Department of Economics
University of Ioannina

Small sample size corrections of the t- and F-econometric tests in the Linear Models with ARMA(1,1) disturbances

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

by

Dimitra Tatsi

June, 2025

Consulting Committee

1. Symeonides Spyridon, Professor, University of Ioannina (Supervisor)
2. Karaganis Anastassios, Associate Professor, Panteion University
3. Karpetis Christos, Professor, University of Macedonia

Examination Committee

1. Goletsis Yorgos, Associate Professor, University of Ioannina
2. Karaganis Anastassios, Associate Professor, Panteion University
3. Karpetis Christos, Professor, University of Macedonia
4. Salamaliki Paraskevi, Assistant Professor, University of Ioannina
5. Simos Theodore, Professor, University of Ioannina
6. Symeonides Spyridon, Professor, University of Ioannina
7. Venetis Ioannis, Professor, University of Patras

DECLARATION

I hereby declare that the thesis entitled "Small sample size corrections of the t- and F-econometric tests in the Linear Models with ARMA(1,1) disturbances" submitted by me, for the award of the degree of *Doctor of Philosophy* to University of Ioannina is a record of bonafide work carried out by me under the supervision of Dr. Symeonides, Department of Economics, School of Economics and Administrative Sciences, University of Ioannina.

I further declare that the work reported in this thesis has not been submitted and will not be submitted, either in part or in full, for the award of any other degree or diploma in this institute or any other institute or university.

Place: Ioannina

Date: 6/2025

ABSTRACT

Using refined asymptotic techniques, we derived small-sample size corrections for the t and F econometric tests in the linear regression model with ARMA(1,1) errors. These size corrections are based on the Edgeworth-corrected critical values and on the Cornish-Fisher-corrected test statistics. In particular, the size correction of the t-test can be derived from the normal or Student-t approximations. Moreover, the small-sample size corrections for the Wald and F tests can be derived from the χ^2 and F approximations, respectively.

Given that the Edgeworth and Cornish-Fisher corrections have an error of order $O(T^{-3/2})$, where T is the sample size, the relative performance of these corrections can be investigated only by means of simulation experiments.

In the context of the linear regression model with ARMA(1,1) errors, the simulation experiment conducted in this thesis seems to confirm the theoretical advantages of the Cornish-Fisher corrections. In almost all cases, the Edgeworth and Cornish-Fisher size corrections seem to improve the small-sample null rejection probabilities of the corrected t and F tests relative to the corresponding uncorrected tests.

Keywords: *ARMA(1,1)*, *Cornish-Fisher corrections*, *Edgeworth approximation*, *Monte Carlo simulation*, *refined asymptotics*, *small sample size corrections*, *t and F econometric tests*.

"*I dedicate this thesis to my Professor, Spyros Symeonides, whose unwavering support and guidance have inspired me throughout this journey, and to my mother, Ismini, and my father, Spyros, for their moral support and encouragement whenever I needed it.*"

ACKNOWLEDGEMENT

In April 2021, I embarked on what would become one of the most challenging and rewarding journeys of my academic and personal life — the pursuit of my doctoral research. What began as a distant goal gradually evolved into a transformative experience, filled with moments of doubt, discovery, growth, and determination. Throughout this path, I was fortunate to be surrounded by people whose support, wisdom, and belief in me made all the difference.

I would like to express my sincere and profound gratitude to my supervisor, Professor **Spyro Symeonide,** who stood by me as a mentor and supporter throughout all stages of this research journey. With his deep, extensive and multifaceted knowledge, which he generously shared with me, he provided not only scientific guidance but also a way of thinking that will accompany me in every step I take.

What I appreciated most in his supervision was the trust he showed me: he gave me the space and freedom to take initiative and manage my research in a creative and independent way, never limiting me. At the same time, he was always present in every decision — with discretion, yet with meaningful and substantial guidance, making me feel I was never alone in this journey.

Even in my mistakes, he encouraged me to continue. Through his unique way, he helped me see them as valuable lessons rather than failures. He motivated me to learn from them, to persist, and to become more determined until I found the solution — a lesson that goes far beyond the boundaries of research.

