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ΤΜΗΜΑ ΜΑΘΗΜΑΤΙΚΩΝ

Αικατερίνη Μαρία Ντάσιου

ΜΙΑ SAT ΔΙΑΤΥΠΩΣΗ ΓΙΑ ΤΗΝ ΕΞΕΤΑΣΗ ΤΗΣ ΕΙΚΑΣΙΑΣ ΤΟΥ BARNETTE

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΑΤΡΙΒΗ

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ENGINEERING BARNETTE'S CONJECTURE WITH SAT

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Αφιερώνεται στην οικογένειά μου

Η παρούσα Μεταπτυχιακή Διατριβή εκπονήθηκε στο πλαίσιο των σπουδών για την απόκτηση του Μεταπτυχιακού Διπλώματος Ειδίκευσης στα Εφαρμοσμένα Μαθηματικά και Πληροφορική, που απονέμει το Τμήμα Μαθηματικών του Πανεπιστημίου Ιωαννίνων.

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ΥΠΕΥΘΥΝΗ ΔΗΛΩΣΗ

"Δηλώνω υπεύθυνα ότι η παρούσα διατριβή εκπονήθηκε κάτω από τους διεθνείς ηθικούς και ακαδημαϊκούς κανόνες δεοντολογίας και προστασίας της πνευματικής ιδιοκτησίας. Σύμφωνα με τους κανόνες αυτούς, δεν έχω προβεί σε ιδιοποίηση ξένου επιστημονικού έργου και έχω πλήρως αναφέρει τις πηγές που χρησιμοποίησα στην εργασία αυτή."

Αικατερίνη Μαρία Ντάσιου

ΕΥΧΑΡΙΣΤΙΕΣ

Με την ολοκλήρωση της παρούσας μεταπτυχιακής διατριβής, θα ήθελα να εκφράσω τις θερμές μου ευχαριστίες σε όλους όσοι στάθηκαν αρωγοί σε αυτή μου την προσπάθειά, συμβάλλοντας καθοριστικά στην επιτυχή περάτωσή της.

Ιδιαίτερη μνεία οφείλω στον επιβλέποντα καθηγητή μου κ. Μιχαήλ Α. Μπέκο, για την καθοδήγηση, την υποστήριξη και τις ουσιαστικές συμβουλές, καθ' όλη τη διάρκεια εκπόνησης της εργασίας. Ευχαριστώ, επίσης, την υποψήφια διδάκτορα του τμήματος Μαρία-Ελένη Παυλίδη για την σημαντική στήριξή της.

Τέλος, οφείλω ένα μεγάλο ευχαριστώ και την ειλικρινή μου ευγνωμοσύνη στην οικογένειά μου και στους φίλους μου, των οποίων η κατανόηση και η ενθάρρυνση - διαρκείς και καίριας σημασίας - συμβάλλουν στην επίτευξη των στόχων μου.

Περιληψη

Ένα από τα πιο γνωστά προβλήματα στη Θεωρία Γραφημάτων αποτελεί η εικασία του Barnette, η οποία αν και διατυπώθηκε πριν αρκετές δεκαετίες, εξακολουθεί να παραμένει ανοικτό πρόβλημα. Η εικασία υποστηρίζει ότι κάθε 3-κανονικό, 3-συνδεδεμένο, διμερές, επίπεδο γράφημα είναι Hamiltonian, δηλαδή, υπάρχει κύκλος που να διέρχεται από όλες τις κορυφές του γραφήματος ακριβώς μία φορά.

Στην παρούσα μεταπτυχιακή διατριβή προτείνουμε μία SAT διατύπωση του προβλήματος του ελέγχου ύπαρξης Hamilton κύκλων σε επίπεδα γραφήματα και την αξιοποιούμε με σκοπό τη διερεύνηση της εικασίας του Barnette. Αξιολογούμε πειραματικά μία υλοποίηση αυτής της διατύπωσης σε τέσσερις κατηγορίες Barnette γραφημάτων που δημιουργήσαμε χρησιμοποιώντας διαφορετικές μεθόδους.

Abstract

One of the most well-known problems in Graph Theory is Barnette's conjecture, which although formulated several decades ago, is still open. The conjecture states that every 3-regular, 3-connected, bipartite, planar graph is Hamiltonian, that is, there is a cycle that passes through all the vertices of the graph exactly ones.

In this thesis, we give a SAT-based formulation for testing the existence of Hamilton cycles in planar graphs and we use it to investigate Barnette's conjecture. We experimentally evaluate an implementation of this formulation on four categories of Barnette graphs generated using different methods.

Contents

CHAPTER 1

Introduction

Barnette's conjecture [4] is a long-standing unsolved problem in graph theory. Formulated by David W. Barnette back in 1969, it asserts that every 3-connected, 3-regular, bipartite, planar graph is Hamiltonian; see Chapter 2 for precise definitions. The conjecture is motivated by two earlier conjectures, proposed by Tait [22] and Tutte [23], both of which were eventually disproven. Nonetheless, these earlier efforts underscore the necessity of each condition in Barnette's formulation, as the omission of any single one may lead to a graph that fails to be Hamiltonian. For instance, there exist 3-connected bipartite planar graphs of degree 4 that are not Hamiltonian [6], while, without the assumption of 3-connectivity, deciding whether a 3-regular, bipartite, planar graph is Hamiltonian is NP-complete [14]. Note that all planar 4-connected graphs are Hamiltonian [24].

