (0 \quad g^n) = d_Z^n \circ m^n
 \end{aligned}$$

i.e.  $m^\bullet$  is a chain map. In addition we have that  $m^\bullet \circ i_f^\bullet = g^\bullet$ . From Lemma 1.4.86, in the commutative diagram

$$\begin{array}{ccc}
 X^\bullet & \xlongequal{\quad} & X^\bullet \\
 \parallel & & \downarrow f^\bullet \\
 X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet
 \end{array}$$

we can glue the chain map  $w^\bullet : C_{1_X}^\bullet \rightarrow C_f^\bullet$  which is defined by

$$w^n = \begin{pmatrix} 1_{X^{n+1}} & 0 \\ 0 & f^n \end{pmatrix} \quad \forall n \in \mathbb{Z}$$

Clearly  $w^\bullet$  is a monomorphism, and

$$\text{Im}(w^n) = X^{n+1} \oplus \text{Im}(f^n) = X^{n+1} \oplus \text{Ker}(g^n) = \text{Ker}(m^n), \quad \forall n \in \mathbb{Z}$$

This means that the sequence

$$0 \longrightarrow C_{1_X}^\bullet \xrightarrow{w^\bullet} C_f^\bullet \xrightarrow{m^\bullet} Z^\bullet \longrightarrow 0$$

is a short exact sequence in  $C^*(\mathcal{A})$ .

From Lemma 1.4.92,  $C_{1_X}^\bullet = 0^\bullet$  in  $K^*(\mathcal{A})$ , and so  $H^n(C_{1_X}^\bullet) = 0$ ,  $\forall n \in \mathbb{Z}$ . Consequently, from the long exact sequence of cohomology of the above short exact sequence, we have that  $H^p(m^\bullet) : H^p(C_f^\bullet) \rightarrow H^p(Z^\bullet)$  is an isomorphism for all  $n \in \mathbb{Z}$ . Thus we have proved the following result.

**Lemma 1.4.126.** *The chain map  $m^\bullet : C_f^\bullet \rightarrow Z^\bullet$  is a quasi-isomorphism.*

The proofs of the following results are straightforward. For more details we refer the reader to [12, Lemma 3.5.1]

**Proposition 1.4.127.** *Let*

$$0 \longrightarrow X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \longrightarrow 0$$

*be a short exact sequence in  $C(\mathcal{A})$ . Then its image under the functor  $C(\mathcal{A}) \rightarrow D(\mathcal{A})$  is a distinguished triangle*

$$X^\bullet \xrightarrow{[f^\bullet]} Y^\bullet \xrightarrow{[g^\bullet]} Z^\bullet \longrightarrow \Sigma(X^\bullet)$$

*in  $D(\mathcal{A})$ .*

Dually, if we have again a short exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \longrightarrow 0$$

we can consider the standard triangle

$$Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{i_g^\bullet} C_g^\bullet \xrightarrow{p_g^\bullet} \Sigma(Y^\bullet)$$

By considering the morphism  $k^\bullet : X^\bullet \rightarrow Y^\bullet \oplus \Sigma^{-1}(Z^\bullet) = C_g^\bullet[-1]$  to be the composition of  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  with the canonical injection we have the following result

**Lemma 1.4.128.** *The chain map  $k^\bullet : X^\bullet \rightarrow C_g^\bullet[-1]$  is a quasi-isomorphism.*

### 1.4.13 The distinguished triangle of truncations

In this subsection, we consider the distinguished triangle associated to a truncation of a complex. It allows us to decompose a complex providing a more concrete way to study it.

Let  $X^\bullet$  be a complex and  $n \in \mathbb{Z}$ . Consider the short exact sequence

$$0 \longrightarrow \tau_{\leq n}(X^\bullet) \longrightarrow X^\bullet \longrightarrow Q^\bullet \longrightarrow 0 .$$

By definition we have that

$$Q^p = \begin{cases} 0, & p < n \\ \text{Coim}(d^n), & p = n \\ X^p, & p > n \end{cases}$$

Therefore  $H^n(Q^\bullet) = 0$ ,  $\forall p < n$  and  $H^p(X^\bullet) = H^p(Q^\bullet)$ ,  $\forall p = n$ . Consider the canonical projection  $Q^\bullet \rightarrow \tau_{\geq n+1}(Q^\bullet) = \tau_{\geq n+1}(X^\bullet)$ , i.e. the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \text{Coim}(d_X^n) & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow & \cdots \\ & & \parallel & & \downarrow & & \downarrow & & \parallel & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \text{Coker}(d_X^n) & \longrightarrow & X^{n+2} & \longrightarrow & \cdots \end{array}$$

we get that the above chain map is a quasi-isomorphism.

From Proposition 1.4.127, we have the distinguished triangle

$$\tau_{\leq n}(X^\bullet) \xrightarrow{i^\bullet} X^\bullet \longrightarrow Q^\bullet \longrightarrow \Sigma(\tau_{\leq n}(X^\bullet))$$

in  $D(\mathcal{A})$ . From the above discussion, the complex  $Q^\bullet$  is isomorphic to  $\tau_{\geq n+1}(X^\bullet)$ . Thus we get the distinguished triangle

$$\tau_{\leq n}(X^\bullet) \xrightarrow{i^\bullet} X^\bullet \xrightarrow{q^\bullet} \tau_{\geq n+1}(X^\bullet) \longrightarrow \Sigma(\tau_{\leq n}(X^\bullet))$$

The above distinguished triangle proves the existence part of the following Proposition.

**Proposition 1.4.129.** *For every complex  $X^\bullet$  and  $n \in \mathbb{Z}$ , there exists a unique chain map  $h^\bullet : \tau_{\geq n+1}(X^\bullet) \rightarrow \Sigma(\tau_{\leq n}(X^\bullet))$  such that the triangle*

$$\tau_{\leq n}(X^\bullet) \xrightarrow{i^\bullet} X^\bullet \xrightarrow{q^\bullet} \tau_{\geq n+1}(X^\bullet) \xrightarrow{h^\bullet} \Sigma(\tau_{\leq n}(X^\bullet))$$

*is distinguished in  $D(\mathcal{A})$ .*

*Proof.* For more details we refer the reader to [12][Proposition 3.6.1.] ■

**Remark 1.4.130.** The proof of the uniqueness of the above Proposition comes from Lemma 1.4.24.

**Lemma 1.4.131.** Let  $X^\bullet, Y^\bullet$  be two complexes such that  $X^p = 0, \forall p \geq n$  and  $Y^p = 0, \forall p < n$ . Then  $\text{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet) = 0$

*Proof.* Let  $\phi \in \text{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet)$ , with a representative

$$\begin{array}{ccc} & Z^\bullet & \\ s^\bullet \swarrow & & \searrow f^\bullet \\ X^\bullet & & Y^\bullet \end{array}$$

$\sim$

Since  $H^p(X^\bullet) = 0, \forall p \geq n$  and  $s^\bullet$  is a quasi-isomorphism, we have that  $H^p(Z^\bullet) = 0, \forall p \geq n$ . Hence  $i^\bullet : \tau_{\leq n-1}(Z^\bullet) \rightarrow Z^\bullet$  is a quasi-isomorphism. By setting  $U^\bullet := \tau_{\leq n-1}(Z^\bullet)$ , we obtain the commutative diagram

$$\begin{array}{ccccc} & & Z^\bullet & & \\ & s^\bullet \swarrow & & \searrow f^\bullet & \\ X^\bullet & & U^\bullet & & Y^\bullet \\ & \nwarrow \sim & \uparrow i^\bullet & & \\ & & U^\bullet & & \\ & s^\bullet \circ i^\bullet \swarrow & \parallel & \searrow f^\bullet \circ i^\bullet & \\ & & U^\bullet & & \end{array}$$

This means that  $\phi$  has as a representative a left roof with  $Z^p = 0, \forall p > n$ . Therefore,  $f^\bullet = 0^\bullet$  ■

#### 1.4.14 Exact sequences and distinguished triangles

In this subsection we discuss the relationship between short exact sequences in an abelian category  $\mathcal{A}$  and the distinguished triangles in the derived category  $\mathcal{D}(\mathcal{A})$ .

Let  $\mathcal{A}$  be an abelian category and

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

a short exact sequence in  $\mathcal{A}$ . Then, from Proposition 1.4.127, we have the distinguished triangle

$$\mathcal{D}(L) \xrightarrow{\mathcal{D}(f)} \mathcal{D}(M) \xrightarrow{\mathcal{D}(g)} \mathcal{D}(N) \longrightarrow \Sigma(\mathcal{D}(L))$$

in  $\mathcal{D}^*(\mathcal{A})$ . Therefore we have the following Proposition.

**Proposition 1.4.132.** There exists a unique morphism  $h$  such that

$$\mathcal{D}(L) \xrightarrow{\mathcal{D}(f)} \mathcal{D}(M) \xrightarrow{\mathcal{D}(g)} \mathcal{D}(N) \xrightarrow{h} \Sigma(\mathcal{D}(L))$$

is a distinguished triangle in  $\mathcal{D}^*(\mathcal{A})$ .

*Proof.* The uniqueness of  $h$  comes from Lemma 1.4.24 and from Lemma 1.4.131. ■



# *t*-STRUCTURES, HRS-TILTING AND QUASI-ABELIAN CATEGORIES

This chapter is dedicated to the basic tools which play an important role in Bridgeland’s paper, namely *t*-structures, Hapel, Reiten and Smalø tilting (HRS-tilting for short), and quasi-abelian categories. Apart from playing a crucial role in the theory of stability conditions on triangulated categories, they were developed independently and have found applications in Representation Theory, Algebraic Geometry and Homological Algebra. In the first section we develop the theory of *t*-structures, presenting some fundamental properties. At the end we introduce the notion of the Grothendieck group of an abelian and a triangulated category. The second section is dedicated to HRS-tilting. We outline the basic theory, and provide a way to construct *t*-structures in the bounded derived category of an abelian category provided that we have a torsion pair in the abelian category. The third and final section of this chapter is about quasi-abelian categories. They are close to being abelian categories, however they have less structure. Nevertheless, they come up naturally in Bridgeland’s theory and play an important role. This chapter follows [1], [7], [12] and [17].

## 2.1 *t*-structures

In this section, we present the main theory of *t*-structures. They were introduced by Beilinson, Bernstein and Deligne in [1], in their effort to study perverse sheaves. *t*-structures are the right tool for finding abelian categories inside triangulated categories. In the first section we give the definition of a *t*-structure and present some basic results about them. Next, we shift our focus to a special of *t*-structures called bounded *t*-structures, which play a crucial role in Bridgeland’s paper. Last but not least, we give the definition of the Grothendieck group of an abelian and a triangulated category and show some basic results. Throughout this section we closely follow Milicic’s notes [12].

### 2.1.1 Truncations on derived categories

In this subsection, we make some observations about a pair of categories that are defined from the cohomological functors of the bounded derived category of an abelian category.

**Definition 2.1.1.** Let  $\mathcal{A}$  be an abelian category and  $D = D^*(\mathcal{A})$ . We define the full subcategory  $\mathcal{T}^{\geq n}$  of  $\mathcal{T}$ , which consists of the complexes  $X^\bullet$  such that  $H^p(X^\bullet) = 0$ , for  $p \leq n$ . Respectively we define  $\mathcal{T}^{\leq n}$ . Then, we have that  $\mathcal{T}^{\leq n} = \Sigma^{-n}(\mathcal{T}^{\leq 0})$  and  $\mathcal{T}^{\geq n} = \Sigma^n(\mathcal{T}^{\geq 0})$ . It is also true that

$$\begin{aligned} \dots \subseteq \mathcal{T}^{\leq n-1} \subseteq \mathcal{T}^{\leq n} \subseteq \mathcal{T}^{\leq n+1} \subseteq \dots \\ \dots \supseteq \mathcal{T}^{\geq n-1} \supseteq \mathcal{T}^{\geq n} \supseteq \mathcal{T}^{\geq n+1} \supseteq \dots \end{aligned}$$

For reference:  $\mathcal{T}^{\leq n}$  and  $\mathcal{T}^{\geq n}$  are called the truncation categories of  $D^*(\mathcal{A})$ .

For the pair  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  in Definition 2.1.1, based on theory of derived categories developed in Chapter ??, we can prove the following:

- $\mathcal{T}^{\leq 0} \subseteq \Sigma^{-1}(\mathcal{T}^0)$  and  $\mathcal{T}^{\geq 0} \supseteq \Sigma^{-1}(\mathcal{T}^{\geq 0})$ ,
- $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \Sigma^{-1}(\mathcal{T}^{\geq 0})) = 0$ ,
- for every  $X$  in  $\mathcal{T}$ , there exists a distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow \Sigma(Y)$$

such that  $Y \in \mathcal{T}^{\leq 0}$  and  $Z \in \Sigma^{-1}(\mathcal{T}^{\geq 0})$ .

The proofs are omitted and for details we refer the reader to [12, Chapter 4, Section 1, Subsection 1]

### 2.1.2 $t$ -structures

Formalizing the observations of subsection 2.1.1 we arrive naturally at the concept of a  $t$ -structure. In this section, we give the definition of a  $t$ -structure and present some basic results.

**Definition 2.1.2.** Let  $\mathcal{T}$  be a triangulated category. A  **$t$ -structure** in  $\mathcal{T}$  is a pair of full subcategories  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  with the following properties:

- ( $t_1$ )  $\mathcal{T}^{\leq 0} \subseteq \Sigma^{-1}(\mathcal{T}^0)$  and  $\mathcal{T}^{\geq 0} \supseteq \Sigma^{-1}(\mathcal{T}^{\geq 0})$ ,

$$(t_2) \operatorname{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \Sigma^{-1}(\mathcal{T}^{\geq 0})) = 0,$$

( $t_3$ ) for every  $X$  in  $\mathcal{T}$ , there exists a distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow \Sigma(Y)$$

such that  $Y \in \mathcal{T}^{\leq 0}$  and  $Z \in \Sigma^{-1}(\mathcal{T}^{\geq 0})$ .

**Notation 2.1.3.** If  $\mathcal{T}$  is a triangulated category and  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is a  $t$ -structure in  $\mathcal{T}$ , we define:

- $\mathcal{T}^{\leq n} := \Sigma^{\leq n}(\mathcal{T}^{\leq 0})$ ,
- $\mathcal{T}^{\geq n} := \Sigma^{\geq n}(\mathcal{T}^{\geq 0})$ ,
- $\mathcal{T}^{[m,n]} = \mathcal{T}^{\geq m} \cap \mathcal{T}^{\leq n}$ .

**Definition 2.1.4.** Let  $\mathcal{T}$  be a triangulated category and  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  a  $t$ -structure in  $\mathcal{T}$ . The **heart** of the  $t$ -structure is the subcategory  $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ .

**Example 2.1.5.** The subcategories  $\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}$  of  $D^*(\mathcal{A})$  of subsection 2.1.1 give rise to a  $t$ -structure, (the canonical  $t$ -structure) in  $D^*(\mathcal{A})$  with heart equivalent to  $\mathcal{A}$ .

Our main goal in this section is to prove that the heart of a  $t$ -structure in a triangulated category is an abelian category. In the following, unless otherwise stated,  $\mathcal{T}$  will be a triangulated category,  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  a  $t$ -structure in  $\mathcal{T}$  and  $\mathcal{A}$  its heart.

The following lemma will be useful in what follows.

**Lemma 2.1.6.** Let  $n, m \in \mathbb{Z}$ ,  $X \in \mathcal{T}^{\leq n}$  and  $Y \in \mathcal{T}^{\geq m}$ . Then  $\operatorname{Hom}_{\mathcal{T}}(X, Y) = 0$ .

*Proof.* By definition we have that

$$\dots \subseteq \mathcal{T}^{\leq n-1} \subseteq \mathcal{T}^{\leq n} \subseteq \mathcal{T}^{n-1} \subseteq \dots$$

and

$$\dots \supseteq \mathcal{T}^{\geq n-1} \supseteq \mathcal{T}^{\geq n} \supseteq \mathcal{T}^{n-1} \supseteq \dots$$

Since  $X \in \mathcal{T}^{\leq n}$ , there exists an object  $X' \in \mathcal{T}^{\leq 0}$  such that  $X = \Sigma^n(X')$ , and since  $Y \in \mathcal{T}^{\geq n+1}$ , there exists an object  $Y' \in \mathcal{T}^{\geq 1}$  such that  $\Sigma^{n+1}(Y') = Y$ . Therefore  $\operatorname{Hom}(X, Y) = \operatorname{Hom}(\Sigma^{-n}(X'), \Sigma^{n+1}(Y')) = \operatorname{Hom}(X', Y') = 0$ . ■

In the following Remark we present an important construction.

**Remark 2.1.7.** Let  $n \in \mathbb{Z}$ . Then, from (TR3), for every object  $X$  in  $\mathcal{T}$  there exists a distinguished triangle

$$\Sigma^n(A) \longrightarrow \Sigma^n(X) \longrightarrow \Sigma^n(B) \longrightarrow \Sigma^{n+1}(Y)$$

where  $\Sigma^n(A) \in \mathcal{T}^{\leq 0}$  and  $\Sigma^n(B) \in \mathcal{T}^{\geq 1}$ . By rotating the triangle  $3n$  times, we get the distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow \Sigma(Y)$$

where  $Y \in \mathcal{T}^{\leq n}$  and  $Z \in \mathcal{T}^{\geq n+1}$ . Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{T}$  and

$$C \longrightarrow Y \longrightarrow D \longrightarrow \Sigma(C)$$

be a distinguished triangle, where  $C \in \mathcal{T}^{\leq n}$  and  $D \in \mathcal{T}^{\geq n+1}$ . Then we get the diagram

$$\begin{array}{ccccccc} A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & \Sigma(A) \\ & & \downarrow f & & & & \\ C & \longrightarrow & Y & \longrightarrow & D & \longrightarrow & \Sigma(C) \end{array}$$

From Lemma 2.1.6, since  $A \in \mathcal{T}^{\leq n}$  and  $D \in \mathcal{T}^{\geq n+1}$ , we have that  $\text{Hom}(A, D) = 0$ . Therefore we have that  $(A \rightarrow D) \circ f \circ (Y \rightarrow D) = 0$  and thus, from Proposition 1.4.23, there exist morphisms  $\phi : A \rightarrow C$  and  $\psi : B \rightarrow D$  such that the following diagram is commutative

$$\begin{array}{ccccccc} A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & \Sigma(A) \\ \downarrow \phi & & \downarrow f & & \downarrow \psi & & \downarrow \Sigma(\phi) \\ C & \longrightarrow & Y & \longrightarrow & D & \longrightarrow & \Sigma(C) \end{array}$$

In addition, since  $A \in \mathcal{T}^{\leq n}$  and  $D[-1] \in \mathcal{T}^{\geq n+2}$ , we have that  $\text{Hom}(A, D[-1]) = 0$ , thus these morphisms are unique. We apply the above situation when  $X = Y$  and  $f = 1_X$ . Then there exist unique morphisms  $\alpha : A \rightarrow C$  and  $\beta : B \rightarrow D$ , such that the following diagram is commutative

$$\begin{array}{ccccccc} A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & \Sigma(A) \\ \alpha \downarrow & & \parallel & & \downarrow \beta & & \downarrow \Sigma(\alpha) \\ C & \longrightarrow & X & \longrightarrow & D & \longrightarrow & \Sigma(C) \end{array}$$

Respectively, there exist unique morphisms  $\gamma : C \rightarrow A$  and  $\delta : D \rightarrow B$ , such that the following diagram is commutative

$$\begin{array}{ccccccc} C & \longrightarrow & X & \longrightarrow & D & \longrightarrow & \Sigma(C) \\ \gamma \downarrow & & \parallel & & \downarrow \delta & & \downarrow \Sigma(\gamma) \\ A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & \Sigma(A) \end{array}$$

The composition of the above morphisms of triangles is the commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & \Sigma(A) \\ \gamma \circ \alpha \downarrow & & \parallel & & \downarrow \delta \circ \beta & & \downarrow \Sigma(\gamma \circ \alpha) \\ A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & \Sigma(A) \end{array}$$

and due to uniqueness, we get that  $\gamma \circ \alpha = 1_A$  and  $\delta \circ \beta = 1_B$ . Analogously, we have that  $\alpha \circ \gamma = 1_C$  and  $\beta \circ \delta = 1_D$ . Thus we have that  $\alpha$  and  $\beta$  are isomorphisms and,  $\gamma$  and  $\delta$  are their inverses. Therefore the objects  $A$  and  $B$  are unique up to unique isomorphism. This leads, for every object  $X$  in  $\mathcal{T}$ , to chose these objects  $A$  and  $B$ , and we denote them by  $\tau_{\leq n}(X)$  and  $\tau_{\geq n+1}(X)$  respectively. If now  $f : X \rightarrow Y$  is a morphism in  $\mathcal{T}$ , from the above, we get the following morphism of distinguished triangles.

$$\begin{array}{ccccccc} \tau_{\leq n}(X) & \longrightarrow & X & \longrightarrow & \tau_{\geq n+1}(X) & \longrightarrow & \Sigma(\tau_{\leq n}(X)) \\ \phi \downarrow & & \downarrow f & & \downarrow \psi & & \downarrow \Sigma(\phi) \\ \tau_{\leq n}(Y) & \longrightarrow & Y & \longrightarrow & \tau_{\geq n+1}(Y) & \longrightarrow & \Sigma(\tau_{\leq n}(Y)) \end{array}$$

where the morphisms  $\phi$  and  $\psi$  are unique, and we denote them by  $\tau_{\leq n}(f)$  and  $\tau_{\geq n+1}(f)$  respectively.

From the above Remark 2.1.7 we present the following Lemma.

**Lemma 2.1.8.** *There exist functors  $\tau_{\leq n} : \mathcal{T} \rightarrow \mathcal{T}^{\leq n}$  and  $\tau_{\geq n} : \mathcal{T} \rightarrow \mathcal{T}^{\geq n}$  such that:*

- (1)  $\tau_{\leq n} : \mathcal{T} \rightarrow \mathcal{T}^{\leq n}$  is a right adjoint to the natural inclusion  $\mathcal{T}^{\leq n} \rightarrow \mathcal{T}$ ,
- (2)  $\tau_{\geq n} : \mathcal{T} \rightarrow \mathcal{T}^{\geq n}$  is a left adjoint to the natural inclusion  $\mathcal{T}^{\geq n} \rightarrow \mathcal{T}$ .

*Proof.* Their existence comes from Remark 2.1.7. It is left to prove the part that has to do with the adjoint functors. Let  $X \in \mathcal{T}^{\leq n}$ . Then we have the distinguished triangle

$$X \rightrightarrows X \longrightarrow 0 \longrightarrow \Sigma(X)$$

which satisfies the above conditions of Remark 2.1.7. Then we have the morphism of distinguished triangles

$$\begin{array}{ccccccc} X & \rightrightarrows & X & \longrightarrow & 0 & \longrightarrow & \Sigma(X) \\ \downarrow \phi & & \downarrow f & & \downarrow & & \downarrow \Sigma(\phi) \\ \tau_{\leq n}(Y) & \longrightarrow & Y & \longrightarrow & \tau_{\geq n+1}(Y) & \longrightarrow & \Sigma(\tau_{\leq n}(Y)) \end{array}$$

where the morphism  $f$  comes from Remark 2.1.7. Since  $\phi$  is uniquely determined by  $f$ , there is a one to one and onto correspondence from  $\text{Hom}(X, Y)$  to  $\text{Hom}(X, \tau_{\leq n}(Y))$ , where  $f$  maps to  $\phi$ . Therefore  $\tau_{\leq n}$  is a right adjoint of the natural inclusion functor  $\mathcal{T}_{\leq n} \hookrightarrow \mathcal{T}$ . The proof of (2) is omitted since it is dual to that of (1). ■

**Remark 2.1.9.** Obviously we have the natural morphisms  $i : \tau_{\leq n}(X) \rightarrow X$  and  $p : X \rightarrow \tau_{\geq n}(X)$ , which are called the truncation morphisms.

**Definition 2.1.10.** The functors  $\tau_{\leq n}$  and  $\tau_{\geq n}$  are called the **truncation functors** of the  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ .

**Lemma 2.1.11.** For every object  $X$  in  $\mathcal{T}$  we have the distinguished triangle

$$\tau_{\leq n}(X) \xrightarrow{i} X \xrightarrow{p} \tau_{\geq n+1}(X) \xrightarrow{q} \Sigma(\tau_{\leq n}(X))$$

where the morphism  $q$  is unique.

*Proof.* This result is immediate from the proof of Lemma 2.1.8 and from Proposition 1.4.23. ■

**Definition 2.1.12.** The above distinguished triangle is called the **truncation distinguished triangle** of  $X$ .

Next we present some Lemmas without proof, about the behaviour of the truncation functors with the shift functor  $\Sigma$ , and about some properties of the truncation functors. For the details of the proofs, we refer the reader to [12, Lemma 1.2.6 - Lemma 1.2.12].

**Lemma 2.1.13.** For every  $n \in \mathbb{Z}$  we have that

$$(1) \tau_{\leq n} \circ \Sigma = \Sigma \circ \tau_{\leq n+1},$$

$$(2) \tau_{\geq n} \circ \Sigma = \Sigma \circ \tau_{\geq n+1}.$$

The following Lemma gives a description of the truncation subcategories associated to a  $t$ -structure

**Lemma 2.1.14.** *Let  $X$  be an object in  $\mathcal{T}$ . Then:*

(1) *The following are equivalent:*

- (i)  $X \in \mathcal{T}^{\leq n}$ ,
- (ii)  $i : \tau_{\leq n}(X) \rightarrow X$  is an isomorphism,
- (iii)  $\tau_{\geq n+1}(X) = 0$ ,

(2) *The following are equivalent:*

- (i)  $X \in \mathcal{T}^{\geq n}$ ,
- (ii)  $p : X \rightarrow \tau_{\geq n}(X)$  is an isomorphism,
- (iii)  $\tau_{\leq n-1}(X) = 0$ ,

**Lemma 2.1.15.** *Let*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

*be a distinguished triangle in  $\mathcal{T}$ . Then*

- (1) *If  $X, Z \in \mathcal{T}^{\leq n}$ , then  $Y \in \mathcal{T}^{\leq n}$ ,*
- (2)  *$X, Z \in \mathcal{T}^{\geq n}$ , then  $Y \in \mathcal{T}^{\geq n}$ .*

**Lemma 2.1.16.** *Let  $n \in \mathbb{Z}$ . Then*

- (1) *The subcategories  $\mathcal{T}^{\leq n}$  and  $\mathcal{T}^{\geq n}$ , are additive subcategories of  $\mathcal{T}$ ,*
- (2) *The functors  $\tau_{\leq n} : \mathcal{T} \rightarrow \mathcal{T}^{\leq n}$  and  $\tau_{\geq n} : \mathcal{T} \rightarrow \mathcal{T}^{\geq n}$  are additive.*

From the first part of the above Lemma we get the following.

**Lemma 2.1.17.** *The heart  $\mathcal{A}$  of a  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is an additive category.*

Next, we present some lemmas, without proof, about the composition of the truncations functors.

**Lemma 2.1.18.** *Let  $m, n \in \mathbb{Z}$ , with  $m \leq n$ . Then:*

- (1)  $\tau_{\leq m} \circ \tau_{\leq n} \cong \tau_{\leq n} \circ \tau_{\leq m} \cong \tau_{\leq m}$ ,

(2)  $\tau_{\geq m} \circ \tau_{\geq n} \cong \tau_{\geq n} \circ \tau_{\geq m} \cong \tau_{\geq n}$ .

**Lemma 2.1.19.** *Let  $m, n \in \mathbb{Z}$ , where  $m < n$ . Then  $\tau_{\leq m} \circ \tau_{\geq n} \cong \tau_{\geq n} \circ \tau_{\leq m} \cong 0$ .*

**Lemma 2.1.20.** *Let  $m, n \in \mathbb{Z}$ , where  $m \geq n$ , and  $X \in \mathcal{T}$ . Then  $(\tau_{\leq m} \circ \tau_{\geq n})(X)$ ,  $(\tau_{\geq n} \circ \tau_{\leq m})(X) \in \mathcal{T}^{[n,m]}$ .*

**Remark 2.1.21.** Let  $X \in \mathcal{T}$ . Then we have the truncation morphisms

$$\tau_{\leq m}(X) \xrightarrow{i} X \xrightarrow{p} \tau_{\geq n}(X)$$

From Lemma 2.1.8, the composition  $p \circ i : \tau_{\leq m}(X) \rightarrow \tau_{\geq n}(X)$  factors uniquely through  $(\tau_{\geq n} \circ \tau_{\leq m})(X)$ , that is, we have the commutative diagram

$$\begin{array}{ccc} \tau_{\leq m}(X) & \xrightarrow{p \circ i} & \tau_{\geq n}(X) \\ \downarrow & \nearrow & \\ \tau_{\geq n}(\tau_{\leq m})(X) & & \end{array}$$

where the vertical arrow is a truncation morphism. From Lemma 2.1.20,  $\tau_{\geq n}(\tau_{\leq m}(X)) \in \mathcal{T}^{\leq n}$ . Therefore, from Lemma 2.1.8, the morphism  $\tau_{\geq n}(\tau_{\leq m}(X)) \rightarrow \tau_{\geq n}(X)$  factors uniquely through  $\tau_{\leq m}(\tau_{\geq n}(X))$ , that is, the following square is commutative

$$\begin{array}{ccc} \tau_{\leq m}(X) & \xrightarrow{p \circ i} & \tau_{\geq n}(X) \\ \downarrow & & \uparrow \\ \tau_{\geq n}(\tau_{\leq m}(X)) & \xrightarrow{\phi} & \tau_{\leq m}(\tau_{\geq n}(X)) \end{array}$$

and the vertical arrows are truncation morphisms.

**Lemma 2.1.22.** *Let  $m, n \in \mathbb{Z}$ , such that  $m \geq n$ , and  $X \in \mathcal{T}$ . Then there exists a unique morphism  $\phi : \tau_{\geq n}(\tau_{\leq m}(X)) \rightarrow \tau_{\leq m}(\tau_{\geq n}(X))$ , such that the following diagram is commutative*

$$\begin{array}{ccccc} \tau_{\leq m}(X) & \xrightarrow{i} & X & \xrightarrow{p} & \tau_{\geq n}(X) \\ \downarrow & & & & \uparrow \\ \tau_{\geq n}(\tau_{\leq m}(X)) & \xrightarrow[\phi]{\cong} & & & \tau_{\leq m}(\tau_{\geq n}(X)) \end{array}$$

In addition, the morphism  $\phi$  is an isomorphism.

*Proof.* We already have the existence of the morphism  $\phi$  from Remark 2.1.21, so it is left to show that it is an isomorphism. Let  $h : \tau_{\leq n-1}(X) \rightarrow X$  be a truncation morphism from Remark 2.1.9. Then, from Lemma 2.1.8, the morphism  $h$  factors uniquely through  $\tau_{\leq m}(X)$ , i.e., the following diagram is commutative

$$\begin{array}{ccc} \tau_{\leq n-1}(X) & \xrightarrow{h} & X \\ f \downarrow & \nearrow g & \\ \tau_{\leq m}(X) & & \end{array}$$

where  $g : \tau_{\leq m}(X) \rightarrow X$  is also a truncation morphism. The above morphisms fit into the following diagram

$$\begin{array}{ccccccc} \tau_{\leq n-1}(X) & \xrightarrow{f} & \tau_{\leq m}(X) & \longrightarrow & \tau_{\geq n}(\tau_{\leq m}(X)) & \longrightarrow & \Sigma(\tau_{\leq n-1}(X)) \\ \parallel & & \downarrow g & & & & \parallel \\ \tau_{\leq n-1}(X) & \xrightarrow{h} & X & \longrightarrow & \tau_{\geq n}(X) & \longrightarrow & \Sigma(\tau_{\leq n-1}(X)) \\ \downarrow f & & \parallel & & & & \downarrow \Sigma(f) \\ \tau_{\leq m}(X) & \xrightarrow{g} & X & \longrightarrow & \tau_{\geq m+1}(X) & \longrightarrow & \Sigma(\tau_{\leq m}(X)) \end{array}$$

where the squares in the first column are commutative and the first row is a truncation distinguished triangle from Lemma 2.1.20. The other two rows are also truncation distinguished triangles. From (TR4), the above diagram can be completed to the octahedral diagram

$$\begin{array}{ccccccc} \tau_{\leq n-1}(X) & \xrightarrow{f} & \tau_{\leq m}(X) & \longrightarrow & \tau_{\geq n}(\tau_{\leq m}(X)) & \longrightarrow & \Sigma(\tau_{\leq n-1}(X)) \\ \parallel & & \downarrow g & & \vdots & & \parallel \\ \tau_{\leq n-1}(X) & \xrightarrow{h} & X & \longrightarrow & \tau_{\geq n}(X) & \longrightarrow & \Sigma(\tau_{\leq n-1}(X)) \\ \downarrow f & & \parallel & & \vdots & & \downarrow \Sigma(f) \\ \tau_{\leq m}(X) & \xrightarrow{g} & X & \longrightarrow & \tau_{\geq m+1}(X) & \longrightarrow & \Sigma(\tau_{\leq m}(X)) \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \tau_{\geq n}(\tau_{\leq m}(X)) & \longrightarrow & \tau_{\leq n}(X) & \longrightarrow & \tau_{\geq m+1}(X) & \longrightarrow & \Sigma(\tau_{\geq n}(\tau_{\leq m}(X))) \end{array}$$

By examining the middle square of the first row, we have that the morphism  $\tau_{\geq n}(\tau_{\leq m}(X)) \rightarrow \tau_{\geq n}(X)$  is the composition of  $\phi$  with the truncation morphism  $\tau_{\leq m}(\tau_{\geq n}(X)) \rightarrow \tau_{\geq n}(X)$ . Hence we have the morphism of triangles

$$\begin{array}{ccccccc}
\tau_{\geq n}(\tau_{\leq m})(X) & \longrightarrow & \tau_{\geq n}(X) & \longrightarrow & \tau_{\geq m+1}(X) & \longrightarrow & \Sigma(\tau_{\geq n}(\tau_{\leq m}(X))) \\
\phi \downarrow & & \parallel & & \parallel & & \downarrow \Sigma(\phi) \\
\tau_{\leq m}(\tau_{\geq n}(X)) & \longrightarrow & \tau_{\geq n-1} & \longrightarrow & \tau_{\geq m+1}(X) & \longrightarrow & \Sigma(\tau_{\leq m}(\tau_{\geq n}(X)))
\end{array}$$

where the first row is the last row of the above octahedral diagram, and the second row is the truncation distinguished triangle from Lemma 2.1.20. Since the identity maps are isomorphisms, from the five lemma for triangulated categories, Lemma 1.4.18, we have that  $\phi$  is also an isomorphism.  $\blacksquare$

**Remark 2.1.23.** The morphism  $\phi : \tau_{\geq n}(\tau_{\leq m}(X)) \xrightarrow{\cong} \tau_{\leq m}(\tau_{\geq n}(X))$  induce natural isomorphisms  $\tau_{\geq n} \circ \tau_{\leq m} \cong \tau_{\leq m} \circ \tau_{\geq n}$  and  $\tau_{\geq n} \circ \tau_{\leq m} \cong \tau_{\leq m} \circ \tau_{\geq n}$ . Therefore we just proved the following Lemma.

