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# Dark Matter and Black Holes in Theories with Extra Dimensions

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**Abstract**

In the present work, we examine a very recent scenario which attempts to explain the nature of dark matter. This scenario suggests that dark matter is composed of primordial five-dimensional spinning black holes within the dark dimension scenario. In the first chapters, we provide an introduction to the basic tools of both string theory and the swampland program, as well as general relativity and black holes in extra dimensions. Subsequently, we study the dark dimension scenario and see that the fact that these black holes perceive the fifth dimension results in the prolongation of their lifespan to such an extent that black holes within a specific mass range can survive until today and thus constitute the entirety of the dark matter we observe today. Finally, we discuss the memory burden effect and see that if we include it in the picture of the black hole's evolution, then we can further open up the mass range limits of the black holes that can constitute the entirety of the universe's dark matter.

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# Chapter 1

# Introduction

Nowadays, string theory is considered the most promising candidate for a theory of quantum gravity. It is the only framework that can unify the previously incompatible theories—the general theory of relativity and quantum field theory (QFT)—within a single, consistent model. However, string theory faces significant challenges.

The first major issue is that it cannot make predictions at low energies, meaning it lacks the ability to produce testable results within the energy scales accessible to current experiments. The second problem stems from this: string theory requires extra spatial dimensions—either ten or eleven in total—while our observable universe appears to be four-dimensional. When attempting to compactify these additional dimensions, the process yields an overwhelming number of possibilities—over $10^{500}$ different ways—making it extremely difficult to identify the specific Calabi-Yau manifold that corresponds to our four-dimensional universe.

In recent years, a new research area called the Swampland program has emerged. Its goal is to distinguish effective field theories (EFTs) that are consistent with quantum gravity (the "Landscape") from those that are not (the "Swampland"). This approach helps identify the essential characteristics that EFTs must possess to be compatible with a complete theory of quantum gravity in the ultraviolet regime.

One notable idea arising from this framework is the concept of a "dark dimension," which proposes that one of the six or seven extra dimensions may be large, characterized by a length scale in the micron range. This idea has intriguing implications, such as describing dark matter as primordial black holes residing in this dark dimension.In this paper, we explore this concept further.

In Chapter 2 we will focus on bosonic string theory and Swampland program highlighting their main features. We will then briefly discuss some basics conjectures for the Swampland program which they are very important for the foundation of dark dimension concept.

In the last chapter we'll examine a very recent scenario for the nature of dark matter. This scenario suggests that dark matter comprises essentially 5-dimensional primordial black holes. This hypothesis falls within the framework of a theoretical model called the dark dimension. According to this model, our universe is a 4-dimensional brane immersed in a 5-dimensional space.This study generalizes the results of work [16] ,for cases where these primordial black holes that make up dark matter have a non-zero angular momentum. More specifically, we will derive the relationship for the lifetime of rotating black holes using semi-classical methods, and we will examine the mass spectrum range of these black holes that can explain the amount of dark matter we observe today.Finaly, we will discuss how the memory burden effect, a phenomenon conjectured to occur after a black hole has lost half of its mass through Hawking radiation and which is a purely quantum phenomenon, alters the picture of the black hole's evolution.

# Chapter 2

# A Brief introduction to String Theory and the Swampland Program

The string theory is a theory which attempts to unify all fundamental forces of nature in a single and well defined frame.Is the only mathematically consistent quantum gravity which exists nowadays.In this chapter we will discuss some basic properties and concepts of the theory.After this,we will make an introduction to the very important concept of the Swampland program.his introduction to string theory and Swampland program based on [1],[2] and [3].

## 2.1 The Nambu-Goto action

We want to write down the Lagrangian describing a relativistic particle of mass m.In anticipation of string theory, we'll consider D-dimensional Minkowski space $R^{1,D-1}$.We work with signature

$$\eta = diag(-1,1,1,1) \tag{2.1}$$

If we fix a frame with coordinates $X^{\mu} = (t, \vec{x})$ the action is simple:

$$S = -m \int \sqrt{-\dot{x}^2} dt (2.2)$$

with

$$\dot{x} = \frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} \tag{2.3}$$

One basic property of the above action is that action is invariant under transformations of this type:

$$\tau \rightarrow \bar{\tau} \tag{2.4}$$

A particle sweeps out a worldline in Minkowski space. A string sweeps out a worldsheet in Minkowski space.We'll parameterize this worldsheet by one timelike coordinate $\tau$ , and one spacelike coordinate $\sigma$ . In this section we'll focus on closed strings and take to be periodic, with range

$$\sigma\epsilon[0, 2\pi) \tag{2.5}$$

In this case the action of a sting which propagate in D dimensional background spacetime is given by Nambu-Gotto action:

$$S = -T\int d\alpha = \int \sqrt{\Gamma}\, d\sigma^2 = -T\int \sqrt{|det\partial_\alpha X^\mu \partial_\beta X_\mu|}\, d\sigma^2 \tag{2.6}$$

We will mostly be concerned with closed, rather than open, strings. We therefore identify

$$\sigma \simeq \sigma + 2\pi \tag{2.7}$$

But in this form the action is impossible to quantize it.For this reason we should to re-write the action in a more convinient form to quantize it.This action is named Polyakov and is given by:

$$S_P = -\frac{T}{2}\int \sqrt{-h}h^{\alpha\beta}(\xi)\partial_\alpha X(\xi)^\mu \partial_\beta X(\xi)^\nu g_{\mu\nu} \tag{2.8}$$

where $h^{\alpha\beta}(\xi)$ is the metric of the worldsheet , $h = deth_{\alpha\beta}$ and $h^{\alpha\beta} = (h^{-1})_{\alpha\beta}$.. T is the string tension, which is often denoted in terms of a parameter $a'$ as:

$$T = \frac{1}{2\pi a'} \tag{2.9}$$

We should think of the worldsheet action (2.15) as specifying a two-dimensional theory with scalar fields $X^\mu(\xi)$. Such theories are called sigma models. The space-time in which the string propagates, parameterised by the $X^\mu(\xi)$ is known as the Target space of the worldsheet theory.The metric on that spacetime, here $\eta_{\mu\nu}$,is the metric on the field space of the scalar fields $X^\mu(\xi)$.So strings propagating in different target spaces have different metrics on the scalar field spaces.

The worldsheet theory (2.15) is invariant under local diffeomorphisms:

$$\xi^a \to \tilde{\xi}^a \tag{2.10}$$

It is also invariant under Weyl transformations, which are defined as:

$$\delta X^\mu = 0, h_{ab} \to \tilde{h}_{ab} = e^{2\Lambda(\xi)}h_{ab} \tag{2.11}$$

The worldsheet symmetries can be used to completely fix the worldsheet metric. It is worth looking at this generally. For a D-dimensional theory, we can count the number of degrees of freedom in the metric $h_{ab}(= \frac{D}{2}(D+1))$, a symmetric tensor, and in the diffeomorphism($= D$) and Weyl symmetries($= 1$). We see that for D = 2, so a string, the number of symmetry parameters is the same as the degrees of freedom of the metric. Using the symmetries we can therefore set:

The coordinates therefore parameterise the string worldsheet.We will often denote

$$[\sigma, \tau] \equiv \xi^a \tag{2.12}$$

with $a = 0, 1$.

The equation of motion for the metric corresponding to the vanishing of the energy momentum tensor:

$$T_{a\beta} = 0 \tag{2.13}$$

where

$$T_{a\beta} = \frac{4\pi}{\sqrt{-deth}} \frac{\delta S_p}{\delta h^{a\beta}} \tag{2.14}$$

The constraint (2.13) its called Virassoro constraint and will play an important role when we quantize the string. In flat gauge the Polyakov action given by:

$$S_p = \int d\sigma d\tau [(\partial_\tau X)^2 - (\partial_\sigma X)^2] \tag{2.15}$$

It is convenient to go to so-called light-cone coordinates:

$$\xi^{\pm} \equiv \tau \pm \sigma, \partial_{\pm} \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{2.16}$$

In light-cone coordinates the Polyakov action read:

$$S_P = T \int_\Sigma d\xi^+ d\xi^- \partial_+ X \partial_- X \tag{2.17}$$

The equations of motion for the $X^\mu$ are readily obtained:

$$\partial_+ \partial_- X^\mu = 0 \tag{2.18}$$

We can therefore write as a sum of left-moving and right-moving waves a long the string:

$$X^\mu = X^\mu_L(\xi^+) + X^\mu_R(\xi^-) \tag{2.19}$$

And we must impose periodic boundary conditions$X^\mu(\tau, \sigma = 0) = X^\mu(\tau, \sigma = 2\pi)$:

$$X^\mu_R(\xi^-) = \frac{1}{2}(x^\mu + c^\mu) + \frac{1}{2} a' p^\mu_R \xi^- + i\sqrt{\frac{a'}{2}} \sum_{neZ, n\neq 0} \frac{1}{n} a^\mu_n e^{-in\xi^-} \tag{2.20}$$

$$X^\mu_R(\xi^-) = \frac{1}{2}(x^\mu - c^\mu) + \frac{1}{2} a' p^\mu_L \xi^+ + i\sqrt{\frac{a'}{2}} \sum_{neZ, n\neq 0} \frac{1}{n} \bar{a}^\mu_n e^{-in\xi^+} \tag{2.21}$$

Here $x^\mu$,$c^\mu$,$p^\mu_L$,$p^\mu_R$,$a^\mu_n$ and $\bar{a}^\mu_n$ are constants.Periodicity in $\sigma$ implies:

$$p^\mu_L = p^\mu_R \equiv p^\mu \tag{2.22}$$

It will be important for later to note that even after the gauge fixing the world sheet metric, there are still residual symmetries:

$$\xi^\pm \to \tilde{\xi}^\pm(\xi^\pm) \tag{2.23}$$

These are associated to so-called conformal Killing vectors.

## 2.2 Quantization of string

So far we have considered the string in a classical sense,but in order to study the spectrum of excitations we need to quantize it.We will do this using so-called light-cone quantization.The starting point is to introduce target-space light-cone coordinate:

$$X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \tag{2.24}$$

The target-space metric then becomes:

$$\eta_{+-} = \eta_{-+} = -1, \eta_{ij} = \delta_{ij} \tag{2.25}$$

And this gives an inner product:

$$X^2 = -2X^+X^- + \dot{X}^i\dot{X}^i \tag{2.26}$$

Consider now the expansion for:

$$X^\mu_R(\xi^-) = x^+ + a'p^+\tau + i\sqrt{\frac{a'}{2}}\sum_{neZ,n\neq 0}\frac{1}{n}\bar{a}^+_n e^{-in\xi^-} + i\sqrt{\frac{a'}{2}}\sum_{neZ,n\neq 0}\frac{1}{n}\bar{a}^+_n e^{-in\xi^-} \tag{2.27}$$

Recall that we have a residual infunite dimensional symmetry after going to light-cone gauge.We can use this to set all the oscillator modes$X^+$. In that gauge then we have:

$$X^+ = x^+ + a'p^+\tau \tag{2.28}$$

Now recall that we must impose the Virasoro constraints(2.21)on the theory. It can be shown that these imply:

$$\partial_\pm X^- = \frac{1}{a'p^+}(\partial_\pm X^i)^2 \tag{2.29}$$

Therefore,we see that also the $X^-$ oscillators are given in terms of the transverse oscillators in $X^i$.Soonly the transverse oscillators are independent degrees of freedom. The usefulness of the target-space light-cone gauge is therefore that only the $X^i$ contain physically independent oscillators.This is useful because it automatically projects out two polarizations of the string which are unphysical.This is completely analogous to how a Maxwell field in four dimensions only has two physical polarizations.

The action in light-cone gauge reads:

$$S_{LC} = \frac{1}{4\pi a'}\int_\Sigma d\tau d\sigma[(\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2 + 2(-\partial_\tau X^+\partial_\tau X^- + \partial_\sigma X^+\partial_\sigma X^-]$$
$$= \frac{1}{4\pi a'}\int_\Sigma d\tau d\sigma[(\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2 + \int_\Sigma d\tau p^+\partial_\tau q^- \tag{2.30}$$

where define:

$$q^- = \frac{1}{2\pi}\int_0^{2\pi} d\sigma X^- \tag{2.31}$$

From this Lagrangian we can define canonical momenta:

$$p_- \equiv \frac{\partial L}{\partial \dot{q}^-} = -p^+, \Pi_i \equiv \frac{\partial L}{\partial \dot{X}^i} = \frac{\dot{X}_i}{2\pi a'} \tag{2.32}$$

We then quantize the theory by introducing the canonical commutation relations:

$$[X^\mu(\tau,\sigma), \Pi^\mu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{2.33}$$

which give:

$$\begin{aligned}
[x^i, p^i] &= i\delta_{ij} \\
[p^+, q^-] &= i \\
[a^i_m, a^j_n] &= m\delta_{n+m,0}\delta_{ij} \\
[\tilde{a^i_m}, \tilde{a^j_n}] &= m\delta_{n+m,0}\delta_{ij}
\end{aligned} \tag{2.34}$$

We therefore follow the usual procedure for quantization,as in quantum field theory,by promoting the oscillator modes to operators acting on a Hilbert space.The $a^i_{-n}$ with $n > 0$ are creation operators acting on a vacuum state.$|0, p>$..While the $a^i_n$ with $n > 0$ are annihilation operators.

Recall that there are no oscillators to quantize for $X^+$, while the $X^-$ oscillators are given in terms of the $X^i$.Explicitly this reads:

$$a^-_n = \frac{1}{2\sqrt{2a'}p^+}\sum_{m=-\infty}^{m=\infty} a^i_{n-m}a^i_m \tag{2.35}$$

When we quantize the theory the ordering of the a's matters,and so we should write things in terms of normal ordered products and a normal ordering constant a which we need to determine:

$$a^-_n = \frac{1}{2\sqrt{2a'}p^+}\sum_{m=-\infty}^{m=\infty} : a^i_{n-m}a^i_m : -a\delta_{n,0} \tag{2.36}$$

where

$$: a^i_{n-m} a^i_m = \begin{cases} a^i_m a^i_n & \text{for} \\ m \leq n \\ a^i_n a^i_m & \text{for} \\ m > n \end{cases} \tag{2.37}$$

This is the canonical quantization procedure.