Conjecture 1. *(Barnette [4]) Every 3-connected, 3-regular, bipartite, planar graph (or* Barnette graph*, for short) is Hamiltonian.*

While the truth of Barnette's conjecture still remains unknown, numerous partial and related results have been proposed in the literature over the years; see [1, 2, 3, 9, 10, 12, 16, 17, 19, 21] and the references therein. It is noteworthy that Barnette's conjecture has been proven for several notable subclasses of Barnette graphs. Back in 1975, Goodey [12] showed that every Barnette graph that contains only faces of degree 4 and 6 (and at most one face of degree 8) is Hamiltonian. Feder and Subi [9] generalized this result by showing that a Barnette graph is Hamiltonian if its faces are 3-colored, with adjacent faces having different color, such that two of the three colors contain faces of degree 4 and 6. Alt, Payne, Schmidt and Wood [3] proved that a Barnette graph is Hamiltonian if the faces of its dual can be (improperly) colored red and blue, such that the blue vertices cover all faces of the dual, there is no blue cycle, and every red cycle contains a vertex of degree at most 4.

Considerable research has also been directed towards establishing strengthened forms of Barnette's conjecture. The first one was given back in 1986 by Kelmans [20], who proved that Barnette's conjecture holds if and only if for every Barnette graph and for every two edges in a common face of it there is a Hamiltonian cycle that contains one but not the other. Since then, several other partial results have been established. E.g., it is known that Barnette's conjecture holds if and only if

(i) any arbitrary edge is part of some Hamiltonian cycle [15],

(ii) any path of length 3 which has two edges of a face and the final edge leaving that face is part of some Hamiltonian cycle [16],

(iii) for any arbitrary path of length 3 on the boundary of a face, there is a Hamiltonian cycle which passes through its middle edge and avoids its leading and trailing edge [16],

(iv) every Barnette graph, in which any two edges of it are part of a perfect matching, is Hamiltonian [13].

Refer to [16] for notable corollaries of these results and related conjectures.

Towards finding a counterexample to Barnette's conjecture, Jensen and Toft [18] proved that Barnette's conjecture does not hold if and only if there exists a planar graph with both chromatic number and point arboricity 3 (the latter term refers to the minimum number of colors required to color the vertices of the graph such that no cycle is monochromatic). Back in 1984 Holton, Manvel and McKay [17] demonstrated that any potential counterexample to Barnette's conjecture must have more than 84 vertices (a lower bound which was improved to 90 recently [7]). Their result is based on the observation that, starting with the cube graph (see Fig. 2.2a), which is the smallest graph in the class of Barnette graphs, every other Barnette graph can be generated by iteratively applying two specific operations, known as cube- and C_4-expansions (see Figs. 2.2b and 2.2c); refer to Chapter 2 for their formal definitions.

1.1 Motivation and our Contribution.

Our work is motivated by the concluding remark in Hertel's seminal work [16]: "Then again, the key to settling Barnette's Conjecture may lie in a different area of graph theory altogether; perhaps we have better tools for tackling one of its equivalent problems.”. Inspired by this perspective, we introduce a novel method that, while not resolving Barnette’s conjecture at this time, offers a promising approach for identifying a potential counterexample, should one exist. Our approach is based on a close connection between the problem of testing whether a planar graph admits a 2-page book embedding and the problem of determining Hamiltonicity in planar graphs of low degree (see Chapter 2). By extending a known SAT formulation for the former [5] to the latter (see Chapter 3), we managed to develop an efficient method that, in practice, can determine the Hamiltonicity of Barnette graphs with up to 300 vertices in approximately half an hour (see Chapter 4), marking another significant computational advance in the area especially given the NP-complete nature of the Hamiltonian problem in planar graphs of low degree [11].

1.2 Thesis Organization.

This thesis is organized as follows:

- In Chapter 2 we introduce the required theoretical foundations, needed for the upcoming chapters.
- Chapter 3 deals with the introduction of the SAT formulation to test Hamiltonicity in planar graphs.
- In Chapter 4, we present the results of our experimental evaluation.
- This thesis is concluded with Chapter 5, where we summarize our conclusions.

CHAPTER 2

Preliminaries

2.1 Definitions and properties

A graph G consists of two sets, vertices - also called nodes - and edges, directed or not, that indicate connections between pairs of vertices. These two sets are denoted as V (or $V(G)$) and E (or $E(G)$), respectively. So, one can write $G = (V, E)$. In this thesis, we consider edges that are not directed. Also, if the two vertices at the ends of an edge e are u and v, then the edge can be described as (u, v). In addition, if two vertices share an edge, they are called neighbors or adjacent vertices. For a vertex u, the set containing all its neighboring vertices is denoted by $N(u)$.

In the rest of this section, we briefly recall a few key notions that are central to this work.

Definition 2.1. *A graph is* planar *if it can be drawn on the Euclidean plane without edge crossings.*

Definition 2.2. *A graph $G = (V, E)$ is called a bipartite graph if its vertex set can be partitioned into two non-empty, non-overlapping sets A and B (that is, $V = A \cup B$ and $A \cap B = \emptyset$) such that each edge (u, v) of it connects vertices of different sets (that is, $u \in A$ and $v \in B$).*

In other words, every vertex of a bipartite graph belongs exactly to one of two sets, and there is no edge between vertices of the same set. So, its vertices can be 2-colored so that adjacent vertices differ in color. An alternative characterization is the following. A graph is bipartite if and only if a cycle of odd length is not part of it.

Definition 2.3. *A graph is k-connected, if there exist k paths between any two vertices of it that pairwise do not share interior vertices.*

Another equivalent definition of the class of k-connected graphs is the following. A graph is k-connected, if it remains connected after removing fewer than k vertices.

