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Θεωρητικές και παρατηρησιακές όψεις βαρυτικών κυματομορφών, που παράγονται από συγχώνευση δυαδικών συστημάτων

Theoretical and observational aspects of gravitational waveforms produced by binary system mergers

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Περίληψη

Η παρούσα διατριβή διερευνά τόσο τις θεωρητικές όσο και τις πειραματικές πτυχές των *βαρυτικών κυμάτων* (GWs), όπως ορίζονται στη γενική θεωρία της σχετικότητας του Albert Einstein και μελετώνται σε διεθνή ερευνητικά κέντρα όπως το LIGO. Αυτά τα κύματα διαδίδονται με την ταχύτητα του φωτός, μεταφέροντας ενέργεια σε όλο το σύμπαν. Η διατριβή ξεκινά με μια ανασκόπηση της εμφάνισης των βαρυτικών κυμάτων στη γενική θεωρία της σχετικότητας, εστιάζοντας στις βασικές τους ιδιότητες και στις μεθόδους που χρησιμοποιούνται για την εις βάθος κατανόησή τους. Διευρευνούνται οι προκλήσεις που προέρχονται από τον ορισμό της ενέργειας και της ορμής που μεταφέρει η ακτινοβολία των βαρυτικών κυμάτων, ενώ παράλληλα αντιμετωπίζονται τυχόν παρανοήσεις και λάθη σε προηγούμενες μελέτες. Εν συνεχεία, εφαρμόζονται τεχνικές της κλασικής θεωρίας πεδίου για την μελέτη της γέννησης των βαρυτικών πεδίων, δίνοντας έμφαση στους όρους του πολυπολικού αναπτύγματος και στους υπολογισμούς που βασίζονται σε αυτά. Επιπλέον, εξετάζονται οι πειραματικές ενδείξεις για την ύπαρξή και παραγωγή των βαρυτικών κυμάτων, εστιάζοντας σε συστήματα διπλών pulsar, διπλών μελανών οπών ή αστέρων νετρονίων και επιβεβαιώνεται η ύπαρξή τους μέσω παρατηρήσεων, που αποτελούν καθοριστικό ρόλο στην κατανόηση του φαινομένου. Τέλος, πραγματοποιείται συζήτηση για τις επιπτώσεις αυτών των ευρημάτων στην αστροφυσική και την κοσμολογία, με ιδιαίτερη αναφορά στην μελέτη των συμπαγών δυικών αστέρων, των μελανών οπών και άλλων ακραίων κοσμολογικών αντικειμένων.

Abstract

This thesis explores the theoretical and experimental aspects of *gravitational waves* (GWs), as predicted by Albert Einstein's general theory of relativity and studied in different research centers like LIGO. These waves propagate outward from their source at the speed of light, carrying energy across the universe. The study begins with a discussion on the emergence of GWs in general relativity, focusing on their properties and the methods used to understand them. We investigate the challenges in defining the energy and momentum of gravitational wave radiation and address misconceptions in earlier studies. Following this, the thesis applies classical field theory techniques to the GW generation study, emphasizing multipole expansion and calculations based on it. In addition, we examine the experimental evidence for GWs, particularly from binary pulsars, binary black holes, or neutron star systems, and discuss their confirmation through observations. The thesis concludes with a discussion of the implications of these findings for astrophysics and cosmology, particularly in the study of compact binaries, black holes, and other extreme environments in the universe.

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1 Introduction

In this master-level thesis, we studied the theoretical background of Gravitational Waves. From now on, the abbreviation GW stands for Gravitational waves.

Before attacking the infamous Einstein's equations for gravity and seeing how we obtain results for GWs, it seems useful to see gravity as explained before the theory of relativity. First, Newton understood that two masses m_1 and m_2, interact via the gravitational force, as given in equation 1:

$$F = G\frac{m_1 m_2}{R^2} \tag{1}$$

This equation provides the relation between the distance separating the two masses and the corresponding gravitational force. Until the 19th century, the gravitational interaction equation was considered the Holy Grail of gravitational physics.

This was until over a century ago that Albert Einstein revolutionized our understanding of gravity with his General Theory of Relativity (GR) [1]. Published in 1915, GR redefined the cosmos, not as a static stage upon which cosmic actors move, but as a dynamic entity in itself: spacetime. In this paradigm, mass and energy dictate the curvature of spacetime, and this curvature, in turn, dictates how mass and energy move. This elegant and profound theory supplanted Newtonian gravity, providing explanations for phenomena that Newton's laws could not, such as the anomalous perihelion precession of Mercury, and making bold new predictions, most notably the bending of starlight by massive objects, famously confirmed by Eddington's 1919 solar eclipse expedition [2].

Perhaps the most dramatic and elusive prediction of GR was the existence of gravitational waves. If spacetime is a fabric, then accelerating massive objects should create ripples in it, propagating outwards at the speed of light. These waves are transverse, stretching and squeezing spacetime in the directions perpendicular to their propagation. In the framework of linearized gravity, where the spacetime metric $g_{\mu\nu}$ is treated as a small perturbation $h_{\mu\nu}$ from the flat Minkowski metric $\eta_{\mu\nu}(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu})$, the Einstein Field Equations can be reduced to a wave equation for the metric perturbation:

$$\Box \bar{h}_{\mu\nu} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{h}_{\mu\nu} = 0$$

where $\bar{h}_{\mu\nu}$ is the trace-reversed metric perturbation. The solutions to this equation are waves that carry energy and momentum away from a source, analogous to electromagnetic waves carrying energy away from an accelerating charge.

For decades, this prediction remained purely theoretical. The fundamental challenge lies in the weakness of gravity. The strain h—the fractional change in length $\Delta L/L$ induced by a passing gravitational wave—is extraordinarily small. For even the most cataclysmic astrophysical events, the expected strain on Earth is on the order of 10^{-21} or less, equivalent to measuring a change in the distance between the Earth and the Sun to less than the width of a single atom.

The first attempts at direct detection, pioneered by Joseph Weber in the 1960s using resonant bar detectors [3], were ultimately unsuccessful but laid the critical groundwork for future efforts. The first compelling evidence for the existence of gravitational waves came indirectly. In 1974, Russell Hulse and Joseph Taylor discovered the first binary pulsar, PSR

B1913+16 [4]. This system, consisting of two neutron stars orbiting each other, proved to be a perfect laboratory for testing GR. Over years of meticulous observation, they demonstrated that the binary's orbit was shrinking at precisely the rate predicted by GR due to the emission of gravitational waves. This landmark discovery, which earned them the 1993 Nobel Prize in Physics, provided irrefutable, albeit indirect, proof that gravitational waves are a physical reality.

The dream of direct detection was finally realized on September 14, 2015. After decades of technological development, the twin detectors of the Laser Interferometer Gravitational-Wave Observatory (LIGO) simultaneously registered a signal, designated GW150914 [5]. The signal was a characteristic "chirp," rising in frequency and amplitude over a fraction of a second, perfectly matching the theoretical waveform predicted for the final moments of a binary black hole (BBH) merger. This event marked not just the first direct detection of gravitational waves but also the first observation of a binary black hole system, heralding the birth of gravitational-wave astronomy. We had, for the first time, heard the universe, opening an entirely new, non-electromagnetic window through which to observe the cosmos's most violent and enigmatic phenomena.

The detection of GW150914 was the opening act in a new era of physics, the gravitational-wave astrophysics. The universe is filled with a symphony of gravitational-wave sources, each with its own characteristic sound, frequency, and duration. The primary observable is the waveform, a time-series of the gravitational-wave strain, from which we can infer the properties of the source. These sources can be broadly classified into four categories.

The first category surveys Compact Binary Coalescences (CBCs): These are the loudest and most frequently observed sources to date. They involve the orbital inspiral and eventual merger of two compact objects: black holes or neutron stars. This category is the central focus of this thesis and includes:

Binary Black Hole (BBH) Systems: The merger of two stellar-mass black holes, like GW150914. These are "dark" events, emitting no electromagnetic radiation, making GWs the only way to observe them. They are the most massive and thus "loudest" stellar-mass sources, providing pristine probes of strong-field gravity.

Binary Neutron Star (BNS) Systems: The merger of two neutron stars. The landmark detection of GW170817 [6] was a BNS merger, accompanied by a host of electromagnetic counterparts, from a short gamma-ray burst (GRB) to a kilonova [7]. This event launched the era of multi-messenger astronomy, where information from both GWs and light is combined to paint a complete picture of an astrophysical event. BNS mergers provide a unique laboratory for studying the equation of state (EoS) of ultra-dense nuclear matter.

Neutron Star-Black Hole (NSBH) Systems: The merger of a neutron star and a black hole. These asymmetric systems are fascinating probes of both strong-field gravity and matter effects. Depending on the masses and black hole spin, the neutron star may be swallowed whole or tidally disrupted before merger, potentially creating an electromagnetic counterpart. The first confident detections of these systems were announced in 2021 [8].

The second category is thought to include Continuous Waves: These are persistent, nearly monochromatic signals emitted by rapidly rotating, asymmetric neutron stars (pulsars). Any non-axisymmetric feature, such as a "mountain" on the star's crust, will generate a continuous train of gravitational waves at twice the star's rotation frequency. Detecting these signals would provide invaluable information about the structure and physics of neutron star interiors. They are, however, expected to be extremely weak, and no confirmed detection has been made to date.

The third category involves theStochastic Gravitational-Wave Background (SGWB). This is an incoherent superposition of gravitational waves from a multitude of unresolved sources, analogous to the Cosmic Microwave Background (CMB). It is expected to have two main components: an astrophysical background from the superposition of countless distant CBC events, and a potential cosmological background generated by physical processes in the very early universe, such as inflation or phase transitions. Detecting the SGWB would provide a unique probe of the universe's first moments.

And the final category has Burst Sources. Burst sources are short-duration, unmodeled transient signals. The canonical example is a core-collapse supernova, where the violent, asymmetric explosion of a massive star could produce a burst of gravitational waves. Detecting such a signal would give us a direct view into the heart of the explosion, a region completely obscured from electromagnetic telescopes.

The theoretical prediction of waveforms is only half the story; the other half is their detection and interpretation. This is the domain of observational gravitational-wave astronomy, a field defined by cutting-edge instrumentation and sophisticated data analysis. Modern GW detectors are giant, L-shaped Michelson interferometers.

A powerful laser is split into two beams that travel down perpendicular arms, each several kilometers long. The beams reflect off mirrors at the ends of the arms and recombine at the beam splitter. In the absence of a gravitational wave, the arm lengths are tuned so that the returning light beams interfere destructively, and no light reaches the output photodetector.

When a gravitational wave passes, it differentially alters the effective lengths of the two arms, stretching one while compressing the other. This minute change in path length breaks the perfect destructive interference, causing a tiny amount of light—a signal proportional to the gravitational-wave strain h(t)—to reach the photodetector. To achieve the required sensitivity, these basic interferometers are enhanced with Fabry-Pérot resonant cavities in the arms to increase the effective path length of the light, and with power and signal recycling mirrors to increase the circulating power and optimize the detector's bandwidth.The current global network of detectors includes:

- The two LIGO detectors in Hanford, Washington, and Livingston, Louisiana (USA).
- The Virgo detector near Pisa, Italy.
- The KAGRA detector in Kamioka, Japan.

Operating as a network provides several key advantages over a single detector. Most importantly, it allows for the sky localization of a source through triangulation, based on the relative arrival time of the signal at different sites. A network also improves the duty cycle and detection confidence, and it allows for the measurement of the wave's polarization, providing additional tests of GR.

The raw data from a GW interferometer is dominated by noise from a myriad of sources: seismic vibrations, thermal fluctuations, quantum shot noise, etc. The GW signal is typically much weaker than the noise. The primary technique for identifying a signal from a known source type, like a CBC, is matched filtering.

In this process, the data stream s(t) is cross-correlated with a theoretical template waveform h(t). The output of the filter is the signal-to-noise ratio (SNR), ρ, which quantifies the likelihood of a signal being present. This is mathematically expressed as an inner product, weighted by the detector's noise power spectral density $S_n(f)$:

$$\rho = \frac{\langle s|h\rangle}{\sqrt{\langle h|h\rangle}} \quad \text{where} \quad \langle a|b\rangle = 4\Re \int_0^\infty \frac{\tilde{a}(f)\tilde{b}^*(f)}{S_n(f)} df \tag{2}$$

Here, $\tilde{a}(f)$ and $\tilde{b}(f)$ are the Fourier transforms of the time-series a(t) and b(t). A detection is claimed when the SNR exceeds a predetermined threshold, corresponding to a very low false-alarm rate. Once a signal is confidently detected, the next step is parameter estimation.

Bayesian inference methods, such as Markov Chain Monte Carlo (MCMC) sampling, are used to compare the data against millions of waveform templates. This process generates posterior probability distributions for the source's physical parameters, such as:

- Component masses (m_1, m_2).
- Component spins (magnitude and orientation).
- Luminosity distance to the source.
- Sky location (right ascension and declination).
- Binary orientation (inclination, polarization angle).
- Tidal deformability (for neutron stars).

These estimated parameters are the fundamental data products of GW astronomy, forming the bridge between observation and astrophysical theory.

The first few observing runs of the LIGO-Virgo-KAGRA (LVK) collaboration have yielded a catalog of nearly one hundred GW events, revolutionizing our understanding of stellar-mass black holes and neutron stars. Yet, this is just the beginning. Each new detection brings new insights and raises new questions. The field is now poised to address some of the most fundamental problems in physics and astronomy.

Key scientific frontiers include probing the Formation of Binary Black Holes: The observed masses and spins of BBH systems challenge existing models of stellar evolution. Are these binaries formed in isolation in the galactic field, or are they assembled dynamically in dense stellar environments like globular clusters? The distribution of observed parameters holds the key.

Another frontier is how can we constrain the Neutron Star Equation of State: The tidal deformation of neutron stars during the final moments of a BNS inspiral leaves a subtle imprint on the gravitational waveform. Measuring this effect, we get results on the tidal deformability parameter, Λ. The Λ-parameter provides a direct constraint on the relationship between pressure and density in nuclear matter, a long-standing problem in nuclear physics. Another key factor is to apply precision Tests of General Relativity. Binary mergers are laboratories for strong-field gravity, a regime previously inaccessible to experimental tests.

By comparing observed waveforms to the predictions of GR, we can place stringent bounds on potential deviations from Einstein's theory. The ringdown phase, in particular, allows for direct tests of the black hole no-hair theorem. Also, GWs can be used in Cosmology Cosmology with Standard Sirens. For events like GW170817 with an electromagnetic counterpart, the host galaxy and its redshift can be identified. Since the GW signal provides a direct measurement of the luminosity distance, the source can be used as a "standard siren" to measure cosmological parameters, most notably the Hubble constant, H_0. This provides a completely independent method to weigh in on the current tension between early- and late-universe measurements of H_0. Finally, they can improve the Waveform Fidelity. As detector sensitivity improves, our theoretical models must keep pace. The next generation of science will require waveforms that include more subtle physical effects, such as orbital eccentricity and the full dynamics of spin precession, as well as the contribution of higher-order emission modes beyond the dominant quadrupole.

The work presented in this thesis is situated at the intersection of these challenges, focusing on the development and application of gravitational waveform models to extract maximum scientific insight from observational data.

The era of gravitational-wave astronomy has transitioned from one of discovery to one of systematic characterization and precision science. The ever-growing catalog of compact binary coalescence events demands a corresponding increase in the accuracy, completeness, and computational efficiency of our theoretical waveform models. Sub-dominant physical effects, once negligible, are now becoming measurable, and their inclusion in our models is essential for avoiding systematic biases in parameter estimation and for unlocking new scientific discoveries.

This thesis is motivated by the need to advance the theoretical and observational toolkit for analyzing signals from binary systems. Specifically, it addresses key limitations in current waveform modeling and data analysis techniques, with the goal of enhancing our ability to test General Relativity and constrain the astrophysical properties of compact binary sources.

The primary objectives of this research are first to develop and implement a more sophisticated theoretical waveform model that incorporates [State the specific improvement, e.g., the effects of orbital eccentricity, higher-order spherical harmonic modes, or improved tidal approximants]. Secondly, to validate this new model against Numerical Relativity simulations and compare its performance to existing phenomenological and effective-one-body models.

Having achieved the above, we can apply this model in a full Bayesian parameter estimation analysis of select GW events from the LVK catalog, quantifying the impact of the new physical effects on inferred source properties and investigate the implications of these refined measurements for specific astrophysical questions, such as [State the specific question, e.g., distinguishing between binary formation channels, placing new constraints on the neutron star equation of state, or performing more stringent tests of GR].

As contemporary literature proposes, the keystone to understanding GWs is to study their expansion around a flat spacetime. Thus, the order we follow is: The first chapter in this work includes a geometric point of view in GWs and a more field-theoretical view of them. Here, we see GWs as a geometric tool for computations, including the different gauges used and the way energy, momentum, and energy flux are carried by GWs. In this chapter, as well as in chapters 2,3,4 and 5, we choose to study Michelle Maggiore's book "Gravitational

Waves, Volume I"[15]. The first part of this textbook contains an extraordinary analysis of any theoretical aspect of GW.

In the second chapter, we study the behavior of GWs in linearized theories. It is known that a linearized theory is used to describe arbitrary systems with different energy-momentum tensors. We leave the geometrical approach and reach a more field-theoretical one. In this chapter, we see a low-velocity expansion of tensorial components, we prove formulas that compute the radiated energy, angular momentum, and power in GW emission in various orders of the multipole expansion, e.g., the mass quadrupole and octupole terms, as well as the current quadrupole term.

The third chapter is a more compact one. Here, we decided to dive into the Symmetric Trace-free Formalism for scalar and vector fields and to produce the tensor components in spherical coordinates. We did not study the STF formalism for tensorial fields because the level of mathematics used in such computations is far beyond the level of any master-level thesis. The fourth chapter is based completely on applications of GWs and the physics behind these.

The fifth chapter of our study has many parts coming from the experimental nature of GWs. Here we see natural objects, like pulsars, and their use in GW physics and astrophysics. Pulsars, due to their rotation, can be used as clocks and produce several timing formulas. Furthermore, based on pulsar physics, we can define some time delays applied to GW propagation and finally see the relativistic correction for binary pulsars and the induced GW physics.

Concluding the first part of our thesis, we see a useful set of equations, used to describe astrophysical objects, that Newtonian theory cannot. This set of equations is the TOV equations, and their main use lies in describing neutron stars (NS). Here we escape the classical Gravitational Waves textbook and find ourselves focusing on the twelfth chapter of the book "General Relativity: An Introduction for Physicists" by Hobson, M.P., Efstathiou, G., and Lasenby, A.N. [21]

In the second part of this thesis, we focus on scientific articles from international literature. The first article studies an analytical model for GW and includes some models for the inspiraling, the merging, and the ringdown phase. The second article is called "Constraining scalar-tensor theories by NS-BH GW events," and to study it, we had to obtain information from a plethora of articles. These are "Testing scalar-tensor gravity with GW observations of inspiraling compact objects" by Will C.M. [126] and "Non-perturbative strong-field effects in scalar-tensor theories of gravitation" written by Damour T. and Esposito-Farese G. [127].

2 Geometrical interpretation and Field Theory of GWs

In this chapter, we review concepts studied in general relativity and classical field theories of gravity. In the general relativistic part, we perturb our theory in first and second order around a flat spacetime and produce the equations of motion of a wave.

Next, we consider General Relativity as a classical field theory and apply the standard methods in our calculations. In this part, we return to the linearized equations of gravity, forgetting that $h_{\mu\nu}$ has a space-metric interpretation. Instead, we treat the perturbation as any other classical field on a flat Minkowski spacetime.

The reason we chose to study these two interpretations in one chapter is that GR complements field theories and vice versa. Some aspects of GWs (e.g. the GW amplitudes) are better understood in the geometric perspective, while others, like the energy-momentum tensor, are easier to comprehend in the field-theory approach.

2.1 GWs as perturbations around a flat spacetime

As stated in the introductive chapter, the main textbook used in this chapter is [1]. Although the steps followed in this section and section 1.2 can be found in any classical textbook, see bibliography [15], [16] , [17], [18], [19], [20], [22], [23]. The methodology used in these two sections is standard and we have studied it thoroughly in the undergraduate-level thesis. For this reason we can skip some trivial algebras and focus more on the ideas behind GW. Thus, we begin with the equations of motion of GW.

To obtain the equations of motion, we need to first consider the weak-field approximation. In this loose limit, we demand that the gravitational field is weak, varies with time, and does not restrict the motion of any test particle.

The weakness of the field allows us to decompose the spacetime in two parts: the background one, which is flat and described in completeness from the known metric $\eta_{\mu\nu}$, and the perturbation, denoted by $h_{\mu\nu}$. With this decomposition, we allow ourselves to study the missing effects of any gravitational theory in the Newtonian limit. For completeness, we demand that the linear metric perturbation $h_{\mu\nu}$ is small enough, meaning that $|h_{\mu\nu}| \ll 1$.

The mathematical expression that expresses the weakness of the field reads:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \;\text{ with }\; |h_{\mu\nu}| \ll 1 \tag{3}$$

At this point, equation 3 can be inverted, with the inverse to be computed as $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. For a specific set of coordinates, the Minkowski metric tensor can be written in its canonical form $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$. Because of the restriction $|h_{\mu\nu}| \ll 1$, we can ignore terms that correspond to second or higher orders in the perturbation theory.

Having defined the context and limitations of our theory, we can now begin the more formalized production of the equations of motion. The first step in our search for an equation that describes the propagation of GWs is to define the total gravitational action S. The Variation calculus of the action S produces the equations of motion of a wave.

The gravitational action is the sum of Einstein's action S_E and matter's action S_M and reads as $S = S_E + S_M$. The Einstein action S_E is defined via Ricci's scalar as

$$S_E = \frac{c^3}{16\pi G} \int \mathrm{d}^4x \, \sqrt{-g} \, R \tag{4}$$

The matter action can be defined by using the energy-momentum tensor $T_{\mu\nu}$ when methods from variational calculus are applied. Mathematically, this translates into the following definition:

$$\delta S_M \equiv \frac{1}{2c}\int \mathrm{d}^4x\,\sqrt{-g}\,T^{\mu\nu}\,\delta g_{\mu\nu} \tag{5}$$

The same methods of variation calculus, when applied to Einstein's action, produce the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{6}$$

At this point, we skip the procedure used to derive Einstein's tensor and equations, because it exists in every classical textbook of General Relativity. Finally, Einstein's equations obey the following expression:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{7}$$

With all the above being said, we need to study the huge symmetry group, under which GR is invariant. As guessed, this group is made up of any coordinate transformation, $x^\mu \to x'^\mu$, where x'^μ is an arbitrary smooth function of x^μ. Specifically, we demand that x'^μ is an invertible and differentiable diffeomorphism, with a differentiable inverse. Under x'^μ, we can see that the full metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma} \tag{8}$$

In international literature, this transformation is called GR's gauge symmetry. In the physical case, there exists a reference frame where equation 3 still holds on a sufficiently large region of space. This exact choice in the reference frame breaks down GR's invariance under coordinate transformations and results in a shortening of the degrees of freedom of the gravitational field. In terms of $x_{\mu\nu}$, we can rewrite the coordinate transformation as

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \tag{9}$$

Applying the transformation rule given in relation 9 on the full metric $g_{\mu\nu}$, we get the following:

$$\begin{aligned}
(7) \Rightarrow g_{\mu\nu} \to g'_{\mu\nu} &= g_{\rho\sigma}(x)\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\\
&= g_{\rho\sigma}(x)\frac{\partial(x'^\rho - \xi^\rho)}{\partial x'^\mu}\frac{\partial(x'^\sigma - \xi^\sigma)}{\partial x'^\nu}\\
&= g_{\rho\sigma}(x)(\delta^\rho_\mu - \partial_\mu\xi^\rho)(\delta^\sigma_\nu - \partial_\nu\xi^\sigma)\\
&= g_{\mu\nu}(x) - g_{\mu\sigma}(x)\partial_n u\xi^\sigma - g_{\rho\nu}(x)\partial_\mu\xi^\sigma + g_{\rho\sigma}(x)\partial_\mu\xi^\rho\partial_\nu\xi^\sigma\\
&= g_{\mu\nu}(x) - g_{\mu\sigma}(x)\partial_\nu\xi^\sigma - g_{\rho\nu}(x)\partial_\mu\xi^\sigma
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\Rightarrow \eta'_{\mu\nu}(x) + h'_{\mu\nu}(x) = \eta_{\mu\nu}(x) + h_{\mu\nu}(x) - (\eta_{\mu\sigma}(x) + h_{\mu\sigma}(x))\partial_\nu\xi^\sigma - (\eta_{\rho\nu}(x) + h_{\rho\nu}(x))\partial_\mu\xi^\rho\\
&\Rightarrow h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu\\
&\Rightarrow h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)}
\end{aligned} \tag{11}$$

The way the perturbation $h_{\mu\nu}$ transforms under these generic coordinate transformations becomes manifested in the last equation.

Having seen the transformation rules for both $g_{\mu\nu}$ and $h_{\mu\nu}$, we can now insert equation 3 in the general formula of Christoffel's connections and Riemann's tensor. We then obtain the following results

$$\begin{aligned}
\Rightarrow \Gamma^{\rho(1)}_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\
&= \frac{1}{2}(\eta^{\rho\lambda} - h^{\rho\lambda})\left[\partial_\mu(\eta_{\nu\lambda} + h_{\nu\lambda}) + \partial_\nu(\eta_{\lambda\mu} + h_{\lambda\mu}) - \partial_\lambda(\eta_{\mu\nu} + h_{\mu\nu})\right] \\
&= \frac{1}{2}\eta^{\rho\lambda}\left[\partial_\mu(\eta_{\nu\lambda} + h_{\nu\lambda}) + \partial_\nu(\eta_{\lambda\mu} + h_{\lambda\mu}) - \partial_\lambda(\eta_{\mu\nu} + h_{\mu\nu})\right] \\
&- h^{\rho\lambda}\left(\partial_\mu(\eta_{\nu\lambda} + h_{\nu\lambda}) + \partial_\nu(\eta_{\lambda\mu} + h_{\lambda\mu}) - \partial_\lambda(\eta_{\mu\nu} + h_{\mu\nu})\right) \\
&= \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right)
\end{aligned} \tag{12}$$

and dropping the second order perturbations in Γ^2 we get

$$\begin{aligned}
(23) \Rightarrow R^{\alpha(1)}_{\sigma\mu\nu} &= \partial_\mu\Gamma^\alpha_{\nu\sigma} - \partial_\nu\Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \\
&= \partial_\mu\Gamma^\alpha_{\nu\sigma} - \partial_\nu\Gamma^\alpha_{\mu\sigma}
\end{aligned} \tag{13}$$

We contract the index α only with the flat metric since a contraction with the perturbation will produce terms of second order in h. The terms in the second order of perturbation will be studied in the following sections.

$$\begin{aligned}
R^{(1)}_{\rho\sigma\mu\nu} &= \eta_{\alpha\rho}R^\alpha{}_{\sigma\mu\nu} = \eta_{\alpha\rho}\partial_\mu\Gamma^\alpha_{\nu\sigma} - \eta_{\alpha\rho}\partial_\nu\Gamma^\alpha_{\mu\sigma} \\
&= \eta_{\alpha\rho}\partial_\mu\left[\frac{1}{2}\eta^{\alpha\lambda}\left(\partial_\nu h_{\sigma\lambda} + \partial_\sigma h_{\lambda\nu} - \partial_\lambda h_{\nu\sigma}\right)\right] - \eta_{\alpha\rho}\partial_\nu\left[\frac{1}{2}\eta^{\alpha\lambda}\left(\partial_\mu h_{\sigma\lambda} + \partial_\sigma h_{\lambda\mu} - \partial_\lambda h_{\mu\sigma}\right)\right] \\
&= \frac{1}{2}\eta_{\alpha\rho}\eta^{\alpha\lambda}\left[\left(\partial_\mu\partial_\nu h_{\sigma\lambda} + \partial_\mu\partial_\sigma h_{\lambda\nu} - \partial_\mu\partial_\lambda h_{\nu\sigma}\right) - \left(\partial_\nu\partial_\mu h_{\sigma\lambda} - \partial_\nu\partial_\sigma h_{\lambda\mu} - \partial_\nu\partial_\lambda h_{\mu\sigma}\right)\right] \\
&= \frac{1}{2}\delta^\lambda_\rho\left(\partial_\mu\partial_\sigma h_{\lambda\nu} - \partial_\mu\partial_\lambda h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\lambda\mu} + \partial_\nu\partial_\lambda h_{\mu\sigma}\right) \\
&= \frac{1}{2}\left(\partial_\mu\partial_\sigma h_{\rho\nu} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\rho\mu} + \partial_\nu\partial_\rho h_{\mu\sigma}\right)
\end{aligned} \tag{14}$$

Another contraction in $R^a_{\sigma\mu\nu}$ produces the $O(h)$ perturbed Ricci's tensor, and a second contraction produces Ricci's scalar. The explicit formulae, after dropping $O(h^2)$ terms are computed as:

$$\begin{aligned}
R^{(1)}_{\mu\sigma} &= R^\nu{}_{\mu\nu\sigma} \\
&= \frac{1}{2}\left(\partial_\mu\partial_\sigma h^\nu{}_\nu - \partial_\mu\partial^\nu h_{\nu\sigma} - \partial_\nu\partial_\sigma h_\mu{}^\nu + \partial_\nu\partial_\nu h_{\mu\sigma}\right) \\
&= \frac{1}{2}\left(\partial_\mu\partial_\sigma h - \partial^\nu\partial_\mu h_{\nu\sigma} + \partial_\nu\partial_\sigma h^\nu_\mu - \Box h_{\mu\sigma}\right)
\end{aligned} \tag{15}$$

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\nu h - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial_\alpha h_\mu{}^\alpha + \Box h_{\mu\nu}\right) \tag{16}$$

$$
\begin{aligned}
R^{(1)} &= \eta^{\mu\nu} R_{\mu\nu} \\
&= \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\partial_\nu h - \eta^{\mu\nu}\partial_\mu\partial^\alpha h_{\alpha\nu} - \eta^{\mu\nu}\partial_\nu\partial_\alpha h_\mu{}^\alpha + \Box\eta^{\mu\nu}h_{\mu\nu}\right) \\
&= \frac{1}{2}\left(\Box h - \partial_\alpha\partial_\mu h^{\alpha\mu} - \partial_\mu\partial_\alpha h^{\mu\alpha} + \Box h\right) \\
&= \Box h - \partial_\alpha\partial_\mu h^{\alpha\mu}
\end{aligned} \tag{17}
$$

The last step before writing down Einstein's equations in tensorial form is to compute Einstein's tensor in the first order of perturbation h. This can be done straightforwardly by substituting Eqs. 16 and 17 in equation 6.

$$
\begin{aligned}
(5) \Rightarrow G_{\mu\nu} &= R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}g_{\mu\nu} \\
&= \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\partial_\mu\partial^\alpha h_{\alpha\nu} - \frac{1}{2}\partial_\nu\partial_\alpha h_\mu{}^\alpha + \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(\Box h - \partial_\alpha\partial_\mu h^{\alpha\mu}\right) \\
&= \frac{1}{2}[\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\alpha\partial_{(\mu}h^\alpha_{\nu)}] - \frac{1}{2}\eta_{\mu\nu}\Box h + \frac{1}{2}\eta_{\mu\nu}\partial_\rho\partial_\alpha h^{\rho\alpha} - \frac{1}{2}h_{\mu\nu}\Box h + \frac{1}{2}h_{\mu\nu}\partial_\rho\partial_\alpha h^{\rho\alpha} \\
&= \frac{1}{2}\left(\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\sigma\partial_{(\mu}h^\sigma{}_{\nu)} + \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} - \eta_{\mu\nu}\Box h\right)
\end{aligned} \tag{18}
$$

Now, the time for Einstein's equation in the first order of perturbation has arrived:

$$
\begin{aligned}
(6) \Rightarrow G_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}g_{\mu\nu} &= \frac{8\pi G}{c^4}T_{\mu\nu} \\
\Rightarrow \Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\sigma\partial_{(\mu}h^\sigma{}_{\nu)} + \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} - \eta_{\mu\nu}\Box h &= \frac{8\pi G}{c^4}T_{\mu\nu}
\end{aligned} \tag{19}
$$

Here we can rewrite 19 more compactly by shifting the field perturbation $h_{\mu\nu}$. This can be achieved by applying the following notation:

$$
h = \eta^{\mu\nu}h_{\mu\nu}
$$

$$
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h
$$

$$
\bar{h} = \eta_{\mu\nu}\bar{h}_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu}h
$$

$$
h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}
$$

Inserting the expressions above, as stated for the field perturbation, we finally see the linearized Einstein's equations take the following form:

$$
\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T^{\mu\nu} \tag{20}
$$

As mentioned before, choosing a reference frame cancels out a few degrees of freedom. The remaining free degrees of freedom of the perturbed field can be tied down by inserting a gauge. This gauge is, exactly like Electromagnetism, called the Lorentz or harmonic gauge and defined by the expression:

$$
\partial^\nu\bar{h}_{\mu\nu} = 0 \tag{21}
$$

The simplified expression of Einstein's equations is obtained since the condition in 21 cancels the last three terms of the expression in 20.

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T^{\mu\nu} \tag{22}$$

2.2 Equations of motion of GWs and the applied gauges

Within the linearized theory, the production of GWs is manifested in 22. When one studies the propagation of GWs, the space needs to be considered, and afterward, the differential equations need to be solved. So, at this point, we need to look at a flat space outside the source, where the energy-momentum tensor reads as $T_{\mu\nu} = 0$.

In this case, the differential equation that governs the propagation of GWs on an empty, flat space and outside the source is given as a wave equation of the form

$$\Box\bar{h}_{\mu\nu} = 0 \Rightarrow \left(-\frac{1}{c^2}\partial_0^2 + \nabla^2\right)\bar{h}_{\mu\nu} = 0 \tag{23}$$

With just a glance, we see that the wave that solves this differential equation travels with the speed of light c. This means that we can treat any gravitational wave in a flat space as we treat electromagnetic waves.

Outside the source, we can simplify the expression of the metric tensor by observing that the gauge condition doesn't completely fix the gauge. It is easier to understand this statement when we observe the way that the perturbation $h_{\mu\nu}$ transforms under the rule given in equation 9 and apply the gauge condition 21. The gauge condition isn't spoiled by the aforementioned coordinate transformation, if and only if $\Box\xi_\mu = 0$.

If we create a new tensor $\xi_{\mu\nu}$, defined as $\xi_{\mu\nu} \equiv \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho$ and differentiate once concerning the D'Alembertian, we get:

$$\begin{aligned}\Box\xi_{\mu\nu} &= \Box(\partial_\mu\xi_\nu) + \Box(\partial_\nu\xi_\mu) - \eta_{\mu\nu}\partial_\rho\Box\xi_\rho \\ &= \partial_\mu(\Box\xi_\nu) + \partial_\nu(\Box\xi_\mu) - \eta_{\mu\nu}\partial_\rho\Box\xi_\rho \\ &= 0\end{aligned} \tag{24}$$

So the new tensor $\xi_{\mu\nu}$ is defined in terms of the arbitrary, harmonic coordinate transformation ξ_μ, that satisfies the condition $\Box\xi_\mu = 0$. Having followed this path, we lowered the original ten independent degrees of freedom into six with the Lorentz gauge, and when the residual gauge was used, the free degrees were lowered into two.

Equation 24 shows that we can add or subtract in the metric perturbation a term, which includes $\xi_{\mu\nu}$, and satisfy the same equation of motion. Thus, we have the liberty of choice in the components of the original $h_{\mu\nu}$ perturbation. The Lorentz condition in terms of the $h_{\mu\nu}$ perturbation and for $\mu = 0$ reads:

$$\partial^0 h_{00} + \partial^i h_{0i} = 0 \Rightarrow \partial^0 h_{00} = 0 \text{ for } h_{0i} = 0$$

The component h_{00} is now constant in time and corresponds to the static part of the gravitational wave interaction. Essentially, this component depicts the Newtonian potential that generated the gravitational wave and can be set to zero as $h_{00} = 0$.

If we read more into this, we can see that all temporal components $h_{0\mu}$ are set to zero, and the gauge condition reads as

$$\partial^j h_{ij} = 0 \tag{25}$$

Furthermore, the freedom of choice allows us to redefine the temporal component of the vector ξ, such that the trace of the perturbation tensor $h_{\mu\nu}$ vanishes. Next, a vanishing trace implies that $h_{\mu\nu} = \bar{h}_{\mu\nu}$ and $h^i{}_i = 0$. In conclusion, when it comes to gravitational wave gauges outside the source, we can always define the transverse-traceless gauge (TT gauge) as given by the following set of equations:

$$h^{0\mu} = 0, \quad h^i{}_i = 0 \text{ and } \partial^j h_{ij} = 0 \tag{26}$$

The TT-gauge, as was previously defined, can be used only outside the source, since when we suppose a source, $T_{\mu\nu} \neq 0$ and as a consequence $Box h_{\mu\nu} \neq 0$. When this gauge is applied to the perturbed metric tensor, it is denoted as $h^{\mathrm{TT}}_{\mu\nu}$, and since the temporal components vanish, we can change the notation to spatial indices as h^{TT}_{ij}.

Having filled our armory with the TT gauge and the differential equation that defines the motion, we can look for solutions. It is obvious, that the solutions of equation 23 are plane waves and on the TT gauge read as follows:

$$h^{\mathrm{TT}}_{ij}(x) = e_{ij}(\vec{k})e^{ikx} \tag{27}$$

Here, the real part of the equation 27 is applied at the end of our computation. The polarization vector is $e_{ij}(\vec{k})$, $\vec{k}$ stands for the wave vector and the direction of propagation is given by $\hat{n} = \vec{k}/|\vec{k}|$. Suppose a monochromatic plane wave, we observe that the non-vanishing components travel on the plane transverse to $\hat{n}$, and the condition $\partial^j h_{ij} = 0$ reads as

$$\begin{aligned}
\partial^j h^{\mathrm{TT}}_{ij} = 0 &\Rightarrow \partial^j [e_{ij}(\vec{k})e^{ik_a x^a}] = 0 \\
&\Rightarrow e_{ij}\partial^j e^{ik_a x^a} = 0 \\
&\Rightarrow e_{ij}e^{ik_a x^a}\partial^j(ik_a x^a) = 0 \\
&\Rightarrow e_{ij}e^{ik_a x^a}\eta^{jk}\partial_k(ik_a x^a) = 0 \\
&\Rightarrow e_{ij}e^{ik_a x^a}\eta^{jk} ik_k = 0 \\
&\Rightarrow ik^j h^{\mathrm{TT}}_{ij} = 0 \Rightarrow k^j h^{\mathrm{TT}}_{ij} = 0 \\
&\Rightarrow \hat{n}^j|\vec{k}|h^{\mathrm{TT}}_{ij} = 0 \\
&\Rightarrow \hat{n}^j h^{\mathrm{TT}}_{ij} = 0
\end{aligned} \tag{28}$$

Also, the plane wave polarization vector in this gauge has to obey:

$$h^{i\,\mathrm{TT}}_{\ i} = 0 \Rightarrow e^i{}_i(\vec{k}) = 0 \tag{29}$$

$$h^{\mathrm{TT}}_{0\mu} = 0 \Rightarrow e_{0\mu}(\vec{k}) = 0 \tag{30}$$

Everything discussed above can be expressed more simply and compactly in matrix notation. The polarization vector, as given on the TT-gauge, takes the form:

$$e_{\mu\nu}(\vec{k}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & e_{ij}(\vec{k}) & \\ 0 & & & \end{pmatrix} \tag{31}$$

The temporal components vanish, and because of the vanishing trace, we have $e^i{}_i = 0 \Rightarrow e^1{}_1 + e^2{}_2 + e^3{}_3 = 0$. If we suppose also the vector along z-axis, as $\hat{n} \equiv \hat{n}^3$, we get the purely spatial expression to be

$$e_{ij}(\vec{k}) = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$

The final solution is obtained when taking the real part of the equation 32:

$$h_{ij}^{\mathrm{TT}} = Re\left[\right]h_{ij}^{\mathrm{TT}}\Big] = Re\left[e^{ik_a x^a}\begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \tag{33}$$

$$\Rightarrow h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} Re[cos(ik \cdot x) + isin(ik \cdot x)] \tag{34}$$

Again, observing that h_{i3} components are zero, we can read an even more compact formula for the GW amplitude:

$$h_{ab}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times \\ h_\times & h_+ \end{pmatrix}_{ab} cos(k \cdot x) = \begin{pmatrix} h_+ & h_\times \\ h_\times & h_+ \end{pmatrix}_{ab} cos[k_0 x^0 + k_i x^i] \tag{35}$$

or equivalently

$$h_{ab}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times \\ h_\times & h_+ \end{pmatrix}_{ab} cos\left[\frac{\omega}{c}ct - \mathtt{k}z\right] \Rightarrow h_{ab}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times \\ h_\times & h_+ \end{pmatrix}_{ab} cos\left[\omega\left(t - \frac{z}{c}\right)\right] \tag{36}$$

Having written the full expression on the perturbed metric, we can now compute the interval $\mathrm{d}s^2$, which expresses the propagation of GWs in a background, flat spacetime.

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + h_{\mu\nu}^{\mathrm{TT}}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + h_+\mathrm{d}x^2 cos\left[\omega\left(t - \frac{z}{c}\right)\right] + \\ &+ h_+\mathrm{d}y^2 cos\left[\omega\left(t - \frac{z}{c}\right)\right] + 2h_+\mathrm{d}x\ \mathrm{d}y\ cos\left[\omega\left(t - \frac{z}{c}\right)\right] \\ &= -c^2\mathrm{d}t^2 + \left[1 + h_+ cos\left(\omega(t - \frac{z}{c})\right)\right]\mathrm{d}x^2 + \left[1 - h_+ cos\left(\omega(t - \frac{z}{c})\right)\right]\mathrm{d}y^2 + \\ &+ \mathrm{d}z^2 + 2h_\times cos\left[\omega(t - \frac{z}{c})\right]\mathrm{d}x\ \mathrm{d}y \end{aligned} \tag{37}$$

2.3 Projection operators on the TT-gauge

Following [17] at Chapter 10, equation 10.4.14 and below we can define an operator, which transforms directly a GW in the TT-gauge, the Λ-operator.

Consider next a plane wave outside the source, but in the Lorentz gauge. By defining some projectors, we can always readjust the amplitude, so it is in the TT gauge. We can achieve this by introducing a new tensor P_{ij}, which is symmetric, transverse, and a projector. We see this projecting operator to have the form:

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j \tag{38}$$

And applying the properties of symmetry, transversality, and projection, we see:

$$\begin{aligned}
&P_{ji}(\hat{n}) = \delta_{ji} - n_j n_i = \delta_{ij} - n_i n_j = P_{ij}(\hat{n}) \\
&n^j P_{ij} = n^j \delta_{ij} - n^j n_i n_j = n_i - n_j n^j n_i \Rightarrow n^j P_{ij} = n_i - n_i = 0 \\
&P_{ij} P_{jk} = (\delta_{ij} - n_i n_j)(\delta_{jk} - n_j n_k) = \delta_{ik} - n_i n_k \\
&P_{ii} = \delta^{ij} P_{ij} = \delta^{ij} \delta_{ij} - \delta^{ij} n_i n_j = 2
\end{aligned} \tag{39}$$

Having understood the projection tensor, we can obtain a new projection tensor in terms of P_{ij}. The expression that suits our needs in the TT-gauge reads:

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \tag{40}$$

This tensor, as defined above, is a projector since:

$$\begin{aligned}
\Lambda_{ij,kl} \Lambda_{kl,mn} &= \left(P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \right) \left(P_{km} P_{ln} - \frac{1}{2} P_{kl} P_{mn} \right) \\
&= P_{ik} P_{jl} P_{km} P_{ln} - \frac{1}{2} P_{ik} P_{kl} P_{jl} P_{mn} - \frac{1}{2} P_{ij} P_{kl} P_{km} P_{ln} + \frac{1}{4} P_{ij} P_{kl} P_{kl} P_{mn} \\
&= P_{im} P_{jn} - \frac{1}{2} P_{ij} P_{mn} - \frac{1}{2} P_{ij} P_{mn} + \frac{1}{4} 2 P_{ij} P_{mn} \\
&= P_{im} P_{jn} - \frac{1}{2} P_{ij} P_{mn} = \Lambda_{ij,mn}
\end{aligned} \tag{41}$$

It is transverse,

$$n^i \Lambda_{ij,kl} = n^i P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} = 0 \tag{42}$$

traceless concerning i,j, and k,l

$$\begin{aligned}
\Lambda_{ii,kl} &= P_{ik} P_{il} - \frac{1}{2} P_{ii} P_{kl} = P_{kl} - \frac{1}{2} 2 P_{kl} = 0 \\
\Lambda_{ij,kk} &= P_{ik} P_{jk} - \frac{1}{2} P_{ij} P_{kk} = P_{ij} - P_{ij} = 0
\end{aligned} \tag{43}$$

and symmetric under the exchange of $(i,j) \leftrightarrow (k,l)$

$$\Lambda_{ij,kl} = P_{ki} P_{lj} - \frac{1}{2} P_{kl} P_{ij} = \Lambda_{ij,kl} \tag{44}$$

The above properties are enough to support the claim that the tensor $\Lambda_{ij,kl}$ projects any tensor in the TT-gauge. When we consider arbitrary, symmetric tensors of the form S_{ij}^{TT}, we see that the projector conserves the symmetry. We can rewrite this tensor in terms of unit vectors $\hat{n}^i$. Inserting equation (37) in (39), we get:

$$\begin{aligned}
\Lambda_{ij,kl}(\hat{n}) &= (\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) - \frac{1}{2} (\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l) \\
&= \delta_{ik} \delta_{jl} - \frac{1}{2} \delta_{ij} \delta_{kl} - \delta_{ik} n_j n_l - \delta_{jl} n_i n_k \\
&+ \frac{1}{2} \delta_{ij} n_k n_l + \frac{1}{2} \delta_{kl} n_i n_j - \frac{1}{2} n_i n_j n_k n_l
\end{aligned} \tag{45}$$

2.3.1 The TT-frame

The perturbed metric tensor h_{ij} in this gauge can be written as $h_{ij}^{\mathrm{TT}} = \Lambda_{ijkl}h_{kl}$ meaning that any amplitude that originally is in the Lorentz gauge can be written in the Transverse-Traceless gauge using the Λ projector. Following this, we see that it also satisfies the same equations of motion, as given in equation (22), and can be Fourier expanded as a monochromatic plane wave, as already done in equation (26):

$$h_{ij}^{\mathrm{TT}}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[A_{ij}(\vec{k})e^{ikx} + A_{ij}^*(\vec{k})e^{-ikx}\right] \tag{46}$$

In polar coordinates, the integration measure reads as

$$\begin{aligned}
\mathrm{d}^3k &= k^2\mathrm{d}k\mathrm{d}\Omega = k^2 sin\theta \mathrm{d}k\mathrm{d}\theta\mathrm{d}\phi \\
&= k^2\mathrm{d}k\mathrm{d}\cos\theta\mathrm{d}\phi = k^2\mathrm{d}k\mathrm{d}^2\hat{n} \\
&= \frac{(2\pi)^2 f^2}{c^2}\mathrm{d}k\mathrm{d}^2\hat{n} = \frac{(2\pi)^2 f^2}{c^2}\frac{2\pi\mathrm{d}f}{c}\mathrm{d}^2\hat{n} \\
&= \left(\frac{2\pi}{c}\right)^3 f^2\mathrm{d}f\mathrm{d}^2\hat{n}
\end{aligned} \tag{47}$$

And the Fourier expansion in equation 46 reads:

$$\begin{aligned}
h_{ij}^{\mathrm{TT}}(x) &= \int \frac{df\mathrm{d}^2\hat{n}}{(2\pi)^3}\frac{(2\pi)^3}{c^3}f^2\left[A_{ij}(f,\hat{n})e^{ikx} + \mathrm{c.c.}\right] \\
&= \frac{1}{c^3}\int \mathrm{d}f f^2 \int \mathrm{d}^2\hat{n}\left[A_{ij}(f,\hat{n})e^{-i(\omega t + \frac{2\pi f}{c}\vec{x}\cdot\hat{n})} + \mathrm{c.c.}\right] \\
&= \frac{1}{c^3}\int \mathrm{d}f f^2 \int \mathrm{d}^2\hat{n}\left[A_{ij}(f,\hat{n})e^{-2\pi i f(t - \frac{1}{c}\hat{n}\cdot\vec{x})} + \mathrm{c.c.}\right]
\end{aligned} \tag{48}$$

When the direction of propagation of a GW is well-defined, we can write equation 48 as

$$\begin{aligned}
h_{ij}^{\mathrm{TT}}(x) &= \frac{1}{c^3}\int \mathrm{d}f f^2 \int \mathrm{d}^2\hat{n}\left[A_{ij}(f)\delta^{(2)}(\hat{n}-\hat{n}_0)e^{-2\pi i f(t - \frac{1}{c}\hat{n}\cdot\vec{x})} + \mathrm{c.c.}\right] \\
&= \frac{1}{c^3}\int \mathrm{d}f\left[f^2 A_{ij}(f)e^{-2\pi i f(t - \frac{1}{c}\hat{n}_0\cdot\vec{x})} + \mathrm{c.c.}\right] \\
&= \int \mathrm{d}f\left[\tilde{h}_{ij}^{\mathrm{TT}}(f,\vec{x})e^{-2\pi i f t} + \tilde{h}_{ij}^{\mathrm{TT}*}(f,\vec{x})e^{2\pi i f t}\right]
\end{aligned} \tag{49}$$

In the last line of 49, we set

$$\tilde{h}_{ij}^{\mathrm{TT}}(f,\vec{x}) \equiv \frac{f^2}{c^3}A_{ij}(f)e^{-2\pi i f/c(\hat{n}_0\cdot\vec{x})} \tag{50}$$

Next, we can impose the gauge properties and rewrite the amplitude as:

$$h_{ab}(t,\vec{x}) = \int_0^\infty \mathrm{d}f[\tilde{h}_{ab}(f,\vec{x})e^{-2\pi i f t} + \tilde{h}_{ab}^*(f,\vec{x})e^{2\pi i f t}] \tag{51}$$

Writing the equations down in the detector frame, we can eliminate the $\vec{x}$ dependence as:

$$h_{ab}(t) = \int_0^\infty \mathrm{d}f[\tilde{h}_{ab}(f)e^{-2\pi ift} + \tilde{h}^*_{ab}(f)e^{2\pi ift}] \tag{52}$$

where $\tilde{h}_{ab}(f)$ is the 2×2 matrix of the $+, \times$ polarization of a GW with physical frequency. These two polarization modes are defined concerning a given choice of axes in the transverse plane.

$$\tilde{h}_{ab}(f) = \begin{pmatrix} \tilde{h}_+(f) & \tilde{h}_\times(f) \\ \tilde{h}_\times(f) & -\tilde{h}_+(f) \end{pmatrix}_{ab} \quad \text{and} \quad \tilde{h}^*_{ab}(f) = \tilde{h}_{ab}(-f) \tag{53}$$

There is another way of reading the physical frequencies. It becomes clear when we insert the property of physical GWs, namely $\tilde{h}^*_{ab}(f) \equiv \tilde{h}_{ab}(-f)$, in the last equation:

$$\begin{aligned} h_{ab}(t) &= \int_0^\infty \mathrm{d}f\tilde{h}_{ab}(f)e^{-2\pi ift} + \int_0^\infty \mathrm{d}f\tilde{h}_{ab}(-f)e^{2\pi ift} \\ &= \int_0^\infty \mathrm{d}f\tilde{h}_{ab}(f)e^{-2\pi ift} - \int_0^{-\infty} \mathrm{d}f\tilde{h}_{ab}(f)e^{-2\pi ift} \\ h_{ab}(t) &= \int_{-\infty}^\infty \mathrm{d}f\tilde{h}_{ab}(f)e^{-2\pi ift} \end{aligned} \tag{54}$$

The last form of the amplitude has the form of a Fourier transform in the space of frequencies. The inverse formula is written in analogy to the Fourier transform:

$$\tilde{h}_{ab}(f) = \int_{-\infty}^\infty \mathrm{d}t h_{ab}(t)e^{2\pi ift} \tag{55}$$

On real axis we can suppose two arbitrary unit vectors $\hat{n}, \hat{v}$ such that $\hat{u} \perp \hat{n}$, $\hat{v} \perp \hat{n}$ and $\hat{u} \perp \hat{v}$ as shown in Figure 1.

The polarization tensors $e^A_{ij}(\hat{n})$ are written for $A = +, \times$ as:

$$\begin{aligned} e^+_{ij}(\hat{n}) &= \hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j, \\ e^\times_{ij}(\hat{n}) &= \hat{u}_i\hat{v}_j + \hat{v}_i\hat{u}_j, \end{aligned}$$

where $\hat{u}$ and $\hat{v}$ are unit vectors orthogonal to the propagation direction $\hat{n}$ and to each other. It follows that $e^A_{ij}(\hat{n})$ are normalized as:

$$e^A_{ij}(\hat{n})e^{A'ij}(\hat{n}) = 2\delta_{AA'}.$$

In the frame where $\hat{n} = \hat{z}$, we get $\hat{u} = \hat{x}$ and $\hat{v} = \hat{y}$. Thus, the polarization tensors are:

$$e^+_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab}, \quad e^\times_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab}.$$

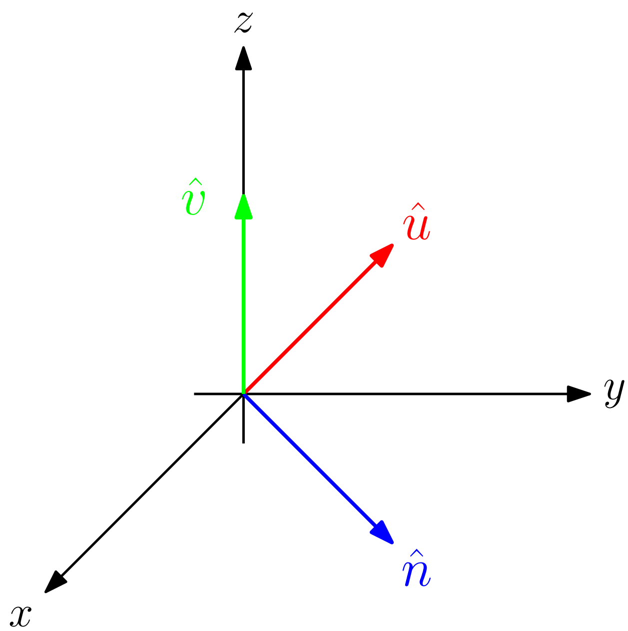

Figure 1: Diagram depicting the x,y,z axes and the "new" unitary vectors $\hat{u}$, $\hat{n}$ and $\hat{v}$, adapted by Maggiore's book, Gravitational Waves [15].

The amplitudes from the $\Box$ equations are given now as:

$$\tilde{h}_{ab}(f,\vec{x}) = \frac{f^2}{c^3}\int d^2\hat{n} A_{ab}(f,\hat{n}) e^{2\pi i f\ \frac{\hat{n}\cdot\vec{x}}{c}} \Rightarrow$$

$$\tilde{h}_{ab}(f,\vec{x}) = \frac{f^2}{c^3}\sum_{A=+,\times} A_{ab}(f) e^A_{ab}(\hat{n}) e^{2\pi i f\ \frac{\hat{n}\cdot\vec{x}}{c}} \Rightarrow$$

$$\tilde{h}_{ab}(f,\vec{x}) = \frac{f^2}{c^3}\sum_{A=+,\times} A_{ab}(f) e^A_{ab}(\hat{n}) \Rightarrow$$

$$\frac{f^2}{c^3} A_{ab}(f) = \sum_{A=+,\times} \tilde{h}_A(f,\vec{n}) e^A_{ab}(\hat{n})$$

So, equation 48 reads:

$$h_{ab}(t,\mathbf{x}) = \sum_{A=+,\times}\int_{-\infty}^{\infty} df \int d^2\hat{n}\ \left[\tilde{h}_A(f,\hat{n}) e^A_{ab}(\hat{n}) e^{-2\pi i f\left(t-\frac{\hat{n}\cdot\mathbf{x}}{c}\right)} + \text{c.c.}\right]$$

2.3.2 The geodesic equation production

This paragraph takes contribution from a variety of sources, namely we see parts from field theoretical actions and their manipulation in [25] and we see the geodesic equation, a basic formula that is produced in any of the following [15], [16] , [17], [18], [19], [20], [22], [23].

Consider a curve x^μ parametrized by an affine parameter λ, as $x^\mu(\lambda)$. Based on this we can write the line element $\mathrm{d}s^2$ as follow:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda^2 \tag{56}$$

A space-like curve satisfies that $\mathrm{d}s^2 > 0 \Rightarrow \mathrm{d}s = (g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu)^{\frac{1}{2}}$, whereas a time-like curve with $\mathrm{d}s^2 < 0$ satisfies that $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ The τ parameter is the proper time, as defined by a clock traveling along the $x^\mu(\lambda)$ curve.

The classical action S , defined by $x^\mu(\lambda)$ trajectory on $x^\mu_A = x^\mu(\tau_A)$ and $x^\mu_B = x^\mu(\tau_B)$ as endpoints is:

$$S = -mc\int_{\tau_A}^{\tau_B} \mathrm{d}s = -mc^2 \int_{\tau_A}^{\tau_B} \mathrm{d}t\sqrt{1-\frac{x^2}{c^2}} \tag{57}$$

And the free-particle Lagrangian is defined as:

$$L = -mc^2\sqrt{1-\frac{x^2}{c^2}} \tag{58}$$

Equation (57) can now be written as:

$$S = -mc\int_{\tau_A}^{\tau_B} \mathrm{d}s = -mc\int_{\tau_A}^{\tau_B} \sqrt{\mathrm{d}x_\mu \mathrm{d}x^\mu} \tag{59}$$

A variation on this gives:

$$\begin{aligned}
\delta S =& -mc\int_{\tau_A}^{\tau_B} \delta\sqrt{\mathrm{d}x_{mu}\mathrm{d}x^\mu} = -mc\int_{\tau_A}^{\tau_B} \frac{\delta(\mathrm{d}x_\mu \mathrm{d}x^\mu)}{2\sqrt{\mathrm{d}x_\mu \mathrm{d}x^\mu}} \Rightarrow \\
\delta S =& -mc\int_{\tau_A}^{\tau_B} \frac{\delta(g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu)}{2\mathrm{d}s} = -mc\int_{\tau_A}^{\tau_B} \frac{g_{\mu\nu}2\mathrm{d}x^\mu \delta \mathrm{d}x^\nu}{2\mathrm{d}s} \Rightarrow \\
\delta S =& -mc\int_{\tau_A}^{\tau_B} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} g_{\mu\nu}\mathrm{d}\delta x^\nu = -mc\int_{\tau_A}^{\tau_B} \mathrm{d}\left[\frac{x^{\mathrm{d}\mu}}{\mathrm{d}s} g_{\mu\nu}\delta x^\nu\right] + mc\int_{\tau_A}^{\tau_B} \mathrm{d}\left[\frac{\mathrm{d}x^\mu}{\mathrm{d}s} g_{\mu\nu}\right]\delta x^\nu \Rightarrow \\
\delta S =& -mc\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\delta x^\nu g_{\mu\nu}|_{\tau_A}^{\tau_B} + mc\int_{\tau_A}^{\tau_B} \mathrm{d}s \frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} g_{\mu\nu}|_{\tau_A}^{\tau_B}\delta x^\nu + \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\mathrm{d}g_{\mu\nu}\delta x^\nu
\end{aligned} \tag{60}$$

Demanding that S has an extreme, we get:

$$\begin{aligned}
\frac{\delta s}{\delta x^\nu} = 0 \Rightarrow & mc\int_{\tau_A}^{\tau_B} \mathrm{d}s\left[\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} g_{\mu\nu} + \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}s}\right] = 0 \Rightarrow \\
& \frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} g_{\mu\nu} + \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}x^\alpha} = 0 \Rightarrow \\
& \frac{\mathrm{d}^2x^\rho}{\mathrm{d}s^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} g^{\rho\nu}\frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}x^\alpha} = 0 \Rightarrow \\
& \frac{\mathrm{d}^2x^\rho}{\mathrm{d}s^2} + \Gamma^\rho_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = 0
\end{aligned} \tag{61}$$

If $u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$ we get in terms of 4−velocity that $\frac{\mathrm{d}u^\rho}{\mathrm{d}s} + \Gamma^\rho_{\mu\alpha}u^\mu u^\alpha$. When we considerate a translation $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ equation 61 gives:

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(x^\rho+\xi^\rho)+\Gamma^\rho_{\mu\alpha}\frac{\mathrm{d}}{\mathrm{d}s}(x^\mu+\xi^\mu)\frac{\mathrm{d}}{\mathrm{d}s}(x^\alpha+\xi^\alpha)=0\Rightarrow$$
$$\frac{\mathrm{d}^2x^\rho}{\mathrm{d}s^2}+\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}s^2}+\left[\Gamma^\rho_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}+\Gamma^\rho_{\mu\alpha}\frac{\mathrm{d}\xi^\mu}{\mathrm{d}s}\right]\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}+\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}s}\right)=0 \tag{62}$$
$$\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}s^2}+\Gamma^\rho_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}s}+\Gamma^\rho_{\mu\alpha}\frac{\mathrm{d}\xi^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}+\Gamma^\rho_{\mu\alpha}\frac{\xi^\mu}{\mathrm{d}s}\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}s}=0$$

Where $\Gamma^\rho_{\mu\alpha}\equiv\Gamma^\rho_{\mu\alpha}(x+\xi)\approx\Gamma^\rho_{\mu\alpha}(x)+\xi^\sigma\partial_\sigma\Gamma^\rho_{\mu\alpha}(x)$

$\Rightarrow \dfrac{\mathrm{d}^2\xi^2}{\mathrm{d}s^2}+2\Gamma^\rho_{\mu\alpha}\dfrac{\mathrm{d}x^\mu}{\mathrm{d}s}\dfrac{\mathrm{d}\xi^\alpha}{\mathrm{d}s}+\xi^\sigma\partial_\sigma\Gamma^\rho_{\mu\alpha}(x)\dfrac{\mathrm{d}x^\mu}{\mathrm{d}s}\dfrac{\mathrm{d}x^\alpha}{\mathrm{d}s}=0$

Next we introduce the covariant derivative for $4-$ vector as

$$\frac{DV^\mu}{D\tau}\equiv\frac{\mathrm{d}V^\mu}{\mathrm{d}\tau}+\Gamma^\mu_{\nu\rho}V^\nu\frac{\mathrm{d}x^\rho}{d\tau}$$

and the geodesic deviation equation is written as:

$$\frac{D^2\xi^\mu}{D\tau^2}=-R^\mu_{\nu\rho\sigma}\xi^\rho\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\Rightarrow\frac{D^2\xi^\mu}{D\tau^2}=-R^\mu_{\nu\rho\sigma}\xi^\rho u^\nu u^\sigma \tag{63}$$

2.4 Energy-momentum tensor in geometric interpretation

The energy-momentum tensor of GW and the short-wave expansion are discussed based on Isaacson's work, see bibliography [30] and [31] and in Thorne's classical textbook [28]. The space-time average over a wave is discussed in Arnowitt, Desser and Misner, [32]. Finally, the geometric optics approximation used in 1.4.2 is referred to [20], [30], [31], [27] and [28].

Until now, we have seen that GWs carry energy and momentum. Furthermore, we saw that GWs set in motion a ring of test masses initially at rest. If these masses are connected by a loose spring with friction, the kinetic energy will be transformed into heat. Thus, GWs produce work, and energy conservation demands that the energy transformed to work must come from the GW energy. We want to check out if GWs curve the background spacetime. This will occur if we allow the background spacetime to be dynamical, meaning we must define GWs over a curved, dynamical background metric $\overline{\mathrm{g}}_{\mu\nu}(x)$ and write the perturbation as:

$$g_{\mu\nu}=\overline{\mathrm{g}}_{\mu\nu}(x)+h_{\mu\nu}(x) \tag{64}$$

where $|h_{\mu\nu}|\ll 1$. The total metric can receive contributions, which change in time and space, on all possible scales, due to growing fields of nearby moving masses. A natural splitting between $\overline{rmg}_{\mu\nu}$ and $h_{\mu\nu}$ arises when there is a clear separation of scales. For example, equation 64 in a coordinate system provides that $\overline{\mathrm{g}}_{\mu\nu}$ has a typical scale of spatial variation L_B and the perturbation has an amplitude proportional to a $\lambda\!\!\!^{-}$ amplitude such that $\lambda\!\!\!^{-}\ll L_B$, where $\lambda\!\!\!^{-}\equiv\frac{\lambda}{2\pi}$. In frequency space $\overline{\mathrm{g}}_{\mu\nu}$ has frequencies f_B (maximum) and $h_{\mu\nu}$ perturbs around the frequencies f such that $f\gg f_B$. In this case, $h_{\mu\nu}$ is a high-frequency perturbation of a static or slowly varying background.

We want to understand how the perturbation $h_{\mu\nu}$ propagates and affects the background spacetime. We begin by expanding the metric tensor as already done in equation 64, $g_{\mu\nu}(x)=\overline{g_{\mu\nu}}+h_{\mu\nu}$ To this expansion, we get two small parameters:

1. the typical amplitude $h \equiv O(|h_{\mu\nu}|)$

2. $\frac{\lambda}{L_B}$ or $\frac{f_B}{f} \ll 1$ (short wave expansion)

We produce, now, the quadratic order of approximation of Christoffel's symbols, Riemann's, Ricci's tensor, and Ricci's scalar.

$$g_{\mu\nu} = \overline{g_{\mu\nu}^{(x)}} + h_{\mu\nu}^{(x)} \Rightarrow g^{\mu\nu} = \overline{g^{\mu\nu}(x)} - h^{\mu\nu}(x) + O(h^2)$$

Christoffel's connections are:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\overline{g^{\mu\sigma}}(D_\nu \overline{g_{\sigma\rho}} + D_\rho \overline{g_{\nu\sigma}} - D_\sigma \overline{g_{\nu\rho}}) + \frac{1}{2}\overline{g^{\mu\sigma}}(D_\nu h_{\rho\sigma} + D_\rho h_{\nu\sigma} - D_\sigma h_{\nu\rho})$$

The Ricci tensor expansion is given as follows:

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + O(R^3) \tag{65}$$

with $\overline{R_{\mu\nu}} \sim \overline{g_{\mu\nu}} > R(1)_{\mu\nu}$ linear in $h_{\mu\nu}$ as:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\bar{D}^\alpha \bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha \bar{D}_\nu h_{\mu\alpha} - \bar{D}^\alpha \bar{D}_\alpha h_{\mu\nu} - \bar{D}_\mu \bar{D}_\nu h\right), \tag{66}$$

and $R^{(2)}_{\mu\nu}$ quadratic in $h_{\mu\nu}$:

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\sigma\rho}\bar{g}^{\alpha\beta}\Bigg[&\frac{1}{2}\bar{D}_\rho h_{\mu\alpha}\bar{D}_\sigma h_{\nu\beta} + \left(\bar{D}_\rho h_{\nu\alpha}\right)\left(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}\right) \\ &+ h_{\rho\alpha}\left(\bar{D}^\beta \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma} - \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu}\right) \\ &+ \left(-\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma}\right)\left(\bar{D}^\beta h_{\mu\beta} + \bar{D}_\nu h_{\beta\sigma} - \bar{D}_\beta h_{\mu\nu}\right)\Bigg]. \end{aligned} \tag{67}$$

The perturbation $h_{\mu\nu}$ depends on the frequency. When we have low frequencies, we define $h^{low}_{\mu\nu}$ and for high frequencies, $h^{high}_{\mu\nu}$. In equation (64) $R_{\mu\nu}$ is separated in three parts:

1. $\overline{R_{\mu\nu}} \sim \overline{g_{\mu\nu}}$

2. $R^{(1)}_{\mu\nu} \sim O(h)$

3. $R^{(2)}_{\mu\nu} \sim O(h^2)$

The part proportional to $O(h)$ contains by definition only high frequencies $f \gg f_B$, When the $O(h^2)$ part may contain terms with $h_{\alpha\beta}h_{\gamma\delta} : f(h_{\alpha\beta}) \gg f_B$ and $f(h_{\gamma\delta}) \sim f_B$. The $R^{(2)}_{\mu\nu}$ components can be divided into low and high-frequency components as:

$$R^{(2)}_{\mu\nu} = R^{(2)low}_{\mu\nu} + R^{(2)high}_{\mu\nu} \tag{68}$$

Substituting Eq. 67 in Einstein's equation alternative form, we get

$$R_{\mu\nu} = \frac{8\pi G}{C^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{69}$$

We get Eq. 68 $\xRightarrow{\text{Eq. 67}}$

$$\overline{R_{\mu\nu}} + R^{(1)\ high}_{\mu\nu} + R^{(2)\ low}_{\mu\nu} + R^{(2)\ high}_{\mu\nu} = \frac{8\pi G}{C^4}\left[(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{low} + (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{high}\right]$$

$$\overline{R_{\mu\nu}} = -R^{(2)\ \text{low}}_{\mu\nu} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)^{\text{low}} \tag{70}$$

$$R^{(1)\ \text{high}}_{\mu\nu} = -R^{(2)\ \text{high}}_{\mu\nu} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)^{\text{high}} \tag{71}$$

At this point, it is useful to see the order of magnitude of all components. In a small region of spacetime $\overline{g_{\mu\nu}}$ can be flat as $h_{\mu\nu}$ and of order $\overline{g_{\mu\nu}} = O(1)$. It can only happen when we are far away from a source, so $T^{\mu\nu} = 0$. Next the perturbation $h_{\mu\nu} = O(|h_{\mu\nu}|)$ and equation 69 tell us for $T_{\mu\nu} = 0$ and $T = 0$ the following:

$$O(\overline{R_{\mu\nu}}) = O(R^{(2)\ low}_{\mu\nu}) = O((\partial h)^2) \tag{72}$$

So the derivatives of $h, \partial h$ affect the spacetime curvature. At the same point $\partial g_{\mu\nu} \sim L_B^{-1}$, since $\overline{g_{\mu\nu}} \simeq O(1)$ and

$$\partial h \sim \frac{h}{\lambda\!\!\!^{-}} \tag{73}$$

So, we take

$$O(R^{(1)}_{\mu\nu}) = O(R^{(2)\ high}_{\mu\nu}) = O(h\partial^2 h) \tag{74}$$

From Eq. 66, we get that

$$\overline{R_{\mu\nu}} \sim (\partial h)^2 \sim \partial^2\overline{g_{\mu\nu}} \sim \frac{1}{L_B^2} \tag{75}$$

eq. 74 $\xRightarrow{\text{eq. 71}}$ $\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda\!\!\!^{-}}\right)^2 \Rightarrow h \sim \frac{\lambda\!\!\!^{-}}{L_B}$ for curvature determined by GWs and $h \ll \frac{\lambda\!\!\!^{-}}{L_B}$ for matter determined curvature.

Eq. 69 is written as:

$$\overline{R_{\mu\nu}} = -R^{(2)\ \text{low}}_{\mu\nu} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{high}}$$

When the length scale $\lambda\!\!\!^{-}$ is separated from the L_B length scale of the background, one can introduce a scale $\bar{l}$ such that $\lambda\!\!\!^{-} \ll \bar{l} \ll L_B$ and average over a spatial volume of side $\bar{l}$. Similarly we can define a time scale $\bar{t} : \frac{1}{f} \ll \bar{t} \ll \frac{1}{f_B}$ and equation (69) can be written as:

$$\begin{aligned}\frac{1}{T}\int d\bar{t}\overline{R_{\mu\nu}} = -\frac{1}{T}\int \mathrm{d}\bar{t}R^{(2)}_{\mu\nu} + \frac{8\pi G}{c^4 T}\int \mathrm{d}\bar{t}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \Rightarrow \\ \overline{R_{\mu\nu}} = -\langle R^{(2)}_{\mu\nu}\rangle + \frac{8\pi G}{c^4}\langle T_{\mu\nu} - \frac{1}{2}T\bar{g_{\mu\nu}}\rangle\end{aligned} \tag{76}$$

We define the effective energy-momentum tensor $\overline{T_{\mu\nu}}$ such that

$$\langle T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\rangle \equiv \overline{T_{\mu\nu}} - \frac{1}{2}\overline{T}\overline{g_{\mu\nu}}. \tag{77}$$

$\overline{T_{\mu\nu}}$ tensor expresses by definition purely low frequencies, as is $\overline{g_{\mu\nu}}$. We define $t_{\mu\nu}$ tensor as:

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\overline{g_{\mu\nu}}R^{(2)}\rangle \tag{78}$$

And its trace:

$$\begin{aligned} t = \bar{g}^{\mu\nu}t_{\mu\nu} &= -\frac{c}{4\pi G}\langle\overline{g^{\mu\nu}}R^{(2)}_{\mu\nu} - \frac{1}{2}\overline{g^{\mu\nu}}\ \overline{g_{\mu\nu}}R^{(2)}\rangle \\ \Rightarrow t &= \frac{c}{4\pi G}\langle R^{(2)} + \frac{1}{2}R^{(2)}\rangle \\ \Rightarrow t &= \frac{c}{4\pi G}\langle R^{(2)}\rangle \end{aligned} \tag{79}$$

Then, Eq. 78 is inserted in Eq. 77, giving:

$$\begin{aligned} t_{\mu\nu} &= -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu}\rangle + \frac{1}{2}\frac{c^4}{8\pi G}\langle R^{(2)}\rangle\overline{g_{\mu\nu}} \\ \Rightarrow t_{\mu\nu} &= -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu}\rangle + \frac{1}{2}t\overline{g_{\mu\nu}} \\ \Rightarrow -\langle R^{(2)}_{\mu\nu}\rangle &= \frac{8\pi G}{c^4}\left(t_{\mu\nu} - \frac{1}{2}t\overline{g_{\mu\nu}}\right) \end{aligned} \tag{80}$$

So Eq. 75 in terms of Eq. 79 provides the following:

$$\begin{aligned} \xrightarrow{\text{Eq. 79}} \overline{R_{\mu\nu}} &= \frac{8\pi G}{c^4}(t_{\mu\nu} - \frac{1}{2}t\overline{g_{\mu\nu}}) + \frac{8\pi G}{c^4}(\overline{T_{\mu\nu}} - \frac{1}{2}\overline{g_{\mu\nu}}T) \\ \overline{R_{\mu\nu}} + \frac{8\pi G}{2c^4}(t\overline{g_{\mu\nu}} - T\overline{g_{\mu\nu}}) &= \frac{8\pi G}{c^4}(t_{\mu\nu} + \overline{T_{\mu\nu}}) \\ \overline{R_{\mu\nu}} - \frac{1}{2}\overline{g_{\mu\nu}}\overline{R} &= \frac{8\pi G}{2c^4}(t_{\mu\nu} + T_{\mu\nu}) \end{aligned} \tag{81}$$

The last equality of 81 hides all the physical meaning in this gauge of Einstein's equations. This form is known as a coarse-grained form of Einstein's equations in quadratic order in $h_{\mu\nu}$, and it is used to determine the dynamics governing $\bar{g}_{\mu\nu}$.

In summary, at a microscopic level, there is no fundamental distinction between the background metric and its perturbation. Moreover, when some fluctuations are distinguishable from the background because $\lambda \ll L_B$, we introduce $\bar{l} : \lambda \ll \bar{l} \ll L_B$ and integrate out the degree of freedom. The result of this integration is shown in equation 80. LHS in equation 70 is Einstein's tensor for slowly varying metrics. RHS is a smoothed version of the matter stress-energy tensor $\overline{T_{\mu\nu}}$. Finally, $t_{\mu\nu}$ comes out in an overlapped form naturally, because we pass from a fundamental microscopic description t to a "coarse-grained", macroscopic description.

2.4.1 Energy-momentum tensor

Now we can compute the explicit form of $t_{\mu\nu}$ (view Eq. 77) when $R^{(2)}_{\mu\nu}$ is given by Eq. 66. In this case, we suppose that the background is flat, so $\overline{g_{\mu\nu}} \to \eta_{\mu\nu}$ and $D_\mu \to \partial_\mu$ and Eq. 66 becomes:

$$
\begin{aligned}
R^{(2)}_{\mu\nu} &= \frac{1}{2}\eta^{\rho\sigma}\eta^{\alpha\beta}\Bigg[\frac{1}{2}\partial_\mu h_{\rho\alpha}\partial_\nu h_{\sigma\beta} + \partial_\rho h_{\nu\alpha}\partial_\sigma h_{\mu\beta} - \partial_\rho h_{\nu\alpha}\partial_\beta h_{\mu\sigma} \\
&\quad + h_{\rho\alpha}\big(\partial_\nu\partial_\mu h_{\sigma\beta} + \partial_\beta\partial_\sigma h_{\mu\nu} - \partial_\beta\partial_\nu h_{\mu\sigma} - \partial_\beta\partial_\mu h_{\nu\sigma}\big) \\
&\quad + \left(\frac{1}{2}\partial_\alpha h_{\rho\sigma} - \partial_\rho h_{\alpha\beta}\right)(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta} - \partial_\beta h_{\mu\nu})\Bigg] \\
\Rightarrow \quad R^{(2)}_{\mu\nu} &= \frac{1}{2}\Bigg[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\beta\partial_\nu h_{\alpha\mu} - h^{\alpha\beta}\partial_\beta\partial_\mu h_{\alpha\nu} \\
&\quad + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h_{\alpha\beta}\partial^\alpha h_{\mu\nu} - \partial^\beta h_{\nu\beta}\partial^\alpha h_{\mu\alpha} - \partial^\beta h_{\mu\beta}\partial^\alpha h_{\nu\alpha} \\
&\quad + \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h\partial_\nu h_{\alpha\mu}\Bigg]
\end{aligned}
\tag{82}
$$

where $h_{\mu\nu}$ is a matrix with 10 degrees of freedom with 8 gauge modes and 2 physical modes, both of them contribute to GWs.