## 2.3 Criticality and Lorentz Invariance

The quantization of the theory was performed in special target-space light-cone coordinates. It is therefore not clear that the quantum theory respects Lorentz invariance.Indeed,we will see that requiring the preservation of target-space Lorentz invariance also in the quantum theory will place rather stringent constraints on the theory

In general,the generators of Lorentz transformations are:

$$J^{\mu\nu} = \int_0^{2\pi} d\sigma (X^\mu \Pi^\nu - X^\nu \Pi^\mu) \equiv l^{\mu\nu} + E^{\mu\nu} + \tilde{E^{\mu\nu}} \tag{2.38}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{2.39}$$

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} (a^\mu_{-n} a^\nu_n - a^\nu_{-n} a^\mu_n) \tag{2.40}$$

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\tilde{a}^\mu_{-n} \tilde{a}^\nu_n - \tilde{a}^\nu_{-n} \tilde{a}^\mu_n) \tag{2.41}$$

Now the Lorentz algebra reads:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho} J^{\nu\sigma} + i\eta^{\nu\sigma} J^{\mu\rho} - i\eta^{\mu\sigma} J^{\nu\rho} - i\eta^{\nu\rho} J^{\mu\sigma} \tag{2.42}$$

In particular,

$$[J^{-i}, J^{-j}] = i\eta^{--} J^{ij} + i\eta^{ij} J^{--} - i\eta^{-j} J^{i-} - i\eta^{i-} J^{-j} \tag{2.43}$$

However, an explicit calculation yields:

$$\Delta_m \equiv m \frac{26-D}{12} + \frac{1}{m} [\frac{D-26}{12} + 2(a-1)] \tag{2.44}$$

From above it is clear that if we want our theory to be Lorentz invariant we must impose:

$$D = 26, a = 1 \tag{2.45}$$

## 2.4 The string spectrum

Having quantized the string we can now examine its spectrum. The classical Hamiltonian is given by:

$$H = \frac{a'}{2} p^i p^i + \frac{1}{2} \sum_{n=-\infty}^{\infty} (a^i_{-n} a^i_n + \tilde{a}^i_{-n} \tilde{a}^i_n) \tag{2.46}$$

We define:

$$N \equiv \sum_{n=1}^{\infty} : a^i_{-n} a^i_n : \tag{2.47}$$

and

$$\tilde{N} \equiv \sum_{n=1}^{\infty} : \tilde{a}^i_{-n} \tilde{a}^i_n : \tag{2.48}$$

The mass in the target-space is given by:

$$M^2 = \frac{2}{a'}(N + \tilde{N} - 2a) \tag{2.49}$$

Finally, we note that for the closed string we have a symmetry of translations along , and this can be shown to imply the level matching condition:

$$N = \tilde{N} \tag{2.50}$$

so the relation for the mass becomes:

$$M^2 = \frac{4}{a'}(N - 1) \tag{2.51}$$

where $a = 1$ as we mention before. From the above relation we can examine the spectrum on the string according to how many oscillators are present:

- For $N = 0$ we have

  $$M^2 = -\frac{4}{a'} \tag{2.52}$$

  which is the tachyonic mode, which means that it is signaling an instability in the bosonic string. For superstrings there is no tachyonic mode , and so such strings are stable.

- For N=1 we have the masless modes of the quantum bosonic string:

  $$M^2 = 0 \tag{2.53}$$

  These states are represented by:

  $$\lambda_{ij} a^i_{-1} a^j_{-1} |0, p >, i, j = 1, ...., 24 \tag{2.54}$$

  We can decompose the tensor ij into irreducible representations of SO(24) as:

  $$\lambda_{ij} = g_{(ij)} + B_{[ij]} + \phi \tag{2.55}$$

where the $g_{(ij)}$ is the symmetric part of the tensor,the $B_{[ij]}$ the symmetric part and $\phi$ is scalar.

As we see above we find a massless symmetric tensor field which we can show that it has spin 2.This field has all the properties of the graviton,the particle of gravitational field.Consequently,we have a self consistent quantum theory which it contain the gravity.

## 2.5 Swampland conjecture

In this section we examine some basic conjectures which they come from some regions of theoretical physic,particularly from the area of quantum gravity,for example string theory.We believe that this conjectures characterize the quantum behavior of gravity and a canditate theory of quantum gravity should obey these relations.

The first conjecture we will see its named distance conjecture and with simple words it says that:

- ***Distance Conjecture***: Consider a theory, coupled to gravity, with a moduli space M which is parametrized by the expectation values of some field $\phi^i$ which have no potential. Starting from any point $P \epsilon M$ there exists another point $Q \epsilon M$ such that the geodesic distance between P and Q, denoted $d(P, Q)$, is infnite.

  Therefore,is exists an infinite tower of states, with an associated mass scale M, such that:

$$M(Q) \sim M(P) e^{-ad(P,Q)} \tag{2.56}$$

We will see an example which it confirms the above statement.We will see the concept of compactification in qft and string theory.First we examine the compactification in qft. We consider D = d+1 dimensional space-time. The spatial direction $X^d$ is taken to be compact in the shape of circle.So the compact dimension is periodically

$$X^d \simeq X^d + 1 \tag{2.57}$$

We are interested for d-dimensional effective fileld theory.Also we are working in the Planck units with d-dimensional Planck mass $M_P^d = 1$.

We can write the metric on the D-dimensional space as:

$$dS^2 = G_{MN} dX^M dX^N = e^{2\alpha\phi} g_{\mu\nu} dX^\mu dX^\nu + e^{2b\phi} (dX^d)^2 \tag{2.58}$$

which $X^M$ is the D-dimensional coordinates wit $M = 0, ..., d$.Also we have the D-dimensional metric $G_{MN}$ and d-dimensional metric $g_{\mu\nu}$ with $\mu, \nu = 0, .., d-1$. The metric has a parameter $\phi$ wich it in d dimensions is a dynamical scalar field.The constants $\alpha$ and b are given by:

$$a^2 = \frac{1}{2(d-2)(d-1)} \tag{2.59}$$

$$\beta = -(d-2)a \tag{2.60}$$

Then the circular dimension can be write as:

$$2\pi R \equiv \int_0^1 \sqrt{G_{dd}} dX^d = e^{\beta\phi} \tag{2.61}$$

We will be interested in the behaviour of the d-dimensional theory under variations of the expectation value of the eld , which amounts to variations of the size of the circle. The rst thing we want to do is decompose the D-dimensional Ricci scalar RD for the metric (2.58). We have:

$$\int d^D X \sqrt{-G} R^D = \int d^d X \sqrt{-g}[R^d + \frac{1}{2}(\partial\phi)^2] \tag{2.62}$$

Now consider introducing a massless D-dimensional scalar field $\Psi$ and is given by:

$$\Psi(X^M) = \sum_{n=-\infty}^{\infty} \psi_n(X^\mu) e^{2i\pi n X^d} \tag{2.63}$$

where $\psi_n$ is the KK modes. The momentum is quantized along the compact direction

$$-i\frac{\partial}{\partial X^d}\Psi = 2\pi n \Psi \tag{2.64}$$

For simplicity we now restrict to $g_{\mu\nu} = \eta_{\mu\nu}$. Since $\Psi$ is massless in D-dimensions, its equation of motion is:

$$\partial^M \partial_M \Psi = (e^{-2a\phi}\partial^\mu\partial_\mu + e^{-2\beta\phi}\partial^2_{X^d})\Psi = 0 \tag{2.65}$$

which gives the equations of motion for the $\psi_n$ modes:

$$[\partial^\mu\partial_\mu - (\frac{1}{2\pi R})^2 (\frac{1}{2\pi R})^{\frac{2}{d-2}} (2\pi n)^2]\psi_n = 0 \tag{2.66}$$

The mass of the KK modes is given by:

$$M_n^2 = (\frac{n}{R})^2 (\frac{1}{2\pi R})^{\frac{2}{d-2}} \tag{2.67}$$

On the other hand in string theory we have the d-th spatial direction to be compact

$$X^d \sim X^d + 2\pi R \tag{2.68}$$

We consider the bosonic mode expansion, as in (2.27)but now we will not impose yet the $X_s^M(\tau, \sigma + 2\pi) = X_s^M(\tau, \sigma)$ on the linear terms in $\sigma$.So we have

$$X_s^M(\tau,\sigma) = x^\mu + a' p^M \tau + \frac{a'}{2}(p_R^M - p_L^M)\sigma + oscillators \tag{2.69}$$

For independent left and right moving momenta ,the overall momentum of string is gven by:

$$p^M = \frac{1}{2}(p_R^M + p_L^M) \tag{2.70}$$

We know tha the dth direction is compact and for that reason the momentum in that direction is quantized:

$$p^d = \frac{n}{R} \tag{2.71}$$

In the non compact directions we imposed $X_s^M(\tau, \sigma + 2\pi) = X_s^M(\tau, \sigma)$ which leads to $p_R^M = p_L^M$, but in the string theory the strings may be winded in the circular dimension

$$X_s^d(t, \sigma + 2\pi) = X_s^d(t, \sigma) + 2w \tag{2.72}$$

with $w\epsilon Z$ wich w is the number which stings be winded around the circular dimension.For such a winding string we therefore have

$$\frac{a'}{2}(p_R^d - p_L^d) = wR \tag{2.73}$$

The Hamiltonian of our theory in this case is

$$H = \frac{a'}{2}[\frac{1}{4}(p_R^d - p_L^d)^2 + p^a p^a + (p^d)^2] + (N + \tilde{N} - 2)] \tag{2.74}$$

where $i = (a, d)$ and we have

$$N - \tilde{N} = nw \tag{2.75}$$

Then the d-dimensional mass is given by $-p^\mu p_\mu = 2p^+ p^- - p^a p^a$ which, for states with no oscillators excited, leads to:

$$(M_{n,w}^s)^2 = (\frac{n}{R})^2 + (\frac{wR}{a'})^2 \tag{2.76}$$

In the Einstein frame the above relation for mass is given by:

$$(M_{n,w})^2 = (\frac{1}{2\pi R})^{\frac{2}{d-2}}(\frac{n}{R})^2 + (2\pi R)^{\frac{2}{d-2}}(\frac{wR}{a_0'}) \tag{2.77}$$

We can now study the d-dimensional effective theory.We're looking at how the spectrum of states changes when the field's expectation value $\phi$ varies. This is straightforward to figure out using relation (2.61), which complements the action (2.62) and spectrum (2.77). The possible values for $\phi$ form a field space $(M_\phi)$ that's infinitely one-dimensional in real numbers:

$$M_\phi : -\infty < \phi < +\infty \tag{2.78}$$

Let us define a variation of $\phi$ from some initial value $\phi_i$ to some final value $\phi_f$ as:

$$\Delta\phi = \phi_f - \phi_i \tag{2.79}$$

There are two infinite towers of massive states in our theory, the KK tower modes with masses $M_{n,0}$ and the tower of winding modes $M_{0,m}$ which given by (2.77). We can associate to each tower a mass scale

$$M_{KK} \sim e^{a\phi} \tag{2.80}$$

and

$$M_w \sim e^{-a\phi} \tag{2.81}$$

where

$$a = \sqrt{2}(\frac{d-1}{d-2})^{\frac{1}{2}} > 0 \tag{2.82}$$

We see that for $\Delta\phi$ there exists an infinite number of states with some associated mass scale M, which becomes light at an exponential rate in $\Delta\phi$:

$$M(\phi + \Delta_\phi) \sim M(\phi)e^{-a|\Delta_\phi|} \tag{2.83}$$

With above picture in our minds, we can make the the following observations First we see that some tower of states always become light no matter what the sign of $\Delta_\phi$ is. Second,the product of the mass scales of the two towers is independent of $\phi$. The third is that, if $|\Delta| \to \infty$, then an infinite number of states become massless, which means that there is no description of that locus in a d-dimensional quantum field theory. To recap, we can summarize the distance conjecture with the following sentence:

**Consider a theory, coupled to gravity, with a moduli space M which is parametrized by the expectation values of some field i which have no potential. Starting from any point P M there exists another point Q M such that the geodesic distance between P and Q, denoted d (P, Q), is infinite.Also,there exists an infinite tower of states, with an associated mass scale M, such that:**

$$M(Q) \sim M(P)e^{-ad(Q,P)} \tag{2.84}$$

where a is some positive constant.

**The weak gravity conjecture:** The weak gravity conjecture which claim that the force of gravity is the weakest force in nature.

Our analysis of string compactifications on a circle revealed the presence of two gauge fields, Aµ and Vµ, which possess gauge couplings. Within the resulting d-dimensional effective theory, states carrying charges for these fields exhibited a mass-charge relationship of m = gq. Crucially, these weren't just isolated states; they represented the foundational elements of an infinite progression, all characterized by a common mass scale, m = g. This "tower" of states implied an alternative description of the effective theory, specifically the D-dimensional theory in the context of the Kaluza-Klein tower. Therefore, this mass scale can be interpreted as a cutoff energy for the d-dimensional effective theory.The aboves can sumerized as:

**Consider a theory, coupled to gravity, with a U(1) gauge symmetry with gauge coupling g**

$$S = \int d^dX\sqrt{-g}[(M_p^d)^{d-2}\frac{R^d}{2} - \frac{1}{4g^2}F^2 + ...] \tag{2.85}$$

**Electric Weak Gravity Conjecture:There is a particle in the theory with mass m and charge which is related with U(1) gauge symmetry and satisfying the relation:**

$$m \leq \sqrt{\frac{d-2}{d-3}} qg(M_p^d)^{\frac{d-2}{2}} \tag{2.86}$$

**Magnetic Weak Gravity Conjecture:There is a cut of scale $\Lambda$ in the effective field theory which satisfying the relation:**

$$\Lambda \leq g(M_p^d)^{\frac{d-2}{2}} \tag{2.87}$$

Also, there is the no global symmetries conjecture which is claim that a theory with a finite number of states, coupled to gravity, can have no exact global symmetries and the species scale conjecture but for the present thesis we will not deal with these conjectures.

# Chapter 3

# Higher Dimensional Black Holes

## 3.1 Mathematical tools of General Relativity

General relativity is the best theory we have for the behavior of gravity and for gravitational systems in large scales like constelations,black holes and the univese itself.It tell us that gravity is the curvature of spacetime itself.The quantity that descibes the curvature of spacetime is called metric tensor and denoted by $g_{\mu\nu}$ where $\mu$ and $\nu$ are the spacetimes indices that run from 0 to 3.

Consequently, to be able to measure distances on a manifold, it is important to define the line-element $ds^2$ by means of the metric, which is written as:

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{3.1}$$

where the metric tensor $g_{\mu\nu}$ provides us with all the information about the geometry of spacetime and is a symmetric tensor $g_{\mu\nu} = g_{\nu\mu}$ with this important property:

$$g^{\mu a} = g_{a\nu} \tag{3.2}$$

Another important quantity in our discusion is the Rieman tensor it's a tensor (1,3) with $n^2(n^2-1)/12$ indepentent components that encodes information about how geodesics (the paths of shortest distance) deviate from being straight lines due to the curvature of the space. In other words is an object which characterize the curvature of a manifold and is given by:

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{3.3}$$

where $\Gamma^{\rho}_{\mu\sigma}$are the Christoffel symbols given by:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}) \tag{3.4}$$

with property $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$. From the Rieman tensor we can construct two importan mathematical objects, the Ricci tensor and the Ricci scalar which they enters in Einstein field equation as we will see bellow and are given by:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{3.5}$$

and

$$R = g^{\mu\nu} R_{\mu\nu} \tag{3.6}$$

At this point it is important to define through the connection, the geodesic equation for a curve $x^{\mu}(\lambda)$:

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^{\rho}}{d\lambda} \frac{dx^{\sigma}}{d\lambda} \tag{3.7}$$

which give us the shortest distance between two points in spacetime.

### 3.1.1 Einstein's Equations

Einstein's equation tells us how the curvature of spacetime is linked to the distribution of energy-momentum, represented in the energy-momentum tensor. Energy and momentum create curvature. Curvature acts as gravity and tells matter how to move via the geodesic equation. We will derive Einstein's equation with the principle of least action. From classical field theory you might be familiar with this method, but to use it in general relativity we first need to generalise it to curved spacetime.