Definition 2.4. *A graph is k-regular if all its vertices are of degree k.*

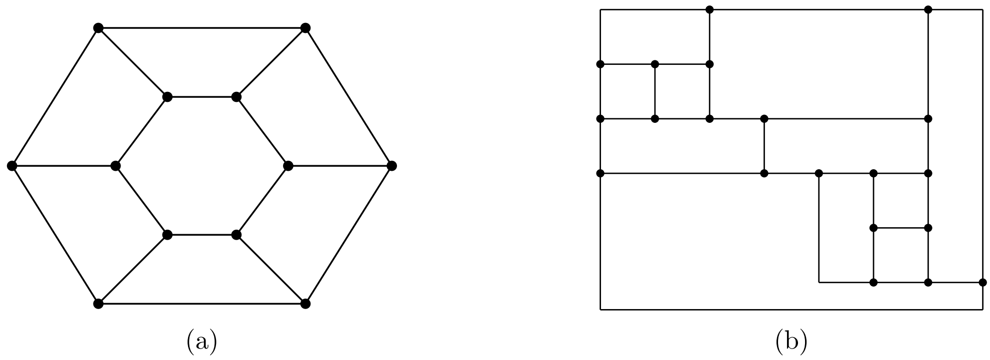


Figure 2.1: Examples of 3-regular, 3-connected, bipartite, planar graphs.

Definition 2.5. *A graph is Hamiltonian if it contains a cycle that visits every vertex exactly once.*

Having formally introduced all necessary ingredients, we are now ready to give the formal definition of Barnette graphs.

Definition 2.6. *We call Barnette a graph that is* 3*-regular,* 3*-connected, bipartite and planar.*

2.2 Barnette graphs and their construction sequences

A decisive result by Holton, Manvel and McKay [17] states that all Barnette graphs can be generated by iteratively applying the operations of cube- and C_4-expansions to the cube graph (see Fig. 2.2).

The former is applied to a vertex x of a Barnette graph and results in substituting vertex x with seven new vertices $x_1, \ldots, x_7$ and the edges (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_1, x_4), (x_1, x_5), (x_4, x_6), (x_5, x_6), (x_2, x_4), (x_2, x_7), (x_3, x_7), (x_6, x_7), (x_3, x_5), where y_1, y_2 and y_3 are neighbors of x.

The later is applied on two edges e_1 and e_2 that have an odd distance along the boundary of a face of a Barnette graph and results in subdividing both

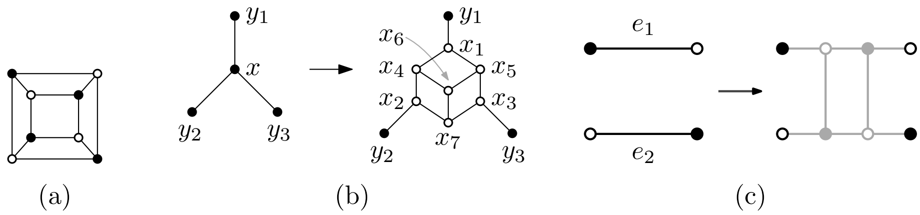


Figure 2.2: Starting from (a) the cube graph all Barnette graphs can be generated with either (b) cube-expansions or (c) C_4-expansions.

edges e_1 and e_2 twice and linking up the new vertices with two edges such that simplicity and bipartiteness are preserved.

Note that Barnette graphs constructed solely through cube-expansions are Hamiltonian, as each cube-expansion preserves Hamiltonicity. Therefore, the main challenge in settling Barnette's conjecture lies in dealing with C_4-expansions.

2.3 A General SAT Formulation for the Book Embedding Problem.

Let X be a set of n boolean variables, that is, $X = \{x_1, x_2, \ldots, x_n\}$. A *literal* over X is either a variable in X or its negation, that is, either x_i or $\neg x_i$ for some i in $\{1, 2, \ldots, n\}$. A *clause* over X is a disjunction of distinct literals of X, e.g., $(x_1 \vee \neg x_5 \vee x_7)$. A *truth assignment* for X is an assignment of either "`true`" or "`false`" to each variable of X. Given a truth assignment for X, a clause over X is *satisfied*, if at least one of its literals has been assigned the value "`true`". A boolean formula Φ in conjunctive normal form over X is a conjunction of clauses over X, that is, $\Phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$, where $C_1, C_2, \ldots, C_m$ are clauses over X. Given a boolean formula Φ over X, the boolean satisfiability problem (or SAT problem for short) asks whether there exists a truth assignment for X that makes Φ satisfiable, that is, all its clauses are satisfied.

When a problem can be expressed as an instance of the SAT problem, modern SAT solvers can be used to efficiently compute corresponding solutions (or to determine that no such solution exists). A notable example is the p-page book embedding problem, which for a given (not necessarily planar) graph asks for an ordering of its vertices and a partition of its edges into p sets, called *pages*, such that the edges of each page are crossing-free, that is, for any two edges of the same page their endpoints do not alternate in the underlying linear order. For this problem, Bekos, Kaufmann and Zielke [5] formulated the problem of testing whether a given graph G admits a book embedding with p pages as a SAT instance $\mathcal{F}(G)$. In the following, we recall the most important aspects of this formulation. More precisely, for every pair of distinct vertices u and v of G, $\mathcal{F}(G)$ has a variable $\sigma(u, v)$, which is `true` if and only if vertex u has precedes v in the underlying vertex order, namely, $u \prec v$. Further, for every edge e of G, $\mathcal{F}(G)$ has a variable $\phi_i(e)$, which is `true` if and only if edge e is assigned to page i with $1 \leq i \leq p$. Finally, for every pair of two distinct edges e and e' of G, $\mathcal{F}(G)$ has a variable $\chi(e, e')$, which is `true` if and only if e and e' are both assigned to the same page. If n and m are the number of vertices and edges of G, respectively, then $\mathcal{F}(G)$ has in total $O(n^2 + pm + m^2)$ variables, while a set of $O(n^3 + m^2)$ clauses ensures that the underlying order is valid, and that no two edges of the same page cross.

