Πρόγραμμα Μεταπτυχιακών Σπουδών στη Φυσική
Τμήμα Φυσικής
Σχολή Θετικών Επιστημών
Πανεπιστήμιο Ιωαννίνων

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ (Μ.Δ.Ε)

Αναλλοίωτες Αρχές Μεταβολών σε Τροποποιημένες Θεωρίες Βαρύτητας: Θεωρία Horndeski

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Παναγιώτα Καντή

Ιωάννινα
Σεπτέμβριος 2025

Postgraduate Studies In Physics
Department of Physics
University of Ioannina

Master’s Thesis (M.Sc.)

Invariant Variational Principles in Modified Theories of Gravity: Horndeski Theory

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September 2025

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In memory of my beloved brother
Loukas

Acknowledgments

First, I would like to express my gratitude to Professor Panagiota Kanti for her supervision of this thesis and for her patience throughout this process. I would also like to thank Emeritus Professor George Leontaris and Professor Leandros Perivolaropoulos for their participation in the three-member examination committee.

I am incredibly grateful to my friends, Katerina Maskanaki and Vagelis Karantanis, for their encouragement throughout this journey. Finally, I owe my greatest debt of gratitude to my family and to Aristea Enezli for their constant support throughout the years.

ABSTRACT

Gravity was the first of the fundamental interactions to be studied, yet it remains the least understood. In the 17th century, Isaac Newton formulated his law of universal gravitation. Although Newton's law could predict how gravity behaved, it offered no explanation for the mechanism behind the attraction of two bodies across empty space. This mystery persisted for centuries until 1915, when Albert Einstein reimagined gravity itself not as a force, but as the curvature of spacetime caused by mass and energy. Despite the successes of General Relativity, numerous alternative theories have since emerged in attempts to modify Einstein's theory. Lovelock's theorem provided a theoretical foundation for this pursuit by constraining the possible forms of such modifications. Among the simplest of these is the introduction of a scalar field, leading to the class of models known as scalar–tensor theories. Scalar-tensor theories can describe both the early inflationary phase of the universe, where a scalar field drives the inflationary expansion, and the late-time accelerated expansion, where the scalar field provides an alternative to the cosmological constant in ΛCDM. In the 1970s, Lovelock and Horndeski formulated the most general scalar–tensor theory in four dimensions.

In this thesis, we study modified theories of gravity using invariant variational principles. In Chapter 2, we apply the methodology originally employed by Horndeski to a simpler model of gravity, whose Lagrangian involves the metric, its first and second derivatives, and a scalar field along with its first derivative. In Chapter 3, we examine the modern formulation of Horndeski's theory, its re-emergence in physics, and derive the corresponding field equations.

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Chapter 1

Introduction

1.1 Mathematical Preliminaries

Within the framework of the General Theory of Relativity (GR), gravity ceases to be understood as a force and is instead conceived as a consequence of the geometry of spacetime. To better comprehend this concept, it is essential to define certain mathematical entities first, in order to describe a curved spacetime. The branch of mathematics that studies these properties is known as Differential Geometry, and the tools required to describe gravity will be introduced in this section [1], [2].

1.1.1 Tensors

A key benefit of classical vector analysis is its ability to represent geometric and physical relationships compactly, independent of any coordinate system. However, the traditional notion of a vector can be restrictive. While a vector in three-dimensional space is fully described by three components relative to a chosen coordinate system, many physical and geometric quantities demand more than three components for their complete characterization (Inertia tensor of a rigid body I_{ij}, Electromagnetic field tensor $F_{\mu\nu}$). Tensor calculus offers a natural extension to address these more complex cases. Moreover, a proper description of nature should not depend on the coordinate system chosen for representation. Mathematically, this is achieved by expressing physical laws in terms of tensorial quantities. Tensors are a generalization of vectors and dual vectors. The components of a tensor can be determined once a specific basis is chosen for the vectors on which the tensor acts.

In a coordinate transformation:

$$\bar{x}^{\mu} = \bar{x}^{\mu}(x^{\nu}) \tag{1.1.1}$$

the components of a vector transform as

$$\bar{A}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}} A^{\nu}. \tag{1.1.2}$$

Similarly, for the components of a dual vector, the transformation rule is

$$\bar{A}_{\mu} = \frac{\partial x^{\nu}}{\partial \bar{x}^{\mu}} A_{\nu}. \tag{1.1.3}$$

Generalizing this, the components of a tensor of rank (κ,ρ)-which we will hereafter equate with the tensor itself-follow the transformation law

$$\bar{A}^{\mu_1\mu_2\ldots\mu_\kappa}{}_{\nu_1\nu_2\ldots\nu_\rho}(\bar{x}^{\sigma}) = \frac{\partial \bar{x}^{\mu_1}}{\partial x^{\lambda_1}} \frac{\partial \bar{x}^{\mu_2}}{\partial x^{\lambda_2}} \cdots \frac{\partial \bar{x}^{\mu_\kappa}}{\partial x^{\lambda_\kappa}} \frac{\partial x^{\tau_1}}{\partial \bar{x}^{\nu_1}} \frac{\partial x^{\tau_2}}{\partial \bar{x}^{\nu_2}} \cdots \frac{\partial x^{\tau_\rho}}{\partial \bar{x}^{\nu_\rho}} A^{\lambda_1\lambda_2\ldots\lambda_\kappa}{}_{\tau_1\tau_2\ldots\tau_\rho}(x^{\xi}). \tag{1.1.4}$$

The way in which a tensor transforms under a change of coordinates ensures that any tensorial relation remains independent of the coordinate system in which it is expressed. Consequently, any equation describing a fundamental law of nature must be a tensorial equation. Otherwise, it would only hold in a specific reference frame and would change under a transformation to another frame.

An important extension of the concept of tensors is the introduction of relative tensors[1] [3]. This significance arises from the fact that the integral of a scalar does not qualify itself as a scalar.[2] To ensure that an integral over a given region remains invariant under arbitrary coordinate transformations, it is necessary for the integrand to be a relative scalar, commonly referred to as a scalar density.

In general, a set of $n^{\kappa+\rho}$ functions

$$A^{\mu_1\ldots\mu_\kappa}{}_{\nu_1\ldots\nu_\rho}(x^{\xi})$$

constitutes the components of a relative tensor field of type (κ,ρ) and weight w if, under the coordinate transformation (1.1.1), these functions transform according to the following relation

$$\bar{A}^{\lambda_1\ldots\lambda_\kappa}{}_{\tau_1\ldots\tau_\rho}(\bar{x}^{\sigma}) = B^{w} \frac{\partial \bar{x}^{\lambda_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \bar{x}^{\lambda_\kappa}}{\partial x^{\mu_\kappa}} \frac{\partial x^{\nu_1}}{\partial \bar{x}\ \ ^{\tau_1}} \cdots \frac{\partial x^{\nu_\rho}}{\partial \bar{x}\ \ ^{\tau_\rho}} A^{\mu_1\ldots\mu_\kappa}{}_{\nu_1\ldots\nu_\rho}(x^{\xi}) \tag{1.1.5}$$

where $B = \det\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}}\right)$ is the Jacobian of the transformation. Similarly, a scalar function $\psi(x^{\mu})$

[1] In physics, the term "tensor density" is common. Here we use the more general mathematical term "relative tensor," of which tensor densities are a special case.

[2] In physics, it is frequently needed to deal with invariant integrals in order to construct actions of physical theories such as General Relativity and Electromagnetism.

is said to represent a relative scalar field of weight w, if under the coordinate transformation (1.1.1), the transformation of this function is given by

$$\bar{\psi}(\bar{x}^{\nu}) = B^{w}\psi(x^{\mu}). \tag{1.1.6}$$

A relative scalar of unit weight is called a scalar density. To highlight the significance of the concept, we present the following example. For the integral of a scalar function $f(x^{\mu})$,

$$I = \int f(x^{\mu})\, dx^1 \ldots dx^n,$$

the corresponding value of the integral in the $\bar{x}$-coordinate system is expressed as

$$\bar{I} = \int \bar{f}(\bar{x}^{\mu}) B^{-1} dx^1 \ldots dx^n$$

where $\bar{f}(\bar{x}^{\mu})$ is the transformation of $f(x^{\mu})$. The B^{-1} term appears due to the transformation properties of the volume element. On the other hand, if the integrand $f(x^{\mu})$ happens to be a scalar density, then it follows immediately from the transformation (1.1.6) that

$$\bar{f}(\bar{x}^{\mu}) B^{-1} = f(x^{\mu})$$

and if this is substituted, one obtains

$$\int \bar{f}(\bar{x}^{\mu}) d\bar{x}^1 \ldots d\bar{x}^n = \int f(x^{\mu}) dx^1 \ldots dx^n$$

which ensures that the integral achieves the desired invariance.

1.1.2 The Metric Tensor

In physics, particularly in differential geometry and GR, the metric, is defined as a $(0,2)$-type tensor on a manifold. The metric provides a means of measuring distances on the manifold and encapsulates all the information about the geometry of the space. By selecting a specific basis, the components of the metric can be represented as a symmetric $n \times n$ matrix, where $n = \dim(M)$ is the dimension of the manifold M. The metric must only satisfy the following conditions:

- **Symmetry:**

$$g_{\mu\nu} = g_{\nu\mu}$$

- **Non-degeneracy:**

$$\det(g_{\mu\nu}) \neq 0$$

The requirement of non-degeneracy allows us to define the inverse metric, as the matrix $g_{\mu\nu}$ is invertible. We denote the inverse metric as $g^{\mu\nu}$, and it satisfies the relation

$$g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu}.$$

Similarly, $g^{\mu\nu}$ can be considered as the components of a symmetric $(2,0)$-type tensor. Using the metric and the inverse metric, we can raise and lower indices of vectors and tensors as follows

$$X_\mu = g_{\mu\nu}X^\nu$$

$$X^\mu = g^{\mu\nu}X_\nu.$$

To gain a clearer understanding of the concepts discussed above, we will now present a few simple examples of manifolds. In a three-dimensional Euclidean space with Cartesian coordinates (x,y,z), the metric takes the form

$$g_{\mu\nu} = \delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the line element, which represents the infinitesimal distance between two points in the space, is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

For the same three-dimensional Euclidean space, performing a coordinate transformation

$$(x,y,z) \to (r,\theta,\phi)$$

where

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

the line element becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

and the components of the metric are

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}.$$

Although the components of the metric in the two previous cases are different, the space and its properties remain the same.

Another example of a manifold is the Minkowski spacetime of the Special Theory of Relativity (SR). In this case, the space is pseudo-Euclidean, and by choosing a suitable Cartesian coordinate system (t,x,y,z) for its description, the metric takes the following form

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The line element, corresponding to the infinitesimal distance between two events on the manifold, is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

or equivalently

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta.$$

Both examples above describe flat spaces. In the first example of three-dimensional Euclidean space, the metric is positive definite, and the corresponding manifold is Riemannian. In the second case, Minkowski space serves as an example of a Lorentzian manifold, where the metric is not positive definite. The negative sign associated with the time coordinate ensures that the spacetime interval ds^2 is invariant under Lorentz transformations. From these examples, we see that the metric essentially provides a way to connect the physical distances between "points" on a manifold to the variations in the coordinates used to describe the manifold.

After formulating the Special Theory of Relativity in 1905, Einstein aimed to reconcile gravity with the principles of SR. This led to General Relativity (GR), where spacetime is modeled as a Lorentzian manifold. The properties of spacetime are also described by a metric. However, in this case, the metric does not necessarily have the form of the Minkowski metric. The space ceases to be flat, and the line element takes the general form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

The deviation from the flat-space metric arises from curvature. This curvature gives rise to the physical phenomena we interpret as gravity. However, as discussed above, the form of the metric depends on the choice of the coordinate system. Therefore, the metric alone is not a reliable criterion for determining the curvature of a manifold. In the following, we will see that the appropriate way to describe the curvature of a manifold is through the Riemann Curvature Tensor.

1.1.3 Covariant Derivative

In an arbitrary manifold, the concept of differentiation is different from the usual one, as defining a derivative requires, in some way, the comparison of vectors or tensors in general, which are defined on different tangent and cotangent spaces of the manifold. Therefore, to define the derivative, we need a way of relating these different spaces. There are various definitions of differentiation, which use different methods to connect the tangent spaces. In this subsection, we will present the covariant derivative [1].

In general, if we try to differentiate a vector or a tensor field in the usual way

$$\partial_\nu a^\mu(x)$$

we can see that under general coordinate transformations (1.1.1), this object ceases to be a tensor. Indeed, we have the relation

$$\partial_{\bar{\nu}}\bar{a}^\mu(\bar{x}) = \frac{\partial x^\sigma}{\partial x^{\bar{\nu}}}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial \bar{x}^\mu}{\partial x^\rho}a_\rho(x)\right) = \frac{\partial x^\sigma}{\partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^\rho}\partial_\sigma a^\rho + \frac{\partial x^\sigma}{\partial \bar{x}^\nu}\frac{\partial^2 \bar{x}^\mu}{\partial x^\sigma \partial x^\rho}a^\rho$$

and comparing with the general tensor transformation relation (1.1.4), we see that the second term, which contains second derivatives, breaks the tensorial character. For transformations where the second derivative vanishes, then this object would indeed be a tensor. This is the case in SR where transformations are restricted to Poincaré (Lorentz boosts, rotations, and translations), which is why there is no issue with differentiating tensors. This reflects the fact that SR operates in flat spacetime, where a single global tangent space suffices, and no additional structure is needed for differentiation. On the other hand, in general spaces with curvature, the definition of different tangent spaces at each point of the manifold is required, and these spaces must somehow be related. The definition of the covariant derivative requires the introduction of an additional structure on the manifold, known as an affine connection. The covariant derivative is typically defined as

$$\nabla_{e_k} e_j = \omega^i_{jk}(x) e_i,$$

where the coefficients ω^i_{jk} represent the connection and e_j is an arbitrary vector base. In general, the choice of connection on a manifold is arbitrary (but it must be chosen in such a way that various compatibility conditions are satisfied). Each different choice of connection leads to a different definition of the covariant derivative. With any given connection, we can associate a tensor called the torsion tensor. The torsion tensor captures the antisymmetry of a connection in its lower indices. In a coordinate basis, it is defined by

$$T^\mu_{\nu\rho} = \omega^\mu_{\nu\rho} - \omega^\mu_{\rho\nu}.$$

In the case of General Relativity, we make a specific choice of connection known as the Levi-Civita connection which, is characterized by two fundamental properties:

- **Absence of torsion**:
  The torsion tensor vanishes ($T^\mu{}_{\nu\rho} = 0$), implying the connection coefficients are symmetric in their lower indices:
  $$\omega^\mu{}_{\nu\rho} = \omega^\mu{}_{\rho\nu}$$
  This symmetry is essential for locally reducing to Cartesian coordinates (where $\omega^\mu_{\nu\rho} = 0$) in small spacetime regions.

- **Metric compatibility**:
  The covariant derivative of the metric vanishes:
  $$\nabla_\rho g_{\mu\nu} = 0, \quad \nabla_\rho g^{\mu\nu} = 0.$$
  This ensures that inner products are preserved under parallel transport.

It is customary, when using the coordinate basis, to denote the connection coefficients by $\Gamma^\mu_{\nu\rho}$, called the Christoffel symbols. Due to the torsion-free property of the Levi-Civita connection, these symbols are symmetric in their lower indices:

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}.$$

However, $\Gamma^\mu_{\nu\rho}$ is *not* a tensor. Under a coordinate transformation $\bar{x}^\mu = \bar{x}^\mu(x^\nu)$, it transforms as:

$$\bar{\Gamma}^\nu_{\mu\lambda} = \frac{\partial x^\sigma}{\partial \bar{x}^\lambda} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial \bar{x}^\nu}{\partial x^\rho} \Gamma^\rho_{\sigma\alpha} - \frac{\partial x^\sigma}{\partial \bar{x}^\mu} \frac{\partial x^\alpha}{\partial \bar{x}^\lambda} \frac{\partial^2 \bar{x}^\nu}{\partial x^\alpha \partial x^\sigma}.$$

The non-tensorial character of the connection is necessary. It is essentially a controlled "breaking" of the tensorial behavior in such a way that makes the covariant derivative itself a tensorial quantity. Based on the above, the covariant derivative acts on vector and dual vector fields as follows:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\rho\mu} V^\rho \tag{1.1.7}$$

for a vector field V^ν, and

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\nu\mu} A_\rho \tag{1.1.8}$$

for a dual vector field A_ν. These definitions extend to arbitrary (κ, ρ)-tensors via the Leibniz rule, with a $+\Gamma$ term for each upper index and a $-\Gamma$ term for each lower index. For scalar fields defined on a manifold, the covariant derivative reduces to the usual partial derivative:

$$\nabla_\mu \Phi = \partial_\mu \Phi \tag{1.1.9}$$

For the Levi-Civita connection, we can derive a relationship between the metric and the Christoffel symbols by enforcing metric compatibility ($\nabla_\rho g_{\mu\nu} = 0$) and the torsion-free condition ($\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$). Starting from the covariant derivative of the metric:

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0.$$

Cyclically permuting the indices ρ, μ, ν, we obtain two additional equations

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0.$$

Combining these, yields the following result

$$\nabla_\rho g_{\mu\nu} - \nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\rho\mu} = \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2 g_{\lambda\rho} \Gamma^\lambda_{\mu\nu} = 0$$

which can be recasted as

$$g_{\lambda\rho} \Gamma^\lambda_{\mu\nu} = \frac{1}{2} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right).$$

Multiplying both sides by $g^{\rho\sigma}$, we get

$$g^{\rho\sigma} g_{\lambda\rho} \Gamma^\lambda_{\mu\nu} = \delta^\sigma_\lambda \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)$$

and, the Christoffel symbols are given by

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right). \tag{1.1.10}$$

This shows that the connection is completely determined by the metric and its first derivatives.

1.1.4 Geodesic Equation

We now consider a curve on a manifold parameterised by λ given by $x^\mu(\lambda)$. Then, a vector V^μ is said to be parallel transported along a curve [1], if its direction remains unchanged as it progresses along the curve. In more precise terms,

$$\frac{dx^\mu}{d\lambda} \nabla_\mu V^\nu = 0, \tag{1.1.11}$$

which can be thought of as projecting the covariant derivative $\nabla_\mu V^\nu$ along the direction of the tangent vector to the curve $\frac{dx^\mu}{d\lambda}$.

If the vector is itself the tangent vector such that $V^\nu = \frac{dx^\nu}{d\lambda}$, then the condition for parallel transport becomes

$$\frac{d^2 x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \tag{1.1.12}$$

This is the geodesic equation, which determines the motion of objects in a curved space in the absence of external forces.[3] Geodesics are the “straightest possible" paths in curved space, generalizing the notion of straight lines in Euclidean geometry. It can also be derived by minimising the action of a free particle in a curved space, given by the integral of the proper time along the path of the particle

$$S = \int d\tau.$$

where,

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu.$$

It can be observed that (1.1.12) recovers the conventional concept of straight lines when the connection coefficients correspond to the Christoffel symbols in Euclidean space. In this case, it is possible to select Cartesian coordinates in which the Christoffel symbols vanish ($\Gamma^\nu_{\mu\sigma} = 0$),

[3]When external forces are present, additional terms appear on the right-hand side of the equation to describe the specific interaction. For example, electromagnetic forces introduce a term proportional to the field strength tensor $\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\sigma}{d\lambda} = (q/m) F^\nu_\mu (dx^\mu/d\lambda)$.

leading to the reduction of the geodesic equation to that of a straight line.

1.1.5 The Riemann Curvature Tensor

The connection contains information about curvature, but only in an indirect way. By itself, the connection is insufficient to describe the geometric notion of curvature, since it is not a tensorial quantity. This means that it will change depending on the coordinate system used even for the same space. The mathematical object that describes the curvature of a manifold is the Riemann curvature tensor, and it is defined as follows [1]

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\lambda\mu\nu}V^\lambda. \tag{1.1.13}$$

This quantity is directly related to how the components of a vector will change when it is parallel transported around a closed curve in the manifold. In the case of a flat space, parallel transporting around a closed curve does not affect the components of a vector, and therefore, $R^\rho_{\lambda\mu\nu} = 0$. Expanding (1.1.13), we obtain the well-known expression for the curvature tensor

$$[\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu(\nabla_\nu V^\rho) - \nabla_\nu(\nabla_\mu V^\rho) \Rightarrow$$

$$[\nabla_\mu, \nabla_\nu]V^\rho = \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}(\nabla_\lambda V^\rho) + \Gamma^\rho_{\mu\sigma}(\nabla_\nu V^\sigma) - \partial_\nu(\nabla_\mu V^\rho) + \Gamma^\lambda_{\nu\mu}(\nabla_\lambda V^\rho) - \Gamma^\rho_{\nu\sigma}(\nabla_\mu V^\sigma)$$

substituting the covariant derivatives in the parentheses based on (1.1.7), we are left with

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]V^\rho = {} & \partial_\mu\left(\cancel{\partial_\nu V^\rho} + \Gamma^\rho_{\nu\sigma}V^\sigma\right) - \Gamma^\lambda_{\mu\nu}\left(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma}V^\sigma\right) \\ & - \partial_\nu\left(\cancel{\partial_\mu V^\rho} + \Gamma^\rho_{\mu\sigma}V^\sigma\right) + \Gamma^\rho_{\mu\sigma}\left(\partial_\nu V^\sigma + \Gamma^\sigma_{\nu\lambda}V^\lambda\right) \\ & + \Gamma^\lambda_{\nu\mu}\left(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma}V^\sigma\right) - \Gamma^\rho_{\nu\sigma}\left(\partial_\mu V^\sigma + \Gamma^\sigma_{\mu\lambda}V^\lambda\right).\end{aligned}$$

Simplifying the terms, the first and fifth term cancel out, because partial derivatives commute $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$. Taking into account the terms that cancel out and renaming some indices $\lambda \leftrightarrow \sigma$ in the two terms where V^λ appears, we arrive at the following relation

$$[\nabla_\mu, \nabla_\nu]V^\rho = \left(\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right)V^\sigma - \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right)\left(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma}V^\sigma\right).$$

Using the definition of the torsion tensor T and noting that the terms in the last parentheses can be written as the covariant derivative of the vector V^ρ according to the definition (1.1.7), we are led to

$$[\nabla_\mu, \nabla_\nu]V^\rho = \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right) V^\sigma - T^\lambda_{\mu\nu}\nabla_\lambda V^\rho. \tag{1.1.14}$$

However, as mentioned before, for the choice of the Levi-Civita connection, the torsion tensor vanishes. Finally, by comparing with (1.1.13), we obtain the final result for the Riemann curvature tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{1.1.15}$$

From (1.1.15), the Riemann curvature tensor satisfies the following symmetries[4]:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$
$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$$
$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

along with the cyclic identity:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0.$$

Another important property that the curvature operator satisfies, is the Bianchi identity:

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\lambda} + \nabla_\nu R_{\rho\sigma\lambda\mu} = 0. \tag{1.1.16}$$

The Riemann curvature tensor fully encodes the geometric curvature of a manifold. By contraction, one obtains the Ricci tensor $R_{\mu\nu}$, a simpler but equally important quantity tensor

$$R_{\sigma\nu} = R^\rho{}_{\sigma\rho\nu},$$

which expands explicitly as:

$$R_{\sigma\nu} = \partial_\rho \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\sigma\rho} + \Gamma^\lambda{}_{\nu\sigma}\Gamma^\rho{}_{\lambda\rho} - \Gamma^\lambda{}_{\rho\sigma}\Gamma^\rho{}_{\nu\lambda}. \tag{1.1.17}$$

Finally, contracting the indices of the Ricci tensor, we can obtain the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.1.18}$$

We can also define the Einstein tensor starting from the Bianchi identity and using the properties of the curvature tensor. Multiplying (1.1.16) by $g^{\mu\rho}$, we get

$$g^{\mu\rho}\left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\lambda} + \nabla_\nu R_{\rho\sigma\lambda\mu}\right) = 0.$$

[4]In n dimensions, these symmetries reduce the Riemann tensor from n^4 to $\frac{1}{12}n^2(n^2-1)$ independent components.

Exploiting the metric compatibility condition $\nabla g = 0$, we simplify this to

$$\nabla_\lambda R_{\sigma\nu} + \nabla_\mu R^\mu{}_{\sigma\nu\lambda} - \nabla_\nu R_{\sigma\lambda} = 0.$$

Contracting with $g^{\sigma\lambda}$ and using $R = g^{\mu\nu}R_{\mu\nu}$ yields

$$\nabla_\lambda R^\lambda{}_\nu + \nabla_\mu R^{\mu\lambda}{}_{\nu\lambda} - \nabla_\nu R = 0. \tag{1.1.19}$$

The second term is simplified as follows

$$\begin{aligned}\nabla_\mu R^{\mu\lambda}{}_{\nu\lambda} &= \nabla_\mu (g^{\mu\sigma} R_{\sigma\nu}) \\ &= \nabla_\mu R^\mu{}_\nu,\end{aligned}$$

where we used the properties of the Riemann tensor. Substituting, we get

$$\nabla_\lambda R^\lambda{}_\nu - \nabla_\nu R + \nabla_\mu R^\mu{}_\nu = 2\nabla_\mu R^\mu{}_\nu - \nabla_\nu R = 0 \Rightarrow$$

$$\nabla_\mu \left(R^\mu{}_\nu - \frac{1}{2}\delta^\mu_\nu R \right) = 0.$$

Finally, multiplying by $g^{\nu\sigma}$, we obtain

$$\nabla_\mu \left(g^{\nu\sigma} R^\mu{}_\nu - \frac{1}{2} g^{\nu\sigma}\delta^\mu_\nu R \right) = 0 \Rightarrow$$

$$\nabla_\mu \left(R^{\mu\sigma} - \frac{1}{2} g^{\mu\sigma} R \right) = 0.$$

The expression within the parentheses is known as the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{1.1.20}$$

and is symmetric. As shown above, we have essentially demonstrated that the Einstein tensor satisfies the relation

$$\nabla_\mu G^{\mu\nu} = 0. \tag{1.1.21}$$

Einstein's fundamental insight was that the geometry of spacetime directly depends on its energy content. In GR, the spacetime metric $g_{\mu\nu}$ is dynamically determined by the distribution of matter and energy. All information about the energy is contained within a symmetric (0, 2)-tensor $T_{\mu\nu}$ called the energy-momentum tensor instead of a mass density scalar function $\rho(\mathbf{x})$ as in Newtonian gravity. With this reasoning, the field equations of gravity should take the form

$$A_{\mu\nu} = \kappa T_{\mu\nu}$$

where $A_{\mu\nu}$ must be some symmetric second-rank tensor and κ is a constant of proportionality. Based on the earlier discussion about curvature, the Riemann curvature tensor is the most suitable choice to describe the geometry of space. However, this cannot be done, as it is a (1, 3) tensor. Therefore, the next logical choice would be the Ricci tensor, which seemingly satisfies all the conditions. Indeed, Einstein's first attempt to formulate the gravitational field equations had the following form

$$R_{\mu\nu} = \kappa T_{\mu\nu}.$$

The problem with the Ricci tensor choice is that, although the conservation of energy requires that

$$\nabla_\mu T^{\mu\nu} = 0.$$

Something similar, as we showed, does not hold in general for the Ricci tensor.

$$\nabla_\mu R^{\mu\nu} \neq 0.$$

A better choice for the tensor $A_{\mu\nu}$, which ultimately turned out to be the correct one, is the Einstein tensor $G_{\mu\nu}$, as it is a symmetric linear combination of the Ricci tensor $R_{\mu\nu}$ and the metric $g_{\mu\nu}$, and as we have shown, it satisfies (1.1.21). Therefore, the correct form of Einstein's field equations is

$$G_{\mu\nu} = \kappa T_{\mu\nu}.$$

The constant κ is determined by requiring that General Relativity reduces to Newtonian gravity in the weak-field limit. This analysis yields the value

$$\kappa = 8\pi G$$

where G is Newton's gravitational constant. The final form of the field equations resulted from Einstein's sustained theoretical work spanning nearly a decade. The critical breakthrough came in mid-1912 when Einstein recognized the fundamental role of the metric tensor in describing gravity. Following this realization, Einstein began a rigorous study of differential geometry with Marcel Grossmann's guidance, who introduced Einstein to Ricci and Levi-Civita's work on tensor calculus [4]. This process was not without its challenges and revisions. A prime example is the Entwurf theory [5], an early, mathematically incomplete attempt at a theory of gravity in collaboration with Grossman.

1.2 Einstein Field Equations

1.2.1 Einstein-Hilbert Action

Like every fundamental physical interaction, gravity can also be described through an action. The action from which the field equations are derived using the calculus of variations must remain invariant under coordinate transformations. Otherwise, the resulting laws would depend on the reference frame chosen in each case. The simplest scalar quantity that can be constructed from the metric $g_{\mu\nu}$ and its derivatives is the Ricci scalar R, which is derived from the curvature tensor. Since gravity is understood as a manifestation of spacetime curvature, it is reasonable to expect that the action will incorporate the curvature tensor in some form. Based on this reasoning, the simplest choice of action is the Einstein-Hilbert action, first proposed in 1915 by David Hilbert [2]. Its form is given by

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int_M \mathrm{d}^4x\sqrt{-g}R \tag{1.2.1}$$

where $\kappa = 8\pi G$. The term $\mathrm{d}^4x\sqrt{-g}$ is the invariant volume element, where g denotes the determinant of the metric tensor, and M is the region of spacetime over which the integral is evaluated. It is straightforward to see from the definitions of the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$, the Ricci tensor $R_{\alpha\beta}$, the Ricci scalar R, and the Christoffel symbols $\Gamma^{\mu}_{\nu\lambda}$ that the Einstein-Hilbert action contains derivatives of the metric $g_{\mu\nu}$ up to second order. The field equations can be derived by considering a variation of the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

for which we assume that both the variation and its first derivative vanish on the boundary of M.

$$\delta g_{\mu\nu}\Big|_{\partial M} = \partial(\delta g_{\mu\nu})\Big|_{\partial M} = 0. \tag{1.2.2}$$

Finally, by setting the variation of the action $\delta S = 0$, we will arrive at the field equations of gravity. Expressing the Ricci scalar using (1.1.18), we obtain

$$\delta S_{\mathrm{EH}} = \frac{1}{2\kappa}\int_M \sqrt{-g}R\,\delta(\sqrt{-g})\,\mathrm{d}^4x + \int_M \delta g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x + \int_M g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x. \tag{1.2.3}$$

It is convenient to use $\delta g^{\mu\nu}$ instead of $\delta g_{\mu\nu}$. These, of course, are not independent, they are related by the well-known relation[5]

$$g^{\mu\nu}g_{\rho\nu} = \delta^{\mu}_{\rho}. \tag{1.2.4}$$

The variation of the first term $\delta\sqrt{-g}$ can be easily computed by considering the identity, according to which, for a square invertible matrix, the following holds

$$\frac{1}{\det(A)}\delta(\det(A)) = \mathrm{Tr}(A^{-1}\delta A).$$

Then, for the metric, we have

$$\delta(\sqrt{-g}) = \frac{1}{2\sqrt{-g}}\delta(-g) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{1.2.5}$$

Based on the above, (1.2.3) can be written as

$$\delta S_{\mathrm{EH}} = \frac{1}{2\kappa}\int_M \mathrm{d}^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \frac{1}{2\kappa}\int_M \sqrt{-g}\,\mathrm{d}^4x g^{\mu\nu}\delta R_{\mu\nu}. \tag{1.2.6}$$

Finally, we compute the variation of the Ricci tensor. Essentially, we will show that the term in the second integral of (1.2.6) is a total derivative and does not contribute to the final result. To proceed, we first derive the variation of the Christoffel symbols. From (1.1.10) we get

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\delta g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) + \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha\delta g_{\beta\nu} + \partial_\beta\delta g_{\alpha\nu} - \partial_\nu\delta g_{\alpha\beta}\right).$$

Using Equation (1.2.4) for the first term and the definition of the Christoffel symbols, we obtain

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha\delta g_{\beta\nu} + \partial_\beta\delta g_{\alpha\nu} - \partial_\nu\delta g_{\alpha\beta}\right) - g^{\mu\rho}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\alpha\beta}. \tag{1.2.7}$$

We can substitute the terms in parentheses using the covariant derivatives

$$\nabla_a\delta g_{\mu\nu} = \partial_\alpha\delta g_{\mu\nu} - \Gamma^{\beta}_{\mu\alpha}\delta g_{\beta\nu} - \Gamma^{\beta}_{\nu\alpha}\delta g_{\mu\beta}.$$

Thus, for the terms with partial derivatives in the parentheses of (1.2.7), we have the following expressions

$$\partial_\alpha\delta g_{\beta\nu} = \nabla_\alpha\delta g_{\beta\nu} + \Gamma^{\rho}_{\alpha\beta}\delta g_{\rho\nu} + \Gamma^{\rho}_{\alpha\nu}\delta g_{\beta\rho},$$

$$\partial_\beta\delta g_{\alpha\nu} = \nabla_\beta\delta g_{\alpha\nu} + \Gamma^{\rho}_{\beta\alpha}\delta g_{\rho\nu} + \Gamma^{\rho}_{\beta\nu}\delta g_{\alpha\rho},$$

$$-\partial_\nu\delta g_{\alpha\beta} = -\nabla_\nu\delta g_{\alpha\beta} - \Gamma^{\rho}_{\nu\alpha}\delta g_{\rho\beta} - \Gamma^{\rho}_{\nu\beta}\delta g_{\alpha\rho}.$$

[5] One can show by considering the variation of this equation that

$$\delta(g^{\mu\nu})g_{\rho\nu} + g^{\mu\nu}\delta g_{\rho\nu} = 0 \Rightarrow$$
$$\delta g_{\rho\lambda} = -g_{\rho\nu}g_{\lambda\mu}\delta g^{\mu\nu}.$$

Substituting we get

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\nabla_{\alpha}\delta g_{\beta\nu} + \Gamma^{\rho}_{\alpha\beta}\delta g_{\rho\nu} + \nabla_{\beta}\delta g_{\alpha\nu} + \Gamma^{\rho}_{\beta\alpha}\delta g_{\rho\nu} - \nabla_{\nu}\delta g_{\alpha\beta}\right) - g^{\mu\rho}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\alpha\beta} \Rightarrow$$

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\nabla_{\alpha}\delta g_{\beta\nu} + \nabla_{\beta}\delta g_{\alpha\nu} - \nabla_{\nu}\delta g_{\alpha\beta}\right) + \cancel{g^{\mu\nu}\Gamma^{\rho}_{\alpha\beta}\delta g_{\rho\nu}} - \cancel{g^{\mu\rho}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\alpha\beta}}.$$

Renaming the summed indices in the third term, $\sigma,\rho \to \rho,\nu$, we observe that the second and third terms cancel, leading to the final result for the variation of the Christoffel symbols

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\nabla_{\alpha}\delta g_{\beta\nu} + \nabla_{\beta}\delta g_{\alpha\nu} - \nabla_{\nu}\delta g_{\alpha\beta}\right). \tag{1.2.8}$$

Next, we compute the variation of the Riemann tensor,

$$(1.1.15) \Rightarrow \delta R^{\mu}_{\alpha\nu\beta} = \partial_{\nu}\delta\Gamma^{\mu}_{\alpha\beta} - \partial_{\beta}\delta\Gamma^{\mu}_{\alpha\nu} + \delta\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\mu}_{\sigma\nu} + \Gamma^{\sigma}_{\alpha\beta}\delta\Gamma^{\mu}_{\sigma\nu} - \delta\Gamma^{\sigma}_{\alpha\nu}\Gamma^{\mu}_{\sigma\beta} - \Gamma^{\sigma}_{\alpha\nu}\delta\Gamma^{\mu}_{\sigma\beta}. \tag{1.2.9}$$

Following our earlier treatment of metric derivatives, we now express the partial derivatives of the Christoffel symbols in terms of covariant derivatives of the Christoffel symbols[6] using the relation

$$\nabla_{\nu}\delta\Gamma^{\mu}_{\alpha\beta} = \partial_{\nu}\delta\Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\nu\sigma}\delta\Gamma^{\sigma}_{\alpha\beta} - \Gamma^{\sigma}_{\alpha\nu}\delta\Gamma^{\mu}_{\sigma\beta} - \Gamma^{\sigma}_{\nu\beta}\delta\Gamma^{\mu}_{\alpha\sigma}.$$

With this approach, the first two terms of (1.2.9) can be written as follows

$$\partial_{\nu}\delta\Gamma^{\mu}_{\alpha\beta} = \nabla_{\nu}\delta\Gamma^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\nu\sigma}\delta\Gamma^{\sigma}_{\alpha\beta} + \Gamma^{\sigma}_{\alpha\nu}\delta\Gamma^{\mu}_{\sigma\beta} + \Gamma^{\sigma}_{\nu\beta}\delta\Gamma^{\mu}_{\alpha\sigma}$$

and

$$-\partial_{\beta}\delta\Gamma^{\mu}_{\alpha\nu} = -\nabla_{\beta}\delta\Gamma^{\mu}_{\alpha\nu} + \Gamma^{\mu}_{\beta\sigma}\delta\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\sigma}_{\alpha\beta}\delta\Gamma^{\mu}_{\sigma\nu} - \Gamma^{\sigma}_{\beta\nu}\delta\Gamma^{\mu}_{\alpha\sigma}.$$

Thus, substituting these expressions, we see that the terms with the products $\Gamma\delta\Gamma$ cancel out, and the final result for the variation of the Riemann tensor is

$$\delta R^{\mu}_{\alpha\nu\beta} = \nabla_{\nu}\delta\Gamma^{\mu}_{\alpha\beta} - \nabla_{\beta}\delta\Gamma^{\mu}_{\alpha\nu}.$$

By summing over the indices μ and ν, we find the variation of the Ricci tensor

$$\delta R_{\alpha\beta} = \nabla_{\mu}\delta\Gamma^{\mu}_{\alpha\beta} - \nabla_{\beta}\delta\Gamma^{\mu}_{\alpha\mu}$$

therefore,

$$g^{\alpha\beta}\delta R_{\alpha\beta} = g^{\alpha\beta}\nabla_{\mu}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\alpha\beta}\nabla_{\beta}\delta\Gamma^{\mu}_{\alpha\mu}.$$

[6] It is worth noting that, although the Christoffel symbols are not tensors, their variation $\delta\Gamma$ transforms as a tensor.

Finally, by renaming the indices $\beta, \mu \to \mu, \rho$ in the second term, we get

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\mu \left(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\rho_{\alpha\rho} \right) = \nabla_\mu A^\mu \tag{1.2.10}$$

where,

$$A^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\rho_{\alpha\rho}.$$

Substituting this into the second integral of (1.2.6), we obtain

$$\delta S_{\text{EH}} = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\nabla_\mu A^\mu.$$

The second integral contains a total derivative. Therefore, using Stokes' theorem, this term reduces to a boundary contribution when integrated, which can be neglected, according to the assumption that the partial derivative of the variation $\delta g_{\mu\nu}$ vanishes at the boundary of the region

$$\frac{1}{2\kappa}\int_M \sqrt{-g}\nabla_\mu A^\mu d^4x = \frac{1}{2\kappa}\oint_{\partial M} A^\mu d\Sigma_\mu.$$

The requirement that the variation of the action vanishes leads to Einstein field equations in vacuum

$$\delta S_{\text{EH}} = \frac{1}{2k}\int_M d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} = 0 \Rightarrow$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{1.2.11}$$

Equivalently, by introducing the Einstein tensor $G_{\mu\nu}$, the gravitational field equations take the following form

$$G_{\mu\nu} = 0.$$

1.2.2 Complete Field Equations

As introduced at the beginning of this section, the Einstein-Hilbert action represents the simplest choice of action we can make. More generally, the complete action includes additional contributions from matter fields and the cosmological constant Λ.

$$S = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(R - 2\Lambda) + S_m, \tag{1.2.12}$$

where

$$S_m = \int_M d^4x\sqrt{-g}\mathcal{L}_m.$$

In this case, the variation of the action is given by

$$\delta S=\frac{1}{2\kappa}\int_M \delta R\sqrt{-g}\,d^4x+\frac{1}{2\kappa}\int_M R\delta(\sqrt{-g})\,d^4x-\frac{1}{\kappa}\int_M \Lambda\delta\sqrt{-g}\,d^4x+\int_M \delta(\sqrt{-g}\mathcal{L}_m)\,d^4x.$$

Based on the calculations for the Einstein-Hilbert action, the expression above can be rewritten as follows

$$\delta S=\frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}+\frac{1}{2\kappa}\int_M \Lambda\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,d^4x+\int_M \delta(\sqrt{-g}\mathcal{L}_m)\,d^4x,$$

which simplifies to

$$\delta S=\frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}+\int_M \delta(\sqrt{-g}\mathcal{L}_m)\,d^4x\Rightarrow$$

$$\delta S=\frac{1}{2}\int_M d^4x\sqrt{-g}\left[\frac{1}{\kappa}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}\right)+\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}=0,$$

and as a result

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=\kappa T_{\mu\nu}, \tag{1.2.13}$$

where $T_{\mu\nu}$ is the energy-momentum tensor introduced in the first section, which is related to the energy and pressure of matter, and is defined as

$$T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}.$$

Lovelock showed [6], [7] that a linear combination of the Einstein tensor $G_{\mu\nu}$ and the metric $g_{\mu\nu}$ represents the most general symmetric second-rank tensor that is divergence-free and can be constructed from the metric and its derivatives up to second order. Thus, (1.2.13) gives the most general modification of Einstein's equations in a four-dimensional space. The Einstein field equations unveil a multitude of extraordinary phenomena, such as the gravitational deflection of light, black holes and the Universe's expansion.[7] In addition, gravitational waves are a natural consequence of Einstein's field equations. The existence of gravitational waves was verified directly for the first time by the LIGO team [8] on September 14, 2015. The cosmological constant Λ was introduced by Einstein himself in an attempt to create static models of the universe, which, however, proved to be unstable. This was, of course, before the expansion of the universe was experimentally confirmed in 1929 by Edwin Hubble. Subsequently, until the 1990s, the idea of the cosmological constant was largely abandoned by most physicists. However, with the discovery of the accelerated expansion of the universe in 1998, the cosmological constant returned to prominence.

[7] It is interesting to note that the first mention of an astronomical object resembling the concept of what we now call “black hole" came in 1784 by John Michell. Michell theorized that, if a star were sufficiently massive and compact, its gravitational pull would be so strong that not even light could escape from it. He referred to these objects as "dark stars". Michell's "dark stars" were an early precursor to the concept of black holes, although his analysis was based on Newtonian physics.

1.2.3 Boundary Contributions and the Gibbons-York-Hawking Term

As discussed above, when the Einstein-Hilbert action variation is considered, surface contributions are encountered that must vanish for the action to be stationary. These surface contributions contain the variation of the metric δg, as well as the variations of the derivatives of the metric, $\delta(\partial g)$. Setting $\delta g = 0$ on the boundary is insufficient to eliminate all surface contributions. It is therefore necessary to fix both the metric and its derivatives on the boundary, as done in the previous section, though this can be a somewhat restrictive condition. The boundary value problem arises, because General relativity exhibits the distinctive feature that the Einstein-Hilbert action includes terms involving second derivatives of the dynamical variable, the metric tensor. These second derivatives do not introduce higher-order equations of motion due to the specific structure of the Einstein-Hilbert action, wherein boundary terms encapsulate the second-derivative contributions. To address this issue and formulate a well-posed variational principle, it is necessary to incorporate additional boundary terms into the Einstein-Hilbert action. These terms preserve the consistency of the theory without modifying its fundamental physics. Gibbons, Hawking, and York (GHY) suggested adding a term proportional to the trace of the extrinsic curvature of the boundary K [9], [10]. With this modification, the action takes the form

$$S_G = \frac{1}{2\kappa}\int_M \mathrm{d}^4x\sqrt{-g}R + \frac{1}{\kappa}\oint_{\partial M}\varepsilon K\sqrt{|h|}d^3y \tag{1.2.14}$$

Based on the findings of the preceding section, the boundary term of the Einstein-Hilbert action can be written as follows

$$\int_M \sqrt{-g}d^4x\nabla_\mu A^\mu = \oint_{\partial M} A^\mu d\Sigma_\mu = \oint_{\partial M}\varepsilon A^\mu n_\mu |h|^{\frac{1}{2}}d^3y$$

where n_μ is the unit normal to ∂M, h is the determinant of the induced metric and

$$\varepsilon = n^\mu n_\mu = \begin{cases} +1 & \text{if } \Sigma \text{ is timelike,} \\ -1 & \text{if } \Sigma \text{ is spacelike.} \end{cases}$$

Considering that $\delta g^{\mu\nu} = 0$ at the boundary, we proceed to compute $A^\mu n_\mu$

$$\delta\Gamma^\mu_{\alpha\beta}|_{\partial M} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha\delta g_{\nu\beta} + \partial_\beta\delta g_{\alpha\nu} - \partial_\nu\delta g_{\alpha\beta}\right)$$

$$\delta\Gamma^\beta_{\alpha\beta}|_{\partial M} = \frac{1}{2}g^{\beta\nu}\left(\partial_\alpha\delta g_{\nu\beta} + \partial_\beta\delta g_{\alpha\nu} - \partial_\nu\delta g_{\alpha\beta}\right)$$

Using (1.2.10),

$$A^\mu|_{\partial M} = \frac{1}{2}\left[g^{\mu\nu}\left(\partial_\alpha\delta g_{\nu\beta} + \partial_\beta\delta g_{\alpha\nu} - \partial_\nu\delta g_{\alpha\beta}\right)g^{\alpha\beta} - g^{\beta\nu}\left(\partial_\alpha\delta g_{\nu\beta} + \partial_\beta\delta g_{\alpha\nu} - \partial_\nu\delta g_{\alpha\beta}\right)g^{\alpha\mu}\right]$$

swapping $\nu \leftrightarrow \alpha$ in the second term and performing some minor computations, we arrive at

$$A_\mu|_{\partial M} = g^{\alpha\beta}\left(\partial_\alpha \delta g_{\mu\beta} - \partial_\mu \delta g_{\alpha\beta}\right)$$

where the inverse of the metric is[8]

$$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{\alpha\beta}.$$

Combining the above results

$$\begin{aligned} n^\mu A_\mu|_{\partial M} &= \left(\varepsilon n^\alpha n^\beta + h^{\alpha\beta}\right)\left(\partial_\alpha \delta g_{\mu\beta} - \partial_\mu \delta g_{\alpha\beta}\right) n^\mu \\ &= \left(\varepsilon n^\alpha n^\beta \partial_\alpha \delta g_{\mu\beta} - \varepsilon n^\alpha n^\beta \partial_\mu \delta g_{\alpha\beta} + h^{\alpha\beta}\partial_\alpha \delta g_{\mu\beta} - h^{\alpha\beta}\partial_\mu \delta g_{\alpha\beta}\right) n^\mu. \end{aligned}$$

By renaming indices on the first term and noting that since $\delta g^{\alpha\beta} = 0$ at the boundary, the tangential derivatives of the metric also vanish $h^{\alpha\beta}\partial_\alpha \delta g_{\mu\beta} = 0$, we arrive at the following result

$$n^\mu A_\mu|_{\partial M} = -h^{\alpha\beta}\partial_\mu \delta g_{\alpha\beta} n^\mu.$$

Taking this into account, the variation of the gravitational action is

$$\delta S = \int_M \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \delta g^{\mu\nu} \sqrt{-g}\, d^4x - \oint_{\partial M} \varepsilon h^{\alpha\beta} \partial_\mu \delta g_{\alpha\beta} n^\mu |h|^{\frac{1}{2}} d^3y. \tag{1.2.15}$$

To offset the second term, it is necessary to incorporate a boundary term into the gravitational action. This boundary term, which should be added to the Einstein-Hilbert action, is not uniquely defined. The most widely adopted boundary term, which ensures that the action remains diffeomorphism invariant and well-posed, is the Gibbons-Hawking-York term

$$S_{GHY} = \frac{1}{\kappa} \oint \varepsilon K |h|^{\frac{1}{2}} d^3y.$$

One can see that the variation of this term leads to the desired result [11]

$$K = h^{\alpha\beta}\left(\partial_\beta n_\alpha - \Gamma^\lambda_{\alpha\beta} n_\lambda\right)$$

therefore,

$$\delta K = -h^{\alpha\beta}\delta\Gamma^\lambda_{\alpha\beta} n_\lambda = -\frac{1}{2} h^{\alpha\beta}\left(\partial_\beta \delta g_{\mu\alpha} + \partial_\alpha \delta g_{\mu\beta} - \partial_\mu \delta g_{\alpha\beta}\right) n^\mu = \frac{1}{2} h^{\alpha\beta} n^\mu \partial_\mu \delta g_{\alpha\beta}$$

which proves that indeed the G.H.Y term cancels the boundary terms that arise from the varia-

[8] Where $h_{\alpha\beta}$ is the induced metric of the hypersurface.

tion of the Ricci Tensor. Before the formal proposal by Gibbons, Hawking, and York, Einstein himself introduced a boundary term in 1916 [12]. He utilized a Lagrangian denoted as H, which is first order in derivatives of the metric, thereby eliminating the necessity to fix the derivative of the metric at the boundary

$$H = g^{\alpha\beta}(\Gamma^{\nu}_{\mu\alpha}\Gamma^{\mu}_{\nu\beta} - \Gamma^{\mu}_{\mu\nu}\Gamma^{\nu}_{\alpha\beta})$$

H differs from R by a total derivative

$$H = R - \nabla_\mu A^\mu$$

where $A^\mu = g^{\alpha\mu}\Gamma^{\beta}_{\beta\alpha} - g^{\mu\beta}\Gamma^{\alpha}_{\alpha\beta}$. This form of the Lagrangian, which is referred to as the gamma-gamma Lagrangian yields the same equations of motion as the Einstein-Hilbert action upon variation, but lacks the clear geometric interpretation of the Ricci scalar.

1.3 Cosmology

1.3.1 FLRW Metric

The distribution of galaxies and, more generally, cosmological structures around us appears isotropic on scales larger than approximately 100 Mpc, suggesting that spacetime possesses spherical symmetry. Combining this observational finding with the Copernican principle, which posits that we do not have a privileged position in the universe, requires that the spatial metric be homogeneous and isotropic. This limits the variety of permissible geometries to being flat $k = 0$, a sphere with positive spatial curvature $k = +1$, or a hyperboloid with constant negative spatial curvature $k = -1$. The general spatial metric is given by:

$$d\Sigma^2 = a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right].$$

The metric for a four-dimensional space with constant spatial curvature can be readily expressed in the following form:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \tag{1.3.1}$$

The FLRW metric, initially introduced by the Soviet mathematician Alexander Friedmann in 1922 [13], [14],[9]is the unique metric of a homogeneous and isotropic space with a time coordinate t, and is therefore useful for describing the Universe on the largest scale. The only dynamical quantity that appears in the FLRW metric is the scale factor $a(t)$, which is determined by solving the Einstein field equations. The coordinates (t, r, θ, ϕ) are called comoving coordinates. Two objects at rest in different spatial comoving coordinates remain at those coordinates (under the influence of cosmic expansion alone) at all times. The physical distance between two bodies at different comoving coordinates depends on the scale factor.

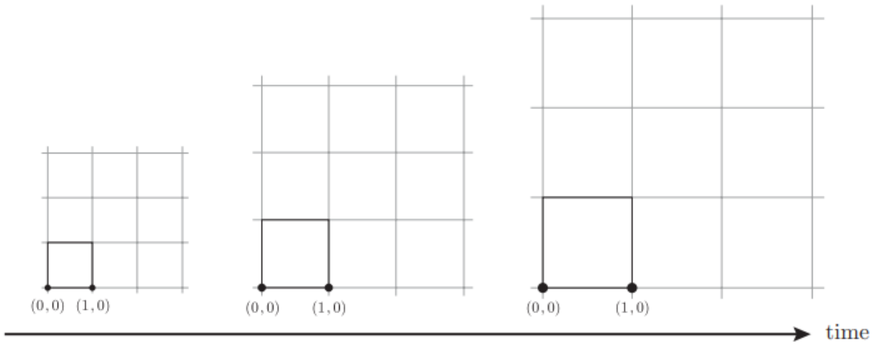


Figure 1.1: The comoving coordinates remain constant during cosmic expansion. However, the physical distance between two points increases [18].

1.3.2 The Hubble-Lemaître Law

The redshift of astronomical spectra was first observed in 1912 by Vesto Slipher. At that time, the interpretation of this phenomenon differed significantly from our modern understanding. Notably, most of the scientific community believed our Galaxy constituted the entire universe. The term "nebulae" was used to describe any extended luminous celestial object beyond stars, whereas today it specifically refers to gas and dust clouds. The nature of redshifted objects remained unclear - whether they belonged to our Galaxy or were separate "island universes" (a term coined by Immanuel Kant). This debate was resolved in 1923 when Edwin Hubble measured Cepheid variable distances to the Andromeda "nebula," proving these were independent galaxies beyond our Milky Way.

[9]FLRW metric was later independently developed by Georges Lemaître in 1927 [15] . In 1935, Robertson and Walker further demonstrated that the FLRW metric uniquely characterizes a spacetime that is both spatially homogeneous and isotropic [16], [17]. Their result was a geometric one and did not depend on the equations of general relativity, which both Friedmann and Lemaître implicitly assumed.

In 1929, Hubble examined the relationship between redshift z and galaxy distances d. Recall that redshift is defined as:

$$z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}}.$$

For relativistic sources moving at velocity u, the special relativistic Doppler effect gives the redshift z as:

$$1+z = \sqrt{\frac{1+u/c}{1-u/c}}.$$

For $u \ll c$, we can Taylor expand to first order yielding

$$z = \frac{u}{c}.$$

Knowing the distances d of 24 galaxies, Hubble observed that there appears to be a linear relationship between the recession velocity of the galaxies and their distances. This relation can be expressed as

$$cz = Hd \quad \text{or} \quad u = Hd. \tag{1.3.2}$$

This same relation had been derived two years earlier, in 1927, by Georges Lemaître [15]. The above equation constitutes the well-known Hubble–Lemaître law, and the quantity H is referred to as the Hubble parameter.[10] Note that H is not a constant, but rather evolves over time. The present-day value of H (like other time-dependent cosmological parameters) is denoted with a subscript zero as H_0, and is referred to as the Hubble constant. It should be clarified that the term recession velocity refers to the rate at which galaxies move away from us due to the homogeneous and isotropic expansion of the universe, and not to their peculiar velocities. Furthermore, due to the assumptions used in deriving Hubble's law, we understand that this relation is an approximation and holds only for galaxies with $u \ll c$. Hubble originally estimated the Hubble constant to be $H_0 = 500\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, a value significantly higher than current measurements, as he underestimated the distances to nearby galaxies.

[10] It should be noted that Hubble never interpreted equation (1.3.2) as evidence for the expansion of the universe.

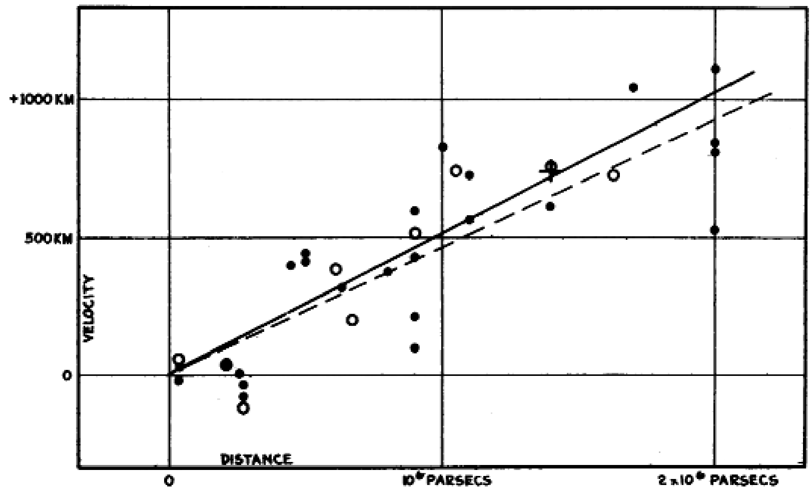


Figure 1.2: The velocity–distance diagram presented in Hubble's publication [19].

The Hubble parameter is defined in terms of the scale factor as:

$$H(t) = \frac{\dot{a}(t)}{a(t)}. \tag{1.3.3}$$

While the Hubble parameter $H(t)$ measures the instantaneous expansion rate, the deceleration parameter $q(t)$ reveals whether this expansion is accelerating or slowing down. The deceleration parameter is defined in terms of the scale factor and its derivatives as follows:

$$q(t) = -\frac{\ddot{a}(t)a(t)}{\dot{a}^2(t)}.$$

The parameter's name and negative sign convention were established during an era when the prevailing view held that the expansion of the universe was decelerating ($\ddot{a} < 0$). This picture changed when, in 1998 the accelerated expansion of the universe ($\ddot{a} > 0$) was experimentally observed by two independent research teams: the High-Z Supernova Search Team and the Supernova Cosmology Project [20], [21]. These two teams used Type Ia supernovae (SN Ia) to measure the parameter. These are stars that belong to the category of white dwarfs and are in binary systems. As these stars gain mass from their companion star, their mass increases and exceeds the Chandrasekhar limit ($1.44 M_{\odot}$), causing them to explode. The intensity of this phenomenon is such that for a few days after the explosion, the brightness of such a star can surpass that of an entire galaxy. Despite the rarity of these explosions, the High-Z Supernova Search Team managed, through observations of 27 SNe Ia to measure the cosmological parameters (H_0, $\Omega_{m,0}$, $\Omega_{\Lambda,0}$, q_0) and also give an estimate for the age of the universe ($t = 14.2 \pm 1.7$ Gyr). The observations ruled out the cases of zero cosmological constant $\Omega_{\Lambda} = 0$ and decelerating expansion $q_0 \geq 0$ with great statistical significance. The value of the Hubble constant resulting from these data is $H_0 = (65 \pm 7)$ km s^{-1} Mpc^{-1}.

1.3.3 Friedmann-Raychaudhuri Equations

As previously discussed, the evolution of the scale factor $a(t)$ is determined by solving the Einstein field equations. In this section, we derive the field equations for a universe that evolves with time, described by the FLRW metric [22]. Starting from the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

and multiplying by $g^{\lambda\mu}$, we obtain

$$g^{\lambda\mu}R_{\mu\nu} - \frac{1}{2}g^{\lambda\mu}g_{\mu\nu}R + \Lambda g^{\lambda\mu}g_{\mu\nu} = \kappa T_{\mu\nu}g^{\lambda\mu}.$$

Thus,

$$R^{\lambda}_{\nu} - \frac{1}{2}\delta^{\lambda}_{\nu}R + \Lambda\delta^{\lambda}_{\nu} = \kappa T^{\lambda}_{\nu},$$

setting $\lambda = \nu$, we obtain

$$R = 4\Lambda - \kappa T. \tag{1.3.4}$$

Substituting (1.3.4) into the initial field equations and performing some algebraic manipulations, we arrive at

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}. \tag{1.3.5}$$

The following analysis is based on this equation. First, we compute the components of the FLRW metric and the corresponding Christoffel symbols. From the line element (1.3.1), we find:

$$g_{00} = -1, \quad g^{00} = -1$$

$$g_{11} = \frac{a^2(t)}{1-kr^2}, \quad g^{11} = \frac{1-kr^2}{a^2(t)}$$

$$g_{22} = a^2(t)r^2, \quad g^{22} = \frac{1}{a^2(t)r^2}$$

$$g_{33} = a^2(t)r^2\sin^2\theta, \quad g^{33} = \frac{1}{a^2(t)r^2\sin^2\theta}.$$

Subsequently, using the definition (1.1.10), we can compute the non-zero components of the Christoffel symbols:

$$\Gamma^0_{11} = \frac{\dot{a}}{a(1-\kappa r^2)}, \quad \Gamma^0_{22} = \frac{\dot{a}r^2}{a}, \quad \Gamma^0_{33} = \frac{ar\dot{r}}{2}\sin^2\theta,$$

$$\Gamma^1_{11} = \frac{\kappa r}{1-\kappa r^2}, \quad \Gamma^1_{22} = -\frac{r}{1-\kappa r^2}, \quad \Gamma^1_{33} = -\frac{r}{1-\kappa r^2}\sin^2\theta,$$

$$\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}, \quad \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}.$$

Next, we compute the components of the Ricci tensor. Using the definition of the curvature tensor (1.1.15) and contracting the first and second indices, we have

$$R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}. \tag{1.3.6}$$

By utilizing the non-zero Christoffel symbols and metric components, it can be shown that only the diagonal elements of the Ricci tensor are non-zero. After substituting the Christoffel symbols and performing straightforward calculations, we obtain the non-zero elements of the Ricci tensor:

$$R_{00} = -3\left(\frac{\ddot{a}}{a}\right),$$

$$R_{11} = \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2k\frac{1}{1-kr^2},$$

$$R_{22} = r^2\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2k\right),$$

$$R_{33} = r^2\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2k\right)\sin^2\theta.$$

The scalar curvature is then given by:

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2(t)}\right). \tag{1.3.7}$$

We now have almost all the necessary tools to solve the field equations and determine the time evolution of the scale factor $a(t)$. The final step is to find the appropriate form for the energy-momentum tensor $T_{\mu\nu}$. Approximately, we assume that the universe contains a uniform background of matter within which galaxies behave as particles in a perfect fluid. In this case, the energy-momentum tensor is given by:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{1.3.8}$$

where ρ and p are the energy density and pressure of the fluid, respectively. Choosing the rest frame of the fluid simplifies our analysis since, in this frame, the fluid 4-velocity vector is

$$u^{\mu} = (1,0,0,0).$$

Thus, only the diagonal components of $T_{\mu\nu}$ contribute, with the following values

$$T_{00} = \rho, \quad T_{11} = pg_{11}, \quad T_{22} = pg_{22}, \quad T_{33} = pg_{33}.$$

Additionally, we can easily compute the trace of T

$$T = g_{\mu\nu}T^{\mu\nu} = -\rho + 3p.$$

Now, we can solve the field equations. From (1.3.5), for $\mu \neq \nu$, all terms vanish, and the non-zero equations are

$$R_{00} = \kappa\left(T_{00} - \frac{1}{2}g_{00}T\right) + \Lambda g_{00},$$

$$R_{ii} = \kappa\left(T_{ii} - \frac{1}{2}g_{ii}T\right) + \Lambda g_{ii}.$$

Due to the homogeneity and isotropy of the FLRW metric, the three spatial equations are equivalent. Thus, we obtain only two independent equations for $\mu, \nu = 0$ and $\mu, \nu = 1,2,3$

$$R_{00} = \kappa\left(T_{00} - \frac{1}{2}g_{00}T\right) + \Lambda g_{00} \Rightarrow$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.3.9}$$

Similarly, for the spatial components

$$R_{11} = \kappa\left(T_{11} - \frac{1}{2}g_{11}T\right) + \Lambda g_{11},$$

after some algebraic manipulations, we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{4}(\rho - p) + \frac{\Lambda}{2} - \frac{\ddot{a}}{2a} - \frac{k}{a^2}.$$

By substituting the term $\frac{\ddot{a}}{a}$ from equation (1.3.9) and performing a few algebraic steps, we arrive at the following equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{1.3.10}$$

Equations (1.3.9) and (1.3.10) are known as the Lemaître-Friedmann equations. If we set the cosmological constant $\Lambda = 0$, we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.3.11}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.3.12}$$

From equation (1.3.12), we observe that a positive spatial acceleration ($\ddot{a} > 0$) in the FLRW metric requires $\rho + 3p < 0$.

1.3.4 Cosmological Fluid Components-Evolution equation

The Friedmann equation alone is not sufficient to determine the scale factor $a(t)$. We need an additional equation that describes how energy density ρ evolves as a function of time t or the scale factor $a(t)$. This equation will be derived from the energy conservation condition

$$\nabla_\mu T^{\mu\nu} = 0.$$

For $\nu = 0$, we have

$$\partial_\mu T^{\mu 0} + \Gamma^\mu{}_\rho T^{\rho 0} + \Gamma^0{}_\rho T^{\rho\mu} = 0$$

considering that only diagonal components of $T^{\mu\nu}$ are non-zero, and given that for $\mu \neq 0$ and $\rho \neq 0$, after some straightforward calculations we derive

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{1.3.13}$$

Equation (1.3.13) is known as the fluid equation, and given that the different components of the cosmological fluid do not interact with each other,[11] it is satisfied independently for each component. The equations (1.3.11), (1.3.12), and (1.3.13) form a system of three differential equations involving the unknown quantities a, ρ, and p. However, only two of these equations are independent, as differentiating equation (1.3.11) with respect to time and substituting the fluid equation leads to the acceleration equation (1.3.12).

The final equation we need to add is the equation of state of the fluid, which relates the energy density to the pressure:

[11] Although matter and radiation interacted in the early universe, this assumption is reasonable for most of the evolution of the universe.

$$p = p(\rho).$$

Knowing this, allows us to solve equation (1.3.13) to find the function $\rho = \rho(a)$. Subsequently, using the Friedmann equation, we determine $a(t)$ by applying an appropriate boundary condition. Typically, we use the present value of the scale factor, assuming that $a(t_0) = 1$. In standard cosmology, each component of the cosmological fluid is assumed to satisfy an equation of state of the form

$$p = w\rho \tag{1.3.14}$$

where w is a dimensionless parameter with a value range $-1 < w < 1$ and is typically treated as constant. Substituting the equation of state into the fluid equation (1.3.13), we obtain the density evolution

$$\rho = \frac{\rho_0}{a^{3(1+w)}}. \tag{1.3.15}$$

Thus, the value of the equation-of-state parameter w determines how the fluid evolves over time. We now have all the necessary equations to compute the evolution of the universe through the scale factor $a(t)$. However, as previously discussed, the solutions depend critically on the composition of the cosmological fluid. The primary cases are:

- **Non-relativistic matter** $(w = 0,\ p = 0)$: Includes baryonic matter (ρ_b) and cold dark matter (ρ_{dm}), with negligible pressure. Observations indicate that dark matter dominates the mass content of the universe, interacting via gravity. From equation (1.3.15), the energy density evolves as

  $$\rho_m(t) = \rho_{m,0}\, a^{-3}(t)$$

  or equivalently, in terms of redshift z (using $a(t_0) = 1$)

  $$\rho_m(z) = \rho_{m,0}(1+z)^3.$$

- **Relativistic matter** $(w = \frac{1}{3},\ p = \frac{1}{3}\rho)$: Includes photons, neutrinos, and other relativistic particles. Their energy density scales as

  $$\rho_r(t) = \rho_{r,0}\, a^{-4}(t)$$

  or

  $$\rho_r(z) = \rho_{r,0}(1+z)^4.$$

  The dominant contribution comes from CMB photons. The additional $(1+z)$ factor (beyond $(1+z)^3$) arises from cosmological redshift.