The classical solutions for a field theory, where the dynamical variables are a set of fields $\Phi^i(x)$ will be the critical points of an action S.The action is the integral of a Lagrange density L, a function of the fields and their covariant derivatives, over space:

$$S = \int d^n x \mathcal{L}(\Phi^i, \nabla_{\mu} \Phi^i) \tag{3.8}$$

To find the critical points of S we vary the fields and require the action to be unchanged. So we begin by varying the fields and making an appropriate Taylor expansion of the Lagrangian:

$$\Phi^i \to \Phi^i + \delta\Phi^i \tag{3.9}$$

$$\nabla_{\mu}\Phi^i \to \nabla_{\mu}\Phi^i + \nabla_{\mu}(\delta\Phi^i) \tag{3.10}$$

$$\mathcal{L}(\Phi^i, \nabla_{\mu}\Phi^i) \to \mathcal{L}(\Phi^i, \nabla_{\mu}\Phi^i) + \frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\nabla_{\mu}\Phi^i)}\nabla_{\mu}(\delta\Phi^i) \tag{3.11}$$

$$= \mathcal{L} + \delta\mathcal{L} \tag{3.12}$$

The action then follows with:

$$S \to S + \delta S \tag{3.13}$$

$$= \int d^n x \mathcal{L}(\Phi^i, \nabla_{\mu}\Phi^i) + \int d^n x [\frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\nabla_{\mu}\Phi^i)}\nabla_{\mu}(\delta\Phi^i)] \tag{3.14}$$

We want $\delta S$ to be zero so the actions remains unchanged under field variations. To reformulate this condition we factor out the term $\delta\Phi^i$ in $\delta S$. We do this by integrating the second term by parts:

$$\int d^n x \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi^i)} \nabla_\mu(\delta\Phi^i) = -\int d^n x \nabla_\mu (\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi^i)})\delta\Phi^i + \int d^n x \nabla_\mu (\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi^i)}\delta\Phi^i) \tag{3.15}$$

$$= -\int d^n x \nabla_\mu (\frac{\partial \mathcal{L}'}{\partial(\nabla_\mu \Phi^i)})\sqrt{-g}\delta\Phi^i + \int d^n x \nabla_\mu (\frac{\partial \mathcal{L}'}{\partial(\nabla_\mu \Phi^i)}\delta\Phi^i)\sqrt{-g} \tag{3.16}$$

$$= -\int d^n x \nabla_\mu (\frac{\partial \mathcal{L}'}{\partial(\nabla_\mu \Phi^i)})\sqrt{-g}\delta\Phi^i \tag{3.17}$$

where we write the e Lagrangian as:

$$\mathcal{L} = \sqrt{-g}\mathcal{L}' \tag{3.18}$$

The Stokes's theorem and our choice to set the variation of the field to zero at the boundary (infinity).Stokes's theorem in curved spacetime is:

$$\int d^n x \nabla_\mu V^\mu \sqrt{|g|} = \int d^{n-1} x n_\mu V^\mu \sqrt{|\gamma|} \tag{3.19}$$

with $V^\mu$ a vector field over a region  with boundary $\partial\Sigma$,$n_\mu$ normal to $\partial\Sigma$,$n_\mu$ and $\gamma_{ij}$ the induced metric on $\partial\Sigma$.Now we have an expression for $\delta S$ and we also know its form by definition:

$$S = \int d^n x \sqrt{-g}[\frac{\partial \mathcal{L}'}{\partial \Phi^i}\delta\Phi^i - \nabla_\mu (\frac{\partial \mathcal{L}'}{\partial(\nabla_\mu \Phi^i)})\delta\Phi^i] \tag{3.20}$$

$$= \int d^n x \frac{\delta S}{\delta \Phi^i}\delta\Phi^i \tag{3.21}$$

again using equation (2.18). For $\delta S$ S to be zero we need $\delta S/\delta\Phi^i$ to be zero, so:

$$\frac{\partial \mathcal{L}'}{\partial \Phi^i} - \nabla_\mu (\frac{\partial \mathcal{L}'}{\partial(\nabla_\mu \Phi^i)}) = 0 \tag{3.22}$$

These are called the Euler-Lagrange equations. Solutions to the field theory also satisfy these equations.

Now we want to apply this method to general relativity, a field theory with the metric as dynamic variable. The only independent scalar you can construct from the metric with no derivatives higher than the first one is the Ricci scalar.So the Lagrangian and corresponding action, called the Hilbert action, are:

$$L = \sqrt{-g}R \tag{3.23}$$

and

$$S = \int d^n x \sqrt{-g} R \tag{3.24}$$

The Hilbert action is not of the same form as equation (2.40) (it can not be written in terms of the metric and its covariant derivative), so we can not simply plug the Lagrangian in the Euler-Lagrange equation to obtain the field equations and instead have to explicitly variate the action with respect to the metric. It turns out to be easier to variate with respect to the inverse metric $g_{\mu\nu}$ and stationary points coming from variations in $g^{\mu\nu}$ are equivalent to ones from variations in $g_{\mu\nu}$. This can be seen from varying the expression

$$\begin{aligned} g^{\mu\lambda} g_{\lambda\nu} =& \delta^\mu_\nu \\ (g^{\mu\lambda} + \delta g^{\mu\lambda})(g_{\lambda\nu} + \delta g_{\lambda\nu}) =& \delta^\mu_\nu \\ =& \delta^\mu_\nu + g^{\mu\lambda} \delta g_{\lambda\nu} + \delta g^{\mu\lambda} g_{\lambda\nu} \\ \rightarrow \delta g_{\mu\nu} = g_{\mu\lambda} g_{\rho\nu} \delta g^{\lambda\rho} \end{aligned} \tag{3.25}$$

We can start by expressing the variation of the Hilbert action as:

$$\delta S_H = (\delta S)_1 + (\delta S)_2 + (\delta S)_3 \tag{3.26}$$

We want each of these to have a separate $\delta g^{\mu\nu}$ term, just as in equation (2.46).$(\delta S)_3$ is already in the right form. We will first look at $(\delta S)_1$ We need to know what the variation of $g = det g_{\mu\nu}$ s. For a general square matrix M with non vanishing determinant the following is true:

$$ln(det M) = Tr(ln M) \tag{3.27}$$

The variation of this is

$$\frac{1}{det M} \delta(det M) = Tr(M^{-1} \delta M) \tag{3.28}$$

So for the metric we have:

$$\delta g = g(g^{\mu\nu} \delta g_{\mu\nu}) = -g(g_{\mu\nu} \delta g^{\mu\nu}) \tag{3.29}$$

using equation().Then we now know

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}} \delta g \\ &= \frac{g}{2\sqrt{-g}} g_{\mu\nu} \delta g^{\mu\nu} \\ &= \frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \end{aligned} \tag{3.30}$$

and $(\delta S)_1$ become:

$$(\delta S)_1 = \int d^n x \sqrt{-g} [R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}] \delta g^{\mu\nu} \tag{3.31}$$

Now we will look at $\delta S_2$. The variation of the Riemann tensor $\delta R_{\mu\nu}$ can be found by varying its definition (equation (2.30)) with respect to the Christoffel symbols and then inserting the variation of the Christoffel symbols with respect to the inverse metric. The expression you get can be converted into a boundary integral by Stokes's theorem and can be set to zero by making the variation vanish at infinity. The remaining terms give the variation of the Hilbert action:

$$\delta S_H = \int d^n x \sqrt{-g}[R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}]\delta g^{\mu\nu} \tag{3.32}$$

At the critical points of the action $\delta S_H/\delta g^{\mu\nu} = 0$ and comparing equation (2.34) to equation (2.21) we see that

$$\frac{1\delta S_H}{2\sqrt{-g}\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{3.33}$$

This is Einstein's equation in vacuum. To get the field equations of general relativity coupled to matter we have to include an extra term $S_M$, representing the matter field, in the action:

$$S = \frac{1}{16\pi G}S_H + S_M \tag{3.34}$$

with G Newton's gravitational constant. We used a normalisation on the Hilbert action we know will yield the right equation. When we apply the same method as above we get

$$\frac{1\delta S_H}{2\sqrt{-g}\delta g^{\mu\nu}} = \frac{1}{16\pi G}[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}] + \frac{1\delta S_M}{2\sqrt{-g}\delta g_{\mu\nu}} = 0 \tag{3.35}$$

Now we need a new (and correct) definition of the energy-momentum tensor:

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}} \tag{3.36}$$

such that equation (2.61) becomes

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.37}$$

This is the complete Einstein's equation where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is fittingly called the Einstein tensor. In four dimensions these are 16 second-order differential equations for the metric. But because both sides of equation (2.39) are symmetric tensors there are only 10 independent equations. This corresponds to the 10 unknown metric components. We can rewrite Einstein's equation, using $R = -8\pi GT$ T, in a form that is very convenient when considering vacuum space:

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{3.38}$$

while in the vacuum the Einstein equation is:

$$R_{\mu\nu} = 0 \tag{3.39}$$

## 3.2 Black holes in 4 dimensions

The most fascinating solutions of Einstein's field equation are the black hole solutions.The most simple black hole described by the Schwarzschild metric.The metric describes spherically symmetric and static sources for gravity in vacuum space, and is a good approximation to describe the field created by the Earth or the Sun at distances far greater than the Schwarzschild radius (defined shortly).The Schwarzschild metric in spherical coordinates is:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{3.40}$$

where M can be interpreted as the mass of the source and $d\Omega^2$ is the metric on a unit two-sphere:

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{3.41}$$

In this section we will derive analytically the Schwarzschild metric.So we assume a metric of the form:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta + sin^2\theta) \tag{3.42}$$

So we assume a metric of the form(2.40)In order to solve equation (37), we will first calculate the Christoffel symbols and the Ricci tensor. We first calculate the Christoffel symbols, using equation(2.4)

$$\Gamma^0_{01} = \frac{A'(r)}{2A(r)}, \Gamma^1_{11} = \frac{B'(r)}{2B(r)}, \Gamma^1_{00} = \frac{A'(r)}{2B(r)}, \Gamma^1_{22} = \frac{-r}{B(r)}, \Gamma^1_{33} = -\frac{r}{B(r)} sin^2\theta$$
$$\Gamma^2_{33} = -sin\theta cos\theta, \Gamma^2_{21} = \Gamma^3_{31} = \frac{1}{r}, \Gamma^3_{23} = \frac{cos\theta}{sin\theta} \tag{3.43}$$

and then, from equations (2.3)(2.5) we can calculate the non-zero Ricci tensor components:

$$R_{00} = -\frac{A''(r)}{2B(r)} + \frac{A'(r)}{4B(r)}[\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}] - \frac{A'(r)}{rB(r)}$$
$$R_{11} = \frac{A''(r)}{2A(r)} - \frac{A'(r)}{4B(r)}[\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}] - \frac{B'(r)}{rB(r)}$$
$$R_{22} = \frac{1}{B(r)} - 1 + \frac{r}{2B(r)}[\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)}]$$
$$R_{33} = sin^2\theta R_{22} \tag{3.44}$$

Then, using the Einstein's equation, we get:

$$R_{00} + \frac{A(r)}{B(r)} R_{11} + \frac{2A(r)}{r^2} R_{22} = 0 \tag{3.45}$$

$$R_{11} + \frac{B(r)}{A(r)} R_{00} - \frac{2B(r)}{r^2} R_{22} = 0 \Rightarrow \frac{A(r)}{B(r)} R_{11} + R_{00} - \frac{2A(r)}{r^2} R_{22} = 0 \tag{3.46}$$

Then, by adding equations (2.45) and (2.46) and employing the expressions (2.44)-(2.46), we get the equation:

$$2R_{00} + \frac{2A(r)}{B(r)}R_{11} = 0 \Rightarrow -\frac{A'}{A} = \frac{B'}{B} \tag{3.47}$$

Also, demanding that $R_{22} = 0$, we obtain:

$$\frac{1}{B(r)} - 1 + \frac{r}{2B(r)}[\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)}] = 0 \tag{3.48}$$

After integrating the equation (2.47), we find the relation:

$$-lnA(r) + k = lnB(r) \Rightarrow A(r)B(r) = e^k \Rightarrow B(r) = \Lambda/A(r) \tag{3.49}$$

with $e^k = \Lambda$ Now, substituting equation (2.49) into equation (2.48), we have:

$$\frac{A(r)}{\Lambda} - 1 + \frac{r}{\Lambda}A'(r) = 0 \Rightarrow rA'(r) + A(r) = \Lambda \Rightarrow \frac{d}{dr}(rA(r)) = \Lambda \Rightarrow A(r) = \Lambda(1 + \frac{C}{r}) \tag{3.50}$$

and then:

$$B(r) = (1 + \frac{C}{r})^{-1} \tag{3.51}$$

Therefore, A(r) and B(r), have the form:

$$A(r) = 1 + \frac{C}{r} \tag{3.52}$$

$$B(r) = (1 + \frac{C}{r})^{-1} \tag{3.53}$$

where C is a constant and  has been absorbed in the time coordinate.

To calculate the constant C, we apply Gauss's law:

$$\int g ds_2 = -4\pi GM \tag{3.54}$$

where g is the intensity of the gravitational field and $ds_2 = r^2 sin\theta d\theta d\phi$ Therefore equation (51) becomes:

$$gr^2 \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\phi = -4\pi GM \Rightarrow g = -\frac{GM}{r^2} \tag{3.55}$$

In order, to find the potential and subsequently the constant C, we use the well-known relation $g = -\nabla\Phi$ and then:

$$\Phi = \int \frac{GM}{r^2} dr = -\frac{GM}{r} \tag{3.56}$$

Taking as an approximation the limit of the weak gravitational field where $A(r) = -1 + 2\Phi$ we have:

$$2\Phi == \frac{C}{r} \Rightarrow \frac{-2GM}{r} = \frac{C}{r} \Rightarrow C = -2GM \tag{3.57}$$

Thus, for equations (2.52) and (2.53), we have:

$$A(r) = 1 - \frac{2GM}{r} \equiv (1 - \frac{r_H}{r}) \tag{3.58}$$

$$B(r) = (1 - \frac{2GM}{r})^{-1} \equiv (1 - \frac{r_H}{r})^{-1} \tag{3.59}$$

where 2GM is the Schwarzschild radius,$r_H = 2GM$ in four dimensions. After all we get the Schwarzschild metric:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{3.60}$$

At the radial coordinate $r = 2GM$, the term $12GM/r$ becomes zero, leading to a coordinate singularity. This is the event horizon, beyond which nothing can escape the gravitational pull of the black hole.At $r = 0$ we have the real spacetime singulaty because the , the curvature invariants become infinite and near $r = 0$.

In this section we saw the simplest solution of Einstein equation of motion which describe a non rotating and electrically neutral black hole.But General relativity includes and other solutions for black holes with charge(NOrdstromm) and angular momentum(Kerr).In the next section we will study the Kerr black hole in four dimensions.

## 3.3 Kerr Black hole

Another black hole in asymptotically flat vacuum spacetime is the rotating Kerr black hole. This solution does not have spherical but axial symmetry around its axis of rotation $\theta = 0, \pi$. The Kerr metric is given by:

$$\begin{aligned} ds^2 = -(1 - \frac{2GMr}{\rho^2})dt^2 - \frac{2GMarsin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 \\ +\rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2}((r^2 + a^2)^2 - a^2\Delta sin^2\theta)d\phi^2 \end{aligned} \tag{3.61}$$

with

$$\Delta(r) = r^2 - 2GMr + a^2 \tag{3.62}$$

and

$$\rho^2(r, \theta) = r^2 + a^2cos^2\theta \tag{3.63}$$

The used coordinates $t, r, \theta, \phi$ are called Boyer-Lindquist coordinates. M is again the mass of the source and a is the angular momentum per unit mass. When M $\rightarrow$ 0 flat spacetime in ellipsoidal coordinates is recovered, and as a $\rightarrow$ 0 the metric reduces to the Schwarzschild metric. The Kerr metric describes a stationary solution; the black hole rotates in exactly the same way for all time. The metric components are independent of t but the metric is not time-reversal invariant. When time is reversed the black hole will spin the other way around.