Thanks to him, I now reach the end of a demanding but truly unique journey, which has taught me a great deal — not only academically, but also on a personal level. A journey that helped me grow in my everyday life and develop a more mature way of facing challenges and difficulties.

Moreover, I would also like to sincerely thank the six distinguished members of my examination committee —**Simos Theodore, Salamaliki Paraskevi, Venetis Ioannis, Karaganis Anastassios, Karpetis Christos, and Goletsis Yorgos** — who honored me with their presence at this important milestone of my academic journey.

Additionally, I wish to extend my profound sense of gratitude to **my parents, Ismini Mousga and Spyro Tatsi,** for all the sacrifices they made during my research and also providing me with moral support and encouragement whenever required.

Last but not the least, I would like to sincerely thank my **friends** who stood by me at every step of this journey, offering unwavering support, encouragement, and precious moments of relief when I needed them the most.

In this research journey, nothing would have been the same without my invaluable companions along the way. Their presence, encouragement, and the shared moments of both anguish and small

victories added meaning and color to every step of the process. As the saying goes, 'it is better to travel with company than alone, even if the road is long' — and for that, I thank them from the bottom of my heart.

Dimitra Tatsi

CONTENTS

Chapter 1

Introduction

1.1 Refined Asymptotic Theory in Econometrics

This study is situated within the framework of refined (i.e., higher-order) asymptotic theory, aiming to develop small-sample corrections for the t and F test statistics in linear regression models with a non-scalar error covariance structure. Refined asymptotic theory constitutes one of the main tools for understanding the small-sample behavior of econometric estimators and test statistics, alongside exact finite-sample theory and Monte Carlo simulations (Magdalinos (1983)).

In econometric theory, two primary schools utilize refined asymptotic methods: the Sargan school and the Nagar school.

The Sargan school relies on representing the estimator or test statistic as a function of random variables whose cumulants can be analytically computed. Using these cumulants and the corresponding partial derivatives, one can derive Edgeworth or Edgeworth-type expansions, which improve the accuracy of the normal approximation (Chambers (1967), Sargan (1975), Sargan (1976), Sargan (1980), Phillips (1977a), Phillips (1977b), Phillips (1978)). The polynomials used in these expansions include Hermite (or their transformations such as Chebyshev-Cramér) for the normal distribution, and Laguerre polynomials in the case of the χ^2 distribution (Chandra and Ghosh (1979)).

A hallmark of the Sargan school is its high level of mathematical rigor. However, empirical applications are limited due to extremely demanding computational requirements. As noted by Magdalinos (1983), computing symmetric acceptance regions accurate to order $O(1/\sqrt{T^3})$ may require millions of third-order derivatives, even for relatively small models.

In contrast, the Nagar school adopts a more computationally accessible approach. It is based on the asymptotic expansion of estimators or statistics in series using statistical differentials. The first few terms of these series are used to approximate the moments or the distribution function, resulting in simpler calculations with reliable results (Nagar (1959), Nagar (1962), Nagar (1970)). This method has been theoretically supported by various researchers (Basmann (1961), Amemiya (1966), Ramage (1971)), while Magdalinos (1992) largely confirmed its validity in econometric contexts. Moreover, the general validity of Edgeworth expansions and similar refined asymptotic techniques has been formally justified in mainstream statistical literature Bhattacharya and Ghosh (1978), Brown et al. (1974), further reinforcing the soundness of the Nagar school approach.

The methodology of this thesis is grounded in the Nagar school, employing Edgeworth-type expansions and moment approximations for correcting the t and F statistics. This approach allows both analytical rigor and practical applicability, with minimal computational burden, making it suitable for empirical studies with small samples.

1.2 Small Sample Issues and Refined Asymptotic Corrections in the Generalized Linear Model

The Generalized Linear Model (GLM) is usually estimated using the Feasible Generalized Least Squares (FGLS) estimator. Although the FGLS estimator has good asymptotic statistical properties, the lack of a general exact inference theory compels researchers to rely on econometric tests that are based on first-order asymptotic approximations of the distributions of the test statistics. However, since the sample size is often small, the actual size of these commonly used asymptotic tests may significantly deviate from the nominal size, which can lead to incorrect conclusions and misspecification of the econometric model.