**Lemma 2.1.24.** *Let  $m, n \in \mathbb{Z}$ , where  $m \geq n$ . Then the functors  $\tau_{\geq n} \circ \tau_{\leq m}$  and  $\tau_{\leq m} \circ \tau_{\geq n}$  are naturally isomorphic.*

One can generalize the notion of cohomology to triangulated categories under the existence of  $t$ -structures. In this direction, we give the following definition.

**Definition 2.1.25.** *We define a functor  $H^0 : \mathcal{T} \rightarrow \mathcal{A}$ , by*

$$H^0(X) := \tau_{\leq 0}(\tau_{\geq 0}(X)) \cong \tau_{\geq 0}(\tau_{\leq 0}(X))$$

for every object  $X \in \mathcal{T}$ .

**Remark 2.1.26.** It follows from Lemma 2.1.16 that the functor  $H$  is additive.

In our attempt to prove that the heart of a  $t$ -structure is an abelian category, we proceed by proving that it has kernels and cokernels.

**Lemma 2.1.27.** *Let  $f : X \rightarrow Y$  be a morphism in the heart of the  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ . Let also*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma(X)$$

be a distinguished triangle, where  $Z$  is the cone of  $f$ . Then  $Z \in \mathcal{T}^{[-1,0]}$ .

*Proof.* By rotating the distinguished triangle, we get the distinguished triangle

$$Y \longrightarrow Z \longrightarrow \Sigma(X) \longrightarrow \Sigma(Y)$$

First, we prove that  $Z \in \mathcal{T}^{\leq 0}$ . Obviously  $Y \in \mathcal{T}^{\leq 0}$ . Since  $X \in \mathcal{T}^{\leq 0}$ , we have that  $\Sigma(X) \in \mathcal{T}^{\leq -1}$ . From the inclusion  $\mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$ , we have that  $\Sigma(X) \in \mathcal{T}^{\leq 0}$ . Hence from Lemma 2.1.15 we have that  $Z \in \mathcal{T}^{\leq 0}$ . Now we prove that  $Z \in \mathcal{T}^{\geq -1}$ . We have that  $X, Y \in \mathcal{T}^{\geq 0}$ , so  $\Sigma(X) \in \mathcal{T}^{\geq -1}$ . However,  $\mathcal{T}^{\geq -1} \supseteq \mathcal{T}^{\geq 0}$ , therefore  $\Sigma(X), Y \in \mathcal{T}^{\geq 1}$ , and from Lemma 2.1.15, we have that  $Z \in \mathcal{T}^{\geq -1}$ . By combining the above, we have shown that  $Z \in \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq -1} = \mathcal{T}^{[-1,0]}$ . ■

We proceed with the construction of kernels and cokernels. From Lemma 2.1.15 we have that  $\Sigma^{-1}(Z) \in \mathcal{T}^{[0,1]}$ . We define

$$K := H^0(\Sigma^{-1}(Z)) = \tau_{\leq 0}(\tau_{\geq 0}(X)) = \tau_{\leq 0}(\Sigma^{-1}(Z)) \in \mathcal{A}$$

and

$$C := H^0(Z) = \tau_{\geq 0}(\tau_{\leq 0}(Z)) = \tau_{\geq 0}(Z) \in \mathcal{A}$$

In addition, we have the natural morphisms

$$k : K = \tau_{\leq 0}(\Sigma^{-1}(Z)) \rightarrow \Sigma^{-1}(Z) \rightarrow X$$

and

$$c : Y \rightarrow Z \rightarrow \tau_{\geq 0}(X) = C$$

which will be denoted by  $k$  and  $c$  respectively. From Lemma 2.1.11, we have the truncation distinguished triangle

$$\tau_{\leq 0}(\Sigma^{-1}(Z)) \longrightarrow \Sigma^{-1}(Z) \longrightarrow \tau_{\geq 1}(\Sigma^{-1}(Z)) \longrightarrow \Sigma(\tau_{\leq 0}(\Sigma^{-1}(Z)))$$

By definition, we have that  $K = \tau_{\leq 0}(\Sigma^{-1}(Z))$ . Since,  $\tau_{\geq 1}(\Sigma^{-1}(Z)) = \Sigma^{-1}(\tau_{\geq 0}(Z)) = \Sigma^{-1}(C)$ . In other words, we just proved the following Lemma.

**Lemma 2.1.28.** *The triangles*

$$K \longrightarrow \Sigma^{-1}(Z) \longrightarrow \Sigma^{-1}(C) \longrightarrow \Sigma(K)$$

and

$$\Sigma^{-1}(K) \longrightarrow Z \longrightarrow C \longrightarrow K$$

are distinguished triangles, and the morphisms that appear are truncation morphisms.

**Lemma 2.1.29.** *Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{A}$*

- (1) *The pair  $(K, k)$  is a kernel of  $f : X \rightarrow Y$ ,*
- (2) *The pair  $(C, c)$  is a cokernel of  $f : X \rightarrow Y$ .*

*Proof.* (1) By definition we have the diagram

$$\begin{array}{ccccc} & & k & & \\ & \searrow & \curvearrowright & \searrow & \\ K & \longrightarrow & \Sigma^{-1}(Z) & \longrightarrow & X \xrightarrow{f} Y \end{array}$$

Since consecutive compositions in a distinguished triangle are zero, we have that

$$\begin{aligned} f \circ k &= f \circ [(\Sigma^{-1}(Z) \rightarrow X) \circ (K \rightarrow \Sigma^{-1}(Z))] = [f \circ (\Sigma^{-1}(Z) \rightarrow X)] \circ (K \rightarrow \Sigma^{-1}(Z)) = \\ &= 0 \circ (K \rightarrow \Sigma^{-1}(Z)) = 0 \end{aligned}$$

From Lemma 2.1.28, we have the distinguished triangle

$$K \longrightarrow \Sigma^{-1}(Z) \longrightarrow \Sigma^{-1}(C) \longrightarrow \Sigma(K)$$

We have that  $\Sigma^{-1}(C) \in \mathcal{T}^{\geq 1}$  and  $\Sigma^{-1}(C) \in \mathcal{T}^{\geq 2}$ . Therefore, for every  $U \in \mathcal{T}^{\leq 0}$ , we get that  $\text{Hom}(U, \Sigma^{-1}(C)) = \text{Hom}(U, \Sigma^{-2}(Z)) = 0$ . From the long exact cohomology sequence of the above triangle we have

$$0 = \text{Hom}(U, \Sigma^{-1}(C)) \longrightarrow \text{Hom}(U, K) \longrightarrow \text{Hom}(U, \Sigma^{-1}(Z)) \longrightarrow \text{Hom}(U, \Sigma^{-1}(C)) = 0$$

Therefore the morphism  $(K \rightarrow \Sigma^{-1}(Z))_* : \text{Hom}(U, K) \xrightarrow{\cong} \text{Hom}(U, \Sigma^{-1}(Z))$  is an isomorphism. Consider now the triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

and the long exact sequence that accompanies it

$$\cdots \longrightarrow \text{Hom}(U, \Sigma^{-1}(Y)) \longrightarrow \text{Hom}(U, \Sigma^{-1}(Z)) \longrightarrow \text{Hom}(U, X) \xrightarrow{f_*} \text{Hom}(U, Y) \longrightarrow \cdots$$

Since  $U \in \mathcal{T}^{\leq 0}$  and  $\Sigma^{-1}(Y) \in \mathcal{T}^{\geq 1}$ , from Lemma 2.1.6, we conclude that  $\text{Hom}(U, \Sigma^{-1}(Y)) = 0$ . Furthermore, from Lemma 2.1.28, we have the exact sequence

$$0 \longrightarrow \text{Hom}(U, K) \xrightarrow{k_*} \text{Hom}(U, X) \xrightarrow{f_*} \text{Hom}(U, Y)$$

It is left to show the universal property of the kernel. So, let  $j : A \rightarrow X$  be a morphism in  $\mathcal{A}$ , such that  $f \circ j = 0$ . Then  $f_*(j) = 0$ , and due to the exactness of the above sequence, we have that  $j = k_*(i) = k \circ i$  for some morphism  $i : A \rightarrow K$ . Therefore the pair  $(K, k)$  is a kernel of  $f$ .

(2) The proof is omitted since it is dual to that of 1. ■

In order to finish the proof that the heart of a  $t$ -structure is abelian, we need to prove that the parallel morphism of a morphism is an isomorphism. We proceed with this proof.

Let  $J$  be the cone of the cokernel  $c : Y \rightarrow C$ . Then we have the distinguished triangle

$$Y \xrightarrow{c} C \longrightarrow J \longrightarrow \Sigma(Y)$$

and from Lemma 2.1.27 we have that  $J \in \mathcal{T}^{[-1,0]}$ . In particular  $J \in \mathcal{T}^{\geq -1} = \Sigma(\mathcal{T}^{\geq 0})$ . Thus there exists an object  $I \in \mathcal{T}^{\geq 0}$ , such that  $J = \Sigma(I)$ . Consider now the natural morphism

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z & \xrightarrow{q} & C \\ & & \searrow & \nearrow & \\ & & & c & \end{array}$$

and the distinguished triangle with base the morphism  $f$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

where  $Z = \text{cone}(f)$ . Then the diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) & \xrightarrow{-\Sigma(f)} & \Sigma(Y) \\ \parallel & & \downarrow q & & & & \parallel \\ Y & \xrightarrow{c} & C & \longrightarrow & \Sigma(I) & \longrightarrow & \Sigma(Y) \\ \downarrow g & & \parallel & & & & \downarrow \\ Z & \longrightarrow & C & \longrightarrow & \Sigma^2(K) & \xrightarrow{-\Sigma(i)} & \Sigma(Z) \end{array}$$

since, the squares in the first column are commutative, can be completed through the octahedral axiom to the commutative diagram

$$\begin{array}{ccccccc}
Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) & \xrightarrow{-\Sigma(f)} & \Sigma(Y) \\
\parallel & & \downarrow q & & \downarrow -\Sigma(u) & & \parallel \\
Y & \xrightarrow{c} & C & \longrightarrow & \Sigma(I) & \longrightarrow & \Sigma(Y) \\
\downarrow g & & \parallel & & \downarrow & & \downarrow \\
Z & \longrightarrow & C & \longrightarrow & \Sigma^2(K) & \xrightarrow{-\Sigma(i)} & \Sigma(Z) \\
\downarrow h & & \downarrow & & \parallel & & \downarrow \Sigma(h) \\
\Sigma(X) & \xrightarrow{-\Sigma(u)} & \Sigma(I) & \xrightarrow{\quad} & \Sigma^2(K) & \xrightarrow{-w} & \Sigma^2(X)
\end{array}$$

Now, the first row is the distinguished triangle with base  $f$  rotated from (TR2), the second row is the distinguished triangle which we defined above, and the third row is the distinguished triangle from Lemma 2.1.28 with the truncation morphism  $i : K[1] \rightarrow Z$  after it has been rotated from (TR2). Therefore, in the last row of the octahedral diagram, we have  $w = \Sigma(h) \circ (-\Sigma(i)) = -\Sigma(h) \circ \Sigma(i) = -\Sigma(h \circ i) = -\Sigma^2(k)$ . By rotating the distinguished triangle in the last row three times, we get the distinguished triangle

$$X \xrightarrow{u} I \longrightarrow \Sigma(K) \xrightarrow{\Sigma(K)} \Sigma(X)$$

Since  $X, K \in \mathcal{A}$  and  $X \in \mathcal{T}^{\leq 0}$ , we get  $\Sigma(K) \in \mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$ . Therefore, from Lemma 2.1.15 we have that  $I \in \mathcal{T}^{\leq 0}$ . We have already shown that  $I \in \mathcal{T}^{\geq 0}$ , therefore we conclude that  $I \in \mathcal{A} = \mathcal{T}^{\geq 0} \cap \mathcal{T}^{\leq 0}$ . By rotating the last triangle, we get the distinguished triangle

$$K \xrightarrow{-k} X \xrightarrow{u} I \longrightarrow \Sigma(K)$$

Consider now the isomorphism of triangles

$$\begin{array}{ccccccc}
K & \xrightarrow{-k} & X & \xrightarrow{u} & I & \xrightarrow{-a} & \Sigma(K) \\
-1_K \downarrow & & \parallel & & \parallel & & \downarrow -1_{\Sigma(K)} \\
K & \xrightarrow{k} & X & \xrightarrow{u} & I & \xrightarrow{-a} & \Sigma(K)
\end{array}$$

and then the triangle

$$K \xrightarrow{k} X \xrightarrow{u} I \longrightarrow \Sigma(K)$$

is distinguished, that is,  $I = \text{cone}(k)$ . From Lemma 2.1.15, the pair  $(I, u)$  is a cokernel of  $k$ . Respectively, from the distinguished triangle

$$Y \xrightarrow{c} C \longrightarrow \Sigma(I) \xrightarrow{\Sigma(p)} \Sigma(Y)$$

we have that the pair  $(I, p)$  is a kernel of  $c$ . Also, from the commutativity of the square

$$\begin{array}{ccc} \Sigma(X) & \xrightarrow{-\Sigma(f)} & \Sigma(Y) \\ -\Sigma(u) \downarrow & & \parallel \\ \Sigma(I) & \xrightarrow{\Sigma(p)} & \Sigma(Y) \end{array}$$

we have that  $f = p \circ u$ . From the last we conclude that  $\text{Coim}(f) = \text{Im}(f) = I$ . Therefore we proved the following theorem.

The following important theorem provides a bridge between the triangulated category  $\mathcal{T}$  and the heart  $\mathcal{A}$  of a  $t$ -structure. It allows us to recover the usual notion of cohomology within a purely triangulated setting.

**Theorem 2.1.30.** *The heart of a  $t$ -structure in a triangulated category is an abelian category.*

**Theorem 2.1.31.** *The functor  $H^0 : \mathcal{T} \rightarrow \mathcal{A}$  is a cohomological functor.*

*Proof.* We need to prove that for every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

the sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is exact in  $\mathcal{A}$ . We will prove the statement for 4 separate occasions before proving it in its generality.

- $X, Y, Z \in \mathcal{T}^{\geq 0}$ : Let  $U \in \mathcal{A}$ . Then we have the exact sequence

$$\text{Hom}(U, \Sigma^{-1}(Z)) \longrightarrow \text{Hom}(U, X) \xrightarrow{f_*} \text{Hom}(U, Y) \xrightarrow{g_*} \text{Hom}(U, Z)$$

Since,  $U \in \mathcal{T}^{\leq 0}$  and  $\Sigma^{-1}(Z) \in \mathcal{T}^{\geq 1}$ , we have that  $\text{Hom}(U, \Sigma^{-1}(Z)) = 0$ , and the above exact sequence becomes

$$0 \longrightarrow \text{Hom}(U, X) \xrightarrow{f_*} \text{Hom}(U, Y) \xrightarrow{g_*} \text{Hom}(U, Z)$$

However, from Lemma 2.1.14, we have that

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(U, H^0(X)) &= \text{Hom}_{\mathcal{T}}(U, H^0(X)) = \text{Hom}_{\mathcal{T}}(U, \tau_{\leq 0}(\tau_{\geq 0}(X))) = \text{Hom}_{\mathcal{T}}(U, \tau_{\leq 0}(X)) = \\ &= \text{Hom}_{\mathcal{T}}(U, X) \end{aligned}$$

by the definition of the functor  $H$ . Respectively, we have

$$\text{Hom}_{\mathcal{A}}(U, H^0(Y)) = \text{Hom}_{\mathcal{T}}(U, Y) \text{ and } \text{Hom}_{\mathcal{A}}(U, H^0(Z)) = \text{Hom}_{\mathcal{T}}(U, Z)$$

Hence, the above exact sequence becomes

$$0 \longrightarrow \text{Hom}(U, H^0(X)) \xrightarrow{H^0(f)_*} \text{Hom}(U, H^0(Y)) \xrightarrow{H^0(g)_*} \text{Hom}(U, H^0(Z))$$

which is exact, since the functor  $\text{Hom}(U, -)$  is cohomological. Therefore the sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is also exact. (This follows from Yoneda's Lemma which is also used in the sequel).<sup>1</sup>

- $X, Y, Z \in \mathcal{T}^{\leq 0}$  : Similar argument to the first part of the proof.
- $Z \in \mathcal{T}^{\geq 0}$  : Let  $W \in \mathcal{T}^{\leq -1}$ . Then  $\text{Hom}(W, \Sigma^{-1}(Z)) = 0$ , since  $\Sigma^{-1}(Z) \in \mathcal{T}^{\geq 1}$ . Therefore, we have the exact sequence

$$0 = \text{Hom}(W, \Sigma^{-1}(Z)) \longrightarrow \text{Hom}(W, X) \xrightarrow{f_*} \text{Hom}(W, Y) \longrightarrow \text{Hom}(W, Z) = 0$$

which implies that  $f_*$  is an isomorphism. Consider now the commutative diagram

$$\begin{array}{ccc} \tau_{\leq -1}(X) & \xrightarrow{\tau_{\leq -1}(f)} & \tau_{\leq -1}(Y) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

<sup>1</sup>For the statement and the proof of Yoneda's Lemma we refer the reader to [13, Chapter IV, Section 2, Lemma 2.1]

where the vertical arrows are truncation morphisms. By applying the functor  $\text{Hom}(W, -)$  to the last diagram, we get the commutative diagram

$$\begin{array}{ccc} \text{Hom}(W, \tau_{\leq -1}(X)) & \xrightarrow{\tau_{\leq -1}(f)_*} & \text{Hom}(W, \tau_{\leq -1}(Y)) \\ \downarrow & & \downarrow \\ \text{Hom}(W, X) & \longrightarrow & \text{Hom}(W, Y) \end{array}$$

Due to adjointness, the vertical arrows are isomorphisms. Since this happens for arbitrary  $W$ , the morphism  $\tau_{\leq -1}(f) : \tau_{\leq -1}(X) \rightarrow \tau_{\leq -1}(Y)$  is an isomorphism, since we have shown that  $f_*$  is an isomorphism. Consider now the diagram

$$\begin{array}{ccccc} \tau_{\leq -1}(X) & \xrightarrow{i} & X & \xrightarrow{f} & Y \\ & & \searrow & \nearrow & \\ & & & c & \end{array}$$

and the composition  $c = f \circ i$ . Then we get the octahedral diagram

$$\begin{array}{ccccccc} \tau_{\leq -1}(X) & \xrightarrow{i} & X & \longrightarrow & \tau_{\geq 0}(X) & \longrightarrow & \Sigma(\tau_{\leq -1}(X)) \\ \parallel & & \downarrow f & & \downarrow & & \parallel \\ \tau_{\leq -1}(X) & \xrightarrow{c} & Y & \longrightarrow & \tau_{\geq 0}(Y) & \longrightarrow & \Sigma(\tau_{\leq -1}(X)) \\ \downarrow i & & \parallel & & \downarrow & & \downarrow \Sigma(i) \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma(X) \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \tau_{\geq 0} & \longrightarrow & \tau_{\geq 0}(Y) & \longrightarrow & Z & \longrightarrow & \Sigma(\tau_{\geq 0}(X)) \end{array}$$

The first row is truncation distinguished triangle of  $X$ , the second row is truncation distinguished triangle of  $Y$ , where we have replaced  $\tau_{\leq -1}(Y)$  with its isomorphic  $\tau_{\leq -1}(X)$ , and lastly, the third row is the distinguished triangle with base  $f$ . The distinguished triangle of the last row has all of its objects in  $\mathcal{T}^{\geq 0}$  and thus we can use 1. By applying the functor  $H^0$  to the last two rows of the above octahedral diagram, we get the commutative diagram

$$\begin{array}{ccccccc} H^0(X) & \xrightarrow{H^0} & H^0(Y) & \xrightarrow{H^0(g)} & H^0 Z & & \\ \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & H^0(\tau_{\geq 0}(X)) & \xrightarrow{H^0(\tau_{\geq 0}(f))} & H^0(\tau_{\geq 0}(Y)) & \xrightarrow{H^0(\tau_{\geq 0}(g))} & H^0(Z) \end{array}$$

where the first two vertical arrows are the induced truncation morphisms. By the definition of the functor  $H^0$ , these morphisms are isomorphisms, because  $X, Y \in \mathcal{A} \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0} \Rightarrow X, Y \in \mathcal{T}^{\geq 0}$ . Thus  $\tau_{\geq 0}(X)$  and  $\tau_{\geq 0}(Y)$  are isomorphisms, which leads to that  $H^0(\tau_{\geq 0}(X))$  and  $H^0(\tau_{\geq 0}(Y))$  are isomorphisms. Therefore the sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is exact in the heart  $\mathcal{A}$ .

- $X \in \mathcal{T}^{\leq 0}$ : Let  $W \in \mathcal{T}^{\geq 1}$ . Then  $\text{Hom}(X, W) = \text{Hom}(X[1], W) = 0$ , since  $X[1] \in \mathcal{T}^{\leq -1}$ . Therefore, from Proposition 1.4.17, we have the exact sequence

$$0 = \text{Hom}(X[1], W) \longrightarrow \text{Hom}(Z, W) \xrightarrow{g^*} \text{Hom}(Y, W) \longrightarrow \text{Hom}(X, W) = 0$$

hence  $g^*$  is an isomorphism. Consider now the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow \\ \tau_{\geq 1}(Y) & \longrightarrow & \tau_{\geq 1}(Z) \end{array}$$

By applying the contravariant functor  $\text{Hom}(-, W)$ , we get the commutative diagram

$$\begin{array}{ccc} \text{Hom}(\tau_{\geq 1}(Z)) & \xrightarrow{\tau_{\geq 1}(g)^*} & \text{Hom}(\tau_{\geq 1}(Y), W) \\ \downarrow & & \downarrow \\ \text{Hom}(Z, W) & \xrightarrow{g^*} & \text{Hom}(Y, W) \end{array}$$

From adjointness Lemma 2.1.8, the vertical arrows are isomorphisms. Since  $W$  is an arbitrary object in  $\mathcal{T}^{\geq 1}$ , the morphism  $\tau_{\geq 1}(g) : \tau_{\geq 1}(Y) \rightarrow \tau_{\geq 1}(Z)$  is an isomorphism. Consider now the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{g} & Z & \xrightarrow{q} & \tau_{\geq 1}(Z) \\ & & \searrow & \nearrow & \uparrow \\ & & & & d \end{array}$$

from which we obtain the octahedral diagram

$$\begin{array}{ccccccc}
 Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) & \xrightarrow{-\Sigma(f)} & \Sigma(Y) \\
 \parallel & & \downarrow q & & \downarrow \Sigma(u) & & \parallel \\
 Y & \xrightarrow{d} & \tau_{\geq 1}(Z) & \longrightarrow & \Sigma(\tau_{\leq 0}(Y)) & \xrightarrow{-\Sigma(i)} & \Sigma(Y) \\
 \downarrow g & & \parallel & & \downarrow \Sigma(v) & & \downarrow \Sigma(g) \\
 Z & \xrightarrow{q} & \tau_{\geq 1}(Z) & \longrightarrow & \Sigma(\tau_{\leq 0}(Z)) & \xrightarrow{-\Sigma(j)} & \Sigma(Z) \\
 \downarrow h & & \downarrow & & \parallel & & \downarrow \Sigma(h) \\
 \Sigma(X) & \xrightarrow{\Sigma(u)} & \Sigma(\tau_{\leq 0}(Y)) & \xrightarrow{\Sigma(v)} & \Sigma(\tau_{\leq 0}(Z)) & \xrightarrow{\Sigma(w)} & \Sigma^2(X)
 \end{array}$$

The first row is the distinguished triangle with base  $f$ , after it has been rotated. The second row is the truncation distinguished triangle of  $Y$  after we have substituted  $\tau_{\geq 1}(Y)$  with  $\tau_{\leq -1}(Z)$ , by using the above isomorphism, and after we have rotated it. The third row is truncation distinguished triangle of  $Z$  after we have rotated it. The morphisms  $i : \tau_{\leq 0}(Y) \rightarrow Y$  and  $j : \tau_{\leq 0}(Z) \rightarrow Z$  are the canonical truncation morphisms. By rotating the distinguished triangle of the last row three times, we get the distinguished triangle

$$X \xrightarrow{-u} \tau_{\leq 0}(Y) \xrightarrow{-v} \tau_{\leq 0}(Z) \xrightarrow{-w} \Sigma(X)$$

whose vertices are in  $\mathcal{F}^{\leq 0}$ . Therefore we can apply the second part of the proof. Consider now the commutative diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{-u} & \tau_{\leq 0}(Y) & \xrightarrow{-v} & \tau_{\leq 0}(Z) & \xrightarrow{-w} & \Sigma(X) \\
 \parallel & & \downarrow -i & & \downarrow j & & \parallel \\
 X & \xrightarrow{f} & Y & \xrightarrow{g'} & Z & \xrightarrow{h'} & \Sigma(X)
 \end{array}$$

where the first row is the above distinguished triangle, and the second row is the distinguished triangle with base  $f$ . From the octahedral axiom, we have that the above diagram is a morphism of triangles (last column of the octehedral diagram). By applying the functor  $H^0$  to the last diagram, we get the commutative diagram

$$\begin{array}{ccccccc}
 H^0(X) & \longrightarrow & H^0(\tau_{\geq 0}(Y)) & \longrightarrow & H^0(\tau_{\geq 0}(Z)) & \longrightarrow & 0 \\
 \parallel & & \downarrow & & \downarrow & & \\
 H^0(X) & \xrightarrow{H^0(f)} & H^0(Y) & \xrightarrow{H^0(g)} & H^0(Z) & & 
 \end{array}$$

where the second and the third vertical arrows are induced from the truncation morphisms. Now, by the definition of  $H$ , these arrows are isomorphisms, hence the sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is exact.

- General case: Consider the diagram

$$\begin{array}{ccccc} \tau_{\leq 0}(X) & \xrightarrow{i} & X & \xrightarrow{f} & Y \\ & & \searrow & \nearrow & \\ & & & c & \end{array}$$

Then we have the octehedral diagram

$$\begin{array}{ccccccc} \tau_{\leq 0}(X) & \xrightarrow{i} & X & \longrightarrow & \tau_{\geq 1}(X) & \longrightarrow & \Sigma(\tau_{\leq 0}(X)) \\ \parallel & & \downarrow f & & \downarrow & & \parallel \\ \tau_{\leq 0}(X) & \xrightarrow{c} & Y & \xrightarrow{u} & W & \longrightarrow & \Sigma(\tau_{\leq 0}(X)) \\ \downarrow i & & \parallel & & \downarrow v & & \downarrow \Sigma(i) \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \tau_{\geq 1}(X) & \longrightarrow & W & \longrightarrow & Z & \longrightarrow & \Sigma(\tau_{\geq 1}(X)) \end{array}$$

The first row is the truncation distinguished triangle of  $X$ . The second and third rows are the distinguished triangles with base  $c$  and  $f$  respectively. From the fourth part of the proof we have that  $X \in \mathcal{T}^{\leq 0}$ , hence the distinguished triangle of the second row gives the following exact sequence

$$H^0(\tau_{\leq 0}(X)) \xrightarrow{H^0(c)} H^0(Y) \xrightarrow{H^0(u)} H^0(W) \longrightarrow 0$$

In addition, we have that  $c = f \circ i$  and  $H(i) : H^0(\tau_{\leq 0}(X)) \rightarrow H^0(X)$  are isomorphisms. Therefore we have the exact sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(u)} H^0(W) \longrightarrow 0$$

Particularly  $H^0(u)$  is an epimorphism. However, by rotating the triangle of the last row, we get the distinguished triangle

$$W \longrightarrow Z \longrightarrow \Sigma(\tau_{\geq 1}(X)) \longrightarrow \Sigma(W)$$

where  $\Sigma(\tau_{\geq 1}(X)) \in \mathcal{T}^{\geq 0}$ . From the third part of the proof, we have the exact sequence

$$0 \longrightarrow H^0(W) \xrightarrow{H^0(u)} H^0(X) \longrightarrow H^0(\Sigma(\tau_{\geq 1}(X)))$$

Particularly  $H^0(u)$  is a monomorphism. From the middle square of the octahedral diagram

$$\begin{array}{ccc} Y & \xrightarrow{u} & W \\ \parallel & & \downarrow v \\ Y & \xrightarrow{g} & Z \end{array}$$

we have that  $g = v \circ u$ , so  $H^0(g) = H^0(v) \circ H^0(u)$  and  $\text{Ker}(H^0(g)) = \text{Ker}(H^0(u))$ . Therefore the sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is exact, and thus we have completed the proof. ■

The next Lemma establishes a relationship between distinguished triangles in  $\mathcal{T}$  and short exact sequences in  $\mathcal{A}$ .

**Lemma 2.1.32.** *Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

*be a short exact sequence in  $\mathcal{A}$ . Then, the cone of  $f : X \rightarrow Y$  is isomorphic to  $Z$ , and we have the distinguished triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

*Proof.* Let

$$X \xrightarrow{f} Y \xrightarrow{h} C \xrightarrow{k} \Sigma(X)$$

be the distinguished triangle with base the morphism  $f$ . Since  $f$  is a monomorphism we have that  $\text{Ker}(f) = 0$ , and from the arguments of the proof of Lemma 2.1.29, we have that  $H^{-1}(C) = 0$ . Therefore, from Lemma 2.1.27, we have that  $C \in \mathcal{A}$ . In addition, the pair  $(C, h)$  is a cokernel of  $f$ . On the other hand, since the above sequence is exact, the pair  $(Z, g)$  is also a cokernel of  $f$ . Therefore, there exists an isomorphism  $j : Z \rightarrow C$ , such that  $h = j \circ g$ . Thus we have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{k \circ j} & \Sigma(X) \\ \parallel & & \parallel & & \downarrow j & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{h} & C & \xrightarrow{k} & \Sigma(X) \end{array}$$

where all the vertical arrows are isomorphisms, hence the above diagram is an isomorphism of triangles. Since the bottom row is a distinguished triangle, so is the top row.  $\blacksquare$

The proofs of the following Lemmas and Corollary are rather technical, therefore we refer the reader to [12, Lemma 1.2.21 - Lemma 1.2.23] for details.

**Lemma 2.1.33.** *Let  $X \in \mathcal{T}$ ,  $n \in \mathbb{Z}$  and  $i : \tau_{\leq n} \rightarrow X$ ,  $q : X \rightarrow \tau_{\geq n}(X)$  be truncation morphisms. Then*

- (1)  $H^p(i) : H^p(\tau_{\leq n}(X)) \rightarrow H^p(X)$  is an isomorphism for  $p \leq n$ , and  $H^p(\tau_{\leq n}(X)) = 0$ , for  $p > n$ ,
- (2)  $H^p(q) : H^p(X) \rightarrow H^p(\tau_{\geq n}(X))$  is an isomorphism for  $p \geq n$ , and  $H^p(\tau_{\geq n}(X)) = 0$ , for  $p < n$ .

**Corollary 2.1.34.** *Let  $n \in \mathbb{Z}$ . Then:*

- (1) if  $X \in \mathcal{T}^{\leq n}$ , then  $H^0(X) = 0$  for  $p > n$ ,
- (2) if  $X \in \mathcal{T}^{\geq n}$ , then  $H^0(X) = 0$  for  $p < n$ .

**Lemma 2.1.35.** *Let  $n \in \mathbb{Z}$  and  $X \in \mathcal{T}$  such that  $H^p(X) = 0$  for all  $p \in \mathbb{Z}$ . Then:*

- (1) if  $X \in \mathcal{T}^{\leq n}$ , then  $X \in \mathcal{T}^{\leq p}$ , for all  $p \in \mathbb{Z}$ ,
- (2) if  $X \in \mathcal{T}^{\geq n}$ , then  $X \in \mathcal{T}^{\geq p}$ , for all  $p \in \mathbb{Z}$ .

### 2.1.3 Nondegenerate and bounded $t$ -structures

In this section, we study nondegenerate and bounded  $t$ -structures. In Bridgeland's paper, bounded  $t$ -structures play a crucial role, since they appear naturally under the existence of slicings. We give their definition, and present some basic results.