- **Cosmological constant** Λ $(w=-1,\ p=-\rho)$: Represents dark energy with constant energy density
$$\frac{d\rho_\Lambda}{dt}=0 \quad \Rightarrow \quad \rho_\Lambda = \text{constant}.$$
The presence of such a type of matter can lead to an accelerated universe expansion due to the negative pressure that characterizes it (1.3.12). Current observations suggest Λ to dominate the cosmological fluid composition today.

The relative dominance of different cosmological fluid components has evolved with the scale factor $a(t)$, defining distinct cosmic epochs. In the early universe $(a(t)\ll 1)$, radiation $(\rho_r \propto a^{-4})$ dominated. As expansion progressed, matter $(\rho_m \propto a^{-3})$ became dominant around $z_{eq}\approx 3400$, driving structure formation. In the current epoch $(a(t)\approx 1)$, dark energy $(\rho_\Lambda = \text{const.})$ prevails.

1.3.5 Cosmological Parameters

In the previous section, we demonstrated that by determining just a few basic quantities and using the equations we have already derived, we can calculate how the universe evolves- in other words, the scale factor $a(t)$. Specifically, the parameters that need to be determined to compute the scale factor are the following:

$$\rho_{m,0}, \quad \rho_{r,0}, \quad \rho_\Lambda.$$

Typically, in cosmology, instead of working with these parameters directly, we use the dimensionless density parameters defined as:

$$\Omega_i \equiv \frac{8\pi G}{3H^2(t)}\rho_i(t) \tag{1.3.16}$$

where the index i denotes the different components of the cosmological fluid. These parameters are used, in part, because they result in equations with simpler form. For example, starting from the Friedmann equation (1.3.10) and dividing by H^2, we obtain

$$1=\frac{8\pi G}{3H^2(t)}\left(\rho_m(t)+\rho_r(t)\right)+\frac{\Lambda}{3H^2(t)}-\frac{k}{a^2(t)H^2(t)}$$

which can be rewritten as

$$1=\Omega_m(t)+\Omega_r(t)+\Omega_\Lambda(t)+\Omega_k,(t) \tag{1.3.17}$$

where, we used the curvature density parameter defined as:

$$\Omega_k(t) \equiv -\frac{k}{H^2(t)a^2(t)} \tag{1.3.18}$$

and the corresponding parameter for the cosmological constant:

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2(t)}. \tag{1.3.19}$$

From the definition of Ω_Λ and equation (1.3.16), we derive the following expression for the energy density of the cosmological constant

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = \text{constant}.$$

Furthermore, from definition (1.3.18), we see that a positive curvature constant ($k > 0$) corresponds to a negative curvature density parameter, while a negative curvature constant ($k < 0$) corresponds to a positive curvature density parameter.

Equation (1.3.17) holds for all cosmic time t. We can distinguish the following cases:

$$\Omega_m + \Omega_r + \Omega_\Lambda < 1 \Rightarrow \Omega_\kappa > 0, \quad k = -1 \quad \text{(open universe)}$$
$$\Omega_m + \Omega_r + \Omega_\Lambda = 1 \Rightarrow \Omega_\kappa = 0, \quad k = 0 \quad \text{(flat universe)}$$
$$\Omega_m + \Omega_r + \Omega_\Lambda > 1 \Rightarrow \Omega_\kappa < 0, \quad k = +1 \quad \text{(closed universe)}$$

From the above, we see that for a cosmological model to be flat, it must satisfy:

$$\Omega_m + \Omega_r + \Omega_\Lambda = 1 \tag{1.3.20}$$

. Using equation (1.3.16), one can determine the energy density required for a flat Universe with $k = 0$. This is known as critical density and is given by

$$\rho_{\text{crit}} = \frac{3H(t)^2}{8\pi G}.$$

From this equation, we can calculate the required total energy density for a flat universe given a specific value of the Hubble parameter. Observations indicate that in our universe the value of the total energy density is very close to critical.

Based on all the considerations mentioned above, we can rewrite the Friedmann equation

(1.3.10) in the following form

$$H^2(t) = \frac{8\pi G}{3}\left(\sum_i \rho_i\right) - \frac{k}{a^2(t)} \Rightarrow$$

$$H^2(t) = \frac{8\pi G}{3H_0^2}\left(\frac{\rho_{m,0}}{a^3(t)} + \frac{\rho_{r,0}}{a^4(t)} + \frac{\Lambda}{8\pi G} - \frac{k}{a^2(t_0)}\frac{1}{a^2(t)}\right)H_0^2.$$

Using the definitions of the density parameters,

$$\frac{H^2}{H_0^2} = \Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0}a^{-2}. \tag{1.3.21}$$

Alternatively, using the redshift relation $a(t) = (1+z)^{-1}$ with the boundary condition $a(t_0) = 1$, we can express this equation in terms of redshift

$$H^2(z) = H_0^2\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2\right].$$

Following the derivation of the Hubble parameter H in terms of density parameters, we can similarly relate the deceleration parameter q to density parameters using the acceleration equation (1.3.12). Dividing by $H^2 = \left(\frac{\dot{a}}{a}\right)^2$ yields

$$-\frac{a\ddot{a}}{\dot{a}^2} = \frac{4\pi G}{3H^2}\sum_i \rho_i(1+3w_i) \Rightarrow -\frac{a\ddot{a}}{\dot{a}^2} = \frac{1}{2}\frac{8\pi G}{3H^2}\sum_i \rho_i(t)(1+3w_i).$$

Using the characteristic equation-of-state parameters w_i for the various components of the cosmological fluid along with the definition of the deceleration parameter q, we arrive at the following equation

$$q(t) = \frac{1}{2}\left(\Omega_m + 2\Omega_r - 2\Omega_\Lambda\right). \tag{1.3.22}$$

1.4 Modifications of Gravity

General Relativity has established itself as an extraordinarily successful model of gravity and cosmology. It is in remarkable agreement with a wealth of Solar System precision tests, such as gravitational redshift, gravitational lensing of light from distant background stars, anomalous perihelion precession of Mercury, Shapiro time-delay effect, and Lunar laser experiments [23]. Despite the significant success of General Relativity, a multitude of theories emerged to develop a more unified version shortly after its publication in 1915:

- Weyl's Unified Theory (1918), introducing non-metricity to unify gravity and electromagnetism
- Kaluza-Klein Theory (1921), extending GR to higher dimensions
- Eddington's Affine Gravity (1924), treating the connection as fundamental
- Cartan's Torsion Theory (1922-1925), enriching spacetime geometry with torsion.

While these early theories addressed theoretical unification, modern motivations stem from cosmological observations. The Universe is homogeneous and isotropic in cosmological scales and is described by the FLRW metric (1.3.1). The ΛCDM model (the standard model of cosmology), is based on the premise that GR describes gravity in cosmological scales and that dark matter and dark energy Λ, account for the majority of the Universe's energy density. To date, these dominant constituents have only been detected indirectly through their gravitational effects.[12] This discrepancy can be resolved by either introducing exotic forms of energy-matter beyond the Standard Model, or modifying GR itself in cosmological regimes. However, the situation is also unsatisfactory from a theoretical standpoint. GR is a classical theory, whereas the standard model is a quantum field theory. As such, we expect Einstein's theory to break down at very high energies close to the Planck scale, where higher order curvature terms can no longer be neglected. Although we still do not have a physically and mathematically consistent theory of quantum gravity, both theory and observations suggest that GR might have significant corrections in the strong gravity regimes.

1.4.1 Ways to modify GR

General Relativity is based on very solid mathematical and physical foundations. This, however does not mean that the Einstein-Hilbert action is the only action constructed from the metric $g_{\mu\nu}$ that results in the Einstein equations. In four dimensions, the most general Lagrangian is the following [3]

$$L = \alpha\sqrt{-g}R - 2\Lambda\sqrt{-g} + \beta\varepsilon^{\mu\nu\rho\lambda}R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta\rho\lambda} + \gamma\sqrt{-g}\left(\frac{1}{2}R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}\right).$$

A variational analysis reveals that the final two terms do not contribute to the field equations. In general, Lovelock's theorem establishes that the Einstein field equations (1.2.13) are

[12] For example, if one tries to understand the galactic rotation curves, which remain flat far away from the galactic center, using GR, a significant amount of dark matter is needed.

the unique second-order local equations of motion for a metric tensor in four dimensions. However, this uniqueness holds only under specific conditions:

1. Second-order equations of motion
2. No additional fields beyond the metric
3. General covariance
4. Locality
5. Four-dimensional spacetime.

In arbitrary D-dimensions, Lovelock derived the most general symmetric, divergence-free second-rank tensor depending only on the metric and its first two derivatives and showed that the Einstein-Hilbert action is no longer unique [6]. This leads to a generalization of the Einstein-Hilbert action through the Lovelock densities:

$$L^{(h)} = \frac{1}{2^h} \delta^{\mu_1 \cdots \mu_{2h}}_{\nu_1 \cdots \nu_{2h}} R^{\mu_1 \mu_2}_{\nu_1 \nu_2} \cdots R^{\mu_{2h-1} \mu_{2h}}_{\nu_{2h-1} \nu_{2h}}$$

where δ represents the generalized Kronecker delta. Lovelock theories are straightforward extensions to GR. They appear as a sum of terms increasing in curvature order

$$L = aL_0 + \beta L_1 + \gamma L_2 + \cdots$$

In this framework, the $L^{(0)}$ and $L^{(1)}$ terms reproduce the cosmological constant Λ and the standard Einstein-Hilbert action, whereas $L^{(2)}$ gives the Gauss-Bonnet invariant

$$L_0 = \sqrt{-g}\Lambda, \quad L_1 = \sqrt{-g}R, \quad L_2 = \sqrt{-g}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}).$$

The complete Lovelock Lagrangian can be expressed as a sum

$$L = \sum_{h=0}^{k} c_h L^{(h)}$$

that has maximal order k determined by the spacetime dimension

$$k = \left\lfloor \frac{D-1}{2} \right\rfloor,$$

and c_h are constant coefficients. D denotes the dimension of spacetime. For spacetime dimensions satisfying $h = D/2$, the Lovelock density reduces to a topological invariant that does not contribute. Lovelock theories possess two particularly attractive features as extensions of GR:

- They maintain second-order field equations (avoiding Ostrogradsky instabilities)
- They automatically reduce to standard GR in $D = 4$ dimensions.

However, it is known that in these theories, gravity does not propagate at the speed of light. Instead, the speed depends on the curvature of the spacetime.

In light of Lovelock's theorem, we may construct alternative relativistic gravity theories through the following approaches:

- Higher-than-second-order derivatives of the metric
- Fields other than the metric (additional dynamical fields)
- Non-Riemannian geometries (torsion, non-metricity)
- Breaking local Lorentz invariance
- Non-locality
- Extra spacetime dimensions ($D > 4$).

These six directions are not independent of each other. Many theories emerging from one approach admit equivalent descriptions using other modification frameworks.

Since Lovelock's theorem restricts pure metric modifications, the simplest extension introduces an additional scalar degree of freedom. Theories involving a scalar field together with the metric are called Scalar-Tensor theories. The prototype of scalar-tensor theories is Brans-Dicke theory [24]:

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\partial_\mu\phi\partial^\mu\phi\right) + S_m(g_{\mu\nu},\psi)$$

where, ϕ is a scalar field and ω is the only parameter of the theory and it is called the Brans-Dicke parameter.[13] Comparing Brans-Dicke theory with the standard Einstein- Hilbert action we find that this model can be considered as a theory with a varying gravitational constant G_{eff}, defined by

$$G_{\text{eff}} = \frac{1}{\phi}.$$

Since solar system tests imply extreme values for ω, Brans-Dicke theory is not considered as a viable alternative to General Relativity. Despite that, it serves as a model theory within the

[13] ϕ is not present in the matter action S_m, therefore it is not coupled to the matter, but it is nonminimally coupled to gravity.

more general class of theories that include a scalar field. Scalar-tensor theories can describe both the early inflationary phase of the universe, where a scalar field (inflaton) drives the inflationary expansion and the late-time accelerated expansion, where the scalar field provides an alternative to the cosmological constant in ΛCDM. In the 1970s, Lovelock and Horndeski conducted a systematic study of scalar-tensor theories expanding upon Rund's earlier work, leading Horndeski to formulate their most general version in four dimensions. This thesis will investigate Horndeski gravity as an extension to GR. Chapter 2 presents its theoretical foundations, while Chapter 3 explores its re-emergence in modern physics.

Chapter 2

Invariant variational principles for Scalar-Tensor field theories

In this chapter we study a Lagrangian density that depends on both the metric tensor $g_{\mu\nu}$, its first and second derivatives and a scalar field ϕ, and the field's first derivative as a simpler case to demonstrate the methods employed by Horndeski for the formulation of the most general Lagrangian that gives rise to second order equations. The analysis of Sections (2.1)-(2.4) follows [25], [26] and it is heavily based on work done by H. Rund, D. Lovelock and G. Horndeski. At the end of this chapter we review Horndeski's theory of gravity.

2.1 Transformations

Our objective is to construct an action based on a Lagrangian density L of the following form

$$L(g_{\mu\nu};\partial_\rho g_{\mu\nu,};\partial_\rho\partial_\sigma g_{\mu\nu,};\phi;\partial_\mu\phi) \tag{2.1.1}$$

that exhibits invariance under arbitrary coordinate transformations

$$\bar{x}^\mu = \bar{x}^\mu(x^\nu).$$

The action of the theory has the following form

$$S = \int L(g_{\mu\nu};\partial_\rho g_{\mu\nu,};\partial_\rho\partial_\sigma g_{\mu\nu,};\phi;\partial_\mu\phi)\, d^4x$$

and the field equations corresponding to L are defined by

$$E^{\mu\nu}(L) = \frac{\partial}{\partial x^\rho}\left[\frac{\partial L}{\partial(\partial_\rho g_{\mu\nu})} - \frac{\partial}{\partial x^\sigma}\left(\frac{\partial L}{\partial(\partial_\rho\partial_\sigma g_{\mu\nu})}\right)\right] - \frac{\partial L}{\partial g_{\mu\nu}}, \tag{2.1.2}$$

$$E(L) = \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\right) - \frac{\partial L}{\partial\phi} \tag{2.1.3}$$

where the first is obtained from the action through a variation with respect to the metric and the second by a variation with respect to the scalar field. If the action is to be invariant under arbitrary coordinate transformations, then L has to be a scalar density, that is, that L satisfies

$$\bar{L}\left(\bar{g}_{\kappa\lambda};\partial_\tau\bar{g}_{\kappa\lambda};\partial_\tau\partial_\alpha\bar{g}_{\kappa\lambda};\bar{\phi};\partial_\kappa\bar{\phi}\right) = BL\left(g_{\mu\nu};\partial_\rho g_{\mu\nu};\partial_\rho\partial_\sigma g_{\mu\nu};\phi;\partial_\mu\phi\right) \tag{2.1.4}$$

where B is the Jacobian

$$B \equiv \det\left(\frac{\partial x^\mu}{\partial\bar{x}^\kappa}\right) > 0.$$

In the subsequent analysis, we will derive several identities related to the Lagrangian and its derivatives. These identities restrict the structure of L. To simplify the calculations, we denote partial differentiation by a comma in this chapter. We also adopt the following standard notation

$$\Lambda^{\mu\nu} = \frac{\partial L}{\partial g_{\mu\nu}}, \quad \Lambda^{\mu\nu,\rho} = \frac{\partial L}{\partial g_{\mu\nu,\rho}}, \quad \Lambda^{\mu\nu,\rho\sigma} = \frac{\partial L}{\partial g_{\mu\nu,\rho\sigma}}, \quad \Phi = \frac{\partial L}{\partial\phi}, \quad \Lambda^\mu = \frac{\partial L}{\partial\phi_{,\mu}}.$$

We note that the first three terms are symmetric in (μ,ν) and the third is also symmetric in (ρ,σ). Using this notation equations (2.1.2), (2.1.3) can be reformulated as

$$E^{\mu\nu}(L) = \frac{\partial}{\partial x^\rho}\left(\Lambda^{\mu\nu,\rho} - \frac{\partial}{\partial x^\sigma}\Lambda^{\mu\nu,\rho\sigma}\right) - \Lambda^{;\mu\nu},$$

$$E(L) = \frac{\partial}{\partial x^\mu}(\Lambda^\mu) - \Phi.$$

We also define the transformation matrix and its derivatives in the following manner

$$B^\mu_\kappa = \frac{\partial x^\mu}{\partial\bar{x}^\kappa}, \quad B^\mu_{\kappa\lambda} = \frac{\partial^2 x^\mu}{\partial\bar{x}^\lambda\partial\bar{x}^\kappa}, \quad \text{and} \quad B^\mu_{\kappa\lambda\tau} = \frac{\partial^3 x^\mu}{\partial\bar{x}^\tau\partial\bar{x}^\lambda\partial\bar{x}^\kappa}.$$

Next, we compute the transformations of the scalar field as well as the transformations of the metric and their derivatives

$$\phi(x) = \bar{\phi}(\bar{x}), \tag{2.1.5}$$

$$\bar{\phi}_{,\mu}(\bar{x}) = \frac{\partial\bar{\phi}(\bar{x})}{\partial\bar{x}^\mu} = \frac{\partial x^\nu}{\partial\bar{x}^\mu}\frac{\partial\phi(x)}{\partial x^\nu} \Rightarrow$$

$$\bar{\phi}_{,\mu} = B^\nu_\mu\phi_{,\nu}. \tag{2.1.6}$$

The metric tensor transforms as

$$\bar{g}_{\mu\nu} = B^{\rho}_{\mu} B^{\sigma}_{\nu} g_{\rho\sigma}. \tag{2.1.7}$$

Similarly, for the second derivative of the scalar field, we obtain the following result

$$\bar{\phi}_{,\mu\nu}(\bar{x}) = \frac{\partial^2\phi(x)}{\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}} = \frac{\partial x^{\rho}}{\partial\bar{x}^{\nu}}\frac{\partial}{\partial x^{\rho}}\left(\frac{\partial x^{\sigma}}{\partial\bar{x}^{\mu}}\frac{\partial}{\partial x^{\sigma}}\phi\right) = \frac{\partial x^{\rho}}{\partial\bar{x}^{\nu}}\frac{\partial x^{\sigma}}{\partial\bar{x}^{\mu}}\phi_{,\sigma\rho} + \frac{\partial x^{\rho}}{\partial\bar{x}^{\nu}}\frac{\partial}{\partial x^{\rho}}\left(\frac{\partial x^{\sigma}}{\partial\bar{x}^{\mu}}\right)\phi_{,\sigma} \Rightarrow$$

$$\bar{\phi}_{,\mu\nu}(\bar{x}) = B^{\rho}_{\nu} B^{\sigma}_{\mu}\phi_{,\sigma\rho} + B^{\sigma}_{\nu\mu}\phi_{,\sigma}. \tag{2.1.8}$$

For the first derivative of the metric we find

$$\bar{g}_{\mu\nu,\rho} = \frac{\partial}{\partial\bar{x}^{\rho}}(\bar{g}_{\mu\nu}) = \frac{\partial x^{\lambda}}{\partial\bar{x}^{\rho}}\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial x^{\sigma}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\kappa}}{\partial\bar{x}^{\nu}}g_{\sigma\kappa}\right) = \frac{\partial x^{\lambda}}{\partial\bar{x}^{\rho}}\frac{\partial x^{\sigma}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\kappa}}{\partial\bar{x}^{\nu}}\frac{\partial}{\partial x^{\lambda}}g_{\sigma\kappa} + g_{\sigma\kappa}\frac{\partial x^{\lambda}}{\partial\bar{x}^{\rho}}\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial x^{\sigma}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\kappa}}{\partial\bar{x}^{\nu}}\right) \Rightarrow$$

$$\bar{g}_{\mu\nu,\rho} = B^{\lambda}_{\rho} B^{\sigma}_{\mu} B^{\kappa}_{\nu} g_{\sigma\kappa,\lambda} + g_{\sigma\kappa} B^{\sigma}_{\mu\rho} B^{\kappa}_{\nu} + g_{\sigma\kappa} B^{\kappa}_{\nu\rho} B^{\sigma}_{\mu}. \tag{2.1.9}$$

Utilizing this result, we can determine the transformation of the second derivative of the metric as follows

$$\begin{aligned}
\bar{g}_{\mu\nu,\rho\tau} &= \frac{\partial}{\partial\bar{x}^{\tau}}\left(\bar{g}_{\mu\nu,\rho}\right) = \frac{\partial x^{\alpha}}{\partial\bar{x}^{\tau}}\frac{\partial}{\partial x^{\alpha}}\left[B^{\lambda}_{\rho} B^{\sigma}_{\mu} B^{\kappa}_{\nu} g_{\sigma\kappa,\lambda} + g_{\sigma\kappa} B^{\sigma}_{\mu\rho} B^{\kappa}_{\nu} + g_{\sigma\kappa} B^{\kappa}_{\nu\rho} B^{\sigma}_{\mu}\right] \Rightarrow \\
\bar{g}_{\mu\nu,\rho\tau} &= B^{\alpha}_{\tau} B^{\lambda}_{\rho} B^{\sigma}_{\mu} B^{\kappa}_{\nu} g_{\sigma\kappa,\lambda\alpha} + g_{\sigma\kappa,\lambda}\left(B^{\lambda}_{\rho\tau} B^{\sigma}_{\mu} B^{\kappa}_{\nu} + B^{\sigma}_{\mu\tau} B^{\kappa}_{\nu} B^{\lambda}_{\rho} + B^{\kappa}_{\nu\tau} B^{\sigma}_{\mu} B^{\lambda}_{\rho} + B^{\sigma}_{\mu\rho} B^{\kappa}_{\nu} B^{\lambda}_{\tau} + B^{\kappa}_{\nu\rho} B^{\sigma}_{\mu} B^{\lambda}_{\tau}\right) \\
&\quad + g_{\sigma\kappa}\left(B^{\sigma}_{\mu\rho\tau} B^{\kappa}_{\nu} + B^{\sigma}_{\mu\rho} B^{\kappa}_{\nu\tau} + B^{\kappa}_{\nu\rho\tau} B^{\sigma}_{\mu} + B^{\sigma}_{\mu\tau} B^{\kappa}_{\nu\rho}\right).
\end{aligned} \tag{2.1.10}$$

Using the transformations (2.1.5)-(2.1.10) one can determine the tensorial character of the derivatives of L. For example differentiating (2.1.4) with respect to ϕ, we find

$$B\frac{\partial L}{\partial\phi} = \frac{\partial\bar{L}}{\partial\phi} \Rightarrow B\Phi = \bar{\Phi}.$$

This indicates that Φ is a scalar density. Similarly, by differentiating equation (2.1.4) with respect to $\phi_{,\mu}$, $g_{\mu\nu,\rho}$, $g_{\mu\nu}$, and $g_{\mu\nu,\rho\sigma}$, one can conclude that $\Lambda^{\mu\nu,\rho\sigma}$ is a tensor density, as is Λ^{μ}. In contrast, $\Lambda^{\mu\nu}$ and $\Lambda^{\mu\nu,\rho}$ do not represent tensorial quantities. Nevertheless, Rund has demonstrated that by suitably combining these quantities with the Christoffel symbol, one can construct the following tensor densities [27]

$$\Pi^{\mu\nu,\rho} = \Lambda^{\mu\nu,\rho} + \Gamma^{\rho}_{\sigma\lambda}\Lambda^{\mu\nu,\sigma\lambda} + 2\Gamma^{\mu}_{\sigma\lambda}\Lambda^{\sigma\nu,\rho\lambda} + 2\Gamma^{\nu}_{\sigma\lambda}\Lambda^{\mu\sigma,\rho\lambda}, \tag{2.1.11}$$

$$\begin{aligned}
\Pi^{\mu\nu} &= \Lambda^{\mu\nu} + \Gamma^{\mu}_{\rho\kappa}\Lambda^{\rho\nu,\kappa\tau} + \Gamma^{\nu}_{\rho\kappa}\Lambda^{\rho\mu,\kappa\tau} \\
&\quad + \Gamma^{\mu}_{\rho\kappa}\left(\Pi^{\rho\nu,\kappa} - \Gamma^{\kappa}_{\tau\xi}\Lambda^{\rho\nu,\tau\xi} - \Gamma^{\rho}_{\tau\xi}\Lambda^{\tau\nu,\kappa\xi} - \Gamma^{\nu}_{\tau\xi}\Lambda^{\tau\rho,\kappa\xi}\right) \\
&\quad + \Gamma^{\nu}_{\rho\kappa}\left(\Pi^{\rho,\mu} - \Gamma^{\kappa}_{\tau\xi}\Lambda^{\rho\mu,\tau\xi} - \Gamma^{\rho}_{\tau\xi}\Lambda^{\tau\mu,\kappa\xi} - \Gamma^{\mu}_{\tau\xi}\Lambda^{\tau\rho,\kappa\xi}\right).
\end{aligned} \tag{2.1.12}$$

We note that both of these tensors are symmetric in the first two indices. Using $\Pi^{\mu\nu}$ and $\Pi^{\mu\nu;\rho}$ Rund showed that the Euler-Lagrange equations corresponding to (2.1.1) can be written as

$$E^{\mu\nu}(L) = -\left(\nabla_\rho \nabla_\sigma \Lambda^{\mu\nu,\rho\sigma} - \nabla_\rho \Pi^{\mu\nu,\rho} + \Pi^{\mu\nu}\right), \tag{2.1.13}$$

$$E(L) = \nabla_\mu \Lambda^\mu - \Phi. \tag{2.1.14}$$

Based on the earlier observations regarding the tensorial nature of the Lagrangian's derivatives, it is evident that the original form of the Euler-Lagrange equations involves terms that are not tensorial. In contrast, Rund's formulas are manifestly tensorial.

In the calculations that follow, we will make extensive use of the identity transformation. It is therefore important to note that, in the case of the identity transformation, the following properties hold:

$$\bar{x}^\mu = x^\mu, \quad B^\mu_\nu = \delta^\mu_\nu, \quad B^\mu_{\nu\rho} = B^\mu_{\nu\rho\sigma} = 0.$$

2.2 Invariance Identities

In this section we derive the invariance Identities associated with the Lagrangian (2.1.1). To achieve this we differentiate L with respect to the Jacobian matrix B^μ_ν along with its derivatives $B^\mu_{\nu\rho}$ and $B^\mu_{\nu\rho\sigma}$, while utilizing the transformations derived in the previous section. As we demonstrate, these Invariance Identities impose severe restrictions on L, which will be crucial in Section (2.4), where we construct the most general form of the Lagrangian (2.1.1) in 4 dimensions that results in equations of motion that are second-order in the derivatives of both the metric $g_{\mu\nu}$ and the scalar field ϕ.

2.2.1 First Invariance Identity

We differentiate equation (2.1.4) with respect to $B^\mu_{\nu\rho\sigma}$. Noting that the right-hand side is independent of $B^\mu_{\nu\rho\sigma}$, we find

$$\frac{\partial \bar{L}}{\partial \bar{g}_{\mu\nu}} \frac{\partial \bar{g}_{\mu\nu}}{\partial B^\lambda_{\beta\gamma\delta}} + \frac{\partial \bar{L}}{\partial \bar{g}_{\mu\nu,\rho}} \frac{\partial \bar{g}_{\mu\nu,\rho}}{\partial B^\lambda_{\beta\gamma\delta}} + \frac{\partial \bar{L}}{\partial \bar{g}_{\mu\nu,\rho\sigma}} \frac{\partial \bar{g}_{\mu\nu,\rho\sigma}}{\partial B^\lambda_{\beta\gamma\delta}} + \frac{\partial \bar{L}}{\partial \phi} \frac{\partial \phi}{\partial B^\lambda_{\beta\gamma\delta}} + \frac{\partial \bar{L}}{\partial \bar{\phi}_{,\mu}} \frac{\partial \bar{\phi}_{,\mu}}{\partial B^\lambda_{\beta\gamma\delta}} = 0.$$

Using the notation we introduced and the fact that $B^{\mu}_{\nu\rho\sigma}$ appears only in the transformation of the second derivative of the metric, we obtain

$$0 = \bar{\Lambda}^{\mu\nu,\rho\sigma} \frac{\partial}{\partial B^{\lambda}_{\beta\gamma\delta}} \left[g_{\kappa\tau} \left(B^{\kappa}_{\mu\rho\sigma} B^{\tau}_{\nu} + B^{\kappa}_{\mu} B^{\tau}_{\nu\rho\sigma} \right) \right].$$

Utilizing the symmetry of $B^{\mu}_{\nu\rho\sigma}$ with respect to all its lower indices,[1] we deduce that

$$\begin{aligned} 0 = \bar{\Lambda}^{\mu\nu,\rho\sigma} \Bigg[& \frac{1}{6} g_{\kappa\tau} B^{\tau}_{\nu} \delta^{\kappa}_{\lambda} \left(\delta^{\beta}_{\mu} \delta^{\gamma}_{\rho} \delta^{\delta}_{\sigma} + \delta^{\beta}_{\mu} \delta^{\gamma}_{\sigma} \delta^{\delta}_{\rho} + \delta^{\beta}_{\rho} \delta^{\gamma}_{\mu} \delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma} \delta^{\gamma}_{\rho} \delta^{\delta}_{\mu} + \delta^{\beta}_{\rho} \delta^{\gamma}_{\sigma} \delta^{\delta}_{\mu} + \delta^{\beta}_{\sigma} \delta^{\gamma}_{\mu} \delta^{\delta}_{\rho} \right) \\ & + \frac{1}{6} g_{\kappa\tau} B^{\kappa}_{\mu} \delta^{\tau}_{\lambda} \left(\delta^{\beta}_{\nu} \delta^{\gamma}_{\rho} \delta^{\delta}_{\sigma} + \delta^{\beta}_{\nu} \delta^{\gamma}_{\sigma} \delta^{\delta}_{\rho} + \delta^{\beta}_{\sigma} \delta^{\gamma}_{\nu} \delta^{\delta}_{\rho} + \delta^{\beta}_{\sigma} \delta^{\gamma}_{\rho} \delta^{\delta}_{\nu} + \delta^{\beta}_{\rho} \delta^{\gamma}_{\sigma} \delta^{\delta}_{\nu} + \delta^{\beta}_{\rho} \delta^{\gamma}_{\nu} \delta^{\delta}_{\sigma} \right) \Bigg]. \end{aligned}$$

Finally, using the μ, ν symmetry of $\Lambda^{\mu\nu,\rho\sigma}$ for the second term, we find

$$\bar{\Lambda}^{\mu\nu,\rho\sigma} g_{\lambda\tau} B^{\tau}_{\nu} \left(\delta^{\beta}_{\mu} \delta^{\gamma}_{\rho} \delta^{\delta}_{\sigma} + \delta^{\beta}_{\mu} \delta^{\gamma}_{\sigma} \delta^{\delta}_{\rho} + \delta^{\beta}_{\rho} \delta^{\gamma}_{\mu} \delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma} \delta^{\gamma}_{\rho} \delta^{\delta}_{\mu} + \delta^{\beta}_{\rho} \delta^{\gamma}_{\sigma} \delta^{\delta}_{\mu} + \delta^{\beta}_{\sigma} \delta^{\gamma}_{\mu} \delta^{\delta}_{\rho} \right) = 0.$$

This equation is valid for arbitrary transformations, making use of the identity transformation[2] yields

$$g_{\lambda\tau} \left(\Lambda^{\beta\tau,\gamma\delta} + \Lambda^{\beta\tau,\delta\gamma} + \Lambda^{\gamma\tau,\beta\delta} + \Lambda^{\delta\tau,\gamma\beta} + \Lambda^{\delta\tau,\beta\gamma} + \Lambda^{\gamma\tau,\delta\beta} \right) = 0.$$

By employing the symmetries of $\Lambda^{\mu\nu,\rho\sigma}$ with respect to both pairs of its indices, we derive the *First Invariance Identity*:

$$\Lambda^{\mu\nu,\rho\sigma} + \Lambda^{\mu\rho,\nu\sigma} + \Lambda^{\mu\sigma,\rho\nu} = 0. \tag{2.2.1}$$

This identity imposes significant constraints on the Lagrangian and will be used frequently in the following calculations. We note that since $\Lambda^{\mu\nu,\rho\sigma}$ is a tensor density, the identity holds for arbitrary coordinate system. Using (2.2.1), it can be readily shown that $\Lambda^{\mu\nu,\rho\sigma}$ also exhibits the following symmetry

$$\begin{aligned} \Lambda^{\mu\nu,\rho\sigma} &= -\Lambda^{\mu\rho,\nu\sigma} - \Lambda^{\mu\sigma,\nu\rho} = -\Lambda^{\rho\mu,\nu\sigma} - \Lambda^{\sigma\mu,\nu\rho} \\ &= \Lambda^{\rho\sigma,\nu\mu} + \Lambda^{\rho\nu,\mu\sigma} + \Lambda^{\sigma\rho,\mu\nu} + \Lambda^{\sigma\nu,\mu\rho} \\ &= 2\Lambda^{\rho\sigma,\nu\mu} + \Lambda^{\nu\rho,\mu\sigma} + \Lambda^{\nu\sigma,\mu\rho} \\ &= 2\Lambda^{\rho\sigma,\nu\mu} - \Lambda^{\nu\mu,\rho\sigma} \end{aligned}$$

which implies that

$$\Lambda^{\mu\nu,\rho\sigma} = \Lambda^{\rho\sigma,\nu\mu}. \tag{2.2.2}$$

[1] $B^{\kappa}_{\mu\rho\sigma} = B^{\kappa}_{(\mu\rho\sigma)} = \frac{1}{6} \left(B^{\kappa}_{\mu\rho\sigma} + B^{\kappa}_{\mu\sigma\rho} + B^{\kappa}_{\rho\mu\sigma} + B^{\kappa}_{\sigma\rho\mu} + B^{\kappa}_{\rho\sigma\mu} + B^{\kappa}_{\sigma\mu\rho} \right)$

[2] At this point we can drop the bar of $\Lambda^{\mu\nu,\rho\sigma}$.

In summary, we have established the following symmetries

$$\Lambda^{\mu\nu,\rho\sigma} = \Lambda^{\nu\mu,\rho\sigma} = \Lambda^{\mu\nu,\sigma\rho} = \Lambda^{\rho\sigma,\mu\nu}. \tag{2.2.3}$$

2.2.2 Second Invariance Identity

In order to derive the second invariance identity we differentiate (2.1.4) with respect to $B^{\mu}_{\nu\rho}$. As in the previous case, the right-hand side contains no explicit dependence on $B^{\mu}_{\nu\rho}$. Therefore

$$\frac{\partial \bar{L}}{\partial \bar{g}_{\mu\nu}}\frac{\partial \bar{g}_{\mu\nu}}{\partial B^{\lambda}_{\beta\gamma}} + \frac{\partial \bar{L}}{\partial \bar{g}_{\mu\nu,\rho}}\frac{\partial \bar{g}_{\mu\nu,\rho}}{\partial B^{\lambda}_{\beta\gamma}} + \frac{\partial \bar{L}}{\partial \bar{g}_{\mu\nu,\rho\sigma}}\frac{\partial \bar{g}_{\mu\nu,\rho\sigma}}{\partial B^{\lambda}_{\beta\gamma}} + \frac{\partial \bar{L}}{\partial \phi}\frac{\partial \phi}{\partial B^{\lambda}_{\beta\gamma}} + \frac{\partial \bar{L}}{\partial \bar{\phi}_{\mu}}\frac{\partial \bar{\phi}_{\mu}}{\partial B^{\lambda}_{\beta\gamma}} = 0.$$

Using the notation we introduced and the fact that $B^{\mu}_{\nu\rho}$ appears only in the transformation of the derivatives of the metric we obtain

$$0 = \bar{\Lambda}^{\mu\nu,\rho}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left(\bar{g}_{\mu\nu,\rho}\right) + \bar{\Lambda}^{\mu\nu,\rho\sigma}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left(\bar{g}_{\mu\nu,\rho\sigma}\right). \tag{2.2.4}$$

We will evaluate each term in the expression individually. For the first term, using the transformation laws established at the beginning of the chapter, we obtain

$$\bar{\Lambda}^{\mu\nu,\rho}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left(\bar{g}_{\mu\nu,\rho}\right) = \bar{\Lambda}^{\mu\nu,\rho}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left[g_{\sigma\kappa}B^{\kappa}_{\nu}B^{\sigma}_{\mu\rho} + g_{\sigma\kappa}B^{\sigma}_{\mu}B^{\kappa}_{\nu\rho}\right].$$

Changing the indices of the first term and using the symmetry of $\Lambda^{\mu\nu,\rho}$ we get

$$\bar{\Lambda}^{\mu\nu,\rho}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left[2g_{\sigma\kappa}B^{\sigma}_{\mu}\left(B^{\kappa}_{\nu\rho}\right)\right] = \bar{\Lambda}^{\mu\nu,\rho}g_{\sigma\kappa}B^{\sigma}_{\mu}\delta^{\kappa}_{\lambda}\left(\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho} + \delta^{\beta}_{\rho}\delta^{\gamma}_{\nu}\right) \Rightarrow$$

$$\bar{\Lambda}^{\mu\nu,\rho}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left(\bar{g}_{\mu\nu,\rho}\right) = \bar{\Lambda}^{\mu\nu,\rho}g_{\sigma\lambda}B^{\sigma}_{\mu}\left(\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho} + \delta^{\beta}_{\rho}\delta^{\gamma}_{\nu}\right). \tag{2.2.5}$$

For the second term of (2.2.4), using the transformation for the second derivative of the metric yields

$$\bar{\Lambda}^{\mu\nu,\rho\sigma}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left(\bar{g}_{\mu\nu,\rho\sigma}\right) = \bar{\Lambda}^{\mu\nu,\rho\sigma}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left[g_{\kappa\tau,\alpha}\left(B^{\kappa}_{\mu\sigma}B^{\tau}_{\nu}B^{\alpha}_{\rho} + B^{\kappa}_{\mu}B^{\tau}_{\nu\sigma}B^{\alpha}_{\rho} + B^{\kappa}_{\mu}B^{\tau}_{\nu}B^{\alpha}_{\rho\sigma} + B^{\kappa}_{\mu\rho}B^{\tau}_{\nu}B^{\alpha}_{\sigma} + B^{\kappa}_{\mu}B^{\tau}_{\nu\rho}B^{\alpha}_{\sigma}\right) + \dots\right].$$

Here, we have omitted terms that, after differentiation, yield linear contributions to $B^{\lambda}_{\mu\nu}$, as these terms do not contribute once the identity transformation is imposed at the end of the calculation. By renaming the indices $(\rho \leftrightarrow \sigma)$, in the fourth and fifth terms we get

$$\bar{\Lambda}^{\mu\nu,\rho\sigma}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left[g_{\kappa\tau,\alpha}\left(2B^{\kappa}_{\mu\sigma}B^{\tau}_{\nu}B^{\alpha}_{\rho}+2B^{\kappa}_{\mu}B^{\tau}_{\nu\sigma}B^{\alpha}_{\rho}\right)+g_{\kappa\tau,\alpha}B^{\kappa}_{\mu}B^{\tau}_{\nu}B^{\alpha}_{\rho\sigma}\right].$$

Exchanging $(\kappa \leftrightarrow \tau)$ and $(\mu \leftrightarrow \nu)$ in the second term of the parenthesis we obtain

$$\bar{\Lambda}^{\mu\nu,\rho\sigma}\frac{\partial}{\partial B^{\lambda}_{\beta\gamma}}\left[g_{\kappa\tau,\alpha}\left(4B^{\kappa}_{\mu\sigma}B^{\tau}_{\nu}B^{\alpha}_{\rho}\right)+g_{\kappa\tau,\alpha}B^{\kappa}_{\mu}B^{\tau}_{\nu}B^{\alpha}_{\rho\sigma}\right]=\frac{1}{2}\bar{\Lambda}^{\mu\nu,\rho\sigma}\left[4g_{\kappa\tau,\alpha}B^{\tau}_{\nu}B^{\alpha}_{\rho}\delta^{\kappa}_{\lambda}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\sigma}+\delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\right)\right.$$

$$\left.+g_{\kappa\tau,\alpha}\delta^{\alpha}_{\lambda}B^{\kappa}_{\mu}B^{\tau}_{\nu}\left(\delta^{\beta}_{\rho}\delta^{\gamma}_{\sigma}+\delta^{\beta}_{\sigma}\delta^{\gamma}_{\rho}\right)\right]=$$

$$\frac{1}{2}\bar{\Lambda}^{\mu\nu,\rho\sigma}\left[4g_{\lambda\tau,\kappa}B^{\tau}_{\nu}B^{\kappa}_{\rho}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\sigma}+\delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\right)+g_{\kappa\tau,\lambda}B^{\kappa}_{\mu}B^{\tau}_{\nu}\left(\delta^{\beta}_{\rho}\delta^{\gamma}_{\sigma}+\delta^{\beta}_{\sigma}\delta^{\gamma}_{\rho}\right)\right] \tag{2.2.6}$$

where we have renamed $(\alpha \leftrightarrow \kappa)$ for the last result.

Combining the previous results, (2.2.4) yields

$$0=\bar{\Lambda}^{\mu\nu,\rho}g_{\sigma\lambda}B^{\sigma}_{\mu}\left(\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}+\delta^{\beta}_{\rho}\delta^{\gamma}_{\nu}\right)+\frac{1}{2}\bar{\Lambda}^{\mu\nu,\rho\sigma}\left[4g_{\lambda\tau,\kappa}B^{\tau}_{\nu}B^{\kappa}_{\rho}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\sigma}+\delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\right)+g_{\kappa\tau,\lambda}B^{\kappa}_{\mu}B^{\tau}_{\nu}\left(\delta^{\beta}_{\rho}\delta^{\gamma}_{\sigma}+\delta^{\beta}_{\sigma}\delta^{\gamma}_{\rho}\right)\right].$$

Following the approach used in the derivation of the first invariance identity, we apply the identity transformation

$$0=\Lambda^{\mu\nu,\rho}g_{\sigma\lambda}\delta^{\sigma}_{\mu}\left(\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}+\delta^{\beta}_{\rho}\delta^{\gamma}_{\nu}\right)+\Lambda^{\mu\nu,\rho\sigma}\left[2g_{\lambda\tau,\kappa}\delta^{\tau}_{\nu}\delta^{\kappa}_{\rho}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\sigma}+\delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\right)+\frac{1}{2}g_{\kappa\tau,\lambda}\delta^{\kappa}_{\mu}\delta^{\tau}_{\nu}\left(\delta^{\beta}_{\rho}\delta^{\gamma}_{\sigma}+\delta^{\beta}_{\sigma}\delta^{\gamma}_{\rho}\right)\right]$$

$$=2g_{\lambda\nu,\rho}\left(\Lambda^{\beta\nu,\rho\gamma}+\Lambda^{\gamma\nu,\rho\beta}\right)+\frac{1}{2}g_{\mu\nu,\lambda}\left(\Lambda^{\mu\nu,\beta\gamma}+\Lambda^{\mu\nu,\gamma\beta}\right)+g_{\mu\lambda}\left(\Lambda^{\mu\beta,\gamma}+\Lambda^{\mu\gamma,\beta}\right).$$

We can simplify this further by the use of a locally inertial frame at an arbitrary spacetime point P. This is defined as the coordinate frame $\{x^{\mu}\}$ satisfying:

$$g_{\mu\nu}(P)=\eta_{\mu\nu}, \quad \left.\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right|_{P}=0, \quad \Gamma^{\mu}_{\nu\rho}(P)=0.$$

Making use of the above and the tensors we indtroduced in the previous section (2.1.11),(2.1.12),which are now simplified since in this coordinate system $\Lambda^{\mu\nu,\rho}=\Pi^{\mu\nu,\rho}$, we obtain

$$g_{\mu\lambda}\Lambda^{\mu\beta,\gamma}+g_{\mu\lambda}\Lambda^{\mu\gamma,\beta}=0\Rightarrow g_{\mu\lambda}\Pi^{\mu\beta,\gamma}+g_{\mu\lambda}\Pi^{\mu\gamma,\beta}=0.$$

Therefore, we have established the following result

$$\Pi^{\mu\nu,\rho}+\Pi^{\mu\rho,\nu}=0. \tag{2.2.7}$$

Despite the fact that we used a specific coordinate system in order to derive this result, due to the tensorial character of $\Pi^{\mu\nu,\rho}$, this equation is valid for all coordinate systems. Given the symmetry properties of $\Pi^{\mu\nu,\rho}$ it is straightforward to show that

$$\Pi^{\mu\nu,\rho} = \Pi^{\nu\mu,\rho} = -\Pi^{\nu\rho,\mu} = \Pi^{\mu\rho,\nu} = -\Pi^{\mu\nu,\rho}$$

which implies

$$\Pi^{\mu\nu,\rho} = 0 \tag{2.2.8}$$

which is the *Second Invariance Identity*. As a direct consequence of this result we have established that there can not exist a Lagrangian density which does not depend on the second derivative of the metric, $g_{\mu\nu,\rho\sigma}$ given by

$$L = L(g_{\mu\nu}; g_{\mu\nu,\rho}; \phi; \phi_{,\mu})$$

If this were the case, then $\Lambda^{\mu\nu,\rho\sigma} = 0$ and by virtue of equations (2.1.11) and (2.2.8), we would have

$$\Pi^{\mu\nu,\rho} = \Lambda^{\mu\nu,\rho} = 0$$

which implies

$$\frac{\partial L}{\partial g_{\mu\nu,\rho}} = 0.$$

This result holds special significance for the framework of general relativity. Any invariant variational principle that is to yield the field equations must be of the second kind, meaning it must depend on the second derivative of the metric. In general, of course, such field equations would be of the fourth order. However, for the special case of the Einstein-Hilbert Lagrangian, the second derivatives in the Ricci scalar R are arranged in a way that they appear as a total divergence. As a result, the boundary terms absorb these second derivatives and do not contribute to the equations of motion.

As a consequence of the second invariance identity the Euler-Lagrange equations (2.1.13) can be further simplified as follows

$$E^{\mu\nu}(L) = -\Big(\nabla_\rho \nabla_\sigma \Lambda^{\mu\nu,\rho\sigma} + \Pi^{\mu\nu}\Big). \tag{2.2.9}$$

2.2.3 Third invariance Identity

For the final invariance identity we differentiate (2.1.4) with respect to B^{μ}_{ν}. Noting that

$$\frac{\partial B}{\partial B^{\lambda}_{\xi}} = B\left(B^{-1}\right)^{\xi}_{\lambda}$$

where $\left(B^{-1}\right)^{\xi}_{\lambda}$ is the inverse of the transformation matrix B^{λ}_{γ}

$$\left(B^{-1}\right)^{\xi}_{\lambda} B^{\lambda}_{\gamma} = \delta^{\xi}_{\gamma}.$$

The differentiation yields

$$B\left(B^{-1}\right)^{\xi}_{\lambda} L = \bar{\Lambda}^{\mu\nu}\frac{\partial \bar{g}_{\mu\nu}}{\partial B^{\lambda}_{\xi}} + \bar{\Lambda}^{\mu\nu,\rho}\frac{\partial \bar{g}_{\mu\nu,\rho}}{\partial B^{\lambda}_{\xi}} + \bar{\Lambda}^{\mu\nu,\rho\sigma}\frac{\partial \bar{g}_{\mu\nu,\rho\sigma}}{\partial B^{\lambda}_{\xi}} + \Phi\frac{\partial \phi}{\partial B^{\lambda}_{\xi}} + \bar{\Lambda}^{\mu}\frac{\partial \bar{\phi}_{,\mu}}{\partial B^{\lambda}_{\xi}}.$$

In this case, only the fourth term does not contribute. Each of the remaining terms is evaluated independently[3]

$$\begin{aligned}
&\bar{\Lambda}^{\mu}\frac{\partial \bar{\Lambda}_{,\mu}}{\partial B^{\lambda}_{\xi}} = \bar{\Lambda}^{\mu}\frac{\partial}{\partial B^{\lambda}_{\xi}}\left(B^{\nu}_{\mu}\phi_{,\nu}\right) = \bar{\Lambda}^{\mu}\delta^{\nu}_{\lambda}\delta^{\xi}_{\mu}\phi_{,\nu},\\
&\bar{\Lambda}^{\mu\nu}\frac{\partial \bar{g}_{\mu\nu}}{\partial B^{\lambda}_{\xi}} = \bar{\Lambda}^{\mu\nu}\frac{\partial}{\partial B^{\lambda}_{\xi}}\left(g_{\rho\sigma}B^{\rho}_{\mu}B^{\sigma}_{\nu}\right) = \bar{\Lambda}^{\mu\nu}g_{\rho\sigma}\left(\delta^{\rho}_{\lambda}\delta^{\xi}_{\mu}B^{\sigma}_{\nu} + \delta^{\sigma}_{\lambda}\delta^{\xi}_{\nu}B^{\rho}_{\mu}\right),\\
&\bar{\Lambda}^{\mu\nu,\rho}\frac{\partial \bar{g}_{\mu\nu,\rho}}{\partial B^{\lambda}_{\xi}} = \bar{\Lambda}^{\mu\nu,\rho}\frac{\partial}{\partial B^{\lambda}_{\xi}}\left(g_{\sigma\kappa,\tau}B^{\sigma}_{\mu}B^{\kappa}_{\nu}B^{\tau}_{\rho} + \dots\right) = \bar{\Lambda}^{\mu\nu,\rho}g_{\sigma\kappa,\tau}\left(\delta^{\sigma}_{\lambda}\delta^{\xi}_{\mu}B^{\kappa}_{\nu}B^{\tau}_{\rho} + \delta^{\kappa}_{\lambda}\delta^{\xi}_{\nu}B^{\sigma}_{\mu}B^{\tau}_{\rho} + \delta^{\xi}_{\rho}\delta^{\tau}_{\lambda}B^{\kappa}_{\nu}B^{\sigma}_{\mu}\right),\\
&\bar{\Lambda}^{\mu\nu,\rho\sigma}\frac{\partial \bar{g}_{\mu\nu,\rho\sigma}}{\partial B^{\lambda}_{\xi}} = \bar{\Lambda}^{\mu\nu,\rho\sigma}\frac{\partial}{\partial B^{\lambda}_{\xi}}\left(g_{\kappa\tau,\alpha\beta}B^{\kappa}_{\mu}B^{\tau}_{\nu}B^{\alpha}_{\rho}B^{\beta}_{\sigma} + \dots\right) = \bar{\Lambda}^{\mu\nu,\rho\sigma}g_{\kappa\tau,\alpha\beta}\left(B^{\kappa}_{\mu}B^{\tau}_{\nu}B^{\alpha}_{\rho}\delta^{\beta}_{\lambda}\delta^{\xi}_{\sigma} + B^{\kappa}_{\mu}B^{\tau}_{\nu}B^{\beta}_{\sigma}\delta^{\alpha}_{\lambda}\delta^{\xi}_{\rho}\right.\\
&\qquad \left. + B^{\kappa}_{\mu}B^{\alpha}_{\rho}B^{\beta}_{\sigma}\delta^{\tau}_{\lambda}\delta^{\xi}_{\nu} + B^{\tau}_{\nu}B^{\alpha}_{\rho}B^{\beta}_{\sigma}\delta^{\kappa}_{\lambda}\delta^{\xi}_{\mu}\right).
\end{aligned}$$

Taking into account the results derived above, we arrive at

$$\begin{aligned}
B\left(B^{-1}\right)^{\xi}_{\lambda} L = {} & \bar{\Lambda}^{\mu\nu,\rho\sigma}g_{\kappa\tau,\alpha\beta}\left(B^{\kappa}_{\mu}B^{\tau}_{\nu}B^{\alpha}_{\rho}\delta^{\beta}_{\lambda}\delta^{\xi}_{\sigma} + B^{\kappa}_{\mu}B^{\tau}_{\nu}B^{\beta}_{\sigma}\delta^{\alpha}_{\lambda}\delta^{\xi}_{\rho} + B^{\kappa}_{\mu}B^{\alpha}_{\rho}B^{\beta}_{\sigma}\delta^{\tau}_{\lambda}\delta^{\xi}_{\nu} + B^{\tau}_{\nu}B^{\alpha}_{\rho}B^{\beta}_{\sigma}\delta^{\kappa}_{\lambda}\delta^{\xi}_{\mu}\right)\\
& + \bar{\Lambda}^{\mu\nu,\rho}g_{\sigma\kappa,\tau}\left(\delta^{\sigma}_{\lambda}\delta^{\xi}_{\mu}B^{\kappa}_{\nu}B^{\tau}_{\rho} + \delta^{\xi}_{\rho}\delta^{\tau}_{\lambda}B^{\sigma}_{\mu}B^{\kappa}_{\nu} + \delta^{\kappa}_{\lambda}\delta^{\xi}_{\nu}B^{\tau}_{\rho}B^{\sigma}_{\mu}\right)\\
& + \bar{\Lambda}^{\mu\nu}g_{\rho\sigma}\left(\delta^{\rho}_{\lambda}\delta^{\xi}_{\mu}B^{\sigma}_{\nu} + \delta^{\sigma}_{\lambda}\delta^{\xi}_{\nu}B^{\rho}_{\mu}\right)\\
& + \bar{\Lambda}^{\xi}\phi_{,\lambda}.
\end{aligned}$$

[3]Where again we omit the terms that won't contribute at the end of the calculations when we impose the identity transformation.

Applying the identity transformation yields

$$\begin{aligned}
\delta^\xi_\lambda L = {} & \Lambda^{\mu\nu,\rho\sigma} g_{\kappa\tau,\alpha\beta} \Big(\delta^\kappa_\mu \delta^\tau_\nu \delta^\alpha_\rho \delta^\beta_\lambda \delta^\xi_\sigma + \delta^\kappa_\mu \delta^\tau_\nu \delta^\beta_\sigma \delta^\alpha_\lambda \delta^\xi_\rho + \delta^\kappa_\mu \delta^\alpha_\rho \delta^\beta_\sigma \delta^\tau_\lambda \delta^\xi_\nu + \delta^\tau_\nu \delta^\alpha_\rho \delta^\beta_\sigma \delta^\kappa_\lambda \delta^\xi_\mu \Big) \\
& + \Lambda^{\mu\nu,\rho} g_{\sigma\kappa,\tau} \Big(\delta^\sigma_\lambda \delta^\xi_\mu \delta^\kappa_\nu \delta^\tau_\rho + \delta^\sigma_\mu \delta^\kappa_\nu \delta^\tau_\lambda \delta^\xi_\rho + \delta^\sigma_\mu \delta^\xi_\nu \delta^\kappa_\lambda \delta^\tau_\rho \Big) \\
& + \Lambda^{\mu\nu} g_{\rho\sigma} \Big(\delta^\rho_\lambda \delta^\xi_\mu \delta^\sigma_\nu + \delta^\sigma_\lambda \delta^\xi_\nu \delta^\rho_\mu \Big) \\
& + \Lambda^\xi \phi_{,\lambda}
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\delta^\xi_\lambda L = {} & \Lambda^{\kappa\tau,\alpha\xi} g_{\kappa\tau,\alpha\lambda} + \Lambda^{\kappa\tau,\xi\beta} g_{\kappa\tau,\lambda\beta} + \Lambda^{\kappa\xi,\alpha\beta} g_{\kappa\lambda,\alpha\beta} + \Lambda^{\xi\tau,\alpha\beta} g_{\lambda\tau,\alpha\beta} \\
& + \Lambda^{\xi\kappa,\tau} g_{\lambda\kappa,\tau} + \Lambda^{\sigma\xi,\tau} g_{\sigma\lambda,\tau} + \Lambda^{\sigma\kappa,\xi} g_{\sigma\kappa,\lambda} \\
& + \Lambda^{\xi\sigma} g_{\lambda\sigma} + \Lambda^{\rho\xi} g_{\rho\lambda} + \Lambda^\xi \phi_{,\lambda}.
\end{aligned}$$

Renaming the indices in the second, fourth, sixth, and eighth terms leads to the following

$$\begin{aligned}
\delta^\xi_\lambda L = {} & 2\Lambda^{\kappa\tau,\alpha\xi} g_{\kappa\tau,\alpha\lambda} + 2\Lambda^{\kappa\xi,\alpha\tau} g_{\kappa\lambda,\alpha\tau} + 2\Lambda^{\xi\kappa,\tau} g_{\lambda\kappa,\tau} \\
& + \Lambda^{\sigma\kappa,\xi} g_{\sigma\kappa,\lambda} + 2\Lambda^{\xi\sigma} g_{\lambda\sigma} + \Lambda^\xi \phi_{,\lambda}.
\end{aligned}$$

Exchanging the indices $(\alpha \leftrightarrow \kappa)$ and using the first invariance identity for the first term we arrive at

$$\begin{aligned}
\delta^\xi_\lambda L = {} & 2\Lambda^{\kappa\xi,\alpha\tau} \Big(g_{\kappa\lambda,\alpha\tau} + g_{\alpha\tau,\kappa\lambda} \Big) + 2\Lambda^{\xi\kappa,\tau} g_{\lambda\kappa,\tau} \\
& + \Lambda^{\sigma\kappa,\xi} g_{\sigma\kappa,\lambda} + 2\Lambda^{\xi\sigma} g_{\lambda\sigma} + \Lambda^\xi \phi_{,\lambda}.
\end{aligned} \tag{2.2.10}$$

Our goal is to express the right-hand side of this equation in terms of tensorial quantities. The idea being that if the final result is expressed in terms of tensorial quantities then it holds for arbitrary coordinate systems. To achieve this we will try to express the first term that involves the second derivatives of the metric, and $\Lambda^{\xi\sigma} g_{\lambda\sigma}$ in terms of the Riemann tensor, $\Lambda^{\mu\nu,\rho\sigma}$ (which as we noted is tensorial) and Rund's tensor $\Pi^{\mu\nu}$. At last, we will work in locally inertial coordinates, where the first derivative of the metric vanishes.

At the spacetime point P of the locally inertial frame the Riemann tensor becomes

$$R_{\tau\lambda\alpha\kappa} = \frac{1}{2} \Big(g_{\tau\kappa,\lambda\alpha} + g_{\lambda\alpha,\tau\kappa} - g_{\lambda\kappa,\tau\alpha} - g_{\tau\alpha,\lambda\kappa} \Big).$$

Multiplying with $\Lambda^{\kappa\xi,\alpha\tau}$

$$\Lambda^{\kappa\xi,\alpha\tau}R_{\tau\lambda\alpha\kappa}=\frac{1}{2}\Big(g_{\tau\kappa,\lambda\alpha}\Lambda^{\kappa\xi,\alpha\tau}+g_{\lambda\alpha,\tau\kappa}\Lambda^{\kappa\xi,\alpha\tau}-g_{\lambda\kappa,\tau\alpha}\Lambda^{\kappa\xi,\alpha\tau}-g_{\tau\alpha,\lambda\kappa}\Lambda^{\kappa\xi,\alpha\tau}\Big). \tag{2.2.11}$$

Using the following identity that allows us to exchange one index of each pair of $\Lambda^{\mu\nu,\rho\sigma}$, when it is contracted with any symmetric tensor $a_{\mu\nu}$

$$a_{\mu\nu}\Lambda^{\rho\mu,\sigma\nu}=\frac{1}{2}a_{\mu\nu}\left(\Lambda^{\rho\mu,\sigma\nu}+\Lambda^{\rho\nu,\sigma\mu}\right)=-\frac{1}{2}a_{\mu\nu}\Lambda^{\rho\sigma,\mu\nu}$$

we can reformulate the first two terms as

$$\Lambda^{\kappa\xi,\alpha\tau}g_{\tau\kappa,\lambda\alpha}=-\frac{1}{2}g_{\tau\kappa,\lambda\alpha}\Lambda^{\kappa\tau,\xi\alpha},$$
$$\Lambda^{\kappa\xi,\alpha\tau}g_{\lambda\alpha,\tau\kappa}=-\frac{1}{2}g_{\lambda\alpha,\tau\kappa}\Lambda^{\kappa\tau,\xi\alpha}.$$

Substituting into (2.2.11), we obtain

$$\Lambda^{\kappa\xi,\alpha\tau}R_{\tau\lambda\alpha\kappa}=-\frac{1}{4}g_{\tau\kappa,\lambda\alpha}\Lambda^{\kappa\tau,\xi\alpha}-\frac{1}{4}g_{\lambda\alpha,\tau\kappa}\Lambda^{\kappa\tau,\xi\alpha}-\frac{1}{2}g_{\lambda\kappa,\tau\alpha}\Lambda^{\kappa\xi,\alpha\tau}-\frac{1}{2}g_{\tau\alpha,\lambda\kappa}\Lambda^{\kappa\xi,\alpha\tau}.$$

By renaming $(\alpha\leftrightarrow\kappa)$ in the terms with the $\frac{1}{2}$ factor and with the use of the first invariance identity

$$\Lambda^{\kappa\xi,\alpha\tau}R_{\tau\lambda\alpha\kappa}=-\frac{3}{4}\Lambda^{\kappa\tau,\xi\alpha}\Big(g_{\tau\kappa,\lambda\alpha}+g_{\lambda\alpha,\tau\kappa}\Big). \tag{2.2.12}$$

Following this, we focus on the fourth term of (2.2.10). At the spacetime point P using Rund's tensor (2.1.12) we obtain

$$\Lambda^{\lambda\xi}=\Pi^{\lambda\xi}-\Gamma^{\lambda}_{\mu\nu,\rho}\Lambda^{\mu\xi,\nu\rho}-\Gamma^{\xi}_{\mu\nu,\rho}\Lambda^{\mu\lambda,\nu\rho}\Rightarrow$$
$$\Lambda^{\lambda\xi}=\Pi^{\lambda\xi}-\frac{1}{2}g^{\lambda\beta}\Lambda^{\mu\xi,\nu\rho}\Big(g_{\mu\beta,\nu\rho}+g_{\nu\beta,\mu\rho}-g_{\mu\nu,\beta\rho}\Big)-\frac{1}{2}g^{\xi\beta}\Lambda^{\mu\lambda,\nu\rho}\Big(g_{\mu\beta,\nu\rho}+g_{\nu\beta,\mu\rho}-g_{\mu\nu,\beta\rho}\Big) \tag{2.2.13}$$

where we have used

$$\Gamma^{\lambda}_{\mu\nu,\rho}=\frac{1}{2}g^{\lambda\beta}\Big(g_{\mu\beta,\nu\rho}+g_{\nu\beta,\mu\rho}-g_{\mu\nu,\beta\rho}\Big),$$
$$\Gamma^{\xi}_{\mu\nu,\rho}=\frac{1}{2}g^{\xi\beta}\Big(g_{\mu\beta,\nu\rho}+g_{\nu\beta,\mu\rho}-g_{\mu\nu,\beta\rho}\Big)$$

for the second term of the first parenthesis of (2.2.13). With the help of the identity introduced previously, we interchange μ with ν and then proceed to rename the two indices. We proceed similarly for the third term of the same parenthesis.

$$\Lambda^{\mu\xi,\nu\rho}g_{\nu\beta,\mu\rho}=\Lambda^{\xi\mu,\nu\rho}g_{\nu\beta,\mu\rho}=-\frac{1}{2}\Lambda^{\xi\nu,\mu\rho}g_{\nu\beta,\mu\rho}=-\frac{1}{2}\Lambda^{\xi\mu,\nu\rho}g_{\mu\beta,\nu\rho}$$

where for the last result we have exchanged indices μ,ν. Similarly

$$-\Lambda^{\mu\xi,\nu\rho}g_{\mu\nu,\beta\rho}=\frac{1}{2}\Lambda^{\xi\rho,\mu\nu}g_{\mu\nu,\beta\rho}=\frac{1}{2}\Lambda^{\xi\mu,\rho\nu}g_{\nu\rho,\beta\mu}.$$

Thus, the first parenthesis of (2.2.13) takes the following form

$$-\frac{1}{4}\Lambda^{\mu\xi,\nu\rho}g^{\lambda\beta}\left(g_{\mu\beta,\nu\rho}+g_{\nu\rho,\mu\beta}\right)$$

Following a similar approach, for the second parenthesis of (2.2.13) we find for the second and third terms

$$\Lambda^{\mu\lambda,\nu\rho}g_{\nu\beta,\mu\rho}=-\frac{1}{2}\Lambda^{\lambda\nu,\mu\rho}g_{\nu\beta,\mu\rho}=-\frac{1}{2}\Lambda^{\mu\lambda,\nu\rho}g_{\mu\beta,\nu\rho},$$

$$-\Lambda^{\mu\lambda,\nu\rho}g_{\mu\nu,\beta\rho}=\frac{1}{2}\Lambda^{\lambda\rho,\mu\nu}g_{\mu\nu,\beta\rho}=\frac{1}{2}\Lambda^{\mu\lambda,\rho\nu}g_{\nu\rho,\mu\beta}.$$

Substituting the results into (2.2.13)

$$\Lambda^{\lambda\xi}=\Pi^{\lambda\xi}-\frac{1}{4}g^{\lambda\beta}\Lambda^{\mu\xi,\nu\rho}\left(g_{\mu\beta,\nu\rho}+g_{\nu\rho,\mu\beta}\right)-\frac{1}{4}g^{\xi\beta}\Lambda^{\mu\lambda,\nu\rho}\left(g_{\mu\beta,\nu\rho}+g_{\nu\rho,\mu\beta}\right)$$

using (2.2.12)

$$\Lambda^{\lambda\xi}=\Pi^{\lambda\xi}+\frac{1}{3}g^{\lambda\beta}\Lambda^{\nu\rho,\xi\mu}R_{\rho\beta\nu\mu}+\frac{1}{3}g^{\xi\beta}\Lambda^{\nu\rho,\lambda\mu}R_{\rho\beta\nu\mu}$$

and multiplying with $g_{\lambda\sigma}g^{\sigma\kappa}$ we get

$$\Lambda^{\kappa\xi}=\Pi^{\kappa\xi}+\frac{1}{3}\Lambda^{\nu\rho,\xi\mu}R_{\rho\beta\nu\mu}g^{\kappa\beta}+\frac{1}{3}g^{\xi\beta}\Lambda^{\nu\rho,\kappa\mu}R_{\rho\beta\nu\mu}.$$

Multiplying again with $g_{\kappa\lambda}$ and then renaming the indices $(\kappa\leftrightarrow\sigma)$ we end up with

$$g_{\sigma\lambda}\Lambda^{\sigma\xi}=g_{\sigma\lambda}\Pi^{\sigma\xi}+\frac{1}{3}\Lambda^{\nu\rho,\xi\mu}R_{\rho\lambda\nu\mu}+\frac{1}{3}\Lambda^{\nu\rho,\sigma\mu}R_{\rho\beta\nu\mu}g^{\xi\beta}g_{\sigma\lambda}. \tag{2.2.14}$$

Substituting (2.2.14), and using the identity (2.2.12), (2.2.10) yields

$$\delta^{\xi}_{\lambda}L=-\frac{8}{3}\Lambda^{\alpha\tau,\xi\kappa}R_{\tau\lambda\alpha\kappa}+2g_{\sigma\lambda}\Pi^{\sigma\xi}+\frac{2}{3}\Lambda^{\nu\rho,\xi\mu}R_{\rho\lambda\nu\mu}+\frac{2}{3}\Lambda^{\nu\rho,\sigma\mu}R_{\rho\beta,\nu\mu}g^{\xi\beta}g_{\sigma\lambda}+\Lambda^{\xi}\phi_{,\lambda}\Rightarrow$$

$$\delta^{\xi}_{\lambda}L=-2\Lambda^{\alpha\tau,\xi\kappa}R_{\tau\lambda\alpha\kappa}+2g_{\sigma\lambda}\Pi^{\sigma\xi}+\frac{2}{3}\Lambda^{\nu\rho,\sigma\mu}R_{\rho\beta\nu\mu}g^{\xi\beta}g_{\sigma\lambda}+\Lambda^{\xi}\phi_{,\lambda}$$

where we have renamed the indices of the third term to obtain the last result. Subsequently, multiplying with $g^{\lambda\chi}$, we obtain

$$g^{\xi\chi}L=-2\Lambda^{\alpha\tau,\xi\kappa}R_{\tau}{}^{\chi}{}_{\alpha\kappa}+2\Pi^{\chi\xi}+\frac{2}{3}\Lambda^{\nu\rho,\chi\mu}R_{\tau}{}^{\xi}{}_{\nu\mu}+g^{\lambda\chi}\Lambda^{\xi}\phi_{,\lambda}. \tag{2.2.15}$$

Throughout the rest of this section we simplify (2.2.15) to derive the final expression for the identity. Utilizing the fact that the left-hand side of the equation is symmetric in ξ,χ, we can accordingly express the right-hand side as follows

$$-2\Lambda^{\alpha\tau,\xi\kappa}R_{\tau}{}^{\chi}{}_{\alpha\kappa}+\frac{2}{3}\Lambda^{\nu\rho,\chi\mu}R_{\tau}{}^{\xi}{}_{\nu\mu}+g^{\lambda\chi}\Lambda^{\xi}\phi_{,\lambda}=-2\Lambda^{\alpha\tau,\chi\kappa}R_{\tau}{}^{\xi}{}_{\alpha\kappa}+\frac{2}{3}\Lambda^{\nu\rho,\xi\mu}R_{\tau}{}^{\chi}{}_{\nu\mu}+g^{\lambda\xi}\Lambda^{\chi}\phi_{,\lambda}.$$

Renaming the indices of the fist term, this yields

$$-2\Lambda^{\nu\rho,\xi\mu}R_{\rho}{}^{\chi}{}_{\alpha\mu}-\frac{2}{3}\Lambda^{\nu\rho,\xi\mu}R_{\rho}{}^{\chi}{}_{\nu\mu}=-\frac{2}{3}\Lambda^{\nu\rho,\chi\mu}R_{\rho}{}^{\xi}{}_{\nu\mu}-g^{\lambda\chi}\Lambda^{\xi}\phi_{,\lambda}+g^{\lambda\xi}\Lambda^{\chi}\phi_{,\lambda}-2\Lambda^{\alpha\tau,\chi\kappa}R_{\tau\alpha\kappa}\Rightarrow$$

$$\Lambda^{\nu\rho,\chi\mu}R_{\rho}{}^{\xi}{}_{\nu\mu}=\Lambda^{\nu\rho,\xi\mu}R_{\rho}{}^{\chi}{}_{\nu\mu}+\frac{3}{8}\left(g^{\lambda\xi}\Lambda^{\chi}\phi_{,\lambda}-g^{\lambda\chi}\Lambda^{\xi}\phi_{,\lambda}\right). \tag{2.2.16}$$

Substituting into (2.2.15)

$$g^{\xi\chi}L=-2\Lambda^{\alpha\tau,\xi\kappa}R_{\tau}{}^{\chi}{}_{\alpha\kappa}+2\Pi^{\chi\xi}+\frac{2}{3}\Lambda^{\nu\rho,\xi\mu}R_{\rho}{}^{\chi}{}_{\nu\mu}+\frac{1}{4}\left(g^{\lambda\xi}\Lambda^{\chi}\phi_{,\lambda}-g^{\lambda\chi}\Lambda^{\xi}\phi_{,\lambda}\right)+g^{\lambda\chi}\Lambda^{\xi}\phi_{\lambda}.$$

Finally, by renaming $\alpha,\tau,\kappa\leftrightarrow\nu,\rho,\mu$ in the first term we arrive at the final result

$$\frac{1}{2}g^{\xi\chi}L=\Pi^{\chi\xi}-\frac{2}{3}\Lambda^{\nu\rho,\xi\mu}R_{\rho}{}^{\chi}{}_{\nu\mu}+\frac{3}{8}g^{\lambda\chi}\Lambda^{\xi}\phi_{,\lambda}+\frac{1}{8}g^{\lambda\xi}\Lambda^{\chi}\phi_{,\lambda}. \tag{2.2.17}$$

This equation is the *Third Invariance Identity*.[4] Given the tensorial nature of the quantities involved, the result obtained using a locally inertial frame remains valid in all coordinate systems. In the following sections, we will use these three identities to derive the most general Lagrangian of the form (2.1.1) in a four-dimensional space.