In the econometric literature on the GLM (see, among others, Rothenberg (1984a), Rothenberg (1984b), Magdalinos and Symeonides (1995), Symeonides et al. (2017)), such problems are addressed through the use of Refined Asymptotic Techniques, which adjust the actual size of the t and F econometric tests in small-sample contexts. Unlike the conventional asymptotic t and F tests, which rely on the normal and chi-squared distributions, respectively, the corrected tests are based either on corrected critical values or on corrected test statistics.

Specifically, the corrected critical values of the t and F tests are derived from Edgeworth approximations to the normal (or Student-t) and chi-squared (or F) distributions, respectively, while the corrected t and F test statistics are obtained using the Cornish-Fisher expansion method. It should be noted that the Edgeworth approximations do not correspond to proper distributions, which means that in the tails of the Edgeworth expansions, negative "probabilities" may appear. On the contrary, the Cornish-Fisher corrected test statistics are properly defined random variables and are therefore theoretically preferable to the Edgeworth-corrected critical values.

It is worth noting that when exact distributions such as the Student-t and F are used as reference distributions, the resulting Edgeworth approximations are locally exact, in the sense that they coincide with the true distribution in simplified versions of the model.

The standard t and F econometric tests are based on consistent (first-order) estimators of the Ω matrix, which captures the covariance structure of the stochastic error term and both the Edgeworth and Cornish-Fisher corrections rely on an asymptotic expansion of the estimated Ω matrix around its true value.

1.3 Local Exactness and Degrees-of-Freedom Adjustments in Refined Asymptotic Approximations

A fundamental issue arising in statistical testing under small sample conditions is the discrepancy between the true and nominal size of usual econometric tests. Classical inferential methods, such as the Wald, likelihood ratio (LR), and Lagrange multiplier (LM) tests, may lead to conflicting conclusions due to these size differences (Rothenberg (1982)). Size correction constitutes an effective strategy to address this problem, as it reduces deviations between the true and theoretical size of tests, with a small cost in terms of power. Thus, even when alternative more efficient second-order tests exist, the use of size-corrected t and F tests enhances the reliability of inferences (Rothenberg (1984b)).

In statistical and econometric research, t and F tests are widely used to test hypotheses concerning model parameters. When the sample size is large, the distributions of these tests are satisfactorily approximated by the normal and χ^2 distributions, respectively. However, in small samples, there is often a significant discrepancy between the true size of a test and the nominal significance level, which may lead to erroneous conclusions and misspecification of the model.

To improve the accuracy of statistical tests in small samples, two main size correction strategies have been proposed:

- Corrected critical values via Edgeworth expansions (Rothenberg (1988)), and
- Corrected test statistics via the Cornish-Fisher method (Cornish and Fisher (1938), Fisher and Cornish (1960)).

Both techniques rely on asymptotic expansions and have an error of order $O(T^{-3/2})$, where T is the sample size. Although considered asymptotically equivalent, they differ with respect to their behaviour in the tails of the distributions. The Cornish-Fisher approach offers a significant practical advantage: the same corrected statistic can be used for any significance level, unlike the Edgeworth approach where critical values must be recalculated for each different level (Rothenberg (1988)).

Moreover, more refined asymptotic techniques apply degrees-of-freedom adjustments, basing the approximations on the exact Student-t and F distributions rather than the asymptotic normal and χ^2 ones (Rothenberg (1984b)). This approach leads to greater accuracy, especially when the standard asymptotic assumptions are not fully met or the convergence rate is slow due to the number of estimated parameters.

In this context, the notion of local exactness is introduced. An asymptotic approximation is said to be locally exact when it coincides with the exact distribution of the test statistic in a sufficiently simplified version of the model (Rothenberg (1984b)). This property makes the approximation theoretically stronger and practically more reliable, as it reduces the discrepancy between the theoretical and the true distribution, thereby enhancing the validity of statistical conclusions.

Finally, for the practical implementation of these methods, unknown parameters and random variables are replaced by consistent estimators or predictors. In this way, the asymptotic validity of the expansions is maintained, allowing their application in empirical contexts with limited sample sizes.