**Lemma 2.1.36.** *Let  $\mathcal{T}$  be a triangulated category and  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  a  $t$ -structure in  $\mathcal{T}$ . Then the following are equivalent:*

$$(1) \bigcap_{n \in \mathbb{Z}} \mathcal{T}^{\leq n} = \{0\} \text{ and } \bigcap_{n \in \mathbb{Z}} \mathcal{T}^{\geq n} = \{0\},$$

(2) for all  $X \in \mathcal{T}$ , if  $H^p(X) = 0$ , for all  $p \in \mathbb{Z}$ , then  $X = 0$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $X \in \mathcal{T}$ , such that  $H^p(X) = 0$ , for all  $p \in \mathbb{Z}$ . Consider the truncation distinguished triangle

$$\tau_{\leq 0}(X) \longrightarrow X \longrightarrow \tau_{\geq 1}(X) \longrightarrow \Sigma(\tau_{\leq 0}(X))$$

From Lemma 2.1.33, we have that  $H^p(\tau_{\leq 0}(X)) = H^p(\tau_{\geq 1}(X)) = 0$ , for all  $p \in \mathbb{Z}$ . Therefore, from Lemma 2.1.35, we have that  $\tau_{\leq 0}(X) \in \mathcal{T}^{\leq p}$  and  $\tau_{\geq 1}(X) \in \mathcal{T}^{\geq p}$ , for all  $p \in \mathbb{Z}$ . By assumption, we have that  $\tau_{\leq 0}(X) = \tau_{\geq 1}(X) = 0$ . By rotating the triangle, we have that  $X$  is isomorphic to the cone of the morphism  $0 \rightarrow 0$ . From (TR1.b), we have that  $X = 0$ .

(2)  $\Rightarrow$  (1) : Let  $X \in \mathcal{T}^{\leq p}$ , for all  $p \in \mathbb{Z}$ . From Lemma 2.1.14, we have that  $\tau_{\geq p+1}(X) = 0$ , for all  $p \in \mathbb{Z}$ . Hence  $H^{p+1}(X) = \tau_{\leq p+1}(\tau_{\geq p+1}(X))[p+1] = \tau_{\geq p+1}(X) = 0$ , for all  $p \in \mathbb{Z}$ . Therefore, by assumption,  $X = 0$ , and thus,  $\bigcap_{n \in \mathbb{Z}} \text{Ob}(\mathcal{T})^{\leq n} = \{0\}$ . Let now  $X \in \mathcal{T}^{\geq p}$ , for all  $p \in \mathbb{Z}$ . Then, from Lemma 2.1.14, we have that  $\tau_{\leq p-1}(X) = 0$ , for all  $p \in \mathbb{Z}$ . Hence  $X = 0$  from the assumption. Thus  $\bigcap_{n \in \mathbb{Z}} \text{Ob}(\mathcal{T})^{\geq n} = \{0\}$ .  $\blacksquare$

**Definition 2.1.37.** A  $t$ -structure is called **nondegenerate**, if it satisfies one of the two equivalent conditions of Lemma 2.1.36.

The importance of nondegenerate  $t$ -structures is clear from the following Theorem.

**Theorem 2.1.38.** Let  $(\mathcal{T}^{\geq 0}, \mathcal{T}^{\leq 0})$  be a nondegenerate  $t$ -structure. Then:

- (1) a morphism  $f : X \rightarrow Y$  in  $\mathcal{T}$  is an isomorphism if and only if  $H^n(f) : H^n(X) \rightarrow H^n(Y)$  is an isomorphism in the heart  $\mathcal{A}$ ,
- (2) for every  $n \in \mathbb{Z}$ ,  $\mathcal{T}^{\leq n}$  is the full subcategory of  $\mathcal{T}$  consisting of objects  $X$  such that  $H^p(X) = 0$ , for all  $p > n$ ,
- (3) for every  $n \in \mathbb{Z}$ ,  $\mathcal{T}^{\geq n}$  is the full subcategory of  $\mathcal{T}$  consisting of objects  $X$  such that  $H^p(X) = 0$ , for all  $p < n$ .

*Proof.* (1)  $(\Rightarrow)$ : Clearly, if  $f$  is an isomorphism, then  $H^p(f)$  is also an isomorphism, for all  $p \in \mathbb{Z}$ .

$(\Leftarrow)$ : Suppose now that  $H^p(f) : H^p(X) \rightarrow H^p(Y)$  is an isomorphism, for all  $p \in \mathbb{Z}$ . Consider the distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma(X)$$

with base  $f$ . The long exact cohomology sequence of the above distinguished triangle is

$$\cdots \longrightarrow H^p(X) \xrightarrow{H^p(f)} H^p(Y) \longrightarrow H^p(Z) \longrightarrow H^{p+1}(X) \xrightarrow{H^{p+1}(f)} H^{p+1}(Y) \longrightarrow \cdots$$

Since  $H^p(f)$  is an isomorphism for all  $p \in \mathbb{Z}$ , then  $H^p(Z) = 0$  for all  $p \in \mathbb{Z}$ . Since the  $t$ -structure is nondegenerate, we have that  $Z = 0$ , hence  $f$  is an isomorphism, because the cone of  $f$  is zero.

(2)  $(\Rightarrow)$ : From Corollary 2.1.34, since  $X \in \mathcal{F}^{\leq n}$ , we have that  $H^p(X) = 0$ , for all  $p \in \mathbb{Z}$ .

$(\Leftarrow)$ : Suppose that  $H^p(X) = 0$ , for all  $p \in \mathbb{Z}$ . We will show that  $X \in \mathcal{F}^{\leq n}$ . From Corollary 2.1.34, the morphism  $H^p(X) \rightarrow H^p(\tau_{\geq n+1}(X))$  is an isomorphism, for all  $p > n$ . In particular  $H^p(\tau_{\geq n+1}(X)) = 0$ , for all  $p > n$ . On the other hand, from Lemma 2.1.33, we have that  $H^p(\tau_{\geq n+1}(X)) = 0$ , for all  $p < n$ . Therefore  $H^p(\tau_{\geq n+1}(X)) = 0$ , for all  $p \in \mathbb{Z}$ . From the first part of the proof we have that  $\tau_{n+1}(X) = 0$ . From the distinguished triangle

$$\tau_{\leq n}(X) \longrightarrow X \longrightarrow \tau_{\geq n+1}(X) \longrightarrow \Sigma(\tau_{\leq n}(X))$$

and from Lemma 1.4.22, the morphism  $\tau_{\leq n}(X) \rightarrow X$  is an isomorphism, hence  $X \in \mathcal{F}^{\leq n}$ .

(3) The proof is omitted since it is similar to that of 2. ■

**Lemma 2.1.39.** *Let  $(\mathcal{F}^{\leq 0}, \mathcal{F}^{\geq 0})$  be a  $t$ -structure in  $\mathcal{T}$ . Then the following are equivalent:*

$$(1) \bigcup_{n \in \mathbb{Z}} \mathcal{F}^{\leq n} = \mathcal{T} \text{ and } \bigcap_{n \in \mathbb{Z}} \mathcal{F}^{\geq n} = \mathcal{T},$$

(2) the  $t$ -structure is nondegenerate, and for every  $X \in \mathcal{T}$ ,  $H^p(X) = 0$ ,  $\forall p \in \mathbb{Z}$  except for finitely many  $p \in \mathbb{Z}$ .

*Proof.* (1)  $\Rightarrow$  (2): We will first show that the  $t$ -structure is nondegenerate. Let  $X \in \mathcal{T}$ , such that  $H^p(X) = 0$ , for all  $p \in \mathbb{Z}$ . From the assumption, there exist  $n, m \in \mathbb{Z}$ , such that  $X \in \mathcal{F}^{\leq n}$  and  $X \in \mathcal{F}^{\geq m}$ . From Lemma 2.1.35, we have that  $X \in \mathcal{F}^{\leq n} \cap \mathcal{F}^{\geq m}$ . In particular  $X \in \mathcal{F}^{\leq n-1}$  and  $X \in \mathcal{F}^{\geq 0}$ . Therefore  $\text{Hom}(X, X) = 0$ , thus  $X = 0$ . Consequently, from Lemma 2.1.36, the  $t$ -structure is nondegenerate. Let now  $X$  be an arbitrary object in  $\mathcal{T}$ . Then,  $X \in \mathcal{F}^{\leq n}$  and  $X \in \mathcal{F}^{\geq m}$ , for some  $n, m \in \mathbb{Z}$ . From Corollary 2.1.34,  $H^p(X) = 0$  for  $p > n$  and  $p < m$ . Hence  $H^p(X) \neq 0$ , for finitely many  $p \in \mathbb{Z}$ .

(2)  $\Rightarrow$  (1): Let  $X \in \mathcal{T}$ . From the assumption, there exists  $n \in \mathbb{Z}_+$ , such that  $H^p(X) = 0$ , for all  $|p| > n$ . From Lemma 2.1.33 we have that  $H^p(\tau_{\leq -n}(X)) = 0$ , and  $H^p(\tau_{\geq n}(X)) = 0$ , for all  $p \in \mathbb{Z}$ . Since the  $t$ -structure is nondegenerate, we have that  $\tau_{\leq -n}(X) = \tau_{\geq n}(X) = 0$ . From Lemma 2.1.14,  $X \in \mathcal{F}^{\geq -n+1}$  and  $X \in \mathcal{F}^{\leq n-1}$ . ■

**Definition 2.1.40.** A nondegenerate  $t$ -structure is called **bounded**, if it satisfies one of the two equivalent conditions of the Lemma 2.1.39.

Let  $\mathcal{T}$  be a triangulated category and  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  a nondegenerate  $t$ -structure in  $\mathcal{T}$ . Let also  $\mathcal{T}^b$  be the full subcategory of  $\mathcal{T}$  whose objects are those  $X$ , such that  $H^p(X) \neq 0$ , for finitely many  $p \in \mathbb{Z}$ . Clearly  $\mathcal{T}^b$  is a strictly full subcategory of  $\mathcal{T}$ . Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

be a distinguished triangle, and suppose that two of its vertices are in  $\mathcal{T}^b$ . Then, from the long exact cohomology sequence of the above distinguished triangle, we have that the third vertex is also in  $\mathcal{T}^b$ . Hence  $\mathcal{T}^b$  is a triangulated subcategory of  $\mathcal{T}$ . Let now  $X \in \mathcal{T}^b$ . Then, from Lemma 2.1.33, we have that  $\tau_{\leq n}(X)$  and  $\tau_{\geq n}(X)$  are in  $\mathcal{T}^b$ , for all  $n \in \mathbb{Z}$ . Therefore  $(\mathcal{T}^b \cap \mathcal{T}^{\geq 0}, \mathcal{T}^b \cap \mathcal{T}^{\leq 0})$  is a  $t$ -structure in  $\mathcal{T}^b$ . The truncation and cohomological functors are the restrictions of the ones in  $\mathcal{T}$ . Also, from the above, this  $t$ -structure is bounded. Hence we have the following definition.

**Definition 2.1.41.** The triangulated subcategory  $\mathcal{T}^b$  of  $\mathcal{T}$  is called the **subcategory of cohomologically bounded objects of  $\mathcal{T}$** .

### 2.1.4 Induced $t$ -structures

In this subsection we provide a criterion ensuring that a full triangulated subcategory of a triangulated category inherits a  $t$ -structure, relative to the  $t$ -structure of the triangulated category.

Let  $\mathcal{C}$  be a full triangulated subcategory of  $\mathcal{T}$  and  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  a  $t$ -structure in  $\mathcal{T}$ . Clearly, the inclusion functor  $\mathcal{C} \hookrightarrow \mathcal{T}$  is exact. We define

$$\mathcal{C}^{\leq 0} := \mathcal{C} \cap \mathcal{T}^{\leq 0} \quad \text{and} \quad \mathcal{C}^{\geq 0} := \mathcal{C} \cap \mathcal{T}^{\geq 0}$$

**Lemma 2.1.42.** Let  $\mathcal{C}$  be a full triangulated subcategory of  $\mathcal{T}$ . Then, the following are equivalent

- (1)  $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$  is a  $t$ -structure in  $\mathcal{C}$ ,
- (2) For the truncation functor  $\tau_{\leq 0}$  in  $\mathcal{T}$ , we have  $\tau_{\leq 0}(\mathcal{C}) \subseteq \mathcal{C}$ ,
- (3) For the truncation functor  $\tau_{\geq 0}$  in  $\mathcal{T}$ , we have  $\tau_{\geq 0}(\mathcal{C}) \subseteq \mathcal{C}$ .

*Proof.* (2)  $\Rightarrow$  (1): We will prove that  $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ .

(t1) Since  $\mathcal{C}$  is invariant under the shift functor, we have that

$$\mathcal{C}^{\geq 1} = \Sigma^{-1}(\mathcal{C}^{\geq 0}) = \mathcal{C} \cap \Sigma^{-1}(\mathcal{T}^{\geq 0}) = \mathcal{C} \cap \mathcal{T}^{\geq 1} \subseteq \mathcal{C} \cap \mathcal{T}^{\geq 0} \subseteq \mathcal{C}^{\geq 0}$$

and

$$\mathcal{C}^{\leq 1} = \Sigma^{-1}(\mathcal{C}^{\leq 0}) = \mathcal{C} \cap \Sigma^{-1}(\mathcal{T}^{\leq 0}) = \mathcal{C} \cap \mathcal{T}^{\leq 1} \supseteq \mathcal{C} \cap \mathcal{T}^{\leq 0} \supseteq \mathcal{C}^{\leq 0}$$

(t2) Immediate.

(t3) We need to prove that for every  $X \in \mathcal{C}$ , there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow \Sigma(A)$$

where  $A \in \mathcal{C}^{\leq 0}$  and  $B \in \mathcal{C}^{\geq 1}$ . If  $\tau_{\leq 0}(X) \in \mathcal{C}$ , we define  $A := \tau_{\leq 0}(X)$ . Since  $\mathcal{C}$  is a triangulated subcategory of  $\mathcal{T}$ , there exists  $B \in \mathcal{C}$  such that

$$\tau_{\leq 0}(X) \longrightarrow X \longrightarrow B \longrightarrow \Sigma(\tau_{\leq 0}(X))$$

is a distinguished triangle. Clearly  $\tau_{\leq 0}(X) \in \mathcal{C} \cap \mathcal{T}^{\leq 0} = \mathcal{C}^{\leq 0}$ , and  $B \cong \tau_{\geq 1}(X) \in \mathcal{T}^{\geq 1}$ . Therefore,  $B \in \mathcal{C} \cap \mathcal{T}^{\geq 1} = \mathcal{C}^{\geq 1}$ . If now  $\tau_{\geq 0}(X) \in \mathcal{C}$ , then  $\Sigma^{-1}(\tau_{\geq 0}(X)) \in \mathcal{C}$ . We define  $B = \Sigma^{-1}(\tau_{\geq 0}(\Sigma(X)))$ . We can easily see, from Lemma 2.1.14, that  $\Sigma^{-1}(\tau_{\geq 0}(\Sigma(X)))$  is isomorphic to  $\tau_{\geq 1}(X)$  in  $\mathcal{T}$ . Hence,

$$\tau_{\leq 0}(X) \longrightarrow X \longrightarrow B \longrightarrow \Sigma(\tau_{\leq 0}(X))$$

is a distinguished triangle in  $\mathcal{T}$ . Because  $\mathcal{C}$  is a triangulated subcategory of  $\mathcal{T}$ , there exists  $A \in \mathcal{C}$  such that

$$A \longrightarrow X \longrightarrow B \longrightarrow \Sigma(A)$$

is a distinguished triangle in  $\mathcal{C}$ . From the later, we get that  $A \cong \tau_{\leq 0}(X) \in \mathcal{T}^{\leq 0}$ , hence  $A \in \mathcal{C} \cap \mathcal{T}^{\leq 0} = \mathcal{C}^{\leq 0}$ . On the other hand  $B \cong \tau_{\geq 1}(X) \in \mathcal{T}^{\geq 1}$ , so  $B \in \mathcal{C} \cap \mathcal{T}^{\geq 1} = \mathcal{C}^{\geq 1}$ .

(3)  $\Rightarrow$  (1): Similar to the proof of (2)  $\Rightarrow$  (1).

(1)  $\Rightarrow$  (2), (3): Now, if  $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$  is a  $t$ -structure in  $\mathcal{C}$ , then from the construction of the truncation functor in  $\mathcal{T}$ , we can construct them so that they leave  $\mathcal{C}$  invariant.  $\blacksquare$

**Definition 2.1.43.** If the pair  $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$  as defined at the beginning of this section is a  $t$ -structure, then it is called the **induced  $t$ -structure** in  $\mathcal{C}$ .

**Remark 2.1.44.** If the  $t$ -structure in  $\mathcal{C}$  is induced, then its heart  $\mathcal{B}$  is a full triangulated subcategory of  $\mathcal{A}$ , which is the heart of the  $t$ -structure in  $\mathcal{T}$ . In addition, the truncation functors and cohomological functors are isomorphic with the restrictions of them in  $\mathcal{T}$ .

### 2.1.5 Extensions in the heart

Extensions in the heart of a  $t$ -structure are fundamental because the heart  $\mathcal{A}$  is an abelian category, and the notion of extensions allows us to study exact sequences, cohomology, etc. In this direction, let  $\mathcal{T}$  be a triangulated category and  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  a  $t$ -structure in  $\mathcal{T}$  with heart  $\mathcal{A}$ . We have shown that  $\mathcal{A}$  is an abelian category. By definition  $\text{Hom}_{\mathcal{A}}(X, Y) = \text{Hom}_{\mathcal{T}}(X, Y)$ , for every  $X, Y \in \mathcal{A}$ .

**Definition 2.1.45.** *Let*

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

be a short exact sequence in  $\mathcal{C}$ . The object  $X$  is called the **extension** of  $Z$  by  $Y$ .

**Definition 2.1.46.** *Two extensions  $Z$  and  $Z'$  of  $X$  by  $Y$  are called **equivalent**, if there exists a morphism  $\alpha : Z \rightarrow Z'$  such that the following diagram is commutative*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \parallel & & \downarrow \alpha & & \parallel & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

**Remark 2.1.47.** From Lemma 1.2.29 we have that  $\alpha$  is an isomorphism. Therefore the equivalence of extensions is an equivalence relation in the set of all extensions.

**Definition 2.1.48.** *We denote with  $\text{Ext}_{\mathcal{A}}(X, Y)$  the **set of all extensions** of  $X$  by  $Y$ .*

**Remark 2.1.49.** We will show that for any two objects  $X, Y \in \mathcal{T}$  there exists a bijection  $\text{Ext}(X, Y) \rightarrow \text{Hom}(X, Y)$ . In this direction, let

$$0 \longrightarrow Y \xrightarrow{i} Z \xrightarrow{p} X \longrightarrow 0$$

be a short exact sequence in  $\mathcal{A}$ . Then, from Lemma 2.1.32, we have the distinguished triangle

$$Y \xrightarrow{i} Z \xrightarrow{p} X \xrightarrow{\phi} \Sigma(Y)$$

in  $\mathcal{T}$ . Clearly  $Y \in \mathcal{T}^{\leq 0}$  and  $X \in \mathcal{T}^{\geq 0}$ . Hence,  $\Sigma^{-1}(X) \in \mathcal{T}^{\geq 1}$ . This means that  $\text{Hom}(Y, \Sigma^{-1}(X)) = 0$  and the morphism  $\phi$  is unique. Let now  $Z'$  be an equivalent extension of  $X$  by  $Y$ , and  $\alpha : Z \rightarrow Z'$  be the respective morphism. Then we have the distinguished triangle

$$Y \xrightarrow{i'} Z' \xrightarrow{p'} X \xrightarrow{\phi'} \Sigma(Y)$$

and the diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{i} & Z & \xrightarrow{p} & X & \xrightarrow{\phi} & \Sigma(Y) \\ \parallel & & \alpha \downarrow \cong & & & & \parallel \\ Y & \xrightarrow{i'} & Z' & \xrightarrow{p'} & X & \xrightarrow{\phi'} & \Sigma(Y) \end{array}$$

where the first square is commutative. From (TR3), the above square can be completed to following morphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{i} & Z & \xrightarrow{p} & X & \xrightarrow{\phi} & \Sigma(Y) \\ \parallel & & \alpha \downarrow \cong & & \downarrow \beta & & \parallel \\ Y & \xrightarrow{i'} & Z' & \xrightarrow{p'} & X & \xrightarrow{\phi'} & \Sigma(Y) \end{array}$$

Thus, we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \xrightarrow{i} & Z & \xrightarrow{p} & X \xrightarrow{\beta} 0 \\ & & \parallel & & \alpha \downarrow \cong & & \\ 0 & \longrightarrow & Y & \xrightarrow{i'} & Z' & \xrightarrow{p'} & X \longrightarrow 0 \end{array}$$

in  $\mathcal{A}$ , and we get  $\beta \circ p = p' \circ \alpha = p$ . Since  $p$  is an epimorphism we have that  $\beta = 1_X$ , which implies that  $\phi = \phi'$ . Therefore  $\phi$  depends only from the equivalence class of the extension of  $X$  by  $Y$ . Therefore we can define a map  $\text{Ext}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{T}}(X, \Sigma(Y))$ . We claim that this map is one-to-one and onto. Indeed, for the one-to-one part, let

$$0 \longrightarrow Y \xrightarrow{i} Z \xrightarrow{p} X \longrightarrow 0$$

and

$$0 \longrightarrow Y \xrightarrow{i'} Z' \xrightarrow{p'} X \longrightarrow 0$$

be two extensions which induce the same morphism  $\phi : X \rightarrow \Sigma(Y)$ . Then, the respective distinguished triangles are

$$Y \xrightarrow{i} Z \xrightarrow{p} X \xrightarrow{\phi} \Sigma(Y)$$

and

$$Y \xrightarrow{i'} Z' \xrightarrow{p'} X \xrightarrow{\phi} \Sigma(Y)$$

and we have the diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{i} & Z & \xrightarrow{p} & X & \xrightarrow{\phi} & \Sigma(Y) \\ \parallel & & & & \parallel & & \parallel \\ Y & \xrightarrow{i'} & Z' & \xrightarrow{p'} & X & \xrightarrow{\phi} & \Sigma(Y) \end{array}$$

where the last square is commutative. The above diagram can be completed to the morphism of triangles

$$\begin{array}{ccccccc} Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & \Sigma(Y) \\ \parallel & & \downarrow \alpha & & \parallel & & \parallel \\ Y & \longrightarrow & Z' & \longrightarrow & X & \longrightarrow & \Sigma(Y) \end{array}$$

Therefore, for the morphism  $\alpha$ , we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X \longrightarrow 0 \\ & & \parallel & & \downarrow \alpha & & \parallel \\ 0 & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & X \longrightarrow 0 \end{array}$$

which means that the extensions are equivalent. Now, for the onto part, let  $\phi : X \rightarrow \Sigma(Y)$  be a morphism in  $\mathcal{T}$ . The latter morphism can be embedded to the distinguished triangle

$$\Sigma^{-1}(X) \xrightarrow{-\Sigma^{-1}(\phi)} Y \longrightarrow U \longrightarrow X$$

By rotating the triangle, we get the distinguished triangle

$$Y \xrightarrow{f} U \xrightarrow{g} X \xrightarrow{\phi} \Sigma(Y)$$

for some  $U \in \mathcal{T}$ . From Lemma 2.1.15 we have that  $U \in \mathcal{A}$ , and from Lemma 2.1.28 the morphism  $f$  is a monomorphism and  $g$  is the cokernel of  $f$ . Therefore the sequence

$$0 \longrightarrow Y \xrightarrow{f} U \xrightarrow{g} X \longrightarrow 0$$

is exact and  $U$  is an extension of  $Y$  from  $X$ . Immediately, the latter extension determines the morphism  $\phi : X \rightarrow \Sigma(X)$ .

**Remark 2.1.50.** Let  $X, Y \in \mathcal{C}$ ,  $i_Y : Y \rightarrow X \oplus Y$  the canonical injection and  $p_X : X \oplus Y \rightarrow X$  the canonical projection. Then we have the short exact sequence

$$0 \longrightarrow Y \xrightarrow{i_Y} X \oplus Y \xrightarrow{p_X} X \longrightarrow 0$$

**Definition 2.1.51.** The above short exact sequence is called the **split extension** of  $X$  by  $Y$ .

**Remark 2.1.52.** The equivalence class of the above extension corresponds to the zero morphism from  $X$  to  $Y[1]$ .

### 2.1.6 Cohomological length

The cohomological length of an object measures the number of nonzero cohomology objects in the heart. It provides a way to measure the complexity of an object with respect to a  $t$ -structure. Via cohomological length we can relate properties of objects in the triangulated category to exact sequences in the heart.

**Definition 2.1.53.** Let  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  be a bounded  $t$ -structure in  $\mathcal{T}$ . For every object  $X \in \mathcal{T}$ , we define:

$$\ell(X) = \text{Card}\{p \in \mathbb{Z} \mid H^p(X) \neq 0\}$$

and we call in the **cohomological length** of  $X$ .

**Remark 2.1.54.** Since we have a bounded  $t$ -structure, therefore a nondegenerate one, if  $\ell(X) = 0$  then  $X = 0$  from Lemma 2.1.36.

**Lemma 2.1.55.** Let  $X \in \mathcal{T}$  with  $\ell(X) > 0$ . Then there exists  $p \in \mathbb{Z}$ , such that for the distinguished triangle

$$\tau_{\leq p}(X) \longrightarrow X \longrightarrow \tau_{\geq p+1}(X) \longrightarrow \Sigma(\tau_{\leq p}(X))$$

we have that  $\ell(\tau_{\leq p}(X)) = \ell(X) - 1$ .

*Proof.* Because  $\ell(X) > 0$ , there exists  $n \in \mathbb{Z}$  such that  $H^{n+1}(X) \neq 0$  and  $H^p(X) = 0$ , for all  $p > n + 1$ . Therefore, from Lemma 2.1.33 we have that  $H^p(q) : H^p(X) \rightarrow H^p(\tau_{\geq n+1}(X))$  is an isomorphism for all  $p > n + 1$ , and  $H^p(\tau_{\geq n+1}(X)) = 0$ , for all  $p \leq n$ . In particular, we have that

$$H^p(\tau_{\geq n+1}(X)) = \begin{cases} 0 & p \neq n+1 \\ H^{n+1}(X) & p = n+1 \end{cases}$$

and  $\ell(\tau_{\geq n+1}(X)) = 1$ . By letting  $Y = \tau_{\geq n+1}(X)[n+1]$ , we get that  $H^p(Y) = H^{p+n+1}(\tau_{\geq n+1}(X)) = 0$ , for  $p \neq 0$  and  $Y \in \mathcal{A}$ . On the other hand, from Lemma 2.1.33, the morphism  $H^p : H^p(\tau_{\leq n}(X)) \rightarrow H^p(X)$  is an isomorphism for  $p \leq n$ , and  $H^p(\tau_{\leq n}(X)) = 0$  for  $p > n$ . Consequently,  $H^p(\tau_{\leq n}(X)) \neq 0$ , which implies that  $p \leq n$  and  $H^p(X) \neq 0$ . Hence,  $\ell(\tau_{\leq n}(X)) = \ell(X) - 1$ . ■

In the following, we present a splitting result.

Let  $X, Y \in \mathcal{C}$ , where  $\mathcal{C}$  is the heart of the  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ . Then  $X \in \mathcal{T}^{\leq 0}$  and  $Y[-n] \in \mathcal{T}^{\geq n}$ , for every  $n \in \mathbb{N}$ . Therefore, from (t2) we have that  $\text{Hom}(X, \Sigma^{-1}(Y)) = 0$ , for every  $n \in \mathbb{N}$ . Clearly  $\text{Hom}_{\mathcal{T}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$ , and from Remark 2.1.49 we have that  $\text{Hom}_{\mathcal{T}}(X, \Sigma(Y)) = \text{Ext}_{\mathcal{C}}(X, Y)$ . In this section, we will study  $\mathcal{T}$  with the following extra assumptions:

- (1) the  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is bounded,
- (2) for every  $X, Y \in \mathcal{C}$ ,  $\text{Hom}_{\mathcal{T}}(X, Y[n]) = 0$ , for every  $n > 1$ .

**Lemma 2.1.56.** *Let  $X \in \mathcal{T}$ . Then*

$$X \cong \bigoplus_{p \in \mathbb{Z}} H^p(X)[-p]$$

*Proof.* We will use induction on the length  $\ell(X)$  of  $X$ .

$n = 1$ : Immediate.

Induction hypothesis: Suppose that the statement is true for  $n$ , that is, suppose that  $\ell(X) = n + 1$ .

Induction step From Lemma 2.1.55, there exists  $p$  such that  $\ell(\tau_{\leq p}(X)) = n$  and  $\ell(\tau_{\geq p+1}(X)) = 1$ . Therefore, there exists an object  $Y \in \mathcal{A}$ , such that  $\tau_{\geq p+1}(X) \cong Y[-p-1]$ , hence, the morphism  $h : \tau_{\geq p+1} \rightarrow \tau_{\leq p}(X)[1]$  induces the morphism  $h[p+1] : Y \rightarrow \tau_{\leq p}(X)[p+1]$ . On the other hand, since  $H^q(\tau_{\leq p}(X))[p+2] = H^{q+p+2}(\tau_{\leq p}(X)) \neq 0$ , we have that  $p+q+2 \leq p \Rightarrow q \leq -2$ . From the induction hypothesis, we have that  $\tau_{\leq p}(X)[p+2] = \bigoplus_{q \leq -2} H^{p+q+2}(X)[-q] = \bigoplus_{q \leq -2} H^{-q+p+2}(X)[-q]$ .

Hence,  $\text{Hom}_{\mathcal{T}}(Y, \tau_{\leq p}(X)[p+2]) = \bigoplus_{q \geq 2} \text{Hom}_{\mathcal{T}}(Y, H^{-p+p+2}(X)[q]) = 0$ . From latter we have that

$h = 0$ , and from Lemma 2.1.33, we get  $X \cong \tau_{\leq p}(X) \oplus \tau_{\geq 1}(X) \cong \bigoplus_{q \in \mathbb{Z}} H^q(X)[-q]$ . ■

**Corollary 2.1.57.** *If  $\mathcal{A}$  is a hereditary abelian category (i.e.  $\text{gldim}(\mathcal{A}) \leq 1$ ), then  $\forall X \in D^b(\mathcal{A}) : X \cong \bigoplus_{p \in \mathbb{Z}} H^p(X)[-p]$ .*

### 2.1.7 Grothendieck groups

We turn now interest into Grothendieck groups. The stability functions that Bridgeland defines in his paper, have domain a Grothendieck group. We start by defining the Grothendieck group of an abelian category.

**Definition 2.1.58.** Let  $\mathcal{A}$  be an abelian category and  $\mathcal{G}(\mathcal{A})$  be the free abelian group generated by isomorphism classes of objects of  $\mathcal{A}$ . Let  $\mathcal{N}$  be the subgroup of  $\mathcal{G}(\mathcal{A})$ , which is generated by elements  $[X] - [Y] - [Z]$  for every short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in  $\mathcal{A}$ . The quotient  $\mathcal{G}(\mathcal{A})/\mathcal{N}$  is called the **Grothendieck group** of the abelian category  $\mathcal{A}$ , and is denoted by  $K_0(\mathcal{A})$ .

For every object  $X \in \mathcal{A}$ , we denote its respective class by  $[X]$  in  $K_0(\mathcal{A})$ .

**Remark 2.1.59.** Let us see two immediate observations from the definition. For a start, from the short exact sequence

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

we get that  $[0] - [0] - [0] = [0] \in \mathcal{N}$ , which means that  $[0] = 0$  in  $K_0(\mathcal{A})$ . Now, if  $X, Y$  are objects of  $\mathcal{A}$  such that  $X \cong Y$ , we have the short exact sequence

$$0 \longrightarrow X \xrightarrow{\cong} Y \longrightarrow 0 \longrightarrow 0$$

so  $[Y] = [X] + [0] \Rightarrow [X] = [Y]$  in  $K_0(\mathcal{A})$ .

**Remark 2.1.60.** Let  $G$  be an abelian group and  $\phi : \text{Iso}(\mathcal{A}) \rightarrow G$  a map, where  $\text{Iso}(\mathcal{A})$  is the collection of all isomorphism classes of objects of  $\mathcal{A}$ , such that for every short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

we have that  $\phi(Y) = \phi(X) + \phi(Z)$ . From the universal property of the free abelian groups, there exists a unique group homomorphism  $\phi' : G \rightarrow \mathcal{G}(\mathcal{A})$  such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\phi} & G \\ i \downarrow & \nearrow \exists! \phi' & \\ \mathcal{G}(\mathcal{A}) & & \end{array}$$

From the above,  $\phi$  is zero for every element of the subgroup  $\mathcal{N}$ . Therefore, we get a unique group homomorphism from  $K_0(\mathcal{A})$  to  $G$ , which we denote again with the letter  $\phi$ .

**Lemma 2.1.61.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two abelian categories, and  $F : \mathcal{A} \rightarrow \mathcal{B}$  an exact functor. Then, the functor  $F$  induces a group homomorphism  $K_0(F) : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$ , which is defined as*

$$K_0(F)([X]) = [F(X)], \text{ for every } X \in \mathcal{A}.$$

*Proof.* We have the natural map  $\phi : \mathcal{A} \rightarrow K_0(\mathcal{B})$ , which is defined by  $\phi(X) = [F(X)]$ , so, it is left to show that the induced map on the Grothendieck groups is nullified for every element of  $\mathcal{N}$ . From the above remark,  $\phi$  can be extended to a unique group homomorphism from  $\mathcal{G}(\mathcal{A})$  to  $K_0(\mathcal{B})$ . Since for every short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in  $\mathcal{A}$ , by applying the functor  $F$  which is exact, we get the short exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0$$

in  $\mathcal{B}$ , and we have that  $[F(Y)] = [F(X)] + [F(Z)]$ , hence,  $\phi(Y) = \phi(X) + \phi(Z)$ . From the latter we have that  $\phi$  is zero for every element of  $\mathcal{N}$ . ■

The following is an important Lemma for the rest of this section, which is called Euler's principle.