Using the third invariance identity (2.2.9) can be rewritten as follows

$$E^{\mu\nu}(L)=-\nabla_{\rho}\nabla_{\sigma}\Lambda^{\mu\nu,\rho\sigma}+\frac{2}{3}R^{\mu}_{\sigma\lambda\tau}\Lambda^{\tau\sigma,\nu\lambda}+\frac{3}{8}\Lambda^{\nu}g^{\mu\tau}\phi_{,\tau}+\frac{1}{8}\Lambda^{\mu}g^{\nu\tau}\phi_{,\tau}-\frac{1}{2}g^{\mu\nu}L. \tag{2.2.18}$$

This form for the Euler-Lagrange equations is useful due to the fact that the only quantities required to be evaluated are $\Lambda^{\mu\nu,\rho\sigma}$, and Λ^{μ}. For example, in the special case where

$$L=L(g_{\mu\nu};g_{\mu\nu,\rho};g_{\mu\nu,\rho\sigma})$$

equation (2.2.17) simplifies to

$$\frac{1}{2}g^{\xi\chi}L=\Pi^{\chi\xi}-\frac{2}{3}\Lambda^{\nu\rho,\xi\mu}R_{\rho}{}^{\chi}{}_{\nu\mu}$$

[4] It is worth noting the asymmetrical way in which the last two terms appear. These two terms constitute the only difference between the third invariance identity for the lagrangian (2.1.1) in which our analysis is based and Rund's original identity for the simpler case of a lagrangian of the form:

$$L=L(g_{\mu\nu};g_{\mu\nu,\rho};g_{\mu\nu,\rho\sigma}).$$

which matches the result obtained by Rund [3]. For the field equations in this case, we find

$$E^{\mu\nu}(L) = -\nabla_\rho \nabla_\sigma \Lambda^{\mu\nu,\rho\sigma} + \frac{2}{3} R^{\mu}_{\sigma\lambda\tau} \Lambda^{\tau\sigma,\nu\lambda} - \frac{1}{2} g^{\mu\nu} L. \tag{2.2.19}$$

If we consider, as a special case, the theory

$$L = \sqrt{g}\Big(\alpha R - 2\Lambda\Big)$$

where α, Λ are constants, we can derive the corresponding Euler-Lagrange equations by simply calculating $\Lambda^{\mu\nu,\rho\sigma}$. Noting that,

$$R = g^{\mu\nu} g^{\rho\sigma} R_{\sigma\mu\rho\nu}$$

and

$$\begin{aligned}\frac{\partial R_{\sigma\mu\rho\nu}}{\partial g_{\tau\lambda,\kappa\chi}} =& \frac{1}{8}\Big[\big(\delta^\tau_\sigma \delta^\lambda_\nu + \delta^\tau_\nu \delta^\lambda_\sigma\big)\big(\delta^\kappa_\mu \delta^\chi_\rho + \delta^\kappa_\rho \delta^\chi_\mu\big) + \big(\delta^\tau_\mu \delta^\lambda_\rho + \delta^\tau_\rho \delta^\lambda_\mu\big)\big(\delta^\kappa_\sigma \delta^\chi_\nu + \delta^\kappa_\nu \delta^\chi_\sigma\big) \\ &- \big(\delta^\tau_\sigma \delta^\lambda_\rho + \delta^\tau_\rho \delta^\lambda_\sigma\big)\big(\delta^\kappa_\mu \delta^\chi_\nu + \delta^\kappa_\nu \delta^\chi_\mu\big) - \big(\delta^\tau_\mu \delta^\lambda_\nu + \delta^\tau_\nu \delta^\lambda_\mu\big)\big(\delta^\kappa_\sigma \delta^\chi_\rho + \delta^\kappa_\rho \delta^\chi_\sigma\big)\Big]\end{aligned}$$

follows that

$$\Lambda^{\tau\lambda,\kappa\chi} = -\alpha\sqrt{g}\frac{2g^{\tau\lambda}g^{\kappa\chi} - g^{\tau\kappa}g^{\lambda\chi} - g^{\tau\chi}g^{\lambda\kappa}}{2}$$

which when substituted to (2.2.19) yields the Einstein field equations with a cosmological term in vacuum

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0.$$

2.3 Generalized Bianchi Identity

The next step in this analysis is to study the divergence of $E^{\mu\nu}(L)$. We can write the Euler-Lagrange equations corresponding to (2.1.1) as follows

$$E^{\mu\nu}(L) = \frac{\partial}{\partial x^\rho}\Big(\Lambda^{\mu\nu,\rho} - \frac{\partial}{\partial x^\sigma}\Lambda^{\mu\nu,\rho\sigma}\Big) - \Lambda^{\mu\nu}.$$

At the spacetime point P of our locally inertial coordinates the covariant derivative reduces to partial, thus

$$\nabla_\nu E^{\mu\nu}(L) = \Lambda^{\mu\nu,\rho}{}_{,\rho\nu} - \Lambda^{\mu\nu,\rho\sigma}{}_{,\sigma\rho\nu} - \Lambda^{\mu\nu}{}_{,\nu}.$$

Using the first invariance identity, it becomes apparent that the second term identically vanishes[5]

$$\nabla_\nu E^{\mu\nu}(L) = \Lambda^{\mu\nu,\rho}{}_{,\rho\nu} - \Lambda^{\mu\nu}{}_{,\nu}. \tag{2.3.1}$$

We compute the remaining two terms of the equation. Using (2.1.11), and invoking the second invariance identity (2.2.8)

$$\Lambda^{\mu\nu,\rho} = -\Gamma^{\rho}_{\sigma\kappa}\Lambda^{\mu\nu,\sigma\kappa} - 2\Gamma^{\mu}_{\sigma\kappa}\Lambda^{\sigma\nu,\rho\kappa} - 2\Gamma^{\nu}_{\sigma\kappa}\Lambda^{\sigma\mu,\rho\kappa}. \tag{2.3.2}$$

Utilizing the identity applied in the previous section $g_{\mu\nu}\Lambda^{\sigma\mu,\rho\nu} = -\frac{1}{2}g_{\mu\nu}\Lambda^{\sigma\rho,\mu\nu}$

$$2\Gamma^{\mu}_{\sigma\kappa}\Lambda^{\sigma\nu,\rho\kappa} = \Gamma^{\mu}_{\sigma\kappa}\Lambda^{\nu\rho,\sigma\kappa},$$
$$-2\Gamma^{\nu}_{\sigma\kappa}\Lambda^{\sigma\mu,\rho\kappa} = \Gamma^{\nu}_{\sigma\kappa}\Lambda^{\mu\rho,\sigma\kappa}.$$

Thus,

$$\Lambda^{\mu\nu,\rho} = -\Gamma^{\rho}_{\sigma\kappa}\Lambda^{\mu\nu,\sigma\kappa} + \Gamma^{\mu}_{\sigma\kappa}\Lambda^{\nu\rho,\sigma\kappa} + \Gamma^{\nu}_{\sigma\kappa}\Lambda^{\mu\rho,\sigma\kappa}. \tag{2.3.3}$$

Using this expression, we compute the first term of (2.3.1)

$$\Lambda^{\mu\nu,\rho}{}_{,\rho} = -\Gamma^{\rho}_{\sigma\kappa,\rho}\Lambda^{\mu\nu,\sigma\kappa} - \Gamma^{\rho}_{\sigma\kappa}\Lambda^{\mu\nu,\sigma\kappa}{}_{,\rho} + \Gamma^{\mu}_{\sigma\kappa,\rho}\Lambda^{\nu\rho,\sigma\kappa} + \Gamma^{\mu}_{\sigma\kappa}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\rho} + \Gamma^{\nu}_{\sigma\kappa,\rho}\Lambda^{\mu\rho,\sigma\kappa} + \Gamma^{\nu}_{\sigma\kappa}\Lambda^{\mu\rho,\sigma\kappa}{}_{,\rho}$$

and

$$\begin{aligned}
\Lambda^{\mu\nu,\rho}{}_{,\rho\nu} = &-\Gamma^{\rho}_{\sigma\kappa,\rho\nu}\Lambda^{\mu\nu,\sigma\kappa} - \Gamma^{\rho}_{\sigma\kappa,\rho}\Lambda^{\mu\nu,\sigma\kappa}{}_{,\nu} - \Gamma^{\rho}_{\sigma\kappa,\nu}\Lambda^{\mu\nu,\sigma\kappa}{}_{,\rho} - \Gamma^{\rho}_{\sigma\kappa}\Lambda^{\mu\nu,\sigma\kappa}{}_{,\rho\nu} \\
&+\Gamma^{\mu}_{\sigma\kappa,\rho\nu}\Lambda^{\nu\rho,\sigma\kappa} + \Gamma^{\mu}_{\sigma\kappa,\rho}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\nu} + \Gamma^{\mu}_{\sigma\kappa,\nu}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\rho} + \Gamma^{\mu}_{\sigma\kappa}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\rho\nu} \\
&+\Gamma^{\nu}_{\sigma\kappa,\rho\nu}\Lambda^{\mu\rho,\sigma\kappa} + \Gamma^{\nu}_{\sigma\kappa,\rho}\Lambda^{\mu\rho,\sigma\kappa}{}_{,\nu} + \Gamma^{\nu}_{\sigma\kappa,\nu}\Lambda^{\mu\rho,\sigma\kappa}{}_{,\rho} + \Gamma^{\nu}_{\sigma\kappa}\Lambda^{\mu\rho,\sigma\kappa}{}_{,\rho\nu}.
\end{aligned}$$

By appropriately renaming indices, the equation simplifies to

$$\Lambda^{\mu\nu,\rho}{}_{,\rho\nu} = \Gamma^{\mu}_{\sigma\kappa,\rho\nu}\Lambda^{\nu\rho,\sigma\kappa} + \Gamma^{\mu}_{\sigma\kappa,\rho}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\nu} + \Gamma^{\mu}_{\sigma\kappa,\nu}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\rho} + \Gamma^{\mu}_{\sigma\kappa}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\rho\nu}.$$

At the pole of our inertial coordinates system where $\Gamma^{\mu}_{\rho\sigma} = 0$ we obtain

$$\Lambda^{\mu\nu,\rho}{}_{,\rho\nu} = \Gamma^{\mu}_{\sigma\kappa,\rho\nu}\Lambda^{\nu\rho,\sigma\kappa} + 2\Gamma^{\mu}_{\sigma\kappa,\rho}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\nu}.$$

[5]

$$\begin{aligned}
&\Lambda^{\mu\nu,\rho\sigma} + \Lambda^{\mu\rho,\nu\sigma} + \Lambda^{\mu\sigma,\rho\nu} = 0 \;\Rightarrow \\
&\Lambda^{\mu\rho,\nu\sigma}_{,\sigma\rho\nu} + \Lambda^{\mu\sigma,\rho\nu}_{,\sigma\rho\nu} + \Lambda^{\mu\nu,\rho\sigma}_{,\sigma\rho\nu} = 0 \;\Rightarrow \\
&3\Lambda^{\mu\rho,\sigma\nu}_{,\rho\sigma\nu} = 0 \;\Rightarrow \\
&\Lambda^{\mu\nu,\rho\sigma}_{,\sigma\rho\nu} = 0
\end{aligned}$$

Substituting the derivatives of the Christoffel symbols, yields

$$\Lambda^{\mu\nu,\rho}{}_{,\rho\nu} = \frac{1}{2}g^{\mu\lambda}\Lambda^{\nu\rho,\sigma\kappa}\left(g_{\sigma\lambda,\kappa\rho\nu} + g_{\kappa\lambda,\sigma\rho\nu} - g_{\sigma\kappa,\lambda\rho\nu}\right) + g^{\mu\lambda}\Lambda^{\rho\nu,\sigma\kappa}{}_{,\nu}\left(g_{\sigma\lambda,\kappa\rho} + g_{\kappa\lambda,\sigma\rho} - g_{\sigma\kappa,\lambda\rho}\right). \tag{2.3.4}$$

The first two terms within the second set of parentheses can be expressed using the identity applied previously, followed by renaming the summed indices as

$$\Lambda^{\rho\nu,\sigma\kappa}{}_{,\nu}\, g_{\sigma\lambda,\kappa\rho} = -\frac{1}{2}\Lambda^{\nu\sigma,\rho\kappa}{}_{,\nu}\, g_{\sigma\lambda,\kappa\rho} = -\frac{1}{2}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\nu}\, g_{\lambda\rho,\sigma\kappa},$$

$$\Lambda^{\rho\nu,\sigma\kappa}{}_{,\nu}\, g_{\kappa\lambda,\sigma\rho} = -\frac{1}{2}\Lambda^{\nu\kappa,\rho\sigma}{}_{,\nu}\, g_{\kappa\lambda,\sigma\rho} = -\frac{1}{2}\Lambda^{\nu\rho,\sigma\kappa}{}_{,\nu}\, g_{\lambda\rho,\sigma\kappa}.$$

Then, we end up with[6]

$$\Lambda^{\mu\nu,\rho}{}_{,\rho\nu} = -\frac{1}{2}g^{\mu\lambda}g_{\sigma\kappa,\lambda\rho\nu}\Lambda^{\nu\rho,\sigma\kappa} - g^{\mu\lambda}\left(g_{\lambda\rho,\sigma\kappa} + g_{\sigma\kappa,\lambda\rho}\right)\Lambda^{\rho\nu,\sigma\kappa}{}_{,\nu}. \tag{2.3.5}$$

In what follows we use (2.2.10) to calculate $\Lambda^{\mu\nu}{}_{,\rho}$. Multiplying with $g^{\lambda\chi}$

$$g^{\xi\chi}L = 2g^{\lambda\chi}\Lambda^{\kappa\xi,\alpha\tau}\left(g_{\kappa\lambda,\alpha\tau} + g_{\alpha\tau,\kappa\lambda}\right) + 2g^{\lambda\chi}\Lambda^{\xi\kappa,\tau}g_{\lambda\kappa,\tau} + g^{\lambda\chi}\Lambda^{\sigma\kappa,\xi}g_{\sigma\kappa,\lambda} + 2\Lambda^{\chi\xi} + g^{\lambda\chi}\Lambda^{\xi}\phi_{,\lambda}$$

solving for $\Lambda^{\chi\xi}$ and using the inertial coordinate system where the first derivative of the metric vanishes, we deduce

$$\begin{aligned}\Lambda^{\chi\xi}{}_{,\xi} = {} & \frac{1}{2}g^{\xi\chi}L_{,\xi} - g^{\lambda\chi}\Lambda^{\kappa\xi,\alpha\tau}\left(g_{\kappa\lambda,\alpha\tau\xi} + g_{\alpha\tau,\kappa\lambda\xi}\right) - g^{\lambda\chi}\Lambda^{\kappa\xi,\alpha\tau}{}_{,\xi}\left(g_{\kappa\lambda,\alpha\tau} + g_{\alpha\tau,\kappa\lambda}\right) \\ & - g^{\lambda\chi}\Lambda^{\xi\kappa,\tau}g_{\lambda\kappa,\tau\xi} - \frac{1}{2}g^{\lambda\chi}\Lambda^{\sigma\kappa,\xi}g_{\sigma\kappa,\lambda\xi} - \frac{1}{2}g^{\lambda\chi}\Lambda^{\xi}{}_{,\xi}\,\phi_{,\lambda} - \frac{1}{2}g^{\lambda\chi}\Lambda^{\xi}\,\phi_{,\lambda\xi}.\end{aligned} \tag{2.3.6}$$

Using the general expression for the Lagrangian (2.1.1) we can calculate $L_{,\xi}$ at the spacetime point P of our coordinate system

$$L_{,\xi} = g_{\mu\nu,\xi}\,\Lambda^{\mu\nu} + g_{\mu\nu,\rho\xi}\,\Lambda^{\mu\nu,\rho} + \Lambda^{\mu\nu,\rho\sigma}\,g_{\mu\nu,\rho\sigma\xi} + \Phi\,\phi_{,\xi} + \Lambda^{\mu}\,\phi_{,\mu\xi} \Rightarrow$$

$$g^{\chi\xi}L_{,\xi} = g^{\chi\xi}g_{\mu\nu,\rho\xi}\,\Lambda^{\mu\nu,\rho} + g^{\chi\xi}g_{\mu\nu,\rho\sigma\xi}\,\Lambda^{\mu\nu,\rho\sigma} + \phi_{,\xi}\,\Phi\,g^{\chi\xi} + \phi_{,\mu\xi}\,\Lambda^{\mu}\,g^{\chi\xi}.$$

Substituting this into (2.3.6) yields

[6]The first two terms of the first parenthesis when contracted with $\Lambda^{\nu\rho,\sigma\kappa}$ do not contribute due to the first invariance identity (2.2.1). For example

$$\Lambda^{\nu\rho,\sigma\kappa}g_{\sigma\lambda,\kappa\rho\nu} = -\Lambda^{\nu\sigma,\rho\kappa}g_{\sigma\lambda,\kappa\rho\nu} - \Lambda^{\nu\kappa,\sigma\rho}g_{\sigma\lambda,\kappa\rho\nu} = -\Lambda^{\kappa\sigma,\rho\nu}g_{\sigma\lambda,\kappa\rho\nu} - \Lambda^{\nu\rho,\sigma\kappa}g_{\sigma\lambda,\kappa\rho\nu} = \Lambda^{\sigma\kappa,\nu\rho}g_{\sigma\lambda,\kappa\rho\nu} \Rightarrow$$

$$\Lambda^{\nu\rho,\sigma\kappa}g_{\sigma\lambda,\kappa\rho\nu} = -\Lambda^{\nu\rho,\sigma\kappa}g_{\sigma\lambda,\kappa\rho\nu}$$

$$
\begin{aligned}
\Lambda^{\chi\xi}{}_{,\xi} = {} & \cancel{\frac{1}{2} g^{\chi\xi} g_{\mu\nu,\rho\xi} \Lambda^{\mu\nu,\rho}} + \frac{1}{2} g^{\chi\xi} g_{\mu\nu,\rho\sigma\xi} \Lambda^{\mu\nu,\rho\sigma} + \frac{1}{2} g^{\chi\xi} \phi_{,\xi} \Phi + \cancel{\frac{1}{2} g^{\chi\xi} \phi_{,\mu\xi}} \Lambda^{\mu} \\
& - g^{\chi\lambda} \Lambda^{\kappa\xi,\alpha\tau} \left(g_{\kappa\lambda,\alpha\tau\xi} + g_{\alpha\tau,\kappa\lambda\xi}\right) - g^{\chi\lambda} \Lambda^{\kappa\xi,\alpha\tau}{}_{,\xi} \left(g_{\kappa\lambda,\alpha\tau} + g_{\alpha\tau,\kappa\lambda}\right) \\
& - g^{\lambda\chi} \Lambda^{\xi\kappa,\tau} g_{\lambda\kappa,\tau\xi} - \cancel{\frac{1}{2} g^{\lambda\chi} \Lambda^{\sigma\kappa,\xi} g_{\sigma\kappa,\lambda\xi}} - \frac{1}{2} g^{\lambda\chi} \Lambda^{\xi}{}_{,\xi} \phi_{,\lambda} - \cancel{\frac{1}{2} g^{\lambda\chi} \Lambda^{\xi} \phi_{,\lambda\xi}}.
\end{aligned}
$$

The last term cancels with the fourth and the first with the eighth by renaming indices, noting also that $\Lambda^{\kappa\xi,\alpha\tau} g_{\kappa\lambda,\alpha\tau\xi}$=0 and $\Lambda^{\xi\kappa,\tau} g_{\lambda\kappa,\tau\xi} = 0$. We end up with the following result

$$
\Lambda^{\chi\xi}{}_{,\xi} = \frac{1}{2} g^{\chi\xi} g_{\mu\nu,\rho\sigma\xi} \Lambda^{\mu\nu,\rho\sigma} + \frac{1}{2} g^{\chi\xi} \phi_{,\xi} \Phi - g^{\chi\lambda} \Lambda^{\kappa\xi,\alpha\tau} g_{\alpha\tau,\kappa\lambda\xi} - g^{\lambda\chi} \Lambda^{\kappa\xi,\alpha\tau}{}_{,\xi} \left(g_{\kappa\lambda,\alpha\tau} + g_{\alpha\tau,\kappa\lambda}\right) - \frac{1}{2} g^{\lambda\chi} \Lambda^{\xi}{}_{,\xi} \phi_{,\lambda}.
$$

Substituting this and (2.3.6) into (2.3.1) we get

$$
\begin{aligned}
\nabla_{\xi} E^{\chi\xi} = {} & -\frac{1}{2} g^{\chi\lambda} g_{\sigma\kappa,\lambda\rho\xi} \Lambda^{\xi\rho,\sigma\kappa} - g^{\chi\lambda} \left(g_{\lambda\rho,\sigma\kappa} + g_{\sigma\kappa,\lambda\rho}\right) \Lambda^{\rho\xi,\sigma\kappa}{}_{,\xi} - \frac{1}{2} g^{\chi\xi} g_{\mu\nu,\rho\sigma\xi} \Lambda^{\mu\nu,\rho\sigma} \\
& - \frac{1}{2} g^{\chi\xi} \phi_{,\xi} \Phi + g^{\chi\lambda} \Lambda^{\kappa\xi,\alpha\tau} g_{\alpha\tau,\kappa\lambda\xi} + g^{\lambda\chi} \Lambda^{\kappa\xi,\alpha\tau}{}_{,\xi} \left(g_{\kappa\lambda,\alpha\tau} + g_{\alpha\tau,\kappa\lambda}\right) + \frac{1}{2} g^{\lambda\chi} \Lambda^{\xi}{}_{,\xi} \phi_{,\lambda}.
\end{aligned}
$$

This expression can be simplified further by renaming indices and applying the first invariance identity

$$
\nabla_{\xi} E^{\chi\xi} = \frac{1}{2} g^{\chi\lambda} \Lambda^{\xi}{}_{,\xi} \phi_{,\lambda} - \frac{1}{2} g^{\chi\xi} \Phi \phi_{,\xi} \Rightarrow
$$

$$
\nabla_{\xi} E^{\chi\xi} = \frac{1}{2} g^{\chi\xi} \phi_{,\xi} \left(\Lambda^{\alpha}{}_{,\alpha} - \Phi\right). \tag{2.3.7}
$$

Noting that

$$
E(L) = \Lambda^{\mu}{}_{,\mu} - \Phi,
$$

we reach the final result

$$
\nabla_{\nu} E^{\mu\nu}(L) = \frac{1}{2} g^{\mu\nu} \phi_{,\nu} E(L). \tag{2.3.8}
$$

This equation is the generalization of the Bianchi Identity for the Einstein Tensor

$$
\nabla_{\mu} G^{\mu\nu} = 0.
$$

The Generalized Bianchi Identity constraints the structure of $E^{\mu\nu}$ in two ways. First, it forbids $E^{\mu\nu}$ to depend on third derivatives of the fields since the right-hand side is manifestly up to second order in derivatives. Secondly, it demands that the divergence is proportional to a gradient of the scalar field, and the components in any other direction must vanish. In our case, we will not need to explicitly use this identity to derive the Lagrangian of the theory however, in the case of Horndeski theory, this identity served as the starting point for the derivation.

Before proceeding, we summarize the results from the previous sections. In Sections (2.1)-(2.3) we showed that for a Lagrangian of the form

$$L = L(g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\sigma}; \phi; \phi_{,\mu})$$

the following three identities hold for L and the tensors produced by its partial derivatives:

- $\Lambda^{\mu\nu,\rho\sigma} + \Lambda^{\mu\rho,\nu\sigma} + \Lambda^{\mu\sigma,\rho\nu} = 0$
- $\Pi^{\mu\nu,\rho} = 0$
- $\frac{1}{2} g^{\xi\chi} L = \Pi^{\chi\xi} - \frac{2}{3} \Lambda^{\nu\rho,\xi\mu} R_\rho{}^\chi{}_{\nu\mu} + \frac{3}{8} g^{\lambda\chi} \Lambda^\xi \phi_{,\lambda} + \frac{1}{8} g^{\lambda\xi} \Lambda^\chi \phi_{,\lambda}$.

We mentioned that the Euler-Lagrange equations of our Lagrangian can be written as:

- $E^{\mu\nu}(L) = -\left(\nabla_\rho \nabla_\sigma \Lambda^{\mu\nu,\rho\sigma} + \Pi^{\mu\nu}\right)$
- $E(L) = \Lambda^\mu_{,\mu} - \Phi$.

At the end of this section, we showed that these two equations are related by the Generalised Bianchi identity

$$\nabla_\nu E^{\mu\nu}(L) = \frac{1}{2} g^{\mu\nu} \phi_{,\nu} E(L).$$

2.4 The Lagrangian

In this section, we derive the necessary and sufficient conditions for the Euler-Lagrange equations to be, at most, second order in both the metric and the scalar field. At the end of the section we derive the most general Lagrangian of the form (2.1.1) in four dimensions.

2.4.1 Restrictions and Conditions

The Euler-Lagrange equations associated with

$$L = L(g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\sigma}; \phi; \phi_{,\mu})$$

are

$$E^{\mu\nu}(L) = \tfrac{\partial}{\partial x^{\kappa}}\left(\Lambda^{\mu\nu,\kappa} - \tfrac{\partial}{\partial x^{\rho}}\Lambda^{\mu\nu,\kappa\rho}\right) - \Lambda^{\mu\nu},$$

$$E(L) = \tfrac{\partial}{\partial x^{\mu}}\Lambda^{\mu} - \Phi.$$

These equations are of the form

$$E^{\chi\xi} = E^{\chi\xi}\left(g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\sigma}; g_{\mu\nu,\rho\sigma\tau}; g_{\mu\nu,\rho\sigma\tau\lambda}; \phi; \phi_{,\mu}; \phi_{,\mu\nu}; \phi_{,\mu\nu\rho}\right) \tag{2.4.1}$$

$$E = E\left(g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\sigma}; g_{\mu\nu,\rho\sigma\tau}; \phi; \phi_{,\mu}; \phi_{,\mu\nu}\right) \tag{2.4.2}$$

meaning that, in general, these will be of the fourth order in terms of the metric $g_{\mu\nu}$ and third order in the derivatives of the scalar field ϕ. Calculating,

$$\begin{aligned}
E^{\mu\nu}(L) = {} & \frac{\partial g_{\alpha\beta}}{\partial x^{\kappa}}\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial g_{\alpha\beta}} + \frac{\partial g_{\alpha\beta,\gamma}}{\partial x^{\kappa}}\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial g_{\alpha\beta,\gamma}} + \frac{\partial g_{\alpha\beta,\gamma\delta}}{\partial x^{\kappa}}\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial g_{\alpha\beta,\gamma\delta}} \\
& + \frac{\partial\phi}{\partial x^{\kappa}}\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial\phi} + \frac{\partial\phi_{,\alpha}}{\partial x^{\kappa}}\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial\phi_{,\alpha}} \\
& - \frac{\partial}{\partial x^{\kappa}}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\rho}}\frac{\partial\Lambda^{\mu\nu,\kappa\rho}}{\partial g_{\alpha\beta}} + \frac{\partial g_{\alpha\beta,\gamma}}{\partial x^{\rho}}\frac{\partial\Lambda^{\mu\nu,\kappa\rho}}{\partial g_{\alpha\beta,\gamma}} + \frac{\partial g_{\alpha\beta,\gamma\delta}}{\partial x^{\rho}}\frac{\partial\Lambda^{\mu\nu,\kappa\rho}}{\partial g_{\alpha\beta,\gamma\delta}}\right. \\
& \left. + \frac{\partial\phi}{\partial x^{\rho}}\frac{\partial\Lambda^{\mu\nu,\kappa\rho}}{\partial\phi} + \frac{\partial\phi_{,\alpha}}{\partial x^{\rho}}\frac{\partial\Lambda^{\mu\nu,\kappa\rho}}{\partial\phi_{,\alpha}}\right) - \Lambda^{\mu\nu} \\
E(L) = {} & \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\frac{\partial\Lambda^{\mu}}{\partial g_{\alpha\beta}} + \frac{\partial g_{\alpha\beta,\gamma}}{\partial x^{\mu}}\frac{\partial\Lambda^{\mu}}{\partial g_{\alpha\beta,\gamma}} + \frac{\partial g_{\alpha\beta,\gamma\delta}}{\partial x^{\mu}}\frac{\partial\Lambda^{\mu}}{\partial g_{\alpha\beta,\gamma\delta}} \\
& + \frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\Lambda^{\mu}}{\partial\phi} + \frac{\partial\phi_{,\alpha}}{\partial x^{\mu}}\frac{\partial\Lambda^{\mu}}{\partial\phi_{,\alpha}} - \Phi.
\end{aligned}$$

At this point, it becomes necessary to define new notation. Following, in the spirit of the above we introduce:

$$\begin{aligned}
\frac{\partial\Lambda^{\mu\nu,\kappa\rho}}{\partial g_{\alpha\beta}} &= \frac{\partial}{\partial g_{\alpha\beta}}\frac{\partial L}{\partial g_{\mu\nu,\kappa\rho}} = \Lambda^{\mu\nu,\kappa\rho;\alpha\beta} \\
\frac{\partial\Lambda^{\mu\nu,\kappa\rho}}{\partial g_{\alpha\beta,\gamma}} &= \frac{\partial}{\partial g_{\alpha\beta,\gamma}}\frac{\partial L}{\partial g_{\mu\nu,\kappa\rho}} = \Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma} \\
\frac{\partial\Lambda^{\mu\nu,\kappa\rho}}{\partial g_{\alpha\beta,\gamma\delta}} &= \frac{\partial}{\partial g_{\alpha\beta,\gamma\delta}}\frac{\partial L}{\partial g_{\mu\nu,\kappa\rho}} = \Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial g_{\alpha\beta}} &= \frac{\partial}{\partial g_{\alpha\beta}}\frac{\partial L}{\partial g_{\mu\nu,\kappa}} = \Lambda^{\mu\nu,\kappa;\alpha\beta} \\
\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial g_{\alpha\beta,\gamma}} &= \frac{\partial}{\partial g_{\alpha\beta,\gamma}}\frac{\partial L}{\partial g_{\mu\nu,\kappa}} = \Lambda^{\mu\nu,\kappa;\alpha\beta,\gamma} \\
\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial g_{\alpha\beta,\gamma\delta}} &= \frac{\partial}{\partial g_{\alpha\beta,\gamma\delta}}\frac{\partial L}{\partial g_{\mu\nu,\kappa}} = \Lambda^{\mu\nu,\kappa;\alpha\beta,\gamma\delta}
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial \Lambda^{\mu}}{\partial g_{\alpha\beta}} &= \frac{\partial}{\partial g_{\alpha\beta}} \frac{\partial L}{\partial \phi_{,\mu}} = \Lambda^{\mu;\alpha\beta} \\
\frac{\partial \Lambda^{\mu}}{\partial g_{\alpha\beta,\gamma}} &= \frac{\partial}{\partial g_{\alpha\beta,\gamma}} \frac{\partial L}{\partial \phi_{,\mu}} = \Lambda^{\mu;\alpha\beta,\gamma} \\
\frac{\partial \Lambda^{\mu}}{\partial g_{\alpha\beta,\gamma\delta}} &= \frac{\partial}{\partial g_{\alpha\beta,\gamma\delta}} \frac{\partial L}{\partial \phi_{,\mu}} = \Lambda^{\mu;\alpha\beta,\gamma\delta}
\end{aligned}
$$

We also introduce a new tensor density that will prove useful for the calculations that follow

$$\chi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta} \equiv \Lambda^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta} + \Lambda^{\mu\nu,\rho\delta;\alpha\beta,\sigma\gamma} + \Lambda^{\mu\nu,\rho\gamma;\alpha\beta,\delta\sigma}.$$

Using this notation, and by renaming some indices in the third term of the parenthesis, we can rewrite the above as follows

$$
\begin{aligned}
E^{\mu\nu}(L) = {} & g_{\alpha\beta,\kappa}\Lambda^{\mu\nu,\kappa;\alpha\beta} + g_{\alpha\beta,\gamma\kappa}\Lambda^{\mu\nu,\kappa;\alpha\beta,\gamma} + g_{\alpha\beta,\gamma\delta\kappa}\left(\Lambda^{\mu\nu,\kappa;\alpha\beta,\gamma\delta} - \Lambda^{\mu\nu,\kappa\delta;\alpha\beta,\gamma} - \frac{\partial}{\partial x^{\rho}}\left(\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}\right)\right) \\
& + \phi_{,\kappa}\frac{\partial \Lambda^{\mu\nu,\kappa}}{\partial \phi} + \phi_{,\alpha\kappa}\Lambda^{\mu\nu,\kappa;\alpha} - g_{\alpha\beta,\rho\kappa}\Lambda^{\mu\nu,\kappa\rho;\alpha\beta} - g_{\alpha\beta,\rho}\,\Lambda^{\mu\nu,\kappa\rho;\alpha\beta}{}_{,\kappa} \\
& - g_{\alpha\beta,\gamma\rho}\,\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma}{}_{,\kappa} - g_{\alpha\beta,\gamma\delta\rho\kappa}\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta} - \phi_{,\rho\kappa}\frac{\partial \Lambda^{\mu\nu,\kappa\rho}}{\partial \phi} - \phi_{,\rho}\frac{\partial}{\partial x^{\kappa}}\left(\frac{\partial \Lambda^{\mu\nu,\kappa\rho}}{\partial \phi}\right) \\
& - \phi_{,\alpha\rho\kappa}\Lambda^{\mu\nu,\kappa\rho;\alpha} - \phi_{,\alpha\rho}\,\Lambda^{\mu\nu,\kappa\rho;\alpha}{}_{,\kappa} - \Lambda^{\mu\nu}, \qquad (2.4.3)
\end{aligned}
$$

$$
\begin{aligned}
E(L) = {} & g_{\alpha\beta,\mu}\,\Lambda^{\mu;\alpha\beta} + g_{\alpha\beta,\gamma\mu}\,\Lambda^{\mu;\alpha\beta,\gamma} + g_{\alpha\beta,\gamma\delta\mu}\,\Lambda^{\mu;\alpha\beta,\gamma\delta} + \phi_{,\mu}\frac{\partial \Lambda^{\mu}}{\partial \phi} \\
& + \phi_{,\alpha\mu}\frac{\partial \Lambda^{\mu}}{\partial \phi_{,\alpha}} - \Phi. \qquad (2.4.4)
\end{aligned}
$$

We will identify the terms that involve fourth-order and third-order derivatives of the metric and the scalar field in equations (2.4.3), (2.4.4). The analysis that follows is similar to [7]. We can reformulate (2.4.3) as

$$E^{\mu\nu}(L) = -g_{\alpha\beta,\gamma\delta\rho\kappa}\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta} - \phi_{,\alpha\rho\kappa}\Lambda^{\mu\nu,\kappa\rho;\alpha} + \Xi^{\mu\nu}$$

where

$$\Xi^{\mu\nu}(g_{\alpha\beta}; g_{\alpha\beta,\gamma}; g_{\alpha\beta,\gamma\delta}; g_{\alpha\beta,\gamma\delta\rho}; \phi; \phi_{,\alpha}; \phi_{,\alpha\beta}).$$

Regarding the scalar field, $E^{\mu\nu}(L)$ does not contain fourth-order derivatives. In addition there is only one term that involves the third derivative of ϕ. Therefore,

$$E^{\mu\nu}(L)_{(4\text{th},g_{\mu\nu})} = -g_{\alpha\beta,\gamma\delta\rho\kappa}\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta}, \qquad (2.4.5)$$

$$E^{\mu\nu}(L)_{(3\text{rd},\phi)} = -\phi_{\alpha\rho\kappa}\Lambda^{\mu\nu,\kappa\rho;\alpha}. \qquad (2.4.6)$$

The scalar field equation of motion, $E(L)$ does not contain terms involving the fourth derivative of the metric or the scalar field. There is only one term of third-order derivative of the metric. Similarrly, we can express (2.4.4) as

$$E(L) = g_{\alpha\beta,\gamma\delta\mu}\Lambda^{\mu;\alpha\beta,\gamma\delta} + X$$

where,

$$X = X(g_{\alpha\beta}; g_{\alpha\beta,\gamma}; g_{\alpha\beta,\gamma\delta}; \phi; \phi_{,\alpha}; \phi_{,\alpha\beta}).$$

Therefore,

$$E(L)_{3\text{rd},g_{\mu\nu}} = g_{\alpha\beta,\gamma\delta\mu}\Lambda^{\mu;\alpha\beta,\gamma\delta}. \tag{2.4.7}$$

We will examine terms involving third-order derivatives of the metric in the term $\Xi^{\mu\nu}$ of $E^{\mu\nu}$ separately as this part demands a more complicated treatment. So far, according to the above, we have established the following results:

- In order for $E^{\mu\nu}(L)$ to be at most of third order in derivatives of the metric:

$$\begin{aligned} &\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta} + \Lambda^{\mu\nu,\kappa\delta;\alpha\beta,\gamma\rho} + \Lambda^{\mu\nu,\kappa\gamma;\alpha\beta,\rho\delta} \\ &+ \Lambda^{\mu\nu,\gamma\delta;\alpha\beta,\kappa\rho} + \Lambda^{\mu\nu,\gamma\rho;\alpha\beta,\kappa\delta} + \Lambda^{\mu\nu,\rho\delta;\alpha\beta,\kappa\gamma} = 0, \end{aligned} \tag{2.4.8}$$

  or equivalently using $\chi^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta}$:

$$\chi^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta} = -\chi^{\alpha\beta,\kappa\rho;\mu\nu,\gamma\delta}. \tag{2.4.8a}$$

- In order for $E^{\mu\nu}(L)$ to be at most of second order in derivatives of the scalar field:

$$\Lambda^{\mu\nu,\kappa\rho;\alpha} + \Lambda^{\mu\nu,\alpha\rho;\kappa} + \Lambda^{\mu\nu,\kappa\alpha;\rho} = 0. \tag{2.4.9}$$

- In order for $E(L)$ to be at most of second order in derivatives of the metric:

$$\Lambda^{\mu;\alpha\beta,\gamma\delta} + \Lambda^{\delta;\alpha\beta,\mu\gamma} + \Lambda^{\gamma;\alpha\beta,\delta\mu} = 0. \tag{2.4.10}$$

Before identifying the terms that involve third-order derivatives of the metric in $E^{\mu\nu}(L)$, we note that the partial derivative of $\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}$ also contains implicit third order terms

$$
\begin{aligned}
&-g_{\alpha\beta,\rho}\,\Lambda^{\mu\nu,\kappa\rho;\alpha\beta}{}_{,\kappa} = -g_{\alpha\beta,\rho}\left(\dots + \Lambda^{\mu\nu,\kappa\rho;\alpha\beta;\chi\xi,\psi\lambda}\,g_{\chi\xi,\psi\lambda\kappa} + \dots\right),\\
&-\phi_{,\alpha\rho}\,\Lambda^{\mu\nu,\kappa\rho;\alpha}{}_{,\kappa} = -\phi_{,\alpha\rho}\left(\dots + \Lambda^{\mu\nu,\kappa\rho;\alpha;\chi\xi,\psi\lambda}\,g_{\chi\xi,\psi\lambda\kappa}\right),\\
&-\phi_{,\rho}\frac{\partial}{\partial x^\kappa}\left(\frac{\partial}{\partial\phi}\Lambda^{\mu\nu,\kappa\rho}\right) = -\phi_{,\rho}\left(\dots + \Phi^{\mu\nu,\kappa\rho;\chi\xi,\psi\lambda}\,g_{\chi\xi,\psi\lambda\kappa}\right),\\
&-g_{\alpha\beta,\gamma\rho}\,\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma}{}_{,\kappa} = -g_{\alpha\beta,\gamma\rho}\left(\dots + \Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma;\chi\xi,\psi\lambda}\,g_{\chi\xi,\psi\lambda\kappa} + \dots\right).
\end{aligned}
$$

Additionally, the third term within the parenthesis of the first line of (2.4.3) can be written as follows

$$
\begin{aligned}
-g_{\alpha\beta,\gamma\delta\kappa}\frac{\partial}{\partial\rho}\left(\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}\right) = -g_{\alpha\beta,\gamma\delta\kappa}\Bigg(& g_{\chi\xi,\psi\lambda\rho}\,\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda} + g_{\chi\xi,\psi\rho}\,\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi\xi,\psi}\\
&+ g_{\chi\xi,\rho}\,\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi\xi} + \phi_{,\rho}\frac{\partial}{\partial\phi}\left(\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}\right) + \phi_{,\chi\rho}\,\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi}\Bigg).
\end{aligned}
$$

One can easily show that by index renaming the third-order terms we identified previously can be written in terms of the following

$$
-g_{\alpha\beta,\gamma\delta\kappa}\frac{\partial}{\partial x^\rho}\left(\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}\right) + g_{\chi\xi,\psi\lambda\rho}\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda}g_{\alpha\beta,\gamma\delta\kappa}
$$

thus, the terms in $E^{\mu\nu}(L)$ that involve the third derivative of the metric are

$$
\begin{aligned}
E^{\mu\nu}(L)_{(3\mathrm{rd})} = {}& g_{\alpha\beta,\gamma\delta\kappa}\,\Lambda^{\mu\nu,\kappa;\alpha\beta,\gamma\delta} - g_{\alpha\beta,\gamma\delta\kappa}\,\Lambda^{\mu\nu,\kappa\delta;\alpha\beta,\gamma}\\
&- 2\,g_{\alpha\beta,\gamma\delta\kappa}\frac{\partial}{\partial x^\rho}\left(\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}\right) + g_{\alpha\beta,\gamma\delta\kappa}\,g_{\chi\xi,\psi\lambda\rho}\,\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda}.
\end{aligned} \tag{2.4.11}
$$

Going forward, we can express the first two quantities in terms of $\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}$. From (2.3.2) we find:

$$
\frac{\partial\Lambda^{\mu\nu,\kappa}}{\partial g_{\alpha\beta,\gamma\delta}} = \Lambda^{\mu\nu,\kappa;\alpha\beta,\gamma\delta} = -\Gamma^\kappa_{\sigma\rho}\Lambda^{\mu\nu,\sigma\rho;\alpha\beta,\gamma\delta} - 2\Gamma^\mu_{\sigma\rho}\Lambda^{\sigma\nu,\kappa\rho;\alpha\beta,\gamma\delta} - 2\Gamma^\nu_{\sigma\rho}\Lambda^{\sigma\mu,\kappa\rho;\alpha\beta,\gamma\delta} \Rightarrow
$$

$$
g_{\alpha\beta,\gamma\delta\kappa}\Lambda^{\mu\nu,\kappa;\alpha\beta,\gamma\delta} = -g_{\alpha\beta,\gamma\delta\kappa}\Gamma^\kappa_{\sigma\rho}\Lambda^{\mu\nu,\sigma\rho;\alpha\beta,\gamma\delta} - 2g_{\alpha\beta,\gamma\delta\kappa}\Gamma^\mu_{\sigma\rho}\Lambda^{\sigma\nu,\kappa\rho;\alpha\beta,\gamma\delta} - 2g_{\alpha\beta,\gamma\delta\kappa}\Gamma^\nu_{\sigma\rho}\Lambda^{\sigma\mu,\kappa\rho;\alpha\beta,\gamma\delta}.
$$

Similarly we find

$$
\begin{aligned}
-g_{\alpha\beta,\gamma\delta\kappa}\Lambda^{\mu\nu,\kappa\delta;\alpha\beta,\gamma} = {}& g_{\alpha\beta,\gamma\delta\kappa}\Gamma^\gamma_{\sigma\rho}\Lambda^{\alpha\beta,\sigma\rho,\mu\nu,\kappa\delta}\\
&+ 2g_{\alpha\beta,\gamma\delta\kappa}\Gamma^\alpha_{\sigma\rho}\Lambda^{\sigma\beta,\gamma\rho;\mu\nu,\kappa\delta} + 2g_{\alpha\beta,\gamma\delta\kappa}\Gamma^\beta_{\sigma\rho}\Lambda^{\sigma\alpha,\gamma\rho;\mu\nu,\kappa\delta}.
\end{aligned}
$$

Substituting the above into (2.4.11) we end up with

$$\begin{aligned}E^{\mu\nu}(L)_{(3\mathrm{rd})} = &-g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\kappa}_{\sigma\rho}\Lambda^{\mu\nu,\sigma\rho;\alpha\beta,\gamma\delta}-2g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\mu}_{\sigma\rho}\Lambda^{\sigma\nu,\kappa\rho;\alpha\beta,\gamma\delta}-2g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\nu}_{\sigma\rho}\Lambda^{\sigma\mu,\kappa\rho;\alpha\beta,\gamma\delta}\\ &+g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\gamma}_{\sigma\rho}\Lambda^{\alpha\beta,\sigma\rho;\mu\nu,\kappa\delta}+2g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\alpha}_{\sigma\rho}\Lambda^{\sigma\beta,\gamma\rho;\mu\nu,\kappa\delta}+2g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\beta}_{\sigma\rho}\Lambda^{\sigma\alpha,\gamma\rho;\mu\nu,\kappa\delta}\\ &-2g_{\alpha\beta,\gamma\delta\kappa}\,g_{\alpha\beta,\gamma\delta\kappa}\frac{\partial}{\partial x^{\rho}}\left(\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}\right)+g_{\alpha\beta,\gamma\delta\kappa}\,g_{\chi\xi,\psi\lambda\rho}\,\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda}.\end{aligned}$$

Employing the symmetries of $\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta}$, and the definition of $\chi^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta}$ we can use a similar identity to the one we used in the derivation of the third invariance identity. Specifically we have:

$$\frac{1}{3}g_{\alpha\beta,\gamma\delta\kappa}\chi^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta}=\frac{1}{3}g_{\alpha\beta,\gamma\delta\kappa}\left(\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta}+\Lambda^{\mu\nu,\rho\delta:\alpha\beta,\kappa\gamma}+\Lambda^{\mu\nu,\rho\gamma;\alpha\beta,\delta\kappa}\right)=g_{\alpha\beta,\gamma\delta\kappa}\Lambda^{\mu\nu,\kappa\rho;\alpha\beta,\gamma\delta}.$$

Substituting, we obtain

$$\begin{aligned}E^{\mu\nu}_{(3\mathrm{rd})}(L)=&-\frac{2}{3}g_{\alpha\beta,\gamma\delta\kappa}\frac{\partial}{\partial x^{\rho}}\left(\chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}\right)+g_{\alpha\beta,\gamma\delta\kappa}\,g_{\chi\xi,\psi\lambda\rho}\,\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda}\\ &-\frac{2}{3}g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\mu}_{\sigma\rho}\chi^{\sigma\nu,\kappa\rho;\alpha\beta,\gamma\delta}-\frac{2}{3}g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\nu}_{\sigma\rho}\chi^{\sigma\mu,\kappa\rho;\alpha\beta,\gamma\delta}+\frac{2}{3}g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\alpha}_{\sigma\rho}\chi^{\sigma\beta,\gamma\rho;\mu\nu,\kappa\delta}\\ &+\frac{2}{3}g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\beta}_{\sigma\rho}\chi^{\sigma\alpha,\gamma\rho;\mu\nu,\kappa\delta}-g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\kappa}_{\sigma\rho}\Lambda^{\mu\nu,\sigma\rho;\alpha\beta,\gamma\delta}+g_{\alpha\beta,\gamma\delta\kappa}\Gamma^{\gamma}_{\sigma\rho}\Lambda^{\alpha\beta,\sigma\rho;\mu\nu,\kappa\delta}.\end{aligned}$$

Noting that $\chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}$ is a tensor density we can express the partial derivative in the first term, using the covariant derivative[7]

$$\begin{aligned}\nabla_{\rho}\chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}=&\partial_{\rho}\chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}+\Gamma^{\mu}_{\lambda\rho}\chi^{\lambda\nu,\rho\kappa;\alpha\beta,\gamma\delta}+\Gamma^{\nu}_{\lambda\rho}\chi^{\mu\lambda,\rho\kappa;\alpha\beta,\gamma\delta}\\ &+\Gamma^{\rho}_{\lambda\rho}\chi^{\mu\nu,\lambda\kappa;\alpha\beta,\gamma\delta}+\Gamma^{\kappa}_{\lambda\rho}\chi^{\mu\nu,\rho\lambda;\alpha\beta,\gamma\delta}+\Gamma^{\alpha}_{\lambda\rho}\chi^{\mu\nu,\rho\kappa;\lambda\beta,\gamma\delta}\\ &+\Gamma^{\beta}_{\lambda\rho}\chi^{\mu\nu,\rho\kappa;\alpha\lambda,\gamma\delta}+\Gamma^{\gamma}_{\lambda\rho}\chi^{\mu\nu,\rho\kappa;\alpha\beta,\lambda\delta}+\Gamma^{\delta}_{\lambda\rho}\chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\lambda}-\Gamma^{\lambda}_{\rho\lambda}\chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}.\end{aligned}$$

Substituting this into the previous result and after much simplification using the symmetries of χ we end up with the following

$$E^{\mu\nu}_{(3\mathrm{rd})}(L)=-\frac{2}{3}g_{\alpha\beta,\gamma\delta\kappa}\,\nabla_{\rho}\chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}+g_{\alpha\beta,\gamma\delta\kappa}\,g_{\chi\xi,\psi\lambda\rho}\,\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda}. \tag{2.4.12}$$

The second term can be shown that vanishes using (2.4.8) and the fact that $g_{\alpha\beta,\gamma\delta\kappa}$ and $g_{\chi\xi,\psi\lambda\rho}$ are symmetric in γ,δ,κ and ψ,λ,ρ respectively. To eliminate third-order derivatives of the metric terms, the above equation must vanish. Thus, we are led to the following:

[7]Note that the last term of the covariant derivative with the minus sign appears due to the tensor density nature of χ.

- In order for $E^{\mu\nu}$ not to have any third-order derivatives of the metric, the Lagrangian (2.1.1) must satisfy the following

$$\nabla_\rho \chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta} = 0 \tag{2.4.13}$$

or equivalently

$$\nabla_\rho \left(\Lambda^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta} + \Lambda^{\mu\nu,\rho\delta;\alpha\beta,\gamma\kappa} + \Lambda^{\mu\nu,\rho\gamma;\alpha\beta,\delta\kappa} \right) = 0.$$

Notice that both are conditions for the same tensor density. We can re-establish all the conditions for $E^{\mu\nu}(L)$ as follows:

- In order for $E^{\mu\nu}(L)$ to be at most of second order in derivatives of the metric, the following must hold

$$\nabla_\rho \chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta} = 0 \tag{2.4.15}$$

and

$$\chi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta} = -\chi^{\alpha\beta,\rho\sigma;\mu\nu,\gamma\delta}. \tag{2.4.16}$$

2.4.2 Lagrangian density and Euler-Lagrange equations

In this section, we derive the most general Lagrangian of the form (2.1.1) in a four-dimensional space. By imposing the conditions derived in the previous section, namely (2.4.10) and (2.4.16), we ensure that the Euler-Lagrange equations are at most second-order in derivatives of both the metric and the scalar field ϕ, and thus we obtain the most general Lagrangian in four dimensions that leads to second-order field equations.

In the case of four-dimensional space, and if $\chi^{\mu\nu,\rho\kappa;\alpha\beta,\gamma\delta}$ satisfies (2.4.16), due to the symmetries of the tensor density and dimensional limitations, it follows that[8]

$$\Lambda^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda} = A\varepsilon^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda} \tag{2.4.18}$$

where A is a scalar function

$$A = A(g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\sigma}; \phi; \phi_{,\mu})$$

[8] A detailed proof of this can be found in Appendix 3 of [7]. The main idea is that since we work on 4 dimensions, some of the twelve indices coincide and by applying the symmetry properties, one finds that there exists only one independent component.

and[9]

$$\varepsilon^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda} = \sum_{\psi\lambda}\sum_{\chi\xi}\sum_{\gamma\delta}\sum_{\alpha\beta}\sum_{\rho\sigma}\sum_{\mu\nu}\frac{\varepsilon^{\mu\rho\alpha\gamma}\varepsilon^{\nu\sigma\chi\psi}\varepsilon^{\delta\beta\xi\lambda}}{g}$$

is quantity which has the same symmetry properties as $\Lambda^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda}$. In order to integrate (2.4.18) we need more information about the structure of A. Using the symmetries of the tensor density involved and the restrictions derived in the previous sections one can easily show that in a four-dimensional space:

$$\Lambda^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda;\kappa\tau,\varepsilon\eta} = 0$$

which implies that the scalar function A does not contain $g_{\mu\nu,\rho\sigma}$. Using the second invariance identity, we can see that

$$A = A(g_{\mu\nu};\phi;\phi_{,\mu}).$$

Following this, by considering the transformation of a scalar function ψ which is a function of the independent variables $g_{\mu\nu}$, ϕ, $\phi_{,\rho}$ under arbitrary coordinate transformations it is possible to show that[10]

$$\psi = \psi(\phi,\rho), \quad \text{where} \quad \rho = g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}.$$

Finally, we make use of the so-called *property S* established by Lovelock [28]. A quantity $B^{i_1 i_2 \cdots i_{2h-1} i_{2h} \cdots i_{2p}}$, where $p > 1$, has the property S if it satisfies the following conditions:

(i) It is symmetric in (i_{2h-1}, i_{2h}) for $h = 1,\ldots,p$

(ii) It is symmetric under the interchange of the pair $(i_1 i_2)$ with the pair $(i_{2h-1} i_{2h})$ for $h = 2,\ldots,p$

(iii) It satisfies the cyclic identity involving any three of the four indices $(i_1 i_2)(i_{2h-1} i_{2h})$ for $h = 2,\ldots.p$.

It is often helpful to define a positive integer m defined by

$$m = \begin{cases} n/2 & \text{if} \quad n \quad \text{is even} \\ (n+1)/2 & \text{if} \quad n \quad \text{is odd} \end{cases}$$

Then it can be shown that if $B^{i_1 \cdots i_{4M+2}}$ enjoys the property S, it vanishes when, $M \geq m$. In our

[9] The symbol $\varepsilon^{\mu\rho\sigma\nu}$ denotes the four-dimensional permutation symbol, which takes the values $0, 1$, or -1. The notation $\sum_{\mu\nu}$ indicates symmetrization, meaning $\sum_{\mu\nu} A^{\mu\nu} = A^{\mu\nu} + A^{\nu\mu}$.

[10] A detailed proof of this can be found in the appendix of [25]. The derivation of this result does not depend on the number of dimensions and is similar to the derivation of the invariance identities.

case, as a result of this, in a four-dimensional space $(n=4)$, we obtain

$$\Lambda^{\alpha;\beta;\mu\nu,\rho\sigma;\gamma\delta,\chi\xi}=0$$

which implies

$$A=A(\phi).$$

Integrating Equation (2.4.18) yields

$$\Lambda^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}=\frac{2}{3}A(\phi)\varepsilon^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta;\chi\xi,\psi\lambda}R_{\psi\chi\xi\lambda}+\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta} \tag{2.4.19}$$

where[11]

$$\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}=\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}(g_{\mu\nu},\phi).$$

Integrating (2.4.19) two more times[12] we end up with an equation for the Lagrangian

$$L=\frac{4}{81}A\varepsilon^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta;\chi\xi,\psi}R_{\psi\chi\xi\lambda}R_{\gamma\alpha\beta\delta}R_{\rho\mu\nu\sigma}+\frac{2}{9}\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}R_{\gamma\alpha\beta\delta}R_{\rho\mu\nu\sigma}+\xi^{\mu\nu,\rho\sigma}R_{\rho\mu\nu\sigma}+\lambda$$

where $\lambda=\lambda(\phi,\rho)$ is a scalar density and $\xi^{\mu\nu,\rho\sigma}=\xi^{\mu\nu,\rho\sigma}(g_{\mu\nu};\phi;\phi_{,\alpha})$ is a tensor density with the same symmetries as $\Lambda^{\mu\nu,\rho\sigma}$. At this point our problem has been reduced to finding the most general form of $\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}(g_{\mu\nu};\phi)$, $\xi^{\mu\nu,\rho\sigma}(g_{\mu\nu};\phi;\phi_{,\alpha})$ and $\lambda=\lambda(\phi,\rho)$. Here, commas and semicolons are used to denote the relevant symmetries. The classification of such quantities follows from the work of Lovelock. Detailed calculations for these terms can be found on lemmas $A.5$, $A.6$ of [7] and lemmas $A.2$ and $A.6$ of [25] as well as in [3]. For example it can be shown that if $\xi^{\mu\nu,\rho\sigma}(g_{\mu\nu};\phi;\phi_{,\alpha})$ is a tensor density for which

$$\xi^{\mu\nu,\rho\sigma}=\xi^{\nu\mu,\rho\sigma}=\xi^{\mu\nu,\sigma\rho}$$

and

$$\xi^{\mu\nu,\rho\sigma}+\xi^{\mu\rho,\nu\sigma+}+\xi^{\mu\sigma,\nu\rho}=0,$$

then for a space with dimension $n\geq 3$ the following holds

$$\xi^{\mu\nu,\rho\sigma}R_{\mu\rho\nu\sigma}=\alpha R^{\mu\nu}\phi_{,\mu}\phi_{,\nu}+\beta R.$$

[11] $\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}$ is a tensor density with the same symmetry properties as $\Lambda^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}$. Using simmilar arguments it can be shown that it depends on $g_{\mu\nu}$ and ϕ.

[12] Note that upon integration of (2.4.19) we make use of a similar technique as the one used in the derivation of the third invariance identity, in order to express the result in tensorial form in terms of the Riemann tensor.

Adopting these results, we find

$$L = \frac{\alpha_1}{g}(*R^{\mu\nu}{}_{\rho\sigma})(*R^{\rho\sigma}{}_{\kappa\lambda})(*R^{\kappa\lambda}{}_{\mu\nu}) + \alpha_2\sqrt{-g}\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} - R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right)$$
$$+\alpha_3(*R^{\mu\nu}{}_{\rho\sigma})(*R^{\rho\sigma}{}_{\mu\nu}) + \alpha_4\sqrt{-g}R^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \alpha_5\sqrt{g}R + \alpha_6\sqrt{-g} \tag{2.4.20}$$

where,

$$\alpha_1 = \alpha_1(\phi), \quad \alpha_2 = \alpha_2(\phi), \quad \alpha_3 = \alpha_3(\phi), \quad \alpha_4 = \alpha_4(\phi,\rho), \quad \alpha_5 = \alpha_5(\phi,\rho), \quad \alpha_6 = \alpha_6(\phi,\rho)$$

and

$$*R^{\mu\nu}{}_{\rho\sigma} = \varepsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}$$

is the dual Riemann tensor. The second and third terms in (2.4.20) arise from the computation of $\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}R_{\gamma\alpha\beta\delta}R_{\rho\mu\nu\sigma}$[13].