In conclusion, the use of locally exact approaches and size correction methods in statistical testing improves both the theoretical foundation and empirical reliability of results, particularly within econometric models of complex structure or small samples.

1.3.1 Stochastic order of our expansions

In this thesis we use the stochastic order $\omega(\cdot)$ defined as follows:

For any collection of real-valued stochastic quantities (scalars, vectors, or matrices), we write Y_τ ($\tau \in I$), in S, which is defined on the probability space (Ω, A, P), and we say that it is of order $\omega(\tau^i)$, and we write $Y_\tau = \omega(\tau^i)$, if for a given $n > 0$, there exists some $0 < \varepsilon < \infty$ such that

$$\Pr\left[\| Y_\tau/\tau^i \| > (-\ln \tau)^\varepsilon\right] = o(\tau^n), \tag{1.1}$$

as $\tau \to 0$, where $\| \cdot \|$ denotes the Euclidean norm. If (1.1) holds for every $n > 0$, then we write $Y_\tau = \omega(\infty)$. The use of this order notation is justified by the fact that if two stochastic quantities differ by a term of order $\omega(\tau^i)$, then under general conditions the distribution function of one is an asymptotic expansion of the distribution function of the other, with an error of order $O(\tau^i)$. Moreover, the orders $\omega(\cdot)$ and $O(\cdot)$ possess the same functional properties Magdalinos (1992).

1.4 Objectives of the Doctoral Thesis

This doctoral dissertation focuses on improving the accuracy of econometric tests under small sample conditions. In particular, classical hypothesis tests such as the t and F tests are based on asymptotic distributions (normal and chi-squared), the accuracy of which decreases significantly when the sample size is limited. The deviation of these tests' actual performance in small samples can lead to substantial discrepancies between the actual and nominal test sizes, resulting in misleading conclusions and incorrect model specifications.

Based on this observation, the objectives of the dissertation are formulated as follows:

1. Specialization of Edgeworth and Cornish-Fisher corrections for small samples in the Generalized Linear Model with ARMA(1,1) errors

The first objective is the theoretical and computational development of size corrections for t and F tests using the Edgeworth and Cornish-Fisher methods, specifically for the Generalized Linear Model (GLM) in the presence of ARMA(1,1) stochastic errors.

The need for this specialization arises from the fact that existing applications of these corrections in the literature are mainly limited to simpler models without internal dependence. The ARMA(1,1) model introduces dynamic dependence and lagged feedback in the error term, which alters the distribution of test statistics. Therefore, adjustments of the corrections to this structure are required, through appropriate expressions for the moments and covariances of the relevant estimators.

The two approaches (Edgeworth and Cornish-Fisher) are selected due to their strong theoretical foundation, their widespread use in statistical contexts, and their ability to produce more accurate finite-sample approximations, maintaining an asymptotic error of order $O(T^{-3/2})$.

2. Comparison of the accuracy of the two corrections using Monte Carlo simulations

The second objective concerns the systematic empirical evaluation of the effectiveness of the aforementioned corrections using Monte Carlo simulation methods. The aim is to estimate, across a range of small sample scenarios, the actual size of the corrected t and F tests under each method and to compare the results with respect to proximity to the nominal significance level.

Particular emphasis will be placed on:

- the effect of dependence in the error term through the parametrization of the ARMA(1,1) process,
- the investigation of local accuracy of the tests after the corrections,

- the performance of the corrected tests at different significance levels ($\alpha = 0.01$, 0.05, 0.10),
- and the stability of the results across different sample sizes ($T = 20, 30$).

Through this comparison, the goal is to draw well-documented conclusions regarding the relative performance and practical usefulness of the two correction methods in the context of small samples.

General Objective

Overall, the dissertation aims to contribute to econometric methodology by providing practically applicable and theoretically well-founded techniques for correcting the size of statistical tests, adapted to small sample conditions. This approach will offer useful tools for empirical researchers, enhancing the accuracy and reliability of econometric inferences in applications with limited data.

1.5 Doctoral Thesis Structure

This thesis is organized into six main chapters, aiming at a thorough investigation and computational evaluation of the Generalized Linear Model (GLM) with ARMA(1,1) type disturbances, as well as the presentation of asymptotic correction theory in small sample cases.