**Lemma 2.1.62** (Euler's Principle). *Let  $\mathcal{A}$  be an abelian category and*

$$\dots \longrightarrow 0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \dots \longrightarrow X^n \longrightarrow 0 \longrightarrow \dots$$

*a bounded cochain complex in  $\mathcal{A}$ . Then*

$$\sum_{p \in \mathbb{Z}} (-1)^p [X^p] = \sum_{p \in \mathbb{Z}} (-1)^p [H^p(X^\bullet)]$$

*in  $K_0(\mathcal{A})$ .*

*Proof.* We will use induction on the length of the complex. Obviously, for  $n = 0$ ,  $[X] = [H^0(X^\bullet)]$ . For the induction step, consider the complex

$$\dots \longrightarrow 0 \longrightarrow X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \dots \xrightarrow{d_X^{n-1}} X^n \longrightarrow 0 \longrightarrow \dots$$

The stupid truncation  $\sigma_{\leq n-1}(X^\bullet)$  is the complex

$$\dots \longrightarrow 0 \longrightarrow X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \dots \xrightarrow{d_X^{n-2}} X^{n-1} \longrightarrow 0 \longrightarrow \dots$$

We have that  $H^p(\sigma_{\leq n-1}(X^\bullet)) = H^p(X^\bullet)$ , for  $p \neq n-1, n$ , and  $H^{n-1}(\sigma_{\leq n-1}(X^\bullet)) = \text{Coker}(d_X^{n-2})$ , plus  $H^n(\sigma_{\leq n-1}(X^\bullet)) = 0$ . From the induction hypothesis, we have that

$$\begin{aligned} \sum_{p=0}^{n-1} (-1)^p [X^p] &= \sum_{p \in \mathbb{Z}} (-1)^p [\sigma_{\leq n-1}(X^\bullet)^p] \\ &= \sum_{p \in \mathbb{Z}} (-1)^p [H^p(\sigma_{\leq n-1}(X^\bullet))] \\ &= \sum_{p=0}^{n-1} (-1)^p [H^p(X^\bullet)] + (-1)^{n-1} [\text{Coker}(d_X^{n-2})] \end{aligned}$$

Now, from the short exact sequences

$$\begin{aligned} 0 &\longrightarrow \text{Im}(d_X^{n-2}) \longrightarrow X^{n-1} \longrightarrow \text{Coker}(d_X^{n-2}) \longrightarrow 0 \\ 0 &\longrightarrow \text{Im}(d_X^{n-2}) \longrightarrow \text{Ker}(d_X^{n-1}) \longrightarrow H^{n-1} \longrightarrow 0 \\ 0 &\longrightarrow \text{Ker}(d_X^{n-1}) \longrightarrow X^{n-1} \longrightarrow \text{Im}(d_X^{n-1}) \longrightarrow 0 \\ 0 &\longrightarrow \text{Im}(d_X^{n-1}) \longrightarrow X^n \longrightarrow H^n(X^\bullet) \end{aligned}$$

we get

$$\begin{aligned} [X^{n-1}] &= [\text{Im}(d_X^{n-2})] + [\text{Coker}(d_X^{n-2})] \\ [\text{Ker}(d_X^{n-1})] &= [\text{Im}(d_X^{n-1})] + [H^{n-1}(X^\bullet)] \\ [X^{n-1}] &= [\text{Ker}(d_X^{n-1})] + [\text{Im}(d_X^{n-1})] \\ [X^n] &= [\text{Im}(d_X^{n-1})] + [H^n(X)] \end{aligned}$$

From the first equation we have:

$$\begin{aligned} [X^{n-1}] = [\text{Im}(d_X^{n-2})] + [\text{Coker}(d_X^{n-2})] &\Rightarrow [\text{Coker}(d_X^{n-2})] = [X^{n-1}] - [\text{Im}(d_X^{n-2})] \\ &= [X^{n-1}] - \{[\text{Ker}(d_X^{n-1})] - [H^{n-1}(X^\bullet)]\} \\ &= [X^{n-1}] - [\text{Ker}(d_X^{n-1})] + [H^{n-1}(X^\bullet)] \\ &= [\text{Im}(d_X^{n-1})] + [H^{n-1}(X^\bullet)] \\ &= [X^n] + [H^{n+1}(X^\bullet)] - [H^n(X^\bullet)] \end{aligned}$$

By combining the above we have:

$$\begin{aligned}
\sum_{p \in \mathbb{Z}} (-1)^p [X^p] &= \sum_{p=0}^{n-1} (-1)^p [X^p] + (-1)^n [X^n] \\
&= \sum_{p=0}^{n-2} (-1)^p [H^p(X^\bullet)] + (-1)^{n-1} [\text{Coker}(d_X^{n-2})] + (-1)^n [X^n] \\
&= \sum_{p=0}^{n-2} (-1)^p [H^p(X^\bullet)] + (-1)^{n-1} [X^n] + (-1)^{n-1} [H^{n-1}(X^\bullet)] - (-1)^{n-1} [H^n(X^\bullet)] + (-1)^n [X^n] \\
&= \sum_{p=0}^{n-1} (-1)^p [H^p(X^\bullet)] + (-1)^{n-1} [H^{n-1}(X^\bullet)] + (-1)^n [H^n(X^\bullet)] + (-1)^{n-1} [X^n] + (-1)^n [X^n] \\
&= \sum_{p=0}^n (-1)^p [H^p(X^\bullet)] + (-1)^{n-1} [X^n] - (-1)^n [X^n] = \sum_{p \in \mathbb{Z}} (-1)^p [H^p(X^\bullet)] \quad \blacksquare
\end{aligned}$$

**Corollary 2.1.63.** *Let  $\mathcal{A}$  be an abelian category and*

$$0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0$$

*an exact sequence in  $\mathcal{A}$ . Then*

$$\sum_{p \in \mathbb{Z}} (-1)^p [X^p] = 0$$

*in  $K_0(\mathcal{A})$ .*

Next, since we have defined the Grothendieck group of an abelian category, we will give the definition of the Grothendieck group of a triangulated category and present some basic results.

**Definition 2.1.64.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{G}(\mathcal{T})$  the free abelian group generated by isomorphism classes of objects of  $\mathcal{T}$ . Let  $\mathcal{M}$  be the subgroup of  $\mathcal{G}(\mathcal{T})$ , which is generated by elements  $[X] - [Y] - [Z]$  for every distinguished triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

*in  $\mathcal{T}$ . The quotient  $\mathcal{G}(\mathcal{T})/\mathcal{M}$  is called the **Grothendieck group** of the triangulated category  $\mathcal{T}$ , and it is denoted by  $K_0(\mathcal{T})$ . For every object  $X \in \mathcal{T}$ , we denote its respective class by  $[X]$  in  $K_0(\mathcal{T})$ .*

**Remark 2.1.65.** Let  $f : X \rightarrow Y$  be an isomorphism. Then, since its cone is 0, we have the distinguished triangle

$$X \longrightarrow Y \longrightarrow 0 \longrightarrow \Sigma(X)$$

Therefore the element  $[Y] - [X] - [0]$  is in  $\mathcal{M}$ , hence  $[X] = [Y]$  in  $K(\mathcal{T})$ . Analogously, if  $X, Y \in \mathcal{T}$ , we have the distinguished triangle

$$X \longrightarrow X \oplus Y \longrightarrow Y \longrightarrow \Sigma(X)$$

in  $\mathcal{T}$ , hence the triplet  $X \oplus [Y] - [X] - [Y]$  is in  $\mathcal{M}$ , so  $[X \oplus Y] = [X] + [Y]$  in  $K(\mathcal{T})$ . Also  $[0] = 0$  in  $K_0(\mathcal{A})$ .

**Lemma 2.1.66.** Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be two triangulated categories, and  $F : \mathcal{T}_1 \rightarrow \mathcal{T}_2$  an exact functor. Then, the functor  $F$  induces a group homomorphism  $K_0(F) : K_0(\mathcal{T}_1) \rightarrow K_0(\mathcal{T}_2)$  which is defined as

$$K_0(F)([X]) = [F(X)], \text{ for every } X \in \mathcal{T}_1$$

*Proof.* The proof is similar with the one on abelian categories so we omit it. ■

We can easily check that the suspension functor automatically induces actions on the Grothendieck groups.

**Lemma 2.1.67.** Let  $\mathcal{T}$  be a triangulated category and  $X \in \mathcal{T}$ . Then

$$[\Sigma^p(X)] = (-1)^p[X], p \in \mathbb{Z}$$

We turn our interest to the Grothendieck group of triangulated categories which are equipped with a bounded  $t$ -structure. So, in this direction, let  $\mathcal{T}$  be a triangulated category and  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  a bounded  $t$ -structure in  $\mathcal{T}$  with heart  $\mathcal{A}$ . Let also  $\iota : \mathcal{A} \hookrightarrow \mathcal{T}$  be the natural inclusion functor. Working similarly, the natural inclusion functor induces a group homomorphism  $\phi : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{T})$ , which is defined as  $\phi([X]) = [\iota(X)]$ ,  $X \in \mathcal{A}$ .

**Lemma 2.1.68.** Let  $X$  be a cohomologically bounded object of  $\mathcal{T}$ . Then

$$[X] = \sum_{p \in \mathbb{Z}} (-1)^p [\iota(H^p(X))]$$

*Proof.* We will use induction on the cohomological length of  $X$ . For a start, if  $\ell(X) = 1$ , then  $X \cong \iota(H^p(X))[-p]$ , for some  $p \in \mathbb{Z}$ . Then we have that  $[X] = [\iota(H^p(X))[-p]] = (-1)^{-p} [\iota(H^p(X))] = \sum_{p \in \mathbb{Z}} (-1)^p [\iota(H^p(X))]$ . Let now  $X$  be an arbitrary object of  $\mathcal{T}$ . Then we have the truncation distinguished triangle

$$\tau_{\leq q}(X) \longrightarrow X \longrightarrow \tau_{\geq p+1}(X) \longrightarrow \Sigma(\tau_{\leq q}(X))$$

Consequently,  $[X] = [\tau_{\leq p}(X)] + [\tau_{\geq p+1}(X)]$ , for every  $p \in \mathbb{Z}$ . Without loss of generality, we can choose  $p$  such that  $\ell(\tau_{\leq p}(X)), \ell(\tau_{\geq p+1}(X)) < \ell(X)$ . From the induction hypothesis, the statement is true for  $\tau_{\leq p}(X)$  and  $\tau_{\geq p+1}(X)$ . Hence, we have that

$$[X] = \sum_{q \in \mathbb{Z}} (-1)^q [\iota(H^q(\tau_{\leq p}(X)))] + \sum_{q \in \mathbb{Z}} (-1)^q [\iota(H^q(\tau_{\geq p+1}(X)))]$$

In addition, we have

$$\sum_{q \in \mathbb{Z}} (-1)^q [\iota(H^q(\tau_{\leq p}(X)))] = \sum_{q \leq p} (-1)^q [\iota(H^q(X))] \text{ and } \sum_{q \in \mathbb{Z}} (-1)^q [\iota(H^q(\tau_{\geq p+1}(X)))] = \sum_{q \geq p} (-1)^q [\iota(H^q(X))]$$

By combining the above we get that  $[X] = \sum_{q \in \mathbb{Z}} (-1)^q [\iota(H^q(X))]$  in  $K_0(\mathcal{T})$ .  $\blacksquare$

We will now construct a group homomorphism from  $K_0(\mathcal{T})$  to  $K_0(\mathcal{A})$ , provided that  $\mathcal{T}$  is equipped with a bounded  $t$ -structure. Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$$

be a distinguished triangle in  $\mathcal{T}$ . Then we have long exact cohomology sequence of groups

$$\cdots \longrightarrow H^{p-1}(X) \longrightarrow H^p(X) \longrightarrow H^p(Y) \longrightarrow H^p(Z) \longrightarrow H^{p+1}(X) \longrightarrow \cdots$$

which can be considered as an exact sequence in the heart of the  $t$ -structure  $\mathcal{A}$ . Since the  $t$ -structure is bounded, only finite amount of cohomology groups are non zero. From Lemma 2.1.33, we get that  $\sum_{p \in \mathbb{Z}} (-1)^p [H^p(Y)] = \sum_{p \in \mathbb{Z}} (-1)^p [H^p(X)] + \sum_{p \in \mathbb{Z}} (-1)^p [H^p(Z)]$  in  $K(\mathcal{A})$ . Therefore the map  $X \mapsto \sum_{p \in \mathbb{Z}} (-1)^p [H^p(X)]$  from  $\mathcal{T}$  to  $K_0(\mathcal{A})$ , induces a group homomorphism  $\psi : K_0(\mathcal{T}) \rightarrow K_0(\mathcal{A})$ .

**Theorem 2.1.69.** *The natural group homomorphism  $\phi : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{T})$ , with  $[X] \mapsto [\iota(X)]$ , is a group isomorphism. Its inverse is the group homomorphism  $\psi : K_0(\mathcal{T}) \rightarrow K_0(\mathcal{A})$  defined as  $[X] \mapsto \sum_{p \in \mathbb{Z}} (-1)^p [H^p(X)]$ .*

*Proof.* By the definition of  $\phi$  and  $\psi$ , we immediately get that  $\psi \circ \phi = 1_{K_0(\mathcal{A})}$ . On the other hand,  $(\phi \circ \psi)([X]) = \phi(\psi([X])) = \phi(\sum_{p \in \mathbb{Z}} (-1)^p [H^p(X)]) = \sum_{p \in \mathbb{Z}} (-1)^p \phi([H^p(X)]) = \sum_{p \in \mathbb{Z}} (-1)^p [\iota(H^p(X))] = [X]$ .  $\blacksquare$

**Remark 2.1.70.** From the above Theorem, we can deduce that we can identify the Grothendieck groups of triangulated categories with the Grothendieck groups of the hearts of bounded  $t$ -structures. As an application of the above Theorem, we have the following Corollary.

**Corollary 2.1.71.** *Let  $\mathcal{A}$  be an abelian category. Then*

$$K_0(\mathcal{A}) \cong K_0(D^b(\mathcal{A}))$$

*Proof.* From Example 2.1.5, consider the standard  $t$ -structure  $\mathcal{T}^{\leq 0} = \{X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0, \text{ for } i > 0\}$  and  $\mathcal{T}^{\geq 0} = \{X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0, \text{ for } i < 0\}$  in the bounded derived category  $D^b(\mathcal{A})$ , of an abelian category  $\mathcal{A}$ . Obviously the standard  $t$ -structure is bounded. By applying the above Theorem for the triangulated category  $D^b(\mathcal{A})$  and the heart of the bounded  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$   $\mathcal{A}$ , we have that  $K_0(\mathcal{A}) \cong K_0(D^b(\mathcal{A}))$ . ■

## 2.2 HRS-tilting

The theory of HRS-tilting was introduced in order to systematically construct new abelian categories inside a triangulated category. In representation theory, this construction arose from the study of tilting modules and torsion pairs, where it provides a powerful method for producing and comparing different abelian hearts associated to a fixed derived category. The first part of this section is dedicated to the construction of a  $t$ -structure in the bounded derived category  $\mathcal{A}$  of an abelian category  $\mathcal{A}$ , if we have a torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{A}$ . Next, we investigate an important construction which is defined in the first subsection. In this section we follow [7].

### 2.2.1 $t$ -structures and torsion pairs

In this subsection we present how a torsion pair in an abelian category  $\mathcal{A}$ , can induce a  $t$ -structure in the bounded derived category  $D^b(\mathcal{A})$  of  $\mathcal{A}$ .

**Definition 2.2.1.** *Let  $(\mathcal{T}, \mathcal{F})$  be a pair of full subcategories closed under morphisms of  $\mathcal{A}$ . The pair  $(\mathcal{T}, \mathcal{F})$  is called a **torsion pair** if:*

- (1)  $\text{Hom}(T, F) = 0$ , for every  $T \in \mathcal{T}$  and  $F \in \mathcal{F}$ ,
- (2) for every  $X \in \mathcal{A}$  there exists a short exact sequence

$$0 \longrightarrow t(X) \longrightarrow X \longrightarrow X/t(X) \longrightarrow 0$$

where  $t(X) \in \mathcal{T}$  and  $X/t(X) \in \mathcal{F}$ .

The subcategory  $\mathcal{T}$  is called the **torsion class**, and the subcategory  $\mathcal{F}$  is called the **torsion free class**. If  $T \in \mathcal{T}$ , then it is called a **torsion object**, and if  $F \in \mathcal{F}$ , then it is called **torsion free object**.

**Remark 2.2.2.** (1) For every torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{A}$  it is true that:

- (i) if  $X \in \mathcal{A}$  and  $\text{Hom}(T, X) = 0$  for every  $T \in \mathcal{T}$ , then  $X \in \mathcal{F}$ ,
- (ii) if  $X \in \mathcal{A}$  and  $\text{Hom}(X, F) = 0$  for every  $F \in \mathcal{F}$ , then  $X \in \mathcal{T}$ .

(2) The full subcategories  $\mathcal{T}$  and  $\mathcal{F}$  are closed under extensions, subobjects and quotient objects.

**Proposition 2.2.3.** Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$ . Let also

$$\begin{aligned}\mathcal{T}^{\leq 0} &= \{X^\bullet \in \text{D}^b(\mathcal{A}) \mid H^i(X^\bullet) = 0 \text{ for } i > 0, H^0(X^\bullet) \in \mathcal{T}\} \\ \mathcal{T}^{\geq 0} &= \{X^\bullet \in \text{D}^b(\mathcal{A}) \mid H^i(X^\bullet) = 0 \text{ for } i < -1, H^{-1}(X^\bullet) \in \mathcal{F}\}\end{aligned}$$

Then the pair  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is a *t-structure* in  $\text{D}^b(\mathcal{A})$

*Proof.* Firstly, we will show that  $\text{Hom}(X^\bullet, Y^\bullet) = 0$ , for every  $X^\bullet \in \mathcal{T}^{\leq 0}$  and  $Y^\bullet \in \mathcal{T}^{\geq 1}$ , where  $\mathcal{T}^{\geq 1} = \{X^\bullet \in \text{D}^b(\mathcal{A}) \mid H^i(X^\bullet) = 0 \text{ for } i < 0, H^0(X) \in \mathcal{F}\}$ . We assume, for the sake of contradiction, that  $\text{Hom}(X^\bullet, Y^\bullet) \neq 0$ , for some  $X^\bullet \in \mathcal{T}^{\leq 0}$  and  $Y^\bullet \in \mathcal{T}^{\geq 1}$ , that is, there exists a  $f^\bullet : X^\bullet \rightarrow Y^\bullet$ , such that  $f^\bullet \neq 0$ . Then  $f^\bullet$  can be represented from the right roof

$$\begin{array}{ccc} & Z^\bullet & \\ f^\bullet \nearrow & & \nwarrow s^\bullet \\ X^\bullet & & Y^\bullet \end{array}$$

~

where  $s^\bullet$  is quasi-isomorphism. Hence, since  $s^\bullet$  is a quasi-isomorphism, the complexes  $Y^\bullet$  and  $Z^\bullet$  have the same cohomology, which means that  $Z^\bullet \in \mathcal{T}^{\geq 1}$ . So  $f^\bullet$  is a chain map, not homotopic to zero. From the truncation functors we have the following morphism of triangles

$$\begin{array}{ccccccc} \tau_{\leq 0}(X^\bullet) & \xrightarrow{\mu^\bullet} & X^\bullet & \longrightarrow & \tau_{\geq 1}(X^\bullet) & \longrightarrow & \Sigma(\tau_{\leq 0}(X^\bullet)) \\ \downarrow & & \downarrow f^\bullet & & \downarrow h^\bullet & & \downarrow \\ \tau_{\leq 0}(Z^\bullet) & \longrightarrow & Z^\bullet & \longrightarrow & \tau_{\geq 1}(Z^\bullet) & \longrightarrow & \Sigma(\tau_{\leq 0}(Z^\bullet)) \end{array}$$

for some cochain map  $h^\bullet$ . From the assumption,  $\tau_{\geq 1}(X^\bullet)$  is acyclic, i.e.,  $H^i(\tau_{\geq 1}(X^\bullet)) = 0$ , for every  $i \in \mathbb{Z}$ . Hence  $\tau_{\geq 1}(X) = 0$  and  $\mu^\bullet$  is an isomorphism in  $\text{D}^b(\mathcal{A})$ . In particular,  $\tau_{\leq 0}(f^\bullet) \neq 0$  in  $\text{K}^b(\mathcal{A})$ . By using the truncations, we get the morphism of triangles

$$\begin{array}{ccccccc}
\sigma(\tau_{\leq 0}(X^\bullet)) & \longrightarrow & \tau_{\leq 0}(X^\bullet) & \longrightarrow & H^0(X^\bullet) & \longrightarrow & \Sigma(\sigma(\tau_{\leq 0}(X^\bullet))) \\
\downarrow \sigma(\tau_{\leq 0}(f^\bullet)) & & \downarrow \tau_{\leq 0}(f^\bullet) & & \downarrow h^\bullet & & \downarrow \\
\sigma(\tau_{\leq 0}(Z^\bullet)) & \longrightarrow & \tau_{\leq 0}(Z^\bullet) & \xrightarrow{\rho^\bullet} & H^0(Z^\bullet) & \longrightarrow & \Sigma(\sigma(\tau_{\leq 0}(Z^\bullet)))
\end{array}$$

for some morphism  $h^\bullet$ . Now, since  $\sigma(\tau_{\leq 0}(Z^\bullet))$  is acyclic,  $\rho^\bullet$  is an isomorphism in  $D^b(\mathcal{A})$ . In addition, because  $H^0(X^\bullet) \in \mathcal{T}$  and  $H^0(Z^\bullet) \cong H^0(Y^\bullet) \in \mathcal{F}$ , we have that  $h = 0$ . This means that  $\tau_{\leq 0}(f^\bullet) = 0$ , which is a contradiction. The second axiom is immediate. For the third axiom, let  $X^\bullet = (X^i, d_X^i) \in D^b(\mathcal{A})$ . Since  $(\mathcal{T}, \mathcal{F})$  is a torsion pair in  $\mathcal{A}$ , we have the short exact sequence

$$0 \longrightarrow t(H^0(X^\bullet)) \xrightarrow{\mu^\bullet} H^0(X^\bullet) \xrightarrow{\pi} H^0(X^\bullet)/t(H^0(X^\bullet)) \longrightarrow 0$$

where  $t(H^0(X^\bullet)) \in \mathcal{T}$  and  $H^0(X^\bullet)/t(H^0(X^\bullet)) \in \mathcal{F}$ . From the pullback of the diagram

$$\begin{array}{ccc}
& t(H^0(X^\bullet)) & \\
& \downarrow \mu & \\
\text{Ker}(d_X^0) & \longrightarrow & H^0(X^\bullet)
\end{array}$$

we have the commutative diagram

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Im}(d_X^{-1}) & \xrightarrow{\mu''} & E & \longrightarrow & t(H^0(X^\bullet)) \longrightarrow 0 \\
& & \parallel & & \downarrow \mu' & & \downarrow \mu \\
0 & \longrightarrow & \text{Im}(d_X^{-1}) & \longrightarrow & \text{Ker}(d_X^0) & \longrightarrow & H^0(X^\bullet) \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \pi \\
& & & & H^0(X^\bullet)/t(H^0(X^\bullet)) & \equiv & H^0(X^\bullet)/t(H^0(X^\bullet)) \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

of short exact sequences. Let  $d_X^{-1} = i \circ \rho$  be the canonical decomposition of  $d_X^{-1}$  through its image  $\text{Im}(d_X^{-1})$ , and  $\widetilde{d_X^{-1}} = \mu'' \circ \rho$ . We define the subcomplex  $X'^\bullet$  of  $X^\bullet$ , whose objects are  $X'^i = X^i$ , for  $i \leq -1$ ,  $X'^0 = E$  and  $X^i = 0$ , for  $i > 0$ , and whose differentials are  $d_{X'}^i = d_X^i$ , for  $i < -1$ ,  $d_{X'}^{-1} = \widetilde{d_X^{-1}}$  and  $d_{X'}^i = 0$ , for  $i > 0$ . From our construction we have that  $X \in \mathcal{T}^{\leq 0}$ . Let now  $X''^\bullet$  be the quotient complex  $X^\bullet/X'^\bullet$ . Thus we get the distinguished triangle

$$X''^\bullet \longrightarrow X'''^\bullet \longrightarrow X'' \longrightarrow \Sigma(X'^\bullet)$$

in  $D^b(\mathcal{A})$ . It is left to prove that  $X'''^\bullet \in \mathcal{T}^{\geq 1}$ . By construction we have that  $H^i(X'''^\bullet) = 0$ , for  $i > 0$ . Furthermore we have  $X''^0 = X^0/E$  and  $X''^1 = X^1$ , hence we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \hookrightarrow & X^0 & \longrightarrow & X^0/E \longrightarrow 0 \\ & & \downarrow & & \downarrow d_X^0 & & \downarrow \widetilde{d}_X^0 \\ 0 & \longrightarrow & 0 & \longrightarrow & X^1 & \xlongequal{\quad} & X^1 \longrightarrow 0 \end{array}$$

Therefore  $H^0(X'''^\bullet) = \text{Ker}(\widetilde{d}_X^0) \cong \text{Ker}(d_X^0)/E \cong H^0(X^\bullet)/t(H^0(X^\bullet)) \in \mathcal{F}$ . ■

**Corollary 2.2.4.** *Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$ . Then:*

- (1) *the category  $\mathcal{B} = \{X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0, \text{ for } i \neq 0, -1, H^0(X^\bullet) \in \mathcal{T}, H^{-1}(X^\bullet) \in \mathcal{F}\}$  is abelian,*
- (2) *The pair  $(\mathcal{X}, \mathcal{Y})$ , where  $\mathcal{X} = \mathcal{F}[1]$  and  $\mathcal{Y} = \mathcal{T}$  are subcategories of  $\mathcal{B}$ , is a torsion pair in  $\mathcal{B}$ ,*
- (3) *For  $X, Y \in \mathcal{B}$  we have*

$$\text{Hom}_{D^b(\mathcal{B})}(X, Y[n]) \cong \text{Hom}_{D^b(\mathcal{A})}(X, Y[n]), \text{ for } n = 0, 1$$

*Proof.* Since the heart of a  $t$ -structure is abelian, from Proposition 2.2.3 the first part is true. From Remark 3.1.17. in [1], the third part is also straightforward. It is left to prove the second part. Clearly  $\mathcal{X} = \mathcal{F}[1]$  and  $\mathcal{Y} = \mathcal{T}$  are subcategories of  $\mathcal{B}$ . If  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ , then  $X \cong F[1]$  for some  $F \in \mathcal{F}$ . Furthermore we have  $\text{Hom}_{\mathcal{B}}(X, Y) = \text{Hom}_{D^b(\mathcal{A})}(F[1], Y) = \text{Ext}_{\mathcal{A}}^{-1}(F, Y) = 0$ , which proves the first part from the definition of a torion pair. Let now  $Z^\bullet \in \mathcal{B}$ . First we show that  $Z^\bullet \cong \widetilde{Z}^\bullet$  in  $D^b(\mathcal{A})$ , for some  $\widetilde{Z}^\bullet$ , where  $\widetilde{Z}^i = 0$ , for  $i \neq 0, 1$ . Since  $H^i(Z^\bullet) = 0$  for  $i > 0$ , we have that  $Z^\bullet \cong \tau_{\leq 0}(Z^\bullet)$  in  $D^b(\mathcal{A})$ . Let  $U^\bullet$  be the subcomplex of  $\tau_{\leq 0}(Z^\bullet)$  defined as  $U^0 = 0$ ,  $U^{-1} = \text{Im}(d^{-2})$ ,  $U^i = Z^i$ , for  $i \leq -2$ , with the induced differentials. Because  $U^\bullet$  is acyclic we have that  $\widetilde{Z}^\bullet = \tau_{\leq 0}(Z^\bullet)/U^\bullet \cong Z^\bullet$  in  $D^b(\mathcal{A})$ . It is also true that  $\widetilde{Z}^i = 0$ , for  $i \neq 0, 1$ . Let now  $Z^\bullet \in \mathcal{B}$ . We can assume, without the loss of generality, that  $Z^i = 0$ , for  $i = 0, -1$ . Then, the following commutative diagram is an exact sequence in  $\mathcal{B}$

$$\begin{array}{cccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & H^{-1}(Z^\bullet) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \parallel & & \downarrow & & \downarrow & & \parallel & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & Z^{-1} & \xrightarrow{d_Z^{-1}} & Z^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \parallel & & \downarrow & & \downarrow & & \parallel & & \\ \cdots & \longrightarrow & 0 & \xlongequal{\quad} & 0 & \longrightarrow & H^0(Z^0) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

which represents an exact sequence in  $\mathcal{B}$

$$0 \longrightarrow X^\bullet \longrightarrow Z^\bullet \longrightarrow Y^\bullet \longrightarrow 0$$

where  $X^\bullet \in \mathcal{X}$  and  $Y^\bullet \in \mathcal{Y}$ . Hence  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair in  $\mathcal{B}$ .  $\blacksquare$

**Remark 2.2.5.** We will use the following notation as far as the last Corollary 2.2.4 is concerned. Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$ . Then  $\Phi(\mathcal{A}; (\mathcal{T}, \mathcal{F})) = (\mathcal{B}; (\mathcal{X}, \mathcal{Y}))$ , where  $\mathcal{B}$  is the category of the first part, and  $(\mathcal{X}, \mathcal{Y})$  is the torsion pair of the second part.

### 2.2.2 Tilting torsion classes

Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$ . In this section we will apply the construction  $\Phi$  to a particular class of torsion pairs.

**Definition 2.2.6.** A torsion class  $\mathcal{T}$  is called **tilting torsion class** if it is a cogenerator of  $\mathcal{A}$ , that is, for every  $X \in \mathcal{A}$ , there exist  $T_X \in \mathcal{T}$  and a monomorphism  $\mu_X : X \rightarrow T_X$ . Dually, a torsion free class  $\mathcal{F}$  is called a **cotilting torsion free class** if it is a generator of  $\mathcal{A}$ , that is, for every  $X \in \mathcal{A}$ , there exists  $F_X \in \mathcal{F}$  and an epimorphism  $\nu_X : F_X \rightarrow X$ .

**Proposition 2.2.7.** Let  $\mathcal{A}$  be an abelian category,  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$ , and consider  $\Phi(\mathcal{A}; (\mathcal{T}, \mathcal{F}))$ . Then:

- (1)  $\mathcal{T}$  is a tilting torsion class if and only if  $\mathcal{Y}$  is a cotilting torsion free class,
- (2)  $\mathcal{F}$  is a cotilting torsion free class if and only if  $\mathcal{X}$  is a tilting torsion class.

*Proof.* (1)  $(\Rightarrow)$ : Suppose that  $\mathcal{T}$  is a tilting torsion class and let  $X^\bullet \in \mathcal{B}$ . We can assume, without loss of generality, that  $X^i = 0$ , for  $i \neq 0, -1$ . Since  $\mathcal{T}$  is a tilting torsion class, hence a cogenerator of  $\mathcal{A}$ , there exists a monomorphism  $\mu_X : X^{-1} \rightarrow T_0$ , for some  $T_0 \in \mathcal{T}$ . Furthermore, because  $\mathcal{T}$  is closed under quotient objects, we get the short exact sequence

$$0 \longrightarrow X^{-1} \xrightarrow{\mu} T_0 \longrightarrow T_1 \longrightarrow 0$$

where  $T_0, T_1 \in \mathcal{T}$ . Hence we have the distinguished triangle

$$X^{-1} \longrightarrow T_0 \longrightarrow T_2 \xrightarrow{f} \Sigma(X^{-1})$$

in  $D^b(\mathcal{A})$ . Then  $d^{-1}[1] \circ f \in \text{Hom}_{D^b(\mathcal{A})}(T_1, X^0[1]) \cong \text{Ext}_{\mathcal{A}}^1(T_1, X^0)$ . Let now

$$0 \longrightarrow X^0 \longrightarrow E \longrightarrow T_1 \longrightarrow 0$$

be the corresponding short exact sequence. From the above, we get the commutative diagram

$$\begin{array}{ccccccc}
X^0 & \xrightarrow{\mu} & E & \xrightarrow{\pi} & T_1 & \xrightarrow{d^{-1}[1] \circ f} & X^0[1] \\
\parallel & & \downarrow g & & \downarrow f & & \parallel \\
X^0 & \xrightarrow{u} & X^\bullet & \xrightarrow{v} & X^{-1}[1] & \xrightarrow{d^{-1}[1]} & X^0[1]
\end{array}$$

in  $D^b(A)$ , for some  $g \in \text{Hom}_{\text{Db}(\mathcal{A})}(E, X^\bullet)$ , whose existence comes from (TR3). Firstly, we will show that  $E \in \mathcal{Y}$ . We have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & X^{-1} & \longrightarrow & T_0 & \longrightarrow & T_1 & \longrightarrow & 0 \\
& & \downarrow d_X^{-1} & & \downarrow e & & \parallel & & \\
0 & \longrightarrow & X^0 & \longrightarrow & E & \longrightarrow & T_1 & \longrightarrow & 0
\end{array}$$

By assumption, we have that  $H^0(X^\bullet) = \text{Coker}(d_X^{-1}) \in \mathcal{T}$ . However,  $\mathcal{T}$  is closed under quotient objects, therefore  $\text{Im}(e) \in \mathcal{T}$ . Consequently, because  $\text{Coker}(e) \cong \text{Coker}(d_X^{-1})$ , we get  $E \in \mathcal{T} = \mathcal{Y}$ . It is left to show that  $g$  is an epimorphism. In this direction, let  $Y^\bullet \in \mathcal{B}$  and  $h \in \text{Hom}_{\mathcal{B}}(X^\bullet, Y^\bullet)$ , such that  $h \circ g = 0$ . Consider the torsion exact sequence

$$0 \longrightarrow t(Y^\bullet) \xrightarrow{\alpha} Y^\bullet \xrightarrow{\beta} Y^\bullet/t(Y^\bullet) \longrightarrow 0$$

where  $t(Y^\bullet) \in \mathcal{X} = \mathcal{F}[1]$  and  $Y^\bullet/t(Y^\bullet) \in \mathcal{Y} = \mathcal{T}$ . Since  $h \circ u = h \circ g \circ \mu = 0$ , there exists a unique morphism  $h' : X^{-1}[1] \rightarrow Y^\bullet$ , such that  $h = h' \circ v$ . Because  $\pi$  is an epimorphism in  $\mathcal{B}$  we have

$$h' \circ f \circ \pi = h' \circ v \circ g = h \circ g = 0$$

hence  $h' \circ f = 0$ . In addition,  $\beta \circ h' \in \text{Hom}_{\text{Db}(\mathcal{A})}(X^{-1}[1], Y^\bullet/t(Y^\bullet)) = \text{Ext}_{\mathcal{A}}^{-1}(X^{-1}, Y^\bullet/t(Y^\bullet)) = 0$ . From the above, there exists  $h'' : X^{-1}[1] \rightarrow Y^\bullet$  such that  $h' = \alpha \circ h''$ , hence  $0 = h' \circ f = \alpha \circ h'' \circ f$ . Now, because  $\alpha$  is a monomorphism in  $\mathcal{B}$  we get  $h'' \circ f = 0$ . By applying the  $\text{Hom}_{\text{Db}(\mathcal{A})}(-, t(Y^\bullet))$  functor to the distinguished triangle

$$T_0 \longrightarrow T_1 \xrightarrow{f} X^{-1}[1] \xrightarrow{w} \Sigma(T_0)$$

we get that there exists a unique morphism  $h''' : T_0[1] \rightarrow t(Y^\bullet)$ , such that  $h'' = h''' \circ w$ . However,  $\text{Hom}_{\text{Db}(\mathcal{A})}(T_0[1], t(Y^\bullet)) = \text{Hom}_{\text{Db}(\mathcal{A})}(T_0, t(Y^\bullet)[-1]) = 0$ , since  $T_0 \in \mathcal{T}$  and  $t(Y^\bullet)[-1] \in \mathcal{F}$ . Consequently,  $h''' = 0$  which implies that  $h = 0$ , hence  $g$  is an epimorphism.