To obtain the final result, we need to apply the conditions derived in the previous section to our Lagrangian. One can already see that the first term of (2.4.20) leads to terms in the equations that involve higher derivatives of the metric, therefore we require $\alpha_1 = 0$. In a similar way, one concludes that $\alpha_3 = \text{constant}$[14] and by the use of (2.4.9) or (2.4.10), we get the second set of conditions below. We show the rest of these calculations in A.1, and note that as a result of those, we have:

$$\alpha_1 = 0 \qquad\qquad \frac{\partial\alpha_4}{\partial\rho} = 0$$
$$\alpha_3 = \text{constant} \qquad\qquad \frac{\partial\alpha_5}{\partial\rho} = -\frac{1}{2}\alpha_4$$

Therefore,

$$\alpha_5 = -\frac{1}{2}\alpha_4\rho + \xi(\phi)$$

This leads us to the final result:

- In a space of four dimensions, the most general Lagrangian of the form $L = L(g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\sigma}; \phi; \phi_{,\mu})$, for which the Euler-Lagrange equations are of second order

[13] This is a lengthy but straightforward calculation. The main steps are presented in A.2

[14] These can be proven by either imposing (2.4.15) or by calculating the contributions of these terms in the Euler-Lagrange equations using (2.2.18).

in both the metric and scalar field is

$$
\begin{aligned}
L = \sqrt{-g}\bigg(&\beta_1\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} - R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right) + \beta_2 G^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \beta_3 R + \beta_4\bigg) \\
&+ c\,(*R^{\mu\nu}{}_{\rho\sigma})\,(*R^{\rho\sigma}{}_{\mu\nu})
\end{aligned} \tag{2.4.21}
$$

where

$$
\beta_1 = \beta_1(\phi), \quad \beta_2 = \beta_2(\phi), \quad \beta_3 = \beta_3(\phi), \quad \beta_4 = \beta_4(\phi,\rho), \quad c = \text{constant}.
$$

Using Equations (2.3.8), (2.2.18) we calculate the Euler-Lagrange equations

$$
\begin{aligned}
E^{\mu\nu}(\beta_1 L_1) = 4\sqrt{-g}\beta_1'\bigg(&\nabla^\mu\nabla^\alpha\phi\, R_\alpha^{\ \nu} + \nabla^\nu\nabla^\alpha\phi\, R_\alpha^{\ \mu} + \frac{1}{2}R(g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi\, g^{\mu\nu} - \nabla^\mu\nabla^\nu\phi) \\
&- g^{\mu\nu}\nabla_\alpha\nabla_\beta\phi\, R^{\alpha\beta} - g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi\, R^{\mu\nu} - \nabla_\alpha\nabla_\beta\phi\, R^{\alpha\mu\nu\beta}\bigg) \\
&+ 4\sqrt{-g}\beta_1''\bigg(\nabla^\mu\phi\,\nabla_\alpha\phi\, R^{\alpha\nu} + \nabla^\nu\phi\,\nabla_\alpha\phi\, R^{\alpha\mu} + \frac{1}{2}R(g^{\mu\nu}\nabla_\alpha\phi\,\nabla^\alpha\phi - \nabla^\mu\phi\,\nabla^\nu\phi) \\
&- g^{\mu\nu}\nabla_\alpha\phi\,\nabla_\beta\phi\, R^{\alpha\beta} - R^{\mu\nu}\nabla_\alpha\phi\,\nabla^\alpha\phi - \nabla_\alpha\phi\,\nabla_\beta\phi\, R^{\alpha\mu\nu\beta}\bigg) \\
E(\beta_1 L_1) = -\beta_1' L_1,&
\end{aligned}
$$

$$
\begin{aligned}
E^{\mu\nu}(\beta_2 L_2) = \sqrt{-g}\,\beta_2\bigg[&\frac{1}{2}g^{\mu\nu}\Big[(g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi)^2 - \nabla_\alpha\nabla_\beta\phi\,\nabla^\alpha\nabla^\beta\phi - 2\nabla_\alpha\phi\,\nabla_\beta\phi\, R^{\alpha\beta} \\
&+ \frac{1}{2}\nabla_\alpha\phi\,\nabla^\alpha\phi\, R\Big] + \nabla^\mu\nabla^\alpha\phi\,\nabla_\alpha\nabla^\nu\phi - g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi\,\nabla^\mu\nabla^\nu\phi \\
&+ \nabla_\alpha\phi\left(\nabla^\mu\phi\, R^{\alpha\nu} + \nabla^\nu\phi\, R^{\alpha\mu}\right) - \frac{1}{2}\nabla^\mu\phi\,\nabla^\nu\phi\, R - \frac{1}{2}\nabla_\alpha\phi\,\nabla^\alpha\phi\, R^{\mu\nu} \\
&- \nabla_\alpha\phi\,\nabla_\beta\phi\, R^{\alpha\mu\nu\beta}\bigg] + \frac{1}{2}\sqrt{-g}\,\beta_2'\bigg[g^{\mu\nu}\Big(\nabla_\alpha\phi\,\nabla^\alpha\phi\, g^{\rho\sigma}\nabla_\rho\nabla_\sigma\phi \\
&- \nabla_\alpha\phi\,\nabla_\beta\phi\,\nabla^\alpha\nabla^\beta\phi\Big) - \nabla^\mu\phi\,\nabla^\nu\phi\, g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi + \nabla_\alpha\phi\Big(\nabla^\mu\phi\,\nabla^\alpha\nabla^\nu\phi + \nabla^\nu\phi\,\nabla^\alpha\nabla^\mu\phi\Big) \\
&- \nabla^\mu\nabla^\nu\phi\,\nabla_\alpha\phi\,\nabla^\alpha\phi\bigg] \\
E(\beta_2 L_2) = \sqrt{-g}G^{\alpha\beta}&\left(\beta_2'\nabla_\alpha\phi\,\nabla_\beta\phi + 2\beta_2\nabla_\alpha\nabla_\beta\phi\right),
\end{aligned}
$$

$$
\begin{aligned}
E^{\mu\nu}(\beta_3 L_3) &= \sqrt{-g}\bigg[\beta_3 G^{\mu\nu} - \beta_3'\nabla^\mu\nabla^\nu\phi - \beta_3''\nabla^\mu\phi\,\nabla^\nu\phi + g^{\mu\nu}\Big(\beta_3''\nabla_\alpha\phi\,\nabla^\alpha\phi + \beta_3' g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi\Big)\bigg] \\
E(\beta_3 L_3) &= -\sqrt{-g}\,\beta_3' R,
\end{aligned}
$$

$$E^{\mu\nu}(\beta_4 L_4) = \sqrt{-g}\left[\frac{\partial\beta_4}{\partial\rho}\nabla^{\mu}\phi\,\nabla^{\nu}\phi - \frac{1}{2}g^{\mu\nu}\beta_4\right]$$

$$E(\beta_4 L_4) = 2\sqrt{-g}\left[\frac{\partial^2\beta_4}{\partial\phi\partial\rho}\nabla_{\alpha}\phi\,\nabla^{\alpha}\phi + \frac{\partial^2\beta_4}{\partial\rho^2}\nabla^{\alpha}\phi\,\nabla^{\beta}\phi\,\nabla_{\alpha}\nabla_{\beta}\phi \right.$$
$$\left. + \frac{\partial\beta_4}{\partial\rho}g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\phi\right] - \sqrt{-g}\frac{\partial\beta_4}{\partial\phi},$$

$$E^{\mu\nu}(cL_5) = 0$$
$$E(cL_5) = 0.$$

Where a prime denotes differentiation with respect to ϕ. These constitute the most general second order Euler-Lagrange equations for a Lagrangian of the form (2.1.1) in a four-dimensional space. Following this analysis for a Lagrangian density of the form

$$L = L(g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\sigma})$$

which satisfies (2.4.16) in a four-dimensional space leads to

$$L = \frac{\alpha}{\sqrt{-g}}\left(*R^{\mu\nu}{}_{\rho\sigma}\right)\left(*R^{\rho\sigma}{}_{\alpha\beta}\right)\left(*R^{\alpha\beta}{}_{\mu\nu}\right) + \beta\sqrt{-g}\Big(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\Big)$$
$$+\gamma\left(*R^{\mu\nu}{}_{\rho\sigma}\right)R^{\rho\sigma}{}_{\mu\nu} + \eta\sqrt{-g}R + \xi\sqrt{-g}$$

where in this case, $\alpha,\beta,\gamma,\eta,\xi$ are constants. If we restrict ourselves to Lagrangians whose corresponding equations are of second order[15], we obtain

$$L = \sqrt{-g}(\eta R + \xi)$$

where we have used the fact that the second and third terms (Gauss-Bonnet term and Pontryagin density) do not contribute to the Euler-Lagrange equations in four dimensions[16]. Consequently, the only second-order Euler-Lagrange equations for a Lagrangian of this form in a four-dimensional space are Einstein's field equations with a cosmological term.

[15]We can do this by either imposing the conditions derived in the previous section or by simply setting $\alpha = 0$.

[16]Notice that in the previous case the Gauss-Bonnet term is no longer topological since it's coupled to the scalar field.

2.5 Review of Horndeski Theory of Gravity

The preceding analysis naturally raises the question: What is the most general scalar-tensor theory in four-dimensional spacetime that yields second-order field equations? Horndeski provided the answer to this question [29], two years after publishing his result for the most general Lagrangian of the form (2.1.1). He generalized his earlier work by constructing the most general second-order Euler-Lagrange equations in a four-dimensional space, derived from a Lagrangian of the following form

$$L = L(g_{\mu\nu}; g_{\mu\nu,\iota_1}; \dots; g_{\mu\nu,\iota_1\dots\dots\iota_\rho}; \phi; \phi_{,\iota_1}; \dots; \phi_{,\iota_1\dots\dots\iota_\kappa}) \tag{2.5.1}$$

where $\rho,\kappa \geq 2$. The Horndeski Lagrangian has the form[17]:

$$\begin{aligned} L = \sqrt{-g}\Big[& \mathcal{K}_1 \delta^{\gamma\delta\varepsilon}_{\eta\iota\tau} \nabla_\gamma \nabla^\eta \phi R_{\delta\varepsilon}{}^{\iota\tau} - \frac{4}{3}\dot{\mathcal{K}}_1 \delta^{\gamma\delta\varepsilon}_{\eta\iota\tau} \nabla_\gamma \nabla^\eta \phi \nabla_\delta \nabla^\iota \phi \nabla_\varepsilon \nabla^\tau \phi + \mathcal{K}_3 \delta^{\gamma\delta\varepsilon}_{\eta\iota\tau} \nabla_\gamma \phi \nabla^\eta \phi R_{\delta\varepsilon}{}^{\iota\tau} \\ & - 4\dot{\mathcal{K}}_3 \delta^{\gamma\delta\varepsilon}_{\eta\iota\tau} \nabla_\gamma \phi \nabla^\eta \phi \nabla_\delta \nabla^\iota \phi \nabla_\varepsilon \nabla^\tau \phi + (\mathcal{F} + 2\mathcal{W}) \delta^{\gamma\delta}_{\lambda\eta} R_{\gamma\delta}{}^{\lambda\eta} + 2\left(2\mathcal{K}_3 - 2\mathcal{K}_1' + 4\rho\dot{\mathcal{K}}_3\right) \delta^{\gamma\delta}_{\lambda\eta} \nabla_\gamma \nabla^\lambda \phi \nabla_\delta \nabla^\eta \phi \\ & - 3(2\mathcal{F}' + 4\mathcal{W}' + \rho\mathcal{K}_8) \nabla_\gamma \nabla^\gamma \phi + 2\mathcal{K}_8 \delta^{\gamma\delta}_{\lambda\eta} \nabla_\gamma \phi \nabla^\lambda \phi \nabla_\delta \nabla^\eta \phi + \left\{4\mathcal{K}_9 - \rho\left(2\mathcal{F}'' + 4\mathcal{W}'' + \rho\mathcal{K}_8' + 2\dot{\mathcal{K}}_9\right)\right\}\Big] \end{aligned} \tag{2.5.2}$$

where $\rho = \nabla_\mu \phi \nabla^\mu \phi$ and $\mathcal{F}$ is given by

$$\mathcal{F} = \int \left\{\mathcal{K}_1'(\phi;\rho) - \mathcal{K}_3(\phi;\rho) - 2\rho\dot{\mathcal{K}}_3(\phi;\rho)\right\} d\rho. \tag{2.5.3}$$

The primes and dots denote partial derivatives with respect to ϕ and ρ, respectively, and $\mathcal{K}_1, \mathcal{K}_3, \mathcal{K}_8, \mathcal{K}_9$ are arbitrary functions of ϕ, ρ.

2.5.1 An outline of the derivation

The Euler-Lagrange tensors corresponding to (2.5.1) are given by

[17] Generalised kronecker deltas are defined as

$$\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n} = \det \begin{bmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_n}_{\nu_1} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_1}_{\nu_n} & \cdots & \delta^{\mu_n}_{\nu_n} \end{bmatrix}$$

$$E^{\mu\nu}(L) = \sum_{\eta=0}^{\rho} (-1)^{\eta+1} \frac{d}{dx^{\iota_1}} \cdots \frac{d}{dx^{\iota_\eta}} \frac{\partial L}{\partial g_{\mu\nu,\iota_1 \dots \iota_\eta}} \tag{2.5.4}$$

$$E(L) = \sum_{\eta=0}^{\kappa} (-1)^{\eta+1} \frac{d}{dx^{\iota_1}} \cdots \frac{d}{dx^{\iota_\eta}} \frac{\partial L}{\partial \phi_{,\iota_1} \dots \iota_\eta} \tag{2.5.5}$$

It is important to note that, in general, $E^{\mu\nu}(L)$ is of 2ρ-th order in the derivatives of $g_{\mu\nu}$ and $(\rho+\kappa)$-th order in the derivatives of ϕ, while $E(L)$ is of 2κ-th order in the derivatives of ϕ and $(\rho+\kappa)$-th order in the derivatives of $g_{\mu\nu}$. To avoid the presence of unphysical degrees of freedom, one can either impose second-order field equations, or enforce degeneracy conditions on the Lagrangian (DHOST's approach[18]). Horndeski's work focused on the former approach. An essential part of the derivation is the Generalised Bianchi Identity satisfied by the field equations of the Lagrangian:

$$\nabla_\nu E^{\mu\nu}(L) = \frac{1}{2}(\nabla^\mu \phi) E(L) \tag{2.5.6}$$

While $\nabla_\nu E^{\mu\nu}(L)$ would generally contain third-order derivatives of both the metric $g_{\mu\nu}$ and scalar field ϕ, the right-hand side of the Bianchi identity is constrained to at most second-order derivatives. This requirement forces $\nabla_\nu E^{\mu\nu}(L)$ to maintain second-order behavior, thereby placing strong constraints on the form of $E^{\mu\nu}$.

This led Horndeski to consider the following problem: In a four-dimensional space, what is the most general second-order symmetric tensor $A^{\mu\nu}$ satisfying the following:

- $A^{\mu\nu}$ is at most second-order in derivatives of the metric $g_{\mu\nu}$ and scalar field ϕ

$$A^{\mu\nu} = A^{\mu\nu}(g_{\alpha\beta}; g_{\alpha\beta,\gamma}; g_{\alpha\beta,\gamma\delta}; \phi; \phi_{,\gamma}; \phi_{,\gamma\delta}) \tag{2.5.7}$$

- Its divergence $\nabla_\nu A^{\mu\nu}$ must also be second-order in derivatives, satisfying

$$\nabla_\nu A^{\mu\nu} = \nabla^\mu \phi A \tag{2.5.8}$$

  where A is a scalar density of the form

$$A = A(g_{\alpha\beta}; g_{\alpha\beta,\gamma}; g_{\alpha\beta,\gamma\delta}; \phi; \phi_{,\gamma}; \phi_{,\gamma\delta}) \tag{2.5.9}$$

Therefore, our objective is to devise a tensor $A^{\mu\nu}$ that satisfies these fundamental requirements.

[18]Decades after Horndeski's original construction, it was realized that the requirement of second-order equations could be relaxed while still avoiding ghosts. This led to the discovery of Degenerate Higher-Order Scalar-Tensor (DHOST) theories [30], which retain only healthy degrees of freedom by enforcing degeneracy conditions on the Lagrangian, even when field equations are higher-order.

Horndeski first constructed the most general tensor $A^{\mu\nu}$ satisfying the derivative-order constraint, using generalized Kronecker deltas and relying on the assumption of a four-dimensional space-time. Then he proceeded to restrict the form of $A^{\mu\nu}$ by requiring that the divergence constraint is satisfied. The tensor obtained following this procedure will be $E^{\mu\nu}(L)$. The final step is to seek the Lagrangian L that yields as its Euler-Lagrange equations $E^{\mu\nu}(L) = 0$ and $E(L) = 0$. Following an approach earlier used by Lovelock, Horndeski tried the following as a possible Lagrangian:

$$L = g_{\mu\nu}A^{\mu\nu}$$

Remarkably, the Euler-Lagrange equations derived from this Lagrangian were found to reproduce the required structure of both $E^{\mu\nu}(L) = 0$ and $E(L) = 0$ exactly, which is how he arrived at the Lagrangian (2.5.2).

2.5.2 The construction of $A^{\mu\nu}$

Following a similar analysis to that of previous sections, we can establish the following result:

- A symmetric tensor density of the form

  $$A^{\mu\nu} = A^{\mu\nu}(g_{\alpha\beta}; g_{\alpha\beta,\kappa}; g_{\alpha\beta,\kappa\rho}; \phi, \phi_\kappa; \phi_{\kappa\rho})$$

  will have its divergence, $\nabla_\nu A^{\mu\nu}$, being at most of second-order in the derivatives of both $g_{\mu\nu}$ and ϕ if and only if[19]

  $$A^{\mu\alpha;\rho\sigma,\beta\kappa} + A^{\mu\beta;\rho\sigma,\kappa\alpha} + A^{\mu\kappa;\rho\sigma,\alpha\beta} = 0$$

  and

  $$A^{\mu\alpha;\beta\kappa} + A^{\mu\beta;\kappa\alpha} + A^{\mu\kappa;\alpha\beta} = 0.$$

As was the case in our analysis earlier, these two equation are tensorial. Also, it is clear that these tensor densities satisfy the requirements of the Property S. In this case using property S

[19] Here we use the notation introduced in the earlier sections of this chapter.

$$A^{\mu\alpha;\rho\sigma,\beta\kappa} \equiv \frac{\partial A^{\mu\alpha}}{\partial g_{\rho\sigma,\beta\kappa}} \quad \text{and} \quad A^{\mu\alpha;\beta\kappa} = \frac{\partial A^{\mu\alpha}}{\partial \phi_{,\beta\kappa}}$$

leads us to the following

$$\frac{\partial}{\partial g_{\alpha\beta,\gamma\delta}}\left(\frac{\partial A^{\mu\nu}}{\partial g_{\rho\lambda,\sigma\tau}}\right) = A^{\mu\nu;\rho\lambda,\sigma\tau;\alpha\beta,\gamma\delta} = 0$$

which when integrated twice with respect to the second derivative of the metric yields

$$A^{\mu\nu} = \Psi^{\mu\nu;\alpha\beta,\gamma\delta}R_{\gamma\alpha\beta\delta} + \Xi^{\mu\nu}$$

where $\Xi^{\mu\nu}$ and $\Psi^{\mu\nu;\alpha\beta,\gamma\delta}$ are tensor densities that enjoy the property S and are functions of $g_{\mu\nu}, \phi, \phi_{,\rho}$. In order to find the general form of $\Psi^{\mu\nu;\alpha\beta,\gamma\delta}$ and $\Xi^{\mu\nu}$, one makes use of the property S and techniques similar to those employed by Lovelock [7],[28],[31]. The result of these calculations is

$$\begin{aligned} A^{\alpha\beta} =& \xi^{\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta}R_{\varepsilon\gamma\delta\zeta}\nabla_\eta\nabla_\theta\phi + \xi^{\alpha\beta\gamma\delta\varepsilon\zeta}R_{\varepsilon\gamma\delta\zeta} \\ &+ \psi^{\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta}\nabla_\gamma\nabla_\delta\phi\,\nabla_\varepsilon\nabla_\zeta\phi\,\nabla_\eta\nabla_\theta\phi + \psi^{\alpha\beta\gamma\delta\varepsilon\zeta}\nabla_\gamma\nabla_\delta\phi\,\nabla_\varepsilon\nabla_\zeta\phi \\ &+ \psi^{\alpha\beta\gamma\delta}\nabla_\gamma\nabla_\delta\phi + \psi^{\alpha\beta}. \end{aligned} \tag{2.5.10}$$

It can be shown that the most general tensor densities of the form

$$\Theta^{\mu_1\cdots\mu_{2k}} = \Theta^{\mu_1\cdots\mu_{2k}}(g_{\mu\nu};\phi;\phi_{,\nu})$$

$(k = 1,\ldots,4)$ which satisfy the property S in a space of four-dimensions are given by:

$$\begin{aligned} \Theta^{\mu_1\mu_2} =& \sqrt{-g}\,(C_1 g^{\mu_1\mu_2} + C_2\phi^{;\mu_1}\phi^{;\mu_2}) \\ \Theta^{\mu_1\mu_2\mu_3\mu_4} =& \sqrt{-g}\,C_3\,(g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3} - 2g^{\mu_1\mu_2}g^{\mu_3\mu_4}) \\ &+ \sqrt{-g}\,C_4\Big(\phi^{;\mu_1}\phi^{\mu_3}g^{\mu_2\mu_4} + \phi^{;\mu_2}\phi^{;\mu_4}g^{\mu_1\mu_3} \\ &+ \phi^{;\mu_1}\phi^{;\mu_4}g^{\mu_2\mu_3} + \phi^{;\mu_2}\phi^{;\mu_3}g^{\mu_1\mu_4} \\ &- 2\,[\phi^{;\mu_1}\phi^{;\mu_2}g^{\mu_3\mu_4} + \phi^{;\mu_3}\phi^{;\mu_4}g^{\mu_1\mu_2}]\Big) \\ \Theta^{\mu_1\cdots\mu_6} =& \frac{1}{\sqrt{-g}}(C_5\phi_{,\rho}\phi_{,\sigma} + C_6 g_{\rho\sigma})\Big\{\varepsilon^{\mu_1\mu_3\mu_5\rho}\varepsilon^{\mu_2\mu_4\mu_6\sigma} \\ &+ \varepsilon^{\mu_1\mu_3\mu_6\rho}\varepsilon^{\mu_2\mu_4\mu_5\sigma} + \varepsilon^{\mu_1\mu_4\mu_5\rho}\varepsilon^{\mu_2\mu_3\mu_6\sigma} \\ &+ \varepsilon^{\mu_1\mu_4\mu_6\rho}\varepsilon^{\mu_2\mu_3\mu_5\sigma}\Big\} \end{aligned}$$

and

$$\begin{aligned} \Theta^{\mu_1\cdots\mu_8} =& \frac{C_7}{\sqrt{-g}}\Big\{\varepsilon^{\mu_1\mu_3\mu_5\mu_7}\varepsilon^{\mu_2\mu_4\mu_6\mu_8} + \varepsilon^{\mu_1\mu_3\mu_5\mu_8}\varepsilon^{\mu_2\mu_4\mu_6\mu_7} \\ &+ \varepsilon^{\mu_1\mu_3\mu_6\mu_7}\varepsilon^{\mu_2\mu_4\mu_5\mu_8} + \varepsilon^{\mu_1\mu_3\mu_6\mu_8}\varepsilon^{\mu_2\mu_4\mu_5\mu_7} \\ &+ \varepsilon^{\mu_2\mu_3\mu_5\mu_7}\varepsilon^{\mu_1\mu_4\mu_6\mu_8} + \varepsilon^{\mu_2\mu_3\mu_5\mu_8}\varepsilon^{\mu_1\mu_4\mu_6\mu_7} \\ &+ \varepsilon^{\mu_2\mu_3\mu_6\mu_7}\varepsilon^{\mu_1\mu_4\mu_5\mu_8} + \varepsilon^{\mu_2\mu_3\mu_6\mu_8}\varepsilon^{\mu_1\mu_4\mu_5\mu_7}\Big\} \end{aligned}$$

where $C_1, \ldots, C_7$ are arbitrary functions of ϕ and $\rho = \phi_{,\mu}\phi_{,\nu}g^{\mu\nu}$. Using these expressions and considering the symmetries of the Riemann curvature tensor, Horndeski demonstrated that (2.5.10) can be written as

$$\begin{aligned}A^{\alpha\beta} = \sqrt{-g}\Big(& K_1\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\theta\rho\kappa}g^{\zeta\beta}\nabla^{\theta}\nabla_{\gamma}\phi\, R_{\delta\varepsilon}{}^{\rho\kappa} + K_2\delta^{\alpha\gamma\delta}_{\varepsilon\zeta\theta}g^{\varepsilon\beta}R_{\gamma\delta}{}^{\zeta\theta} \\ &+ K_3\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\theta\rho\kappa}g^{\zeta\beta}\nabla_{\gamma}\phi\,\nabla^{\theta}\phi\, R_{\delta\varepsilon}{}^{\rho\kappa} + K_4\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\theta\rho\kappa}g^{\zeta\beta}\nabla^{\theta}\nabla_{\gamma}\phi\,\nabla^{\rho}\nabla_{\delta}\phi\,\nabla^{\kappa}\nabla_{\varepsilon}\phi \\ &+ K_5\delta^{\alpha\gamma\delta}_{\varepsilon\zeta\theta}g^{\varepsilon\beta}\nabla^{\zeta}\nabla_{\gamma}\phi\,\nabla^{\theta}\nabla_{\delta}\phi + K_6\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\theta\rho\kappa}g^{\zeta\beta}\nabla_{\gamma}\phi\,\nabla^{\theta}\phi\,\nabla^{\rho}\nabla_{\delta}\phi\,\nabla^{\kappa}\nabla_{\varepsilon}\phi \\ &+ K_7\delta^{\alpha\gamma}_{\delta\varepsilon}g^{\delta\beta}\nabla^{\varepsilon}\nabla_{\gamma}\phi + K_8\delta^{\alpha\gamma\delta}_{\varepsilon\zeta\theta}g^{\varepsilon\beta}\nabla_{\gamma}\phi\,\nabla^{\zeta}\phi\,\nabla^{\theta}\nabla_{\delta}\phi + K_9 g^{\alpha\beta} \\ &+ K_{10}\nabla^{\alpha}\phi\,\nabla^{\beta}\phi\Big)\end{aligned} \tag{2.5.11}$$

where $K_{1,\ldots,10}$ are arbitrary functions of ϕ and ρ. We have thus established that in a space of four dimensions the most general symmetric tensor of the form (2.5.7) which is such that its covariant divergence is at most of second-order in the derivatives of both $g_{\mu\nu}$ and ϕ is given by (2.5.11). At this point, we know that the symmetric rank 2 tensor densities that satisfy (2.5.8) for some scalar density A will be contained in (2.5.11). In order to proceed we need to calculate the divergence of $A^{\alpha\beta}$ and determine the conditions under which this occurs. These calculations, after much simplification[20], yield

$$\begin{aligned}\nabla_{\beta}A^{\alpha\beta} = \sqrt{-g}\Big(& \alpha\delta^{\alpha\gamma\varepsilon}_{\eta\rho\kappa}\nabla^{\delta}\phi\,\nabla^{\eta}\nabla_{\gamma}\phi\, R_{\delta\varepsilon}{}^{\rho\kappa} + \beta\delta^{\alpha\delta\varepsilon}_{\eta\rho\kappa}\nabla^{\gamma}\phi\,\nabla^{\eta}\nabla_{\gamma}\phi\, R_{\delta\varepsilon}{}^{\rho\kappa} \\ &- \gamma\delta^{\alpha\gamma\varepsilon}_{\zeta\rho\kappa}\nabla_{\lambda}\phi\,\nabla^{\zeta}\nabla^{\lambda}\phi\,\nabla_{\gamma}\phi\,\nabla^{\delta}\phi\, R_{\delta\varepsilon}{}^{\rho\kappa} + \varepsilon\delta^{\alpha\beta\gamma\varepsilon}_{\eta\rho\kappa\tau}\nabla^{\lambda}\phi\, R_{\lambda\beta}{}^{\rho\eta}\,\nabla^{\kappa}\nabla_{\gamma}\phi\,\nabla^{\zeta}\nabla_{\varepsilon}\phi \\ &+ \mu\delta^{\alpha\gamma}_{\delta\varepsilon}\nabla^{\tau}\phi\, R_{\tau\gamma}{}^{\varepsilon\delta} + \nu\delta^{\alpha\delta\varepsilon}_{\zeta\rho\kappa}\nabla^{\lambda}\phi\,\nabla^{\zeta}\nabla_{\lambda}\phi\,\nabla^{\rho}\nabla_{\delta}\phi\,\nabla^{\kappa}\nabla_{\varepsilon}\phi \\ &+ 2\omega\delta^{\alpha\gamma}_{\delta\varepsilon}\nabla^{\lambda}\phi\,\nabla^{\delta}\nabla_{\lambda}\phi\,\nabla^{\varepsilon}\nabla_{\gamma}\phi + \xi\nabla_{\lambda}\phi\,\nabla^{\lambda}\nabla^{\alpha}\phi\Big) \\ &+ \sqrt{-g}\nabla^{\alpha}\phi\, Q\end{aligned} \tag{2.5.12}$$

where

$$\begin{aligned}&\alpha = 2K_1' - 2K_3 + K_5 + \rho K_6, \quad \beta = 2\dot{K}_2 - K_1' + K_3 + 2\rho\dot{K}_3, \\ &\gamma = 4\dot{K}_3 + K_6, \quad \varepsilon = 2\dot{K}_1 + \frac{3}{2}K_4, \quad \mu = 2K_2' + \frac{1}{2}K_7 + \frac{1}{2}\rho K_8, \\ &\nu = 2\dot{K}_5 + 3K_6 - 3K_4' + 2\rho\dot{K}_6, \quad \omega = \dot{K}_7 - K_5' + K_8 + \rho\dot{K}_8, \quad \xi = 2\dot{K}_9 + K_{10} - K_7'\end{aligned} \tag{2.5.13}$$

[20] In a four-dimensional space the following identities hold that are crucial for this simplification

$$\delta^{\alpha\beta\gamma\delta\lambda}_{\sigma\eta\iota\kappa\mu}\nabla^{\sigma}\phi R_{\beta\gamma}{}^{\eta\iota}R_{\delta\lambda}{}^{\kappa\mu} = 0,$$
$$\delta^{\alpha\beta\gamma\delta\lambda}_{\sigma\eta\iota\kappa\mu}\nabla^{\sigma}\phi\nabla^{\eta}\nabla_{\beta}\phi\nabla^{\iota}\nabla_{\gamma}\phi R_{\delta\lambda}{}^{\kappa\mu} = 0,$$
$$\delta^{\alpha\beta\gamma\delta\lambda}_{\sigma\eta\iota\kappa\mu}\nabla^{\sigma}\phi\nabla^{\eta}\nabla_{\beta}\phi\nabla^{\iota}\nabla_{\gamma}\phi\nabla^{\kappa}\nabla_{\delta}\phi\nabla^{\mu}\nabla_{\lambda}\phi = 0.$$

and,

$$
\begin{aligned}
Q =&\dot{K}_1\delta^{\beta\gamma\delta\varepsilon}_{\eta\rho\kappa\tau}\nabla^\eta\nabla_\beta\phi\nabla^\rho\nabla_\gamma\phi R_{\delta\varepsilon}{}^{\kappa\tau}\\
&+(K_1'-K_3)\delta^{\gamma\delta\varepsilon}_{\zeta\rho\kappa}\nabla^\zeta\nabla_\gamma\phi R_{\delta\varepsilon}{}^{\rho\kappa}-2\dot{K}_3\delta^{\gamma\delta\varepsilon}_{\zeta\rho\kappa}\nabla^\lambda\phi\nabla^\zeta\nabla_\lambda\phi\nabla_\gamma\phi R_{\delta\varepsilon}{}^{\rho\kappa}\\
&-K_6\delta^{\gamma\delta\varepsilon}_{\zeta\rho\kappa}\nabla_\gamma\phi\nabla^\tau\phi R_{\tau\delta}{}^{\rho\zeta}\nabla^\kappa\nabla_\varepsilon\phi-\frac{1}{8}K_1\delta^{\beta\gamma\delta\varepsilon}_{\eta\rho\kappa\tau}R_{\beta\gamma}{}^{\eta\rho}R_{\delta\varepsilon}{}^{\kappa\tau}\\
&+K_2'\delta^{\gamma\delta}_{\zeta\eta}R_{\gamma\delta}{}^{\zeta\eta}-\frac{1}{2}K_8\delta^{\gamma\delta}_{\varepsilon\eta}\nabla_\gamma\phi\nabla^\tau\phi R_{\tau\delta}{}^{\eta\varepsilon}\\
&-2\dot{K}_8\delta^{\gamma\delta}_{\varepsilon\eta}\nabla_\lambda\phi\nabla^\lambda\nabla^\varepsilon\phi\nabla_\gamma\phi\nabla^\eta\nabla_\delta\phi+\frac{1}{2}\dot{K}_4\delta^{\beta\gamma\delta\varepsilon}_{\eta\rho\kappa\tau}\nabla^\eta\nabla_\beta\phi\nabla^\rho\nabla_\gamma\phi\nabla^\kappa\nabla_\delta\phi\nabla^\tau\nabla_\varepsilon\phi\\
&-2\dot{K}_6\delta^{\gamma\delta\varepsilon}_{\zeta\rho\kappa}\nabla^\lambda\phi\nabla^\zeta\nabla_\lambda\phi\nabla_\gamma\phi\nabla^\rho\nabla_\delta\phi\nabla^\kappa\nabla_\varepsilon\phi\\
&+(K_4'-K_6)\delta^{\gamma\delta\varepsilon}_{\eta\rho\kappa}\nabla^\eta\nabla_\gamma\phi\nabla^\rho\nabla_\delta\phi\nabla^\kappa\nabla_\varepsilon\phi+(K_5'-K_8)\delta^{\gamma\delta}_{\zeta\eta}\nabla^\zeta\nabla_\gamma\phi\nabla^\eta\nabla_\delta\phi\\
&+K_9'+\rho K_{10}'+2\dot{K}_{10}\nabla^\beta\phi\nabla^\gamma\phi\nabla_\beta\nabla_\gamma\phi+(K_{10}+K_7')\nabla_\gamma\nabla^\gamma\phi.
\end{aligned}
\tag{2.5.14}
$$

It is clear that in order for $\nabla_\beta A^{\alpha\beta}$ to satisfy (2.5.8) there should exist a scalar density B of the form

$$B=B(g_{\mu\nu};g_{\mu\nu,\rho};g_{\mu\nu,\rho\sigma};\phi;\phi_{,\mu};\phi_{,\mu\nu})$$

which such that

$$
\begin{aligned}
\nabla^\alpha\phi B =\sqrt{g}\Bigg(&\alpha\delta^{\alpha\gamma\varepsilon}_{\eta\rho\kappa}\nabla^\delta\phi\nabla^\eta\nabla_\gamma\phi R_{\delta\varepsilon}{}^{\rho\kappa}+\beta\delta^{\alpha\delta\varepsilon}_{\eta\rho\kappa}\nabla^\gamma\phi\nabla^\eta\nabla_\gamma\phi R_{\delta\varepsilon}{}^{\rho\kappa}\\
&-\gamma\delta^{\alpha\gamma\varepsilon}_{\zeta\rho\kappa}\nabla_\lambda\phi\nabla^\lambda\nabla^\zeta\phi\nabla_\gamma\phi\nabla^\delta\phi R_{\delta\varepsilon}{}^{\rho\kappa}+\varepsilon\delta^{\alpha\beta\gamma\varepsilon}_{\eta\rho\kappa\tau}\nabla^\lambda\phi R_{\lambda\beta}{}^{\rho\eta}\nabla^\kappa\nabla_\gamma\phi\nabla^\zeta\nabla_\varepsilon\phi\\
&+\mu\delta^{\alpha\gamma}_{\delta\varepsilon}\nabla^\tau\phi R_{\tau\gamma}{}^{\varepsilon\delta}+\nu\delta^{\alpha\delta\varepsilon}_{\zeta\rho\kappa}\nabla^\lambda\phi\nabla^\zeta\nabla_\lambda\phi\nabla^\rho\nabla_\delta\phi\nabla^\kappa\nabla_\varepsilon\phi\\
&+2\omega\delta^{\alpha\gamma}_{\delta\varepsilon}\nabla^\lambda\phi\nabla^\delta\nabla_\lambda\phi\nabla^\varepsilon\nabla_\gamma\phi+\xi\nabla_\lambda\phi\nabla^\alpha\nabla^\lambda\phi\Bigg).
\end{aligned}
\tag{2.5.15}
$$

We need to solve this equation for B. Horndeski showed that (2.5.15) admits a solution if and only if

$$\alpha=\beta=\gamma=\varepsilon=\mu=\nu=\omega=\xi=0$$

and in this case, the solution is

$$B=0.$$

As a result, $\nabla_\beta A^{\alpha\beta}$ will be of the form (2.5.8) if and only if the ten scalar functions K_i, introduced in (2.5.13) satisfy the following partial differential equations. These equations comprise a system of eight partial differential equations, among which only six are linearly independent, as the remaining two can be expressed as linear combinations of the others. The functions K_i, must satisfy the following:

$$K_4 = -\frac{4}{3}\dot{K}_1, \qquad K_5 = 2K_3 - 2K_1' - 4\rho\dot{K}_3, \qquad K_6 = -4\dot{K}_3,$$
$$K_2 = \frac{1}{2}F + W, \qquad K_7 = -2F^{'} - 4W^{'} - \rho K_8,$$
$$K_{10} = -2F'' - 4W'' - \rho K_8' - 2\dot{K}_9.$$

where K_1, K_8, K_9, K_3 are functions of ϕ and ρ, W is an arbitrary function of ϕ and F is given by the following integral:

$$F = F(\varphi;\rho) = Z\int \left(K_1'(\phi;\rho) - K_3(\phi;\rho) - 2\rho\dot{K}_3(\phi;\rho)\right) d\rho. \tag{2.5.16}$$

Therefore we have established that in four-dimensional space, the most general symmetric tensor density which is of the form (2.5.7) and its divergence satisfies (2.5.8) is

$$\begin{aligned} A^{\alpha\beta} = \sqrt{g}\Bigg[& K_1\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\eta\rho\kappa} g^{\zeta\beta}\nabla^{\eta}\nabla_{\gamma}\phi R_{\delta\varepsilon}{}^{\rho\kappa} + \left(\frac{1}{2}F + W\right)\delta^{\alpha\gamma\delta}_{\varepsilon\zeta\eta} g^{\varepsilon\beta} R_{\gamma\delta}{}^{\zeta\eta} \\ & + K_3\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\eta\rho\kappa} g^{\zeta\beta}\nabla_{\gamma}\phi\nabla^{\eta}\phi R_{\delta\varepsilon}{}^{\rho\kappa} - \frac{4}{3}\dot{K}_1\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\eta\rho\kappa} g^{\zeta\beta}\nabla^{\eta}\nabla_{\gamma}\phi\nabla^{\rho}\nabla_{\delta}\phi\nabla^{\kappa}\nabla_{\varepsilon}\phi \\ & + \left(2K_3 - 2K_1' - 4\rho\dot{K}_3\right)\delta^{\alpha\gamma\delta}_{\varepsilon\zeta\eta} g^{\varepsilon\beta}\nabla^{\zeta}\nabla_{\gamma}\phi\nabla^{\eta}\nabla_{\delta}\phi \\ & - 4\dot{K}_3\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\eta\rho\kappa} g^{\zeta\beta}\nabla_{\gamma}\phi\nabla^{\eta}\phi\nabla^{\rho}\nabla_{\delta}\phi\nabla^{\kappa}\nabla_{\varepsilon}\phi \\ & - \left(2F' + 4W' + \rho K_8\right)\delta^{\alpha\gamma}_{\delta\varepsilon} g^{\delta\beta}\nabla^{\varepsilon}\nabla_{\gamma}\phi \\ & + K_8\delta^{\alpha\gamma\delta}_{\varepsilon\zeta\eta} g^{\varepsilon\beta}\nabla_{\gamma}\phi\nabla^{\zeta}\phi\nabla^{\eta}\nabla_{\delta}\phi + K_9 g^{\alpha\beta} \\ & - \left(2F'' + 4W'' + \rho K_8' + 2\dot{K}_9\right)\nabla^{\alpha}\phi\nabla^{\beta}\phi\Bigg]. \end{aligned} \tag{2.5.17}$$

2.5.3 Construction of the Lagrangian

The final step in the analysis involves finding a suitable Lagrangian L that yields this tensor density as its Euler-Lagrange tensor. Lovelock has shown [28], that the Lagrangian that yields

$E^{\mu\nu} = A^{\mu\nu}$ as its equations of motion can be constructed using $g_{\mu\nu}A^{\mu\nu}$. Therefore we find

$$\begin{aligned}
g_{\alpha\beta}A^{\alpha\beta} = \sqrt{g}\Bigg[& K_1\delta^{\alpha\gamma\delta\varepsilon}_{\eta\rho\kappa}\nabla^{\eta}\nabla_{\gamma}\phi R_{\delta\varepsilon}{}^{\rho\kappa} - \frac{4}{3}\dot{K}_1\delta^{\gamma\delta\varepsilon}_{\eta\rho\kappa}\nabla^{\eta}\nabla_{\gamma}\phi\nabla^{\rho}\nabla_{\delta}\phi\nabla^{\kappa}\nabla_{\varepsilon}\phi \\
& + K_3\delta^{\gamma\delta\varepsilon}_{\eta\rho\kappa}\nabla_{\gamma}\phi\nabla^{\eta}\phi R_{\delta\varepsilon}{}^{\rho\kappa} - 4\dot{K}_3\delta^{\gamma\delta\varepsilon}_{\eta\rho\kappa}\nabla_{\gamma}\phi\nabla^{\eta}\phi\nabla^{\rho}\nabla_{\delta}\phi\nabla^{\kappa}\nabla_{\varepsilon}\phi \\
& + (F+2W)\delta^{\gamma\delta}_{\zeta\eta}R_{\gamma\delta}{}^{\zeta\eta} + 2(2K_3 - 2K_1' + 4\rho\dot{K}_3)\delta^{\gamma\delta}_{\zeta\eta}\nabla^{\zeta}\nabla_{\gamma}\phi\nabla^{\eta}\nabla_{\delta}\phi \\
& - 3(2F' + 4W' + \rho K_8)\nabla^{\gamma}_{\gamma}\phi + 2K_8\delta^{\gamma\delta}_{\zeta\eta}\nabla_{\gamma}\phi\nabla^{\zeta}\phi\nabla^{\eta}\nabla_{\delta}\phi \\
& + 4K_9 - \rho(2F'' + 4W'' + \rho K_8' + 2\dot{K}_9)\Bigg].
\end{aligned} \tag{2.5.18}$$

A similar analysis to that performed in Sections (2.1)-(2.2), for the simpler case of a Lagrangian of the form (2.1.1), can also be applied to this case. Consequently, it can be shown that for a Lagrangian of the form

$$L = L(g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\sigma}; \phi; \phi_{\mu}\phi_{,\mu\nu}).$$

The Euler-Lagrange equations, using Rund's tensors, can be expressed as

$$\begin{aligned}
E^{\alpha\beta}(L) &= -\nabla_{\eta}\nabla_{\kappa}\Lambda^{\alpha\beta,\kappa\eta} + \nabla_{\eta}\Pi^{\alpha\beta,\eta} - \Pi^{\alpha\beta}, \\
E(L) &= -\nabla_{\eta}\nabla_{\kappa}\zeta^{\kappa\eta} + \nabla_{\eta}\zeta^{\eta} - \zeta
\end{aligned} \tag{2.5.19}$$

where,

$$\pi^{\alpha\beta,\eta\kappa} = \frac{\partial L}{\partial g_{\alpha\beta,\eta\kappa}}, \quad \zeta = \frac{\partial L}{\partial\phi}, \quad \zeta^{\alpha\beta} = \frac{\partial L}{\partial_{\phi,\alpha\beta}}, \quad \zeta^{\alpha} = \frac{\partial L}{\partial\phi_{,\alpha}} + \Gamma^{\alpha}_{\rho\sigma}\zeta^{\rho\sigma} \tag{2.5.20}$$

and,[21]

$$\pi^{\alpha\beta} = \frac{1}{3}R_{\kappa}{}^{\beta}{}_{\tau\eta}\pi^{\eta\kappa,\alpha\mu} - R_{\kappa}{}^{\alpha}{}_{\tau\eta}\pi^{\eta\kappa,\beta\mu} - \frac{1}{2}\nabla^{\alpha}\phi\nabla^{\beta}\zeta - \zeta^{\beta\sigma}\nabla^{\alpha}\nabla_{\sigma}\phi + \frac{1}{2}g^{\alpha\beta}L \tag{2.5.21}$$

$$\pi^{\alpha\beta,\eta} = \frac{1}{2}\left(\zeta^{\alpha\beta}\nabla^{\eta}\phi - \zeta^{\eta\beta}\nabla^{\alpha}\phi - \zeta^{\eta\alpha}\nabla^{\beta}\phi\right) \tag{2.5.22}$$

[21] Note that, there appears to be a mistake in (2.5.21) in [29] that involves the third and fourth terms of this equation.

Using these and (2.5.16), we find

$$
\begin{aligned}
E^{\alpha\beta}(g_{\gamma\delta}A^{\gamma\delta}) = \sqrt{g}\Big[& \rho\dot{K}_1\delta^{\alpha\delta\varepsilon\zeta}_{\eta\kappa\lambda\chi}g^{\eta\beta}\nabla^{\kappa}\nabla_{\delta}\phi R_{\varepsilon\zeta}{}^{\lambda\chi} + \rho\dot{K}_3\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\eta\rho\kappa}g^{\zeta\beta}\nabla_{\gamma}\phi\nabla^{\eta}\phi R_{\delta\varepsilon}{}^{\rho\kappa} \\
& + \left(\frac{1}{2}J - W\right)\delta^{\alpha\gamma\delta}_{\varepsilon\zeta\eta}g^{\varepsilon\beta}R_{\gamma\delta}{}^{\zeta\eta} - \frac{4}{3}\frac{\partial}{\partial\rho}(\rho\dot{K}_1)\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\eta\rho\kappa}g^{\zeta\beta}\nabla^{\eta}\nabla_{\gamma}\phi\nabla^{\rho}\nabla_{\delta}\phi\nabla^{\kappa}\nabla_{\varepsilon}\phi \\
& - 4\frac{\partial}{\partial\rho}(\rho\dot{K}_3)\delta^{\alpha\gamma\delta\varepsilon}_{\zeta\eta\rho\kappa}g^{\zeta\beta}\nabla_{\gamma}\phi\nabla^{\eta}\phi\nabla^{\rho}\nabla_{\delta}\phi\nabla^{\kappa}\nabla_{\varepsilon}\phi \\
& + \left(-2\rho\dot{K}_1' + 6\rho\dot{K}_3 + 4\rho^2\ddot{K}_3\right)\delta^{\alpha\delta\varepsilon}_{\zeta\rho\kappa}g^{\zeta\beta}\nabla^{\rho}\nabla_{\delta}\phi\nabla^{\kappa}\nabla_{\varepsilon}\phi \\
& + \rho\dot{K}_8\delta^{\alpha\gamma\delta}_{\varepsilon\zeta\eta}g^{\varepsilon\beta}\nabla_{\gamma}\phi\nabla^{\zeta}\phi\nabla^{\eta}\nabla_{\delta}\phi + \left(-2J' + 4W' - \rho^2\dot{K}_8\right)\delta^{\alpha\gamma}_{\delta\varepsilon}g^{\delta\beta}\nabla^{\varepsilon}\nabla_{\gamma}\phi \\
& + \left(\rho\dot{K}_9 - 2K_9\right)g^{\alpha\beta} + \left(-J'' + 4W'' + 2\dot{K}_9 - 2\rho\ddot{K}_9 - \rho^2\dot{K}_8'\right)\nabla^{\alpha}\phi\nabla^{\beta}\phi\Big]
\end{aligned} \tag{2.5.23}
$$

where,

$$
J = \int\left(\frac{\partial}{\partial\varphi}(\rho\dot{K}_1) - \rho\dot{K}_3 - 2\rho\frac{\partial}{\partial\rho}(\rho\dot{K}_3)\right)d\varphi.
$$

Integration by parts has been used to show that

$$
-F + \rho\dot{F} = J.
$$

Comparing the equations (2.5.17), (2.5.18) we deduce the Lagrangian:

$$
\begin{aligned}
L = \sqrt{-g}\Big[& \mathcal{K}_1\delta^{\gamma\delta\varepsilon}_{\eta\iota\tau}\nabla_{\gamma}\nabla^{\eta}\phi R_{\delta\varepsilon}{}^{\iota\tau} - \frac{4}{3}\dot{\mathcal{K}}_1\delta^{\gamma\delta\varepsilon}_{\eta\iota\tau}\nabla_{\gamma}\nabla^{\eta}\phi\nabla_{\delta}\nabla^{\iota}\phi\nabla_{\varepsilon}\nabla^{\tau}\phi \\
& + \mathcal{K}_3\delta^{\gamma\delta\varepsilon}_{\eta\iota\tau}\nabla_{\gamma}\phi\nabla^{\eta}\phi R_{\delta\varepsilon}{}^{\iota\tau} - 4\dot{\mathcal{K}}_3\delta^{\gamma\delta\varepsilon}_{\eta\iota\tau}\nabla_{\gamma}\phi\nabla^{\eta}\phi\nabla_{\delta}\nabla^{\iota}\phi\nabla_{\varepsilon}\nabla^{\tau}\phi \\
& + (\mathcal{F} + 2\mathcal{W})\delta^{\gamma\delta}_{\lambda\eta}R_{\gamma\delta}{}^{\lambda\eta} + 2\left(2\mathcal{K}_3 - 2\mathcal{K}_1' + 4\rho\dot{\mathcal{K}}_3\right)\delta^{\gamma\delta}_{\lambda\eta}\nabla_{\gamma}\nabla^{\lambda}\phi\nabla_{\delta}\nabla^{\eta}\phi \\
& - 3(2\mathcal{F}' + 4\mathcal{W}' + \rho\mathcal{K}_8)\nabla_{\gamma}\nabla^{\gamma}\phi + 2\mathcal{K}_8\delta^{\gamma\delta}_{\lambda\eta}\nabla_{\gamma}\phi\nabla^{\lambda}\phi\nabla_{\delta}\nabla^{\eta}\phi \\
& + \left\{4\mathcal{K}_9 - \rho\left(2\mathcal{F}'' + 4\mathcal{W}'' + \rho\mathcal{K}_8' + 2\dot{\mathcal{K}}_9\right)\right\}\Big]
\end{aligned} \tag{2.5.24}
$$

where,

$$
\begin{aligned}
&\mathcal{K}_1 = \int_1 \frac{K_1}{\rho}d\rho, \qquad \mathcal{K}_3 = \int_1 \frac{K_3}{\rho}d\rho, \\
&\mathcal{K}_8 = \int_1 \frac{K_8}{\rho}d\rho, \qquad \mathcal{K}_9 = \rho^2\int_1 \frac{K_9}{\rho^3}d\rho, \\
&\mathcal{W} = -W,
\end{aligned}
$$

and

$$
\mathcal{F} = \int\left(\mathcal{K}_1' - \mathcal{K}_3 - 2\rho\dot{\mathcal{K}}_3\right)d\rho.
$$

Following the construction of the most general single-scalar–tensor theory yielding second-order field equations, it is natural to extend the analysis to theories involving multiple scalar fields. Although several attempts were made, a fully general multi-scalar Horndeski theory had not been achieved. Notably, Ohashi et al. [32], followed Horndeski's original derivation and obtained the most general second-order equations of motion for a bi-scalar–tensor theory, but the corresponding action remained unknown. It was only recently that Horndeski published his bi-scalar version of the theory, providing the complete formulation [33]. An analogous extension of gravity theories involves introducing a vector field rather than a scalar field. Motivated by the earlier work of Horndeski, Deffayet et al. [34] attempted to construct the most general vector theory on flat spacetime with second-order field equations, referred to as vector-Galileon theory.

Chapter 3

Horndeski Theory

3.1 The Rediscovery of Horndeski Theory

For decades, Horndeski's theory remained obscure, as research focused on simpler scalar-tensor models. Ironically, Horndeski and Lovelock themselves initially dismissed the theory as impractical due to its complexity. As Horndeski later remarked, "There were just too many of them, and they are way too complicated... We wondered who would be crazy enough to work with such equations. Then crazy showed up!" [35]. The theory's unexpected revival began with the Dvali–Gabadadze–Porrati (DGP) model [36], a braneworld model of gravity by which our observed four-dimensional Universe resides in a larger, five-dimensional space. The decoupling limit of the DGP model revealed an effective scalar field, π, governed by nontrivial self-interactions invariant under Galilean symmetry [37]:

$$\pi \to \pi + b_\mu x^\mu + c$$

The most general Galileon theory in a four-dimensional flat spacetime, that possesses the Galilean shift symmetry and is second order in ϕ's equation of motion, is given by

$$\begin{aligned} L = c_1\phi + c_2 X - c_3 X\Box\phi + c_4 X\left[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right] \\ - \frac{c_5}{3}X\left[(\Box\phi)^3 - 3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi\right], \end{aligned}$$

where $X = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi$ and $c_1, \cdots, c_5$ are constants. The combination of Galilean symmetry and the requirement of second-order equations of motion imposes strong constraints on the allowed Lagrangian terms. Specifically, in a D-dimensional spacetime, the Lagrangian admits exactly D+1 nontrivial terms of the general form $\partial\phi(\partial^2\phi)^{n-2}$ with $n = 1, \cdots, D+1$. Notably, while Galilean-invariant combinations exist for $n > D+1$, just like in Lovelock gravity, these reduce to boundary terms that vanish in the equations of motion. Introducing gravity requires promoting the theory to curved spacetime. The naive replacement of partial derivatives with covariant derivatives $\partial_\mu \to \nabla_\mu$ in the Galileon Lagrangian introduces higher-derivative terms. To preserve second-order equations of motion, these terms must be precisely canceled by the addition of curvature-dependent counterterms such as couplings to R and $G_{\mu\nu}$. The covariantization of Galileons was achieved by Deffayet [38] who introduced the following Lagrangian that maintains second-order equations of motion

$$\begin{aligned}\mathcal{L} = c_1\phi + c_2 X - c_3 X\Box\phi + \frac{c_4}{2}X^2 R + c_4 X\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ + c_5 X^2 G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{c_5}{3}X\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\nu\phi)(\nabla^\nu\nabla_\lambda\phi)(\nabla^\lambda\nabla_\mu\phi)\right].\end{aligned}$$

However, the price of maintaining second-order field equations in curved spacetime is the loss of Galilean symmetry. Further generalization of the theory led to what is known as Generalised Galileon theory [39]:

$$S = \sum_{i=2}^{5}\int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}_i$$

$$\begin{aligned}&\mathcal{L}_2 = K(\phi, X)\\ &\mathcal{L}_3 = -G_3(\phi, X)\Box\phi\\ &\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]\\ &\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]\end{aligned} \tag{3.1.1}$$

where

$$\begin{aligned}&(\nabla_\mu\nabla_\nu\phi)^2 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi,\\ &(\nabla_\mu\nabla_\nu\phi)^3 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi, \nabla_\lambda\nabla^\mu\phi.\end{aligned}$$

Generalised Galileon theory permits four free functions, $G_i(\phi, X)$, of ϕ and its canonical kinetic term $X = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ in the Lagrangian. For the functions G_i, we denote their partial derivatives with respect to ϕ and X by

$$G_{i\phi} \equiv \frac{\partial G_i}{\partial\phi} \quad \text{and} \quad G_{iX} \equiv \frac{\partial G_i}{\partial X},$$

respectively.

Much like in the covariant Galileon case, the non-minimal couplings to curvature in $\mathcal{L}_4$ and

$\mathcal{L}_5$ play the same essential role: they precisely cancel the higher-derivative terms that would otherwise arise from the non-commutativity of covariant derivatives in the field equations. As shown in [40] the Generalised Galileons are equivalent to the Horndeski theory. Specifically, the Lagrangian (3.1.1), can be expressed in the form (2.5.24) by making the following choices for the free functions G_i:

$$
\begin{aligned}
G_2 &= \mathcal{K}_9 + \rho \int d\rho \left(\mathcal{K}_8' - 2\mathcal{K}_3''\right), \\
G_3 &= 6\left(F' + 2\mathcal{W}'\right) + \rho\mathcal{K}_8 + 4\rho\mathcal{K}_{3\phi} - \int d\rho \left(\mathcal{K}_8 - 2\mathcal{K}_3'\right), \\
G_4 &= 2\left(F + 2\mathcal{W} + \rho\mathcal{K}_3\right), \\
G_5 &= -4\mathcal{K}_1.
\end{aligned}
$$

In what follows, we will take Horndeski theory to mean the Lagrangian in (3.1.1), which will serve as the starting point for our calculations.

Horndeski gravity includes the majority of proposed models of dark energy and modified gravity that have been studied. For example, by making the following choices for the functions G_i:

1. **Quintessence:**

   $$K = X - V(\phi), \quad G_3 = G_5 = 0, \quad G_4 = \frac{1}{2}.$$

   Quintessence models correspond to a canonical scalar field minimally coupled to gravity, with potential $V(\phi)$, and represent the simplest model of dark energy [41].

2. **k-essence:**

   $$K = K(\phi, X), \quad G_3 = G_5 = 0, \quad G_4 = \frac{1}{2}.$$

   Here, the scalar field has a non-canonical kinetic term $K(\phi, X)$, allowing a richer variety of cosmological dynamics and serving as a generalization of quintessence [42].

3. **Non-minimally coupled scalar field:**

   $$K = 2\omega(\phi)X - V(\phi), \quad G_3 = 0, \quad G_4 = F(\phi), \quad G_5 = 0.$$

   In this case, the scalar field couples directly to the Ricci scalar through $F(\phi)R$. Scalar–tensor theories such as Brans–Dicke gravity, are recovered for the specific choice $F(\phi) = \phi$ and $V(\phi) = 0$.

In the particular case where $G_2 = -\Lambda$, $G_3 = G_5 = 0$, and $G_4 = \frac{1}{2\kappa}$, we recover standard General Relativity with a cosmological constant. Although the non-minimal coupling to the Gauss–Bonnet term does not appear explicitly in the standard G_i functions, it can nonetheless be reproduced within the Horndeski theory[1]. This requires carefully choosing the functions $G_i(\phi, X)$, involving logarithmic forms of the kinetic term, as demonstrated below:

$$\begin{aligned}
K &= 8\xi^{(4)}X^2(3-\ln X),\\
G_3 &= 4\xi^{(3)}X(7-3\ln X),\\
G_4 &= 4\xi^{(2)}X(2-\ln X),\\
G_5 &= -4\xi^{(1)}\ln X,
\end{aligned}$$

or, equivalently,

$$\kappa_1 = \xi^{(1)}\ln X, \quad \kappa_3 = \xi^{(2)}\ln X, \quad \kappa_8 = 0, \quad \kappa_9 = 16\xi^{(4)}X^2,$$

where

$$\xi^{(n)} := \frac{\partial^n \xi}{\partial \phi^n}.$$

3.2 Equations of motion

In this section, we present the gravitational and scalar field equations for Horndeski theory (3.1.1). The results are derived so as to reproduce those obtained by Kobayashi et al. [40], and the detailed calculations are provided in B.1.1, B.1.2.

3.2.1 Gravitational Field Equations

Naturally, the equations of motion for Horndeski theory are quite complicated, yet they can be derived directly from the standard application of variational calculus. The variation of the Horndeski action is given by:

$$\delta\left(\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i\right) = \sqrt{-g}\left[\sum_{i=2}^{5}\mathcal{G}^{(i)}_{\mu\nu}\,\delta g^{\mu\nu} + \sum_{i=2}^{5}\left(P^{\phi}_{(i)} - \nabla^{\mu}J^{(i)}_{\mu}\right)\delta\phi\right] \tag{3.2.1}$$

[1] Similarly, one can recover $F(R)$ theories as special cases of Horndeski theory.

The gravitational field equations are obtained by requiring the variation of the action with respect to the metric to vanish, $\delta S/\delta g^{\mu\nu}=0$, holding the scalar field ϕ fixed. From the general variation (3.2.1), this immediately yields:

$$\sum_{i=2}^{5}\mathcal{G}^{(i)}_{\mu\nu}=0.$$

The tensors $\mathcal{G}^{(i)}_{\mu\nu}$ encode the contribution of each Lagrangian $\mathcal{L}_i$ to the total gravitational field equations. Their explicit forms are given below. The relatively simple forms for $i=2,3$ are:

$$\mathcal{G}^{(2)}_{\mu\nu}=-\frac{1}{2}K_X\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{2}Kg_{\mu\nu}, \tag{3.2.2}$$

$$\mathcal{G}^{(3)}_{\mu\nu}=\frac{1}{2}G_{3X}\Box\phi\nabla_\mu\phi\nabla_\nu\phi+\nabla_{(\mu}G_3\nabla_{\nu)}\phi-\frac{1}{2}g_{\mu\nu}\nabla_\lambda G_3\nabla^\lambda\phi. \tag{3.2.3}$$

Note that the notation $\nabla_{(\mu}G_3\nabla_{\nu)}\phi$ indicates symmetrization over the indices μ and ν. The expressions for $i=4,5$ are significantly longer than those for $i=2,3$. They also contain derivative couplings to both the Ricci and Riemann tensors. The full form of $\mathcal{G}^{(4)}_{\mu\nu}$ is:

$$\begin{aligned}
\mathcal{G}^{(4)}_{\mu\nu}=\;&G_4G_{\mu\nu}-\tfrac{1}{2}G_{4X}R\nabla_\mu\phi\nabla_\nu\phi-\tfrac{1}{2}G_{4XX}\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right]\nabla_\mu\phi\nabla_\nu\phi\\
&-G_{4X}\Box\phi\,\nabla_\mu\nabla_\nu\phi+G_{4X}\nabla_\lambda\nabla_\mu\phi\,\nabla^\lambda\nabla_\nu\phi+2\nabla^\lambda G_{4X}\nabla_\lambda\nabla_{(\mu}\phi\,\nabla_{\nu)}\phi\\
&-\nabla^\lambda G_{4X}\nabla_\lambda\phi\,\nabla_\mu\nabla_\nu\phi+g_{\mu\nu}\big(G_{4\phi}\Box\phi-2XG_{4\phi\phi}\big)\\
&+g_{\mu\nu}\Big\{-2G_{4\phi X}\nabla^\alpha\nabla^\beta\phi\,\nabla_\alpha\phi\nabla_\beta\phi+G_{4XX}\nabla^\alpha\nabla^\lambda\phi\,\nabla^\beta\nabla_\lambda\phi\,\nabla_\alpha\phi\nabla_\beta\phi\\
&+\tfrac{1}{2}G_{4X}\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right]\Big\}\\
&+2\big(G_{4X}R_{\lambda(\mu}\nabla_{\nu)}\phi\nabla^\lambda\phi-\nabla_{(\mu}G_{4X}\nabla_{\nu)}\phi\Box\phi\big)\\
&-g_{\mu\nu}\big(G_{4X}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi-\nabla^\lambda G_{4X}\nabla_\lambda\phi\Box\phi\big)+G_{4X}R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi\\
&-G_{4\phi}\nabla_\mu\nabla_\nu\phi-G_{4\phi\phi}\nabla_\mu\phi\nabla_\nu\phi+2G_{4\phi X}\nabla^\lambda\phi\nabla_\lambda\nabla_{(\mu}\phi\,\nabla_{\nu)}\phi\\
&-G_{4XX}\nabla^\alpha\phi\nabla_\alpha\nabla_\mu\phi\,\nabla^\beta\phi\nabla_\beta\nabla_\nu\phi.
\end{aligned} \tag{3.2.4}$$

The Riemann and Ricci tensors appear as a consequence of third-order covariant derivatives of ϕ encountered in the variation. These terms are simplified through commutator relations of covariant derivatives, which convert them into curvature tensors. For example, we have:

$$[\nabla_\mu,\nabla_\nu]\nabla_\alpha\nabla^\beta\phi=R_{\mu\nu\alpha}{}^{\lambda}\,\nabla_\lambda\nabla^\beta\phi-R_{\mu\nu\lambda}{}^{\beta}\,\nabla_\alpha\nabla^\lambda\phi,$$

$$\nabla^\alpha\big(\nabla_\alpha\Box-\Box\nabla_\alpha\big)\phi=-R_{\lambda\alpha}\nabla^\alpha\nabla^\lambda\phi-\frac{1}{2}\nabla_\lambda R,$$

$$[\nabla_\mu,\Box]\,\phi\;=R_{\mu\nu}\nabla^\nu\phi.$$

As for the G_5 contribution, the tensor $\mathcal{G}^{(5)}_{\mu\nu}$ is given by the following expression:

$$
\begin{aligned}
\mathcal{G}^{(5)}_{\mu\nu} = {} & G_{5X}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\nabla_{(\mu}\phi\nabla_{\nu)}\phi - G_{5X}R_{\alpha(\mu}\nabla_{\nu)}\phi\nabla^\alpha\phi\Box\phi - \frac{1}{2}G_{5X}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi \\
& - \frac{1}{2}G_{5X}R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi\Box\phi + G_{5X}R_{\alpha\lambda\beta(\mu}\left(\nabla_{\nu)}\phi\nabla^\lambda\phi\nabla^\alpha\nabla^\beta\phi + \nabla_{\nu)}\nabla^\lambda\phi\nabla^\alpha\phi\nabla^\beta\phi\right) \\
& - \frac{1}{2}\nabla_{(\mu}(G_{5X}\nabla_\alpha\phi)\nabla^\alpha\nabla_{\nu)}\phi\Box\phi + \frac{1}{2}\nabla_{(\mu}(G_{5\phi}\nabla_{\nu)}\phi)\Box\phi - \nabla_\lambda(G_{5\phi}\nabla_{(\mu}\phi)\nabla_{\nu)}\nabla^\lambda\phi \\
& + \frac{1}{2}\left[\nabla_\lambda(G_{5\phi}\nabla^\lambda\phi) - \nabla_\alpha(G_{5X}\nabla_\beta\phi)\nabla^\alpha\nabla^\beta\phi\right]\nabla_\mu\nabla_\nu\phi \\
& + \nabla^\alpha G_5\nabla^\beta\phi R_{\alpha(\mu\nu)\beta} - \nabla_{(\mu}G_5 G_{\nu)\lambda}\nabla^\lambda\phi + \frac{1}{2}\nabla_{(\mu}G_{5X}\nabla_{\nu)}\phi\left[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right] \\
& - \nabla^\lambda G_5 R_{\lambda(\mu}\nabla_{\nu)}\phi + \nabla_\alpha(G_{5X}\nabla_\beta\phi)\nabla^\alpha\nabla_{(\mu}\phi\nabla^\beta\nabla_{\nu)}\phi \\
& - \nabla^\beta G_{5X}\left[\Box\phi\nabla_\beta\nabla_{(\mu}\phi - \nabla_\alpha\nabla_\beta\phi\nabla^\alpha\nabla_{(\mu}\phi\right]\nabla_{\nu)}\phi \\
& + \frac{1}{2}\nabla^\alpha\phi\nabla_\alpha G_{5X}\left[\Box\phi\nabla_\mu\nabla_\nu\phi - \nabla^\beta\nabla_\mu\phi\nabla_\beta\nabla_\nu\phi\right] \\
& - \frac{1}{2}G_{5X}G_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}G_{5X}\Box\phi\nabla^\alpha\nabla_\mu\phi\nabla_\alpha\nabla_\nu\phi + \frac{1}{2}G_{5X}(\Box\phi)^2\nabla_\mu\nabla_\nu\phi \\
& + \frac{1}{12}G_{5XX}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\alpha\nabla_\beta\phi)^2 + 2(\nabla_\alpha\nabla_\beta\phi)^3\right]\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}\nabla^\lambda G_5 G_{\mu\nu}\nabla_\lambda\phi \\
& + g_{\mu\nu}\Big(-\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\alpha\nabla_\beta\phi)^2 + 2(\nabla_\alpha\nabla_\beta\phi)^3\right] + \nabla^\alpha G_5 R_{\alpha\beta}\nabla^\beta\phi \\
& - \frac{1}{2}\nabla^\alpha(G_{5\phi}\nabla_\alpha\phi)\Box\phi + \frac{1}{2}\nabla^\alpha(G_{5\phi}\nabla_\beta\phi)\nabla_\alpha\nabla^\beta\phi - \frac{1}{2}\nabla^\alpha G_{5X}\nabla_\alpha X\Box\phi + \frac{1}{2}\nabla^\alpha G_{5X}\nabla^\beta X\nabla_\alpha\nabla_\beta\phi \\
& - \frac{1}{4}\nabla^\lambda G_{5X}\nabla_\lambda\phi\left[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right] + \frac{1}{2}G_{5X}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi\Box\phi - \frac{1}{2}G_{5X}R_{\alpha\lambda\beta\rho}\nabla^\alpha\nabla^\beta\phi\nabla^\lambda\phi\nabla^\rho\phi\Big).
\end{aligned}
\tag{3.2.5}
$$

As in the fourth part, the variation produces higher-order derivatives of the scalar field. In this case, however, the terms involve fourth-order covariant derivatives of ϕ. These contributions are likewise reduced to curvature tensors by the use of commutator identities and the relations they satisfy. The field equations presented in (3.2.2), (3.2.3), (3.2.4), (3.2.5), describe the dynamics of the gravitational sector. When matter fields are included via a matter action S_m, the principle of least action yields a source term described by the energy-momentum tensor.