The Second Chapter focuses on the theoretical foundation of the Generalized Least Squares (GLS) model with ARMA(1,1) type disturbances. The basic AR, MA, and ARMA models are presented, followed by their incorporation within the GLS framework.

In the Third Chapter, the previous analysis is extended to the Generalized Linear Model (GLM) with ARMA(1,1) disturbances. A detailed presentation of the t and F tests and their respective corrections is given, as well as the quantities required for their implementation.

The Fourth Chapter focuses on computational techniques necessary for the estimation of quantities without closed-form expressions, which are critical for applying corrections to the t and F tests. Due to the complexity of the ARMA procedures, numerical methods are used to estimate the relevant quantities.

Two fundamental methodological approaches are also presented: the Gradient Descent algorithm and L2 Regularization. These techniques contribute to the stability of computations, preventing instability that arises when model coefficients approach extreme theoretical values (e.g., -1 or 1).

The chapter includes a basic Monte Carlo experiment, implemented with the purpose of estimating the necessary quantities without yet applying the corrections. Special emphasis is placed on the sensitivity analysis of results with respect to the number of repetitions, to ensure the reliability of computations.

Thus, the Monte Carlo experiment serves a dual purpose: it functions both as a parameter estimation tool and as a foundation for developing more accurate inference procedures in small samples. The purpose of the simulation is to quantify the behavior of the maximum likelihood estimators (MLE) under different combinations of parameters and sample sizes, as well as to compute the asymptotic moments — specifically the means, variances, and covariances — of the parameter estimators.

The Fifth Chapter uses the methodology of the previous chapter, in order to develop the full implementation of the Monte Carlo simulation for the evaluation of corrected t, Wald, and F tests. The experiment builds on the techniques of the Fourth Chapter, integrating them into a broader application framework.

The simulation results are extensively analyzed, and the behavior of the statistics is examined under various scenarios. Special attention is given to cases of negative values in the in the Cornish-Fisher corrected Wald and F statistics.

The chapter concludes with a comparative assessment of the effectiveness of the corrections and substantially contributes to the general conclusions of the thesis regarding the validity of statistical tests under ARMA(1,1) disturbances.

Finally, in the Sixth Chapter, the main findings of the thesis are summarized and the general conclusions derived from the theoretical and computational investigation are stated.

The thesis is accompanied by appendices, which include theoretical proofs, derivatives of the variance-covariance matrix, analysis of initial values, as well as regression results and visualizations. The bibliography is provided at the end.

Chapter 2

Generalized Least Squares (GLS) with Arma(1,1) disturbances

Time series analysis is a fundamental tool in econometrics, finance, data science and many other scientific fields, as it allows understanding and predicting the future behaviour of time-dependent variables. One of the most popular and widely used models for time series analysis is the Autoregressive Moving Average (ARMA), which combines two basic elements: autoregressive (AR) and moving average (MA). This chapter presents the basic characteristics of AR(p), AR(1), MA(q), MA(1), ARMA(p,q), ARMA(1,1) and GLS (Generalized Least Squares) models with ARMA(1,1) disturbances and their usefulness in time series analysis and forecasting.

2.1 Autoregressive Model AR(p) and AR(1)

The Autoregressive (AR) model is a workhorse of time series analysis and is extensively used to model and predict the pattern of stationary time series data (Hamilton (1994) and Box and Jenkins (1976)). It is based on the assumption that the current value of a time series is a linear function of previous values, with an additional random error term. In essence, the AR model attempts to capture the dependency of each data point on its past observations, making it particularly suited to modelling temporal dynamics in fields such as economics, finance, and the natural sciences. The mathematical equation of the AR model of order p, the AR(p) model, is expressed as:

$$Y_t = \rho_0 + \rho_1 Y_{t-1} + \rho_2 Y_{t-2} + \ldots + \rho_p Y_{t-p} + \varepsilon_t, \tag{2.1}$$

where Y_t is the value of the time series at time t, ρ_0 is a constant, $\rho_1, \rho_2, \ldots, \rho_p$ are the autoregressive coefficients, and ε_t is the error term assumed to be white noise.