CHAPTER 3

A SAT Formulation for Testing Hamiltonicity in Planar Graphs

In this section, we present a SAT-based formulation for the problem of deciding whether a given planar graph is Hamiltonian, that is, whether it contains a simple cycle passing through every vertex exactly once. If such a cycle exists, it is returned in the output from the solution of the SAT instance.

This problem is closely related to another well-studied graph theoretic problem; the one of testing whether a planar graph admits 2-page book embedding. Recall that a p-page book embedding of a given (not necessarily planar) graph defines an ordering of its vertices and a partition of its edges into p sets, called *pages*, such that the edges of each page are crossing-free, that is, for any two edges of the same page their endpoints do not alternate in the underlying linear order. For a given planar graph, testing for Hamiltonicity is equivalent to determining whether the graph admits a 2-page book embedding such that every pair of consecutive vertices in the linear order—including the first and last that we call *extreme*—are adjacent in the graph.

Note that, in general, graphs that admit 2-page book embeddings are not necessarily Hamiltonian, e.g., the complete bipartite graph $K_{2,3}$ is not Hamiltonian but it does admit a 2-page book embedding. Conversely, any graph that admits a 2-page book embedding can be made Hamiltonian by adding edges, namely, the ones between consecutive vertices of the underlying vertex order—including again the one between the first and last vertex. Because of this relationship, graphs admitting 2-page book embeddings are referred to as *subhamiltonian* in the literature. Likewise, the vertex order of a 2-page book embedding is referred to as *subhamiltonian cycle* since it specifies the edges that are needed in an augmentation of the graph to Hamiltonian.

Note that subhamiltonian graphs are, by definition, planar. Furthermore, a subhamiltonian cycle with n edges in a planar graph with n vertices corresponds to a Hamiltonian cycle, and vice versa. In the following, we will leverage this relationship to devise a SAT-formulation of the problem of deciding whether a given planar graph is Hamiltonian. Since the formulation is of independent interest (not solely for verifying Barnette's conjecture), we present it in the most general form possible.

3.1 Description of the SAT extension

Let $G = (V, E)$ be a planar graph with n vertices and m edges, that is, $n = |V|$ and $m = |E|$. To determine whether G is Hamiltonian, we first extend the SAT instance $\mathcal{F}(G)$ presented in Section 2.3 with $p = 2$ by introducing a $O(n^3)$ variables. More specifically, for every triplet $\langle x, y, z\rangle$ of distinct vertices of G, if variable $\psi(x, y, z)$ is `true`, then x precedes z, and y appears between x and z in the underlying vertex order (note that the other direction does not necessarily hold). The later can be guaranteed by introducing the following $O(n^3)$ clauses to $\mathcal{F}(G)$:

$$\psi(x, y, z) \longrightarrow \sigma(x, y) \wedge \sigma(y, z), \quad \forall x, y, z \in V \text{ with } x \neq y \neq z \neq x \tag{3.1}$$

which can be easily rewritten in conjunctive normal form as:

$$(\neg\psi(x, y, z) \vee \sigma(x, y)) \wedge (\neg\psi(x, y, z) \vee \sigma(y, z)), \forall x, y, z \in V \text{ with } x \neq y \neq z \neq x$$

We further ensure compatibility between the σ-variables and the ψ-variables by introducing $O(n^3)$ additional clauses to $\mathcal{F}(G)$:

$$\begin{aligned} &\sigma(x, y) \longrightarrow \neg\psi(y, z, x) \\ &\sigma(y, x) \longrightarrow \neg\psi(x, z, y), \quad \forall x, y, z \in V \text{ with } x \neq y \neq z \neq x \end{aligned} \tag{3.2}$$

which can also be easily written in conjunctive normal form as:

$$(\neg\sigma(x, y) \vee \neg\psi(y, z, x)) \wedge (\neg\sigma(y, x) \vee \neg\psi(x, z, y)), \forall x, y, z \in V \text{ with } x \neq y \neq z \neq x$$

In the following, we demonstrate how to use these variables to ensure that (i) no two non-adjacent vertices of G are consecutive in the linear order, and that (ii) the first and last vertices in the linear order are adjacent in G. Properties (i) and (ii) imply that if two vertices either are consecutive or extreme in the linear order, then there is an edge between them, which in turn implies that the calculated embedding (if any) yields a Hamiltonian cycle.

To guarantee Property (i), for each pair of non-adjacent, distinct vertices x and y of G with w.l.o.g. $\deg(x) \leq \deg(y)$, we introduce the following clause to $\mathcal{F}(G)$:

$$\bigvee_{(x,z)\in E} (\psi(x,z,y) \vee \psi(y,z,x)) \tag{3.3}$$

This clause guarantees that between x and y there exists a neighbor of x, or equivalently, that x and y cannot be consecutive along the Hamiltonian cycle (if any). It is worth noting that, to guarantee Property (i), it would also suffice to ensure the presence of any vertex of G between x and y (i.e., not necessarily a neighbor of x or of y). However, in practice, this approach proved to be rather inefficient, since it results in unnecessarily long clauses.