For this metric the curvature scalar $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ diverges at $\rho = 0$, so at that point there is a curvature singularity. $\rho$ only vanishes when both $r = 0$ and $\theta = \phi/2$ this describes a ring on the edge of the disk $r = 0$ (as can be seen by plugging $t = r = 0$ and $\theta = \phi/2$ into the metric).

When searching for event horizons, we again look for those values of r for which $g^{rr} = 0$s (we have chosen the right coordinates to use this method).Here $g^{rr} = \Delta/\rho^2$,and since $^2 \geq 0$ we want:

$$\Delta(r) = r^2 - 2GMr + a^2 = 0 \tag{3.64}$$

There are three possibilities:$GM < a$,$GM = 0$ and $GM > a$.The first one describes a naked singularity; a singularity without an event horizon around it.The second one is an unstable solution.We will focus on the case $GM > a$a because this one is of the most physical interest.It has been proven that for$GM > a$ the Kerr metric is the unique stationary solution to Einstein's equation in asymptotically flat vacuum spacetime.The mass and angular momentum of the system uniquely determine the stationary solution. The two event horizons are in this case:

$$r_{H_\pm} = GM \pm \sqrt{G^2M^2 - a^2} \tag{3.65}$$

A new feature of rotating black holes is the existence of an ergosphere; a space around the black hole where it is impossible to stand still. A necessary condition for an observer to stand still is that its world line, his path in space time, is timelike. Suppose we try to stay at one fixed point of the coordinate system, the world line is then:

$$X^\mu(t) = (t, r_0, \theta_0, \phi_0) \tag{3.66}$$

and the corresponding tangent vector is

$$T^\mu = \frac{dX^\mu}{dt} = (1, 0, 0, 0) \tag{3.67}$$

For the world line to be timelike we need

$$g_{\mu\nu}T^\mu T^\nu = g_{tt} < 0 \tag{3.68}$$

or specifically for the Kerr metric

$$g_{tt} = -(1 - \frac{2GMr}{\rho^2}) < 0 \tag{3.69}$$

The solutions to $g_{tt} = 0$ are

$$r_{E\pm} = GM \pm \sqrt{G^2M^2 - a^2cos^2\theta} \tag{3.70}$$

and are called stationary limit surfaces.

## 3.4 Schwarzchild solution in higher dimemsions

In this chapter we will deal with the generalization of the Schwarzchild black hole in higher dimensions. Schwarzchild black hole is a static and sperically symmetric solution of Einstein field equation.The static property states that all components are time independent and there are no space-time cross terms ( $dtdx^i + dx^i dt$).A first, basic problem is how we can measure the mass and angular momentum of the solutions. Black holes in general are solutions to the Einstein equations that do not have any sources of mass; the matter stress tensor is zero. However, we can also identify the mass and angular momenta of isolated systems (such as black holes) from the asymptotic behavior of their gravitational field.The asymptotic behavior of an (N+1)-dimesional metric means that the system is weakly gravitating($r \gg r_H$) and the metric is only modified from flat spacetime value:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.71}$$

with $|h_{\mu\nu}| \ll 1$.The metric inverse is:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{3.72}$$

On the other hand,the condition that the system is non-relativistic means the time derivatives can be considered much smaller than spatial derivatives which implies that components of the stress energy tensor may be ordered $|T_{00}| \gg |T_{0i}| \gg |T_{ij}|$.This relation tell us that the dominant source for the gravitational field is the energy density and that the momentum density provides the next most important contribution.

For the principle of least action in higher dimensional Einstein-Hilbert action we get:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{3.73}$$

From the above two relations and using the harmonic gauge condition:

$$(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h^{\alpha}_{\alpha})_{,\nu} = 0 \ , \tag{3.74}$$

we get the e.o.m for $h_{\mu\nu}$:

$$\nabla^2 h_{\mu\nu} = -16\pi G(T_{\mu\nu} - \frac{1}{N-1}\eta_{\mu\nu}T) = -16\pi G\bar{T}_{\mu\nu} \tag{3.75}$$

where $T = T^{\mu}_{\mu} \approx -T_{00}$ because as we said before the dominant source for the gravitational field is the energy density.The Green's function for the N-dimensional Laplacian may be used to solve for $h_{\mu\nu}$:

$$h_{\mu\nu}(x^i) = \frac{16\pi G}{(N-2)A_{N-1}} \int \frac{\bar{T}_{\mu\nu}}{|x-y|^{N-2}} d^N y \qquad (3.76)$$

where $A_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$:is the (N-1)-sphere area.

Expanding (3.76) in the asymptotic region far from any sources with $r = |x| \gg |y|$ gives:

$$h_{\mu\nu}(x^i) = \frac{16\pi G}{(N-2)A_{N-1}} \frac{1}{r^{N-2}} \int \bar{T}_{\mu\nu} d^N y + \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \int y^k \bar{T}_{\mu\nu} d^N y + ... \Rightarrow$$

$$\Rightarrow h_{\mu\nu}(x^i) = \frac{16\pi G}{(N-2)A_{N-1}} \frac{1}{r^{N-2}} \int (T_{\mu\nu} - \frac{1}{N-1}\eta_{\mu\nu} T) d^N y + \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \int y^k (T_{\mu\nu} - \frac{1}{N-1}\eta_{\mu\nu} T) d^N y + ...$$

with $\int T_{00} d^N x = M$ ,$\int T_{0i} d^N x = 0$,$\int x^k T_{00} d^N x = 0$,$J^{0k} = 0$ and $J^{kl}$

From the above relations we can calculate the components of $h_{\mu\nu}$:

$$h_{00} \approx \frac{16\pi G}{(N-2)A_{N-1}} \frac{1}{r^{N-2}} \int (T_{00} - \frac{1}{N-1}\eta_{00} T_{00}) d^N y + \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \int y^k (T_{00} - \frac{1}{N-1}\eta_{00} T_{00}) d^N y$$

$$\approx \frac{16\pi G}{(N-2)A_{N-1}} \frac{1}{r^{N-2}} \int (\frac{(N-1)T_{00} - T_{00}}{N-1}) d^N y \approx \frac{16\pi G}{(N-2)A_{N-1}} \frac{1}{r^{N-2}} \int (\frac{(N-2)T_{00}}{N-1}) d^N y$$

$$\approx \frac{16\pi G}{(N-1)A_{N-1}} \frac{1}{r^{N-2}} \int T_{00} d^N y \approx \frac{16\pi G}{(N-1)A_{N-1}} \frac{M}{r^{N-2}}$$

where M is the mass of the higher dimensional black hole.

$$\begin{aligned} h_{0i} \approx & \frac{16\pi G}{(N-2)A_{N-1}} \frac{1}{r^{N-2}} \int (T_{0i} - \frac{1}{N-1}\eta_{0i} T_{0i}) d^N y \\ & + \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \int y^k (T_{0i} - \frac{1}{N-1}\eta_{0i} T_{0i}) d^N y \\ & \approx \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \int y^k (T_{0i} - \frac{1}{N-1}\eta_{0i} T_{0i}) d^N y \\ & \approx \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \int y^k T_{0i} - \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \int \frac{1}{N-1}\eta_{0i} T_{0i} d^N y \\ & \approx \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \int -y^k T^{0i} \approx \frac{16\pi G}{A_{N-1}} \frac{x^k}{r^N} \frac{J^{ki}}{2} \approx \frac{-8\pi G}{A_{N-1}} \frac{x^k}{r^N} J^{ki} \end{aligned} \qquad (3.77)$$

With the same way we have:

$$h_{ij} \approx \frac{16\pi G}{(N-2)(N-1)A_{N-1}} \frac{M}{r^{N-2}} \delta_{ij} \tag{3.78}$$

From (2.1) and (2.7) we get:

$$g_{00} = \eta_{00} + h_{00} = -1 + \frac{16\pi G}{(N-1)A_{N-1}} \frac{M}{r^{N-2}} \tag{3.79}$$

These results may be used to define the mass for the Schwarzchild black hole in its center of mass frame.

The general line element of the higher dimensional Schwarzchild black hole is:

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + r^2 d\Omega^2_{N-1} \tag{3.80}$$

where r is a radial coordinate, $\Omega^2_{N-1}$ is the line element on the unit N-sphere, and f and g are functions of $r$ only. The vacuum Einstein equations then imply that:

$$f = g^{-1} = (1 - \frac{C}{r^{N-2}})^{1/2} \tag{3.81}$$

Correlating the relation (2.9) with the (2.11) we come to the conclusion that:

$$C = \frac{16\pi GM}{(N-1)A_{N-1}} \tag{3.82}$$

Consequently we have:

$$f = g^{-1} = (1 - \frac{\frac{16\pi GM}{(N-1)A_{N-1}}}{r^{N-2}})^{1/2} = (1 - \frac{\frac{16\pi GM}{(N-1)\frac{2\pi^{N/2}}{\Gamma(N/2)}}}{r^{N-2}} = (1 - \frac{16\pi GM\Gamma(N/2)}{(N-1)r^{N-2}2\pi^{N/2}})^{1/2}$$

$$= (1 - \frac{16\pi GM\Gamma(n+3/2)}{(n+2)r^{n+1}2\pi^{(n+3)/2}})^{1/2} = (1 - \frac{8\pi GM\Gamma((n+3)/2)}{(n+2)r^{n+1}\pi^{(n+3)/2}})^{1/2} \tag{3.83}$$

where n is the extra spatial dimensions and $N = n+3$.Now we know that the higher dimensional Newton constant can be written functions of the mass Planck in higher dimensions like that $G = \frac{1}{M_\star^{2+n}}$.Therefore,from relation (2.13) we get:

$$f = g^{-1} = 1 - \frac{8\pi \frac{M}{M_\star^{2+n}} \Gamma((n+3)/2)}{(n+2)r^{n+1}\pi^{(n+3)/2}} \tag{3.84}$$

$$= 1 - \frac{(8\pi \frac{M}{M_\star^{2+n}} \Gamma((n+3)/2)/(n+2)\pi^{(n+3)/2})^{\frac{n+1}{n+1}}}{r^{n+1}} \tag{3.85}$$

Therefore we have:

$$f = g^{-1} = 1 - (\frac{r_H}{r})^{n+1} \tag{3.86}$$

where

$$r_H =(\frac{8\pi\Gamma(\frac{n+3}{2})}{n+2})^{\frac{1}{n+1}}(\frac{M}{M_\star^{2+n}})^{\frac{1}{n+1}}(\frac{1}{\pi^{\frac{n+3}{2}}})^{\frac{1}{n+1}}$$
$$= (\frac{8\Gamma(\frac{n+3}{2})}{n+2})^{\frac{1}{n+1}}(\frac{M}{M_\star^{2+n}})^{\frac{1}{n+1}}(\frac{1}{\pi^{\frac{n+1}{2}}})^{\frac{1}{n+1}}$$

$$= (\frac{8\Gamma(\frac{n+3}{2})}{n+2})^{\frac{1}{n+1}}\frac{(M)^{\frac{1}{n+1}}}{M_\star^{\frac{1}{n+1}}(M_\star^{1+n})^{\frac{1}{n+1}}}(\frac{1}{\pi^{\frac{n+1}{2}}}) =$$
$$\frac{1}{\sqrt{\pi}M_\star}(\frac{M}{M_\star})^{\frac{1}{n+1}}(\frac{8\Gamma(\frac{n+3}{2})}{n+2})^{\frac{1}{n+1}} \tag{3.87}$$

Therefore ,the higher dimensional Schwarzchild metric is given by:

$$ds^2 = -[1-(\frac{r_H}{r})^{n+1}]dt^2 + [1-(\frac{r_H}{r})^{n+1}]^{-1}dr^2 + r^2 d\Omega^2_{2+n} \tag{3.88}$$

where

$$r_H = \frac{1}{\sqrt{\pi}M_\star}(\frac{M}{M_\star})^{\frac{1}{n+1}}(\frac{8\Gamma(\frac{n+3}{2})}{n+2})^{\frac{1}{n+1}} \tag{3.89}$$

After all these,we can calculate the size of (4+n)-dimensional black hole and compare with the size of 4 dimensional analog.The radius of 4-dimensional black hole is given by (2.18) if we put the n=0:

$$r_H \sim \frac{1}{M_{pl}}(\frac{M}{M_{pl}}) \sim \frac{M}{M_{pl}^2} \sim \frac{M}{M_\star^2(M_\star R)^2} \tag{3.90}$$

where we use: $M_{pl}^2 \approx R^n M_\star^{n+2}$ On the other hand the size of (4+n)-dimensional B.H is given by:

$$r_H \sim \frac{1}{M_\star}(\frac{M}{M_\star})^{\frac{1}{n+1}} \tag{3.91}$$

From (3.89) and (3.90) we relate the two radii:

$$r_{H(4)}R^n M_\star^n \sim r_{H(4+n)}^{n+1}M_\star^n \Rightarrow r_{H(4)}R^n M_\star^n \sim r^n_{H(4+n)}r_{H(4+n)}M_\star^n$$
$$\Rightarrow \frac{r_{H(4)}}{r_{(4+n)}} \sim \frac{r^n_{(4+n)}M_\star^n}{R^n M_\star^n} = \frac{r^n_{(4+n)}}{R^n} = (\frac{r_{H(4+n)}}{R})^n$$
$$\Rightarrow r_{H(4)} \leq r_{H(4+n)} \leq R \tag{3.92}$$

because we know that: $r_{H(4+n)} \leq R$ .The Hawking temperature $T_{4+n}$ of a (4+n)-dimensional Schwarzchild black hole can be easily estimated from the first law of black hole thermodynamics.Because the higher dimensional black hole is larger than 4-d analog we expect that the there are implications in temperature and lifetime of higher dimensional black hole in relation to 4D analog.