Gauge modes are associated with ripples in spacetime, are coordinate-dependent, and can be gauged away. Physical modes produce an energy-momentum tensor of GWs and cannot be gauged away. These are found using Lorentz gauge condition $\partial_\mu h^{\mu\nu} = 0$

$$
\begin{aligned}
\text{Eq. 81} \xRightarrow[\Box h_{\alpha\beta}=0]{h=0} \langle R^{(2)}_{\mu\nu}\rangle &= \left\langle \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + \frac{1}{2}h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta}\right\rangle, \\
&= \frac{1}{4}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle + \frac{1}{2}\left\langle \partial_\mu\left(h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\right)\right\rangle - \frac{1}{2}\left\langle \partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\right\rangle, \\
\langle R^{(2)}_{\mu\nu}\rangle &= -\frac{1}{4}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle.
\end{aligned}
\tag{83}
$$

$$
\langle R^{(2)} = 0\rangle \tag{84}
$$

because of integration by parts.

$$
\begin{aligned}
\text{From Eq. 77, using Eqs. 82 and 83,} \quad t_{\mu\nu} &= -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu}\rangle - \frac{1}{2}\overline{g_{\mu\nu}}\langle R^{(2)}\rangle, \\
t_{\mu\nu} &= \frac{c^4}{8\pi G}\left(\frac{1}{4}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle\right), \\
\Rightarrow t_{\mu\nu} &= \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle.
\end{aligned}
\tag{85}
$$

To see that gauge modes do not contribute to Eq. 84, we vary it with $\delta h^{\alpha\beta}$ and use eq. 11. So Eq. 84 can be written as

$$
\begin{aligned}
\delta t_{\mu\nu} &= \frac{c^4}{32\pi G}\left[\langle(\partial_\mu \delta h_{\alpha\beta})\partial_\nu h^{\alpha\beta}\rangle + \langle\partial_\mu h_{\alpha\beta}\partial_\nu \delta h^{\alpha\beta}\rangle\right] \Rightarrow \\
\delta t_{\mu\nu} &= \frac{c^4}{32\pi G}\left[\langle\partial_\mu(\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha)\partial_\nu h^{\alpha\beta}\rangle + \langle\partial_{mu} h_{\alpha\beta}\partial_\nu(\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha)\rangle\right] \xrightarrow{\alpha\longleftrightarrow\beta} \\
\delta t_{\mu\nu} &= \frac{c^4}{16\pi G}\left[\langle\partial_\mu\partial_\alpha \xi_\beta \partial_\nu h^{\alpha\beta}\rangle + \langle\partial_\mu h_{\alpha\beta}\partial_\nu \partial^\alpha \xi^\beta\rangle\right] \Rightarrow \\
\delta t_{\mu\nu} &= \frac{c^4}{16\pi G}\left[\langle\partial_\mu\partial_\alpha \xi_\beta \partial_\nu h^{\alpha\beta}\rangle + \langle\partial_\mu h^{\alpha\beta}\partial_\nu \partial_\alpha \xi_\beta\rangle\right] \Rightarrow \\
\delta t_{\mu\nu} &= \frac{c^4}{16\pi G}\Big[\partial_\alpha\langle\partial_\mu \xi_\beta \partial_\nu h^{\alpha\beta}\rangle - \langle\partial_\mu \xi_\beta \partial_\nu \partial_\alpha h^{\alpha\beta}\rangle + \\
&\quad \partial_\alpha\langle\partial_\mu h^{\alpha\beta}\partial_\nu \xi_\beta\rangle - \langle\partial_\mu\partial_\alpha h^{\alpha\beta}\partial_\nu \xi_\beta\rangle + \langle\partial_\mu h^{\alpha\beta}\partial_\nu\partial_\alpha \xi_\beta\rangle\Big] \Rightarrow \\
\delta t_{\mu\nu} &= \frac{c^4}{16\pi G}\left[\partial_\alpha\langle\partial_\mu(\xi_\beta \partial_\nu h^{\alpha\beta}) - \xi_\beta \partial_\mu\partial_\nu h^{\alpha\beta} + \partial_\mu(h^{\alpha\beta}\partial_\nu \xi_\beta) - h_{\alpha\beta}\partial_\mu\partial_\nu \xi^\beta\rangle\right] \Rightarrow \\
\delta t_{\mu\nu} &= \frac{c^4}{16\pi G}\left[\partial_\alpha\partial_\mu\partial_\nu\langle\xi_\beta h^{\alpha\beta}\rangle\right] = 0
\end{aligned}
\tag{86}
$$

Since equation 84 holds, $t_{\mu\nu}$ does not depend on gauge modes and only on physical modes h_{ij}^{TT}, we can rewrite equation 83 as:

$$
t_{\mu\nu} = \frac{c^4}{32\pi G}\langle\partial_\mu h_{ij}^{\mathrm{TT}}\partial_\nu h_{\mathrm{TT}}^{ij}\rangle \tag{87}
$$

$$
\texttt{with } h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} cos\left[w\left(\epsilon - \frac{2}{\epsilon}\right)\right]
$$

All components from equation 85 are:

$$
t_{00} = \frac{c^2}{32\pi G}\langle\frac{1}{c}\partial_t h_{ij}^{\mathrm{TT}}\frac{1}{c}\partial_t h_{\mathrm{TT}}^{ij}\rangle \Rightarrow t_{00} = \frac{c^2}{32\pi G}\langle\dot{h}_{ij}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{ij}\rangle \Rightarrow
$$

$$
\begin{aligned}
t_{00} = \frac{c^2}{32\pi G}\langle & \dot{h}_{11}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{11} + \dot{h}_{12}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{12} + \dot{h}_{13}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{13} + \\
& \dot{h}_{21}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{21} + \dot{h}_{22}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{22} + \dot{h}_{23}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{23} + \\
& \dot{h}_{31}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{31} + \dot{h}_{32}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{32} + \dot{h}_{33}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{33}\rangle \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
t_{00} &= \frac{c^2}{32\pi G}\langle \dot{h}_{11}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{11} + \dot{h}_{12}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{12} + 0 + \dot{h}_{21}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{21} + \dot{h}_{22}^{\mathrm{TT}}\dot{h}_{\mathrm{TT}}^{22} + 0 + 0 + 0 + 0\rangle \Rightarrow \\
t_{00} &= \frac{c^2}{32\pi G} = \langle \dot{h}_{+}^2 + \dot{h}_{x}^2 + \dot{h}_{x}^2 + \dot{h}_{+}^2\rangle \Rightarrow \\
t_{00} &= \frac{c^2}{16\pi G} = \langle \dot{h}_{+}^2 + \dot{h}_{x}^2\rangle
\end{aligned} \tag{88}
$$

When applied the covariant derivative D_μ in equation 80, yields

$$
\begin{aligned}
&D^\mu \left[\overline{R_\mu\nu} - \frac{1}{2}\overline{R}\overline{g_{\mu\nu}}\right] = D^\mu \left(\overline{T_{\mu\nu}} + t_{\mu\nu}\right) \Rightarrow \\
&D^\mu\overline{T_{\mu\nu}} + D^\mu t_{\mu\nu} = D^\mu\overline{R_{\mu\nu}} - \frac{1}{2}(D^\mu\overline{R})\overline{g_{\mu\nu}} - \frac{1}{2}\overline{R}D^\mu\overline{g_{\mu\nu}} \Rightarrow \\
&D^\mu\overline{T_{\mu\nu}} + D^\mu t_{\mu\nu} = D^\mu\overline{R_{\mu\nu}} - \frac{1}{2}(D^\mu\overline{R_{\alpha\beta}})\overline{g^{\alpha\beta}}\overline{g_{\mu\nu}} - \frac{1}{2}\overline{R_{\alpha\beta}}D^\mu\overline{g^{\alpha\beta}}\overline{g_{\mu\nu}} \Rightarrow \\
&D^\mu(\overline{T_{\mu\nu}}t_{\mu\nu}) = 0 \text{ due to Bianchi identity}
\end{aligned} \tag{89}
$$

All the large distance limits, the background spacetime can be approximated by a flat spacetime and $D^\mu \rightarrow \partial_m u$ with $\overline{T}^{\mu\nu} = 0$. So, equation 88 reads as $\partial^\mu t_{\mu\nu}$.

2.4.2 Energy flux radiated by GWs

Energy flows per unit of time through a unit surface at large distances from the source. Conservation of Equation $\partial^\mu t_{\mu\nu}$ shows:

$$v = 0 : \partial^\mu t_{\mu\nu} = 0 \Rightarrow \partial^0 t_{00} + \partial^i t_{i0} = 0$$

When integrated over a spatial volume V in the far region, we get:

$$\int_v d^3x(\partial^0 t_{00} + \partial^i t_{i0}) = 0 \tag{90}$$

The energy of GW is defined as:

$$E_v = \int_v d^3x t^{00}$$

Equation 90 can be written as:

$$
\begin{aligned}
\frac{d}{dx^0}E_V &= -\int_V d^3x\, \partial_i t^{0i} \Rightarrow \\
\frac{d}{dx^0}E_V &= -\int_{S\equiv\partial V} dA\, n_i t^{0i} \Rightarrow \\
\frac{1}{c}\frac{d}{dt}E_V &= -\int_S dA\, n_i t^{0i} \Rightarrow \\
\frac{d}{dt}E_V &= -c\int_S dA\, n_i t^{0i}
\end{aligned} \tag{91}
$$

If V is a spherical shell as $V = S^2$, then far away from the source the outer normal vector to V is simply $\hat{n} \equiv \hat{r}$, the unitary radial vector and $dA \equiv r^2 d\Omega$. So, 91 reads:

$$\frac{dE_V}{dt} = -c \int_S r^2 d\Omega \, t^{0r}, \tag{92}$$

where

$$t^{0r} = \frac{c^4}{32\pi G} \left\langle \partial_0 h_{ij}^{\mathrm{TT}} \frac{\partial}{\partial r} h_{ij}^{\mathrm{TT}} \right\rangle. \tag{93}$$

A radially propagating GW far away from the source can be described in general as:

$$h_{ij}^{\mathrm{TT}}(t,r) = \frac{1}{r} f_{ij}(t - r/c), \tag{94}$$

where $f_{ij}(t - r/c)$ is a function of the retarded time $t_{\mathrm{ret}} = t - r/c$.

Taking derivative ∂_r:

$$\frac{\partial}{\partial r} h_{ij}^{\mathrm{TT}}(t,r) = -\frac{1}{r^2} f_{ij}(t - \frac{r}{c}) + \frac{1}{r}\frac{\partial}{\partial r} f_{ij}(t - \frac{r}{c}). \tag{95}$$

Since $dr = -\frac{dr}{c} = d\left(t - \frac{r}{c} = dt_{\mathrm{ret}}\right) \Rightarrow \partial_r = -\partial_{t_r}$ we get

$$\frac{\partial}{\partial r} f_{ij}(t - \frac{r}{c}) = -\frac{1}{c}\frac{\partial}{\partial t} f_{ij}(t - \frac{r}{c}) \tag{96}$$

and therefore:

$$\frac{\partial}{\partial r} h_{ij}^{\mathrm{TT}}(t,r) = -\partial_0 h_{ij}^{\mathrm{TT}}(t,r) \tag{97}$$

Substituting into Eq. 93, we see that at large distances:

$$\begin{aligned} t^{0r} &= -\frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{\mathrm{TT}} \partial_0 h_{ij}^{\mathrm{TT}} \rangle \Rightarrow \\ t^{0r} &= \frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{\mathrm{TT}} \partial^0 h_{ij}^{\mathrm{TT}} \rangle = t^{00} \end{aligned}$$

And the energy flow from 92 is decreasing since

$$\frac{dE_V}{dt} = -c \int dA \, t^{0r} = -c \int dA \, t^{00} \tag{98}$$

This decrease shows that the outgoing GW carries energy flux:

$$\begin{aligned} \frac{dE}{dA\,dt} &= c\,t^{00} = \frac{c^3}{32\pi G} \left\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}} \right\rangle \Rightarrow \\ \frac{dE}{dt} &= \frac{c^3}{32\pi G} \int dA \left\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}} \right\rangle \Rightarrow \\ \frac{dE}{dt} &= \frac{c^3 r^2}{32\pi G} \int d\Omega \left\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}} \right\rangle \end{aligned} \tag{99}$$

In terms of h_+ and $h_\times$, we can rewrite the result as:

$$\frac{dE}{dA\,dt} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle . \tag{100}$$

The total energy flowing through dA between $t = -\infty$ and $t = +\infty$ is therefore:

$$\frac{dE}{dA} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} dt \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \tag{101}$$

or in terms of $\frac{dE}{dA}$ we get:

$$\frac{dE}{dA} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} dt \left(\dot{h}_+^2 + \dot{h}_\times^2 \right) \tag{102}$$

Because of Parseval's theorem, we get:

$$\begin{aligned}
\int_{-\infty}^{+\infty} dt |\dot{h}_{+,\times}(t)|^2 &= \int_{-\infty}^{+\infty} df |\dot{h}_{+,\times}(t)|^2 \\
&= \int_{-\infty}^{+\infty} df |\tilde{h}_{+,\times}(f)\frac{\partial}{\partial_t}e^{-2\pi i f t}|^2 \\
&= \int_{-\infty}^{+\infty} df |\tilde{h}_{+,\times}(f)(2\pi f)e^{-2\pi i f t}|^2 \\
&= \int_{-\infty}^{+\infty} df (2\pi f)^2 |\tilde{h}_{+,\times}(f)|^2
\end{aligned} \tag{103}$$

Combining the two equations above, we get:

$$\begin{aligned}
\frac{dE}{dA} &= \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} df\ (2\pi f)^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right) \Rightarrow \\
\frac{dE}{dA} &= \frac{\pi c^3}{4G}\int_{-\infty}^{\infty} df\ f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)
\end{aligned} \tag{104}$$

Since the integrand is even under $f \to -f$, we can restrict it to physical frequencies $f > 0$, writing:

$$\begin{aligned}
\frac{dE}{dA} &= \frac{\pi c^3}{2G}\int_{0}^{\infty} df\ f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right) \Rightarrow \\
\frac{dE}{dA\,df} &= \frac{\pi c^3}{2G} f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)
\end{aligned} \tag{105}$$

The energy spectrum on a sphere of constant radius r is written as:

$$\frac{dE}{df} = \frac{\pi c^3}{2G} f^2 r^2 \int d\Omega \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right) . \tag{106}$$

2.4.3 Momenta carried by GWs

To calculate momenta, we have to take the equation $\partial^\mu t_{\mu\nu} = 0$, which illustrates the energy-momentum conservation, and choose the $\nu = i$ component.

$$\partial^\mu t_{\mu\nu} = 0 \xRightarrow{\nu=i} \partial^\mu t_{\mu i} = 0 \Rightarrow \partial_0 t^{0i} + \partial_j t^{ji} = 0 \tag{107}$$

Integration on a volume V gives:

$$\int_\nu \mathrm{d}^3 x[\partial_0 t^{0i} + \partial_j t^{ji}] = 0 \tag{108}$$

A GW carries momentum given by:

$$P^k_\nu = \frac{1}{c}\int \mathrm{d}^3 x t^{0k} \tag{109}$$

With the energy being expressed as the derivative of equation 109

$$\frac{1}{c}\frac{\mathrm{d}P^k_\nu}{\mathrm{d}t} = \frac{1}{c}\int \mathrm{d}^3 x \partial_0 t^{0k} \tag{110}$$

When the integrated equation 108 produces a form of momentum conservation for the gravitational waveform

$$\begin{aligned}
&\int_\nu \mathrm{d}^3 x[\partial_0 t^{0i} + \partial_j t^{ji}] = 0 \Rightarrow \int_\nu \mathrm{d}^3 x[\partial_0 t^{0i} = -\int_\nu \mathrm{d}^3 x \partial_j t^{ji}] \xRightarrow{equation84} \\
&\frac{1}{c}\frac{\mathrm{d}P^i_\nu}{\mathrm{d}t} = -\int_\nu \mathrm{d}^3 x \partial_j t^{ji} \overset{Stoke's_theorem}{=} -\int_S \mathrm{d}A n_j t^{ij} \Rightarrow \\
&\frac{\mathrm{d}P^i_v}{\mathrm{d}t} = -c\int_S \mathrm{d}A n_j t^{ji}
\end{aligned} \tag{111}$$

$$\begin{aligned}
\xRightarrow{equation84} \frac{1}{c}\frac{\mathrm{d}P^i_\nu}{\mathrm{d}t} &= -\int \mathrm{d}^3 x \partial_0 t^{0i} = -\int \mathrm{d}A n_0 t^{0i} \Rightarrow \\
\frac{\mathrm{d}P^i_\nu}{\mathrm{d}t} &= -c\int \mathrm{d}A n_0 t^{0i} \Rightarrow \frac{\mathrm{d}P^i_\nu}{\mathrm{d}A\mathrm{d}t} = -ct^{0i}
\end{aligned} \tag{112}$$

$$\Rightarrow \frac{\mathrm{d}P^i_\nu}{\mathrm{d}A\mathrm{d}t} = -c\frac{c^2}{32\pi G}\langle \dot{h}^{\mathrm{TT}}_{jk}\partial^i h^{\mathrm{TT}}_{jk}\rangle \tag{113}$$

$$\Rightarrow \frac{\mathrm{d}P^i_\nu}{\mathrm{d}t} = \frac{c^3}{32\pi G}r^2\int \mathrm{d}\Omega\langle \dot{h}^{\mathrm{TT}}_{jk}\partial^i\partial^i h^{\mathrm{TT}}_{jk}\rangle \tag{114}$$

2.5 Energy-momentum tensor in field theoretical approach

2.5.1 Energy-momentum tensor produced in field theory models

More details and proofs about the action and the energy-momentum tensor in field theoretical models can be found in Maggiore's book [36], as well as [25] and [19].

The action that describes any physical system is

$$S = \int \mathrm{d}^4 x \mathcal{L} = \int \mathrm{d}t \int \mathrm{d}^3 x \mathcal{L}(\phi_i, \partial\phi_i) \tag{115}$$

All ϕ^i fields are components of the perturbation metric $h_{\mu\nu}$ and are denoted simply as $\phi_i \equiv \phi$.

A coordinate transformation bets on $x^\mu \to x'^\mu$ and on $\phi(x) \to \phi'(x')$. An infinitesimal transformation for the coordinate and the field is given as:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\alpha A^\mu_\alpha(x) \tag{116}$$

$$\phi_i(x) \to \phi'_i(x') = \phi_i(x) + \epsilon^\alpha F_{i,a}(\phi, \partial\phi) \; \forall\alpha = 1, \ldots, \mu \tag{117}$$

Eqs. 116 and 117 leave the action invariant and define a symmetry of $S(\phi)$.

- When the symmetric transformation leaves the action invariant and the ϵ^α parameters are constant, we get a global transformation.

- When the symmetry leaves the action invariant and the ϵ^α parameters are allowed to be arbitrary functions of x, we have a local transformation.

Noether's theorem states that for each generator of a global symmetry (meaning ϵ^α, $\forall\alpha$) there is a current $j^\mu_\alpha[\phi, \partial_\phi]$, that is conserved as

$$\partial_\mu j^\mu_a = 0 \tag{118}$$

and a corresponding conserved charged Q_a such as:

$$Q_\alpha \equiv \int \mathrm{d}^3 x j^0_\alpha(x, t) \tag{119}$$

The conservation of Q_α is given as

$$\partial_0 Q_\alpha = -\int \mathrm{d}^3 x \partial_i j^i_\alpha(x, t) \Rightarrow \partial_0 Q_\alpha = -j^i_\alpha(x, t)\hat{x}_i|_{\partial_V} = 0 \tag{120}$$

Equation 100 vanishes, since we demand $\phi|_{\partial_V} \to 0$. The generic formula of j^μ_α is given as:

$$j^\mu_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\left[A^\nu_\alpha(x)\partial_\nu\phi_i - F_{i,\alpha}(\phi, \partial\phi)\right] - A^\mu_\alpha(x)\mathcal{L} \tag{121}$$

Symmetry under spacetime translations

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu + \epsilon^\mu = x^\mu + \epsilon^\nu \delta^\mu_\nu \\ \phi(x) &\to \phi'_i(x') = \phi_i(x) \end{aligned} \tag{122}$$

So we have $A^\mu_\nu \equiv \delta^\mu_\nu$ and $F_{i,\alpha} = 0$ and the conserved current is

$$\begin{aligned} j^\mu_\alpha &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}[\delta^\nu_\alpha \partial_\nu \phi_i - F_{i,\alpha}(\theta, \phi)] - A^\mu_\alpha(x)\mathcal{L} \\ j^\mu_\alpha &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\alpha \phi_i - \delta^\mu_\alpha \mathcal{L} \end{aligned} \tag{123}$$

The energy-momentum tensor is defined as:

$$\theta^\mu{}_\nu \equiv -j^\mu_\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\nu \phi_i + \delta^\mu_\nu \mathcal{L}$$

Or with all indices raised:

$$\theta^{\mu\nu} = \eta^{\nu\rho}\theta^\mu{}_\rho = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i + \eta^{\mu\nu}\mathcal{L} \tag{124}$$

with conservation

$$\partial_\mu \theta^{\mu\nu} = 0 \tag{125}$$

The conserved charge is the four-momentum P^ν defined as

$$cP^\nu \equiv \int \mathrm{d}^3x \theta^{0\nu} \tag{126}$$

with components

$$E \equiv cP^0 = \int \mathrm{d}^3x \theta^{00} \tag{127}$$

$$P^i \equiv \frac{1}{c}\int \mathrm{d}^3x \theta^{0i} \tag{128}$$

In the case of electrodynamics, the Lagrangian density can be expressed as:

$$L_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{129}$$

and the electric and magnetic fields can be written as the components of the EM tensor $E^i \equiv F^{oi}$ and $F^{ij} \equiv E^{ijk}B^k$. Thus, the $F_{\mu\nu}$ Maxwell tensor will be:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{130}$$

and its "square" $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$ will read:

$$\begin{aligned} F^2 = F_{\mu\nu}F^{\mu\nu} &= F_{oi}F^{oi} + F_{io}F^{io} + F_{ij}F^{ij} \\ &= E^iE_i + E_iE^i + E_{ij}B_kE^{ijl}B^l = 2\vec{E}^2 + 2\vec{B}^2 \Rightarrow \\ F_{\mu\nu}F^{\mu\nu} &= (-\vec{E}^2 + \vec{B}^2) \end{aligned} \tag{131}$$

The Lagrangian density in terms of the electromagnetic fields will be written as:

$$L_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{132}$$

To retrieve a closed form for the EM energy-momentum tensor, we need to compute the following quantity:

$$\begin{aligned}
\frac{\partial L_{EM}}{\partial(\partial_\mu A_\rho)} &= \frac{\partial}{\partial(\partial_\rho A_\rho)}[-\frac{1}{4}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)2F^{\alpha\beta}] \Rightarrow \\
\frac{\partial L_{EM}}{\partial(\partial_\mu A_\rho)} &= -\frac{1}{2}F^{\alpha\beta}\frac{\partial(\partial_{[\alpha}A_{\beta]})}{\partial(\partial_\mu A_\rho)} = -\frac{1}{2}F^{\alpha\beta}(\delta^\mu_{[\alpha}\delta^\rho_{\beta]}) \Rightarrow \\
\frac{\partial L_{EM}}{\partial(\partial_\mu A_\rho)} &= -F^{\mu\rho}
\end{aligned} \tag{133}$$

Having the full expression of equation 133, we can now write down the full expression of the EM energy-momentum tensor. Following the steps below, we get:

$$\begin{aligned}
\theta^{\mu\nu}_{em} &= -\frac{\partial L_{EM}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho + \eta^{\mu\nu}L_{EM} \Rightarrow \\
\theta^{\mu\nu}_{em} &= F^{\mu\rho}\partial^\nu A_\rho + \eta^{\mu\nu}[-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}] \\
\theta^{\mu\nu}_{em} &= F^{\mu\rho}\partial^\nu A_\rho - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}
\end{aligned} \tag{134}$$

Since classical electrodynamics is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu\theta$ We see that Eqs. 132 and 130 have:

$$\begin{aligned}
F^{'}_{\mu\nu} &= \partial_\mu A^{'}_\nu - \partial_\nu A^{'}_\mu \Rightarrow \\
F^{'}_{\mu\nu} &= \partial_\mu A_\nu \partial_\mu\partial_\nu\theta - \partial_\nu A_\mu + \partial_\nu\partial_\mu\theta \Rightarrow \\
F^{'}_{\mu\nu} &= F_{\mu\nu}
\end{aligned} \tag{135}$$

$$L^{'}_{EM} = -\frac{1}{4}F^{'}_{\mu\nu}F^{\mu\nu\ '} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = L_{EM} \tag{136}$$

Although $\theta^{\mu\nu}_{EM}$ via Equation 134 is:

$$\begin{aligned}
\theta'^{\mu\nu}_{EM} &= F'^{\ \mu\rho}\partial^\nu A'_\rho - \frac{1}{4}\eta^{\mu\nu}F'_{\alpha\beta}F^{\alpha\beta\ '} \Rightarrow \\
\theta'^{\mu\nu}_{EM} &= F^{\mu\rho}\partial^\nu A_\rho - F^{\mu\rho}\partial^\nu\partial_\rho\theta - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \Rightarrow \\
\theta'^{\mu\nu}_{EM} &= \theta^{\mu\nu}_{EM} - F^{\mu\rho}\partial^\nu\partial_\rho\theta
\end{aligned} \tag{137}$$

$$\begin{aligned}
\theta'^{\mu\nu}_{EM} &= F^{\mu\rho}(\partial^\nu A_\rho - \partial_\rho A^\nu + \partial_\rho A^\nu) - \frac{1}{4}\eta^{\mu\nu}F'_{\alpha\rho}F'^{\ \alpha\beta} \Rightarrow \\
\theta'^{\mu\nu}_{EM} &= F^{\mu\rho}F^\nu_\rho - \frac{1}{4}\eta^{\mu\nu}F'_{\alpha\beta}F'^{\ \alpha\beta} + F^{\mu\rho}\partial_\rho A^\nu \xRightarrow{\partial_\rho F^{\mu\rho}=0} \\
\theta'^{\mu\nu}_{EM} &= F^{\mu\rho}F^\nu_\rho - \frac{1}{4}\eta^{\mu\nu}F^2 + \partial_\rho(F^{\mu\rho}A^\nu) \Rightarrow \\
\theta'^{\mu\nu}_{EM} &= T^{\mu\nu}_{EM} + \partial_\rho(F^{\mu\rho}A^\nu)
\end{aligned} \tag{138}$$

Next we can define $C^{\rho\mu\nu}$ as $C^{\rho\mu\nu} \equiv F^{\mu\rho A^{\nu}}$ an antisymmetric tensor in $\rho \leftrightarrow \mu$, with $\partial_\rho C^{\rho\mu\nu}$ not gauge invariant. Also we set $T^{\mu\nu}_{EM} \equiv F^{\mu\rho F^{\nu}_{\rho} - \frac{1}{4}}\eta^{\mu\nu}F^2$, the improved energy-momentum tensor with the 00−component to be given as the energy density:

$$\mathcal{E} = T^{00}_{EM} = \frac{1}{2}(\vec{E}^2(x) + \vec{B}^2(x)) \tag{139}$$

We notice the following:

- $\partial_\mu \partial_\rho C^{\rho\mu\nu} = \partial_\mu \partial_\rho (E^{\mu\rho} A^{\nu}) = 0$, so $\partial_\mu \theta^{\mu\nu}_{em} = 0 \Rightarrow \partial_\mu (T^{\mu\nu}_{em} + C^{\rho\mu\nu}) = 0 \Rightarrow \partial_\mu T^{\mu\nu}_{EM} = 0$
- The conserved charge for $\theta^{\mu\nu}_{EM}$ is:

$$cP^{\nu}_1 = \int_V \mathrm{d}^3x \theta^{0\nu}_{EM} = \int \mathrm{d}^3x (F^{0\rho}\partial^{\nu}A_\rho - \frac{1}{4}\eta^{0\nu}F^2) \tag{140}$$

- while for $T^{\mu\nu}_{EM}$ will be:

$$cP^{\nu}_2 = \int_V \mathrm{d}^3x T^{0\nu}_{EM} = \int_V \mathrm{d}^3x (F^{0\rho}F^{\nu}_{\rho} - \frac{1}{4}\eta^{0\nu}F^2) \tag{141}$$

The two forms of the energy-momentum tensor, as stated in Eqs. 140 and 141 differ by a factor. This factor, when applied to some algebra, transforms into the following expression:

$$\int_V \mathrm{d}^3x \partial_\rho C^{\rho 0\nu} = \int_V \mathrm{d}^3x (\partial_0 C^{00\nu} + \partial_i C^{i0\nu}) \Rightarrow \int_V \mathrm{d}^3x \partial_\rho C^{\rho 0\nu} = \int_V \mathrm{d}^3x \partial_i C^{i0\nu} \tag{142}$$

If $A^\mu \to 0$ is fast enough at the boundary, we get from equation 142 the following.

$$\xRightarrow{equation 119} \int_V \mathrm{d}^3x \partial_\rho C^{\rho 0\nu} = \int_V \mathrm{d}^3x \partial_i C^{i0\nu} = C^{i0\nu} n_i|_{\partial_V} \Rightarrow$$
$$\int_V \mathrm{d}^3x \partial_\rho C^{\rho 0\nu} = F^{i0}A^{\nu} n_i|_{\partial_V} \to 0$$

Since equation 142 vanishes, we get: $CP^{\nu}_1 = CP^{\nu}_2$ and P^{ν} is gauge invariant.

2.5.2 Energy-momentum tensor of GWs

The quantum field theoretical approach to gravitation is explicitly studied in Feynman's Lectures n Gravitation [37]. Also, in [38] and [39].

To find an expression for the energy-momentum tensor of GWs, we must begin studying Einstein's action as given in equation 1 of Chapter 1: $S_E = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R$, Expand the metric as per usual $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and compute the Ricci scalar as:

$$R = g^{\mu\nu}R_{\mu\nu} = [\eta^{\mu\nu} - h^{\mu\nu} + O(h^2)][R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + O(h^3)] \tag{143}$$

The $\sqrt{-g}$ term is expanded as follows:

$$-g = -detg_{\mu\nu} = -det(\eta_{\mu\rho}g^{\rho}_{\nu}) \Rightarrow -g = detg^{\rho}_{\nu} \tag{144}$$

The permutation $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$ can be written as

$$\eta^{\rho\mu} g_{\mu\nu} = \eta^{\rho\mu}\eta_{\mu\nu} + \eta^{\rho\mu} h_{\mu\nu} \Rightarrow g^{\rho}_{\nu} = \delta^{\rho}_{\nu} + h^{\rho}_{\nu} \tag{145}$$

Inserting equation 144 into 145, we find the first-order expansion of the scalar $-g$ to be:

$$\begin{aligned}
\text{eq. } 121 \xRightarrow{\text{eq. } 122} & - g = det(\delta^{\rho}_{\nu} + h^{\rho}_{\nu}) = det(I + H) \Rightarrow \\
& - g = e^{lndet(I+H)} = e^{Tr[ln(I+H)]} \Rightarrow \\
& - g \approx e^{Tr[(1+H)-\frac{1}{2}(I+H^2)+O(H^3)]} = e^{Tr[1+H]} \Rightarrow \\
& - g \approx 1 + Tr(1 + H) + O(H^2) \Rightarrow \\
& - g \approx 1 + h + O(h^2)
\end{aligned} \tag{146}$$

Thus, Einstein's action S_E can be written as:

$$\begin{aligned}
S_E &= \int \mathrm{d}^4x \frac{c^3}{16\pi G}\sqrt{-g}R \xRightarrow[equation123]{equation120} \\
S_E &= \int \mathrm{d}^4x \frac{c^3}{16\pi G}\sqrt{1+h}(\eta^{\mu\nu} - h^{\mu\nu} + O(h^2))[R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + O(h^3)]
\end{aligned}$$

We ignore the term $\sqrt{1+h}$ because it is of order $O(h^3)$.

$$\begin{aligned}
S_E = & \int \mathrm{d}^4x \frac{c^3}{16\pi G}[R^(1) + R^(2) - h^{\mu\nu}R^{(1)}_{\mu\nu} - h^{\mu\nu}R^{(2)}_{\mu\nu} + O(h^3)] \\
& \frac{c^3}{16\pi G}\int \mathrm{d}^4x[\eta^{\mu\nu}\frac{1}{2}(\partial^{\alpha}\partial_{\mu}h_{\nu\alpha} + \partial^{\alpha}\partial_{\nu}h_{\mu\alpha}) - \partial^{\alpha}\partial_{\alpha}h_{\mu\nu} - \partial_{\nu}\partial_{\mu}h) \\
& + \eta^{\mu\nu}\frac{1}{2}(\frac{1}{2}\partial_{\mu}h_{\alpha\beta}\partial_{\nu}h^{\alpha\beta} + h^{\alpha\beta}\partial_{\mu}\partial_{\nu}h_{\alpha\beta} - h^{\alpha\beta}\partial_{\nu}\partial_{\beta}h_{\alpha\mu} - h^{\alpha\beta}\partial_{\mu}\partial_{\beta}h_{\alpha\nu} \\
& + h^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h_{\mu\nu} + \partial^{\beta}h^{\alpha}_{\nu}\partial_{\beta}h_{\alpha\mu} - \partial_{\beta}h^{\alpha\beta}\partial_{\nu}h_{\alpha\mu} - \partial^{\alpha}h^{\beta}_{\nu}\partial_{\alpha}h_{\beta\mu} + \partial_{\beta}h^{\alpha\beta}\partial_{\alpha}h_{\mu\nu} \\
& - \partial_{\beta}h^{\alpha\beta}\partial_{\mu}h_{\alpha\nu} - \frac{1}{2}\partial^{\alpha}h\partial_{\alpha}h_{\mu\nu} + \frac{1}{2}\partial^{\alpha}h\partial_{\nu}h_{\alpha\mu} + \frac{1}{2}\partial^{\alpha}h\partial_{\mu}h_{\alpha\nu}) \\
& - h_{\mu\nu}\frac{1}{2}(\partial^{\alpha}\partial_{\mu}h_{\nu\alpha} + \partial^{\alpha}\partial_{\nu}h_{\mu\alpha} - h_{\mu\nu} - \partial_{\nu}\partial_{\mu}h)] \Rightarrow \\
S_E = & -\frac{c^3}{64\pi G}\int \mathrm{d}^4x[(\partial_{\mu}h_{\alpha\beta})^2 - (\partial_{\mu}h)^2 + 2\partial_{\mu}h^{\mu\nu}(\partial_{\nu}h - \partial_{\rho}h^{\rho}\nu)]
\end{aligned} \tag{147}$$

And the Lagrangian density of the Gravitational theory will be

$$L_E \equiv -\frac{c^4}{64\pi G}[(\partial_{\mu}h_{\alpha\beta})^2 - (\partial_{\mu}h)^2 + 2\partial_{\mu}h^{\mu\nu}(\partial_{\nu}h - \partial_{\rho}h^{\rho}\nu)] \tag{148}$$

In the Lorentz gauge, the Lagrangian density can be written as:

$$L_E = \frac{c^4}{64\pi G}(\partial_{\mu}h_{\alpha\beta}\partial^{\mu}h^{\alpha\beta}) \tag{149}$$

And the stress-energy tensor is:

$$\begin{aligned}\theta^{\mu\nu} &= -\frac{\partial L_E}{\partial(\partial_\mu h_{\alpha\beta})}\partial^\nu h_{\alpha\beta} + \eta^{\mu\nu} L_E \Rightarrow \\ \theta^{\mu\nu} &= -\frac{c^4}{64\pi G} 2\partial^\mu h^{\alpha\beta}\partial^\nu h_{\alpha\beta} + \eta^{\mu\nu}\left(-\frac{c^3}{64\pi G}\right)\left(\partial_\rho h_{\alpha\beta}\partial^\rho h^{\alpha\beta}\right)\end{aligned} \tag{150}$$

Since

$$\Box h_{\mu\nu} = 0 \Rightarrow h_{\alpha\beta}\partial_\rho\partial^\rho h_{\alpha\beta} = 0 \Rightarrow \partial_\rho[h_{\alpha\beta}\partial^\rho h_{\alpha\beta}] - \partial_\rho h_{\alpha\beta}\partial^\rho h^{\alpha\beta} = 0 \tag{151}$$

And since $\partial_\rho[h_{\alpha\beta}\partial^\rho h_{\alpha\beta}]$ does not add in action, due to boundary conditions, we get:

$$\partial_\rho h_{\alpha\beta}\partial^\rho h^{\alpha\beta} = 0 \tag{152}$$

The stress-energy tensor will take the final form stated below:

$$Eq.127 \xrightarrow{Eq.129} \theta^{\mu\nu} = -\frac{c^4}{32\pi G}(\partial^\mu h^{\alpha\beta}\partial^\nu h_{\alpha\beta}) \tag{153}$$

If we evaluate the mean value of $\theta^{\mu\nu}$ on several wavelength $\lambdabar$, we get the " macroscopic" stress-energy tensor $t^{\mu\nu}$ as:

$$t^{\mu\nu} = -\frac{1}{\lambdabar}\int_0^{\lambdabar} \mathrm{d}\lambdabar' \theta^{\mu\nu} = \langle -\theta^{\mu\nu}\rangle \Rightarrow$$

$$t^{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial^\mu h^{\alpha\beta}\partial^\nu h_{\alpha\beta}\rangle \tag{154}$$

2.5.3 Angular momentum carried by GWs

Angular momentum carried by GWs comes as the conserved charge under spatial rotations. A symmetric tensor $h_{\mu\nu}$, when rotated, decomposes into h_{00} and the spatial trace h^i_i, which are scalars (spin-O fields), to h_{0i} a spin-1 spatial vector and a purely spatial tensor h_{ij} with spin-2. When a GW is expressed in the TT-gauge, we have $h_{0\mu} = 0, h^i_i = h = 0$, and $\partial^j h_{ij} = 0$. In this gauge, the total d.o.f. reduce from 10 to 2 and equation 147, that describes the gravitational wave action, yields:

$$S_E = \int \left[\mathrm{d}^4x \frac{-c^3}{64\pi G}(\partial\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta})\right] \Rightarrow S_E = \int \mathrm{d}^4x \left[-\frac{c^3}{64\pi G}(\partial_\mu h^{\mathrm{TT}}_{ij}\partial^\mu h^{ij}_{\mathrm{TT}}\right] \tag{155}$$

With the corresponding Lagrangian density:

$$\Rightarrow L_E = -\frac{c^4}{64\pi G}(\partial\mu h^{\mathrm{TT}}_{ij}\partial^\mu h^{ij}_{\mathrm{TT}}) \tag{156}$$

The h^{TT}_{ij} fields describe the two physical degrees of freedom. The conserved current under rotations uses the Lagrangian density, given by equation 156. The rotations on three-dimensional spaces are described by $3x3$ matrices R, such that:

$$x^i \rightarrow x^i = R^{ij}x^j \tag{157}$$

For infinitesimal rotations, we can write

$$R^{ij} = \delta^{ij} + w^{ij}, \forall w^{ij} \ antisymmetric \tag{158}$$

Antisymmetricity of w^{ij} is derived from the R^{ij} matrices' orthogonality. Because of it we get $w^{ij} = -w^{ji}$ and w^{ij} as a matrix is:

$$w_{ij} = \begin{pmatrix} 0 & w_{12} & w_{13} \\ w_{21} & 0 & w_{23} \\ w_{31} & w_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{pmatrix} \tag{159}$$

Thus, the transformation rules of coordinates are:

$$x^i \to x'^i = x^i + \sum_{k<l} w^{kl} A^i_{kl} \tag{160}$$

$$\xRightarrow{\phi_i = h^{\mathrm{TT}}_{ij}} h^{TT}_{ij} \to h'^{TT}_{ij} = h^{TT}_{ij} + \sum_{k<l} w^{kl} F_{ij,kl} \tag{161}$$

The conserved current by Noether's theorem is given as:

$$\begin{aligned} j^\mu_{kl} &= \frac{\partial L_E}{\partial(\partial_\mu h^{\mathrm{TT}}_{ij})} \left(\partial_\nu h^{\mathrm{TT}}_{ij} A^\nu_{kl} - F_{ij,kl}\right) - A^\mu_{kl} L \\ &= \left[\partial^\mu h^{\mathrm{TT}}_{ij} \left(-\frac{2c^4}{64\pi G}\right) \partial_\nu h^{\mathrm{TT}}_{ij}\right] \left[(\delta^{\nu k} x^l - \delta^{\nu l} x^k) - \delta_{ji} h^{\mathrm{TT}}_{kl} - \delta_{ij} h^{\mathrm{TT}}_{kl} - \delta_{il} h^{\mathrm{TT}}_{jk} - \delta_{ij} h^{\mathrm{TT}}_{ik}\right] \\ &+ (\delta^{\mu k} x^l - \delta^{\mu l} x^k) \frac{c^4}{64\pi G} (\partial_\rho h^{\mathrm{TT}}_{ij} \partial^\rho h^{\mathrm{TT}}_{ij}) = \\ &- \frac{c^4}{32\pi G} [\partial^\mu h^{\mathrm{TT}}_{ij} (\partial^k h^{\mathrm{TT}}_{ij} x^l - \partial^l h^{\mathrm{TT}}_{ij} x^k) - \partial^\mu h^{\mathrm{TT}}_{ij} (\delta_{jk} h^{\mathrm{TT}}_{il} + \delta_{ik} h^{\mathrm{TT}}_{jl} + \delta_{jl} h^{\mathrm{TT}}_{ik}) - \\ &\frac{1}{2} (\delta^{\mu k} x^l - \delta^{\mu l} x^k) \partial_\rho h^{\mathrm{TT}}_{ij} \partial^\rho h^{\mathrm{TT}}_{ij}] \end{aligned} \tag{162}$$

For $\mu = 0$ we get

$$\begin{aligned} j^0_{kl} &= -\frac{c^3}{32\pi G} \left[\frac{1}{c} \dot{h}^{TT}_{ij} \partial^k h^{TT}_{ij} x^l - \frac{1}{c} 2\dot{h}^{TT}_{ik} h^{TT}_{il} - \frac{1}{c} 2\dot{h}^{TT}_{il} h^{TT}_{ik}\right] \\ j^0_{kl} &= \frac{c^3}{32\pi G} \left[-\dot{h}^{TT}_{\alpha\beta} \partial^{[k} h^{TT}_{\alpha\beta} x^{l]} + 2\dot{h}^{TT}_{\alpha k} h^{TT}_{\alpha l} + 2\dot{h}^{TT}_{al} h^{TT}_{\alpha k}\right] \\ j^0_{kl} &= \frac{c^3}{32\pi G} \left[-2\dot{h}^{TT}_{\alpha\beta} (\frac{1}{2} \partial^{[k} h^{TT}_{\alpha\beta} x^{l]} - \delta_{bk} h^{TT}_{\alpha l} - \delta_{bl} h^{TT}_{\alpha k})\right] \end{aligned} \tag{163}$$

With the corresponding conserved charge:

$$\begin{aligned} J_{kl} &= \frac{1}{c} \int \mathrm{d}^3 x j^0_{kl} \\ \xRightarrow{Eq.140} J_{kl} &= \frac{c^2}{32\pi G} \int \mathrm{d}^3 x - \dot{h}^{TT}_{ab} (x^{[k} \partial^{l]} h^{TT}_{ab} - \delta_{bk} h^{TT}_{al} - \delta_{bl} h^{TT}_{ak}) \end{aligned} \tag{164}$$

Or via Poincaré duality, we obtain:

$$\begin{aligned} J^i &= \frac{1}{2}\epsilon^{ikl} J_{kl} = \frac{c^2}{64\pi G}\int \mathrm{d}^3x \left[-2\dot{h}^{TT}_{ab}\epsilon^{ikl}x^k\partial^l h^{TT}_{ab} + 2\epsilon^{ikl}\dot{h}^{TT}_{ak}h^{TT}_{al} + 2\epsilon^{ikl}\dot{h}^{TT}_{ak}h^{TT}_{al}\right] \\ J^i &= \frac{c^2}{32\pi G}\int \mathrm{d}^3x \left[-\epsilon^{ikl}\dot{h}^{TT}_{ab}x^k\partial^l h^{TT}_{ab} + 2\epsilon^{ikl}\dot{h}^{TT}_{ak}h^{TT}_{al}\right] \end{aligned} \tag{165}$$

The physical density of angular momentum is the localized current over a few wavelengths, such that:

$$\frac{j^i}{c} = \frac{c^2}{32\pi G}\langle -\epsilon^{ikl}\dot{h}^{TT}_{ab}x^k\partial^l h^{TT}_{ab} + 2\epsilon^{ikl}\dot{h}^{TT}_{ak}h^{TT}_{al}\rangle \tag{166}$$

which can be interpreted as the angular momentum per volume. The total angular momentum carried by GWs is

$$\mathrm{d}J^i = \int \mathrm{d}^3x\mathrm{d}t j^i \Rightarrow \frac{\mathrm{d}J^i}{\mathrm{d}t} = \int_V \mathrm{d}^3x\frac{j^i}{c} \tag{167}$$

The volume of integration can be a sphere of radius r, so we get:

$$\xRightarrow{Eq.141} \frac{\mathrm{d}J^i}{\mathrm{d}t} = \frac{c^3r^2}{32\pi G}\int \mathrm{d}\Omega\langle -\epsilon^{ikl}\dot{h}^{TT}_{ab}x^k\partial^l h^{TT}_{ab} + 2\epsilon^{ikl}\dot{h}^{TT}_{ak}h^{TT}_{al}\rangle \tag{168}$$

The rate of $\frac{\mathrm{d}J^i}{\mathrm{d}t}$ expresses the emission rate of angular momentum due to GWs.

3 GWs in linearized theory

We consider the generation of GW in the context of linearized theory. In this type of theory, the generated gravitational field produced by the source is weak, and as a result, we can expand over a flat spacetime. In a two-body system with reduced mass μ and total mass m, we get,

$$E_{kin} = -\frac{1}{2}U \Rightarrow \frac{1}{2}\mu u^2 = \frac{1}{2}\mu m\frac{G}{r} \overset{\frac{1}{c^2}}{\Longrightarrow} \frac{u^2}{c^2} = \frac{2Gm}{2rc^2} = \frac{Rs}{2r} \tag{169}$$

When the gravity field is weak, we get:

$$\frac{Rs}{r} \ll 1 \Rightarrow \frac{u^2}{c^2} \ll 1 \Rightarrow \frac{u}{c} \ll 1 \tag{170}$$

This means that in a weak gravity field, the velocities that concern a self-gravitating system are small. Because of $\frac{u}{c} \ll 1$, we can expand in powers of $\frac{u}{c}$. Neutron Stars (NS), Black Holes (BH), or compact binaries are self-gravitating systems, but because of spherical symmetry, we cannot consider a flat spacetime expansion beyond the lowest order.

3.1 Energy and power spectra equations for arbitrary systems

This subsection is analysed in a methodical way in Sean Carroll's book [16]. The Fourier transform used in points is similar to Weinberg's [17], the main difference is that we transform with respect to $\frac{d\omega}{2\pi}$, while Weinberg uses just $d\omega$. The spatial component transform is the same in both analysis.

In Chapter 1, we wrote down the linearised field equations as

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{171}$$

This set of differential equations can be solved with Green's functions $G(x-x')$ as follows: First, we must remember the generalized Green's function for the $\Box$ operator, as given in 172:

$$\Box_x G(x-x') = \delta^{(4)}(x-x') \tag{172}$$

and implement it in equation 171, as following:

$$\begin{aligned}
&-\frac{16\pi G}{c^4}\Box_x G(x-x') = -\frac{16\pi G}{c^4}\delta^{(4)}(x-x') \Rightarrow \\
&-\frac{16\pi G}{c^4}\Box_x G(x-x')T_{\mu\nu}(x') = --\frac{16\pi G}{c^4}\delta(x-x')T_{\mu\nu}(x') \Rightarrow \\
&\int d^4x' \left[\left(-\frac{16\pi G}{c^4}\right)\Box_x G(x-x')T_{\mu\nu}(x')\right] = -\frac{16\pi G}{c^4}\int d^4x'\delta(x-x')T_{\mu\nu}(x') \Rightarrow \\
&\Box_x\left[-\frac{16\pi G}{c^4}\int d^4x' G(x-x')T_{\mu\nu}(x')\right] = -\frac{16\pi G}{c^4}T_{\mu\nu}(x)
\end{aligned} \tag{173}$$

After some algebra is done, we can compare the result of equation 173 with equation 171 and obtain:

$$\bar{h}_{\mu\nu}(x) \equiv -\frac{16\pi G}{c^4}\int d^4x' G(x-x')T_{\mu\nu}(x') \tag{174}$$

Thus, the generic solution to equation 172 is given as:

$$G(\vec{x},t) \equiv -\frac{1}{4\pi c}\frac{\delta\left(t-\frac{|\vec{x}|}{c}\right)}{|\vec{x}|} \tag{175}$$

where in

$$G(x-x') \equiv G(\vec{x},t;\vec{x'},t) = -\frac{\delta\left(t-t'-\frac{|\vec{x}-\vec{x'}|}{c}\right)}{4\pi c|\vec{x}-\vec{x'}|} \tag{176}$$

is used in the retarded time solution. The advanced time solution eliminates causality between events. The retarded and advanced time is defined as:

$$t_{ret} \equiv t - \frac{|\vec{x}-\vec{x'}|}{c} \ \& \ t_{adv} \equiv t + \frac{|\vec{x}-\vec{x'}|}{c} \tag{177}$$

Equation 176 when substituted in equation 173 yields:

$$\begin{aligned}
\bar{h}_{\mu\nu}(x) &= \frac{16\pi G}{c^4}\int \mathrm{d}^4x' \frac{T_{\mu\nu}(x')}{|\vec{x}-\vec{x'}|}\frac{1}{4\pi c}\delta\left[t-t'-\frac{\vec{x}-\vec{x'}}{c}\right] \\
\bar{h}_{\mu\nu}(x) &= \frac{4G}{c^4}\int \mathrm{d}^3\vec{x'}\int \mathrm{d}t'c\frac{1}{c}\frac{T_{\mu\nu}(\vec{x'},t')}{|\vec{x}-\vec{x'}|}\delta\left[t-t'-\frac{|\vec{x}-\vec{x'}|}{c}\right] \\
\bar{h}_{\mu\nu}(x) &= \frac{4G}{c^4}\int \mathrm{d}^3\vec{x'}T_{\mu\nu}\left(\vec{x'},t-\frac{|\vec{x}-\vec{x'}|}{c}\right)\frac{1}{|\vec{x}-\vec{x'}|}
\end{aligned} \tag{178}$$

with spatial components

$$\bar{h}_{ij}(x) = \frac{4G}{c^4}\int \mathrm{d}^3\vec{x}T_{ij}\left(\vec{x'},t-\frac{|\vec{x}-\vec{x'}|}{c}\right)\frac{1}{|\vec{x}-\vec{x'}|} \tag{179}$$

Outside the source, we can project equation 177 in the TT-gauge, using the projection operator (defined in Chapter 1)

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl}h_{kl} \tag{180}$$

The equations above produce:

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl}(\hat{n})h_{kl} = \frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n})\int \frac{\mathrm{d}^3\vec{x'}}{|\vec{x}-\vec{x'}|}T_{kl}\left(\vec{x'},t-\frac{|\vec{x}-\vec{x'}|}{c}\right) \tag{181}$$

Denoting $|\vec{x}-\vec{x'}| \equiv R$ we get

$$h_{ij}^{\mathrm{TT}} = \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\int \mathrm{d}^3\vec{x'}T_{kl}\left(\vec{x'},t-\frac{|\vec{x}-\vec{x'}|}{c}\right) \tag{182}$$

If we expand for $R \gg r$ we get:

$$R = |\vec{x}-\vec{x'}| = |\vec{x}-\vec{x'}| - \vec{\bigtriangledown}R\cdot\vec{x'} \tag{183}$$

where

$$
\begin{aligned}
(\vec{\bigtriangledown}R)_i &= \partial_i\sqrt{|\vec{x}-\vec{x'}|^2} = \frac{1}{2\sqrt{|\vec{x}-\vec{x'}|^2}}\partial_i(|\vec{x}-\vec{x'}|^2) \\
(\vec{\bigtriangledown}R)_i &= \frac{1}{2R}2|\vec{x}-\vec{x'}|_j\partial_i|\vec{x}-\vec{x'}|_j = \frac{R_j}{R}\partial_i R_j \\
(\vec{\bigtriangledown}R)_i &= \frac{R_i}{R} = \hat{n}_i = \hat{n}
\end{aligned}
\tag{184}
$$

Thus, from equation 183 and equation 184 we get:

$$
R = R - \hat{n}\cdot\vec{x'} \tag{185}
$$

and from equation 182:

$$
h_{ij}^{\mathrm{TT}} = \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\int \mathrm{d}^3\vec{x'}T_{kl}\left(\vec{x'}, t-\frac{r}{c}+\frac{1}{c}\hat{n}\vec{x'}\right) \tag{186}
$$

Next, we can transform by Fourier the energy-momentum tensor $T_{kl}(t,\vec{x})$ as following:

$$
T_{kl}(t,\vec{x}) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{T}_{kl}(w,\vec{k})e^{-i(\omega t-\vec{k}\vec{x})} \tag{187}
$$

And substitute in equation 186:

$$
\begin{aligned}
h_{ij}^{\mathrm{TT}}(t,\vec{x}) &= \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\int \mathrm{d}^3\vec{x'}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{T}_{kl}(w,\vec{k})e^{-i(\omega t-\vec{k})\vec{x}} \\
h_{ij}^{\mathrm{TT}} &= \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\int\frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\vec{k})\int \mathrm{d}^3\vec{x'}e^(-i\omega(t-\frac{r}{c}))e^{i(\vec{k}-\frac{k\hat{n}}{c})} \\
h_{ij}^{\mathrm{TT}} &= \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\int\frac{\mathrm{d}k^0}{2\pi}\tilde{T}_{kl}(\omega,\vec{k})e^{-i\omega(t-\frac{R}{c})}(2\pi)^3\delta^{(3)}(\vec{k}-\frac{w\hat{n}}{c}) \\
h_{ij}^{\mathrm{TT}} &= \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\int\frac{\mathrm{d}\omega}{2\pi c}\tilde{T}_{kl}(\omega,\omega\frac{\hat{n}}{c})e^{-i\omega\left(t-\frac{R}{c}\right)} \\
h_{ij}^{\mathrm{TT}} &= \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\int \mathrm{d}\omega\tilde{T}_{kl}\left(\omega,\omega\frac{\hat{n}}{c}\right)e^{-i\omega\left(t-\frac{R}{c}\right)}
\end{aligned}
\tag{188}
$$

In general, around value $\omega_s, \tilde{T}_{kl}$ takes large values, and the characteristic speed of movement of mass across the source is $u \sim \omega_s$. Equation 188 applies for both relativistic and non-relativistic systems, as long as the weak-field approximation applies and we are at large R away from the source. In Chapter 1, we have proved the following:

$$
\frac{1}{2}\dot{h}_{ij}^{\mathrm{TT}}\dot{h}_{ij}^{\mathrm{TT}} = \dot{h}_t^2 + \dot{h}_x^2 \ \& \ \frac{\mathrm{d}E}{\mathrm{d}A} = \frac{c^3}{16\pi G}\int_{-\infty}^{+\infty}\mathrm{d}t(\dot{h}_t^2+\dot{h}_x^2)
$$

Using the set of Eqs in 3.1, we can produce an expression for the total radiated energy per solid angle:

$$
\frac{\mathrm{d}E}{\mathrm{d}A} = \frac{c^3}{16\pi G}\int_{-\infty}^{+\infty}\mathrm{d}t\frac{1}{2}\dot{h}_{ij}^{\mathrm{TT}}\dot{h}_{ij}^{\mathrm{TT}} = \frac{c^3}{32\pi G}\int_{\infty}^{\infty}\mathrm{d}t\dot{h}_{ij}^{\mathrm{TT}}\dot{h}_{ij}^{\mathrm{TT}} \tag{189}
$$

Since $\mathrm{d}A = R^2\mathrm{d}\Omega$ we get

$$\frac{\mathrm{d}E}{\mathrm{d}A} = \frac{R^2c^3}{32\pi G}\int_{-\infty}^{+\infty}\mathrm{d}t\dot{h}_{ij}^{\mathrm{TT}}\dot{h}_{ij}^{\mathrm{TT}} \tag{190}$$

Inserting equation 188 to equation 190 we get:

$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{R^2c^3}{32\pi G}\int_{-\infty}^{+\infty}\mathrm{d}t\left[\frac{4G}{Rc^5}\Lambda_{ij,kl}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\tilde{T}_{kl}\left(w, w\frac{\hat{n}}{c}\right)e^{-iw\left(t-\frac{R}{c}\right)}\right]\cdot$$
$$\left[\frac{4G}{Rc^5}\Lambda_{ij,pr}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{2\pi}\tilde{T}_{pr}\left(\omega', \omega'\frac{\hat{n}}{c}\right)e^{i\omega'\left(t-\frac{R}{c}\right)}\right]\Rightarrow$$
$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{G}{2\pi c^7}\int_{-\infty}^{+\infty}\mathrm{d}t\Lambda_{ij,kl}\Lambda_{ij,pr}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{2\pi}w\omega'\tilde{T}_{kl}\left(w, w\frac{\hat{n}}{c}\right)\tilde{T}_{pr}\left(w, \omega'\frac{\hat{n}}{c}\right)\cdot$$
$$e^{-i(w+\omega')t}e^{i\frac{R}{c}(w+\omega')t}\Rightarrow$$
$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{G\Lambda_{ij,kl}}{c^7}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{2\pi}\tilde{T}_{kl}\left(w, w\frac{\hat{n}}{c}\right)\tilde{T}_{ij}\left(\omega', \omega'\frac{\hat{n}}{c}\right)e^{i\frac{R}{c}(w+\omega')}w\omega'\int_{-\infty}^{+\infty}\mathrm{d}te^{-i(w+\omega')t}$$
$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = -\frac{G}{4\pi^2c^7}\Lambda_{ij,kl}\int_{-\infty}^{+\infty}\mathrm{d}\omega w^2\tilde{T}_{ij}\left(-w, -w\frac{\hat{n}}{c}\right)\tilde{T}_{kl}\left(w, w\frac{\hat{n}}{c}\right)$$
$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = -\frac{G}{4\pi^2c^7}\Lambda_{ij,kl}\int_{-\infty}^{0}\mathrm{d}\omega w^2\tilde{T}_{ij}\left(-w, -w\frac{\hat{n}}{c}\right)\tilde{T}_{kl}\left(w, w\frac{\hat{n}}{c}\right)-$$
$$-\frac{G}{4\pi^2c^7}\Lambda_{ij,kl}\int_{0}^{+\infty}\mathrm{d}\omega w^2\tilde{T}_{ij}\left(-w, -w\frac{\hat{n}}{c}\right)\tilde{T}_{kl}\left(w, w\frac{\hat{n}}{c}\right)$$
$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = -\frac{G}{4\pi^2c^7}\Lambda_{ij,kl}(\hat{n})\int_{0}^{+\infty}\mathrm{d}\omega w^2\tilde{T}_{ij}\left(w, w\frac{\hat{n}}{c}\right)\tilde{T}_{kl}\left(-w, -w\frac{\hat{n}}{c}\right)$$

Since $\tilde{T}\left(-w, -w\frac{\hat{n}}{c}\right) = \tilde{T}^*\left(w, w\frac{\hat{n}}{c}\right)$ we get:

$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{G}{2\pi^2c^7}\int_{-\infty}^{+\infty}\mathrm{d}\omega\Lambda_{ij,kl}(\hat{n})w^2\tilde{T}_{ij}\left(w, w\frac{\hat{n}}{c}\right)\tilde{T}^*_{kl}\left(w, w\frac{\hat{n}}{c}\right) \tag{191}$$
$$\frac{\mathrm{d}E}{\mathrm{d}\omega\mathrm{d}\Omega} = \frac{G}{2\pi^2c^7}\Lambda_{ij,kl}(\hat{n})w^2\tilde{T}_{ij}\tilde{T}^*_{kl} \tag{192}$$
$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \frac{Gw^2}{2\pi^2c^7}\int\mathrm{d}\Omega\Lambda_{ij,kl}(\hat{n})\tilde{T}_{ij}\left(w, w\frac{\hat{n}}{c}\right)\tilde{T}^*_{kl}\left(w, w\frac{\hat{n}}{c}\right) \tag{193}$$

Now, equation 193 produces the energy spectrum of GWs. A typical source radiates for a characteristic time Δ_t. Ideally, the monochromatic source radiates for $\Delta_t \to +\infty$ and $E_{rad} \to +\infty$. We define the instantaneously radiated power for a source that radiates at w_0, $\tilde{T}_{ij}(w, \vec{k})$ is written as

$$\tilde{T}_{ij}(w, \vec{k}) = \theta_{ij}(w, \vec{k})2\pi\delta(w - w_0) \tag{194}$$

and Eq. 193 yield:

$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{G}{2\pi^2 c^7}\Lambda_{ij,kl}(\hat{n})\int_{-\infty}^{+\infty}\mathrm{d}\omega w^2\theta_{ij}\left(w, w\frac{\hat{n}}{c}\right)\theta^*_{kl}\left(w, w\frac{\hat{n}}{c}\right)(2\pi)^2\delta(w-w_0)\delta(w-w_0)$$
$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{Gw_0^2}{\pi c^7}\Lambda_{ij,kl}(\hat{n})\theta_{ij}\left(w_0, w_0\frac{\hat{n}}{c}\right)\theta^*_{kl}\left(w, w\frac{\hat{n}}{c}\right)(2\pi)\delta(w_-w_0) \tag{195}$$
$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{Gw_0^2}{\pi c^7}\Lambda_{ij,kl}(\hat{n})T\theta_{ij}\left(w_0, w_0\frac{\hat{n}}{c}\right)\theta^*_{kl}\left(w, w\frac{\hat{n}}{c}\right)$$

The power radiated in an instant is given as:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} \equiv \frac{1}{T}\frac{\mathrm{d}E}{\mathrm{d}\omega} = \frac{G\omega_0^2}{\pi c^7}\Lambda_{ij,kl}(\hat{n})\theta_{ij}\left(\omega_0, \omega_0\frac{\hat{n}}{c}\right)\theta^*_{kl}\left(\omega, \omega\frac{\hat{n}}{c}\right) \tag{196}$$

The total radiated power is:

$$P = \int \mathrm{d}\Omega\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \int \mathrm{d}\Omega\frac{G\omega_0^2}{\pi c^7}\Lambda_{ij,kl}(\hat{n})\theta_{ij}\left(\omega_0, \omega_0\frac{\hat{n}}{c}\right)\theta^*_{kl}\left(\omega, \omega\frac{\hat{n}}{c}\right) \tag{197}$$

Next, we can substitute equation 29, which gives the analytic formula of $\Lambda_{ij,kl}(\hat{n})$ as:

$$\Lambda_{ij,kl}(\hat{n}) = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l\delta_{ik} - n_i n_k\delta_{jl} + \frac{1}{2}n_k n_l\delta_{ij} + \frac{1}{2}n_i n_j\delta_{kl} - \frac{1}{2}n_i n_j n_k n_l$$

Take the following:

$$\begin{aligned}
P &= \int \mathrm{d}\Omega\left(\frac{G\omega_0^2}{\pi c^7}\right)\Bigg[\theta_{ij}\theta^*_{ij} - \frac{1}{2}\theta_{ii}\theta^*_{kk} - n_j n_l\theta_{ij}\theta^*_{il} - n_i n_k\theta_{ij}\theta^*_{kj} \\
&\quad + \frac{1}{2}n_k n_l\theta_{ii}\theta^*_{kl} + \frac{1}{2}n_i n_j\theta_{ij}\theta^*_{kk} - \frac{1}{2}n_i n_j n_k n_l\theta_{ij}\theta^*_{kl}\Bigg] \\
\Rightarrow P &= \frac{G\omega_0^2}{\pi c^7}\Bigg[\int \mathrm{d}\Omega\left(|\theta_{ij}|^2 - \frac{1}{2}|\theta_{ii}|^2\right) - \int \mathrm{d}\Omega\left(\frac{4\pi}{3}|\theta_{ii}|^2 - \frac{4\pi}{3}|\theta_{ij}|^2\right. \\
&\quad \left. + \frac{2\pi}{3}|\theta_{ii}|^2 + \frac{2\pi}{3}|\theta_{ii}|^2\right) - \int \mathrm{d}\Omega\frac{2\pi}{15}(|\theta_{ii}|^2 + |\theta_{ij}|^2 + |\theta_{ij}|^2)\Bigg] \\
P &= \frac{G\omega_0^2}{\pi c^7}\int \mathrm{d}\Omega\left[|\theta_{ij}|^2\left(1 + \frac{4\pi}{3} - \frac{4\pi}{15}\right) + |\theta_{ii}|^2\left(-\frac{1}{2} - \frac{8\pi}{3} - \frac{2\pi}{15}\right)\right] \\
P &= \frac{G\omega_0^2}{\pi c^7}\left(1 + \frac{16\pi}{15}\right)\left[|\theta_{ij}|^2 - \frac{1}{2}|\theta_{ii}|^2\right]
\end{aligned} \tag{198}$$

3.1.1 Low-velocity expansion

Since GW are a form of waves, general relativity can be treated as a field theory, it is useful to apply techniques from electromagnetism in GW. The first example is the low-velocity expansion; here, we use [19] and [25]. Another example computed based on the above is the GW amplitude and the angular distribution. These two can be employed by Weinberg's Chapter 10 [17].

The equations for radiation generation are simplified when typical velocities inside the source are small compared to the speed of light, c. Consider a source of size d and the typical frequency of motion ω_s inside the source, then the typical velocities u will be of order:

$$u \sim \omega_s d \tag{199}$$

Radiational frequency ω is of order ω_s, except some factors, so we get $\omega \sim \omega_s$, and the reduced wavelength will be:

$$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} \Rightarrow \lambdabar = \frac{\lambda}{2\pi} = \frac{c}{\omega} \sim \frac{c}{u} d \tag{200}$$

In a non-relativistic system, $u \ll c \Rightarrow \frac{c}{u} \gg 1$ and equation 200 shows that $\lambdabar \gg d$, meaning that the reduced wavelengths $\lambdabar$ generated by a non-relativistic source are much bigger than the size of the system. Since $\lambdabar \gg d$, we do not need to know in full detail the internal motion of the source, but only the course features. This means that the radiation emitted is governed by the lowest multipole moments. We begin with equation 186 stated, below:

$$\xRightarrow{186} h_{ij}^{\mathrm{TT}} = \frac{4G}{Rc^4} \Lambda_{ij,kl}(\hat{n}) \int \mathrm{d}^3\vec{x'} T_{kl}\left(\vec{x'}, t - \frac{R}{c} + \frac{\hat{n}\cdot\vec{x'}}{c}\right)$$

and Fourier transform T_{kl} as:

$$\xRightarrow{186} T_{kl}\left(\vec{x'}, t - \frac{R}{c} + \frac{\hat{n}\cdot\vec{x'}}{c}\right) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} T_{kl}(\omega, \vec{k}) e^{-i\omega\left(t - \frac{R}{c} + \frac{\hat{n}\cdot\vec{x'}}{c}\right)}$$

For a non-relativistic source, $T_{kl}(\omega, \vec{k})$ peaks around ω_s typical frequency with $\omega_s d \ll c$. Equation's 186 integral is restricted to $|\vec{x}| \leq d$, since outside the source $T_{kl}(\vec{x}, t_{ret}) = 0$. This means that the dominant contribution to h_{ij}^{TT} comes from frequencies ω, such that

$$\frac{\omega}{c}\vec{x'} \cdot \hat{n} \preceq \frac{\omega_s d}{c} \ll 1 \tag{201}$$

and the exponential $exp\left[-i\omega\left(t - \frac{R}{c} + \frac{\hat{n}\cdot\vec{x'}}{c}\right)\right]$ can be expanded as following:

$$e^{-i\omega\left(t-\frac{R}{c}\right)} e^{-i\omega\frac{\hat{n}\cdot\vec{x'}}{c}} = e^{-i\omega\left(t-\frac{R}{c}\right)}\left[1 - \frac{i\omega(\hat{n}\cdot\vec{x'})}{c} + \frac{1}{2}\left(-i\frac{\omega}{c}\right)^2 \hat{n}_i \hat{n}_j \vec{x^{i'}} \vec{x^{j'}} + O[(\hat{n}\cdot\vec{x'})^3]\right] \tag{202}$$

Equivalently we can Taylor expand the energy-momentum tensor $T_{kl}\left(\vec{x'}, t - \frac{R}{c} + \frac{\hat{n}\cdot\vec{x'}}{c}\right)$ around $\frac{1}{c}\hat{n}\cdot\vec{x'} \ll 1$ as following:

$$T_{kl}\left(\vec{x'}, t - \frac{R}{c} + \frac{\hat{n}\cdot\vec{x'}}{c}\right) \simeq T_{kl}\left(t - \frac{R}{c}, \vec{x'}\right) + \frac{\hat{n^i}\vec{x^{i'}}}{c}\partial_0 T_{kl} + \frac{x'^j x'^i \hat{n^i}\hat{n^j}}{2c^2}\partial_0^2 T_{kl} + O(\partial_0^3)$$

The momenta of the stress tensor T^{ij} (spatial components) are defined in the following way:

$$S^{ij}(t) = \int \mathrm{d}^3x' T^{ij}(t, \vec{x'}) \tag{203}$$

$$S^{ij,k}(t) = \int \mathrm{d}^3x' x'^k T^{ij}(t, \vec{x'}) \tag{204}$$

$$S^{ij,kl}(t) = \int \mathrm{d}^3x' x'^k x'^l T^{ij}(t, \vec{x'}) \tag{205}$$

Note commas separate spatial ij given in T^{ij} from $k, kl, klm, \ldots$ given by $x^N|_{k,\ldots}$. They do not denote derivatives.The $S^{ij,\cdots}(t)$ tensors are symmetric in $i \leftrightarrow j$ and $k \leftrightarrow l, \ldots$, but not in $i \not\leftrightarrow k$ and $j \not\leftrightarrow k$. Inserting the previous results in equation 186 and after applying some algebra, we get the following results:

$$\begin{aligned} \xrightarrow{186} h_{ij}^{\mathrm{TT}} &= \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n}) \int \mathrm{d}^3\vec{x'}\ [T_{kl}\left(t - \frac{R}{c}, \vec{x'}\right) + \frac{1}{c}x'^m \hat{n}^m \partial_0 T_{kl} \\ &+ \frac{1}{2c^2}x'^m x'^n \hat{n}^m \hat{n}^n \partial_0^2 T_{kl} + \ldots] \end{aligned} \tag{206}$$

$$\Rightarrow h_{ij}^{\mathrm{TT}} = \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})[S_{kl}(t) + \frac{1}{c}\hat{n}_m \dot{S}^{kl,m} + \frac{1}{2c^2}\hat{n}^m \hat{n}^n \ddot{S}^{kl,mn}] \tag{207}$$

Dimensional analysis in equation 207 tells us the following:

$$\frac{1}{c}\left[S^{\dot{kl,m}}\right] = \left[\frac{1}{c}\partial_0 S^{k,l} x^m\right] = \left[\frac{1}{c}\omega S^{kl}\right][x^m] = \frac{1}{c}\omega d\left[S^{kl}\right] = \frac{u}{c}[S^{kl}] \tag{208}$$

$$\left[\frac{1}{c^2}\ddot{S}^{kl,mn}\right] = \frac{1}{c^2}\left[\partial_0^2 S^{kl}\right][x^m x^n] = \frac{\mathrm{d}^2}{c^2}\left[\omega^2 S^{kl}\right] = \frac{\omega^2 \mathrm{d}^2}{c^2}[S^{kl}] = \frac{u^2}{c^2}[S^{kl}] \tag{209}$$

So, Eqs. 208 and 209 give an expansion of the typical velocity of the source, and equation 208 is the first correction of $O(\frac{u}{c})$ order. Next, we can define the momenta if T^{00} and T^{0i} as follows. For $\frac{1}{c^2}T^{00}$ we get:

$$M = \frac{1}{c^2}\int \mathrm{d}^3x\, T^{00}(t, \mathbf{x}), \tag{210}$$

$$M^i = \frac{1}{c^2}\int \mathrm{d}^3x\, T^{00}(t, \mathbf{x})\, x^i, \tag{211}$$

$$M^{ij} = \frac{1}{c^2}\int \mathrm{d}^3x\, T^{00}(t, \mathbf{x})\, x^i x^j, \tag{212}$$

$$M^{ijk} = \frac{1}{c^2}\int \mathrm{d}^3x\, T^{00}(t, \mathbf{x})\, x^i x^j x^k \ldots \tag{213}$$

For $\frac{1}{c}T^{0i}$

$$P^i = \frac{1}{c}\int \mathrm{d}^3x\, T^{0i}(t, \mathbf{x}), \tag{214}$$

$$P^{i,j} = \frac{1}{c}\int \mathrm{d}^3x\, T^{0i}(t, \mathbf{x})\, x^j, \tag{215}$$

$$P^{i,jk} = \frac{1}{c}\int \mathrm{d}^3x\, T^{0i}(t, \mathbf{x})\, x^j x^k \ldots. \tag{216}$$

Conservation of energy-momentum tensor is given as $\partial_\mu T^{\mu\nu} = 0$ for $v = 0 : \partial_\mu T^{\mu 0} = 0 \Rightarrow \partial_0 T^{00} = -\partial_i T^{i0}$ The time derivative of M is given as:

$$\begin{aligned}
&\frac{1}{c}\partial_t M = \partial_0 M \Rightarrow \partial_t M = c\partial_0 M = c\dot{M} = \int \mathrm{d}^3x \partial_0 T^{00} \Rightarrow \\
&c\dot{M} = -\int_v \mathrm{d}^3x \partial_i T^{0i} = -\int_{\partial_v} \mathrm{d}S_i T^{0i} = 0 \\
&\dot{M} = 0
\end{aligned} \tag{217}$$

Integration in the second line is on a volume V bigger than the sources with $T^{\mu\nu}|_{\partial_v} = 0$. Similarly, we can find:

$$\begin{aligned}
c\dot{M}^i &= \int \mathrm{d}^3x x^i \partial_0 T^{00} = -\int_v \mathrm{d}^3x x^i \partial_j T^{0j} \\
c\dot{M}^i &= \int_v \mathrm{d}^3x T^{0i} = cP^i
\end{aligned} \tag{218}$$

and

$$\dot{M}^{ij} = P^{i,j} + P^{j,i} \tag{219}$$

$$\dot{M}^{ijk} = P^{i,jk} + P^{j,ki} + P^{k,ij} \tag{220}$$

On the other hand, the time derivative of Eqs. 216 yields:

$$\dot{P}^i = \frac{1}{c}\int \mathrm{d}^3x\, \partial_0 T^{0i} = 0 \tag{221}$$

$$\dot{P}^{i,j} = S^{ij} \tag{222}$$

$$\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j} \tag{223}$$

Combining Eqs. 219 and 221 and remembering that S^{ij} is symmetric as

$$S^{ij} + S^{ji} = 2S^{ij}$$

We get:

$$\begin{aligned}
\ddot{M}^{ij} &= \dot{P}^{i,j} + \dot{P}^{j,i} = S^{ij} + S^{ji} = 2S^{ij} \Rightarrow \\
S^{ij} &= \frac{1}{2}\ddot{M}^{ij}
\end{aligned} \tag{224}$$

And from EQ. 220:

$$\begin{aligned}
\ddot{M}^{ijk} &= \dot{P}^{i,jk} + \dot{P}^{j,ki} + \dot{P}^{k,ij} = \\
&\quad S^{ij,k} + S^{ik,j} + S^{jk,i} + S^{ji,k} + S^{ki,j} + S^{kj,i} \\
\dddot{M}^{ijk} &= 2(\dot{S}^{ij,k} + \dot{S}^{ik,j} + \dot{S}^{jk,i})
\end{aligned} \tag{225}$$

From the above Eqs:

$$\begin{aligned}
\dot{S}^{ij,k} &= \frac{1}{2}\dddot{M}^{ijk} - \dot{S}^{ik,j} - \dot{S}^{jk,i} \\
&= \frac{1}{6}\dddot{M}^{ijk} + \frac{2}{6}(2\dot{S}^{ij,k} + 2\dot{S}^{ik,j} + 2\dot{S}^{jk,i}) - \dot{S}^{ik,j} - \dot{S}^{jk,i}
\end{aligned}$$

$$\dot{S}^{ijk} = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}(\ddot{P}^{i,jk} + \ddot{P}^{j,ki} - 2\ddot{P}^{k,ij}) \tag{226}$$

The derivatives of M, P, $\dot{M} = 0$ and $\dot{P}^i = 0$, give out the mass and total momentum conservation for the source. Since $\ddot{M}^{ij} = \dot{P}^{i,j} + \dot{P}^{j,i} = S^{ij} + S^{ji} = 0$, we get the conservation of angular momentum for the source.