For reasons of efficiency, we can further impose that whenever two vertices are consecutive in the linear order, then the edge connecting them is in the first page of the layout. Note that, although this requirement is not necessary for the correctness of our approach, in practice it eliminates symmetric solutions and thus leads to faster computations. This requirement can be guaranteed by introducing for each pair of adjacent vertices x and y of G the following clause to $\mathcal{F}(G)$:

$$\left(\bigwedge_{(x,z)\in E} \neg\psi(x,z,y) \wedge \neg\psi(y,z,x) \right) \rightarrow \phi_1((x,y)) \tag{3.4}$$

To guarantee Property (ii), for each pair of non-adjacent, distinct vertices x and y of G, we need to ensure that if x is the first (last) vertex in the order of the vertices, then y is not the last (first, respectively), and vice versa. To translate these requirements into corresponding clauses efficiently, we will leverage the following lemma.

Lemma 3.1. *A vertex x is first in the vertex order if and only if it precedes all its neighbors in the vertex order.*

Proof. Assume first that x precedes all its neighbors in the vertex order. To derive a contradiction, suppose that x is not the first vertex in this order, that is, there exists at least one vertex that comes before x. Let y be the last vertex preceding x, that is, between x and y there is no vertex of G, or equivalently, x and y are consecutive. It follows that y is not a neighbor of x, that is, $y \notin N(x)$. Assume without loss of generality that $\deg(x) \leq deg(y)$. Since $(x, y) \notin E$, by Property (i), there exists at least one neighbor of x, say z, such that $\psi(y, z, x) = \texttt{true}$, which implies $\sigma(y, z) \wedge \sigma(z, x)$ (by the definition of variable $\psi(y, z, x)$). However, this is a contradiction to the fact that x and y are consecutive in the order of the vertices. This completes the proof, since the other direction of the lemma is trivial. □

Symmetrically, one can prove the following lemma.

Lemma 3.2. *A vertex x is last in the vertex order if and only if it follows all its neighbors in the vertex order.*

In view of Lemmas 3.1 and 3.2, for each pair of non-adjacent, distinct vertices x and y of G, we can guarantee that if vertex x is the first in the order of the vertices, then y is not the last by the following clause to $\mathcal{F}(G)$:

$$\sigma(x, y) \bigwedge_{(x,z) \in E} \sigma(x, z) \longrightarrow \neg \left(\bigwedge_{(y,z) \in E} \sigma(z, y) \right) \tag{3.5}$$

which can be easily rewritten in conjunctive normal form as:

$$\neg\sigma(x, y) \bigvee_{(x,z) \in E} \neg\sigma(x, z) \bigvee_{(y,z) \in E} \neg\sigma(z, y)$$

The clauses for the remaining three cases needed to guarantee Property (ii) can be obtained symmetrically. Observe that, in total, we introduced $O(n^2)$ additional clauses for both properties and that the length of each clause is proportional to the degree of the two involved vertices. We summarize these findings in the following theorem, which is the main theorem of this section.

Theorem 3.1. *Let $G = (V, E)$ be a planar graph with n vertices. Then, G is Hamiltonian if and only if formula $\mathcal{F}(\mathcal{G})$ is satisfiable. In addition, $\mathcal{F}(\mathcal{G})$ has $O(n^3)$ variables and clauses.*

Proof. The numbers of σ-, ϕ-, χ- and ψ-variables are $O(n^2)$, $O(m)$, $O(m^2)$ and $O(n^3)$, respectively. The number of clauses of $\mathcal{F}(G)$ is $O(n^3 + m^2)$ due to the original formulation plus $O(n^3)$ additional ones to ensure Hamiltonicity (refer to the clauses of Eq. (3.1) and (3.2)). Since G is planar, it follows that $\mathcal{F}(G)$ has $O(n^3)$ variables and clauses. So, to prove the theorem it remains to show that (i) a satisfying assignment of $\mathcal{F}(G)$ yields a Hamiltonian cycle of G, and that (ii) a Hamiltonian cycle in G yields a satisfying assignment of $\mathcal{F}(G)$.

(i) *From a satisfying assignment to a Hamiltonian cycle:* Let $(\hat{\sigma}, \hat{\phi}, \hat{\chi}, \hat{\psi})$ be a satisfying assignment to $\mathcal{F}(G)$. Bekos, Kaufmann and Zielke [5] showed that $\hat{\sigma}$ yields a total order of the vertices of G, while $\hat{\phi}$ and $\hat{\chi}$ an assignment of the edges of G to the two pages of the book embedding such that no two edges of the same page cross. We next prove that every pair of consecutive vertices in the total order are joined by an edge. Assume for a contradiction that there exists a pair of consecutive vertices in the total order, say u and v, that are not joined by an edge. Without loss of generality, we may further assume that u precedes v. Therefore, $\hat{\sigma}(u, v) = \texttt{true}$. By Eq. (3.2), it follows that for every vertex $z \notin \{u, v\}$, it holds that $\hat{\psi}(v, x, u) = \texttt{false}$. Since $(\hat{\sigma}, \hat{\phi}, \hat{\chi}, \hat{\psi})$ is a satisfying assignment to $\mathcal{F}(G)$, by Eq. (3.3) the later implies that there exists a vertex, say w with $w \notin \{u, v\}$, such that $\hat{\psi}(u, w, v) = \texttt{true}$, which by Eq. (3.1) implies that $\hat{\sigma}(u, w) \wedge \hat{\sigma}(w, v)$. Therefore, w appears between u and v in the total order, which is a contradiction to our assumption that u and v are consecutive. Hence, every pair of consecutive vertices in the total order are joined by an edge, as desired. Since the proof that the extreme vertices of the total order are also joined by an edge is symmetric, the calculated book embedding yields a Hamiltonian cycle.