3.2.2 Scalar Field Equations

The equation of motion for the scalar field is obtained by requiring the variation of the action with respect to ϕ to vanish, $\delta S/\delta\phi = 0$, while holding the metric fixed. In this case the general variation (3.2.1) yields:

$$\sum_{i=2}^{5}\left(P^i_\phi - \nabla^\mu J^i_\mu\right) = 0 \quad \text{or, equivalently,} \quad \nabla^\mu\left(\sum_{i=2}^{5}J^i_\mu\right) = \sum_{i=2}^{5}P^i_\phi.$$

The contributions for each of the terms of the Lagrangian (3.1.1) are

$$P_\phi^{(2)} = K_\phi, \tag{3.2.6}$$

$$P_\phi^{(3)} = \nabla_\mu G_3 \nabla^\mu \phi, \tag{3.2.7}$$

$$P_\phi^{(4)} = G_{4\phi} R + G_{4X\phi}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \tag{3.2.8}$$

$$P_\phi^{(5)} = -\nabla_\mu G_{5\phi} G^{\mu\nu}\nabla_\nu\phi - \frac{1}{6}G_{5\phi X}\Big[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\Big], \tag{3.2.9}$$

and the currents are

$$J_\mu^{(2)} = -\mathcal{L}_{2X}\nabla_\mu\phi, \tag{3.2.10}$$

$$J_\mu^{(3)} = -\mathcal{L}_{3X}\nabla_\mu\phi + G_{3X}\nabla_\mu X + 2G_{3\phi}\nabla_\mu\phi, \tag{3.2.11}$$

$$\begin{aligned} J_\mu^{(4)} &= -\mathcal{L}_{4X}\nabla_\mu\phi + 2G_{4X}R_{\mu\nu}\nabla^\nu\phi \\ &\quad - 2G_{4XX}\left(\Box\phi\nabla_\mu X - \nabla^\nu X\nabla_\mu\nabla_\nu\phi\right) - 2G_{4\phi X}\left(\Box\phi\nabla_\mu\phi + \nabla_\mu X\right), \end{aligned} \tag{3.2.12}$$

$$\begin{aligned} J_\mu^{(5)} &= -\mathcal{L}_{5X}\nabla_\mu\phi - 2G_{5\phi}G_{\mu\nu}\nabla^\nu\phi \\ &\quad - G_{5X}\Big[G_{\mu\nu}\nabla^\nu X + R_{\mu\nu}\Box\phi\nabla^\nu\phi - R_{\nu\lambda}\nabla^\nu\phi\nabla^\lambda\nabla_\mu\phi - R_{\alpha\mu\beta\nu}\nabla^\nu\phi\nabla^\alpha\nabla^\beta\phi\Big] \\ &\quad + G_{5XX}\left[\frac{1}{2}\nabla_\mu X\left((\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right) - \nabla_\nu X\left(\Box\phi\nabla_\mu\nabla^\nu\phi - \nabla_\alpha\nabla_\mu\phi\nabla^\alpha\nabla^\nu\phi\right)\right] \\ &\quad + G_{5\phi X}\left[\frac{1}{2}\nabla_\mu\phi\left((\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right) + \Box\phi\nabla_\mu X - \nabla^\nu X\nabla_\nu\nabla_\mu\phi\right]. \end{aligned} \tag{3.2.13}$$

3.3 Cosmological Background Equations

We consider the case of a flat FLRW spacetime with a lapse function $N(t)$ described by the line element

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\,d\mathbf{x}^2 \tag{3.3.1}$$

where $a(t)$ is the scale factor. We assume a homogeneous scalar field, $\phi = \phi(t)$ and derive the background evolution of the Horndeski theory. The following results are necessary for this analysis:

$$R = 6\Big(\frac{\ddot{a}}{N^2 a} - \frac{\dot{a}\dot{N}}{aN^3} + \frac{\dot{a}^2}{a^2N^2}\Big)$$

and

$$\Box\phi = -\frac{\ddot{\phi}}{N^2} - \frac{3H\dot{\phi}}{N^2} + \frac{\dot{N}\dot{\phi}}{N^3}.$$

The details of these calculations can be found in B.2.1, B.2.2, B.2.3.

3.3.1 Generalised Friedmann Equation

Varying the action with respect to $N(t)$, and then setting $N = 1$, gives the generalized Friedmann equation

$$\sum_{i=2}^{5} \mathcal{E}_i = -\frac{1}{a^3}\frac{\delta S}{\delta N} = 0$$

where

$$\mathcal{E}_2 = 2XK_X - K, \tag{3.3.2}$$

$$\mathcal{E}_3 = 6X\dot{\phi}HG_{3X} - 2XG_{3\phi}, \tag{3.3.3}$$

$$\mathcal{E}_4 = -6H^2G_4 + 24H^2X(G_{4X} + XG_{4XX}) - 12HX\dot{\phi}G_{4\phi X} - 6H\dot{\phi}G_{4\phi}, \tag{3.3.4}$$

$$\mathcal{E}_5 = 2H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) - 6H^2X(3G_{5\phi} + 2XG_{5\phi X}), \tag{3.3.5}$$

$H = \frac{\dot{a}}{a}$, being the Hubble parameter.

3.3.2 Evolution Equation

Variation with respect to $a(t)$ yields the evolution equation

$$\sum_{i=2}^{5} \mathcal{P}_i = \frac{1}{3a^2}\frac{\delta S}{\delta a} = 0$$

where

$$\mathcal{P}_2 = K, \tag{3.3.6}$$

$$\mathcal{P}_3 = -2X(G_{3\phi} + \ddot{\phi}G_{3X}), \tag{3.3.7}$$

$$\begin{aligned}\mathcal{P}_4 = {} & 2(3H^2 + 2\dot{H})G_4 - 12H^2XG_{4X} - 4H\dot{X}G_{4X} - 8\dot{H}XG_{4X} \\ & - 8HX\dot{X}G_{4XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4\phi} + 4XG_{4\phi\phi} \\ & + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4\phi X},\end{aligned} \tag{3.3.8}$$

$$\begin{aligned}\mathcal{P}_5 = {} & -2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5X} - 4H^2X^2\ddot{\phi}G_{5XX} \\ & + 4HX(\dot{X} - HX)G_{5\phi X} + 2[2(HX)^{\cdot} + 3H^2X]G_{5\phi} + 4HX\dot{\phi}G_{5\phi\phi}.\end{aligned} \tag{3.3.9}$$

3.3.3 Scalar Field Equation

Variation with respect to $\phi(t)$

$$\mathcal{E}_\phi = \frac{1}{a^3}\frac{\delta S}{\delta \phi} = 0$$

gives the scalar-field equation of motion

$$\frac{1}{a^3}\frac{d}{dt}(a^3 J) = P_\phi \tag{3.3.10}$$

where

$$\begin{aligned} J &= \dot{\phi}K_X + 6HXG_{3X} - 2\dot{\phi}G_{3\phi} + 6H^2\dot{\phi}(G_{4X} + 2XG_{4XX}) \\ &\quad - 12HXG_{4\phi X} + 2H^3X(3G_{5X} + 2XG_{5XX}) - 6H^2\dot{\phi}(G_{5\phi} + XG_{5\phi X}), \end{aligned} \tag{3.3.11}$$

$$\begin{aligned} P_\phi &= K_\phi - 2X(G_{3\phi\phi} + \ddot{\phi}G_{3\phi X}) + 6(2H^2 + \dot{H})G_{4\phi} + 6H(\dot{X} + 2HX)G_{4\phi X} \\ &\quad - 6H^2XG_{5\phi\phi} + 2H^3X\dot{\phi}G_{5\phi X}. \end{aligned} \tag{3.3.12}$$

As expected, all higher-derivative terms cancel, resulting in equations that are at most second order. The difference with GR is that in the case of Horndeski theory, the scalar field equations and the evolution equation depend on both $\dot{H}$, $\ddot{\phi}$. This implies a kinetic mixing of gravity and the scalar field which does not occur in GR [43]. This feature arises once one adds the $\mathcal{L}_3$ part of (3.1.1). When matter is included, the background equations take the form

$$\mathcal{E} = -\rho, \quad \mathcal{P} = -p, \quad \mathcal{E}_\phi = 0,$$

where ρ and p denote the energy density and pressure of the matter component, respectively. Setting $G_4 = \frac{1}{2\kappa}$ recovers the standard background equations of General Relativity.

3.4 Horndeski after GW

To derive the quadratic actions for scalar and tensor perturbations, it is convenient to employ the Arnowitt–Deser–Misner (ADM) decomposition

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{3.4.1}$$

where N is the lapse function, N^i is the shift vector, and γ_{ij} is the induced metric on the spatial hypersurfaces. One then expands the ADM quantities around the FLRW background as

$$N = 1 + a, \qquad N_i = \partial_i \beta, \qquad \gamma_{ij} = a^2(t)\, e^{2\zeta}\left(\delta_{ij} + h_{ij} + \tfrac{1}{2} h_{ik} h_{kj}\right),$$

where a, β and ζ describe scalar perturbations, and h_{ij} is the transverse-traceless tensor perturbation. Substituting these expressions into the Horndeski action and expanding to second order in perturbations, we obtain[2]

$$S^{(2)} = S^{(2)}_{tensor} + S^{(2)}_{scalar}.$$

Since we are interested in gravitational-wave propagation, we focus on the tensor sector, whose quadratic action is [40]:

$$S^{(2)}_{tensor} = \frac{1}{8} \int dt\, d^3x\, a^3 \left[\mathcal{G}_T\, \dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2} (\vec{\nabla} h_{ij})^2 \right], \tag{3.4.2}$$

where

$$\mathcal{G}_T = 2\Big[G_4 - 2X G_{4X} - X\left(H\dot{\phi}\, G_{5X} - G_{5\phi}\right)\Big], \tag{3.4.3}$$

$$\mathcal{F}_T = 2\Big[G_4 - X\left(\ddot{\phi}\, G_{5X} + G_{5\phi}\right)\Big]. \tag{3.4.4}$$

The propagation speed of the tensor modes is then given by

$$c_T^2 = \frac{\mathcal{F}_T}{\mathcal{G}_T}$$

or equivalently

$$c_{\rm GW}^2 = \frac{G_4 - X(\ddot{\phi} G_{5X} + G_{5\phi})}{G_4 - 2X G_{4X} - X(H\dot{\phi} G_{5X} - G_{5\phi})}. \tag{3.4.5}$$

The simultaneous detection of GW170817 and its electromagnetic counterpart GRB170817A established that the speed of gravitational waves is equal to the speed of light to an extremely high precision, with deviations constrained to be of order 10^{-15} [44]. This constraint rules out a significant portion of the Horndeski theory. For (3.4.5) to be equal to the speed of light, we require that

$$G_{4X} = 0, \quad G_5 = 0.$$

Consequently, models with $G_{4X} \neq 0$ or $G_5 \neq 0$ have been severely constrained. Thus, the viable

[2] In principle, one may also include vector perturbations. However, in scalar–tensor theories such as Horndeski, these modes are non-dynamical.

subclass within Horndeski is described by the Lagrangian

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi)R.$$

3.5 Kinetic Gravity Braiding

Dark energy can be modeled within scalar field theories of the Kinetic Gravity Braiding (KGB) type, which do not require a cosmological constant. The general action of the KGB theory, can be obtained from Horndeski when $G_2 \neq 0$, $G_3 \neq 0$, $G_4 = 1/(2\kappa)$, $G_5 = 0$

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa}R + K(\phi, X) - G(\phi, X)\Box\phi + \mathcal{L}_m\right], \tag{3.5.1}$$

where $K(\phi, X)$ and $G(\phi, X)$ are arbitrary functions of the scalar field ϕ and $\mathcal{L}_m$ is the matter Lagrangian. This class of models can exhibit many interesting properties, such as attractor solutions [45]. From sections 3.2.1, 3.2.2, we deduce the equations of motion of the theory

$$G_{\mu\nu} = \kappa\left(T^{(\varphi)}_{\mu\nu} + T^{(m)}_{\mu\nu}\right), \tag{3.5.2}$$

where $T^{(m)}_{\mu\nu}$ is the energy-momentum tensor of the matter section and

$$T^{(\varphi)}_{\mu\nu} = K_X \nabla_\mu\varphi\nabla_\nu\varphi + g_{\mu\nu}K + g_{\mu\nu}\nabla_\lambda G\nabla^\lambda\varphi - (\nabla_\nu G\nabla_\mu\varphi + \nabla_\mu G\nabla_\nu\varphi) - G_X\Box\varphi\nabla_\mu\varphi\nabla_\nu\varphi. \tag{3.5.3}$$

The Scalar field equation is given by

$$\begin{aligned}
&K_\phi + \nabla_\alpha(K_X\nabla^\alpha\phi) + 2G_{\phi X}X - 2G_{\phi\phi}\Box\phi + G_{X\phi}\left(2X\Box\phi + 2\nabla_\mu\phi\,\nabla^\mu\nabla^\nu\phi\,\nabla_\nu\phi\right)\\
&\quad + G_X\left[-(\Box\phi)^2 + (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi\right]\\
&\quad - G_{XX}\left[(\nabla_\nu X)(\nabla^\nu X) - \nabla_\mu\phi\,\nabla^\mu\nabla^\nu\phi\,\nabla_\nu X\right] = 0.
\end{aligned} \tag{3.5.4}$$

For the Friedmann equation, the results from the previous section reduce to

$$H^2 = \frac{\kappa}{3}\left(\rho_m + K_X\dot{\phi}^2 - \dot{\phi}^2 G_\phi + 3G_X\dot{\phi}^3 H - K\right). \tag{3.5.5}$$

The shift-symmetric sector of the theory corresponds to the choice $G_i = G_i(X)$. The presence of the shift symmetry of the scalar field,

$$\phi \;\rightarrow\; \phi + c,$$

allows the existence of a Noether current, which is given by

$$J^{\mu} = \left(K_X - 2G_{\phi} - G_X \Box\phi\right) \nabla^{\mu}\phi + G_X \nabla^{\mu}\nabla^{\nu}\phi \nabla_{\nu}\phi. \tag{3.5.6}$$

In this case, the scalar field equation (3.5.4) reduces to

$$\nabla_{\mu} J^{\mu} = 0.$$

As a subclass of the broader Horndeski framework, KGB models yield second-order field equations. Moreover, they naturally ensure that gravitational waves propagate at the speed of light. The KGB framework has been studied in the context of cosmology, including its potential to alleviate the Hubble tension [46].

Chapter 4

Summary

In this thesis, we initially investigated the structure of gravitational Lagrangians that depend on the metric tensor and its first two derivatives, along with a scalar field and its first derivative. The foundational principle of our analysis is the requirement that the action remains invariant under arbitrary coordinate transformations, which forces the Lagrangian to transform as a scalar density:

$$\bar{L}\Big(\bar{g}_{\kappa\lambda};\partial_\tau\bar{g}_{\kappa\lambda};\partial_\tau\partial_\alpha\bar{g}_{\kappa\lambda};\bar{\phi};\partial_\kappa\bar{\phi}\Big)=BL\big(g_{\mu\nu};\partial_\rho g_{\mu\nu};\partial_\rho\partial_\sigma g_{\mu\nu};\phi;\partial_\mu\phi\big).$$

As we demonstrated, the requirement of invariance severely restricts the possible form of the Lagrangian. Utilizing this, we constructed tensorial quantities that involve partial derivatives of the Lagrangian with respect to the metric, its derivatives, the scalar field, and its first derivative, and we derived the following three invariance identities:

- $\Lambda^{\mu\nu,\rho\sigma}+\Lambda^{\mu\rho,\nu\sigma}+\Lambda^{\mu\sigma,\rho\nu}=0,$
- $\Pi^{\mu\nu,\rho}=0,$
- $\frac{1}{2}g^{\xi\chi}L=\Pi^{\chi\xi}-\frac{2}{3}\Lambda^{\nu\rho,\xi\mu}R_\rho{}^\chi{}_{\nu\mu}+\frac{3}{8}g^{\lambda\chi}\Lambda^\xi\phi_{,\lambda}+\frac{1}{8}g^{\lambda\xi}\Lambda^\chi\phi_{,\lambda}.$

We showed how these identities can be used to identify and restrict the structure of the Lagrangian. In particular, the second invariance identity demonstrates that in order to construct a Lagrangian for a theory of gravity, the inclusion of second derivatives of the metric is necessary. Furthermore, the third invariance identity provides a practical expression for the equations of motion:

$$E^{\mu\nu}(L) = -\nabla_\rho \nabla_\sigma \Lambda^{\mu\nu,\rho\sigma} + \frac{2}{3} R^{\mu}_{\sigma\lambda\tau} \Lambda^{\tau\sigma,\nu\lambda} + \frac{3}{8} \Lambda^{\nu} g^{\mu\tau} \phi_{,\tau} + \frac{1}{8} \Lambda^{\mu} g^{\nu\tau} \phi_{,\tau} - \frac{1}{2} g^{\mu\nu} L,$$

where the only quantities to be evaluated are $\Lambda^{\mu\nu,\rho\sigma}$, and Λ^{μ}. Using the three invariance identities and the symmetries that $\Lambda^{\mu\nu,\rho\sigma}$, $\Pi^{\mu\nu,\rho}$ and $\Pi^{\mu\nu}$ have, we constructed the most general Lagrangian in four spacetime dimensions

$$\begin{aligned} L = & \frac{\alpha_1}{g} (*R^{\mu\nu}{}_{\rho\sigma})(*R^{\rho\sigma}{}_{\kappa\lambda})(*R^{\kappa\lambda}{}_{\mu\nu}) + \alpha_2 \sqrt{-g} \left(R^2 - 4R^{\mu\nu}R_{\mu\nu} - R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right) \\ & + \alpha_3 (*R^{\mu\nu}{}_{\rho\sigma})(*R^{\rho\sigma}{}_{\mu\nu}) + \alpha_4 \sqrt{-g} R^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \alpha_5 \sqrt{g} R + \alpha_6 \sqrt{-g} \end{aligned} \tag{2.4.20}$$

where, $\alpha_1, \alpha_2, \alpha_3$, are functions of ϕ and $\alpha_4, \alpha_5, \alpha_6$ are functions of ϕ and ρ. To ensure that the resulting Euler-Lagrange equations remain second order, we derived a set of conditions on the tensors $\chi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}$ and $\Lambda^{\mu\nu,\rho\sigma;\alpha}$, specifying the symmetries they must satisfy under index permutations. Applying these conditions yields constraints on the free functions of the theory, namely $\alpha_1 = 0$, $\frac{\partial \alpha_4}{\partial \rho} = 0$, $\alpha_3 = \text{constant}$, and $\frac{\partial \alpha_5}{\partial \rho} = -\frac{1}{2}\alpha_4$. Consequently, we constructed the most general four-dimensional Lagrangian whose Euler–Lagrange equations are second order in both the metric and the scalar field:

$$\begin{aligned} L = & \sqrt{-g} \left(\beta_1 \left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right) + \beta_2 G^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \beta_3 R + \beta_4 \right) \\ & + c\,(*R^{\mu\nu}{}_{\rho\sigma})\,(*R^{\rho\sigma}{}_{\mu\nu}), \end{aligned}$$

where $\beta_1, \beta_2, \beta_3$ are functions of ϕ, β_4 is a function of ϕ, ρ and c is a constant.

The inclusion of the scalar field introduces additional terms beyond those of General Relativity. In particular, the Gauss–Bonnet term $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, which is topological in 4D, becomes dynamical when coupled to the function $\beta_1(\phi)$. The final term, proportional to the constant c, does not contribute to the equations of motion and therefore does not affect the dynamics. Moreover, by setting $\beta_1 = \beta_2 = \beta_4 = 0$ and $\beta_3 = 1/2\kappa$, the Lagrangian reduces to standard General Relativity. This work can be extended by incorporating additional scalar fields in the Lagrangian. This generalization would lead to significantly more complex invariance identities. Furthermore, the results used to constrain the Lagrangian's form, such as those based on the work of Lovelock, would need to be appropriately generalized to accommodate the additional field. Another route to expand on this work would be to study the resulting theory within the Palatini formalism where the metric and connection are treated as independent variables. A further generalization would be to derive the most general Lagrangian of this form in a spacetime with torsion.

In Chapter 3, we discussed the modern formulation of Horndeski theory. Using the calculus of variations, we reproduced the equations of motion as derived in [40] and demonstrated how the higher-order derivatives arising in the process can be rewritten in terms of curvature tensors using commutator relations. We then specialized to a flat FLRW metric and derived the corresponding cosmological background equations. Furthermore, we analyzed the implications of recent gravitational-wave observations for Horndeski theory, focusing on the constraints they impose on the free functions of the theory. In this context, we examined the Kinetic Gravity Braiding model as a particular subtheory of Horndeski theory, discussing its potential applications in cosmology.

Appendices

Appendix A

Calculations of CH.2

A.1 Conditions

In order for the Lagrangian, (2.4.20) to yield equations of motion that are at most of second order in derivatives, we impose the identity (2.4.9)[1]

$$\Lambda^{\kappa\lambda,\tau\sigma;\rho} + \Lambda^{\kappa\lambda,\tau\rho;\sigma} + \Lambda^{\kappa\lambda,\sigma\rho;\tau} = 0$$

Since we already know that in order for (2.4.20) to be second order in derivatives $\alpha_1 = 0$, $\alpha_3 = c$, we only need to focus our attention on the fourth and fifth terms of the Lagrangian (2.4.20). Therefore, we consider the following Lagrangian:

$$L = \sqrt{-g}\Big(\alpha_4(\phi,\rho)R_{\mu\nu}\nabla^{\mu}\phi\nabla^{\nu}\phi + \alpha_5(\phi,\rho)R\Big), \tag{A.1.1}$$

and we calculate the partial derivatives

$$\Lambda^{\kappa\lambda,\tau\sigma;\rho} = \frac{\partial}{\partial(\nabla_{\rho}\phi)}\frac{\partial}{\partial g_{\kappa\lambda,\tau\sigma}}(L)\,. \tag{A.1.2}$$

To proceed, we must first calculate

$$\frac{\partial R_{\mu\nu}}{\partial g_{\kappa\lambda,\tau\sigma}} \tag{A.1.3}$$

and

$$\frac{\partial R}{\partial g_{\kappa\lambda,\tau\sigma}}. \tag{A.1.4}$$

The Ricci tensor can be expressed as

$$R_{\sigma\nu} = \partial_{\rho}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\rho\sigma} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho}_{\lambda\rho} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\rho}_{\lambda\nu}.$$

Since the last two terms do not contain second-order derivatives of the metric, we omit them in the following calculations. Therefore,

$$R_{\sigma\nu} = \frac{1}{2}g^{\rho\chi}\left(\partial_{\rho}\partial_{\sigma}g_{\nu\chi} - \partial_{\rho}\partial_{\chi}g_{\nu\sigma} - \partial_{\nu}\partial_{\sigma}g_{\rho\chi} + \partial_{\nu}\partial_{\chi}g_{\sigma\rho}\right).$$

For (A.1.3) we have

$$\frac{\partial R_{\mu\nu}}{\partial g_{\kappa\lambda,\tau\sigma}} = \frac{1}{2}g^{\rho\chi}\frac{\partial}{\partial g_{\kappa\lambda,\tau\sigma}}\Big[\partial_{\rho}\partial_{\mu}g_{\nu\chi} - \partial_{\rho}\partial_{\chi}g_{\nu\mu} - \partial_{\nu}\partial_{\mu}g_{\rho\chi} + \partial_{\nu}\partial_{\chi}g_{\mu\rho}\Big].$$

[1] In fact the identities (2.4.9), (2.4.10) are the same condition. Thus, it is sufficient to impose only one of them.

Using the following result [2]

$$\frac{\partial\left(\partial_\alpha\partial_\beta g_{\rho\sigma}\right)}{\partial(\partial_\tau\partial_\eta g_{\kappa\lambda})}=\frac{1}{4}\frac{\partial}{\partial\left(\partial_\tau\partial_\eta g_{\kappa\lambda}\right)}\left[\partial_\alpha\partial_\beta g_{\rho\sigma}+\partial_\alpha\partial_\beta g_{\sigma\rho}+\partial_\beta\partial_\alpha g_{\rho\sigma}+\partial_\beta\partial_\alpha g_{\sigma\rho}\right],$$

we find

$$\begin{aligned}\frac{\partial R_{\mu\nu}}{\partial(g_{\kappa\lambda,\tau\sigma})}=\frac{1}{8}g^{\rho\chi}\Big[&\delta^\kappa_\rho\delta^\lambda_\mu\delta^\tau_\nu\delta^\sigma_\chi+\delta^\kappa_\rho\delta^\lambda_\mu\delta^\tau_\chi\delta^\sigma_\nu+\delta^\kappa_\mu\delta^\lambda_\rho\delta^\tau_\nu\delta^\sigma_\chi+\delta^\kappa_\mu\delta^\lambda_\rho\delta^\tau_\chi\delta^\sigma_\nu\\&-\delta^\kappa_\rho\delta^\lambda_\chi\delta^\tau_\nu\delta^\sigma_\mu-\delta^\kappa_\rho\delta^\lambda_\chi\delta^\tau_\mu\delta^\sigma_\nu-\delta^\kappa_\chi\delta^\lambda_\rho\delta^\tau_\nu\delta^\sigma_\mu-\delta^\kappa_\chi\delta^\lambda_\rho\delta^\tau_\mu\delta^\sigma_\nu\\&-\delta^\kappa_\nu\delta^\lambda_\mu\delta^\tau_\rho\delta^\sigma_\chi-\delta^\kappa_\nu\delta^\lambda_\mu\delta^\tau_\chi\delta^\sigma_\rho-\delta^\kappa_\mu\delta^\lambda_\nu\delta^\tau_\rho\delta^\sigma_\chi-\delta^\kappa_\mu\delta^\lambda_\nu\delta^\tau_\chi\delta^\sigma_\rho\\&+\delta^\kappa_\nu\delta^\lambda_\chi\delta^\tau_\mu\delta^\sigma_\rho+\delta^\kappa_\nu\delta^\lambda_\chi\delta^\tau_\rho\delta^\sigma_\mu+\delta^\kappa_\chi\delta^\lambda_\nu\delta^\tau_\mu\delta^\sigma_\rho+\delta^\kappa_\chi\delta^\lambda_\nu\delta^\tau_\rho\delta^\sigma_\mu\Big].\end{aligned}$$

Simplifying the above, we obtain

$$\begin{aligned}\frac{\partial(R_{\mu\nu})}{\partial(g_{\kappa\lambda,\tau\sigma})}\nabla^\mu\phi\nabla^\nu\phi=\frac{1}{4}\Big[&g^{\kappa\sigma}\nabla^\lambda\phi\nabla^\tau\phi+g^{\kappa\tau}\nabla^\lambda\phi\nabla^\sigma\phi\\&+g^{\lambda\sigma}\nabla^\kappa\phi\nabla^\tau\phi+g^{\lambda\tau}\nabla^\kappa\phi\nabla^\sigma\phi\\&-2g^{\kappa\lambda}\nabla^\sigma\phi\nabla^\tau\phi-2g^{\tau\sigma}\nabla^\kappa\phi\nabla^\lambda\phi\Big].\end{aligned}\tag{A.1.5}$$

Following a similar approach for (A.1.4) we find

$$\frac{\partial(R)}{\partial(g_{\kappa\lambda,\tau\sigma})}=\frac{1}{4}\left[2g^{\kappa\sigma}g^{\lambda\tau}+2g^{\lambda\sigma}g^{\kappa\tau}-4g^{\sigma\tau}g^{\kappa\lambda}\right]\tag{A.1.6}$$

therefore,

$$\begin{aligned}\frac{\partial}{\partial g_{\kappa\lambda,\tau\sigma}}\left(\alpha_4R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi+\alpha_5R\right)=\frac{\alpha_4}{4}\Big[&g^{\kappa\sigma}\nabla^\lambda\phi\nabla^\tau\phi+g^{\kappa\tau}\nabla^\lambda\phi\nabla^\sigma\phi\\&+g^{\lambda\sigma}\nabla^\kappa\phi\nabla^\tau\phi+g^{\lambda\tau}\nabla^\kappa\phi\nabla^\sigma\phi\\&-2g^{\kappa\lambda}\nabla^\sigma\phi\nabla^\tau\phi-2g^{\tau\sigma}\nabla^\kappa\phi\nabla^\lambda\phi\Big]\\&+\frac{\alpha_5}{2}\left[g^{\kappa\sigma}g^{\lambda\tau}+g^{\lambda\sigma}g^{\kappa\tau}-2g^{\sigma\tau}g^{\kappa\lambda}\right].\end{aligned}\tag{A.1.7}$$

Substituting these (A.1.2) can be written as

$$\begin{aligned}\Lambda^{\kappa\lambda,\tau\sigma;\rho}=\frac{\partial}{\partial(\phi_{,\rho})}\frac{\partial}{\partial g_{\kappa\lambda,\tau\sigma}}(L)=\sqrt{-g}\,\frac{1}{4}\Bigg\{&\frac{\partial\alpha_4}{\partial(\phi_{,\rho})}\Big(g^{\kappa\sigma}\nabla^\lambda\phi\nabla^\tau\phi+g^{\kappa\tau}\nabla^\lambda\phi\nabla^\sigma\phi+g^{\lambda\sigma}\nabla^\kappa\phi\nabla^\tau\phi+g^{\lambda\tau}\nabla^\kappa\phi\nabla^\sigma\phi\\&-2g^{\kappa\lambda}\nabla^\sigma\phi\nabla^\tau\phi-2g^{\tau\sigma}\nabla^\kappa\phi\nabla^\lambda\phi\Big)\\&+\alpha_4\frac{\partial}{\partial(\phi_{,\rho})}\Big(g^{\kappa\sigma}\nabla^\lambda\phi\nabla^\tau\phi+g^{\kappa\tau}\nabla^\lambda\phi\nabla^\sigma\phi+g^{\lambda\sigma}\nabla^\kappa\phi\nabla^\tau\phi+g^{\lambda\tau}\nabla^\kappa\phi\nabla^\sigma\phi\\&-2g^{\kappa\lambda}\nabla^\sigma\phi\nabla^\tau\phi-2g^{\tau\sigma}\nabla^\kappa\phi\nabla^\lambda\phi\Big)\\&+\frac{\partial\alpha_5}{\partial(\phi_{,\rho})}\Big(2g^{\kappa\sigma}g^{\lambda\tau}+2g^{\lambda\sigma}g^{\kappa\tau}-4g^{\sigma\tau}g^{\kappa\lambda}\Big)\Bigg\}.\end{aligned}\tag{A.1.8}$$

[2] The factor $\frac{1}{4}$ comes from the symmetrization of $g_{\kappa\lambda,\tau\sigma}$.

Noting that[3]

$$\frac{\partial \alpha_{4,5}}{\partial(\phi_{,\rho})} = \frac{\partial \alpha_{4,5}}{\partial \rho}\frac{\partial \rho}{\partial(\phi_{,\rho})} = \frac{\partial \alpha_{4,5}}{\partial \rho}\frac{\partial(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi)}{\partial(\nabla_\rho\phi)} = 2\frac{\partial \alpha_{4,5}}{\partial \rho}\nabla^\rho\phi$$

and calculating the partial derivatives for the second parenthesis of (A.1.8) we find

$$\begin{aligned}\Lambda^{\kappa\lambda,\tau\sigma;\rho} = \sqrt{-g}\,\frac{1}{4}\Bigg\{ & 2\frac{\partial \alpha_4}{\partial \rho}\nabla^\rho\phi\Big(g^{\kappa\sigma}\nabla^\lambda\phi\nabla^\tau\phi + g^{\kappa\tau}\nabla^\lambda\phi\nabla^\sigma\phi + g^{\lambda\sigma}\nabla^\kappa\phi\nabla^\tau\phi + g^{\lambda\tau}\nabla^\kappa\phi\nabla^\sigma\phi \\ & \quad -2g^{\kappa\lambda}\nabla^\sigma\phi\nabla^\tau\phi - 2g^{\tau\sigma}\nabla^\kappa\phi\nabla^\lambda\phi\Big) \\ & + \alpha_4\Big(g^{\kappa\sigma}g^{\lambda\rho}\nabla^\tau\phi + g^{\kappa\sigma}g^{\tau\rho}\nabla^\lambda\phi + g^{\kappa\tau}g^{\lambda\rho}\nabla^\sigma\phi + g^{\kappa\tau}g^{\sigma\rho}\nabla^\lambda\phi \\ & \quad + g^{\lambda\sigma}g^{\kappa\rho}\nabla^\tau\phi + g^{\lambda\sigma}g^{\tau\rho}\nabla^\kappa\phi + g^{\tau\lambda}g^{\kappa\rho}\nabla^\sigma\phi + g^{\tau\lambda}g^{\sigma\rho}\nabla^\kappa\phi \\ & \quad -2g^{\kappa\lambda}g^{\sigma\rho}\nabla^\tau\phi - 2g^{\kappa\lambda}g^{\tau\rho}\nabla^\sigma\phi - 2g^{\tau\sigma}g^{\kappa\rho}\nabla^\lambda\phi - 2g^{\tau\sigma}g^{\lambda\rho}\nabla^\kappa\phi\Big) \\ & + \frac{\partial \alpha_5}{\partial \rho}\nabla^\rho\phi\Big(4g^{\kappa\sigma}g^{\lambda\tau} + 4g^{\lambda\sigma}g^{\kappa\tau} - 8g^{\sigma\tau}g^{\kappa\lambda}\Big).\end{aligned} \tag{A.1.9}$$

Using this expression and after some lengthy but straightforward calculations, we find

$$\begin{aligned}&\Lambda^{\kappa\lambda,\tau\sigma;\rho} + \Lambda^{\kappa\lambda,\tau\rho;\sigma} + \Lambda^{\kappa\lambda,\sigma\rho;\tau} \\ &= \sqrt{-g}\,\frac{1}{4}\Bigg\{2\frac{\partial \alpha_4}{\partial \rho}\Big[2g^{\kappa\sigma}\nabla^\lambda\phi\nabla^\tau\phi\nabla^\rho\phi + 2g^{\kappa\tau}\nabla^\lambda\phi\nabla^\sigma\phi\nabla^\rho\phi + 2g^{\lambda\sigma}\nabla^\kappa\phi\nabla^\tau\phi\nabla^\rho\phi + 2g^{\lambda\tau}\nabla^\kappa\phi\nabla^\sigma\phi\nabla^\rho\phi \\ &\quad -4g^{\kappa\lambda}\nabla^\sigma\phi\nabla^\tau\phi\nabla^\rho\phi - 2g^{\tau\sigma}\nabla^\kappa\phi\nabla^\lambda\phi\nabla^\rho\phi + 2g^{\kappa\rho}\nabla^\sigma\phi\nabla^\lambda\phi\nabla^\tau\phi + 2g^{\kappa\tau}\nabla^\lambda\phi\nabla^\rho\phi\nabla^\sigma\phi \\ &\quad +2g^{\lambda\rho}\nabla^\kappa\phi\nabla^\tau\phi\nabla^\sigma\phi - 2g^{\kappa\lambda}\nabla^\rho\phi\nabla^\tau\phi\nabla^\sigma\phi - 2g^{\tau\rho}\nabla^\kappa\phi\nabla^\lambda\phi\nabla^\sigma\phi - 2g^{\sigma\rho}\nabla^\kappa\phi\nabla^\lambda\phi\nabla^\tau\phi\Big] \\ &\quad + \Big(2\frac{\partial \alpha_5}{\partial \rho} + \alpha_4\Big)\Big[2g^{\kappa\sigma}g^{\lambda\rho}\nabla^\tau\phi + 2g^{\kappa\tau}g^{\lambda\rho}\nabla^\sigma\phi + 2g^{\kappa\rho}g^{\lambda\sigma}\nabla^\tau\phi + 2g^{\tau\lambda}g^{\kappa\rho}\nabla^\sigma\phi \\ &\quad -4g^{\kappa\lambda}g^{\sigma\rho}\nabla^\tau\phi - 4g^{\kappa\lambda}g^{\tau\rho}\nabla^\sigma\phi + 2g^{\kappa\tau}g^{\lambda\sigma}\nabla^\rho\phi + 2g^{\tau\lambda}g^{\kappa\sigma}\nabla^\rho\phi - 4g^{\kappa\lambda}g^{\sigma\tau}\nabla^\rho\phi\Big]\Bigg\} = 0,\end{aligned}$$

which leads to the following result

$$\frac{\partial \alpha_4}{\partial \rho} = 0 \qquad \text{and} \qquad \frac{\partial \alpha_5}{\partial \rho} = -\frac{1}{2}\alpha_4. \tag{A.1.10}$$

A.2 Computation of $\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}(g_{\kappa\lambda},\phi)$

To compute the term $\Psi^{\mu\nu,\rho\sigma;\alpha\beta,\gamma\delta}R_{\gamma\alpha\beta\delta}R_{\rho\mu\nu\sigma}$, we utilize the following results from [7]. If $\phi_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon} = \phi_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon}(g_{\alpha\beta},\phi)$ is a tensor and

$$\phi_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon} = \phi_{\rho\sigma,\tau\upsilon;\mu\nu,\kappa\eta} = \phi_{\nu\mu,\kappa\eta;\rho\sigma,\tau\upsilon} = \phi_{\mu\nu,\eta\kappa;\rho\sigma,\tau\upsilon},$$

together with

$$\phi_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon} + \phi_{\mu\eta,\nu\kappa;\rho\sigma,\tau\upsilon} + \phi_{\mu\kappa,\eta\nu;\rho\sigma,\tau\upsilon} = 0,$$

and

$$\begin{aligned}&\phi_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon} + \phi_{\mu\nu,\kappa\upsilon;\rho\sigma,\eta\tau} + \phi_{\mu\nu,\kappa\tau;\rho\sigma,\upsilon\eta} \\ &\quad + \phi_{\rho\sigma,\kappa\eta;\mu\nu,\tau\upsilon} + \phi_{\rho\sigma,\kappa\upsilon;\mu\nu,\eta\tau} + \phi_{\rho\sigma,\kappa\tau;\mu\nu,\upsilon\eta} = 0,\end{aligned}$$

then, for spacetime dimension $n > 3$,

[3] Note that in the notation used in Chapter 2, $\rho = g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.

$$\phi_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon}R^{\kappa\mu\nu\eta}R^{\tau\rho\sigma\upsilon} = \frac{(2n-5)\alpha_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon} + 2\alpha_{\mu\nu,\rho\sigma;\kappa\eta,\tau\upsilon}}{(2n-3)(n-3)} R^{\kappa\mu\nu\eta}R^{\tau\rho\sigma\upsilon}, \tag{A.2.1}$$

where

$$\begin{aligned}\alpha_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon} = g^{\lambda\chi}\Big[&g_{\mu\nu}\phi_{\chi\lambda,\kappa\eta;\rho\sigma,\tau\upsilon} + g_{\mu\nu}\phi_{\chi\lambda,\kappa\eta;\rho\sigma,\tau\upsilon}\\ &+ g_{\mu\nu}\phi_{\chi\lambda,\kappa\eta;\rho\sigma,\tau\upsilon} + g_{\mu\nu}\phi_{\chi\lambda,\kappa\eta;\rho\sigma,\tau\upsilon}\\ &+ g_{\mu\nu}\phi_{\chi\lambda,\kappa\eta;\rho\sigma,\tau\upsilon} + g_{\mu\nu}\phi_{\chi\lambda,\kappa\eta;\rho\sigma,\tau\upsilon}\\ &+ g_{\mu\nu}\phi_{\chi\lambda,\kappa\eta;\rho\sigma,\tau\upsilon}\Big].\end{aligned} \tag{A.2.2}$$

Under the same conditions for $\phi_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon}$ we also have

$$g^{\tau\upsilon}\phi_{\eta\nu,\kappa\varepsilon;\rho\sigma,\tau\upsilon} = \frac{1}{n-2}\beta_{\eta\nu,\kappa\varepsilon;\rho\sigma}, \tag{A.2.3}$$

and

$$\begin{aligned}g^{\tau\upsilon}\phi_{\kappa\tau,\mu\eta;\lambda\upsilon,\rho\sigma} = -\frac{1}{2}g^{\tau\upsilon}&\phi_{\mu\eta,\kappa\lambda;\rho\sigma,\tau\upsilon}\\ &+ n\mu\delta_n^4\sqrt{-g}\big(g_{\rho\eta}\varepsilon_{\mu\kappa\lambda\sigma} + g_{\rho\eta}\varepsilon_{\mu\kappa\lambda\sigma}\\ &+ g_{\rho\eta}\varepsilon_{\mu\kappa\lambda\sigma} + g_{\rho\eta}\varepsilon_{\mu\kappa\lambda\sigma}\big),\end{aligned} \tag{A.2.4}$$

where

$$\begin{aligned}\beta_{\eta\nu,\kappa\varepsilon;\rho\sigma} = \lambda\Big(&g_{\eta\nu}g_{\kappa\varepsilon\rho\sigma} - \tfrac{1}{2}g_{\eta\nu}g_{\kappa\varepsilon\rho\sigma}\\ &- \tfrac{1}{2}g_{\eta\nu}g_{\kappa\varepsilon\rho\sigma} - \tfrac{1}{2}g_{\eta\nu}g_{\kappa\varepsilon\rho\sigma} - \tfrac{1}{2}g_{\eta\nu}g_{\kappa\varepsilon\rho\sigma}\Big),\end{aligned}$$

and

$$g_{\kappa\eta\rho\sigma} = g_{\kappa\eta}g_{\rho\sigma} - \tfrac{1}{2}\big(g_{\kappa\sigma}g_{\varepsilon\rho} + g_{\kappa\rho}g_{\varepsilon\sigma}\big),$$

while λ and μ are functions of ϕ. In summary, our approach is as follows. First, we compute $\alpha_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon}$ and $\alpha_{\mu\nu,\rho\sigma;\kappa\eta,\tau\upsilon}$ using the relations (A.2.2)–(A.2.4). We then substitute these expressions into (A.2.1), and subsequently simplify the result by exploiting the symmetries of both the Riemann tensor and the tensor $\phi_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon}$. We have

$$\begin{aligned}\alpha_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon} = g_{\mu\nu}\,&g^{\lambda\chi}\,\phi_{\chi\lambda,\kappa\eta;\rho\sigma,\tau\upsilon}\\ &+ g_{\mu\kappa}\,g^{\lambda\chi}\,\phi_{\chi\nu,\lambda\eta;\rho\sigma,\tau\upsilon} + g_{\mu\eta}\,g^{\lambda\chi}\,\phi_{\chi\nu,\kappa\lambda;\rho\sigma,\tau\upsilon}\\ &+ g_{\mu\rho}\,g^{\lambda\chi}\,\phi_{\chi\nu,\kappa\eta;\lambda\sigma,\tau\upsilon} + g_{\mu\sigma}\,g^{\lambda\chi}\,\phi_{\chi\nu,\kappa\eta;\rho\lambda,\tau\upsilon}\\ &+ g_{\mu\tau}\,g^{\lambda\chi}\,\phi_{\chi\nu,\kappa\eta;\rho\sigma,\lambda\upsilon} + g_{\mu\upsilon}\,g^{\lambda\chi}\,\phi_{\chi\nu,\kappa\eta;\rho\sigma,\tau\lambda}.\end{aligned} \tag{A.2.5}$$

For the first term we use (A.2.3). The second and the third term can be written as

$$g^{\lambda\chi}\phi_{\chi\nu,\lambda\eta;\rho\sigma,\tau\upsilon} = -\frac{1}{2}g^{\lambda\chi}\phi_{\chi\lambda,\nu\eta;\rho\sigma,\tau\upsilon} = -\frac{1}{4}\beta_{\rho\sigma,\tau\upsilon;\nu\eta}$$

and

$$g^{\lambda\chi}\phi_{\chi\nu,\kappa\lambda;\rho\sigma,\tau\upsilon} = -\frac{1}{2}g^{\lambda\chi}\phi_{\chi\lambda,\nu\kappa;\rho\sigma,\tau\upsilon} = -\frac{1}{4}\beta_{\rho\sigma,\tau\upsilon;\nu\kappa}$$

For the rest of the terms we use (A.2.4). We find

$$\begin{aligned}
\alpha_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon} = &\frac{1}{2}g_{\mu\nu}\beta_{\rho\sigma,\tau\upsilon;\kappa\eta} - \frac{1}{4}g_{\mu\kappa}\beta_{\rho\sigma,\tau\upsilon;\nu\eta} - \frac{1}{4}g_{\mu\eta}\beta_{\rho\sigma,\tau\upsilon;\nu\kappa} \\
&- \frac{1}{4}g_{\mu\rho}\beta_{\kappa\eta,\nu\sigma;\tau\upsilon} - \frac{1}{4}g_{\mu\sigma}\beta_{\kappa\eta,\nu\rho;\tau\upsilon} \\
&- \frac{1}{4}g_{\mu\tau}\beta_{\kappa\eta,\nu\upsilon;\rho\sigma} - \frac{1}{4}g_{\mu\upsilon}\beta_{\kappa\eta,\nu\tau;\rho\sigma} \\
&+ 4\mu\sqrt{-g}\Bigg[g_{\mu\rho}\Big(g_{\tau\eta}\varepsilon_{\kappa\nu\sigma\upsilon} + g_{\upsilon\eta}\varepsilon_{\kappa\nu\sigma\tau} + g_{\tau\kappa}\varepsilon_{\eta\nu\sigma\upsilon} + g_{\upsilon\kappa}\varepsilon_{\eta\nu\sigma\tau}\Big) \\
&\quad + g_{\mu\sigma}\Big(g_{\tau\eta}\varepsilon_{\kappa\nu\rho\upsilon} + g_{\upsilon\eta}\varepsilon_{\kappa\nu\rho\tau} + g_{\tau\kappa}\varepsilon_{\eta\nu\rho\upsilon} + g_{\upsilon\kappa}\varepsilon_{\eta\nu\rho\tau}\Big) \\
&\quad + g_{\mu\tau}\Big(g_{\rho\eta}\varepsilon_{\kappa\nu\upsilon\sigma} + g_{\sigma\eta}\varepsilon_{\kappa\nu\upsilon\rho} + g_{\rho\kappa}\varepsilon_{\eta\nu\upsilon\sigma} + g_{\sigma\kappa}\varepsilon_{\eta\nu\upsilon\rho}\Big) \\
&\quad + g_{\mu\upsilon}\Big(g_{\rho\eta}\varepsilon_{\kappa\nu\tau\sigma} + g_{\sigma\eta}\varepsilon_{\kappa\nu\tau\rho} + g_{\rho\kappa}\varepsilon_{\eta\nu\tau\sigma} + g_{\sigma\kappa}\varepsilon_{\eta\nu\tau\rho}\Big)\Bigg]. \qquad \text{(A.2.6)}
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\alpha_{\mu\nu,\rho\sigma;\kappa\eta,\tau\upsilon} = &\frac{1}{2}g_{\mu\nu}\beta_{\kappa\eta,\tau\upsilon;\rho\sigma} - \frac{1}{4}g_{\mu\rho}\beta_{\kappa\eta,\tau\upsilon;\nu\sigma} - \frac{1}{4}g_{\mu\sigma}\beta_{\kappa\eta,\tau\upsilon;\nu\rho} \\
&- \frac{1}{4}g_{\mu\kappa}\beta_{\rho\sigma,\nu\eta;\tau\upsilon} - \frac{1}{4}g_{\mu\eta}\beta_{\rho\sigma,\nu\kappa;\tau\upsilon} \\
&- \frac{1}{4}g_{\mu\tau}\beta_{\rho\sigma,\nu\upsilon;\kappa\eta} - \frac{1}{4}g_{\mu\upsilon}\beta_{\rho\sigma,\nu\tau;\kappa\eta} \\
&+ 4\mu\sqrt{-g}\Bigg[g_{\mu\kappa}\Big(g_{\tau\sigma}\varepsilon_{\rho\nu\eta\upsilon} + g_{\upsilon\sigma}\varepsilon_{\nu\eta\tau} + g_{\tau\rho}\varepsilon_{\sigma\nu\eta\upsilon} + g_{\upsilon\rho}\varepsilon_{\sigma\nu\eta\tau}\Big) \\
&\quad + g_{\mu\eta}\Big(g_{\tau\sigma}\varepsilon_{\rho\nu\kappa\upsilon} + g_{\upsilon\sigma}\varepsilon_{\rho\nu\kappa\tau} + g_{\tau\rho}\varepsilon_{\sigma\upsilon\kappa\nu} + g_{\upsilon\rho}\varepsilon_{\sigma\nu\kappa\tau}\Big) \\
&\quad + g_{\mu\tau}\Big(g_{\kappa\sigma}\varepsilon_{\rho\nu\upsilon\eta} + g_{\sigma\eta}\varepsilon_{\nu\upsilon\kappa} + g_{\rho\kappa}\varepsilon_{\sigma\nu\upsilon\eta} + g_{\eta\rho}\varepsilon_{\sigma\nu\upsilon\kappa}\Big) \\
&\quad + g_{\mu\upsilon}\Big(g_{\kappa\sigma}\varepsilon_{\rho\nu\tau\eta} + g_{\sigma\eta}\varepsilon_{\rho\nu\tau\kappa} + g_{\rho\kappa}\varepsilon_{\sigma\nu\tau\eta} + g_{\eta\rho}\varepsilon_{\sigma\nu\tau\kappa}\Big)\Bigg]. \qquad \text{(A.2.7)}
\end{aligned}$$

Note that from the definition of $\beta_{\eta\nu,\kappa\varepsilon;\rho\sigma}$, the following symmetry holds that can be used in order to simplify the calculations

$$\beta_{\eta\nu,\kappa\varepsilon;\rho\sigma} = \beta_{\eta\nu,\rho\sigma;\kappa\varepsilon}.$$

Substituting (A.2.6) and (A.2.7) into (A.2.1) yields

$$
\begin{aligned}
&\Big(3\alpha_{\mu\nu,\kappa\eta;\rho\sigma,\tau\upsilon}+2\alpha_{\mu\nu,\rho\sigma;\kappa\eta,\tau\upsilon}\Big)R^{\kappa\mu\nu\eta}R^{\tau\rho\sigma\upsilon}/5\\
&=\Big(\frac{1}{2}g_{\mu\nu}\beta_{\rho\sigma,\tau\upsilon;\kappa\eta}-\frac{1}{4}g_{\mu\kappa}\beta_{\rho\sigma,\tau\upsilon;\nu\eta}-\frac{1}{4}g_{\mu\eta}\beta_{\rho\sigma,\tau\upsilon;\nu\kappa}\\
&\quad-\frac{1}{4}g_{\mu\rho}\beta_{\kappa\eta,\nu\sigma;\tau\upsilon}-\frac{1}{4}g_{\mu\sigma}\beta_{\kappa\eta,\nu\rho;\tau\upsilon}-\frac{1}{4}g_{\mu\tau}\beta_{\kappa\eta,\nu\upsilon;\rho\sigma}\\
&\quad-\frac{1}{4}g_{\mu\nu}\beta_{\kappa\eta,\nu\tau;\rho\sigma}\Big)R^{\kappa\mu\nu\eta}R^{\tau\rho\sigma\upsilon}\\
&\quad+\mu\sqrt{-g}\Big\{12\Big[g_{\mu\rho}\left(g_{\tau\eta}\varepsilon_{\kappa\nu\sigma\upsilon}+g_{\upsilon\eta}\varepsilon_{\kappa\nu\sigma\tau}+g_{\tau\kappa}\varepsilon_{\eta\nu\sigma\upsilon}+g_{\upsilon\kappa}\varepsilon_{\eta\nu\sigma\tau}\right)\\
&\quad+g_{\mu\sigma}\left(g_{\tau\eta}\varepsilon_{\kappa\nu\rho\upsilon}+g_{\upsilon\rho}\varepsilon_{\kappa\nu\rho\tau}+g_{\tau\kappa}\varepsilon_{\eta\upsilon\rho\nu}+g_{\upsilon\kappa}\varepsilon_{\eta\nu\rho\tau}\right)\\
&\quad+g_{\mu\tau}\left(g_{\rho\eta}\varepsilon_{\kappa\nu\upsilon\sigma}+g_{\sigma\eta}\varepsilon_{\kappa\nu\upsilon\rho}+g_{\rho\kappa}\varepsilon_{\eta\nu\upsilon\sigma}+g_{\sigma\kappa}\varepsilon_{\eta\nu\upsilon\rho}\right)\\
&\quad+g_{\mu\upsilon}\left(g_{\rho\eta}\varepsilon_{\kappa\nu\tau\sigma}+g_{\sigma\eta}\varepsilon_{\kappa\nu\tau\rho}+g_{\rho\kappa}\varepsilon_{\eta\nu\tau\sigma}+g_{\sigma\kappa}\varepsilon_{\eta\nu\tau\rho}\right)\Big]\\
&\quad+8\Big[g_{\mu\kappa}\left(g_{\tau\sigma}\varepsilon_{\rho\nu\eta\upsilon}+g_{\upsilon\sigma}\varepsilon_{\rho\nu\eta\tau}+g_{\tau\rho}\varepsilon_{\sigma\nu\eta\upsilon}+g_{\upsilon\rho}\varepsilon_{\sigma\nu\eta\tau}\right)\\
&\quad+g_{\mu\eta}\left(g_{\tau\sigma}\varepsilon_{\rho\nu\kappa\upsilon}+g_{\upsilon\sigma}\varepsilon_{\rho\nu\kappa\tau}+g_{\tau\rho}\varepsilon_{\sigma\upsilon\kappa\nu}+g_{\upsilon\rho}\varepsilon_{\sigma\nu\kappa\tau}\right)\\
&\quad+g_{\mu\tau}\left(g_{\kappa\sigma}\varepsilon_{\rho\nu\upsilon\eta}+g_{\eta\sigma}\varepsilon_{\rho\nu\upsilon\kappa}+g_{\kappa\rho}\varepsilon_{\sigma\nu\upsilon\eta}+g_{\eta\rho}\varepsilon_{\sigma\nu\upsilon\kappa}\right)\\
&\quad+g_{\mu\nu}\left(g_{\kappa\sigma}\varepsilon_{\rho\nu\tau\eta}+g_{\eta\sigma}\varepsilon_{\rho\nu\tau\kappa}+g_{\kappa\rho}\varepsilon_{\sigma\nu\tau\eta}+g_{\eta\rho}\varepsilon_{\sigma\nu\tau\kappa}\right)\Big]\Big\}R^{\kappa\mu\nu\eta}R^{\tau\rho\sigma\upsilon}.
\end{aligned}
\tag{A.2.8}
$$

The expression (A.2.8) can be separated into two distinct contributions. The first three lines, upon simplification, reproduce the Gauss–Bonnet term in (2.4.21). The remaining lines correspond precisely to the fifth term of (2.4.21). In order to achieve this, we substitute the terms that involve the tensor β, and use the symmetries of the Riemann tensor. Many of these terms vanish as a result of these symmetries; for example, the second term vanishes because it is symmetric in the indices μ and κ, whereas the Riemann tensor is antisymmetric in the same indices.

A.3 Equations of Motion

A.3.1 Scalar Field Equations

In the notation used in Chapter 2

$$E(L)=\frac{\partial}{\partial x^{\mu}}(\Lambda^{\mu})-\Phi \tag{A.3.2.1}$$

or equivalently

$$E(L)=-\frac{\partial L}{\partial\phi}+\nabla_{\mu}\left(\frac{\partial L}{\partial(\nabla_{\mu}\phi)}\right)$$

- $\mathbf{L_1}=\sqrt{-g}\beta_{\mathbf{1}}(\phi)\left(\mathbf{R^2}-\mathbf{4R^{\mu\nu}R_{\mu\nu}}-\mathbf{R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}}\right)$

The first term of (A.3.2.1) for this part of the Lagrangian is[4]

$$\frac{\partial L_1}{\partial\phi}=\beta_1'(\phi)\mathcal{G}\nabla_{\alpha}\phi\nabla_{\beta}\phi.$$

While the second term vanishes:

$$\frac{\partial L_1}{\partial(\nabla_{\mu}\phi)}=0.$$

Therefore,

$$E_1(L_1)=-\sqrt{-g}\beta_1'(\phi)\mathcal{G} \tag{A.3.2.2}$$

[4] We denote the Gauss Bonnet term by $\mathcal{G}$

- $\mathbf{L_2} = \sqrt{-g}\beta_2(\phi)\mathbf{G}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$

The first term of (A.3.2.1) is

$$\frac{\partial L_2}{\partial \phi} = \beta_2'(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi.$$

The second term is

$$\begin{aligned}\frac{\partial L_2}{\partial(\nabla_\mu\phi)} &= \beta_2(\phi)\,G^{\alpha\beta}\,\frac{\partial}{\partial(\nabla_\mu\phi)}\left(\nabla_\alpha\phi\,\nabla_\beta\phi\right)\\ &= \beta_2(\phi)\,G^{\alpha\beta}\left[\,\delta^\mu_\alpha\,\nabla_\beta\phi+\delta^\mu_\beta\,\nabla_\alpha\phi\right]\\ &= 2\,\beta_2(\phi)\,G^{\mu\alpha}\,\nabla_\alpha\phi.\end{aligned}$$

Thus,

$$\nabla_\mu\left(\frac{\partial L_2}{\partial(\nabla_\mu\phi)}\right) = 2\beta_2(\phi)G^{\mu\alpha}\nabla_\mu\nabla_\alpha\phi+2G^{\mu\alpha}\nabla_\alpha\phi\nabla_\mu\left(\beta_2(\phi)\right) =$$
$$2\beta_2(\phi)G^{\mu\alpha}\nabla_\mu\nabla_\alpha\phi+2G^{\mu\alpha}\nabla_\alpha\phi\nabla_\mu\phi\beta_2'(\phi)$$

Combining the results above, we find

$$E_2(L) = \sqrt{-g}G^{\alpha\beta}\left[2\beta_2(\phi)\nabla_\alpha\nabla_\beta\phi+\beta_2'(\phi)\nabla_\alpha\phi\nabla_\beta\phi\right]. \tag{A.3.2.3}$$

- $\mathbf{L_3} = \sqrt{-g}\beta_3(\phi)\mathbf{R}$

For the third part of the Lagrangian, we find

$$\frac{\partial L_3}{\partial\phi} = \sqrt{-g}\beta_3'(\phi)R$$

and

$$\frac{\partial L_3}{\partial(\nabla_\mu\phi)} = 0.$$

Substituting into (A.3.2.1) yields

$$E_3 = \sqrt{-g}\beta_3'(\phi)R. \tag{A.3.2.4}$$

- $\mathbf{L_4} = \sqrt{-g}\beta_4(\phi,\rho)$

For the fourth part of the Lagrangian, we find

$$\frac{\partial L_4}{\partial\phi} = \frac{\partial\beta_4}{\partial\phi},$$

and

$$\begin{aligned}\frac{\partial L_4}{\partial(\nabla_\mu\phi)} &= \frac{\partial\beta_4}{\partial\rho}\,\frac{\partial\rho}{\partial(\nabla_\mu\phi)}\\ &= \frac{\partial\beta_4}{\partial\rho}\,\frac{\partial}{\partial(\nabla_\mu\phi)}\left(g^{\alpha\beta}\nabla_\alpha\phi\,\nabla_\beta\phi\right)\\ &= \frac{\partial\beta_4}{\partial\rho}\left(g^{\alpha\beta}\,\delta^\mu_\alpha\nabla_\beta\phi+g^{\alpha\beta}\,\delta^\mu_\beta\nabla_\alpha\phi\right)\\ &= 2\,\frac{\partial\beta_4}{\partial\rho}\,\nabla^\mu\phi.\end{aligned}$$

The second term of (A.3.2.1) is

$$\begin{aligned}
\nabla_\mu\Big(\frac{\partial L_4}{\partial(\nabla_\mu\phi)}\Big) &= 2\,\nabla_\mu\Big(\frac{\partial\beta_4}{\partial\rho}\,\nabla^\mu\phi\Big)\\
&= 2\frac{\partial\beta_4}{\partial\rho}\,\nabla_\mu\nabla^\mu\phi + 2\nabla^\mu\phi\,\nabla_\mu\Big(\frac{\partial\beta_4}{\partial\rho}\Big)\\
&= 2\frac{\partial\beta_4}{\partial\rho}\,\Box\phi + 2\frac{\partial^2\beta_4}{\partial\rho^2}\nabla_\mu\rho + 2\frac{\partial^2\beta_4}{\partial\phi\,\partial\rho}\nabla_\mu\phi\,\nabla^\mu\phi\\
&= 2\frac{\partial\beta_4}{\partial\rho}\,\Box\phi + 4\frac{\partial^2\beta_4}{\partial\rho^2}\,\nabla^\mu\phi\,\nabla^\beta\phi\,\nabla_\mu\nabla_\beta\phi + 2\frac{\partial^2\beta_4}{\partial\phi\,\partial\rho}\,\nabla_\mu\phi\,\nabla^\mu\phi.
\end{aligned}$$

Substituting into (A.3.2.1) we obtain

$$E(L_4) = \sqrt{-g}\Bigg[-\frac{\partial\beta_4}{\partial\phi} + 2\frac{\partial\beta_4}{\partial\rho}\Box\phi + 4\frac{\partial^2\beta_4}{\partial\rho^2}\nabla^\mu\phi\nabla^\beta\phi\nabla_\mu\nabla_\beta\phi + 2\frac{\partial^2\beta_4}{\partial\phi\partial\rho}\nabla_\mu\phi\nabla^\mu\phi\Bigg]. \tag{A.3.2.5}$$

The calculations for the equations of motion for the metric $g_{\mu\nu}$, can be obtained using the results from B.1.1

Appendix B

Equations of Motion for Horndeski Gravity

B.1 Equations of Motion

We note here some general results that will be used extensively in the following calculations:

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu},$$

$$\delta X = -\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu},$$

$$\delta(G_i(\phi,X)) = \frac{\partial G_i}{\partial X}\delta X = -G_{iX}\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu},$$

$$g^{\mu\nu}g^{\lambda\sigma}\delta g_{\nu\sigma} = -\delta g^{\mu\lambda},$$

$$\nabla_\mu X = -(\nabla^\alpha\phi)\nabla_\mu\nabla_\alpha\phi,$$

The variation of the Ricci tensor is given by:

$$\delta R^\mu{}_{\sigma\mu\nu} \equiv \delta R_{\sigma\nu} = \nabla_\mu(\delta\Gamma^\mu{}_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\mu{}_{\mu\sigma}).$$

This result is sometimes referred to as the *Palatini Identity*. A more applicable form of this result for the subsequent calculations is given by:

$$\delta R_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left[\nabla_\alpha\nabla_\nu\delta g_{\mu\beta} + \nabla_\alpha\nabla_\mu\delta g_{\nu\beta} - \nabla_\mu\nabla_\nu\delta g_{\alpha\beta} - \nabla_\alpha\nabla_\beta\delta g_{\mu\nu}\right].$$

In the calculations that follow, we will need to simplify terms using commutators of covariant derivatives. For this reason, we first present some useful results that will be applied repeatedly throughout the derivation. Starting from the general commutator relation for covariant derivatives acting on a vector field (1.1.13):

$$[\nabla_\alpha,\nabla_\mu]\nabla^\nu\phi = R^\nu_{\lambda\alpha\mu}\nabla^\lambda\phi,$$

we contract the indices α and ν to obtain

$$[\nabla_\nu,\nabla_\mu]\nabla^\nu\phi = R^\nu_{\lambda\nu\mu}\nabla^\lambda\phi = R_{\lambda\mu}\nabla^\lambda\phi.$$

Expanding the commutator yields the identity

$$\nabla_\nu\nabla_\mu\nabla^\nu\phi - \nabla_\mu\Box\phi = R_{\lambda\mu}\nabla^\lambda\phi,$$

or equivalently

$$[\Box,\nabla_\mu]\phi = R_{\mu\lambda}\nabla^\lambda\phi.$$

Commutators of covariant derivative acting on dual vectors give:

$$[\nabla_\alpha, \nabla_\mu]\nabla_\beta \phi = -R^{\sigma}_{\beta\alpha\mu}\nabla_{\sigma\phi}$$

while for mixed tensors $X^{\mu}{}_{\nu}$, we have

$$[\nabla_\rho, \nabla_\sigma]X^{\mu}{}_{\nu} = R^{\mu}_{\lambda\rho\sigma}X^{\lambda}{}_{\nu} - R^{\lambda}_{\nu\rho\sigma}X^{\mu}{}_{\lambda}$$

Using the above we can also prove the following:

$$\nabla^\alpha \left(\nabla_\alpha \Box - \Box \nabla_\alpha\right)\phi = -R_{\lambda\alpha}\nabla^\alpha\nabla^\lambda\phi - \frac{1}{2}\nabla_\lambda R$$

where in the last equality we have utilized the contracted Bianchi identity.

B.1.1 Gravitational Field Equations

- $\mathcal{L}_\mathbf{2} = \mathbf{K}(\phi, \mathbf{X})$

We start by considering the simplest term in the Horndeski Lagrangian.

$$S_2 = \int d^4x\sqrt{-g}K(\phi, X).$$

Variation with respect to the metric yields

$$\delta S_2 = \int d^4x\delta(\sqrt{-g})K + \int d^4x\sqrt{-g}\delta(K) =$$

$$\int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}K - \frac{1}{2}K_X\nabla_\mu\phi\nabla_\nu\phi\right)\delta g^{\mu\nu}.$$

Therefore,

$$G^{(2)}_{\mu\nu} = -\frac{1}{2}K_X\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}K \tag{B.1.1.1}$$

- $\mathcal{L}_\mathbf{3} = -\mathbf{G_3}(\phi, \mathbf{X})\Box\phi$

The action is

$$S_3 = \int d^4x\sqrt{-g}(-G_3(\phi, X)\Box\phi).$$

Variation with respect to the metric leads to

$$\begin{aligned}\delta S_3 &= \int d^4x\,\delta(\sqrt{-g})(-G_3\Box\phi) + \int d^4x\sqrt{-g}\,\delta(-G_3)\Box\phi + \int d^4x\sqrt{-g}(-G_3)\delta\left(g^{\mu\nu}\nabla_\mu\nabla_\nu\phi\right)\\ &= \int d^4x\left(\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\right)G_3\Box\phi\,\delta g^{\mu\nu} + \int d^4x\sqrt{-g}\,G_{3X}\left(\frac{1}{2}\nabla_\mu\phi\,\nabla_\nu\phi\,\Box\phi\right)\delta g^{\mu\nu}\\ &\quad - \int d^4x\sqrt{-g}\,G_3\,\nabla_\mu\nabla_\nu\phi\,\delta g^{\mu\nu} - \int d^4x\sqrt{-g}g^{\mu\nu}\,G_3\,\delta\left(\nabla_\mu\nabla_\nu\phi\right).\end{aligned} \tag{B.1.1.2}$$

For the last term, we have

$$\delta(\nabla_\mu\nabla_\nu\phi) = -\delta\Gamma^{\lambda}_{\mu\nu}\nabla_\lambda\phi = -\frac{1}{2}g^{\lambda\sigma}\left(\nabla_\mu\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu}\right)\nabla_\lambda\phi.$$

Using this, it can be recast in the following form

$$\int d^4x\sqrt{-g}G_3g^{\mu\nu}\frac{1}{2}g^{\lambda\sigma}\left(\nabla_\mu\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu}\right)\nabla_\lambda\phi =$$

$$\int d^4x\sqrt{-g}G_3 g^{\mu\nu}g^{\lambda\sigma}\left(\nabla_\mu\delta g_{\nu\sigma}-\frac{1}{2}\nabla_\sigma\delta g_{\mu\nu}\right)\nabla_\lambda\phi.$$

Where, for the last line we have exchanged $\mu\leftrightarrow\nu$ in the first term. Note that in order to write the result in terms of the variation of the inverse metric, we get an extra minus sign for each of the terms. The result can be written as

$$\int d^4x\sqrt{-g}G_3\left(-\nabla_\mu\delta g^{\mu\lambda}+\frac{1}{2}g_{\mu\nu}\nabla^\lambda\delta g^{\mu\nu}\right)\nabla_\lambda\phi.$$

Integrating by parts, discarding the surface term which vanishes under the assumption of vanishing variations at infinity, and exchanging $\lambda\leftrightarrow\nu$, we obtain

$$\int d^4x\sqrt{-g}\left[\nabla_\mu(G_3\nabla_\nu\phi)-\frac{1}{2}g_{\mu\nu}\nabla_\lambda(G_3\nabla^\lambda\phi)\right]\delta g^{\mu\nu}.$$

Substituting into (B.1.1.2) yields

$$\begin{aligned}\delta S_3=\int d^4x\sqrt{-g}\Big[&\cancel{\frac{1}{2}g_{\mu\nu}G_3\Box\phi}+\frac{1}{2}G_{3X}\nabla_\mu\phi\nabla_\nu\phi\Box\phi-G_3\nabla_\mu\nabla_\nu\phi+\nabla_\mu\left(G_3\nabla_\nu\phi\right)\\&-\frac{1}{2}g_{\mu\nu}\nabla^\lambda G_3\nabla_\lambda\phi-\cancel{\frac{1}{2}G_3\Box\phi g_{\mu\nu}}\Big]\delta g^{\mu\nu}.\end{aligned}$$

Simplifying further,

$$\delta S_3=\int d^4x\sqrt{-g}\left[\frac{1}{2}G_{3X}\nabla_\mu\phi\nabla_\nu\phi\Box\phi+\nabla_\mu G_3\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}\nabla^\lambda G_3\nabla_\lambda\phi\right]\delta g^{\mu\nu}$$

which yields

$$G^{(3)}_{\mu\nu}=\frac{1}{2}G_{3X}\nabla_\mu\phi\nabla_\nu\phi\Box\phi+\nabla_{(\mu}G_3\nabla_{\nu)}\phi-\frac{1}{2}g_{\mu\nu}\nabla_\lambda G_3\nabla^\lambda\phi. \tag{B.1.1.3}$$

The need for symmetrization of the second term stems from the fact that the tensor $(\nabla_\mu G_3)\nabla_\nu\phi$ is not manifestly symmetric in μ and ν, but it is contracted with the symmetric tensor $\delta g^{\mu\nu}$, which is symmetric in μ and ν. As a result, only its symmetric part contributes to the variation while the antisymmetric part vanishes. We can therefore replace it with its symmetric part, $\nabla_{(\mu}G_3\nabla_{\nu)}\phi$, without loss of generality.