3.1.2 Amplitude and angular distribution

In eq. 207 we have found the expansion of $h^{\rm TT}_{ij}$. When substituted Eq. 224 to Eq. 207 yields for the leading term $O(\frac{u}{c})$ the following:

$$\begin{aligned}
h^{\rm TT}_{ij}(t,\vec{x})\Big|_{\rm quad} &= \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})S^{kl}(t) \Rightarrow \\
h^{\rm TT}_{ij}(t,\vec{x})\Big|_{\rm quad} &= \frac{4G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\frac{1}{2}\ddot{M}^{kl}\left(t-\frac{R}{c}\right) \Rightarrow \\
h^{\rm TT}_{ij}(t,\vec{x})\Big|_{\rm quad} &= \frac{2G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}\left(t-\frac{R}{c}\right)
\end{aligned} \tag{227}$$

M_{ij} tensor, when under rotation transformation, can be decomposed as any symmetric tensor:

$$M^{kl} = M^{kl} - \frac{1}{3}\delta^{kl}M_{ii} + \frac{1}{3}\delta^{kl}M_{ii} \tag{228}$$

The first part $M^{kl} - \frac{1}{3}\delta^{kl}M_{ii}$ is traceless, since:

$$M^{kk} - \frac{1}{3}\delta^{kk}M_{ii} = M^{kk} - \frac{1}{3}3M_{ii} = 0$$

and by construction is a spin-2 tensor.
The second part $\frac{1}{3}\delta^{kl}M_{ii}$ is the trace part, and it is a scalar. We denote by

$$\rho \equiv \frac{1}{c^2}T^{00} \tag{229}$$

the density and at lowest order in $\frac{u}{c}$ expansion it gives the mass density. Also, we can rewrite the quadrupole moment as:

$$Q^{ij} \equiv M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} \Rightarrow \tag{230}$$

$$\Rightarrow Q^{ij} \equiv \frac{1}{c^2}\int \mathrm{d}^3x T^{00}(t,\vec{x})x^i x^j - \frac{1}{3c^2}\delta^{ij}\int \mathrm{d}^3x T^{00}x^k x^k$$

$$Q^{ij} = \int \mathrm{d}^3x \rho(t,\vec{x})\left[x^i x^j - \frac{1}{3}\delta^{ij}R^2\right] \tag{231}$$

And after applying twice the time derivative, we get:

$$\ddot{Q}^{ij} = \ddot{M}^{ij} - \frac{1}{3}\delta^{ij}\ddot{M}_{kk} \tag{232}$$

and from Eq. 228:

$$\begin{aligned}
M^{ij} &= M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} + \frac{1}{3}\delta^{ij}M_{kk} \Rightarrow \\
M^{ij} &= Q^{ij} + \frac{1}{3}\delta^{ij}M_{kk} \Rightarrow \\
\ddot{M}^{ij} &= \ddot{Q}^{ij} + \frac{1}{3}\delta^{ij}\ddot{M}_{kk}
\end{aligned} \tag{233}$$

And finally, Eq. 227 is written as:

$$h_{ij}^{\mathrm{TT}}\Big|_{\text{quad}} = \frac{2G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}\left(t - \frac{R}{c}\right) \tag{234}$$

$$h_{ij}^{\mathrm{TT}}\Big|_{\text{quad}} = \frac{2G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\ddot{Q}_{kl}\left(t - \frac{R}{c}\right) \tag{235}$$

$$h_{ij}^{\mathrm{TT}}\Big|_{\text{quad}} = \frac{2G}{Rc^4}\ddot{Q}_{ij}^{\mathrm{TT}}\left(t - \frac{R}{c}\right) \tag{236}$$

We can produce an expression for the angular distribution of the quadrupole term of radiation as follows. We first consider the waveform emitted in an arbitrary direction $\hat{n}$, which can be obtained by substituting the explicit expression of $\Lambda_{ij,kl}$ in equation 236. Because of the strenuous algebra, we may use an alternative way. We consider a GW traveling along the z-axis, so $\hat{z} = \hat{n}$. The projector, in this case, P_{ij}, is an operator that projects everything included on the (x, y) plane.

An arbitrary 3×3 matrix A_{ij} will be written as:

$$A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{237}$$

The operator Λ_{ijkl} when applied on this matrix yields:

$$\Lambda_{ijkl}A_{ij} = \left[P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right]A_{kl} \tag{238}$$

$$\Rightarrow \Lambda_{ijkl}A_{ij} = P_{ik}A_{kl}P_{jl} - \frac{1}{2}P_{ij}P_{kl}A_{kl}$$

$$\Rightarrow \Lambda_{ijkl}A_{ij} = (PAP)_{ij} - \frac{1}{2}P_{ij}\,\mathrm{tr}(PA) \tag{239}$$

Since $\hat{n} = \hat{z}$, the projector in matrix form will be:

$$P_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{240}$$

With $(PAP)_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ik} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_{kl} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$

$$\Rightarrow (PAP)_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix}_{il} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{lj}$$

$$(PAP)_{ij} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

$$\text{and } PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$\text{and } PA = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \operatorname{tr}(PA) = A_{11} + A_{22} \tag{241}$$

$$\Lambda_{ij,kl}A_{kl} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} - \frac{1}{2} \begin{pmatrix} A_{11} + A_{22} & 0 & 0 \\ 0 & A_{11} + A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

$$\Lambda_{ij,kl}A_{kl} = \begin{pmatrix} (A_{11} - A_{22})/2 & A_{12} & 0 \\ A_{21} & -(A_{11} - A_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{242}$$

So when $\hat{z} = \hat{n}$ and $A_{kl} \equiv \ddot{M}_{kl}$ we get:

$$\Lambda_{ij,kl}\ddot{M}_{kl} = \begin{pmatrix} (\ddot{M}_{11} - \ddot{M}_{22})/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -(\ddot{M}_{11} - \ddot{M}_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{243}$$

Based on equation 243, we can rewrite it in equation 244 as:

$$h_{ij}^{\mathrm{TT}}\Big|_{\mathrm{quad}} = \frac{2G}{Rc^4} \begin{pmatrix} (\ddot{M}_{11} - \ddot{M}_{22})/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -(\ddot{M}_{11} - \ddot{M}_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{244}$$

And when compared to the equation that gives the generic formula of h_{ij}^{TT} (see Chapter 1), we get

$$h_{ij}^{\mathrm{TT}}\Big|_{\mathrm{quad}} = \frac{2G}{Rc^4} \begin{pmatrix} (\ddot{M}_{11} - \ddot{M}_{22})/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -(\ddot{M}_{11} - \ddot{M}_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

$$h_+ = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) \tag{245}$$

$$h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12} \tag{246}$$

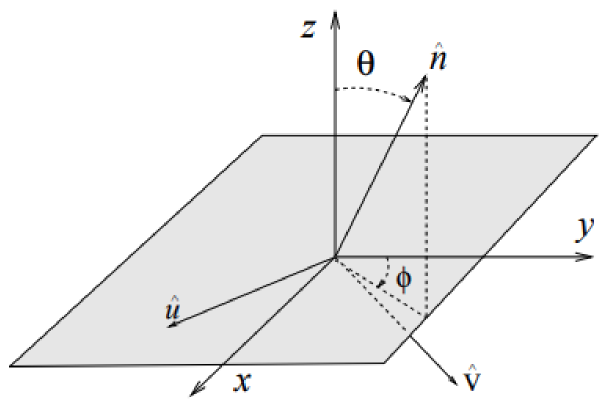

Figure 2: The relation between the $(\hat{x}, \hat{y}, \hat{z})$ frame and the $(\hat{u}, \hat{v}, \hat{n})$ frame. The vector $\hat{u}$ is in the $(\hat{x}, \hat{y})$ plane, while $\hat{v}$ points downward, with respect to the $(\hat{x}, \hat{y})$ plane, adapted by Maggiore's book, Gravitational Waves [15].

with $\ddot{M}_{ij} \equiv \ddot{M}_i j\left(t - \frac{R}{c}\right)$ computed on retarded time. To generalize these results, we must compute the amplitude in a generic direction $\hat{n}$. We introduce two more unitary vectors $\hat{v}$ & $\hat{u}$, orthogonal to $\hat{n}$ and to each other, so $\hat{u} \times \hat{v} = \hat{n}$. These vectors live on a new frame $(x', y'z')$ on which the following are valid:

We take $\hat{u}$ to be on the (x, y) plane. Eqs. 243 and 244 are written in the primed frame as follows, and the components of the second mass moment are

$$h_+(t, \hat{n}) = \frac{G}{Rc^4}(\ddot{M'}_{11} - \ddot{M'}_{22}) \tag{247}$$

$$h_\times(t, \hat{n}) = \frac{2G}{Rc^4}\ddot{M'}_{12} \tag{248}$$

Here M_{ij} and M'_{ij} are related via the $\hat{n}_i$ and $\hat{n'}_i$ components. On (x, y, z) frame we can write $\hat{n}$ as $\hat{n}_i = (0, 0, 1)$, when on (x', y', z') we have $\hat{n'}_i = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$ and $n'_i = R_{ij}n_j$ with

$$R_{ij} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{249}$$

The M_{ij} tensor components will transform under the following

$$M'_{ij} = R_{ik}R_{jl}M_{kl} \Rightarrow (M')_{ij} = (R^T M R)_{ij} \tag{250}$$

and the components $h_+(t, \theta, \phi)$ and $h_\times(t, \theta, \phi)$ will be:

$$\begin{aligned} h_+(t, \theta, \phi) =& \frac{G}{Rc^4}\Big[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta) + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) - \ddot{M}_{33}\sin^2\theta \\ & -\ddot{M}_{12}\sin 2\phi(1 + \cos^2\phi) + \ddot{M}_{13}\sin\phi\sin 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta\Big] \end{aligned} \tag{251}$$

$$h_\times(t,\theta,\phi) = \frac{2G}{Rc^4}\Big[(\ddot{M}_{11}-\ddot{M}_{22})\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta - 2\ddot{M}_{13}\cos\phi\sin\theta - 2\ddot{M}_{23}\sin\phi\sin\theta\Big] \tag{252}$$

Once M_{ij} is given by Eqs. 251 and 252, it provides the angular distribution of quadrupole radiation. It becomes evident that the leading term in the multipole expansion is the mass quadrupole. There are no monopole or dipole terms because M and P^i can be set to zero by appropriately shifting the coordinate system.

Radiated energy

Delving into the radiated energy and angular momentum by a GW, we aimed to find and produce expressions of it in various scientific articles and reports. The main route followed is in [15], also in [20], [44],[45],[46],[47] and [48] We begin with the following equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{c^3 r^2}{32\pi G}\int \mathrm{d}\Omega \langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}}\rangle.$$

And apply the derivative on solid angle dΩ, so we have:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} = P = \frac{c^3R^2}{32\pi G}\int \mathrm{d}\Omega\langle \dot{h}_{ij}^{\mathrm{TT}}\dot{h}_{ij}^{\mathrm{TT}}\rangle \\
\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c^3R^2}{32\pi G}\langle \dot{h}_{ij}^{\mathrm{TT}}\dot{h}_{ij}^{\mathrm{TT}}\rangle \\
\xRightarrow{\text{Eq. 236}} \frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c^3R^2}{32\pi G}\left(\frac{2G}{Rc^4}\right)^2\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{ij}^{\mathrm{TT}}\rangle \\
\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{4G^2c^3}{32\pi Gc^5}\Lambda_{ij,kl}(\hat{n})\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{ij}^{\mathrm{TT}}\rangle \\
\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}\Omega}\bigg|_{\mathrm{quad}} = \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{n})\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{ij}^{\mathrm{TT}}\rangle
\end{aligned} \tag{253}$$

The double brackets $\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{ij}^{\mathrm{TT}}\rangle$ denote the temporal average over several periods of GWs and the derivative $\dddot{Q}_{ij}^{\mathrm{TT}}$ is evaluated at the retarded time $t-\frac{R}{c}$. The dependence of $\hat{n}$ on solid angles, lies only on $\Lambda_{ij,kl}(\hat{n})$, so we can perform an integration with respect to $d\Omega$ as following:

$$P_{quad} = \int \mathrm{d}\Omega \frac{\mathrm{d}P}{\mathrm{d}\Omega}\bigg|_{\mathrm{quad}} = \frac{G}{8\pi c^5}\int \mathrm{d}\Omega\Lambda_{ij,kl}(\hat{n})\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{kl}^{\mathrm{TT}}\rangle \tag{254}$$

We have proved in subsection 2.1 that

$$\int \mathrm{d}\omega\Lambda_{ij,kl}(\hat{n}) = \frac{2\pi}{15}\left(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\right) \tag{255}$$

So, combining Eqs. 254 and 255 we get:

$$\begin{aligned}
P_{quad} &= \frac{G2\pi}{8\pi c^5 15}\left[11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\right]\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{kl}^{\mathrm{TT}}\rangle \\
\Rightarrow P_{quad} &= \frac{G}{60c^5}\left[11\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{ij}^{\mathrm{TT}}\rangle + \langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{ij}^{\mathrm{TT}}\rangle\right] \\
\Rightarrow P_{quad} &= \frac{G}{5c^5}\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{ij}^{\mathrm{TT}}\rangle
\end{aligned} \tag{256}$$

Inserting Eq. 232 we get:

$$\begin{aligned}
P_{quad} &= \frac{G}{5c^5}\langle\left(\dddot{M}_{ij} - \frac{1}{3}\delta_{ij}\dddot{M}_{kk}\right)\left(\dddot{M}_{ij} - \frac{1}{3}\delta_{ij}\dddot{M}_{kk}\right)\rangle \\
&= \frac{G}{5c^5}\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{2}{3}\delta_{ij}\dddot{M}_{ij}\dddot{M}_{kk} + \frac{1}{9}3(\dddot{M}_{kk})^2\rangle \\
P_{quad} &= \frac{G}{5c^5}\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}(\dddot{M}_{kk})^2\rangle
\end{aligned} \tag{257}$$

In Astrophysics, the total radiated power as produced in equation 233 above, is called the total gravitational luminosity L of the source. Based on equation 253, this result can be expressed in terms of the radiated energy as:

$$\begin{aligned}
\left.\frac{\mathrm{d}E}{\mathrm{d}t\mathrm{d}\Omega}\right|_{\mathrm{quad}} &= \frac{R^2c^3}{32\pi G}\langle \ddot{h}_{ij}^{\mathrm{TT}}\ddot{h}_{ij}^{\mathrm{TT}}\rangle \\
\Rightarrow \left.\frac{\mathrm{d}E}{\mathrm{d}t\mathrm{d}\Omega}\right|_{\mathrm{quad}} &= \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{n})\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{kl}^{\mathrm{TT}}\rangle \\
\Rightarrow \left.\frac{\mathrm{d}E}{\mathrm{d}\omega}\right|_{\mathrm{quad}} &= \frac{G}{8\pi c^5}\int dt\Lambda_{ij,kl}(\hat{n})\langle \dddot{Q}_{ij}^{\mathrm{TT}}\dddot{Q}_{kl}^{\mathrm{TT}}\rangle
\end{aligned} \tag{258}$$

Here, it is useful to insert the Fourier transform of Q_{ij}, its third derivative by time, and rewrite the radiated energy as follows:

$$\begin{aligned}
Q_{ij}(\omega) &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\tilde{Q}_{ij}(\omega)e^{-i\omega t} \\
\Rightarrow \ddot{Q}_{ij}(t) &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\tilde{Q}_{ij}(\omega)\frac{\mathrm{d}^3}{\mathrm{d}t^3}(e^{-i\omega t}) \\
\Rightarrow \ddot{Q}_{ij}(t) &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\tilde{Q}_{ij}(\omega)\omega^3 e^{-i\omega t}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \frac{\mathrm{d}E}{\mathrm{d}\Omega}\bigg|_{\text{quad}} &= \frac{G}{8\pi c^5}\int \mathrm{d}t \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{2\pi}\Lambda_{ij,kl}(\hat{n})\langle \tilde{Q}_{ij}(\omega)\omega^3\tilde{Q}_{kl}(\omega')\omega'^3 e^{-i(\omega+\omega')t}\rangle \\
\Rightarrow \frac{\mathrm{d}E}{\mathrm{d}\Omega}\bigg|_{\text{quad}} &= \frac{G}{8\pi c^5}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{2\pi}\Lambda_{ij,kl}(\hat{n})\tilde{Q}_{ij}(\omega)\omega^3\omega'^3\tilde{Q}_{kl}(\omega')\int_{-\infty}^{+\infty}\mathrm{d}t e^{-i(\omega+\omega')t} \\
\Rightarrow \frac{\mathrm{d}E}{\mathrm{d}\Omega}\bigg|_{\text{quad}} &= \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{n})\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\tilde{Q}_{ij}(\omega)\tilde{Q}_{kl}(-\omega)\omega^6 \\
\Rightarrow \frac{\mathrm{d}E}{\mathrm{d}\Omega}\bigg|_{\text{quad}} &= \frac{G}{16\pi^2 c^5}\Lambda_{ij,kl}(\hat{n})\left[\int_0^{+\infty}\mathrm{d}\omega\ \omega^6\tilde{Q}_{ij}(\omega)\tilde{Q}_{kl}(-\omega) + \int_{-\infty}^{0}\mathrm{d}\omega\ \omega^6\tilde{Q}_{ij}(\omega)\tilde{Q}_{kl}(-\omega)\right] \\
\Rightarrow \frac{\mathrm{d}E}{\mathrm{d}\Omega}\bigg|_{\text{quad}} &= \frac{G}{8\pi^2 c^5}\Lambda_{ij,kl}(\hat{n})\int_0^{\infty}\mathrm{d}\omega\omega^6\tilde{Q}_{ij}(\omega)\tilde{Q}^*_{kl}(\omega)
\end{aligned} \tag{259}
$$

Finally, the total radiated energy is given as:

$$
\begin{aligned}
E_{\text{quad}} &= \int \frac{\mathrm{d}E}{\mathrm{d}\omega}d\Omega\bigg|_{\text{quad}} = \frac{G}{8\pi^2 c^5}\int \mathrm{d}\Omega\Lambda_{ij,kl}(\hat{n})\int_0^{\infty}\mathrm{d}\omega\omega^6\tilde{Q}_{ij}(\omega)\tilde{Q}^*_{kl}(\omega) \\
\Rightarrow E_{\text{quad}} &= \frac{G}{8\pi^2 c^5}\frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})\int_0^{\infty}\mathrm{d}\omega\omega^6\tilde{Q}_{ij}\tilde{Q}^*_{kl} \\
\Rightarrow E_{\text{quad}} &= \frac{G}{60\pi c^5}\int_0^{\infty}\mathrm{d}\omega\omega^6\left[11\tilde{Q}_{ij}\tilde{Q}^*_{ij} + \tilde{Q}_{ij}\tilde{Q}^*_{ij}\right] \\
\Rightarrow E_{\text{quad}} &= \frac{G}{5\pi c^5}\int_0^{\infty}\mathrm{d}\omega\left[\omega^6\tilde{Q}_{ij}(\omega)\tilde{Q}^*_{ij}(\omega)\right]
\end{aligned} \tag{260}
$$

We suppose a monochromatic source, with radiating frequency $\omega_0 > 0$ and $\tilde{Q}_{ij}(\omega) = q_{ij}(2\pi)\delta(\omega-\omega_0)$ and equation 259 will be for $E_{\text{quad}} = \frac{\mathrm{d}P}{\mathrm{d}t}\big|_{\text{quad}}$

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}\Omega\mathrm{d}\omega}\bigg|_{\text{quad}} &= \frac{G}{8\pi^2 c^5}\Lambda_{ij,kl}(\hat{n})\omega^6 q_{ij}q^*_{kl}(2\pi^2)\delta(\omega-\omega_0)\delta(\omega-\omega_0) \\
\frac{\mathrm{d}E}{\mathrm{d}\Omega\mathrm{d}\omega}\bigg|_{\text{quad}} &= \frac{G\omega_0^6}{4\pi c^5}\Lambda_{ij,kl}(\hat{n})q_{ij}q^*_{kl}[2\pi\delta(\omega-\omega_0)\delta(\omega-\omega_0)]
\end{aligned} \tag{261}
$$

$$
\frac{\mathrm{d}E}{\mathrm{d}\Omega\mathrm{d}\omega}\Big|_{\text{quad}} = T\frac{G\omega_0^6}{4\pi c^5}\Lambda_{ij,kl}(\hat{n})q_{ij}q^*_{kl}\delta(\omega-\omega_0) \tag{262}
$$

In equation 262 T is the total infinite time interval and $\frac{\mathrm{d}E}{\mathrm{d}\Omega\mathrm{d}\omega}\big|_{\text{quad}}$ denotes the instantaneous energy radiated by a monochromatic source.

$$
\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}\Omega\mathrm{d}\omega}\Big|_{\text{quad}} = \frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}E}{\mathrm{d}\omega\omega}\right) = \frac{Gw_0^6}{4\pi c^5}\Lambda_{ij,kl}(\hat{n})q_{ij}q^*_{kl}\delta(\omega-\omega_0) \tag{263}
$$

In this study, the linear momentum radiated is :

$$
\begin{aligned}
\frac{\mathrm{d}P^i}{\mathrm{d}t} &= -\frac{c^3}{32\pi G}R^2\int \mathrm{d}\Omega\langle \dot{h}^{\mathrm{TT}}_{kl}\partial^i h^{\mathrm{TT}}_{kl}\rangle \\
\Rightarrow \frac{\mathrm{d}P^i}{\mathrm{d}t} &= -\frac{G}{8\pi c^5}\int \mathrm{d}\Omega\dddot{Q}^{\mathrm{TT}}_{kl}\partial^i\ddot{Q}^{\mathrm{TT}}_{kl}
\end{aligned}
$$

Under $\vec{x} \to -\vec{x}$ reflection we get:

$$\frac{\mathrm{d}P^{i'}}{\mathrm{d}t} = -\frac{G}{8\pi c^5}\int \mathrm{d}\Omega\, \dddot{Q}^{\mathrm{TT}}_{kl}(-\partial^i)\ddot{Q}^{\mathrm{TT}}_{kl} = \frac{\mathrm{d}P^i}{\mathrm{d}t} \tag{264}$$

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = 0 \tag{265}$$

Thus, we have no loss of linear momentum in the quadrupole approximation.

Radiated angular momentum

The angular momentum radiated per unit time by gravitational waves can be determined by substituting the expression for h^{TT}_{ij} in the quadrupole approximation into the general formula for the rate of angular momentum loss. It is important to remember that the first term in

$$\frac{\mathrm{d}J^i}{\mathrm{d}t} = \frac{c^3}{32\pi G}\int r^2\, \mathrm{d}\Omega \left\langle -\epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{ab}x^k\partial^l h^{\mathrm{TT}}_{ab} + 2\epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{al}h^{\mathrm{TT}}_{ak}\right\rangle.$$

represents the contribution from the orbital angular momentum L_i of the gravitational waves, while the second term accounts for the contribution from the spin S_i of the field configuration. Separating these two terms using the additive property of integrals, we get:

$$\frac{\mathrm{d}J^i}{\mathrm{d}t} = \frac{c^3R^2}{32\pi G}\int \mathrm{d}\Omega\langle -\epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{ab}x^k\partial^l h^{\mathrm{TT}}_{ab}\rangle + \frac{c^3R^2}{16\pi G}\int \mathrm{d}\Omega\langle -\epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{ak}x^k\partial^l h^{\mathrm{TT}}_{al}\rangle$$

Which, when compared to the time derivative of the total orbital contribution:

$$\frac{\mathrm{d}J^i}{\mathrm{d}t} = \frac{\mathrm{d}L^i}{\mathrm{d}t} + \frac{\mathrm{d}S^i}{\mathrm{d}t} \tag{266}$$

Yields the following:

$$\frac{\mathrm{d}L^i}{\mathrm{d}t} = -\frac{c^3R^2}{32\pi G}\int \mathrm{d}\Omega\langle \epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{ab}x^k\partial^l h^{\mathrm{TT}}_{ab}\rangle \tag{267}$$

$$\frac{\mathrm{d}S^i}{\mathrm{d}t} = \frac{c^3R^2}{16\pi G}\int \mathrm{d}\Omega\langle \epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{ak}x^k\partial^l h^{\mathrm{TT}}_{al}\rangle \tag{268}$$

Eq. 267 represents the contribution of the orbital angular momentum of gravitational waves (GWs), while equation 268 accounts for the contribution from the spin S^i of the field configuration. The orbital part of the equation 267 gives:

$$\begin{aligned}
\frac{\mathrm{d}L^i}{\mathrm{d}t} &= -\frac{c^3R^2}{32\pi G}\int \mathrm{d}\Omega\langle \epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{ab}x^k\partial^l h^{\mathrm{TT}}_{ab}\rangle \\
\left.\frac{\mathrm{d}L^i}{\mathrm{d}t}\right|_{\mathrm{quad}} &= -\frac{c^3R^2}{32\pi G}\int \mathrm{d}\Omega\langle \epsilon^{ikl}\frac{2G}{Rc^4}\Lambda_{ab,cd}x^k\partial^l\frac{2G}{Rc^4}\Lambda_{ab,gh}\ddot{Q}_{gh}\rangle \\
\left.\frac{\mathrm{d}L^i}{\mathrm{d}t}\right|_{\mathrm{quad}} &= -\frac{G\epsilon^{ikl}}{8\pi c^5}\int \mathrm{d}\Omega\left[\Lambda_{ab,cd}(\hat{n})\Lambda_{ab,gh}(\hat{n})\langle\dddot{Q}_{cd}x^k\partial^l\ddot{Q}_{gh}\rangle + \Lambda_{ab,cd}(\hat{n})\langle\dddot{Q}_{cd}x^k(\partial^l\Lambda_{ab,gh}(\hat{n}))\ddot{Q}_{gh}\rangle\right]
\end{aligned} \tag{269}$$

Furthermore, we demand

$$\partial^l \ddot{Q}_{gh} = \left(\frac{\partial r}{\partial x^l}\right)\frac{\partial \ddot{Q}_{gh}}{\partial r} = \left(-\frac{x^l}{r}\right)\frac{\partial \ddot{Q}_{gh}}{\partial (t-\frac{r}{c})} = -\frac{x^l}{r}\dddot{Q}_{gh} \tag{270}$$

$$\begin{aligned}\partial^l \Lambda_{ab,gh}(\hat{n}) &= \frac{\partial n^m}{\partial x^l}\frac{\partial \Lambda_{ab,gh}(\hat{n})}{\partial n^m}\\ \partial^l \Lambda_{ab,gh}(\hat{n}) &= -\frac{1}{r}(n_f \Lambda_{ab,lg} + n_g \Lambda_{ab,lf} + n_a \Lambda_{lb,fg} + n_b \Lambda_{al,fg})\\ \Rightarrow \Lambda_{ab,cd}(\hat{n})\partial^l \Lambda_{ab,gh}(\hat{n}) &= -\frac{1}{r}\left(\Lambda_{ab,cd}(\hat{n})_f \Lambda_{ab,lg} + \Lambda_{ab,cd}(\hat{n})_g \Lambda_{ab,fl}\right)\end{aligned} \tag{271}$$

Combining the three equations above, we take:

$$\begin{aligned}\left.\frac{\mathrm{d}L^i}{\mathrm{d}t}\right|_{\text{quad}} &= \frac{G\epsilon^{ikl}}{8c^5\pi}\int \mathrm{d}\Omega \langle \dddot{Q}_{cd}\ddot{Q}_{fg}\rangle \left(\Lambda_{ab,cd}n^k n^f \Lambda_{ab,lg} + \Lambda_{ab,cd}n^k n^g \Lambda_{ab,fl}\right)\\ \left.\frac{\mathrm{d}L^i}{\mathrm{d}t}\right|_{\text{quad}} &= \frac{G\epsilon^{ikl}}{8c^5\pi}\langle \dddot{Q}_{cd}\ddot{Q}_{fg}\rangle \int \frac{\mathrm{d}\Omega}{4\pi}\left(\Lambda_{ab,cd}n^k n^f \Lambda_{ab,lg} + \Lambda_{ab,cd}n^k n^g \Lambda_{ab,fl}\right)\\ \left.\frac{\mathrm{d}L^i}{\mathrm{d}t}\right|_{\text{quad}} &= \frac{G\epsilon^{ikl}}{8c^5\pi}\left[\int \frac{\mathrm{d}\Omega}{4\pi}\Lambda_{cd,lg}n^k n^f + \int \frac{\mathrm{d}\Omega}{4\pi}\Lambda_{cd,fl}n^k n^g\right]\\ \left.\frac{\mathrm{d}L^i}{\mathrm{d}t}\right|_{\text{quad}} &= \frac{2G}{15c^5}\epsilon^{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle\end{aligned} \tag{272}$$

Similarly, the spin part of equation 268 gives:

$$\begin{aligned}\left.\frac{\mathrm{d}S^i}{\mathrm{d}t}\right|_{\text{quad}} &= \frac{c^3 R^2}{16\pi G}\int \mathrm{d}\Omega \epsilon^{ikl}\left(\frac{2G}{Rc^4}\right)^2 \Lambda_{ak,mn}\Lambda_{al,cd}\langle \dddot{Q}_{mn}\ddot{Q}_{cd}\rangle\\ \left.\frac{\mathrm{d}S^i}{\mathrm{d}t}\right|_{\text{quad}} &= \frac{G}{4\pi c^5}\epsilon^{ikl}\langle \dddot{Q}_{mn}\ddot{Q}_{cd}\rangle \int \mathrm{d}\Omega \Lambda_{ak,mn}\Lambda_{al,cd}\end{aligned} \tag{273}$$

We write $\Lambda_{ak,mn}\Lambda_{al,cd} = (P_{am}P_{kn} - \frac{1}{2}P_{ak}P_{mn})\Lambda_{al,cd} = P_{kn}\Lambda_{ml,cd} - \frac{1}{2}P_{mn}\Lambda_{kl,cd}$ (274)

Furthermore, multiplying equation 274 with ϵ^{ikl} we get:

$$\epsilon^{ikl}\Lambda_{ak,mn}\Lambda_{al,cd} = \epsilon^{ikl}P_{kn}\Lambda_{ml,cd} - \frac{1}{2}P_{mn}\epsilon^{ikl}\Lambda_{kl,cd} = \epsilon^{ikl}P_{kn}\Lambda_{ml,cd} \tag{275}$$

Inserting equation 273 in equation 275 and after some really long algebra, we get:

$$\left.\frac{\mathrm{d}S^i}{\mathrm{d}t}\right|_{\text{quad}} = \frac{G\epsilon^{ikl}}{c^5}\langle \dddot{Q}_{mn}\ddot{Q}_{cd}\rangle \int \frac{\mathrm{d}\Omega}{4\pi}P_{ln}\Lambda_{mk,cd} \tag{276}$$

$$\cdots \Rightarrow \left.\frac{\mathrm{d}S^i}{\mathrm{d}t}\right|_{\text{quad}} = \frac{4G}{15c^5}\epsilon^{ikl}\langle \dddot{Q}_{al}\ddot{Q}_{ak}\rangle \tag{277}$$

Finally, the total angular momentum carried away by GWs in quadrupole order of expansion can be expressed as:

$$\left.\frac{\mathrm{d}J^i}{\mathrm{d}t}\right|_{\text{quad}} = \frac{2G}{15c^5}\epsilon_{ikl}\langle \dddot{Q}_{al}\ddot{Q}_{ak}\rangle \tag{278}$$

3.2 Mass quadrupole and octupole radiation

The study and comprehension of mass quadrupole and octupole terms, as well as the current quadrupole term of radiation, is studied thoroughly in a plethora of textbooks and articles. We follow the logic of [15], but in order to completely understand the physics and to obtain the full picture behind the expansion, we used [40], [41], [42]. These articles provided us with details about the radiated energy and momentum as well. Furthermore, a physical discussion of current quadrupole radiation is given in [43].

3.2.1 Review of the mass quadrupole term

In this section, we study the next-to-leading terms of the mass-term expansion, i.e., we study the mass quadrupole and octupole terms. The mass quadrupole term is already been described in the previous section and can be summarized in the following expressions:

- For the GW amplitude we have:

$$h_{ij}^{\mathrm{TT}}(t,\vec{x})\Big|_{\mathrm{quad}} = \frac{2G}{Rc^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}\left(t-\frac{R}{c}\right)$$

- for the mass quadrupole tensor, we proved that:

$$M^{kl} = M^{kl} - \frac{1}{3}\delta^{kl}m_{ii} + \frac{1}{3}\delta^{kl}Mii$$

- and

$$Q^{ij} = \int \mathrm{d}^3x\rho(t,\vec{x})\left[x^ix^j - \frac{1}{3}\delta^{ij}R^2\right]$$

3.2.2 Mass octupole

The next expansion term of the GW amplitude in the TT gauge reads as:

$$h_{ij}^{\mathrm{TT}} = \frac{4G}{Rc^5}\Lambda_{ij,kl}(\hat{n})n_m\dot{S}^{kl,m}\left(t-\frac{r}{c}\right) \tag{279}$$

The term $\dot{S}^{kl,m}$ is symmetric in $k \leftrightarrow l$. From equation 226 we get:

$$\begin{aligned}
\dot{S}^{kl,m} &= \frac{1}{6}\dddot{M}^{klm} + \frac{1}{3}(\ddot{P}^{k,lm} + \ddot{P}^{l,mk} - 2\ddot{P}^{m,kl}) \\
h_{ij}^{\mathrm{TT}} &= \frac{4G}{6Rc^5}\Lambda_{ij,kl}n_m\dddot{M}^{klm} + \frac{4G}{3Rc^5}\Lambda_{ij,kl}n_m(\ddot{P}^{kl,m} + \ddot{P}^{l,mk} + \ddot{P}^{m,kl})
\end{aligned} \tag{280}$$

The $\Lambda_{ij,kl}n_m\dddot{M}^{klm}$ is a symmetric term that produces mass octupole terms, while $\ddot{P}^{kl,m} + \ddot{P}^{l,mk} + \ddot{P}^{m,kl}$ presents mixed symmetry terms, which produce the current quadrupole expansion terms. We denote with

$$O^{klm} \equiv M^{klm} - \frac{1}{5}(\delta^{kl}M^{k'k'm} + \delta^{km}M^{k'lk'} + \delta^{lm}M^{kk'k'}) \tag{281}$$

The mass octupole term without traces. Since $\Lambda_{ij,kl}(\hat{n})$ is traceless and the trace part in equation (230) vanishes, we can interchange:

$$h_{ij}^{TT}\big|_{\text{oct}} = \frac{2G}{3Rc^5}\Lambda_{ij,kl}(\hat{n})\dddot{O}^{klm}(\hat{n})_m \tag{282}$$

Similarly to the case of quadrupole radiation, the use of O^{klm} is preferable from a group-theoretical perspective, as it represents a pure spin-3 tensor.

Note that for quantities quadratic in h_{ij}^{TT}, such as the radiated energy, there is no interference between the mass quadrupole and mass octupole terms due to their differing parity. Under a parity transformation, $\mathbf{x} \to -\mathbf{x}$, the mass density remains a true scalar. Consequently, the quadrupole remains invariant, while the octupole changes sign. For the same reason, in electrodynamics, there is no interference between dipole and quadrupole radiation.

When comparing the mass quadrupole and mass octupole, we observe that while the contribution to the gravitational wave (GW) amplitude from the mass quadrupole is dominant, the mass octupole provides a smaller, higher-order correction.

Every time derivative carries a factor $O(\omega_s)$, so we get

$$\frac{1}{c}\dddot{M}^{ijk} = O\left(\frac{\omega_s d}{c}\right)\ddot{M}^{ij} = O(\frac{v}{c})\ddot{M}^{ij} \tag{283}$$

This means that $\dddot{M}^{ijk}$ is smaller than $\ddot{M}^{ij}$ by a factor of $O(\frac{v}{c})$.

The power emitted per unit solid angle is obtained by inserting equation 282 into Eq. ?? as follows:

$$\begin{aligned}
P_{\text{oct}} = \frac{\mathrm{d}E_{\text{oct}}}{\mathrm{d}t} &= \frac{c^3R^2}{32\pi G}\int \mathrm{d}\Omega\langle \dot{h}_{ij}^{\text{TT}}\dot{h}_{ij}^{\text{TT}}\rangle \\
\Rightarrow P_{\text{oct}} &= \frac{R^2c^3}{32\pi G}\int \mathrm{d}\Omega\left(\frac{2G}{3Rc^5}\right)^2\langle \Lambda_{ij,kl}n_on_m\frac{\mathrm{d}^4O^{klm}}{\mathrm{d}t^4}\Lambda_{ij,np}\frac{\mathrm{d}^4O^{npo}}{\mathrm{d}t^4}\rangle \\
\Rightarrow P_{\text{oct}} &= \frac{4G}{8\cdot 9\pi c^7}\langle\frac{\mathrm{d}^4O^{klm}}{\mathrm{d}t^4}\frac{\mathrm{d}^4O^{npo}}{\mathrm{d}t^4}\rangle\int \mathrm{d}\Omega\Lambda_{ij,kl}\Lambda_{ij,np}n_on_m \\
\Rightarrow P_{\text{oct}} &= \frac{G}{18c^7}\langle\frac{\mathrm{d}^4O^{klm}}{\mathrm{d}t^4}\frac{\mathrm{d}^4O^{npo}}{\mathrm{d}t^4}\rangle\int \frac{\mathrm{d}\Omega}{4\pi}\Lambda_{ij,kl}\Lambda_{ij,np}n_on_m \\
\Rightarrow P_{\text{oct}} &= \frac{G}{18c^7}\langle\frac{\mathrm{d}^4O^{klm}}{\mathrm{d}t^4}\frac{\mathrm{d}^4O^{npo}}{\mathrm{d}t^4}\rangle\int \frac{\mathrm{d}\Omega}{4\pi}\Lambda_{ij,np}n_on_m
\end{aligned} \tag{284}$$

Direct computation of $\int \frac{\mathrm{d}\Omega}{4\pi}\Lambda_{ij,np}n_on_m$ gives:

$$\int \frac{\mathrm{d}\Omega}{4\pi}\Lambda_{ij,np}n_on_m = \int \frac{\mathrm{d}\Omega}{4\pi}\big(P_{in}P_{jp} - \frac{1}{2}P_{ij}P_{np}\big)n_on_m \tag{285}$$

The power radiated due to the octupole moment is given by:

$$P_{\text{oct}} = \frac{G}{72\pi c^7}\int \mathrm{d}\Omega\, n_in_jn_kn_mn_nn_p\left\langle\frac{\mathrm{d}^4O^{ijk}}{\mathrm{d}t^4}\frac{\mathrm{d}^4O^{mnp}}{\mathrm{d}t^4}\right\rangle, \tag{286}$$

where O^{ijk} is the mass octupole moment, and $n_i, n_j, \ldots$ are components of the unit vector $\mathbf{n}$ in spherical coordinates.

The integral over the unit sphere for $n_i n_j n_k n_m n_n n_p$ is evaluated using symmetry arguments. The result is:

$$\int \frac{\mathrm{d}\Omega}{4\pi} n_i n_j n_k n_m n_n n_p = \frac{1}{105}\left(\delta_{ij}\delta_{km}\delta_{np} + \text{other permutations}\right), \tag{287}$$

where the permutations distribute contributions equally among all pairwise indices.

The fourth time derivative of the mass octupole moment $\frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4}$ is symmetric and trace-free. Using these properties, the contraction:

$$\left\langle \frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4}\frac{\mathrm{d}^4 O^{mnp}}{\mathrm{d}t^4} \right\rangle \tag{288}$$

With the solid angle integral simplified to:

$$\int \frac{\mathrm{d}\Omega}{4\pi} n_i n_j n_k n_m n_n n_p \left\langle \frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4}\frac{\mathrm{d}^4 O^{mnp}}{\mathrm{d}t^4} \right\rangle = \frac{1}{105}\left\langle \frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4}\frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4} \right\rangle. \tag{289}$$

Substituting this result into the power expression:

$$P_{\text{oct}} = \frac{G}{72\pi c^7}\cdot\frac{1}{105}\left\langle \frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4}\frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4} \right\rangle. \tag{290}$$

Simplifying the constants:

$$P_{\text{oct}} = \frac{G}{189 c^7}\left\langle \frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4}\frac{\mathrm{d}^4 O^{ijk}}{\mathrm{d}t^4} \right\rangle. \tag{291}$$

3.2.3 Current quadrupole and loss in linear momentum

We saw that the sum of power emitted is given by the equation $\dddot{P}^{kl,m} + \dddot{P}^{l,mk} + \dddot{P}^{m,kl}$, similarly, it produces the current quadrupole term. Based on definition of $P^{k,lm}$ we get:

$$P^{i,jk} = \frac{1}{c}\int d^3x T^{oi}(t,\vec{x})x^j x^k$$

And the sum of Ps is:

$$\begin{aligned} P^{k,lm} + P^{l,km} - 2P^{m,kl} &= \frac{1}{c}\int \mathrm{d}^3x\left[T^{0k}x^l x^m + T^{0l}x^k x^m - 2T^{0m}x^k x^l\right] \\ &= \frac{1}{c}\int \mathrm{d}^3x\left[x^l\left(x^m T^{0k} - x^k T^{0m}\right) + x^k\left(x^m T^{0l} - x^l T^{0m}\right)\right] \\ &= \frac{1}{c}\int \mathrm{d}^3x\left[x^l j^{mk} + x^k j^{ml}\right] \end{aligned} \tag{292}$$

where

$$j^{jk} = \frac{1}{c}\left(x^j T^{0k} - x^k T^{0j}\right) \tag{293}$$

is the angular density associated to the (j,k) plane. Based on 293 we define j^l the l-th component of the angular momentum density vector as:

$$j^l = \epsilon^{ijl} j^{ij} \tag{294}$$

and the first moment of angular momentum density as

$$J^{i,j} = \int \mathrm{d}^3 x j^i x^j \tag{295}$$

.

Inserting Eqs. 293, 294 and 295 in 292 we get:

$$P^{k,lm} + P^{l,mk} - 2R^{m,kl} = \epsilon^{mkp} J^{p,l} + \epsilon^{mlp} J^{p,k}$$

Then equation $\ddot{P}^{kl,m} + \ddot{P}^{l,mk} + \ddot{P}^{m,kl}$ writes as:

$$h_{ij}^{\mathrm{TT}}\Big|_{\text{curr. quad.}} = \frac{4G}{3Rc^5} \Lambda_{ij,kl} n_m (\epsilon^{mkp} \ddot{J}^{p,l} + \epsilon^{mlp} + \ddot{J}^{p,k}) \tag{296}$$

The associated power to the current quadrupole is given again, as already done in the mass quadrupole and octupole:

$$P_{\text{curr. quad}} = \frac{c^3 R^2}{32\pi G} \int d\Omega \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle$$

$$\begin{aligned} P_{\text{curr. quad}} &= \frac{c^3 R^2}{8G} \int \frac{d\Omega}{4\pi} \left(\frac{4G}{3c^5 R} \right)^2 \langle \Lambda_{ij,kl} n^m \dddot{J}^{P,l} + \epsilon^{mlp} \dddot{J}^{p} \Lambda_{ij,or} n_q (\epsilon^{qoa} \dddot{J}^{a,r} + \epsilon^{qra} \dddot{J}^{aq})\rangle \\ P_{\text{curr. quad}} &= \frac{2G}{9c^7} \int \frac{d\Omega}{4\pi} \Lambda_{kl,or} n_m n_q \langle (\epsilon^{mkp} \dddot{J}^{p,l} + \epsilon^{mlp} \dddot{J}^{m,k})(\epsilon^{qoa} \dddot{J}^{a,r} + \epsilon^{qra} \dddot{J}^{a,o})\rangle \end{aligned} \tag{297}$$

In performing contractions, we use the property that $J^{i,i}$ is traceless. Specifically, this is given by:

$$J^{i,i} = \int \mathrm{d}^3 x \, x^i j^i = \frac{1}{c} \int \mathrm{d}^3 x \, x^i \epsilon^{ijk} x^j T^{0k} = 0 \tag{298}$$

where the sum over i is implied.

Next, we compute the power radiated by the current quadrupole, which is expressed as:

$$P_{\text{curr quad}} = \frac{16G}{45c^7} \left\langle \dddot{\mathcal{J}}^{ij} \dddot{\mathcal{J}}^{ij} \right\rangle, \tag{3.154}$$

where the traceless symmetric matrix $\mathcal{J}^{ij}$ is defined as:

$$\mathcal{J}^{ij} \equiv \frac{J^{i,j} + J^{j,i}}{2} \tag{299}$$

representing the symmetrization of the dipole moment of the angular momentum density.

Combining the contributions from the mass quadrupole, current quadrupole, and mass octupole, the total power radiated is:

$$P = \frac{G}{c^5} \left[\frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle + \frac{1}{c^2} \frac{16}{45} \left\langle \dddot{\mathcal{J}}^{ij} \dddot{\mathcal{J}}^{ij} \right\rangle + \frac{1}{c^2} \frac{1}{189} \left\langle \frac{\mathrm{d}^4 O_{ijk}}{\mathrm{d}t^4} \frac{\mathrm{d}^4 O_{ijk}}{\mathrm{d}t^4} \right\rangle + \mathcal{O}\left(\frac{v^4}{c^4} \right) \right] \tag{300}$$

where Q_{ij}, $\mathcal{J}^{ij}$, and O_{ijk} refer to the mass quadrupole, current quadrupole, and mass octupole moments, respectively. Higher-order terms $\mathcal{O}(v^4/c^4)$ represent negligible corrections.

It is significant to note that the primary term responsible for the loss of linear momentum arises from the interference between the quadrupole term and the next-to-leading term, which is the octupole combined with the current quadrupole. We restate an expression of the radiated power, which states that:

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} \propto \int d\Omega\, h_{ij}^{TT} \partial_t h_{ij}^{TT} \tag{301}$$

The h_{ij}^{TT} term can be expressed as follows:

$$h_{ij}^{TT} = \left(h_{ij}^{TT}\right)_{\text{quad}} + \left(h_{ij}^{TT}\right)_{\text{next-to-leading}} \tag{302}$$

Where:

$$\left(h_{ij}^{TT}\right)_{\text{quad}} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{TT}, \quad \left(h_{ij}^{TT}\right)_{\text{next-to-leading}} = \frac{1}{r}\frac{2G}{c^5} n_m \dot{S}_{ij}^{TT}.$$

In the product $h_{ij}^{TT} h_{kl}^{TT}$, we identify diagonal terms and interference terms between the quadrupole and the next-leading term.

For the quadrupole contributions, the diagonal terms are proportional to:

$$\int \mathrm{d}\Omega\, Q_{ij}^{TT} \ddot{Q}_{ij}^{TT} \tag{303}$$

While for the next-to-leading term, they are proportional to:

$$\int \mathrm{d}\Omega\, (n_m S_{ij}^{TT}) \dot{S}_{ij}^{TT} \tag{304}$$

The interference terms involve cross-products, such as:

$$\int \mathrm{d}\Omega\, Q_{ij}^{TT} n_m \dot{S}_{ij}^{TT} \tag{305}$$

These integrals are non-zero in general and do not vanish due to their parity properties. The angular integral will vanish if the product involves an odd number of factors n_m. Hence, only products with an even number of such factors contribute to the integration.

For a deeper insight, the projection Q_{ij}^{TT} onto the transverse-traceless space involves the Lamb shift tensor, ensuring the evenness of the contributions. Furthermore, since the derivative operator ∂_k acts to increase or decrease the even number of factors n_m, the leading quadrupole and interference terms remain significant in determining the loss of linear momentum.

3.3 Application on different examples

In the next three sections, we study three useful examples on quadrupole and octupole radiation. All of these can be found in Maggiore's Book [15], in Section 3.6.

The first part includes the quadrupole radiation produced by an oscillating mass, whereas the second and third paragraphs include the expression of radiation for a mass in circular orbit in the mass quadrupole and octupole order of multipole. expansion and the current octupole term of this system.

3.3.1 Quadrupole Radiation from an Oscillating Mass

Computation of the quadrupole gravitational radiation emitted by a non-relativistic system with a degree of freedom $z_0(t)$ that performs harmonic oscillations along the z-axis, described by $z_0(t) = a\cos(\omega_s t)$, where $\omega_s \ll c$. Computations on radiation emission consist only of closed systems, and no external forces exist. We have a spring with a rest length of zero that connects two masses.

The mass density is given by $\rho(t, \mathbf{x}) = \mu\delta(x)\delta(y)\delta(z - z_0(t))$, where μ is the reduced mass. The second mass moment, as produced in equation 210

$$
\begin{aligned}
M^{ij} &= \frac{1}{c^2}\int \mathrm{d}^3x\, T^{oo}(t,\vec{x})\, x^i x^j = \int \mathrm{d}^3x\, \rho(t,\vec{x})\, x^i x^j \\
\Rightarrow M^{ij} &= \int \mathrm{d}^3x\, \mu\, \delta(x)\, \delta(y)\, \delta[z - z_0(t)]\, x^i x^j \\
\Rightarrow M^{ij} &= \int \mathrm{d}x\, \delta(x) \int \mathrm{d}y\, \delta(y) \int \mathrm{d}z\, \delta[z - z_0(t)]\, \mu\, x^i x^j \\
\Rightarrow_{x=0,y=0} M^{ij} &= \int \mathrm{d}z\, \mu\, z^2\, \delta[z - z_0(t)]\, \delta^{i3}\, \delta^{j3} \\
\Rightarrow M^{ij} &= \mu\, z_0^2(t)\, \delta^{i3}\, \delta^{j3} \\
\Rightarrow M^{ij} &= \mu\, a^2\, \frac{1 + \cos(2\omega_s t)}{2}\, \delta^{i3}\, \delta^{j3}
\end{aligned}
\tag{306}
$$

Substituting into the quadrupole formula yields the plus-polarized wave component:

$$
h_+(t;\theta,\phi) = -\frac{G}{Rc^4}\ddot{M}_{33}\sin^2\theta \tag{307}
$$

Substituting into the quadrupole formula yields the cross-polarized wave component:

$$
\begin{aligned}
h_\times(t;\theta,\phi) = \frac{G}{Rc^4}&[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta - 2\ddot{M}_{13}\cos\phi\sin\theta \\
&+ 2\ddot{M}_{23}\sin\phi\sin\theta] \\
h_\times(t;\theta,\phi) = 0&
\end{aligned}
\tag{308}
$$

Eqs. 306, 307 and 308 yield:

$$
\begin{aligned}
\ddot{M}^{ij} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}[\mu a^2 \frac{1}{2}(1 + \cos 2\omega_s t)\delta^{i3}\delta^{j3}] \\
\Rightarrow \ddot{M}^{ij} &= \frac{\mu a^2}{2}\delta^{i3}\delta^{j3}\frac{\mathrm{d}}{\mathrm{d}t}[-2\omega_s \sin 2\omega_2 t] \\
\Rightarrow \ddot{M}^{ij} &= -\frac{\mu a^2}{2}\delta^{i3}\delta^{j3}(2\omega_s)^2\cos 2\omega_s t \\
\Rightarrow \ddot{M}^{ij} &= -2\mu a^2\omega_s^2\delta^{i3}\delta^{j3}\cos 2\omega_s t
\end{aligned}
\tag{309}
$$

$$
\text{eq. } 307 \xRightarrow{\text{eq. } 309} h_+(t;\theta,\phi) = \frac{G}{Rc^4}(2\mu a^2\omega_s^2)\ddot{M}^{33} \tag{310}
$$

Finally, we have:

$$h_+(t;\theta,\phi) = \frac{2G\mu a^2\omega_s^2}{Rc^4}\cos 2\omega_s t\sin^2\theta \tag{311}$$
$$h_\times(t;\theta,\phi) = 0 \tag{312}$$

So we have a monochromatic radiation at a frequency $\omega = 2\omega_s$ with a pure (+)-polarisation. The angular distribution reflects the cylindrical symmetry of the source, and therefore it is independent of ϕ with a maximum at $\theta = \frac{\pi}{2}$. Along the z-axis, radiation vanishes, so only components transverse to the line-of-sight contribute to Gω's production. We denoted this polarization as plus because of our choice of $(\hat{u},\hat{v})$ axes. In our definition $(\hat{u},\hat{v})$ are obtained from $(\hat{x},\hat{y})$ applying the rotation matrix R as given in equation 249:

$$R_{ij} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi 0 \end{pmatrix}$$

So for a direction along $\hat{n}$ where $\theta = \frac{\pi}{2}$ we get $\hat{v} = -\hat{z}$, while $\hat{u}$ is the intersection of this transverse plane with the original $(\hat{x},\hat{y})$ plane. If we rotate the $(\hat{u},\hat{v})$ axes by 45°, we could call this polarization a purely cross one. For a generic ψ angle, we would have a mixture of plus and cross-polarisation. The radiated power in terms of $h_+, h_\times$ is:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} = \frac{R^2c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle \tag{313}$$

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} &= \frac{R^2c^3}{16\pi G}\langle \dot{h}_+^2\rangle = \frac{R^2c^3}{16\pi G}\frac{4G\mu^2a^4\omega_s^4}{R^2c^8}\sin^4\theta(2\omega_s)^2\langle\sin^2 2\omega_s t\rangle \\ \frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} &= \frac{G\mu^2a^4\omega_s^6}{c^5\pi}\sin^4\theta\langle\sin^2 2\omega_s t\rangle \\ \frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} &= \frac{G\mu^2a^4\omega_s^6}{2c^5\pi}\sin^4\theta \end{aligned} \tag{314}$$

If we use equation 310 in equation 313 we get:

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} &= \frac{G}{Rc^4}(-\langle sin^2\theta\rangle)\langle \dddot{M}^{33}\rangle = -\frac{G}{2c^4}\langle\dddot{M}^{33}\rangle \\ \frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} &= \frac{G}{8\pi c^5}\Lambda_{33,33}(\hat{n})\langle\dddot{M}_{33}^2\rangle \end{aligned} \tag{315}$$

$$\text{where}\Lambda_{33,33} = \frac{1}{2}(1-n_3^2)^2 = \frac{1}{2}(1-\cos^2\theta)^2 = \frac{1}{2}sin^4\theta \tag{316}$$

and integrating equation 314 over a solid angle we get:

$$\begin{aligned} P|_{\text{quad}} &= \int \mathrm{d}\Omega\frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} = \frac{G\mu^2a^4\omega_s^6}{2\pi c^5}\int_0^\pi \mathrm{d}\theta\sin^2\theta\int_0^{2\pi} d\phi\sin^4\theta \\ \Rightarrow P|_{\text{quad}} &= \frac{16}{15}\frac{G\mu^2a^4\omega_s^6}{c^5} \end{aligned} \tag{317}$$

The total energy radiated over a period $T = \frac{2\pi}{\omega_s}$ is:

$$\begin{aligned}\langle E_{\text{quad}}\rangle_T &= \int_0^T \mathrm{d}t \frac{16}{15}\frac{G\mu^2 a^4 \omega_s^6}{c^5} = \frac{16}{15}\frac{G\mu^2 a^4\omega_s^5}{c^5} T \frac{2\pi}{\omega_s} \\ \Rightarrow \langle E_{\text{quad}}\rangle_T &= \frac{32\pi G}{15c^5}\mu^2 \frac{a^5\omega_s^5}{a} = \frac{32\pi G}{15c^5}\frac{\mu^2}{a}\left(\frac{v}{c}\right)^5\end{aligned} \tag{318}$$

where $v = a\omega_s$ is the source's maximum speed.

So in a complete cycle, the radiated energy is suppressed by a factor $\left(\frac{v}{c}\right)^5$.

3.3.2 Quadrupole Radiation from a Mass in Circular Orbit

We analyze the gravitational wave emission from a binary system with masses m_1 and m_2 in a circular orbit. For a given orbital motion and negligible back reaction and beyond lowest order at $\frac{v}{c}$, we cannot keep a flat spacetime for our description. For the moment we choose a trajectory on the (x, y, z) frame, so the orbit lies on (x, y) described by:

$$x_0(t) = R\cos(\omega_s t + \frac{\pi}{2}) \tag{319}$$

$$y_0(t) = R\sin(\omega_s t + \frac{\pi}{2}) \tag{320}$$

$$z_0(t) = 0 \tag{321}$$

In the CM frame, the second mass moment is:

$$M^{ij} = \mu x_0^i(t) x_0^j(t)$$

Thus, the second moments are:

$$\begin{aligned} M^{11} &= \mu x_0^1 x_0^1 = \mu R^2 \cos^2\left(\omega_s t + \frac{\pi}{2}\right) = \mu\frac{R^2}{2}\left(1 - \cos\left(2\omega_s t\right)\right) \\ M^{12} = M^{21} &= \mu x_0^1 x_0^2 = \mu R^2 \cos\left(\omega_s t + \frac{\pi}{2}\right)\sin\left(\omega_s t + \frac{\pi}{2}\right) \\ &= \frac{1}{2}\mu R^2 \sin\left(2\omega_s t + \frac{\pi}{2}\right) = -\frac{1}{2}\mu R^2 \sin\left(2\omega_s t\right) \\ M^{22} &= \mu x_0^2 x_0^2 = \mu R^2 \sin^2\left(\omega_s t + \frac{\pi}{2}\right) = \mu R^2[1 - \cos^2\left(\omega_s t + \frac{\pi}{2}\right)] \\ &= \mu R^2 \frac{1 + \cos\left(2\omega_s t\right)}{2}\end{aligned}$$

For the M^{13}, M^{23}, M^{33} we have that $\sim x_0^3 = z_0(t) = 0$ for $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

The second time derivative yields:

$$\ddot{M}_{11} = \frac{\mu R^2}{2}(2\omega_s)^2\cos(2\omega_s t) = 2\mu R^2\omega_s^2\cos(2\omega_s t) \tag{322}$$

$$\ddot{M}_{22} = -\ddot{M}_{11} \tag{323}$$

$$\ddot{M}_{12} = 2\mu R^2 \omega_s^2 \sin(2\omega_s t) \tag{324}$$

Substituting to equation 251 for the plus-component we get:

$$\begin{aligned}
h_+(t;\theta,\phi) &= 2\mu R^2\omega_s^2\frac{G}{rc^4}[\cos^2\phi(1+\cos^2\theta)-\sin 2\phi(1+\cos^2\theta)] \\
&\quad + [\sin 2\phi(1+\cos^2\theta)]\sin 2\omega_s t+\cos 2\omega_s t \\
\Rightarrow h_+(t;\theta,\phi) &= \frac{4GR^2\omega_s\mu}{2rc^4}(1+\cos^2\theta)[\cos 2\phi\cos 2\omega_s t+\sin 2\phi\sin 2\omega_s t] \\
\Rightarrow h_+(t;\theta,\phi) &= \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\frac{1+\cos^2\theta}{2}\cos(2\omega_s t+2\phi)
\end{aligned} \tag{325}$$

While for the cross component of the radiation, as given in equation 252, is:

$$\begin{aligned}
h_\times(t;\theta,\phi) &= \frac{2}{r}\frac{G}{c^4}\cos 2\phi\cos\theta\frac{2G}{rc^4}2\mu R^2\omega_s^2\sin 2\omega_s+\cos 2\phi\cos\theta \\
h_\times(t;\theta,\phi) &= \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos\theta\sin(2\omega_s t+2\phi)
\end{aligned} \tag{326}$$

In this example, we have a dependence on ϕ since the system is not invariant under rotation of the $\hat{z}$ axis. The orbit edge is for $\theta = i$ angle at $i=\frac{\pi}{2}$. We get for i from Eqs. 325 and 326:

$$h_+(t) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\left(\frac{1+\cos^2 i}{2}\right)\cos 2\omega_s t \tag{327}$$

$$h_\times(t) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos i\sin 2\omega_s t \tag{328}$$

The angular distribution of the radiated power in quadrupole approximation, given in equation 253, is:

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} &= \frac{r^2c^3}{16\pi G}\langle\dot{h}_+^2+\dot{h}_\times^2\rangle \\
\frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}} &= \frac{2G\mu^2R^4\omega_s^6}{\pi c^5}\left[\left(\frac{1+\cos^2\theta}{2}\right)^2+\cos^2\theta\right]
\end{aligned} \tag{329}$$

Integrating over all angles gives the total power:

$$\begin{aligned}
P|_{\text{quad}} &= \int\frac{\mathrm{d}P}{\mathrm{d}\Omega}\Big|_{\text{quad}}\mathrm{d}\Omega = \frac{2G\mu^2R^4\omega_s^6}{\pi c^5} = 2\pi\int_0^\pi\left[\mathrm{d}\theta\left(\frac{1+\cos^2\theta}{2}\right)^2+\cos^2\theta\right] \\
P|_{\text{quad}} &= \frac{32}{5}\frac{G\mu^2R^4\omega_s^6}{c^5}
\end{aligned} \tag{330}$$

Finally, the total energy radiated by a GW reads:

$$\begin{aligned}
\langle E_{\text{quad}}\rangle_T &= \frac{32}{5}\frac{G\mu^2}{c^5}R^4\omega_s^6T\frac{2\pi}{\omega_s} \\
\langle E_{\text{quad}}\rangle_T &= \frac{64\pi}{5}\frac{G\mu^2}{c^5R}(\omega_s^5R^5) \\
\langle E_{\text{quad}}\rangle_T &= \frac{64\pi G}{5}\frac{\mu^2}{R}\left(\frac{v^5}{c^5}\right)
\end{aligned} \tag{331}$$

3.3.3 Mass Octupole and Current Quadrupole Radiation from a mass in circular motion

Computation of mass octupole and current quadrupole radiation generated by a binary system, with the center of mass (CM) coordinate to be described by a circular trajectory. We want to compute the radiation emitted from the star in the observer's direction. We consider the observer along the $\hat{z}$ axis. The equations of orbit in this frame are:

$$x_0(t) = R\cos\omega_s t \tag{332}$$
$$y_0(t) = R\cos i\sin\omega_s t \tag{333}$$
$$z_0(t) = R\sin i\sin\omega_s t \tag{334}$$

We can set the observer along $\hat{z}$ and compute the radiation emitted along $\hat{n} = (0, 0, 1)$. Equation 282 provides a useful formula for the mass octupole radiation. Since $\hat{n} = (0, 0, 1)$ we get $n_m = n_3 \neq 0 \Rightarrow m = 3$. So equation 282 can be written as:

$$h_{ij}^{\mathrm{TT}}\big|_{\mathrm{oct}} = \frac{2G}{3c^5 r}\Lambda_{ij,kl}(\hat{n})\dddot{M}^{kl3} \tag{335}$$

Because $\dddot{M}^{kl3}$ contracts with $\Lambda_{ij,kl}$, when $k = 3$ or $l = 3$ the corresponding components vanishes. So we are left with $\dddot{M}^{123}$ and $\dddot{M}^{213}$ component to calculate. Since we have fixed the third index, we can write

$$\begin{aligned}
\dddot{M}^{kl3} &\equiv \dddot{M}^{kl} z(t) \equiv \mu x_k(t) x_l(t) z(t)\\
\ddot{M}^{113} &= \mu x_0^1(t) x_0^1(t) z(t) = \mu R^2\cos^2\omega_s t R\sin i\sin\omega_s t\\
\dddot{M}^{123} &= \mu R^2\cos i\sin\omega_s t\cos\omega_s t R\sin i\sin\omega_s t\\
\dddot{M}^{213} &= \dddot{M}^{123}\\
\dddot{M}^{223} &= \mu R^2\cos^2 i\sin^2\omega_s t R\sin i\sin\omega_s t
\end{aligned}$$

or in matrix form

$$M^{ij3} = \mu R^3\sin i\sin\omega_s t\begin{pmatrix}\cos^2\omega_s t & \cos i\sin\omega_s t\cos\omega_s t\\ \cos i\sin\omega_s t\cos\omega_s t & \cos^2 i\sin^2\omega_s t\end{pmatrix}^{ij} \tag{336}$$

When equation 336 is contracted to $\Lambda^{kl,ij}$ we get:

$$\Lambda^{kl,ij}M^{kl3} = \mu R^3\sin i sin\omega_s t\begin{pmatrix}\frac{1}{2}(\cos^2\omega_s t - \cos^2 i\sin^2\omega_s t) & \cos i\sin\omega_s t\cos\omega_s t\\ \cos i\sin\omega_s t\cos\omega_s t & \frac{1}{2}(\cos^2\omega_s t - \cos^2 i\sin^2\omega_s t)\end{pmatrix}^{ij} \tag{337}$$

In Eq. 337, after applying three times the derivative concerning time, we get:

$$h_+|_{\mathrm{oct}} = \frac{1}{r}\frac{G\mu R^3\omega_s^3}{12c^5}\sin i[(3\cos^2 i - 1)\cos\omega_s t - 27(1+\cos^2 i)\cos(3\omega_s t)] \tag{338}$$

$$h_\times|_{\mathrm{oct}} = \frac{1}{r}\frac{G\mu R^3\omega_s^3}{12c^5}\sin 2i[\sin\omega_s t - 27\sin 3\omega_s t] \tag{339}$$

The current quadrupole radiation is the sum of $h_{ij}^{\mathrm{TT}}|_{\mathrm{oct}} + h_{ij}^{\mathrm{TT}}|_{\mathrm{cq}}$. Or in mathematical form, as given in equation 204:

$$S^{ij,k}(t) = \int \mathrm{d}^3x' x'^k T^{ij}(t, \vec{x}') = \mu x_0^k \dot{x}_0^i \dot{x}_0^j \tag{340}$$

Along the z-axis, the radiation is, as mentioned, the sum, or in mathematical form, we get:

$$\begin{aligned} h_{ij}^{\mathrm{TT}}|_{\mathrm{oct+cq}} &= \frac{2G}{3c^5 r}\Lambda_{ij,kl}(\hat{n})(\dddot{M}^{kl3} + \epsilon^{kl3}\ddot{J}^{p,l} + \epsilon^{3lp}\ddot{J}^{p,k}) \\ \Rightarrow h_{ij}^{\mathrm{TT}}|_{\mathrm{oct+cq}} &= \frac{4G}{c^5 r}\Lambda_{ij,kl}(\hat{n})\dot{S}^{kl,3} \end{aligned} \tag{341}$$

Similarly to mass octupole radiation, we get:

$$(h_+)_{\mathrm{oct+cq}} = \frac{G\mu R^3\omega_s^3}{2rc^5}\sin i[(cos^2 i - 3)\cos\omega_s t - 3(1 + cos^2 i)\cos 3\omega_s t] \tag{342}$$

$$(h_\times)_{\mathrm{oct+cq}} = \frac{G\mu R^3\omega_s^3}{2rc^5}\sin 2i[\sin\omega_s t - 3\sin 3\omega_s t] \tag{343}$$

The difference between $h_{+,\times}|_{\mathrm{oct+cq}} - h_{+,\times}|_{\mathrm{oct}}$ yields the current quadrupole amplitude as:

$$h_+|_{\mathrm{cq}} = \frac{G\mu R^3\omega_s^3}{12rc^5}\sin i[(3\cos^2 i - 17)\cos\omega_s t + 9(1 + cos^2 i)\cos 3\omega_s t] \tag{344}$$

$$h_\times|_{\mathrm{cq}} = \frac{G\mu R^3\omega_s^3}{12rc^5}\sin i[5\sin\omega_s t + 9\sin 3\omega_s t] \tag{345}$$

So, the current quadrupole contribution is a sum of terms with frequencies ω_s and $3\omega_s$ The total radiated power in the mixed current quadrupole and octupole term is given in:

$$\begin{aligned} \frac{\mathrm{d}P_{\mathrm{oct+cq}}}{\mathrm{d}\Omega} &= -\frac{r^2c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \Rightarrow \\ P_{\mathrm{oct+cq}} &= \frac{r^2c^3}{16\pi G}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}i\sin i\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \\ P_{\mathrm{oct+cq}} &= \frac{r^2c^3}{8G}\int_{-1}^{1} d\cos i\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \\ P_{\mathrm{oct+cq}} &= \frac{424}{105}\frac{G\mu^2}{c^7}R^6\omega_s^8 \end{aligned} \tag{346}$$

For $P(2_{\omega_s}) = P_{\mathrm{quad}} = \frac{32G\mu^2}{5c^5}R^4\omega_s^6$ we get the power at $\omega_s; P(\omega_s)$ as

$$P(\omega_s) = \frac{19}{672}\left(\frac{v}{c}\right)^2 P(2\omega_s) \tag{347}$$

and for the frequency $3\omega_s$

$$P(3\omega_s) = \frac{135}{224}\left(\frac{v}{c}\right)^2 P(2\omega_s) \tag{348}$$

4 Symmetric trace-free and spherical tensor components formalisms

The first section of this chapter is devoted to a useful formalism, used in the multipole expansion the Symmetric Trace-Free formalism (STF). In order to understand this new formalis, we used the articles in [26], [44], [49], [50], [51], [52], [53], [54] and [55].

From a group theoretical point of view, we separated the next-to-leading order into irreducible representations of the rotation group. To generalize such a construction to an arbitrary order of the multipole expansion, we introduce a complete set of representations in two ways:

1. The first approach considers symmetric and trace-free tensors (SFT formalism).
2. The second approach introduces spherical components of tensors and the tensorial form of spherical harmonics.

We begin by recalling the workings of the multipole expansion in a static situation, governed by Poisson's equation:

$$\nabla^2\phi = -4\pi\rho \tag{349}$$

where ϕ is the scalar potential, and $\rho(\vec{x})$ is the source density.

As in the static case of electrodynamics, we consider a source density $\rho(\vec{x})$ localized in space, so that:

$$\rho(\vec{x}) = 0, \quad \forall\, r > d \tag{350}$$

where $r = |\vec{x}|$ and d is the characteristic size of the localized source. The Laplacian operator in spherical coordinates is written as

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) - \frac{L^2}{r^2}\phi \tag{351}$$

while $\phi(r,\theta,\phi)$ can be written without loss of generality as:

$$\phi(r,\theta,\phi) = Q_{\mathrm{lm}}(r)Y_{\mathrm{lm}}(\theta,\phi) \tag{352}$$

Inserting Eqs. 351 and 352 in equation 349, we get:

$$\begin{aligned}
&\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\partial r) - \frac{L^2}{r^2}\right]\phi = -4\pi\rho \\
&\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\partial r Q_{\mathrm{lm}}Y_{\mathrm{lm}}) - \frac{1}{r^2}L^2Q_{\mathrm{lm}}(r)Y_{\mathrm{lm}}(\theta,\phi) = -4\pi\rho \\
&\frac{Y_{\mathrm{lm}}}{r^2}\partial r(r^2\partial r Q_{\mathrm{lm}}(r)) - \frac{\mathrm{lm}(r)}{r^2}l(l+1)Y_{\mathrm{lm}} = -4\pi\rho
\end{aligned} \tag{353}$$

We are interested in results outside the source $r > 0$, so,

$$\xrightarrow[\rho=0]{r>0} \frac{1}{Q_{\mathrm{lm}}(r)}\left[\partial_r\left(r^2\frac{\partial Q_{\mathrm{lm}}(r)}{\partial r}\right)\right] - l(l+1) = 0 \tag{354}$$

The differential equation as expressed in 354 has known solutions given as:

$$Q_{\mathrm{lm}}(r) = Ar^l + \frac{B}{r^{l+1}} \tag{355}$$

Next, we insert $Q_{\mathrm{lm}}(r)$ into equation 352 and get the most generic solution given as:

$$\phi(\vec{x}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Q_{\mathrm{lm}}}{2l+1} \frac{Y_{\mathrm{lm}}(\theta, \phi)}{r^{l+1}} \tag{356}$$

where

$$\nabla^2 \left[\frac{Y_{\mathrm{lm}}(\theta, \phi)}{r^{l+1}} \right] = 0. \tag{357}$$

Similarly, $\phi(r, \theta, \phi)$ can be recovered as the general Green's function of the Laplacian operator. For the case where x lies inside to outside the source, we get:

$$\phi(\vec{x}) = \int \mathrm{d}^3 y \frac{1}{|\vec{x} - \vec{y}|} \rho(\vec{y}) \tag{358}$$

with $\frac{1}{|\vec{x}-\vec{y}|}$ being according to additional theorem for spherical harmonics:

$$\frac{1}{|\vec{x} - \vec{y}|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r'^l}{r^{l+1}} Y^*_{\mathrm{lm}}(\theta', \phi') Y_{\mathrm{lm}}(\theta, \phi) \tag{359}$$

where $r \equiv |\vec{x}| \& r' \equiv |\vec{y}|$, (θ, ϕ) are the polar angles of $\vec{x}$ and (θ', ϕ') are the polar angles of $\vec{y}$. Similarly equation 357 in terms of equation 359 is:

$$\phi(\vec{x}) = \int \mathrm{d}^3 y \rho(\vec{y}) 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r'^l}{r^{l+1}} Y^*_{\mathrm{lm}} Y_{\mathrm{lm}} \tag{360}$$

and equating to equation 355 we get:

$$Q_{\mathrm{lm}} = \int \mathrm{d}^3 y Y^*_{lm}(\rho', \theta') \rho(\vec{y}) r'^l \tag{361}$$

An alternative way of performing the multipole expansion is to write:

$$\frac{1}{|\vec{x} - \vec{y}|} = \frac{1}{|\vec{x}|} - y^i \partial_i \frac{1}{|\vec{x}|} + \frac{1}{2} y^i y^j \partial_i \partial_j \frac{1}{|\vec{x}|} + \dots \tag{362}$$

$$\frac{1}{|\vec{x} - \vec{y}|} = \sum_{l=0}^{infty} \frac{(-)^l}{l!} y^{i_1} y^{i_2} \dots y^{i_l} \partial_{i_1} \partial_{i_2} \dots \partial_{i_l} \frac{1}{|\vec{x}|} \tag{363}$$

In equation 362 and 363 we get:

$$\phi(\vec{x}) = \int \mathrm{d}^3 y \rho(y) \left(\frac{1}{|\vec{x}|} - y^i \partial_i \frac{1}{|\vec{x}|} + \frac{1}{2} y^i y^j \partial_i \partial_j \frac{1}{|\vec{x}|} + \dots \right) \tag{364}$$

$$
\begin{aligned}
\phi(\vec{x}) &= \int \mathrm{d}^3 y \rho(y) \sum_{l=0}^{\infty} \frac{(-)^l}{l!} y^{i_1} \dots y^{i_l} \partial_{i_1} \dots \partial_{i_l} \frac{1}{|\vec{x}|} \\
\phi(\vec{x}) &= \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \int \mathrm{d}^3 y \rho(y) y^{i_1} \dots y^{i_l} \partial_{i_1} \dots \partial_{i_l} \frac{1}{|\vec{x}|}
\end{aligned} \tag{365}
$$

The trace-less combination for $y^{i_1} \dots y^{i_l}$ is

$$
\begin{aligned}
y^{\langle i_1} y^{i_2 \rangle} &\to y^{i_1} y^{i_2} - \frac{1}{2} \delta^{i,i_2} |\vec{y}|^2 \\
\Rightarrow \phi(\vec{x}) &= \sum_{l=0}^{\infty} \frac{(-)^l}{l!} Q_{i_1 \dots i_l} \partial_{i_1} \dots \partial_{i_l} \frac{1}{|\vec{x}|}
\end{aligned} \tag{366}
$$

with

$$
Q_{i_1 \dots i_l} = \int \mathrm{d}^3 y^{\langle i_1} \dots y^{i_l \rangle} \rho(y) \tag{367}
$$

The brackets $y^{\langle i_1} \dots y^{i_l \rangle}$ in equation 367 mean that we get the symmetric and trace-free part of the tensor $y^{i_1} \dots y^{i_l}$.

4.1 Symmetric-Trace-Free (STF) Formalism

We introduce a helpful multi-index notation developed by Blanchet and Damour. In this notation, a tensor with l indices, $i_1 i_2 \dots i_l$, is compactly represented by a single capital letter L:

$$
F_L \equiv F_{i_1 i_2 \dots i_l}
$$

Similarly, a tensor with $l+1$ indices is denoted by $G_{i_L} \equiv G_{i_1 i_2 \dots i_l}$. For example, $F_{i_L - 1}$ represents $F_{i_1 i_2 \dots i_{l-1}}$. Additionally, ∂_L is shorthand for $\partial_{i_1} \dots \partial_{i_l}$, and x_L and n_L denote $x_{i_1} x_{i_2} \dots x_{i_l}$ and $n_{i_1} n_{i_2} \dots n_{i_l}$, respectively, where $n_i = x_i / r$ is the radial unit vector. When expressions like $F_L G_L$ appear, the summation over all indices $i_1, i_2, \dots, i_l$ is implicit:

$$
F_L G_L = \sum_{i_1 \dots i_l} F_{i_1 \dots i_l} G_{i_1 \dots i_l} \tag{368}
$$

Symmetrisation is denoted by round brackets as

$$
a_{(ij)} = \frac{1}{2} (a_{ij} + a_{ji}).
$$

Finally, the symmetric-trace-free (STF) projection is denoted by a hat, as in $\hat{K}_L$, which indicates that all indices of the tensor $K_{i_1 \dots i_l}$ are symmetrized, and all traces are removed. Alternatively, this operation can be denoted with angle brackets $K_{<L>}$, so $\hat{K}_L \equiv K_{<L>}$. This notation allows a compact representation of STF operations, such as $\epsilon_{ij} (k A_{L-1})_{i_1}$, where STF symmetrization applies to the index k of ϵ_{ij} and the first $l-1$ indices of $A_{i_1 \dots I_{l-1}}$.

A rank-l STF tensor $A_{i_1 \dots i_l}$ has $2l+1$ independent components, forming an irreducible representation of dimension $2l+1$ under the rotation group $SO(3)$. The complete set of STF tensors for all ranks l provides a full set of $SO(3)$ representations.