(ii) *From a Hamiltonian cycle to a satisfying assignment:* Assume that G has a Hamiltonian cycle, which implies that G has a book embedding $\mathcal{E}(G)$ on two pages, in which each pair of consecutive vertices of $\mathcal{E}(G)$ as well as the extreme vertices of $\mathcal{E}(G)$ are joined by an edge. We define an assignment $(\hat{\sigma}, \hat{\phi}, \hat{\chi}, \hat{\psi})$ to the σ-, ϕ-, χ- and ψ-variables of $\mathcal{F}(G)$ consistent with the intended meaning of these variables as follows: (a) for each pair of vertices u and v of G, $\hat{\sigma}(u, v) = \texttt{true}$ if and only if u is before v in $\mathcal{E}(G)$, (b) for each edge e of G, $\hat{\phi}_q(e) = \texttt{true}$ if and only if edge e is assigned to the q-th page of $\mathcal{E}(G)$, where $q \in \{1, 2\}$, (c) for each pair of edges e and e' of G, $\hat{\chi}(e, e') = \texttt{true}$ if and only if e and e' are assigned to the same page of $\mathcal{E}(G)$, and (d) for each triple $\langle u, v, w \rangle$ of distinct vertices of G, $\hat{\psi}(u, v, w) = \texttt{true}$ if and only if u precedes w, and v appears between v and w in $\mathcal{E}(G)$. We next prove that the assignment $(\hat{\sigma}, \hat{\phi}, \hat{\chi}, \hat{\psi})$ satisfies $\mathcal{F}(G)$. Bekos, Kaufmann and Zielke [5] show that $(\hat{\sigma}, \hat{\phi}, \hat{\chi}, \hat{\psi})$ yields a valid 2-page book embedding, that is, all clauses of $\mathcal{F}(G)$ except those given by Eq. (3.1), (3.2), (3.3) and (3.5) are satisfied by $(\hat{\sigma}, \hat{\phi}, \hat{\chi}, \hat{\psi})$. The clauses given in Eq. (3.1) and (3.2) are also satisfied by $(\hat{\sigma}, \hat{\phi}, \hat{\chi}, \hat{\psi})$, since $\hat{\sigma}$ is a total order over the vertices of G. Since every pair of consecutive vertices of $\mathcal{E}(G)$ are joined by an edge, the clauses described in Eq. (3.3) are also satisfied. The same holds for the clauses described by Eq. (3.5), since the extreme vertices of $\mathcal{E}(G)$ are also joined by an edge. Hence, the assignment $(\hat{\sigma}, \hat{\phi}, \hat{\chi}, \hat{\psi})$ satisfies $\mathcal{F}(G)$, as desired. This completes the proof of this theorem. □

CHAPTER 4

Experimental Evaluation

In this section, we present the results of our experimental evaluation. In addition to evaluating our own algorithm, we also compared it against the algorithm by Brinkmann, Goedgebeur and McKay [7], called *cubhamg*, which is tailored for finding Hamiltonian cycles in cubic graphs. Our experiment was conducted on a single-node machine with a Intel(R) Core(TM) i7-13700 processor at 2.10GHz with 16 cores and 64GB of RAM. More specifically, the experiment was conducted as follows:

(i) We implemented four different methods for generating Barnette graphs based on the operations of cube- and C_4-expansions [17]; see Section 4.1.

(ii) Using each of these methods, we generated in total $2,760$ graphs, the number of vertices of which was ranging in $[100, 500]$.

(iii) These graphs were provided as input to our algorithm, so as to measure the computation time for solving each of them; see Section 4.2.

(iv) Finally, we also installed *cubhamg* and using it we tested for Hamiltonicity each of the graphs of our experiment; see Section 4.3.

In the following, we present the details and results of our experiment. Note that the code of our SAT formulation as well as the code for the aforementioned methods for generating Barnette graphs is available to the community at a github repository (`https://github.com/linear-layouts/SAT`); see also `http://alice.math.uoi.gr`.

4.1 Procedures for Generating Barnette Graphs

For our experiment, we developed four different methods for generating Barnette graphs with a certain number of vertices. More specifically, starting from the cube graph, each of these methods generates a Barnette graph using the operations of cube- and C_4-expansions (see Fig. 2.2). The four methods differ in the order in which these operations are applied. More precisely:

- **M.1**: According to this method, every fourth operation is a cube-expansion; the remaining ones are C_4-expansions. More precisely, every three consecutive C_4-expansions (applied to two randomly chosen edges belonging to a face of the graph) are followed by a cube-expansion (also applied to a randomly chosen vertex) and this pattern is repeated until the obtained graph has the desired number of vertices.

- **M.2**: According to this method, the first one quarter of the operations are cube-expansions, while the remaining ones are C_4-expansions; the exact number of these operations is determined based on the desired number of vertices the obtained graph must have. Each cube-expansion is applied to a randomly chosen vertex. The C_4-expansions are applied to edges belonging to randomly chosen faces among those having length greater than or equal to 8. Furthermore, the edges on which each C_4-expansion is applied are required to be at least three edges apart along the face.

- **M.3**: This method is a modification of M.2. As before, the first quarter of the operations are cube-expansions, while the remaining ones are C_4-expansions. However, in this variant, the cube-expansions are applied in groups of five (each to a randomly selected vertex), as illustrated in Fig. 4.1. More specifically, in such a group, one cube-expansion is applied to a randomly chosen vertex v and immediately after four cube-expansions are applied to the new vertices introduced by the first cube-expansion that are not neighboring vertices which are also neighbors of v.

- **M.4**: According to this method, all operations performed are C_4-expansions until the obtained graph has the desired number of vertices. Note that we did not implement a fifth method that generates graphs using only cube-expansions, as it is already known that such graphs are Hamiltonian.