( $\Leftarrow$ ): Dual to ( $\Rightarrow$ ).

(2) The proof is omitted since it is dual to that of 1. ■

Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$ , and  $\Phi(\mathcal{A}; (\mathcal{T}, \mathcal{F})) = (\mathcal{B}; (\mathcal{X}, \mathcal{Y}))$ . [1] proves some general results not concerning the setting of HRS. Applying the results of [1] in the

particular situation of HRS, it follows that there exists an exact functor  $G : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$  such that  $G|_{\mathcal{B}}$  is the canonical inclusion  $\mathcal{B} \hookrightarrow D^b(\mathcal{A})$ .

In the above context, suppose that  $\mathcal{T}$  is a tilting torsion class. We will construct an exact functor  $G : D^b(\mathcal{B}) \rightarrow D^b(\mathcal{A})$  when  $\mathcal{B}$  has enough projectives, and an exact functor  $F : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$  when  $\mathcal{A}$  has enough projectives. In this direction, let  ${}_{\mathcal{B}}\mathcal{P}$  be the subcategory of projective objects in  $\mathcal{B}$ . Then we have an equivalence of triangulated categories  $K^{-,b}({}_{\mathcal{B}}\mathcal{P}) \xrightarrow{\cong} D^b(\mathcal{B})$ . Since  $\mathcal{T}$  is a tilting torsion class in  $\mathcal{A}$ ,  $\mathcal{Y}$  is cotilting torsion free class in  $\mathcal{B}$ , from Proposition 2.2.7. In the case where  $\mathcal{B}$  has enough projectives, we have  ${}_{\mathcal{B}}\mathcal{P} \subseteq \mathcal{Y} \cong \mathcal{T}$ , hence we have the inclusion functor  $K^{-,b}({}_{\mathcal{B}}\mathcal{P}) \hookrightarrow K^{-,b}(\mathcal{A})$ . Consider now the restriction of  $Q : K(\mathcal{A}) \rightarrow D(\mathcal{A})$  in  $K^{-,b}(\mathcal{A})$ , which gives an exact functor  $K^{-,b}({}_{\mathcal{A}}\mathcal{P}) \rightarrow D^b(\mathcal{A})$ . Thus we get the functor  $G : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$ .

Working dually, we can construct the functor  $F : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$ .

**Theorem 2.2.8.** *Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$  like above, where  $\mathcal{T}$  is a tilting torsion class. Then:*

- (1) *if  $\mathcal{B}$  has enough projectives, then the functor  $G : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$  is an equivalence of triangulated categories, and  $G|_{\mathcal{B}}$  is the identity functor of  $\mathcal{B}$ ,*
- (2) *if  $\mathcal{A}$  has enough injectives, then the functor  $F : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$  is an equivalence of triangulated categories, and  $F|_{\mathcal{A}}$  is the identity functor of  $\mathcal{B}$ .*

*Proof.* (1) From Proposition 3.1.16 in [1], it is enough to prove that for every  $B, B' \in \mathcal{B}$  the induced morphism  $G_n : \text{Hom}_{D^b(\mathcal{B})}(B, B'[n]) \rightarrow \text{Hom}_{D^b(\mathcal{A})}(B, B'[n])$  is a bijection, for every  $n \in \mathbb{Z}$ . From Proposition 3.1.17, the above is true for  $n \leq 1$ , and if  $G_{n-1}$  is a bijection, then  $G_n$  is an injection. The above is true even without the assumption in [1] (that is, the category  $\mathcal{A}$  has enough injectives). We will use induction on  $n$ . So, it is left to prove that  $G_n$  is surjective. In this direction, let  $B, B' \in \mathcal{B}$  and  $f \in \text{Hom}_{D^b(\mathcal{A})}(B, B'[n])$ . Therefore  $f$  can be represented by the right roof

$$\begin{array}{ccc} & X^\bullet & \\ f^\bullet \nearrow & & \nwarrow s^\bullet \\ B & & B'[n] \end{array}$$

for some  $X^\bullet \in D^b(\mathcal{A})$ ,  $f \in \text{Hom}_{K^b(\mathcal{A})}(B, X^\bullet)$ , and quasi-isomorphism  $s^\bullet$ . Hence, there exist  $Y^\bullet \in D^b(\mathcal{A})$ , where  $Y^i \in \mathcal{T}$  for every  $i$ , and a quasi-isomorphism  $t^\bullet : X^\bullet \rightarrow Y^\bullet$ . Consider now the morphism which is represented by the right roof

$$\begin{array}{ccc} & Y^\bullet & \\ \tilde{f}^\bullet \nearrow & & \nwarrow \tilde{s}^\bullet \\ B & & B'[n] \end{array}$$

where  $\widetilde{f}^\bullet = t^\bullet \circ f^\bullet$  and  $\widetilde{s}^\bullet = t^\bullet \circ s^\bullet$ . Now, from the commutative diagram

$$\begin{array}{ccccc}
 & & X^\bullet & & \\
 & f^\bullet \nearrow & \downarrow t^\bullet & \nwarrow s^\bullet & \\
 B & \xrightarrow{\widetilde{f}^\bullet} & Y^\bullet & \xleftarrow{\widetilde{s}^\bullet} & B'[n] \\
 & \searrow \widetilde{f}^\bullet & \parallel & \swarrow \widetilde{s}^\bullet & \\
 & & Y^\bullet & & 
 \end{array}$$

we have that the two right roofs are equivalent. Therefore,  $\widetilde{f} \in \text{Hom}_{\text{D}^b(\mathcal{B})}(B, B'[n])$ , where  $G_n(\widetilde{f}) = \widetilde{f}$ . Consequently,  $G_n$  is surjective.

(2) The proof is omitted since it is dual to that of (1). ■

Let  $G : \text{D}^b(\mathcal{B}) \rightarrow \text{D}^b(\mathcal{A})$  be the equivalence of triangulated categories from the Theorem 2.2.8. Let also  $(\mathcal{X}, \mathcal{Y})$  be the torsion pair in  $\mathcal{B}$ , which is induced from the torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{A}$ . By construction,  $G(\mathcal{X}) = \mathcal{F}[1]$  and  $G(\mathcal{Y}) = \mathcal{T}$ . In the following Proposition, we describe the inverse process.

**Proposition 2.2.9.** *Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$  as above, where  $\mathcal{T}$  is a tilting torsion class, and  $\Phi(\mathcal{B}; (\mathcal{T}, \mathcal{F})) = (\mathcal{B}; (\mathcal{X}, \mathcal{Y}))$ . Then:*

(1) *if  $\mathcal{B}$  has enough projectives, then*

$$\Phi(\mathcal{B}; (\mathcal{X}, \mathcal{Y})) \cong (\mathcal{A}; (\mathcal{T}, \mathcal{F}))$$

(2) *if  $\mathcal{A}$  has enough injectives, then*

$$\Phi(\mathcal{B}; (\mathcal{X}, \mathcal{Y})) \cong (\mathcal{A}; (\mathcal{T}, \mathcal{F}))$$

*Proof.* (1) Suppose that  $\Phi(\mathcal{B}; (\mathcal{X}, \mathcal{Y})) = (\mathcal{A}'; (\mathcal{T}', \mathcal{F}'))$ . By definition

$$\mathcal{A}' = \{X^\bullet \in \text{D}^b(\mathcal{A}) \mid H^0(X^\bullet) \in \mathcal{X} \text{ and } H^{-1}(X^\bullet) \in \mathcal{Y}, H^i(X^\bullet) = 0, i \neq 0, 1\}$$

and  $\mathcal{T}' = \mathcal{Y}[1]$ ,  $\mathcal{F}' = \mathcal{X}$ . From Theorem 3.3 in [1], we have the equivalence of triangulated categories  $G : \text{D}^b(\mathcal{B}) \rightarrow \text{D}^b(\mathcal{A}')$ , where  $G(\mathcal{X}) = \mathcal{F}[1]$  and  $G(\mathcal{Y}) = \mathcal{T}$ . We will show that its restriction to  $\mathcal{A}'$  is an equivalence of categories. Let  $X' \in \mathcal{A}'$ . Then, we have the distinguished triangle

$$U \longrightarrow X' \longrightarrow V \longrightarrow \Sigma(U)$$

in  $D^b(\mathcal{B})$ , where  $U \in \mathcal{T}'$  and  $V \in \mathcal{F}'$ . By applying  $G$ , we get the distinguished triangle

$$G(U) \longrightarrow G(X) \longrightarrow G(V) \longrightarrow \Sigma(G(U))$$

in  $D^b(\mathcal{A})$ . From the last distinguished triangle, we get the long exact cohomology sequence

$$\cdots \longrightarrow H^{i-1}(G(V)) \longrightarrow H^i(G(U)) \longrightarrow H^i(G(X)) \longrightarrow H^i(G(V)) \longrightarrow H^{i+1}(G(U)) \longrightarrow \cdots$$

Since  $G(U) \in \mathcal{T}[1]$  and  $G(V) \in \mathcal{F}[1]$ , then  $H^i(G(X)) = 0$ , for  $i \neq -1$ , so  $G(X) \in \mathcal{A}[1]$ . From the later, we get that the functor  $G|_{\mathcal{A}'} : \mathcal{A}' \rightarrow \mathcal{A}[1]$  is fully faithful. It is left to show that it dense. Let  $X \in \mathcal{A}$ . Then we have the distinguished triangle

$$t(X)[1] \longrightarrow X[1] \longrightarrow (X/t(X))[1] \xrightarrow{w} \Sigma(t(X)[1])$$

where  $t(X) \in \mathcal{T}$  and  $V \in \mathcal{F}'$ . We choose now  $U \in \mathcal{Y}'$ , where  $G(U) = t(X)[1]$ , and  $V \in \mathcal{F}'$ , where  $G(V) = (X/t(X))[1]$ . Because  $G$  is dense, there exists a morphism  $w' \in \text{Hom}_{D^b(\mathcal{B})}(V, U[1])$ , such that  $G(w') = w$ . Thus, we get the distinguished triangle

$$U \longrightarrow X' \longrightarrow V \xrightarrow{w'} \Sigma(U)$$

in  $D^b(\mathcal{B})$ , for some  $X' \in D^b(\mathcal{B})$ . From the long exact cohomology sequence of the above triangle, we again have that  $X' \in \mathcal{A}'$ . Applying  $G$  to the above triangle we have the commutative diagram

$$\begin{array}{ccccccc} G(U) & \longrightarrow & G(X') & \longrightarrow & G(V) & \longrightarrow & G(U[1]) \\ \parallel & & \downarrow h & & \parallel & & \parallel \\ t(X)[1] & \longrightarrow & X[1] & \longrightarrow & (X/t(X))[1] & \longrightarrow & t(X)[2] \end{array}$$

which yields that  $h$  is an isomorphism. Therefore,  $G|_{\mathcal{A}'}$  is dense.

(2) The proof is omitted since it is dual to that of (1). ■

**Remark 2.2.10.** *In this context, if  $\Phi(\mathcal{A}; (\mathcal{T}, \mathcal{F})) = (\mathcal{B}; (\mathcal{X}, \mathcal{Y}))$ , then there exists an equivalence of categories  $\mathcal{A}' \xrightarrow{\sim} \mathcal{A}$  which induces equivalences of (sub)categories  $\mathcal{T}' \approx \mathcal{T}$  and  $\mathcal{F}' \approx \mathcal{F}$ .*

## 2.3 Quasi-abelian categories

In the third and last section of this chapter, we provide an introduction to the fundamental concepts of quasi-abelian categories. They were introduced by Schneiders in [17]. They provide the necessary concept for categories where kernels and cokernels, however not all morphisms are strict leading to

weaker notions of exactness. Quasi-abelian categories arise naturally in functional analysis and in categories of topological vector spaces, where many important examples fail to be abelian but still retain enough structure to carry a meaningful homological theory. Here, we present basic definitions and properties of quasi-abelian categories. For this section we follow [17].

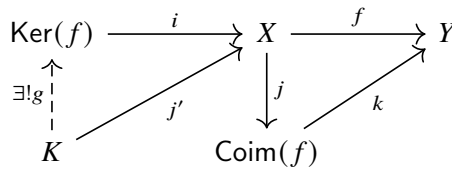
**Definition 2.3.1.** A **pre-abelian** category  $\mathcal{A}$  is an additive category with kernels and cokernels.

In what follows the notions of **strict epimorphisms** and **strict monomorphisms** will be crucial. Strict monomorphisms coincide with kernels and strict epimorphisms coincide with cokernels.

**Proposition 2.3.2.** For any morphism  $f : X \rightarrow Y$ ,

- (1) the canonical morphism  $\text{Ker}(f) \rightarrow X$  is a strict monomorphism,
- (2) the canonical morphism  $Y \rightarrow \text{Coker}(f)$  is a strict epimorphism.

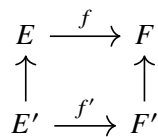
*Proof.* If  $i : \text{Ker}(f) \rightarrow X$  is the canonical morphism and  $j : X \rightarrow \text{Coim}(f)$ , it is enough to prove that  $\text{coker}(i) = j$ . Let  $i' : K \rightarrow X$  be a morphism such that  $i' \circ j = 0$ . Then we have  $f \circ i = k \circ j \circ i = k \circ 0 = 0$ . Therefore, from the universal property of the kernel of  $f$ , there exists a unique morphism  $g : K \rightarrow \text{Ker}(f)$  such that  $i' = i \circ g$ . The above fit into the following commutative diagram



■

**Definition 2.3.3.** A pre-abelian category  $\mathcal{A}$  is called **quasi-abelian** if:

(QA) In the following pullback diagram



if the morphism  $f$  is a strict epimorphism, then the morphism  $f'$  is a strict epimorphism.

(QA\*) In the following pushout diagram

$$\begin{array}{ccc} E' & \xrightarrow{f'} & F' \\ \uparrow & & \uparrow \\ E & \xrightarrow{f} & F \end{array}$$

if the morphism  $f$  is a strict monomorphism, then the morphism  $f'$  is a strict monomorphism.

**Remark 2.3.4.** We can give an equivalent definition of a quasi-abelian category, without the notions of strict monomorphisms and strict epimorphisms, but with kernels and cokernels. Therefore we have the following equivalent definition.

**Definition 2.3.5.** A pre-abelian category  $\mathcal{A}$  is called **quasi-abelian** if:

(QA) Pull-back diagrams preserve cokernels. That is, in the pull-back diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \uparrow & & \uparrow \\ E' & \xrightarrow{f'} & F' \end{array}$$

if  $f$  is a cokernel, then  $f'$  is also a cokernel.

(QA\*) Push-out diagrams preserve kernels. That is, in the push-out diagram

$$\begin{array}{ccc} E' & \xrightarrow{f'} & F' \\ \uparrow & & \uparrow \\ E & \xrightarrow{f} & F \end{array}$$

if  $f$  is a kernel, then  $f'$  is also a kernel.

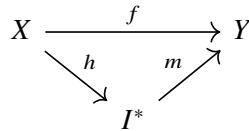
Until the end of this section,  $\mathcal{A}$  will be a quasi-abelian category.

**Proposition 2.3.6.** Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{A}$ .

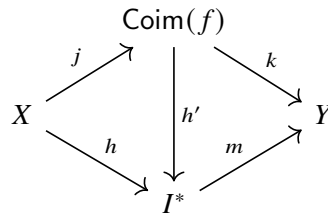
(1) In the canonical decomposition of  $f$  through its coimage

$$\begin{array}{ccc} & \text{Coim}(f) & \\ j \nearrow & & \searrow k \\ X & \xrightarrow{f} & Y \end{array}$$

$j$  is a strict epimorphism and  $k$  is a monomorphism. Furthermore, for every decomposition of  $f$  of the form

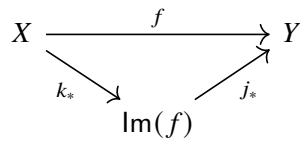


where  $m$  is a monomorphism, there exists a unique morphism  $h' : \text{Coim}(f) \rightarrow I^*$  making the following diagram

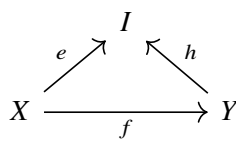


commutative.

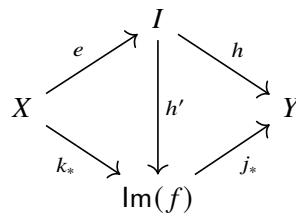
(2) Dually, in the canonical decomposition of  $f$  through its image



$k_*$  is an epimorphism and  $j_*$  is a strict monomorphism. Furthermore, for every decomposition of  $f$  of the form



where  $e$  is an epimorphism, there exists a unique morphism  $h' : I \rightarrow \text{Im}(f)$  making the following diagram



commutative.

*Proof.* 1) Let  $i : \text{Ker}(f) \rightarrow E$  be the kernel of  $f$ . Since  $j$  is the cokernel of  $i$ , it is a strict epimorphism. It is left to show that  $k$  is a monomorphism. Let  $x : X \rightarrow \text{Coim}(f)$  be a morphism such that  $k \circ x = 0$ . We will show that  $x = 0$ . Consider the following pullback digram

$$\begin{array}{ccc} E & \xrightarrow{j} & \text{Coim}(f) \\ x' \uparrow & & \uparrow x \\ X' & \xrightarrow{j'} & X \end{array}$$

Since  $\mathcal{A}$  is a quasi-abelian category,  $j'$  is a strict epimorphism. Furthermore we have  $f \circ x' = k \circ j \circ x = k \circ x \circ j' = 0 \circ j = 0$ , and thus, from the universal property of the kernel of  $f$ , there exists a unique morphism  $x'' : X' \rightarrow \text{Ker}(f)$  such that  $i \circ x'' = x'$ . We also have  $x \circ j' = j \circ x' = j \circ i \circ x'' = 0 \circ x'' = 0$  or  $x \circ j' = 0$ , and since  $j'$  is an epimorphism, we have that  $x = 0$ .

For the second part, since  $m$  is a monomorphism, and from  $m \circ h \circ i = f \circ i = 0$ , we have that  $h \circ i = 0$ . In addition, since  $j$  is the cokernel of  $i$ , there exists a unique morphism  $h' : \text{Coim}(f) \rightarrow I^*$  such that  $h' \circ j = h$ . From the equality  $f = f \Leftrightarrow k \circ j = m \circ h = m \circ h' \circ j \Rightarrow k \circ j = m \circ h' \circ j$ , and since  $j$  is an epimorphism we have  $k = m \circ h'$ . ■

**Proposition 2.3.7.** *The class of strict epimorphisms (strict monomorphisms) in a quasi-abelian category is closed under composition.*

*Proof.* Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be two strict epimorphisms. We denote by  $i_f : \text{Ker}(f) \rightarrow X$ ,  $i_g : \text{Ker}(g) \rightarrow Y$  and  $i_{g \circ f} : \text{Ker}(g \circ f) \rightarrow X$  the natural inclusions of each kernel respectively. Since  $g \circ f \circ i_{g \circ f} = 0$ , from the universal property of the kernel of  $g$ , there exists a unique morphism  $h : \text{Ker}(g \circ f) \rightarrow \text{Ker}(g)$  such that  $f \circ i_{g \circ f} = i_g \circ h$ , i.e., the following diagram is commutative

$$\begin{array}{ccccc} \text{Ker}(g) & \xrightarrow{i_g} & Y & \xrightarrow{g} & Z \\ \uparrow \exists! h & \nearrow f \circ i_{g \circ f} & & & \\ \text{Ker}(g \circ f) & & & & \end{array}$$

Obviously from the above, the following square is commutative

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_{g \circ f} \uparrow & & \uparrow i_g \\ \text{Ker}(g \circ f) & \xrightarrow{h} & \text{Ker}(g) \end{array}$$

We will show that it is a pullback. Let  $\alpha : W \rightarrow X$  and  $\beta : W \rightarrow \text{Ker}(g)$  be two morphisms such that  $f \circ \alpha = i_g \circ \beta$ . Then we have  $g \circ f \circ \alpha = g \circ i_g \circ \beta = 0 \circ \beta = 0$ . Therefore, from the universal property of the kernel of  $g \circ f$ , there exists a unique morphism  $\gamma : W \rightarrow \text{Ker}(g \circ f)$  such that  $\alpha = i_{g \circ f} \circ \gamma$ , i.e., the following diagram is commutative

$$\begin{array}{ccc} \text{Ker}(g \circ f) & \xrightarrow{i_{g \circ f}} & X & \xrightarrow{g \circ f} & Z \\ \uparrow \exists! \gamma & \nearrow \alpha & & & \\ W & & & & \end{array}$$

Since  $i_g \circ h \circ \gamma = f \circ i_{g \circ f} \circ \gamma = f \circ \alpha = i_g \circ \beta$  we have that  $i_g \circ h \circ \gamma = i_g \circ \beta$ , and since  $i_g$  is a monomorphism we get  $h \circ \gamma = \beta$ . Hence, the following diagram is commutative

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y \\ & \nearrow \alpha & \uparrow i_{g \circ f} & & \uparrow i_g \\ & & \text{Ker}(g \circ f) & \xrightarrow{h} & \text{Ker}(g) \\ W & \nearrow \gamma & & & \\ & \searrow \beta & & & \end{array}$$

and it is indeed a pullback. Since  $\mathcal{A}$  is a quasi-abelian category and  $f$  is a strict epimorphism,  $h$  is also a strict epimorphism. To conclude, it is enough to show that  $g \circ f = \text{coker}(i_{g \circ f})$ . Clearly,  $g \circ f \circ i_{g \circ f} = 0$ . To prove the universal property, let  $t : X \rightarrow H$  be a morphism such that  $t \circ i_{g \circ f} = 0$ . Since  $g \circ f \circ i_f = 0$ , from the universal property of the kernel of  $g \circ f$ , there exists a unique morphism  $z : \text{Ker}(f) \rightarrow \text{Ker}(g \circ f)$  such that  $i_f = i_{g \circ f} \circ z$ , i.e., the following square is commutative

$$\begin{array}{ccc} \text{Ker}(g \circ f) & \xrightarrow{i_{g \circ f}} & X & \xrightarrow{g \circ f} & Z \\ \uparrow \exists! z & \nearrow i_f & & & \\ \text{Ker}(f) & & & & \end{array}$$

Therefore  $t \circ i_f = t \circ i_{g \circ f} \circ z = 0 \circ z = 0$ . Because  $f$  is the cokernel of  $i_f$ , there exists a unique morphism  $t' : Y \rightarrow H$  such that  $t = t' \circ f$ , i.e., the following diagram is commutative

$$\begin{array}{ccccc}
 \text{Ker}(f) & \xrightarrow{i_f} & X & \xrightarrow{f} & Y \\
 & & & \searrow t & \downarrow \exists! t' \\
 & & & & H
 \end{array}$$

Therefore  $t' \circ i_g \circ h = t' \circ f \circ i_{g \circ f} = t \circ i_{g \circ f} = 0$ , and because  $h$  is an epimorphism, we get  $t' \circ i_g = 0$ . Now, because  $g$  is the cokernel of  $i_g$ , there exists a unique morphism  $t'' : Z \rightarrow H$  such that  $t' = t'' \circ g$ , i.e., the following diagram is commutative

$$\begin{array}{ccccc}
 \text{Ker}(g) & \xrightarrow{i_g} & Y & \xrightarrow{g} & Z \\
 & & & \searrow t' & \downarrow \exists! t'' \\
 & & & & H
 \end{array}$$

So we have  $t'' \circ g \circ f = t' \circ f = t$ . It is left to prove that  $t''$  is unique. Let's assume that there exists another morphism  $s : Z \rightarrow H$  such that  $s \circ g \circ f = t$ . Then  $t'' \circ g \circ f = t = s \circ g \circ f = t \Rightarrow t'' \circ g \circ f = s \circ g \circ f$  and since  $g \circ f$  is an epimorphism we have that  $t'' = s$  which concludes the proof. ■

**Proposition 2.3.8.** *In a quasi-abelian category if*

$$\begin{array}{ccc}
 & Y & \\
 f \nearrow & & \searrow g \\
 X & \xrightarrow{g \circ f} & Z
 \end{array}$$

*is a commutative diagram and  $g \circ f$  is a strict epimorphism, then  $g$  is a strict epimorphism. Also, if  $g \circ f$  is a strict monomorphism, then  $f$  is a strict monomorphism.*

*Proof.* We will show it for the case when  $g \circ f$  is a strict epimorphism. The other one is a dual. Also, we will use the morphisms and the diagrams from the previous proof. We will show first that the diagram

$$\begin{array}{ccc}
 X \oplus \text{Ker}(g) & \xrightarrow{(f, i_g)} & Y \\
 (1_X, 0) \downarrow & & \downarrow g \\
 X & \xrightarrow{g \circ f} & Z
 \end{array}$$

is a pullback. Because  $g \circ \begin{pmatrix} f \\ i_g \end{pmatrix} = (g \ 0) \circ \begin{pmatrix} f \\ i_g \end{pmatrix} = g \circ f$  and  $(g \circ f) \circ \begin{pmatrix} 1_X \\ 0 \end{pmatrix} = (g \circ f \ 0) \circ \begin{pmatrix} 1_X \\ 0 \end{pmatrix} = g \circ f \circ 1_X = g \circ f$  the above square is commutative. Let  $\alpha : W \rightarrow X$  and  $\beta : W \rightarrow Y$  be two morphisms such that  $g \circ f \circ \alpha = g \circ \beta$ . Then  $g \circ (\beta - f \circ \alpha) = g \circ \beta - g \circ f \circ \alpha = g \circ \beta - g \circ \beta = 0$ . Therefore, from the universal property of the kernel of  $g$ , there exists a unique morphism  $h : W \rightarrow \text{Ker}(g)$  such that  $i_g \circ h = \beta - f \circ \alpha$ . We define the morphism  $\begin{pmatrix} \alpha \\ h \end{pmatrix} : W \rightarrow X \oplus \text{Ker}(g)$ . We see that  $(1_X \ 0) \circ \begin{pmatrix} \alpha \\ h \end{pmatrix} = 1_X \circ \alpha + 0 \circ h = \alpha$ , and  $(f \ i_g) \circ \begin{pmatrix} \alpha \\ h \end{pmatrix} = f \circ \alpha + i_g \circ h = \beta$ . It remains to show that the morphism  $\begin{pmatrix} \alpha \\ h \end{pmatrix} : W \rightarrow X \oplus \text{Ker}(g)$  is unique. By contradiction, let's assume that it is not unique and let  $\begin{pmatrix} x \\ y \end{pmatrix} : W \rightarrow X \oplus \text{Ker}(g)$  be a morphism such that  $(1_X \ 0) \begin{pmatrix} x \\ y \end{pmatrix} = \alpha$  and  $(f \ i_g) \begin{pmatrix} x \\ y \end{pmatrix} = \beta$ . From the first equation we have that  $1_X \circ x = \alpha \Rightarrow x = \alpha$ . From the second equation we get that  $f \circ x + i_g \circ y = \beta \Rightarrow f \circ \alpha + i_g \circ y = \beta \Rightarrow i_g \circ y = \beta - f \circ \alpha \Rightarrow i_g \circ y = i_g \circ h$  and since  $i_g$  is a monomorphism we have  $y = h$ . It is left to prove that  $g$  is the cokernel of  $i_g$ . Clearly  $g \circ i_g = 0$ . Let  $z : Y \rightarrow K$  be a morphism such that  $z \circ i_g = 0$ . Then  $z \circ f \circ i_{g \circ f} = z \circ i_g \circ h = 0 \circ h = 0$  and since  $g \circ f$  is the cokernel of  $i_{g \circ f}$ , there exists a unique morphism  $z' : Z \rightarrow K$  such that  $z \circ f = z' \circ g \circ f \Rightarrow z \circ f - z' \circ g \circ f = 0 \Rightarrow (z - z' \circ g) \circ f = 0$ . Also  $(z - z' \circ g) \circ i_g = 0$  because  $(z - z' \circ g) \circ i_g = 0 = z \circ i_g - z' \circ g \circ i_g = 0 - z' \circ 0 = 0$ . Therefore we have  $(z - z' \circ g) \circ f + (z - z' \circ g) \circ i_g = 0 \Rightarrow (z - z' \circ g) \begin{pmatrix} f \\ i_g \end{pmatrix} = 0$  and because  $\begin{pmatrix} f \\ i_g \end{pmatrix}$  is an epimorphism we have that  $z - z' \circ g = 0 \Rightarrow z = z' \circ g$ . Using the same argument from the last proof, since  $g$  is an epimorphism, we have uniqueness of  $z'$ . ■

**Definition 2.3.9.** A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

where  $g \circ f = 0$  is called **strictly exact** (respectively **strictly coexact**) if  $f$  (respectively  $g$ ) is a strict morphism and the canonical morphism  $\text{Im}(f) \rightarrow \text{Ker}(g)$  is an isomorphism.

In general, a sequence

$$X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \dots \xrightarrow{f^{n-1}} X^n$$

where  $f^{i+1} \circ f^i = 0$  for  $i = 1, 2, \dots, n-1$ , is called a **strictly exact** (respectively **strictly coexact**) sequence if every subsequence

$$X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} X^{i+2}$$

is strictly exact for every  $i = 1, 2, \dots, n - 2$ .

**Proposition 2.3.10.** *Let  $\mathcal{A}$  be a quasi-abelian category. Then:*

(1) *The sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \tag{2.1}$$

*is strictly exact if and only if  $f$  is a monomorphism,*

(2) *The sequence*

$$X \xrightarrow{f} Y \longrightarrow 0 \tag{2.2}$$

*is strictly exact if and only if  $f$  is a strict epimorphism,*

(3) *The sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \tag{2.3}$$

*is strictly exact if and only if the pair  $(X, f)$  is a kernel of  $g$ ,*

(4) *The sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

*is strictly exact if and only if the pair  $(X, f)$  is a kernel of  $f$  and the pair  $(Z, g)$  is a cokernel of  $f$*

*Proof.* (1) For the forward direction, since (2.1) is strictly exact, we have that  $\text{Im}(0 \rightarrow X) = \text{Ker}(f)$ . However, the morphism  $0 \rightarrow X$  is the zero morphism, hence  $\text{Im}(0 \rightarrow X) = 0$ , which implies that  $\text{Ker}(f) = 0$ , and since  $\mathcal{A}$  is quasi-abelian, hence additive, the morphism  $f$  is a monomorphism. For the converse, suppose that  $f$  is a monomorphism and we will show that  $0 \rightarrow X$  is strict and  $\text{Im}(0 \rightarrow X) = \text{Ker}(f)$ . The morphism  $0 \rightarrow X$  is strict. In addition  $f$  is a monomorphism, hence  $\text{Ker}(f) = 0$ . In addition, since  $\text{Im}(0 \rightarrow X)$ , we have that  $\text{Im}(0 \rightarrow X) = \text{Ker}(f)$ .

(2) For the forward direction, since (2.2) is strictly exact, the morphism  $f$  is strict and  $\text{Im}(f) = \text{Ker}(Y \rightarrow 0)$ . However  $\text{Ker}(Y \rightarrow 0) = Y$ , therefore  $\text{Im}(f) = Y$ . Because  $f$  is strict, we have that  $\text{Coim}(f) = \text{Im}(f)$ , which implies that  $\text{Coim}(f) = Y$ , which completes the proof. For the other direction, suppose that  $f$  is a strict epimorphism. For a start, we will show that  $\text{Im}(f) = \text{Ker}(Y \rightarrow 0)$ . The morphism  $f$  is an epimorphism since it is also a strict epimorphism, therefore  $\text{Im}(f) = Y$ . However  $\text{Ker}(Y \rightarrow 0) = Y$ , hence  $\text{Im}(f) = \text{Ker}(Y \rightarrow 0)$ . To prove that  $f$  is strict, it is enough to prove that  $\text{Coim}(f) = \text{Im}(f)$ . We know that  $\text{Im}(f) = Y$ . However, since  $f$  is a strict epimorphism, then  $\text{Coim}(f) = Y$ , hence  $\text{Im}(f) = Y = \text{Coim}(f)$ , which implies that  $\text{Im}(f) = \text{Coim}(f)$ .

(3) For the forward direction, suppose that (2.3) is strictly exact. Then we have that

$$0 \longrightarrow X \xrightarrow{f} Y$$

is strictly exact, therefore  $f$  is a monomorphism. In addition

$$X \longrightarrow Y \longrightarrow Z$$

is strictly exact, therefore  $f$  is strict and  $\text{Im}(f) = \text{Ker}(f)$ . Since  $f$  is a monomorphism, we have that  $\text{Ker}(f) = 0$ . Therefore we have that  $\text{Coim}(f) = \text{Coker}(\text{Ker}(f)) = \text{Coker}(0 \rightarrow X) = X$ . However  $\text{Coim}(f) = \text{Im}(f)$ , because  $f$  is strict. This implies that  $X = \text{Ker}(g)$ , hence the pair  $(X, f)$  is a kernel of  $g$ .