Consider a static observer at radius $r_1 > r_H$ outside of (4+n)-dimensional Schwarzchild black hole.Such an observer moves along orbits of the timelike killing vector $k = \partial_t$. For our occasion we have the timelike vector in the following:

$$k^\mu = (1, 0, 0, 0) \tag{3.93}$$

and the four-velocity in this form:

$$U^\mu = [(1 - (\frac{r_H}{r})^{n+1}))^{-1/2}, 0, 0, 0] \tag{3.94}$$

So,we know for the 4 dimensional process that the redshift factor given by:

$$V = \sqrt{1 - (\frac{r_H}{r})} \tag{3.95}$$

Also we have the relations for four-acceleration,magnitude of four-acceleration and surface gravity which is associated with the black hole horizon:

$$\alpha_\mu = \nabla_\mu lnV \tag{3.96}$$

$$\alpha = \sqrt{\alpha_\mu \alpha^\mu} = V^{-1}\sqrt{\nabla_\mu V \nabla^\mu V} \tag{3.97}$$

$$\kappa = V\alpha = \sqrt{\nabla_\mu V \nabla^\mu V} \tag{3.98}$$

We use all that to higher dimensional case and we get:

$$V = \sqrt{1 - (\frac{r_H}{r})^{n+1}}$$

$$\Rightarrow \nabla_\mu V = \frac{(n+1)(\frac{r_H}{r})^n(\frac{r_H}{r})'}{2\sqrt{1-(\frac{r_H}{r})^{n+1}}} = \frac{(n+1)\frac{r_H^{n+1}}{r_H^{n+2}}}{2\sqrt{1-(\frac{r_H}{r})^{n+1}}}$$

$$\Rightarrow \nabla_\mu V \nabla^\mu V = \frac{(n+1)^2(r_H^{n+1})^2}{4(r^{n+2})^2(1-(\frac{r_H}{r})^{n+1})} \tag{3.99}$$

$$\alpha = V^{-1}\sqrt{\nabla_\mu V \nabla^\mu V} = \frac{1}{\sqrt{1-(\frac{r_H}{r})^{n+1}}}\frac{(n+1)r_H^{n+1}}{2r^{n+2}\sqrt{1-(\frac{r_H}{r})^{n+1}}} \Rightarrow$$

$$\Rightarrow V\alpha = \frac{(n+1)r_H^{n+1}}{2r^{n+2}} \tag{3.100}$$

From above equation,if we put $r = r_H$ we have the surface gravity:

$$\kappa = \frac{(n+1)}{2r_H} \tag{3.101}$$

Finally we can write the expression of Hawking temperature of (4+n)-dimensional Schwarzchild black hole as follows:

$$T = \frac{\kappa}{2\pi} = \frac{n+1}{4\pi r_H} \tag{3.102}$$

We know that the entropy of the black hole is propotional to the area of the event horizon:

$$S = \frac{A}{4G} \rightarrow S_{(4+n)} = \frac{A_{(4+n)}}{4G_D} = \frac{2\pi^{\frac{n+3}{2}}}{4G_D\Gamma(\frac{n+3}{2})}[\frac{1}{M_\star^{n+1}}(\frac{M}{M_\star})^{\frac{1}{n+1}}(\frac{2^n\pi^{\frac{(n-3)}{2}}\Gamma(\frac{n+3}{2})}{n+2})]^{n+2} =$$

$$= \frac{2\pi^{\frac{n+3}{2}}}{4G_D\Gamma(\frac{n+3}{2})}\frac{1}{M_\star^{n+1}}(\frac{M}{M_\star})(\frac{2^n\pi^{\frac{(n-3)}{2}}\Gamma(\frac{n+3}{2})}{n+2})r_H = \frac{2^{n+1}\pi^n}{4G_D}\frac{M}{M_\star^{n+2}}\frac{r_H}{n+2} =$$

$$= \frac{2^{n+1}\pi^n}{\frac{(2\pi)^n}{2\pi}}\frac{M}{n+1}r_H = \frac{2(2\pi)^{n+1}M}{(2\pi)^n(n+2)}r_H = \frac{4\pi M}{n+2}r_H \Rightarrow$$

$$.S_{(4+n)} = \frac{4\pi M}{n+2}r_H \tag{3.103}$$

## 3.5 Myers-Perry Black Hole

In 3 + 1 dimensions an uncharged black hole is completely characterized by two parameters:

- 1)Mass
- 2)Angular momentum

The situation in higher dimensions is little more complicated.We consider the Poincare group which in higer dimensions include the space-tme translations and Lorrentz boosts represented by SO(N,1) group.In this case the parameters are:

- 1)Mass
- 2)[N/2] invariants(Casimir invariants) of the little group SO(N) which represent the different angular momentums.

In other words,the higher dimensional spinning black hole which named Myers-Perry black hole(from the physicists who discover this solution) is very complicated because it can be rotating in many different planes and has many different angular momentums. The general metric solution for this higher dimensional spinning black hole can be written in Kerr-Schild form:

$$g_{\mu\nu} = \eta_{\mu\nu} + hk_\mu k_\nu \tag{3.104}$$

where $k_\mu$ is a null vector field.It is null with respect to the flat metric $\eta_{\mu\nu}$ and hence $g_{\mu\nu}$

$$k^\mu = g^{\mu\nu}k_\nu = \eta^{\mu\nu}k_\nu \tag{3.105}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - hk^\mu k^\nu \tag{3.106}$$

Odd and even dimensional cases have seperate solutions.

- For even :

$$k_\mu dx^\mu = dt + \frac{r(x^i dx^i + y^i dy^i) + a_i(x^i dy^i - y^i dx^i)}{r^2 + a_i^2} \tag{3.107}$$

and

$$h = \frac{\mu r^2}{\Pi F} \tag{3.108}$$

where

$$F = 1 - \frac{a_i^2(x_i^2 + y_i^2)}{(r^2 + a_i)^2} \tag{3.109}$$

$$\Pi = \prod_{i=1}^{(N-1)/2} (r^2 + a_i^2) \tag{3.110}$$

where i=1,2,......,N/2.

The condition that $k^\mu$ is null and can be written as:

$$k^\mu k_\mu = 0 \Rightarrow \frac{(x_i^2 + y_i^2)}{(r^2 + a_i)^2} = 1 \tag{3.111}$$

This condition defines the radial coordinate.With the same way

- For odd N:

$$k_\mu dx^\mu = dt + \frac{r(x^i dx^i + y^i dy^i) + a_i(x^i dy^i - y^i dx^i)}{r^2 + a_i^2} + \frac{zdz}{r} \tag{3.112}$$

and

$$h = \frac{\mu r}{\Pi F} \tag{3.113}$$

with F a $\Pi$ are the same as previous.The condition

$$k^\mu k_\mu = 0 \Rightarrow \frac{(x_i^2 + y_i^2)}{(r^2 + a_i)^2} + \frac{z^2}{r^2} = 1 \tag{3.114}$$

defines the radial cordinate in that case.For N=3 we get the familiar Kerr metric but in this chapter we are looking for $N > 3$ solutions.

Now next up,we want to determine the mass and the angular momentum of the new metric. Examining the asymptotic form shows they are not in a form suitable to compare with(2.7).Therefore a transformation to a set of Boyer-Lindquist coordinates will be made where the desired quantities can be determined.

First for even N, angular coordinates are introduced with the following transformation:

$$x^i = \sqrt{(r^2 + a_i^2)}\mu_i \cos[\phi_i - \tan^{-1}(\frac{a_i}{r})] \tag{3.115}$$

$$y^i = \sqrt{(r^2 + a_i^2)}\mu_i \sin[\phi_i - \tan^{-1}(\frac{a_i}{r})] \tag{3.116}$$

Then the metric becomes:

$$ds^2 = -dt^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i d\phi_i^2) + 2\mu_i^2 a_i d\phi dr + \frac{\mu r^2}{\Pi F}(dt + dr + a_i \mu_i^2 d\phi_i)^2 \tag{3.117}$$

with

$$F = 1 - \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \tag{3.118}$$

.The first four terms are actually a metric on flat space, and the remaining term involves a null vector field squared. This metric is constructed here simply to introduce angular coordinates. Note that the $p_i$ are not all independent such

$$p_i = 1 \tag{3.119}$$

Now we can use these tranformations:

$$d\bar{t} = dt - \frac{\Pi F}{\Pi - \mu r^2} dr \tag{3.120}$$

$$d\bar{\phi}_i = d\phi_i + \frac{a_i dr}{r^2 + a_i^2} \frac{\Pi}{\Pi - \mu r^2} \tag{3.121}$$

and the metric becomes:

$$ds^2 = -d\bar{t}^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i d\bar{\phi}_i^2) + \frac{\mu r^2}{\Pi F}(d\bar{t}^2 + a_i \mu_i^2 d\bar{\phi}_i^2)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 \tag{3.122}$$

which is the metric an odd D-dimensional spinning black hole(or N even)

These steps are slightly modified for odd N. Angular coordinates are introduced as in (3.12-13) with the additional transformation:

$$z = ra \tag{3.123}$$

with $-1 < a < 1$ we have:

$$ds^2 = -dt^2 + dr^2 + r^2 da^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i d\phi_i^2) + 2\mu_i^2 a_i d\phi dr + \frac{\mu r}{\Pi F}(dt + dr + a_i \mu_i^2 d\phi_i)^2 \tag{3.124}$$

with i=1,....,(N-1)/2 and $\mu_i^2 + a = 1$

The transformation to BL coordinates for odd N is :

$$d\bar{t} = dt - \frac{\Pi F}{\Pi - \mu r} dr \tag{3.125}$$

and

$$d\bar{\phi_i} = d\phi_i + \frac{a_i dr}{r^2 + a_i^2} \frac{\Pi}{\Pi - \mu r} \tag{3.126}$$

yields

$$ds^2 = -d\bar{t} + r^2 da^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i d\bar{\phi_i^2}) + \frac{\mu r}{\Pi F}(d\bar{t}^2 + a_i \mu_i^2 d\bar{\phi_i^2})^2 + \frac{\Pi F}{\Pi - \mu r} dr^2 \tag{3.127}$$

which is the metric of even D-dimensional spinning black hole(or N odd).This new transformed metric have a suitable asymptotic form to compare to previous relation:

$$h_{00} \approx \frac{16\pi GM}{(N-1)A_{N-1} r^{N-2}} \tag{3.128}$$

One finds $g_{00} \approx -1 + \frac{\mu}{r^(N-2)}$ for both odd and even metric.Therefore,the black hole mass and angular momentum for both cases are:

$$M = \frac{(N-1)A_{N-1}}{16\pi G} \mu \tag{3.129}$$

and

$$J = \frac{A_{N-1}}{8\pi G} \mu = \frac{2}{N-1} M a_i \tag{3.130}$$

The $x^i - y^i$ planes are the planes in which the black hole is spinning.When the spin parameters $a_i$are zero,the previous metric reduces to the D-dimensional Schwarzchild-Tangherlini metric.When both $a_i = 0$ and $\mu = 0$ the flat metric is recovered.

We will now focus on the solution in five dimensions(N=4) with n=2d+1,$d \geq 2$ which n=dimensions.Our case is d=2(i=1,2) and in this case there are two Casimir invariant and therefore two angular momentum parameters($a_1, a_2$).We choose for convinience $a_1 = a, a_2 = b, \mu_1 = sin\theta, \mu_2 = cos\theta, \phi_1 = \psi, \phi_2 = \phi$.Then the Myers-Perry metric in five dimensions is given by:

$$ds^2 = -d\bar{t}^2 + \frac{\mu}{\Sigma}(d\bar{t} + asin^2\theta d\psi + bcos^2\theta d\phi)^2 + \frac{r^2\Sigma}{\Pi - \mu r^2} dr^2 + \Sigma d\theta^2 + (r^2+a^2)sin^2\theta d\psi^2 + (r^2+b^2)cos^2\theta d\phi^2 \tag{3.131}$$

where $\Sigma = r^2 + a^a cos^2\theta + b^2 sin^2\theta$ and $\Pi = (r^2 + a^2)(r^2 + b^2)$.

We see that various metric components diverge for $\Sigma = 0$ and $\Pi - \mu r^2 = 0$, and suspect this metric contains singularities, event horizons and ergospheres. The situation will be different depending on whether a spin parameter vanishes or not. We will go over all possibilities now.

**Singularities** To find a singularity we examine where the curvature scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges. It is explicitly given by:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{24\mu^2}{\Sigma^6}(4r^2 - 3\Sigma)(4r^2 - \Sigma) \tag{3.132}$$

- When one of the spin parameters vanishes, say b = 0, the curvature scalar becomes:

  $$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}|_{b=0} = \frac{24\mu^2}{(r^2 + a^2cos^2\theta)^6}(r^2 - 3a^2cos^2\theta)(3r^2 - a^2cos^2\theta) \tag{3.133}$$

  At $r = 0$ we see that the curvature scalar diverges for $\theta \to \pi/2$.If we choose a=0 the singularity would be at r=0,$\theta = 0$.

- When neither spin parameter vanishes($a, b \neq 0$),the curvature scalar remains finite.But we can introduce the coordinate change $\rho = r^2$ and explore negative values of $r^2$.When $a = b$ the curvature scalar becomes

  $$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}|_{b=0} = \frac{24\mu^2}{(r^2 + a^2)^6}(r^2 - 3a^2)(3r^2 - a^2) \tag{3.134}$$

  and we see the entire surface $\rho = -a^2$ is singular. If $a < b$ the curvature scalar diverges for $\Sigma = 0$, or

  $$\sin^2\theta = \frac{-\rho - a^2}{b^2 - a^2} \tag{3.135}$$

  where $\rho$ is required to stay in the domain $-b^2 \leq \rho \leq -a^2$

**Event Horizons** The event horizon is determined by the condition $g^{rr} = 0$ or in this specific case:

$$\Pi - \mu r^2 = (r^2 + a^2)(r^2 + b^2) - \mu r^2 = 0 \tag{3.136}$$

This is a quadratic equation in $r^2$ and its solution is

$$r^2_{H\pm} = \frac{1}{2}(\mu - a^2 - b^2 \pm \sqrt{(\mu - a^2 - b^2)^2 - 4a^2b^2} \tag{3.137}$$

We want real solutions,so we require that

$$\mu \geq a^2 + b^2 + 2ab \tag{3.138}$$

- $b = 0$:In this case,the condition becomes $\mu \geq a^2$ and the event horizon are:

  $$r^2_{H+} = \mu - a^2 \geq 0 \tag{3.139}$$

  and

  $$r^2_{H-} = 0 \tag{3.140}$$

- $a, b \neq 0$ :We can extend the coordinates to negative values of  in this case and that the singularities are in that region. To avoid naked singularities we need $\rho_H = r^2_{H\pm} > -a^2$.The mass condition ensures that $\rho_H > 0$ thus there are no naked singularities[].We can rewrite the mass condition(3.35) in terms of the (real) mass M and angular momenta J like that:

  $$M^3 \geq \frac{27\pi}{32G}(J^2_\psi + J^2_\phi + 2J_\psi J_\phi) \tag{3.141}$$

  Just like the Kerr black hole the angular momentum is restricted by the mass. In dimensions greater than five this restriction is not present and there are event horizons for arbitrarily large angular momentum. These are called ultra-spinning black holes.