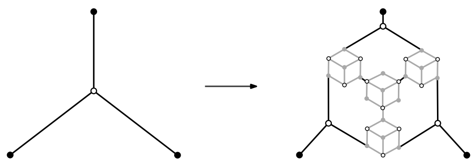


Figure 4.1: Illustration for method M.3.

Using each of the methods describe above, and for each value of n in $\{100, 120, 140, \ldots, 500\}$, we generated 50 Barnette graphs with n vertices for $100 \leq n \leq 320$, and 10 graphs for larger values of n. This resulted in a test-set of $4 \cdot (600 + 90) = 2,760$ graphs for our experiment. Note that besides these graphs, we also tested all Barnette graphs available from the *House of Graphs* [8], as well as several Barnette graphs that we crafted. **All the graphs that we tested were Hamiltonian and thus Barnette's conjecture still remains open.**

4.2 Experimental results

As already stated, each of the graphs generated with the methods described in Section 4.1 was given as input to our algorithm. To increase our algorithm's efficiency, we managed to avoid symmetric solutions by additionally requiring:

(i) a specific vertex to be first in the layout;

(ii) two specific vertices (different than the previous one) to have a specific order in the layout, that is, one of them to precede the other;

(iii) a specific edge to be assigned to the first page of the layout.

Note that each of these requirements is without loss of generality.

We evaluated our algorithm's performance on each of the four categories of graphs generated using the methods described in Section 4.1. Among them, M.4 was the fastest to process; our algorithm required in total 15d 23h 6m 53s to process all graphs of this category. It seems that using solely C_4-expansions in the generation procedure simplifies Hamiltonicity detection for our algorithm. This also raises an intriguing theoretical question, whether the subclass of Barnette graphs generated solely with C_4-expansions starting from the cube graph is Hamiltonian; we leave this as an open problem in the conclusions of this thesis. The performance of our algorithm for graphs of the remaining categories were as follows. The graphs of M.2 were processed in 18d 22h 26m 4s, those of M.3 in 21d 15h 39m 22s, while those of M.4 in 15d 23h 6m 53s. In total, the algorithm ran for 76 days, 17 hours, 2 minutes and 54 seconds across all tested instances. Fig. 4.2 shows how our algorithm scales with input size for the graphs of each of the four categories of our experiment. These results highlight that while the runtime slightly varies by category, our method consistently completes the analysis within a manageable timeframe.

4.3 Comparison against Cubhamg

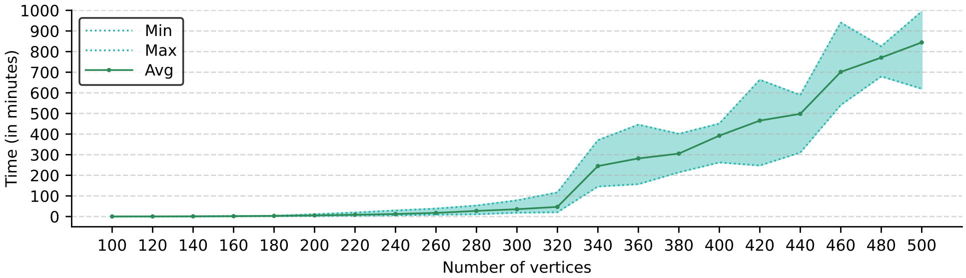
As previously mentioned, we compared our algorithm with *cubhamg*, developed by Brinkmann, Goedgebeur, and McKay [7]. This algorithm is specifically designed to determine Hamiltonicity in cubic graphs via a complete search using backtracking. For comparison, we ran *cubhamg* on the same dataset (i.e., the one described in Section 4.1).

We can summarize the outcome of our comparison, in short, as follows. When the *cubhamg* algorithm terminates successfully, it determines Hamiltonicity in under a second, even for graphs with 500 vertices. However, we observed an unexpected and unpredictable behavior: in certain instances, the computation freezes (or enters an infinite loop or fails to progress), and no result is returned, even after several hours of execution. Initially, the source of this issue was unclear to us, and we were unsure how to address it. After some investigation, we found that re-executing the algorithm multiple times often resolves the problem, suggesting that the search procedure may involve some form of randomization.

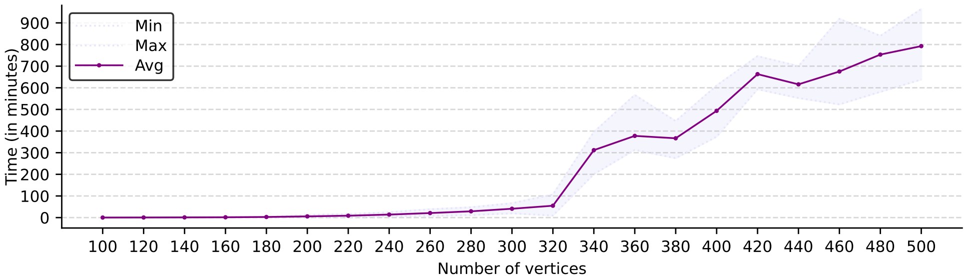
To conduct the experiment, we imposed a time limit on each execution of *cubhamg*, terminating any process that exceeded this limit and re-executing the algorithm until a result was obtained. For our experiments, the time limit was set to 5 minutes. Each graph was allowed up to 100 attempts.

A particularly notable outcome of our experiment is the contrast in reliability across the different methods for generating Barnette graphs. All graphs generated using methods M.2 and M.3 were successfully verified for Hamiltonicity on the first attempt, without a single failure. The former ones were processed in 1.77 sec, while the latter in 0.67 sec. In contrast, the graphs generated with methods M.1 and M.4 exhibited instability. Specifically, the former required 48 failed attempts across 24 different instances and were processed in 4 h and 6 m. The latter produced 7 failed attempts across 6 instances and were processed in 38 m and 20 sec (see Fig. 4.3). In any case, these execution times indicate a substantial difference in runtime compared to our algorithm.