For the converse, suppose that the pair  $(X, f)$  is a kernel of  $g$ . Then  $f$  is a monomorphism since it is a kernel, therefore  $\text{Ker}(f) = 0$  and  $\text{Im}(f) = X$ . In addition  $\text{Ker}(g) = X$ , hence  $\text{Im}(f) = \text{Ker}(g)$ . It is left to prove that  $f$  is strict. We have that  $\text{Coim}(f) = \text{Coker}(\text{Ker}(f)) = \text{Coker}(0 \rightarrow X)$ . Therefore, from the above we have  $\text{Coim}(f) = \text{Im}(f)$ .

(4) The proof is straightforward by combining part (3) of the proof and its dual. ■

Since in quasi-abelian categories we have two notions of exactness, it is logical to expect to have more than one notion of exact functors. In this section we study these functors. For a start, we focus on left exactness.

**Definition 2.3.11.** Let  $F : \mathcal{E} \rightarrow \mathcal{F}$  be an additive functor.

(1) It is called **left exact** if for every strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

in  $\mathcal{E}$ , the sequence

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'')$$

is a strictly exact sequence in  $\mathcal{F}$ ,

(2) It is called **strongly left exact** if for every strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

in  $\mathcal{E}$ , the sequence

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'') \longrightarrow 0$$

is a strictly exact sequence in  $\mathcal{F}$ ,

(3) It is called **regular** if it maps strict morphisms to strict morphisms, and **regularizing** if it maps any morphism to a strict morphism.

**Remark 2.3.12.** (1) A functor  $F$  is left exact if and only if it preserves kernels of strict morphisms.

(2) A functor  $F$  is strongly left exact if and only if it preserves kernels of morphisms (not necessarily strict epimorphisms).

**Definition 2.3.13.** Let  $F : \mathcal{E} \rightarrow \mathcal{F}$  be an additive functor. Consider the chain complex

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E''$$

and its image from the functor  $F$

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'')$$

We introduce four notions of left exactness depending on the exactness of the above two chain complexes, which are shown in the following table.

$F$	$S$	$F(S)$
LL left exact	strictly exact	strictly exact
LR left exact	strictly exact	strictly coexact
RL left exact	strictly coexact	strictly exact
RR left exact	strictly coexact	strictly coexact

The following proposition gives a characterization of the above functors.

**Proposition 2.3.14.** Let  $\mathcal{E}, \mathcal{F}$  be quasi-abelian categories and  $F : \mathcal{E} \rightarrow \mathcal{F}$  be an additive functor. Then:

- (1)  $F$  is LL left exact if and only if  $F$  is strongly left exact,
- (2)  $F$  is LR left exact if and only if  $F$  is strongly left exact and regularizing,
- (3)  $F$  is RL left exact if and only if  $F$  is left exact,
- (4)  $F$  is RR left exact if and only if  $F$  is left exact and regular.

Analogously, we have the notion of right exact functors.

**Definition 2.3.15.** Let  $F: \mathcal{E} \rightarrow \mathcal{F}$  be an additive functor.

(1) It is called **right exact** if for every strictly coexact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

in  $\mathcal{E}$ , the sequence

$$F(E') \longrightarrow F(E) \longrightarrow F(E'') \longrightarrow 0$$

is strictly (co)exact in  $\mathcal{F}$ ,

(2) It is called **strongly right exact** if for every strictly coexact sequence

$$E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

in  $\mathcal{E}$ , the sequence

$$F(E') \longrightarrow F(E) \longrightarrow F(E'') \longrightarrow 0$$

is a strongly exact sequence in  $\mathcal{F}$ .

**Remark 2.3.16.** (1) A functor  $F$  is right exact if it preserves kernels of strict monomorphisms,

(2) A functor  $F$  is strongly right exact if it preserves cokernels of morphisms (not necessarily strict).

**Definition 2.3.17.** Let  $\mathcal{E}, \mathcal{F}$  be quasi-abelian categories and  $F: \mathcal{E} \rightarrow \mathcal{F}$  be an additive functor.

(1) It is called **exact** if for every strictly (co)exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

in  $\mathcal{E}$ , the sequence

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'') \longrightarrow 0$$

is strictly (co)exact in  $\mathcal{F}$ ,

(2) It is called **strictly exact** if for every strictly exact sequence

$$E' \longrightarrow E \longrightarrow E''$$

in  $\mathcal{E}$ , the sequence

$$F(E') \longrightarrow F(E) \longrightarrow F(E'')$$

is a strictly exact sequence in  $\mathcal{F}$ ,

(3) It is called **strongly exact** if it is strictly exact and strictly coexact.

Lastly, before we move on to the next chapter, we give a characterization of quasi-abelian categories

**Theorem 2.3.18.** *An additive category  $\mathcal{A}$  is quasi-abelian if and only if there exist abelian categories  $\mathcal{A}_1, \mathcal{A}_2$ , and fully faithful embeddings  $\mathcal{A} \subset \mathcal{A}_1$  and  $\mathcal{A} \subset \mathcal{A}_2$  such that:*

- (1) *If  $A \rightarrow E$  is a monomorphism in  $\mathcal{A}_1$  with  $E \in \mathcal{A}$ , then  $A \in \mathcal{A}$  also.*
- (2) *If  $E \rightarrow B$  is an epimorphism in  $\mathcal{A}_2$  with  $E \in \mathcal{A}$ , then  $B \in \mathcal{A}$  also.*

*Proof.* For the proof we refer to [17, Lemma 1.2.34] ■

For more information, details and proofs, we refer to [17].

## BRIDGELAND STABILITY CONDITIONS

This chapter is dedicated to Bridgeland's theory of stability conditions developed in [2]. Similar notions of stability have been introduced from A.King in [8], and from A.Rudakov in [16]. For the purposes of this chapter we will use a less general, relative relative to that in the papers of King and Rudakov, notion of stability in abelian categories. Our main goal is to prove the main Theorem of [2] which is the following

**Theorem 3.0.1.** *Let  $\mathcal{T}$  be a triangulated category. For each connected component  $\Sigma \subset \text{Stab}(\mathcal{T})$  there are a linear subspace  $V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C})$ , with a well-defined linear topology, and a local homeomorphism  $\mathcal{Z} : \Sigma \rightarrow V(\Sigma)$  which maps a stability condition  $(Z, \mathcal{P})$  to its central charge  $Z$ .*

After making some assumptions about the triangulated category  $\mathcal{T}$ , we have the following Corollary

**Theorem 3.0.2.** *Suppose that  $\mathcal{T}$  is numerically finite. For each connected component  $\Sigma \subset \text{Stab}_{\mathcal{N}}(\mathcal{T})$  there are a subspace  $V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C})$  and a local homeomorphism  $\mathcal{Z} : \Sigma \rightarrow V(\Sigma)$  which maps a stability condition to its central charge  $Z$ . In particular  $\Sigma$  is a finite-dimensional complex manifold, with dimension  $n = \text{rank}(\mathcal{N}(\mathcal{T}))$ .*

### 3.1 Stability Functions and the Harder-Narasimhan Filtrations

We begin this section by giving some basic definitions, which are all generalizations of concepts from algebraic geometry, to a categorical context.

**Definition 3.1.1.** *Let  $\mathcal{A}$  be an abelian category. A **stability function** on  $\mathcal{A}$  is a group homomorphism*

$$Z : K_0(\mathcal{A}) \rightarrow \mathbb{C}$$

*such that every non-zero object  $E$  of  $\mathcal{A}$  lies in the upper half of the complex plane  $H$ , where*

$$H = \{re^{i\pi\phi} \mid r > 0 \text{ and } 0 < \phi \leq 1\}$$

**Definition 3.1.2.** Let  $\mathcal{A}$  be an abelian category and  $Z : K(\mathcal{A}) \rightarrow \mathbb{C}$  a stability function on  $\mathcal{A}$ . The **phase** of a non-zero object  $E$  of  $\mathcal{A}$  is defined to be

$$\phi(E) = \frac{1}{\pi} \arg(Z(E)) \in [0, 1]$$

The map  $\phi : \text{Ob}(\mathcal{A}) \rightarrow [0, 1]$  allows one to order the non-zero objects of an abelian  $\mathcal{A}$ .

**Definition 3.1.3.** Let  $\mathcal{A}$  be an abelian category and  $Z : K(\mathcal{A}) \rightarrow \mathbb{C}$  a stability function on  $\mathcal{A}$ . A non-zero object  $E$  of  $\mathcal{A}$  is called **semistable** if for every non-zero subobject of  $A \subset E$  we have

$$\phi(A) \leq \phi(E)$$

**Remark 3.1.4.** One could also define an object to be semistable if for every non-zero quotient  $E \rightarrow B$  we have

$$\phi(E) \leq \phi(B)$$

We will use both of them in what follows.

**Definition 3.1.5.** Let  $\mathcal{A}$  be an abelian category and  $Z : K(\mathcal{A}) \rightarrow \mathbb{C}$  a stability function on  $\mathcal{A}$ . A **Harder-Narasimhan filtration** of a non-zero object  $E$  of  $\mathcal{A}$  is a finite sequence of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

where the factors  $F_j = E_j/E_{j-1}$  are semistable objects and

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$$

A stability function is said to have the **Harder-Narasimhan property** if every non-zero object of  $\mathcal{A}$  has a Harder-Narasimhan filtration.

**Remark 3.1.6.** 1. From now on we will write HN-filtration instead of Harder-Narasimhan filtration.

2. The fact that an object might be semistable or not depends on the stability function.

**Lemma 3.1.7.** Let  $\mathcal{A}$  be an abelian category and  $Z : K(\mathcal{A}) \rightarrow \mathbb{C}$  be a stability function on  $\mathcal{A}$ . The HN filtration of an object, if it exists, is unique up to isomorphism.

*Proof.* We will first show that if  $E, F$  are two semistable objects and  $f : E \rightarrow F$  is a non-zero morphism, then  $\phi(E) \leq \phi(F)$ . Consider the canonical decomposition of the morphism  $f$ :

$$\begin{array}{ccccccc}
K & \xrightarrow{k} & E & \xrightarrow{f} & F & \xrightarrow{c} & C \\
& & \downarrow & & \uparrow & & \\
& & \text{Coim}(f) & \xrightarrow[\cong]{\bar{f}} & \text{Im}(f) & & 
\end{array}$$

The image  $\text{Im}(f)$  is a subobject of  $F$  because the morphism  $\text{Im}(f) \rightarrow F$  is a kernel of  $f$ . Since  $F$  is semistable, we have that  $\phi(\text{Im}(f)) \leq \phi(F)$ . Dually, the coimage  $\text{Coim}(f)$  is a quotient object of  $E$  because the morphism  $E \rightarrow \text{Im}(f)$  is a cokernel of  $f$ . Again, because  $E$  is semistable, we have that  $\phi(E) \leq \phi(\text{Coim}(f))$ . Since  $\text{Coim}(f) \cong \text{Im}(f)$ , we get that  $\phi(\text{Coim}(f)) = \phi(\text{Im}(f))$ . Therefore we have

$$\phi(E) \leq \phi(\text{Coim}(f)) = \phi(\text{Im}(f)) \leq \phi(F) \Rightarrow \phi(E) \leq \phi(F) \quad \blacksquare$$

The next proposition provides some conditions for the H-N filtration to exist. However, before we proceed, we give the definition of a maximally destabilizing quotient, which will help us with the proof.

**Definition 3.1.8.** Let  $\mathcal{A}$  be an abelian category and  $Z : K(\mathcal{A}) \rightarrow \mathbb{C}$  a stability function on  $\mathcal{A}$ . For an object  $E$ , a **maximally destabilizing quotient** (mdq for short) is a non-zero quotient object  $E \twoheadrightarrow B$ , where for every other non-zero quotient  $E \twoheadrightarrow B'$  we have that  $\phi(B') \geq \phi(B)$ . The equality holds only when  $E \twoheadrightarrow B'$  factors through  $E \twoheadrightarrow B$ .

**Proposition 3.1.9.** Let  $\mathcal{A}$  be an abelian category and  $Z : K(\mathcal{A}) \rightarrow \mathbb{C}$  be a stability function on  $\mathcal{A}$  which satisfies the following conditions:

(1) There is no infinite sequence of subobjects

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

such that  $\phi(E_{j+1}) > \phi(E_j)$  for all  $j$ ,

(2) There is no infinite sequence of quotient objects

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots$$

where  $\phi(E_j) > \phi(E_{j+1})$  for all  $j$ .

*Proof.* We will first show the existence of some auxiliary objects. Let  $E$  be a non-zero object of  $\mathcal{A}$ . Then  $E$  will either be semistable or not. If  $E$  is not semistable, by definition, there exists a non-zero subobject  $E_1 \subset E$ , such that  $\phi(E_1) > \phi(E)$ . Again, the object  $E_1$  will either be semistable or not. If  $E_1$  is not semistable, then by definition, there exists a non-zero subobject  $E_2 \subset E_1$ , such that  $\phi(E_2) > \phi(E_1)$ . By continuing this process, we get the following sequence of subobjects

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1 \subset E$$

From the first condition of the proposition, the above sequence has to stop at some  $j$ . By letting  $A := E_j$ , we proved that for every non-zero object  $E$  of  $\mathcal{A}$ , there exists a non-zero semistable subobject of  $A \subset E$ , such that  $\phi(A) \geq \phi(E)$ , where the equality holds only when  $E$  is semistable, and then  $A = E$ . Dually, by using the second condition, we have that for every non-zero object  $E$  of  $\mathcal{A}$ , there exists a non-zero semistable quotient object  $E \twoheadrightarrow B$ , such that  $\phi(E) \geq \phi(B)$ , and again the equality is true only when  $E$  is semistable, where again  $E = B$ .

It will be crucial to prove that mdqs always exist. However, before we do that, we will first prove that for every non-zero object  $E$  in  $\mathcal{A}$  where  $E \twoheadrightarrow B$  is a mdq, then  $B$  is semistable and  $\phi(E) \geq \phi(B)$ . Let  $B \twoheadrightarrow B'$  be a quotient object of  $B$ . Then  $B'$  is also a quotient object of  $E$ , and since  $B$  is a mdq, we have that  $\phi(B) \leq \phi(B')$ , which proves that  $B$  is semistable. In addition, the identity map  $E \rightarrow E$  is a mdq, hence by definition, since  $E \twoheadrightarrow B$  is a quotient object of  $E$ , we have that  $\phi(E) \geq \phi(B)$ .

We now prove the existence of mdqs. If  $E$  is semistable, then the identity map is a mdq. Indeed, if  $E \twoheadrightarrow B$  is a quotient object of  $E$ , then we have that  $\phi(E) \leq \phi(B)$ . If  $E$  is not semistable, then we have the short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow E' \longrightarrow 0$$

where  $A \subset E$  is a non-zero subobject of  $E$  which is semistable, and we have that  $\phi(A) > \phi(E) > \phi(E')$ . The inequalities are strict since  $E$  is not semistable.

Claim: If  $E' \twoheadrightarrow B$  is a mdq of  $E'$ , then the induced quotient object  $E \twoheadrightarrow B$  is a mdq of  $E$ .

Proof: Let  $E \twoheadrightarrow B'$  be a quotient of  $E$  where  $B'$  is semistable. We have to prove that  $\phi(B') \geq \phi(B)$ . Suppose, for contradiction, that  $\phi(B') < \phi(B)$ . Since  $E' \twoheadrightarrow B$  is a mdq, we have that  $\phi(E') \geq \phi(B)$ . By combining the above inequality with the last one, we have that

$$\phi(A) > \phi(E) > \phi(E') \geq \phi(B) \geq \phi(B') \Rightarrow \phi(A) > \phi(B')$$

Since  $A, B'$  are semistable, there is no morphism  $A \rightarrow B'$ . This means that the morphism  $E \twoheadrightarrow B'$  factors through  $E'$ , hence  $\phi(E') \leq \phi(B') \leq \phi(B) \Rightarrow \phi(E') \leq \phi(B)$ , which is a contradiction.

By doing the same thing but for  $E'$  now, we end up with an infinite sequence of quotient objects of  $E$ , which has to stop from the second condition of the proposition. This proves that mdqs always exist.

Let now  $E$  be a non-zero object of  $\mathcal{A}$ . If  $E$  is semistable, trivially  $0 \subset E$  is a H-N filtration. If  $E$  is not semistable, consider the short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow B \longrightarrow 0$$

where  $E \twoheadrightarrow B$  is a mdq and  $\phi(E') > \phi(E)$ , where the inequality is strict because  $E$  is not semistable. Suppose the  $E' \twoheadrightarrow B'$  is a mdq. Considering the push-out of  $E' \rightarrow E$  along  $E' \twoheadrightarrow B'$ , we have the

following exact commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & B' \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & \\
 & & & & B & \xlongequal{\quad} & B & \\
 & & & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 & 
 \end{array} \tag{3.1}$$

Since  $E \twoheadrightarrow B$  is a mdq and  $E \twoheadrightarrow Q$  is a quotient object of  $E$ , we have that  $\phi(Q) > \phi(B)$ . In addition, since  $\phi(B') \geq \phi(Q)$ , we have that  $\phi(B') > \phi(B)$ . By repeating this process for the object  $E'$ , we have the infinite sequence of subobjects

$$\dots \subset E^i \subset E^{i-1} \subset \dots \subset E^1 \subset E^0 = E$$

where  $\phi(E^i) > \phi(E^{i-1})$ . The above sequence, from the first condition, has to terminate, and since the factors  $E^i/E^{i-1}$  are semistable with decreasing phase, we obtain a H-N filtration. ■

## 3.2 Slicings

In this section, we give the definition of a slicing which is a generalization of a  $t$ -structure. However, before we do that, and to be compatible with Bridgeland's notation, we present some known definitions from previous sections of the thesis.

**Definition 3.2.1.** Let  $\mathcal{T}$  be a triangulated category. A  **$t$ -structure** on  $\mathcal{T}$  is a full subcategory  $\mathcal{F}$  of  $\mathcal{T}$  such that:

1.  $\mathcal{F}[1] \subset \mathcal{F}$ ,
2. for every  $E \in \mathcal{T}$ , there exists a distinguished triangle

$$F \longrightarrow E \longrightarrow G \longrightarrow F[1]$$

such that  $F \in \mathcal{F}$  and  $G \in \mathcal{F}^\perp$ , where  $\mathcal{F}^\perp$  is defined as

$$\mathcal{F}^\perp = \{G \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(F, G) = 0, \forall F \in \mathcal{F}\}$$

**Example 3.2.2.** With respect to the notation of Definition 3.2.1, if  $\mathcal{A}$  is an abelian category and  $D^b(\mathcal{A})$  is the bounded derived category of  $\mathcal{A}$ , then the standard  $t$ -structure on  $D^b(\mathcal{A})$  is the following:

$$\mathcal{F} = \{X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0, i > 0\}$$

**Remark 3.2.3.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{F}$  a  $t$ -structure on  $\mathcal{T}$ . If the  $t$ -structure is bounded, then  $\mathcal{F}$  is purely determined by its heart, as we have seen. In fact, it is the extension-closed subcategory which is generated by  $\mathcal{A}[j]$ , for all  $j \in \mathbb{Z}_+$ , where  $\mathcal{A}$  is the heart of the bounded  $t$ -structure  $\mathcal{F}$ . The following lemma gives a characterization of bounded  $t$ -structures from their hearts.

**Lemma 3.2.4.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{A}$  an additive subcategory of  $\mathcal{T}$ . Then  $\mathcal{A}$  is the heart of a bounded  $t$ -structure if and only if:

- (1) if  $k_1 > k_2$ , where  $k_1, k_2 \in \mathbb{Z}$ , then  $\text{Hom}_{\mathcal{T}}(A[k_1], A[k_2]) = 0$ , for every  $A, B \in \mathcal{A}$ ,
- (2) for every non-zero object  $E$  of  $\mathcal{T}$ , there exists a finite sequence of integers

$$k_1 > k_2 > \cdots > k_n$$

and a collection of distinguished triangles

$$\begin{array}{ccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\
 & & \swarrow [1] & & \swarrow [1] & & & & \swarrow [1] & & \swarrow \\
 & & A_1 & & A_2 & & & & A_n & & 
 \end{array}$$

where  $A_j \in \mathcal{A}[k_j]$ , for all  $j$ .

*Proof.* ( $\Rightarrow$ ): Let  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  be a bounded  $t$ -structure on  $\mathcal{T}$  with heart  $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ . Let  $k_1, k_2 \in \mathbb{Z}$  be such that  $k_1 > k_2$ , and let  $A, B \in \mathcal{A}$ . Then we have that  $A[k_1] \in \mathcal{T}^{\leq -k_1}$  and  $B[k_2] \in \mathcal{T}^{\geq -k_2}$ . Since  $k_1 > k_2$ , we have  $-k_1 < -k_2$ , and then, since  $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq n}, \mathcal{T}^{\geq m}) = 0$ ,  $\forall n < m$ , it follows that  $\text{Hom}_{\mathcal{T}}(A[k_1], B[k_2]) = 0$ . Now, since the  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is bounded, the number of non-zero cohomology objects  $H^k(E) \in \mathcal{A}$  is finite,  $\forall E \in \mathcal{T}$ . Therefore, for every non-zero  $E \in \mathcal{T}$ , there exist integers  $m < M$  such that  $H^i(E) = 0$ ,  $\forall i < m$  and  $\forall i > M$ . To construct the filtration we use the truncation functors  $\tau_{\leq n}$ . Let the non-zero cohomology objects of  $E$  appear in the indices  $k_1, k_2, \dots, k_n$ , where  $k_n < k_{n-1} < \cdots < k_1$ . We define inductively objects of  $\mathcal{T}$  as follows

$$E_0 = 0, E_1 = \tau_{\leq -k_1}(E), E_2 = \tau_{\leq -k_2}(E), \dots, E_n = \tau_{\leq -k_n}(E)$$

Since  $\forall l \in \mathbb{Z}$  we have a natural morphism  $\tau_{\leq -k_{l-1}} \rightarrow \tau_{\leq -k_l}$ , then when evaluated at  $E$ , gives rise to a sequence of distinguished triangles

$$E_{l-1} \longrightarrow E_l \longrightarrow A_l \longrightarrow E_{l-1}[1]$$

By construction the cone  $A_l$  of  $E_{l-1} \rightarrow E_l$  coincides with the  $k_l$ -cohomology object of  $E$  shifted by  $[-k_l]$ , that is  $A_l \cong H^{k_l}(E)[-k_l]$ ,  $\forall l$ . By definition of the heart  $\mathcal{A}$  of the  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ , we have that  $H^{k_l}(E) \in \mathcal{A}$ , hence each  $A_l \in \mathcal{A}[-k_l]$ .

( $\Leftarrow$ ): Suppose that (1) and (2) hold. We define full subcategories of  $\mathcal{T}$  as follows:

$$\mathcal{T}^{\leq 0} = \{E \in \mathcal{T} \mid k_j \geq 0, \forall j\} \quad \text{and} \quad \mathcal{T}^{\geq 0} = \{E \in \mathcal{T} \mid k_j \leq 0, \forall j\}$$

Then we have obviously that  $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$ . Let  $X \in \mathcal{T}^{\leq 0}$  and  $Y \in \mathcal{T}^{\geq 1}$ . Then  $X$  is constructed from the shifted copy  $\mathcal{A}[l]$  of the heart, where  $l \geq 0$  and  $Y$  is constructed from the shifted cope  $\mathcal{A}[r]$ , where  $r \leq -1$ , so  $l > r$ . Since, by (1), we have that  $\text{Hom}_{\mathcal{T}}(\mathcal{A}[l], \mathcal{A}[r]) = 0$ , since  $l > r$ . Applying induction on the length of the filtration (2), and using the above observation, it follows that  $\text{Hom}_{\mathcal{T}}(X, Y) = 0$ . Hence  $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) = 0$ . Finally let  $E \in \mathcal{T}$ , and consider the filtration given in (2). We set  $X := E_i$ , where  $k_i$  is the smallest positive integer in the finite set  $\{k_1, k_2, \dots, k_n\}$ . Then we have non-zero morphism  $X \rightarrow E$  which induces a distinguished triangle

$$X \longrightarrow E \longrightarrow Y \longrightarrow X[1]$$

Observe that by construction  $X \in \mathcal{T}^{\leq 0}$ , and using the filtration in (2) it follows that  $Y \in \mathcal{T}^{\geq 1}$ . Hence  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is a  $t$ -structure, which is clearly bounded, since the filtration (2) is finite. By construction, it follows that  $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ .  $\blacksquare$

The above Lemma gives the motivation for the definition of a slicing.

**Definition 3.2.5.** Let  $\mathcal{T}$  be a triangulated category. A **slicing**  $\mathcal{P}$  of  $\mathcal{T}$  is a collection of full additive subcategories  $\mathcal{P}(\phi)$  of  $\mathcal{T}$ , for every  $\phi \in \mathbb{R}$ , such that:

- (s1)  $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ , for every  $\phi \in \mathbb{R}$ ,
- (s2)  $\text{Hom}_{\mathcal{T}}(A_1, A_2) = 0$ , for ever  $\phi_1 > \phi_2$  and  $A_j \in \mathcal{P}(\phi_j)$ ,
- (s3) for every non zero object  $E$  of  $\mathcal{T}$ , there exists a finite sequence of real numbers

$$\phi_1 > \phi_2 > \dots > \phi_n$$

and a collection of distinguished triangles

$$\begin{array}{ccccccc}
0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots \longrightarrow E_{n-1} & \longrightarrow & E_n = E \\
& & \swarrow & & \swarrow & & \swarrow & & \swarrow \\
& & A_1 & & A_2 & & & & A_n
\end{array}$$

(The above diagram is a sequence of triangles where each triangle has a horizontal top edge and a diagonal bottom edge. The top edge of the  $i$ -th triangle is  $E_{i-1} \rightarrow E_i$  and the bottom edge is  $A_i$ . The diagonal edge is  $E_{i-1} \rightarrow A_i$ . The label  $[1]$  is placed near the diagonal edge of each triangle.)

where  $A_j \in \mathcal{P}(\phi_j)$ , for all  $j$ . This condition is called the Harder-Narasimhan filtration (HN-filtration).

**Lemma 3.2.6.** *The HN-filtration in Definition 3.2.5 is unique.*

*Proof.* Note that,  $\forall A \in \mathcal{P}( > \phi )$  and  $\forall B \in \mathcal{P}( \leq \phi )$ , we have that  $\text{Hom}_{\mathcal{T}}(A, B) = 0$ . This follows from the orthogonality condition of the slicing and induction on the length of the HN-filtration. Similarly we have  $\text{Hom}_{\mathcal{T}}(A, B[-1]) = 0$ . Assume now that the non-zero object  $E$  admits another filtration with corresponding phases  $\phi'_i$  and corresponding semistable factors  $A'_i$ . Denote by  $\phi_*$  the maximum phase of the first HN-filtration and by  $\phi'_*$  the maximum phase of the second filtration. We assume that  $\phi_1 > \phi'_1$ . Then the existence of the second filtration shows that  $E \in \mathcal{P}( \leq \phi'_1 )$ . Since there exists a morphism  $E_1 \rightarrow E$  where  $E_1 \in \mathcal{P}( \phi )$ , the orthogonality induction shows that  $\text{Hom}_{\mathcal{T}}(E_1, E) = 0$  because  $\phi_1 > \phi'_1$ . This is a contradiction since there exists a non-zero morphism  $E_1 \rightarrow E$ . Therefore, using also the dual argument interchanging the roles of  $\phi_1$  and  $\phi'_1$ , we have that  $\phi_1 = \phi'_1$ . At this stage we have the triangles

$$E_1 \xrightarrow{u} E \xrightarrow{v} F_1 \longrightarrow E_1[1]$$

where  $E_1 \in \mathcal{P}(\phi_1)$  and  $F_1 \in \mathcal{P}( < \phi_1 )$ , and

$$E'_1 \xrightarrow{u'} E \xrightarrow{v'} F'_1 \longrightarrow E'_1[1]$$

where  $E'_1 \in \mathcal{P}(\phi_1)$  and  $F'_1 \in \mathcal{P}( < \phi_1 )$ . Since  $\text{Hom}_{\mathcal{T}}(E_1, F'_1) = 0 = \text{Hom}_{\mathcal{T}}(E_1, F'_1[-1])$ , the morphism  $E_1 \rightarrow E$  factors uniquely through  $E'_1$ . Dually, interchanging the roles of the  $E_1, F'_1$ , the morphism  $E'_1 \rightarrow E$  factors uniquely through  $E$ .

Claim: We have that  $E_1 \cong E'_1$  via a unique isomorphism.

Proof: By the above remarks there exists a morphism  $\alpha : E_1 \rightarrow E'_1$  such that  $u' \circ \alpha = u$ , and  $\alpha$  the unique morphism factorizing  $u$ . Dually there exists a unique morphism  $\beta : E'_1 \rightarrow E_1$ , such that  $u \circ \beta = u'$ . Since  $u \circ (\beta \circ \alpha) = u$ , uniqueness gives  $\beta \circ \alpha = \text{Id}$ , and dually  $\alpha \circ \beta = \text{Id}$ . Hence  $\alpha : E_1 \xrightarrow{\cong} E'_1$  is the unique isomorphism such that  $u' \circ \alpha = u$ .

Since  $E_1 \cong E'_1$  in a compatible way, by the above triangles, we also have  $F_1 \cong F'_1$ . The object  $F$  has a HN-filtration of length  $n - 1$ . Hence the above analysis can be repeated and induction on  $n$  gives that  $\phi_i = \phi'_i$  and  $A_i \cong A'_i, \forall i$ . Hence the HN-filtrations give the same phases and isomorphic semistable factors.  $\blacksquare$

**Definition 3.2.7.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{P}$  a slicing of  $\mathcal{T}$ . For every non-zero object  $E$  of  $\mathcal{T}$  we define*

$$\phi_{\mathcal{P}}^+(E) = \phi_1 \quad \text{and} \quad \phi_{\mathcal{P}}^-(E) = \phi_n$$

where  $\phi_1, \phi_n$  come from (s3) of the definition of a slicing.

If the slicing  $\mathcal{P}$  is clear from the context, we will write  $\phi^+(E)$  and  $\phi^-(E)$ , instead of  $\phi_{\mathcal{P}}^+(E)$  and  $\phi_{\mathcal{P}}^-(E)$  respectively.

**Remark 3.2.8.** Clearly, from the definition of a slicing, we have that

$$\phi^-(E) \leq \phi^+(E)$$

The equality holds when  $E \in \mathcal{P}(\phi)$ , for some  $\phi \in \mathbb{R}$ .

**Definition 3.2.9.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{P}$  a slicing of  $\mathcal{T}$ . For every interval  $I$  of  $\mathbb{R}$ , we define  $\mathcal{P}(I)$  to be the extension-closed subcategory of  $\mathcal{T}$  which is generated by the subcategories  $\mathcal{P}(\phi)$ , for every  $\phi \in I$ .

**Example 3.2.10.** If  $I = (a, b)$ , then  $\mathcal{P}((a, b))$  consists of the zero objects of  $\mathcal{T}$  together with the non-zero objects  $E$  such that

$$a < \phi^-(E) \leq \phi^+(E) < b$$

**Lemma 3.2.11.** Let  $\mathcal{T}$  be a triangulated category, a slicing  $\mathcal{P}$  of  $\mathcal{T}$  and  $I$  an interval with length at most one. If

$$A \longrightarrow E \longrightarrow B \longrightarrow A[1]$$

is a distinguished triangle where  $A, E, B \in \mathcal{P}(I)$ , and are non-zero, then we have the following inequalities:

$$\phi^+(A) \leq \phi^+(E) \quad \text{and} \quad \phi^-(E) \leq \phi^-(B)$$

*Proof.* We will prove only the first inequality, since the second one is proved dually. We can assume, without the loss of generality, that  $I = [t, t + 1]$ , for some  $t \in \mathbb{R}$ . For  $A$ , from the definition of a slicing, there exists a sequence of distinguished triangles of the form

$$\begin{array}{ccccccc}
 0 = A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_{n-1} & \longrightarrow & A_n = A \\
 & & \swarrow & & \swarrow & & & & \swarrow & & \swarrow \\
 & & C_1 & & C_2 & & & & C_n & & 
 \end{array}$$

where  $C_j \in \mathcal{P}(\phi_j)$ , for all  $j$ . Since we have a distinguished triangle

$$0 \longrightarrow A_1 \longrightarrow C_1 \longrightarrow 0$$

it follows that the map  $A_1 \rightarrow C_1$  is an isomorphism. Thus we have a morphism  $f : C_1 \rightarrow A$  which is the composition of the morphisms

$$C_1 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A$$

and is non-zero. Suppose, for contradiction, that  $\phi_1 > \phi^+(E)$ . Then from (s2) of Definition 3.2.5 we have that  $\text{Hom}_{\mathcal{T}}(C_1, E) = 0$ . By applying the functor  $\text{Hom}_{\mathcal{T}}(C_1, -)$  to the distinguished triangle

$$A \longrightarrow E \longrightarrow B \longrightarrow A[1]$$

we get the long exact cohomology sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(C_1, B[-1]) \longrightarrow \text{Hom}_{\mathcal{T}}(C_1, A) \longrightarrow \text{Hom}_{\mathcal{T}}(C_1, E) \longrightarrow \cdots$$

Since  $\text{Hom}_{\mathcal{T}}(C_1, E) = 0$  the group homomorphism  $\text{Hom}_{\mathcal{T}}(C_1, B[-1]) \rightarrow \text{Hom}_{\mathcal{T}}(C_1, A)$  is an epimorphism, hence the morphism  $f$  factors through  $B[-1]$ . In addition, since  $B \in \mathcal{P}(I) = \mathcal{P}([t, t+1])$  we have that  $B[-1] \in \mathcal{P}([t, t-1])$  from (s1). Since now there exists a non-zero morphism  $C_1 \rightarrow B[-1]$ , from (s2) we can not have  $\phi_1 > t$ , so  $\phi_1 \leq t$ . Therefore we have shown that  $\phi^+(E) < \phi_1 \leq t$  which implies that  $\phi^+(E) < t$ . This is a contradiction since  $\phi^+(E) \geq t$ .  $\blacksquare$

**Lemma 3.2.12.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{P}$  a slicing of  $\mathcal{T}$ . Then  $\mathcal{P}( > \phi )$  is a bounded  $t$ -structure in  $\mathcal{T}$ .*

*Proof.* We set  $\mathcal{T}^{\leq 0} := \mathcal{P}( > \psi )$  and  $\mathcal{T}^{\geq 1} := \mathcal{P}( \leq \psi )$ , so  $\mathcal{T}^{\leq 0}$  is the full subcategory of  $\mathcal{T}$  generated by extensions of objects in  $\mathcal{P}(\psi)$  for any  $\psi > \phi$ , and  $\mathcal{T}^{\geq 1}$  is the full subcategory of  $\mathcal{T}$  generated by extensions of objects in  $\mathcal{P}(\psi)$  for any  $\psi \leq \phi$ .