- $\mathcal{L}_\mathbf{4}=\mathbf{G_4}(\phi,\mathbf{X})\mathbf{R}+\mathbf{G_{4X}}\left[(\Box\phi)^\mathbf{2}-(\nabla_\mu\nabla_\nu\phi)^\mathbf{2}\right]$

The action for this part of the Horndenski theory is

$$S_4=\int d^4x\sqrt{-g}\Big[G_4(\phi,X)R+G_{4X}\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right]\Big]. \tag{B.1.1.4}$$

The calculations for $\mathcal{L}_4,\mathcal{L}_5$ are lengthy so we will split them into multiple parts. Variation with respect to the metric for (B.1.1.4) yields

$$\begin{aligned}\delta S_4=&\int d^4x\delta(\sqrt{-g})G_4R+\int d^4x\sqrt{-g}\delta(G_4)R+\int d^4x\sqrt{-g}G_4\delta(R)\\&+\int d^4x\delta(\sqrt{-g})G_{4X}\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right]+\int d^4x\sqrt{-g}\delta(G_{4X})\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right]\\&+\int d^4x\sqrt{-g}G_{4X}\delta\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right].\end{aligned} \tag{B.1.1.5}$$

The first, second, fourth, and fifth terms are straightforward.

$$\int d^4x\delta(\sqrt{-g})G_{4X}\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right]=-\frac{1}{2}\int d^4x\sqrt{-g}g_{\mu\nu}G_{4X}\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right]\delta g^{\mu\nu},$$

$$\int d^4x\sqrt{-g}\delta(G_{4X})\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right]=-\frac{1}{2}\int d^4x\sqrt{-g}G_{4XX}\nabla_\mu\phi\nabla_\nu\phi\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right]\delta g^{\mu\nu},$$

$$\int d^4x\sqrt{-g}\delta(G_4)R = -\frac{1}{2}\int d^4x\sqrt{-g}\left(G_{4X}\nabla_\mu\phi\nabla_\nu\phi R\right)\delta g^{\mu\nu},$$
$$\int d^4x\delta(\sqrt{-g})G_4R = -\frac{1}{2}\int d^4x\sqrt{-g}g_{\mu\nu}G_4R\delta g^{\mu\nu}. \qquad \text{(B.1.1.6)}$$

We calculate the remaining two terms in detail below. For the third term,

$$\int d^4x\sqrt{-g}\,G_4\,\delta R = \int d^4x\sqrt{-g}\,G_4\,\delta(g^{\mu\nu}R_{\mu\nu})$$
$$= \int d^4x\sqrt{-g}\,G_4\,g^{\mu\nu}\delta R_{\mu\nu} + \int d^4x\sqrt{-g}\,G_4\,R_{\mu\nu}\delta g^{\mu\nu}. \qquad \text{(B.1.1.7)}$$

In contrast to the simple case of Einstein-Hilbert action, where the first integral reduces to a surface term, the presence of G_4 requires computing the full variation. Since this is the first time we encounter this variation, we compute it below, in detail. Later on, when terms like these appear in the calculations for $\mathcal{L}_5$, we will substitute the result from the introduction of this section. Using (1.2.10) for $\delta R_{\mu\nu}$, we find,

$$\int d^4x\sqrt{-g}G_4g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}G_4g^{\mu\nu}\left(\nabla_\rho\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho_{\mu\rho}\right).$$

Substituting the variation of the Christoffel symbols,

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\nabla_\mu\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu}\right)$$

$$\delta\Gamma^\rho_{\mu\rho} = \frac{1}{2}g^{\rho\sigma}\left(\nabla_\mu\delta g_{\sigma\rho} + \nabla_\rho\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\rho}\right)$$

we obtain,

$$\int \mathrm{d}^4x\sqrt{-g}\,G_4\,g^{\mu\nu}\Bigg[\frac{1}{2}g^{\rho\sigma}\big(\nabla_\rho\nabla_\mu\delta g_{\nu\sigma} + \nabla_\rho\nabla_\nu\delta g_{\mu\sigma} - \nabla_\rho\nabla_\sigma\delta g_{\mu\nu} - \nabla_\nu\nabla_\mu\delta g_{\sigma\rho}$$
$$- \nabla_\nu\nabla_\rho\delta g_{\sigma\mu} + \nabla_\nu\nabla_\sigma\delta g_{\mu\rho}\big)\Bigg].$$

Which, after renaming some indices and simplifying, can be reformulated as

$$\int d^4x\sqrt{-g}G_4\Big[g_{\mu\nu}\Box(\delta g^{\mu\nu}) - \nabla_\mu\nabla_\nu(\delta g^{\mu\nu})\Big].$$

We now use integration by parts to shift around the covariant derivatives (discarding the resulting boundary terms each time as justified previously). Note that there is no overall sign change since we perform the integration twice on each term. Also, the order of differentiation on each term changes (even though, in this case, it makes no difference).

$$\int d^4x\sqrt{-g}\Big[g_{\mu\nu}\Box G_4 - \nabla_\nu\nabla_\mu G_4\Big]\delta g^{\mu\nu}. \qquad \text{(B.1.1.8)}$$

Now we turn our attention to the last term of (B.1.1.5)

$$\int \mathrm{d}^4x\sqrt{-g}\,G_{4\mathrm{X}}\,\delta\Big[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\Big] = \int \mathrm{d}^4x\sqrt{-g}\,G_{4\mathrm{X}}\Big[2\Box\phi\,\nabla_\mu\nabla_\nu\phi\,\delta g^{\mu\nu}$$
$$+ 2\Box\phi\,\nabla_\nu\phi\,\nabla_\mu\delta g^{\mu\nu} - g_{\mu\nu}\Box\phi\,\nabla_\rho\phi\,\nabla^\rho\delta g^{\mu\nu} - 2\nabla_\rho\nabla_\mu\phi\,\nabla_\nu\phi\,\nabla^\rho\delta g^{\mu\nu}$$
$$+ \nabla_\mu\nabla_\nu\phi\,\nabla_\rho\phi\,\nabla^\rho\delta g^{\mu\nu} - 2\nabla_\mu\nabla^\rho\phi\,\nabla_\nu\nabla_\rho\phi\,\delta g^{\mu\nu}\Big].$$

Where we used

$$\delta\left((\nabla_\mu\nabla_\nu\phi)^2\right) = 2\nabla^\rho\nabla_\nu\phi\nabla_\rho\nabla_\mu\phi\delta g^{\mu\nu} + 2\nabla_\rho\nabla_\nu\phi\nabla_\mu\phi\nabla^\rho\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\phi\nabla_\rho\phi\nabla^\rho\delta g^{\mu\nu},$$

$$\delta\left((\Box\phi)^2\right) = 2\Box\phi\nabla_\mu\nabla_\nu\phi\delta g^{\mu\nu} + 2\Box\phi\nabla_\nu\phi\nabla_\mu\delta g^{\mu\nu} - g_{\mu\nu}\Box\phi\nabla_\rho\phi\nabla^\rho\delta g^{\mu\nu}.$$

Integration by parts, yields

$$
\begin{aligned}
\int d^4x\sqrt{-g}\Big[& 2G_{4\mathrm{X}}\Box\phi\nabla_\mu\nabla_\nu\phi - 2\nabla_\mu G_{4\mathrm{X}}\Box\phi\nabla_\nu\phi - 2G_{4\mathrm{X}}\nabla_\mu(\Box\phi)\nabla_\nu\phi \\
& - 2G_{4\mathrm{X}}\Box\phi\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\Big(\nabla^\rho G_{4\mathrm{X}}\nabla_\rho\phi\Box\phi + G_{4\mathrm{X}}\nabla^\rho(\Box\phi)\nabla_\rho\phi \\
& + G_{4\mathrm{X}}(\Box\phi)^2\Big) + 2(\nabla^\rho G_{4\mathrm{X}})\nabla_\rho\nabla_\mu\phi\nabla_\nu\phi \\
& + 2G_{4\mathrm{X}}\Box(\nabla_\mu\phi)\nabla_\nu\phi + 2G_{4\mathrm{X}}\nabla_\rho\nabla_\mu\phi\nabla^\rho\nabla_\nu\phi \\
& - (\nabla^\rho G_{4\mathrm{X}})\nabla_\mu\nabla_\nu\phi\nabla_\rho\phi - G_{4\mathrm{X}}\nabla^\rho(\nabla_\mu\nabla_\nu\phi)\nabla_\rho\phi \\
& - G_{4\mathrm{X}}\nabla_\mu\nabla_\nu\phi\Box\phi - 2G_{4\mathrm{X}}\nabla_\mu\nabla^\rho\phi\nabla_\nu\nabla_\rho\phi\Big]\delta g^{\mu\nu}.
\end{aligned}
\tag{B.1.1.9}
$$

This concludes the variational calculations for the individual terms of (B.1.1.5). The final result is obtained by summing the contributions from equations (B.1.1.6), (B.1.1.7), (B.1.1.8), (B.1.1.9) and simplifying the resultant expression.

$$
\begin{aligned}
\delta S_4 = \int d^4x\sqrt{-g}\Big\{ & -\frac{1}{2}g_{\mu\nu}G_{4X}\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right] - \frac{1}{2}G_{4XX}\nabla_\mu\phi\nabla_\nu\phi\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right] \\
& - \frac{1}{2}G_{4X}\nabla_\mu\phi\nabla_\nu\phi R - \frac{1}{2}g_{\mu\nu}G_4R + G_4R_{\mu\nu} + g_{\mu\nu}\Box G_4 - \nabla_\mu\nabla_\nu G_4 \\
& + \cancel{2G_{4X}\Box\phi\nabla_\mu\nabla_\nu\phi} - 2\Box\phi\nabla_\mu G_{4X}\nabla_\nu\phi - 2G_{4X}\nabla_\mu(\Box\phi)\nabla_\nu\phi - \cancel{2\Box\phi G_{4X}\nabla_\mu\nabla_\nu\phi} \\
& + \nabla^\rho G_{4X}\, g_{\mu\nu}\nabla_\rho\phi\,\Box\phi + G_{4X}\nabla^\rho(\Box\phi)\, g_{\mu\nu}\nabla_\rho\phi + G_{4X}\, g_{\mu\nu}(\Box\phi)^2 \\
& + 2(\nabla^\rho G_{4X})\nabla_\rho\nabla_\mu\phi\,\nabla_\nu\phi + 2G_{4X}\Box(\nabla_\mu\phi)\nabla_\nu\phi + \cancel{2G_{4X}\nabla_\rho\nabla_\mu\phi\nabla^\rho\nabla_\nu\phi} \\
& - (\nabla^\rho G_{4X})\nabla_\mu\nabla_\nu\phi\,\nabla_\rho\phi - G_{4X}\nabla^\rho(\nabla_\mu\nabla_\nu\phi)\nabla_\rho\phi - G_{4X}\nabla_\mu\nabla_\nu\phi\,\Box\phi \\
& - \cancel{2G_{4X}\nabla_\mu\nabla^\rho\phi}\,\nabla_\nu\nabla_\rho\phi\Big\}\delta g^{\mu\nu}.
\end{aligned}
$$

The terms that involve third-order covariant derivatives of the metric are arranged in such a way that they will be replaced using commutators of covariant derivatives acting on vectors or dual vectors that will eventually reduce to Riemann or Ricci terms.
We can reformulate some of the terms as follows,

$$
\begin{aligned}
-\nabla_\mu\nabla_\nu G_4 &= -\nabla_\mu\left(\nabla_\nu\phi G_{4\phi} + \nabla_\nu X G_{4X}\right) \\
&= -G_{4\phi}\nabla_\mu\nabla_\nu\phi - 2\nabla_\nu\phi\nabla_\mu X G_{4\phi X} - \nabla_\nu\phi\nabla_\mu\phi G_{4\phi\phi} - \nabla_\mu\nabla_\nu X G_{4X} - \nabla_\nu X\nabla_\mu X G_{4XX} \\
&= -G_{4\phi}\nabla_\mu\nabla_\nu\phi + 2G_{4\phi X}\nabla^\rho\phi\nabla_\rho\nabla_\mu\phi\nabla_\nu\phi - \nabla_\nu\phi\nabla_\mu\phi G_{4\phi\phi} + G_{4X}\nabla_\rho\nabla_\mu\phi\nabla^\rho\nabla_\nu\phi \\
&\quad + G_{4X}\nabla^\rho\phi(\nabla_\mu\nabla_\nu\nabla_\rho\phi) - G_{4XX}\nabla^\alpha\phi\nabla_\alpha\nabla_\mu\phi\nabla^\beta\phi\nabla_\beta\nabla_\nu\phi.
\end{aligned}
$$

In a similar way, we find,

$$
\begin{aligned}
g_{\mu\nu}\Box G_4 = & -g_{\mu\nu}G_{4X}(\nabla_\alpha\nabla_\beta\phi)^2 - g_{\mu\nu}\nabla^\alpha\phi\,\Box(\nabla_\alpha\phi)G_{4X} + 2g_{\mu\nu}\nabla_\rho X\nabla^\rho\phi G_{4X\phi} \\
& + g_{\mu\nu}\nabla_\alpha\nabla_\lambda\phi\,\nabla^\beta\nabla^\lambda\phi\,\nabla^\alpha\phi\nabla^\beta\phi\, G_{4XX} + g_{\mu\nu}\left(G_{4\phi}\Box\phi - 2XG_{4\phi\phi}\right),
\end{aligned}
$$

and

$$
2G_{4X}\left(\Box(\nabla_\mu\phi) - \nabla_\mu(\Box\phi)\right)\nabla_\nu\phi = 2G_{4X}R_{\mu\lambda}\nabla^\lambda\phi\nabla_\nu\phi.
$$

Therefore

$$
\begin{aligned}
\delta S_4 = \int \mathrm{d}^4x\sqrt{-g}\Bigg\{ & -\frac{1}{2}g_{\mu\nu}G_{4X}\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right]-\frac{1}{2}G_{4XX}\nabla_\mu\phi\nabla_\nu\phi\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right] \\
& -\frac{1}{2}G_{4X}\nabla_\mu\phi\nabla_\nu\phi R+G_4G_{\mu\nu}-g_{\mu\nu}G_{4X}(\nabla_\alpha\nabla_\beta\phi)^2-g_{\mu\nu}\nabla^\alpha\phi\,\Box(\nabla_\alpha\phi)G_{4X}+2g_{\mu\nu}\nabla_\rho X\nabla^\rho\phi G_{4X\phi} \\
& +g_{\mu\nu}\nabla_\alpha\nabla_\lambda\phi\,\nabla_\beta\nabla^\lambda\phi\,\nabla^\alpha\phi\nabla^\beta\phi\,G_{4XX}+g_{\mu\nu}\left(G_{4\phi}\Box\phi-2XG_{4\phi\phi}\right)-G_{4\phi}\nabla_\mu\nabla_\nu\phi \\
& +2G_{4\phi X}\nabla^\lambda\phi\nabla_\lambda\nabla_\mu\phi\nabla_\nu\phi-\nabla_\nu\phi\nabla_\mu\phi G_{4\phi\phi}+G_{4X}\nabla_\lambda\nabla_\mu\phi\nabla^\lambda\nabla_\nu\phi \\
& +G_{4X}\nabla^\rho\phi(\nabla_\mu\nabla_\nu\nabla_\rho\phi)-G_{4XX}\nabla^\alpha\phi\nabla_\alpha\nabla_\mu\phi\nabla^\beta\phi\nabla_\beta\nabla_\nu\phi \\
& -2\Box\phi\nabla_\mu G_{4X}\nabla_\nu\phi+2G_{4X}R_{\mu\lambda}\nabla^\lambda\phi\nabla_\nu\phi \\
& +\nabla^\rho G_{4X}\,g_{\mu\nu}\,\nabla_\rho\phi\,\Box\phi+G_{4X}\,\nabla^\rho(\Box\phi)\,g_{\mu\nu}\,\nabla_\rho\phi+G_{4X}\,g_{\mu\nu}\,(\Box\phi)^2 \\
& -(\nabla_\lambda G_{4X})\nabla_\mu\nabla_\nu\phi\,\nabla^\lambda\phi-G_{4X}\nabla^\lambda(\nabla_\mu\nabla_\nu\phi)\,\nabla_\lambda\phi-G_{4X}\nabla_\mu\nabla_\nu\phi\,\Box\phi \\
& +2(\nabla_\lambda G_{4X})\nabla^\lambda\nabla_\mu\phi\,\nabla_\nu\phi\Bigg\}\delta g^{\mu\nu}.
\end{aligned}
$$

Simplifying further using

$$
\begin{aligned}
g_{\mu\nu}G_{4X}\,\nabla^\rho(\Box\phi)\,\nabla_\rho\phi-g_{\mu\nu}\nabla^\alpha\phi\,\Box(\nabla_\alpha\phi)G_{4X} &= g_{\mu\nu}G_{4X}\nabla_\rho\phi\Big(\nabla^\rho(\Box\phi)-\Box(\nabla^\rho\phi)\Big) \\
&= -g_{\mu\nu}G_{4X}R^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi, \\
G_{4X}\nabla^\rho\phi\Big(\nabla_\mu\nabla_\nu\nabla_\rho\phi-\nabla_\rho\nabla_\mu\nabla_\nu\phi\Big) &= G_{4X}R^\alpha_{\ \nu\rho\mu}\nabla_\alpha\phi\nabla^\rho\phi \\
&= G_{4X}R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi,
\end{aligned}
$$

and,

$$2g_{\mu\nu}\nabla_\rho X\,\nabla^\rho\phi\,G_{4X\phi}=-2g_{\mu\nu}G_{4\phi X}\nabla_\alpha\nabla_\beta\phi\,\nabla^\alpha\phi\,\nabla^\beta\phi.$$

We obtain

$$
\begin{aligned}
\delta S_4 = \int \mathrm{d}^4x\sqrt{-g}\Bigg\{ & G_4G_{\mu\nu}-\frac{1}{2}G_{4X}\nabla_\mu\phi\nabla_\nu\phi R \\
& -\frac{1}{2}G_{4XX}\nabla_\mu\phi\nabla_\nu\phi\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right]-G_{4X}\nabla_\mu\nabla_\nu\phi\,\Box\phi \\
& +G_{4X}\nabla_\lambda\nabla_\mu\phi\nabla^\lambda\nabla_\nu\phi+2(\nabla_\lambda G_{4X})\nabla^\lambda\nabla_{(\mu}\phi\,\nabla_{\nu)}\phi \\
& -(\nabla_\lambda G_{4X})\nabla_\mu\nabla_\nu\phi\,\nabla^\lambda\phi+g_{\mu\nu}\left(G_{4\phi}\Box\phi-2XG_{4\phi\phi}\right) \\
& +g_{\mu\nu}\Big[-2G_{4\phi X}\nabla_\alpha\nabla_\beta\phi\nabla^\alpha\phi\nabla^\beta\phi \\
& \quad +G_{4XX}\nabla_\alpha\nabla_\lambda\phi\,\nabla_\beta\nabla^\lambda\phi\,\nabla^\alpha\phi\nabla^\beta\phi \\
& \quad +\frac{1}{2}G_{4X}\left[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\right]\Big] \\
& +2\Big[G_{4X}R_{\lambda(\mu}\nabla_{\nu)}\phi\nabla^\lambda\phi-\nabla_{(\mu}G_{4X}\nabla_{\nu)}\phi\Box\phi\Big] \\
& -g_{\mu\nu}\Big[G_{4X}R^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi-\nabla_\lambda G_{4X}\,\nabla^\lambda\phi\,\Box\phi\Big] \\
& +G_{4X}R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi \\
& -G_{4\phi}\nabla_\mu\nabla_\nu\phi-G_{4\phi\phi}\nabla_\nu\phi\nabla_\mu\phi \\
& +2G_{4\phi X}\nabla^\lambda\phi\nabla_\lambda\nabla_{(\mu}\phi\nabla_{\nu)}\phi \\
& -G_{4XX}\nabla^\alpha\phi\nabla_\alpha\nabla_\mu\phi\nabla^\beta\phi\nabla_\beta\nabla_\nu\phi\Bigg\}\delta g^{\mu\nu}
\end{aligned}
$$

Requiring $\delta S_4 = 0$ yields the final result:

$$
\begin{aligned}
G^{(4)}{}_{\mu\nu} =& G_4 G_{\mu\nu} - \frac{1}{2} G_{4X} \nabla_\mu \phi \nabla_\nu \phi R \\
& - \frac{1}{2} G_{4XX} \nabla_\mu \phi \nabla_\nu \phi \left[(\Box\phi)^2 - (\nabla_\alpha \nabla_\beta \phi)^2 \right] - G_{4X} \nabla_\mu \nabla_\nu \phi \Box\phi \\
& + G_{4X} \nabla_\lambda \nabla_\mu \phi \nabla^\lambda \nabla_\nu \phi + 2(\nabla_\lambda G_{4X}) \nabla^\lambda \nabla_{(\mu} \phi \nabla_{\nu)} \phi \\
& - (\nabla_\lambda G_{4X}) \nabla_\mu \nabla_\nu \phi \nabla^\lambda \phi + g_{\mu\nu} \left(G_{4\phi} \Box\phi - 2X G_{4\phi\phi} \right) \\
& + g_{\mu\nu} \Big[-2G_{4\phi X} \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \phi \nabla^\beta \phi \\
& \quad + G_{4XX} \nabla_\alpha \nabla_\lambda \phi \nabla_\beta \nabla^\lambda \phi \nabla^\alpha \phi \nabla^\beta \phi \\
& \quad + \frac{1}{2} G_{4X} \left[(\Box\phi)^2 - (\nabla_\alpha \nabla_\beta \phi)^2 \right] \Big] \\
& + 2 \Big[G_{4X} R_{\lambda(\mu} \nabla_{\nu)} \nabla^\lambda \phi \phi - \nabla_{(\mu} G_{4X} \nabla_{\nu)} \phi \Box\phi \Big] \\
& - g_{\mu\nu} \Big[G_{4X} R^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - \nabla_\lambda G_{4X} \nabla^\lambda \phi \Box\phi \Big] \\
& + G_{4X} R_{\mu\alpha\nu\beta} \nabla^\alpha \phi \nabla^\beta \phi \\
& - G_{4\phi} \nabla_\mu \nabla_\nu \phi - G_{4\phi\phi} \nabla_\nu \phi \nabla_\mu \phi \\
& + 2G_{4\phi X} \nabla^\lambda \phi \nabla_\lambda \nabla_{(\mu} \phi \nabla_{\nu)} \phi \\
& - G_{4XX} \nabla^\alpha \phi \nabla_\alpha \nabla_\mu \phi \nabla^\beta \phi \nabla_\beta \nabla_\nu \phi
\end{aligned}
\tag{B.1.1.10}
$$

- $\mathcal{L}_5 = \mathbf{G_5(\phi, X) G_{\mu\nu}} \nabla^\mu \nabla^\nu \phi - \frac{\mathbf{G_{5X}}}{\mathbf{6}} \left[(\Box\phi)^{\mathbf{3}} - \mathbf{3}(\Box\phi)(\nabla_\mu \nabla_\nu \phi)^{\mathbf{2}} + \mathbf{2}(\nabla_\mu \nabla_\nu \phi)^{\mathbf{3}} \right]$

Following the same procedure as for $\mathcal{L}_4$, we split the variation of the fifth part of the Horndeski Lagrangian into four parts:

$$
\begin{aligned}
\delta S_5 =& \int d^4x \, \delta(\sqrt{-g}) \left\{ G_5 G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{G_{5X}}{6} \left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \right] \right\} \\
& - \int d^4x \sqrt{-g} \, \delta \left(\frac{G_{5X}}{6} \right) \left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \right] \\
& - \int d^4x \sqrt{-g} \, \frac{G_{5X}}{6} \delta \Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \Big] \\
& + \int d^4x \sqrt{-g} \, \delta \left(G_5 G_{\mu\nu} \nabla^\mu \nabla^\nu \phi \right).
\end{aligned}
\tag{B.1.1.11}
$$

The first, and second terms are

$$
\begin{aligned}
& \int d^4x \, \delta(\sqrt{-g}) \left\{ G_5 G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{G_{5X}}{6} \left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \right] \right\} \\
& = - \int d^4x \frac{1}{2} \sqrt{-g} \, g_{\mu\nu} \left\{ G_5 G_{\alpha\beta} \nabla^\alpha \nabla^\beta \phi - \frac{G_{5X}}{6} \left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\alpha \nabla_\beta \phi)^2 + 2(\nabla_\alpha \nabla_\beta \phi)^3 \right] \right\} \delta g^{\mu\nu},
\end{aligned}
$$

and

$$
\begin{aligned}
& - \int d^4x \sqrt{-g} \, \delta \left(\frac{G_{5X}}{6} \right) \Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \Big] \\
& = \int d^4x \sqrt{-g} \, \frac{G_{5XX}}{12} (\nabla_\mu \phi \nabla_\nu \phi) \Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\alpha \nabla_\beta \phi)^2 + 2(\nabla_\alpha \nabla_\beta \phi)^3 \Big] \delta g^{\mu\nu}.
\end{aligned}
\tag{B.1.1.12}
$$

For the third term,

$$\begin{aligned}
\delta S_{5,3} &= \int d^4x\sqrt{-g}\,\frac{G_{5X}}{6}\,\delta\Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\Big] \\
&= \int d^4x\sqrt{-g}\,\frac{G_{5X}}{6}\Big[\delta\big((\Box\phi)^3\big) - 3(\nabla_\mu\nabla_\nu\phi)^2\,\delta(\Box\phi) - 3(\Box\phi)\,\delta\big((\nabla_\mu\nabla_\nu\phi)^2\big) + 2\,\delta\big((\nabla_\mu\nabla_\nu\phi)^3\big)\Big] \\
&= \int d^4x\sqrt{-g}\,G_{5X}\Bigg\{-\frac{1}{2}(\Box\phi)^2\Big(\nabla_\mu\nabla_\nu\phi\,\delta g^{\mu\nu} + \nabla_\mu\phi\,\nabla_\nu\delta g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,\nabla_\lambda\phi\,\nabla^\lambda\delta g^{\mu\nu}\Big) \\
&\quad + \frac{1}{2}(\nabla_\alpha\nabla_\beta\phi)^2\Big(\nabla_\mu\nabla_\nu\phi\,\delta g^{\mu\nu} + \nabla_\nu\phi\,\nabla_\mu\delta g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,\nabla_\lambda\phi\,\nabla^\lambda\delta g^{\mu\nu}\Big) \\
&\quad + (\Box\phi)\Big(\nabla^\alpha\nabla_\nu\phi\,\nabla_\alpha\nabla_\mu\phi\,\delta g^{\mu\nu} + \nabla_\lambda\nabla_\nu\phi\,\nabla_\mu\phi\,\nabla^\lambda\delta g^{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu\phi\,\nabla_\lambda\phi\,\nabla^\lambda\delta g^{\mu\nu}\Big) \\
&\quad - \nabla_\nu\nabla_\kappa\phi\,\nabla_\mu\nabla_\lambda\phi\,\nabla^\lambda\nabla^\kappa\phi\,\delta g^{\mu\nu} - \nabla_\mu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\alpha\phi\,\nabla_\nu\phi\,\nabla^\alpha\delta g^{\mu\nu} \\
&\quad + \frac{1}{2}\nabla_\nu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\mu\phi\,\nabla_\lambda\phi\,\nabla^\lambda\delta g^{\mu\nu}\Bigg\}.
\end{aligned}$$

Where we our earlier result for $\delta\big(\nabla_\mu\nabla_\nu\phi)^2\big)$ in addition with:

$$\begin{aligned}
\delta\big((\nabla_\mu\nabla_\nu\phi)^3\big) &= 3\,\nabla_\nu\nabla_\kappa\phi\,\nabla_\mu\nabla_\lambda\phi\,\nabla^\lambda\nabla^\kappa\phi\,\delta g^{\mu\nu} \\
&\quad + 3\,\nabla_\mu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\alpha\phi\,\nabla_\nu\phi\,\nabla^\alpha\delta g^{\mu\nu} \\
&\quad - \frac{3}{2}\,\nabla_\nu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\mu\phi\,\nabla_\lambda\phi\,\nabla^\lambda\delta g^{\mu\nu},
\end{aligned}$$

and

$$\begin{aligned}
\delta\big((\Box\phi)^3\big) &= 3(\Box\phi)^2\,\nabla_\mu\nabla_\nu\phi\,\delta g^{\mu\nu} \\
&\quad + 3(\Box\phi)^2\,\nabla_\mu\phi\,\nabla_\nu\delta g^{\mu\nu} \\
&\quad - \frac{3}{2}(\Box\phi)^2\,g_{\mu\nu}\,\nabla_\lambda\phi\,\nabla^\lambda\delta g^{\mu\nu}.
\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}
\delta S_{5,3} = \int d^4x\sqrt{-g}\Bigg\{&\frac{1}{2}G_{5X}\,\nabla_\mu\phi\,\nabla_\nu((\Box\phi)^2) - \frac{1}{4}g_{\mu\nu}G_{5X}\,\nabla_\lambda\phi\nabla^\lambda((\Box\phi)^2) \\
&- \frac{1}{4}g_{\mu\nu}\,\nabla^\lambda G_{5X}\,\nabla_\lambda\phi\Big[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\Big] - \frac{1}{4}g_{\mu\nu}G_{5X}(\Box\phi)^3 \\
&+ \frac{1}{2}\nabla_\mu G_{5X}\,\nabla_\nu\phi\Big[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\Big] - \frac{1}{2}G_{5X}\,\nabla_\nu\phi\,\nabla_\mu\big((\nabla_\alpha\nabla_\beta\phi)^2\big) \\
&+ \frac{1}{4}g_{\mu\nu}G_{5X}(\Box\phi)(\nabla_\alpha\nabla_\beta\phi)^2 + \frac{1}{4}g_{\mu\nu}G_{5X}\,\nabla_\lambda\phi\nabla^\lambda\big((\nabla_\alpha\nabla_\beta\phi)^2\big) \\
&- \nabla_\beta G_{5X}\Big[\Box\phi\,\nabla^\beta\nabla_\mu\phi - \nabla^\alpha\nabla^\beta\phi\,\nabla_\alpha\nabla_\mu\phi\Big]\nabla_\nu\phi \\
&- G_{5X}\,\Box(\nabla_\nu\phi)\,\nabla_\mu\phi\,\Box\phi - G_{5X}\,\nabla^\lambda(\Box\phi)\,\nabla_\lambda\nabla_\nu\phi\,\nabla_\mu\phi \\
&+ \frac{1}{2}\nabla^\alpha\phi\,\nabla_\alpha G_{5X}\Big[(\Box\phi)\,\nabla_\mu\nabla_\nu\phi - \nabla_\beta\nabla_\mu\phi\,\nabla^\beta\nabla_\nu\phi\Big] \\
&+ \frac{1}{2}G_{5X}\,\nabla^\lambda(\Box\phi)\,\nabla_\mu\nabla_\nu\phi\,\nabla_\lambda\phi + \frac{1}{2}G_{5X}(\Box\phi)\,\nabla^\lambda\nabla_\mu\nabla_\nu\phi\,\nabla_\lambda\phi \\
&+ \frac{1}{2}(\Box\phi)^2\,\nabla_\mu\nabla_\nu\phi + G_{5X}\,\nabla_\alpha\nabla_\mu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\alpha\phi\,\nabla_\nu\phi \\
&+ G_{5X}\,\nabla_\mu\nabla_\kappa\phi\,\Box(\nabla^\kappa\phi)\,\nabla_\nu\phi - G_{5X}\,\nabla^\lambda\nabla_\nu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\mu\phi\,\nabla_\lambda\phi \\
&- \frac{1}{2}G_{5X}(\Box\phi)\,\nabla_\alpha\nabla_\mu\phi\,\nabla^\alpha\nabla_\nu\phi\Bigg\}\,\delta g^{\mu\nu}.
\end{aligned} \tag{B.1.1.13}$$

Since all of them are multiplied by the symmetric variation $\delta g^{\mu\nu}$, we are free to exchange the indices $\mu \leftrightarrow \nu$ in each expression. Furthermore, because ϕ is a scalar field, the commutator of covariant derivatives vanishes, $[\nabla_\alpha,\nabla_\beta],\phi=0$, allowing the order of any two covariant derivatives acting on ϕ to be interchanged. Using these observations, we can rewrite the following terms.

$$\begin{aligned}
&G_{5X}\,\nabla_\mu\nabla_\kappa\phi\,\Box(\nabla^\kappa\phi)\,\nabla_\nu\phi-G_{5X}\,\nabla^\lambda(\Box\phi)\,\nabla_\lambda\nabla_\nu\phi\,\nabla_\mu\phi\\
&\qquad=G_{5X}\,R_{\alpha\beta}\,\nabla^\alpha\phi\,\nabla^\beta\nabla_\mu\phi\,\nabla_\nu\phi,\\
&\frac{1}{2}G_{5X}\,\nabla_\mu\phi\,\nabla_\nu\left[(\Box\phi)^2\right]=G_{5X}\,\nabla_\mu\phi\,\nabla_\nu(\Box\phi)\,\Box\phi,\\
&-\frac{1}{4}g_{\mu\nu}G_{5X}\,\nabla_\lambda\phi\,\nabla^\lambda\left[(\Box\phi)^2\right]=-\frac{1}{2}g_{\mu\nu}G_{5X}\,\nabla^\lambda\phi\,\Box\phi\,\nabla_\lambda(\Box\phi),\\
&-\frac{1}{2}G_{5X}\,\nabla_\nu\phi\,\nabla_\mu\left[(\nabla_\alpha\nabla_\beta\phi)^2\right]+G_{5X}\,\nabla_\alpha\nabla_\mu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\alpha\phi\,\nabla_\nu\phi\\
&\qquad=-G_{5X}\,\nabla_\nu\phi\,\nabla^\alpha\nabla^\beta\phi\,\nabla_\mu\nabla_\alpha\nabla_\beta\phi+G_{5X}\,\nabla_\alpha\nabla_\mu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\alpha\phi\,\nabla_\nu\phi\\
&\qquad=G_{5X}\,R_{\alpha\lambda\beta\mu}\,\nabla_\nu\phi\,\nabla^\lambda\phi\,\nabla^\alpha\nabla^\beta\phi.
\end{aligned}$$

Substituting into (B.1.1.13) we obtain

$$\begin{aligned}
\delta S_{5,3}=\int d^4x\sqrt{-g}\Big\{&G_{5X}\,\nabla_\mu\phi\,\nabla_\nu(\Box\phi)\Box\phi-\frac{1}{2}g_{\mu\nu}G_{5X}\,\nabla^\lambda\phi\,\Box\phi\,\nabla_\lambda(\Box\phi)\\
&-\frac{1}{4}g_{\mu\nu}\,\nabla^\lambda G_{5X}\,\nabla_\lambda\phi\Big[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\Big]-\frac{1}{4}g_{\mu\nu}G_{5X}(\Box\phi)^3\\
&+\frac{1}{2}\nabla_\mu G_{5X}\,\nabla_\nu\phi\Big[(\Box\phi)^2-(\nabla_\alpha\nabla_\beta\phi)^2\Big]\\
&+G_{5X}R_{\alpha\lambda\beta\mu}\,\nabla_\nu\phi\,\nabla^\lambda\phi\,\nabla^\alpha\nabla^\beta\phi\\
&+\frac{1}{4}g_{\mu\nu}G_{5X}(\Box\phi)(\nabla_\alpha\nabla_\beta\phi)^2+\frac{1}{4}g_{\mu\nu}G_{5X}\,\nabla_\lambda\phi\,\nabla^\lambda\big((\nabla_\alpha\nabla_\beta\phi)^2\big)\\
&-\nabla_\beta G_{5X}\Big[\Box\phi\,\nabla^\beta\nabla_\mu\phi-\nabla^\alpha\nabla^\beta\phi\,\nabla_\alpha\nabla_\mu\phi\Big]\nabla_\nu\phi\\
&-G_{5X}\,\Box(\nabla_\nu\phi)\,\nabla_\mu\phi\,\Box\phi+G_{5X}R_{\alpha\beta}\,\nabla^\alpha\phi\,\nabla^\beta\nabla_\mu\phi\,\nabla_\nu\phi\\
&+\frac{1}{2}\nabla^\alpha\phi\,\nabla_\alpha G_{5X}\Big[(\Box\phi)\,\nabla_\mu\nabla_\nu\phi-\nabla_\beta\nabla_\mu\phi\,\nabla^\beta\nabla_\nu\phi\Big]\\
&+\frac{1}{2}G_{5X}\,\nabla^\lambda(\Box\phi)\,\nabla_\mu\nabla_\nu\phi\,\nabla_\lambda\phi+\frac{1}{2}G_{5X}(\Box\phi)\,\nabla^\lambda\nabla_\mu\nabla_\nu\phi\,\nabla_\lambda\phi\\
&+\frac{1}{2}G_{5X}(\Box\phi)^2\,\nabla_\mu\nabla_\nu\phi-G_{5X}\,\nabla^\lambda\nabla_\nu\nabla_\kappa\phi\,\nabla^\kappa\nabla_\mu\phi\,\nabla_\lambda\phi\\
&-\frac{1}{2}G_{5X}(\Box\phi)\,\nabla_\alpha\nabla_\mu\phi\,\nabla^\alpha\nabla_\nu\phi\Big\}.
\end{aligned}\tag{B.1.1.14}$$

The fourth term is

$$\begin{aligned}
\delta S_{5,4}&=\int d^4x\sqrt{-g}\,G_5\,\delta\big(G_{\mu\nu}\nabla^\mu\nabla^\nu\phi\big)+\int d^4x\sqrt{-g}\,G_{\mu\nu}\nabla^\mu\nabla^\nu\phi\,\delta(G_5)\\
&=\int d^4x\sqrt{-g}\left(-\frac{1}{2}G_{5X}\,\nabla_\mu\phi\nabla_\nu\phi\,G_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi\right)\delta g^{\mu\nu}\\
&\quad+\int d^4x\sqrt{-g}\,G_5\left(\delta R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}\,\delta g^{\alpha\beta}-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\,\delta R_{\alpha\beta}-\frac{1}{2}R\,\delta g_{\mu\nu}\right)\nabla^\mu\nabla^\nu\phi\\
&\quad+\int d^4x\sqrt{-g}\,G_5\,G_{\mu\nu}\,\delta\big(g^{\mu\alpha}g^{\nu\beta}\nabla_\alpha\nabla_\beta\phi\big).
\end{aligned}$$

Using the Palatini identity for the terms that involve the variation of the Ricci tensor we find[1]

$$
\begin{aligned}
\delta S_{5,4} = \int d^4x \sqrt{-g}\, G_5 \Bigg\{ & \frac{1}{2} g^{\alpha\beta} \Big[\nabla_\alpha \nabla_\nu \delta g_{\mu\beta} + \nabla_\alpha \nabla_\mu \delta g_{\nu\beta} - \nabla_\mu \nabla_\nu \delta g_{\alpha\beta} - \nabla_\alpha \nabla_\beta \delta g_{\mu\nu} \Big] \nabla^\mu \nabla^\nu \phi \\
& - \frac{1}{2} g_{\alpha\beta} R_{\mu\nu} \nabla^\alpha \nabla^\beta \phi \, \delta g^{\mu\nu} - \frac{1}{2} R \nabla^\mu \nabla^\nu \phi \, \delta g_{\mu\nu} \\
& - \frac{1}{4} \Box\phi \, g^{\alpha\beta} g^{\rho\sigma} \Big[\nabla_\rho \nabla_\beta \delta g_{\alpha\sigma} + \nabla_\rho \nabla_\alpha \delta g_{\beta\sigma} - \nabla_\alpha \nabla_\beta \delta g_{\rho\sigma} - \nabla_\rho \nabla_\sigma \delta g_{\alpha\beta} \Big] \\
& + G_{\mu\alpha} \nabla_\nu \nabla^\alpha \phi \, \delta g^{\mu\nu} + G_{\beta\nu} \nabla^\beta \nabla_\mu \phi \, \delta g^{\nu\mu} \\
& - \frac{1}{2} G_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} g^{\lambda\sigma} \big(\nabla_\alpha \delta g_{\beta\sigma} + \nabla_\beta \delta g_{\alpha\sigma} - \nabla_\sigma \delta g_{\alpha\beta} \big) \nabla_\lambda \phi \\
& - \frac{1}{2} G_{5X} \nabla_\mu \phi \nabla_\nu \phi \, G_{\alpha\beta} \nabla^\alpha \nabla^\beta \phi \, \delta g^{\mu\nu} \Bigg\}.
\end{aligned}
$$

Simplifying and expressing the equation in terms of the variation of the inverse metric, we obtain:

$$
\begin{aligned}
\delta S_{5,4} = \int d^4x \sqrt{-g}\, G_5 \Bigg\{ & - \nabla_\mu \nabla^\alpha \phi \, \nabla_\nu \nabla_\alpha \delta g^{\mu\nu} + \frac{1}{2} g_{\mu\nu} \nabla^\alpha \nabla^\beta \phi \, \nabla_\alpha \nabla_\beta \delta g^{\mu\nu} \\
& + \frac{1}{2} \nabla_\mu \nabla_\nu \phi \, \Box \delta g^{\mu\nu} - \frac{1}{2} R_{\mu\nu} \Box\phi \, \delta g^{\mu\nu} + \frac{1}{2} R \nabla_\mu \nabla_\nu \phi \, \delta g^{\mu\nu} \\
& + \frac{1}{2} \Box\phi \, \nabla_\mu \nabla_\nu \delta g^{\mu\nu} - \frac{1}{2} \Box\phi \, g_{\mu\nu} \Box \delta g^{\mu\nu} \\
& + 2 G_{\mu\alpha} \nabla_\nu \nabla^\alpha \phi \, \delta g^{\mu\nu} + G_{\sigma\nu} \nabla_\mu \phi \, \nabla^\sigma \delta g^{\mu\nu} - \frac{1}{2} G_{\mu\nu} \nabla_\sigma \phi \, \nabla^\sigma \delta g^{\mu\nu} \\
& - \frac{1}{2} G_{5X} \nabla_\mu \phi \nabla_\nu \phi \, G_{\alpha\beta} \nabla^\alpha \nabla^\beta \phi \, \delta g^{\mu\nu} \Bigg\}.
\end{aligned}
$$

Integrating by parts and discarding the boundary terms yields

$$
\begin{aligned}
\delta S_{5,4} = \int d^4x \sqrt{-g} \Bigg\{ & \nabla_\alpha \nabla_\nu \big(G_5 \nabla_\mu \nabla^\alpha \phi \big) + \frac{1}{2} g_{\mu\nu} \nabla_\beta \nabla_\alpha \Big(G_5 \nabla^\alpha \nabla^\beta \phi \Big) \\
& + \frac{1}{2} \Box \big(G_5 \nabla_\mu \nabla_\nu \phi \big) - \frac{1}{2} G_5 R_{\mu\nu} \Box\phi + \frac{1}{2} G_5 R \nabla_\mu \nabla_\nu \phi \\
& + \frac{1}{2} \nabla_\nu \nabla_\mu (G_5 \Box\phi) - \frac{1}{2} g_{\mu\nu} \Box (G_5 \Box\phi) + 2 G_{\mu\alpha} \nabla_\nu \nabla^\alpha \phi \, G_5 \\
& - \nabla^\sigma \big(\nabla_\mu \phi \, G_5 \big) G_{\sigma\nu} + \frac{1}{2} \nabla^\sigma \big(G_5 G_{\mu\nu} \nabla_\sigma \phi \big) \\
& - \frac{1}{2} G_{5X} \nabla_\mu \phi \nabla_\nu \phi \, G_{\alpha\beta} \nabla^\alpha \nabla^\beta \phi \Bigg\} \delta g^{\mu\nu}
\end{aligned}
\tag{B.1.1.15}
$$

In this part of the variation, fourth-order covariant derivatives of the scalar field naturally arise. Although their appearance is anticipated, such terms require careful treatment. The standard procedure involves employing commutators of covariant derivatives to express these contributions in terms of the Riemann and Ricci tensors, together with their covariant derivatives. The identities satisfied by these tensors then enable the resulting terms to be simplified. Below, we show how the terms from (B.1.1.15) are handled. The specific terms we are referring to are as follows.

[1] For $\delta(\nabla^\mu \nabla^\nu \phi)$:

$$
\begin{aligned}
\delta(\nabla^\mu \nabla^\nu \phi) = \delta(g^{\mu\alpha} g^{\nu\beta} \nabla_\alpha \nabla_\beta) = {} & \nabla_\alpha \nabla^\nu \phi \, \delta g^{\mu\alpha} + \nabla_\beta \nabla^\mu \phi \, \delta g^{\nu\beta} \\
& - \frac{1}{2} g^{\mu\alpha} g^{\nu\beta} g^{\lambda\sigma} \big(\nabla_\alpha \delta g_{\beta\sigma} + \nabla_\beta \delta g_{\alpha\sigma} - \nabla_\sigma \delta g_{\alpha\beta} \big) \nabla_\lambda \phi.
\end{aligned}
$$

$$
\begin{aligned}
-\nabla_\alpha\nabla_\nu\big(G_5\nabla_\mu\nabla^\alpha\phi\big) &= -\nabla_\alpha\nabla_\nu G_5\,\nabla_\mu\nabla^\alpha\phi-\nabla_\nu G_5\,\Box(\nabla_\mu\phi)-\nabla_\alpha G_5\,\nabla_\nu\nabla_\mu\nabla^\alpha\phi-G_5\,\nabla_\alpha\nabla_\nu\nabla_\mu\nabla^\alpha\phi,\\
\frac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha\big(G_5\nabla^\alpha\nabla^\beta\phi\big) &= \frac{1}{2}g_{\mu\nu}\Big(\nabla_\beta\nabla_\alpha G_5\,\nabla^\alpha\nabla^\beta\phi+G_5\,\nabla_\beta\Box(\nabla^\beta\phi)+2\nabla^\alpha G_5\,\Box(\nabla^\alpha\phi)\Big),\\
\frac{1}{2}\Box\big(G_5\nabla_\mu\nabla_\nu\phi\big) &= \Big(2\nabla^\alpha G_5\,\nabla_\alpha\nabla_\mu\nabla_\nu\phi+\frac{1}{2}\Box G_5\,\nabla_\mu\nabla_\nu\phi+G_5\,\Box(\nabla_\mu\nabla_\nu\phi)\Big),\\
\frac{1}{2}\nabla_\nu\nabla_\mu\big(G_5\Box\phi\big) &= \frac{1}{2}\nabla_\nu\nabla_\mu G_5\,\Box\phi+\frac{1}{2}G_5\,\nabla_\nu\nabla_\mu(\Box\phi)+\nabla_\mu G_5\,\nabla_\nu(\Box\phi),\\
-\frac{1}{2}g_{\mu\nu}\Box\big(G_5\Box\phi\big) &= -\frac{1}{2}g_{\mu\nu}\Big(\Box G_5\,\Box\phi+2\nabla^\alpha G_5\,\nabla_\alpha(\Box\phi)+G_5\,\Box(\Box\phi)\Big).
\end{aligned}
\tag{B.1.1.16}
$$

From these, we simplify the following terms and then substitute the combined results back in (B.1.1.15)

$$
\begin{aligned}
T_1 &= -G_5\nabla_\alpha\nabla_\nu\nabla_\mu\nabla^\alpha\phi+\frac{1}{2}G_5\Box(\nabla_\mu\nabla_\nu\phi)+\frac{1}{2}G_5\nabla_\mu\nabla_\nu(\Box\phi),\\
T_2 &= \frac{1}{2}g_{\mu\nu}G_5\left(\nabla_\beta\Box(\nabla^\beta\phi)-\Box(\Box\phi)\right).
\end{aligned}
$$

The first one can be written as

$$
\begin{aligned}
T_1 &= G_5\left(-\nabla_\alpha\nabla_\mu\nabla_\nu\nabla^\alpha\phi+\frac{1}{2}\nabla^\alpha\nabla_\alpha\nabla_\mu\nabla_\nu\phi+\frac{1}{2}\nabla_\mu\nabla_\nu\nabla^\alpha\nabla_\alpha\phi\right)\\
&= G_5\left(-\nabla_\alpha\nabla_\mu\nabla_\nu\nabla^\alpha\phi+\frac{1}{2}\nabla_\alpha\nabla_\mu\nabla^\alpha\nabla_\nu\phi+\frac{1}{2}\nabla^\alpha\big([\nabla_\alpha,\nabla_\mu]\nabla_\nu\phi\big)+\frac{1}{2}\nabla_\mu\nabla_\nu\nabla^\alpha\nabla_\alpha\phi\right)\\
&= \frac{G_5}{2}\left(-\nabla_\alpha\nabla_\mu\nabla_\nu\nabla^\alpha\phi-\nabla^\alpha\left(R_{\sigma\nu\alpha\mu}\nabla^\sigma\phi\right)+\nabla_\mu\nabla_\nu\nabla^\alpha\nabla_\alpha\phi\right)\\
&= \frac{G_5}{2}\left(-\nabla_\mu\nabla_\alpha\nabla_\nu\nabla^\alpha\phi-[\nabla_\alpha,\nabla_\mu]\big(\nabla_\nu\nabla^\alpha\phi\big)-\nabla^\alpha\left(R_{\sigma\nu\alpha\mu}\nabla^\sigma\phi\right)+\nabla_\mu\nabla_\nu\nabla^\alpha\nabla_\alpha\phi\right)\\
&= \frac{G_5}{2}\left(-[\nabla_\alpha,\nabla_\mu]\big(\nabla_\nu\nabla^\alpha\phi\big)-\nabla^\alpha\left(R_{\sigma\nu\alpha\mu}\nabla^\sigma\phi\right)+\nabla_\mu\left([\nabla_\nu,\nabla_\alpha]\nabla^\alpha\phi\right)\right)\\
&= \frac{G_5}{2}\left(R_{\sigma\nu\alpha\mu}\nabla^\sigma\nabla^\alpha\phi-R_{\sigma\mu}\nabla_\nu\nabla^\sigma\phi-\nabla^\alpha\left(R_{\sigma\nu\alpha\mu}\nabla^\sigma\phi\right)-\nabla_\mu\left(R_{\sigma\nu}\nabla^\sigma\phi\right)\right).
\end{aligned}
\tag{B.1.1.17}
$$

As for the second, we have

$$
\begin{aligned}
T_2 &= \frac{1}{2}g_{\mu\nu}G_5\Big(\nabla_\beta\nabla_\alpha\nabla^\alpha\nabla^\beta\phi-\nabla^\alpha\nabla_\alpha\nabla^\beta\nabla_\beta\phi\Big)=\frac{1}{2}g_{\mu\nu}G_5\Big(-\nabla^\alpha\big([\nabla_\alpha,\nabla_\beta]\nabla^\beta\phi\big)\Big)\\
&= -\frac{1}{2}g_{\mu\nu}G_5\nabla^\alpha\big(R^\beta{}_{\sigma\alpha\beta}\nabla^\sigma\phi\big)=\frac{1}{2}g_{\mu\nu}G_5\nabla^\alpha\big(R_{\sigma\alpha}\nabla^\sigma\phi\big)\\
&= \frac{1}{2}g_{\mu\nu}G_5\Big(R_{\sigma\alpha}\nabla^\alpha\nabla^\sigma\phi+\nabla^\sigma\phi\,\nabla^\alpha R_{\sigma\alpha}\Big).
\end{aligned}
\tag{B.1.1.18}
$$

Where in the first line we exchanged $\alpha\leftrightarrow\beta$ after commuting the covariant derivatives.

Substituting (B.1.1.16), (B.1.1.17), (B.1.1.18), into (B.1.1.15), and simplifying we obtain

$$\begin{aligned}\delta S_{5,4} = \int d^4x\sqrt{-g}\Big\{ &-\nabla_\alpha\nabla_\nu G_5\,\nabla_\mu\nabla^\alpha\phi - \nabla_\mu G_5\,G_{\nu\lambda}\nabla^\lambda\phi + \nabla^\alpha G_5\,R_{\alpha\mu\nu\beta}\nabla^\beta\phi \\ &+\frac{1}{2}g_{\mu\nu}\,\nabla_\beta\nabla_\alpha G_5\,\nabla^\alpha\nabla^\beta\phi + g_{\mu\nu}\,\nabla^\alpha G_5\,R_{\alpha\beta}\nabla^\beta\phi + \frac{1}{2}\Box G_5\,\nabla_\mu\nabla_\nu\phi \\ &+\frac{1}{2}\nabla_\nu\nabla_\mu G_5\,\Box\phi - \frac{1}{2}g_{\mu\nu}\,\Box G_5\,\Box\phi - \nabla^\lambda G_5\,R_{\lambda\mu}\nabla_\nu\phi \\ &+\frac{1}{2}\nabla_\lambda G_5\,G_{\mu\nu}\nabla^\lambda\phi - \frac{1}{4}G_5\,g_{\mu\nu}\,R\,\Box\phi \\ &+\frac{1}{2}g_{\mu\nu}\,G_5\,R_{\sigma\alpha}\nabla^\alpha\nabla^\sigma\phi\Big\}\delta g^{\mu\nu}. \end{aligned} \tag{B.1.1.19}$$

During this calculation, we used the contracted Bianchi identity for the covariant derivative of the Riemann tensor appearing in T_1 and T_2:

$$\nabla^\alpha R_{\sigma\nu\alpha\mu} = \nabla_\sigma R_{\nu\mu} - \nabla_\nu R_{\sigma\mu}.$$

Note that the last two terms of (B.1.1.19) cancel with the contribution arising from the variation of the first term in (B.1.1.12), $-\frac{1}{2}g_{\mu\nu}G_5G_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi$ as they can be equivalently expressed as

$$\frac{1}{2}g_{\mu\nu}G_5R_{\sigma\alpha}\nabla^\alpha\nabla^\sigma\phi - \frac{1}{4}G_5g_{\mu\nu}R\Box\phi = \frac{1}{2}g_{\mu\nu}G_5\nabla^\alpha\nabla^\sigma\phi\left(R_{\sigma\alpha} - \frac{1}{2}g_{\sigma\alpha}R\right) = \frac{1}{2}g_{\mu\nu}G_5G_{\alpha\sigma}\nabla^\alpha\nabla^\sigma\phi$$

In fact, similar simplifications arise in the derivation of (B.1.1.18). We can re-write some of the terms of (B.1.1.19) as follows

$$\begin{aligned} -\frac{1}{2}g_{\mu\nu}\Box G_5\,\Box\phi &= -g_{\mu\nu}\left[\frac{1}{2}\nabla_\alpha\big(G_{5\phi}\nabla^\alpha\phi\big)\,\Box\phi + \frac{1}{2}\nabla_\alpha G_{5X}\,\nabla^\alpha X\,\Box\phi + \frac{1}{2}G_{5X}\,\Box\phi\,\Box X\right] \\ &= g_{\mu\nu}\Big[-\frac{1}{2}\nabla_\alpha\big(G_{5\phi}\nabla^\alpha\phi\big)\,\Box\phi - \frac{1}{2}\nabla_\alpha G_{5X}\,\nabla^\alpha X\,\Box\phi + \frac{1}{2}G_{5X}\,\Box\phi\,(\nabla_\alpha\nabla_\beta\phi)^2 \\ &\qquad + \frac{1}{2}G_{5X}\,\Box\phi\,\Box(\nabla^\beta\phi)\,\nabla_\beta\phi\Big], \\ \frac{1}{2}\nabla_\nu\nabla_\mu G_5\,\Box\phi &= \frac{1}{2}\nabla_\mu\big(G_{5\phi}\nabla_\nu\phi\big)\Box\phi + \frac{1}{2}\nabla_\mu\big(G_{5X}\nabla_\nu X\big)\Box\phi. \end{aligned} \tag{B.1.1.20}$$

What remains at this point is to combine the results above and simplify the final expression.