For example, a $(0,2)$ rank in SFT mode T_{ij} is a generic tensor and can be written as:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) \tag{369}$$

$$T_{ij} =\equiv S_{ij} + A_{ij} \tag{370}$$

Here, S_{ij} is the symmetric part, and A_{ij} is the antisymmetric part. Using $A_k = \epsilon_{ijk} A_{ij}$, we find $A_{ij} = \frac{1}{2}\epsilon_{ijk} A_k$. Defining $S = S_{ii}$ as the trace of S_{ij}, equation 370 becomes:

$$T_{ij} = \frac{1}{3} S\delta_{ij} + \frac{1}{2}\epsilon_{ijk} A_k + \left(S_{ij} - \frac{1}{3} S\delta_{ij} \right) \tag{371}$$

This explicitly separates T_{ij} into a scalar S, a vector A_k, and a $(0,2)$ rank STF tensor $S_{ij} - \frac{1}{3}S\delta_{ij}$. These components are used in the STF formalism to analyze a scalar S, a vectorA_k, and a tensor $S_{ij} - \frac{1}{3}S\delta_{ij}$ field in multipole expansions.

4.1.1 SFT Formalism for Scalar Fields

We consider a scalar field ϕ governed by the relativistic wave equation:

$$\Box \phi = -4\pi\rho, \tag{372}$$

where the source $\rho(t, \mathbf{x})$ is generally time-dependent but localized in space, so for $|\vec{x}| > d \Rightarrow \rho(t, \vec{x}) = 0$. The generic solution of equation 372 outside the source is given as the expansion around $|\vec{x}| \gg d$ is:

$$\phi(t, \vec{x}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{F_L\left(t - \frac{r}{c}\right)}{r} \right], \tag{373}$$

where L comes from the multi-index notation and $F_L\left(t - \frac{r}{c}\right)$ is calculated on the retarded time $t_{\text{ret}} = t - \frac{r}{c}$. This result relies on the fact that, for $r > 0$, F_L can be any function of the retarded time $u = t - r/c$, satisfying:

$$\Box \left[\frac{F_L\left(t - \frac{r}{c}\right)}{r} \right] = 0, \tag{374}$$

and each term is solution of $\Box\phi(x) = 0$. Equation 373 comes as the most generic solution, since all F_L tensors, with all possible ranks l, provide an irreducible representation of the $SO(3)$ group. The appropriate Green's function for a radiation problem is:

$$\phi(t, \vec{x}) = \int \mathrm{d}^3x' \, \frac{1}{|\vec{x} - \vec{x}'|} \rho \left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}' \right), \tag{375}$$

where this integral form holds both inside and outside the source region.

Comparing this expression with equation 375 and equation 373, we get full computation in

$$F_L(u) = \int \mathrm{d}^3y\, \hat{y}_L \int_{-1}^{1} \mathrm{d}z\, \delta_l(z) \rho\left(u + z\frac{|\vec{y}|}{c}, vecy\right), \tag{376}$$

where $\hat{y}_L$ is the symmetric-trace-free (STF) projection of y_L and $F_L(u)$ is in fact a multipole moment for the source ρ. The function $\delta_l(z)$ is a weight function and is defined as:

$$\delta_l(z) = \frac{(2l+1)!!}{2^l l!}(1-z^2)^l, \tag{377}$$

With normalization condition:

$$\int_{-1}^{1} \delta_l(z)\,\mathrm{d}z = 1, \tag{378}$$

The $\int_{-1}^{1} \delta_l(z)\,\mathrm{d}z \rho(u + \frac{z}{c}|\vec{y}|, \vec{y})$ is the average over time $(t - \frac{|\vec{x}-\vec{x}'|}{c} + \frac{z}{c}|\vec{y}|)$, the retarded time, Integration over $\mathrm{d}z$ creates a different weight function for each multipole moment l.

4.1.2 SFT formalism for a Vector Field

We consider the electromagnetic field A^μ (Lorentz gauge $\partial \mu A^\mu = 0$), that satisfies the wave equation:

$$\Box A^\mu = -\frac{4\pi}{c} J^\mu \tag{379}$$

where the role of a source plays $J^\mu = (c\rho, \mathbf{J})$, again is time-dependent and localized as $J^\mu = 0$ if $|\vec{x}| > d$. Each component of A^μ can be treated as a scalar field, and therefore, in the external source region, we get:

$$A^0(t,\mathbf{x}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{F_L(u)}{r}\right] \tag{380}$$

$$A^i(t,\mathbf{x}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{G_{iL}(u)}{r}\right] \tag{381}$$

where $u = t - r/c$ and $F_L(u), G_{iL}(u)$ represent relativistic multipole moments. The explicitly expressions of $F_L(u)$ and $G_{iL}(u)$,are given as:

$$F_L\left(t - \frac{r}{c}\right) = \int \mathrm{d}^3y \hat{y}_L \int_{-1}^{1} \delta_l(z) \rho\left(t\frac{r}{c} + \frac{z|\vec{y}|, \vec{y}}{c},\right) \tag{382}$$

$$G_{ik}\left(t - \frac{r}{c}\right) = \int \mathrm{d}^3 \hat{y}_L \int_{-1}^{1} \delta_l(z) J_i\left(t\frac{r}{c} + \frac{z|\vec{y}|, \vec{y}}{c},\right) \tag{383}$$

G_{iL} is symmetric under $i, \ldots i_l$ exchanges, but it is not symmetric under $i \leftrightarrow i_1 \mathrm{or} i_2 \ldots \mathrm{or} i_l$, and tracefree since it depends on $\hat{y}_L$. So we can define an irreducible representation as:

$$G_{iL} = U_{iL+1} + \frac{l}{l+1}\epsilon_{ia\langle\langle i_l} C_{L-1\rangle a} + \frac{2l-1}{2l+1}\delta_{i\langle i_l} D_{L-1\rangle} \tag{384}$$

where $U_{iL} \equiv G_{\langle L+1\rangle}$, $C_L \equiv G_{ab\langle L-1}\epsilon_{il\rangle ab}$, $D_{L-1} = G_{aaL-1}$.

Finally equation 381 and 382 are written as:

$$381 \xRightarrow{377} A^0(t,\vec{x}) = \sum_{l=0}^{\infty} \frac{(-)^l}{l!}\partial_L \left[\frac{Q_L\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{r}\right] \tag{385}$$

$$382 \xRightarrow{383} A^i(t,\vec{x}) = \sum_{l=0}^{\infty} \frac{(-)^l}{l!}\partial_{L-1} \left[\frac{Q^{(1)}_{iL-1}\left(t-\frac{r}{c}\right)}{r} + \frac{l}{l+1}\epsilon_{iab}\partial_a \frac{M_{bL-1}\left(t-\frac{r}{c}\right)}{r}\right] \tag{386}$$

where

$$\begin{aligned} Q_L\left(t-\frac{r}{c}\right) = \int \mathrm{d}^3y \int_{-1}^{1} \mathrm{d}z \Bigg[\delta_l(z)\hat{y}_L \rho\left(t-\frac{r+z|\vec{y}|}{c},\vec{y}\right) \\ - \frac{1}{c^2}\frac{2l+1}{(l+1)(2l+3)}\delta_{l+1}(z)\hat{y}_{iL}J_i^{(1)}\left(t-\frac{r-z|\vec{y}|}{c},\vec{y}\right)\Bigg], \forall l \geq 1 \end{aligned} \tag{387}$$

$$M_L\left(t-\frac{r}{c}\right) = \int \mathrm{d}^3y \int_{-1}^{1} \mathrm{d}z \delta_l(z)\hat{y}_{\langle L-1}m_{il\rangle}\left(t-\frac{r-z|\vec{y}|}{c},\vec{y}\right), \forall l \geq 1 \tag{388}$$

where $m_i \equiv \epsilon_{ijk}y_jJ_k$ is the magnetization density.

4.2 Spherical tensor components form

In this subsection we detail the decomposition of a symmetric, trace-free (STF) tensor into its spherical tensor components. Explicitly we construct the basis of tensor spherical harmonics for the l=2 case and generalize to arbitrary l. This is a cornerstone of gravitational wave theory for analyzing radiation patterns and decomposing waveforms. This analysis is derived by [15], but more information can be found in [56], [57], [58], [59], [60].

Generalization of spherical harmonics to a spin-2 field. We consider any traceless, asymmetric tensor with two indices. The Cartesian components of the tensor are denoted by Q_{ij}. First, we introduce a basis in the space of traceless symmetric tensors with two indices, chosen to have a single relation with the $l=2$ spherical harmonies. Spherical harmonies for $l=2$ are:

$$Y^{22}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}e^{2i\phi}\sin^2\theta \tag{389}$$

$$Y^{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}e^{i\phi}\sin\theta\cos\theta \tag{390}$$

$$Y^{20}(\theta,\phi) = \sqrt{\frac{15}{16\pi}}(3\cos^2\theta - 1) \tag{391}$$

since $Y^{l,-m} = (-)^m Y^{lm*}$ we get

$$Y^{2,-1} = \left(\sqrt{\frac{15}{8\pi}}\right) e^{i\phi} \sin\theta \cos\theta \tag{392}$$

$$Y^{2,-2} = \left(\sqrt{\frac{15}{16\pi}}\right) (3\cos^2\theta - 1) \tag{393}$$

In polar coordinates the unit vector $\hat{n}$ has the following components:

$$n_x = \sin\theta \cos\phi \tag{394}$$

$$n_y = \sin\theta \sin\phi \tag{395}$$

$$n_z = \cos\theta \tag{396}$$

θ is measured with respect to $\hat{2}$ axis and ϕ is measured from $\hat{x} - axis$. Therefore,

$$n_x + in_y = e^{i\phi} \sin\theta \tag{397}$$

$$n_z = \cos\theta \tag{398}$$

So the above set of equations can be written as follows:

$$Y^{2,2} = Y^{2,-2} = \sqrt{\frac{15}{32\pi}}(n_x + in_y)^2 \tag{399}$$

$$Y^{2,1} = -\sqrt{\frac{15}{8\pi}}(n_x + in_y)n_z \tag{400}$$

$$Y^{2,0} = \sqrt{\frac{5}{16\pi}}(3n_z^2 - 1) = \sqrt{\frac{5}{4\pi}}(2n_z^2 - n_x^2 - n_y^2) \tag{401}$$

$$Y^{2,-1} = \sqrt{\frac{15}{8\pi}}(n_x + in_y)n_z \tag{402}$$

$$Y^{2,-2} = \sqrt{\frac{15}{32\pi}}(n_x + in_y)^2 \tag{403}$$

$$\tag{404}$$

We can write $Y^{l,m}$ in terms of $Y_{ij}^{l,m}$ components, independent of (θ, ϕ) as followed:

$$Y^{l,m} \equiv Y_{ij}^{l,m} n_i n_j \tag{405}$$

Since, $n_i \equiv \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$, $Y_{ij}^{l,m}$ are 3×3 matrices, symmetric in $I \leftrightarrow j$ and with vanishing antisymmetric part, explicitly, we can write:

$$Y_{ij}^{2,2} = \sqrt{\frac{15}{32\pi}} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{406}$$

$$Y^{2,1}_{ij} = -\sqrt{\frac{15}{8\pi}} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}_{ij} = -\sqrt{\frac{15}{32\pi}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}_{ij} \tag{407}$$

$$Y^{2,0}_{ij} = \sqrt{\frac{5}{4\pi}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{ij} \tag{408}$$

$$Y^{2,-1}_{ij} = \sqrt{\frac{15}{8\pi}} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix}_{ij} = \sqrt{\frac{15}{32\pi}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}_{ij} \tag{409}$$

$$Y^{2,-2}_{ij} = \sqrt{\frac{15}{32\pi}} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{410}$$

Matrices in Eqs. 406 - 410 , are traceless and symmetric. We can also see it by integrating equation 405 and seeing that $\int d\Omega\, Y^{l,m} = 0$ as:

$$\begin{aligned}
&\int Y^{2,m}_{ij} n^i n^j \,\mathrm{d}\Omega = \int \mathrm{d}\Omega\, Y^{l,m} \Rightarrow \\
&\int \hat{Y}^{2,m}_{ij} n^i n^j \,\mathrm{d}\Omega = 0 \Rightarrow \\
&\hat{Y}^{2m}_{ij} \int \mathrm{d}\Omega\, n^i n^j \;= 0, \text{ where } \mathrm{d}\Omega\, n^i n^j = \frac{1}{3}\delta^{ij} \Rightarrow \\
&Y^{2m}_{ij} = 0 \quad \text{(traceless)}.
\end{aligned} \tag{411}$$

From the explicit form of Eqs. 406 - 410, we get that Y^{lm}_{il} constitute an orthogonal basis as:

$$\sum_{ij} \left(Y^{2m'}_{ij}\right)^* Y^{2m}_{ij} = \frac{15}{8\pi}\delta^{mm'} \tag{412}$$

We can insert equation 405, and get:

$$n_i n_j - \frac{1}{3}\delta_{ij} = \sum_{m=-2}^{2} c^m_{ij}\; Y^{2m}\;(\theta, \phi) \tag{413}$$

$$\text{where } \; c^m_{ij} = \frac{8\pi}{15}\left(Y^{2m}_{ij}\right)^* \tag{414}$$

The coefficient given in equation 414 is fixed by observing that in equation 413 the LHS is traceless. The matrices in Eqs. 406 - 410 constitute a basis for the five-dimensional space of traceless, symmetric tensors Q_{ij}.We can expand any symmetric, traceless Q_{ij} tensor as:

$$Q_{ij} = \sum_{m=-2}^{2} Q_m Y^{2m}_{ij} \tag{415}$$

where Q_m are the spherical components of Q_{ij}

$$\begin{aligned}
415 \xrightarrow{\cdot n_i n_j} n_i n_j\ Q_{ij} &= \sum_{m=-2}^{2} Q_m n_i n_j Y_{ij}^{2m} \\
\xrightarrow{405} n_i n_j\ Q_{ij} &= \sum_{m=-2}^{2} Q_m Y^{2m}\ (\theta, \phi)
\end{aligned} \tag{416}$$

Now equation 415 can be inverted by following the known rules, as:

$$\begin{aligned}
415 \xrightarrow{\left(Y_{ij}^{2m}\right)^*} \left(Y_{ij}^{2m}\right)^*\ Q_{ij} &= \sum_{m=-2}^{2} Q_m \left(Y_{ij}^{2m}\right)^* Y_{ij}^{2m} \\
\xrightarrow{413} Q_m &= \frac{8\pi}{15} Q_{ij} \left(Y_{ij}^{2m}\right)^*
\end{aligned} \tag{417}$$

$$\text{with } Q_m^* = (-)^m Q_{-m} \tag{418}$$

Explicit:

$$\begin{aligned}
m = \pm 2 : Q_{\pm 2} &= \frac{8\pi}{15} Q_{ij} \left(Y_{ij}^{2,\pm 2}\right)^* \\
\Rightarrow Q_{\pm 2} &= \frac{8\pi}{15}\sqrt{\frac{15}{32\pi}}(Q_{11}\ Y_{11}^{2,\pm 2} + Q_{22}^{2,\pm 2} Y_{22}^{2,\pm 2} + 2Q_{12} Y_{12}^{2,\pm 2}) \\
\Rightarrow Q_{\pm 2} &= \sqrt{\frac{2\pi}{15}}(Q_{11} - Q_{22} \mp 2iQ_{12})
\end{aligned} \tag{419}$$

$$\begin{aligned}
m = \pm 1 : Q_{\pm 1} &= \frac{8\pi}{15} Q_{ij} \left(Y_{ij}^{2,\pm 1}\right)^* \\
\Rightarrow Q_{\pm 1} &= \mp \frac{8\pi}{15}\sqrt{\frac{15}{32\pi}} \left(2Q_{13} \left(Y_{13}^{2,\pm 1}\right)^* + 2Q_{23} \left(Y_{23}^{2,\pm 1}\right)^*\right) \\
\Rightarrow Q_{\pm 1} &= \mp \sqrt{\frac{2\pi}{15}}(2Q_{13} \mp 2iQ_{23}) \\
\Rightarrow Q_{\pm 1} &= \mp \sqrt{\frac{2\pi}{15}}(Q_{13} \mp iQ_{23})
\end{aligned} \tag{420}$$

$$\begin{aligned}
m = 0 : Q_0 &= \frac{8\pi}{15} Q_{ij} \left(Y_{ij}^{2,0}\right)^* \\
\Rightarrow Q_0 &= \frac{8\pi}{15}\sqrt{\frac{5}{16\pi}}(-Q_{11} - Q_{22} + 2Q_{33}) \\
\Rightarrow Q_0 &= -\sqrt{\frac{4\pi}{45}}(Q_{11} + Q_{22} - 2Q_{33}) \\
\Rightarrow Q_0 &= -\sqrt{\frac{4\pi}{45}}(3Q_{11} + 3Q_{22}) \\
\Rightarrow Q_0 &= -\sqrt{\frac{4\pi}{45}}(Q_{11} + Q_{22})
\end{aligned} \tag{421}$$

Applying thrice a time derivative on equation 416 we get:

$$\dddot{Q}_{ij}n_in_j = \sum_{m=-2}^{2} \dddot{Q}_m Y^{2m}(\theta,\phi) \tag{422}$$

And then taking modulus squared, we get:

$$\begin{aligned}
||\dddot{Q}_{ij}n_in_j||^2 &= \sum_{m=-2}^{2} ||\dddot{Q}_m Y^{2m}(\theta,\phi)||^2 \\
\Rightarrow \dddot{Q}_{ij}n_in_j\dddot{Q}_{lm}n_ln_m &= \sum_{m=-2}^{2} |\dddot{Q}_m\dddot{Q}_{m'}\left(Y^{2m'}(\theta,\phi)\right)^2 Y^{2m}(\theta,\phi)| \\
\Rightarrow 2\dddot{Q}_{ij}\dddot{Q}_{ij} &= \sum_{m=-2}^{2}\left[\frac{15}{8\pi}|\dddot{Q}_m|^2 + \frac{15}{8\pi}|\dddot{Q}_m|^2\right] \\
\Rightarrow \dddot{Q}_{ij}\dddot{Q}_{ij} &= \sum_{m=-2}^{2}\frac{15}{8\pi}|\dddot{Q}_m|^2 \\
\Rightarrow |\dddot{Q}_m|^2 &= \frac{15}{8\pi}\dddot{Q}_{ij}\dddot{Q}_{ij}
\end{aligned} \tag{423}$$

Substituting everything in equation 423 we get:

$$\begin{aligned}
P_{\text{quad}} &= \frac{G}{5c^5}\langle\dddot{Q}_{ij}\dddot{Q}_{ij}\rangle \\
\Rightarrow P_{\text{quad}} &= \frac{G}{5c^5}\frac{15}{8\pi}\sum_{m=-2}^{2}\langle|\dddot{Q}_m|^2\rangle \\
\Rightarrow P_{\text{quad}} &= \frac{3G}{8\pi c^5}\sum_{m=-2}^{2}\langle|\dddot{Q}_m|^2\rangle
\end{aligned} \tag{424}$$

We can generalize the above to traceless, symmetric tensors with an arbitrary number of indices. We begin by considering a STF tensor with indices $T_{i_1,\dots i_l}$ and write the spherical harmonies as:

$$Y^{l,m}(\theta,\phi) = C^{lm}e^{im\phi}P^{lm}(\cos(\theta)) \tag{425}$$

In equation 424, the term $P^{lm}\cos\theta$ expresses the associated Legendre polynomials, given by the following formula for $m \geq 0$

$$P^{lm}(x) = (-)^m 2^l (1-x^2)^{\frac{m}{2}} \sum_{k=m}^{l} \frac{k!}{(k-m)!} x^{k-m} \binom{l}{k} \binom{\frac{l-k+l}{2}}{l}^m \tag{426}$$

or in terms of $\cos\theta$

$$P^{lm}(\cos\theta) = (-)^m(\sin\theta)^m \frac{d^m}{d(\cos\theta)^m}[P^l(\cos\theta)] \tag{427}$$

Orthogonality relations read for:

$$\text{fixed m}: \int_o^\pi \mathrm{d}\theta P^{km}(\cos\theta)P^{lm}(\cos\theta)\sin\theta = \frac{2(l+m)!}{(2l+1)(l-m)!}\delta^{kl} \tag{428}$$

$$\text{fixed l}: \int_o^\pi \mathrm{d}\theta P^{lm}(\cos\theta)P^{ln}(\cos\theta)\csc\theta = \begin{cases} 0, & m \neq n \\ \infty, & m = n = 0 \\ \frac{(l+m)!}{m(l-m)!}, & m = n \neq 0 \end{cases} \tag{429}$$

Eqs. 428 and 429 normalize the C^{lm} components as following:

$$\begin{aligned} &\int \mathrm{d}\Omega Y^{lm*}Y^{lm} = 1 \Rightarrow \\ \Rightarrow &\int \mathrm{d}\theta\mathrm{d}\phi \sin\theta |C^{lm}|^2 P^{lm}P^{lm*} = 1 \\ \Rightarrow &2\pi|C^{lm}|^2\frac{2(l+m)!}{(2l+1)(l-m)!} = 1 \\ \Rightarrow &|C^{lm}| = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \\ \Rightarrow &C^{lm} = (-)^m\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \end{aligned} \tag{430}$$

$$\text{and} \quad a_k^{lm} = \frac{(-)^k}{2^l k!(l-k)!}\frac{(2l-2k)!}{(l-m-2k)!} \tag{431}$$

For $m < 0$ we may use $Y^{lm} = (-)^m(Y^(l,-m))^*$. The final formula for the arbitrary spherical harmonics is:

$$Y^{lm}(\theta,\phi) = C^{lm}(e^{i\phi}\sin\theta)^m \sum_{k=0}^{\lceil\frac{l-m}{2}\rceil} a_k^{lm}(\cos\theta)^{l-m-2k} \tag{432}$$

where $\lceil\frac{l-m}{2}\rceil$ denotes the largest integer smaller than or equal to $\frac{l-m}{2}$. In equation 432, we can substitute Eqs. 397 and 398 and get:

$$Y^{lm}(\theta,\phi) = C^{lm}(n_x + in_y)^m \sum_{k=0}^{\lceil\frac{l-m}{2}\rceil} a_k^{lm}(n_z)^{l-m-2k} \tag{433}$$

because of $(n_x+in_y)^m$ and $\sum_{k=0}^{\lceil\frac{l-m}{2}\rceil}(n_z)^{l-m-2k}$, we get the sum of a term containing l factors $n_i, l-2$ factors $n_i, l-4$ factors n_i etc. We can use $n_in_i = 1$ and write the $l-2$ factors as:

$$\delta_i j n_{i_1}\ldots n_{i_{l-2}}n_in_j\text{etc}$$

So finally we can write

$$Y^{lm}(\theta,\phi) = Y^{lm}_{i_1,\ldots i_l}n_{i_1}\ldots n_{i_l} \tag{434}$$

Again, $Y^{lm}_{i_1,\ldots i_l}$ tensors form a basis in the space of traceless symmetric tensors with l indices. So we can expand any arbitrary tensor $T_{i_1,\ldots i_l}$ as:

$$T_{i_1,\ldots i_l} = \sum_{m=-l}^{l} T_{lm} Y^{lm}_{i_1,\ldots i_l} \tag{435}$$

In terms of the spherical components. Or in terms of l unitary vectors, we get:

$$n_{i_1}\ldots n_{i_l} T_{i_1,\ldots i_l} = \sum_{m=-l}^{l} T_{lm} Y^{lm}(\theta,\phi) \tag{436}$$

And we can insert the last equation as:

$$\begin{aligned}
&\int \mathrm{d}\Omega T_{lm} \left(Y^{l'm'}_{j_1,\ldots j_l}\right)^*_{n_{j_1}\ldots n_{j_l}} \left(Y^{lm}_{i_1 i_2\ldots i_l} n_{i_1}\ldots n_{i_l}\right)\\
\Rightarrow & T_{l'm'} = T_{i_1,\ldots i_l}\left(Y^{l'm'}_{j_1,\ldots j_l}\right)^* \int \mathrm{d}\Omega n_{i_1}\ldots n_{i_l} n_{j_1}\ldots n_{j_l}\\
\xRightarrow[m'\to m]{l'\to l} & T_{lm} = 4\pi \frac{l!}{(2l+1)!!} T_{i_1\ldots i_l} (\mathcal{Y}^{lm}_{i_1\ldots i_l})^*
\end{aligned} \tag{437}$$

In the last line, we use the total symmetry of $T_{i_1\ldots i_l}$ and vanishing Kronecker delta, so the only contributions will come from $\delta_{i_1 j_1}\ldots\delta_{i_l j_l}$ and symmetric l' permutations. Again, we can take the modulus squared of equation 436 and we get:

$$\begin{aligned}
&\|n_{i_1}\ldots n_{i_l} T_{i_1,i_2,\ldots,i_l}\|^2 = \sum_{m=-l}^{l} \|T_{lm} Y^{lm}\|^2\\
\Rightarrow & \int \mathrm{d}n_{i_1}\ldots \mathrm{d}n_{i_l}\, n_{j_1}\ldots n_{j_l} T_{i_1,\ldots,i_l} T_{j_1,\ldots,j_l}\\
&= \sum_{m=-l}^{l} \int \mathrm{d}T_{lm} T_{no} (Y^{lm})^* Y^{no} = \sum_{m=-l}^{l} |T_{lm}|^2\\
\Rightarrow & \sum_{m=-l}^{l} |T_{lm}|^2 = \frac{4\pi l!}{(2l+1)!!} T_{i_1 i_2\ldots i_l} T^{i_1\ldots i_l}.
\end{aligned} \tag{438}$$

Consider a rotation by an angle φ around the z-axis, such that $\phi \to \phi+\varphi$. In equation 435, the left-hand side (LHS) is a scalar and, therefore, remains invariant under such rotations. On the other hand, the right-hand side (RHS) transforms according to the spherical harmonics $Y^{lm} \to e^{im\varphi} Y^{lm}$. Consequently, the coefficients T_{lm} must transform as

$$T_{lm} \to e^{-im\varphi} T_{lm}$$

To preserve the equality.

More generally, the $2l+1$ components of T_{lm} (with $m = -l,\ldots,l$ for a given l) transform among themselves in the same way as the conjugate spherical harmonics $Y^{lm}(\theta,\phi)^*$.

5 Applications

5.1 Radiation from a closed system of masses

In this section, we derive the foundational quadrupole formula for gravitational radiation from a closed system of masses. The main reference is again [15].The energy-momentum tensor for a particle moving on an $x_0(t)$ trajectory is:

$$T^{\mu\nu}(t,\vec{x}) = \frac{p^\mu p^\nu}{\gamma m}\delta^{(3)}(\vec{x}-\vec{x}^0(t)) \tag{439}$$

where for a set of particles labelled by A we get:

$$\begin{aligned} T^{\mu\nu}_{\text{tot}}(t,\vec{x}) &= \sum_A \frac{p^\mu_A p^\nu_A}{\gamma m_A}\delta^{(3)}(\vec{x}-\vec{x}_A(t)) \Rightarrow \\ T^{\mu\nu}_{\text{tot}}(t,\vec{x}) &= \sum_A \gamma_A m_A \frac{dx^\mu_A}{dt}\frac{dx^\nu_A}{dt}\delta^{(3)}(\vec{x}-\vec{x}_A(t)) \Rightarrow \\ T^{00}_{\text{tot}}(t,\vec{x}) &= \sum_A \gamma m_A \frac{dx^0_A}{dt}\frac{dx^0_A}{dt}\delta^{(3)}(\vec{x}-\vec{x}_A(t)) \Rightarrow \\ T^{00}(t,\vec{x})_{\text{tot}} &= \sum_A \gamma m_A c^2 \delta^{(3)}(\vec{x}-\vec{x}_A(t)) \end{aligned} \tag{440}$$

Since conservation of $T^{\mu\nu}_{\text{tot}}$ on a flat space-time is a consequence of the invariance under spacetime translation, the effect on multipole moments by shifting the origin of the coordinate system. The second moment of T^{00}/c^2 or abusively speaking the second mass moment is:

$$\begin{aligned} M^{ij}(t) &= \frac{1}{c^2}\int d^3x' T^{00}(t,\vec{x'})x'^i x'^j \\ M^{ij}(t) &= \frac{1}{c^2}\int d^3x' \sum_A \gamma m_a c^2 \delta^{(3)}(\vec{x'}-\vec{x}_A)x'^i x'^j \\ M^{ij}(t) &= \sum_A \gamma m_a \int d^3x x^i x^j \delta^{(3)}(\vec{x}-\vec{x}_A) \\ M^{ij}(t) &= \sum_A \gamma m_a x^i_A(t) x^j_A(t) \end{aligned} \tag{441}$$

Under translation we get $x^i \to x^i + \alpha^i$.

$$\begin{aligned} M^{ij}(t) \to M'^{ij}(t) &= \sum_A \gamma m_A (x^i_A + a^i_A)(x^j_A + a^j_A) \Rightarrow \\ M'^{ij}(t) &= \sum_A \gamma m_A x^i_A x^j_A + \sum_A \gamma m_A (x^i_A + a^i_A)(x^j_A + a^j_A) + \sum_A \gamma m_A a^i_A a^j_A \Rightarrow \\ M'^{ij}(t) &= M^{ij}(t) + a^i \sum_A \gamma m_A x^j_A(t) + a^j \sum_A \gamma m_A x^i_A(t) + a^i a^j \sum_A \gamma m_A \end{aligned} \tag{442}$$

With time derivative:

$$
\begin{aligned}
&M'^{ij}(t) = \dot{M}^{ij}(t) + a^i \sum_A \gamma m_A \dot{x}^j_A(t) + a^j \sum_A \gamma m_A \dot{x}^i_A(t) \Rightarrow \\
&M'^{ij}(t) = \dot{M}^{ij}(t) + a^i P^j_{\text{tot}} + a^i P^i_{\text{tot}}
\end{aligned}
\tag{443}
$$

Where $P^i_{\text{tot}} \equiv \sum_A \gamma m_A \dot{x}^i_A$ total momentum of a non-relativistic system and its constant. The second derivative yields:

$$\ddot{M'}^{ij} = \ddot{M}^{ij} + a^j \dot{P}^i_{\text{tot}} a^i \dot{P}^j_{\text{tot}} \Rightarrow \ddot{M'}^{ij} = \ddot{M}^{ij}(\text{invariant}) \tag{444}$$

Since $h^{TT}_{ij} \sim \ddot{M}^{ij}$, gravitational radiation is not affected by the shift of the origin. All of the above are valid only for closed systems and of no external forces are present. The procedure described above is correct when $\vec{x}_0$ is the relative coordinate of an isolated two-body system in the center of mass frame, with $\vec{x}_0(t)$ the time evolved of $\vec{x}_0$.

Center of mass coordinate is:

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{445}$$

Second mass moment

$$
\begin{aligned}
&M^{ij} = \gamma m_1 x^i_1 x^j_1 + \gamma m_2 x^i_2 x^j_2 \\
&M^{ij} = \gamma m x^i_{\text{CM}} x^j_{\text{CM}} + \gamma \mu (x^i_{\text{CM}} x^j_0 + x^j_{\text{CM}} x^i_0) + \gamma \mu x^i_0 x^j_0
\end{aligned}
\tag{446}
$$

If we choose $x_{\text{CM}} = 0$, we have:

$$M^{ij} = \gamma \mu x^i_0 x^j_0 \tag{447}$$

Eq. 447 shows the effective particle's mass moment, with mass μ, and described by coordinates $x_0(t)$. Since $P^i_{\text{tot}} = \sum_A \gamma m_A \dot{x}^i_A$ is conserved, and $\ddot{M}^{ij}$ is the quadrupole moment, it will be invariant under the $x^i \to a^i + x^i$ shifts. If we describe the system with $\vec{x}_1$ and $\vec{x}_2$, it is consistent with working in the center-of-mass frame, where $\vec{x}_{\text{CM}} = 0$. This description is valid only in the CM frame. The mass density is

$$p(t, \vec{*}) = \mu \delta^{(3)}(\vec{x} - \vec{x}_0(t)) \tag{448}$$

and the second moment

$$M^{ij}(t) = \mu x^i_0(t) x^j_o(t) \tag{449}$$

And the mass-quadrupole:

$$Q^{ij}(t) = \mu \left(x^i_0(t) x^j_o(t) - \frac{1}{3} R^2_0(t) \delta^{ij} \right) \tag{450}$$

We want radiation emitted by a two-body system, whose relative coordinates have harmonic ways of motion (e.g., harmonic oscillation). We suppose that the relative coordinate $x_0(t)$ is periodic along the z direction with frequency ω_s. Then:

$$x^3_0(t) = z_0(t) = a_1 \cos \omega_s t \tag{451}$$

$$
\begin{aligned}
M^{ij}(t) &= \mu a_1 \cos\omega_s t \delta^{i3} a_1 \cos\omega_s t \delta^{j3} \\
M^{ij}(t) &= \mu a_1^2 \cos^2\omega_s t \delta^{i3}\delta^{j3} \\
M^{ij}(t) &= \mu a_1^2 (\cos 2\omega_s t + 1)\frac{1}{2}\delta^{i3}\delta^{j3}
\end{aligned}
\tag{452}
$$

GWs' amplitude depends on $\ddot{M}^{ij}(t)$ so we have:

$$
\begin{aligned}
\ddot{M}^{ij}(t) &= -\mu a_1^2 4\omega_s^2 \cos 2\omega_s t \delta^{i3}\delta^{j3} \Rightarrow \\
\ddot{M}^{ij}(t) &= -4\mu(a_1\omega_s)^2 \delta^{i3}\delta^{j3} \cos 2\omega_s t
\end{aligned}
\tag{453}
$$

We see that the corresponding GW oscillates as $\cos 2\omega_s t$.

A non-relativistic source performing simple harmonic oscillations with a frequency ω_s emits a monochromatic quadrupole radiation at $\omega = 2\omega_s$

Last is true only for simple harmonic motion If the system performs a superposition of periodic motion and higher harmonies, e.g., if:

$$
\begin{aligned}
z_0(t) &= a_1 \cos(\omega_s t) + a_2 \cos(2\omega_s t) + \dots \\
z_0^2(t) &= a_1^2 \cos^2(\omega_s t) + a_2^2 \cos(2\omega_s t) + 2a_1 a_2 \cos(\omega_s t)\cos(2\omega_s t) \\
z_0^2(t) &= \frac{a_1^2}{2}(1+\cos(2\omega_s t)) + a_2^2\frac{1}{2}(1+\cos(4\omega_s t)) + a_1 a_2 \cos 3\omega_s t + a_1 a_2 \cos(\omega_s t)
\end{aligned}
\tag{454}
$$

Radiation

$$
\begin{aligned}
a_1 \text{ emits at} \cos(2\omega_s t) &\quad \omega_{\text{gw}} = 2\omega_s \\
a_2 \text{ emits at} \cos(4\omega_s t) &\quad \omega_{\text{gw}} = 4\omega_s \\
a_1 a_2 \text{ emits at} \cos(\omega_s t) + \cos(3\omega_s t) &\quad \omega_{\text{gw}} = \omega_s, \omega_{\text{gw}} = 3\omega_s
\end{aligned}
$$

There exists every n $n\omega_s$ frequency in quadrupole radiation.

5.2 Application on inspiral of compact binaries

Here we follow again the methodology and idea in Maggiore's book [15], but also take pieces of information from [50], [28], [61], [65], [66], [67] and [68], we derive the foundational quadrupole formula for gravitational radiation from a closed system of compact binaries. This directly connects to the radiated wave amplitude and power.

We analyze a binary system consisting of two compact objects, such as neutron stars or black holes. We consider these objects as point masses, with masses m_1 and m_2, and positions $\mathbf{r}_1$ and $\mathbf{r}_2$. In Newtonian approximation and the center-of-mass (CM) frame, the dynamics reduce to a one-body problem with a reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$, and the equation of motion is given by:

$$
\vec{r} = \left(\frac{Gm}{r^3}\right)\vec{r}
\tag{455}
$$

where $m = m_1 + m_2$ is the total mass, and $\vec{r} = \vec{r}_2 - \vec{r}_1$ is the relative position vector.

We first examine the case of circular orbits. The orbital velocity v is related to the orbital radius R by:

$$v^2 = \frac{Gm}{R}$$

where $v = \omega R$, with ω being the orbital frequency. Using this relation, we derive Kepler's law:

$$\omega_s^2 = \frac{Gm}{R^3} \tag{456}$$

In Eqs. 251 and 252, we eliminate R in terms of ω_s^2, by using 456 and we get:

$$\begin{aligned} h_+ &= \frac{4}{r}\frac{G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\omega_s t + 2\phi\right) \\ h_+ &= \frac{4}{r}\frac{G\mu\omega_s^2}{c^4}\frac{G^{\frac{2}{3}}m^{\frac{2}{3}}}{\omega_s^{\frac{4}{3}}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\omega_s t + 2\phi\right) \end{aligned} \tag{457}$$

$$\begin{aligned} h_\times &= \frac{4}{r}\frac{G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos\theta\sin\left(2\omega_s t + 2\phi\right) \\ h_\times &= \frac{4}{r}\frac{G\mu\omega_s^2}{c^4}\frac{G^{\frac{2}{3}}m^{\frac{2}{3}}}{\omega_s^{\frac{4}{3}}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\theta\sin\left(2\omega_s t + 2\phi\right) \end{aligned} \tag{458}$$

We define the chirp mass M_c as:

$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \mu^{3/5} m^{2/5} \tag{459}$$

And we write Eqs. 819 and 820 as:

$$\begin{aligned} h_+ &= \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\omega_s}{c}\right)^{\frac{2}{3}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\omega_s t + 2\phi\right) \\ h_+ &= \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{\mathrm{GW}}}{c}\right)^{\frac{2}{3}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\pi f_{\mathrm{GW}} t + 2\phi\right) \end{aligned} \tag{460}$$

$$\begin{aligned} h_\times &= \frac{4}{r}\frac{G\mu\omega_s^2}{c^4}\frac{G^{\frac{2}{3}}m^{\frac{2}{3}}}{\omega_s^{\frac{4}{3}}}\cos\theta\sin\left(2\omega_s t + 2\phi\right) \\ h_\times &= \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\omega_s}{c}\right)^{\frac{2}{3}}\cos\theta\sin\left(2\omega_s t + 2\phi\right) \\ h_\times &= \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{\mathrm{GW}}}{c}\right)^{\frac{2}{3}}\cos\theta\sin\left(2\pi f_{\mathrm{GW}} t + 2\phi\right) \end{aligned} \tag{461}$$

In this lowest order of Newtonian approximation, the h_+ and $h_\times$ amplitudes of the GWs emitted depend on the masses m_1 and m_2 through M_c. Now, we can introduce the Schwarzschild radius in terms of the chirp mass:

$$R_c \equiv \frac{2GM_c}{c^2} \tag{462}$$

Then we can write Eqs. 822 and 823 as follows:

$$\begin{aligned}
h_+ &= \frac{4}{r}\left(\frac{R_c}{2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{GW}}{c}\right)^{\frac{2}{3}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\pi f_{GW}t+2\phi\right)\\
\xrightarrow[f_{GW}=\frac{c}{\lambda_{GW}}]{\lambda\!\!\!^{-}_{GW}=\frac{\lambda_{GW}}{2\pi}} h_+ &= \frac{4}{r}\left(\frac{R_c}{2}\right)^{\frac{5}{3}}\left(\frac{1}{2\lambda\!\!\!^{-}}\right)^{\frac{2}{3}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\pi f_{GW}t+2\phi\right)\\
h_+ &= \frac{4}{2^{7/3}}\left(\frac{R_c}{r}\right)\left(\frac{R_c}{\lambda\!\!\!^{-}}\right)^{\frac{2}{3}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\pi f_{GW}t+2\phi\right)\\
h_+ &= \frac{1}{2^{1/3}}\left(\frac{R_c}{r}\right)\left(\frac{R_c}{\lambda\!\!\!^{-}}\right)^{\frac{2}{3}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\pi f_{GW}t+2\phi\right)\\
h_+ &= A\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\pi f_{GW}t+2\phi\right)
\end{aligned} \tag{463}$$

$$\begin{aligned}
h_\times &= \frac{4}{r}\left(\frac{R_c}{2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{GW}}{c}\right)^{\frac{2}{3}}\cos\theta\sin\left(2\pi f_{GW}t+2\phi\right)\\
h_\times &= \frac{4}{r}\left(\frac{R_c}{2}\right)^{\frac{5}{3}}\left(\frac{1}{2\lambda\!\!\!^{-}_{GW}}\right)^{\frac{2}{3}}\cos\theta\sin\left(2\pi f_{GW}t+2\phi\right)\\
h_\times &= \frac{4}{2^{7/3}}\left(\frac{R_c}{r}\right)\left(\frac{R_c}{\lambda\!\!\!^{-}_{GW}}\right)^{\frac{2}{3}}\cos\theta\sin\left(2\pi f_{GW}t+2\phi\right)\\
h_\times &= A\cos\theta\sin\left(2\pi f_{GW}t+2\phi\right)
\end{aligned} \tag{464}$$

where

$$A \equiv \frac{1}{2^{1/3}}\frac{R_c}{r}\left(\frac{R_c}{\lambda\!\!\!^{-}}\right)^{2/3} \tag{465}$$

Next, we compute the quadrupole radiated power. By defining the angular dependence as

$$g(\theta) = \left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right],$$

The power can be expressed as:

$$
\begin{aligned}
\left.\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right|_{\text{quad}} &= \frac{2G\mu^2R^4\omega_s^6}{\pi c^5}g(\theta) \\
\Rightarrow \left.\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right|_{\text{quad}} &= \frac{2}{\pi}\frac{G\mu^2}{c^5}\frac{G^{4/3}m^{4/3}}{(\omega_s^2)^{4/3}}\omega_s^6 g(\theta) \\
\Rightarrow \left.\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right|_{\text{quad}} &= \frac{2}{\pi}\left(\frac{G^{7/3}\mu^2m^{4/3}}{c^5}\omega_s^{10/3}\right)g(\theta) \\
\Rightarrow \left.\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right|_{\text{quad}} &= \frac{2c^5}{\pi G}\left(\frac{G^{10/3}\mu^{6/3}m^{4/3}}{c^{10}}\omega_s^{10/3}\right)g(\theta) \\
\Rightarrow \left.\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right|_{\text{quad}} &= \frac{2c^5}{\pi G}\left(\frac{GM_c\omega_{\text{gw}}}{2c^3}\right)^{10/3} g(\theta)
\end{aligned}
\tag{466}
$$

The integral $\int \mathrm{d}\Omega g(\theta)$ is calculated as:

$$
\begin{aligned}
\int \mathrm{d}\Omega\, g(\theta) &= \int \mathrm{d}\Omega \left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right] \\
\int \mathrm{d}\Omega\, g(\theta) &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin\theta\cos^2\theta + \frac{1}{4}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin\theta(1+2\cos^2\theta+\cos^4\theta) \\
\frac{1}{2\pi}\int \mathrm{d}\Omega\, g(\theta) &= -\left.\frac{\cos^3\theta}{3}\right|_0^{\pi} - \frac{1}{4}\int_0^{\pi}\mathrm{d}(\cos\theta) - \frac{1}{2}\int_0^{\pi}\mathrm{d}(\cos\theta)\cos^2\theta - \frac{1}{4}\int_0^{\pi}\mathrm{d}(\cos\theta)\cos^4\theta \\
\frac{1}{2\pi}\int \mathrm{d}\Omega\, g(\theta) &= 1+\frac{1}{2}+\frac{1}{10} = 1+\frac{1}{2}(1+\frac{1}{5}) = 1+\frac{3}{5} = \frac{8}{5} \\
\int \frac{\mathrm{d}\Omega}{4\pi} &= \frac{4}{5}
\end{aligned}
\tag{467}
$$

We get the total radiated power $P\big|_{\text{quad}}$ by integrating over all solid angles $\mathrm{d}\Omega$, equation 466:

$$
\begin{aligned}
P_{\text{quad}} &= \int \mathrm{d}\Omega \left.\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right|_{\text{quad}} = \frac{2c^5}{\pi G}\left(\frac{GM_c\omega_{\text{gw}}}{2c^3}\right)^{10/3}\int \mathrm{d}\Omega g(\theta) \\
P_{\text{quad}} &= \frac{32c^5}{5G}\left(\frac{GM_c\omega_{\text{gw}}}{2c^3}\right)^{10/3}
\end{aligned}
\tag{468}
$$

5.3 Application on elliptic orbits

Further information for Kepplerian orbits can be found in any astrophysical textbook. GW production and emission in these orbits is anaytically derived in [50] and [61].

5.3.1 Total power and frequency spectrum of the emitted radiation

We now analyze the gravitational radiation emitted by two masses moving in an elliptical Keplerian orbit. Let m_1 and m_2 represent the masses of the stars, with their total mass given by $m = m_1 + m_2$ and their reduced mass denoted as μ. To proceed, we revisit the classical mechanics solution for the equation of motion, which describes an elliptical orbit, and subsequently evaluate the gravitational wave power output and its spectral properties.

The general solution to this motion equation relies on the existence of two conserved quantities: the angular momentum L and the total energy E. Conservation of L dictates that the motion is confined to a single plane. Using this insight, we employ polar coordinates (r, ψ) in the orbital plane, where r represents the radial distance and ψ is the angular position along the orbit. Additionally, the angular variable ϕ is introduced to describe the directional dependence of the emitted radiation.

For the equation of motion $\ddot{\vec{r}} = -\frac{Gm}{r^2}\hat{r}$, we take the solution for elliptic orbits. The general solution is obtained by the integral of motions, where the angular momentum $\hat{L}$, with

$$L = \mu r^2 \dot{\psi} \tag{469}$$

while the energy is given by:

$$\begin{aligned} E &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\psi}^2) - \frac{Gm\mu}{r} \\ E &= \frac{1}{2}\mu\dot{r}^2 + \frac{L}{2\mu r^2} - \frac{Gm\mu}{r} \end{aligned} \tag{470}$$

Solving equation 470 we take

$$\left(\frac{\mathrm{d}\dot{r}}{\mathrm{d}t}\right) = \sqrt{\frac{2E}{\mu} - \frac{L}{2\mu r^2} + \frac{G\mu m}{r}} \tag{471}$$

From the energy equation, we get $\dot{r}$ as a function of r. Using the angular momentum equation, we find $\dot{\psi}$ as a function of r. By integrating these expressions, the equation of the orbit can be expressed as:

$$\begin{aligned} &\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{L}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\psi} = \left(\frac{2E}{\mu} - \frac{L}{2\mu r^2} + \frac{G\mu m}{r}\right)^{1/2} \\ \Rightarrow &\frac{\mathrm{d}r}{\mathrm{d}\psi} = \left(\frac{2m^2r^4E}{\mu L} - \frac{m^2r^2}{2\mu L} + \frac{G\mu m^3 r}{L^2}\right)^{1/2} \\ \Rightarrow &\frac{1}{r} = \frac{1}{R}(1 + e\cos\psi) \end{aligned} \tag{472}$$

where R (the length scale) and e (the eccentricity) are constants of motion. They are related to the energy E of the system ($E < 0$ for a bound orbit) and the angular momentum L by:

$$R = \frac{L^2}{Gm\mu^2} \tag{473}$$

and

$$e^2 = 1 + \frac{2EL^2}{G^2m^2\mu^3} \tag{474}$$

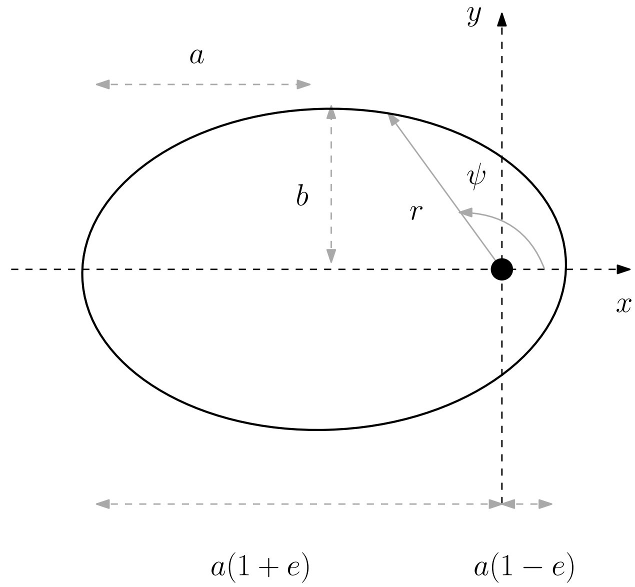

Figure 3: The definitions used for an elliptic orbit: the polar coordinates (r, ψ), as well as the Cartesian coordinates (x, y), are centered on a focus of the ellipse (dark blob). The angle ψ is measured counterclockwise from the x-axis. The semiaxes a and b are indicated. The focus splits the major axis into two segments of length $a(1 + e)$ and $a(1 - e)$, respectively, adapted by Maggiore's book, Gravitational Waves [15].

The eccentricity e of an ellipse satisfies $0 \leq e < 1$. When $e = 0$, the ellipse becomes a perfect circle, while as e approaches 1, the ellipse transitions into a parabola. The semi-axes of the ellipse are

$$a = \frac{R}{1 - e^2} \tag{475}$$

$$b = \frac{R}{\sqrt{1 - e^2}} \tag{476}$$

Inserting equation 474 in equation (429)

$$a = \frac{R}{1 - e^2} = \frac{RG^2m^2\mu^3}{2|E|L^2} = \frac{Gm\mu}{2|E|} \tag{477}$$

Similarly, we can insert equation (429) in 472 and we get:

$$\begin{aligned} \frac{1}{r} &= \frac{1}{R}(1 + e\cos\psi) \\ \frac{1}{r} &= \frac{1}{a(1 - e^2)}(1 + e\cos\psi) \\ r &= \frac{a(1 - e^2)}{1 + e\cos\psi} \end{aligned} \tag{478}$$

Combining the angular momentum and energy equations, $r(t)$ and $\dot{\psi}(t)$ satisfy:

$$\dot{\psi} = \frac{(GmR)^{1/2}}{r^2} \tag{479}$$

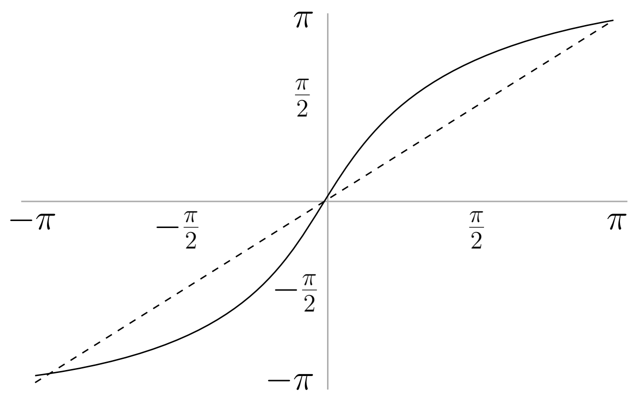

Figure 4: The function $\psi(u)$ for $e = 0.2$ (dashed line) and for $e = 0.75$ (solid line), adapted by Maggiore's book, Gravitational Waves [15].

The explicit time dependence of $r(t)$ and $\dot{\psi}(t)$ is obtained by integrating these equations. Using parametric form:

$$r = a(1 - e\cos u), \tag{480}$$

where u is the eccentric anomaly, related to t by Kepler's equation:

$$\beta = u - e\sin u = \omega_0 t, \tag{481}$$

with $\omega_0^2 = \frac{Gm}{a^3}$.

Using trigonometric identities, $\cos\psi$ can be rewritten as:

$$\begin{aligned}
\cos\psi &= \frac{\cos u - e}{1 - e\cos u} \\
\tan\frac{\psi}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2} \\
\frac{\psi}{2} &= \arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right) \\
\psi &= 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right)
\end{aligned} \tag{482}$$

where we can set $\psi \equiv A_e(u)$, the true anomaly. In the equation 481 we can set $t \to t + \frac{2\pi}{\omega_0}$ and get:

$$\beta = \omega_0 t + 2\pi = \beta + 2\pi \tag{483}$$

$$\begin{aligned}
&\Rightarrow u - e\sin u = u - e\sin u + 2\pi \\
&\Rightarrow u = u + 2\pi
\end{aligned} \tag{484}$$

So the coordinates r and ψ are periodic functions of t, with period $T = \frac{2\pi}{\omega_0}$. As u rubs between $[-\pi, \pi]$, also does ψ. Graphically, we get:

In Cartesian coordinates (x, y) with the center at the focus of the ellipse, we get:

$$x = r\cos\psi = a[\cos u(t) - e] \tag{485}$$

$$y = r\sin\psi = b\sin u(t) \tag{486}$$

For a 2-body problem, the focus of the ellipse coincides with $x_{\rm CM} = 0$.

In order to compute the radiated power of this system, we transfer in the CM frame, the second mass moment is given as $M^{ij} = \mu x_0^i(t) x_0^j(t)$ in the coordinates Eqs. (441) and (442) we have:

$$M^{11} = \mu x^1(t) x^1(t) = \mu r^2 \cos^2\psi \tag{487}$$
$$M^{12} = \mu x^1 x^2 = \mu r^2 \cos\psi \sin\psi \tag{488}$$
$$M^{21} = \mu x^2 x^1 = \mu r^2 \cos\psi \sin\psi \tag{489}$$
$$M^{22} = \mu x^2 x^2 = \mu r^2 \sin^2\psi \tag{490}$$

Since r and ψ are time-dependent we eliminate $r(t)$ in M^{ij} with equation 478

$$M^{11} = \mu \frac{a^2(1-e^2)^2}{(1+e\cos\psi)^2} \cos^2\psi \tag{491}$$
$$M^{12} = M^{21} = \mu \frac{a^2(1-e^2)^2}{(1+e\cos\psi)^2} \cos\psi \sin\psi \tag{492}$$
$$M^{22} = \mu \frac{a^2(1-e^2)^2}{(1+e\cos\psi)^2} \sin^2\psi \tag{493}$$

or in matrix form

$$M_{ab} = \mu \frac{a^2(1-e^2)^2}{(1+e\cos\psi)^2} \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix} \tag{494}$$

Before attacking the time derivatives, we get:

$$\begin{aligned} \dot\psi &= \frac{(GmR)^{1/2}}{r^2} = (Gm)^{1/2} a^{1/2} (1-e^2)^{1/2} \frac{(1+e\cos\psi)^2}{a^2(1-e^2)^2} \\ \dot\psi &= \sqrt{Gm} \frac{(1+e\cos\psi)^2}{[a(1-e^2)]^{3/2}} \\ \dot\psi &= \left(\frac{Gm}{a^3}\right)^{1/2} \frac{(1+e\cos\psi)^2}{(1-e^2)^{3/2}} \end{aligned} \tag{495}$$

Setting

$$\beta \equiv \frac{4G^3\mu^2 m^3}{a^5(1-e^2)^5} \tag{496}$$

We get:

$$\dddot{M}_{11} = \beta(1+e\cos\psi)^2 (2\sin 2\psi + 3e\sin\psi\cos^2\psi) \tag{497}$$
$$\dddot{M}_{22} = \beta(1+e\cos\psi)^2 [-2\sin 2\psi - e\sin\psi(1+3\cos^2\psi)] \tag{498}$$
$$\ddot{M}_{12} = \dddot{M}_{21} = \beta(1+e\cos\psi)^2 [-2\cos 2\psi + e\cos\psi(1-3\cos^2\psi)] \tag{499}$$

The radiated power in the quadrupole approximation is

$$
\begin{aligned}
P(\psi) &= \frac{G}{5c^5}\left[\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 2\dddot{M}_{12}^2 - \frac{1}{3}\left(\dddot{M}_{11} + \dddot{M}_{22}\right)^2\right] \\
&= \frac{2G}{15c^5}\left[\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 3\dddot{M}_{12}^2 - \dddot{M}_{11}\dddot{M}_{22}\right] \\
&= \frac{8G^4}{15c^5}\frac{\mu^2 m^3}{a^5(1-e^2)^5}(1+e\cos\psi)^4\left[12(1+e\cos\psi)^2 + e^2\sin^2\psi\right].
\end{aligned} \tag{500}
$$

In Chapter 1, we explained that the energy of GWs becomes well-defined only when averaged over several wave periods. As we will show, a particle in a Keplerian elliptical orbit generates GWs with frequencies that are integer multiples of the frequency ω_0, as defined in equation (4.59). This means that the GW period is a fraction of the orbital period T. Therefore, the appropriate well-defined quantity is the average of $P(t)$ over one orbital period T. We will now proceed to compute this time average.

$$
\begin{aligned}
P &\equiv \frac{1}{T}\int_0^T P(\psi)\,dt \\
&= \frac{\omega_0}{2\pi}\int_0^{2\pi}\frac{d\psi}{\dot{\psi}}P(\psi) \\
&= (1-e^2)^{3/2}\int_0^{2\pi}\frac{d\psi}{2\pi}(1+e\cos\psi)^{-2}P(\psi) \\
&= \frac{8G^4\mu^2 m^3}{15c^5 a^5}(1-e^2)^{-7/2} \\
&\quad\times\int_0^{2\pi}\frac{d\psi}{2\pi}\left[12(1+e\cos\psi)^4 + e^2(1+e\cos\psi)^2\sin^2\psi\right]
\end{aligned} \tag{501}
$$

The total radiated power is:

$$
P = \frac{32G^4\mu^2 m^3}{5c^5 a^5} f(e) \tag{502}
$$

with

$$
f(e) = \frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \tag{503}
$$

In the special case where $e = 0 \Rightarrow f(e=0) = 1$ and we take

$$
P = \frac{32}{5}\frac{G\mu^2}{c^5}a^4\omega_0^6 \tag{504}
$$

When $a \to R \& \omega_o \to \omega_s$. So we get

$$
P = \frac{32}{5}\frac{G\mu^2}{c^5}R^4\omega_s^5 \ \text{ circular orbit} \tag{505}
$$

We can rewrite the orbital period T as:

$$\begin{aligned} T &= \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{a^3}{Gm}} = \frac{2\pi}{\sqrt{Gm}}\left(\frac{Gm\mu}{2|E|}\right)^{3/2} \\ T &= 2\pi\left(\frac{\mu}{2}\right)^{2/3}(Gm)E^{-3/2} \end{aligned} \tag{506}$$

We notice that the orbital period T is related to the orbital energy E, and thus we take

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &= \#\frac{\mathrm{d}|E|^{-3/2}}{\mathrm{d}t} = -\frac{3}{2}\#E^{-5/2}\dot{E} \\ \dot{T} &= -\frac{3}{2}TE^{-1}\dot{E} \\ \frac{\dot{T}}{T} &= -\frac{3}{2}\frac{\dot{E}}{E} \\ \frac{\dot{T}}{T} &= -\frac{96G^3}{5c^5}\frac{\mu m^2}{a^4}f(e) \\ \frac{\dot{T}}{T} &= -\frac{96}{5c^5}G^{5/3}m^{2/3}\mu\left(\frac{T}{2\pi}\right)^{-8/3}f(e) \end{aligned} \tag{507}$$

In equation 502, when $e \to t^-$ and a is fixed, the radiated power diverges. This means that the point-like mass approximation ceases to be valid. When $e \to 1^-$ we get equation 472 as a parabole:

$$r = \frac{R}{1+\cos\psi} \tag{508}$$

Since $\psi \in [-\pi, \pi]$ we get:

$$\psi = -\pi \Rightarrow r \to \infty \tag{509}$$

$$\psi = 0 \Rightarrow r = \frac{R}{2} \tag{510}$$

$$\psi = \pi \Rightarrow r \to \infty \tag{511}$$

For $e \to 1^-$ we get equation 500 to read:

$$\begin{aligned} P(\psi) &= \frac{8G^4\mu^2m^3}{15c^5a^5(1-e^2)^5}(1+\cos\psi)^4[12(1+\cos\psi)^2+\sin^2\psi] \\ P(\psi) &= \frac{8G^4\mu^2m^3}{15c^5R^5}[12(1+\cos\psi)^6+(1+\cos\psi)^4\sin^2\psi] \end{aligned} \tag{512}$$

Equation 512 produces the radiation emitted along the trajectory in terms of r reads:

$$\begin{aligned} P(\psi) &= \frac{8G^4\mu^2m^3}{15c^5}\frac{R}{r^4}\left[\frac{12(1+\cos\psi)^2+2-\cos^2\psi-1-2\cos\psi+2\cos\psi}{R^2}\right] \\ P(\psi) &= \frac{8G^4\mu^2m^3}{15c^5}\left(\frac{12R}{r^6}+\frac{R}{r^4}\left[\frac{-(1+\cos\psi)^2}{R^2}\right]+\frac{2R}{r^4}\frac{\cos\psi+1}{R^2}\right) \\ P(\psi) &= \frac{8G^4\mu^2m^3}{15c^5}\left(\frac{11R}{r^6}+\frac{2}{r^5}\right) \\ P(\psi) &= \frac{16G^4\mu^2m^3}{15c^5}\frac{1}{r^5}\left(1+\frac{11R}{2r}\right) \end{aligned} \tag{513}$$

Finally, the total energy radiated in GWs is finite

$$
\begin{aligned}
E_{\rm rad} &= \int_{-\infty}^{+\infty} \mathrm{d}t P(\psi(t)) = \int_{-\pi}^{\pi} \mathrm{d}\psi \frac{\mathrm{d}t}{\mathrm{d}\psi} P(\psi) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\psi}{\dot{\psi}} P(\psi) \\
E_{\rm rad} &= \frac{85 \cdot 2G\mu^2 \pi}{3c^5 R} \left(\frac{Gm}{R}\right)^{5/2} = \frac{85 G\mu^2 \pi}{48R} \left[\frac{2}{c}\left(\frac{Gm}{R}\right)^{1/2}\right]^5 \\
E_{\rm rad} &== \frac{85\pi}{48} \frac{G\mu^2}{R} \left(\frac{v_0}{c}\right)^5
\end{aligned}
\tag{514}
$$

where $v_0^2 \equiv 4\frac{Gm}{R}$.

v_0 is the velocity at $\psi = 0 \iff r = \frac{R}{2}$, it defines the maximum velocity attained along the trajectory.

Next, we want to compute the frequency spectrum of the radiated power for a Keplerian elliptic orbit. The trajectory as a function of time is not a harmonic motion when described by Eqs. (441), (442) 483. The first to compute $\frac{\mathrm{d}P}{\mathrm{d}\Omega}$ is to Fourier transform the trajectory. Observing that $x(t)\&y(t)$ are periodic functions of β with period 2π, we can restrict β to $-\pi \leq \beta \leq \pi$ and perform a discrete Fourier transform as:

$$x(\beta) = \sum_{n=-\infty}^{\infty} \tilde{x}_n e^{-in\beta}, \tag{515}$$

$$y(\beta) = \sum_{n=-\infty}^{\infty} \tilde{y}_n e^{-in\beta}. \tag{516}$$

with $\tilde{x^*_{-n}} = \tilde{x}$ and $\tilde{y^*_{-n}} = \tilde{y}$ We can choose the origin of time so that at $t = 0$ or equivalently at $\beta = 0$, to be $x = a(1-e)$ and $y = 0$. With this choice, we get

$$
\begin{aligned}
& x(\beta) = x(-\beta) \Rightarrow \\
& \sum_{n=0}^{\infty} \tilde{x_n} e^{e^{-in\beta}} = \sum_{n=0}^{\infty} \tilde{x_n} e^{e^{in\beta}} \Rightarrow \\
& \cos n\beta - i \sin n\beta = \cos n\beta + i \sin n\beta \Rightarrow \\
& \sin n\beta = 0
\end{aligned}
\tag{517}
$$

and

$$
\begin{aligned}
& y(-\beta) = -y(\beta) \Rightarrow \\
& \sum_{n=0}^{\infty} \tilde{y_n} e^{e^{-in\beta}} = -\sum_{n=0}^{\infty} \tilde{y_n} e^{e^{in\beta}} \Rightarrow \\
& \cos n\beta - i \sin n\beta = -\cos n\beta - i \sin n\beta \Rightarrow \\
& \cos n\beta = 0
\end{aligned}
\tag{518}
$$

The exponential $e^{-in\beta}$ is written as $e^{-in\beta} = \cos n\beta - i \sin n\beta$. Therefore, the expansion of $x(\beta)$, contributes only $\cos(n\beta)$ while the expansion of $y(\beta)$, contributes only $\sin(n\beta)$, we can simplify the Eqs. (471) and (472)

$$
\begin{aligned}
x(\beta) &= \sum_{n=-\infty}^{\infty} \tilde{x_n} \cos n\beta \\
x(\beta) &= \sum_{n=-\infty}^{\infty} 2\tilde{x_n} \cos n\beta \qquad (519) \\
x(\beta) &= \sum_{n=0}^{\infty} a_n \cos(n\beta)
\end{aligned}
$$

$$
\begin{aligned}
y(\beta) &= \sum_{n=-\infty}^{\infty} -i\tilde{y_n} \sin n\beta \\
y(\beta) &= \sum_{n=-\infty}^{\infty} -2i\tilde{y_n} \sin n\beta \qquad (520) \\
y(\beta) &= \sum_{n=1}^{\infty} b_n \sin(n\beta)
\end{aligned}
$$

where, for $n \geq 1$, $a_n = 2\tilde{x}_n$ and $b_n = -2i\tilde{y}_n$, while $a_0 = \tilde{x}_0$. Since $\beta \equiv \omega_0 t$ and $n\beta = n\omega_0 t = \omega_n t$, we get:

$$x(t) = \sum_{n=0}^{\infty} a_n \cos \omega_n t \qquad (521)$$

$$y(t) = \sum_{n=1}^{\infty} b_n \sin \omega_n t \qquad (522)$$

where

$$\omega_n = n\omega_0 \qquad (523)$$

The coefficients a_n and b_n are obtained by inverting Eqs. (477) and (478), which gives, for $n \neq 0$,

$$
\begin{aligned}
x(\beta) &= \sum_{n=0}^{\infty} a_n \cos(n\beta) \\
\int_0^{\pi} \mathrm{d}\beta \cos(n\beta) x(\beta) &= \sum_{n=0}^{\infty} a_n \int_0^{\pi} \mathrm{d}\beta \cos^2(n\beta) \\
&= \sum_{n=0}^{\infty} a_n \int_0^{\pi} \mathrm{d}\beta \frac{1}{2}(1 + \cos(2\beta)) \qquad (524) \\
&= \sum_{n=0}^{\infty} \frac{a_n}{2}[\pi] \Rightarrow \\
a_n &= \frac{2}{\pi} \int_0^{\pi} d\beta x(\beta) \cos(n\beta)
\end{aligned}
$$

$$y(\beta) = \sum_{n=1}^{\infty} b_n \sin(n\beta)$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} d\beta y(\beta)\sin(n\beta) \tag{525}$$

while, for $n = 0 : b_n = 0$,

$$x(\beta) = a_0 \cos 0 \Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} d\beta x(\beta) \tag{526}$$

The integrals above are solved in terms of Bessel's functions :

$$a_n = -J_{n-1}(ne) - J_{n+1}(ne), \tag{4.94}$$
$$b_n = \frac{1}{ne}\big[J_{n-1}(ne) + J_{n+1}(ne)\big]. \tag{4.95}$$

For $n \neq 0$, and $a_0 = -(3/2)ae$.

To compute the second mass moment we need $x^2(t), y^2(t) \& x(t)y(t)$. The inverse Direct Fourier Transform for these is computed to be

$$x^2(t) = \sum_{n=0}^{\infty} A_n \cos\omega_n t \tag{527}$$
$$y^2(t) = \sum_{n=0}^{\infty} B_n \cos\omega_n t \tag{528}$$
$$x(t)y(t) = \sum_{n=1}^{\infty} C_n \sin\omega_n t \tag{529}$$

where

$$A_n = \frac{a^2}{n}\left[J_{n-2}(ne) - J_{n+2}(ne) - 2e\big(J_{n-1}(ne) - J_{n+1}(ne)\big)\right] \tag{530}$$
$$B_n = \frac{b^2}{n}\left[J_{n+2}(ne) - J_{n-2}(ne)\right] \tag{531}$$
$$C_n = \frac{ab}{n}\left[J_{n+2}(ne) + J_{n-2}(ne) - e\big(J_{n-1}(ne) + J_{n+1}(ne)\big)\right] \tag{532}$$

Therefore, the second moment reads:

$$M_{ab}(t) = \sum_{n=0}^{\infty}\begin{pmatrix} A_n\cos\omega_n t & C_n\sin\omega_n t \\ C_n\sin\omega_n t & B_n\cos\omega_n t\end{pmatrix} \tag{533}$$
$$= \sum_{n=0} M_{ab}^{(n)}(t) \tag{534}$$

where $M_{ab}^{(n)}(t)$ represents the n-th harmonic term. Furthermore,

$$\dddot{M}_{ab}^{(n)}(t) = \mu\omega_n^3 \begin{pmatrix} A_n\sin\omega_n t & -C_n\cos\omega_n t \\ -C_n\cos\omega_n t & B_n\sin\omega_n t\end{pmatrix}_{ab} \tag{535}$$

To compute P_n, we use the quadrupole formula, written in the form:

$$P = \frac{2G}{15c^5}\langle \dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 3\dddot{M}_{12}^2 - \dddot{M}_{11}\dddot{M}_{22}\rangle \tag{536}$$

$$\begin{aligned}
P_n &= \frac{2G}{15c^5}\mu^2\omega_n^6\langle A_n^2 \sin^2\omega_n t + B_n^2 \sin^2\omega_n t + 3C_n^2\cos^2\omega_n t - A_n B_n \sin^2\omega_n t\rangle \\
P_n &= \frac{G\mu^2\omega_0^6 n^6}{15c^5}\langle A_n^2 + B_n^2 + 3C_n^2 - A_n B_n\rangle
\end{aligned} \tag{537}$$

where $\langle\sin^2\omega_n t\rangle = \langle\cos^2\omega_n t\rangle = \frac{1}{2}$

inserting $\omega_0^2 = \frac{Gm}{a^3}$ and writing $\frac{1}{15} = \frac{32}{5\cdot 96}$ we get:

$$\begin{aligned}
P_n &= \frac{32G^4\mu^2 m^3}{5c^5a^5}\frac{n^6}{96a^4}\langle A_n^2 + B_n^2 + 3C_n^2 - A_n B_n\rangle \\
P_n &= \frac{32G^4\mu^2 m^3}{5c^5a^5} g(n,e)
\end{aligned} \tag{538}$$

with $g(n,e) \equiv \frac{n^6}{96a^4}\langle A_n^2 + B_n^2 + 3C_n^2 - A_n B_n\rangle$ In equation 538 we see that the coefficients A_n, B_n and C_n are functions of the eccentricity and total power $P \equiv \sum_{n=0}^{\infty} P_n$.

5.3.2 Evolution of the orbit under back-reaction

A binary system following a Keplerian orbit radiates both energy and angular momentum. Assuming the bodies are point-like and lack intrinsic spin, this radiation drains energy and angular momentum from the orbital dynamics. As a result, the orbit undergoes progressive modifications, notably in its semi-major axis and eccentricity, until the system eventually transitions into the merging phase and collapses. This section explores the evolution of the orbital parameters, specifically the size and shape, for a general elliptical orbit. The energy radiated in the quadrupole approximation is given as:

$$P = \frac{32G^4\mu^2 m^3}{5c^5a^5} f(e)$$

where

$$f(e) = (1-e^2)^{-7/2}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \tag{539}$$

The angular momentum radiated in quadrupole approximation reads as:

$$\frac{dL^i}{dt} = -\frac{2G}{5c^5}\epsilon^{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle \tag{540}$$

since $\epsilon^{ikl}\dddot{Q}_{ka} = \epsilon^{ikl}\delta_{ka}\dddot{Q}_{la} = 0$. We change $Q \to M$ and without loss of generality we get:

$$\frac{dL^i}{dt} = -\frac{2G}{5c^5}\epsilon^{ikl}\langle \ddot{M}_{ka}\dddot{M}_{la}\rangle \tag{541}$$

As in the computation of radiated energy, we put the orbit in the (x, y) plane, and M_{ab} is given by Eqs. (443)-(446) or in matrix form

$$M_{ab} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}_{ab} \tag{542}$$

Inside the bracket, we can integrate by parts equation 542 and get:

$$
\begin{aligned}
\frac{dL^i}{dt} &= -\frac{2G}{5c^5}\epsilon^{ikl}\langle\frac{d}{dt}(\ddot{M}_{ka}\dddot{M}_{la}) - \ddot{M}_{ka}M^{(4)}_{la}\rangle \Rightarrow \\
\frac{dL^i}{dt} &= -\frac{2G}{5c^5}\epsilon^{ikl}\langle\ddot{M}_{ka}\dddot{M}_{la}\rangle \Rightarrow^{\text{(x,y)-plane}}_{i=z,\ L_z\equiv L} \\
\frac{dL}{dt} &= -\frac{2G}{5c^5}\epsilon^{3kl}\left\langle\ddot{M}_{ka}\dddot{M}_{la}\right\rangle \Rightarrow \\
\frac{dL}{dt} &= -\frac{2G}{5c^5}\langle\ddot{M}_{1a}\dddot{M}_{2a} - \ddot{M}_{2a}\dddot{M}_{1a}\rangle \Rightarrow \\
\frac{dL}{dt} &= -\frac{2G}{5c^5}\langle\ddot{M}_{12}\dddot{M}_{22} + \ddot{M}_{11}\dddot{M}_{12} - \ddot{M}_{12}\dddot{M}_{11} - \ddot{M}_{22}\dddot{M}_{12}\rangle \\
\frac{dL}{dt} &= -\frac{4G}{5c^5}\langle\ddot{M}_{12}(\dddot{M}_{11} - \dddot{M}_{22})\rangle
\end{aligned}
\tag{543}
$$

The derivative $\ddot{M}_{12}$ reads:

$$
\begin{aligned}
M_{12} &= \mu\frac{a^2(1-e^2)^2}{(1+e\cos\psi)^2}\cos\psi\sin\psi \\
\dot{M}_{12} &= \mu a^2(1-e^2)^2\left[\frac{-\sin^2\psi\dot{\psi} + \cos^2\psi\dot{\psi}}{(1+e\cos\psi)^2} + \frac{2(1+e\cos\psi)e\sin\psi}{(1+e\cos\psi)^4}\dot{\psi}\cos\psi\sin\psi\right] \\
\dot{M}_{12} &= \mu[Gma(1-e^2)]^{1/2}\left(\frac{\cos^2\psi - \sin^2\psi + e\cos^3\psi - e\sin^2\psi\cos\psi + 2e\sin^2\psi\cos\psi}{1+e\cos\psi}\right) \\
\dot{M}_{12} &= \mu[Gma(1-e^2)]^{1/2}\left(\frac{\cos^2\psi + \cos^2\psi - 1 + e\cos^3\psi - e\cos^3\psi + e\cos\psi}{1+e\cos\psi}\right) \\
\dot{M}_{12} &= \mu[Gma(1-e^2)]^{1/2}\left(\frac{2\cos^2\psi + e\cos\psi - 1}{1+e\cos\psi}\right) \\
\ddot{M}_{12} &= \mu[Gma(1-e^2)]^{1/2}\left[\frac{-4\cos\psi\sin\psi\dot{\psi} - e\sin\psi\dot{\psi}}{1+e\cos\psi} + \frac{(2\cos^2\psi + e\cos\psi - 1)(e\sin\psi\dot{\psi})}{(1+e\cos\psi)^2}\right] \\
\ddot{M}_{12} &= \frac{G\mu\sin\psi}{a(1-e^2)}(-4\cos\psi - 4e^2\cos\psi - 8e\cos^2\psi + 6e\cos^2\psi - 2e + 4e^2\cos^2\psi) \\
\ddot{M}_{12} &= \frac{G\mu\sin\psi}{a(1-e^2)}(-4(1+e\cos\psi)^2\cos\psi + 2e(3\cos^2\psi - 1 + 2e\cos^3\psi))
\end{aligned}
\tag{544}
$$

For periodic nations, the average of the GW is the average over one orbital period T and we get:

$$
\int_0^T dt(\ldots) = \int_0^{2\pi}\frac{dt}{d\psi}d\psi\frac{\omega_0}{2\pi} = \int_0^{2\pi}\frac{d\psi}{2\pi}\frac{(1-e^2)^{3/2}}{(1+e\cos^2\psi)}(\ldots) \tag{545}
$$

Then $\frac{dL}{dt}$ reads:

$$
\frac{dL}{dt} = \frac{4G}{5c^5}\int_0^{2\pi}\frac{dt}{2\pi}\frac{(1-e^2)^{3/2}}{(1+e\cos\psi)^2}[\ddot{M}_{12}(\dddot{M}_{11} - \dddot{M}_{22})] \Rightarrow
$$

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{4G}{5c^5}(1-e^2)^{3/2}\frac{Gm\mu}{a(1-e^2)}4\mu\left(\frac{G^3m^3}{a^5(1-e)^5}\right)^{1/2}\times \\
&\int_0^{2\pi}\frac{d\psi\sin\psi}{(1+e\cos\psi)^2}[-4(1+e\cos\psi)^2\cos\psi+2e(3\cos^3\psi-1+2e\cos^3\psi)]\times \\
&[(1+e\cos\psi)^4(4\sin 2\psi+6e\sin\psi\cos^2\psi+e\sin\psi)] \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{16}{5c^5}G^{7/2}\frac{(1-e^2)^{-2}}{a^{7/2}}m^{5/2}\mu^2\times \\
&\int_0^{2\pi}d\psi\left[-4\cos\psi+\frac{2e(3\cos^2\psi-1+2e\cos^3\psi)}{(1+e\cos\psi)^2}\right]\times\sin\psi\times \\
&[(1+e\cos\psi)^2(8\cos\psi+6e\cos^2\psi+e)]
\end{aligned}
\tag{546}
$$

$$
\begin{aligned}
I =& \int_0^{2\pi}d\psi\Bigg[-4\cos\psi+4\cos^3\psi+\frac{2e(3\cos^2\psi-1+2e\cos^3\psi)}{(1+e\cos\psi)^2} \\
&-\frac{2e(3\cos^4\psi-\cos^2\psi+2e\cos^5\psi)}{(1+e\cos\psi)^2}\Bigg]\times(1+e\cos\psi)^2(8\cos\psi+6e\cos^2\psi+e) \\
=& \int_0^{2\pi}\Big[-4\cos\psi(1-\cos^2\psi)(1+e\cos\psi)^2(8\cos\psi+6e\cos^2\psi+e) \\
&+2e(2e\cos^3\psi+3\cos^2\psi-1-3\cos^4\psi+\cos^2\psi-\cos^5\psi)\Big] \\
=& \int_0^{2\pi}d\psi[(-4\cos\psi-4e^2\cos^3\psi-8e\cos^2\psi+4\cos^3\psi+4e^2\cos^5\psi+8e\cos^4\psi)\times \\
&(8\cos\psi+6e\cos^2\psi+e)+2e(4\cos^2\psi-3\cos^4\psi)] \\
I =& \int_0^{2\pi}d\psi[\cos^2\psi(-32-8e^2+8e)+\cos^4\psi(-72e^2+32-6e)+\cos^6\psi(80e^2)] \\
I =& -8\pi(e^2-e+4)-\frac{3\pi}{2}(36e^2-3e+16)+50e^2\pi \\
I =& -8\pi e^2-54\pi e^2+50\pi e^2+\pi e+\frac{9\pi e}{2}-56\pi \\
I =& -12\pi e^2+\frac{25}{2}\pi e-56\pi
\end{aligned}
\tag{547}
$$

Thus, we take

$$
\frac{dL}{dt} = -\frac{32}{5}\frac{G^{7/2}\mu^2m^{5/2}}{c^5a^{7/2}}\frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right) \tag{548}
$$

Going back to Eqs. 474 and 478 we get:

$$
E = \frac{Gm\mu}{2a} \Rightarrow \frac{dE}{dt} = \frac{Gm\mu}{2}\frac{1}{a^2}\frac{da}{dt} \tag{549}
$$

$$
\begin{aligned}
e^2 &= 1\frac{2}{Gm\mu^2}L^2 \Rightarrow \\
2e\frac{de}{dt} &= \frac{2}{Gm\mu^2}2L\frac{dL}{dt} \Rightarrow \\
e\frac{de}{dt} &= \frac{2}{Gm\mu^2}\mu r^2\dot{\psi}\frac{dL}{dt} \Rightarrow \\
e\frac{de}{dt}\frac{Gm\mu}{2} &= \frac{dL}{dt}\left[\frac{a^2(1-e^2)^2}{(1+e\cos\psi)^2}\right]\left(\frac{Gm}{a^3(1-e^2)^3}\right)\times(1+e\cos\psi)^2 \Rightarrow \\
e\frac{de}{dt} &= \frac{2}{Gm\mu}\frac{dL}{dt}[(Gm)^{-1}a^7(1-e^2)^9]^{1/2}
\end{aligned} \tag{550}
$$

$$
\frac{dL}{dt} = \frac{de}{dt}e\left[\frac{4a^7(1-e^2)^5}{(Gm)^3\mu^2}\right]^{1/2} \tag{551}
$$

Substituting Eqs. 550, 551 from 538 and 549 respectively we see evolution of e and a as:

$$
\begin{aligned}
\frac{dE}{dt} &= -\frac{32}{5}\frac{G^4\mu^2m^3}{c^5a^5}(1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \\
\frac{Gm\mu}{2a^2}\frac{da}{dt} &= -\frac{32}{5}\frac{G^4\mu^2m^3}{c^5a^5}(1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \\
\frac{da}{dt} &= -\frac{64}{5}\frac{G^3\mu m^2}{a^3c^5}(1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)
\end{aligned} \tag{552}
$$

$$
\begin{aligned}
\frac{de}{dt} &= \frac{dL}{dt}\frac{1}{e}\left(\frac{4a^7(1-e^2)^5}{G^3m^3\mu^2}\right)^{-1/2} \\
\frac{de}{dt} &= -\frac{304}{15}\frac{G^3\mu m^2}{c^5a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)
\end{aligned} \tag{553}
$$

equation 553 $e = 0 \Rightarrow \frac{de}{dt} = 0$ so we get a circular orbit. Dividing Eqs. 552 and 553 we get:

$$
\begin{aligned}
\frac{da}{de} &= \frac{192}{304}a(1-e^2)^{-1}e^{-1}\left(\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{1+\frac{121}{304}e^2}\right) \\
\frac{da}{de} &= \frac{12}{19}\frac{a}{e(1-e^2)}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{1+\frac{121}{304}e^2} \\
\frac{da}{dt} &= \frac{12}{19}de\left(\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{e-e^3+\frac{121}{304}e^3-\frac{121}{304}e^5}\right)
\end{aligned} \tag{554}
$$

$$
\begin{aligned}
\ln a &= \frac{-2299\ln(e+1)+1452\ln e+870\ln(121e^2+304)-2299\ln(e-1)}{2299} \\
\ln a &= -\ln(e+1)-\ln(e-1)+\frac{12\ln e}{19}+\frac{870}{2299}\ln(121e^2+304) \\
a &= C_0\frac{1}{e^2-1}e^{12/19}\left(1+\frac{121}{304}e^2\right)^{870/2299}\cdot 304^{870/2299}
\end{aligned} \tag{555}
$$

For $e = e_0$, we get $a_0 = C_0 g(e_0)$, so we have

$$\begin{aligned}
a(e) &= \frac{a_0}{g(e_0)} \frac{e^{12/19}}{e^2 - 1} 304^{870/2299} \left(1 + \frac{121}{304} e^2\right)^{870/2299} \\
a(e) &= \frac{a_0}{g(e_0)} \frac{e^{12/19}}{e^2 - 1} \left(1 + \frac{121}{304} e^2\right)^{870/2299}
\end{aligned} \tag{556}$$

Eqs. 538 and 559 for $e = 0$ produce:

$$\frac{dE}{dt} = -\frac{32}{5} \frac{G^4 \mu^2 m^3}{c^5 a^5} \xrightarrow{\omega_0 = \sqrt{\frac{Gm}{a^3}}} \frac{dE}{dt} = -\frac{32}{5} \frac{G^{7/2} m^{5/2}}{c^5 a^{7/2}} \left(\frac{Gm}{a^3}\right)^{1/2} \tag{557}$$

$$\frac{dL}{dt} = -\frac{32}{5} \frac{G^{7/2} \mu^2 m^{5/2}}{c^5 a^{7/2}} \xrightarrow{\omega_0 = \sqrt{\frac{Gm}{a^3}}} \frac{dE}{dt} = \omega_0 \frac{dL}{dt} \tag{558}$$

Equation 558 relates the energy and angular momentum of circular motion ($e = 0$). For ($e > 0$), we get the equation 555 $\frac{de}{dt} < 0$ and on an elliptic orbit becomes more and more circular because of Gω's emission.