- Let  $A \in \mathcal{T}^{\leq 0}$  and  $B \in \mathcal{T}^{\geq 1}$ . Then, by definition  $A$  admits a HN-filtration with phases  $\psi_i > \phi$ , and  $B$  admits an HN-filtration with phases  $\psi_j \leq \phi$ . Since  $\psi_i > \psi_j$ , we have that  $\text{Hom}_{\mathcal{T}}(\mathcal{P}(\psi_i), \mathcal{P}(\psi_j)) = 0$ . Since  $A$  is constructed by objects having phases which are greater than those of  $B$ , by induction on the length of the filtrations it follows that  $\text{Hom}_{\mathcal{T}}(A, B) = 0$ . Hence  $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$ .
- By definition of the slicing  $\mathcal{P}$ , we have that  $\mathcal{P}(\psi)[1] = \mathcal{P}(\psi + 1)$ . Then if  $A \in \mathcal{P}( > \phi ) = \mathcal{T}^{\leq 0}$ , the object  $A[1]$  is an extension of objects of  $\mathcal{P}(\psi + 1)$ . Since  $\psi + 1 > \psi > \phi$ , it follows that  $A[1] \in \mathcal{P}( > \phi ) = \mathcal{T}^{\leq 0}$ . Hence  $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$ .

- Let  $E \in \mathcal{T}$ . Then  $E$  has an HN-filtration as in Lemma 3.2.6. We set  $k$  to be the largest index such that  $\phi_k > \phi$ , and if no such index exists we set  $E_k = 0$ . Then  $E_k$  is an iterated extension of the semistable factors  $A_1, A_2, \dots, A_k$ , all of which have phases greater than  $\phi$ . This means that  $E_k \in \mathcal{P}( > \phi ) = \mathcal{T}^{\leq 0}$ . Considering the triangle with base the induced morphism  $E_k \rightarrow E$

$$E_k \longrightarrow E \longrightarrow B \longrightarrow E_k[1]$$

it follows that  $B$  is an iterated extension of the semistable factors  $A_{k+1}, \dots, A_n$ . By the definition of  $k$ , we have that  $\phi \geq \phi_{k+1}$  and then all phases in the HN-filtration of  $B$  are less or equal than  $\phi$ . Therefore  $B \in \mathcal{P}( \leq \phi ) = \mathcal{T}^{\geq 1}$ . Setting  $A = E_k$ , we have a distinguished triangle

$$A \longrightarrow E \longrightarrow B \longrightarrow A[1]$$

where  $A \in \mathcal{T}^{\leq 0}$  and  $B \in \mathcal{T}^{\geq 1}$ . Hence  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is a  $t$ -structure. Observe that its heart is  $\mathcal{A} = \mathcal{P}((\phi, \phi + 1])$ .

Finally, we have to show that  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is bounded. It suffices to show that, for some  $n \in \mathbb{Z}$ , we have that  $E \in \mathcal{T}^{\leq n} \cap \mathcal{T}^{\geq n}$ , for any object  $E$ . Consider the HN-filtration of  $E$ , where  $\phi^+ = \phi_1 > \phi_2 > \dots > \phi_n = \phi^-$ . Then  $\mathcal{T}^{\leq n} = \mathcal{P}( > \phi - n )$  and  $\mathcal{T}^{\geq -n} = \mathcal{P}( \leq \phi + n + 1 )$ . Then for any  $E \in \mathcal{T}$ , we choose a sufficiently large integer  $n$  such that  $\phi - n < \phi^-(E)$  and  $\phi + n + 1 > \phi^+(E)$ . This choice of  $n$  implies that  $E \in \mathcal{P}( > \phi - n ) = \mathcal{T}^{\leq n}$  and  $E \in \mathcal{P}( \leq \phi + n + 1 ) = \mathcal{T}^{\geq -n}$ . Hence the  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is bounded. ■

**Remark 3.2.13.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{P}$  a slicing of  $\mathcal{T}$ . Then for every  $\phi \in \mathbb{R}$ , the pairs  $(\mathcal{P}( > \phi ), \mathcal{P}( \leq \phi ))$  and  $(\mathcal{P}( \geq \phi ), \mathcal{P}( < \phi ))$  are orthogonal, with respect to Hom, subcategories of  $\mathcal{T}$ . In addition, the categories  $\mathcal{P}( > \phi )$  and  $\mathcal{P}( \geq \phi )$ , and one can prove that they define  $t$ -structures in  $\mathcal{T}$ . Note that if for real numbers  $\phi, \psi$  we have  $\phi \geq \psi$ , then  $\mathcal{P}( > \phi ) \subset \mathcal{P}( > \psi )$  and  $\mathcal{P}( \geq \phi ) \subset \mathcal{P}( \geq \psi )$ . The hearts of the  $t$ -structures  $\mathcal{P}( > \phi )$  and  $\mathcal{P}( \geq \phi )$  are, by definition, the subcategories  $\mathcal{P}((\phi, \phi + 1])$  and  $\mathcal{P}([\phi, \phi + 1))$  respectively, for  $\phi \in \mathbb{R}$ . Thus we have the definition of the heart of a slicing  $\mathcal{P}$  of  $\mathcal{T}$ .

**Definition 3.2.14.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{P}$  a slicing of  $\mathcal{T}$ . We define the **heart** of the slicing  $\mathcal{P}$  to be the abelian subcategory

$$\mathcal{P}((0, 1])$$

If we have a slicing  $\mathcal{P}$ , the next Lemma characterizes the subcategories  $\mathcal{P}(I)$ , where the length of the interval  $I \subset \mathbb{R}$  is less than 1.

**Lemma 3.2.15.** Let  $\mathcal{P}$  be a slicing of a triangulated category  $\mathcal{T}$ . For any interval  $I \subset \mathbb{R}$  of length  $< 1$ , the full subcategory  $\mathcal{P}(I) \subset \mathcal{T}$  is quasi-abelian. The strict short exact sequences in  $\mathcal{P}(I)$  are in one-to-one correspondence with the triangles in  $\mathcal{T}$  all of whose vertices are objects of  $\mathcal{P}(I)$

*Proof.* Assume for definiteness that  $I = (a, b)$  with  $0 < b - a < 1$ . The other cases are proved with similar arguments. By applying Theorem 2.3.18 to the embeddings  $\mathcal{P}((a, b)) \subset \mathcal{P}((a, a + 1])$  and  $\mathcal{P}((a, b)) \subset \mathcal{P}([b - 1, b))$ , and Lemma 3.2.11  $\blacksquare$

Using the notion of a strict subobject in a quasi-abelian category, we introduce the notion of a skewed stability function

**Definition 3.2.16.** A **skewed stability function** on a quasi-abelian category  $\mathcal{A}$  is a group homomorphism  $Z : K_0(\mathcal{A}) \rightarrow \mathbb{C}$  such that there is a strict half-plane

$$H_a = \{re^{i\pi\phi} \mid r > 0 \text{ and } a < \phi \leq a + 1\} \subset \mathbb{C}$$

defined by some  $a \in \mathbb{R}$ , such that  $Z(E) \in H_a$  for all non-zero objects  $E \in \mathcal{A}$ .

### 3.3 Stability Conditions

In this section we give the definition of a Bridgeland Stability Condition on a triangulated category and show its relation to  $t$ -structures.

**Definition 3.3.1.** Let  $\mathcal{T}$  be a triangulated category. A **stability condition** on  $\mathcal{T}$  is a pair  $\sigma = (Z, \mathcal{P})$ , where

$$Z : K_0(\mathcal{T}) \rightarrow \mathbb{C}$$

is a group homomorphism and  $\mathcal{P}$  is a slicing of  $\mathcal{T}$ , such that for every non-zero object  $E$  of  $\mathcal{T}$  then

$$Z(E) = m(E)e^{i\pi\phi}$$

for some  $m(E) \in \mathbb{R}_+$ .

The group homomorphism  $Z$  is called the **central charge** of the stability condition.

**Lemma 3.3.2.** Let  $\mathcal{T}$  be a triangulated category and  $\sigma = (Z, \mathcal{P})$  a stability condition on  $\mathcal{T}$ . Then for every  $\phi \in \mathbb{R}$  the subcategory  $\mathcal{P}(\phi)$  is an abelian category.

*Proof.* We use that  $\forall \phi \in \mathbb{R}$ , the full subcategory  $\mathcal{P}((\phi, \phi + 1])$  is an abelian category, being the heart of a  $t$ -structure on  $\mathcal{T}$ . In particular  $\mathcal{P}((\phi - 1, \phi])$  is an abelian category. Observe that  $\mathcal{P}(\phi) \subset \mathcal{P}((\phi - 1, \phi]) := \mathcal{A}$ , and it suffices to show that  $\mathcal{P}(\phi)$  is closed under kernels and cokernels in  $\mathcal{A}$ . [We use that any full additive subcategory of an abelian category which is closed under kernels and cokernels is abelian]. So let  $f : A \rightarrow B$  be a morphism between objects of  $\mathcal{P}(\phi) \subset \mathcal{A}$ . We have to show that the kernel  $K = \text{Ker}(f)$  and the cokernel  $C = \text{Coker}(f)$  in  $\mathcal{A}$ , belong to  $\mathcal{P}(\phi)$ .

We use the construction of the kernel and cokernel of a morphism in the heart of a general  $t$ -structure. Recall that every non-zero  $E \in \mathcal{A}$  has a maximum phase  $\phi^+(E)$  and a minimum phase  $\phi^-(E)$  uniquely determined by the HN-filtration. Since  $\mathcal{A} = \mathcal{P}((\phi - 1, \phi])$ , we also have that  $\phi - 1 < \phi^-(E) \leq \phi^+(E) \leq \phi$ . Since  $A, B \in \mathcal{P}(\phi)$ , we have that  $\phi^+(A) = \phi^-(A) = \phi$  and  $\phi^+(B) = \phi^-(B) = \phi$ . We have to show that  $\phi^+(K) = \phi^-(K) = \phi$  (if  $K \neq 0$ ). Since  $\in \mathcal{P}(\phi)$ , we assume from now on that  $K \neq 0$ . Consider the canonical analysis of the morphism  $f : A \rightarrow B$  in  $\mathcal{A}$

$$0 \longrightarrow K \xrightarrow{k} A \xrightarrow{f} B \xrightarrow{p} C \longrightarrow 0$$

$$\begin{array}{c} \searrow \quad \nearrow \\ \quad I \end{array}$$

where  $K = \text{Ker}(f)$ ,  $C = \text{Coker}(f)$ , and  $I = \text{Im}(f)$ , all computed in  $\mathcal{A}$ . Since  $K \in \mathcal{A}$ , we have that  $\phi^+(K)$  satisfies  $\phi^+(K) \leq \phi$ . Indeed, if  $\phi^+(K) > \phi$ , then using that  $\text{Hom}(\mathcal{P}(\psi), \mathcal{P}(\phi)) = 0$ , for  $\psi > \phi$ , it would follow that the inclusion  $K \hookrightarrow A$  is zero. Equivalently  $K = 0$  and this is not the case.

Claim:  $\phi^-(K) \geq \phi$ .

Proof of claim: Suppose that  $\phi^-(K) < \phi$ , and let  $K \twoheadrightarrow S$  be the last factor in the HN-filtration of  $K$ ,  $S$  being a semistable object of phase  $\phi^-(K) = \phi < \phi$ . The phase of  $A \in \mathcal{P}(\phi) \subseteq \mathcal{A}$  is  $\phi$ . As above the image  $I \hookrightarrow B$  of  $f$  has the property that  $\phi^+(I) \leq \phi$  and  $\phi^-(I) \geq \phi$ . Since

$$0 \longrightarrow K \longrightarrow A \longrightarrow I \longrightarrow 0$$

is a short exact sequence in  $\mathcal{A}$ , the central charge  $Z$  satisfies  $Z(A) = Z(K) + Z(I)$ . If any semistable factor of the HN-filtration of  $K$  or of  $I$ , had phase less than  $\phi$ , the  $Z(K)$  and  $Z(I)$  would necessarily lie in the region of the complex plane which corresponds to phases  $\leq \phi$ . Then in order for their sum  $Z(A)$  to lie exactly on the ray with phase  $\phi$ , both  $Z(K)$  and  $Z(I)$  either should be zero or must lie on the same ray of the phase  $\phi$ . As a consequence  $Z(K)$  must have phase  $\phi$ , which implies that  $\phi^-(K) \geq \phi$  and therefore  $\phi^+(K) = \phi = \phi^-(K)$ , i.e  $K \in \mathcal{P}(\phi)$ .

A dual argument, working with the short exact sequence

$$0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$$

in  $\mathcal{A}$ , shows that  $C \in \mathcal{P}(\phi)$ . We infer that  $\mathcal{P}(\phi)$  is closed under kernels and cokernels in the abelian category  $\mathcal{A}$ , hence  $\mathcal{P}(\phi)$  is abelian.  $\blacksquare$

**Definition 3.3.3.** Let  $\mathcal{T}$  be a triangulated category and  $\sigma = (Z, \mathcal{P})$  a stability condition on  $\mathcal{T}$ . For every  $\phi \in \mathbb{Z}$ :

- (1) the non-zero objects of  $\mathcal{P}(\phi)$  are called **semistable** in  $\sigma$  of phase  $\phi$ ,
- (2) the simple objects of  $\mathcal{P}(\phi)$  are said to be **stable**.

We continue by giving further definitions concerning a stability condition.

**Definition 3.3.4.** Let  $\mathcal{T}$  be a triangulated category and  $\sigma = (Z, \mathcal{P})$  a stability condition on  $\mathcal{T}$ . Then, for every non-zero object  $E$  of  $\mathcal{T}$  we define:

(1) the **semistable factors** of  $E$ , to be the objects  $A_j$  from (s3) of Definition 3.2.5,

(2) the **mass** of  $E$  to be the real number  $m_\sigma(E) = \sum_i |Z(A_i)|$

**Remark 3.3.5.** (1) The definition of the semistable objects is good, by Lemma 3.2.6 the decomposition of (s3) in the definition of a slicing is unique up to isomorphism. We define  $\phi_\sigma^\pm(E) := \phi_{\mathcal{P}}^\pm(E)$ . Then we have  $\phi_\sigma^-(E) \leq \phi_\sigma^+(E)$  and the equality holds whenever  $E$  is semistable in  $\sigma$ . We will often drop the stability condition  $\sigma$  and write just  $\phi^\pm(E)$  if it is clear from the context.

(2) From the triangle inequality, one has that  $m_\sigma(E) = \sum_i |Z(A_i)| \geq |Z(E)|$ . Again, if the stability condition  $\sigma$  is clear from the context, we will usually drop it and just write  $m(E)$ .

The following proposition gives the relation between stability conditions and  $t$ -structures on a triangulated category.

**Proposition 3.3.6.** To give a stability condition on a triangulated category  $\mathcal{T}$  is equivalent to giving a bounded  $t$ -structure on  $\mathcal{T}$  and a stability function on its heart with the HN property.

*Proof.* ( $\Rightarrow$ ): Assume that  $\sigma = (Z, \mathcal{P})$  is a given stability condition on  $\mathcal{T}$ . We set  $\mathcal{A} = \mathcal{P}((0, 1])$  and then we know that  $\mathcal{A}$  since it appears as the heart of a bounded  $t$ -structure on  $\mathcal{T}$ , by Lemma 3.2.4. We check the conditions of the Lemma. First note that by the properties of the slicing  $\mathcal{P}$  we have  $\text{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0, \forall \phi_1 > \phi_2$ . Now let  $A, B \in \mathcal{A}$ , so  $A$  and  $B$  have phases in  $(0, 1]$ . Consider the objects  $A[k_1]$  and  $B[k_2]$  for  $k_1 > k_2$ . Then by the properties of the slicing we have that  $A[k_1] \in \mathcal{P}((k_1, k_1 + 1])$  and  $B[k_2] \in \mathcal{P}((k_2, k_2 + 1])$ . Clearly, since  $k_1 > k_2$ , the smallest possible phase for  $A[k_1]$  is  $k_1$  and the largest possible phase for  $B[k_2]$  is  $k_2 + 1$ . Since  $k_1 \geq k_2 + 1$ , we have, by the properties of the slicing  $\mathcal{P}$ , that  $\text{Hom}(A[k_1], B[k_2]) = 0$ . By the properties of the slicing  $\mathcal{P}$ , it follows that every non-zero object  $E$  has a finite filtration with factors  $A_i \in \mathcal{P}(\phi_i)$  where  $\phi_1 > \phi_2 > \dots > \phi_n$ . Then for small  $\epsilon$ , we have that some suitable shifts of  $A_i$  lie in  $\mathcal{A}[k_i]$ , by taking  $k_i$  to be the largest integer that is less than  $\phi_i - \epsilon$  [using that  $\mathcal{P}(\phi + k) = \mathcal{P}(\phi)[k]$ , if an object has phase  $\phi$ , we search for an integer  $k_i$  such that  $\phi_i - k_i \in [0, 1]$  and the above choice of  $k_i$  does the job]. The above constructed finite sequence of shifted factors from  $\mathcal{A}$  gives rise to the required filtration for the existence of a bounded  $t$ -structure in Lemma 3.2.4, with heart  $\mathcal{A}$ . The stability function on  $\mathcal{A}$  is given by restricting  $Z$  to  $K_0(\mathcal{A})$ . This works, since any non-zero object  $E$  of  $\mathcal{A}$  has HN-filtration with phase in  $(0, 1]$  and linearity of  $Z$  shows that  $Z(E)$  lies in the upper semi-half plane. Finally and non-zero  $E \in \mathcal{A}$ , as an object of  $\mathcal{T}$ , has a unique (by Lemma 3.2.6) HN-filtration in  $\mathcal{T}$ . Since

$E \in \mathcal{P}((0, 1])$ , its semistable factors must have phases in  $(0, 1]$  and therefore they form semistable objects in  $\mathcal{A}$ . So  $Z|_{\mathcal{K}_0(\mathcal{A})}$  is a stability function satisfying the HN-property in  $\mathcal{A}$ .

( $\Leftarrow$ ): Assume that a heart  $\mathcal{A}$  of a bounded  $t$ -structure on  $\mathcal{T}$  is give, together with a stability function  $Z$  satisfying the HN-property. First we define the slicing  $\mathcal{P}$ :  $\forall \phi \in (0, 1]$ , set  $\mathcal{P}(\phi)$  to be the category of semistable objects in  $\mathcal{A}$  with respect to  $Z$  with phase  $\phi$ . Then,  $\forall \phi \in \mathbb{R}$  we have that  $\mathcal{P}(\phi + n) = \mathcal{P}(\phi)[n]$ , using that any  $X \in \mathbb{R}$  can be written as  $x = \phi + n$ ,  $\phi \in (0, 1]$ ). The first two axioms of slicing for  $\mathcal{P}$  are satisfied by construction.

- Let  $E \in \mathcal{P}(\phi_1)$  and  $F \in \mathcal{P}(\phi_2)$ , where  $\phi_1 > \phi_2$ . If  $\phi_1, \phi_2 \in (0, 1]$ , then  $\text{Hom}(E, F) = \text{Hom}_{\mathcal{A}}(E, F) = 0$ , as semistable objects in the abelian category  $\mathcal{A}$ . If  $\phi_1, \phi_2$  lie in different intervals, then, by the properties of  $t$ -structures, we have that  $\text{Hom}(\mathcal{A}[k_1], \mathcal{A}[k_2]) = 0, \forall k_1 > k_2$  so also in this case orthogonality holds.
- Finally, for every non-zero  $E \in \mathcal{T}$ , consider the cohomology objects  $H^i(E) \in \mathcal{A}$  with respect to a given  $t$ -structure. Since each  $H^i(E)$  has a unique HN-filtration in  $\mathcal{A}$  with respect to  $Z$ , and all cohomology objects  $H^i(E)$  are zero, except of finitely many of them, we may refine the HN-filtration of  $E$  to an HN-filtration on  $\mathbb{T}$  as follows. Let  $n_j$  be the integers such that the cohomology objects of  $E$  are non-zero, and set  $C_j := H^{n_j}(E)[-n_j], j = 1, 2, \dots, k$ . These objects are included in a sequence of triangles

$$E_{j-1} \longrightarrow E_j \longrightarrow C_j \longrightarrow E_{j-1}[1] \quad (3.2)$$

where  $E_0 = 0, E_k = E$ , and  $E_j$  is the  $j$ -th truncation with respect to the given  $t$ -structure. For each  $j = 1, \dots, k$ , let  $S_1^j, S_2^j, \dots, S_{m_j}^j$  be the semistable factors in  $\mathcal{A}$  with phases  $\phi(S_1^j) > \phi(S_2^j) > \dots > \phi(S_{m_j}^j)$ . Since  $S_t^j \in \mathcal{A}$ , their phases in  $\mathcal{T}$  are by construction  $\phi(S_t^j) + n_j$ . Using the octahedral axiom, we can insert the filtration with the above semistable factors in  $\mathcal{A}$  into the triangles (3.2), giving rise to the required HN-filtration in  $\mathcal{T}$ : for any short exact sequence in the heart  $\mathcal{A}$  of the form

$$0 \longrightarrow A \longrightarrow N \longrightarrow C \longrightarrow 0$$

and a given filtration of  $\mathcal{A}$ , the octahedral axiom can be used to construct the corresponding filtration of  $B$  involving also the semistable factors of  $A$  and  $C$ . Iterating this construction: combining the filtration coming from the  $t$ -structure and the cohomology objects with the filtration of the heart of the induced by the stability function, we arrive at the filtration we are looking for in  $\mathcal{T}$ . Note that the rising filtration has semistable factors, say  $A_1, A_2, \dots, A_m$ , where each  $A_i \in \text{mathscrP}(\phi)$ , and the phases are  $\phi_i = \phi(S_r^j) + n_j$ , so the remain strictly decreasing. The orthogonality of homs i both filtrations, implies that the resulting HN-filtration on  $\mathcal{T}$  is unique.  $\blacksquare$

We say that an abelian category  $\mathcal{A}$  is **Noetherian** if for every  $X \in \mathcal{A}$ , any ascending chain of subobjects stabilizes, that is, if

$$X \subset X_1 \subset X_2 \subset \dots$$

then there exists  $n \in \mathbb{Z}$  such that  $X_n = X_{n+1} = \dots$ . Respectively, it is called **Artinian** if for every  $X \in \mathcal{A}$ , any descending chain of subobjects stabilizes, that is, if

$$X \supset X_1 \supset X_2 \supset \dots$$

then there exists  $n \in \mathbb{N}$  such that  $X_n = X_{n+1} = \dots$ . Now, if  $\mathcal{A}$  is both Noetherian and Artinian, then it is called **of finite length**. By using strict monomorphisms and strict epimorphisms we can define strict subobjects and strict quotient objects in quasi-abelian categories. Therefore we have the following definitions.

**Definition 3.3.7.** Let  $\mathcal{A}$  be a quasi-abelian category and  $A, B, E \in \mathcal{A}$ .

- (1) The object  $A$  is **strict subobject** of  $E$  if there exists a strict monomorphism  $i : A \rightarrow E$ , and we denote it by  $A \subset E$ . We will also denote  $E/A$  the cokernel of  $i$ .
- (2) The object  $B$  is a **strict quotient** of  $E$  if there exists a strict epimorphism  $E \rightarrow B$ , and we denote it by  $E \twoheadrightarrow B$ .
- (3) The category  $\mathcal{A}$  is called **Noetherian** if for every  $E \in \mathcal{A}$  every ascending chain of strict subobjects stabilizes.
- (4) The category  $\mathcal{A}$  is called **Artinian** if for every  $E \in \mathcal{A}$  every descending chain of strict subobjects stabilizes.
- (5) The category  $\mathcal{A}$  is called **of finite length** if it is Noetherian and Artinian.

We proceed with a definition which will allow us to build a richer theory about stability conditions.

**Definition 3.3.8.** A slicing  $\mathcal{P}$  of a triangulated category  $\mathcal{T}$  is **locally-finite** if there exists a real number  $\eta > 0$  such that for all  $t \in \mathbb{R}$  the quasi-abelian category  $\mathcal{P}((t - \eta, t + \eta)) \subseteq \mathcal{T}$  is of finite length. A stability condition  $(Z, \mathcal{P})$  is locally-finite if the slicing  $\mathcal{P}$  is.

### 3.4 The space of stability conditions

Let  $\mathcal{T}$  be a triangulated category and denote by  $\text{Slice}(\mathcal{T})$  the set of locally-finite slicings of  $\mathcal{T}$  and by  $\text{Stab}(\mathcal{T})$  the set of locally-finite stability conditions of  $\mathcal{T}$ . Our main goal in this section is to define topologies on the above spaces. These topologies do not depend on the locally-finite condition.

Before we start equipping  $\text{Slice}(\mathcal{T})$  and  $\text{Stab}(\mathcal{T})$  with topologies, we need to introduce the notion of a generalized metric.

**Definition 3.4.1.** Let  $X$  be a set and  $d: X \times X \rightarrow [0, \infty]$ . The function  $d$  is called a **generalized metric** if for every  $x, y, z \in X$  the following conditions hold:

- (1)  $d(x, y) \geq 0$ ,
- (2)  $d(x, y) = 0 \iff x = y$ ,
- (3)  $d(x, y) = d(y, x)$ ,
- (4)  $d(x, z) \leq d(x, y) + d(y, z)$

**Remark 3.4.2.** Note that the definition of a generalized metric is the same as the definition of a metric, except that it can also take the value  $\infty$ .

**Proposition 3.4.3.** Let  $\mathcal{T}$  be a triangulated category and  $\text{Slice}(\mathcal{T})$  the set of all locally-finite slicings of  $\mathcal{T}$ . The function

$$d : \text{Slice}(\mathcal{T}) \times \text{Slice}(\mathcal{T}) \rightarrow [0, \infty]$$

$$d(\mathcal{P}, \mathcal{Q}) = \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{Q}}^-(E)|, |\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{Q}}^+(E)|\} \in [0, \infty]$$

is a generalised metric on  $\text{Slice}(\mathcal{T})$ .

*Proof.* (1) For any two slicings  $\mathcal{P}, \mathcal{Q} \in \text{Slice}(\mathcal{T})$  we have that  $d(\mathcal{P}, \mathcal{Q}) \geq 0$  since it is the supremum of non-negative real numbers.

(2) Let  $\mathcal{P} \in \text{Slice}(\mathcal{T})$ . Then

$$\begin{aligned} d(\mathcal{P}, \mathcal{P}) &= \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{P}}^-(E)|, |\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{P}}^+(E)|\} \\ &= \sup_{0 \neq E \in \mathcal{T}} \{0, 0\} \\ &= 0 \end{aligned}$$

For the converse, let  $\mathcal{P}, \mathcal{Q} \in \text{Slice}(\mathcal{T})$  such that  $d(\mathcal{P}, \mathcal{Q}) = 0$ . We need to prove that  $\mathcal{P} = \mathcal{Q}$ . This means that we have to show that  $\mathcal{P}(\phi) = \mathcal{Q}(\phi), \forall \phi \in \mathbb{R}$ . Let  $\phi \in \mathbb{R}$  and  $0 \neq E \in \mathcal{P}(\phi)$ . We have  $d(\mathcal{P}, \mathcal{Q}) = 0 \implies \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{Q}}^-(E)|, |\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{Q}}^+(E)|\} = 0$ , which implies that  $|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{Q}}^-(E)| = 0$  and  $|\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{Q}}^+(E)| = 0$ , hence  $\phi_{\mathcal{P}}^-(E) = \phi_{\mathcal{Q}}^-(E)$  and  $\phi_{\mathcal{P}}^+(E) = \phi_{\mathcal{Q}}^+(E)$ . This means that  $0 \neq E \in \mathcal{Q}(\phi)$ , thus we have  $\mathcal{P}(\phi) \subseteq \mathcal{Q}(\phi)$ . The reverse inclusion is proved via a similar argument.

(3) Let  $\mathcal{P}, \mathcal{Q} \in \text{Slice}(\mathcal{T})$ . We have

$$\begin{aligned} d(\mathcal{P}, \mathcal{Q}) &= \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{Q}}^-(E)|, |\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{Q}}^+(E)|\} \\ &= \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{Q}}^-(E) - \phi_{\mathcal{P}}^-(E)|, |\phi_{\mathcal{Q}}^+(E) - \phi_{\mathcal{P}}^+(E)|\} \\ &= d(\mathcal{Q}, \mathcal{P}) \end{aligned}$$

(4) Let  $\mathcal{P}, \mathcal{R} \in \text{Slice}(\mathcal{T})$ . Then we have

$$\begin{aligned} d(\mathcal{P}, \mathcal{R}) &= \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{R}}^-(E)|, |\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{R}}^+(E)|\} \\ &= \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{Q}}^-(E) + \phi_{\mathcal{Q}}^-(E) - \phi_{\mathcal{R}}^-(E)|, |\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{Q}}^+(E) + \phi_{\mathcal{Q}}^+(E) - \phi_{\mathcal{R}}^+(E)|\} \\ &\leq \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{Q}}^-(E)| + |\phi_{\mathcal{Q}}^-(E) - \phi_{\mathcal{R}}^-(E)|, |\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{Q}}^+(E)| + |\phi_{\mathcal{Q}}^+(E) - \phi_{\mathcal{R}}^+(E)|\} \\ &\leq \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{Q}}^-(E)|, |\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{Q}}^+(E)|\} \\ &\quad + \sup_{0 \neq E \in \mathcal{T}} \{|\phi_{\mathcal{Q}}^-(E) - \phi_{\mathcal{R}}^-(E)|, |\phi_{\mathcal{Q}}^+(E) - \phi_{\mathcal{R}}^+(E)|\} \\ &= d(\mathcal{P}, \mathcal{Q}) + d(\mathcal{Q}, \mathcal{R}) \end{aligned}$$

for every  $\mathcal{Q} \in \text{Slice}(\mathcal{T})$ . ■

The above metric can have another description which is given by the following lemma.

**Lemma 3.4.4.** *The metric in Proposition 3.4.3 is also given by*

$$d(\mathcal{P}, \mathcal{Q}) = \inf \{\epsilon \in \mathbb{R}_{\geq 0} \mid \mathcal{Q}(\phi) \subseteq \mathcal{P}([\phi - \epsilon, \phi + \epsilon]) \text{ for all } \phi \in \mathbb{R}\}$$

*Proof.* Let  $d' = d'(\mathcal{P}, \mathcal{Q}) = \inf \{\epsilon \in \mathbb{R}_{\geq 0} \mid \mathcal{Q}(\phi) \subseteq \mathcal{P}([\phi - \epsilon, \phi + \epsilon]), \forall \phi \in \mathbb{R}\}$ . We will prove that  $d = d'$ . We start by showing that  $d(\mathcal{P}, \mathcal{Q}) \leq d'(\mathcal{P}, \mathcal{Q})$ . From the definition of  $d(\mathcal{P}, \mathcal{Q})$ , we have to prove that  $|\phi_{\mathcal{P}}^-(E) - \phi_{\mathcal{Q}}^-(E)| \leq d'(\mathcal{P}, \mathcal{Q})$  and  $|\phi_{\mathcal{P}}^+(E) - \phi_{\mathcal{Q}}^+(E)| \leq d'(\mathcal{P}, \mathcal{Q})$ , for every  $E \in \mathcal{T}$ . Since  $\mathcal{Q}$  is a slicing of  $\mathcal{T}$ , let

$$\begin{array}{ccccccc} 0 = E_0 & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & E_2 & \longrightarrow \cdots \longrightarrow & E_{n-1} & \xrightarrow{\quad} & E_n = E \\ & \swarrow [1] & \searrow & \swarrow [1] & \searrow & & \swarrow [1] & \searrow & \\ & & A_1 & & A_2 & & & & A_n \end{array}$$

where  $A_j \in \mathcal{Q}(\phi_j)$  are the semistable factors of  $E$  for all  $j$ , be the HN-filtration of  $E$  with respect to  $\mathcal{Q}$ . Then by definition we have that  $\phi_{\mathcal{Q}}^-(E) = \phi_{\mathcal{Q}}(A_1)$  and  $\phi_{\mathcal{Q}}^+(E) = \phi_{\mathcal{Q}}(A_n)$ . Hence  $\phi_{\mathcal{P}}^+(A_i) \leq \phi_{\mathcal{Q}}(A_i) +$

$d'(\mathcal{P}, \mathcal{Q})$  and  $\phi_{\mathcal{P}}^-(A_i) \geq \phi_{\mathcal{Q}}(A_i) + d'$ . However  $\phi_{\mathcal{P}}^+(E) \leq \max_i \{\phi_{\mathcal{P}}^-(A_i)\}$  and  $\phi_{\mathcal{P}}^-(E) \geq \min_i \{\phi_{\mathcal{P}}^-(A_i)\}$ . Therefore we have that  $\phi_{\mathcal{P}}^+(E) \leq \phi_{\mathcal{Q}}^+(E) + d'$  and  $\phi_{\mathcal{P}}^-(E) \geq \phi_{\mathcal{Q}}^-(E) - d'$ , which proves the first inequality. The inequality  $d'(\mathcal{P}, \mathcal{Q}) \leq d(\mathcal{P}, \mathcal{Q})$  is straightforward. ■

Consider the inclusion of sets

$$\text{Stab}(\mathcal{T}) \subseteq \text{Slice}(\mathcal{T}) \times \text{Hom}_{\mathbb{Z}}(\mathbb{K}_0(\mathcal{T}), \mathbb{C})$$

We would like to define a topology on  $\text{Stab}(\mathcal{T})$ , by defining a topology on  $\text{Slice}(\mathcal{T}) \times \text{Hom}_{\mathbb{Z}}(\mathbb{K}_0(\mathcal{T}), \mathbb{C})$ . We have already defined a topology on  $\text{Slice}(\mathcal{T})$  via Proposition 3.4.3, so we need to equip the  $\text{Hom}_{\mathbb{Z}}(\mathbb{K}_0(\mathcal{T}), \mathbb{C})$  with a topology. For every  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$ , we define a function

$$\|\cdot\|_{\sigma} : \text{Hom}_{\mathbb{Z}}(\mathbb{K}_0(\mathcal{T}), \mathbb{C}) \rightarrow [0, \infty]$$

which sends a group homomorphism  $U : \mathbb{K}_0(\mathcal{T}) \rightarrow \mathbb{C}$  to

$$\|U\|_{\sigma} := \sup \left\{ \frac{|U(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\}$$

**Lemma 3.4.5.** *The function  $\|\cdot\|_{\sigma}$  is a norm, hence the pair  $(\text{Hom}_{\mathbb{Z}}(\mathbb{K}_0(\mathcal{T}), \mathbb{C}), \|\cdot\|_{\sigma})$  is a normed complex vector space for every stability condition  $\sigma = (Z, \mathcal{P})$ .*

*Proof.* Let  $\sigma = (Z, \mathcal{P})$  be a stability condition.