From (B.1.1.12), (B.1.1.14), (B.1.1.19), (B.1.1.20) we have

$$\begin{aligned}
\delta S_5 = \int d^4x\sqrt{-g}\Big\{ & G_{5X}\nabla_\mu\phi\nabla_\nu(\Box\phi)\Box\phi - \frac{1}{2}g_{\mu\nu}G_{5X}\nabla^\lambda\phi\Box\phi\nabla_\lambda(\Box\phi) \\
& - \frac{1}{4}g_{\mu\nu}\nabla^\lambda G_{5X}\nabla_\lambda\phi\Big[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\Big] - \frac{1}{4}g_{\mu\nu}G_{5X}(\Box\phi)^3 \\
& + \frac{1}{2}\nabla_\mu G_{5X}\nabla_\nu\phi\Big[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\Big] + G_{5X}R_{\alpha\lambda\beta\mu}\nabla_\nu\phi\nabla^\lambda\phi\nabla^\alpha\nabla^\beta\phi \\
& + \frac{1}{4}g_{\mu\nu}G_{5X}(\Box\phi)(\nabla_\alpha\nabla_\beta\phi)^2 + \frac{1}{4}g_{\mu\nu}G_{5X}\nabla_\lambda\phi\nabla^\lambda((\nabla_\alpha\nabla_\beta\phi)^2) \\
& - \nabla_\beta G_{5X}\Big[\Box\phi\nabla^\beta\nabla_\mu\phi - \nabla^\alpha\nabla^\beta\phi\nabla_\alpha\nabla_\mu\phi\Big]\nabla_\nu\phi \\
& - G_{5X}\Box(\nabla_\nu\phi)\nabla_\mu\phi\Box\phi + G_{5X}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\nabla_\mu\phi\nabla_\nu\phi \\
& + \frac{1}{2}\nabla^\alpha\phi\nabla_\alpha G_{5X}\Big[(\Box\phi)\nabla_\mu\nabla_\nu\phi - \nabla_\beta\nabla_\mu\phi\nabla^\beta\nabla_\nu\phi\Big] \\
& + \frac{1}{2}G_{5X}\nabla^\lambda(\Box\phi)\nabla_\mu\nabla_\nu\phi\nabla_\lambda\phi + \frac{1}{2}G_{5X}(\Box\phi)\nabla^\lambda\nabla_\mu\nabla_\nu\phi\nabla_\lambda\phi \\
& + \frac{1}{2}G_{5X}(\Box\phi)^2\nabla_\mu\nabla_\nu\phi - G_{5X}\nabla^\lambda\nabla_\nu\nabla_\kappa\phi\nabla^\kappa\nabla_\mu\phi\nabla_\lambda\phi \\
& - \frac{1}{2}G_{5X}(\Box\phi)\nabla_\alpha\nabla_\mu\phi\nabla^\alpha\nabla_\nu\phi - \nabla_\alpha\nabla_\nu G_5\nabla_\mu\nabla^\alpha\phi - \nabla_\mu G_5 G_{\nu\lambda}\nabla^\lambda\phi \\
& + \nabla^\alpha G_5 R_{\alpha\mu\nu\beta}\nabla^\beta\phi + \frac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha G_5\nabla^\alpha\nabla^\beta\phi + g_{\mu\nu}\nabla^\alpha G_5 R_{\alpha\beta}\nabla^\beta\phi \\
& + \frac{1}{2}\Box G_5\nabla_\mu\nabla_\nu\phi + \frac{1}{2}\nabla_\mu(G_{5\phi}\nabla_\nu\phi)\Box\phi + \frac{1}{2}\nabla_\mu(G_{5X}\nabla_\nu X)\Box\phi \\
& + g_{\mu\nu}\Big[-\frac{1}{2}\nabla_\alpha(G_{5\phi}\nabla^\alpha\phi)\Box\phi - \frac{1}{2}\nabla_\alpha G_{5X}\nabla^\alpha X\Box\phi + \frac{1}{2}G_{5X}\Box\phi(\nabla_\alpha\nabla_\beta\phi)^2 - \nabla^\lambda G_5 R_{\lambda\mu}\nabla_\nu\phi\Big] \\
& + \frac{1}{2}\nabla_\lambda G_5 G_{\mu\nu}\nabla^\lambda\phi - \frac{1}{2}G_{5X}\nabla_\mu\phi\nabla_\nu\phi G_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi \\
& + \frac{G_{5XX}}{12}\nabla_\mu\phi\nabla_\nu\phi\Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\alpha\nabla_\beta\phi)^2 + 2(\nabla_\alpha\nabla_\beta\phi)^3\Big] \\
& + g_{\mu\nu}\frac{G_{5X}}{12}\Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\alpha\nabla_\beta\phi)^2 + 2(\nabla_\alpha\nabla_\beta\phi)^3\Big]\Big\}.
\end{aligned} \tag{B.1.1.21}$$

Reformulating the following terms

$$\frac{1}{2}g_{\mu\nu}G_{5X}\Box\phi\nabla^\beta\phi\Big(\Box(\nabla_\beta\phi) - \nabla_\beta(\Box\phi)\Big) = \frac{1}{2}g_{\mu\nu}G_{5X}\Box\phi\nabla^\beta\phi R_{\alpha\beta}\nabla^\alpha\phi,$$

$$G_{5X}\Box\phi\nabla_\mu\phi\Big(\nabla_\nu(\Box\phi) - \Box(\nabla_\nu\phi)\Big) = -G_{5X}\Box\phi\nabla_\mu\phi R_{\nu\alpha}\nabla^\alpha\phi = -G_{5X}\Box\phi\nabla_\nu\phi R_{\mu\alpha}\nabla^\alpha\phi,$$

$$-\nabla_\alpha\nabla_\nu G_5\nabla_\mu\nabla^\alpha\phi = -\nabla_\lambda\Big[G_{5\phi}\nabla_\mu\phi\Big]\nabla_\nu\nabla^\lambda\phi - \nabla_\lambda\Big[G_{5X}\nabla_\mu X\Big]\nabla_\nu\nabla^\lambda\phi,$$

$$\frac{1}{4}g_{\mu\nu}G_{5X}\nabla_\lambda\phi\nabla^\lambda\Big((\nabla_\alpha\nabla_\beta\phi)^2\Big) = \frac{1}{2}g_{\mu\nu}G_{5X}\nabla^\lambda\phi\nabla^\alpha\nabla^\beta\phi\nabla_\lambda\nabla_\alpha\nabla_\beta\phi.$$

$$\begin{aligned}
\frac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha G_5\nabla^\alpha\nabla^\beta\phi &= \frac{1}{2}g_{\mu\nu}\nabla_\alpha G_{5X}\nabla^\beta\nabla^\alpha\phi\nabla_\beta X + \frac{1}{2}g_{\mu\nu}G_{5X}\nabla_\beta\nabla_\alpha X\nabla^\alpha\nabla^\beta\phi \\
&\quad + \frac{1}{2}g_{\mu\nu}\nabla_\alpha\left(G_{5\phi}\nabla_\beta\phi\right)\nabla^\alpha\nabla^\beta\phi \\
&= \frac{1}{2}g_{\mu\nu}\nabla_\alpha G_{5X}\nabla^\beta\nabla^\alpha\phi\nabla_\beta X + \frac{1}{2}g_{\mu\nu}\nabla_\alpha\left(G_{5\phi}\nabla_\beta\phi\right)\nabla^\alpha\nabla^\beta\phi \\
&\quad - \frac{1}{2}g_{\mu\nu}G_{5X}(\nabla_\alpha\nabla_\beta\phi)^3 - \frac{1}{2}g_{\mu\nu}G_{5X}\nabla^\lambda\phi\nabla_\beta\nabla_\alpha\nabla_\lambda\phi\nabla^\alpha\nabla^\beta\phi.
\end{aligned}$$

And combining some of the terms of (B.1.1.21) yields

$$
\begin{aligned}
&\frac{1}{2}\Box G_5 \nabla_\mu\nabla_\nu\phi + \frac{1}{2}G_{5X}\nabla^\lambda(\Box\phi)\nabla_\mu\nabla_\nu\phi\nabla_\lambda\phi = \\
&\frac{1}{2}\nabla_\lambda\left(G_{5\phi}\nabla^\lambda\phi\right)\nabla_\mu\nabla_\nu\phi - \frac{1}{2}\nabla_\alpha\left(G_{5X}\nabla_\beta\phi\right)\nabla^\alpha\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi \\
&\quad + \frac{1}{2}G_{5X}\nabla_\mu\nabla_\nu\phi\nabla^\lambda\phi\left(\nabla_\lambda(\Box\phi) - \Box(\nabla_\lambda\phi)\right) \\
&= -\frac{1}{2}G_{5X}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi + \frac{1}{2}\nabla_\lambda\left(G_{5\phi}\nabla^\lambda\phi\right)\nabla_\mu\nabla_\nu\phi \\
&\quad - \frac{1}{2}\nabla_\alpha\left(G_{5X}\nabla_\beta\phi\right)\nabla^\alpha\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\nabla_\mu(G_{5X}\nabla_\nu X)\Box\phi + \frac{1}{2}G_{5X}(\Box\phi)\nabla_\lambda\nabla_\mu\nabla_\nu\phi\nabla^\lambda\phi = -\frac{1}{2}\nabla_\mu\Big[G_{5X}\nabla^\alpha\phi\Big]\nabla_\alpha\nabla_\nu\phi\Box\phi \\
&\quad - \frac{1}{2}G_{5X}\nabla^\alpha\phi\nabla_\mu\nabla_\nu\nabla_\alpha\phi\Box\phi + \frac{1}{2}G_{5X}(\Box\phi)\nabla_\lambda\nabla_\mu\nabla_\nu\phi\nabla^\lambda\phi = \\
&-\frac{1}{2}\nabla_\mu\Big[G_{5X}\nabla^\alpha\phi\Big]\nabla_\alpha\nabla_\nu\phi\Box\phi - \frac{1}{2}G_{5X}R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi\Box\phi.
\end{aligned}
$$

Substituting in we obtain

$$
\begin{aligned}
\delta S_5 = \int d^4x\sqrt{-g}\Bigg\{ &-G_{5X}\Box\phi\,\nabla_\nu\phi\,R_{\mu\alpha}\nabla^\alpha\phi + \frac{1}{2}g_{\mu\nu}G_{5X}\Box\phi\,\nabla^\beta\phi\,R_{\alpha\beta}\nabla^\alpha\phi \\
&-\frac{1}{4}g_{\mu\nu}\nabla^\lambda G_{5X}\nabla_\lambda\phi\Big[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\Big] + \frac{1}{2}\nabla_\mu G_{5X}\nabla_\nu\phi\Big[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\Big] \\
&+G_{5X}R_{\alpha\lambda\beta\mu}\nabla_\nu\phi\nabla^\lambda\phi\nabla^\alpha\nabla^\beta\phi + \frac{1}{2}g_{\mu\nu}G_{5X}\nabla^\lambda\phi\nabla^\alpha\nabla^\beta\phi\nabla_\lambda\nabla_\alpha\nabla_\beta\phi \\
&-\nabla_\beta G_{5X}\Big[\Box\phi\nabla^\beta\nabla_\mu\phi - \nabla^\alpha\nabla^\beta\phi\nabla_\alpha\nabla_\mu\phi\Big]\nabla_\nu\phi + G_{5X}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\nabla_\mu\phi\nabla_\nu\phi \\
&+\frac{1}{2}\nabla^\alpha\phi\nabla_\alpha G_{5X}\Big[\Box\phi\nabla_\mu\nabla_\nu\phi - \nabla_\beta\nabla_\mu\phi\nabla^\beta\nabla_\nu\phi\Big] + \frac{1}{2}\nabla_\lambda\left(G_{5\phi}\nabla^\lambda\phi\right)\nabla_\mu\nabla_\nu\phi \\
&-\frac{1}{2}\nabla_\alpha\left(G_{5X}\nabla_\beta\phi\right)\nabla^\alpha\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi - \frac{1}{2}G_{5X}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi \\
&-\frac{1}{2}\nabla_\mu\left(G_{5X}\nabla^\alpha\phi\right)\nabla_\alpha\nabla_\nu\phi\,\Box\phi - \frac{1}{2}G_{5X}R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi\,\Box\phi + \frac{1}{2}G_{5X}(\Box\phi)^2\nabla_\mu\nabla_\nu\phi \\
&-G_{5X}\nabla^\lambda\nabla_\nu\nabla_\kappa\phi\nabla^\kappa\nabla_\mu\phi\nabla_\lambda\phi - \frac{1}{2}G_{5X}(\Box\phi)\,\nabla_\alpha\nabla_\mu\phi\nabla^\alpha\nabla_\nu\phi \\
&-\frac{1}{6}g_{\mu\nu}G_{5X}\Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\alpha\nabla_\beta\phi)^2 + 2(\nabla_\alpha\nabla_\beta\phi)^3\Big] \\
&-\frac{1}{2}G_{5X}\nabla_\mu\phi\nabla_\nu\phi\,G_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi + \frac{1}{12}G_{5XX}\Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\Big]\nabla_\mu\phi\nabla_\nu\phi \\
&-\nabla_\lambda\left(G_{5\phi}\nabla_\mu\phi\right)\nabla_\nu\nabla^\lambda\phi - \nabla_\lambda\left(G_{5X}\nabla_\mu X\right)\nabla_\nu\nabla^\lambda\phi - \nabla_\mu G_5\,G_{\nu\lambda}\nabla^\lambda\phi \\
&+\nabla^\alpha G_5 R_{\alpha\mu\nu\beta}\nabla^\beta\phi + \frac{1}{2}g_{\mu\nu}\nabla_\alpha G_{5X}\nabla_\beta X\nabla^\beta\nabla^\alpha\phi + \frac{1}{2}g_{\mu\nu}\nabla_\alpha\left(G_{5\phi}\nabla_\beta\phi\right)\nabla^\alpha\nabla^\beta\phi \\
&-\frac{1}{2}g_{\mu\nu}G_{5X}\nabla^\lambda\phi\nabla_\beta\nabla_\alpha\nabla_\lambda\phi\nabla^\alpha\nabla^\beta\phi + g_{\mu\nu}\nabla^\alpha G_5 R_{\alpha\beta}\nabla^\beta\phi \\
&-g_{\mu\nu}\Big[\frac{1}{2}\nabla_\alpha\left(G_{5\phi}\nabla^\alpha\phi\right)\Box\phi + \frac{1}{2}\nabla_\alpha G_{5X}\nabla^\alpha X\Box\phi\Big] + \frac{1}{2}\nabla_\mu\left(G_{5\phi}\nabla_\nu\phi\right)\Box\phi \\
&-g_{\mu\nu}\nabla^\lambda G_5 R_{\lambda\mu}\nabla_\nu\phi + \frac{1}{2}\nabla_\lambda G_5 G_{\mu\nu}\nabla^\lambda\phi\Bigg\}\,\delta g^{\mu\nu}.
\end{aligned}
\tag{B.1.1.22}
$$

Finally, we combine and reformulate the following terms:

$$
\begin{aligned}
&+\frac{1}{2}g_{\mu\nu}G_{5X}\nabla^{\lambda}\phi\,\nabla^{\alpha}\nabla^{\beta}\phi\,\nabla_{\lambda}\nabla_{\alpha}\nabla_{\beta}\phi-\frac{1}{2}g_{\mu\nu}G_{5X}\nabla^{\lambda}\phi\,\nabla^{\alpha}\nabla^{\beta}\phi\,\nabla_{\beta}\nabla_{\alpha}\nabla_{\lambda}\phi\\
&=-\frac{1}{2}g_{\mu\nu}G_{5X}\nabla^{\lambda}\phi\,\nabla^{\alpha}\nabla^{\beta}\phi\,R_{\alpha\lambda\beta\rho}\nabla^{\rho}\phi,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\nabla_{\lambda}\big[G_{5X}\nabla_{\mu}X\big]\nabla_{\nu}\nabla^{\lambda}\phi-G_{5X}\nabla^{\lambda}\nabla_{\nu}\nabla_{\kappa}\phi\,\nabla^{\kappa}\nabla_{\mu}\phi\nabla_{\lambda}\phi\\
&=\nabla_{\alpha}\big[G_{5X}\nabla_{\beta}\phi\big]\nabla^{\alpha}\nabla_{\mu}\phi\,\nabla^{\beta}\nabla_{\nu}\phi+G_{5X}\nabla^{\beta}\phi\,\nabla^{\alpha}\nabla_{\nu}\phi\,(\nabla_{\alpha}\nabla_{\beta}-\nabla_{\beta}\nabla_{\alpha})\nabla_{\mu}\phi\\
&=\nabla_{\alpha}\big[G_{5X}\nabla_{\beta}\phi\big]\nabla^{\alpha}\nabla_{\mu}\phi\,\nabla^{\beta}\nabla_{\nu}\phi+G_{5X}\nabla^{\beta}\phi\,\nabla^{\alpha}\phi\,\nabla_{\nu}\nabla^{\lambda}\phi\,R_{\alpha\lambda\beta\mu}.
\end{aligned}
$$

Substituting into (B.1.1.22) and imposing $\delta S=0$ yields[2]:

$$
\begin{aligned}
G^{(5)}_{\mu\nu}=&-G_{5X}\Box\phi\,\nabla_{(\nu}\phi\,R_{\mu)\alpha}\nabla^{\alpha}\phi+\frac{1}{2}g_{\mu\nu}G_{5X}\Box\phi\,\nabla^{\beta}\phi\,R_{\alpha\beta}\nabla^{\alpha}\phi\\
&-\frac{1}{4}g_{\mu\nu}\nabla^{\lambda}G_{5X}\nabla_{\lambda}\phi\Big[(\Box\phi)^2-(\nabla_{\alpha}\nabla_{\beta}\phi)^2\Big]+\frac{1}{2}\nabla_{(\mu}G_{5X}\nabla_{\nu)}\phi\Big[(\Box\phi)^2-(\nabla_{\alpha}\nabla_{\beta}\phi)^2\Big]\\
&+G_{5X}R_{\alpha\lambda\beta(\mu}\nabla_{\nu)}\phi\nabla^{\lambda}\phi\nabla^{\alpha}\nabla^{\beta}\phi-\nabla_{\beta}G_{5X}\Big[\Box\phi\nabla^{\beta}\nabla_{\mu}\phi-\nabla^{\alpha}\nabla^{\beta}\phi\nabla_{\alpha}\nabla_{(\mu}\phi\Big]\nabla_{\nu)}\phi\\
&+G_{5X}R_{\alpha\beta}\nabla^{\alpha}\phi\nabla^{\beta}\nabla_{\mu}\phi\nabla_{\nu}\phi+\frac{1}{2}\nabla^{\alpha}\phi\nabla_{\alpha}G_{5X}\Big[\Box\phi\nabla_{\mu}\nabla_{\nu}\phi-\nabla_{\beta}\nabla_{\mu}\phi\nabla^{\beta}\nabla_{\nu}\phi\Big]\\
&+\frac{1}{2}\nabla_{\lambda}\left(G_{5\phi}\nabla^{\lambda}\phi\right)\nabla_{\mu}\nabla_{\nu}\phi-\frac{1}{2}\nabla_{\alpha}\left(G_{5X}\nabla_{\beta}\phi\right)\nabla^{\alpha}\nabla^{\beta}\phi\nabla_{\mu}\nabla_{\nu}\phi-\frac{1}{2}G_{5X}R_{\alpha\beta}\nabla^{\alpha}\phi\nabla^{\beta}\phi\nabla_{\mu}\nabla_{\nu}\phi\\
&-\frac{1}{2}\nabla_{(\mu}\big(G_{5X}\nabla^{\alpha}\phi\big)\nabla_{\alpha}\nabla_{\nu)}\phi\Box\phi-\frac{1}{2}G_{5X}R_{\mu\alpha\nu\beta}\nabla^{\alpha}\phi\nabla^{\beta}\phi\Box\phi+\frac{1}{2}G_{5X}(\Box\phi)^2\nabla_{\mu}\nabla_{\nu}\phi\\
&-\frac{1}{2}G_{5X}(\Box\phi)\nabla_{\alpha}\nabla_{\mu}\phi\nabla^{\alpha}\nabla_{\nu}\phi-\frac{1}{6}g_{\mu\nu}G_{5X}\Big[(\Box\phi)^3-3(\Box\phi)(\nabla_{\alpha}\nabla_{\beta}\phi)^2+2(\nabla_{\alpha}\nabla_{\beta}\phi)^3\Big]\\
&-\frac{1}{2}G_{5X}\nabla_{\mu}\phi\nabla_{\nu}\phi\,G_{\alpha\beta}\nabla^{\alpha}\nabla^{\beta}\phi+\frac{1}{12}G_{5XX}\Big[(\Box\phi)^3-3(\Box\phi)(\nabla_{\mu}\nabla_{\nu}\phi)^2+2(\nabla_{\mu}\nabla_{\nu}\phi)^3\Big]\nabla_{\mu}\phi\nabla_{\nu}\phi\\
&-\nabla_{\lambda}\left(G_{5\phi}\nabla_{(\mu}\phi\right)\nabla_{\nu)}\nabla^{\lambda}\phi+\nabla_{\alpha}\big[G_{5X}\nabla_{\beta}\phi\big]\nabla^{\alpha}\nabla_{(\mu}\phi\,\nabla^{\beta}\nabla_{\nu)}\phi\\
&+G_{5X}\nabla^{\beta}\phi\,\nabla^{\alpha}\phi\,\nabla_{(\nu}\nabla^{\lambda}\phi\,R_{\alpha\lambda\beta\mu)}-\nabla_{(\mu}G_5\,G_{\nu)\lambda}\nabla^{\lambda}\phi+\nabla^{\alpha}G_5R_{\alpha(\mu\nu)\beta}\nabla^{\beta}\phi\\
&+\frac{1}{2}g_{\mu\nu}\nabla_{\alpha}G_{5X}\nabla_{\beta}X\nabla^{\beta}\nabla^{\alpha}\phi+\frac{1}{2}g_{\mu\nu}\nabla_{\alpha}\big(G_{5\phi}\nabla_{\beta}\phi\big)\nabla^{\alpha}\nabla^{\beta}\phi-\frac{1}{2}g_{\mu\nu}G_{5X}\nabla^{\lambda}\phi\,\nabla^{\alpha}\nabla^{\beta}\phi\,R_{\alpha\lambda\beta\rho}\nabla^{\rho}\phi\\
&+g_{\mu\nu}\nabla^{\alpha}G_5R_{\alpha\beta}\nabla^{\beta}\phi-g_{\mu\nu}\Big[\frac{1}{2}\nabla_{\alpha}\big(G_{5\phi}\nabla^{\alpha}\phi\big)\Box\phi+\frac{1}{2}\nabla_{\alpha}G_{5X}\nabla^{\alpha}X\Box\phi\Big]\\
&+\frac{1}{2}\nabla_{(\mu}\big(G_{5\phi}\nabla_{\nu)}\phi\big)\Box\phi-g_{\mu\nu}\nabla^{\lambda}G_5R_{\lambda(\mu}\nabla_{\nu)}\phi+\frac{1}{2}\nabla_{\lambda}G_5G_{\mu\nu}\nabla^{\lambda}\phi. \qquad \text{(B.1.1.23)}
\end{aligned}
$$

B.1.2 Scalar Field Equations

In this section, we present a detailed derivation of the scalar field equation arising from the Horndeski action. We employ the generalization of the standard Euler–Lagrange equation for a scalar field ϕ:

$$
\delta\mathcal{L}_i=\frac{\partial\mathcal{L}_i}{\partial\phi}-\nabla_{\mu}\left(\frac{\partial\mathcal{L}_i}{\partial(\nabla_{\mu}\phi)}\right)+\nabla_{\mu}\nabla_{\nu}\left(\frac{\partial\mathcal{L}_i}{\partial(\nabla_{\mu}\nabla_{\nu}\phi)}\right) \qquad \text{(B.1.2.1)}
$$

- $\mathcal{L}_2=\mathbf{K}(\phi,\mathbf{X})$

For the first part of the Horndenski Lagrangian, the third term of (B.1.2.1) vanishes. Therefore

$$
\delta\mathcal{L}_2=\frac{\partial\mathcal{L}_2}{\partial\phi}-\nabla_{\mu}\left(\frac{\partial\mathcal{L}_2}{\partial(\nabla_{\mu}\phi)}\right)=K_{\phi}-\nabla_{\mu}\left(K_X\frac{\partial X}{\partial(\nabla_{\mu}\phi)}\right)=K_{\phi}-\nabla_{\mu}\left(-K_X\nabla^{\mu}\phi\right)\Rightarrow
$$

[2] As in the previous cases, symmetrization of the μ,ν indices is applied where required.

$$\delta\mathcal{L}_2 = K_\phi - \nabla^\mu \left(-\mathcal{L}_{2X}\nabla_\mu\phi\right)$$

where

$$\begin{aligned} P^2_\phi &= K_\phi \\ J^2_\mu &= -\mathcal{L}_{2X}\nabla_\mu\phi \end{aligned} \tag{B.1.2.2}$$

- $\mathcal{L}_\mathbf{3} = -\mathbf{G_3}(\phi,\mathbf{X})\Box\phi$

$$\delta\mathcal{L}_3 = \frac{\partial\mathcal{L}_3}{\partial\phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}_3}{\partial(\nabla_\mu\phi)}\right) + \nabla_\mu\nabla_\nu\left(\frac{\partial\mathcal{L}_3}{\partial(\nabla_\mu\nabla_\nu\phi)}\right) \tag{B.1.2.3}$$

The first term is

$$\frac{\partial\mathcal{L}_3}{\partial\phi} = -G_3\Box\phi.$$

For the second term, we have

$$\frac{\partial\mathcal{L}_3}{\partial(\nabla_\mu\phi)} = -G_{3X}\frac{\partial X}{\partial(\nabla_\mu\phi)}\Box\phi = G_{3X}\nabla^\mu\phi\Box\phi \Rightarrow$$

$$-\nabla_\mu\left(\frac{\partial\mathcal{L}_3}{\partial(\nabla_\mu\phi)}\right) = -\nabla_\mu\left(G_{3X}\nabla^\mu\phi\Box\phi\right) = -\nabla^\mu\left(-\mathcal{L}_{3X}\nabla_\mu\phi\right)$$

For the final term,

$$\frac{\partial\mathcal{L}_3}{\partial(\nabla_\mu\nabla_\nu\phi)} = -G_3\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi\right) = -G_3 g^{\mu\nu}.$$

Substituting into (B.1.2.3),

$$\begin{aligned} \delta\mathcal{L}_3 &= -G_3\Box\phi - \nabla^\mu\left(-\mathcal{L}_{3X}\nabla_\mu\phi\right) - \nabla_\mu\nabla_\nu\left(G_3 g^{\mu\nu}\right) \\ &= -G_3\Box\phi - \nabla^\mu\left(-\mathcal{L}_{3X}\nabla_\mu\phi\right) - \nabla^\mu\left(\nabla_\mu X\, G_{3X} + \nabla_\mu\phi\, G_{3\phi}\right) \\ &= -\nabla^\mu\left(G_{3\phi}\nabla_\mu\phi\right) + \nabla_\mu G_{3\phi}\nabla^\mu\phi - \nabla^\mu\left(-\mathcal{L}_{3X}\nabla_\mu\phi + G_{3X}\nabla_\mu X + \nabla_\mu\phi\, G_{3\phi}\right) \\ &= \nabla_\mu G_{3\phi}\nabla^\mu\phi - \nabla^\mu\left[-\mathcal{L}_{3X}\nabla_\mu\phi + G_{3X}\nabla_\mu X + 2G_{3\phi}\nabla_\mu\phi\right] \end{aligned}$$

where

$$\begin{aligned} P^\phi_3 &= \nabla_\mu G_{3\phi}\nabla^\mu\phi \\ J^3_\mu &= -\mathcal{L}_{3X}\nabla_\mu\phi + G_{3X}\nabla_\mu X + 2G_{3\phi}\nabla_\mu\phi \end{aligned} \tag{B.1.2.4}$$

- $\mathcal{L}_\mathbf{4} = \mathbf{G_4}(\phi,\mathbf{X})\mathbf{R} + \mathbf{G_{4X}}\left[(\Box\phi)^\mathbf{2} - (\nabla_\mu\nabla_\nu\phi)^\mathbf{2}\right]$

$$\delta\mathcal{L}_4 = \frac{\partial\mathcal{L}_4}{\partial\phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}_4}{\partial(\nabla_\mu\phi)}\right) + \nabla_\mu\nabla_\nu\left(\frac{\partial\mathcal{L}_4}{\partial(\nabla_\mu\nabla_\nu\phi)}\right) \tag{B.1.2.5}$$

The first two terms are

$$\frac{\partial\mathcal{L}_4}{\partial\phi} = G_{4\phi}R + G_{4X\phi}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right],$$

and

$$\begin{aligned} \frac{\partial\mathcal{L}_4}{\partial(\nabla_\mu\phi)} &= \frac{\partial G_4}{\partial X}\frac{\partial X}{\partial(\nabla_\mu\phi)}R + \frac{\partial G_{4X}}{\partial X}\frac{\partial X}{\partial(\nabla_\mu\phi)}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ &= -G_{4X}\nabla^\mu\phi\, R - G_{4XX}\nabla^\mu\phi\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ &= -\mathcal{L}_{4X}\nabla^\mu\phi. \end{aligned}$$

Therefore,

$$-\nabla_\mu\left(\frac{\partial\mathcal{L}_4}{\partial(\nabla_\mu\phi)}\right) = -\nabla_\mu\left(-\mathcal{L}_{4X}\nabla^\mu\phi\right)$$

For the last term
$$\frac{\partial \mathcal{L}_4}{\partial(\nabla_\mu\nabla_\nu\phi)} = G_{4X}\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left[(\Box\phi)^2 - (\nabla_\rho\nabla_\sigma\phi)^2\right].$$
Using the following results
$$\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(\nabla_\rho\nabla^\rho\phi\nabla_\sigma\nabla^\sigma\phi\right) = 2\Box\phi g^{\mu\nu},$$
$$-\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left((\nabla_\rho\nabla_\sigma\phi)^2\right) = -2\nabla^\mu\nabla^\nu\phi,$$
we find,
$$\frac{\partial \mathcal{L}_4}{\partial(\nabla_\mu\nabla_\nu\phi)} = G_{4X}2\Box\phi g^{\mu\nu} - 2G_{4X}\nabla^\mu\nabla^\nu\phi.$$
Thus,
$$\nabla_\mu\nabla_\nu\left(\frac{\partial \mathcal{L}_4}{\partial(\nabla_\mu\nabla_\nu\phi)}\right) = \nabla_\mu\Big[2(\nabla_\nu G_{4X})\Box\phi\, g^{\mu\nu} + 2G_{4X}\nabla_\nu(\Box\phi)\, g^{\mu\nu} - 2(\nabla_\nu G_{4X})\nabla^\mu\nabla^\nu\phi - 2G_{4X}\nabla_\nu\nabla^\mu\nabla^\nu\phi\Big]$$
$$= \nabla^\mu\Big[2(\nabla_\mu G_{4X})\Box\phi + 2G_{4X}\nabla_\mu(\Box\phi) - 2(\nabla_\nu G_{4X})\nabla_\mu\nabla^\nu\phi - 2G_{4X}\nabla_\nu\nabla_\mu\nabla^\nu\phi\Big]$$
$$= \nabla^\mu\Big[2\nabla_\mu X\, G_{4XX}\Box\phi + 2\nabla_\mu\phi\, G_{4X\phi}\Box\phi + \underline{2G_{4X}\nabla_\mu(\Box\phi)} - 2\nabla_\nu X\, G_{4XX}\nabla_\mu\nabla^\nu\phi$$
$$- 2\nabla_\nu\phi\, G_{4X\phi}\nabla_\mu\nabla^\nu\phi - \underline{2G_{4X}\nabla_\nu\nabla_\mu\nabla^\nu\phi}\Big]$$
Using the methods employed previously in the derivation of the metric equations of motion, the two underlined terms can be rewritten as follows
$$2G_{4X}\nabla_\mu(\Box\phi) - 2G_{4X}\nabla_\nu\nabla_\mu\nabla^\nu\phi = -2G_{4X}R_{\lambda\mu}\nabla^\lambda\phi$$
Therefore,
$$\nabla_\mu\nabla_\nu\left(\frac{\partial \mathcal{L}_4}{\partial(\nabla_\mu\nabla_\nu\phi)}\right) = \nabla^\mu\Big[2G_{4XX}\left(\Box\phi\,\nabla_\mu X - \nabla_\nu X\,\nabla_\mu\nabla^\nu\phi\right)$$
$$+2G_{4X\phi}\left(\Box\phi\,\nabla_\mu\phi - \nabla_\nu\phi\,\nabla_\mu\nabla^\nu\phi\right) - 2G_{4X\phi}R_{\mu\nu}\nabla^\nu\phi\Big]$$
Substituting these results into (B.1.2.5) yields
$$\delta\mathcal{L}_4 = G_{4\phi}R + G_{4X\phi}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] - \nabla^\mu\left[-\mathcal{L}_{4X}\nabla_\mu\phi - 2G_{4XX}\left(\Box\phi\,\nabla_\mu X - \nabla_\nu X\,\nabla_\mu\nabla^\nu\phi\right)\right.$$
$$\left. - 2G_{4X\phi}\left(\Box\phi\,\nabla_\mu\phi + \nabla_\mu X\right) + 2G_{4X}R_{\mu\nu}\nabla^\nu\phi\right.$$
where
$$\begin{aligned} J^4_\mu &= -\mathcal{L}_{4X}\nabla_\mu\phi + 2G_{4X}R_{\mu\nu}\nabla^\nu\phi \\ &\quad - 2G_{4XX}\left(\Box\phi\nabla_\mu X - \nabla_\nu X\nabla_\mu\nabla^\nu\phi\right) \\ &\quad - 2G_{4\phi X}\left(\Box\phi\nabla_\mu\phi + \nabla_\mu X\right), \\ P^\phi_4 &= G_{4\phi}R + G_{4\phi X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]. \end{aligned} \tag{B.1.2.6}$$

- $\boldsymbol{\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]}$

We now consider the variation of the final term in the Horndeski Lagrangian, which can be written as
$$\delta\mathcal{L}_5 = \frac{\partial \mathcal{L}_5}{\partial\phi} - \nabla_\mu\left(\frac{\partial \mathcal{L}_5}{\partial(\nabla_\mu\phi)}\right) + \nabla_\mu\nabla_\nu\left(\frac{\partial \mathcal{L}_5}{\partial(\nabla_\mu\nabla_\nu\phi)}\right). \tag{B.1.2.7}$$
The first term is
$$\frac{\partial \mathcal{L}_5}{\partial\phi} = G_{5\phi}G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X\phi}}{6}\Big[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\Big]. \tag{B.1.2.8}$$

The second term is

$$\frac{\partial\mathcal{L}_5}{\partial(\nabla_\mu\phi)}=\frac{\partial}{\partial(\nabla_\mu\phi)}\left(G_5G_{\rho\sigma}\nabla^\rho\nabla^\sigma\phi\right)-\frac{1}{6}\frac{\partial G_{5X}}{\partial X}\frac{\partial X}{\partial(\nabla_\mu\phi)}\left[(\Box\phi)^3-3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2+2(\nabla_\mu\nabla_\nu\phi)^3\right]$$

$$=-G_{5X}\nabla^\mu\phi\, G_{\rho\sigma}\nabla^\rho\nabla^\sigma\phi+\frac{G_{5XX}}{6}\nabla^\mu\phi\left[(\Box\phi)^3-3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2+2(\nabla_\mu\nabla_\nu\phi)^3\right]$$

$$=-\mathcal{L}_{5X}\nabla^\mu\phi$$

or equivalently,

$$-\nabla_\mu\left(\frac{\partial\mathcal{L}_5}{\partial(\nabla_\mu\phi)}\right)=-\nabla^\mu\left(-\mathcal{L}_{5X}\nabla_\mu\phi\right). \tag{B.1.2.9}$$

For the last term of (B.1.2.7), we find

$$\frac{\partial\mathcal{L}_5}{\partial(\nabla_\mu\nabla_\nu\phi)}=-\frac{G_{5X}}{6}\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left[(\Box\phi)^3-3(\Box\phi)(\nabla_\rho\nabla_\sigma\phi)^2+2(\nabla_\rho\nabla_\sigma\phi)^3\right]$$
$$+\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(G_5G_{\rho\sigma}\nabla^\rho\nabla^\sigma\phi\right)$$

Utilizing the following results

$$\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left((\Box\phi)^3\right)=3g^{\mu\nu}(\Box\phi)^2,$$
$$\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left((\nabla_\rho\nabla_\sigma\phi)^3\right)=3\nabla^\nu\nabla_\rho\phi\,\nabla^\rho\nabla^\mu\phi,$$
$$\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(\Box\phi\right)=g^{\mu\nu},$$
$$\frac{\partial}{\partial(\nabla_\mu\nabla_\nu\phi)}\left((\nabla_\rho\nabla_\sigma\phi)^2\right)=2\nabla^\mu\nabla^\nu\phi.$$

leads to

$$\frac{\partial\mathcal{L}_5}{\partial(\nabla_\mu\nabla_\nu\phi)}=G_5G^{\mu\nu}-\frac{G_{5X}}{6}\left[3g^{\mu\nu}(\Box\phi)^2-3g^{\mu\nu}(\nabla_\rho\nabla_\sigma\phi)^2-6(\nabla^\mu\nabla^\nu\phi)\Box\phi+6\nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla^\mu\phi\right] \tag{B.1.2.10}$$

Substituting (B.1.2.10), (B.1.2.9), (B.1.2.8) into (B.1.2.7)

$$\delta\mathcal{L}_5=G_{5\phi}\,G_{\mu\nu}\nabla^\mu\nabla^\nu\phi-\frac{G_{5X\phi}}{6}\left[(\Box\phi)^3-3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2+2(\nabla_\mu\nabla_\nu\phi)^3\right]$$
$$-\nabla^\mu\left(-\mathcal{L}_{5X}\nabla_\mu\phi\right)+\nabla_\mu\nabla_\nu\left(G_5G^{\mu\nu}\right)$$
$$-\nabla_\mu\nabla_\nu\left(\frac{G_{5X}}{6}\right)\left[3g^{\mu\nu}(\Box\phi)^2-3g^{\mu\nu}(\nabla_\rho\nabla_\sigma\phi)^2\right.$$
$$\left.-6(\nabla^\mu\nabla^\nu\phi)\Box\phi+6\nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla^\mu\phi\right]$$
$$-\nabla_\mu\left\{\frac{G_{5X}}{6}\nabla_\nu\left[3g^{\mu\nu}(\Box\phi)^2-3g^{\mu\nu}(\nabla_\rho\nabla_\sigma\phi)^2\right.\right.$$
$$\left.\left.-6(\nabla^\mu\nabla^\nu\phi)\Box\phi+6\nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla^\mu\phi\right]\right\}\Rightarrow$$

$$
\begin{aligned}
\delta\mathcal{L}_5 = {} & G_{5\phi}\, G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X\phi}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right] \\
& - \nabla^\mu\left(-\mathcal{L}_{5X}\nabla_\mu\phi\right) + \nabla_\mu\left(\nabla_\nu X\, G_{5X}G^{\mu\nu} + \nabla_\nu\phi\, G_{5\phi}G^{\mu\nu}\right) \\
& - \nabla^\mu\Bigg\{G_{5XX}\Bigg[\frac{1}{2}\nabla_\mu X\left((\Box\phi)^2 - (\nabla_\rho\nabla_\sigma\phi)^2\right) \\
& \quad - (\nabla_\mu\nabla^\nu\phi)\Box\phi\nabla_\nu X + \nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla^\mu\phi\nabla_\nu X\Bigg]\Bigg\} \\
& - \nabla^\mu\Bigg\{G_{5X\phi}\Bigg[\frac{1}{2}\nabla_\mu\phi\left((\Box\phi)^2 - (\nabla_\rho\nabla_\sigma\phi)^2\right) \\
& \quad - \left(\nabla_\mu\nabla^\nu\phi\right)\Box\phi\nabla_\nu\phi + \nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla_\mu\phi\nabla_\nu\phi\Bigg]\Bigg\} \\
& - \nabla^\mu\Bigg\{G_{5X}\Bigg[\frac{1}{2}\nabla_\mu\left((\Box\phi)^2 - (\nabla_\rho\nabla_\sigma\phi)^2\right) \\
& \quad - \nabla_\nu\left(\nabla_\mu\nabla^\nu\phi\right)\Box\phi - \left(\nabla_\mu\nabla^\nu\phi\right)\left(\nabla_\nu(\Box\phi)\right) \\
& \quad + \nabla_\nu\nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla_\mu\phi + \nabla^\nu\nabla_\rho\phi\nabla_\nu\nabla^\rho\nabla_\mu\phi\Bigg]\Bigg\}.
\end{aligned}
$$

By renaming indices and simplifying terms we obtain

$$
\begin{aligned}
\delta\mathcal{L}_5 = {} & \underline{G_{5\phi}\, G_{\mu\nu}\nabla^\mu\nabla^\nu\phi} - \frac{G_{5X\phi}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right] \\
& - \nabla^\mu\left(-\mathcal{L}_{5X}\nabla_\mu\phi\right) + \underline{\nabla_\mu\left(\nabla_\nu X\, G_{5X}G^{\mu\nu} + \nabla_\nu\phi\, G_{5\phi}G^{\mu\nu}\right)} \\
& - \nabla^\mu\Bigg\{G_{5XX}\Bigg[\frac{1}{2}\nabla_\mu X\left((\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right) \\
& \quad - \nabla_\nu X\left(\Box\phi\nabla_\mu\nabla^\nu\phi - \nabla_\alpha\nabla_\mu\phi\nabla^\alpha\nabla^\nu\phi\right)\Bigg]\Bigg\} \\
& - \nabla^\mu\Bigg\{G_{5X\phi}\Bigg[\frac{1}{2}\nabla_\mu\phi\left((\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right) \\
& \quad + \Box\phi\nabla_\mu X - \nabla^\nu X\nabla_\nu\nabla_\mu\phi\Bigg]\Bigg\} \\
& - \nabla^\mu\Bigg\{G_{5X}\Bigg[\frac{1}{2}\nabla_\mu\left((\Box\phi)^2 - (\nabla_\rho\nabla_\sigma\phi)^2\right) \\
& \quad - \nabla_\nu\left(\nabla_\mu\nabla^\nu\phi\right)\Box\phi - \left(\nabla_\mu\nabla^\nu\phi\right)\left(\nabla_\nu(\Box\phi)\right) \\
& \quad + \nabla_\nu\nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla_\mu\phi + \nabla^\nu\nabla_\rho\phi\nabla_\nu\nabla^\rho\nabla_\mu\phi\Bigg]\Bigg\}
\end{aligned}
\tag{B.1.2.11}
$$

The underlined terms can be written as:

$$
\nabla^\mu\left(2G_{5\phi}G_{\mu\nu}\nabla^\nu\phi\right) - \nabla_\mu G_{5\phi}G^{\mu\nu}\nabla_\nu\phi + \nabla^\mu\left(\nabla^\nu X G_{5X}G_{\mu\nu}\right) \tag{B.1.2.12}
$$

In addition, the terms in the last bracket involving G_{5X} can be reformulated as:

$$
\begin{aligned}
& \frac{1}{2}\nabla_\mu\left((\Box\phi)^2 - (\nabla_\rho\nabla_\sigma\phi)^2\right) - \nabla_\nu\left(\nabla_\mu\nabla^\nu\phi\right)\Box\phi - \left(\nabla_\mu\nabla^\nu\phi\right)\left(\nabla_\nu(\Box\phi)\right) \\
& \quad + \nabla_\nu\nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla_\mu\phi + \nabla^\nu\nabla_\rho\phi\nabla_\nu\nabla^\rho\nabla_\mu\phi \\
& = \nabla_\mu(\nabla^\alpha\nabla_\alpha\phi\nabla^\beta\nabla_\beta\phi) - \nabla_\mu(\nabla_\rho\nabla_\sigma\phi)\nabla^\rho\nabla^\sigma\phi \quad - \nabla_\nu(\nabla_\mu\nabla^\nu\phi)\Box\phi - \nabla_\mu\nabla^\nu\phi\nabla_\nu(\Box\phi) \\
& \quad + \Box(\nabla_\rho\phi)\nabla^\rho\nabla_\mu\phi + \nabla^\nu\nabla_\rho\phi\nabla_\nu(\nabla^\rho\nabla_\mu\phi) \\
& = R_{\alpha\mu\beta\nu}\nabla^\nu\phi\nabla^\alpha\nabla^\beta\phi - R_{\mu\nu}\Box\phi\nabla^\nu\phi + R_{\lambda\nu}\nabla^\nu\phi\nabla^\lambda\nabla_\mu\phi
\end{aligned}
\tag{B.1.2.13}
$$

where for the last line in detail we have:

$$
\begin{aligned}
\nabla_\mu(\nabla^\alpha\nabla_\alpha\phi)\Box\phi - \nabla_\alpha(\nabla_\mu\nabla^\alpha\phi)\Box\phi &= \nabla_\mu(\nabla^\alpha\nabla_\alpha\phi)\Box\phi - \nabla_\mu(\Box\phi)\Box\phi - R_{\lambda\mu}\nabla^\lambda\phi\Box\phi \\
&= -R_{\mu\nu}\Box\phi\nabla^\nu\phi,
\end{aligned}
$$

$$
\begin{aligned}
&\Box(\nabla_\rho\phi)\nabla^\rho\nabla_\mu\phi - \nabla_\mu\nabla^\nu\phi\nabla_\nu(\Box\phi)\\
&= \Box(\nabla_\nu\phi)\nabla^\nu\nabla_\mu\phi - \nabla^\nu\nabla_\mu\phi\nabla_\nu(\Box\phi)\\
&= \left(\nabla^\alpha(\nabla_\alpha\nabla_\nu\phi) - \nabla_\nu(\nabla_\alpha\nabla^\alpha\phi)\right)\nabla^\nu\nabla_\mu\phi\\
&= \left(\nabla_\alpha(\nabla_\nu\nabla^\alpha\phi) - \nabla_\nu(\nabla_\alpha\nabla^\alpha\phi)\right)\nabla^\nu\nabla_\mu\phi\\
&= [\nabla_\alpha,\nabla_\nu]\nabla^\alpha\phi\nabla^\nu\nabla_\mu\phi\\
&= R^\alpha{}_{\lambda\alpha\nu}\nabla^\lambda\phi\nabla^\nu\nabla_\mu\phi\\
&= R_{\lambda\nu}\nabla^\nu\phi\nabla^\lambda\nabla_\mu\phi,
\end{aligned}
$$

and

$$
\begin{aligned}
&\nabla^\nu\nabla_\rho\phi\nabla_\nu\nabla^\rho\nabla_\mu\phi - \nabla_\mu\nabla_\rho\nabla_\sigma\phi\nabla^\rho\nabla^\sigma\phi\\
&= \nabla^\nu\nabla_\rho\phi\nabla_\nu\nabla^\rho\nabla_\mu\phi - \nabla_\mu\nabla_\sigma\nabla^\rho\phi\nabla^\sigma\nabla_\rho\phi\\
&= \nabla^\nu\nabla_\rho\phi\nabla_\nu\nabla^\rho\nabla_\mu\phi - \nabla_\mu\nabla_\nu\nabla^\rho\phi\nabla^\nu\nabla_\rho\phi\\
&= \nabla^\nu\nabla_\rho\phi\left(\nabla_\nu\nabla_\mu\nabla^\rho\phi - \nabla_\mu\nabla_\nu\nabla^\rho\phi\right)\\
&= \nabla^\nu\nabla_\rho\phi[\nabla_\nu,\nabla_\mu]\nabla^\rho\phi\\
&= R_{\rho\lambda\nu\mu}\nabla^\lambda\phi\nabla^\nu\nabla^\rho\phi\\
&= R_{\alpha\mu\beta\nu}\nabla^\nu\phi\nabla^\alpha\nabla^\beta\phi
\end{aligned}
$$

Substituting (B.1.2.12), (B.1.2.13) into the previous result (B.1.2.11), leads to

$$
\begin{aligned}
\delta\mathcal{L}_5 = &-\nabla_\mu G_{5\phi}G^{\mu\nu}\nabla_\nu\phi - \frac{G_{5X\phi}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]\\
&-\nabla^\mu\Bigg\{-G_{5X}\left[G_{\mu\nu}\nabla^\nu X + R_{\mu\nu}\Box\phi\nabla^\nu\phi - R_{\lambda\nu}\nabla^\nu\phi\nabla^\lambda\nabla_\mu\phi - R_{\alpha\mu\beta\nu}\nabla^\nu\phi\nabla^\alpha\nabla^\beta\phi\right]\\
&\quad + G_{5XX}\left[\frac{1}{2}\nabla_\mu X\left((\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right) - \nabla_\nu X\left(\Box\phi\nabla_\mu\nabla^\nu\phi - \nabla_\alpha\nabla_\mu\phi\nabla^\alpha\nabla^\nu\phi\right)\right]\\
&\quad + G_{5X\phi}\left[\frac{1}{2}\nabla_\mu\phi\left((\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right) + \Box\phi\nabla_\mu X - \nabla^\nu X\nabla_\nu\nabla_\mu\phi\right]\\
&\quad - \mathcal{L}_{5X}\nabla_\mu\phi - 2G_{5\phi}G_{\mu\nu}\nabla^\nu\phi\Bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
P^5_\phi &= -\nabla_\mu G_{5\phi}G^{\mu\nu}\nabla_\nu\phi - \frac{1}{6}G_{5\phi X}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]\\
J^5_\mu &= -G_{5X}\left[G_{\mu\nu}\nabla^\nu X + R_{\mu\nu}\Box\phi\nabla^\nu\phi - R_{\lambda\nu}\nabla^\nu\phi\nabla^\lambda\nabla_\mu\phi - R_{\alpha\mu\beta\nu}\nabla^\nu\phi\nabla^\alpha\nabla^\beta\phi\right]\\
&\quad + G_{5XX}\left[\frac{1}{2}\nabla_\mu X\left((\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right) - \nabla_\nu X\left(\Box\phi\nabla_\mu\nabla^\nu\phi - \nabla_\alpha\nabla_\mu\phi\nabla^\alpha\nabla^\nu\phi\right)\right]\\
&\quad + G_{5X\phi}\left[\frac{1}{2}\nabla_\mu\phi\left((\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right) + \Box\phi\nabla_\mu X - \nabla^\nu X\nabla_\nu\nabla_\mu\phi\right]\\
&\quad - \mathcal{L}_{5X}\nabla_\mu\phi - 2G_{5\phi}G_{\mu\nu}\nabla^\nu\phi
\end{aligned}
\tag{B.1.2.14}
$$

B.2 Cosmological Background Equations

B.2.1 Generalized Friedmann Equation

We derive the generalized Friedmann equations by considering a homogeneous scalar field, $\phi = \phi(t)$, and varying the action with respect to the lapse function $N(t)$. After performing the variation, we impose the condition $N = 1$ to obtain the final result. Note that in flat FLRW spacetime (3.3.1), the following holds:

$$
\sqrt{-g} = Na^3, \quad g^{00} = -\frac{1}{N^2}
$$

$$
\frac{d}{dt}(X) = \frac{\dot\phi\ddot\phi}{N^2}, \qquad X = \frac{\dot\phi^2}{2N^2}
$$

In addition, the following expression are used extensively in simplifying the subsequent calculations

$$\frac{\ddot{a}}{a} = \dot{H} + H^2$$

together with,

$$R = 6\left(\frac{\ddot{a}}{N^2 a} - \frac{\dot{a}\dot{N}}{aN^3} + \frac{\dot{a}^2}{N^2 a^2}\right).$$

- $\mathcal{L}_\mathbf{2} = \mathbf{K}(\phi, \mathbf{X})$

The first part of the Horndenski action in this spacetime is given by

$$S_2 = \int \mathrm{d}^3x\,\mathrm{d}t\, Na^3 K(\phi, X)$$

Therefore,

$$\delta_N S_2 = \int d^3x dt a^3 K(\phi, X)\delta_N N + \int d^3x dt N a^3 \delta_N K(\phi, X).$$

The variation of the kinetic term $\delta_N K(\phi, X)$ is calculated as follows. Since $K = K(\phi, X)$ and ϕ is unvaried with respect to N, we have $\delta_N K = K_X\, \delta_N X$. Recalling that $X = \frac{\dot{\phi}^2}{2N^2}$ in this metric, we find

$$\delta_N K(\phi, X) = K_X\, \delta_N(X) = K_X\, \delta_N\left(\frac{\dot{\phi}^2}{2N^2}\right) = \frac{-K_X\dot{\phi}^2}{N^3}\delta_N N$$

Substituting this result back into the variation yields

$$\delta_N S_2 = \int d^3x dt a^3 \left[K - \frac{K_X\dot{\phi}^2}{N^3}\right]\delta_N N$$

Finally, we impose the condition $N(t) = 1$ which leads

$$-\frac{1}{a^3}\frac{\delta_N S_2}{\delta_N N} = \mathcal{E}_2 = 2XK_X - K \tag{B.2.1.1}$$

- $\mathcal{L}_\mathbf{3} = -\mathbf{G_3}(\phi, \mathbf{X})\Box\phi$

The action for this part of the Lagrangian is

$$S_3 = \int d^3x dt N a^3 G_3\left[\frac{3H\dot{\phi}}{N^2} + \frac{\ddot{\phi}}{N^2} - \frac{\dot{N}\dot{\phi}}{N^3}\right]$$

Variation of this action with respect to the lapse function $N(t)$ yields the following:

$$\begin{aligned}\delta_N S_3 = &\int \mathrm{d}^3x\,\mathrm{d}t\, a^3 G_3\left[\frac{3H\dot{\phi}}{N^2} + \frac{\ddot{\phi}}{N^2} - \frac{\dot{N}\dot{\phi}}{N^3}\right]\delta_N N + \int \mathrm{d}^3x\,\mathrm{d}t\, Na^3\left[\frac{3H\dot{\phi}}{N^2} + \frac{\ddot{\phi}}{N^2} - \frac{\dot{N}\dot{\phi}}{N^3}\right]\delta_N G_3\\ &\int \mathrm{d}^3x\,\mathrm{d}t\, Na^3 G_3\,\delta_N\left[\frac{3H\dot{\phi}}{N^2} + \frac{\ddot{\phi}}{N^2} - \frac{\dot{N}\dot{\phi}}{N^3}\right]\end{aligned} \tag{B.2.1.2}$$

Using

$$\delta_N G_3 = G_{3X}\,\delta_N X = -\frac{G_{3X}\dot{\phi}^2}{N^3}\delta_N N,$$

the second term of (B.2.1.2) can be written as

$$\delta_N S_{3,2} = \int d^3x dt N a^3\left[-\frac{3H\dot{\phi}}{N^2} - \frac{\ddot{\phi}}{N^2} + \frac{\dot{N}\dot{\phi}}{N^3}\right]\frac{G_{3X}\dot{\phi}^2}{N^3}\delta_N N \tag{B.2.1.3}$$

For the final term we have[3]

$$\begin{aligned}\delta_N S_{3,3} = \int \mathrm{d}^3x\,\mathrm{d}t\, Na^3 G_3\,\delta_N\left[\frac{3H\dot{\phi}}{N^2} + \frac{\ddot{\phi}}{N^2} - \frac{\dot{N}\dot{\phi}}{N^3}\right] &= \int \mathrm{d}^3x\mathrm{d}t\left[-a^3 G_3\dot{\phi}\frac{6H}{N^2} - 2a^3 G_3\frac{\ddot{\phi}}{N^2}\right]\delta_N N\\ &- \int \mathrm{d}^3x\,\mathrm{d}t\, a^3 G_3\frac{\dot{\phi}}{N^2}\,\delta_N\left(\frac{\mathrm{d}}{\mathrm{d}t}N\right).\end{aligned}$$

[3]Note that the terms that end up involving $\dot{N}$ do not contribute to the final result of the variation after setting N=1.

Integrating by parts the last term can be written as follows

$$-\int \mathrm{d}^3x\,\mathrm{d}t\,a^3G_3\frac{\dot\phi}{N^2}\delta_N\left(\frac{\mathrm{d}}{\mathrm{d}t}N\right) = \int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3G_3\frac{\dot\phi}{N^2}\right)\delta_N N$$
$$= \int \mathrm{d}^3x\,\mathrm{d}t\left[3a^2\dot a G_3\frac{\dot\phi}{N^2}+\frac{a^3\dot\phi}{N^2}\left(G_{3X}\dot X+G_{3\phi}\dot\phi\right)+a^3G_3\frac{\ddot\phi}{N^2}\right]\delta_N N.$$

Combining the results above and setting $N=1$ we obtain

$$\delta_N S_3 = \int \mathrm{d}^3x\,\mathrm{d}t\,a^3\Bigg[\cancel{G_3 3H\dot\phi}+\cancel{G_3\ddot\phi}-3G_{3X}H\dot\phi^3-G_{3X}\dot\phi^2\ddot\phi-\cancel{6G_3H\dot\phi}-\cancel{2G_3\ddot\phi}+\cancel{3HG_3\dot\phi}+$$
$$\dot\phi G_{3X}\dot X+\dot\phi G_{3\phi}\dot\phi+\cancel{G_3\ddot\phi}\Bigg]\delta_N N.$$

Therefore

$$\delta_N S_3 = \int d^3x\,dt\,a^3\left[-3H\dot\phi G_{3X}\dot\phi^2+\dot\phi^2G_{3\phi}\right]\delta_N N$$

from which follows,

$$-\frac{1}{a^3}\frac{\delta_N S_3}{\delta_N N} = \mathcal{E}_3 = 6HX\dot\phi G_{3X}-2XG_{3\phi} \tag{B.2.1.4}$$

- $\mathcal{L}_\mathbf{4} = \mathbf{G_4(\phi,X)R}+\mathbf{G_{4X}}\left[(\Box\phi)^\mathbf{2}-(\nabla_\mu\nabla_\nu\phi)^\mathbf{2}\right]$

The next part of the action when evaluated in this spacetime, takes the form

$$S_4 = \int d^3xdtNa^3\left[6G_4\Big(\frac{\ddot a}{N^2a}-\frac{\dot a\dot N}{aN^3}+\frac{\dot a^2}{N^2a^2}\Big)+G_{4X}\Big(\frac{6H\dot\phi\ddot\phi}{N^4}+\frac{6H^2\dot\phi^2}{N^4}-\frac{6H\dot\phi^2\dot N}{N^5}\Big)\right]$$

where we have used

$$(\nabla_\mu\nabla_\nu\phi)^2 = \frac{\ddot\phi}{N^4}-\frac{2\dot N\ddot\phi\dot\phi}{N^5}+\frac{\dot N^2\dot\phi^2}{N^6}+\frac{3H^2\dot\phi^2}{N^4},$$
$$(\Box\phi)^2 = \frac{\ddot\phi}{N^4}+\frac{6H\dot\phi\ddot\phi}{N^4}-\frac{2\dot\phi\ddot\phi\dot N}{N^5}+\frac{9H^2\dot\phi^2}{N^4}-\frac{6H\dot\phi^2\dot N}{N^5}+\frac{\dot N^2\dot\phi^2}{N^6}$$

We split the variation of this part of the action into four terms

$$\delta_N S_4 = \delta_N S_{4,1}+\delta_N S_{4,2}+\delta_N S_{4,3}+\delta_N S_{4,4}. \tag{B.2.1.5}$$

The first term is

$$\delta_N S_{4,1} = \int d^3xdta^3\left[6G_4\Big(\frac{\ddot a}{N^2a}-\frac{\dot a\dot N}{aN^3}+\frac{\dot a^2}{N^2a^2}\Big)+G_{4X}\Big(\frac{6H\dot\phi\ddot\phi}{N^4}+\frac{6H^2\dot\phi^2}{N^4}-\frac{6H\dot\phi^2\dot N}{N^5}\Big)\right]\delta_N N.$$

Setting N=1, we obtain

$$-\frac{1}{a^3}\frac{\delta_N S_{4,1}}{\delta_N N} = -6G_4\Big(\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\Big)-G_{4X}\Big(6H\dot\phi\ddot\phi+6H^2\dot\phi^2\Big) \tag{B.2.1.6}$$

The second term of (B.2.1.5) is

$$\delta_N S_{4,2} = \int d^3xdtNa^3\left[6\Big(\frac{\ddot a}{N^2a}-\frac{\dot a\dot N}{aN^3}+\frac{\dot a^2}{N^2a^2}\Big)\delta_N(G_4)+\Big(\frac{6H\dot\phi\ddot\phi}{N^4}+\frac{6H^2\dot\phi^2}{N^4}-\frac{6H\dot\phi^2\dot N}{N^5}\Big)\delta_N(G_{4X})\right] =$$

$$\delta_N S_{4,2} = \int d^3xdta^3N\left[6\Big(-\frac{G_{4X}\dot\phi^2}{N^3}\ \Big)\Big(\frac{\ddot a}{N^2a}-\frac{\dot a\dot N}{aN^3}+\frac{\dot a^2}{N^2a^2}\Big)-\frac{G_{4XX}\dot\phi^2}{N^3}\Big(\frac{6H\dot\phi\ddot\phi}{N^4}+\frac{6H^2\dot\phi^2}{N^4}-\frac{6H\dot\phi^2\dot N}{N^5}\Big)\right]\delta_N N$$

where we used

$$\delta_N G_{4X} = -\frac{G_{4XX}\dot\phi^2}{N^3}\delta_N N$$

$$\delta_N G_4 = -\frac{G_{4X}\dot{\phi}^2}{N^3}\delta_N N$$

Setting $N=1$, and simplifying we obtain

$$-\frac{1}{a^3}\frac{\delta_N S_{4,2}}{\delta_N N} = 6G_{4X}\dot{\phi}^2\left(\frac{\ddot{a}}{a}+H^2\right)+G_{4XX}\dot{\phi}^2\left(6H\dot{\phi}\ddot{\phi}+6H^2\dot{\phi}^2\right) \tag{B.2.1.7}$$

For the third term of the variation we have[4]

$$\begin{aligned}
\delta_N S_{4,3} &= \int d^3x\,dt\,Na^3 6G_4\,\delta_N\left(\frac{\ddot{a}}{aN^2}-\frac{\dot{a}\dot{N}}{aN^3}+\frac{\dot{a}^2}{a^2N^2}\right)\\
&= \int d^3x\,dt\,Na^3 6G_4\left[\frac{\ddot{a}}{a}\,\delta_N\left(\frac{1}{N^2}\right)-H\,\delta_N\left(\frac{\dot{N}}{N^3}\right)+H^2\,\delta_N\left(\frac{1}{N^2}\right)\right]\\
&= \int d^3x\,dt\,(-12a^2G_4)\frac{\ddot{a}}{N^2}\delta_N N+\int d^3x\,dt\,(-12a^3G_4)\frac{H^2}{N^2}\delta_N N\\
&\quad+\cancel{\int d^3x\,dt\,(-18G_4H)\frac{a^3\dot{N}}{N^3}}\delta_N N-\int d^3x\,dt\,a^3 6G_4\frac{H}{N^2}\delta_N\left(\frac{dN}{dt}\right)
\end{aligned}$$

Using integration by parts and simplifying the resulting expression, we obtain

$$\delta_N S_{4,3} = \int d^3x dt\left(-\frac{12G_4a^2\ddot{a}}{N^2}-\frac{12a^3G_4H^2}{N^2}\right)+\int d^3x dt\frac{6}{N^2}\left(3a^2\dot{a}HG_4+a^3G_4\dot{H}+a^3G_{4\phi}\dot{\phi}H+a^3G_{4X}\dot{X}H\right).$$

Finally, setting $N=1$, yields

$$-\frac{1}{a^3}\frac{\delta_N S_{4,3}}{\delta_N N} = 12\frac{\ddot{a}}{a}G_4-6H^2G_4-6G_4\dot{H}-6G_{4\phi}\dot{\phi}H-6G_{4X}\dot{X}H \tag{B.2.1.8}$$

For the last part of (B.2.1.5)

$$\begin{aligned}
\delta_N S_{4,4} &= \int d^3x\,dt\,Na^3G_{4X}\,\delta_N\left(\frac{6H\dot{\phi}\ddot{\phi}}{N^4}+\frac{6H^2\dot{\phi}^2}{N^4}-\frac{6H\dot{\phi}^2\dot{N}}{N^5}\right)\\
&= \int d^3x\,dt\,a^3G_{4X}\left(-\frac{24H\dot{\phi}\ddot{\phi}}{N^4}-\frac{24H^2\dot{\phi}^2}{N^4}\right)\delta_N N\\
&\quad+\int d^3x\,dt\,\frac{6}{N^4}\frac{d}{dt}\left(a^3G_{4X}H\dot{\phi}^2\right)\delta_N N
\end{aligned}$$

where we have used integration by parts for the last term of the first line. Setting $N=1$, and performing straightforward calculations, we obtain

$$\begin{aligned}
-\frac{1}{a^3}\frac{\delta_N S_{4,4}}{\delta_N N} &= (24H\dot{\phi}\ddot{\phi}+24H^2\dot{\phi}^2)G_{4X}-18H^2G_{4X}\dot{\phi}^2-6G_{4X}\dot{H}\dot{\phi}^2\\
&\quad-12G_{4X}H\dot{\phi}\ddot{\phi}-6H\dot{\phi}^2G_{4XX}\dot{X}-6H\dot{\phi}^3G_{4X\phi}
\end{aligned} \tag{B.2.1.9}$$

Combining the results above, (B.2.1.6), (B.2.1.7), (B.2.1.8), (B.2.1.9)

$$\begin{aligned}
\mathcal{E}_4 = &-6G_4\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)-G_{4X}\left(6H\dot{\phi}\ddot{\phi}+6H^2\dot{\phi}^2\right)+6G_{4X}\dot{\phi}^2\left(\frac{\ddot{a}}{a}+H^2\right)\\
&+G_{4XX}\dot{\phi}^2\left(6H\dot{\phi}\ddot{\phi}+6H^2\dot{\phi}^2\right)+12\frac{\ddot{a}}{a}G_4-6H^2G_4-6G_4\dot{H}\\
&-6G_{4\phi}\dot{\phi}H-6G_{4X}\dot{X}H+24\left(H\dot{\phi}\ddot{\phi}+H^2\dot{\phi}^2\right)G_{4X}\\
&-18H^2G_{4X}\dot{\phi}^2-6G_{4X}\dot{H}\dot{\phi}^2-12G_{4X}H\dot{\phi}\ddot{\phi}\\
&-6H\dot{\phi}^2G_{4XX}\dot{X}-6H\dot{\phi}^3G_{4X\phi}
\end{aligned}$$

The above when simplified yields

$$-\frac{1}{a^3}\frac{\delta_N S_4}{\delta_N N} = \mathcal{E}_4 = -6H^2G_4+24H^2X\left(G_{4X}+XG_{4XX}\right)-12HX\dot{\phi}G_{4\phi X}-6H\dot{\phi}G_{4\phi} \tag{B.2.1.10}$$

[4] Since the first term of the last line does not contribute to the final result after setting $N=1$, it is omitted.

- $\mathcal{L}_5 = \mathbf{G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]}$

The final part of the Horndeski action in this spacetime takes the following form,

$$S_5 = \int d^3xdtNa^3\left[G_5\Big(\frac{6\ddot{a}H\dot{\phi}}{aN^4} + \frac{3H^3\dot{\phi}}{N^4} + \frac{3H^2\ddot{\phi}}{N^4} - \frac{9H^2\dot{N}\dot{\phi}}{N^5}\Big) + G_{5X}\Big(\frac{3H^2\dot{\phi}^2\ddot{\phi}}{N^6} - \frac{3H^2\dot{\phi}^3\dot{N}}{N^7} + \frac{H^3\dot{\phi}^3}{N^6}\Big)\right] \tag{3.3.14}$$

where we have used that in flat FLRW spacetime, the following relations hold[5]

$$-\frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right] = G_{5X}\Big(\frac{3H^2\dot{\phi}^2\ddot{\phi}}{N^6} - \frac{3H^2\dot{\phi}^3\dot{N}}{N^7} + \frac{H^3\dot{\phi}^3}{N^6}\Big)$$

and

$$G_{\mu\nu}\nabla^\mu\nabla^\nu\phi = \frac{6\ddot{a}H\dot{\phi}}{aN^4} + \frac{3H^3\dot{\phi}}{N^4} + \frac{3H^2\ddot{\phi}}{N^4} - \frac{9H^2\dot{N}\dot{\phi}}{N^5}$$

As in the previous calculation, we split the variation of the action into four parts.

$$\delta_N S_5 = \delta_N S_{5,1} + \delta_N S_{5,2} + \delta_N S_{5,3} + \delta_N S_{5,4} \tag{B.2.1.11}$$

For the first part,

$$\delta_N S_{5,1} = \int d^3xdta^3\left[G_5\Big(\frac{6\ddot{a}H\dot{\phi}}{aN^4} + \frac{3H^3\dot{\phi}}{N^4} + \frac{3H^2\ddot{\phi}}{N^4} - \frac{9H^2\dot{N}\dot{\phi}}{N^5}\Big) + G_{5X}\Big(\frac{3H^2\dot{\phi}^2\ddot{\phi}}{N^6} - \frac{3H^2\dot{\phi}^3\dot{N}}{N^7} + \frac{H^3\dot{\phi}^3}{N^6}\Big)\right]\delta_N N$$

By setting $N = 1$, we find

$$-\frac{1}{a^3}\frac{\delta S_{5,1}}{\delta N} = -G_5\left(6\frac{\ddot{a}}{a}H\dot{\phi} + 3H^3\dot{\phi} + 3H^2\ddot{\phi}\right) - G_{5X}\left(3H^2\dot{\phi}^2\ddot{\phi} + H^3\dot{\phi}^3\right) \tag{B.2.1.12}$$

For the second part,

$$\begin{aligned}\delta_N S_{5,2} = \int d^3x\,dt\,Na^3\Bigg[&\delta_N(G_5)\left(\frac{6\ddot{a}H\dot{\phi}}{aN^4} + \frac{3H^3\dot{\phi}}{N^4} + \frac{3H^2\ddot{\phi}}{N^4} - \frac{9H^2\dot{N}\dot{\phi}}{N^5}\right)\\ &+\delta_N(G_{5X})\left(\frac{3H^2\dot{\phi}^2\ddot{\phi}}{N^6} - \frac{3H^2\dot{\phi}^3\dot{N}}{N^7} + \frac{H^3\dot{\phi}^3}{N^6}\right)\Bigg]\end{aligned}$$

using

$$\delta_N G_5 = -\frac{G_{5X}\dot{\phi}^2}{N^3}\,\delta_N N$$
$$\delta_N G_{5X} = -\frac{G_{5XX}\dot{\phi}^2}{N^3}\,\delta_N N$$

we find after setting $N = 1$

$$-\frac{1}{a^3}\frac{\delta S_{5,2}}{\delta N} = G_{5X}\left(\frac{6\ddot{a}H\dot{\phi}^3}{a} + 3H^3\dot{\phi}^3 + 3H^2\dot{\phi}^2\ddot{\phi}\right) + G_{5XX}\left(3H^2\dot{\phi}^4\ddot{\phi} + H^3\dot{\phi}^5\right) \tag{B.2.1.13}$$

For the third term of (B.2.1.11) we have

$$\delta_N S_{5,3} = \int d^3x\,dt\,Na^3G_{5X}\,\delta_N\left(\frac{3H^2\dot{\phi}^2\ddot{\phi}}{N^6} - \frac{3H^2\dot{\phi}^3\dot{N}}{N^7} + \frac{H^3\dot{\phi}^3}{N^6}\right)$$

Using integration by parts for the second term and evaluating the remaining two terms, we find

$$\delta_N S_{5,3} = \int d^3xdtNa^3G_{5X}\left(-\frac{18H^2\dot{\phi}^2\ddot{\phi}}{N^7} - \frac{6H^3\dot{\phi}^3}{N^7}\right)\delta_N N + \int d^3xdt\frac{d}{dt}\left(\frac{Na^3G_{5X}3H^2\dot{\phi}^3}{N^7}\right)\delta_N N.$$

[5] The derivation of these relations involves straightforward but lengthy calculations. Therefore, only the resulting expressions are presented.

Calculating the derivatives and setting $N=1$ yields

$$-\frac{1}{a^3}\frac{\delta S_{5,3}}{\delta N} = -9H^3G_{5X}\dot{\phi}^3 - 3G_{5XX}\dot{X}H^2\dot{\phi}^3 - 3H^2\dot{\phi}^4G_{5X\phi} - 6G_{5X}H\dot{H}\dot{\phi}^3$$
$$-9G_{5X}H^2\dot{\phi}^2\ddot{\phi} + 18H^2\dot{\phi}^2\ddot{\phi}G_{5X} + 6H^3\dot{\phi}^3G_{5X} \qquad \text{(B.2.1.14)}$$

For the last term of (B.2.1.11) we have

$$\delta_N S_{5,4} = \int d^3xdtNa^3G_5\delta_N\left(\frac{6\ddot{a}H\dot{\phi}}{aN^4} + \frac{3H^3\dot{\phi}}{N^4} + \frac{3H^2\ddot{\phi}}{N^4} - \frac{9H^2\dot{N}\dot{\phi}}{N^5}\right) =$$

$$\int d^3xdta^3G_5\left(-\frac{24\ddot{a}H\dot{\phi}}{aN^4} - \frac{12H^3\dot{\phi}}{N^4} - \frac{12H^2\ddot{\phi}}{N^4}\right)\delta_N N + \int d^3xdt\frac{9}{N^4}\frac{d}{dt}\left(a^3G_5H^2\dot{\phi}\right)\delta_N N$$

After computing the derivatives and imposing $N=1$, the result is

$$-\frac{1}{a^3}\frac{\delta S_{5,4}}{\delta N} = G_5\left(12H^3\dot{\phi} + 24\frac{\ddot{a}}{a}H\dot{\phi} + 12H^2\ddot{\phi} - 27H^3\dot{\phi} - 9H^2\ddot{\phi} - 18H\dot{H}\dot{\phi}\right)$$
$$+G_{5\phi}\left(-18XH^2\right) + G_{5X}\left(-9\dot{X}H^2\dot{\phi}\right). \qquad \text{(B.2.1.15)}$$

Combining the results (B.2.1.12), (B.2.1.13), (B.2.1.14), (B.2.1.15) we have

$$\begin{aligned}
-\frac{1}{a^3}\frac{\delta S_5}{\delta N} = &-G_5\left(6\frac{\ddot{a}}{a}H\dot{\phi} + 3H^3\dot{\phi} + 3H^2\ddot{\phi}\right) - G_{5X}\left(3H^2\dot{\phi}^2\ddot{\phi} - H^3\dot{\phi}^3\right)\\
&+G_{5X}\left(\frac{6\ddot{a}H\dot{\phi}^3}{a} + 3H^3\dot{\phi}^3 + 3H^2\dot{\phi}^2\ddot{\phi}\right) + G_{5XX}\left(3H^2\dot{\phi}^4\ddot{\phi} + H^3\dot{\phi}^5\right)\\
&-9H^3G_{5X}\dot{\phi}^3 - 3G_{5XX}\dot{X}H^2\dot{\phi}^3 - 3H^2\dot{\phi}^4G_{5X\phi} - 6G_{5X}H\dot{H}\dot{\phi}^3\\
&+G_5\left(12H^3\dot{\phi} + 24\frac{\ddot{a}}{a}H\dot{\phi} + 12H^2\ddot{\phi} - 27H^3\dot{\phi} - 9H^2\ddot{\phi} - 18H\dot{H}\dot{\phi}\right)\\
&-9G_{5X}H^2\dot{\phi}^2\ddot{\phi} + 18H^2\dot{\phi}^2\ddot{\phi}G_{5X} + 6H^3\dot{\phi}^3G_{5X} + G_{5\phi}\left(-18XH^2\right)\\
&+G_{5X}\left(-9\dot{X}H^2\dot{\phi}\right)
\end{aligned}$$

which when simplified yields

$$\mathcal{E}_5 = 2H^3X\dot{\phi}\left(5G_{5X} + 2XG_{5XX}\right) - 6H^2X\left(3G_{5\phi} + 2XG_{5\phi X}\right) \qquad \text{(B.2.1.16)}$$

B.2.2 Scalar Field Equation

In this section, we derive the scalar field equation of motion for the Horndeski action in flat FLRW metric.