We can rewrite the equations 552 and 553 in dimensionless form, by introducing a length scale R_*, given as

$$R_*^3 = \frac{4G^3 \mu m^2}{c^6} \tag{559}$$

and the dimensionless variable;

$$\tau = \frac{ct}{R_*} = \frac{ctc^6}{4G^3 \mu m^2} = \frac{c^7 t}{4G^3 \mu m^2} \tag{560}$$

If $\tilde{a}(\tau) = \frac{a(\tau)}{R_*}$, we get:

$$\begin{aligned}
R_* \frac{d\tilde{a}(\tau)}{dt} &= -\frac{16}{5} \frac{4G^3 \mu m^2}{c^5} \frac{c}{R_*^3 \tilde{a}^3(\tau)} \frac{1}{(1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \\
\frac{R_*}{c} \frac{d\tau}{dt} \frac{d\tilde{a}}{d\tau} &= -\frac{16}{5} \frac{1}{\tilde{a}^3} \frac{1}{(1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \\
\frac{d\tilde{a}}{d\tau} &= -\frac{16}{5} \frac{1}{\tilde{a}^3} \frac{1}{(1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right)
\end{aligned} \tag{561}$$

$$\begin{aligned}
\frac{d\tau}{dt} \frac{de}{d\tau} &= -\frac{76}{15} \frac{4G^3 \mu m^2}{c^5} \frac{1}{R_*^4 \tilde{a}^4} \frac{e}{(1 - e^2)^{5/2}} \left(1 + \frac{121}{304} e^2\right) \\
\frac{de}{d\tau} &= -\frac{76}{15} \frac{1}{\tilde{a}^4} \frac{e}{(1 - e^2)^{5/2}} \left(1 + \frac{121}{304} e^2\right)
\end{aligned} \tag{562}$$

Where $\tau = \frac{ct}{R_*}$ is the natural adimensional time-scale in the D.Eqs.

The rapid reduction in orbital eccentricity due to the back-reaction of gravitational waves is a key outcome of the system's evolution. This effect drives the orbit towards circularity over time.

Take, for example, a compact binary system like a neutron star-neutron star (NS-NS) pair. At the early stages, when the orbital separation is much larger than the radius of a neutron star, the system is far from merging, and the initial eccentricity, e_0, is relatively high. In the regime where e is small, the relationship simplifies to:

$$a(e) \approx a_0 \left(\frac{e}{e_0}\right)^{12/19} \tag{563}$$

Which leads to:

$$e \approx \left(\frac{a}{a_0} g(e_0)\right)^{19/12} \tag{564}$$

This substantial reduction indicates that the orbit becomes nearly circular unless external influences disturb the system before the merger phase. As a result, the two stars settle into an almost perfectly circular trajectory, gradually shrinking in separation.

The time to coalescence is $\tau(a_0, e_0)$, and it represents the time required for a binary system with an initial semi-major axis a_0 and eccentricity e_0 to merge. For a circular orbit ($e_0 = 0$), the expression simplifies to:

$$\tau(a_0, e_0 = 0) = \tau_0(a_0) = \frac{5}{256} \frac{c^5 a_0^4}{G^3 m_1 m_2 \mu} \tag{565}$$

where μ is the reduced mass of the system.

For systems with elliptical orbits, $\tau(a_0, e_0)$ can be determined by integrating the governing equations. Integration is performed such that $a(t) = 0$ when $t = \tau(a_0, e_0)$, or equivalently, $e(t) = 0$ at $t = \tau(a_0, e_0)$, as the eccentricity approaches zero at coalescence. Using the form of $a(e)$, the coalescence time is given by:

$$\tau(a_0, e_0) = \int_0^{\tau(a_0,e_0)} dt = -\frac{15}{304} \frac{c^5}{G^3 m^2 \mu} \int_{e_0}^{0} de \frac{a(e)^4 (1 - e^2)^{5/2}}{e\left(1 + \frac{121}{304} e^2\right)} \tag{566}$$

Substituting $a(e)$, this becomes:

$$\tau(a_0, e_0) = \tau_0(a_0) \frac{48}{19 g^4(e_0)} \int_{e_0}^{0} de \frac{g^4(e)(1 - e^2)^{5/2}}{e\left(1 + \frac{121}{304} e^2\right)} \tag{567}$$

Using the orbital period relation, this simplifies to:

$$\tau(a_0, e_0) \approx 9.829\,\mathrm{Myr} \left(\frac{\tau_0}{1\,\mathrm{hr}}\right) \left(\frac{M_\odot}{m}\right)^{8/3} \left(\frac{M_\odot}{\mu}\right) F(e_0) \tag{568}$$

Where $F(e_0)$ is a function of the initial eccentricity.

The function $F(e_0)$ is given by:

$$F(e_0) = \frac{48}{19} \frac{1}{g^4(e_0)} \int_0^{e_0} \frac{g^4(e)(1 - e^2)^{5/2}}{e\left(1 + \frac{121}{304} e^2\right)} de \tag{569}$$

with

$$g(e) = \left(1 + \frac{121}{304}e^2\right) \tag{570}$$

For small initial eccentricities ($e_0 \to 0$), the result converges to that of a circular orbit, where $F(0) = 1$. In the limit $e_0 \to 1$, the integral becomes dominated by values near $e = 1$, resulting in:

$$F(e_0) \propto \frac{48}{19}\frac{1}{g^4(e_0)} \int_0^{e_0} \frac{1}{\left(1 + \frac{121}{304}e^2\right)} g^4(e)(1 - e^2)^{5/2}\, de \tag{571}$$

This further simplifies to:

$$F(e_0) \propto G(e_0)(1 - e_0^2)^{7/2} \tag{572}$$

and after some strenuous numerical calculations, we get that

$$F(e_0) \approx 1$$

. This last statement provides the evidence that for $e_0 \ll 1$, we have exactly circular orbits. When $G(e_0)$ is a slowly varying function that approaches a finite limit as $e_0 \to 1$. Numerically, $G(1) \approx 1.80$. A plot of $G(e_0)$ shows that it remains close to unity for most values of e_0.

5.4 Application in Cosmological Distances

The application in cosmological distances has many aspects included. Here we study GW aspects in Friedmann-Robertson-Walker spacetime in distances of several gigaparcecs. Further information can be found in [27] and [69].

Until now, our discussions have assumed that the merging binary systems are sufficiently close to Earth, such that the effect of the Universe's expansion on the gravitational waves (GWs) traveling to the detector could be ignored. However, advanced gravitational-wave detectors are expected to observe merging binaries at cosmological distances.

On scales of several gigaparsecs (Gpc), the Universe can be treated as isotropic and homogeneous, approximated by the Friedmann–Robertson–Walker (FRW) metric:

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right] \tag{573}$$

A light signal follows a null curve, so it obeys:

$$ds^2 = 0 \tag{574}$$

Because of homogeneity, we choose the starting point of the trajectory r_0 to be $r_0 = 0$. Isotropy makes the choice of θ_0 and ϕ_0 irrelevant. All geodesics that pass through $r_0 = 0$ are times a constant θ, ϕ so we get $d\theta = 0 = d\phi$ and equation 573 reads:

$$ds^2 = 0 \Rightarrow c^2dt^2 = a^2(t)\frac{dr^2}{1 - kr^2} \Rightarrow \int_0^t \frac{cdt'}{a(t')} = \int_0^{r_H} \frac{dr}{\sqrt{1 - kr^2}} \tag{575}$$

The proper distance to the horizon measured at time t is:

$$d_H(t) = \int_0^{r_H} \sqrt{g_{rr}} dr = \int_0^{r_H} a(t) \frac{dr}{\sqrt{1-kr^2}} \tag{576}$$

$$d_H(t) = a(t) \int_{-}^{t} \frac{cdt'}{a(t')} \tag{577}$$

where $d_H(t) = r_{\mathrm{phys}(t)}$.

The proper distance to the horizon defines the physical distance for a flat universe ($k = 0$), and then we get:

$$r_{\mathrm{phys}} = a(t) r \tag{578}$$

Next, if we consider a source located at the comoving distance r, that emits signals traveling at the speed of light c, and is received by an observer located at $r = 0$. Under this assumption, equation 575 reads as:

$$\int_{t_{\mathrm{obs}}}^{t_{\mathrm{emis}}} dt \frac{c}{a(t)} = \int_0^r \frac{dr}{\sqrt{1-kr^2}} \tag{579}$$

Suppose that $t_{\mathrm{emis}} + \Delta t_{\mathrm{emis}}$ is emitted a second wavecrest and received at $t_{\mathrm{obs}} + \Delta t_{\mathrm{obs}}$. So we get:

$$\int_{t_{\mathrm{emis}}+\Delta t_{\mathrm{emis}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} \frac{c\,dt}{a(t)} = \int_0^r \frac{dr}{\sqrt{1-kr^2}} \tag{580}$$

$$\int_{t_{\mathrm{emis}}}^{t_{\mathrm{obs}}} \frac{c\,dt}{a(t)} = \int_{t_{\mathrm{emis}}+\Delta t_{\mathrm{emis}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} \frac{c\,dt}{a(t)} \tag{581}$$

$$\begin{gathered}\left[\int_{t_{\mathrm{em}}+\Delta t_{\mathrm{em}}} + \int_{t_{\mathrm{em}}} - \int^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} + \int^{t_{\mathrm{obs}}}\right] \frac{cdt}{a(t)} = 0 \Rightarrow \\ \int_{t_{\mathrm{em}}}^{t_{\mathrm{em}}+\Delta t_{\mathrm{em}}} \frac{cdt}{a(t)} = \int_{t_{\mathrm{obs}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} \frac{cdt}{a(t)}\end{gathered} \tag{582}$$

RHS in equation 580 remains the same since the source is fixed in the comoving coordinate system. Finally we impose that the wavelength λ is much smaller than ct the time interval between signals, so:

$$\lambda \equiv c\delta_t \ll ct \tag{583}$$

and because of condition 583 we see that $a(t) \approx a =$constant and equation 582 yields:

$$\frac{1}{a(t_{\mathrm{em}})} \int_{t_{\mathrm{em}}}^{t_{\mathrm{em}}+\Delta t_{\mathrm{em}}} cdt \simeq \frac{1}{a(t_{\mathrm{obs}})} \int_{t_{\mathrm{obs}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} cdt \tag{584}$$

$$\frac{\Delta t_{\mathrm{em}}}{a(t_{\mathrm{em}})} \simeq \frac{\Delta t_{\mathrm{obs}}}{a(t_{\mathrm{obs}})} \Rightarrow \Delta t_{\mathrm{obs}} = \frac{a(t_{\mathrm{obs}})}{a(t_{\mathrm{em}})} \Delta t_{\mathrm{em}} \tag{585}$$

Based on 585, we define the redshift of the source as

$$1 + z = \frac{a(t_{\mathrm{obs}})}{a(t_{\mathrm{em}})} \tag{586}$$

From 585 we get:

$$dt_{\text{obs}} = \frac{a(t_{\text{obs}})}{a(t_{\text{em}})} dt_{\text{em}} \Rightarrow dt_{\text{obs}} = (1+z) dt_s \tag{587}$$

for infinitesimal Δt and t_s the time measured by the source. Inversely, the frequency redshift is:

$$\frac{1}{dt_{\text{obs}}} = \frac{1}{(1+z)dt_s} \Rightarrow f_{\text{obs}} = \frac{f_s}{1+z} \tag{588}$$

$$\xRightarrow{c=\lambda f} \frac{c}{\lambda_{\text{obs}}} = \frac{c}{\lambda_s(1+z)} \Rightarrow \lambda_{\text{obs}} = (1+z)\lambda_s \tag{589}$$

The absolute luminosity $\mathcal{L}$ of the source in the rest frame is the radiated power and is defined as:

$$\mathcal{L} = \frac{dE_s}{dt_s} \tag{590}$$

The energy flux is defined by:

$$\mathcal{F} = \frac{L}{4\pi d_L^2} \tag{591}$$

Where d_L is the luminosity distance. The observed energy is redshifted in an expanding universe as

$$E_{\text{obs}} = \frac{E_s}{1+z} \tag{592}$$

$$\begin{aligned} \frac{dE_{\text{obs}}}{dt_{\text{obs}}} &= \frac{1}{1+z} \frac{dE_s}{dt_{\text{obs}}} \\ \frac{dE_{\text{obs}}}{dt_{\text{obs}}} &= \frac{1}{1+z} \frac{1}{1+z} \frac{dE_s}{dt_s} \\ \frac{dE_{\text{obs}}}{dt_{\text{obs}}} &= \frac{1}{(1+z)^2} \frac{dE_s}{dt_s} \end{aligned} \tag{593}$$

At time t, the surface of a sphere with comoving radius r is $4\pi a^2(t) r^2$. When the radiation arrives at the observer after t_{obs}, it is spread over an area $4\pi a^2(t_{\text{obs}}) r^2$. Therefore, the luminosity flux is:

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi a^2(t_{\text{obs}}) r^2 (1+z)^2} = \frac{\mathcal{L}}{4\pi d_L^2} \Rightarrow \tag{594}$$

$$d_L = a(t_{\text{obs}}) r (1+z) \tag{595}$$

d_L in terms of z is expresses as following. We Taylor expend $a(t)$ around the present epoch $t = t_0$

$$\frac{a(t)}{a(t_0)} = 1 + \frac{\dot{a}(t_0)}{a(t_0)}(t-t_0) - \frac{1}{2}\frac{\ddot{a}(t_0)}{a(t_0)}(t-t_0)^2 + O(3) \tag{596}$$

We set the Humble parameter in the present epoch to be:

$$H(t_0) \equiv H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)} \tag{597}$$

And the deceleration parameters are:

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)} \frac{1}{H_0^2} = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}^2(t_0)} \tag{598}$$

So we write 596 as:

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 + O(t - t_0)^3 \tag{599}$$

or inverted using $z = 1 - \frac{a(t)}{a(t_0)}$ is:

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2(t_0 - t)^2 + \dots \tag{600}$$

$$(t_o - t) = H_o^{-1}\left[z - \left(1 + \frac{q_o}{2}\right) z^2 + \dots\right] \tag{601}$$

We can rewrite equation 599 with 601 as:

$$\frac{a(t)}{a(t_0)} = 1 - H_0 H_0^{-1}\left[z - \left(1 + \frac{q_o}{2}\right) z^2 + \dots\right] - \frac{1}{2}q_0 H_0^2 H_0^{-2}\left[z - \left(1 + \frac{q_o}{2}\right) z^2\right]^2 + O(z^3)$$

and

$$\frac{H_0 d_L}{c} = z + \frac{1}{2}(1 - q_0)z^2 + \dots \tag{602}$$

The first term of the expansion in equation 602 is Hubble's law $z \simeq \frac{d_L H_0}{c}$, which states that redshifts are proportional to distances. For example, we apply $k = 0$ for a flat universe in equation 579 and then we get:

$$\int_{t_{\rm em}+\Delta t_{\rm em}}^{t_{\rm obs}+\Delta t_{\rm obs}} \frac{cdt}{a(t)} = \int_0^r dr' = r \tag{603}$$

$$\begin{aligned}
1 + z(t) &= \frac{a(t_0)}{a(t)} \Rightarrow \\
\frac{dz(t)}{dt} &= a(t_0)\left(-\frac{1}{a^2(t)}\frac{da(t)}{dt}\right) \Rightarrow \\
\frac{dz(t)}{dt} &= -\frac{a(t_0)}{a(t)}\frac{\dot{a}(t)}{a(t)} \\
\frac{dt}{a(t)} &= -\frac{1}{a(t_0)}\frac{dz}{H(z)}
\end{aligned} \tag{604}$$

$$\int_{t_{\rm em}}^{t_{\rm obs}} \frac{cdt}{a(t)} = -\int_z^0 \frac{cdz'}{a(t_0)H(z')} = r \Rightarrow a(t_0)r = c\int_0^z \frac{dz'}{H(z')} \tag{605}$$

$$\xRightarrow{(1+z)} d_L = (1 + z)a(t_0)r = (1 + z)c\int_0^z \frac{dz'}{H(z')} \tag{606}$$

$$\xRightarrow{\frac{d}{dz}} \frac{c}{H(z)} = \frac{d}{dz}\left[\frac{d_L}{1 + z}\right] \tag{607}$$

We want to see the way a propagating waveform at cosmological distances is modified. We define a local wave zone as the region where the distance t_0 to the source is sufficiently large, so the grows. Field goes as $\frac{1}{r}$, but also sufficiently small so the expansion of the universe is negligible. In the local wave zone during the propagation, the scale factor $a(t)$ does not change appreciably, so physical distances in this zone are written as:

$$r_{\text{phys}} = a(t_{\text{emis}})r \tag{608}$$

Where r is the commoving distance.

Eqs. 251 and 252 read in the local wave zone as:

$$h_+(t_s) = h_c(t_s^{\text{ret}}) \left(\frac{1 + cos^2 i}{2}\right) \cos\left[2\pi \int^{t_s^{\text{ret}}} dt'_s f_{\text{gw}}^{(s)}(t'_s)\right] \tag{609}$$

where

- t_s is the time measured by a clock in the source
- t_s^{ret} is the corresponding retarded time of the source
- $f_{\text{gw}}^{(s)}$ is the associated GW frequency to t_s

And we define

$$h_c(t_s^{\text{ret}}) \equiv \frac{4}{a t_{\text{emis}} r} \left(\frac{GM_c}{c^2}\right)^{5/3} \left(\frac{\pi f_{\text{gw}} t_s^{\text{ret}}}{c}\right)^{2/3} \tag{610}$$

$$h_\times(t_s) = h_c(t_s^{\text{ret}}) \cos i \; sin \left[2\pi \int^{t_s^{\text{ret}}} dt'_s f_{\text{gw}}^{(s)}(t'_s)\right] \tag{611}$$

In terms if time to coalescence $\tau_s = t_{\text{cod}}^s - t_s$, we read the dependence of $f_{\text{gw}}^{(s)}$ on t_s as:

$$f_{\text{gw}}^{(s)} = \frac{1}{\pi} \left(\frac{5}{256} \frac{1}{\tau_s}\right)^{3/8} \left(\frac{GM_c}{c^3}\right)^{-5/8} \tag{612}$$

$\bar{H}$ scalar perturbation ϕ propagates in a FRW metric, following

$$\Box\phi = 0 \Rightarrow \frac{1}{\sqrt{-g}} \partial\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0 \tag{613}$$

This wave equation is solved on a FRW metric, by introducing the conformal time η, given as:

$$d\eta = \frac{dt}{a(t)} \tag{614}$$

$$\Rightarrow \eta = \int d\eta = \int_0^t \frac{dt'}{a(t')} \tag{615}$$

The line element (given in equation 573) is written in terms of the conformal tie as:

$$
\begin{aligned}
ds^2 &= -c^2dt^2 + a^2(t)[dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2] \Rightarrow \\
ds^2 &= a^2(\eta)[-c^2d\eta^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2]
\end{aligned} \tag{616}
$$

Here

$$
g_{\mu\nu} = \begin{pmatrix} -c^2a^2 & & & \varnothing \\ & a^2 & & \\ & & a^2r^2 & \\ \varnothing & & & a^2r^2sin^2\theta \end{pmatrix} \tag{617}
$$

and $\sqrt{-g} = -acr^2\sin\theta$ and equation 613 is:

$$
\begin{aligned}
&\frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\big) = 0 \Rightarrow \\
&\partial_o\big(\sqrt{-g}\, g^{o\nu}\partial_\nu\phi\big) + \partial_i\big(\sqrt{-g}\, g^{i\nu}\partial_\nu\phi\big) = 0 \Rightarrow \\
&\partial_o\big(\sqrt{-g}\, g^{oo}\partial_o\phi\big) + \partial_o\big(\sqrt{-g}\, g^{oi}\partial_i\phi\big) \\
&\quad + \partial_i\big(\sqrt{-g}\, g^{io}\partial_o\phi\big) + \partial_i\big(\sqrt{-g}\, g^{ii}\partial_i\phi\big) = 0 \Rightarrow \\
&-\frac{1}{c^2}\partial_\eta\big[a^2(\eta)r^2\partial_\eta\phi\big] + \partial_r\big[a^2(\eta)r^2\partial_r\phi\big] = 0 \Rightarrow \\
&-\frac{1}{c^2}\big[2a(\eta)\partial_\eta a(\eta)r^2\partial_\eta\phi + a^2(\eta)r^2\partial^2_\eta\phi\big] \quad + a^2(\eta)\big[2r\partial_r\phi + r^2\partial^2_r\phi\big] = 0.
\end{aligned}
$$

or for $f' = \frac{1}{c}\partial_\eta f$ we get:

$$
\partial^2_r f - f'' - \frac{2a'}{a}f' = 0 \tag{618}
$$

If we write $f(r,\eta) = \frac{1}{a(\eta)}g(r,\eta)$, we have:

- The first derivative:

$$
f' = -\frac{1}{a^2}a'(\eta)g(r,\eta) + \frac{1}{a(\eta)}g'(r,\eta) \tag{619}
$$

- The second derivative:

$$
f'' = \frac{2}{a^3}\big(a'(\eta)\big)^2 g(r,\eta) - \frac{1}{a^2}a''(\eta)g(r,\eta) - 2\frac{a'(\eta)}{a^2}g'(r,\eta) + \frac{1}{a(\eta)}g''(r,\eta) \tag{620}
$$

-

$$
\partial^2_r f(r,\eta) = \frac{1}{a(\eta)}\partial^2_r g(r,\eta) \tag{621}
$$

Thus, equation 618 can be written as:

$$
\begin{aligned}
&\frac{1}{a(\eta)}\partial^2_r g - \frac{2a'^2}{a^3}g + \frac{2a'}{a^2}g' + \frac{2(a')^2}{a^3}g + \frac{a''}{a^2}g - \frac{1}{a}g'' - \frac{2a'}{a^2}g' = 0 \Rightarrow \\
&\partial^2_r g - \frac{a''}{a}g - g'' = 0
\end{aligned} \tag{622}
$$

We approximate the solutions in the limit $\omega^2 \gg \frac{a''}{a}$ we see that $\frac{a''}{a} \sim \frac{1}{\eta^2} \Rightarrow \omega^2 \gg \frac{a''}{a}$ and then equation 622 reads:

$$\partial_r^2 g - g'' = 0 \tag{623}$$

with solution

$$g(r,\eta) \simeq exp\left[\pm i\omega\left(\eta - \frac{r}{c}\right)\right] \tag{624}$$

$$f(r,\eta) \simeq \frac{1}{a(\eta)} \exp\left[\pm i\omega\left(\eta - \frac{r}{c}\right)\right] \tag{625}$$

$$\begin{aligned}\phi(r,\eta) &\simeq \frac{1}{r} f(r,\eta)\\ \phi(r,\eta) &\simeq \frac{1}{ra(\eta)} \exp\left[\pm i\omega\left(\eta - \frac{r}{c}\right)\right]\end{aligned} \tag{626}$$

In the present epoch we normalize the conformal time η to be $\eta = t$, so we get:

$$\phi(r,t) \simeq \frac{1}{ra(t_0)} \exp\left[\pm i\omega\left(\eta - \frac{r}{c}\right)\right] \tag{627}$$

So a scalar wave through an FRW background simply follows equation 627, a plane wave. For tensor perturbations $h_{\mu\nu}$ we have the propagation equation to read:

$$D_\rho D^\rho h_{\mu\nu} = 0 \tag{628}$$

Following the same steps as in scalar ϕ, we get the same equation:

$$h_{\mu\nu}(r,t) \simeq \frac{A_{\mu\nu}}{ra(t_0)} e^{\pm i\omega\left(t - \frac{r}{c}\right)} \tag{629}$$

In the analysis between Eqs. 628 and 629 we need to impose the condition $\omega^2 \gg \frac{1}{\eta^2}$. This condition defines the background geometrical optics approximation, where ω is large w.r.t η^{-1}. In the BG geometrical optics approximation, all massless particles follow null geodesics. To leading order:

i) The two polarizations $h_\times$ and h_+ decouple, so each one satisfies a wave equation independently.

ii) Both $h_\times$ and h_+ satisfy the same Eqs. (618 - 627) as ϕ.

The conclusion is that after propagation from source to detector, the GW amplitude from a binary is given by Eqs. 609, 610 and 611.

In equation 610, though, we must write:

$$h_c(t_s^{t_{\text{ret}}}) = \frac{4}{a(t_0)r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\text{gw}}^{(s)} t_{t_{\text{ret}}}^s}{c}\right)^{2/3} \tag{630}$$

The geometrical optics condition today is given by $2\pi f_{\text{gw}} \gg t_0^{-1}$, where t_0 is the age of the Universe today. This condition is satisfied by all gravitational waves (GWs) with wavelengths smaller than the present Hubble size of the Universe.

Following equation 588, we can write the observed frequency as:

$$f_{\mathrm{gw}}^{(s)} = f_{\mathrm{gw}}^{(\mathrm{obs})}(1+z), \tag{631}$$

where z is the redshift.

$$h_c(t_{\mathrm{obs}}^{\mathrm{ret}}) = \frac{4}{a(t_0)r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}^{\mathrm{obs}}(1+z)}{c}\right)^{2/3} \tag{632}$$

$$h_c(t_{\mathrm{obs}}^{\mathrm{ret}}) = \frac{4(1+z)^{5/3}}{d_L}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}^{\mathrm{obs}}}{c}\right)^{2/3} \tag{633}$$

If we define $\mathcal{M}_c = (1+z)M_c$, we get the following:

$$h_c(t_{\mathrm{obs}}^{\mathrm{ret}}) = \frac{4}{d_L}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}^{\mathrm{obs}}}{c}\right)^{2/3} \tag{634}$$

The same form as for non-expansible universe but with $r \to d_L$ and $M_c \to \mathcal{M}_c$. In the case of nonvanishing redshift, we reserve the name "chirp mass" for $\mathcal{M}_c$, not M_c. The dependence of $f_{\mathrm{gw}}^{(\mathrm{obs})}$ on t_{obs} is given by

$$\begin{aligned} f_{\mathrm{gw}}^{(\mathrm{obs})} &= \frac{1}{1+z} f_{\mathrm{gw}}^{(s)} = \frac{1}{1+z}\frac{1}{\pi}\left(\frac{5}{256}\frac{1+z}{t_{\mathrm{obs}}}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8} \Rightarrow \\ f_{\mathrm{gw}}^{(\mathrm{obs})} &= \frac{1}{\pi}\left(\frac{5}{256}\frac{1+z}{\tau_{\mathrm{obs}}}\right)^{3/8}\left(\frac{G\mathcal{M}_c}{c^3}\right)^{-5/8} \end{aligned} \tag{635}$$

with

$$\begin{aligned} \dot{f}_{\mathrm{gw}}^{(\mathrm{obs})} &= \frac{d}{dt}\left(\frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8}\tau_{\mathrm{obs}}^{-3/8}\left(\frac{G}{c^3}\right)^{-5/8}\mathcal{M}_c^{-5/8}(z)\right) \Rightarrow \\ \dot{f}_{\mathrm{gw}}^{(\mathrm{obs})} &= \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}_c(z)}{c^3}\right)^{5/3}\left[f_{\mathrm{gw}}^{(\mathrm{obs})}\right]^{11/3} \end{aligned} \tag{636}$$

Compared to the $z=0$ case, we get the modification of fields as below:

i) $f^{(\mathrm{obs})} = \frac{f^{(s}}{1+z}$

ii) $\frac{1}{r}$ in amplitude is replaced by $\frac{1}{d_L(z)}$

iii) M_c is replaced by $\mathcal{M}_c(z) = (1+z)M_c$.

5.5 Radiation from rigid bodies

The production of GWs from rotating and precessing rigid bodies is computed in [29]. The radiation from non-axisymmetric bodies is discussed in [70] and [71]. In [72], the rotating fluid stars are extensively studied, while in [73] and [74] the back-reaction due to wobble radiation is discussed.

The generation of gravitational waves (GWs) from the rotation of a rigid body is of significant importance, particularly when applied to isolated neutron stars. In classical mechanics, the inertia tensor is introduced as a fundamental quantity that characterizes the rotational properties of a rigid body. It is expressed as

$$I^{ij} = \int d^3x \, \rho(\vec{x}) \left(r^2 \delta^{ij} - x^i x^j \right), \tag{637}$$

where $\rho(\vec{x})$ represents the mass density of the body.

The inertia tensor I^{ij} is a symmetric Hermitian matrix. Through an appropriate rotation, it can always be diagonalized in an orthogonal coordinate system. This results in a frame where the components of the inertia tensor become the principal moments of inertia, denoted by I_1, I_2, I_3. The coordinate system in which the tensor is diagonal is referred to as the "body frame," and its axes are called the principal axes of the body. The corresponding coordinates are denoted by x_i'. The diagonal components are:

$$I_{11} = \int d^3x' \, \rho(\vec{x'}) \left(r^2 \delta^{11} - x_1' x_1' \right) = \int d^3x' \, \rho(\vec{x'}) \left(x_2^{2'} + x_3^{2'} \right) \tag{638}$$

$$I_{22} = \int d^3x' \, \rho(\vec{x'}) \left(r^2 \delta^{22} - x_2' x_2' \right) = \int d^3x' \, \rho(\vec{x'}) \left(x_1^{'2} + x_3^{'2} \right) \tag{639}$$

$$I_{33} = \int d^3x' \, \rho(\vec{x'}) \left(r^2 \delta^{33} - x_3' x_3' \right) = \int d^3x' \, \rho(\vec{x'}) \left(x_1^{'2} + x_2^{'2} \right) \tag{640}$$

Adding Eqs. 638 and 639 we get:

$$I_{11}+I_{22} = \int d^3x' \rho(\vec{x'})[x_1^{'2}+x_2^{'2}+2x_3^{'2}] \geq \int d^3x' \rho(x')(x_1^{'2}+x_2^{'2}) \equiv I_{33} \Rightarrow I_{11}+I_{22} \geq I_{33} \tag{641}$$

The quality in Eqs. 641 holds only if $\rho(\vec{x'}) = \delta(x_3^{'})$. We consider a simple geometry of an ellipsoid with semiaxes a,b,c, and uniform mass density $\rho(\vec{x'})$. We denote the density as:

$$\rho(\vec{x'}) = \rho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi abc} \tag{642}$$

And the ellipsoidal equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \tag{643}$$

We set $x_1^{'} \equiv \frac{x_1}{a}$, $x_2^{'} \equiv \frac{x_2}{b}$ and $x_3^{'} \equiv \frac{x_3}{c}$ and rewrite the above equations as:

$$
\begin{aligned}
I_{11} &= \int d^3x' \frac{m}{\frac{4\pi}{3}abc}(b^2x_2^{'2} + c^2x_3^{'2}) \Rightarrow \\
I_{11} &= \frac{m}{\frac{4\pi}{3}abc}\int\int\int abcdx_1^{'}dx_2^{'}dx_3^{'}(b^2x_2^{'2} + c^2x_3^{'2}) \Rightarrow \\
I_{11} &= \frac{3m}{4\pi}\left[\int dx_1^{'}\int dx_2^{'}\int dx_3^{'}\ b^2x_2^{'2} + \int dx_1^{'}\int dx_2^{'}\int dx_3^{'}\ c^2x_3^{'2}\right] \Rightarrow \\
I_{11} &= \frac{3m}{4\pi}\left[\int_0^R dr\int_0^\pi d\theta\int_0^{2\pi} d\phi\ b^2r^2\sin\theta r^2\sin^2\theta\sin^2\phi + \int_0^R dr\int_0^\pi d\theta\int_0^{2\pi} d\phi\ c^2r^2\sin\theta r^2\cos^2\theta\right] \Rightarrow \\
I_{11} &= \frac{R^5mb^2}{5} + \frac{R^5mc^2}{5} \xRightarrow{R=1} \\
I_{11} &= \frac{m}{5}(b^2 + c^2)
\end{aligned}
\tag{644}
$$

$$
\begin{aligned}
I_{22} &= \int d^3x' \frac{m}{\frac{4\pi}{3}abc}(a^2x_1^{'2} + c^2x_3^{'2}) \xRightarrow{R=1} \\
I_{22} &= \frac{3m}{4\pi}\int_0^1 dr\ r^2\int_0^\pi d\theta\ \sin\theta\int_0^{2\pi} d\phi\ (a^2r^2\sin^2\theta cos^2\phi + c^2r^2\cos^2\theta) \Rightarrow \\
I_{22} &= \frac{3m}{4\pi}\frac{1}{5}\left[a^2\int_{-1}^1 d\cos\theta(1-\cos^2\theta)\int_0^{2\pi} d\phi\ \cos^2\phi + c^2\int_{-1}^1 d\cos\theta\ \cos^2\theta\int_0^{2\pi} d\phi\right] \Rightarrow \\
I_{22} &= \frac{m}{5}(a^2 + c^2)
\end{aligned}
\tag{645}
$$

$$
\begin{aligned}
I_{33} &= \int d^3x' \frac{3m}{4\pi abc}(a^2x_1^{'2} + b^2x_2^{'2}) \xRightarrow{R=1} \\
I_{33} &= \frac{3m}{4\pi}\int_0^1 dr\ r^2\int_0^\pi d\cos\theta\int_0^{2\pi} d\phi\ (a^2r^2\sin^2\theta cos^2\phi + b^2r^2\sin^2\theta\sin^2\phi) \Rightarrow \\
I_{33} &= \frac{m}{5}(a^2 + b^2)
\end{aligned}
\tag{646}
$$

If we consider a rotating body with angular velocity ω, it will have angular momentum given as:

$$
J_i = I_{ij}\omega_j \tag{647}
$$

In the body frame, we denote by $J_i^{'}$ and $\omega_i^{'}$ the components of angular momentum and velocity, respectively. So we have:

$$
\begin{aligned}
J_1^{'} &= I_{11}\omega_1^{'} \\
J_2^{'} &= I_{22}\omega_2^{'} \\
J_3^{'} &= I_{33}\omega_3^{'}
\end{aligned}
\tag{648}
$$

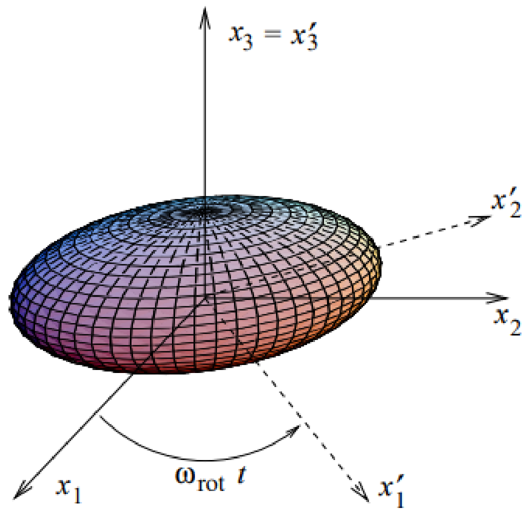

Figure 5: The principal axes (x_1', x_2', x_3'), which rotate with the rigid body, and the fixed axes (x_1, x_2, x_3), adapted by Maggiore's book, Gravitational Waves [15].

The direction of $\vec{\omega}$ is different from the direction of $\vec{J}$ unless either $I_1 = I_2 = I_3$ (spherical objects) or the rotation is around one of the principal axes, e.g., when $\omega_1 = \omega_2 = 0$. The rotational kinetic energy is

$$E_{\rm rot} = \frac{1}{2} I_{ij} \omega_i \omega_j \tag{649}$$

So, in the body frame, it is given simply by

$$E_{\rm rot} = \frac{1}{2} \left(I_{11} \omega_1'^2 + I_{22} \omega_2'^2 + I_{33} \omega_3'^2 \right) \tag{650}$$

Finally, we can define the moment of inertia about the axis of rotation as follows:

If $\hat{\omega}$ is the unit vector in the direction of the axis of rotation so that $\vec{\omega} = \omega \hat{\omega}$, we can write equation 649 as:

$$\begin{aligned} E_{\rm rot} &= \frac{1}{2} I_{ij} \hat{\omega}_i \hat{\omega}_j \omega^2 \Rightarrow \\ E_{\rm rot} &= \frac{1}{2} \vec{I} \omega^2 \end{aligned} \tag{651}$$

with

$$\vec{I} \equiv I_{ij} \hat{\omega}_i \hat{\omega}_j \tag{652}$$

Rotation around a principle axis

We now consider a rotating rigid body around one of the principal axes. The coordinates in the body frame are x_i'. The body frame, by definition, is attached to the body and rotates with it. And the origin of the fixed and the principal axes coincide with the center of mass of the body.

As shown in the figure above, the axes between the time frames are related by the following rule:

$$
\begin{aligned}
x_3^{'} &= x_3 \\
x_2^{'} &= x_1(-\sin(\omega_{\rm rot}t)) + x_2(\cos(\omega_{\rm rot}t)) \\
x_1^{'} &= x_1(\cos(\omega_{\rm rot}t)) + x_2(\sin(\omega_{\rm rot}t))
\end{aligned}
$$

or by the time-dependent rotation matrix

$$
R_{ij} = \begin{pmatrix} \cos(\omega_{\rm rot}t) & \sin(\omega_{\rm rot}t) & 0 \\ -\sin(\omega_{\rm rot}t) & \cos(\omega_{\rm rot}t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{653}
$$

used in

$$
x_i^{'} = R_{ij}x_j \tag{654}
$$

we denote by $I_{ij} \equiv \mathrm{diag}(I_1, I_2, I_3)$ the inertia tensor in the $x_1^{'}, x_2^{'}, x_3^{'}$ coordinates system and by I_{ij} the components in the x_1, x_2, x_3 coordinate frame.

$I_{ij}^{'}$ is a constant matrix and I_{ij} is time-dependent. The moment of inertia is a tensor and implies that:

$$
I_{ij}^{'} = R_{ik}R_{jl}I_{kl} \tag{655}
$$

$$
I_{ij}^{'} = (RIR^T)_{ij} \tag{656}
$$

So in the matrix form, we have

$$
I' = RIR^T \Rightarrow I = R^T I' R \tag{657}
$$

$$
\begin{aligned}
x_1^{'} &= x_1 \cos\omega_{\rm rot}t + x_2 \sin\omega_{\rm rot}t \\
x_2^{'} &= -x_1 \sin\omega_{\rm rot}t + x_2 \cos\omega_{\rm rot}t \\
x_3^{'} &= x_3
\end{aligned}
$$

$$
\begin{aligned}
I_{11}^{'} &= R_{1k}I_{kl}R_{l1} = R_{11}I_{11}R_{11} + R_{12}I_{22}R_{21} \\
I_{11}^{'} &= I_{11}\cos^2\omega_{\rm rot}t + I_{22}\sin^2\omega_{\rm rot}t \\
I_{11}^{'} &= I_{11}\cos^2\omega_{\rm rot}t + I_{22}(1-\cos^2\omega_{\rm rot}t) \\
I_{11}^{'} &= (I_{11}-I_{22})\cos^2\omega_{\rm rot}t + I_{22} \\
I_{11}^{'} &= (I_{11}-I_{22})\left(\frac{1}{2} - \frac{1}{2}\cos 2\omega_{\rm rot}t\right) + I_{22} \\
I_{11}^{'} &= \frac{1}{2}I_{11} - \frac{1}{2}I_{22} + I_{22} - \frac{1}{2}I_{11}\cos 2\omega_{\rm rot}t + \frac{1}{2}I_{22}\cos 2\omega_{\rm rot}t \\
I_{11}^{'} &= \frac{1}{2}(I_{11}+I_{22}) - \frac{1}{2}(I_{11}-I_{22})\cos 2\omega_{\rm rot}t
\end{aligned} \tag{658}
$$

$$
\begin{aligned}
I'_{12} &= R_{1k}I_{kl}R_{l2} = R_{11}I_{11}R_{12} + R_{12}I_{22}R_{22} \\
I'_{12} &= I_{11}\cos\omega_{\rm rot}t\sin\omega_{\rm rot}t - I_{22}\sin\omega_{\rm rot}t\cos\omega_{\rm rot}t \\
I'_{12} &= \frac{1}{2}(I_{11} - I_{22})\sin 2\omega_{\rm rot}t
\end{aligned}
\tag{659}
$$

$$
\begin{aligned}
I'_{22} &= R_{2k}I_{kl}R_{l2} = R_{21}I_{11}R_{12} + R_{22}I_{22}R_{22}, \\
I'_{22} &= -I_{11}\sin^2\omega_{\rm rot}t + I_{22}\cos^2\omega_{\rm rot}t, \\
I'_{22} &= 1 - \frac{I_{11} - I_{22}}{2}\cos 2\omega_{\rm rot}t.
\end{aligned}
\tag{660}
$$

$$I'_{33} = I_{33} \tag{661}$$

while

$$I_{13} = 0 = I_{23} \tag{662}$$

In the Quadrupole approximation, the GW amplitudes depend on the second mass moment M^{ij}.Comparing the formula that define M^{ij} and I^{ij}, we see that

$$M^{ij} = -I^{ij} + c^{ij} \tag{663}$$

and M^{ij}, I^{ij} are traceless. Based on equation 663, we rewrite Eqs. 659, 660 and 661 in terms of M^{ij} as:

$$M_{11} = -\frac{I_{11} - I_{22}}{2}\cos 2\omega_{\rm rot}t + c_{11} \tag{664}$$

$$M_{12} = -\frac{I_{11} - I_{22}}{2}\sin 2\omega_{\rm rot}t + c_{12} \tag{665}$$

$$M_{22} = \frac{I_{11} - I_{22}}{2}\cos 2\omega_{\rm rot}t + c_{22} \tag{666}$$

$$M_{33} = -I_{33} + c'_{33} = c_{33} \tag{667}$$

$$M_{13} = c_{13} \ \& \ M_{23} = c_{23} \tag{668}$$

with c_{i3} constants. The second time derivative of Eqs. 664-668 yield:

$$\ddot{M}_{11} = \frac{I_{11} - I_{22}}{2}\cos 2\omega_{\rm rot}t(4\omega^2_{\rm rot}) \tag{669}$$

$$\ddot{M}_{12} = \frac{I_{11} - I_{22}}{2}\sin 2\omega_{\rm rot}t(4\omega^2_{\rm rot}) \tag{670}$$

$$\ddot{M}_{22} = -\frac{I_{11} - I_{22}}{2}\sin 2\omega_{\rm rot}t(4\omega^2_{\rm rot}) \tag{671}$$

$$\ddot{M}_{33} = \ddot{M}_{23} = \ddot{M}_{13} = 0 \tag{672}$$

Eqs. 251 and 252 read:

$$h_+(t;\theta,\phi) = \frac{G}{Rc^4}\Big[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2 i) + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2 i) - \ddot{M}_{33}\sin^2 i - \\ \ddot{M}_{12}\sin 2\phi(1+\cos^2 i) + \ddot{M}_{13}\sin\phi\sin 2i + \ddot{M}_{23}\cos\phi\sin 2i\Big] \tag{673}$$

$$h_+ = \frac{4G\omega_{\rm rot}^2}{Rc^4}\frac{I_{11}-I_{22}}{2}\Bigg[\cos 2\omega_{\rm rot}t(cos^2\phi - \sin^2\phi\cos^2 i - \sin^2\phi + \cos^2\phi\cos^2 i) - \\ \sin 2\omega_{\rm rot}t\sin 2\phi(1+cos^2 i)\Bigg]$$

$$h_+ = \frac{4G\omega_{\rm rot}^2}{Rc^4}\frac{I_{11}-I_{22}}{2}\Bigg[\cos 2\omega_{\rm rot}t\left(\cos^2\phi(1+\cos^2 i) - \sin^2\phi(1+\cos^2 i)\right) \\ -\sin 2\omega_{\rm rot}t\sin^2\phi(1+\cos^2 i)\Bigg]$$

$$h_+ = \frac{4G\omega_{\rm rot}^2}{Rc^4}\frac{I_{11}-I_{22}}{2}(1+\cos^2 i)\Bigg[\cos 2\omega_{\rm rot}t(\cos^2\phi - \sin^2\phi) \\ -\sin 2\omega_{\rm rot}t\sin 2\phi\Bigg]$$

$$h_+ = \frac{4G\omega_{\rm rot}^2}{Rc^4}(I_{11}-I_{22})\frac{1+cos^2 i}{2}\Big[cos2\omega_{\rm rot}t\cos 2\phi - \sin 2\omega_{\rm rot}t\sin 2\phi\Big]$$

$$h_+(t;\theta,\phi) = \frac{4G\omega_{\rm rot}^2}{Rc^4}(I_{11}-I_{22})\frac{1+\cos^2 i}{2}\cos\left(2\omega_{\rm rot}t+\phi\right) \tag{674}$$

$$h_\times = \frac{2G}{Rc^4}\Big[(\ddot{M}_{11}-\ddot{M}_{22})\sin 2\phi\cos i + 2\ddot{M}_{12}\cos 2\phi\cos i - 2\ddot{M}_{13}\cos\phi\sin i - 2\ddot{M}_{23}\sin\phi\sin i\Big] \\ h_\times = \frac{4G\omega_{\rm rot}^2}{Rc^4}(I_1 - I_2)\cos i[2\sin(2\omega_{\rm rot}t+2\phi)] \tag{675}$$

In Eqs. 674 and 675 we see GW amplitudes with period $\omega_{\rm gw} = 2\omega_{\rm rot}$. The fact that $h_+ \sim \frac{1}{2}(1+\cos^2 i)$ and $h_\times \sim \cos i$ is a generic property of Eqs. 251 and 252, whenever

$$\ddot{M}_{11} = -\ddot{M}_{22} \ \&\ \ddot{M}_{i3} = 0\ \forall i = 1,2,3.$$

Next, we define ellipticity

$$\epsilon \equiv \frac{I_{11}-I_{22}}{I_{33}} \tag{676}$$

For a homogenous ellipsoid with semiaxes a,b,c, we have:

$$\epsilon \equiv \frac{a-b}{c}$$

and in the small asymmetry limit, Eqs. 638 - 640 produce:

$$\begin{aligned}
\epsilon &= \int d^3x' \left(\frac{b^2 x_2'^2 + c^2 x_2'^3 - a^2 x_a'^2 - c^2 x_2'^3}{a^2 x_2'^1 + b^2 x_2'^2} \right) \Rightarrow \\
\epsilon &= \int d^3x' \left(\frac{b^2 x_2'^2 - a^2 x_a'^2}{a^2 x_1'^2 + b^2 x_2'^2} \right) \Rightarrow \\
\epsilon &= \frac{b^2 - a^2}{b^2 + a^2} = \frac{(b-a)(b+a)}{b^2 + a^2} = \frac{(b-a)2a}{2a^2} \Rightarrow \\
\epsilon &= \frac{b-a}{a}
\end{aligned} \tag{677}$$

Finally, Eqs. 674 and 675 read:

$$\begin{aligned}
h_+ &= \frac{4G\omega_{\mathrm{rot}}^2}{Rc^4}(I_{11} - I_{22})\frac{1+\cos^2 i}{2}\cos(2\omega_{\mathrm{rot}}t) \Rightarrow \\
h_+ &= \frac{4G\omega_{\mathrm{rot}}^2}{Rc^4}\left(\frac{b-a}{a}\right) I_{33} \left(\frac{1+\cos^2 i}{2}\right)\cos(2\omega_{\mathrm{rot}}t) \Rightarrow \\
h_+ &= \frac{4\pi^2 f_{\mathrm{gw}}^2 G}{Rc^4}\left(\frac{b-a}{a}\right) I_{33} \left(\frac{1+\cos^2 i}{2}\right)\cos(2\omega_{\mathrm{rot}}t) \Rightarrow \\
h_+ &= h_0 \left(\frac{1+\cos^2 i}{2}\right)\cos(2\omega_{\mathrm{rot}}t)
\end{aligned} \tag{678}$$

with $h_0 \equiv \frac{4\pi^2 f_{\mathrm{gw}}^2 G}{Rc^4}\left(\frac{b-a}{a}\right) I_{33}$ and Eq. 675

$$\begin{aligned}
h_\times &= \frac{4G\omega_{\mathrm{rot}}^2}{Rc^4}(I_{11} - I_{22})\cos i \sin(2\omega_{\mathrm{rot}}t) \Rightarrow \\
h_\times &= h_0 \cos i \sin(2\omega_{\mathrm{rot}}t)
\end{aligned} \tag{679}$$

The radiated power in GWs is given if we insert equations 669 - 672 in the radiated power expression of Chapter 3:

$$\begin{aligned}
P_{\mathrm{quad}} &= \frac{G}{5c^5}\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}(\dddot{M}_{kk})^2 \rangle \\
P_{\mathrm{quad}} &= \frac{G}{5c^5}\langle \dddot{M}_{1j}\dddot{M}_{1j} - \frac{1}{3}(\dddot{M}_{11}^2 + \dddot{M}_{22}^2 \dddot{M}_{33}^2)\rangle + \dddot{M}_{2j}\dddot{M}_{2j} + \dddot{M}_{3j}\dddot{M}_{3j} \\
P_{\mathrm{quad}} &= \frac{G}{5c^5}\langle \dddot{M}_{11}\dddot{M}_{11} + \dddot{M}_{12}\dddot{M}_{12} + \dddot{M}_{21}\dddot{M}_{21} + \dddot{M}_{22}\dddot{M}_{22} \rangle - \frac{1}{3}\dddot{M}_{11}^2 - \frac{1}{3}\dddot{M}_{22}^2 \\
P_{\mathrm{quad}} &= \frac{16G\omega_{\mathrm{rot}}^6}{5c^5}(I_{11} - I_{22})^2 \langle 2\cos^2(2\omega_{\mathrm{rot}}) + 2\sin^2(2\omega_{\mathrm{rot}})\rangle \\
P_{\mathrm{quad}} &= \frac{16G\omega_{\mathrm{rot}}^6}{5c^5} 2\epsilon^2 I_{33}^2 \\
P_{\mathrm{quad}} &= -\frac{32G\epsilon^2}{5c^5} I_{33}^2 \omega_{\mathrm{rot}}^6
\end{aligned} \tag{680}$$

$$\frac{dE_{\mathrm{rot}}}{dt} = -\frac{32G}{5c^5}E^2 I_{33}^2 \omega_{\mathrm{rot}}^6 \tag{681}$$

Eq. 681 produces the rotation energy decrease of a star because of GW emission.

$$\dot{\omega}_{\mathrm{rot}} = -\frac{32G}{5c^5}\epsilon^2 I_{33}\omega_{\mathrm{rot}}^5 \tag{682}$$

Eq. 682 shows the decrease in rotational frequency of a star because of GW emission.

5.5.1 GWs from freely precessing rigid bodies

In astronomical objects, the rotation axis does not coincide with a principal axis is and the motion is a combination of rotation and precession. We introduce a fixed reference frame with axes (x_1, x_2, x_3). In this inertial frame, the angular momentum of the rigid body $\vec{J}$ is conserved, and we choose the x_3 axis in the direction of $\vec{J}$. Next, we introduce the body frame, a reference frame attached to the rotating body with axes (x_1', x_2', x_3') that coincide with the principal axes of the body. The two frames are related by the Euler angles (α, β, γ). The pass from fixed to body frame is done as follows: We perform a counterclockwise rotation by an ample β around the x_3 axis on the (x_1, x_2) plane. This way, we bring the x_1 axis on the line of nodes. The line of nodes is the intersection of the (x_1, x_2) and (x_1', x_2') planes. Next, we rotate around the line of nodes by an ample α and bring the x_3 axis to x_3'. Finally, we rotate around x_3' by an ample γ, so we bring the line of nodes to the x_1' axis. Therefore, we have $x_i' = R_{ij}x_j$, but now the rotation matrix is more complicated. The rotation matrix R is given by:

$$R = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{683}$$

The full motion of the rigid body is specified once we know how α, β, γ evolve with time. The fixed frame, the angular momentum $\vec{J}$ is conserved, but in the non-inertial body frame is not. We orient $\vec{J}$ along x_3 axis, so $\vec{J} = (0, 0, J)$ In body frame we have (x_1', x_2', x_3') and the components of angular momentum are: (J_1', J_2', J_3')

$$\begin{pmatrix} J_1' \\ J_2' \\ J_3' \end{pmatrix} = R\begin{pmatrix} 0 \\ 0 \\ J \end{pmatrix} = \begin{cases} J_1' = J\sin\alpha\sin\gamma, \\ J_2' = J\sin\alpha\cos\gamma, \\ J_3' = J\cos\alpha. \end{cases} \tag{684}$$

To compute the components of angular velocity ω_j' in terms of Eulerian angles and their derivatives, we need to compute analytic expressions for $\dot{\alpha}, \dot{\beta}$ and $\dot{\gamma}$. The $\dot{\alpha}$ angular velocity is orthogonal to the (x_3, x_3') plane, so it lies along the line of nodes and has components in the body frame that read:

$$\frac{d\vec{\alpha}}{dt} = (\dot{\alpha}\cos\gamma, -\alpha\sin\gamma, 0) \tag{685}$$

Similarly $\dot{\vec{\beta}}$ has components:

$$\frac{d\vec{\beta}}{dt} = (\dot{\beta}\sin\alpha\sin\gamma, \dot{\beta}\sin\alpha\cos\gamma, \dot{\beta}\cos\alpha) \tag{686}$$

and $\frac{d\vec{r}}{dt} = (0, 0, \dot{\gamma})$. The total angular velocity is:

$$\vec{\omega} = \frac{d\vec{\alpha}}{dt} + \frac{d\vec{\beta}}{dt} + \frac{d\vec{\gamma}}{dt} \tag{687}$$

So in the body frame, the components read:

$$\omega_1' = \dot{\alpha}\cos\gamma + \dot{\beta}\sin\alpha\sin\gamma \tag{688}$$
$$\omega_2' = -\dot{\alpha}\sin\gamma + \dot{\beta}\sin\alpha\cos\gamma \tag{689}$$
$$\omega_3' = \dot{\beta}\cos\alpha + \gamma \tag{690}$$
$$\tag{691}$$

The angular momentum in terms of the inertia tensor components in the body frame is written as:

$$J_1' = I_{11}\omega_1' = I_{11}(\dot{\alpha}\cos\gamma + \dot{\beta}\sin\alpha\sin\gamma) \tag{692}$$
$$J_2' = I_{22}\omega_2' = I_{22}(\dot{\beta}\sin\alpha\cos\gamma - \dot{\alpha}\sin\gamma) \tag{693}$$
$$J_3' = I_{33}\omega_3' = I_{33}(\dot{\gamma} + \dot{\beta}\cos\alpha) \tag{694}$$

Comparing Eq. 683 with Eqs. 692-694 we get:

$$a : I_{11}(\dot{\alpha}\cos\gamma + \dot{\beta}\sin\alpha\sin\gamma) = J\sin\alpha\sin\gamma \tag{695}$$
$$b : I_{22}(-\dot{\alpha}\sin\gamma + \dot{\beta}\sin\alpha\cos\gamma) = J\sin\alpha\sin\gamma \tag{696}$$
$$c : I_{33}(\dot{\gamma} + \dot{\beta}\cos\alpha) = J\cos\alpha \tag{697}$$

Eqs. 695-697 provide the first order of equations for (α, β, γ) variables and constitute the first integral of motion provided the angular momentum conservation.

Wobble radiation from an axisymmetric rigid body

For an antisymmetric body, with longitudinal axis x_3' that makes an angle α with the angular momentum axis x_3, we get the angle α to be called "wobble" angle and the corresponding GW emission called "wobble radiation". Since the rigid body is axisymmetric, we have $I_{11} = I_{22}$ and in Eqs. 695-696 we get:

$$I_{11}(\dot{\alpha}\cos^2\gamma + \dot{\beta}\sin\alpha\sin\gamma\cos\gamma) = J\sin\alpha\sin\gamma\cos\gamma)$$
$$I_{11}(-\dot{\alpha}\sin^2\gamma + \dot{\beta}\sin\alpha\sin\gamma\cos\gamma) = J\sin\alpha\sin\gamma\cos\gamma)$$

And subtracting:

$$\begin{aligned} &I_{11}(-\dot{\alpha}\cos^2\gamma + \dot{\alpha}\sin^2\gamma + \dot{\beta}\sin\alpha\sin\gamma\cos\gamma - \dot{\beta}\sin\alpha\sin\gamma\cos\gamma) = 0 \\ &I_{11}(\dot{\alpha}) = 0 \Rightarrow \dot{\alpha} = 0 \end{aligned} \tag{698}$$

Eq. 698 tells that the inclination of the x_3' axis with respect to the angular momentum $\vec{J}$ is constant.

Again

$$I_{11}(\dot{\alpha}\cos\gamma\sin\gamma + \dot{\beta}\sin\alpha\sin^2\gamma]) = J\sin\alpha\sin^2\gamma$$
$$I_{11}(-\dot{\alpha}\sin\gamma\cos\gamma + \dot{\beta}\sin\alpha\cos^2\gamma]) = J\sin\alpha\cos^2\gamma$$

Adding now we get:

$$I_{11}\dot{\beta}\sin\alpha = J\sin\alpha. \tag{699}$$

In Eq. 699, if $\alpha \neq 0$, $I_{11} = \frac{J}{\dot{\beta}} \Rightarrow \dot{\beta} = \frac{J}{I_{11}}$ so the angular velocity of x_3' rotation is constant about the direction of $\vec{J}$. We define Ω as

$$\Omega \equiv \dot{\beta} = \frac{J}{I_{11}}. \tag{700}$$

In Eq. 699 we supposed that $\alpha \neq 0$ and $\dot{\alpha} = 0$ thus α is a constant and $\cos\alpha$ and $\sin\alpha$ are constants as well and since $\dot{\beta}$ is constant (Eq. 700) we get from Eq. 696 we get:

$$I_{33}(\dot{\gamma} + \dot{\beta}\cos\alpha) = J\cos\alpha \Rightarrow \dot{\gamma} = \text{constant}$$

Again if $J = I_{11}\dot{\beta}$ is inserted in 696 we get:

$$\begin{aligned}
I_{33}(\dot{\gamma} + \dot{\beta}\cos\alpha) &= I_{11}\dot{\beta}\cos\alpha \\
I_{33}\dot{\gamma} &= (I_{11} - I_{33})\dot{\beta}\cos\alpha \\
\dot{\gamma} &= \frac{I_{11} - I_{33}}{I_{33}}\dot{\beta}\cos\alpha \\
\dot{\gamma} &= \frac{I_{11} - I_{33}}{I_{33}}\Omega\cos\alpha \\
-\dot{\gamma} &= \frac{I_{11} - I_{33}}{I_{33}}\Omega\cos\alpha \equiv \omega_p
\end{aligned} \tag{701}$$

We define $\omega_p \equiv -\dot{\gamma}$, since oblate objects satisfy $I_{33} > I_{11}$, which is the normals shape of astromical objects, so $\omega_p > 0$ Inserting $\dot{\alpha} = 0, \dot{\beta} = \frac{J}{I_{11}} \equiv \Omega$ and $-dot\gamma \equiv \omega_p$ in Eqs. 688-690 a,b,c we get:

$$\omega_1' = \dot{\alpha}\cos\gamma + \dot{\beta}\sin\alpha\sin\gamma = \frac{J}{I_{11}}\sin\alpha\sin\gamma = \Omega\sin\alpha\cos(\omega_p t) \tag{702}$$
$$\omega_2' = \Omega\sin\alpha\cos(\omega_p t) \tag{703}$$
$$\omega_3' = \Omega\cos\alpha - \omega_p \tag{704}$$

In the body frame, the angular velocity rotates in the (x_1', x_2') plane, so it precesses around the x_3' axis with angular velocity ω_p. In $\omega_p > 0$, then precession is counterclockwise. We observe that

$$|I_{33} - I_{11}| \ll I_{33} \Rightarrow \frac{|I_{33} - I_{11}|}{I_{33}} \ll 1 \Rightarrow \frac{|\omega_p|}{\Omega} \ll 1 \Rightarrow \Omega \gg |\omega_p| \tag{705}$$

Condition 705, when satisfied, describes a free precession that takes place in the absence of external torques: The components of the inertia tensor read

$$I'_{11} = \frac{1}{2}(I_{11} - I_{33}) \sin^2 \alpha \cos 2\beta + \text{constant} \tag{706}$$

$$I'_{12} = \frac{1}{2}(I_{11} - I_{33}) \sin^2 \alpha \sin 2\beta \tag{707}$$

$$I'_{22} = -\frac{1}{2}(I_{11} - I_{33}) \sin^2 \alpha \cos 2\beta + \text{constant} \tag{708}$$

$$I'_{13} = -(I_{11} - I_{33}) \sin \alpha \cos \alpha \sin \beta \tag{709}$$

$$I'_{23} = (I_{11} - I_{33}) \sin \alpha \cos \alpha \cos \beta \tag{710}$$

$$I'_{33} = I_{11} \sin^2 \alpha + I_{33} \cos^2 \alpha = \text{constant} \tag{711}$$

Observation 1

i. γ does not enter Eqs. 706-711.

ii. α is a constant, so time-dependence manifests in $\beta(t)$.

We choose the origin of $\beta(t)$ to be at

$$t = 0 \Rightarrow \beta(t = 0) = 0 \tag{712}$$

In Eq. 676 we have $M_{ij} = -I'_{ij} + c_{ij}$ So Eq. 711 gives

$$M_{11} = \frac{1}{2}(I_{33} - I_{11}) \sin^2 \alpha \cos 2\beta + \text{constant} \tag{713}$$

$$M_{12} = \frac{1}{2}(I_{33} - I_{11}) \sin^2 \alpha \sin 2\beta + \text{constant} \tag{714}$$

$$M_{22} = -\frac{1}{2}(I_{33} - I_{11}) \sin^2 \alpha \cos 2\beta + \text{constant} \tag{715}$$

$$M_{13} = -(I_{33} - I_{11}) \sin \alpha \cos \alpha \sin \beta \tag{716}$$

$$M_{23} = (I_{33} - I_{11}) \sin \alpha \cos \alpha \cos \beta \tag{717}$$

$$M_{33} = \text{constant} \tag{718}$$

And the second time derivative produces:

$$\ddot{M}_{11} = 2\Omega^2(I_{11} - I_{33}) \sin^2 \alpha \cos(2\Omega t) \tag{719}$$

$$\ddot{M}_{12} = 2\Omega^2(I_{11} - I_{33}) \sin^2 \alpha \sin(2\Omega t) \tag{720}$$

$$\ddot{M}_{22} = -2\Omega^2(I_{11} - I_{33}) \sin^2 \alpha \cos(2\Omega t) \tag{721}$$

$$\ddot{M}_{13} = -\Omega^2(I_{11} - I_{33}) \sin \alpha \cos \alpha \sin(\Omega t) \tag{722}$$

$$\ddot{M}_{23} = \Omega^2(I_{11} - I_{33}) \sin \alpha \cos \alpha \cos(\Omega t) \tag{723}$$

$$\ddot{M}_{33} = 0 \tag{724}$$

Note that some terms oscillate as $\cos 2\Omega t$ or $\sin 2\Omega t$ and a few as $\sin \Omega t$ or $\cos \Omega t$. This means that GWs are emitted in two frequencies $\omega_{\rm gw} = 2\Omega$ and $\omega_{\rm gw} = \Omega$. The emission in $\omega_{\rm gw} = \Omega$ frequencies is due to the motion of precession.

The GW amplitudes h_+ and $h_\times$ are calculated using Eqs. 251 and 252:

$$\begin{aligned}
h_+(t;i;\phi=0) &= \frac{G}{Rc^4}[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2 i) + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2 i) - \\
&\quad \ddot{M}_{33}\sin^2 i - \ddot{M}_{12}\sin 2\phi(1+\cos^2 i) + \ddot{M}_{13}\sin\phi\sin 2i + \ddot{M}_{23}\cos\phi\sin 2i] \\
h_+ &= \frac{2G\Omega^2}{Rc^4}(I_{11} - I_{33})[\sin^2\alpha\cos(2\Omega t) + \sin^2\alpha\cos(2\Omega t)\cos^2 i + \\
&\quad \frac{1}{2}\sin\alpha\cos\alpha\cos(\Omega t)\sin 2i] \\
h_+ &= 2h_0'\sin^2\alpha\cos(2\Omega t)(1+\cos^2 i) + 2\sin\alpha\cos\alpha\sin i\cos i\cos(\Omega t) \\
h_+ &= A_{+,2}\cos 2\Omega t + A_{1,+}\cos\Omega t
\end{aligned} \tag{725}$$

with

$$h_0' \equiv \frac{G\Omega^2}{Rc^4}(I_{11} - I_{33}) \tag{726}$$

$$A_{+,1} \equiv h_0'\sin 2\alpha\sin i\cos i \tag{727}$$

$$A_{+,2} \equiv 2h_0'\sin^2\alpha(1+\cos^2\alpha i) \tag{728}$$

$$\begin{aligned}
h_\times(t;i;\phi=0) &= \frac{G}{Rc^4}\Big[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos i + 2\ddot{M}_{12}\cos 2\phi\cos i \\
&\quad - 2\ddot{M}_{13}\cos\phi\sin i + 2\ddot{M}_{23}\sin\phi\sin i\Big], \\
h_\times &= \frac{2G\Omega^2}{Rc^4}(I_{11} - I_{33})\Big[2\cos i\sin^2\alpha\sin(2\Omega t) + \sin i\sin\alpha\cos\alpha\sin(\Omega t)\Big], \\
h_\times &= h_0'\Big[4\cos i\sin^2\alpha\sin 2\Omega t + \sin i\sin 2\alpha\sin(\Omega t)\Big] \\
h_\times &= A_{\times,1}\sin(\Omega t) + A_{\times,2}\sin 2\Omega t
\end{aligned} \tag{729}$$

with

$$A_{\times,1} \equiv h_0'\sin 2\alpha\sin i \tag{730}$$

$$A_{\times,2} \equiv h_0'\sin^2\alpha\cos i \tag{731}$$

In Eq. 729 t corresponds to the rotated time, and we get GWs radiated in both $\omega_{\rm gw} = \Omega$ and $\omega_{\rm gw} = 2\Omega$

The ratio

$$\frac{A_{+,1}}{A_{\times,1}} = \frac{h_0'\sin 2\alpha\sin i\cos i}{h_0'\sin 2\alpha\sin i} = \cos i \tag{732}$$

and the ratio

$$\frac{A_{+,2}}{A_{\times,2}} = \frac{2h'_0 \sin^2\alpha(1+\cos^2 i)}{4h'_0 \sin^2\alpha \cos i} = \frac{3+\cos 2i}{2\cos i} \tag{733}$$

The ratios 732, 731 produce the inclination angle i, which by definition is $0 \leq i \leq \pi$. Given i, we next determine α and fix $|h'_0|$.