We can now choose a new coordinate frame, co–rotating with the black hole horizon, to eliminate the dragging motion on the rotating degrees of freedom of a tunneling particle by using the following coordinate transformations:

$$d\bar{\phi} = d\acute{\phi} + \Omega_a \bar{d}t \tag{3.142}$$

$$d\bar{\psi} = d\acute{\psi} + \Omega_b \bar{d}t \tag{3.143}$$

in which the corresponding angular momentum velocities is given by:

$$\Omega_a = \frac{a}{r^2 + a^2} \tag{3.144}$$

and

$$\Omega_b = \frac{b}{r^2 + b^2} \tag{3.145}$$

Then the metric becomes:

$$ds^2 = -G_{tt}(r, \theta, \phi, \psi)dt^2 + \frac{r^2}{\Pi F - \mu r^2}dr^2 + \Sigma d\theta^2 + [(r^2 + a^2 + \frac{\mu a^2 sin^2\theta}{\Sigma^2} sin^2\theta d\acute{\phi}^2] +$$
$$+ [(r^2 + b^2 + \frac{\mu b^2 cos^2\theta}{\Sigma^2} cos^2\theta d\acute{\psi}^2] + \frac{2ab\mu}{\Sigma^2} sin^2\theta cos^2\theta d\acute{\phi} d\acute{\psi} \tag{3.146}$$

We know that in the coordinate wich co-rotate with event horizon $g_{t\acute{\phi}} = g_{t\phi} - g_{\phi_i \phi_j}\Omega_j = 0$ at the horizon

$$g_{t\acute{\phi}_i} = 0 \tag{3.147}$$

After that we have:

$$G_{tt}(r_+) = g_{tt} + g_{t\phi}\Omega_a + g_{t\psi}\Omega_b \tag{3.148}$$

with

$$g_{rr} = \frac{r^{2\mu}}{\Pi F - \mu r^2} \tag{3.149}$$

$$g_{tt} = 1 - \frac{\mu}{\Sigma^2} \tag{3.150}$$

$$g_{t\phi} = \frac{a\mu \sin^2\theta}{\Sigma^2} \tag{3.151}$$

$$g_{t\psi} = \frac{b\mu \cos^2\theta}{\Sigma^2} \tag{3.152}$$

Finally,the relation of the Hawking temperature for 5D Myers-Perry black hole is given by:

$$T_H = \frac{\kappa(r_+)}{2\pi} = \lim_{r\to r_+} \frac{\partial_r \Pi - 2\mu r}{4\pi\mu r^2} \tag{3.153}$$

Also,we can derive the expression of the entropy for that case.The area of the event horizon of MP black hole is(for odd number of dimensions):

$$A = \frac{\Omega_{d-2}}{r_+}\Pi(r_i^2 + a_i^2) \to^{d=5}\to A = \frac{2\pi^2}{r_+}(r^2+a^2)(r^2+b^2) \tag{3.154}$$

So the Bekenstein-Hawking entropy for MP black hole is given by:

$$S = \frac{2\pi^2}{4G_D r_+}(r^2+a^2)(r^2+b^2) \tag{3.155}$$

Consider the spinning black hole solutions with single nonvanishing spin parameter:

$$\begin{aligned} ds^2 = (1 - \frac{\mu}{\Sigma r^{n-1}})dt^2 + \frac{2a\mu\sin^2\theta}{\Sigma r^{n-1}}dt\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 \\ -(r^2 + a^2 + \frac{a^2\mu\sin^2\theta}{\Sigma r^{n-1}})\sin^2\theta d\phi^2 - r^2\cos^2\theta d\Omega_n^2 \end{aligned} \tag{3.156}$$

with $\Delta = r^2 + a^2 - \frac{\mu}{r^{n-1}}$ and $\Sigma = r^2 + a^2 cos^2\theta$ Again

$$M = \frac{(n+2)A_{n+2}}{16\pi G}\mu \tag{3.157}$$

and

$$J = \frac{2}{n+2}Ma \tag{3.158}$$

The line element (3.155) describes successfully a small higher dimensional rotating black hole in spin down phase.After the end of this phase when all the angular momentum of black hole vanished($a \to 0$) the black hole is described by the Schwarzschil-Tagherlini line element.

The black hole horizon is given by:

$$\Delta(r) = 0 \Rightarrow ..... \Rightarrow r_H^{n+1} = \frac{\mu}{1+a_\star^2} \tag{3.159}$$

with $a_\star = \frac{a}{r_H}$.

The Hawking temperature of the (n+4)-dimensional rotating black hole is found to be:

$$T_H = \frac{(n+1) + (n-1)a_\star^2}{4\pi(1 + a_\star^2)r_H} \tag{3.160}$$

In the limit $a \to 0 \Rightarrow a_\star \to 0$ we get the relation for Hawking temperature for static black hole we found previously

# Chapter 4

# The Dark Dimension and the Dark Matter fraction composed of Rotating Primordial Black Holes

## 4.1 The Dark Dimension

In previous chapters, we introduced the necessary tools that we'll need to study a new scenario proposed by Luis A. Anchordoqui, Ignatios Antoniadis and Dieter Lust[16]. This scenario attempts to explain the nature of dark matter in the universe. Until recently, several scenarios had been proposed for what this mysterious dark matter could be, with the most prevalent being supersymmetry, which suggests that dark matter is composed of supersymmetric particles. Physicists in the past decade believed it was only a matter of time until these particles appeared in the Large Hadron Collider (LHC) at CERN. However, this has not happened, at least not yet. The absence of these particles has led physicists to search for new scenarios regarding the nature of dark matter.

In this chapter, we'll examine a very recent scenario for the nature of dark matter. This scenario suggests that dark matter comprises essentially 5-dimensional primordial black holes. This hypothesis falls within the framework of a theoretical model called the dark dimension. According to this model, our universe is a 4-dimensional brane embeded in a 5-dimensional space. This fifth dimension has a size on the order of a micrometer. Let's now look in more detail at this scenario and how, and to what extent, it can explain the amount of dark matter we observe in our universe.

Recently, it has been proven that by combining the cosmological hierarchy problem, i.e., the smallness of dark energy in Planck units, and the distance conjecture, we deduce that the extra dimension is on the order of a micron.As we explained in detail in a previous chapter, the distance conjecture essentially predicts the existence of infinite towers of states that are massless at an infinite distance in moduli space, which means that the EFT (Effective Field Theory) description breaks down at infinite distance in moduli space.

On the other hand,the anti-de Sitter (AdS) distance conjecture [3], it suggests that there should be an infinite tower of states, whose mass is related to the magnitude of the cosmological constant.More precisely, the mass scale m behaves as $m \sim |\Lambda|^a$, as the negative AdS vacuum energy $\Lambda \to 0$, with a positive constant of $a$ O(1).

At this point, we should mention that m essentially represents the mass scale of the tower of states, while $M_*$ is a limit beyond which the EFT (Effective Field Theory) description breaks down. This is called the species scale and corresponds to the Planck mass in higher dimensions, given by the following relation $M_* = m^{n/(n+2)} M_{Pl}^{2/(n+2)}$ where $M_{Pl}$ is the Planck mass in 4 dimensions and n is the number of effective dimensions decompactifying.

When we reconcile the experimental constraints on how much gravity might deviate from Newton's inverse-square law [55] with the theoretical bounds imposed by swampland conjectures, it leads us to set $a = 1/4$. This implies that the characteristic mass of the Kaluza-Klein (KK) modes forming the tower is estimated as $m \sim \lambda^{-1}\Lambda^{1/4}$ . Moreover, observations like the heating of neutron stars [17] are consistent with n=1 [8], and the sharp upper limit observed in the cosmic ray spectrum suggests that $\lambda \sim 10^{-3}$ [18]. Taken together, the interplay of swampland considerations and real-world observational data strongly indicates the existence of a single, extra dimension, roughly a micrometer in size $R \sim \lambda\Lambda^{-1/4} \sim 1\mu m$ where $\Lambda^{1/4} = 2.31 meV$. This additional dimension, often referred to as the dark dimension, becomes relevant at the energy scale m of the particle tower. At and above this scale, a higher-dimensional Effective Field Theory (EFT) framework is needed to describe physical phenomena, extending up to a species scale of $M \sim 10^{10} GeV$.

## 4.2 The formation of 5 dimensional PBHs

It was proposed[16] that primordial black holes (PBHs) could have been created by the collapse of large fluctuations in the very early universe [22–25]. On a cosmic scale, these PBHs would behave just like typical cold dark matter, even though their exact mass distribution is not yet known.

The proposal that PBHs could be a form of dark matter dates back to at least 1975 [26]. Interest in this idea has been renewed at different times, particularly following the 1997 microlensing searches for massive compact halo objects

(MACHOs) [27] and the 2016 LIGO/Virgo detections of merging binary black holes [28]. The initial microlensing studies suggested that dark matter might consist of MACHOs with masses around half that of the sun, which aligns with the predicted mass of PBHs formed during the quark-hadron phase transition [29]. However, more recent observations have since ruled out MACHOs as a major component of dark matter across most of the plausible mass range [30–36]. [htbp]

The total fraction of dark matter consisting of primordial black holes is given by relation:

$$f_{PBHs} \equiv \frac{\rho_{PBHs}}{\rho_{CDM}} = \int \psi(M_{BH}) dM_{BH} \tag{4.1}$$

where

$$\psi(M_{BH}) = \frac{M_{BH}}{\rho_{CDM}} \frac{dn_{PBHs}}{dM_{BH}} \tag{4.2}$$

is the mass distribution of PBHs, $dn_{PBHs}$ is the number density of PBHs within the mass $(M_{BH}, M_{BH} + dM_{BH})$ range, $\rho_{CDM}$ is the energy density of cold dark matter[16].Also $\rho_{PBHs} = \int M_{BH} dn_{PBHs}$ is the energy density of PBHs.

We assume that the classical black hole production cross section is a good approximation for the collision of two partons with $\sqrt{s}$ sufficiently larger than $M_*$.Imagine two massless particles undergoing a collision. With an impact parameter b and a center-of-mass energy of $\sqrt{s} = M_i$ each particle possesses a momentum of $M_i/2$ in that frame. Disregarding their spins, the system's initial angular momentum before impact is $J_i = bM_i/2$ A black hole is supposed to emerge when the initial configuration of these two particles (defined by their mass $M_i$ and angular momentum $J_i$) can be entirely encompassed by the event horizon of a black hole with the same mass $M = M_i$ and $J = J_i$:

$$b < 2r_h(M, J) = 2r_h(M_i, bM_i/2) \tag{4.3}$$

where $r_h(M, J)$ is the horizon radius of the higher dimensional Kerr black hole.There is a maximum value in impact parameter b which saturate the above inequality:

$$b_{max} = 2[1 + (\frac{n+2}{2})^2]^{-\frac{1}{n+1}} r_h \tag{4.4}$$

where $r_h$ is the radius of rotating black hole.

Below, we will study the physics of rotating primordial black holes within the framework of the dark dimension scenario. We will specifically examine the lifespan of rotating black holes in five dimensions and observe that they annihilate at a much slower rate than black holes in four dimensions. This obviously means that primordial black holes of this type can live much longer and ultimately constitute a large part of the observed dark matter in the universe.

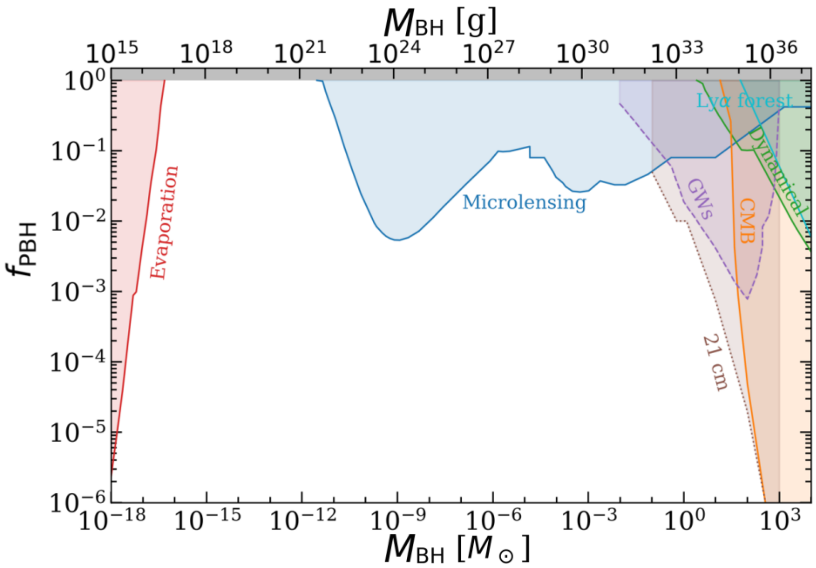


Figure 4.1: FIG. 1: (Taken from [40].) Compilation of contraints on fPBH as a function of the PBH mass MBH, assuming a monochromatic mass function. The different probes considered are: the impact of PBH evaporation (red) on the extragalactic -ray background [41] and on the CMB spectrum [42]; non-observation of microlensing events (blue) from the MACHO [27], EROS [28], Kepler [29], Icarus [30], OGLE [35] and Subaru-HSC [43] collaborations; PBH accretion signatures on the CMB (orange), assuming spherical accretion of PBHs within halos [43]; dynamical constraints, such as disruption of stellar systems by the presence of PBHs (green), on wide binaries [44] and on ultra-faint dwarf galaxies [45]; power spectrum from the Ly forest (cyan) [46]; merger rates from gravitational waves (purple), either from individual mergers [38, 47] or from searches of stochastic gravitational wave background [48]. Gravitational wave limits, denoted by dashed lines, are model dependent [51]. The dotted brown line corresponds to forecasts from the 21 cm power spectrum with SKA sensitivities [49] and from 21 cm forest prospects [50]

## 4.3 The lifetime of 5 dimenional PBHs

In this section we will discuss the case where the primordial 5 dimensional black hole is rotating.This case generalize the previous results of paper[16]. In this paper we present an analytic expression for the greybody factors of brane fields for an $n = 1$ black hole within the low frequency expansion. Here we outline our procedure.

In this paper, we will make the following assumption: that the emission of particles from a black hole during the process of Hawking radiation occurs mainly on the brane. We will ignore particles with degrees of freedom that can also travel outside the brane into the bulk, such as the graviton and the dilaton.Below, we will study the brane field emission from a higher-dimensional black hole, calculate the mass decay rate of a rotating black hole in 5 dimensions, and aim to calculate the lifetime of the black hole.

In order to do this, we Will make the following ansatz for the Newman-Penrose null tetrads[12]:

$$n = dt - asin^2\theta d\phi - \frac{\Sigma}{\Delta}dr \tag{4.5}$$

$$n' = \frac{\Delta}{2\Sigma}(dt - asin^2\theta d\phi) + \frac{1}{2}dr \tag{4.6}$$

$$m = \frac{isin\theta}{2^{1/2}(r + iacos\theta)}[adt - (a^2 + r^2)d\phi] - \frac{r - iacos\theta}{2^{1/2}} \tag{4.7}$$

$$m' = \bar{m} \tag{4.8}$$

where

$$\Sigma = r^2 + a^2cos^2\theta \tag{4.9}$$

$$\Delta = r^2 + a^2 - \mu r^{1-n} \tag{4.10}$$

Assuming the previous equations, we can show that the brane field equations for a massless field with spin 1, 1/2, and 0 can be written in a separated form as follows:

$$\frac{1}{sin\theta}\frac{d}{d\theta}(sin\theta\frac{dS}{d\theta}) + [(s - a\omega cos\theta)^2 - (scot\theta + mscs\theta)^2 - s(s-1) + A]S = 0 \tag{4.11}$$

and

$$\Delta^{-s}\frac{d}{dr}(\Delta^{s+1}\frac{dR}{dr}) + [\frac{K^2}{\Delta} + s(4i\omega r - i\frac{\Delta_r K}{\Delta} + \Delta_{rr} - 2) - A + 2ma\omega - a^2\omega^2]R = 0 \tag{4.12}$$

where

$$K = (r^2 + a^2)\omega - ma \tag{4.13}$$

and A is the angular eigenvalue, and the following decomposition is employed:

$$\Phi = \int d\omega e^{-i\omega t} \sum_m e^{im\phi} \sum_l R_{\omega lm}(r) S_{\omega lm}(\theta) \tag{4.14}$$

The angular part (4.11) is not modified from four dimensions and can be solved in terms of the spin-weighted spheroidal harmonics ${}_sS_{lm}$(which reduces to the spin-weighted spherical harmonics in the limit $\alpha\omega \ll 1$) with angular eigenvalue:

$$A = l(l+1) - s(s+1) - \frac{2ms^2}{l(l+1)} a\omega + O((a\omega)^2) = A_0 + O((a\omega)^2 \tag{4.15}$$

Next, we will derive the relation that describes the grey body factor for our case.But what is the grey body factor.As the Hawking radiation particles propagate away from the event horizon, they must travel through the curved spacetime surrounding the black hole. This curved spacetime acts like a potential barrier. This barrier reflects some of the particles back into the black hole and allows others to escape to infinity.The grey body factor represents the probability that a particle created at the horizon will successfully escape the gravitational potential barrier and be detected by an observer at a great distance.The greybody factor is essentially the transmission probability for the Hawking radiation.