- $\mathcal{L}_2 = \mathbf{K}(\phi, \mathbf{X})$

The action for the first part of the Lagrangian evaluated in flat FLRW spacetime takes the following form

$$S_2 = \int \mathrm{d}^3x\,\mathrm{d}t\,Na^3K(\phi, X).$$

Variation with respect to ϕ of K yields

$$\delta_\phi S_2 = \int \mathrm{d}^3x\,\mathrm{d}t\,Na^3K_X\frac{1}{2N}\delta_\phi\dot{\phi}^2 + \int \mathrm{d}^3x\,\mathrm{d}t\,Na^3K_\phi\delta\phi$$

where we used

$$\delta_\phi K = K_X\delta_\phi X + K_\phi\delta\phi = K_X\dot{\phi}\frac{d}{dt}\left(\delta_\phi\phi\right) + K_\phi\delta_\phi\phi$$

Using integration by parts and setting $N=1$, we obtain

$$\delta_\phi S_2 = \int \mathrm{d}^3x\,\mathrm{d}t\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3K_X\dot{\phi}\right) + Na^3K_\phi\right)\delta_\phi\phi.$$

Therefore,

$$\frac{1}{Na^3}\frac{\delta S_2}{\delta\phi} = 0 \Rightarrow$$
$$\frac{1}{a^3}\frac{d}{dt}\left(a^3 K_X \dot{\phi}\right) = K_\phi . \tag{B.2.2.1}$$

- $\mathcal{L}_3 = -\mathbf{G_3}(\phi, \mathbf{X})\Box\phi$

The action for the second part of the Lagrangian is

$$S_3 = \int \mathrm{d}^3x\,\mathrm{d}t\, Na^3(-G_3)\left(-\frac{\ddot{\phi}}{N^2} - \frac{3\dot{a}\dot{\phi}}{aN^2} + \frac{\dot{N}}{N^3}\dot{\phi}\right)$$

Variation with respect to ϕ yields

$$\delta_\phi S_3 = \int d^3x dt Na^3\left(\delta_\phi G_3\right)\left(\frac{\ddot{\phi}}{N^2} + \frac{3\dot{a}\dot{\phi}}{aN^2} - \frac{\dot{N}}{N^3}\dot{\phi}\right) + \int d^3x dt Na^3 G_3 \delta_\phi\left(\frac{\ddot{\phi}}{N^2} + \frac{3\dot{a}\dot{\phi}}{aN^2} - \frac{\dot{N}}{N^3}\dot{\phi}\right) =$$

$$\int \mathrm{d}^3x\,\mathrm{d}t\, Na^3\left(G_{3X}\dot{\phi}\frac{d}{dt}(\delta_\phi\phi) + G_{3\phi}\,\delta_\phi\phi\right)\left(\frac{\ddot{\phi}}{N^2} + \frac{3\dot{a}\dot{\phi}}{aN^2}\right) + \int d^3x dt Na^3 G_3 \delta_\phi\left(\frac{\ddot{\phi}}{N^2} + \frac{3\dot{a}\dot{\phi}}{aN^2}\right)$$

Note that we omit the term involving $\dot{N}$ since it won't contribute when we set $N = 1$. Therefore,

$$\begin{aligned}\delta_\phi S_3 = &\int \mathrm{d}^3x\,\mathrm{d}t\, a^3\left(\ddot{\phi}G_{3\phi} + 3G_{3\phi}H\dot{\phi}\right)\delta_\phi\phi + \int \mathrm{d}^3x\,\mathrm{d}t\, a^3 G_{3X}\dot{\phi}\ddot{\phi}\frac{d}{dt}\left(\delta_\phi\phi\right)\\ &+ \int \mathrm{d}^3x\,\mathrm{d}t\, 3a^3 H\dot{\phi}^2 G_{3X}\frac{d}{dt}\left(\delta_\phi\phi\right) + \int \mathrm{d}^3x\,\mathrm{d}t\, a^3 G_3\,\delta_\phi\ddot{\phi}\\ &+ \int \mathrm{d}^3x\,\mathrm{d}t\, 3a^3 G_3 H\,\delta_\phi\dot{\phi}\end{aligned}$$

In the following calculations, we employ integration by parts to express all variational terms in the form $\int(...)\delta_\phi\phi$ thereby eliminating derivatives of the field variation $\delta_\phi\phi$ in the action. Note that the term involving the second derivative of the variation of the scalar field does not change sign, since we perform integration by parts twice.

$$\begin{aligned}\int \mathrm{d}^3x\,\mathrm{d}t\, a^3 G_{3X}\ddot{\phi}\dot{\phi}\frac{\mathrm{d}}{\mathrm{d}t}(\delta_\phi\phi) &= -\int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(a^3 G_{3X}\ddot{\phi}\dot{\phi}\Big)\delta_\phi\phi\\ \int \mathrm{d}^3x\,\mathrm{d}t\, 3a^3 H\dot{\phi}^2 G_{3X}\frac{\mathrm{d}}{\mathrm{d}t}(\delta_\phi\phi) &= -\int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(3a^3 H\dot{\phi}^2 G_{3X}\Big)\delta_\phi\phi\\ \int \mathrm{d}^3x\,\mathrm{d}t\, a^3 G_3\delta_\phi\ddot{\phi} &= \int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(a^3 G_3\Big)\delta_\phi\phi\\ \int \mathrm{d}^3x\,\mathrm{d}t\, a^3 3G_3 H\delta_\phi\dot{\phi} &= -\int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(3a^3 G_3 H\Big)\delta_\phi\phi\end{aligned}$$

Substituting, we obtain

$$\begin{aligned}\delta_\phi S_3 = \int \mathrm{d}^3x\,\mathrm{d}t\Big[&\ddot{\phi}a^3 G_{3\phi} + 3H\dot{\phi}a^3 G_{3\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(a^3 G_{3X}\ddot{\phi}\dot{\phi}\right)\\ &- \frac{\mathrm{d}}{\mathrm{d}t}\left(3a^3 H\dot{\phi}^2 G_{3X}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(a^3 G_3\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(3a^3 G_3 H\right)\Big]\delta_\phi\phi.\end{aligned} \tag{B.2.2.2}$$

The term that involves the second derivative can be written as

$$\int d^3x dt\frac{d}{dt}\Big(3a^3 H G_3 + a^3 G_{3\phi}\dot{\phi} + a^3 G_{3X}\dot{\phi}\ddot{\phi}\Big)$$

In this form, one can easily see that its first and third terms cancel out the third and last term of (B.2.2.2). Using $X = \frac{\dot{\phi}}{2}$ we end up with

$$\int \mathrm{d}^3x\,\mathrm{d}t\left[\ddot{\phi}a^3 G_{3\phi} + 3H\dot{\phi}a^3 G_{3\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\Big(6a^3 HXG_{3X}\Big) + \frac{\mathrm{d}}{\mathrm{d}t}\Big(a^3 G_{3\phi}\dot{\phi}\Big)\right] =$$

$$\int \mathrm{d}^3x\,\mathrm{d}t\left[2\ddot{\phi}a^3G_{3\phi}+6H\dot{\phi}a^3G_{3\phi}-\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3HXG_{3X}\right)+a^3G_{3\phi\phi}\dot{\phi}^2+a^3G_{3\phi X}\dot{\phi}\dot{X}\right].$$

Employing $\dot{X}=\dot{\phi}\ddot{\phi}$, $\frac{\dot{\phi}^2}{2}=X$ and noticing that

$$2Xa^3G_{3\phi\phi}+2Xa^3G_{3\phi X}\ddot{\phi}+6a^3HG_{3\phi}\dot{\phi}+2a^3G_{3\phi}\ddot{\phi}=\frac{d}{dt}\left(2a^3\dot{\phi}G_{3\phi}\right)-2Xa^3G_{3\phi\phi}-2Xa^3G_{3\phi X}\ddot{\phi},$$

we obtain

$$\delta_\phi S_3=\int \mathrm{d}^3x\,\mathrm{d}t\left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3HXG_{3X}\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(2a^3\dot{\phi}G_{3\phi}\right)-2Xa^3\left(G_{3\phi\phi}+G_{3\phi X}\ddot{\phi}\right)\right]\delta_\phi\phi$$

Therefore,

$$\frac{1}{a^3N}\frac{\delta S_3}{\delta_\phi\phi}=0\Rightarrow$$

$$\frac{1}{a^3}\frac{d}{dt}\left(6a^3HXG_{3X}-2a^3\dot{\phi}G_{3\phi}\right)=-2X\left(G_{3\phi\phi}+G_{3\phi X}\ddot{\phi}\right) \tag{B.2.2.3}$$

- $\mathbf{\mathcal{L}_4=G_4(\phi,X)R+G_{4X}\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right]}$

The action is

$$S_4=\int d^3xdtNa^3\left[6G_4\left(\frac{\ddot{\alpha}}{N^2\alpha}-\frac{H\dot{N}}{N^3}+\frac{H^2}{N^2}\right)+G_{4X}\left(\frac{6H\dot{\phi}\ddot{\phi}}{N^4}+\frac{6H^2\dot{\phi}^2}{N^4}\right)\right].$$

Variation with respect to ϕ yields

$$\delta_\phi S_4=\int d^3x\,dt\,Na^3 6\left(\frac{\ddot{\alpha}}{N^2\alpha}+\frac{H^2}{N^2}\right)\delta_\phi G_4+\int d^3x\,dt\,Na^3\left(\frac{6H\dot{\phi}\ddot{\phi}}{N^4}+\frac{6H^2\dot{\phi}^2}{N^4}\right)\delta_\phi G_{4X}$$
$$+\int d^3x\,dt\,Na^3G_{4X}\delta_\phi\left(\frac{6H\dot{\phi}\ddot{\phi}}{N^4}+\frac{6H^2\dot{\phi}^2}{N^4}\right). \tag{B.2.2.4}$$

In what follows, we calculate each term individually. We begin with the first term.

$$\begin{aligned}\delta_\phi S_{4,1}&=\int \mathrm{d}^3x\,\mathrm{d}t\,Na^3 6\left(\frac{\ddot{\alpha}}{N^2\alpha}+\frac{H^2}{N^2}\right)\delta_\phi G_4\\&=\int \mathrm{d}^3x\,\mathrm{d}t\,a^3 6\left(\frac{\ddot{a}}{a}+H^2\right)\left(G_{4\phi}\delta_\phi\phi+G_{4X}\dot{\phi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_\phi\phi\right)\right)\\&=\int \mathrm{d}^3x\,\mathrm{d}t\left(6a^3\frac{\ddot{a}}{a}G_{4\phi}+6a^3H^2G_{4\phi}\right)\delta_\phi\phi\\&\quad-\int \mathrm{d}^3x\,\mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3\frac{\ddot{a}}{a}G_{4X}\dot{\phi}\right)\delta_\phi\phi-\int \mathrm{d}^3x\,\mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3H^2G_{4X}\dot{\phi}\right)\delta_\phi\phi.\end{aligned} \tag{B.2.2.5}$$

For the second term of (B.2.2.4) we have

$$\begin{aligned}&\int \mathrm{d}^3x\,\mathrm{d}t\,Na^3\left(\frac{6H\dot{\phi}\ddot{\phi}}{N^4}+\frac{6H^2\dot{\phi}^2}{N^4}\right)\delta_\phi G_{4X}\\&=\int \mathrm{d}^3x\,\mathrm{d}t\,a^3\left(\frac{6H\dot{\phi}\ddot{\phi}}{N^3}+\frac{6H^2\dot{\phi}^2}{N^3}\right)\left(G_{4X\phi}\delta_\phi\phi+G_{4XX}\dot{\phi}\frac{\mathrm{d}}{\mathrm{d}t}(\delta_\phi\phi)\right)\\&=\int \mathrm{d}^3x\,\mathrm{d}t\left(6a^3H\dot{\phi}\ddot{\phi}G_{4X\phi}+6a^3H^2\dot{\phi}^2G_{4X\phi}\right)\delta_\phi\phi\\&\quad-\int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3H\dot{\phi}^2\ddot{\phi}G_{4XX}\right)\delta_\phi\phi-\int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3H^2\dot{\phi}^3G_{4XX}\right)\delta_\phi\phi.\end{aligned} \tag{B.2.2.6}$$

For the third term,

$$\begin{aligned}
&\int \mathrm{d}^3x\,\mathrm{d}t\, Na^3 G_{4X}\,\delta_\phi\left(\frac{6H\dot\phi\ddot\phi}{N^4}+\frac{6H^2\dot\phi^2}{N^4}\right)\\
&=\int \mathrm{d}^3x\,\mathrm{d}t\,a^3G_{4X}\,6H\,\delta_\phi(\dot\phi\ddot\phi)+\int \mathrm{d}^3x\,\mathrm{d}t\,a^3G_{4X}\,6H^2\,2\dot\phi\frac{\mathrm{d}}{\mathrm{d}t}(\delta_\phi\phi)\\
&=-\int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(6a^3HG_{4X}\ddot\phi\Big)\delta_\phi\phi+\int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(6a^3HG_{4X}\dot\phi\Big)\delta_\phi\phi\\
&\quad-\int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(12a^3H^2G_{4X}\dot\phi\Big)\delta_\phi\phi. \qquad \text{(B.2.2.7)}
\end{aligned}$$

Substituting the results into (B.2.2.4) we find

$$\begin{aligned}
\delta_\phi S_4=\int \mathrm{d}^3x\,\mathrm{d}t\,&\Bigg[6a^3\left(\frac{\ddot a}{a}+H^2\right)G_{4\phi}-\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3\frac{\ddot a}{a}G_{4X}\dot\phi\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3H^2G_{4X}\dot\phi\right)\\
&+6a^3G_{4X\phi}H\,\dot\phi\ddot\phi\,\phi+6a^3G_{4X\phi}H^2\dot\phi^2-\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3H\dot\phi^2\ddot\phi G_{4XX}\right)\Bigg]\\
&-\int \mathrm{d}^3x\,\mathrm{d}t\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3HG_{4X}\ddot\phi\right)-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(6a^3HG_{4X}\dot\phi\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(12a^3H^2G_{4X}\dot\phi\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3H^2\dot\phi^3G_{4XX}\right)\right]\delta_\phi\phi.
\end{aligned}$$

Using $\frac{\ddot a}{a}=H^2+\dot H$, and calculating the first derivative of the term $\frac{d^2}{dt^2}$ leads to

$$\begin{aligned}
\delta_\phi S_4=\int \mathrm{d}^3x\,\mathrm{d}t\,a^3&\Bigg[6\left(2H^2+\dot H\right)G_{4\phi}+6H\left(\dot X+2HX\right)G_{4\phi X}\\
&-\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a^3\dot\phi 12H^2X\,G_{4XX}+6H^2\dot\phi a^3\,G_{4X}-12a^3HX\,G_{4\phi X}\Big)\Bigg]\delta_\phi\phi
\end{aligned}$$

Therefore,

$$\frac{1}{a^3}\frac{\delta S_4}{\delta\phi}=0\Rightarrow$$

$$\frac{1}{a^3}\frac{d}{dt}\Big[a^3\big(6H^2\dot\phi\left(G_{4X}+2XG_{4XX}\right)-12HXG_{4\phi X}\big)\Big]=6\left(2H^2+\dot H\right)G_{4\phi}+6H\left(\dot X+2HX\right)G_{4\phi X} \qquad \text{(B.2.2.8)}$$

- $\mathcal{L}_\mathbf{5}=\mathbf{G_5(\phi,X)G}_{\mu\nu}\nabla^\mu\nabla^\nu\phi-\frac{\mathbf{G_{5X}}}{\mathbf{6}}\left[(\Box\phi)^\mathbf{3}-\mathbf{3}(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^\mathbf{2}+\mathbf{2}(\nabla_\mu\nabla_\nu\phi)^\mathbf{3}\right]$

The action for the last part of the Horndeski Lagrangian evaluated in flat FLRW spacetime is

$$S_5=\int d^3x\,dt\,Na^3\left[G_5\left(\frac{6\ddot\alpha H\dot\phi}{aN^4}+\frac{3H^3\dot\phi}{N^4}+\frac{3H^2\ddot\phi}{N^4}\right)+G_{5X}\left(\frac{3H^2\dot\phi^2\ddot\phi}{N^6}+\frac{H^3\dot\phi^3}{N^6}\right)\right].$$

Variation with respect to ϕ yields

$$\begin{aligned}
\delta_\phi S_5=&\int \mathrm{d}^3x\,\mathrm{d}t\,Na^3\delta_\phi(G_5)\left(\frac{6\ddot\alpha H\dot\phi}{aN^4}+\frac{3H^3\dot\phi}{N^4}+\frac{3H^2\ddot\phi}{N^4}\right)+\int \mathrm{d}^3x\,\mathrm{d}t\,Na^3G_5\delta_\phi\left(\frac{6\ddot\alpha H\dot\phi}{aN^4}+\frac{3H^3\dot\phi}{N^4}+\frac{3H^2\ddot\phi}{N^4}\right)\\
&+\int \mathrm{d}^3x\,\mathrm{d}t\,Na^3\delta_\phi(G_{5X})\left(\frac{3H^2\dot\phi^2\ddot\phi}{N^6}+\frac{H^3\dot\phi^3}{N^6}\right)+\int \mathrm{d}^3x\,\mathrm{d}t\,Na^3G_{5X}\delta_\phi\left(\frac{3H^2\dot\phi^2\ddot\phi}{N^6}+\frac{H^3\dot\phi^3}{N^6}\right)
\end{aligned} \qquad \text{(B.2.2.9)}$$

We calculate each line individually. The first line is

$$\begin{aligned}
\delta_\phi S_{5,1}=&\int d^3x\,dt\,Na^3G_5\left(\frac{6\ddot aH}{aN^4}\frac{d}{dt}(\delta_\phi\phi)+\frac{3H^3}{N^4}\frac{d}{dt}(\delta_\phi\phi)+\frac{3H^2}{N^4}\frac{d^2}{dt^2}(\delta_\phi\phi)\right)\\
&+\int d^3x\,dt\,Na^3\left(G_{5X}\dot\phi\frac{d}{dt}(\delta_\phi\phi)+G_{5\phi}\delta_\phi\phi\right)\left(\frac{6\ddot\alpha H\dot\phi}{aN^4}+\frac{3H^3\dot\phi}{N^4}+\frac{3H^2\ddot\phi}{N^4}\right)
\end{aligned}$$

which, after some simple calculations and by performing integration by parts, becomes

$$\delta_\phi S_{5,1} = \int \mathrm{d}^3x\,\mathrm{d}t \Bigg[-\frac{\mathrm{d}}{\mathrm{d}t}\left(6a^3 G_5 H \frac{\ddot{a}}{a} + 3a^3 H^3 G_5 + 6a^3 G_{5X} H \dot{\phi}^2 \frac{\ddot{\alpha}}{a} + 3a^3 G_{5X} H^3 \dot{\phi}^2 + 3a^3 G_{5X} H^2 \dot{\phi}\ddot{\phi} \right)$$
$$+ 6a^3 G_{5\phi} H \dot{\phi} \frac{\ddot{a}}{a} + 3a^3 G_{5\phi} H^3 \dot{\phi} + 3a^3 G_{5\phi} H^2 \ddot{\phi} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(3a^3 G_5 H^2\right) \Bigg] \delta_\phi \phi. \quad \text{(B.2.2.10)}$$

For the second line, we get

$$\delta_\phi S_{5,2} = \int \mathrm{d}^3x\,\mathrm{d}t\, N a^3 \left(G_{5XX} \dot{\phi} \frac{\mathrm{d}}{\mathrm{d}t}(\delta_\phi \phi) + G_{5X\phi} \delta_\phi \phi \right) \left(\frac{3H^2 \dot{\phi}^2 \ddot{\phi}}{N^6} + \frac{H^3 \dot{\phi}^3}{N^6} \right)$$
$$+ \int \mathrm{d}^3x\,\mathrm{d}t\, N a^3 G_{5X} \frac{3H^3 \dot{\phi}^2}{N^6} \frac{\mathrm{d}}{\mathrm{d}t}(\delta_\phi \phi) + \int \mathrm{d}^3x\,\mathrm{d}t\, N a^3 G_{5X} \frac{3H^2}{N^6} \delta_\phi (\dot{\phi}^2 \ddot{\phi}).$$

Setting $(N = 1)$, after some simple calculations, we obtain the following

$$\delta_\phi S_{5,2} = \int \mathrm{d}^3x\,\mathrm{d}t \Bigg[-\frac{\mathrm{d}}{\mathrm{d}t}\left(3a^3 G_{5XX} H^2 \dot{\phi}^3 \ddot{\phi} + a^3 G_{5XX} H^3 \dot{\phi}^4 + 3a^3 G_{5X} H^3 \dot{\phi}^2 + 6a^3 G_{5X} H^2 \dot{\phi}\ddot{\phi} \right)$$
$$+ 3a^3 G_{5X\phi} H^2 \dot{\phi}^2 \ddot{\phi} + a^3 G_{5X\phi} H^3 \dot{\phi}^3 + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(3a^3 G_{5X} H^2 \dot{\phi}^2 \right) \Bigg] \delta_\phi \phi. \quad \text{(B.2.2.11)}$$

Substituting equations (B.2.2.6), (B.2.2.7) into (B.2.2.5)

$$\delta_\phi S_5 = -\int d^3x\, dt \frac{d}{dt}\Big(6a^3 G_5 H^3 + \cancel{6a^3 G_5 H \dot{H}} + 3a^3 G_5 H^3 + \underline{6a^3 G_{5X} H^3 \dot{\phi}^2}$$
$$+ \cancel{6a^3 G_{5X} H \dot{H} \dot{\phi}^2} + \underline{3a^3 G_{5X} H^3 \dot{\phi}^2} + \cancel{3a^3 G_{5X} H^2 \dot{\phi}\ddot{\phi}} \Big) \delta_\phi \phi$$
$$+ \int d^3x\, dt \frac{d}{dt}\Big(9a^3 G_5 H^3 + \cancel{3a^3 G_{5X} \dot{X} H^2} + 3a^3 G_{5\phi} H^2 \dot{\phi} + \cancel{6a^3 G_5 H \dot{H}} \Big) \delta_\phi \phi$$
$$+ \int d^3x\, dt \Big(6a^3 G_{5\phi} H^3 \dot{\phi} + 6a^3 G_{5\phi} H \dot{H} \dot{\phi} + 3a^3 G_{5\phi} H^3 \dot{\phi}$$
$$+ 3a^3 G_{5\phi} H^2 \ddot{\phi} + 3a^3 G_{5\phi X} H^2 \dot{\phi}^2 \ddot{\phi} + a^3 G_{5X\phi} H^3 \dot{\phi}^3 \Big) \delta_\phi \phi$$
$$- \int d^3x\, dt \frac{d}{dt}\Big(\cancel{3a^3 G_{5XX} H^2 \dot{\phi}^3 \ddot{\phi}} + a^3 G_{5XX} H^3 \dot{\phi}^4 + \underline{3a^3 G_{5X} H^3 \dot{\phi}^2} + \cancel{6a^3 G_{5X} H^2 \dot{\phi}\ddot{\phi}} \Big) \delta_\phi \phi$$
$$+ \int d^3x\, dt \frac{d}{dt}\Big(\underline{9a^3 G_{5X} H^3 \dot{\phi}^2} + \cancel{6a^3 G_{5X} H \dot{H} \dot{\phi}^2} + \cancel{6a^3 G_{5X} H^2 \dot{\phi}\ddot{\phi}}$$
$$+ \cancel{3a^3 G_{5XX} H^2 \dot{X} \dot{\phi}^2} + 3a^3 G_{5X\phi} H^2 \dot{\phi}^3 \Big) \delta_\phi \phi$$

where we used,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(3a^3 G_{5X} H^2 \dot{\phi}^2 \right) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(9a^3 G_{5X} H^3 \dot{\phi}^2 + 6a^3 G_{5X} H \dot{H} \dot{\phi}^2 + 6a^3 G_{5X} H^2 \dot{\phi}\ddot{\phi}$$
$$+ 3a^3 G_{5XX} H^2 \dot{X} \dot{\phi}^2 + 3a^3 G_{5X\phi} H^2 \dot{\phi}^3 \Big)$$

and,

$$\dot{X} = \dot{\phi}\ddot{\phi}.$$

Simplifying,

$$\delta_\phi S_5 = -\int d^3x\, dt \frac{d}{dt}\Big(\cancel{6a^3 G_5 H^3} + \cancel{3a^3 G_5 H^3} \Big) \delta_\phi \phi + \int d^3x\, dt \frac{d}{dt}\Big(\cancel{9a^3 G_5 H^3} + 3a^3 G_{5\phi} H^2 \dot{\phi} \Big) \delta_\phi \phi$$
$$+ \int d^3x\, dt \Big(9a^3 G_{5\phi} H^3 \dot{\phi} + 6a^3 G_{5\phi} H \dot{H} \dot{\phi} + 3a^3 G_{5\phi} H^2 \ddot{\phi} + 3a^3 G_{5\phi X} H^2 \dot{\phi}^2 \ddot{\phi} + a^3 G_{5X\phi} H^3 \dot{\phi}^3 \Big) \delta_\phi \phi$$
$$- \int d^3x\, dt\, \underline{\frac{d}{dt}\Big(a^3 G_{5XX} H^3 \dot{\phi}^4 \Big) \delta_\phi \phi} + \int d^3x\, dt\, \underline{\frac{d}{dt}\Big(3a^3 G_{5X\phi} H^2 \dot{\phi}^3 \Big) \delta_\phi \phi} - \int d^3x dt \frac{d}{dt}\Big(3a^3 G_{5X} H^3 \dot{\phi}^2 \Big) \delta_\phi \phi.$$

The underlined terms can be written as

$$\frac{d}{dt}\left(3a^3 G_{5X\phi} H^2 \dot{\phi}^3\right) = \frac{d}{dt}\left(6a^3 G_{5X\phi} H^2 X \dot{\phi}\right)$$

$$\frac{d}{dt}\left(a^3 G_{5XX} H^3 \dot{\phi}^4\right) = \frac{d}{dt}\left(2Xa^3 H^3 (2XG_{5XX})\right)$$

Also, the second line can be rewritten as

$$\begin{aligned} &9a^3 G_{5\phi} H^3 \dot{\phi} + 6a^3 G_{5\phi} H \dot{H} \dot{\phi} + 3a^3 G_{5\phi} H^2 \ddot{\phi} + 3a^3 G_{5\phi X} H^2 \dot{\phi}^2 \ddot{\phi} + a^3 G_{5X\phi} H^3 \dot{\phi}^3 \\ &= \frac{d}{dt}\left(3a^3 H^2 \dot{\phi} G_{5\phi}\right) + a^3 H^3 \dot{\phi}^3 G_{5\phi X} - 3a^3 G_{5\phi\phi} H^2 \dot{\phi}^2. \end{aligned}$$

Using the above, we find

$$\begin{aligned} \delta_\phi S_5 = &\int d^3x\,dt\,\frac{d}{dt}\left(6a^3 G_{5\phi} H^2 \dot{\phi}\right) + \int d^3x\,dt\left(a^3 G_{5\phi X} H^3 \dot{\phi}^3 - 3a^3 G_{5\phi\phi} H^2 \dot{\phi}^2\right) \\ &- \int d^3x\,dt\,\frac{d}{dt}\left(2Xa^3 H^3 2XG_{5XX}\right) + \int d^3x\,dt\,\frac{d}{dt}\left(6a^3 G_{5X\phi} H^2 X \dot{\phi}\right) - \int d^3x\,dt\,\frac{d}{dt}\left(3a^3 G_{5X} H^3 \dot{\phi}^2\right) \Rightarrow \end{aligned}$$

$$\begin{aligned} \delta_\phi S_5 = \int d^3x\,dt \Bigg[&\frac{d}{dt}\left[a^3\left(6G_{5\phi} H^2 \dot{\phi} - 6G_{5X} H^3 X + 6G_{5X\phi} H^2 \dot{\phi} X - 2XH^3 2XG_{5XX}\right)\right] \\ &- 3a^3 2XH^2 G_{5\phi\phi} + 2a^3 X G_{5X\phi} H^3 \dot{\phi}\Bigg] \delta_\phi \phi = 0. \end{aligned}$$

Therefore,

$$\frac{1}{a^3}\frac{\delta_\phi S_5}{\delta_\phi \phi} = 0 \Rightarrow$$

$$\frac{1}{a^3}\frac{d}{dt}\left[a^3\left(2H^3 X(3G_{5X} + 2XG_{5XX}) - 6H^2 \dot{\phi}(G_{5\phi} + XG_{5X\phi})\right)\right] = -6XH^2 G_{5\phi\phi} + 2XG_{5X\phi} H^3 \dot{\phi} \tag{B.2.2.12}$$

B.2.3 Evolution Equation

In this section, we consider the variation of the Horndeski Lagrangian with respect to the scale factor $a(t)$, in order to derive the evolution equation in a flat FLRW metric.

- $\mathcal{L}_2 = \mathbf{K}(\phi, \mathbf{X})$

The first part of the theory in this spacetime takes the following form:

$$S_2 = \int \mathrm{d}^3x\,\mathrm{d}t\, N a^3 K(\phi, X).$$

Variation with respect to $a(t)$ is straightforward,

$$\delta_a S_2 = \int \mathrm{d}^3x\,\mathrm{d}t\, 3Na^2 K(\phi, X) \delta a$$

Setting $N = 1$ we obtain

$$\frac{1}{3a^2}\frac{\delta S_2}{\delta a} = P_2 = K(\phi, X) \tag{B.2.3.1}$$

- $\mathcal{L}_3 = -\mathbf{G_3}(\phi, \mathbf{X})\Box\phi$

The action for the second part of the Lagrangian is

$$S_3 = \int d^3xdtNa^3(G_3)\left(\frac{\ddot{\phi}}{N^2} + \frac{3\dot{a}\dot{\phi}}{aN^2} - \frac{\dot{N}}{N^3}\dot{\phi}\right).$$

Variation with respect to $a(t)$ yields[6]

$$\delta_a S_3 = \int d^3xdt3G_3Na^2\left(\frac{\ddot{\phi}}{N^2} + \frac{3H\dot{\phi}}{N^2}\right)\delta a + \int d^3xdtNa^3G_3\frac{3\dot{\phi}}{N^2}\delta_a\left(\frac{\dot{a}}{a}\right) =$$

$$\int d^3xdt3G_3Na^2\left(\frac{\ddot{\phi}}{N^2} + \frac{3H\dot{\phi}}{N^2}\right)\delta a - \int d^3xdtNa^3G_3\frac{3\dot{\phi}}{N^2}\frac{\dot{a}}{a^2}\delta a + \int d^3xdtNa^3G_3\frac{3\dot{\phi}}{N^2}\frac{1}{a}\frac{d}{dt}(\delta a).$$

Integrating by parts the last term we find

$$\delta_a S_3 = \int \mathrm{d}^3x\,\mathrm{d}t\, G_3\left(\frac{3a^2\ddot{\phi}}{N} + \frac{9H\dot{\phi}a^2}{N} - \frac{3a\dot{a}\dot{\phi}}{N}\right)\delta a - \int \mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(3a^2G_3\dot{\phi}\right)\frac{1}{N}\delta a =$$

$$\int \mathrm{d}^3x\,\mathrm{d}t\left[\cancel{3G_3a^2\ddot{\phi}} + \cancel{9G_3H\dot{\phi}a^2} - \cancel{3G_3a\dot{a}\dot{\phi}} - \cancel{6a\dot{a}G_3\dot{\phi}} - \cancel{3a^2G_3\ddot{\phi}} - 3a^2G_{3\phi}\dot{\phi}^2 - 3a^2G_{3X}\dot{X}\dot{\phi}\right]\frac{1}{N}\delta a.$$

Setting $N = 1$ and simplifying the previous equation we obtain, the following

$$\frac{1}{3a^2}\frac{\delta S_3}{\delta a} = P_3 = -2X(G_{3\phi} + G_{3X}\ddot{\phi}). \tag{B.2.3.2}$$

- $\mathcal{L}_4 = \mathbf{G_4}(\phi, \mathbf{X})\mathbf{R} + \mathbf{G_{4X}}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$

The next part of the theory, evaluated in a flat FLRW spacetime is[7]

$$S_4 = \int \mathrm{d}^3x\,\mathrm{d}t\,Na^3\left[6G_4\left(\frac{\ddot{a}}{N^2a} + \frac{\dot{a}^2}{N^2a^2}\right) + G_{4X}\left(\frac{6\dot{a}\dot{\phi}\ddot{\phi}}{aN^4} + \frac{6\dot{a}^2\dot{\phi}^2}{a^2N^4}\right)\right].$$

Variation with respect to $a(t)$ yields

$$\begin{aligned}\delta_a S_4 = &\int \mathrm{d}^3x\,\mathrm{d}t\,3Na^2\left[6G_4\left(\frac{\ddot{a}}{N^2a} + \frac{\dot{a}^2}{N^2a^2}\right) + G_{4X}\left(\frac{6\dot{a}\dot{\phi}\ddot{\phi}}{aN^4} + \frac{6\dot{a}^2\dot{\phi}^2}{a^2N^4}\right)\right]\delta a + \int \mathrm{d}^3x\,\mathrm{d}t\,Na^3\,6G_4\,\delta_a\left[\frac{\ddot{a}}{N^2a} + \frac{\dot{a}^2}{N^2a^2}\right]\\ &+ \int \mathrm{d}^3x\,\mathrm{d}t\,Na^3G_{4X}\,\delta_a\left(\frac{6\dot{a}\dot{\phi}\ddot{\phi}}{aN^4} + \frac{6\dot{a}^2\dot{\phi}^2}{a^2N^4}\right).\end{aligned} \tag{B.2.3.3}$$

With the first term brought to its final form, we can proceed to calculate the second and third terms. For the second term, we have

$$\delta S_{4,2} = \delta S_{4,21} + \delta S_{4,22} \tag{B.2.3.4}$$

where,

$$\delta S_{4,21} = \int d^3xdt6Na^3G_4\delta_a\left(\frac{\ddot{a}}{N^2a}\right)$$

and

$$\delta S_{4,22} = \int d^3xdt6Na^3G_4\delta_a\left(\frac{\dot{a}^2}{N^2a^2}\right)$$

These two terms are computed individually. For the simpler $\delta S_{4,22}$, we obtain

[6]The third term is omitted, as it does not contribute when $N = 1$.

[7]As done previously, we omit terms that involve derivatives of N.

$$\delta_a S_{4,22} = \int d^3x dt (-12) N a^3 G_4 \Big(\frac{\dot{a}^2}{N^2 a^3}\Big)\delta a + \int d^3x dt 12 N a^3 G_4 \frac{\dot{a}}{N^2 a^2}\delta(\frac{d}{dt}a)$$

Using integration by parts, simplifying the first integral, and setting $N = 1$ we obtain

$$\delta_a S_{4,22} = \int d^3x dt (-12 G_4 \dot{a}^2) - \int d^3x dt 12 \Big(G_{4X}\dot{X} a\dot{a} + G_{4\phi}\dot{\phi} a\dot{a} + G_4\dot{a}^2 + G_4 a\ddot{a}\Big).$$

It then follows that

$$\frac{1}{3a^2}\frac{\delta_a S_{4,22}}{\delta a} = -4G_4H^2 - 4G_{4X}\dot{X}H - 4G_{4\phi}\dot{\phi}H - 4G_4H^2 - 4G_4\frac{\ddot{a}}{a} \tag{B.2.3.5}$$

Following the same procedure, we compute the remaining term of (B.2.3.4)

$$\delta S_{4,21} = \int d^3x dt 6 N a^3 G_4 \frac{\ddot{a}}{N^2}\Big(-\frac{1}{a^2}\Big)\delta a + \int d^3x dt N a^3 6 G_4 \frac{1}{N^2 a}\frac{d^2}{dt^2}(\delta a) =$$

$$\int d^3x dt (-6G_4\ddot{a}a)\delta a + \int d^3x dt \frac{d^2}{dt^2}(6G_4a^2)\delta a$$

where for the last line we have set $N = 1$. Calculating the derivatives and simplifying, we end up with

$$\begin{aligned}\frac{1}{3a^2}\frac{\delta_a S_{4,21}}{\delta a} = &-2G_4\frac{\ddot{a}}{a} - 2G_{4XX}\dot{X}^2 - 2G_{4X\phi}\dot{X}\dot{\phi} - 2G_{4X}\ddot{X} - G_{4X}\dot{X}H\\ &-2G_{4\phi\phi}\dot{\phi}^2 - 2G_{4\phi X}\dot{X}\dot{\phi} - 2G_{4\phi}\ddot{\phi} - 4G_{4\phi}\dot{\phi}H\\ &-4G_{4X}\dot{X}H \quad -4G_{4\phi}\dot{\phi}H \quad -4G_4H^2 - 4G_4\frac{\ddot{a}}{a}\end{aligned} \tag{B.2.3.6}$$

In a similar manner, we decompose the final term of (B.2.3.3) into two parts:

$$\delta_a S_{4,3} = \delta_a S_{4,31} + \delta_a S_{4,32} \tag{B.2.3.7}$$

where,

$$\delta_a S_{4,31} = \int d^3x dt N a^3 G_{4X}\delta_a\Big(\frac{6\dot{a}\dot{\phi}\ddot{\phi}}{aN^4}\Big)$$

and

$$\delta_a S_{4,32} = \int d^3x dt N a^3 G_{4X}\delta_a\Big(\frac{6\dot{a}^2\dot{\phi}^2}{a^2N^4}\Big).$$

For the first integral, we have

$$\delta_a S_{4,31} = \int \mathrm{d}^3x\,\mathrm{d}t\, N a^3 G_{4X}\left(-\frac{6\dot{a}\dot{\phi}\ddot{\phi}}{a^2N^4}\right)\delta a + \int \mathrm{d}^3x\,\mathrm{d}t\, N a^3 G_{4X}\frac{6\dot{\phi}\ddot{\phi}}{aN^4}\frac{\mathrm{d}}{\mathrm{d}t}(\delta a)$$

Setting $N = 1$,

$$\begin{aligned}&\int d^3x\, dt\, G_{4X} a\left(-6\dot{a}\dot{\phi}\ddot{\phi}\right)\delta a + \int d^3x\, dt\, 6a^2 G_{4X}\dot{\phi}\ddot{\phi}\,\frac{d}{dt}(\delta a)\\ &= \int d^3x\, dt\,\left(-6G_{4X}a\dot{a}\dot{\phi}\ddot{\phi}\right)\delta a - \int d^3x\, dt\, 6\frac{d}{dt}\left(a^2G_{4X}\dot{\phi}\ddot{\phi}\right)\delta a\\ &= \int d^3x\, dt\Big[-6G_{4X}a\dot{a}\dot{\phi}\ddot{\phi} - 6G_{4XX}\dot{X}\dot{\phi}\ddot{\phi}a^2 - 6G_{4X\phi}a^2\dot{\phi}^2\ddot{\phi}\\ &\qquad -6G_{4X}a^2\ddot{\phi}^2 - 6G_{4X}a^2\dot{\phi}\,\dddot{\phi} - 12a\dot{a}G_{4X}\dot{\phi}\ddot{\phi}\Big]\delta a.\end{aligned}$$

Therefore,

$$\frac{1}{3a^2}\frac{\delta S_{4,31}}{\delta a} = -2G_{4X}H\dot{\phi}\ddot{\phi} - 2G_{4XX}\dot{X}\dot{\phi}\ddot{\phi} - 2G_{4X\phi}\dot{\phi}^2\ddot{\phi} - 2G_{4X}\ddot{\phi}^2 - 2G_{4X}\dot{\phi}\dddot{\phi} - 4HG_{4X}\dot{\phi}\ddot{\phi} \quad \text{(B.2.3.8)}$$

For the second term of (B.2.3.7) we have

$$\delta_a S_{4,32} = \int \mathrm{d}^3x\,\mathrm{d}t\, Na^3 G_{4X}\delta_a\left(\frac{6\dot{a}^2\dot{\phi}^2}{a^2N^4}\right) = \int \mathrm{d}^3x\,\mathrm{d}t\, G_{4X}(-12)\dot{a}^2\dot{\phi}^2\delta a + \int \mathrm{d}^3x\,\mathrm{d}t\, aG_{4X}12\dot{a}\dot{\phi}^2\frac{\mathrm{d}}{\mathrm{d}t}(\delta a).$$

Using integration by parts we obtain

$$\delta S_{4,32} = \int d^3xdt\left[-12G_{4X}\dot{a}^2\dot{\phi}^2 - 12G_{4XX}a\dot{a}\dot{X}\dot{\phi}^2 - 12G_{4X\phi}\dot{a}a\dot{\phi}^3 - 12G_{4X}\ddot{a}a\dot{\phi}^2 - 12G_{4X}\dot{a}^2\dot{\phi}^2 - 24G_{4X}\dot{a}a\dot{\phi}\ddot{\phi}\right]\delta a \Rightarrow$$

$$\frac{1}{3a^2}\frac{\delta S_{4,32}}{\delta a} = -4G_{4X}H^2\dot{\phi}^2 - 4G_{4XX}H\dot{X}\dot{\phi}^2 - 4G_{4X\phi}H\dot{\phi}^3 - 4G_{4X}\frac{\ddot{a}}{a}\dot{\phi}^2 - 4G_{4X}H^2\dot{\phi}^2 - 8G_{4X}H\dot{\phi}\ddot{\phi}. \quad \text{(B.2.3.9)}$$

Combining the results (B.2.3.3), (B.2.3.5), (B.2.3.6), (B.2.3.8), (B.2.3.9) we obtain

$$\begin{aligned}\frac{1}{3a^2}\frac{\delta S_4}{\delta a} = &-\cancel{2G_4\frac{\ddot{a}}{a}} + 2G_{4XX}\dot{X}^2 + 2G_{4X\phi}\dot{X}\dot{\phi} + 2G_{4X}\ddot{X} + \cancel{4G_{4X}\dot{X}H} + 2G_{4\phi\phi}\dot{\phi}^2 + 2G_{4\phi X}\dot{X}\dot{\phi} + 2G_{4\phi}\ddot{\phi}\\ &+\cancel{4G_{4\phi}\dot{\phi}H} + 4G_{4X}\dot{X}H + 4G_{4\phi}H\dot{\phi} + \cancel{4G_4H^2} + 4G_4\frac{\ddot{a}}{a} - 4G_4H^2 - \cancel{4G_{4X}\dot{X}H} - \cancel{4G_{4\phi}\dot{\phi}H}\\ &-\cancel{4G_4H^2} - \cancel{4G_4\frac{\ddot{a}}{a}} - \cancel{2G_{4X}H\dot{\phi}\ddot{\phi}} - 2G_{4XX}\dot{X}\dot{\phi}\ddot{\phi} - 2G_{4X\phi}\dot{\phi}^2\ddot{\phi} - 2G_{4X}\ddot{\phi}^2 - 2G_{4X}\dot{\phi}\dddot{\phi}\cancel{-4G_{4X}H\dot{\phi}\ddot{\phi}}\\ &-4G_{4X}H^2\dot{\phi}^2 - 4G_{4XX}H\dot{X}\dot{\phi}^2 - 4G_{4X\phi}H\dot{\phi}^3 - 4G_{4X}\dot{\phi}^2\frac{\ddot{a}}{a} - 4G_{4X}H^2\dot{\phi}^2 - 8G_{4X}H\dot{\phi}\ddot{\phi} + \cancel{6G_4\frac{\ddot{a}}{a}}\\ &+6G_4H^2 + \cancel{6G_{4X}H\dot{\phi}\ddot{\phi}} + 6G_{4X}H^2\dot{\phi}^2\end{aligned}$$

By substituting the expressions for X, $\dot{X}$, and $\frac{\ddot{a}}{a}$ and simplifying, we arrive at

$$\begin{aligned}\frac{1}{3a^2}\frac{\delta S_4}{\delta a} = P_4 = &\,2G_4(3H^2+2\dot{H}) + 4XG_{4\phi\phi} - 8HX\dot{X}G_{4XX} + 4X(\ddot{\phi}-2H\dot{\phi})G_{4\phi X}\\ &+2G_{4\phi}(\ddot{\phi}+2H\dot{\phi}) - G_{4X}(12H^2X + 8\dot{H}X + 4H\dot{X})\end{aligned} \quad \text{(B.2.3.10)}$$

- $\mathcal{L}_5 = \mathbf{G_5}(\phi, \mathbf{X})\mathbf{G}_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{\mathbf{G_{5X}}}{\mathbf{6}}\left[(\Box\phi)^3 - \mathbf{3}(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + \mathbf{2}(\nabla_\mu\nabla_\nu\phi)^3\right]$

The final part of the action is

$$S_5 = \int d^3x\,dt\,Na^3\left[G_5\left(\frac{6\ddot{a}\dot{a}\dot{\phi}}{a^2N^4} + \frac{3\dot{a}^3\dot{\phi}}{a^3N^4} + \frac{3\dot{a}^2\ddot{\phi}}{a^2N^4}\right) + G_{5X}\left(\frac{3\dot{a}^2\dot{\phi}^2\ddot{\phi}}{a^2N^6} + \frac{\dot{a}^3\dot{\phi}^3}{a^3N^6}\right)\right]. \tag{B.2.3.11}$$

Following the same approach as before, we decompose the variation into three parts

$$\delta_a S_5 = \delta_a S_{5,1} + \delta_a S_{5,2} + \delta_a S_{5,3} \tag{B.2.3.12}$$

where,

$$\delta_a S_{5,1} = \int d^3x dt 3Na^2\left[G_5\left(\frac{6\ddot{a}\dot{a}\dot{\phi}}{a^2N^4} + \frac{3\dot{a}^3\dot{\phi}}{a^3N^4} + \frac{3\dot{a}^2\ddot{\phi}}{a^2N^4}\right) + G_{5X}\left(\frac{3\dot{a}^2\dot{\phi}^2\ddot{\phi}}{a^2N^6} + \frac{\dot{a}^3\dot{\phi}^3}{a^3N^6}\right)\right]\delta a \tag{B.2.3.13}$$

$$\delta_a S_{5,2} = \int d^3x dt Na^3 G_5 \delta_a\left(\frac{6\ddot{a}\dot{a}\dot{\phi}}{a^2N^4} + \frac{3\dot{a}^3\dot{\phi}}{a^3N^4} + \frac{3\dot{a}^2\ddot{\phi}}{a^2N^4}\right) \tag{B.2.3.14}$$

and

$$\delta_a S_{5,3} = \int d^3x dt Na^3 G_{5X} \delta_a\left(\frac{3\dot{a}^2\dot{\phi}^2\ddot{\phi}}{a^2N^6} + \frac{\dot{a}^3\dot{\phi}^3}{a^3N^6}\right) \tag{B.2.3.15}$$

The first part yields

$$\frac{1}{3a^2}\frac{\delta S_{5,1}}{\delta a} = 6G_5\frac{\ddot{a}}{a}H\dot{\phi} + 3G_5H^3\dot{\phi} + 3H^2\ddot{\phi}G_5 + 3G_{5X}H^2\dot{\phi}^2\ddot{\phi} + G_{5X}H^3\dot{\phi}^3. \tag{B.2.3.16}$$

For the second term (B.2.3.14) we have

$$\delta_a S_{5,2} = \delta_a S_{5,21} + \delta_a S_{5,22} + \delta_a S_{5,23}$$

where

$$\delta_a S_{5,21} = \int d^3x\,dt\,Na^3G_5\frac{6\dot{\phi}}{N^4}\delta_a\left(\frac{\ddot{a}\dot{a}}{a^2}\right) = \int d^3x\,dt\,a^3(-12)G_5\dot{\phi}\dot{a}\ddot{a}\delta a - \int d^3x\,dt\,6\frac{d}{dt}\left(\frac{a^3G_5\dot{\phi}\ddot{a}}{a^2}\right)\delta a$$
$$+ \int d^3x\,dt\,6\frac{d^2}{dt^2}\left(\frac{a^3G_5\dot{\phi}\dot{a}}{a^2}\right)\delta a.$$

Calculating the derivatives, we obtain the following for the second and third terms

$$\int d^3x\,dt\,6\frac{d^2}{dt^2}\left(\frac{a^3G_5\dot{\phi}\dot{a}}{a^2}\right)\delta a = \int d^3x\,dt\,6\Big(2\dot{a}\ddot{a}G_5\dot{\phi} + \dot{a}^2G_{5X}\dot{X}\dot{\phi} + \dot{a}^2G_{5\phi}\dot{\phi}^2 + \dot{a}^2G_5\ddot{\phi}$$
$$\dot{a}^2G_{5X}\dot{X}\dot{\phi} + aG_{5XX}\dot{X}^2\dot{\phi}\dot{a} + aG_{5X\phi}\dot{X}\dot{\phi}^2\dot{a} + aG_{5X}\ddot{X}\dot{\phi}\dot{a} + aG_{5X}\dot{X}\ddot{\phi}\dot{a}$$
$$+ aG_{5X}\dot{X}\dot{\phi}\ddot{a} + \dot{a}^2G_{5\phi}\dot{\phi}^2 + aG_{5\phi\phi}\dot{\phi}^3\dot{a} + aG_{5\phi X}\dot{X}\dot{\phi}^2\dot{a} + 2aG_{5\phi}\dot{\phi}\ddot{\phi}\dot{a}$$
$$+ aG_{5\phi}\dot{\phi}^2\ddot{a} + \dot{a}^2G_5\ddot{\phi} + aG_{5\phi}\dot{\phi}\ddot{\phi}\dot{a} + aG_{5X}\dot{X}\ddot{\phi}\dot{a} + aG_5\dddot{\phi}\dot{a}$$
$$+ aG_5\ddot{\phi}\ddot{a} + \dot{a}G_5\dot{\phi}\ddot{a} + aG_{5\phi}\dot{\phi}^2\ddot{a} + aG_{5X}\dot{\phi}\dot{X}\ddot{a} + aG_5\ddot{\phi}\ddot{a} + aG_5\dot{\phi}\dddot{a}\Big)\delta a$$

and

$$-\int d^3x\,dt\,6\frac{d}{dt}\left(\frac{a^3G_5\dot{\phi}\ddot{a}}{a^2}\right)\delta a = -\int d^3xdt6\Big(\dot{a}G_5\dot{\phi}\ddot{a} + aG_{5X}\dot{X}\dot{\phi}\ddot{a} + aG_{5\phi}\dot{\phi}^2\ddot{a} + aG_5\ddot{\phi}\ddot{a} + aG_5\dot{\phi}\dddot{a}\Big)$$

From which follows

$$\begin{aligned}\frac{1}{3a^2}\frac{\delta S_{5,21}}{\delta a} = & -4G_5\dot{\phi}H\frac{\ddot{a}}{a} - 2HG_5\dot{\phi}\frac{\ddot{a}}{a} - 2G_{5X}\dot{X}\dot{\phi}\frac{\ddot{a}}{a} - 2G_{5\phi}\dot{\phi}^2\frac{\ddot{a}}{a} - 2G_5\ddot{\phi}\frac{\ddot{a}}{a} - 2G_5\dot{\phi}\frac{\dddot{a}}{a}\\ & + 4H\frac{\ddot{a}}{a}G_5\dot{\phi} + 2H^2G_{5X}\dot{X}\dot{\phi} + 2H^2G_{5\phi}\dot{\phi}^2 + 2H^2G_5\ddot{\phi} + 2G_{5XX}\dot{X}^2\dot{\phi}H\\ & + 2G_{5X\phi}\dot{X}\dot{\phi}^2H + 2G_{5X}\ddot{X}\dot{\phi}H + 2G_{5X}\dot{X}\ddot{\phi}H + 2G_{5X}\dot{X}\dot{\phi}\frac{\ddot{a}}{a} + 2G_{5\phi\phi}\dot{\phi}^3H\\ & + 2G_{5\phi X}\dot{\phi}^2\dot{X}H + 4G_{5\phi}\dot{\phi}\ddot{\phi}H + 2G_{5\phi}\dot{\phi}^2\frac{\ddot{a}}{a} + 2G_{5\phi}\dot{\phi}\ddot{\phi}H + 2G_{5X}\dot{X}\ddot{\phi}H\\ & + 2G_5\dddot{\phi}H + 2G_5\ddot{\phi}\frac{\ddot{a}}{a} + 2HG_5\dot{\phi}\frac{\ddot{a}}{a} + 2G_{5\phi}\dot{\phi}^2\frac{\ddot{a}}{a} + 2G_{5X}\dot{\phi}\dot{X}\frac{\ddot{a}}{a} + 2G_5\dot{\phi}\frac{\dddot{a}}{a}\\ & + 2H^2G_{5X}\dot{X}\dot{\phi} + 2H^2G_{5\phi}\dot{\phi}^2 + 2H^2G_5\ddot{\phi} + 2G_5\ddot{\phi}\frac{\ddot{a}}{a}\end{aligned} \tag{B.2.3.17}$$

Similarly, we find

$$
\begin{aligned}
\delta S_{5,22} &= \int d^3x\,dt\,Na^3 G_5 \frac{3\dot{\phi}}{N^4}\delta_a\left(\frac{\dot{a}^3}{a^3}\right) \\
&= \int d^3x\,dt\left(-\frac{9G_5\dot{\phi}\dot{a}^3}{aN^3}\right)\delta a + \int d^3x\,dt\,G_5\frac{9\dot{\phi}\dot{a}^2}{N^3}\frac{d}{dt}(\delta a) \\
&= \int d^3x\,dt\left(-9G_5\dot{\phi}\frac{\dot{a}^3}{a}\right)\delta a - \int d^3x\,dt\,\frac{d}{dt}\left(9G_5\dot{\phi}\dot{a}^2\right)\delta a.
\end{aligned}
$$

Calculating the derivatives yields

$$
\frac{1}{3a^2}\frac{\delta S_{5,22}}{\delta a} = -3G_5\dot{\phi}H^3 - 3G_{5\phi}\dot{\phi}^2H^2 - 3G_{5X}\dot{X}\dot{\phi}H^2 - 3G_{5\phi}\ddot{\phi}H^2 - 6G_5\dot{\phi}H\frac{\ddot{a}}{a} \tag{B.2.3.18}
$$

For the final part of $\delta S_{5,2}$ we have

$$
\delta S_{5,23} = \int d^3xdtNa^3G_5\frac{3\ddot{\phi}}{N^4}\delta_a\left(\frac{\dot{a}^2}{a^2}\right) = \int d^3xdt(-6)G_5\ddot{\phi}\dot{a}^2\delta a + \int d^3xdtaG_5 6\ddot{\phi}\dot{a}\frac{d}{dt}(\delta a)
$$

which, after using integration by parts and evaluating the derivatives, yields

$$
\frac{1}{3a^2}\frac{\delta S_{5,23}}{\delta a} = -2G_5\ddot{\phi}H^2 - 2H^2G_5\ddot{\phi} - 2G_{5X}\dot{X}\ddot{\phi}H - 2G_{5\phi}\dot{\phi}\ddot{\phi}H - 2G_5\dddot{\phi}H - 2G_5\ddot{\phi}\frac{\ddot{a}}{a}. \tag{B.2.3.19}
$$

For the final term of (B.2.3.12), using a similar approach as above, we decompose the variation into two parts

$$
\delta_a S_{5,3} = \delta_a S_{5,31} + \delta_a S_{5,32}
$$

where,

$$
\delta S_{5,31} = \int d^3xdta^3G_{5X}\frac{3\dot{\phi}^2\ddot{\phi}}{N^5}\delta_a\left(\frac{\dot{a}^2}{a^2}\right) = \int d^3xdt\left(-6G_{5X}\dot{\phi}^2\ddot{\phi}\dot{a}^2\right)\delta a - \int d^3xdt6\frac{d}{dt}\left(aG_{5X}\dot{\phi}^2\ddot{\phi}\dot{a}\right)\delta a
$$

from which we obtain

$$
\begin{aligned}
\frac{1}{3a^2}\frac{\delta S_{5,31}}{\delta a} = &-2G_{5X}\dot{\phi}^2\ddot{\phi}H^2 - 2G_{5X}\dot{\phi}^2\ddot{\phi}H^2 - 2G_{5XX}\dot{X}\dot{\phi}^2\ddot{\phi}H - 2G_{5X\phi}\dot{\phi}^3\ddot{\phi}H \\
&- 4G_{5X}\dot{\phi}\ddot{\phi}^2H - 2G_{5X}\dot{\phi}^2\dddot{\phi}H - 2G_{5X}\dot{\phi}^2\ddot{\phi}\frac{\ddot{a}}{a}.
\end{aligned} \tag{B.2.3.20}
$$

As for the second part we have

$$
\delta S_{5,32} = \int d^3x\,dt\,a^3G_{5X}\frac{\dot{\phi}^3}{N^5}\delta_a\left(\frac{\dot{a}^3}{a^3}\right) = \int d^3x\,dt\left(-\frac{3G_{5X}\dot{\phi}^3\dot{a}^3}{a}\right)\delta a - \int d^3x\,dt\,\frac{d}{dt}\left(3G_{5X}\dot{\phi}^3\dot{a}^2\right)\delta a
$$

which yields

$$
\frac{1}{3a^2}\frac{\delta S_{5,32}}{\delta a} = -G_{5X}\dot{\phi}^3H^3 - G_{5XX}\dot{X}\dot{\phi}^3H^2 - G_{5X\phi}\dot{\phi}^4H^2 - 3G_{5X}\dot{\phi}^2\ddot{\phi}H^2 - 2G_{5X}\dot{\phi}^3H\frac{\ddot{a}}{a} \tag{B.2.3.21}
$$

Combining the above expressions, (B.2.3.16), (B.2.3.17), (B.2.3.18), (B.2.3.19), (B.2.3.20), (B.2.3.21), we obtain

$$
\begin{aligned}
\frac{1}{3a^2}\frac{\delta S_5}{\delta a} = {} & \cancel{G_5\left(6\frac{\ddot a}{a}H\dot\phi+3H^3\dot\phi+3H^2\ddot\phi\right)}+3G_{5X}H^2\dot\phi^2\ddot\phi+G_{5X}H^3\dot\phi^3-4G_{5X}\dot\phi^2\ddot\phi H^2 \\
& -2G_{5XX}\dot X\dot\phi^2\ddot\phi H-2G_{5X\phi}\dot\phi^3\ddot\phi H-4G_{5X}\dot\phi\ddot\phi^2H-2G_{5X}\dot\phi^2\dddot\phi H-2G_{5X}\dot\phi^2\ddot\phi\frac{\ddot a}{a} \\
& -G_{5X}\dot\phi^3H^3-G_{5XX}\dot X\dot\phi^3H^2-G_{5X\phi}\dot\phi^4H^2-3G_{5X}\dot\phi^2\ddot\phi H^2-2G_{5X}\dot\phi^3H\frac{\ddot a}{a} \\
& \cancel{-2G_5\ddot\phi H^2}\cancel{-2H^2G_5\ddot\phi}-2G_{5X}\dot X\ddot\phi H-2G_{5\phi}\dot\phi\ddot\phi H\cancel{-2G_5\dddot\phi H} \\
& \cancel{-2G_5\ddot\phi\frac{\ddot a}{a}}\cancel{-3G_5\dot\phi H^3}-3G_{5\phi}\dot\phi^2H^2-3G_{5X}\dot X\dot\phi H^2\cancel{-3G_5\ddot\phi H^2} \\
& \cancel{-6G_5\dot\phi H\frac{\ddot a}{a}}\cancel{-4G_5\dot\phi H\frac{\ddot a}{a}}\cancel{-2HG_5\dot\phi\frac{\ddot a}{a}}-2G_{5X}\dot X\dot\phi\frac{\ddot a}{a}-2G_{5\phi}\dot\phi^2\frac{\ddot a}{a} \\
& \cancel{-2G_5\ddot\phi\frac{\ddot a}{a}}\cancel{-2G_5\dot\phi\frac{\dddot a}{a}}\cancel{+4H\frac{\ddot a}{a}G_5\dot\phi}+2H^2G_{5X}\dot X\dot\phi+2H^2G_{5\phi}\dot\phi^2 \\
& \cancel{+2H^2G_5\ddot\phi}+2G_{5XX}\dot X^2\dot\phi H+2G_{5X\phi}\dot X\dot\phi^2H+2G_{5X}\ddot X\dot\phi H+2G_{5X}\dot X\ddot\phi H \\
& +2G_{5X}\dot X\dot\phi\frac{\ddot a}{a}+2G_{5\phi\phi}\dot\phi^3H+2G_{5\phi X}\dot\phi^2\dot XH+4G_{5\phi}\dot\phi\ddot\phi H+2G_{5\phi}\dot\phi^2\frac{\ddot a}{a} \\
& +2G_{5\phi}\dot\phi\ddot\phi H+2G_{5X}\dot X\ddot\phi H\cancel{+2G_5\dddot\phi H}\cancel{+2G_5\ddot\phi\frac{\ddot a}{a}}\cancel{+2HG_5\dot\phi\frac{\ddot a}{a}} \\
& +2G_{5\phi}\dot\phi^2\frac{\ddot a}{a}+2G_{5X}\dot\phi\dot X\frac{\ddot a}{a}\cancel{+2G_5\dot\phi\frac{\dddot a}{a}}+2H^2G_{5X}\dot X\dot\phi+2H^2G_{5\phi}\dot\phi^2 \\
& \cancel{+2H^2G_5\ddot\phi}\cancel{+2G_5\ddot\phi\frac{\ddot a}{a}}
\end{aligned}
$$

All the terms that involve the G_5 function cancel. We simplify the rest. For example, for the terms that involve $G_{5XX}, G_{5\phi}, G_{5\phi X}$ and $G_{5\phi}$, we have:

$$
G_{5XX}\left(-2\dot X\dot\phi^2\ddot\phi H-\dot X\dot\phi^3H^2+2\dot X^2\dot\phi H\right)=G_{5XX}\left(-2\dot X\dot\phi^2\ddot\phi H-\dot X\dot\phi^3H^2+2\dot X\dot\phi^2\ddot\phi H\right)=G_{5XX}\left(-4H^2X^2\ddot\phi\right),
$$

$$
G_{5\phi X}\left(-2\dot\phi^3\ddot\phi H-\dot\phi^4H^2+4\dot X\dot\phi^2H\right)=G_{5\phi X}\left(-4\dot XXH-4X^2H^2+8X\dot XH\right)=4HX\left(\dot X-XH\right)G_{5\phi X},
$$

$$
G_{5\phi}\left(\dot\phi^2H^2+4\dot\phi\ddot\phi H+2\dot\phi^2\frac{\ddot a}{a}\right)=G_{5\phi}\left(3H^2\dot\phi^2+4\dot XH+4X\dot H\right)=G_{5\phi}\left(3H^2X+2\dot{(XH)}\right)2,
$$

and

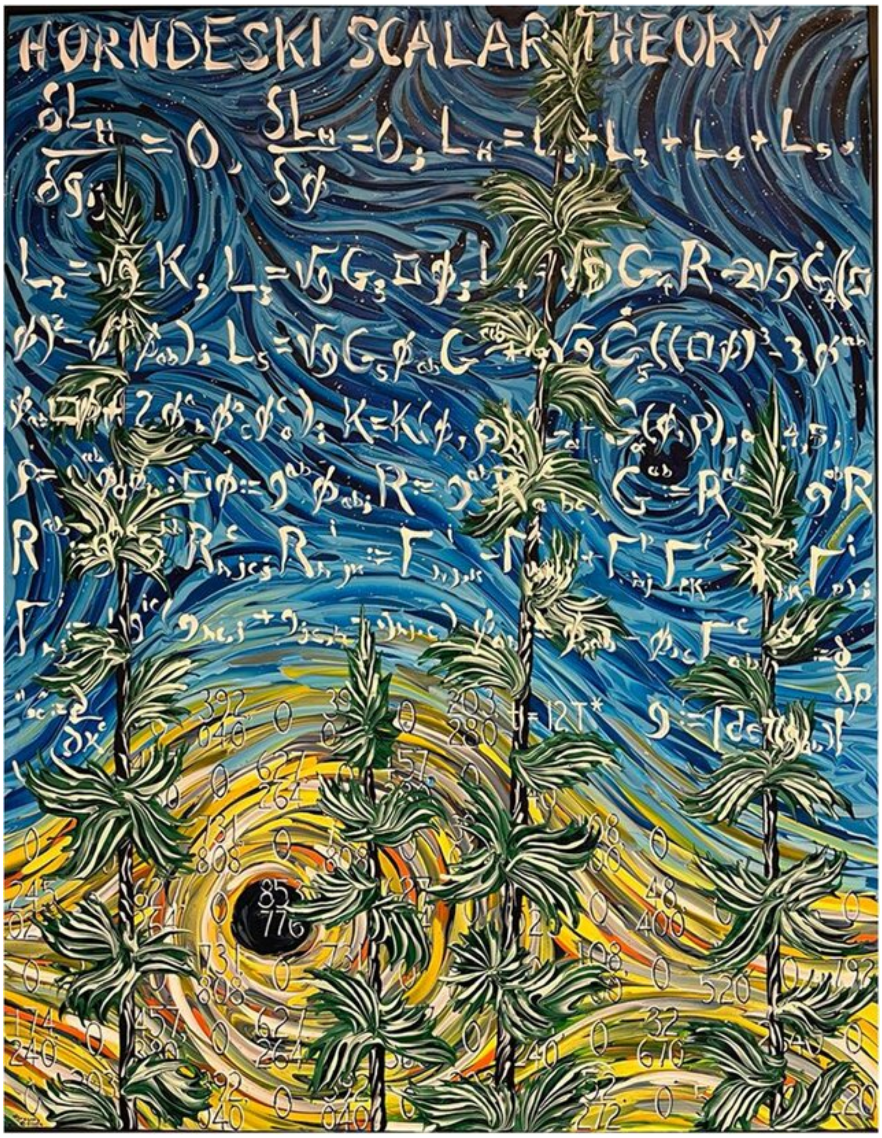
$$
G_{5\phi\phi}\dot\phi^3 2H=4X\dot\phi HG_{5\phi\phi}.
$$

For the G_{5X} terms, a lengthy but straightforward calculation shows that

$$
G_{5X}\Big(-2\dot\phi^3H\dot H-2\dot\phi^3H^3-3\dot X\dot\phi H^2\Big)=-2X\Big(2H^3\dot\phi+2H\dot H\dot\phi+3H^2\ddot\phi\Big)G_{5X}
$$

Combining these results, we obtain:

$$
\begin{aligned}
\frac{1}{3a^2}\frac{\delta S_5}{\delta a}=P_5= {} & -2X\left(2H^3\dot\phi+2H\dot H\dot\phi+3H^2\ddot\phi\right)G_{5X}-4H^2X^2\ddot\phi G_{5XX} \\
& +4HX\left(\dot X-HX\right)G_{5\phi X}+2\left(\frac{d}{dt}(HX)+3H^2X\right)G_{5\phi} \\
& +4HX\dot\phi G_{5\phi\phi}.
\end{aligned}
\tag{B.2.3.22}
$$



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