Finally, if we know the distance R of the source, we can determine $|I_{11} - I_{33}|$. The power radiated is given by Eq. ??, and taking another time derivative in Eqs. 719-724 we get:

$$\dddot{M}_{11} = -4\Omega^3(I_{11} - I_{33})\sin^2\alpha \sin 2\Omega t \tag{734}$$
$$\dddot{M}_{21} = 4\Omega^3(I_{11} - I_{33})\sin^2\alpha \cos 2\Omega t \tag{735}$$
$$\dddot{M}_{22} = 4\Omega^3(I_{11} - I_{33})\sin^2\alpha \sin 2\Omega t \tag{736}$$
$$\dddot{M}_{13} = -\Omega^3(I_{11} - I_{33})\sin\alpha\cos\alpha\cos(\Omega t) \tag{737}$$
$$\dddot{M}_{23} = \Omega^3(I_{11} - I_{33})\sin\alpha\cos\alpha\sin(\Omega t) \tag{738}$$
$$\dddot{M}_{33} = 0 \tag{739}$$

Observe here that $\dddot{M}_{11} = -\dddot{M}_{22}$, so

$$\sum_{\kappa} = \dddot{M}_{11} + \dddot{M}_{22} + \dddot{M}_{33} = 0 \tag{740}$$

And the relation of emitted power reads:

$$\begin{aligned}
P_{\text{quad}} &= \frac{G}{5c^5}\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}(\dddot{M}_{\kappa\kappa})^2\rangle, \\
P_{\text{quad}} &= \frac{G}{5c^5}\Big\langle (\dddot{M}_{11})^2 + (\dddot{M}_{22})^2 + (\dddot{M}_{33})^2 + 2(\dddot{M}_{12})^2 + 2(\dddot{M}_{13})^2 + 2(\dddot{M}_{23})^2 \Big\rangle, \\
P_{\text{quad}} &= \frac{G\Omega^6}{5c^5}(I_{11} - I_{33})^2\Big(32\sin^4\alpha + 2\cos^2\alpha\sin^2\alpha\Big), \\
P_{\text{quad}} &= \frac{2G\Omega^6}{5c^5}(I_{11} - I_{33})^2\sin^2\alpha\Big(16\sin^2\alpha + \cos^2\alpha\Big).
\end{aligned} \tag{741}$$

<u>The back-reaction of GWs</u>

Based on Eq. 741, we write the radiated energy in GWs supplied by the rotational energy E_{rot} of the rigid body as:

$$\frac{dE_{\text{rot}}}{dt} = \frac{2G}{5c^5}(I_1 - I_3)^2\sin^2\alpha\Big(\cos^2\alpha + 16\sin^2\alpha\Big). \tag{742}$$

The angular momentum radiated is given for $Q_{ij} \to M_{ij}$ and $J' \equiv J^3 \equiv J$, we get:

$$
\begin{aligned}
\frac{dJ^3}{dt}\Big|_{\text{quad}} &= \frac{2G}{5c^5}\epsilon^{3kl}\langle \dddot{Q}_{al}\ddot{Q}_{ak}\rangle \\
\frac{dJ}{dt}\Big|_{\text{quad}} &= \frac{2G}{5c^5}\left[\langle \dddot{Q}_{2a}\ddot{Q}_{1a}\rangle + \langle \dddot{Q}_{1a}\ddot{Q}_{2a}\rangle\right] \\
\frac{dJ}{dt}\Big|_{\text{quad}} &= \frac{2G}{5c^5}\langle \dddot{M}_{2a}\ddot{M}_{1a} - \dddot{M}_{1a}\ddot{M}_{2a}\rangle \\
\frac{dJ}{dt}\Big|_{\text{quad}} &= -\frac{4G}{5c^5}\langle \dddot{M}_{2a}\ddot{M}_{1a}\rangle
\end{aligned}
\tag{743}
$$

Inserting Eqs. 732 we get:

$$
\begin{aligned}
\frac{dJ}{dt}\Big|_{\text{quad}} &= -\frac{4G}{5c^5}\langle \ddot{M}_{11}\dddot{M}_{21} + \ddot{M}_{12}\dddot{M}_{22} + \ddot{M}_{13}\dddot{M}_{23}\rangle, \\
&= -\frac{4G}{5c^5}\langle 8\Omega^5(I_{11}-I_{33})^2\sin^4\alpha\cos^2(2\Omega t) + 8\Omega^5(I_{11}-I_{33})^2\sin^4\alpha\sin^2(2\Omega t) \\
&\quad + \Omega^5(I_{11}-I_{33})^2\sin^2\alpha\cos^2\alpha\sin^2(\Omega t)\rangle, \\
&= -\frac{4G\Omega^5}{5c^5}(I_{11}-I_{33})^2\sin^2\alpha\big(\langle 8\sin^2\alpha\rangle + \cos^2\alpha\langle\sin^2(\Omega t)\rangle\big), \\
&= -\frac{2G\Omega^5}{5c^5}(I_{11}-I_{33})^2\sin^2\alpha\big(16\sin^2\alpha + \cos^2\alpha\big).
\end{aligned}
\tag{744}
$$

Comparing eqs. (744) and (742), we see:

$$
\frac{dE_{\text{rot}}}{dt}\Big|_{\text{quad}} = \Omega\frac{dJ}{dt}\Big|_{\text{quad}} \tag{745}
$$

and Eq. (744) for $J = I_{11}\dot{\beta}$ produces:

$$
\xRightarrow{\Omega=\dot{\beta}} \ddot{\beta} = -\frac{2G}{5c^5}\frac{(I_{11}-I_{33})^2}{I_{11}}\dot{\beta}^5\sin^2\alpha(16\sin^2\alpha + \cos^2\alpha) \tag{746}
$$

The D.E. for a reads from Eq. (704) for

$$
\begin{aligned}
\omega_1' &= \frac{J_1'}{I_{11}} \equiv \frac{J}{I_{11}}\sin\alpha\sin\gamma \\
\omega_2' &= \frac{J}{I_{21}}\sin\alpha\cos\gamma \\
\omega_3' &= \frac{J}{I_{33}}\sin\alpha
\end{aligned}
$$

So we have

$$
\begin{aligned}
E_{\text{rot}} &= \frac{1}{2}(I_{11}w_1'^2 + I_{22}w_2'^2 + I_{33}w_3'^2) \\
E_{\text{rot}} &= \frac{1}{2}J^2\left(\frac{\sin^2\alpha\sin^2\gamma}{I_{11}} + \frac{\sin^2\alpha\cos^2\gamma}{I_{11}} + \frac{\cos^2\alpha}{I_{33}}\right) \\
E_{\text{rot}} &= \frac{J^2}{2}\left(\frac{\sin^2\alpha}{I_{11}} + \frac{\cos^2\alpha}{I_{33}}\right)
\end{aligned}
\tag{747}
$$

With time derivative:

$$\begin{aligned}
\frac{dE_{\text{rot}}}{dt} &= J\dot{J}\left(\frac{\sin^2\alpha}{I_{11}}+\frac{\cos^2\alpha}{I_{33}}\right)+\frac{J^2\dot{\alpha}}{2}\left(\frac{2\sin\alpha\cos\alpha\dot{\alpha}}{I_{11}}+\frac{2\cos\alpha\sin\alpha}{I_{33}}\right)\\
&-\frac{2G}{5c^5}(I_{11}-I_{33})^2\dot{\beta}^6\sin^2\alpha(\cos^2\alpha+16\sin^2\alpha)\\
&= I_{11}\dot{\beta}\left(-\frac{2G}{5c^5}(I_{11}-I_{33})^2\dot{\beta}^5\sin^2\alpha(\cos^2\alpha+16\sin^2\alpha)\right)\left(\frac{\sin^2\alpha}{I_{11}}+\frac{\cos^2\alpha}{I_{33}}\right)\\
&+\frac{I_{11}^2\dot{\beta}^2\dot{\alpha}}{2}\left[\frac{\sin\alpha\cos\alpha(I_{33}-I_{11})}{I_{11}I_{33}}\right]\Rightarrow\\
\dot{a} &= -\frac{2G^5}{5c^5}\dot{\beta}^4\frac{I_{11}^2-I_{33}^2}{I_{11}}\sin\alpha\cos\alpha(\cos^2\alpha+16\sin^2\alpha)
\end{aligned}\tag{748}$$

Next, we see that due to the back-reaction of GWs, both the inclination angle α and the angular velocity $\dot{\beta}$ decrease. The term $\frac{d}{dt}(J\cos\alpha)$ is:

$$\begin{aligned}
\frac{d'}{dt}(J\cos\alpha) &= \frac{dJ}{dt}\cos\alpha - J\sin\alpha\dot{\alpha}\\
&= -\frac{2G}{5c^5}(I_{11}-I_{33})^2\Omega^5\sin^2\alpha\cos\alpha(\cos^2\alpha+16\sin^2\alpha)\\
&+\frac{2G}{5c^5}(I_{11}-I_{33})^2\frac{I_{11}\Omega}{I_{11}}\Omega^4\sin^2\alpha\cos\alpha(\cos^2\alpha+16\sin^2\alpha)\Rightarrow\\
&\frac{d(J\cos\alpha)}{dt}=0
\end{aligned}\tag{749}$$

$$J\cos\alpha = ct \tag{750}$$

which shows that as $J\downarrow$ then $\cos\alpha\uparrow$ and $J\cos\alpha$ is constant.

The projection of angular momentum on x_3' is denoted by $J\cos\alpha$, and we see that the rigid body rotates around its longitudinal axis with constant velocity

$$\omega_3' = \frac{J}{I_3}\cos\alpha$$

And the rotation remains unaffected by GW backreaction. Based on the above, we introduce a timescale:

$$\tau_o \equiv \left[\frac{2G}{5c^5}\frac{(I_{11}-I_{33})^2}{I_{11}}\dot{\beta}_o^4\right]^{-1} \tag{751}$$

$$\tau_o \equiv \left[\frac{2G}{5c^5}\frac{(I_{11}-I_{33})^2}{I_{11}}\left(\frac{f_0}{2\pi}\right)^4\right]^{-1} \tag{752}$$

For this time scale the rigid body aligns its rotation axis with the angular momentum's direction for $\alpha\to 0$, while the rotational angular velocity $\dot{\beta}=\Omega$ around x_3 decreases toward the constant value $\Omega_0\cos\alpha_o$ and the rotational velocity $\omega_3'=\Omega\cos\alpha$ is constant.

5.6 Radial infall into a black hole

GWs can also be produced after an object falls into a black hole radially. This subject is discussed in reports [75], [76], and [77]. The complete energy spectrum is computed in [78] and [79].

5.6.1 Radiation from an infalling point-like mass

We want an expression for the radiation generated by a point-like mass m, radially falling into a BH of mass M with $m \ll M$. We super-simplify this example by using linearized equations for GW production and Newtonian equations of motion. In general, this is not the correct way of doing it, since in linearized theory we expand around a flat space instead of a Schwarzschild and Newtonian equations of motion should be the Schwarzschild geodesics. In this case, for a particle coming from the positive values of the z axis, with zero velocity at infinity, we write:

$$\frac{1}{2} m\dot{z}^2 - \frac{GmM}{z} = 0 \tag{753}$$

$$\Rightarrow \dot{z}^2 = \frac{2GM}{z} \Rightarrow |\dot{z}| = c\left(\frac{2GM}{c^2 z}\right)^{1/2} \Rightarrow |\dot{z}^2| = c\sqrt{\frac{R_s}{z}} \Rightarrow \dot{z} = -c\sqrt{\frac{R_s}{z}} \tag{754}$$

Here we also assume that most of the radiation is emitted when the particle is non-relativistic, and we therefore use the quadrupole formula. The above assumptions are valid for distances $z \gg R_s$ since at $z = R_s$ or z close to the BH horizon, they break down. Equation 754 is incompatible with non relativistic particle assumption at $z = R_s$ produces:

$$\dot{z} = -c \tag{755}$$

However, at large distances, the flat space Newtonian approximation is correct, and the above equations become legitimate. We compute radiation emitted from $z = +\infty$ to $z = R \gg R_s$. Here we have only one condition $x_i \equiv z(t)$ and the second mass moment reads:

$$\begin{aligned} M^{ij} &= m x^i(t) x^j(t) \Rightarrow \\ M^{33} &= m z^2(t) \end{aligned} \tag{756}$$

Then we can produce the following:

$$\begin{aligned} P_{\text{quad}} &= \frac{2}{15}\frac{G}{c^5}\langle \dddot{M}_{ij}\dddot{M}_{ij}\rangle \Rightarrow \\ P_{\text{quad}} &= \frac{2}{15}\frac{G}{c^5}\langle \dddot{M}_{33}^2\rangle \Rightarrow \\ P_{\text{quad}} &= \frac{2Gm^2}{15c^5}\left\langle \left(\frac{d^3}{dt^3} z^2(t)\right)^2 \right\rangle \\ P_{\text{quad}} &= \frac{2Gm^2}{15c^5}\langle (6\dot{z}\ddot{z} + 2z\dddot{z})^2\rangle \end{aligned} \tag{757}$$

Equation 757 produces the total radiated power. In the quadrupole approximation, the total radiated energy is:

$$E=\int_{-\infty}^{t_{\max}} dt\frac{dE}{dt}=-\int_{-\infty}^{t_{\max}} dtP_{\text{quad}}=-\frac{2Gm^2}{15c^5}\int_{-\infty}^{t_{\max}} dt\,(6\dot{z}\ddot{z}+2z\dddot{z})^2 \tag{758}$$

with $t_{\max}: z(t_{\max})=R$. Using chain rule for $dt\equiv\frac{dz}{\dot{z}}$ and Equation 754 and the third derivative of it:

$$\dot{z}=-c\sqrt{\frac{R_s}{z}}\Rightarrow \dot{z}^2=c^2\frac{R_s}{z} \tag{759}$$

$$\Rightarrow 2\dot{z}\ddot{z}=-\frac{R_sc^2}{z^2}\Rightarrow \dot{z}\ddot{z}=-\frac{c^2R_s}{2z^2}\dot{z} \tag{760}$$

$$\begin{aligned}\Rightarrow \dddot{z}&=\frac{c^2R_s}{2}\frac{2}{z^3}=\frac{c^2R_s}{z^3}\dot{z}\Rightarrow\\ \dddot{z}&=\frac{c^2R_s}{z^3}(-c)\left(\frac{R_s}{z}\right)\Rightarrow\\ \dddot{z}&=-\frac{c^3R_s^{3/2}}{z^{7/2}}\end{aligned} \tag{761}$$

Equation 758 reads in terms of Eqs. 759 - 761 as follows:

$$\begin{aligned}E&=-\frac{2Gm^2}{15c^5}\int_{-\infty}^{t_{\max}}\frac{dz}{\dot{z}}\left[6c\left(\frac{R_s}{z}\right)^{\frac{1}{2}}c^2R_s\frac{1}{2z^2}+2z(-c^3R^{3/2})\frac{1}{z^{7/2}}\right]^2\\ &=-\frac{2Gm^2}{15c^5}c^6R_s^3\int_{-\infty}^{t_{\max}}\frac{dz}{\dot{z}}\left[\frac{3}{z^{5/2}}-\frac{2}{z^{5/2}}\right]^2\\ &=-\frac{2Gm^2}{15c^5}c^6R_s^3\int_{-\infty}^{t_{\max}}\frac{dz}{\dot{z}}\frac{1}{z^5}=\frac{2Gm^2c}{15}R_s^3\int_{-\infty}^{t_{\max}}dz\frac{z^{1/2}}{cR^{1/2}}\frac{1}{z^5}\\ &=-\frac{2Gm^2R^{5/2}}{15}\int_{-\infty}^{t_{\max}}dz\frac{1}{z^{9/2}}\end{aligned}$$

Setting $v=\frac{z}{R_s}\Rightarrow z=vR_s$ with

$$\begin{cases}z(t=-\infty)=+\infty \quad\Rightarrow\quad v\to+\infty,\\ z(t_{\max})=R \quad\Rightarrow\quad v\to\frac{R}{R_s}.\end{cases}$$

$$\begin{aligned}E&=-\frac{2Gm^2R_s^{5/2}}{15}\int_{-\infty}^{R/R_s}\frac{R_s}{v^{9/2}R_s^{9/2}}=\frac{2Gm^2}{15R_s}\int_{-\infty}^{R/R_s}duu^{-9/2}\\ E&=\frac{2Gm^2}{15R_s}\frac{2}{7}\left(\frac{R}{R_s}\right)^{-7/2}\\ E&=\frac{4Gm^2}{105R_s}\left(\frac{R_s}{R}\right)^{7/2}\end{aligned} \tag{762}$$

We can extrapolate the result of Eq. 762 at $R = R_s$.

$$\begin{aligned} E|_{R=R_s} &= \frac{4}{105}\frac{Gm^2}{R_s} = \frac{4}{105}\frac{Gm^2c^2}{2MG} = \frac{2}{105}mc^2\frac{m}{M} \\ E|_{R=R_s} &\approx 0,0019mc^2\frac{m}{M} \end{aligned} \tag{763}$$

The extrapolation of equation 763 is remarkably close to the relativistic results.

$$E_{relat} = 0.010\, mc^2\,\frac{m}{M} \tag{764}$$

This is possible because outside the BH horizon. The particle's motion is dominated by the lowest orders in the multipole expansion. Also, the rest energy inside the horizon is decreased by a factor $\frac{m}{M}$. With equation 764, we can calculate the radiated energy in GWs in the head-on collision of the BHs with equal masses M. The reduced mass in this case is $M/2$ and equation 764 produces:

$$\begin{aligned} E_{\text{relat}} &= 0,010\frac{M}{2}c^2\frac{\frac{M}{2}}{M} \\ E_{\text{relat}} &= 0,0025Mc^2 = 2,5\cdot 10^{-3}Mc^2 \end{aligned} \tag{765}$$

The result is quite close to the expected value

$$E' = (1-2)\times 10^{-3}Mc^2 \tag{766}$$

The frequency spectrum of the radiation emitted by a radially infalling particle is the Fourier transform of a function $F(t)$, well-defined on the interval $-\infty < t + \infty$ The Newtonian approximately is valid up to a value $t_{\max}$, sud that 2 $t_{\max} = R \gg R_s$. Therefore, the Newtonian approximation does not represent the full form of the spectrum. A typical system with size d and velocity v radiates GWs with reduced wavelength $\lambdabar \sim \frac{dc}{v}$. When the particle approaches the horizon, the size d is of order R_s and $v \sim c$, so $\lambdabar \sim R_s$. On the other hand, at $R \gg R_s$ the length-scale is of order R and $v \ll c$, so the system radiates at $\lambdabar \sim \frac{R_c}{v} \gg R \gg R_s$. With the Newtonian trajectory, we compute only the part of the spectrum at $\lambdabar \gg R_s \Rightarrow \frac{\lambda}{2\pi} \gg R_s \Rightarrow \frac{cd}{2\pi f} \gg R_s \Rightarrow wR_s \gg c$. The complete spectrum peaks at $wR_s \sim c$, with the radiation at these frequencies being generated close to the horizon and cut off at $wR_s > c$, because there is no length-scale smaller than R_s. The first step is to solve the equation of motion in equation 754

$$\begin{aligned} \frac{dz}{dt}z^{1/2} &= -cR_s^{1/2} \Rightarrow \\ \int_{t_0}^{t} dz\ z^{1/2} &= -cR_s^{1/2}\int_{t_0}^{t} dt \Rightarrow \\ \left.\frac{z^{3/2}}{3/2}\right|_{t_0}^{t} &= -cR_s^{1/2}(t-t_0) = cR_s^{1/2}(t_0 - t) \Rightarrow \\ z^{3/2}(t) - z^{3/2}(t_0) &= \frac{3c}{2}R_s^{1/2}(t-t_0) \end{aligned} \tag{767}$$

$$
\begin{aligned}
z^{3/2}(t) &= \frac{3}{2}cR_s^{1/2}t_0 + z^{3/2}(t_0) - \frac{3c}{2}R_s^{1/2}t \Rightarrow \\
z^{3/2}(t) &= \frac{3}{2}cR_s^{1/2}(\bar{t} - t)
\end{aligned}
\tag{768}
$$

Again at $t \to -\infty \Rightarrow z(t \to -\infty) \to +\infty$ and

$$
t = t_{\max} \Rightarrow z(t_{\max}) = R \tag{769}
$$

$$
R^{3/2} = \frac{3}{2}cR_s^{1/2}(\bar{t} - t_{\max}) \Rightarrow t_{\max} = \bar{t} - \frac{2R^{3/2}}{3cR_s^{1/2}} \tag{770}
$$

If we insert the variable $\tau = \bar{t} - t$ we get:

$$
z^{3/2}(\tau) = \frac{3cR_s^{1/2}}{2}\tau \Rightarrow z(\tau) = \left(\frac{3}{2}R_s^{1/2}c\tau\right)^{2/3} \tag{771}
$$

Since $-\infty < t \leq t_{\max} \Rightarrow +\infty > \bar{t} - t \geq \bar{t} - t_{\max}$. From eq. 770 we get:

$$
\frac{2R^{3/2}}{R_s^{1/2}} \leq \tau < +\infty \tag{772}
$$

The total radiated energy is in terms of $\dddot{M}_{ij}$

$$
E = \frac{2G}{15c^5}\int_{-\infty}^{t_{\max}} dt\langle \dddot{M}_{33}^2\rangle = \frac{2G}{15c^5}\int_{-\infty}^{t_{\max}} dt\dddot{M}_{33}^2 \tag{773}
$$

we can write $M_{33}(t) = mz^2(t) \Rightarrow \tilde{M}_{33}(\omega) = m\int_{-\infty}^{t_{\max}} dt z^2(t)e^{i\omega t}$ as a Fourier transform. So,

$$
\begin{aligned}
\dddot{M}_{33} &= m\omega^6 \int_{-\infty}^{t_{\max}} dt|z^2(t)e^{i\omega t}|^2 = \int_{-\infty}^{\omega_{\max}} \frac{d\omega}{2\pi}\omega^6|\tilde{M}_{33}(\omega)|^2 \Rightarrow \\
\dddot{M}_{33} &= 2\int_0^{\omega_{\max}} \frac{d\omega}{2\pi}\omega^6|\tilde{M}_{33}(\omega)|^2
\end{aligned}
\tag{774}
$$

and Eq. 773 reads:

$$
E = \frac{4G}{15c^5}\int_0^{\omega_{\max}} \frac{2\omega}{2\pi}\omega^2|\tilde{\ddot{M}}_{33}(\omega)|^2 \tag{775}
$$

Recalling eqs. 759 - 760 we see:

$$
\begin{aligned}
\ddot{M}_{33}(t) &= m(\dot{z}^2 + z\ddot{z}) = m\left(c^2\frac{R_s}{z} + z\frac{-c^2R_s}{2z^2}\right) \Rightarrow \\
\ddot{M}_{33}(t) &= mc^2R_s\frac{1}{2z} = \frac{mc^2R_s}{2}\left(\frac{2}{3R_s^{1/2}c\tau}\right)^{2/3} \Rightarrow \\
\ddot{M}_{33}(t) &= m\left(\frac{2R_s^{1/2}c^2}{3\tau}\right)^{2/3}
\end{aligned}
\tag{776}
$$

with $\tau \in [\tau_{\min}, +\infty)$. The Fourier transform of $\ddot{M}_{33}(t)$ is:

$$\tilde{\ddot{M}}_{33}(\omega) = m \left(\frac{2R_s c^2}{3}\right)^{2/3} \int_{\tau_{\min}}^{+\infty} d\tau e^{-i\omega\tau} \tau^{-2/3} \tag{777}$$

When $v = \omega\tau$ and $dv = \omega d\tau$ we take:

$$\tilde{\ddot{M}}_{33} = m\omega^{-1/3} \left(\frac{2R_s c^2}{3}\right)^{2/3} \int_{\omega\tau_{\min}}^{\infty} dv\ v^{-2/3} e^{-iv} \tag{778}$$

The leading term is obtained by approximating $\omega_{\min}$ to zero and setting $I = \int_{\omega\tau_{\min}}^{\infty} dv\ v^{-2/3} e^{-iv}$ so we get:

$$\begin{gathered} \int_{\omega\tau\min}^{\infty} dv\ v^{-2/3} e^{-iv} = -i \int_{0}^{\infty} dv\ v^{-2/3}\ (-i)^{-2/3} e^{-v} \\ I = -e^{-\frac{i\pi}{6}} \frac{3}{2} \Gamma(\frac{1}{3}) \end{gathered} \tag{779}$$

Thus, eq. 778 we get:

$$\tilde{\ddot{M}}_{33} = -m\omega^{-1/3} \left(\frac{2R_s c^2}{3}\right)^{-1/3} \Gamma(1/3) e^{-\frac{i\pi}{6}} \tag{780}$$

And eq. 775 reads:

$$\begin{aligned} E &= \frac{4G}{15c^5} \int_{0}^{\omega_{\max}} \frac{d\omega}{2\pi} \omega^2 m^2 \omega^{-2/3} \left(\frac{2R_s c^2}{3}\right)^{-2/3} \Gamma^2(\frac{1}{3}) \Rightarrow \\ \frac{dE}{d\omega} &= \frac{Gm^2}{5\pi c} \Gamma^2(\frac{1}{3}) \left(\frac{2}{3}\right)^{7/3} \left(\frac{\omega R_s}{c}\right)^{4/3} \end{aligned} \tag{781}$$

5.7 Tidal disruption of a real star falling into a BH. Coherent and incoherent radiation

The suppression due to tidal disruption is analyzed in [80] and [81]. A point-like particle is an idealization, and in astrophysical applications, we are interested in the infall of an extended object, i.e., a main sequence star, a white dwarf, a dwarf, or a NS. Because of tidal disruption of the star falling into a black hole, the radiation can be emitted incoherently, and this reduces the GW amplitude by many orders of magnitude. Qualitatively, the difference between coherent and incoherent radiation is understood in equation ??. The radiated energy E is $E \sim \frac{m^2}{M}$, where in the reduced mass of the particle-BH system. If an extended object has N particles of mass δm, we get $m = N\ N\delta_m$. If the N particles radiate in a coherent way as a single object of mass m, we see:

$$E^{\text{coher.}} \sim \frac{m^2}{M} = N^2 (\frac{\delta m^2}{M})$$

The N^2 dependence can be understood by observing that the total amplitude of the GWs is the sum of separate amplitudes as: $h_{\text{tot}} = \sum_{i=1}^{N} h_i$. When the radiation is coherent, we get

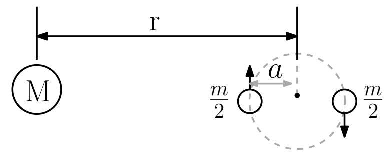

Figure 6: An infalling star of radius a is tidally deformed by the black hole when it enters within the tidal radius r_{tidal}. By the time the horizon is approached, the star is an ellipsoid with semimajor axis a_h, adapted by Maggiore's book, Gravitational Waves [15].

the same phase in $hi, \forall i$ so $h_{tot2} \sim N$ and the radiated energy is $E_{rad} \sim h_{tot} \sim O(N^2)$. On the other hand, incoherent radiation comes from the destructive interference of off-diagonal terms, leaving only diagonal terms $\sum h^2 i$. In this case, the incoherent radiated energy is:

$$E^{coher.} \sim N\frac{(\delta m)^2}{M} = \frac{I}{N}\frac{(N\delta m)^2}{M} = \frac{I}{N}\frac{m^2}{M} \tag{782}$$

So E incoherent is smaller by a factor N than Ecoherent. Whether a distribution radiates coherently or not depends on:

i. the wavelength of the GW we consider

ii. the linear size α of the system.

If $\alpha \ll \lambdabar$, the phase of GW does not change appreciably over the source, and the radiation is coherent. If $\alpha \gg \lambdabar$, the phase of each simple consistent oscillates strongly over the system, and the mixed terms cancel. (averaging to zero), So the radiation is coherent. The transition between the two regimes is governed by a form factor. The distortion of the shape of an infalling star by the tidal grow field of a BH. Any star is held together by self-gravity. We model a star of mass m as two particles of mass $\frac{m}{2}$ orbiting in circular orbit of radius α as shown below:

where the BH has mass M at distance r from the c.o.m. The tidal force that disrupts the star is:

$$\begin{aligned} &F_{\text{tidal}} = \frac{GM(\frac{m}{2})}{(r-\alpha)^2} - \frac{GM(\frac{m}{2})}{(r+\alpha)^2} \Rightarrow F_{tidal} = GM(\frac{m}{2})\Big[\frac{(r+\alpha)^2-(r-\alpha)^2}{(r-\alpha)^2(r+\alpha)^2}\Big] \Rightarrow \\ &F_{\text{tidal}} \simeq 2GMm\frac{\alpha}{r^3} \end{aligned} \tag{783}$$

When F_{tidal} is bigger than F_{grow}, the star breaks down so

$$\begin{aligned} 2GMm\frac{\alpha}{r^3} &> \frac{G(\frac{m}{2})^2}{(2\alpha)^2} \Rightarrow \\ r^3\frac{m^2}{4} &< 8\alpha^3 Mm \Rightarrow \\ r^3 &< 32\alpha^3\frac{M}{m} \end{aligned} \tag{784}$$

$$r < r_{\text{tidal}} = \sqrt[3]{32}\alpha \left(\frac{M}{m}\right)^{1/3} \tag{785}$$

The numerical coefficient depends on the schematization of the extended object.
When we consider a sphere of mass m, mean radius a, and constant density, we get:

$$r_{\text{tidal}} \simeq 2,2 \left(\frac{M}{m}\right)^{1/3} a \tag{786}$$

The star's radius is far away the from BH, when it is near the horizon, it has radius a_h. We estimate the order of magnitude of a_h using the Newtonian trajectory for a radially infalling particle along the z-axis. From 767 we get

$$z(t) = \left[z_0^{3/2} + \frac{3}{2}R_s^{1/2}c(t_0 - t)\right]^{2/3}$$

Variation w.r.t. z produces:

$$\begin{aligned}\delta(z^{3/2}) &= \delta\left[z_0^{3/2} + \frac{3}{2}R_s^{1/2}c(t_0 - t)\right] \\ \delta z(t) &= \left(\frac{z_0}{z(t)}\right)^{1/2} \delta z_0\end{aligned} \tag{787}$$

Eq. 787 shows that two points at time t_0, that are separated by δz_0 radial distance, at time t will separate by $\delta z(t)$. Then t_0 is the time when the star is an ellipsoid with semimajor axis a_h given as:

$$a_h \equiv \left(\frac{r_{\text{tidal}}}{R_s}\right)^{1/2} \alpha \tag{788}$$

The evolution of the shape of the star as it plunges toward the BH is shown below:
When a main sequence star of a star of mass $1M_\odot$ has a radius $a = 7.105$ km. If it falls into a BH of $10M_\odot$ and $R_S = 30$ km, we have the tidal radius is given by:

$$r_{\text{tidal}} = 4,7a \tag{789}$$

and

$$(r_{\text{tidal}}R_S)^{1/2} = 300. \tag{790}$$

Most of the radiation is emitted while the star is close to the horizon, so it has a a_h size in the radial direction. A source radiates coherently only when $\lambda\!\!\!^{-} \gg a_h \Rightarrow a_r$ equivallently $\frac{c}{2\pi f} \gg a_h \Rightarrow W \ll \frac{c}{a_h}$. We define a parameter that governs the loss of coherence as

$$A(\omega) \equiv \frac{\omega a_h}{c} = \frac{\omega a}{c}\left(\frac{r_{\text{tidal}}}{R_S}\right)^{1/2} \tag{791}$$

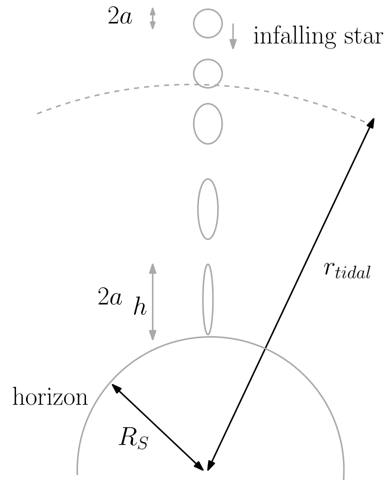

Figure 7: An infalling star of radius a is tidally deformed by the black hole when it enters within the tidal radius r_{tidal}. By the time the horizon is approached, the star is an ellipsoid with semimajor axis a_h, adapted by Maggiore's book, Gravitational Waves [15].

When $A(\omega) \gg 1$, we have incoherent radiation, while for $A(\omega) \gg 1$ we get coherent radiation. In the $\alpha \to 0$ limit, we get $A(\omega) \to 0$ and the point-like result. Based on Shapiro's calculations, we see that the peak in frequency is at $\omega = \bar{\omega} \simeq 0,84c/R_s$. When $A(\omega) \ll 1$ we get:

$$\begin{aligned} \omega &\ll \frac{c}{a_h} \Rightarrow \\ \frac{0,64c}{R_s} &\ll \frac{c}{a_h} \Rightarrow \\ \frac{0,64c}{R_s} &\ll \frac{1}{a_h} \end{aligned} \tag{792}$$

When condition 792 applies, only the high-frequency tails of the point-like spectrum are suppressed. Although the contribution of high-frequency tails is negligible, since they are exponentially suppressed. Therefore, when $\frac{0,64}{R_s} \ll a_h^{-1}$, we get the same total power radiated as in the point-like mass case. On the other hand, when $\frac{0,64}{R_s} \gg a_h^{-1}$, we get incoherent radiation suppressing the coherent parts, where the peak lies and the power is concentrated. We can redefine $A(\omega)$ as:

$$\bar{A}(\bar{\omega}) \equiv \frac{\bar{\omega} a_h}{c} \cong \frac{0,64 a_h}{R_s} \tag{793}$$

$$\begin{aligned} \bar{A}(\bar{\omega}) &\equiv \frac{\bar{\omega} a_h}{c} \left(\frac{r_{\text{tidal}}}{R_s} \right)^{1/2} \\ \bar{A}(\bar{\omega}) &\cong \frac{a}{R_s} \left(\frac{M}{m} \right)^{1/2} \end{aligned} \tag{794}$$

Parameter $\bar{A}$ in 794 shows the suppression of total radiated power:

When $\bar{A} \succeq 1$, the radiated power is strongly suppressed, where $\left(\frac{r_{\text{tidal}}}{R_s}\right)^{1/2}$ is the dilatation factor.

Stars with larger radii have weaker self-gravity and thus resistance to tidal forces of BHs. The solution of Newtonian equations of motion for a particle falling along the z axis (at $t = 0$ has $z(t) \equiv z_i$) reads as:

$$z^{3/2}(t; \bar{t_i}) = \frac{3}{2} R_s^{1/2} c(\bar{t_i} - t) \tag{795}$$

where

$$\bar{t_i} \equiv t_0 + \frac{2z_i^{3/2}}{3cR_s^{1/2}} \tag{796}$$

If we consider a swarm of N particles with mass δ_m we get

$$\tilde{M}_{33}(\omega) = \delta_m \sum_{i=1}^{N} \int_{-\infty}^{t_{\max}} dt z^2(t; \bar{t_i}) e^{i\omega t} \tag{797}$$

$$\tilde{M}_{33} = \delta m \sum_{i=1}^{N} \int_{-\infty}^{t_{\max}} z^2(t, 0) e^{t+\bar{t_i}} \tag{798}$$

$$\tilde{M}_{33} = \left[N\delta m \int_{-\infty}^{t_{\max}} dt z^2(t, 0) \right] \left[\frac{1}{N} \sum_{i=1}^{N} e^{i\omega(t+\bar{t_i})} \right] \tag{799}$$

We define the form factor as the second bracket:

$$F(\omega) \equiv \frac{1}{N} \sum_{i=1}^{N} e^{i\omega(t+\bar{t_i})} \tag{800}$$

The COM crosses the tidal radius at $t = t_0$, so the constituent of the sphere is located at $z_i = r_{\text{tidal}} + \delta z_i$ with $-a < \delta z_i < a$ and $|\delta z_i| \ll r_{\text{tidal}}$. Eq. 796 for $z_i \equiv r_{\text{tidal}} + \delta z_i$ yields:

$$\bar{t_i} \approx t_0 + \frac{2r_{\text{tidal}}^{3/2}}{3cR_s^{1/2}} + \left(\frac{r_{\text{tidal}}}{R_s}\right)^{1/2} \frac{\delta z_i}{c} \tag{801}$$

Now we can rewrite the form factor as:

$$F(\omega) = \frac{1}{N} exp \left[i\omega \left(t_0 + \frac{2r_{\text{tidal}}^{3/2}}{3cR_s^{1/2}} \right) \right] \sum_{i=1}^{N} exp \left[i\omega \left(\frac{r_{\text{tidal}}}{R_s}\right)^{1/2} \frac{\delta z_i}{c} \right] \tag{802}$$

In the continuous limit we get $\delta z_i \to \delta z$ and

$$F(\omega) = \frac{1}{v} exp \left\{ i\omega \left[t_0 + \frac{2r_{\text{tidal}}^{3/2}}{3cR_s^{1/2}} \right] \right\} \pi \int_V d(\delta z) exp \left[i\omega \left(\frac{r_{\text{tidal}}}{R_s}\right)^{1/2} \frac{\delta z_i}{c} \right] \tag{803}$$

V is the volume of the system at $t = t_0$, for a sphere of uniform density, radius α, we have

$$V = \frac{4}{3}\pi\alpha^3. \tag{804}$$

For $\delta z = \alpha u$, the transerve direction, is x with:

$$|x_1|^2 = \alpha^2 - (\delta z)^2 = \alpha^2(1-u^2) \tag{805}$$

And eq. 803 writes:

$$F(\omega) = \frac{3}{4} exp\left[i\omega \left(t_0 + \frac{2r_{\text{tidal}}^{3/2}}{3cR_s^{1/2}} \right) \right] \int_{-1}^{1} du(1-u^2)e^{iA(\omega)u} \tag{806}$$

Setting $x = A(\omega)u$ we get:

$$F(\omega) = \frac{3}{4} exp\left[i\omega \left(t_0 + \frac{2r_{\text{tidal}}^{3/2}}{3cR_s^{1/2}} \right) \right] \int_{-A(\omega)}^{A(\omega)} \frac{dx}{A(\omega)} \left[\left(1 - \frac{x^2}{A^2(\omega)}\right) \cos x + i \left(1 - -\frac{x^2}{A^2(\omega)}\right) \sin x \right] \tag{807}$$

set $\phi = t_0 + \frac{2r_{\text{tidal}}^{3/2}}{3cR_s^{1/2}}$

$$F(\omega) = \frac{3}{4} \int_{1}^{1} du\, (1-u^2) e^{iA(\omega)u} = \frac{3e^{i\omega\phi}}{2A^2(\omega)} \left(\sin A(\omega) - A(\omega) \cos A(\omega)\right) \tag{808}$$

The real star spectrum is:

$$\left.\frac{dE}{d\omega}\right|_{\text{real star}} = F(\omega)^2 \left.\frac{dE}{d\omega}\right|_{\text{point-like}} \tag{809}$$

where we set the point like spectrum to be:

$$\frac{dE}{d\Omega} = \frac{2G\omega^2}{15\pi c^5} N\delta m \int_{-\infty}^{t_{max}} dt z^2(t;0) \tag{810}$$

6 Experimental observations of GW emission

In this section we review experimental evidences for the existence of GWs, as derived by the Hulse-Taylor binary pulsars. A general introduction in pulsars is conducted in [82] and [83]. A complete and regularly updated catalogue of Pulsars existing in Cosmos can be found in [84]. For events before the discovery of PSR B1913+16, see Hulse and Taylor's article in [85], [86] and [87], while in [88], [89] and [90] we get the expected results of observations of gravitational radiation from PSR B1913+16. Finally, in [91] and [92] we have an update on these results.

Pulsar timing formula analyzed in section 5.1 and time delays due to GR in section 5.2, are explicitly derived in the work of Backer and Hellings, see [93], as well as in Stairs' article in [94]. Many classical textbooks also analyze pulsar timing, with [95], [82] and [23]. A comparison with General Relativistic effects, such as Shapiro time delay, is given in [96].

Pulsars are identified by the prefix **PSR**, followed by their equatorial coordinates (α, δ), where α represents the right ascension, expressed in hours and minutes and δ represents the declination, or inclination angle, with $\delta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The Hulse-Taylor binary pulsar, denoted as **PSR B1913+16**, was first detected in 1974. Observations revealed significant secular changes in the pulsar's period, with variations of approximately $10\,\mu$s per year. Furthermore, day-to-day changes in the period were observed, reaching up to $\sim 80\,\mu$s. These daily fluctuations were attributed to Doppler shifts caused by the pulsar's orbital motion around a companion star.

The table below summarizes the measured orbital parameters of the Hulse-Taylor binary pulsar along with their experimental uncertainties:

Parameter	**Value**	**Error**
$(1/c)a_p \sin i$ (s)	2.3417725(8)	(8)
e	0.6171338(4)	(4)
T_0 (MJD)	52144.90097844(5)	(5)
P_b (days)	0.322997448930(4)	(4)
ω_0 (deg)	292.54487(8)	(8)
$\langle\dot{\omega}\rangle$ (deg/yr)	4.226595(5)	(5)
γ (s)	0.0042919(8)	(8)
$\dot{P}_b$	$-2.4184(9) \times 10^{-12}$	(9)

Table 1: Measured orbital parameters of the Hulse-Taylor binary pulsar, along with their errors, adapted by Gravitational Waves [15].

The orbital period of the system is less than 8 days, with an orbital velocity on the order of $v \sim 10^{-3}c$. The geometry of the system is illustrated in the figure below:

The relative coordinate between the pulsar and its companion, $\vec{r} = \vec{r}_c - \vec{r}_p$, describes an elliptical orbit with eccentricity e. The normal to the plane of the orbit forms an angle i with respect to the line of sight (assumed to be the z-axis).

The orbit intersects the (x, y)-plane at two points, known as the "nodes." The line connecting these two nodes is called the line of nodes. The node where the coordinate $\vec{r}$ transitions from the lower hemisphere to the upper hemisphere is referred to as the ascending

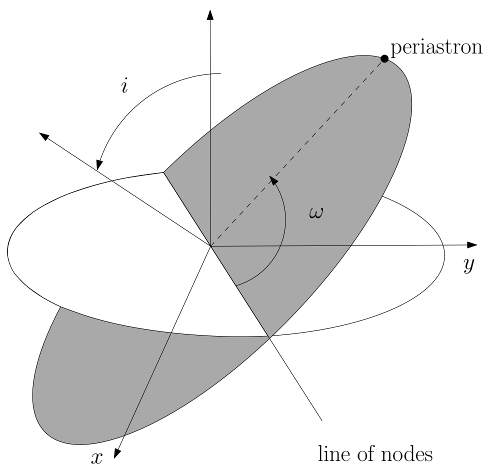

Figure 8: The geometry of the orbit. The plane of the orbit is in gray, adapted by Maggiore's book, Gravitational Waves [15].

node. The angular position of the periastron, measured from the ascending node, is denoted by ω.

The advance of the periastron is represented by $\langle\dot{\omega}\rangle$, while the Einstein parameter is denoted by γ. The system's dynamics depend on the masses of the pulsar (m_p) and its companion (m_c). All astrophysical quantities can be expressed through the above as follows:

$$\dot{\omega} = \frac{3G^{\frac{2}{3}}}{c^2}\left(\frac{P_b}{2\pi}\right)^{-\frac{5}{3}}\frac{(m_p+m_c)^{\frac{2}{3}}}{1-e} = 2.11353\left(\frac{m_p+m_c}{M_\odot}\right)^{\frac{2}{3}}\frac{deg}{y_r} \tag{811}$$

$$\gamma = \frac{G^{\frac{2}{3}}e}{c^2}\left(\frac{P_b}{2\pi}\right)^{\frac{1}{3}}\frac{m_c(m_p+2m_c)}{(m_p+m_c)^{\frac{4}{3}}} = 0.00293696\left(\frac{m_c}{M_\odot}\right)\left(\frac{m_p+2m_c}{M_\odot}\right)\left(\frac{m_p+m_c}{M_\odot}\right)^{-\frac{4}{3}} \tag{812}$$

$$\alpha = \frac{G^{\frac{1}{3}}}{c}\left(\frac{P_b}{2\pi}\right)^{\frac{2}{3}}(m_p+m_c)^{\frac{1}{3}} \tag{813}$$

$$\sin i = \frac{c}{G^{\frac{1}{3}}}\left(\frac{\alpha_p \sin i}{m_c}\right)\left(\frac{P_b}{2\pi}\right)^{-\frac{2}{3}}(m_p+m_c)^{\frac{2}{3}} \tag{814}$$

$$\alpha_p = \alpha m_c(m_p+m_c)^{-1} \tag{815}$$

$$\alpha_c = \alpha m_p(m_p+m_c)^{-1} \tag{816}$$

$$\text{So } m_p = 1.44214M_\odot \text{ and } m_c = 1.3867M_\odot \tag{817}$$

and $\alpha \approx 2.2\times10^9$ m and $R_\odot \approx 7\times10^8$ m. The compactness of the orbit, combined with the absence of any observed eclipse, suggests that the companion is likely a compact star (neutron star or black hole). The dynamics of the binary system can be studied by treating the two stars as point-like bodies, ignoring tidal effects. The orbital period appears to decrease due to gravitational wave (GW) emission. The decrease in the orbital period is given by the following formula:

$$\dot{P}_b = -\frac{192\pi G^{\frac{5}{3}}}{5c^5} \frac{m_p m_c}{(m_p + m_c)^{\frac{1}{3}}} \left(\frac{2\pi}{P_b}\right)^{\frac{5}{3}} \frac{1}{(1-e^2)^{\frac{7}{2}}} \left(1 + \frac{73}{2}e^2 + \frac{37}{96}e^4\right) \tag{818}$$

If we include a Doppler correction due to the relative velocity between us and the pulsar, induced by the differential rotation of the galaxy, we see that the ratio between the GR-predicted and corrected expected values, $(\dot{P}_b)_{\text{cor}}$ and $(\dot{P}_b)_{\text{GR}}$, the relation is given by:

$$\frac{\dot{P}_b|_{\text{corr}}}{\dot{P}_b|_{\text{GR}}} = 1.0013\ (21) \tag{819}$$

6.1 Pulsar timing formula

Neutron stars are rapidly spinning with periods as small as 1.5 milliseconds. This comes as a consequence of angular momentum conservation during the collapse. The term ωr^2 must remain constant, while r decreases from the typical stellar size of the original star core to a radius of just 10 km. The supernova collapse can spin NSs up to 10 milliseconds. In binary systems, NSs can spin up further by the creation of mass from the companion. Similarly, conservation of magnetic flux during the collapse results in strong magnetic fields, reaching 10^{12} Gauss or more, though accretion may slightly weaken these fields.

The magnetic field is generally misaligned with the rotation axis, forming a rotating dipole structure. At a critical distance $\rho_c = c/\Omega$ (where Ω is the angular velocity of the pulsar), magnetic field lines open up and extend to infinity, while those within ρ_c remain closed. This ρ_c marks the furthest distance at which objects can co-rotate with the pulsar without exceeding the speed of light.

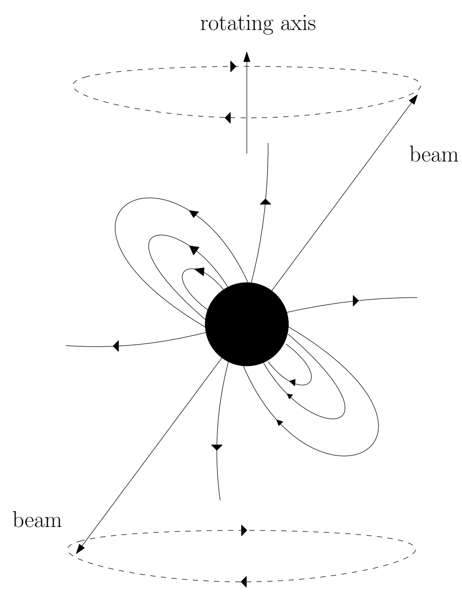

Figure 9: The pulsar magnetosphere and the outgoing beams of radiation, adapted by Maggiore's book, Gravitational Waves [15].

Within ρ_c lies the "magnetosphere," a region filled with ionized plasma that co-rotates with the neutron star. High-energy particles travel along magnetic field lines, emitting radiation near the magnetic poles. This radiation, narrowly focused in the radio spectrum, forms beams that sweep across the sky as the star rotates, creating a lighthouse-like effect. Observers detect these beams as short radio pulses, with the period of the pulses matching the neutron star's rotational period. Given the immense moment of inertia of neutron stars ($\sim 10^{45}\,\mathrm{g\,cm}^2$), their rotation is highly stable.

Interestingly, individual pulses from a given pulsar can vary significantly due to fluctuations in the magnetosphere's dynamics. However, averaging many pulses reveals a stable pattern unique to each pulsar. This averaged pattern, or "template," enables highly precise timing measurements. The times of arrival (TOAs) of individual pulses, compared against the template, can be determined with extraordinary precision, often within 20 microseconds for modern measurements. This precision persists even after long observational gaps, such as during the Arecibo telescope's upgrades in the 1990s. With rotation periods as short as 59 milliseconds, pulsars produce approximately 5×10^8 pulses annually, underscoring their reliability as cosmic clocks.

Despite their stability, TOAs are influenced by time-dependent factors. These include the Earth's motion around the Sun (and the solar system's barycenter) and general relativistic effects due to the solar system's gravitational field. Pulsars in binary systems experience additional modulations from their orbital motion and the gravitational interactions with their companions. These "timing residuals," deviations from perfect periodicity, provide valuable insights into binary system parameters, such as the masses of the stars. The timing formula, discussed in subsequent sections, accounts for these corrections.

6.2 Roemer, Shapiro, and Einstein time delays

We consider a pulsar emitting a sequence of pulses, which are modified by the motion of the Earth and the gravitational field of the solar system, affecting the electromagnetic waves. The corrections to the time of arrival (TOA) are divided into three contributions: the Roemer delay, the Shapiro delay, and the Einstein delay.

6.2.1 Roemer time delay

For simplicity, we assume that Earth performs a circular orbit around the Sun, t_0 is the time that a light beam needs to run from the Sun to Earth, and Earth's angular velocity around the Sun is Ω.

Since light takes approximately 500 seconds to travel from the Sun to the Earth, there is an annual modulation in the arrival times of pulses. For a pulsar located in the plane of the ecliptic, with ecliptic longitude λ, it can be observed from a pulsar in the plane of the ecliptic, at ecliptic longitude.

This modulation is given by:

$$\Delta_{R,\odot} = t_0 \cos(\Omega t - \lambda) \tag{820}$$

Where Ω is the angular velocity of the Earth around the Sun, t_0 is the travel time of light from the Sun to the Earth, and a circular orbit is assumed for simplicity.

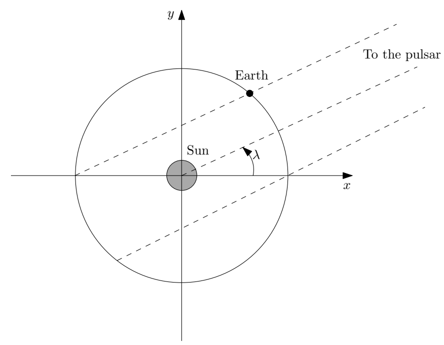

Figure 10: The (x, y) plane is the plane of the orbit of the Earth around the Sun. The angle λ is the ecliptic longitude of the pulsar, adapted by Maggiore's book, Gravitational Waves [15].

Thus, when the Earth is in the same direction as the pulsar,

$$\Omega t - \lambda = 0 \Rightarrow \Delta_{R,\odot} = t_0, \tag{821}$$

The pulse arrives earlier by an amount t_0. Conversely, when the Earth is on the opposite side of its orbit, $\Omega t - \lambda = \pi$, the pulse arrives later by t_0, compared to the arrival time at the Sun. This effect is referred to as the *Roemer time delay*.

If the pulsar is not located in the plane of the ecliptic but has an ecliptic latitude β, the modulation instead becomes:

$$\Delta_{R,\odot} = t_0 \cos(\Omega t - \lambda) \cos\beta \tag{822}$$

With the maximum amplitude to appear in the ecliptic plane at

$$\beta = 0 \Rightarrow \cos\beta = 1 \Rightarrow \Delta_{R,\odot}^{\max} = t_0 \cos(\Omega t - \lambda) \tag{823}$$

and vanishing for pulsars in the direction of the poles of the ecliptic ($\cos\beta = 0$).

The variation of $\Delta_{R,\odot}$ falls to the variation of the angles λ and β so we get:

$$\delta(\Delta_{R,\odot}) = t_0 \delta\lambda \sin(\Omega t - \lambda) \cos\beta - t_0 \delta\beta \cos(\Omega t - \lambda) \sin\beta \tag{824}$$

For precise pulsar timing, additional corrections must be accounted for, as the Earth's orbit cannot be treated as perfectly circular. The Earth's axial rotation introduces a daily modulation with an amplitude $R_\odot/c \approx 21$ ms. The motion of the Sun around the solar system barycenter (SSB), influenced by planets like Jupiter, also contributes to the observed modulation. Therefore, pulse arrival times must be referred to the SSB.

Let $\vec{r}_{\odot e}$ represent the vector from the SSB to the Earth, $\vec{r}_{es}$ the vector from the Earth's center to the observer, and $\vec{r}_{\odot s}$ the vector from the SSB to the Sun. Then, the distance from the observer to the SSB is:

$$\vec{r}_{ob} = \vec{r}_{oe} + \vec{r}_{es} + \vec{r}_{sb} \tag{825}$$

To calculate barycentric arrival times, the observed times must include the term:

$$\Delta_{R,\odot} = -\vec{r}_{ob} \cdot \frac{\hat{n}}{c} \tag{826}$$

Where $\hat{n}$ is the unit vector pointing to the pulsar. The vectors $\vec{r}_{es}$ and $\vec{r}_{\odot s}$ can be determined with sufficient accuracy, while $\vec{r}_{\odot e}$ requires precise measurements due to its dependence on the Earth-Moon system's barycenter.

Barycentric arrival times serve as a critical reference for pulsar timing analyses, accounting for additional effects such as gravitational propagation and interstellar medium interactions.

6.2.2 Shapiro time delay

Roemer's time delay computation neglects all GR effects of the gravitational field in the solar system. To include these effects, recall from equation ?? that the spacetime interval induced by a weak, quasi-static Newtonian source is expressed, to first-order perturbations in the metric ϕ, as

$$ds^2 = -[1 + 2\phi(x)]c^2dt^2 + [1 - 2\phi(x)]d\vec{x}^2. \tag{827}$$

In the solar system, the magnitude of $|\phi(x)|$ is approximately 10^{-6}, ensuring that the weak-field approximation remains valid. Light travels along null geodesics ($ds^2 = 0$), reducing the path integral to

$$\begin{aligned}
&ds^2 = 0 \Rightarrow c^2\mathrm{d}t^2[1 + 2\phi(\vec{x})] = \mathrm{d}\vec{x}^2[1 - 2\phi(\vec{x})] \Rightarrow \\
&c^2\mathrm{d}t^2 = \mathrm{d}\vec{x}^2\frac{[1 - 2\phi(\vec{x})]}{[1 + 2\phi(\vec{x})]} \Rightarrow \\
&c\mathrm{d}t = \pm\mathrm{d}\vec{x}\sqrt{\frac{1 - 2\phi(\vec{x})}{1 + 2\phi(\vec{x})}} \xRightarrow{\phi \ll 1} \\
&c\mathrm{d}t = \pm[1 + 2\phi(\vec{x})]\mathrm{d}\vec{x}
\end{aligned} \tag{828}$$

Let r_p represent the fixed location of the pulsar and r_{obs} denote the observer's position at the time of light's arrival t_{obs}. The coordinate time difference between t_{obs} and the emission time t_e is

$$\begin{aligned}
c(t_{\text{obs}} - t_e) &= \int_{r_e}^{r_p} |dx|\big[1 - 2\phi(x)\big] \Rightarrow \\
c(t_{\text{obs}} - t_e) &= |r_p - r_{\text{obs}}| - 2\int_{r_{\text{obs}}}^{r_p} \phi(x)|dx|
\end{aligned} \tag{829}$$

If $\vec{r}_b$ is the position of SSB and $\hat{n}$ is the unit vector from SSB to the pulsar, then we have:

$$\begin{aligned}
|\vec{r}_p - \vec{r}_{\text{obs}}| &= |\vec{r}_l - \vec{r}_b + \vec{r}_b - \vec{r}_{\text{obs}}| \\
|\vec{r}_p - \vec{r}_{\text{obs}}| &\approx |\vec{r}_p - \vec{r}_b| + (\vec{r}_b - \vec{r}_{\text{obs}}) \cdot \hat{n}
\end{aligned} \tag{830}$$

$$\text{for } \hat{n} \equiv \frac{\vec{r}_p - \vec{r}_b}{|\vec{r}_p - \vec{r}_b|} \tag{831}$$

and $|\vec{r}_p - \vec{r}_b| \gg |\vec{r}_b - \vec{r}_{\text{obs}}|$. Using this, the arrival time becomes

$$t_{\text{obs}} \approx \left(t_e + \frac{|r_p - r_b|}{c}\right) + \frac{(r_b - r_{\text{obs}}) \cdot \hat{n}}{c} - \frac{2}{c}\int_{r_{\text{obs}}}^{r_p} \phi(x)|dx| \tag{832}$$

The second term t_{SSB}, representing the arrival time at the barycenter in the absence of gravitational effects, is defined as

$$t_{\text{SSB}} = t_{\text{obs}} - \frac{r_{\text{obs}} \cdot \hat{n}}{c} + \frac{2}{c}\int_{r_{\text{obs}}}^{r_p} \phi(x)|dx| \tag{833}$$

t_{SSB} is the fictitious time at which the pulse arrives at the SSB without GR effects of the Solar System. The second term is called the solar system Shapiro timedelay, denoted by $\Delta_{S,\odot}$. So equation 833 reads:

$$t_{\text{SSB}} = t_{\text{obs}} - \Delta_{R,\odot} + \Delta_{S,\odot} \tag{834}$$

The Shapiro time delay is dominated by the Sun's gravitational field. We consider a photon emitted by a pulsar that reaches the observer on Earth when the pulsar Sun-Earth angle has a value θ as depicted:

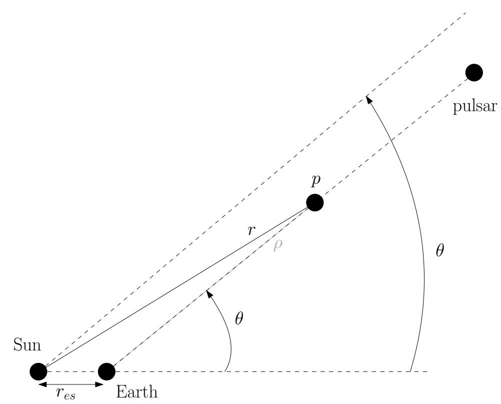

Figure 11: The geometry for the computation of the Shapiro delay, adapted by Maggiore's book, Gravitational Waves [15].

ρ: distance P to Earth
r: distance P to Sun
$r_{\text{es}} = 1\ \ \text{au}$

$$\rho = (\rho\cos\theta + \vec{r}_{\text{es}}, \rho\sin\theta)$$

$$u = \frac{e}{r_{\text{es}}} \tag{835}$$
$$r^2 = (\rho\cos\theta + |r_{\text{es}}|^2) + \rho^2\sin^2\theta \tag{836}$$

$$
\begin{aligned}
r^2 &= \rho^2 \cos^2\theta + r_{\text{es}}^2 + 2r_{\text{es}}\rho\cos\theta + \rho^2\sin^2\theta \\
r^2 &= \rho^2 + r_{\text{es}}^2 + 2r_{\text{es}}\rho\cos\theta \\
r^2 &= r_{\text{es}}^2\left(1 + \left(\frac{\rho}{r_{\text{es}}}\right)^2 + 2\frac{\rho}{r_{\text{es}}}\cos\theta\right) \\
r^2 &= r_{\text{es}}^2(1 + u^2 + 2u\cos\theta) \\
r &= r_{\text{es}}(1 + u^2 + 2u\cos\theta)^{\frac{1}{2}}
\end{aligned}
\tag{837}
$$

$$
\text{Since } \phi = \frac{1}{c^2}\left(-\frac{GM_\odot}{r}\right) \tag{838}
$$

The Shapiro time delay is

$$
\begin{aligned}
\Delta_{S,\odot} &= -\frac{2}{c}\int_{\vec{r}_p}^{\vec{r}_{obs}} \mathrm{d}p\phi = \frac{2GM_\odot}{c^3}\int_0^d \frac{\mathrm{d}p}{r} \\
\Rightarrow \Delta_{S,\odot} &= \frac{2GM_\odot}{c^3}\int_0^d \frac{\mathrm{d}u\, r_{\text{es}}}{r} = \frac{2GM_\odot}{c^3}\int_0^{\bar{u}} \frac{\mathrm{d}u}{(1 + u^2 + 2u\cos\theta)^{\frac{1}{2}}}
\end{aligned}
\tag{839}
$$

$$
\text{where } \bar{u} \equiv \frac{d}{r_{\text{es}}} \tag{840}
$$

In equation 839 we add and subtract a term at a given angle, say $\cos\theta = 0$, so we have:

$$
\Delta_{S,\odot} = \frac{2GM_\odot}{c^3}\int_0^{\bar{u}} \frac{1}{(u^2+1)^{1/2}}\,du - \frac{2GM_\odot}{c^3}\int_0^{\bar{u}} \frac{1}{\left[u^2 + 1 + 2u\cos\theta\right]^{1/2}}\,du \tag{841}
$$

The first term

$$
\frac{2GM_\odot}{c^3}\int_0^{\bar{u}} \frac{\mathrm{d}u}{(1+u^2)^{\frac{1}{2}}} = \frac{2GM_\odot}{c^3}\sin^{-1}h(\bar{u}) \approx \frac{2GM_\odot}{c^3}\log\left(\frac{2u}{r_{\text{es}}}\right) \tag{842}
$$

is a constant logarithmic correction, while the second depends on the Sun-Earth-pulsar geometry, specifically the angle θ. Applying limit $\bar{u} = \frac{\mathrm{u}}{r_{\text{es}}} \to \infty$ and integral converges to

$$
\begin{aligned}
&\int \mathrm{d}u\left[\frac{1}{(u^2 + 1 + 2u\cos\theta)^{\frac{1}{2}}} - \frac{1}{(u^2+1)^{\frac{1}{2}}}\right] = -\log(1+\cos\theta) \\
&\Rightarrow \Delta_{S,\odot}(\theta) \approx \frac{2GM_\odot}{c^3}\left[\log\left(\frac{2d}{r_\odot}\right) - \log(1+\cos\theta)\right]
\end{aligned}
\tag{843}
$$

Equation 843 formally diverges when $\theta = \pi$, that is, when the signal crosses the center of the Sun before reaching Earth, so it is absorbed.

For a pulsar just grazing Sun's surface of θ_{grazing}, we have

$$
\theta \approx \theta_{\text{grazing}} \approx \pi - \frac{R_\odot}{r_{\text{es}}} \tag{844}
$$

$$
\begin{aligned}
1+\cos\theta_{\text{grazing}} &= 1+\cos\left(\pi-\frac{R_\odot}{r_{\text{es}}}\right)\\
1+\cos\theta_{\text{grazing}} &= 1-\cos\left(\frac{R_\odot}{r_{\text{es}}}\right)\\
1+\cos\theta_{\text{grazing}} &= 1-1+\left(\frac{R_\odot}{r_{\text{es}}}\right)^2\\
1+\cos\theta_{\text{grazing}} &= \left(\frac{R_\odot}{r_{\text{es}}}\right)^2
\end{aligned}
\tag{845}
$$

And Shapiro's time delay reads:

$$
\begin{aligned}
\Delta_{S,\odot} &= \frac{2GM_\odot}{c^3}\left[\log\frac{2d}{r_{\text{es}}}-2\log\frac{R_\odot}{r_{\text{es}}}\right]\\
\Delta_{S,\odot}^{\theta_{\text{grazing}}} &= \frac{2GM_\odot}{c^3}\left[\log\left(\frac{\frac{2d}{r_{\text{es}}}}{\frac{R_\odot}{r_{\text{es}}}}\right)\right]\\
\Delta_{S,\odot}^{\theta_{\text{grazing}}} &= \frac{2GM_\odot}{c^3}\log\left(\frac{2dr_{\text{es}}}{R_\odot^2}\right)
\end{aligned}
\tag{846}
$$

The maximum modulation induced by the Shapiro time delay is

$$
\begin{aligned}
\Delta_{S,\odot}^{\max} &= \Delta_{S,\odot}^{\theta_{\text{grazing}}}-\Delta_{S,\odot}(\theta=0)=\frac{2GM_\odot}{c^3}\left[\log\left(\frac{2dr_{\text{es}}}{R^2\odot}\right)-\log\left(\frac{2d}{r_{\text{es}}}+\log 2\right)\right]\\
\Delta_{S,\odot}^{\max} &= \frac{2GM_\odot}{c^3}\left[\log\left(\frac{2r_{\text{es}}}{R_\odot^2}\right)\right]\\
\Delta_{S,\odot}(\theta=0) &\approx \frac{4GM_\odot}{c^3}\log\left(\frac{2r_{\text{es}}}{r_\odot}\right)
\end{aligned}
\tag{847}
$$

6.2.3 Einstein time delay

The Roemer and Shapiro time delays are computed in the coordinate time t. A laboratory clock at a position $\vec{x}_{\text{obs}}$ measures its own proper time τ which is related t_0 as:

$$
c^2d\tau^2=\left[1+2\phi(x_{\text{obs}})\right]c^2dt^2-\left[1-2\phi(x_{\text{obs}})\right]dx_{\text{obs}}^2
\tag{848}
$$

so to first order in small parameters $\phi(x_{\text{obs}})$ and $v_{\text{obs}}=d\vec{x}_{\text{obs}}/dt$, we have:

$$
\begin{aligned}
\frac{\mathrm{d}\tau^2}{dt^2} &= 1+2\phi(x_{\text{obs}})-\frac{\mathrm{d}\vec{x}_{\text{obs}}^2}{\mathrm{dt}}+O(\vec{x}^2)\\
\frac{\mathrm{d}\tau}{dt} &= \sqrt{1+2\phi(x_{\text{obs}})-\frac{\vec{v}_{\text{obs}}}{c^2}}\\
\frac{\mathrm{d}\tau}{dt} &\approx 1+\frac{1}{2}2\phi(x_{\text{obs}})-\frac{1}{2}\frac{v_{\text{obs}}^2}{c^2}\\
\frac{\mathrm{d}\tau}{dt} &\approx 1+\phi(x_{\text{obs}})-\frac{v_{\text{obs}}^2}{2c^2}
\end{aligned}
\tag{849}
$$

Physically, the term $-\frac{v_{\text{obs}}^2}{2c^2}$ in equation 849 produces the transverse Doppler shift, while $\phi(x_{\text{obs}})$ contributes to the gravitational redshift. Integrating this relation, we find:

$$\begin{aligned}
\mathrm{d}\tau &\simeq \mathrm{d}t + \mathrm{d}t\phi(x_{\text{obs}}) - \mathrm{d}t\frac{1}{2}\frac{v_{\text{obs}}^2}{c^2} \\
\Rightarrow \tau &\simeq t + \int \mathrm{d}t' \left[\phi(x_{\text{obs}}(t')) - \frac{v_{\text{obs}}^2(t')}{2c^2}\right]
\end{aligned} \tag{850}$$

Or we can rewrite it as:

$$\tau = t - \Delta_{E,\odot} \tag{851}$$

$$\text{where } \Delta_{E,\odot} \equiv \int^t \mathrm{d}t' \left[\phi(x_{\text{obs}}(t')) + \frac{v_{\text{obs}}^2(t')}{2c^2}\right] \tag{852}$$

The modulating given in 852 is called the **Einstein time delay** and takes into consideration the motion of the Earth around the Sun with $v_\oplus$ velocity and the Earth's rotation around its axis. Also, we can apply the elliptic orbit of the Earth instead of a circle. If the lower limit of the integral is arbitrary (as it corresponds to an arbitrary constant shift), we can rewrite τ as:

$$E = -\frac{GM_\odot \mu}{2a} \tag{853}$$

where μ is the reduced mass of the Earth-Sun system, and $M \simeq M_\odot$ with excellent accuracy. Using this relation, we find:

$$E = \frac{1}{2}\mu v_\odot^2 - \frac{GM_\odot \mu}{r} \tag{854}$$

and:

$$\begin{aligned}
-\frac{GM\mu}{2a} &= \frac{1}{2}\mu v_\odot^2 - \frac{GM\mu}{r} \\
\frac{1}{2}v_\oplus^2 &= \frac{GM_\odot}{r} - \frac{GM_\odot}{2a} \\
\Rightarrow \frac{1}{2}v_\oplus^2 &= GM_\odot\left(\frac{1}{r} - \frac{1}{2a}\right)
\end{aligned} \tag{855}$$

From these expressions, we derive the rate of change of the Einstein delay:

$$\begin{aligned}
\frac{\mathrm{d}\Delta_{E,\odot}}{\mathrm{d}t} &= -\phi(x_{\text{obs}}) + \frac{v_{\text{obs}}^2}{2c^2} \\
\frac{\mathrm{d}\Delta_{E,\odot}}{\mathrm{d}t} &= -\phi(x_{obs}) + \frac{v_{obs}^2}{2c^2} \\
\frac{d\Delta_{E,\odot}}{dt} &\approx \frac{v_\odot^2}{2c^2} - \phi = \frac{2GM_\odot}{c^2}\left(\frac{1}{r} - \frac{1}{4a}\right)
\end{aligned} \tag{856}$$

We recall that μ is the reduced mass

$$\mu = \frac{m_\oplus M}{m_\oplus + M_\odot} \simeq \frac{m_\oplus M_\odot}{M_\odot} \cong m_\oplus \tag{857}$$

and M is the total mass

$$M = m_\oplus + M_\odot \cong M_\odot \tag{858}$$

Also, we have that $v_{\rm obs} \cong v_\oplus$, because Earth's rotation around its axis gives a small correction. A constant part in this expression is absorbed into the clock's time definition, as atomic clocks are adjusted to minimize systematic shifts due to Earth's motion.

6.2.4 Dispersion in the interstellar medium and relation to the intrinsic pulsar signal

Dispersion in the interstellar medium

There is also a correction due to the propagation of radio waves through the ionized interstellar gas. Interstellar gases act as a medium with a reflection under different from unity. The component of a radio pulse with frequency v travels with a group velocity

$$v_g \simeq c\left(1 - \frac{n_e e^2}{2\pi m_e}\frac{1}{\nu^2}\right), \tag{859}$$

where e is the charge, and m_e the mass of the electron, and n_e is the electron number density.

The travel time over a distance L is:

$$\begin{aligned}
\int_0^L \frac{dl}{v_g} &= \int_0^L dl \frac{1}{c}\left(1 - \frac{n_e e^2}{2\pi m_e}\frac{1}{v^2}\right)^{-1} \\
\int_0^L \frac{dl}{v_g} &= \int_0^L dl \left[\frac{1}{c} + \frac{e^2}{2\pi m_e}\frac{1}{v^2} n_e\right] \\
\int_0^L \frac{dl}{v_g} &\simeq \frac{L}{c} + \frac{1}{\nu^2}\left(\frac{e^2}{2\pi m_e c}\right)\int_0^L n_e\, dl
\end{aligned} \tag{860}$$

We denote with

$$\mathrm{DM} \equiv \int_0^L n_e\, dl \tag{861}$$

the *dispersion measure*, and is typically quoted in cm^{-3} pc. Measuring the TOAs (times of arrival) at different frequencies, we can compute the dispersion measure. This procedure is called *de-dispersion*, and goes like:

1. Separate the bandwidth of the receiver into many channels, such that in each channel the effect of dispersion is negligible.

2. The output of the channels operating at different frequencies is then automatically corrected and superimposed, so the signal-to-noise ratio is enhanced

The size of this effect is given by observing that the Hulse–Taylor binary pulsar has a relatively large dispersion measure, DM $\simeq 169\,\mathrm{cm}^{-3}$ pc, at frequencies near 430 MHz.

Relation to the Intrinsic Pulsar Signal

Since all corrections are small, we can put them together by simply adding them linearly:

$$t_{\rm SSB} = \tau_{\rm obs} + \Delta_{E,\odot} + \Delta_{R,\odot}, -\Delta_{S,\odot} \tag{862}$$

where the corrections $\Delta_{E,\odot}, \Delta_{R,\odot}, \Delta_{S,\odot}$ are defined in earlier equations.

In Eq. 860 we must subtract the time delay due to the interaction with the interstellar medium, so we get:

$$t_{\mathrm{SSB}} = \tau_{\mathrm{obs}} - \frac{D}{\nu^2} + \Delta_{E,\odot} + \Delta_{R,\odot} - \Delta_{S,\odot} \tag{863}$$

Where:

$$D = \left(\frac{e^2}{2\pi m_e c}\right) \mathrm{DM}. \tag{864}$$

So t_{SSB} is the coordinate time at which the signal, recorded by τ_{obs}, would have arrived at a fixed point in space such as the solar system barycenter if there were no gravitational potential of the solar system or interaction with the interstellar medium. It therefore depends only on the intrinsic properties of the source. The emission mechanism of the pulsar is not yet completely understood, but is believed to be related to some “hot spot” co-rotating with the pulsar. Denoting by Φ the accumulated phase of the spinning pulsar, we observe a pulse whenever the phase Φ returns to the same value $\Phi_0 \bmod 2\pi$, at which the radiated beam sweeps across the Earth.

If T is the proper time in the pulsar frame, then the phase for a perfectly periodic pulsar will be:

$$\Phi(T) = 2\pi\nu T. \tag{865}$$

The evolution of the pulsar frequency is modeled by Taylor expanding around the source reference value $T_0 = 0$ of the pulsar proper time as:

$$\nu(T) = \nu_0 + \dot{\nu}_0 T + \frac{1}{2}\ddot{\nu}_0 T^2 + \cdots , \tag{866}$$

And the accumulated phase is then:

$$\begin{aligned} \frac{1}{2\pi}\Phi(T) &= \int_0^T d\tau \nu(\tau) \\ \frac{1}{2\pi}\Phi(T) &= \nu_0 T + \frac{1}{2}\dot{\nu}_0 T^2 + \frac{1}{6}\ddot{\nu}_0 T^3 + \cdots \end{aligned} \tag{867}$$

Emission takes place at T_n such that

$$\Phi(T_n) = \phi_0 + 2\pi n$$

With proper times T_n to be given by:

$$\nu_0 T_n + \frac{1}{2}\dot{\nu}_0 T_n^2 + \frac{1}{6}\ddot{\nu}_0 T_n^3 + \cdots = n + \frac{\Phi_0}{2\pi}. \tag{868}$$

This model assumes that the evolution of the pulsar frequency is smooth. Most pulsars exhibit "glitches", sudden jumps in their rotational periods. Related to the source form of rearrangement of internal structure.

6.3 Relativistic corrections for binary pulsars

For a pulsar in a binary system, we proceed similarly to what’s done for the Earth-Sun system and transform from the pulsar’s proper time to the pulsar-companion system coordinate time. Further reading in the following subsections can be found in Reports [97]-[103].

6.3.1 Einstein time delay

We proceed as in Einstein's time delay in SSB, but instead of reduced and total mass, we use the masses m_p and m_c of the pulsar and its companion. The beam is radiated by some "hot spot" at a position $\vec{x}$ on the pulsar's surface.

The Newtonian field ϕ at $\vec{x}$ is therefore:

$$\phi(x) = -\frac{Gm_p}{c^2|x - x_p|} - \frac{Gm_c}{c^2|x - x_c|}, \tag{869}$$

where x_p is the position of the center of the pulsar and x_c is the position of the companion.

$Gm_c/(c^2 a) \sim 10^{-6}$ and therefore is small, so the weak-field approximation is legitimate.

Pulsar's self-gravity is for a typical pulsar of mass $m_p \simeq 1.4M_\odot$, and NS radius $r_{\rm NS} \simeq 10\,\text{km}$, we have $Gm_p/(c^2 r_{\rm NS}) \simeq 0.2$, which is strong on the surface. However, this term does not change along the pulsar's trajectory, so it does not modulate the time of arrivals.

Thus, the time-dependent part of the Einstein time delay is:

$$\phi(x) = -\frac{Gm_c}{c^2|x - x_c|} \tag{870}$$

Inserting Equation 870 in 849 we get:

$$\frac{d\tau}{dt} = 1 - \frac{Gm_c}{c^2|x_p - x_c|} - \frac{v_p^2}{2c^2} \tag{871}$$

In the center-of-mass system ν_p is given as:

$$\nu_p = \frac{m_c}{m_p + m_c} v \tag{872}$$

where ν is the relative velocity.

Here we begin from Eqs. 852 and 853, but $M \to m_p$ and $\mu \Rightarrow m_c$, so:

$$\begin{aligned} E = -\frac{G(m_p + m_c)}{2a} &= \frac{1}{2}\frac{m_p m_c}{m_p + m_c}\nu_p^2 - \frac{Gm_p m_c(m_p + m_c)}{r(m_p + m_c)} \\ \frac{1}{2}v^2 - \frac{G(m_p + m_c)}{r} &= -\frac{G(m_p + m_c)}{2a} \end{aligned} \tag{873}$$

Now, Equation 871 reads:

$$\begin{aligned} \frac{dT}{dt} &= 1 - \frac{Gm_c}{c^2 r} - \frac{1}{2c^2}\frac{m_c^2}{(m_p + m_c)^2}\nu^2 \\ \frac{dT}{dt} &= 1 - \frac{Gm_c}{c^2 r} - \frac{1}{c^2}\frac{m_c^2}{(m_p + m_c)^2}\left[\frac{G(m_p + m_c)}{r} - \frac{G(m_p + m_c)}{2a}\right] \\ \frac{dT}{dt} &= 1 - \frac{G}{c^2}\left[\frac{m_c}{r} + \frac{m_c^2}{(m_p + m_c)r} - \frac{m_c^2}{m_p + m_c}\frac{1}{2a}\right] \\ \frac{dT}{dt} &= 1 - \frac{G}{c^2}\left[\frac{m_c(m_p + 2m_c)}{m_p + m_c}\frac{1}{r} - \frac{m_c^2}{m_p + m_c}\frac{1}{2a}\right] \end{aligned} \tag{874}$$

The parametrization of Keplerian orbit in terms of the eccentric anomaly u is related to t by:

$$u - e\sin u = \frac{2\pi}{P_b}(t - t_0), \tag{875}$$

With t_0 to be a reference time of periastron passage. Differentiating, we have:

$$\frac{du}{dt}(1 - e\cos u) = \frac{2\pi}{P_b}, \tag{876}$$

And therefore:

$$\begin{aligned}
\frac{dT}{dt}\cdot\frac{2\pi}{P_b} &= \left(1 - \frac{G}{c^2}\left(\frac{m_c}{m_p + m_c}\right)\left[(m_p + 2m_c)\frac{1}{r} - m_c\frac{1}{2a}\right]\right) \times (1 - e\cos u) \\
\frac{2\pi}{P_b}\frac{dT}{du} &= 1 - e\cos u - \frac{G}{c^2}\left(\frac{m_c}{m_p + m_c}\right)(1 - e\cos u)\left[(m_p + 2m_c)\frac{1}{a(1 - e\cos u)} - m_c\frac{1}{2a}\right] \\
\frac{2\pi}{P_b}\frac{dT}{du} &= 1 - e\cos u - \frac{Gm_c}{c^2(m_p + m_c)}\left[\frac{1}{a}(m_p + 2m_c) - m_c\frac{1}{2a} + m_c e\cos u\frac{1}{2a}\right] \\
\frac{2\pi}{P_b}\frac{dT}{du} &= \left[1 - \frac{G}{c^2}\frac{2m_c m_p + 3m_c^2}{2a(m_p + m_c)}\right] - e\cos u\left[1 + \frac{G}{c^2}\frac{m_c^2}{2a(m_p + m_c)}\right] \\
\frac{dT}{du} &\approx \frac{P_b}{2\pi}\left[1 - \frac{G}{c^2}\frac{2m_c m_p + 3m_c^2}{2a(m_p + m_c)}\right]\left[1 - e\cos u\left(1 + \frac{G}{c^2}\frac{m_c(m_p + 2m_c)}{a(m_p + m_c)}\right)\right]
\end{aligned} \tag{877}$$

The only observable correction proportional to $\cos u$, since it produces a modulation along the orbit. We redefine

$$T' = \left[1 - \frac{G}{c^2}\frac{2m_c m_p + 3m_c^2}{2a(m_p + m_c)}\right] T \Rightarrow dT' = constant \times dT \tag{878}$$

So we take:

$$\frac{dT}{du} = \frac{P_b}{2\pi}(1 - e\cos u) - \gamma\cos u, \tag{879}$$

where the Einstein parameter γ is given by:

$$\gamma = e\left(\frac{P_b}{2\pi}\right)\frac{G}{c^2}\frac{m_c(m_p + 2m_c)}{a(m_p + m_c)} = e\left(\frac{P_b}{2\pi}\right)^{1/3}\frac{G^{2/3}}{c^2}\frac{m_c(m_p + 2m_c)}{(m_p + m_c)^{4/3}}, \tag{880}$$

We rewrite 879 with $\frac{2\pi}{P_b}\frac{dt}{du} = 1 - e\cos u$, we find:

$$\frac{d\Delta_E}{du} = \gamma\cos u \tag{881}$$

So, the Einstein delay reads:

$$\Delta_E = \gamma\sin u \tag{882}$$

Roemer time delay and Post-Newtonian orbits

Referring the emission time to the barycenter of the pulsar-companion system, we encounter the Roemer and Shapiro time delays, similar to the solar system corrections. The Roemer delay is given by:

$$\Delta_R = \frac{\hat{z} \cdot x_1(t)}{c} \tag{883}$$

where x_1 is the distance of the pulsar from the center of mass of the pulsar-companion system.