First we obtain the "near horizon"(NH) and "far field" (FF) solutions in the following limits.Then we match these two solutions at the "overlapping region" in which both limits are consistently satisfied.Finally we impose the "purely ingoing" boundary condition at the near horizon side and then read the coeffcients of "outgoing" and "ingoing" modes at the far field side, the ratio of these two coefficients can be translated into the absorption probability of the mode, which is nothing but the greybody factor itself.

First, we defne the dimensionless quantities:

$$\xi = \frac{r - r_h}{r_h} \tag{4.16}$$

$$\tilde{\omega} = r_h \omega \tag{4.17}$$

$$\tilde{Q} = \frac{\omega - m\Omega}{2\pi T} = (1 + a_*^2)\tilde{\omega} - ma_* \tag{4.18}$$

Then the radial equation (4.12) becomes:

$$\xi^2(\xi+2)^2 R_{\xi\xi} + 2(s+1)\xi(\xi+1)(\xi+2)R_\xi + \tilde{V}R = 0 \tag{4.19}$$

where

$$\begin{aligned} \tilde{V} = [\tilde{\omega}\xi(\xi+2) + \tilde{Q}] + 2is\tilde{\omega}\xi(\xi+1)(\xi+2) \\ -2is\tilde{Q}(\xi+1) - [A_0 + O(a_*, \tilde{\omega})]\xi(\xi+2) \end{aligned} \tag{4.20}$$

In the near horizon limit $\tilde{\omega}\xi \ll 1$ the potential V becomes:

$$\tilde{V} = \tilde{Q}^2 - 2is(\xi+1)\tilde{Q} - A_0\xi(\xi+2) + O(\tilde{\omega},\xi) \tag{4.21}$$

and the solution of Eq. (4.19) with the potential (4.21) is obtained with the hypergeometric function

$$R_{NH} = C_1(\frac{\xi}{2})^{-s-i\tilde{Q}/2}(1+\frac{\xi}{2})^{-s+i\tilde{Q}/2} \times_2 F_1(-l-s, l-s+1, 1-s-i\tilde{Q}; \frac{\xi}{2}) \tag{4.22}$$

$$+C_2(\frac{\xi}{2})^{i\tilde{Q}/2}(1+\frac{\xi}{2})^{-s+i\tilde{Q}/2} \times_2 F_1(-\lambda+\tilde{Q}, l+1+i\tilde{Q}, 1+s+i\tilde{Q}; -\frac{\xi}{2}) \tag{4.23}$$

To impose the ingoing boundary condition at the horizon i.e:

$$R \sim \ \xi^{-s}e^{-ikr_*}, k\frac{dr_*}{d\xi} \sim \frac{\tilde{Q}}{2\xi} \tag{4.24}$$

we put $C_2 = 0$ and normalize $C = 1$ and then we get:

$$R_{NH} = (\frac{\xi}{2})^{-s-i\tilde{Q}/2}(1+\frac{\xi}{2})^{-s+i\tilde{Q}/2} \times_2 F_1(-l-s, l-s+1, 1-s-i\tilde{Q}; \frac{\xi}{2}) \tag{4.25}$$

In the far field limit $\xi \gg 1 + |\tilde{Q}|$ Eq (4.8) becomes:

$$0 = R_{\xi\xi} + \frac{2(s+1)}{\xi}R_\xi + [\tilde{\omega}^2 + \frac{2i\tilde{\omega}}{\xi}(s-2i\tilde{\omega}) - \frac{1}{\xi^2}[A_0 + O(\tilde{\omega})] + O(\xi^{-3})]R \tag{4.26}$$

and the solution is obtained via Kummer's confluent hypergeometric function

$$R_{FF} = B_1 e^{-i\tilde{\omega}\xi}(\frac{\xi}{2})^{l-s}{}_1F_1(l-s+1, 2l+2; 2i\tilde{\omega}\xi) \tag{4.27}$$

$$+B_2 e^{-i\tilde{\omega}\xi}(\frac{\xi}{2})^{-l-s-1}{}_1F_1(-l-s, -2l; 2i\tilde{\omega}\xi) \tag{4.28}$$

where singularity from $2l$ being integer is regularized by the higher order terms in $\tilde{\omega}$.

Matching the NH and FF solutions (4.25) and (4.28) in the overlapping region $1 + |\tilde{Q}| \ll \xi \ll \frac{1}{\tilde{\omega}}$,we obtain:

$$B_1 = \frac{\Gamma(2l+1)\Gamma(1-s-i\tilde{Q})}{\Gamma(l-s+1)\Gamma(l+1-i\tilde{Q})}$$

$$B_2 = \frac{\Gamma(-2l-1)\Gamma(1-s-i\tilde{Q})}{\Gamma(-l-s)\Gamma(-l-i\tilde{Q})} \tag{4.29}$$

Then we extend the obtained FF solution toward the region $\xi \gg 1/\tilde{\omega}$

$$R_\infty = Y_{in} e^{-i\tilde{\omega}\xi} (\frac{\xi}{2})^{-1} + Y_{out} e^{i\tilde{\omega}\xi} (\frac{\xi}{2})^{-2s-1} \tag{4.30}$$

where

$$\begin{aligned} Y_{in} = & \frac{\Gamma(2l+1)\Gamma(2l+1)\Gamma(1-s-i\tilde{Q})}{\Gamma(l-s+1)\Gamma(l+s+1)\Gamma(l+1-i\tilde{Q})} (-4i\tilde{\omega})^{-l+s-1} \\ & + \frac{\Gamma(-2l-1)\Gamma(-2l)\Gamma(1-s-i\tilde{Q})}{\Gamma(-l-s)\Gamma(-l+s)\Gamma(-l-i\tilde{Q})} (-4i\tilde{\omega})^{l+s} \end{aligned} \tag{4.31}$$

$$Y_{out} = \frac{\Gamma(2l+1)\Gamma(2l+1)\Gamma(1-s-i\tilde{Q})}{[\Gamma(l-s+1)]^2\Gamma(l+1-i\tilde{Q})} (-4i\tilde{\omega})^{-l-s-1} \tag{4.32}$$

$$+ \frac{\Gamma(-2l-1)\Gamma(-2l)\Gamma(1-s-i\tilde{Q})}{[\Gamma(-l-s)]^2\Gamma(-l-i\tilde{Q})} (-4i\tilde{\omega})^{l-s} \tag{4.33}$$

Let us define $R_{-s}$ as the solution of the equation obtained by a flip of the sign of s

Therefore, we may calculate the greybody factor $\Gamma$ the absorption probability in the same way as Page's trick:

$$\Gamma = 1 - |\frac{Y_{out} Z_{out}}{Y_{in} Z_{in}}| = 1 - |\frac{1-C}{1+C}|^2 \tag{4.34}$$

where

$$C = \frac{(4i\tilde{\omega})^{2l+1}}{4} (\frac{(l+s)!(l-s)!}{(2l)!(2l+1)!})^2 (-i\tilde{Q} - l)_{2l+1} \tag{4.35}$$

with $(a)_n = \Pi_{n'}^n (a + n' - 1))$ being the Pochhammer symbol.

The time dependence of the mass of the black hole is given by:

$$\begin{aligned} \frac{dM}{dt} &= -\frac{1}{2\pi} \int_0^\infty d\omega \sum_{s,l,m} \frac{{}_s\Gamma_{lm}\omega}{e^{(\omega - m\Omega)/T} - (-1)^s} \\ &= -\frac{1}{2\pi} \int_0^\infty d\omega \sum_{s,l,m} \frac{{}_s\Gamma_{lm}\omega}{e^{2\pi[(1+a_*^2)\tilde{\omega} - m a_*]} - (-1)^s} \end{aligned} \tag{4.36}$$

because we know that:$\tilde{\omega} = r_h\omega$ and $\tilde{Q} = \frac{\omega - m\Omega}{2\pi T} = (1+a_*^2)\tilde{\omega} - ma_*$ If we use the above relations and for $n = 1$ and $a_* = 1$ we can rewrite the relation as follows:

$$
\begin{aligned}
\frac{dM}{dt} &= -\frac{1}{2\pi}\int_0^\infty d\omega \sum_{s,l,m} \frac{{}_s\Gamma_{lm}\omega}{e^{2\pi[2\tilde{\omega}-m]} - (-1)^s} \\
&= -\frac{1}{2\pi r_h^2}[g_0 \int_0^\infty d\tilde{\omega} \sum_m \frac{{}_0\Gamma_{00}\tilde{\omega}}{e^{2\pi(2\tilde{\omega})} - 1} \qquad (4.37) \\
&+ g_0 \int_0^\infty d\tilde{\omega} \sum_m \frac{{}_0\Gamma_{1m}\tilde{\omega}}{e^{2\pi(2\tilde{\omega}-m)} - 1} + g_0 \int_0^\infty d\tilde{\omega} \sum_m \frac{{}_0\Gamma_{2m}\tilde{\omega}}{e^{2\pi(2\tilde{\omega-m})} - 1} \\
&+ g_{1/2} \int_0^\infty d\tilde{\omega} \sum_m \frac{{}_{1/2}\Gamma_{1/2m}\tilde{\omega}}{e^{2\pi(2\tilde{\omega}-m)} + 1} + g_{1/2} \int_0^\infty d\tilde{\omega} \sum_m \frac{{}_{1/2}\Gamma_{3/2m}\tilde{\omega}}{e^{2\pi(2\tilde{\omega}-m)} + 1} + \\
&g_1 \int_0^\infty d\tilde{\omega} \sum_m \frac{{}_1\Gamma_{1m}\tilde{\omega}}{e^{2\pi(2\tilde{\omega}-m)} - 1} + g_1 \int_0^\infty d\tilde{\omega} \sum_m \frac{{}_1\Gamma_{2m}\tilde{\omega}}{e^{2\pi(2\tilde{\omega}-m)} - 1}] \qquad (4.38)
\end{aligned}
$$

(4.39)

where from(4.19) we write down the explicit expansion of Eq.(4.19) up to $O(\tilde{\omega}^6)$ terms:

$$
\begin{aligned}
\Gamma &= 1 - |\frac{1-C}{1+C}|^2 = 1 - \frac{|1-C|^2}{|1+C|^2} \\
&= 1 - \frac{(1-C)(1-C^*)}{(1+C)(1+C^*)} = 1 - \frac{1 - C^* - C + C^*C}{1 - C^* - C + C^*C} \\
&= 1 - \frac{1 - (C+C^*) + |C|^2}{1 + (C+C^*) + |C|^2} = \frac{2(C+C^*)}{1 + (C+C^*) + |C|^2} \\
\Gamma &= \frac{4Re(C)}{1 + 2Re(C) + |C|^2} \qquad (4.40)
\end{aligned}
$$

where $Re(C)$ the real part of the complex number C.

If we choose $x = 2Re(C) + |C|^2$ can we use the expansion:

$$\frac{1}{1+x} \simeq 1 - x + x^2 - ..... \qquad (4.41)$$

and rewrite the eq.(4.29) as such:

$$\Gamma = 4Re(C) - 8Re^2(C) - 4Re(C)|C|^2 + 4Re(C)[2Re(C) + |C|^2]^2 - .... \qquad (4.42)$$

If we choose the quantum numbers $s = l = m = 0$ then from (4.19) we get:

$$C = \tilde{\omega}\tilde{Q} \qquad (4.43)$$

and

$$Re(C) = \frac{C^* + C}{2} = \tilde{\omega}\tilde{Q} \qquad (4.44)$$

From (4.3) :

$$\tilde{Q} = (1 + a_*^2)\tilde{\omega} \tag{4.45}$$

Now from (4.26),(4.27),(4.28) and (4.29) we have:

$$\begin{aligned}\Gamma &= 4\tilde{\omega}\tilde{Q} - 8(\tilde{\omega}\tilde{Q})^2 - 4(\tilde{\omega}\tilde{Q})(\tilde{\omega}\tilde{Q})^2 + ... \\ &= 4\tilde{\omega}(1 + a_*^2)\tilde{\omega} - 8[\tilde{\omega}(1 + a_*^2)\tilde{\omega}]^2 - ... \\ &= 4\tilde{\omega} + 4\tilde{\omega}a_*^2 - 8\tilde{\omega}^4(1 + a_*^2)^2 - ...\end{aligned} \tag{4.46}$$

In the low frequency limit we have:

$$a_*\tilde{\omega} < 1, \tilde{\omega} < 1 \tag{4.47}$$

and for that we can neglect the terms $\tilde{\omega}^6$ and beyond. Finally for $s = l = m = 0$ we get:

$$\Gamma = 4\tilde{\omega}^2 - 8\tilde{\omega}^4 \tag{4.48}$$

Finally, we write down the explicit expansion of (4.34)

$${}_0\Gamma_{00} = 4\tilde{\omega}^2 - 8\tilde{\omega}^4 + O(\tilde{\omega}^2) \tag{4.49}$$

$${}_0\Gamma_{1m} = \frac{4\tilde{Q}\tilde{\omega}^3}{9}(1 + \tilde{Q}^2) + O(\tilde{\omega}^6) \tag{4.50}$$

$${}_0\Gamma_{2m} = \frac{16\tilde{Q}\tilde{\omega}^5}{2025}(1 + \frac{5\tilde{Q}^2}{4} + \frac{\tilde{Q}^4}{4}) + O(\tilde{\omega}^{10}) \tag{4.51}$$

$${}_{1/2}\Gamma_{1/2m} = \tilde{\omega}^2(1 + 4\tilde{Q}^2) - \frac{\tilde{\omega}^4}{36}(1 + 4\tilde{Q}^2)^2 + O(\tilde{\omega^6}) \tag{4.52}$$

$${}_{1/2}\Gamma_{3/2m} = \frac{\tilde{\omega}^4}{36}(1 + \frac{40\tilde{Q}^2}{9} + \frac{16\tilde{Q}^4}{9}) + O(\tilde{\omega}^6) \tag{4.53}$$

$${}_1\Gamma_{1m} = \frac{16\tilde{Q}\tilde{\omega}^3}{9}(1 + \tilde{Q}^2) + O(\tilde{\omega}^6) \tag{4.54}$$

$${}_1\Gamma_{2m} = \frac{4\tilde{Q}\tilde{\omega}^5}{225}(1 + \frac{5\tilde{Q}^2}{4} + \frac{\tilde{Q}^4}{4}) + O(\tilde{\omega}^{10}) \tag{4.55}$$

and m:$-l \leq m \leq l$. For the calculation of the above integrals we use the mathematica and we find that:

$$\begin{aligned}-\frac{dM}{dt} &= \frac{1}{2\pi r_h^2}[4(0.004 + 0.003 + 0.001) + 90(0.011 + 0.002) + 24(0.011 + 0.002)] \\ &= \frac{1}{2\pi r_h^2}(0.032 + 1.17 + 0.312) \\ &= \frac{01.514}{2\pi r_h^2} \Rightarrow\end{aligned}$$

$$\frac{dM}{dt} = -\frac{0.241}{r_h^2} \tag{4.56}$$

We know that:

$$r_h = (1 + a_*^2)^{-\frac{1}{n+1}} r_s \tag{4.57}$$

with $r_s$ the Schwarzschild radius.

$$r_h^2 = \frac{1}{2\pi M_*}(\frac{M_{BH}}{M_*})(\frac{8\Gamma(2)}{3}) \tag{4.58}$$

with $M_*$ is the 5 dimensional Planck mass .From (4.11) and (4.12) we get

$$\frac{dM}{dt} = \frac{0.241}{\frac{1}{2\pi M_*^2}(\frac{M_{BH}}{M_*})(\frac{8\Gamma(2)}{3})} = -\frac{0.241}{8\Gamma(2)}\frac{M_*^2}{M_{BH}} \tag{4.59}$$

From Swampland arguments we had defined the planck mass $M_* \sim 10^{10}$ GeV. Then we have:

$$\frac{dM}{dt} \simeq -5.68 \times 10^{29} \frac{GeV^3}{M} \tag{4.60}$$

We solve for t and we get:

$$\tau_{BH}^{s.d} \simeq -0.88 \times 10^{-31}(M_f^2 - M_i^2)\text{GeV}^{-3} \tag{4.61}$$

For the realistic case(Standard Model particles),black hole loses roughly 70% to 80% of her mass in $D = 5$ before it stops rotation when starting from the maximum rotation and therefore we have $M_f \simeq 0.3M_i - 0.4M_i$[13]

Suppose to $M_f \simeq 0.3M_i$ then from (4.16) we get the period of spin-down phase of 5D rotating black hole:

$$\tau_{BH}^{s.d} \simeq 5.2 \times 10^{-15}(\frac{M_i}{g})^2 y \tag{4.62}$$

with g:grams and y:years.