(1) Let  $U, V : \mathbb{K}_0(\mathcal{T}) \rightarrow \mathbb{C}$  be two group homomorphisms. We need to show that  $\|U + V\|_{\sigma} \leq \|U\|_{\sigma} + \|V\|_{\sigma}$ . We compute

$$\begin{aligned} \|U + V\|_{\sigma} &= \sup \left\{ \frac{|(U + V)(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\ &= \sup \left\{ \frac{|U(E) + V(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\ &\leq \sup \left\{ \frac{|U(E)| + |V(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\ &= \sup \left\{ \frac{|U(E)|}{|Z(E)|} + \frac{|V(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\ &\leq \sup \left\{ \frac{|U(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} + \sup \left\{ \frac{|V(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\ &= \|U\|_{\sigma} + \|V\|_{\sigma} \end{aligned}$$

(2) Let  $U : K_0(\mathcal{T}) \rightarrow \mathbb{C}$  be a group homomorphism and  $z \in \mathbb{C}$ . We compute

$$\begin{aligned}
\|zU\|_\sigma &= \sup \left\{ \frac{|(zU)(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\
&= \sup \left\{ \frac{|zU(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\
&= \sup \left\{ \frac{|z||U(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\
&= |z| \sup \left\{ \frac{|U(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\
&= |z| \|U\|_\sigma
\end{aligned}$$

(3) For the one direction if  $U$  is the zero group homomorphism then have

$$\begin{aligned}
\|U\|_\sigma &= \sup \left\{ \frac{|U(E)|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\
&= \sup \left\{ \frac{|0|}{|Z(E)|} \mid E \text{ is semistable in } \sigma \right\} \\
&= \sup \{0 \mid E \text{ is semistable in } \sigma\} \\
&= 0
\end{aligned}$$

On the other hand, suppose that we have  $\|U\|_\sigma = 0$  for some group homomorphism  $U : K_0(T) \rightarrow \mathbb{C}$ . We will prove that  $U$  is the zero group homomorphism. Since the supremum of the non-negative quantities  $\frac{|U(E)|}{|Z(E)|}$  is zero, then for every  $\sigma$ -semistable object we have that  $|U(E)| = 0$ , hence  $U(E) = 0$ . It remains to show that  $U(E) = 0$  for all the non- $\sigma$ -semistable stable objects of  $\mathcal{T}$ . Let  $E$  be a non- $\sigma$ -semistable object. From its HN-filtration, since its semistable factors all vanish under  $U$ , we have that  $U(E) = 0$ . ■

The function  $\|\cdot\|_\sigma$  is a norm, not necessarily finite, in the complex vector space  $\text{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$ . We proceed by defining a basis for our topology. Define

$$B_\epsilon(\sigma) := \{\tau := (W, \mathcal{Q}) \mid \|W - Z\|_\sigma < \sin(\pi\epsilon) \text{ and } d(\mathcal{P}, \mathcal{Q}) < \epsilon\} \subseteq \text{Stab}(\mathcal{T})$$

To get a better grasp of the definition, note that if  $\|Z - W\|_\sigma < \sin(\pi\epsilon)$  then for all semistable objects  $E$  of  $\sigma$ , the phase of  $W(E)$  differs from the phase of  $Z(E)$  by less than  $\epsilon$ . Before we prove that as  $\sigma$  varies, the subsets  $B_\epsilon(\sigma)$  form a basis for our topology, we will need to prove the following crucial lemma.

**Lemma 3.4.6.** *If  $\tau = (W, \mathcal{Q}) \in B_\epsilon(\sigma)$  then there exist constants  $k_i, i \in \mathbb{Z}$ , such that*

$$k_1 \|U\|_\sigma < \|U\|_\tau < k_2 \|U\|_\sigma$$

for every  $U \in \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C})$ .

*Proof.* First, before we proceed with the general case, note that for every stability condition  $\sigma = (Z, \mathcal{P})$  on  $\mathcal{T}$ , and every real number  $0 \leq \eta < \frac{1}{2}$ , we have that

$$|U(E)| < \frac{\|U\|_\sigma}{\cos(\pi\eta)} |Z(E)|$$

for every non-zero object  $E \in \mathcal{T}$  which satisfies  $\phi_\sigma^+(E) - \phi_\sigma^-(E) < \eta$ , and for every group homomorphism  $U : K_0(\mathcal{T}) \rightarrow \mathbb{C}$ . Now consider the general case. It is true that  $d(\mathcal{P}, \mathcal{Q}) < \epsilon$  and  $\|W - Z\|_\sigma < \sin(\pi\epsilon)$ . Hence we can apply the above inequality with  $U := Z - W$  and  $\eta = 2\epsilon$  to obtain

$$|W(E) - Z(E)| < \frac{\sin(\pi\epsilon)}{\cos(2\pi\epsilon)} |Z(E)|$$

for every semistable object  $E \in \mathcal{T}$ . Therefore there exists a constant  $k > 0$  such that  $|Z(E)| < k|W(E)|$  for every semistable objects  $E \in \tau$ . Now, for every group homomorphism  $U : K_0(\mathcal{T}) \rightarrow \mathbb{C}$ , by applying the above inequality again and combining it with the previous result, we have that  $\|U\|_\tau < k_2 \|U\|_\sigma$ . The other inequality follows from similar arguments. ■

Consider the space  $\text{Stab}(\mathcal{T})$ , with the topology generated by the basis of open sets  $B_\epsilon(\sigma)$ . Let  $\Sigma$  be a connected component of  $\text{Stab}(\mathcal{T})$ , and  $V(\Sigma)$  the subspace

$$V(\Sigma) := \{U \in \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C}) \mid \|U\|_\sigma < \infty\} \subseteq \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C})$$

With the above notation we can state and prove the first part of the first main result of Bridgeland's paper.

**Theorem 3.4.7.** *Let  $\mathcal{T}$  be a triangulated category. For each connected component  $\Sigma \subseteq \text{Stab}(\mathcal{T})$  there exist a linear subspace  $V(\Sigma) \subseteq \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}), \mathbb{C})$ , with a well-defined linear topology, and a continuous map  $\mathcal{Z} : \Sigma \rightarrow V(\Sigma)$  which maps a stability condition  $(Z, \mathcal{P})$  to its central charge  $Z$ .*

*Proof.* From Lemma 3.4.6 we have that  $V(\Sigma)$  is locally constant on  $\text{Stab}(\mathcal{T})$  and thus constant on  $\Sigma$ . Also, if  $\sigma = (Z, \mathcal{P}) \in \Sigma$  then  $Z \in V(\Sigma)$ . In addition, for every  $\sigma \in \Sigma$ , the function  $\|\cdot\|_\sigma$  defines a norm on  $V(\Sigma)$ , and by Lemma 3.4.6, all these norms are equivalent. ■

To complete the proof of Theorem 3.0.1, It remains to show that the map  $\mathcal{Z}$  is a local homeomorphism, which will be proved in the next section. In this direction, the following proves that  $\mathcal{Z}$  is at least locally injective.

**Lemma 3.4.8.** *Let  $\sigma = (Z, \mathcal{P})$  and  $\tau = (Z, \mathcal{Q})$  be two stability conditions on  $\mathcal{T}$  with the same central charge  $Z$ , such that  $d(\mathcal{P}, \mathcal{Q}) < 1$ . Then  $\sigma = \tau$ .*

*Proof.* Suppose for contradiction that  $\sigma \neq \tau$ . Then, since they have the same central charge  $Z$ , there exists a  $\phi \in \mathbb{R}$  such that  $\mathcal{P}(\phi) \setminus \mathcal{Q}(\phi) \neq \emptyset \implies \exists E \in \mathcal{P}(\phi) \setminus \mathcal{Q}(\phi)$ .

- (1) If  $E \in \mathcal{Q}(> \phi)$ , then from our hypothesis  $d(\mathcal{P}, \mathcal{Q}) < 1$  we have that  $E \in \mathcal{Q}((\phi, \phi + 1))$ . This contradicts the fact that  $\sigma$  and  $\tau$  have the same central charge.
- (2) If  $E \in \mathcal{Q}(< \phi)$  then the argument is similar to (1).
- (3) If  $E \in \mathcal{Q}((\phi - 1, \phi + 1))$ , then there exists a distinguished triangle

$$A \longrightarrow E \longrightarrow B \longrightarrow A[1]$$

where  $A \in \mathcal{Q}(\phi, \phi + 1)$  and  $B \in \mathcal{Q}((\phi - 1, \phi])$ . If  $A \in \mathcal{P}(\leq \phi)$ , then  $A \in \mathcal{P}((\phi - 1, \phi])$ , thus we have the same contradiction as in (1). Hence, there exists  $0 \neq C \in \mathcal{P}(\psi)$ , with  $\psi > \phi$  and a non-zero morphism  $f : C \rightarrow A$  which factors via  $B[-1]$  as  $E \in \mathcal{P}(\phi)$ . The above fit into the following commutative diagram.

$$\begin{array}{ccccc} & & C & & \\ & \swarrow g & \downarrow f & \searrow 0 & \\ B[-1] & \longrightarrow & A & \longrightarrow & E \longrightarrow B \end{array}$$

However,  $B[-1] \in \mathcal{Q}((\phi - 2, \phi - 1]) \subseteq \mathcal{P}((\phi - 3, \phi])$ , therefore  $g = 0$ , which is a contradiction. ■

### 3.5 Deformations of stability conditions

Our main goal in this section is to complete the proof of the first main result, that is, to prove that the map  $\mathcal{Z} : \Sigma \rightarrow V(\Sigma)$  is a local homeomorphism. This will be done by proving a result that allows us to lift deformations of the central charge  $Z$  of a stability condition  $\sigma = (Z, \mathcal{P})$  to deformations of stability conditions.

**Theorem 3.5.1.** *Let  $\sigma = (Z, \mathcal{P})$  be a locally-finite stability condition on a triangulated category  $\mathcal{T}$ . Then there exist a  $\epsilon_0 > 0$  such that if  $0 < \epsilon < \epsilon_0$  and  $W : K(\mathcal{T}) \rightarrow \mathbb{Z}$  is a group homomorphism satisfying*

$$|W(E) - Z(E)| < \sin(\pi\epsilon)|Z(E)|$$

*for every semistable object  $E \in \mathcal{T}$  in  $\sigma$ , then there exists a locally-finite stability condition  $\tau = (W, \mathcal{Q})$  on  $\mathcal{T}$  such that  $d(\mathcal{P}, \mathcal{Q}) < \epsilon$ .*

After what was said in the previous section, the above theorem will be enough to prove the first main result. Note that from Lemma 3.4.6, if we assume that  $\epsilon_0 < \frac{1}{2}$ , then the stability condition  $\tau$  of the above theorem is unique. To proceed, it will be enough to assume that  $\epsilon_0 < \frac{1}{8}$  and that each of the quasi-abelian categories  $\mathcal{P}((t - 4\epsilon_0, t + 4\epsilon_0))$  has finite length. Since  $\mathcal{Q}((t - \epsilon, t + \epsilon)) \subseteq \mathcal{P}((t - 2\epsilon, t + 2\epsilon))$  for every  $t$ , the stability condition  $\tau$  is locally-finite. The proof of the theorem will be broken down into several steps. We begin by giving an auxiliary definition.

**Definition 3.5.2.** *Let  $\mathcal{T}$  be a triangulated category. A **thin** subcategory of  $\mathcal{T}$  is a full subcategory of the form  $\mathcal{P}((a, b)) \subseteq \mathcal{T}$  where  $a, b \in \mathbb{R}$  such that  $0 < b - a < 1 - 2\epsilon$ , where  $\epsilon$  is as in the conditions of Theorem 3.5.1.*

**Lemma 3.5.3.** *Let  $E$  be a  $W$ -semistable object in some thin subcategory  $\mathcal{A} \subseteq \mathcal{T}$ , and  $\psi = \psi(E)$ . Then  $E \in \mathcal{P}((\psi - \epsilon, \psi + \epsilon))$ .*

*Proof.* There exists a short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

in  $\mathcal{A}$  such that  $A \in \mathcal{P}(\phi)$  and  $B \in \mathcal{P}(< \phi)$ . Since  $E$  is  $W$ -semistable, we have that  $\psi(A) \leq \psi(E)$ . However, from the above, we have that  $\psi(A) \in (\phi - \epsilon, \phi + \epsilon)$ . Hence  $\phi < \psi + \epsilon$ . From similar arguments we obtain that  $\phi^-(E) > \psi - \epsilon$ . ■

**Definition 3.5.4.** *Let  $\mathcal{A} = \mathcal{P}((a, b))$  be a thin subcategory  $\mathcal{T}$ . A non-zero object  $E \in \mathcal{A}$  is called **enveloped** by  $\mathcal{A}$  if  $a + \epsilon \leq \psi(E) \leq b - \epsilon$ .*

The above definition gives rise to the following lemma which provides a notion of semistability without a particular choice of a thin subcategory.

**Lemma 3.5.5.** *Let  $E \in \mathcal{T}$  be an object which is enveloped by thin subcategories  $\mathcal{B}$  and  $\mathcal{C}$  of  $\mathcal{T}$ . Then  $E$  is  $W$ -semistable in  $\mathcal{C}$  if and only if it is  $W$ -semistable in  $\mathcal{B}$ .*

*Proof.* From Lemma 3.5.3 we can assume that  $E$  is enveloped by the thin subcategory  $\mathcal{P}((\psi(E) - \epsilon, \psi(E) + \epsilon))$ . Therefore it is enough to prove it for the case where  $\mathcal{B} \subseteq \mathcal{C}$ , and from the "symmetry of the situation", we can assume that  $\mathcal{B} = \mathcal{P}((a, b))$  and  $\mathcal{C} = \mathcal{P}((a, c))$  for real numbers  $a < b < c$ . For the one direction, if  $E$  is  $W$ -semistable in  $\mathcal{C}$  then it is also  $W$ -semistable in  $\mathcal{B}$ , since any strict short sequence in  $\mathcal{B}$  is also a strict short exact sequence in  $\mathcal{C}$ . For the converse, suppose that  $E$  is "unstable" in  $\mathcal{C}$  such that there exists a strict short exact sequence in  $\mathcal{C}$

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

with  $\psi(A) > \psi(E) > \psi(B)$ . Then, from Lemma 3.2.11, we have that  $\phi^+(A) \leq \phi^+(E)$ , and since  $E \in \mathcal{B}$ , we have that  $A \in \mathcal{B}$ . In addition, there exists a short exact sequence

$$0 \longrightarrow B_1 \longrightarrow B \longrightarrow B_2 \longrightarrow 0$$

where  $B_1 \in \mathcal{P}([b, c))$  and  $B_2 \in \mathcal{B}$ . Because  $E$  is enveloped by  $\mathcal{B}$ , we have that  $\psi(E) \leq b - \epsilon < \psi(B_1)$ . Now, consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & A & \xlongequal{\quad} & A & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & B_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & B_1 & \longrightarrow & B & \longrightarrow & B_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Then from Lemma 3.2.11 again, we have that  $\phi^+(K) \leq \phi^+(E)$ . Therefore

$$0 \longrightarrow K \longrightarrow E \longrightarrow B_2 \longrightarrow 0$$

is a short exact sequence in  $\mathcal{B}$ . However  $\psi(K) > \psi(E)$ , hence  $E$  is not  $W$ -semistable in  $\mathcal{B}$ .  $\blacksquare$

To prove Theorem 3.5.1 we have to show that the pair  $(W, \mathcal{Q})$  defines a stability condition on  $\mathcal{T}$ . In this direction, for every  $\psi \in \mathbb{R}$ , we define  $\mathcal{Q}(\psi) \subseteq \mathcal{T}$  to be the full additive subcategory of  $\mathcal{T}$  consisting of the zero objects of  $\mathcal{T}$  together with those objects  $E \in \mathcal{T}$  which are  $W$ -semistable of phase  $\psi$  in some thin enveloping subcategory  $\mathcal{P}(a, b)$ . The following lemma proves the first axiom of Definition 3.3.1.

**Lemma 3.5.6.** *If  $E \in \mathcal{D}(\psi_1)$  and  $F \in \mathcal{D}(\psi_2)$  and  $\psi_1 > \psi_2$  then  $\text{Hom}_{\mathcal{T}}(E, F) = 0$ .*

*Proof.* Suppose, for contradiction, that  $\text{Hom}_{\mathcal{T}}(E, F) \neq 0$ . Therefore there exists a non-zero map  $f : E \rightarrow F$ . From Lemma 3.5.3, we have that  $\psi_1 - \psi_2 < 2\epsilon$ . Let  $a = \frac{\psi_1 + \psi_2}{2} - \frac{1}{2}$  and consider the abelian subcategory  $\mathcal{A} = \mathcal{P}((a, a + 1]) \subseteq \mathcal{T}$  which contains  $E$  and  $F$ . The short exact sequences

$$\begin{aligned} 0 &\longrightarrow \text{Ker}(f) \longrightarrow E \longrightarrow \text{Im}(f) \longrightarrow 0 \\ 0 &\longrightarrow \text{Im}(f) \longrightarrow F \longrightarrow \text{Coker}(f) \longrightarrow 0 \end{aligned}$$

lie in the abelian category  $\mathcal{A}$ . From Lemmas 3.2.11 and 3.5.3 we have that  $\text{Ker}(f) \in \mathcal{P}((a, \psi_1 + \epsilon))$  and  $\text{Coker}(f) \in \mathcal{P}((\psi_2 - \epsilon, a + 1])$  and  $\text{Im}(f) \in \mathcal{P}((\psi_1 - \epsilon, \psi_2 + \epsilon))$ . For small enough  $\epsilon$ , say  $\epsilon < 1/8$ , there exists a thin subcategory of  $\mathcal{T}$  enveloping  $E$  in which the first sequence is a strict short exact sequence, and similarly a thin subcategory enveloping  $F$  in which the second sequence is a strict short exact sequence. However, the objects  $E$  and  $F$  are  $W$ -semistable in every enveloping category, it follows that  $\psi_1 < \psi(\text{Im}(f)) \leq \psi_2$ , a contradiction.  $\blacksquare$

Next we want to prove that HN-filtrations always exist. In order to do that, we first prove the following Lemma.

**Lemma 3.5.7.** *Let  $\mathcal{A} = \mathcal{P}((a, b)) \subseteq \mathcal{T}$  be a thin subcategory of finite length. Then every non-zero object of  $\mathcal{P}((a + 2\epsilon, b - 4\epsilon))$  has a finite HN-filtration whose factors are  $W$ -semistable objects of  $\mathcal{A}$  which are enveloped by  $\mathcal{A}$ .*

*Proof.* The proof is similar to that of Proposition 3.1.9, when we replace subobjects by strict subobjects and quotients by strict quotients. In the following we provide the necessary changes. The chain conditions still hold because  $\mathcal{A}$  has finite length. If an object  $E \in \mathcal{P}(a + 2\epsilon, b - 4\epsilon)$  has a HN-filtration with  $W$ -semistable factors  $F_1, F_2, \dots, F_n$  then  $\psi(F_1) \geq \psi(E) > a + \epsilon$ . Since there exists a non-zero morphism  $F_1 \rightarrow E$ , by applying Lemma 3.5.3 we have that  $\psi(F_1) < b - 3\epsilon$ . This means that the factors of  $E$  are enveloped by  $\mathcal{A}$ . To prove the existence of HN-filtrations, it is enough their existence in the class  $\mathcal{G}$  which is defined to be the class of all non-zero objects  $E \in \mathcal{P}((a, b - 4\epsilon))$ , where every non-zero strict quotient  $E \rightarrow B$  in  $\mathcal{A}$  satisfies the condition  $\psi(B) > a + \epsilon$ . That is because the class  $\mathcal{G}$  contains all non-zero objects of  $\mathcal{P}((a + 2\epsilon, b - 4\epsilon))$ , by Lemma 3.2.11. From the diagram (3.1) and Lemma 3.2.11 we have that if

$$0 \longrightarrow E' \longrightarrow E \longrightarrow B \longrightarrow 0$$

is a strict short exact sequence in  $\mathcal{A}$ , where  $E \rightarrow B$  is a mdq and  $E \in \mathcal{G}$ , the  $E' \in \mathcal{G}$ . Therefore the induction step in the Proposition 3.1.9 stays within the class  $\mathcal{G}$ . Therefore we have to show that every object in  $\mathcal{G}$  has an mdq. Before we proceed with the induction step, it will be useful to prove the

existence of mdqs to a larger class of objects  $\mathcal{H}$  which contains every non-zero object  $E \in \mathcal{A}$  with  $\psi(E) < b - 3\epsilon$ , such that every non-zero quotient  $E \twoheadrightarrow B$  in  $\mathcal{A}$  satisfies  $\psi(B) > a + \epsilon$ . If  $E \in \mathcal{H}$  and the non-zero strict quotient  $E \twoheadrightarrow E'$  satisfies  $\psi(E) \geq \psi(E')$ , then  $E' \in \mathcal{H}$ . Let  $E \in \mathcal{H}$  and we will prove that  $\phi^+(E) < \psi(E) + \epsilon$ . Suppose, for contradiction, that  $\phi^+(E) \geq \psi(E) + \epsilon$ . Then there exists a strict short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow E' \longrightarrow 0$$

where  $A \in \mathcal{P}(\geq \psi(E) + \epsilon)$  and  $E' \in \mathcal{P}(\psi(E) + \epsilon)$ . Then we have that  $\psi(A) > \psi(E) > \psi(E')$ . We will prove that if  $E' \twoheadrightarrow B$  is an mdq for  $E'$ , then the composition  $E \twoheadrightarrow B$  is an mdq for  $E$ . In this direction, if  $E \twoheadrightarrow B'$  is a  $W$ -semistable quotient in  $\mathcal{A}$ , where  $\psi(B') \leq \psi(B)$ , then  $\psi(B') \leq \psi(E)$ . Therefore by Lemma 3.5.3 we have that  $\phi^+(B') < \psi(E) + \epsilon$ . This means that  $\text{Hom}_{\mathcal{A}}(A, B') = 0$ , which implies that  $E \twoheadrightarrow B'$  factors through  $E'$ . By combining Lemmas 3.5.3 and 3.2.11, and the inequalities  $\phi^+(E) < \psi(E) + \epsilon$  and  $\psi(E) < b - 3\epsilon$ , we have that every  $W$ -semistable strict subobject of  $E$  is enveloped by  $\mathcal{A}$ . To conclude the proof that  $E$  has an mdq, we apply the argument of Proposition 3.1.9. This is feasible since by the definition of the class  $\mathcal{H}$ , every  $W$ -semistable strict quotient of  $E$  is also enveloped in  $\mathcal{A}$ . Hence we can apply Lemma 3.5.6  $\blacksquare$

Now, for each real number  $t$  we define  $\mathcal{Q}( > t)$  to be the full extension-closed subcategory of  $\mathcal{T}$  generated by the subcategories  $\mathcal{Q}(\psi)$  for  $\psi > t$ . We also define the full subcategories  $\mathcal{Q}(\leq t)$  and  $\mathcal{Q}(t)$  similarly. To conclude with the proof of the existence of the HN-filtrations, we have to prove the following Lemma.

**Lemma 3.5.8.** *The category  $\mathcal{Q}( > t)$  is a  $t$ -structure in  $\mathcal{T}$ .*

*Proof.* We have to show that for every  $E \in \mathcal{T}$  there exists a distinguished triangle

$$A \longrightarrow E \longrightarrow B \longrightarrow \Sigma(A)$$

where  $A \in \mathcal{Q}( > t)$  and  $B \in \mathcal{Q}(\leq t)$ . From Lemmas 3.5.3 and 3.5.7, we have that  $\mathcal{P}(s) \subset \mathcal{Q}( > t)$  for  $s \geq t + \epsilon$  and  $\mathcal{P}(s) \subset \mathcal{Q}(< t)$  for  $s \leq t - \epsilon$ . Therefore we can restrict ourselves to the case where  $E \in \mathcal{P}((t - \epsilon, t + \epsilon))$ . The object  $E$  also belongs in the category  $\mathcal{P}((t - 3\epsilon, t + 5\epsilon))$ , which has finite length by the assumptions of Theorem 3.5.1. By applying Lemma 3.5.7 we have a HN-filtration of  $E$  which concludes the proof.  $\blacksquare$

In order to finish the proof of Theorem 3.5.1, we have to show that every non-zero object of  $\mathcal{T}$  has a finite filtration by objects of the subcategories  $\mathcal{Q}(\psi)$ . However, it is enough to prove this for objects in each of the full subcategories  $\mathcal{Q}((t, t + \delta)) = \mathcal{Q}( > t) \cap \mathcal{Q}(< t + \delta)$ , since then we can concatenate these filtrations and obtain a filtration in  $\mathcal{Q}(\psi)$ . In order to prove this, we embed the category  $\mathcal{Q}((t, t + \delta))$  in the finite-length, quasi-abelian subcategory  $\mathcal{P}((t - 3\epsilon, t + 5\epsilon + \delta))$  and we apply Lemma 3.5.7.

In order to proceed with the proof of Corollary 3.0.2, we first need to introduce some definitions

**Definition 3.5.9.** A triangulated category  $\mathcal{T}$  is **linear over a field**  $\mathbb{K}$  if the morphisms of  $\mathcal{T}$  have the structure of a vector space over  $\mathbb{K}$ , with respect to which the composition is bilinear.

**Definition 3.5.10.** A triangulated category  $\mathcal{T}$  which is linear over a field  $\mathbb{K}$  is **of finite type** if for every pair of objects  $E, F \in \mathcal{T}$  the vector space  $\bigoplus_i \text{Hom}_{\mathcal{T}}(E, F[i])$  is finite dimensional. In this case the free abelian group  $\mathcal{N}(\mathcal{T}) = K_0(\mathcal{T})/K_0(\mathcal{T})^\perp$  is called the **numerical Grothendieck group** of  $\mathcal{T}$ . Then  $\mathcal{T}$  is called **numerically finite** if the group  $\mathcal{T}$  has finite rank.

**Definition 3.5.11.** Let  $\mathcal{T}$  be of finite type over a field. A stability condition  $\sigma = (Z, \mathcal{P})$  in  $\mathcal{T}$  is called **numerical** if the central charge  $Z: K_0(\mathcal{T})$  factors through the quotient group  $\mathcal{N}(\mathcal{T})$ . The space of all numerical conditions is denoted by  $\text{Stab}_{\mathcal{N}}(\mathcal{T})$ .

The proof of Bridgeland's Corollary 3.0.2 comes from applying the above Definitions 3.5.9, 3.5.10 and 3.5.11, and by applying Theorem 3.0.1. Since  $\mathcal{T}$  is numerically finite the numerical Grothendieck group  $\mathcal{N}(\mathcal{T})$  has finite rank, that is, there exists  $n \in \mathbb{N}$  such that  $\mathcal{N}(\mathcal{T}) \cong \mathbb{Z}^n$ . This implies that  $\text{Hom}_{\mathbb{Z}}(\mathcal{N}(\mathcal{T}), \mathbb{C}) \cong \mathbb{C}^n$ . Therefore by applying Theorem 3.0.1, every connected component  $\Sigma \subset \text{Stab}_{\mathcal{N}}(\mathcal{T})$  is locally homeomorphic to a linear subspace  $V(\Sigma)$ . This yields that  $\Sigma$  is a complex manifold.



# CHAPTER 4

## APPLICATIONS

In this chapter we present some applications of Bridgeland's theory of stability conditions in Algebraic Geometry. In general, the computation of  $\text{Stab}(\mathcal{T})$  presents significant difficulties. It has been proved that the bounded derived category of coherent sheaves over a surface is nonempty. Here we focus on smooth projective curves over  $\mathbb{C}$ . We omit the proofs, as they lie beyond the scope of this Thesis.

Let  $X$  be a smooth projective curve over  $\mathbb{C}$ . We denote with  $D^b(X)$  the bounded derived category of coherent sheaves of  $X$  and let  $\text{Stab}(X)$  be the space of locally-finite stability conditions of  $D^b(X)$ . In [2] Bridgeland proves the following proposition

**Proposition 4.0.1.** *The generalised metric space  $\text{Stab}(\mathcal{T})$  carries a right action of the group  $\tilde{\text{GL}}^+(2, \mathbb{R})$ , the universal covering space of  $\text{GL}^+(2, \mathbb{R}) = \{A \in \text{GL}(2, \mathbb{R}) \mid |A| > 0\}$ , and a left action by isometries of the group  $\text{Aut}(\mathcal{T})$  of exact autoequivalences of  $\mathcal{T}$ . These two actions commute.*

*Proof.* For the proof, we refer the reader to [2]. ■

Now, let  $g(X)$  be the genus of the curve  $X$ . Then we have the following Theorem

**Theorem 4.0.2.** *Let  $X$  be a smooth projective curve over  $\mathbb{C}$ . Then we have:*

- *If  $g(X) = 0$ , then  $\text{Stab}(X) \cong \mathbb{C}^2$ . (Bridgeland)*
- *If  $g(X) = 1$ , then  $\text{Stab}(X) \cong \mathbb{C} \times H$ . (Bridgeland)*
- *If  $g(X) > 1$ , then  $\text{Stab}(X) \cong \mathbb{C} \times H$ . (Macri)*

where  $H$  is the upper-half complex plane.

*Proof.* In [2] Bridgeland proved the cases for  $g(X) = 0, 1$  and Macri in [11] proved the cases for  $g(X) > 1$ . ■

*Chapter 4*

## ΠΕΡΙΛΗΨΗ

Ο Tom Bridgeland το 2007, βλέπε την εργασία [2], έχοντας ως κίνητρο εργασίες του Michael Douglas στη Θεωρητική Φυσική, βλέπε για παράδειγμα την εργασία [4], αναφορικά με μεμβράνες Dirichlet (Dirichlet branes ή D-branes) σε συνδυασμό με την εικασία Ομολογικής Κατοπτρικής Συμμετρίας (Homological Mirror Symmetry) του Kontsevich στη Θεωρία Χορδών (String Theory), εισήγαγε την έννοια του *χώρου συνθηκών ευστάθειας* (space of stability conditions)  $\text{Stab}(\mathcal{T})$  μιας τριγωνισμένης κατηγορίας  $\mathcal{T}$ , και απέδειξε ότι ο χώρος  $\text{Stab}(\mathcal{T})$  έχει τη δομή ενός μιγαδικού πολυπύγματος (complex manifold). Έκτοτε, ο χώρος συνθηκών ευστάθειας  $\text{Stab}(\mathcal{T})$  μιας τριγωνισμένης κατηγορίας  $\mathcal{T}$ , διαδραματίζει σημαντικό ρόλο στην Αλγεβρική Γεωμετρία, όπου για παράδειγμα  $\mathcal{T}$  είναι η παραγόμενη κατηγορία (derived category)  $D^b(\text{coh}X)$  των coherent sheaves υπεράνω μιας λείας προβολικής ποικιλότητας (smooth projective variety  $X$ ), στη Θεωρία Αναπαραστάσεων, όπου  $\mathcal{T}$  είναι η παραγόμενη κατηγορία  $D^b(\mathcal{A})$  μιας αβελιανής κατηγορίας  $\mathcal{A}$  αναπαραστάσεων μιας φαρέτρας ή γενικότερα μιας άλγεβρας, κλπ. Ο κυριότερος λόγος είναι ότι ο χώρος συνθηκών ευστάθειας μας εφοδιάζει με νέους λεπτότερους χώρους παραμέτρων (moduli spaces) στη μελέτη μιας ποικιλότητας ή άλγεβρας.

Ο κύριος σκοπός της Μεταπτυχιακής Διατριβής είναι να παρουσιάσει, έχοντας ως βασικό οδηγό την εργασία [2], μια εισαγωγή στην θεωρία των χώρων συνθηκών ευστάθειας μιας τριγωνισμένης κατηγορίας, αναλύοντας κάποιες από τις εφαρμογές της. Σε αυτό το πλαίσιο, σημαντικό ρόλο στην ανάλυση της δομής των χώρων συνθηκών ευστάθειας διαδραματίζει η θεωρία των  $t$ -structures, βλέπε την εργασία [1], η θεωρία Tilting, βλέπε την εργασία [7], και η θεωρία των Quasi-abelian κατηγοριών, βλέπε την εργασία [17], τα βασικά στοιχεία των οποίων θα αναλυθούν στη Διατριβή.



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