In a Keplerian orbit, neglecting general-relativistic corrections in the plane of the orbit, using polar coordinates (r_1, ψ) in the plane of the orbit, the Keplerian equation of motion is given in parametric form, in terms of the eccentric anomaly u, by:

$$r_1(u) = a_1[1 - e\cos u] \tag{884}$$

$$\cos\psi(u) = \frac{\cos u - e}{1 - e\cos u} \tag{885}$$

Where a_1 is the semimajor axis of the pulsar orbit. At $u = 0$, r_1 is max and in this case $\psi = 0$. Therefore, the angle ψ is measured from periastron, and the angle measured from the line of nodes is $\omega + \psi(u)$.

The Roemer delay is:

$$\begin{aligned} \Delta_R &= r_1(u)\sin i \sin[\omega + \psi(u)] \Rightarrow \\ \Delta_R &= r_1(u)\sin i(\cos\psi\sin\omega + \sin\psi\cos\omega) \end{aligned} \tag{886}$$

since

$$\sin\psi(u) = \frac{\sqrt{1-e^2}\sin u}{1 - e\cos u}, \tag{887}$$

We get:

$$\begin{aligned} \Delta_R &= r_1(u)\sin i\left[\frac{\cos u - e}{1 - e\cos u}\sin\omega + \sqrt{1-e^2}\frac{\sin u}{1 - e\cos u}\cos\omega\right] \\ \Delta_R &= \frac{r_1(u)}{1 - e\cos u}\sin i\left[(\cos u - e)\sin\omega + \sqrt{1-e^2}\sin u\cos\omega\right] \\ \Delta_R &= a_1(u)\sin i\left[(\cos u - e)\sin\omega + \sqrt{1-e^2}\sin u\cos\omega\right] \end{aligned} \tag{888}$$

Numerically, the effect is quite large, and it is necessary to go beyond the Keplerian orbit and include the post-Newtonian corrections to 1PN order. This computation has been performed by Damour and Deruelle (1985, 1986).

Conservation of angular momentum leads to motion in a plane, and the conserved quantities, total energy E and angular momentum J, are given by:

$$\epsilon = \frac{E}{\mu} = \frac{1}{2}v^2 - \frac{Gm}{r} + \frac{3}{8}(1-3\nu)\frac{v^4}{c^2} + \frac{Gm}{2rc^2}\left[(3+\nu)v^2 + \nu(\hat{r}\cdot v)^2 + \frac{Gm}{r}\right] \tag{889}$$

$$\vec{j} = \frac{\vec{J}}{\mu} = \left[1 + \frac{1}{2}(1-3\nu)\frac{v^2}{c^2} + (3+\nu)\frac{Gm}{rc^2}\right]\vec{r}\times\vec{v} \tag{890}$$

In polar coordinates (r, ψ) in the plane of the orbit, the first of these conserved quantities leads to:

$$\left(\frac{dr}{dt}\right)^2 = A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D}{r^3} \tag{891}$$

$$\frac{d\psi}{dt} = \frac{H}{r^2} + \frac{I}{r^3} \tag{892}$$

where A, B, C, D, H, I are polynomials pf ϵ and $\vec{j}$

$$A = 2\epsilon\left[1 + \frac{3}{2}(3\nu - 1)\frac{\epsilon}{c^2}\right] \tag{893}$$

$$B = Gm\left[1 + (7\nu - 6)\frac{\epsilon}{c^2}\right] \tag{894}$$

$$C = -j^2\left[1 + 2(3\nu - 1)\frac{\epsilon}{c^2}\right] + (5\nu - 10)\frac{G^2m^2}{c^2} \tag{895}$$

$$D = (8 - 3\nu)\frac{GM_{j^2}}{c^2} \tag{896}$$

$$H = j\left[1 + (3\nu - 1)\frac{\epsilon}{c^2}\right] \tag{897}$$

$$I = (2\nu - 4)\frac{GM_j}{c^2} \tag{898}$$

In $c \to +\infty$ we have $A = 2\epsilon, B = Gm, C = -j^2, D = 0 = I, H = j$ the Newtonian values. We insert $\bar{r} = r + \frac{D}{2j}$ and Equation 891 reads:

$$\left(\frac{d\bar{r}}{dt}\right)^2 = A + \frac{2B}{\bar{r}} + \frac{\bar{C}}{\bar{r}^2} + \mathcal{O}(v^4/c^4) \tag{899}$$

with $\bar{C} = C + (BD/j^2)$.

Similarly in Equation 892 we set $\tilde{r} \equiv r - \frac{I}{2H}$ and have

$$\frac{d\psi}{dt} = \frac{H}{\tilde{r}^2} \tag{900}$$

In polar coordinates (r, ψ) in the plane of the orbit, the first of these conserved quantities leads to:

$$\left(\frac{dr}{dt}\right)^2 = A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D}{r^3} \tag{901}$$

$$\frac{d\psi}{dt} = \frac{H}{r^2} + \frac{I}{r^3} \tag{902}$$

where A, B, C, D, H, I are polynomials pf ϵ and $\vec{j}$

The 1PN Equations of motion can be integrated, and we get:

$$u - e_t \sin u = \frac{2\pi}{P_b}t \tag{903}$$

$$r = a_r[1e_r \sin u] \tag{904}$$

with

$$a_r = -\frac{Gm}{2\epsilon}\left[1-(\nu-7)\frac{\epsilon}{2c^2}\right] \tag{905}$$

radial eccentricity

$$e_r^2 = 1+\frac{2\epsilon}{Gm^2}\left[1+(5\nu-15)\frac{\epsilon}{2c^2}\right]\left[j^2+(\nu-6)\frac{G^2m^2}{c^2}\right] \tag{906}$$

time eccentricity

$$e_t^2 = 1+\frac{2\epsilon}{Gm^2}\left[1+(17-7\nu)\frac{\epsilon}{2c^2}\right]\left[j^2+(2-2\nu)\frac{G^2m^2}{c^2}\right] \tag{907}$$

$$\frac{2\pi}{P_b} = \frac{(-2\epsilon)^{5/2}}{Gm}\left[1-(\nu-15)\frac{\epsilon}{4c^2}\right] \tag{908}$$

Similarly $\psi(u)$ solution in terms of angular electricity reads:

$$\psi = \omega_0 + (1+k)A_{e_\theta}(u) \tag{909}$$

where

$$k \equiv \frac{3Gm}{c^2a(1-e^2)} \tag{910}$$

and

$$e_\theta^2 = 1+\frac{2\epsilon}{G^2m^2}\left[1+(\nu+5)\frac{\epsilon}{2c^2}\right]\left[j^2-6\frac{G^2m^2}{c^2}\right] \tag{911}$$

and

$$A_{e\theta}(u) \equiv 2arctan\left[\sqrt{\frac{1+e_\theta}{1-e_\theta}}tan\frac{u}{2}\right] \tag{912}$$

This representation is parametric and "quasi-Newtonian," accurate to 1PN order.

7 TOV equations for stellar objects and their structure

7.1 Production of TOV equations

In the following chapter we conduct an explicit derivation of the Tollmann- Oppenheimer-Volkov equations. A complete set of equations, which completely describe a neutron star, when combined with its equation of state. Here we follow the minset, as given in Hobson, Efstathiou and Lasenby's book [18].

Most astrophysical objects never evolve into objects that are not adequately described by the Newtonian theory of stellar structure. Neutron Stars (NS), although, involve extremely high densities, which means that the internal growth forces will be very strong, and we can expect that General Relativistic effects will have a significant role in the stars structure and stability. There is a huge interest in achieving relativistic equations that govern the equilibrium of symmetric gravitating matter distributions.

The demand of spherical symmetry and static matter distribution in the star yields the metric:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\ d\phi^2) \tag{913}$$

where $A(r)$ and $B(r)$ are functions of radius r and can be determined in the interior of the object.

To proceed, we rewrite Eq. 913 in matrix form to obtain the covariant components of the metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} A(r) & & \emptyset & \\ & -B(r) & & \\ & & -r^2 & \\ \emptyset & & & -r^2\sin^2\theta \end{pmatrix} \tag{914}$$

Correspondingly, the contravariant metric tensor, which is the inverse of the above, takes the form:

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{A(r)} & & \emptyset & \\ & -\frac{1}{B(r)} & & \\ & & -\frac{1}{r^2} & \\ \emptyset & & & -\frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{915}$$

There is a standard procedure when one wants to derive the Einstein field equations. The first step is to compute the Christoffel symbols or connections, which depend on derivatives of the metric components. The general expression for the Christoffel symbols is:

$$\boxed{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}).} \tag{916}$$

Since the metric is diagonal and static, many components of the Christoffel symbols vanish due to symmetry and time independence. Specifically, using equation 914, we note:

$$914 \Rightarrow \begin{cases} g_{0i} = 0 = g_{j0} \\ g_{ij} = 0, \quad \forall i \neq j \\ \partial_t g_{\rho\sigma} = 0 \quad \text{no dependance on t or } \phi \\ \partial_\phi g_{\rho\sigma} = 0, \quad \forall \rho, \sigma = 0,1,2,3 \end{cases} \tag{917}$$

Let us now compute the non-zero components of the Christoffel symbols. Starting with the Γ^{μ}_{00} components:

$$\begin{aligned}\Gamma^{0}_{00} &= \frac{1}{2}g^{0\sigma}(\partial_0 g_{\sigma 0} + \partial_0 g_{0\sigma} - \partial_\sigma g_{00})\\ \Gamma^{0}_{00} &= \frac{1}{2}g^{00}\partial_t g_{00} = 0\end{aligned} \tag{918}$$

Since the metric is time-independent, all time derivatives vanish, which leads to $\Gamma^{0}_{00} = 0$.

$$\begin{aligned}\Gamma^{i}_{00} &= \frac{1}{2}g^{i\sigma}(\partial_0 g_{\sigma 0} + \partial_0 g_{0\sigma} - \partial_\sigma g_{00}) \quad \forall i \neq 0\\ \Gamma^{i}_{00} &= -\frac{1}{2}g^{ii}\partial_i g_{00}\end{aligned}$$

We now evaluate this expression for each spatial component i:
For $i = 1$ we get:

$$\Gamma^{1}_{00} = \frac{A'}{2B}, \tag{919}$$

$$\Gamma^{2}_{00} = \Gamma^{3}_{00} = 0. \tag{920}$$

These results reflect the fact that A depends only on r, while there is no angular dependence.

$$\begin{aligned}i = 2: \quad \Gamma^{2}_{00} &= -\frac{1}{2}g^{22}\partial_2 g_{00} = -\frac{1}{2}g^{22}\partial_\theta(A(r)) = 0\\ i = 3: \quad \Gamma^{3}_{00} &= -\frac{1}{2}g^{33}\partial_3 g_{00} = -\frac{1}{2}g^{33}\partial_\phi(A(r)) = 0\end{aligned}$$

Again, since $A(r)$ does not depend on the angular coordinates, these Christoffel symbols vanish.

We now consider mixed components involving time and space indices. These are also expected to vanish for a static metric:

$$\begin{aligned}\Gamma^{1}_{0i} = \Gamma^{1}_{j0} &= \frac{1}{2}g^{1\sigma}(\partial_i g_{0\sigma} + \partial_0 g_{i\sigma} - \partial_\sigma g_{i0}), \quad \forall i \neq 0\\ &= \frac{1}{2}g^{11}\partial_t g_{i1} = 0, \quad \forall i = 1, 2, 3.\end{aligned}$$

Thus, all such terms vanish due to the time-independence of the metric components.

$$\Gamma^{1}_{01} = \Gamma^{1}_{02} = \Gamma^{1}_{03} = \Gamma^{1}_{30} = \Gamma^{1}_{20} = \Gamma^{1}_{10} = 0. \tag{921}$$

We now compute the Christoffel symbols with upper index 2. For all $i \neq 0$, we apply the standard formula for Christoffel symbols:

$$\Gamma^2_{oi} = \Gamma^2_{jo} = \frac{1}{2} g^{2\sigma}(\partial_i g_{o\sigma} + \partial_o g_{i\sigma} - \partial_\sigma g_{io})$$
$$= \frac{1}{2} g^{22} \partial_t g_{i2} = 0$$

All these components vanish since the metric components do not depend on time and $g_{02} = 0$. We therefore have:

$$\Gamma^2_{01} = \Gamma^2_{02} = \Gamma^2_{03} = \Gamma^2_{30} = \Gamma^2_{20} = \Gamma^2_{10} = 0. \tag{922}$$

We repeat the same analysis for the Christoffel symbols with upper index 3. Again, we use the formula:

$$\Gamma^3_{oi} = \Gamma^3_{jo} = \frac{1}{2} g^{3\sigma}(\partial_i g_{o\sigma} + \partial_o g_{i\sigma} - \partial_\sigma g_{io})$$
$$= \frac{1}{2} g^{33} \partial_t g_{i3} = 0$$

As before, all terms vanish because of the time-independence of the metric and the absence of mixed components. Thus:

$$\Gamma^3_{01} = \Gamma^3_{02} = \Gamma^3_{03} = \Gamma^3_{30} = \Gamma^3_{20} = \Gamma^3_{10} = 0. \tag{923}$$

Next, we compute the Christoffel symbols with upper index 0. Using the general formula, we obtain:

$$\Gamma^0_{oi} = \Gamma^0_{jo} = \frac{1}{2} g^{0\sigma}(\partial_i g_{\sigma o} + \partial_o g_{i\sigma} - \partial_\sigma g_{oi}), \quad \forall i \neq 0$$
$$= \frac{1}{2} \rho^{00} \partial_i g_{00}$$

To evaluate these components explicitly, we consider each value of i separately.

$$i = 1: \quad \Gamma^0_{10} = \Gamma^0_{01} = \frac{1}{2} g^{00} \partial_1 g_{00} = \frac{1}{2} \frac{1}{A(r)} \partial_r A(r) = \frac{A'}{2A}$$
$$i = 2: \quad \Gamma^0_{20} = \Gamma^0_{02} = \frac{1}{2} g^{00} \partial_2 g_{00} = \frac{1}{2} \frac{1}{A(r)} \partial_\theta A(r) = 0$$
$$i = 3: \quad \Gamma^0_{30} = \Gamma^0_{03} = \frac{1}{2} g^{00} \partial_3 g_{00} = \frac{1}{2} \partial_\phi g_{00} = 0$$

We conclude that the only nonzero Christoffel symbols in this set are:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{A'(r)}{2A(r)} \tag{924}$$

While the others vanish:

$$\Gamma^0_{02} = \Gamma^0_{20} = \Gamma^0_{03} = \Gamma^0_{30} = 0 \tag{925}$$

We now turn to Christoffel symbols with repeated spatial indices. For the component Γ^1_{11}, we compute:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{1\sigma} (\partial_1 g_{\sigma 1} + \partial_1 g_{1\sigma} - \partial_\sigma g_{11}) \\
&= \frac{1}{2} g^{11} (\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \\
&= \frac{1}{2} \left(-\frac{1}{B(r)} \right) \partial_r (B(r)) \\
\Gamma^1_{11} &= -\frac{B'}{2B}
\end{aligned} \tag{926}$$

Finally, we evaluate Γ^1_{22}. Using the same formula:

$$\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2} g^{1\sigma} (\partial_2 g_{\sigma 2} + \partial_2 g_{2\sigma} - \partial_\sigma g_{22}) \\
&= \frac{1}{2} \frac{1}{B(r)} \partial_r (-r^2) \\
\Gamma^1_{22} &= \frac{-r}{B(r)}
\end{aligned} \tag{927}$$

$$\begin{aligned}
\Gamma^1_{33} &= \frac{1}{2} g^{1\sigma} \left(\partial_3 g_{\sigma 3} + \partial_3 g_{3\sigma} - \partial_\sigma g_{33} \right) \\
&= \frac{1}{2} g^{11} \left(\partial_\phi g_{13} + \partial_\phi g_{31} - \partial_1 g_{33} \right) \\
&= \frac{1}{2} \left(\frac{1}{B(r)} \right) \left[\partial_r \left(-r^2 \sin^2 \theta \right) \right] \\
\Gamma^1_{33} &= \frac{r \sin^2 \theta}{B(r)}
\end{aligned} \tag{928}$$

Here, we compute Γ^1_{33} using the definition of the Christoffel symbol. After substituting the appropriate components and simplifying, we arrive at the final expression.

$$\begin{aligned}
\Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2} g^{1\sigma} \left(\partial_2 g_{\sigma 1} + \partial_1 g_{2\sigma} - \partial_\sigma g_{12} \right) \\
&= \frac{1}{2} g^{11} \left(\partial_2 g_{11} + \partial_1 g_{21} \right) = \frac{1}{2} g^{11} \partial_\theta g_{11} = 0 \\
\Gamma^1_{12} = \Gamma^1_{21} &= 0
\end{aligned} \tag{929}$$

Since g_{11} is independent of θ, its derivative vanishes, leading to a zero Christoffel symbol for Γ^1_{12} and Γ^1_{21}.

$$\Gamma^1_{31} = \Gamma^1_{13} = \frac{1}{2} g^{1\sigma} \left(\partial_3 g_{\sigma 1} + \partial_1 g_{3\sigma} - \partial_\sigma g_{13} \right) \Rightarrow \Gamma^1_{13} = \Gamma^1_{31} = 0 \tag{930}$$

Next, we evaluate Γ^1_{13} and Γ^1_{31}. Since the metric components involved are constants or zero, this leads directly to zero.

$$\begin{aligned}
&\Gamma^1_{32} = \Gamma^1_{23} = \frac{1}{2} g^{1\sigma} \left(\partial_3 g_{\sigma 2} + \partial_2 g_{3\sigma} - \partial_\sigma g_{23}\right) = \frac{1}{2} g^{11} \partial_2 g_{31} = 0 \\
&\Gamma^1_{23} = \Gamma^1_{32} = 0
\end{aligned} \tag{931}$$

For the remaining mixed terms involving indices 2 and 3, we find that the derivatives again vanish, resulting in zero Christoffel symbols.

$$\Gamma^2_{ii} = \frac{1}{2} g^{2\sigma} \left(\partial_i g_{\sigma i} + \partial_i g_{i\sigma} - \partial_\sigma g_{ii}\right) = \frac{1}{2} g^{22} \left(\partial_i g_{2i} + \partial_i g_{i2} - \partial_2 g_{ii}\right)$$

To compute Γ^2_{ii}, we analyze the cases for $i = 1, 2, 3$ individually.

$$\begin{aligned}
i = 1: \quad & \Gamma^2_{11} = \frac{1}{2} \left(-\frac{1}{r^2}\right) \Big[- \partial_\theta(-B(r)) \Big] = 0 \\
i = 2: \quad & \Gamma^2_{22} = \frac{1}{2} g^{22} \partial_\theta(-r^2) = 0 \\
i = 3: \quad & \Gamma^2_{33} = \frac{1}{2} \left(-\frac{1}{r^2}\right) \left(\partial_\theta (r^2 \sin^2 \theta)\right) = -\sin\theta\cos\theta
\end{aligned}$$

We observe that the Christoffel symbols vanish for $i = 1$ and $i = 2$, while for $i = 3$ the non-zero derivative leads to a non-trivial result.

$$\Gamma^2_{11} = 0 = \Gamma^2_{22} \tag{932}$$

Thus, both Γ^2_{11} and Γ^2_{22} vanish.

$$\Gamma^2_{33} = -\sin\theta\cos\theta \tag{933}$$

Only Γ^2_{33} yields a non-zero expression involving trigonometric functions of θ.

We now continue computing additional Christoffel symbols. Let's begin with Γ^2_{12} and Γ^2_{21}.

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2} g^{2\sigma} (\partial_2 g_{\sigma 1} + \partial_1 g_{2\sigma} - \partial_\sigma g_{12}) = \frac{1}{2} \left(\frac{1}{r^2}\right) \partial_r r^2 = \frac{1}{r} \tag{934}$$

Next, we evaluate Γ^2_{13} and Γ^2_{31}. Here, we observe that the relevant metric derivatives vanish.

$$\Gamma^2_{31} = \Gamma^2_{13} = \frac{1}{2} g^{2\sigma} (\partial_3 g_{1\sigma} + \partial_1 g_{3\sigma} - \partial_\sigma g_{13}) = \frac{1}{2} g^{22} (\partial_3 g_{12} + \partial_1 g_{32}) = 0 \tag{935}$$

We now compute Γ^2_{32} and Γ^2_{23}. As with the previous case, these components vanish due to the absence of θ dependence.

$$\Gamma^2_{32} = \Gamma^2_{23} = \frac{1}{2} g^{22} \partial_2 g_{32} = 0 \tag{936}$$

Next, we look at the time components Γ^0_{ii} for spatial indices $I \neq 0$. Since the metric components do not depend on time, these derivatives vanish.

$$\begin{aligned}\Gamma^0_{ii} &= \frac{1}{2}g^{0\sigma}(\partial_i g_{\sigma i} + \partial_i g_{i\sigma} - \partial_\sigma g_{ii}), \quad \forall i \neq 0 \\ &= -\frac{1}{2}g^{00}\partial_t g_{ii} = 0\end{aligned} \tag{937}$$

We now consider mixed time-space Christoffel symbols of the form Γ^0_{2i} and Γ^0_{i2}. Again, these vanish due to the time independence of the metric components.

$$\Gamma^0_{2i} = \Gamma^0_{i2} = \frac{1}{2}g^{0\sigma}(\partial_2 g_{\sigma i} + \partial_i g_{2\sigma} - \partial_\sigma g_{i2}) = \frac{1}{2}g^{00}(\partial_2 g_{0i} + \partial_i g_{20} - \partial_0 g_{i2})$$

We verify the vanishing components for specific values of i.

$$\begin{aligned} i = 1: \quad & \Gamma^0_{21} = \Gamma^0_{12} = \frac{1}{2}g^{00}(\partial_2 g_{01} - \partial_0 g_{12}) = 0 \\ i = 3: \quad & \Gamma^0_{23} = \Gamma^0_{32} = \frac{1}{2}g^{00}(\partial_2 g_{03} - \partial_0 g_{32}) = 0\end{aligned}$$

This confirms the result:

$$\Gamma^0_{21} = \Gamma^0_{12} = \Gamma^0_{23} = \Gamma^0_{32} = 0 \tag{938}$$

Lastly, we evaluate the component Γ^0_{31} and its symmetric counterpart. As before, since the metric components do not depend on time, all time derivatives vanish, and the remaining terms are zero as well.

$$\Gamma^0_{31} = \Gamma^0_{13} = \frac{1}{2}g^{0\sigma}(\partial_3 g_{\sigma 1} + \partial_1 g_{3\sigma} - \partial_\sigma g_{31}) = \frac{1}{2}g^{00}(\partial_3 g_{01} + \partial_1 g_{30}) = 0 \tag{939}$$

We now move on to compute Γ^3_{ii} for spatial indices $i = 1, 2, 3$. These components describe how the ϕ-coordinate changes along the directions of r, θ, and ϕ, respectively.

$$\begin{aligned} & \Gamma^3_{ii} = \frac{1}{2}g^{3\sigma}(\partial_i g_{\sigma i} + \partial_i g_{i\sigma} - \partial_\sigma g_{ii}) \\ i = 1: \quad & \Gamma^3_{11} = \frac{1}{2}g^{33}(\partial_1 g_{31} + \partial_1 g_{13} - \partial_3 g_{11}) \Rightarrow \Gamma^3_{11} = 0 \\ i = 2: \quad & \Gamma^3_{22} = \frac{1}{2}g^{33}(\partial_2 g_{32} + \partial_2 g_{23} - \partial_3 g_{22}) \Rightarrow \Gamma^3_{22} = 0 \\ i = 3: \quad & \Gamma^3_{33} = \frac{1}{2}g^{33}(\partial_3 g_{33} + \partial_3 g_{33} - \partial_3 g_{33}) \Rightarrow \Gamma^3_{33} = 0\end{aligned} \tag{940}$$

As expected, these Christoffel symbols vanish because there is no ϕ-dependence in the metric tensor components g_{11}, g_{22}, or g_{33}.

Next, we examine the mixed second-order Christoffel symbols Γ^3_{12} and Γ^3_{21}. These, too, turn out to be zero, as none of the metric components involved depend on both r and θ in the relevant way.

$$\Gamma^3_{12} = \Gamma^3_{21} = \frac{1}{2} g^{3\sigma}(\partial_2 g_{\sigma 1} + \partial_1 g_{2\sigma} - \partial_\sigma g_{12}) = \frac{1}{2} g^{33}(\partial_2 g_{31} + \partial_1 g_{23}) = 0 \tag{941}$$

We now compute the remaining Christoffel symbols $\Gamma^3_{i3} = \Gamma^3_{3i}$ for $i = 1, 2$. These terms capture how the ϕ direction varies concerning changes in r and θ.

$$\Gamma^3_{i3} = \Gamma^3_{3i} = \frac{1}{2} g^{3\sigma}(\partial_3 g_{\sigma i} + \partial_i g_{3\sigma} - \partial_\sigma g_{i3}) = \frac{1}{2} g^{33} \partial_i g_{33} \tag{942}$$

For $i = 1$, the derivative is concerning r. We compute:

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} \frac{1}{r^2 \sin^2 \theta} \partial_r (r^2 \sin^2 \theta) = \frac{1}{r} \tag{943}$$

For $i = 2$, the derivative is with respect to θ, giving:

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} \frac{1}{r^2 \sin^2 \theta} \partial_\theta (r^2 \sin^2 \theta) = \cot \theta \tag{944}$$

These are the only non-zero Christoffel symbols with upper index 3, and they reflect the spherical symmetry of the metric.

Expressions for $\Gamma^\rho_{\mu\nu}$ are summarized below:

$$\begin{array}{l|l}
\Gamma^0_{01} = \Gamma^0_{10} = \dfrac{A'}{2A} & \Gamma^2_{12} = \Gamma^2_{21} = \dfrac{1}{r} \\
\Gamma^1_{00} = \dfrac{A'}{2B} & \Gamma^2_{33} = -\sin\theta\cos\theta \\
\Gamma^1_{11} = \dfrac{B'}{2B} & \Gamma^3_{13} = \Gamma^3_{31} = \dfrac{1}{r} \\
\Gamma^1_{22} = -\dfrac{r}{B} & \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \\
\Gamma^1_{33} = -\dfrac{r \sin^2\theta}{B} &
\end{array}$$

We now turn our attention to the Ricci tensor, which encodes how volumes deform under parallel transport and plays a central role in formulating Einstein's field equations. In the context of our metric, many of the Christoffel symbols either vanish or simplify considerably. As a result, the Ricci tensor components also exhibit simplifications. **In particular, all off-diagonal components of the Ricci tensor ($R_{\mu\nu}$ for $\mu \neq \nu$) vanish** due to the symmetries present in the spacetime and the structure of the connection coefficients.

To compute the Ricci tensor components, we use the general expression:

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma}$$

In the next terms, most Christoffel symbols either vanish or are independent of time. Consequently, many of the terms above vanish identically or cancel out. Thus, we take:

The R_{00} component reads:

$$R_{00} = \partial_0 \Gamma^\sigma_{0\sigma} - \partial_\sigma \Gamma^\sigma_{00} + \Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0} - \Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma}$$
$$R_{00} = -\partial_r \left(\frac{A'}{2B}\right) + \frac{A'}{2A}\frac{A'}{2B} - \frac{A'}{2B}\frac{B'}{2B} - \frac{A'}{2B}\frac{1}{r} - \frac{A'}{2B}\frac{1}{r}$$

$$\boxed{R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}} \tag{945}$$

The R_{11} component reads:

$$R_{11} = \partial_1 \Gamma^{\sigma}_{1\sigma} - \partial_\sigma \Gamma^{\sigma}_{11} + \Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1} - \Gamma^{\rho}_{11}\Gamma^{\sigma}_{\rho\sigma}$$
$$R_{11} = \frac{A''}{2A} - \frac{(A')^2}{4A^2} - \frac{2}{r^2} + \frac{(A')^2}{4A^2} + \frac{1}{r^2} + \frac{1}{r^2} - \frac{B'}{2B}\frac{A'}{2A} - \frac{B'}{2Br}$$

$$\boxed{R_{11} = -\frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{2Br}} \tag{946}$$

Next, the R_{22} component reads:

$$R_{22} = \partial_2 \Gamma^{\sigma}_{2\sigma} - \partial_\sigma \Gamma^{\sigma}_{22} + \Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2} - \Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma}$$
$$R_{22} = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} + \frac{1}{B} - \frac{2rB'}{2B^2} - \frac{2r}{B}\frac{1}{r} + \frac{\cos^2\theta}{\sin^2\theta} + \frac{rA'}{2BA} + \frac{rB'}{2B^2} + \frac{1}{B} + \frac{1}{B}$$
$$R_{22} = -1 + \frac{3}{B} - \frac{2}{B} - \frac{rB'}{2B^2} + \frac{rA'}{2BA}$$

$$\boxed{R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} + \frac{B'}{B}\right)} \tag{947}$$

Finally, the R_{33} component reads:

$$R_{33} = \partial_3 \Gamma^{\sigma}_{3\sigma} - \partial_\sigma \Gamma^{\sigma}_{33} + \Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3} - \Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma}$$
$$R_{33} = \sin^2\theta\Big[\frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)\Big]$$

$$\boxed{R_{33} = R_{22}\sin^2\theta} \tag{948}$$

The matrix of $R_{\mu\nu}$ components based on the above equations is:

$$R_{\mu\nu} = \begin{pmatrix} R_{00} & 0 & 0 & 0 \\ 0 & R_{11} & 0 & 0 \\ 0 & 0 & R_{22} & 0 \\ 0 & 0 & 0 & R_{22}\sin^2\theta \end{pmatrix} \tag{949}$$

In order to continue our computations and make our life easier, we have to consider the following:

1. The field is **static** and matter obeys a spherically symmetric distribution, implying that there is no evolution with time.

2. The matter distribution is described by a perfect fluid and the energy- momentum tensor is explicitly written as:

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p g_{\mu\nu}. \tag{950}$$

3. The field equations are solutions for the object's interior, the exterior is not taken under consideration, since it is supposed to be empty space.

4. The **Ricci tensor** is assumed to be in **diagonal form**, that is,

$$R_{\mu\nu} = 0 \quad \text{for all } \mu \neq \nu.$$

From assumptions 1 and 3, we get:

$$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} T_{g_{\mu\nu}} \right) \tag{951}$$

$$R_{\mu\nu} = -\kappa \Big[\left(\rho + \frac{p}{c^2} \right) u_\mu u_\nu - p g_{\mu\nu} - \frac{1}{2} T_{g_{\mu\nu}} \Big] \tag{952}$$

$$\begin{aligned} &T \equiv T^\mu_\mu = g^{\mu\nu} T_{\mu\nu} = \Big[\left(\rho + \frac{p}{c^2} \right) g^{\mu\nu} u_\mu u_\nu - p g^{\mu\nu} g_{\mu\nu} \Big] \xRightarrow{g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu} \\ &T = \Big[\left(\rho + \frac{p}{c^2} \right) u_\mu u_\nu - p \delta^\mu_\mu \Big] \xRightarrow[\delta^\mu_\mu = 4]{u_\mu u_\nu = c^2} \\ &T = -\kappa(\rho c^2 + \rho - 4\rho) = -\kappa(\rho c^2 - 3\rho) \end{aligned} \tag{953}$$

Inserting Eq. 953 in Eq. 952 we get

$$\begin{aligned} R_{\mu\nu} &= -\kappa \Big[\left(\rho + \frac{p}{c^2} \right) u_\mu u_\nu - p g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\rho c^2 - 3\rho) \Big] \Rightarrow \\ R_{\mu\nu} &= -\kappa \Big[\left(\rho + \frac{p}{c^2} \right) u_\mu u_\nu - \frac{1}{2} g_{\mu\nu} \rho c^2 + \frac{1}{2} p g_{\mu\nu} - p g_{\mu\nu} \Big] \Rightarrow \\ R_{\mu\nu} &= -\kappa \Big[\left(\rho + \frac{p}{c^2} \right) u_\mu u_\nu - \frac{1}{2} g_{\mu\nu} (\rho c^2 - p) \Big] \end{aligned} \tag{954}$$

From assumption 4, we get:

$$R_{oi} = 0 \Rightarrow (\rho + \frac{p}{c^2}) u_0 u_i - \frac{1}{2} g_{oi} (\rho c^2 - p) = 0 \Rightarrow u_0 u_i = 0 \tag{955}$$

Thus,

$$u_\mu u^\mu = c^2 \quad \text{with} \quad u_\mu = c\sqrt{A}(1, 0, 0, 0) \tag{956}$$

Where this holds without assuming staticity.
Components of $R_{\mu\nu}$ as produced by eq. 954:

$$\begin{aligned} R_{00} &= -\kappa \left[\left(\rho + \frac{p}{c^2} \right) u_0 u_0 - \frac{1}{2} A (\rho c^2 - p) \right] \\ R_{00} &= -\kappa A \left[\rho c^2 + p - \frac{1}{2} \rho c^2 + \frac{1}{2} p \right] \\ &\boxed{R_{00} = -\frac{\kappa}{2} A (\rho c^2 + 3p)} \end{aligned} \tag{957}$$

$$
\begin{aligned}
R_{11} &= \kappa\left[\left(\rho+\frac{p}{c^2}\right)u_1u_1+\frac{1}{2}(-B)(\rho c^2-p)\right]\\
&\boxed{R_{11}=-\frac{\kappa}{2}(\rho c^2-p)B}
\end{aligned}
\tag{958}
$$

$$
\begin{aligned}
R_{22} &= \kappa\left[\left(\rho+\frac{p}{c^2}\right)u_2u_2+\frac{1}{2}(-r^2)(\rho c^2-p)\right]\\
&\boxed{R_{22}=-\frac{\kappa}{2}r^2(\rho c^2-p)}
\end{aligned}
\tag{959}
$$

Lastly,

$$
\boxed{R_{33}=R_{22}\sin^2\theta=-\frac{\kappa}{2}r^2\sin^2\theta(\rho c^2-p)}
\tag{960}
$$

One can rewrite the Eqs. 957 - 959 as following:

$$
\frac{R_{00}}{A}=-\frac{\kappa}{2}(\rho c^2+3p) \tag{961}
$$

$$
\frac{R_{11}}{B}=-\frac{\kappa}{2}(\rho c^2-p) \tag{962}
$$

$$
\frac{2R_{22}}{r^2}=-\kappa(\rho c^2-p) \tag{963}
$$

$$
\tag{964}
$$

And produce the sum of the primed ones to be:

$$
\boxed{\frac{R_{00}}{A}+\frac{R_{11}}{B}+\frac{2R_{22}}{r^2}=-2\kappa\rho c^2}
\tag{965}
$$

By substituting eqs 949 into eq 965, we take:

$$
\begin{aligned}
&\frac{2B'}{rB^2}-\frac{2}{r^2B}+\frac{2}{r^2}=2\kappa\rho c^2\Rightarrow\\
&\frac{B'}{rB^2}-\frac{1}{r^2B}+\frac{1}{r^2}=\kappa\rho c^2\\
&\boxed{\frac{rB'}{B^2}-\frac{1}{B}+1=\kappa r^2\rho c^2}
\end{aligned}
\tag{966}
$$

$$
\begin{aligned}
&1-\frac{1}{B}+\frac{rB'}{B^2}=\kappa r^2\rho c^2\\
&\frac{dr}{dr}\left(1-\frac{1}{B}\right)+r\frac{d}{dr}\left(1-\frac{1}{B}\right)=\kappa r^2\rho c^2\\
&\boxed{\frac{d}{dr}\left[r\left(1-\frac{1}{B}\right)\right]=\kappa r^2\rho c^2}
\end{aligned}
\tag{967}
$$

$$
\begin{aligned}
r\left(1-\frac{1}{B}\right) &= \kappa c^2 \int_0^r dr' r'^2 \rho(r') \\
&= \frac{8\pi G c^2}{c^4} \int_0^r dr' r'^2 \rho(r') \\
&= \frac{2G}{c^2} 4\pi \int_0^r dr' r'^2 \rho^2 \rho(r') \\
&= \frac{2m(r)G}{c^2} \\
B-1 &= B\frac{2m(r)G}{rc^2} \\
1 &= B\left(1-\frac{2Gm(r)}{rc^2}\right) \\
&\boxed{B=\left[1-\frac{2Gm(r)}{rc^2}\right]^{-1}}
\end{aligned}
\tag{968}
$$

with

$$
\boxed{m(r) \equiv 4\pi \int_0^r dr' (r')^2 \rho(r')} \tag{969}
$$

Eq. 969 does not produce the contained mass in coordinate radius r. The proper volume element is expressed in this metric coordinates as $d^3V = \sqrt{B(r)} r^2 \sin^2\theta dr d\theta d\phi$.

The star's proper "mass" is now expressed by the following:

$$
\begin{gathered}
\tilde{m}(r) = 4\pi \int_0^r \rho(r') \sqrt{B(r')} r'^2 dr \Rightarrow \\
\boxed{\tilde{m}(r) = 4\pi \int_0^r \rho(r') \left(1-\frac{2Gm(r')}{r'c^2}\right)^{-1/2} r^{1/2} dr'}
\end{gathered}
\tag{970}
$$

where $B(r) \sim m(r)$ and not $B(r) \nsim \tilde{m}(r)$. When the object extends to $r = R$, and for $r > R$ we suppose an empty space, then outside the spacetime geometry is described by a Schwarzschild metric with mass parameter $M = m(r)$. The difference $E = \tilde{M} - M$ corresponds to the gravitational binding energy. The gravitational binding energy $E = \tilde{M} - M$ is the energy needed to disperse the material of which the object consists to infinite spatial separation.

Differential equation for $A(r)$. Conservation of stress-energy tensor yields:

$$
\boxed{\nabla\mu T^{\mu\nu} = 0} \tag{971}
$$

Substituting 950 to 971 we get:

$$
\boxed{\Gamma^{\mu}_{\sigma\mu} = \Gamma^{\mu}_{\mu\sigma} = \partial_\sigma \ln\sqrt{-g} = \frac{1}{\sqrt{-g}} \partial_\sigma \sqrt{-g}} \Rightarrow \tag{972}
$$

$$\partial_\mu T^{\mu\nu} + \frac{1}{\sqrt{-g}}(\partial_\sigma\sqrt{-g})T^{\sigma\nu} + \Gamma^\nu_{\sigma\mu}T^{\mu\sigma} = 0 \Rightarrow$$
$$\partial_\mu T^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}T^{\sigma\nu}) - \frac{1}{\sqrt{-g}}\sqrt{-g}\partial_\sigma\Gamma^{\sigma\nu} + \Gamma^\nu_{\sigma\mu}T^{\mu\sigma} = 0 \Rightarrow$$
$$\frac{1}{\sqrt{-g}}\partial_\sigma\left[\sqrt{-g}\left(\rho + \frac{p}{c^2}\right)u^\sigma u^\nu\right] + \left(\rho + \frac{p}{c^2}\right)\Gamma^\nu_{\sigma\mu}u^\mu u^\sigma - g^{\mu\nu}(\nabla_\mu p) = 0 \Rightarrow$$
$$\frac{c^2}{A}\left(\rho + \frac{p}{c^2}\right)\left[\frac{1}{2}g^{\nu\sigma}(\partial_o g_{\sigma 0} + \partial_o g_{0\sigma} - \partial_\sigma g_{00]})\right] - g^{\mu\nu}\partial_\mu P = 0 \Rightarrow$$
$$\frac{c^2}{A}\left(\rho + \frac{p}{c^2}\right)\left(-\frac{1}{2}g^{\nu\sigma}\partial_\sigma g_{00}\right) - g^{\mu\nu}\partial_\mu P = 0 \Rightarrow$$
$$\frac{c^2}{2A}\left(\rho + \frac{p}{c^2}\right)g^{\mu\nu}\partial_\mu A + \partial_\mu p g_{\nu\sigma}g^{\mu\nu} = 0 \Rightarrow$$
$$\frac{c^2}{2A}\left(\rho + \frac{p}{c^2}\right)\partial_\sigma A + \partial_\sigma p = 0 \Rightarrow$$

$$\boxed{\partial_\sigma p + \frac{\rho c^2 + p}{2A}\partial_\sigma A = 0} \tag{973}$$

In Eq. 973 σ is a free index, so for:

$$\sigma = 0: \quad \partial_t p + \frac{\rho c^2 + p}{2A}\partial_t A = 0 \Rightarrow \partial_t p = 0 \Rightarrow \boxed{p \neq p(t)} \tag{974}$$
$$\sigma = 2: \quad \partial_\theta p + \frac{\rho c^2 - p}{2A}\partial_\theta A(r) = 0 \Rightarrow \partial_\theta p = 0 \Rightarrow \boxed{p \neq p(\theta)} \tag{975}$$
$$\sigma = 3: \quad \partial_\phi p + \frac{\rho c^2 - p}{2A}\partial_\phi A(r) = 0 \Rightarrow \partial_\phi p = 0 \Rightarrow \boxed{p \neq p(\phi)} \tag{976}$$
$$\sigma = 1: \quad \partial_r p + \frac{\rho c^2 - p}{2A}\partial_r A(r) = 0 \Rightarrow \boxed{\frac{1}{A}\frac{dA(r)}{dr} = -\frac{2}{pc^2 - p}\frac{dp}{dr}} \tag{977}$$

Eq. 977 is the D.E. $A(r)$ must satisfy. In simpler terms, it reads as

$$\boxed{\frac{A'(r)}{A} = -\frac{2p'}{\rho c^2 - p}} \tag{978}$$

The relativistic equations of stellar structure

Exact form of $A(r)$ and $B(r)$ is given if and only if $p = p(\rho)$ is given as an equation of state (E.o.S.). When $\rho(r)$ and $p(r)$ are arbitrarily chosen, they give results unrealistic. Moving on we will produce the first equation of stellar structure.

$$m(r) = 4\pi\int_0^r dr' r'^2 \rho(r') \Rightarrow \boxed{\frac{dm(r)}{dr} = 4\pi r^2\rho(r)} \tag{979}$$

Eq. 979 relates $m(r)$ and $\rho(r)$. The $\theta\theta$ component of Ricci's tensor is given as:

$$\left.\begin{aligned} R_{22} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \\ R_{22} &= -\frac{\kappa}{2} r^2(\rho c^2 - p) \end{aligned}\right\} \Rightarrow \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = -\frac{1}{2}\kappa r^2(\rho c^2 - p)$$

Substituting 977 into the above equation, we get:

$$1 - \frac{2Gm(r)}{rc^2} - 1 + \frac{r}{2}\left(1 - \frac{2Gm(r)}{rc^2}\right)\left(-\frac{2p'}{\rho c^2 + p} - \frac{B'}{B}\right) = -\frac{1}{2}\kappa r^2(\rho c^2 - p) \tag{980}$$

We now recall the expression for $B(r)$ given in equation 968, and proceed to differentiate it:

$$\begin{aligned} B(r) &= \frac{1}{1 - \frac{2Gm(r)}{rc^2}} \Rightarrow \\ \frac{dB(r)}{dr} &= B^2(r)\frac{2G}{c^2}\left(\frac{\frac{dm(r)}{dr}}{r} - \frac{m(r)}{r^2}\right) \Rightarrow \\ \frac{dB(r)}{dr}\frac{1}{B^2(r)} &= \frac{2G}{c^2}\left(\frac{4\pi r^2\rho(r)}{r} - \frac{m(r)}{r^2}\right) \Rightarrow \\ \frac{1}{B^2}\frac{dB}{dr} &= \frac{2G}{r^2c^2}\left[4\pi r^3 p(r) - m(r)\right] \Rightarrow \end{aligned}$$

Which yields the compact form:

$$\boxed{\frac{B'}{B^2} = \frac{2G}{r^2c^2}\left[4\pi r^3\rho(r) - m(r)\right]} \tag{981}$$

Next, using equation 968 again, we derive an identity for $\frac{1}{B} - 1$:

$$\boxed{\frac{1}{B} - 1 = 1 - \frac{2Gm(r)}{rc^2} - 1 = -\frac{2Gm}{rc^2}} \tag{982}$$

We also substitute 968 into 977 to isolate the term $\frac{A'}{A}$:

$$\boxed{\frac{r}{2B}\frac{A'}{A} = \frac{dp}{dr}\left(\frac{1}{\rho c^2 + p}\right)\left(r - \frac{2Gm}{c^2}\right)} \tag{983}$$

Then, from equation 981, we compute:

$$\boxed{-\frac{r}{2}\frac{B'}{B^2} = \frac{mG}{rc^2} - \frac{4\pi\rho G}{c^2}r^2} \tag{984}$$

Finally, rewriting the right-hand side of equation 980 using the definition of κ, we find:

$$\boxed{\frac{1}{2}\kappa(\rho c^2 - p)r^2 = \frac{4\pi G r^2}{c^4}(\rho c^2 - p)} \tag{985}$$

Now, substituting equations 982 through 985 into equation 980, we obtain:

$$\begin{aligned}
&-\frac{mG}{r^2} - \frac{4\pi G r^2}{c^4}p = \frac{dp}{dr}\left(\frac{1}{\rho c^2 + p}\right) r \left(1 - \frac{2Gm}{rc^2}\right) \Rightarrow \\
&-\frac{1}{r}\left(\frac{Gm}{c^2} + \frac{4\pi G r^3}{c^4}p\right) = r\left(\frac{1}{\rho c^2 + p}\right)\frac{dp}{dr}\left(1 - \frac{2Gm}{rc^2}\right)
\end{aligned}$$

Rearranging and simplifying, we finally arrive at the Tolman–Oppenheimer–Volkoff (TOV) equation:

$$\boxed{\frac{dp}{dr} = -\frac{1}{r^2}(\rho c^2 + p)\left[\frac{4\pi G}{c^4}p(r)r^3 + \frac{Gm(r)}{c^2}\right]\left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1}} \tag{986}$$

Equations 979, 986 and $p = p(\rho)$ create a closed system of eqs, needed to define the E.o.S. for the matter. The set is called TOV eqs. Given in the form:

i) $p = p(\rho)$ (E.o.S.), links $p(r)$ and $\rho(r)$

ii) $\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$, links $m(r)$ and $\rho(r)$

iii) $\frac{dp(r)}{dr} = -\frac{1}{r^2}(\rho c^2 + p(r))\left[\frac{4\pi G}{c^4}p(r)r^3 + \frac{Gm(r)}{c^2}\right]\left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1}$, links $m(r)$ and $p(r)$

The above set of equations provides the equations of stellar structure. The matter included in NSs obeys a polytropic E.o.S. of form

$$p = \mathcal{K}\rho^\gamma \tag{987}$$

with

$$\gamma = 1 + \frac{1}{n} \tag{988}$$

Both $\mathcal{K}$ and γ are constants, and n is called the polytropic index.

Set (i-iii) contains two coupled differential equations of first order, which gives a unique solution that is obtained by two boundary conditions.

1. First boundary condition: $m(0) - 0$.
2. Second boundary condition: central pressure $p(0)$ or central density $\rho(0)$.

Set (i-iii) is integrated numerically on a computer, beginning point $r = 0$ and integrating outwards until the pressure drops to zero. Pressure drops to zero when $r = R$ at the star's surface. In a more compact form we write:

$$\texttt{when} \quad r > R \Rightarrow \begin{cases} \rho(r) = 0 = p(r) \\ m(r) = m(R) \equiv M \end{cases} \tag{989}$$

where the spacetime is described by the Schwarzschild metric with mass M.

In the Newtonian limit:

i) $p \ll \rho$

ii) $4\pi r^3 p \ll mc^2$

iii) $g_{\mu\nu} \sim \eta_{\mu\nu} \Rightarrow \frac{2Gm}{rc^2}$

Substituting the above equations in 986:

$$\frac{dP}{dr} \approx -\frac{1}{r^2}\rho c^2 \left(\frac{Gm(r)}{c^2}\right) \Rightarrow \frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{990}$$

Eq. 990 denotes hydrostatic equilibrium.

The Schwarzschild constant-density interior solution

The simplest analysis solution for a relativistic star is obtained by assuming that the density ρ is constant as:

$$\rho = constant \tag{991}$$

Eq. 991 constitutes an E.o.S., borderline of being realististic, corresponds to an ultra-stiff E.o.S. and represents an incompressible fluid.

$$\begin{aligned}
\frac{dm(r)}{dr} &= 4\pi r^2 \Rightarrow \\
\int_0^r dm(r') &= 4\pi\rho \int_0^r dr' r'^2 \Rightarrow \\
m(r) &= 4\pi r \frac{r'^3}{3}\Big|_0^r = \frac{4}{3}\pi\rho r^3, \quad r \leq R \Rightarrow \\
\int_0^R dm(r) &= 4\pi\rho \int_R^0 dr' r'^2 = \frac{4}{3}\pi\rho R^3, \quad r > R \Rightarrow
\end{aligned}$$

$$\boxed{m(r) = \begin{cases} \frac{4}{3}\pi\rho r^3, & r \leq R \\ \frac{4}{3}\pi\rho R^3, & r > R \end{cases}} \tag{992}$$

where R is the radius of the star and M is the mass parameter for the Schwarzchild metric and describes the geometry outside the star.

Substituning eq. 986 to eq. 992 we get:

$$\begin{aligned}
\frac{dp}{dr} &= -\frac{1}{r^2}(\rho c^2 + p)\left(\frac{4\pi G}{c^4}p(r)r^3 + \frac{4\pi G p}{3c^2}r^3\right)\left(1 - \frac{8\pi G p r^3}{3rc^2}\right)^{-1} \\
\frac{dp}{dr} &= -\frac{4\pi G}{3c^4}r(\rho c^2 + p)(\rho c^2 + 3p)\left(1 - \frac{8\pi G p r^3}{3rc^2}\right)^{-1}
\end{aligned} \tag{993}$$

$$\boxed{\int_{p_0}^{p(r)} \frac{d\bar{p}}{(\rho c^2+\bar{p})(\rho c^2+3\bar{p})} = -\frac{4\pi G}{3c^4}\int_0^r d\bar{r}\left(\frac{\bar{r}}{1-\frac{8\pi Gp}{3c^2}\bar{r}^2}\right)} \tag{994}$$

Eq. 994 LHS integrand is written as:

$$\begin{aligned}
&\frac{A}{(\rho c^2+p)}+\frac{B}{(\rho c^2+3p)} = \frac{1}{(\rho c^2+p)(\rho c^2+3p)} \Rightarrow \\
&A(\rho c^2+3p)+B(\rho c^2+p) = 1Op_1 \Rightarrow \\
&\begin{cases} A\rho c^2+B\rho c^2 = 1 \\ 3Ap+Bp = Op \end{cases} \Rightarrow \begin{cases} A+B = \dfrac{1}{\rho c^2} \\ B = -3A \end{cases} \Rightarrow \begin{cases} A = -\dfrac{1}{2\rho c^2} \\ B = \dfrac{3}{2\rho c^2} \end{cases}
\end{aligned}$$

Thus,

$$\boxed{\int_{p_0}^{p(r)} \frac{d\bar{p}}{(\rho c^2+p)(\rho c^2+3p)} = \frac{1}{2\rho c^2}\int_{p_0}^{p(r)} d\bar{p}\left[\frac{3}{\rho c^2+3\bar{p}}-\frac{1}{\rho c^2+\bar{p}}\right]} \tag{995}$$

Instead of integrating RHS, we apply the derivative w.r.t. $\bar{r}$ of the following expression:

$$\boxed{-\frac{4\pi G}{3c^4}\frac{\bar{r}}{1-\frac{8\pi Gp}{3c^2}\bar{r}^2} = \frac{d}{dr}\left[\frac{1}{4c^2\rho}\ln\left(1-\frac{8\pi G\rho}{3c^2}\bar{r}^2\right)\right]} \tag{996}$$

Integrating straightforwardly eq. 995 one yields:

$$\begin{aligned}
&\int_{p_0}^{p(r)} \frac{d\bar{p}}{(\rho c^2+p)(\rho c^2+3p)} = \frac{1}{2pc^2}\int_{p_0}^{p(r)} \left[d\ln\left(\rho c^2+3\bar{p}\right)-d\ln\left(\rho c^2+\bar{p}\right)\right] \\
&\boxed{\int_{p_0}^{p(r)} \frac{d\bar{p}}{(\rho c^2+\bar{p})(\rho c^2+3\bar{p})} = \frac{1}{2\rho c^2}\left[\ln\left(\frac{\rho c^2+3p(r)}{\rho c^2+p(r)}\right)-\ln\left(\frac{\rho c^2+3p_0}{\rho c^2+\bar{p}_0}\right)\right]}
\end{aligned} \tag{997}$$

Finally, eq. 994 is written as:

$$\begin{aligned}
&\ln\left(\frac{\frac{\rho c^2+3p(r)}{\rho c^2+p(r)}}{\frac{\rho c^2+3p_0}{\rho c^2+p_0}}\right) = \frac{1}{2}\ln\left(1-\frac{8\pi Gp}{3c^2}r^2\right) \Rightarrow \\
&\ln\left(\frac{\frac{\rho c^2+3p(r)}{\rho c^2+p(r)}}{\frac{\rho c^2+3p_0}{\rho c^2+p_0}}\right) = \ln\left(1-\frac{8\pi Gp}{3c^2}r^2\right)^{1/2} \Rightarrow
\end{aligned}$$

$$\boxed{\frac{\rho c^2+3p(r)}{\rho c^2+p(r)} = \frac{\rho c^2+3p_0}{\rho c^2+p_0}\left(1-\frac{8\pi G\rho}{3c^2}r^2\right)^{1/2}} \tag{998}$$

On the star's surface, the following arguments are valid without proof:

1. $r = R$
2. $p(R) = 0$
3. $\frac{\rho c^2+3p(R)}{\rho c^2+p(R)} = \frac{\rho c^2}{pc^2} = 1$
4. From Eq. 998 $1 = \frac{\rho c^2+3p_0}{\rho c^2 p_0}\left(1 - \frac{8\pi G\rho}{3c^2}r^2\right)^{1/2} \Rightarrow$

$$\boxed{R^2 = \frac{3c^2}{8\pi Gp}\left[1 - \left(\frac{\rho c^2 + p_0}{\rho c^2 + 3p_0}\right)^2\right]} \tag{999}$$

R is the radius of a star with uniform density ρ at central pressure p_0. Set

$$\mu \equiv \frac{GM}{c^2} = \frac{4\pi G\rho R^3}{3c^2} \tag{1000}$$

$$\begin{aligned}
&R^2 = \frac{R^3}{2\mu}\left[1 - \left(\frac{\rho c^2 + p_0}{\rho c^2 + 3p_0}\right)^2\right] \Rightarrow \\
&\frac{2\mu}{R} = 1 - \left(\frac{\rho c^2 + p_0}{\rho c^2 + 3p_0}\right)^2 \Rightarrow \left(\frac{\rho c^2 + p_0}{\rho c^2 + 3p_0}\right)^2 = 1 - \frac{2\mu}{R} \Rightarrow \\
&\rho c^2 + p_0 = \rho c^2\left(1 - \frac{2\mu}{R}\right)^{1/2} + 3p_0\left(1 - \frac{2\mu}{R}\right)^{1/2} \Rightarrow \\
&p_0\left[-1 + 3\left(1 - \frac{2\mu}{R}\right)^{1/2}\right] = \rho c^2\left[1 - \sqrt{1 - \frac{2\mu}{R}}\right]
\end{aligned}$$

$$\boxed{p_0 = \rho c^2\left(\frac{1 - \sqrt{1 - \frac{2\mu}{R}}}{3\sqrt{1 - \frac{2\mu}{R}} - 1}\right)} \tag{1001}$$

Replacing p_0 in the expression eq. 998 yields:

$$\begin{aligned}
&\frac{\rho c^2 + 3p(r)}{\rho c^2 + p(r)} = \frac{\rho c^2 + 3p_0}{\rho c^2 + p_0}\left(1 - \frac{4\pi\rho R^3}{3c^2}\frac{2r^2}{R^3}\right)^{1/2} \Rightarrow \\
&\frac{\rho c^2 + 3p(r)}{\rho c^2 + p(r)} = \frac{\rho c^2 + 3p_0}{\rho c^2 + p_0}\left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2} \Rightarrow \\
&\frac{\rho c^2 + 3p(r)}{\rho c^2 + p(r)} = \frac{2}{2\sqrt{1 - \frac{2\mu}{R}}}\sqrt{1 - \frac{2\mu r^2}{R^3}} \Rightarrow \\
&(\rho c^2 + 3p(r))\sqrt{1 - \frac{2\mu}{R}} = (\rho c^2 + p(r))\sqrt{1 - \frac{2\mu r^2}{R^3}} \Rightarrow \\
&\left[3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}\right]p(r) = \rho c^2\left[\sqrt{1 - \frac{2\mu r^2}{R^3}} - \sqrt{1 - \frac{2\mu}{R}}\right] \Rightarrow
\end{aligned}$$

$$\boxed{p(r) = \rho c^2 \left[\frac{\left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2} - \left(1 - \frac{2\mu}{R}\right)^{1/2}}{3\left(1 - \frac{2\mu}{R}\right)^{1/2} - \left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2}} \right]} \tag{1002}$$

Lastly, we need an explicit form for $A(r)$ and $B(r)$. For $B(r)$, when in the internal radii we get:

$$B(r) = \left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1} = \left(1 - \frac{8\pi G\rho r^3}{3rc^2}\right)^{-1} = \left(1 - \frac{4\pi\rho G R^3}{3}\frac{2r^2}{R^3}\right)^{-1} \Rightarrow$$

$$\boxed{B(r) = \left(1 - \frac{2\mu r^2}{R^3}\right)^{-1}} \tag{1003}$$

Eq. 1003 for $r = R$ yields:

$$B(R) = \left(1 - \frac{2\mu R^2}{R^3}\right)^{-1} = \frac{1}{1 - \frac{2\mu}{R}} = B_S(R)$$

Solution matches the expression from the Schwarzschild metric outside the sphere.

For $A(r)$ we have:

$$\begin{aligned}
\frac{1}{A}\frac{dA}{dr} &= -\frac{2}{\rho c^2 + p}\frac{dp}{dr} \Rightarrow \frac{dA}{A} = -\frac{2}{\rho c^2 + p}\frac{dp}{dr}dr \Rightarrow \\
\frac{dA}{A} &= dr\frac{8\pi G r}{3c^4}(\rho c^2 + 3p)\left(1 - \frac{2\mu r^2}{R^3}\right)^{-1} \Rightarrow \\
\frac{dA}{A} &= dr\frac{8\pi G r}{3c^4}\left[\rho c^2 + 3\rho c^2\left(\frac{\sqrt{1 - \frac{2\mu r^2}{R^3}} - \sqrt{1 - \frac{2\mu}{R}}}{3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}}\right)\right]\left(\frac{1}{1 - \frac{2\mu r^2}{R^3}}\right) \Rightarrow \\
\frac{1}{A}\frac{dA}{A} &= \frac{8\pi G\rho R^3 r}{3R^3c^2}\left[\frac{3\sqrt{1 - \frac{2\mu r^2}{R^3}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}}{3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}}\right]\frac{1}{\left(\sqrt{1 - \frac{2\mu r^2}{R^3}}\right)^2}
\end{aligned}$$

$$\boxed{\frac{1}{A}\frac{dA}{dr} = \frac{\frac{4\mu r}{R^3}}{\sqrt{1 - \frac{2\mu r^2}{R^3}}\left(3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}\right)}} \tag{1004}$$

$$\frac{d}{dr}\left[\ln\left(3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}\right)\right] = \frac{2\mu r}{R^3}\frac{1}{\sqrt{1 - \frac{2\mu r^2}{R^3}}}\frac{1}{\left(3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}\right)} \tag{1005}$$

Substituting eq. 1005 in eq. 1004 we get:

$$\boxed{A(r) = e^c \left(3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}} \right)^2} \tag{1006}$$

At

$$r = R \Rightarrow A(R) = c^2 \left(1 - \frac{2\mu}{R} \right) \tag{1007}$$

Combining the above, we get:

$$\boxed{A = \frac{c^2}{4}} \tag{1008}$$

Finally the expression for $A(R)$ will be:

$$\boxed{A(r) = \frac{c^2}{4} \left(3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}} \right)^2} \tag{1009}$$

7.2 Buchdahl's theorem

Theorem 2 *Given a static, spherically symmetric solution to Einstein's equations with matter confined to a spatial radius R, that behaves as a perfect fluid with non-increasing density outwards, then the mass of the solution to the field equations must satisfy the upper bound:*

$$\boxed{\frac{GM}{Rc^2} < \frac{4}{9}} \tag{1010}$$

This constraint is derived by the behaviour of p_0, when $\frac{\mu}{R} \to \frac{4}{9}$, then from eq. 1001

$$p_0 = \rho c^2 \frac{1 - \sqrt{1 - \frac{8}{9}}}{3\sqrt{1 - \frac{8}{9}} - 1} = \rho c^2 \frac{\frac{2}{3}}{1 - 1} \to +\infty \tag{1011}$$

Thus, $\frac{GM}{Rc^2} < \frac{4}{9}$.

8 A complete analytical GW model for undergraduates

In this chapter we follow the train of thought of Dillon Buskirk and Maria C. Babiuc Hamilton's article, named "A Complete Analytic Gravitational Wave Model for Undergraduates", as cited in [104], in the final section the images were contructed by the Wolfram Mathematica Coding given by the same author as in [124].

An accurate waveform template is constructed by two specific parts. The first part always includes analytical models of inspirals and mergers, while the second part is obtained with numerical calculations. In this chapter, the Implicit Rotating Source (IRS) is used as the analytical model, and various numerical simulations are performed using Wolfram Mathematica Coding.

In most cases, we apply Post-Newtonian calculations and analytically compute the inspiral phases. A useful parameter in PN theory is χ, given by $\chi = \frac{u^2}{c^2}$. Despite the many advantages PN theory offers, it is not valid for relativistic and near-relativistic cases, such as the merger and ringdown phases. Its validity lies in weak fields, where $\chi \ll 1$, and not on ringdown phases.

The second last part, of which an accurate template is constructed, is the numerical computations of the ringdown phase. In this case, instead of calculating the energy flux through equations, the energy loss is approximated with numerical methods and modeled by an ansatz. In our case, the ansatz is depicted as a generic IRS model tuned to numerical GR. The reason behind using numerical and not exact solutions of Einstein's equations on the merger phase comes from the fact that the gravitational field, and as a result, the gravitational forces, take extreme values reaching relativistic limits. In this regime, PN theory loses validity, since all phenomena are relativistic and the χ-parameter reaches an outlier ($\chi \to 1$).

8.1 Post-Newtonian expansions in Relativity and useful quantities

As mentioned in previous chapters, the energy loss in every merger is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} \tag{1012}$$

where the energy flux is denoted by $\mathcal{F}$. In terms of the PN parameter χ and using the derivative chain rule, equation (1012) is rewritten as

$$\begin{aligned} ((1012)) \Rightarrow \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}E}{\mathrm{d}\chi}\frac{\mathrm{d}\chi}{\mathrm{d}t} = -\mathcal{F} \\ \Rightarrow \frac{\mathrm{d}\chi}{\mathrm{d}t} &= -\frac{\mathcal{F}}{\mathrm{d}E/\mathrm{d}\chi} \end{aligned} \tag{1013}$$

There are several ways to solve equation (1012). The most useful one includes the usage of the T1-T5 Taylor approximants. To use the approximants, it is needed to write down the power expansion of equation (1012) for n terms:

$$(???) \Rightarrow \frac{\mathrm{d}\chi}{\mathrm{d}t} = \frac{\mathrm{d}\chi^{0PN}}{\mathrm{d}t}\chi^5 + \frac{\mathrm{d}\chi^{1PN}}{\mathrm{d}t}\chi^6 + \frac{\mathrm{d}\chi^{2PN}}{\mathrm{d}t}\chi^7 + \frac{\mathrm{d}\chi^{3PN}}{\mathrm{d}t}\chi^8 + \frac{\mathrm{d}\chi^{HT}}{\mathrm{d}t} \tag{1014}$$

where $\frac{\mathrm{d}\chi^{HT}}{\mathrm{d}t}$ sums up all hereditary terms.

Instead of the parameter χ, we can obtain corrections in PN approximation by expanding in terms of $\frac{u}{c}^n$. Each order of this expansion counts as $\frac{n}{2}$ order. Namely, every expression can be expanded based on the following:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{GM}{r^2}[1 + \frac{1PN}{c^2} + \frac{1,5PN}{c^3} + \frac{2PN}{c^4} + \frac{2.5PN}{c^5} + ...] \tag{1015}$$

In equation (1015) The 1PN term expresses the orbit precession, the 1,5PN term provides information about the spin-orbit interaction, and the 2PN term describes the spin-to-spin coupling. Finally, the 2,5PN order of approximation gives the orbital decay that occurs with GW emissions. Furthermore, when substituted Kepler's third law and the expression of orbital velocity, as stated below:

- Keppler's third law: $\omega^2 r^3 = GM \Rightarrow r = \left(\frac{GM}{\omega^2}\right)^{1/3}$
- orbital velocity: $u = \omega r = \omega \left(\frac{GM}{\omega^2}\right)^{1/3} = (GM\omega)^{1/3}$

At this point, it is relatively easy to see, that the $\chi-$parameter can be expressed as:

$$\chi = \frac{u^2}{c^2} = \frac{(GM\omega)^{2/3}}{c^2} \rightarrow \chi = (M\omega^{2/3}) \tag{1016}$$

For our analysis to be complete, we need to see the T4- approximant in a quasicircular limit expressed in 6PN order:

$$M\frac{\mathrm{d}\chi}{\mathrm{d}t}|^{6\mathrm{PN}} = \frac{64}{5}\eta\chi^5\left(1 + \sum_{k=2}^{12} a_{k/2}\chi^{k/2}\right) \tag{1017}$$

and the $\alpha_{k/2}$ coefficients are expressed for every k, that belongs between 2 and 12. Below are shown the first four:

- $a_0 = 153.8803$
- $a_1 = -55.83$
- $a_2 = 588$
- $a_3 = -1144$

The next coefficients $a_4, a_5, a_6, a_{9/2}$ and $a_{11/2}$ are analytically expressed in Appendix B of [104].

8.1.1 The analytical models

Kepler's third law, when written as an expression of r and the $\chi-$parameter's expression is inserted, we can obtain an expression of r in terms of χ, as:

$$r = \frac{M}{\omega^2 r^2} = \chi^{-1}(t)M \tag{1018}$$

Based on this form, we can take a third form of the Post-Newtonian expression in terms of radii r up to 3PN order as:

$$r = M[r^{0PN}\chi^{-1}(t) + r^{1PN} + r^{2PN}\chi(t) + r^{3PN}\chi^2(t)] \tag{1019}$$

with the coefficients to read

- $r^{0PN} = 1$
- $r^{1PN} = -1 + \frac{1}{3}\eta$
- $r^{2PN} = \frac{19}{4}\eta + \frac{1}{9}\eta^2$
- $r^{3PN} = -7.51822\eta - 3.08333\eta^2 + 0.0246914\eta^3$

Finally, if we demand an optimal orientation of the detector normal to the orbital plane ($\theta = 0$), we can rewrite equations. as

$$h_+ = -\frac{2M\eta}{R}\left[\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)cos(2\phi) + 2r\dot{r}\dot{\phi}sin(2\phi)\right] \tag{1020}$$

$$h_\times = -\frac{2M\eta}{R}\left[\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)sin(2\phi) + 2r\dot{r}\dot{\phi}cos(2\phi)\right] \tag{1021}$$

Note here, that instead of the original expressions, as produced in Chapter 2, we substituted the symmetric mass ratio defined as $\eta = \frac{m_1 m_2}{M^2}$ and the total mass of the system $M = m_1 + m_2$.

Concluding this section, we can now produce the waveform that expresses the strain. The strain applied by a gravitational wave gives information about the whole inspiral and it is denoted as $h^{insp}(t)$. Namely, it is the complex sum of the h_+ and $h_\times$ polarizations, with the second being the imaginary part. Applying the above, it is fairly easy to write:

$$\begin{aligned} h^{insp}(t) &= h_+(t) + ih_\times(t) \\ &= -\frac{2M\eta}{R}\left[\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)(cos(2\phi) + isin(2\phi)) + 2r\dot{r}\dot{\phi}(sin(2\phi) + icos(2\phi))\right] \\ &= -\frac{2M\eta}{R}\left[\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}\right)e^{2\phi i} + 2r\dot{r}\dot{\phi}ie^{2\phi i})\right] \\ &= -\frac{2M\eta}{R}\left[\left(-\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r} + 2r\dot{r}\dot{\phi}i\right)e^{2\phi i}\right] \end{aligned} \tag{1022}$$

Finally we derive the expression:

$$h^{insp}(t) = -\frac{2M\eta}{R}\left[A_1(t) + iA_2(t)\right]e^{2\phi i} \tag{1023}$$

where

$$A_1(t) = -\dot{r}^2 + r^2\dot{\phi}^2 + \frac{M}{r}$$

and

$$A_2(t) = 2r\dot{r}\dot{\phi}$$

In a similar way, we can rewrite $A_1(t)$ and $A_2(t)$ in a complex function in form defined as:

$$A(t) = A_1(t) + iA_2(t) \tag{1024}$$

8.1.2 The merger model

One crucial phase of a collision of two celestial objects is the merger phase. The merger, as it is commonly known, begins when the two objects pass the Innermost Stable Circular Orbit (ISCO). As ISCO, we define the stable radius, where two or more objects perform a circular orbit. In Schwartzschild geometry, we obtain as ISCO the radius:

$$r_{ISCO} = \frac{6GM}{c^2} = 3R_{Sch} \Rightarrow r_{ISCO} \to 6M \tag{1025}$$

Based on the definition above, it is easy to understand that at $r = R_{ISCO}$ the inspiral phase ends. Similarly, we can define the frequency where the inspiral ends as:

$$f_{ISCO} = \frac{1}{6(2\pi)\sqrt{6}}\frac{1}{M} \tag{1026}$$

Several semi-analytical models have been developed to fully explain these phenomena, the most successful of which is the Implicit Rotating Source Model. In this case, the amplitude is assumed to be circularly polarized. This model is not valid for the merger and ring-down phases, but provides an excellent approximation for our results.

To obtain a clear picture of the above and have a more exact theory, we follow the way of William East in the article "Observing complete GW signals from dynamical capture binaries" [108]. For this case, we approach the phase evolution to the least damped Quasi-Normal Mode (QNM) frequency of the final black hole, denoted a ω_{QNM} via the expression:

$$\omega(t) = \omega_{QNM}(1 - \hat{f}) \tag{1027}$$

where

$$\hat{f} = \frac{c(\eta)}{2}\left(\left(1 + \frac{1}{\kappa(\eta)}\right)^{1+\kappa(\eta)}\left[1 - \left(1 + \frac{1}{\kappa(\eta)}e^{-\frac{2t}{b(\eta)}}\right)^{-\kappa(\eta)}\right]\right) \tag{1028}$$

Here the amplitude will be:

$$A(t) = \frac{A_0}{\omega(t)}\left[\frac{|\dot{\hat{f}}|}{1 + a(\eta)(\hat{f}^2 - \hat{f}^4)}\right]^{1/2} \tag{1029}$$

where $a = \frac{72.3}{Q^2}$, $b = \frac{2Q}{\omega_{QNM}}$ and $\omega_{QNM} = 1 - 0.63(1 - \hat{S}_{fin})^{0.3}$ The spin of a black hole can be expressed in terms of η according to equation 1030. The complete expression is produced by numerically approximating the problem in hand. Numerically, a black hole's spin can be expressed as:

$$\hat{S}_{fin} = 2\sqrt{3}\eta - \frac{390}{79}\eta^2 + \frac{2379}{287}\eta^3 - \frac{4621}{276}\eta^4 \tag{1030}$$

This derivation of a GWs amplitude and frequency in this model, is complete only if we write down the numerical expressions of the functions $a(\eta), b(\eta), c(\eta)$ and $\kappa(\eta)$. After some strenuous algebra done with the help of Wolfram Mathematica Coding, we get:

$$\begin{aligned}
a(\eta) &= \frac{1}{Q^2(\hat{S}_{fin})}\left[\frac{16313}{562} + \frac{21345}{124}\eta\right] \\
b(\eta) &= \frac{16014}{979} - \frac{29132}{1343}\eta^2 \\
c(\eta) &= \frac{206}{903} + \frac{180}{1141}\sqrt{\eta} + \frac{424}{1205}\frac{\eta^2}{\log\eta} \\
\kappa(\eta) &= \frac{713}{1056} - \frac{23}{193}\eta \\
Q^2(\hat{S}_{fin}) &= \frac{2}{(1-\hat{S}_{fin})^{0.45}}
\end{aligned} \tag{1031}$$

The last piece that completes the merger puzzle is to compute the merger phase. This can be done by integrating the orbital angular velocity for a given time interval:

$$\Phi_{gIRS}(t) = \int_{t_0}^{t} \mathrm{d}t\omega(t) \tag{1032}$$

Concluding this model applies to the merger of non-spinning compact binaries. The elements of the binary may have the same or different mass ratios and the full model describing the merger phase is called the generic Implicit Rotating Source model (gIRS).

8.1.3 Implementation of the models and matching techniques

Before applying the models mentioned in the previous paragraphs, it is useful to define the domain of integration. As the domain of integration, we define the range between the initial and final value of $\chi-$parameter. The lower boundary of χ is denoted as χ_0 and dictated by the seismic background threshold. The seismic background threshold is the minimum constant frequency created by the movement of tectonic plates. This frequency comes to be around $f_{GW}^{low} = 10Hz$.

Knowing the background frequency, we can compute the corresponding angular velocity and the $\chi-$parameter as:

$$\omega_{GW}^{low} = \pi f_{GW}^{low} \tag{1033}$$

$$\chi_0 = \left(\frac{u_0^2}{c^2}\right) = \frac{(GM\omega_{GW}^{low})^{2/3}}{c^2} = \left(\frac{\pi GM f_{GW}^{low}}{c^3}\right)^{2/3} \tag{1034}$$

The upper boundary is defined by the radius of the last stable orbit, namely the ISCO. Consequently, the value for each binary system differs and is frequently needed in the second PN order (dependent on the symmetric mass ratio η). We write the following:

- $\chi_{ISCO}^{0PN} = \frac{1}{6}$
- $\chi_{ISCO}^{2PN} = \frac{1}{6}\left(1 + \frac{7}{18}\eta\right)$

Following the logical process of the article "A complete analytic GW model for undergraduates", we begin with a paradigm of two configurations with equal masses $m_1 = m_2 = 20M$, total m, ass $M = 40M$ and symmetric mass ratio $\eta = 0.25M$. Implementing these values in the Wolfram Mathematica format provided by the author, see bibliography [124], we produce datasets of useful data and images. Since we are dealing with numerical analysis problems, there is a catch: There must exist an upper bound to our frequencies defined by the bin configuration to exclude any stiffness in our differential equations. Stiffnesses in numerical approximation can be caused by factors, initial conditions and singularities. Any stiffness happens at a fixed time called time of stiffness and denoted by t_s. Similarly, instead of limiting the frequency, we can limit the time axis. Graphs producedby the the coding in [124], in the case where each BH has mass equal to 20 solar masses are shown below.

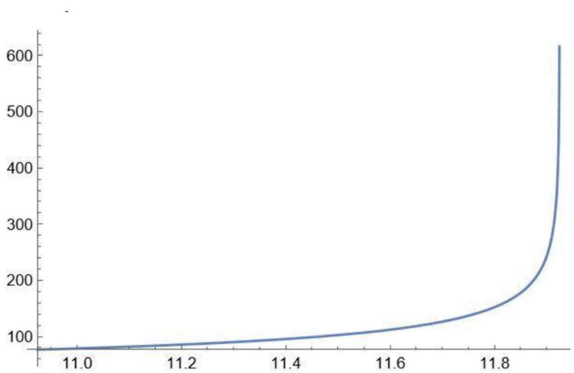

Figure 12: PN x-parameter as a function of time, created with [124].

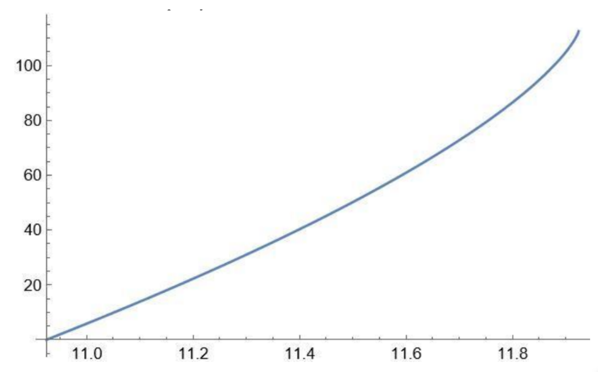

Figure 13: Evolution of Orbital Frequency, created with [124].

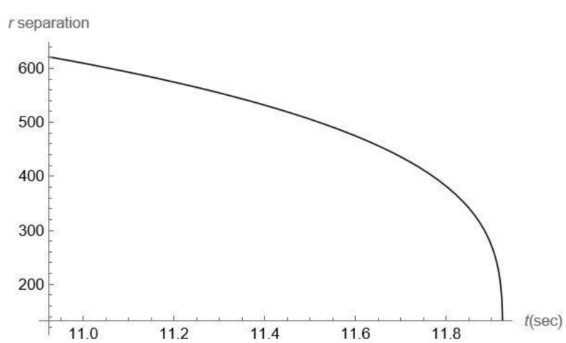

Figure 14: R separation between stars up to 3PN order and as a function of time, created with [124].

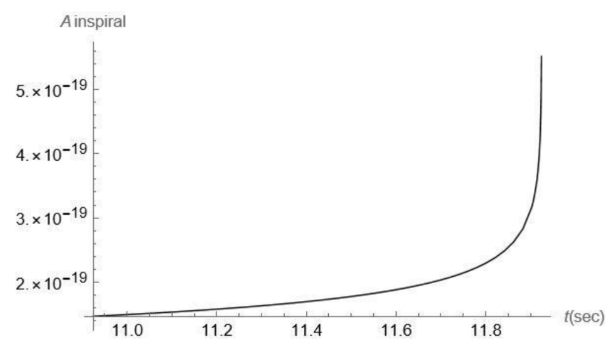

Figure 15: GW Amplitude evolution in time, created with [124].