If the initial mass of primordial rotating black hole is $M_i \simeq 5 \times 10^{11}$g then the spin-down phase period is

$$\tau_{BH}^{s.d} \simeq 1.3 \times 10^9 y \tag{4.63}$$

Now after the spin-down phase the black hole stops the rotation and become a non rotating black hole which described by Schwarzschild-Tangherlini equation.Then the mass of black hole which it remains,evolves according to following relation:

$$\tau_{BH}^{S.T} \simeq 9 \times 10^{-15}(\frac{M_f}{g})^2 y \tag{4.64}$$

If the remaining black hole mass is $M_f \sim 1.5 \times 10^{11}$g then:

$$\tau_{BH}^{S.T} \simeq 2 \times 10^8 y \tag{4.65}$$

Finally the lifetime of primordial 5D rotating black hole with initial mass $M_i \sim 5 \times 10^{11} g$ is given by:

$$\tau_{M.P}^{n=1} = \tau_{BH}^{s.d} + \tau_{BH}^{S.T} \tag{4.66}$$

$$\tau_{M.P}^{n=1} \simeq 1.5 \times 10^9 y \tag{4.67}$$

Now for $a_* = 1.5$ we have:

$$\tilde{\omega} = r_h \omega \tag{4.68}$$

$$\tilde{Q} = \frac{\omega - \frac{3}{2}\Omega}{2\pi T} = \frac{13}{4}\tilde{\omega} - \frac{3}{2}m \tag{4.69}$$

If we use the relation (4.2),(4.23) and (4.24) we get:

$$-\frac{dM}{dt} = \frac{1}{2\pi r_h^2}[4(0.001 + 0,001 + 0.0005) + 90(0.008 + 0.001) + 24(0.005 + 0.015)] \Rightarrow$$

$$\Rightarrow -\frac{dM}{dt} = \frac{0.21}{r_h^2} \tag{4.70}$$

From (4.13),(4.14) and (4.26) we have:

$$\frac{dM}{dt} = -\frac{0.21}{8\Gamma(2)}\frac{M_*^3}{M_{BH}}6\pi \tag{4.71}$$

As we said before $M_* \sim 10^{10} GeV$ then:

$$\frac{dM}{dt} = -4.9 \times 10^{-29}\frac{GeV^3}{M_{BH}} \tag{4.72}$$

From above relation we can find the time which black spend in the spi-down phase:

$$\tau_{s.d} \simeq -1 \times 10^{-30} GeV^{-3}(M_f^2 - M_i^2) \tag{4.73}$$

As before from the paper[] the remaining black hole after spin-down phase is $M_f \simeq 0.3 M_i$.Therefore:

$$\tau_{s.d} \simeq 5.94 \times 10^{-15}(\frac{M_i}{g})^2 y \tag{4.74}$$

If $M_i \sim 5 \times 10^{11} g$ then:

$$\tau_{s.d} \simeq 1.49 \times 10^9 y \tag{4.75}$$

We find before that the non rotating period of remaining mass $M_f \sim 1.5 \times 10^{11} g$ is given by:

$$\tau_{s.d} \simeq 2 \times 10^8 y \tag{4.76}$$

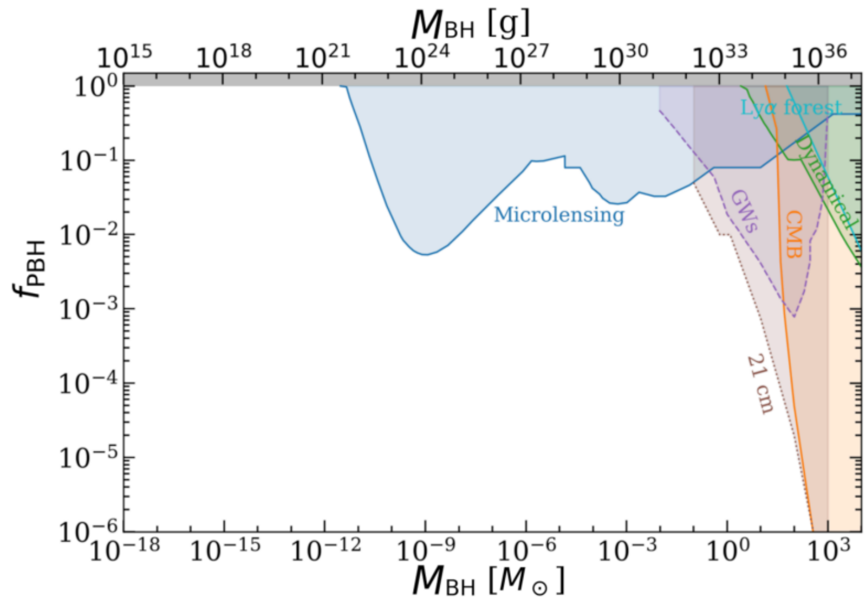


Figure 4.2: Caption

Finally the lifetime of black hole with $a_* = 1.5$ and $n = 1$ is:

$$\tau^{n=1}_{M.P} = \tau^{n=1}_{s.d} + \tau^{n=1}_{S.T} = 1.7 \times 10^9 y \tag{4.77}$$

We have shown that the rate of Hawking radiation slows down for 5-dimensional black holes and thereby an all-dark-matter interpretation in terms of PBHs for $10^{14} \leq M_{BH}/g \leq 10^{21}$.

For $M_{BH} \approx 5\times10^{11} g$ the radius is $r_h \approx 5\times10^{-5}\mu m$ whereas for $M_{BH} \approx 10^{17} g$ we have $r_h \approx 2\times10^{-2}\mu m$ ustifying our assumption that these black holes are 5-dimensional objects.It is noteworthy that a black hole with $M_{BH} \approx 10^{21}$ g has a horizon radius $r_h \approx 2\mu m$ saturating the range of validity of our 5D description.

## 4.4 Memory Burden Effect

The memory burden effect is a proposed phenomenon wherein a system's capacity to store information actively resists its decay and proposed by Dvali [14]. Essentially, the information "loaded" into a system acts as a stabilizing force. This effect is particularly pronounced in systems capable of storing a vast amount of information, like black holes, which possess maximal microstate degeneracy

This is obvious from Bekenstein-Hawking entropy:

$$S_{BH} = \frac{A}{4G_N} = \pi R^2 M_p^2 \tag{4.78}$$

where A is the area of the black hole horizon, $M_P$ is the Planck mass. This effect explains why at the early stages of Hawking's decay, the information stored in a black hole cannot be released together with radiation.

In the process of a black hole decay, the memory burden grows, and after a certain characteristic time $t_M$ when black hole has lost aproximatelly the half of its initial mass, reaches a critical value.We should mention that the time $t_M$ is bounded from above by the age of the black hole. After this time $t_M$ the black hole evaporation slow down dramatically.So dramatically that at this point the black hole has evolved into a "remnant" that cannot continue an ordinary quantum decay.The fate of this remnant can not be determined by the standard semi-classical methods.This remnant can be descibed only by the laws of the quantum gravity.

But black holes are not the only objects with maximal information storage capacity.For that reason the memory burden effect it is not a property that describes only black holes.It was shown recently [15] that in some QFT's there are some systems named saturons which saturates the QFT upper bound on the microstate degeneracy.The bound can be given as the bound of entropy:[14]

$$S \equiv ln(n_{st}) \tag{4.79}$$

where $n_{st}$ is number of degenerate microstates.

Equivalently, the bound can be written in terms of the Goldstone scale, f, of the spontaneously broken Poincare symmetry:

$$S \leq \pi R^2 f^2 \tag{4.80}$$

The above bound set the maximal degeneracy reachable within the validity of the QFT description.If in the above relation $f = M_P$ then we get the Bekenstein-Hawking entropy of a black hole.We observe similarities between black holes and saturons of renormalizable QFTs. Also it was shown recently that the the time $t_M$ which is the start of the memory burden effect is equal to Page time in the case of black holes[39].

The above correspondence makes the study of saturons important due to the following reasons. First, it shows that the black hole properties are not specific to gravity and can be understood within calculability domains of renormalizable QFTs. Secondly, saturons can serve as laboratories for understanding the microscopic nature of known black hole properties and for discovering new features.

Now we will apply the above phainomenon for the behavior of the black hole in our discusion for the lifetime of the five diemnsional primordial black holes. The Page time is defined by the condition that the mass of a black hole has decreased to half its original value via Hawking radiation,$t_{half} \sim M_{BH}/2$[D. N. Page, Information in black hole radiation].Within this time $r_h \to r_h/2$ and $S_{BH} \to S_{BH}/4$.Until this time information remain encoded inside the black hole because the emitted radiation is thermal in character.However, after ,$t_{half}$ the remaining black hole has only 1/4 of its initial entropy and so much less information storage capacity. After this time the memory burden effect starts.We know that the quantum decay rate(with burden effect) is given by[17]:

$$\frac{dM_{M.B}}{dt} = \frac{1}{S_{BH}^k}\frac{dM_{S.C}}{dt} \tag{4.81}$$

where $S_{BH}$ is the entropy of 5D primordial black hole, $dM_{M.B}/dt$ is the mass decay rate of black hole under memory burden effect and $dM_{S.C}/dt$ is the mass decay rate which we calculate in the previous section via semi-classical methods. Throughout, n is a non-negative integer parametrizing the quantum suppression when the black hole enters the memory burden phase. The entropy of 5D rotating black hole given by:

$$S_{BH} = \frac{\pi^2}{2G_D}\mu\sqrt{\mu - a^2} \tag{4.82}$$

with $\mu$ mass parameter and $a$ rotational parameter which are given by (3.156) and (3.157).If we want to calculate the quantum mass decay rate, first we shold calculate the.We know from the previous section that the value of the angular momentum is given by:

$$J = \frac{bM_{BH}}{2} \tag{4.83}$$

where b is impact parameter.The maximum value of this parameter is given by $J = \frac{b_{max}M_{BH}}{2}$

When $a_* = 1.5$ and if we combine the relations(3.156),(3.157),(4.70) and (4.71), we get:

$$S_{BH} = \frac{\pi^2}{2G_D}\mu\sqrt{\mu - \frac{9}{13}\mu} = \frac{\pi^2}{2G_D}\mu\sqrt{\frac{4}{13}\mu} \tag{4.84}$$

We can write the higher dimensional Newton constant in terms of higher dimensional Planck mass as follows: $G_D = \frac{1}{M^{2+n}}$.For our case n=1 and D=5 we have:$G_5 = \frac{1}{M_*^3}$.Then if we use this with relation (4.68) we get:

$$S_{BH} = 3.8\sqrt{\frac{M_{BH}^3}{M_*^3}} \tag{4.85}$$

Then, from relation (4.65) we have:

$$\frac{dM_{M.B}}{dt} = \frac{1}{[3.8 \times (\frac{M_{BH}}{M_*})^{3/2}]^k}(-4.9 \times 10^{-29}\frac{GeV^3}{M_{BH}})$$
$$= 1.3 \times 10^{-29} GeV^3 (M_*)^{3k/2} M_{BH}^{-(\frac{3k}{2}+1)} \Rightarrow$$
$$dt = -0.8 \times 10^{-29} Gev^{-3} (M_*)^{-3k/2} M^{\frac{3k}{2}+1} dM \qquad (4.86)$$

For this work we choose $k = 1$[15].Also $M_* = 10^{10}$ as we know from we get:

$$\tau_{BH} = -8 \times 10^{-45} GeV^{-3} GeV^{-3/2} M_i^{7/2} \qquad (4.87)$$

Then

$$\tau_{BH} = 2.2 \times 10^9 (\frac{M_i}{g})^{7/2} y \qquad (4.88)$$

where g is gramms and y are the years. In turn, by requiring that such holes should live at least as long as the age of the Universe, yields:

$$M \geq 10^{-2.6} g \qquad (4.89)$$

We see that from the moment the black hole enters the memory burden effect phase, it loses mass at an extremely slow rate. In fact, the rate is so slow that the black hole essentially stabilizes during this phase.

# Chapter 5

# Conclusions

In the present work, we provided an introduction to the basic concepts of string theory as well as some of the most significant conjectures of the Swampland program, such as the distance conjecture and the weak gravity conjecture. We then studied the fundamental properties of black holes in higher dimensions, such as the existence and number of event horizons and the singularities in these solutions. We also derived the expressions for entropy and Hawking temperature, both for the case of a static black hole in (n+4) dimensions and for the case of a rotating black hole in (n+4) dimensions.

Next, we discussed the idea of the Dark Dimension Scenario, that is, the idea which suggests that one of the extra dimensions predicted by string theory is decompactified and is of the order of a micron. This scenario arises if we combine some of the most basic conjectures of the Swampland program, such as the ADS conjecture and the distance conjecture, with some cosmological observations. Within the framework of this work, we hypothesized that the dark matter we observe in the universe essentially consists of primordial rotating black holes that perceive this dark dimension. We derived the lifetime of 5-dimensional rotating black holes.We have shown that the rate of Hawking radiation slows down for 5-dimensional black holes and thereby an all-dark-matter interpretation in terms of PBHs for $10^{14} \leq M_{BH}/g \leq 10^{21}$ should be possible.

Finally, we discussed a phenomenon called the memory burden effect and its implications for the lifetime of a black hole. We showed that from the moment the memory burden effect begins (which is approximately when the black hole has lost about half of its mass), we see the black hole loses mass at an extremely slow rate, so slow that the black hole essentially stabilizes during this phase.

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