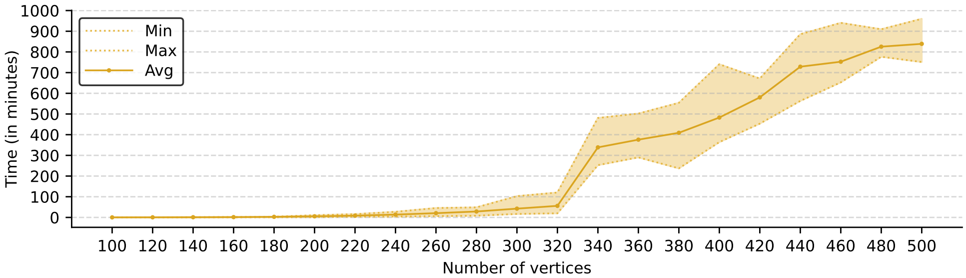
On the other hand, our findings also highlight some limitations. A striking example of us was a non-Hamiltonian cubic graph with approximately 450 vertices, that we crafted using multiple copies of Tait's fragment [22]. For this instance, *cubhamg* required more than 30 attempts before successfully certifying its non-Hamiltonicity. It seems to us that the algorithm struggles with certain graphs and this behavior seems to become more evident when the input graph is non-Hamiltonian. In contrast to *cubhamg*, our algorithm consistently delivers correct and conclusive results. It fully certifies whether a graph is Hamiltonian or non-Hamiltonian and is guaranteed to terminate for any input, thanks to the robustness of modern SAT solvers. While it operates more slowly, this is a deliberate trade-off that favors reliability over speed. Most importantly, it always produces an answer, making it a robust tool for Hamiltonicity testing. Notably it also extends to graphs of higher degree.



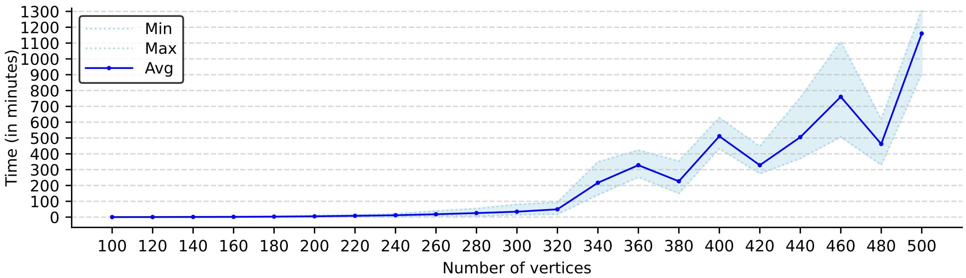
(a)



(b)

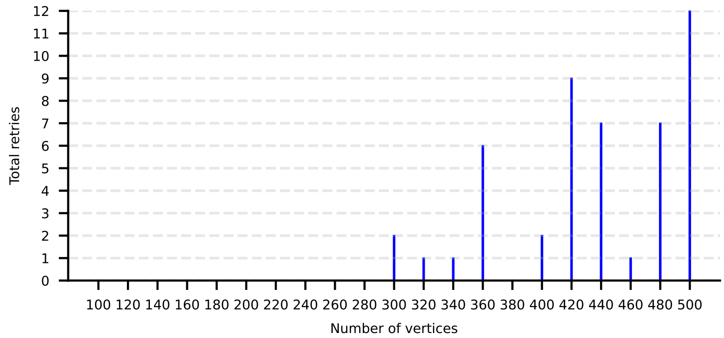


(c)

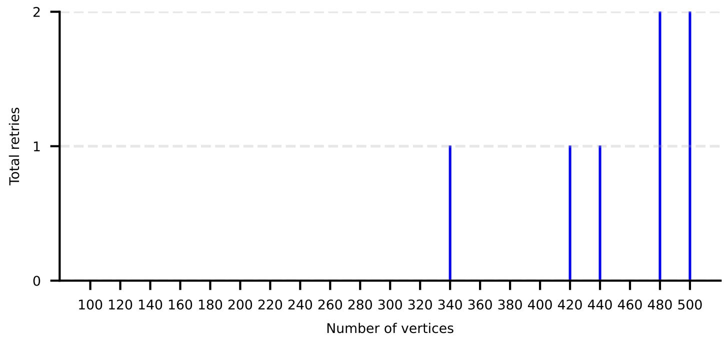


(d)

Figure 4.2: Illustrating how our algorithm scales with input size across each category of graphs generated by methods (a) M.1 (b) M.2 (c) M.3 (d) M.4.



(a)



(b)

Figure 4.3: Illustration of the number of failed attempts of *cubhamg* for the graphs of our experiment. The graphs generated with (a) M1 yielded 48 failed attempts across 24 different instances, (b) M4 yielded 7 failed attempts across 6 different instances.

CHAPTER 5

CONCLUSIONS

In this thesis, we have continued the study of Barnette's conjecture through the lens of 2-page book embeddings and SAT. We modified a known SAT formulation for the latter problem, which allowed us to search for the existence of Hamiltonian cycles in planar graphs, and thus further investigate Barnette's conjecture. Unfortunately, our approach turned out to be slower than the backtracking search of *cubhamg* but it scales to planar graphs of higher degree.

We conclude our work with an interesting question, of theoretical nature, that stems from our work. We believe that this problem is an important milestone in settling Barnette's conjecture.

- Is it true that all Barnette graphs generated solely by C_4-expansions are Hamiltonian?

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