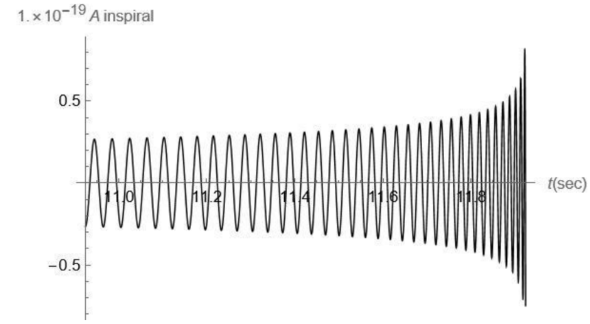

Figure 16: Graph of real part of inspiral amplitude, created with [124].

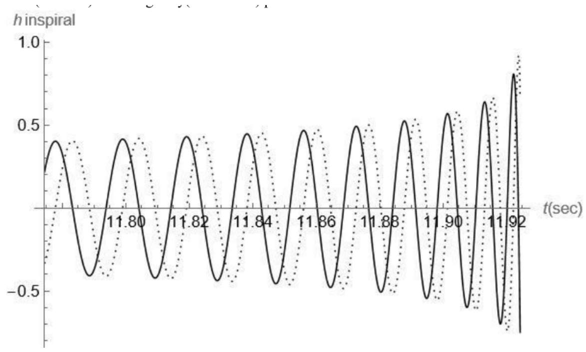

Figure 17: Real (full line) and imaginary (dotted line) part of GW strain, created with [124].

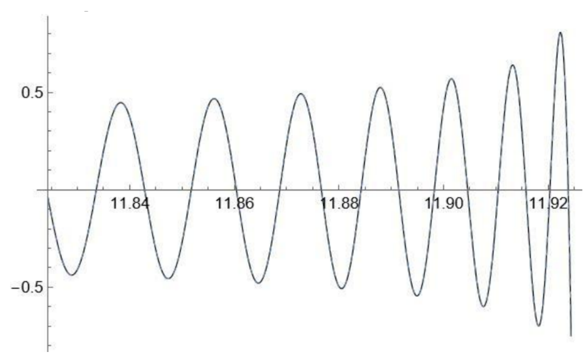

Figure 18: 22-component of GW strain for the event, created with [124].

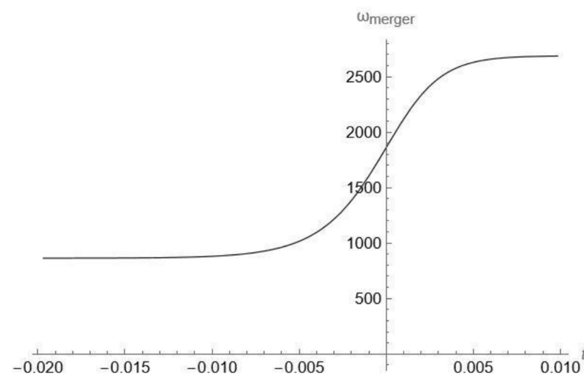

Figure 19: Orbital frequency evolution of the merger phase, created with [124].

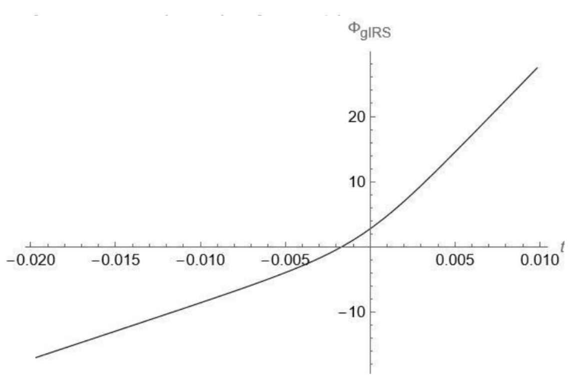

Figure 20: GW phase evolution during the merger as predicted by the generic IRS model, created with [124].

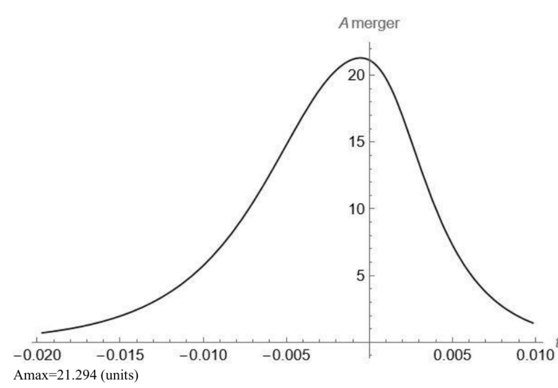

Figure 21: Maximum value of Amplitude of binary merger, created with [124].

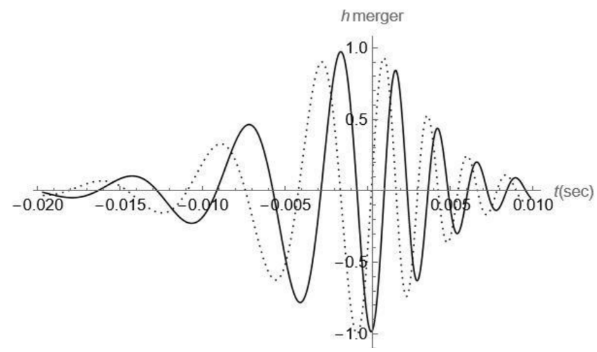

Figure 22: Strain's waveform during the whole event (real and imaginary part), created with [124].

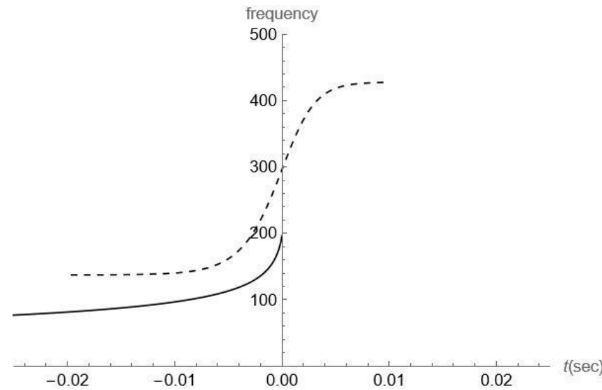

Figure 23: Overlapping the frequency, created with [124].

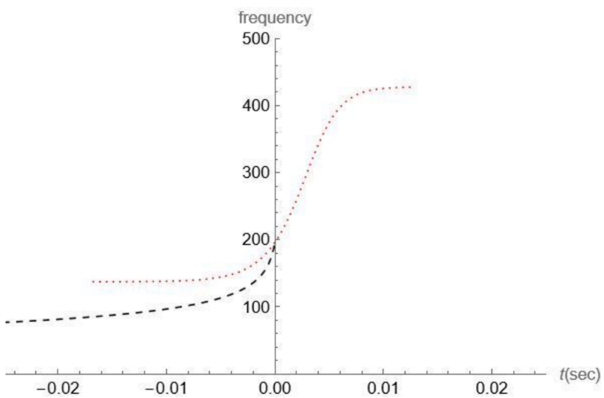

Figure 24: Overlapping the frequency, created with [124].

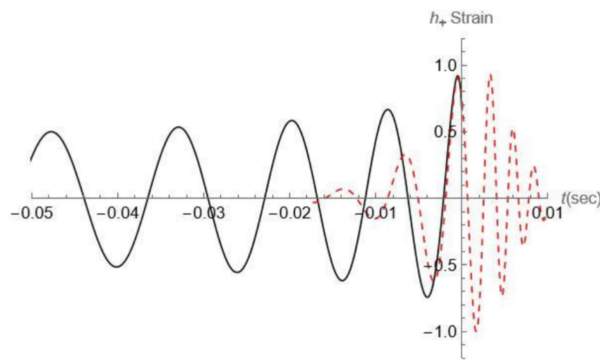

Figure 25: Overlapping the waveform, created with [124].

Furthermore, escaping the article's boundaries, we explored the program's capability of producing approximated waveforms for different pairs of masses. Taking different mass values for each of the objects, we concluded that each one must be $m_i \geq 5M_\odot$ for the program to function properly. Meanwhile, the total mass must be of order $M \geq 12M_\odot$.

Using the data stated above, the program allowed us to compute the total time of the coalescence. The results are shown in Table 2.

M_1 ($M_\odot$)	M_2 ($M_\odot$)	M_{total} ($M_\odot$)	η ($M_\odot$)	t_{total} (s)
10	5	15	3.33	70.0906
8	4	12	2.66	70.0897
6	6	12	3.00	No wave
7	5	12	2.92	93.1185
75	75	150	37.5	1.0988
76	75	151	37.75	1.07886
77	76	153	38.95	1.05158
76.5	76.5	153	38.95	1.05165
150	3	153	2.94	11.3613

Table 2: Table of individual star masses M_1 and M_2, the total mass $M_\odot$, the mass ratio $\eta = \frac{m_1 m_2}{m_1+m_2}$ and total time of coalescence t_{total}. Data created and obtained by [130].

9 Constraining scalar-tensor theories by NS-BH GW events

In this chapter we delve into an extremely important article written by Rui Niu, Xing Zhang, Bo Wang and Wen Zhao, named "Constraining scalar-tensor theorie by neutron star-black hole gravitational wave events", see bibliography [125]. In rder to fully comprehend the algebra performed in this article, as well as the different modified gravity theories implemented, we refered to two more articles [126] and [127].

9.1 Introduction

General Relativity (GR), proposed by Einstein in 1915, is fundamental to modern physics, successfully tested across various scales from laboratory experiments to cosmological observations. However, GR faces challenges such as singularities, the lack of a quantum formulation, and the need for dark matter and energy, prompting the development of alternative theories like scalar-tensor models.

Scalar-tensor theories, which address some of GR's limitations, have their origins in early unification attempts by Kaluza and Klein, with further development by Jordan, Fierz, and Brans-Dicke. In this study, we focus on three models of scalar-tensor theories, the Brans-Dicke theory (BD), the Damour–Esposito-Farèse (DEF) theory, and Screened Modified Gravity (SMG).

Brans-Dicke theory introduces a varying gravitational constant through a scalar field coupled to the Einstein-Hilbert action. DEF theory reveals strong-field deviations from GR, such as spontaneous scalarization in neutron stars, which is a nonperturbative effect emerging in strong-field conditions. Additionally, dynamical scalarization occurs in binary systems but is not the focus of this study. SMG theories include screening mechanisms, like the chameleon, Vainshtein, and symmetron mechanisms, and suppress deviations from GR on small scales, while still allowing for cosmological effects such as dark energy.

Gravitational wave (GW) detections, such as GW150914, offer a new way to test GR in the strong-field regime. The LIGO-Virgo Collaboration (LVC) has conducted model-independent tests, but specific modified gravity models can provide additional constraints. Scalar-tensor theories are particularly relevant in systems, like neutron star-black hole (NSBH) binaries, where asymmetry between the components enhances deviations from GR.

In this study, we test BD, DEF, and SMG theories using GW data from NSBH systems, specifically GW200105, GW200115, and GW190426 152155. We exclude GW190814 due to uncertainty in its secondary component and GW200105 due to waveform systematics. Our analysis will focus on dipole radiation deviations in the GW signal and compare our findings to LVC constraints.

The theory of Brans-Dicke takes Mach's principle as the starting point. Mach's principle states that the phenomenon of inertia depends on the mass distribution of the universe. Because of Mach's principle, we promote the gravitational constant to a variable and demand that it couples to the Einstein-Hilbert Lagrangian as a scalar field. This theory is the simplest scalar tensor theory, it is very well constrained and in general it is considered as a prototype in scalar-tensor theories. Its most stringeent constrain s the measurement of Shapiro time delay, an experiment conducted by Cassini- Huygens spacecraft, results are given in [128].In Brans-Dicke (BD) theory tight bound requires deviations from GR in gravitational experiments to be very small in both weak and strong fields.

Damour and Esposito-Farese showed in their articles [127] [130], that non-perturbative effects can emerge in strong-field conditions. When the object's compactness exceeds a critical point,occurs a phenomenon called spontaneous scalarisation. This allows the behavior in gravitational experiments involving compact objects (like NS), to differ from experiments in the weak field regime. Models with non perturbative strong field effects may develop a deviation proportional to O(1), if and only if the most stringent weak-field constraint is bypassed. Induced scalarisation occurs when the scalar field produced by a scalarised component, induces the scalarisation of another component, which is not scalarised initially. Although, we do not concern ourselves with induced scalarisation, since we deal with NS-BH events and it concerns only events of NS binaries.

Dynamical scalarisation occurs in a binary system that is being scalarised due to the gravitational binding energy of orbit, but the two components cannot be scalarised separately. It is hardly detected by current experiments. This fact shows that non-perturbative strong field effects are constrained by pulsar timing experiments. Precise measurement technology and decades of data provide highly precise measurements of orbital decay rates in binary pulsar systems. This is a good test of gravitational theories in string-field regimes.

SMG evades tight solar system constraints by introducing screening mechanisms (Chameleon, Vainshtein, and Symmetron). The scalar field plays the role of dark energy, driving the cosmic expansion. Screening mechanisms suppress deviations from GR on small scales to circumvent stringent constraints from solar system tests and laboratory experiments. For a given specific modified gravity model, independent parameters cannot always completely describe deviations of GWs. The deviations depend on the physical character of NS and or BH in the theory. Testing S-T theories by GW has been conducted since the 90s. More and more detections allow us to constrain S-T theories with real GW data. In S-T gravities, the deviation of GW from that in GR depends on the sensitivity difference between the two stars. The asymmetric binaries NS-BH, NS-white dwarf are excellent candidates for model tests.

Up until 2021, there where four candidates (NS-BH events): $GW200105, GW200115$, $GW190426-152155$, and $GW190814$. The GW200105 and GW200115 events are thought to be confident observations of NS-BH binaries. The component masses are consistent with observations of BHs and NSs. However, there are not any information on spin and tidal deformation, also there is no electromagnetic counterpart detection. This means that we have no information about the secondaries being a NS and perhaps they are an exotic object.The event $GW190814$ is characterized as plausible, since the secondary mass is $M = 2.6\mathrm{M}_{\odot}$. This mass corresponds to a small BH or a heavy NS and thus we assume that $GW190426-152155$ is an NSBH coalescence event, with a high false-alarm rate (FAR). Finally the two events: $GW200105$ and $GW190814$, are excluded and we use only $GW190426-15155$ and $GW200115$ events.

In order to constrain scalarisation effects, we use a modification of dipole radiation and perform the full Bayesian inference .

9.2 Scalar-tensor theories

We consider a class of scalar-tensor theories described by the action

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g_*}\Big[R_* - 2g_*^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big] + S_m[\psi_m, A^2(\phi)g_*^{\mu\nu}] \tag{1035}$$

9.2.1 Equations of motion in scalar-tensor theories

Written in Einstein's frame, where G_* denotes the bare gravitational coupling constant, which is approximated by $G_* \sim G$ when solving TOV equations. The metric tensor in this frame is denoted by $g_*^{\mu\nu}$ and its determinant by g_*, while $R_* = g_*^{\mu\nu}R^*_{\mu\nu}$ is the Ricci scalar. In the last term of this action the field ψ_m collectively denotes various matter fields and $A(\phi)$ is the conformal coupling function.

The field equations in Brans-Dicke theory can be derived by varying the action 1035 with respect to the metric $g_*^{\mu\nu}$ and the scalar field ϕ. The full expression of this action reads as following:

$$S = \frac{1}{16\pi G_*}\int d^4x[\sqrt{-g_*}g_*^{\mu\nu}R^*_{\mu\nu} - 2\sqrt{-g_*}g_*^{\mu\nu}\partial_\mu\phi\partial_\nu\phi] + S_m[\psi_m, A^2(\phi)g_*^{\mu\nu}] \tag{1036}$$

In order to easy things, we define the following terms:

$$S_1 \equiv \int d^4x\sqrt{-g_*}g_*^{\mu\nu}R^*_{\mu\nu}$$

with variation

$$\delta S_1 = \int d^4x[(\delta\sqrt{-g_*})g_*^{\mu\nu}R^*_{\mu\nu} + \sqrt{-g_*}(\delta g_*^{\mu\nu})R^*_{\mu\nu} + \sqrt{-g_*}g_*^{\mu\nu}(\delta R^*_{\mu\nu})]. \tag{1037}$$

The variation of the Einstein frame metric and inverse metric tensors read:

$$\delta g^*_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g_*^{\rho\sigma}$$

and the field equations are given concerning $g_*^{\mu\nu}$ and ϕ.

Explicit calculations begin with the assumption that $g^*_{\mu\nu}$ is a square matrix with:

$$det|g^*_{\mu\nu}| = g \tag{1038}$$

For any square matrix M we get $ln(detM) = Tr(lnM)$, when variated yields

$$\begin{aligned}\delta[ln(detM)] &= \delta[Tr(lnM)] \Rightarrow\\ \frac{\delta(detM)}{detM} &= Tr\left(\frac{\delta M}{M}\right) \Rightarrow\\ \frac{1}{detM}\delta(detM) &= Tr(M^{-1}\delta M)\end{aligned} \tag{1039}$$

If $g^*_{\mu\nu} = M, g_*^{\mu\nu} = M^{-1}$ and $det(g^*_{\mu\nu}) = det(g_*^{\mu\nu}) = g_*$, then from Eq. 1039 we get:

$$\frac{1}{g_*}\delta(g_*) = Tr(g_*^{\mu\nu}\delta g^*_{\mu\nu}) \tag{1040}$$

also,
$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} \tag{1041}$$

Combining the two above Eqs, we get

$$\delta g_* = -g_* Tr(g^*_{\mu\nu}\delta g^{\mu\nu}_*) = -g_* g^*_{\mu\nu}\delta g^{\mu\nu}_* \tag{1042}$$

Variation on $\sqrt{-g_*}$ yields:

$$\delta\sqrt{-g_*} = \frac{1}{2\sqrt{-g_*}}\delta g_* = -\frac{g_*}{2\sqrt{-g_*}}g^{\mu\nu}_*\delta g^*_{\mu\nu} = -\frac{1}{2}\sqrt{-g_*}g^{\mu\nu}_*\delta g^*_{\mu\nu} \tag{1043}$$

The first term in equation 1037, when variated with respect to $\delta\sqrt{-g_*}g^{\mu\nu}_*R^*_{\mu\nu}$ and along with 1043 reads

$$-\int d^4x R_*\frac{\sqrt{-g_*}}{2}(g^*_{\mu\nu}\delta g^{\mu\nu}_*) \tag{1044}$$

The second term in equation 1037, when variated with respect to $\delta\sqrt{-g_*}g^{\mu\nu}_*R^*_{\mu\nu}$ and along with 1043 reads

$$\int d^4x\sqrt{-g_*}R^*_{\mu\nu}(\delta g^{\mu\nu}_*) \tag{1045}$$

We see that this term remains as it is. While the third term is rewritten as following:

$$\sqrt{-g_*}g^{\mu\nu}_*\delta R^*_{\mu\nu} : \tag{1046}$$

Consider two arbitrary variations of the connections given by replacing

$$\Gamma^\sigma_{\mu\nu} \to \Gamma^{\sigma'}_{\mu'\nu'} = \Gamma^\sigma_{\mu\nu} + \delta\Gamma^\sigma_{\mu\nu} \Rightarrow \delta\Gamma^\sigma_{\mu\nu} = \Gamma^{\sigma'}_{\mu'\nu'} - \Gamma^\sigma_{\mu\nu} \tag{1047}$$

The variation on Christoffel's connections $\delta\Gamma^\sigma_{\mu\nu}$ is given by Eq. 1047 and is a tensor, since it is defined by a difference of two connections. Acting on 1047, the covariant derivative D_λ yields:

$$D_\lambda\delta\Gamma^\sigma_{\mu\nu} = \partial_\lambda\delta\Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\lambda\alpha}\delta\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\lambda\mu}\delta\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\lambda\nu}\delta\Gamma^\sigma_{\mu\alpha} \tag{1048}$$

and

$$D_\lambda\delta\Gamma^\sigma_{\lambda\nu} = \partial_\mu\delta\Gamma^\sigma_{\lambda\nu} + \Gamma^\sigma_{\mu\alpha}\delta\Gamma^\alpha_{\lambda\nu} - \Gamma^\alpha_{\lambda\mu}\delta\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\delta\Gamma^\sigma_{\lambda\alpha} \tag{1049}$$

Taking the difference of 1048 - 1049 we get:

$$D_\lambda\delta\Gamma^\sigma_{\mu\nu} - D_\mu\delta\Gamma^\sigma_{\lambda\nu} = \partial_\lambda\delta\Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\lambda\alpha}\delta\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\lambda\nu}\delta\Gamma^\sigma_{\mu\alpha} - \partial_\mu\delta\Gamma^\sigma_{\lambda\nu} - \Gamma^\sigma_{\mu\alpha}\delta\Gamma^\alpha_{\lambda\nu} + \Gamma^\alpha_{\mu\nu}\delta\Gamma^\sigma_{\lambda\alpha} \tag{1050}$$

As usual Riemann's tensor is defined as:

$$R^{\sigma\;*}_{\nu\lambda\mu} = \partial_\lambda\Gamma^\sigma_{\nu\mu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} + \Gamma^\sigma_{\lambda\alpha}\Gamma^\alpha_{\mu\nu} - \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\lambda\nu}$$

and when variated, it yields:

$$\delta R^{\sigma\;*}_{\nu\lambda\mu} = \partial_\lambda\delta\Gamma^\sigma_{\nu\mu} - \partial_\mu\delta\Gamma^\sigma_{\lambda\nu} + \delta\Gamma^\sigma_{\lambda\alpha}\Gamma^\alpha_{\mu\nu} + \Gamma^\sigma_{\lambda\alpha}\delta\Gamma^\alpha_{\mu\nu} - \delta\Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\lambda\nu} - \Gamma^\sigma_{\mu\alpha}\delta\Gamma^\alpha_{\lambda\nu} \tag{1051}$$

A closer look on Eqs. 1050 and 1051 produces:

$$\delta R^{\sigma}{}_{\nu\lambda\mu}^{*} = D_{\lambda}\delta\Gamma^{\sigma}_{\mu\nu} - D_{\mu}\delta\Gamma^{\sigma}_{\lambda\nu}$$

A contraction on σ, λ yields:

$$\delta R^{\sigma}{}_{\nu\lambda\mu}^{*} = \delta^{\lambda}_{\sigma} R^{\sigma}{}_{\nu\lambda\mu}^{*} = D_{\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - D_{\mu}\delta\Gamma^{\sigma}_{\sigma\nu} \tag{1052}$$

So Eq. 1046 is written as:

$$\int d^4x\, \sqrt{-g_*}\, g_*^{\mu\nu}\, \delta R^*_{\mu\nu} = \int d^4x\, \sqrt{-g_*}\, g_*^{\mu\nu} \left(D_{\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - D_{\mu}\delta\Gamma^{\sigma}_{\nu 6} \right) \tag{1053}$$

We know $D_{\lambda} g^*_{\mu\nu} = 0 = D_{\sigma} g^*_{\mu\nu}$, so:

$$\begin{aligned}
&= \int d^4x\, \sqrt{-g_*} \left[D_{\sigma} g_*^{\mu\nu} \delta\Gamma^{\sigma}_{\nu\mu} - g_*^{\mu\nu} D_{\mu}\delta\Gamma^{\sigma}_{\sigma\nu} \right] \\
&= \int d^4x\, \sqrt{-g_*}\, D_{\sigma} \left[g_*^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu} - g_*^{\sigma\nu}\delta\Gamma^{\mu}_{\mu\nu} \right] \\
&= \int d^4x\, D_{\sigma} \left\{ \sqrt{-g_*} \left[g_*^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g_*^{\sigma\nu}\delta\Gamma^{\mu}_{\nu\mu} \right] \right\}
\end{aligned}$$

Variation of connections produces the following expression:

$$\delta\Gamma^{\sigma}_{\mu\nu} = -\frac{1}{2} \left[g^*_{\alpha\mu} D_{\nu}(\delta g_*^{\alpha\sigma}) + g^*_{\alpha\nu} D_{\lambda}(\delta g_*^{\alpha\sigma}) - g^*_{\mu\alpha} g^*_{\nu\beta} D^{\sigma}(\delta g_*^{\alpha\beta}) \right] \tag{1054}$$

$$\delta\Gamma^{\lambda}_{\nu\lambda} = -\frac{1}{2} \left[g^*_{\alpha\lambda} D_{\nu}(\delta g_*^{\alpha\lambda}) + g^*_{\alpha\nu} D_{\lambda}(\delta g_*^{\alpha\lambda}) - g^*_{\lambda\alpha} g^*_{\nu\beta} D^{\lambda}\delta g_*^{\alpha\beta} \right] \tag{1055}$$

Substitute into Eq. 1053:

$$\begin{aligned}
&= -\frac{1}{2} \int d^4x \sqrt{-g_*}\, D_{\sigma} \Big[D_{\alpha}(\delta g_*^{\alpha\sigma}) + D_{\alpha}(\delta g_*^{\alpha\sigma}) - g_*^{\alpha\beta} D^{\sigma}(\delta g_*^{\alpha\beta}) - g_*^{\alpha\beta} D^{\sigma}(\delta g_*^{\alpha\beta}) - D_{\alpha}(\delta g_*^{\alpha\sigma}) + D_{\alpha}(\delta g^{\alpha\sigma *}) \Big] \\
&= \int d^4x \sqrt{-g_*}\, D_{\sigma} D_{\alpha}(\delta g_*^{\alpha\sigma}) - g^*_{\alpha\beta} D_{\sigma} D^{\sigma}(\delta g_*^{\alpha\beta}) \\
&= \int d^4x \sqrt{-g_*}\, D_{\sigma} \left[D_{\alpha}(\delta g_*^{\alpha\sigma}) - g^*_{\alpha\beta} D^{\sigma}(\delta g_*^{\alpha\beta}) \right] = 0.
\end{aligned} \tag{1056}$$

Defined on the boundary, so that this term does not contribute to the total action. Consequently, δS_1 has the form:

$$\begin{aligned}
\delta S_1 &= \int d^4x \left[(\delta\sqrt{-g_*}) R^* + \sqrt{-g_*}(\delta g_*^{\mu\nu}) R^*_{\mu\nu} + \sqrt{-g_*} g^*_{\mu\nu} (\delta R^*_{\mu\nu}) \right] \\
\Rightarrow \delta S_1 &= \int d^4x \left[-R_* \frac{\sqrt{-g_*}}{2} g^*_{\mu\nu} + \sqrt{-g_*} R^*_{\mu\nu} \right] \delta g_*^{\mu\nu}
\end{aligned} \tag{1057}$$

We define the second term as:

$$S_2 \equiv \int d^4x \sqrt{-g_*} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

with variation

$$\delta S_2 = \int d^4x \left(\delta\sqrt{-g_*} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \sqrt{-g_*} \delta g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \sqrt{-g_*} g_*^{\mu\nu} \delta(\partial_\mu \phi \partial_\nu \phi) \right) \tag{1058}$$

The third term does not contribute when variated with respect to $g_*^{\mu\nu}$. Analytically the terms, when variated produce the results shown in equations 1059 and 1060.

$$\int d^4x \left(\delta\sqrt{-g_*} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) = \int d^4x \left(-\frac{\sqrt{-g_*}}{2} \partial^\mu \phi \partial_\mu \phi (\delta g_{\mu\nu}^*) \right) \tag{1059}$$

The second term:

$$\int d^4x \sqrt{-g_*} (\delta g_*^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi \tag{1060}$$

remains the same, when variated w.r.t. $\delta g_*^{\mu\nu}$

The second field equation is derived, when we define $S_3 \equiv S_M(\Psi_M, A^2(\phi) g_*^{mu\nu}$ and variation $\frac{\delta S_3}{\delta g_*^{\mu\nu}}$

$$\frac{\delta S_3}{\delta_*^{\mu\nu}} = \frac{\delta S_M}{\delta_*^{\mu\nu}} \equiv \frac{\delta}{\delta g_*^{\mu\nu}} \left[S_M(\psi_M, A^2 \phi g_*^{\mu\nu}) \right] \tag{1061}$$

Equation 1038 when variated w.r.t. $g_*^{\mu\nu}$ yields the sum of variations $\delta S_1, \delta S_2, \delta S_3$.

$$\begin{aligned} 0 &= \frac{2}{\sqrt{-g_*}} \frac{\delta S}{\delta g_*^{\mu\nu}} = \frac{2}{\sqrt{-g_*}} \left[\frac{\delta S_1}{\delta g_*^{\mu\nu}} + \frac{\delta S_2}{\delta g_*^{\mu\nu}} + \frac{\delta S_3}{\delta g_*^{\mu\nu}} \right] \frac{1}{16\pi G_*} \\ &\Rightarrow T_{\mu\nu}^* = \frac{1}{16\pi G_*} \left[\frac{1}{2} R_* g_{\mu\nu}^* - R_{\mu\nu}^* - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \right] \\ &\Rightarrow \left(T_{\mu\nu}^* - \frac{1}{2} T^* g_{\mu\nu}^* \right) 8\pi G + 2\partial_\mu \phi \partial_r \phi = R_{\mu\nu}^* \quad q.e.d. \end{aligned} \tag{1062}$$

When varying w.r.t scalar field ϕ, one can write the action as:

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g_*} \left[R_* - 2g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + L_M \right]$$

The new $\mathcal{L}'$ satisfies equations Euler- Lagrange w.r.t. ϕ and we get the second equation of motion:

$$\frac{\partial \mathcal{L}'}{\partial_\phi} - \partial_\| \left(\frac{\partial \mathcal{L}'}{\partial(\partial_\| \phi)} \right) = 0 \tag{1063}$$

The term $\frac{\partial \mathcal{L}'}{\partial_\phi}$ can be written as:

$$\frac{\partial \mathcal{L}'}{\partial \phi} \equiv \frac{\partial \mathcal{L}'}{\partial A(\phi)} \frac{\partial A(\phi)}{\partial \phi} = \frac{\partial A(\phi)}{\partial \phi} \frac{\partial \mathcal{L}_M}{\partial A(\phi)} = \frac{\sqrt{-g_*}}{2} T^{\mu\nu} \frac{\partial A(\phi)}{\partial \phi} 2A^{-1}(\phi) g_{\mu\nu}^*$$

when $\frac{\partial \mathcal{L}'}{\partial_{A(\phi)}}$ yields:

$$\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial(\partial_{\mathcal{K}}\phi)} &= \sqrt{-g_*}(-2g_*^{\mu\nu})\left[\frac{\partial(\partial_\mu\phi)}{\partial(\partial_{\mathcal{K}}\phi)}\partial_\nu\phi + \partial_\mu\phi\frac{\partial(\partial_\nu\phi)}{\partial(\partial_{\mathcal{K}}\phi)}\right] \\
&= \sqrt{-g_*}(-2g_*^{\mu\nu}\partial_\nu\phi - 2g_*^{\mu k}\partial_\mu\phi) \\
&= \sqrt{-g_*}(-4g_*^{k\nu}\partial_\nu\phi)
\end{aligned}$$

the derivative:

$$\begin{aligned}
\partial_{\mathcal{K}}\left(\frac{\partial \mathcal{L}'}{\partial(\partial_{\mathcal{K}}\phi)}\right) &= -4\sqrt{-g_*}\partial_{\mathcal{K}}(g^{k\nu}\partial_\nu\phi)\frac{1}{16\pi G_*} \\
&= \frac{-4}{16\pi G_*}\partial_\mu(\sqrt{-g_*}g^{\mu\nu}\partial_\nu\phi) \\
&= \frac{-4}{16\pi G_*}\Box g_*\phi\sqrt{-g_*}
\end{aligned}$$

and the equations of motion are given by substitution in 1063 as:

$$\begin{aligned}
&\Box_{g*}\phi = -4\pi G_* T_*^{\mu\nu} g_{\mu\nu}^* \frac{\partial A(\phi)}{\partial_\phi}\frac{1}{A(\phi)} \\
&\Box_{g*}\phi = -4\pi G_* T_*^{\mu\nu} \frac{\partial \ln A(\phi)}{\partial\phi} g_{\mu\nu}^* = -4\pi G_* T^* \frac{\partial \ln A(\phi)}{\partial\phi} \\
&\Box_{g*}\phi = -4\pi G_* \alpha(\phi) T^*
\end{aligned}$$

where $\alpha(\phi) \equiv \frac{\partial \ln A(\phi)}{\partial\phi}$ and $T^* = T_*^{\mu\nu} g_{\mu\nu}^*$. The last term is the energy-momentum tensor of matter fields. Lastly , $\Box_{g*}$ is the curved spacetime D'Alenbertian $\Box_{g*} \equiv \frac{1}{\sqrt{-g_*}}\partial_\mu(\sqrt{-g_*}g_*^{\mu\nu}\partial_\nu)$ and $\alpha(\phi)$ is the coupling constant between scalar and matter. Since $A \equiv A(\phi)$, we can expand $\ln A(\phi)$ as power series around ϕ_o as

$$\ln A(\phi) = \alpha_o(\phi - \phi_o) + \frac{1}{2}\beta_o(\phi - \phi_o)^2 + O(\phi - \phi_o)^3$$

where α_o and β_o are related to γ_{PPN} and β_{PPN} as

$$\begin{aligned}
\gamma^{PPN} - 1 &= \frac{-2a_o^2}{1 + a_o^2} \\
\beta^{PPN} - 1 &= \frac{1}{2}\cdot\frac{a_o^2\beta_o}{(1 + a_o^2)^n}
\end{aligned}$$

in parameterised post-Newtonian formalism.

The coupling between a scalar field and a star A is described by a parameter known as the scalar charge α_A. This parameter can determine the equations of motion and the gravitational wave (GW) emission of binary systems and is defined as:

$$\alpha_A \equiv \left.\frac{\partial \ln m_A}{\partial\phi}\right|_{\phi=\phi_0} \tag{1064}$$

In scalar-tensor theories for compact binaries, the center of gravitational binding energy does not coincide with that of the inertial mass. This difference between the two centers results in a varying dipole moment and induces extra energy loss through dipole radiation.

In order to further understand the energy lost by dipole radiation, we will follow Clifford M. Will's article "Testing Scalar-tensor gravity with gravitational wave observations of inspiraling compact binaries", see reference [126].

Promising sources for detection are inspiraling compact-A binaries. As an inspiraling compact binary, we define any binary system of neutron stars (NSs) or black holes (BHs), with a decaying orbit toward a final coalescence. This occurs under the dissipative influence of gravitational radiation reaction. In addition to the simple detection of the waves, we can determine important parameters of the inspiraling system, such as the masses and spins of the celestial bodies.

The term spin of a celestial body refers to a rotational motion around an imaginary axis that runs through its center. This axis is known as the body's rotational or spin axis. The speed of spin varies widely, from hours to hundreds of years. The direction is defined by the right-hand rule.

The celestial body's spin implies many consequences on physical properties. It affects the shape, since faster spin leads to a bulging equator and flattened poles, forming an oblate spheroid shape. Also the night-day cycle and the direction of the magnetic field are affected.This type of spin helps define the characteristics and behavior of the celestial body. The late-time evolution of such systems yields an accurately calculable gravitational wave signal. It is a chirp signal, that increases in amplitude over time.

The chirp signal is a type of signal that changes in frequency over time, often increasing in frequency as it scavenges. In GW context, a chirp signal is emitted when two massive objects (BHs or NSs) orbit and merge. During the orbit, the two semitropical GWs, which cause a detectable chirp signal, start at low frequencies and gradually increase until the objects merge. At the point of merge, this signal stops abruptly. This is a distinctive feature of compact binary objects and provides information about the properties of the binary (mass, spin, distance).

The chirp signal sweeps in the detectors' typical sensitivity bandwidth between 10 to 1000 Hz. Determining parameters is done by matched filtering of theoretical wave templates (dependent on system parameters) against the broadband output of the detector. The evolution of the frequency depends on the parameters of the system.

In the slow-motion, weak field, non-radiative limit appropriate to solar-system dynamics, most alternative theories of gravity can be accomplished by one simple framework, the PPN formalism. We focus on the BD scalar-tensor theory. This theory augments GR by adding a scalar gravitational field, that couples universally to matter and the gravitational coupling strength is determined as G via $G \sim \phi^{-1}$.

Relative importance of scalar field is parameterized by a constant ω_{BD}, which in generalized scalar tensor theories may be defined as $\omega_{BD} = \omega_{BD}(\phi)$. In the large limit of ω_{BD}, the relative difference between effects in GR and BD is of order $O\left(\frac{1}{\omega_{BD}}\right)$. As $\omega_{BD} \to +\infty \Rightarrow BD \to GR$. An empirical bound on $\omega_{BD} > 500$ is imposed for scalar-system measurements of Shapiro time-delay.

Shapiro time-delay or Shapiro effect, is a phenomenon in which the time it takes for a

$M/M_{\odot}$	S
0,132	0,01
0,167	0,02
0,244	0,05
0,424	0,10
0,516	0.13
1,25	0,49
1,41 (max mass)	0,78

Table 3: Table of sensitivities of neutron stars. Adapted by [126]

radio signal to travel through a gravitational field is affected by spacetimes' curvature. This effect results that the signal takes a longer path to reach its destination, than if it would travel through a flat spacetime.

The Shapiro time delay has a small effect, but huge implications for astronomy and astrophysics. It is widely used in measurements of masses and densities of massive objects. For systems with gravitational radiation and compact objects, BD theory introduces the following three effects.

First effect: It implies modifications to the effective masses of the bodies. These depend on the internal structure of bodies and they are parameterized by sensitivities S_A. S_A sensitivity is a measure of the gravitational binding per unit mass. For example, NSs have $S_{NS} \approx 0,1 - 0,2$, where BHs have $S_{BH} = 0,5$.

This type of event violates strong equivalence principle, in the notion that the motion of these bodies depends on their structure and tidal interaction.

Tidal interaction is a reference to gravitational forces that celestial bodies exert on each other when in proximity. Tidal interaction causes deformation of body's shape and change in orbital and rotational motion. Tidal locking refers to the locking effect on the rotation of a moon or a planet because of tidal interactions. Similarly tidal heating is when gravitational forces heat the interior of the body.

Second effect: It implies modifications on the quadrupole gravitational radiation. BD theory predicts monopoles and quadrupole gravitational combined diation. The combine effect modifies GR's effective quadrupole formula for two body energy loss by a term:

$$\frac{dE}{dt} = -\frac{8}{15}\frac{\mu^2 m^2}{r^4}\left(12v^2 - 11\dot{r}^2\right) \tag{1065}$$

where:

- μ: reduced mass of the binary system,
- m: total mass of the binary system,
- r: orbital separation,
- $\dot{r}$: radial velocity,
- v: relative orbital velocity.

More precisely, the orbital separation is the distance between two objects in motion around each other, where this is equal to the distance between their centers. Moreover, it depends on mass, shape of the objects and the gravitational forces. Affects phys. properties, behavior, gravitational interactions, and tidal heating.

Third effect: It implies dipole gravitational radiation. The center of gravity is different form the center of inertial mass. In BD, the dipole moment equals source of scalar radiation. It depends on $S \equiv s_1 - s_2$ and is larger than quadrupole contribution by an order $O\left(\frac{1}{v^2}\right)$.

$$\frac{dE}{dt}|_{\text{dipole}} = -\frac{2}{3}\frac{\mu^2 m^2}{r^4}\left(\frac{S^2}{\omega_{BD}}\right). \tag{1066}$$

This effect modifies the evolution of orbital radius and GW frequency f, by an accumulated phase of GW:

$$\phi_{GW} = \int_{t_{in}}^{t_{out}} dt 2\pi f = \int_{f_{in}}^{f_{out}} \frac{dt}{df} df 2\pi f = \int_{f_{in}}^{f_{out}} df 2\pi \frac{f}{\dot{f}} \tag{1067}$$

where:

- in denotes the signal that enters the detector

- out denotes the signal that leaves the detector

These two together form the detector's bandwidth. Since

$$\frac{\dot{f}}{f} = -\frac{3}{2}\frac{\dot{r}}{r} = -\frac{3\dot{E}}{2|E|} \tag{1068}$$

In the dipole term, we get:

$$\frac{S^2}{w_{BD}} < \frac{5376\pi}{25}\left(\pi f_{in} M\right)^{7/3} \eta^{-2/5} \tag{1069}$$

where

$$\eta = \frac{\mu}{m} \tag{1070}$$

$M = \eta^{3/5} m$ the chirp mass and f_{in} is typically chosen $\approx 30Hz$. The chirp mass is the mass that determines the lowest order of quadrupole effects. It is defined as

$$\begin{aligned}
M &= \eta^{3/5} m = \frac{\mu^{3/5}}{m^{3/5}} m = \frac{\mu^{3/5}}{m^{3/5-1}} = \frac{\mu^{3/5}}{m^{-2/5}} \Rightarrow \\
M &= \left(\frac{m_1 m_2}{m_1 + m_2}\right)^{3/5} \frac{1}{m^{-2/5}} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{3/5}} \frac{1}{(m_1 + m_2)^{-2/5}} \Rightarrow \\
M &= \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}
\end{aligned} \tag{1071}$$

Formalism of matched filtering with post Newtonian effects reduces $\frac{S^2}{\omega_{BD}}$ as

$$\frac{S^2}{\omega_{BD}} < 1,46 \cdot 10^{-5} \left(\frac{M}{M_{\odot}}\right)^{7/3} \eta^{-2/5} \left(\frac{10}{S/N}\right) \tag{1072}$$

where S/N is the signal-to-noise ratio, which measures the strength of a GW signal relative to the background noise of the detector.

9.2.2 Frequencies for three types of celestial bodies

I) NS and BH binary. $S_{NS} \leq 0,2$ and $S_{BH} = 0,5 \Rightarrow S = s_1 - s_2 \geq 0,5 - 0,2 = 0,3$. Restriction:

$$\begin{aligned} \omega_{BD} &> \frac{1}{1,46 \cdot 10^{-5}} \left(\frac{M}{M_{\odot}}\right)^{7/3} \eta^{2/5} \left(\frac{S/N}{10}\right) \left(\frac{S}{0,3}\right)^2 \Rightarrow \\ \omega_{BD} &> 68493 \left(\frac{M}{M_{\odot}}\right)^{7/3} \eta^{2/5} \left(\frac{S/N}{10}\right) \left(\frac{S}{0,3}\right)^2 \end{aligned} \tag{1073}$$

II) Two NSs. S_{NS} varies with mass. Because of S_{NS}, δ is pretty small with $\delta \leq 0,05$. Bound on ω_{BD} weaker than solar-system results. Exception when masses are $0,7M_{\odot}$ and $1,4M_{\odot}$ extreme bound 1100.

III) Two BHs. There, the difference in sensitivities is $\delta = 0,5 - 0,5 = 0$ and ω_{BD} yields no dipole radiation. Bounds on ω_{BD} are placed if both chirp mass M and reduced mass parameter η are measured with accuracy and with the components mass we decide in which case we lie. Dipole radiation effects vary as $v^{-2} \approx \frac{r}{m}$ relative to quadrupole radiation and PPN corrections as $\frac{m}{r}$. These two are not correlated in the matched-filtering formalism.

9.3 Compact objects and gravitational radiation in scalar tensor gravity

The lowest order in power expansion of $v^2 \approx \frac{m}{r}$. This corresponds to Newtonian order for orbital motion. In GR it is called a quadrupole order for gravitational radiation. These equations include contributions due to self-gravitational binding energy. These contribution factors are determined by the sensitivity of the inner tail mass of each body A to changes in the local value of G_{eff}:

$$S_A \equiv -\frac{\partial ln m_A}{\partial ln G}. \tag{1074}$$

If we suppose two-body orbits, then Keppler's third law reads:

$$P^2 = \frac{4\pi^2}{G}\alpha^3 = \frac{4\pi^2}{G}\frac{r^3}{m}. \tag{1075}$$

where P is the orbital period. The corresponding frequency ω is given as

$$\omega = \frac{2\pi}{P} = \sqrt{\frac{Gm}{r^3}}. \tag{1076}$$

And the equations of motion transform as:

$$\frac{d^2\vec{x}}{dt^2} = -\frac{Gm\vec{x}}{r^3} \tag{1077}$$

where

$$G \equiv 1 - \xi(s_1 + s_2 - 2s_1 s_2) \tag{1078}$$

Here s_1, s_2 denotes the sensitivity of the two objects and

$$\xi \equiv \frac{1}{2 + W_{BD}} \tag{1079}$$

The energy of a circular orbit is:

$$E = -\frac{1}{2}\frac{G\mu m}{r} \tag{1080}$$

while the velocity of the same circular orbit reads:

$$v^2 = \frac{Gm}{r} \tag{1081}$$

The rate of energy loss for a quasi-circular orbit

$$\begin{aligned} \frac{dE}{dt} &= -\frac{8}{15}\frac{\mu^2 m^2}{r^4}(12kv^2 + \frac{5}{8}k_D S^2) \Rightarrow \\ \frac{dE}{dt} &= -\frac{32}{5}\frac{\mu^2 m^2}{r^4}kv^2 - \frac{1}{3}\frac{\mu^2 m^2}{r^4}k_D S^2 \end{aligned} \tag{1082}$$

with

$$k \equiv G^2\left(1 - \frac{1}{2}\xi + \frac{1}{12}\xi\Gamma^2\right)$$

,

$$k_D = 2G^2\xi$$

$$S = s_1 s_2$$

$$\Gamma = 1 - 2\frac{m_1 s_2 + m_2 s_1}{m}$$

In Equation 1082, the first term represents a combination of quadrupole and monopole contributions, while the second term corresponds to the dipole contribution. The dominant frequency of the emitted gravitational waves is given by:

$$f = \frac{\omega}{\pi}$$

In gravitational wave-forms the radiative metric pertubation reads:

$$\bar{h}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu} \tag{1083}$$

The spatial components:

$$\bar{h}^{ij} = \theta^{ij} - \frac{1}{2}\theta\delta^{ij} - \frac{\phi}{\phi_o}\delta^{ij} \tag{1084}$$

where $\phi = \phi_0 + \frac{1}{2}\phi|_{\phi_0}$ and ϕ is the perturbation of the scalar field ϕ about it's asymptotic, cosmological value ϕ_0

$$\begin{aligned} \theta_{ij} &= 2\left(1 - \frac{1}{2}\xi\right)\frac{1}{R}\frac{d^2}{dt^2}\left(\sum_A m_a x^i_A x^j_A\right) \Rightarrow \\ \theta_{ij} &= \frac{4\mu}{R}\left(1 - \frac{1}{2}\xi\right)(v^i v^j - \frac{Gm}{r^3}x^i x^j) \end{aligned} \tag{1085}$$

and

$$\frac{\phi}{\phi_0} = \xi\frac{\mu}{R}\left\{\Gamma\left[(\hat{N}\cdot\vec{v})^2 - \frac{Gm}{r^3}(\hat{N}\cdot\vec{\chi})^2\right] - G\Gamma + 2\Lambda\frac{m}{r} - 2S(\hat{N}\cdot\vec{v})\right\} \tag{1086}$$

where:

- R: distance to the observer,
- $\hat{N}$: unit direction vector,
- $\Lambda = 1 - s_1 - s_2 + O(\xi)$

Components of Riemann tensor

$$R^{oioj} = \frac{-1}{2}\frac{d^2h^{ij}}{dt^2} \tag{1087}$$

with h^{ij} the effective gravitational waveform given by the formula.

$$h^{ij} = \theta^{ij}_{TT} - \frac{1}{2}\frac{\phi}{\phi_0}(\delta^{ij} - \hat{N}^i\hat{N}^j) \tag{1088}$$

The effective gravitational waveform is a representation of a GW signal, that is observed by ground-based detectors. This effectiveness comes from the simplified representation, as derived by taking into account only the dominant features of the signal. The term matched filtering refers to the use of effective gravitational waveforms in the detection and analysis of signals by matching observed waveforms and determining the properties of sources, where TT is the transverse-transverse gauge. A full GW waveform is transverse, not traceless, because of scalar contribution.

This way the waveform becomes:

$$h^{ij} = \frac{2\mu}{R}[Q^{ij}_{TT} + S(\delta^{ij} - N^iN^j)] \tag{1089}$$

with

$$Q^{ij}_{TT} \equiv 2\left(1 - \frac{1}{2}\xi\right)\frac{Gm}{r}(\hat{\lambda}^i\hat{\lambda}^j - \hat{\eta}^i\hat{\eta}^j), \tag{1090}$$

$$S = -\frac{1}{4}\xi\left\{r\frac{Gm}{r}\left[(\hat{N}\cdot\hat{\lambda})^2 - (\hat{N}\cdot\hat{\eta})^2\right] - (Gr + 2\Lambda)\frac{m}{r} - 2S\sqrt{\frac{Gm}{r}}\hat{N}\hat{\lambda}\right\} \tag{1091}$$

where $\hat{n} \equiv \frac{\vec{x}}{r}$ and $\hat{\lambda} = \frac{\hat{v}}{u}$.

9.4 Testing scalar-tensor gravity with matched-filtering

We study this subsection in two parts. The first includes the phase-shift estimation and the second the matched-filter analysis.

A. Phase-shift estimate

Combining eqs.1076 and 1080 - 1082 one can get:

$$\dot{f} = \frac{96}{5}\eta\frac{G^{1/2}}{\pi m^2}\left(\frac{m}{r}\right)^{11/2}(k + \frac{5}{96}\frac{k_D}{G}\frac{r}{m}S^2) \tag{1092}$$

Now we can set

$$M \equiv \frac{k^{3/5}}{G^{4/5}}\eta^{3/5}m \tag{1093}$$

and

$$b \equiv \frac{5}{96}k^{-3/5}G^{-6/5}k_D S^2 \tag{1094}$$

and

$$u \equiv \pi M f \tag{1095}$$

with M the BD chirp mass and b the bipolar parameter.

Finally, combining the above equations we get:

$$\dot{u} = M^{-1}\frac{96}{5}u^{11/3}(1 + b\eta^{2/5}u^{-2/3}) \tag{1096}$$

Integration gives:

$$u^{-8/3}[1 - \frac{4}{5}b\eta^{2/5}u^{-2/3}] = \frac{256}{5}\frac{t_c - t}{M} \tag{1097}$$

where t_c is at $v \to +\infty$.

In Eq. 1097 we used the expansion to first order of $b\eta^{2/5}u^{-2/3}$ and used the fact that it is bounded as

$$b\eta^{2/5}u^{-2/3} \leq 5\cdot 10^{-3}\left(\frac{500}{\omega_{BD}}\right)\left(\frac{S}{0,5}\right)^2\left(\frac{M_\odot}{M}\right)\left(\frac{30Hz}{f}\right)^{2/3}$$

The number of cycles observed in a given bandwidth:

$$\begin{aligned}\phi_{GW} &= \frac{2}{M}\int_{u_{in}}^{u_{out}} du\frac{u}{\dot{u}} \\ \phi_{GW} &= \frac{1}{16}(u_{in}^{-5/3} - u_{out}^{-5/3}) - \frac{5}{112}b\eta^{2/5}(u_{in}^{-7/3} - u_{out}^{-7/3})\end{aligned} \tag{1098}$$

The dipole parameter b characterizes the polarization of GW by describing the direction and strength of the wave's distortion of the spacetime. As it is known, any GW has two polarizations, the called plus $(+)$ and cross $(\times)$ polarizations. The dipole parameter describes the relative amplitude and phase of the two polarizations states. Computing this, one can extract information about masses and orbital parameters in a binary system.

B. Matched-filler analysis

For a more accurate bound on the dipole parameter b, one needs to carry out a matched filter analysis. We approximate equations 1089- 1091, that produce a given observed gravitational waveform by

$$h(t) \approx Re[h^o(t)e^{i\phi(t)}] \tag{1099}$$

with $h^o(t)$ the slowly, varying Newtonian order contribution to the waveform amplitude. It depends on the distance of the source, location to the sky, orientation of the detector, and on source parameters M, η, and r. $\phi(t)$ is the gravitational wave phase, dominant at twice the orbital phase. It induces the dipole and higher-order post-Newtonian corrections.

The Fourier transform of $h(t)$ is given as:

$$\tilde{h}(f) = \begin{cases} \mathcal{A}f^{-7/6}e^{i\psi}, & \text{if } 0 < f < f_{\max} \\ 0, & \text{if } f > f_{\max} \end{cases} \tag{1100}$$

where

$$\mathcal{A} \sim R^{-1}M^{-5/6} \times \rho(\theta, \phi) \tag{1101}$$

$$f_{max} = O\left(\frac{1}{m}\right) \tag{1102}$$

and

$$\psi(f) = 2\pi f t_c - (\phi_c + \frac{\pi}{4}) + \frac{3}{128}v^{-5/3}\left(1 - \frac{4}{7}b\eta^{2/5}v^{-2/3}\right) \tag{1103}$$

Analysis of the equation:

1. ϕ_c: GW phase at t_c,

2. $\rho(\theta, \phi)$: arbitrary function of angles θ, ϕ and detector orientation,

3. f_{max}: corresponds to the frequency when the inspiral turns into a coalescence.

With a given noise-to-signal ratio (noise spectrum) $S_n(f)$ one defines the inner product of two signals h_1 and h_2 as

$$(h_1, h_2) \equiv 2\int_0^\infty df \frac{\tilde{h}_1^*\tilde{h}_2 + \tilde{h}_2^*\tilde{h}_1}{S_n(f)} \tag{1104}$$

For a given signal h, the signal-to-noise ratio is

$$\rho(h) \equiv \frac{S}{N(h)} = (h, h)^{1/2} \tag{1105}$$

The noise spectrum is given as the analytic fit used in the LIGO detector:

$$S_n(f) = \begin{cases} \infty, & \text{if } f < 10Hz \\ \frac{S_0}{5}\Big[\left(\frac{f_0}{f^u}\right)^4 + 2 + 2\left(\frac{f_0}{f^u}\right)^2\Big], & \text{if } f > 10Hz \end{cases} \tag{1106}$$

where,

1. $s_0 = 3 \cdot 10^{-48}\frac{1}{Hz}$ and $f_0 = 70Hz$,

2. The cutoff $f = 10Hz$ corresponds to seismic noise.

3. f^{-4} dependence corresponds to thermal noise.

4. f^2 dependence corresponds to photon-shot noise.

We have 5 parameters to estimate:

$$ln\mathcal{A} : \frac{\partial \tilde{h}(f)}{\partial ln\mathcal{A}} = \tilde{h}(f) \tag{1107}$$

$$\phi_c : \frac{\partial \tilde{h}(f)}{\partial \phi_c} = -i\tilde{h}(f) \tag{1108}$$

$$f_0 t_c : \frac{\partial \tilde{h}(f)}{\partial f_0 t_c} = 2\pi i \frac{f}{f_0}\tilde{h}(f) \tag{1109}$$

$$lnM : \frac{\partial \tilde{h}(f)}{\partial lnM} = -\frac{5i}{128}v^{-5/3}\tilde{h}(f)\left(1 - \frac{4}{5}\tilde{b}v^{-2/3}\right) \tag{1110}$$

$$\tilde{b} \equiv b\eta^{2/5} : \frac{\partial \tilde{h}(f)}{\partial \tilde{b}} = -\frac{3i}{224}v^{-7/3}\tilde{h}(f) \tag{1111}$$

The signal-to-noise ratio is given by:

$$\rho^2 = 20|\mathcal{A}|^2 f_0^{-4/3}\frac{I(7)}{S_0} \tag{1112}$$

with

$$I(q) \equiv \int_{1/7}^{\infty} dx\chi^{-9/3}(x^{-4} + 2x^2 + 2)^{-1} \tag{1113}$$

and $B_q \equiv \frac{I(q)}{I(7)}$. A priori we expect validity of GR and search for a bound on $\tilde{b} \equiv b\eta^{2/5}$.

Consider a gravitational waveform with leading order of the modification being a dipole term in the phase:

$$h(f) = h_{GR}(f)exp\Big[i\frac{3}{128\eta}\phi_{-2}(\pi GMf)^{-7/3}\Big] \tag{1114}$$

where

$$\phi_{-2} = -\frac{5}{168}(\Delta\alpha)^2 \tag{1115}$$

$$h_{GR}(f) \equiv \mathcal{A} f^{-7/6} exp(2\pi f t_c - \phi_c - \frac{\pi}{4})$$

when equated with Eq. 1100.

All coefficients are chosen in agreement with LVC's convention.

$$\Delta_\alpha \equiv \alpha_1 - \alpha_2 = \frac{\partial \ln m_1}{\partial \phi}\Big|_{\phi=\phi_0} - \frac{\partial \ln m_2}{\partial \phi}\Big|_{\phi=\phi_0} \tag{1116}$$

is the difference between the scalar charges of two bodies in the binary system. For black-holes, no-hair theorem prevents having scalar charges. In many scalar-tensor theories, where the no-hair theorem can be applied, the scalar charges of BHs are taken as zero. For NSs, scalar charges are given as a solution to TOV equations.

Based on two neutron star–black hole merger events (excluding two others due to possible anomalies), we analyze the inspiral phase with dipole radiation modifications. Scalar charges for neutron stars are calculated by solving the Tolman-Oppenheimer-Volkoff equations for different equations of state. Bayesian inference is performed using the Bilby software. Results show that gravitational wave data yield constraints comparable to those from pulsar timing for the DEF theory, but remain less stringent than solar system constraints for BD and SMG.

We use four commonly employed equations of state (EoS): SLY, ALF2, H4, and MPA1, with data available from public sources.

To solve the differential equations (A3), the initial conditions

$$\mu(0) = 0, \quad \nu(0) = 0, \quad \varphi(0) = \varphi_c, \quad \psi(0) = 0, \quad p(0) = p_c \tag{1117}$$

need to be provided to the differential equation solver. In practice, these initial conditions are set near the center to avoid division by zero. The initial pressure p_c is taken on a dense grid for interpolation, while φ_c is determined using the shooting method. Different values of φ_c are tested iteratively until the desired φ_0 is reached.

To implement Monte Carlo sampling efficiently, the scalar charge needs to be computed quickly. It is impractical to solve the TOV equations for each likelihood evaluation, so we solve the TOV equations once for a dense grid of model parameters and p_c to obtain mass and scalar charge values. During Monte Carlo sampling, the sampled model parameters and p_c are mapped to mass and scalar charge using linear interpolation, and these results will be discussed in subsequent subsections.

9.5 Brans-Dicke Theory

We begin by considering the Brans-Dicke theory, a prototype of scalar-tensor theories, which is widely studied. The theory is characterized by the linear coupling function

$$A(\varphi) = \exp(-\alpha_0 \varphi), \tag{1118}$$

resulting in a field-independent coupling strength

$$\alpha(\varphi) = \alpha_0. \tag{1119}$$

An alternative convention commonly used is

$$\alpha_0^2 = \frac{1}{3 + 2\omega_{\rm BD}}. \tag{1120}$$

Using the coupling function (1118), we can compute the scalar charge of a neutron star by applying the methods described in the previous subsection. By solving the TOV equations with initial conditions and a given EoS, we obtain numerical solutions for the neutron star structure. The scalar charge α_A, the asymptotic scalar field φ_0, and the mass m_A are extracted from the solutions.

The initial conditions p_c and the model parameter α_0 are chosen on a dense grid for interpolation, and the asymptotic scalar field φ_0 is set to 0. The last degree of freedom is the asymptotic scalar field φ_c and it is determined using the shooting method. To reduce computational costs, we use an interpolated relation $\alpha_A(\alpha_0, m_A)$ during Monte Carlo sampling.

Another commonly used parameter is the sensitivity s_A, which is related to the scalar charge by

$$\alpha_A = \frac{1 - 2s_A}{\sqrt{3 + 2\omega_{\rm BD}}}. \tag{1121}$$

In some studies, $s_A = 0.2$ is used as an approximation to the results obtained from solving the TOV equations.

9.6 Screened Modified Gravity (SMG)

Screened Modified Gravity (SMG) theories introduce a scalar field that interacts with matter via a coupling function $A(\phi)$, and evolves under the influence of a self-interaction potential $V(\phi)$. These two functions determine an effective potential $V_{\rm eff}(\phi)$, which governs the scalar field's behavior.

In high-density environments, the effective potential causes the scalar field to become massive, which suppresses the associated fifth force—a phenomenon known as *screening*. Conversely, in low-density cosmological settings, the scalar field is light and can affect large-scale dynamics, such as galactic motion and the accelerated expansion of the Universe (see Ishak's work (2018)) for a comprehensive review of screening mechanisms).

The general action for SMG with a canonical kinetic term is given by:

$$S = \int d^4x \sqrt{-g_*} \left[\frac{1}{16\pi G} R_* - \frac{1}{2} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_m[\psi_m, A^2(\phi) g^*_{\mu\nu}], \tag{1122}$$

where $V(\phi)$ defines the scalar self-interaction and determines the scalar field mass.

Popular SMG models include the chameleon, symmetron, dilaton, and $f(R)$ gravity, each specified by particular forms of $A(\phi)$ and $V(\phi)$. The equation of motion for the scalar field is:

$$\Box_{g_*} \phi = \frac{\partial V_{\rm eff}(\phi)}{\partial \phi}, \quad \text{with} \quad V_{\rm eff}(\phi) = V(\phi) - T_*. \tag{1123}$$

Gravitational waves from compact binaries in SMG exhibit leading-order dipole radiation. Due to screening, scalar charges in neutron stars are expected to be small. Instead of solving the full TOV equations, we use a simplified model where the neutron star is treated as a static, constant-density sphere. This yields an analytic expression for the scalar charge:

$$\alpha_A = \frac{\phi_{\rm VEV}}{M_{\rm Pl}\Phi_A}, \tag{1124}$$

where $\phi_{\rm VEV}$ is the vacuum expectation value of the scalar field, $M_{\rm Pl} = \sqrt{1/8\pi G}$ is the reduced Planck mass, and $\Phi_A = \frac{Gm}{R}$ is the star's surface gravitational potential.

10 Summary

Albert Einstein's 1915 General Theory of Relativity (GR) fundamentally reshaped our understanding of the cosmos, recasting spacetime as a dynamic fabric whose curvature is dictated by mass and energy. Among its most profound predictions was the existence of gravitational waves (GWs)—ripples in spacetime propagating at the speed of light, generated by accelerating massive objects. For decades, this prediction remained an elusive theoretical concept due to the incredible weakness of the gravitational force. The predicted strain, caused by even the most cataclysmic cosmic events, was expected to be on the order of 10^{-21} on Earth, a scale once thought to be immeasurable.

The first indirect, yet compelling, evidence for GWs came from the Hulse-Taylor binary pulsar system (PSR B1913+16). Meticulous observations revealed its orbit was shrinking at precisely the rate predicted by GR due to energy loss from GW emission, a discovery that earned the 1993 Nobel Prize in Physics. However, the ultimate goal was direct detection.

This was spectacularly achieved on September 14, 2015, when the Laser Interferometer Gravitational-Wave Observatory (LIGO) registered the signal GW150914. This characteristic "chirp" perfectly matched the theoretical waveform for the inspiral and merger of a binary black hole (BBH) system. This singular event did more than confirm a century-old prediction; it launched the revolutionary field of gravitational-wave astronomy, opening a non-electromagnetic window to observe the universe's most violent and hidden phenomena.

This thesis is situated at the forefront of this new era, where the focus has shifted from initial discovery to systematic characterization and precision science. The ever-growing catalog of GW events from the LIGO-Virgo-KAGRA (LVK) collaboration demands a corresponding increase in the accuracy and completeness of our theoretical waveform models. Sub-dominant physical effects, once negligible, are now becoming measurable and are essential for unlocking new scientific discoveries and avoiding systematic biases in our analysis.

The thesis begins by establishing the theoretical bedrock of GW physics through two complementary lenses: the geometric interpretation of GR and the framework of classical field theory.

Geometric Approach: This view treats GWs as small perturbations, $h_{\mu\nu}$, on a flat Minkowski spacetime background, $\eta_{\mu\nu}$, under the weak-field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. By linearizing the Einstein Field Equations, the complex dynamics of spacetime are reduced to a wave equation. A crucial step is managing the coordinate freedom of GR through gauge fixing. The imposition of the Lorentz gauge and subsequently the Transverse-Traceless (TT) gauge strips away non-physical degrees of freedom, revealing the true nature of GWs: they are transverse waves with two independent polarization states, known as plus (h_+) and cross ($h_\times$).

Field Theory Approach: Here, the perturbation $h_{\mu\nu}$ is treated as a classical field propagating on a fixed background. This perspective allows the powerful machinery of field theory, such as the Lagrangian formalism and Noether's theorem, to be applied. A key challenge in GR is defining the energy carried by GWs, as this energy itself gravitates.

The thesis demonstrates that both the geometric approach (via Isaacson's method of averaging over short wavelengths) and the field theory approach yield the same effective energy-momentum tensor, for the gravitational field. This tensor quantifies the energy, momentum, and angular momentum carried by the waves and demonstrates how GWs can source the curvature of the large-scale background spacetime.

Having established that GWs exist and carry energy, the thesis moves to the mechanism of their generation by astrophysical sources. The primary tool for this is the multipole expansion, which is applicable to sources with slow internal motions compared to the speed of light ($v/c \ll 1$). This is analogous to the multipole expansion in electromagnetism.

The wave equation for the metric perturbation is solved using a Green's function approach, leading to a retarded-time solution where the GW signal observed today was generated by the source at an earlier time. The core result of this analysis is the celebrated mass quadrupole formula, which states that the leading-order GW emission is proportional to the second time derivative of the source's quadrupole moment.

This formula encapsulates a fundamental principle: to generate GWs, a system must have a changing mass quadrupole moment. A perfectly spherical, pulsating star (monopole) or a rigidly rotating axisymmetric body (which has a constant quadrupole moment) will not radiate GWs. The total power, or luminosity, radiated by a source is then shown to be proportional to the time-averaged square of the third time derivative of the quadrupole moment.

The thesis further develops this expansion to include next-to-leading order terms, which become important for precision modeling and for sources with more complex dynamics. These include the mass octupole and the current quadrupole moments, which are suppressed relative to the mass quadrupole by factors of v/c.

To handle the complexity of the multipole expansion in a rigorous and generalizable way, the thesis introduces advanced mathematical frameworks that are central to modern gravitational theory.

The first framework developed is the Symmetric-Trace-Free (STF) Formalism. This is the natural mathematical language for describing multipoles. The STF formalism provides a systematic way to decompose any tensor into a sum of parts that are symmetric and trace-free in their indices. These STF tensors form irreducible representations of the rotation group SO(3) and correspond directly to the physical multipole moments of the source (monopole, dipole, quadrupole, etc.).

Then the Spherical Tensor Components are inserted, so to connect the abstract STF tensors to observable quantities, the thesis details the formalism of spherical tensor components. This method relates the STF tensors to the well-known spherical harmonics, which describe the angular dependence of fields. This connection is essential for calculating the radiation pattern—how the power and polarization of the emitted GWs vary across the sky—and for decomposing the observed waveforms into different modes. This chapter provides the mathematical rigor needed to move from basic formulas to sophisticated waveform models.

This chapter applies the theoretical machinery developed previously to a diverse range of astrophysical scenarios, with a primary focus on compact binary systems, the most important sources for current ground-based detectors.

The classic case of two point masses in a circular orbit is analyzed in detail. The GWs are shown to be monochromatic, with a frequency twice the orbital frequency. The analysis introduces the crucial concept of the "chirp mass," M_c, a specific combination of the two component masses that is the most easily measured parameter from the inspiral signal. The radiated waveform exhibits the characteristic "chirp" where both the frequency and amplitude increase as the orbit shrinks.

The more general and complex case of elliptical orbits is also explored. Here, the radiation is no longer monochromatic but is emitted at a spectrum of frequencies corresponding to integer multiples (harmonics) of the orbital frequency. The thesis derives the radiated power as a function of the orbit's eccentricity, e. A key physical insight is the effect of radiation back-reaction: the energy and angular momentum carried away by the GWs are drained from the orbit itself. This causes the orbit's semi-major axis to shrink and, importantly, its eccentricity to decrease. This demonstrates that astrophysical binary systems naturally circularize as they inspiral, explaining why many observed sources have nearly circular orbits by the time they enter the sensitive frequency band of detectors.

To illustrate the breadth of the formalism, several other systems are analyzed:

The radiation from a non-axisymmetric rotator (e.g., a neutron star with a "mountain" on its crust) is calculated, showing it emits continuous, monochromatic waves. The more complex case of a freely precessing body is also considered, which produces "wobble radiation" at different frequencies.

A simplified Newtonian model of a particle falling radially into a black hole is used to calculate the burst of GWs produced. This problem highlights the important concept of tidal disruption, where an extended object (like a star) gets torn apart by tidal forces. This can cause the radiation to transition from coherent (where the object acts as a whole) to incoherent (where different parts radiate out of phase), significantly suppressing the total power.

The effects of the universe's expansion on GWs are incorporated. As waves travel over cosmological distances, their frequencies are redshifted, and their amplitude decreases with the luminosity distance (dL), which depends on the cosmological model. This establishes the framework for using GWs as "standard sirens" to measure cosmic expansion.

Before the era of direct detection, the most compelling evidence for the existence of GWs came from observations of binary pulsars. This chapter details the physics of these extraordinary natural laboratories.

The Hulse-Taylor Binary (PSR B1913+16) is studied first. The thesis recounts the landmark discovery and subsequent decades-long monitoring of this system, which consists of two neutron stars. Pulsars are incredibly stable cosmic clocks. By precisely measuring the arrival times of the pulses from PSR B1913+16, astronomers were able to track its orbit with astonishing precision. They observed that the orbital period was decreasing over time. This orbital decay was found to match the predictions of General Relativity for energy loss due to the emission of gravitational waves via the quadrupole formula to within a fraction of a percent. This provided the first, albeit indirect, confirmation of GWs.

The remarkable precision of these tests is only possible after accounting for several subtle timing effects. The thesis details the essential corrections in the pulsar timing formula: The classical light-travel-time delay due to the motion of the Earth and the pulsar around their respective barycenters.

A general relativistic effect where the pulse's travel time is increased as it passes through the curved spacetime near a massive object (like the Sun or the pulsar's companion).

A combination of gravitational redshift (clocks run slower in a gravitational potential) and special relativistic time dilation (moving clocks run slower). The delay caused by the interaction of radio waves with the ionized interstellar medium.

By meticulously modeling these effects, the intrinsic orbital decay can be isolated, providing a powerful test of GR in the strong-field regime. To fully interpret GW signals from sources involving neutron stars, an understanding of their internal structure is required. This chapter shifts focus from the waves themselves to the objects that create them.

The thesis provides a derivation of the TOV equations, which describe the structure of a static, spherically symmetric, self-gravitating body in General Relativity. They are the relativistic generalization of the Newtonian equations of hydrostatic equilibrium.

The TOV equations must be supplemented with an Equation of State (EoS), $p=p(\rho)$, which describes the relationship between pressure and density for the ultra-dense matter inside a neutron star. The EoS is a key unknown in nuclear physics.

For a given EoS, the TOV equations can be solved numerically to yield a unique mass-radius relation for neutron stars. This theoretical prediction can be tested by astrophysical observations, and GWs provide a powerful new tool for this. The tidal deformability of a neutron star during a binary inspiral, which can be measured from the waveform, depends sensitively on its EoS.

The chapter concludes with a discussion of Buchdahl's theorem, a fundamental result in GR which establishes an absolute upper limit on the compactness of any static fluid star. This demonstrates that objects cannot be arbitrarily compact without collapsing into a black hole.

With the foundations laid, the thesis turns to the practical challenge of constructing accurate waveform templates for data analysis. A complete GW signal from a binary coalescence is typically divided into three phases: inspiral, merger, and ringdown. No single analytical method can describe all three phases accurately. The thesis explores a modern "hybrid" approach.

The early, slow orbital decay is well-described by the Post-Newtonian (PN) expansion, an approximation to GR valid for weak fields and low velocities. The thesis discusses the various PN orders and their physical meaning (e.g., spin effects, orbital decay).

As the objects approach their final collision, velocities become relativistic and the gravitational fields become extremely strong, invalidating the PN approximation. This regime is modeled using Numerical Relativity (NR), where the full Einstein equations are solved on a supercomputer. The thesis then presents a semi-analytical fitting formula, the Implicit Rotating Source (IRS) model, which is designed to capture the essential physics of the merger and the subsequent "ringdown" (where the final black hole settles into equilibrium) found in NR simulations.

The process of smoothly stitching the PN inspiral model to the merger-ringdown model at an intermediate frequency is discussed. This hybridization results in a complete, analytical waveform template that is both accurate and computationally efficient enough for use in matched-filtering data analysis pipelines.

The final chapter moves to the research frontier: using GWs to test the foundations of gravity itself. While GR has passed every test to date, it has theoretical limitations, motivating the study of alternative theories.

The thesis focuses on a leading class of alternative theories, including Brans-Dicke (BD), Damour–Esposito-Farèse (DEF), and Screened Modified Gravity (SMG). These theories augment GR by introducing a new scalar field that mediates a component of the gravitational force.

A key prediction of many scalar-tensor theories is the emission of dipole gravitational radiation from binary systems. This is strictly forbidden in GR, where the lowest-order radiation is quadrupolar. Dipole radiation is strongest in asymmetric binaries, such as Neutron Star-Black Hole (NS-BH) systems, because the neutron star can acquire a "scalar charge" (its mass depends on the local scalar field), while a black hole (in many of these theories) cannot, due to no-hair theorems.

The thesis outlines the methodology for testing these theories. The predicted dipole radiation term introduces a characteristic modification to the phase evolution of the GW signal. By performing a full Bayesian inference on real GW data from NS-BH events (like GW200115), one can search for this deviation.

The absence of a detected deviation allows one to place upper limits on the parameters of the scalar-tensor theory, e.g., on the difference in scalar charges, or on the BD parameter. This analysis demonstrates how GW observations are becoming one of our most powerful tools for probing gravity in the strong-field regime and testing the validity of General Relativity.

This thesis presents a comprehensive journey through the physics of gravitational waves. It begins with the fundamental theoretical principles derived from General Relativity and classical field theory, develops the mathematical formalisms for describing wave generation via multipole expansions, and applies this theory to a wide array of realistic astrophysical sources. It further connects theory with observation by detailing the experimental evidence from pulsar timing and the physics governing the structure of neutron stars. Finally, the work culminates in an exploration of modern waveform modeling techniques and their application at the cutting edge of science: using gravitational wave data to test the very foundations of Einstein's theory of gravity.

A Useful notes and mathematical proofs

In this section of the Appendix, several different proofs are studied. First, we will see the solution of an important differential equation by applying Green's function. Next, we will see the way a metric tensor transforms under Lorentz t, transformations and finally, the transformation of the perturbed metric after coordinate transformations are applied.

A.1 Green's function and solutions to the equations

In equation 22 of paragraph 1.1, we saw the differential form of Einstein's equations. We can obtain solutions to these differential equations by using the Green's function and the solving method, as presented below. We begin by writing the closed geometric form of eq. 22, as:

$$\Box_x G(x-x') = \delta^{(4)}(x-x') \tag{1125}$$

Multiplying each side with a generic 4-function $f_\mu(x')$ and integrating over all four-dimensional $x'-$space, we get:

$$\Rightarrow \int \frac{d^4x'}{(2\pi)^4} \Box_x G(x-x') f_\mu(x') = \int \frac{d^4x'}{(2\pi)^4} \delta^{(4)}(x-x') f_\mu(x') \tag{1126}$$

$$\Rightarrow \Box_x \int d^4x' G(x-x') f_\mu(x') = f_\mu(x) \tag{1127}$$

Equation 1127 provides the generic solution to equation 1125 and can be used in various problems of theoretical physics.

A.2 Lorentz transformation of the metric tensor

Under a Lorentz transformation, the metric can be written as below:

$$g_{\mu\nu} \rightarrow go_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma}(x)$$

$$g'_{\mu\nu}(x') = \eta_{\rho\sigma}(x) \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma + \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x)$$

Finally, for a flat metric, ow that under Lorentz transformations remain invariant, so the following is satisfied:

$$\eta_{\mu\nu}(x) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma}(x)$$

The transformation rule is

$$g'_{\mu\nu}(x') = \eta_{\mu\nu}(x) + \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x)$$

Concluding we see the transformation rule of the background metric to be one of a (0,2) tensor, such as:

$$h'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x)$$

and the full metric tensor will obey,

$$g'_{\mu\nu}(x') = \eta_{\mu\nu}(x) + h'_{\mu\nu}(x')$$

while the flat metric tensor remains invariant

A.3 Coordinate transformation of the perturbed metric

Following the procedure described in section 1.1, but for the pe, perturbed metric, we get:

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)}$$

$$\Rightarrow \bar{h}'_{\mu\nu}(x') - \frac{1}{2}\eta_{\mu\nu}\bar{h}' = \bar{h}_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

$$\Rightarrow \bar{h}'_{\mu\nu}(x') = \bar{h}_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\left(\bar{h} - \bar{h}'\right)$$

$$\Rightarrow \bar{h}'_{\mu\nu}(x') = \bar{h}_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)} + \frac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu}\partial_{(\mu}\xi_{\nu)}$$

$$\Rightarrow \bar{h}'_{\mu\nu}(x') = \bar{h}_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\partial_{\rho}\xi^{\rho}$$

The last ex-derived, when derived, produces:

$$\left(\partial^{\nu}\bar{h}_{\mu\nu}\right)' = \partial^{\nu}\bar{h}_{\mu\nu} - \Box\xi_{\mu}$$

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