One of the main requirements of the AR model is that the series must be stationary. A stationary series has a constant mean, variance, and autocorrelation over time. The roots of the characteristic equation of the model must lie inside the unit circle for the AR process to be stationary. Simply put, this means that the coefficients of the model must not produce explosive behaviour, so the series oscillates around a constant mean.

If the series is not stationary, it is typically transformed into a stationary form before an AR model can be fitted.

A key characteristic of the AR model is that the value at time t is a linear combination of past values. Specifically, an AR(1) model (references), an autoregressive process of order 1, can be expressed as:

$$Y_t = \rho_0 + \rho Y_{t-1} + \varepsilon_t \tag{2.2}$$

Here, the value at time t depends on the value at time $t-1$ and a random shock (error term) ε_t. The parameter ρ determines how sensitive the current value is to its past value. If $\mid \rho \mid < 1$, then the series is stationary, meaning that values will converge to a mean, fluctuating around it. If $\mid \rho \mid \geq 1$, the series becomes non-stationary and may diverge without bound, making it unsuitable for prediction.

The AR model also serves as a building block for more advanced time series models. For example, the ARMA (Autoregressive Moving Average) model combines autoregressive and moving average components to capture more complex behaviours. Furthermore, the ARIMA (Autoregressive Integrated Moving Average) model incorporates differencing to handle non-stationary data. These extended models, which are based on the AR framework, are widely used in practice to model and forecast time series data exhibiting trends or seasonality.

Though conceptually simple, the AR model provides a highly effective way of modelling time series data, especially when the data are stationary or can be transformed into a stationary form. It preserves temporal dependencies and offers a parsimonious method for forecasting future observations. However, for the AR model to perform well, the time series must be well-behaved in terms of stationarity. When used appropriately, the AR model is a valuable tool for uncovering the underlying structure of a time series and making reasonable predictions.

In brief, the AR model is a powerful and essential time series method that reveals internal dependencies within a dataset by describing its values as functions of lagged observations. Its simplicity and ability to capture temporal dependencies make it an indispensable tool in domains as diverse as finance, economics, and meteorology.

2.2 Moving Average Model MA(q) and MA(1)

The Moving Average (MA) model is one of the prominent techniques in time series analysis, focusing on modelling error terms or random shocks that affect the data (Hamilton (1994) and Box and Jenkins (1976)). While the Autoregressive (AR) model primarily captures the influence of past observations on present values, the MA model represents a process in which the current value of a time series is affected by past error terms. The most essential characteristic of the MA model is its ability to account for short-term behaviour and randomness in a time series — not explained by past observations themselves, but by shocks (or noise) to the system.

The MA(q) model, where q is the model order, is mathematically represented as:

$$Y_t = \mu + \varepsilon_t + \phi_1 + \varepsilon_{t-1} + \phi_2\varepsilon_{t-2} + \ldots + \phi_q\varepsilon_{t-q}, \tag{2.3}$$

where Y_t is the value of the time series at time t, μ is the mean of the time series, $\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q}$ are the error terms at concequtive time periods, and $\phi_1, \phi_2, \ldots, \phi_q$ are the moving average coefficients.

The assumption of stationarity is a key feature of the MA model. In time series modelling, stationarity refers to the statistical properties of the series — such as mean and variance — remaining constant over time. Because the MA model is a weighted sum of past error terms, and those errors are assumed to have constant mean and variance, the model is inherently stationary. This means that, unlike other models, it typically does not require differencing or other transformations to achieve stationarity.

A key characteristic of the Moving Average (MA) model is that the value at time t depends on past error terms (shocks), rather than past values of the series itself. Specifically, an MA(1) model (references) — a moving average process of order 1 — can be expressed as:

$$Y_t = \mu + \varepsilon_t + \phi\varepsilon_{t-1}. \tag{2.4}$$

Here, the value at time t is determined by the current shock ε_t and the previous period's shock ε_{t-1}. The parameter ϕ controls how much influence the past shock has on the current value.

The MA(1) process is always stationary, assuming the error terms ε_t are white noise — that is, they have zero mean, constant variance, and no autocorrelation.

However, for the model to be invertible, which ensures a unique and stable representation of the process and allows it to be expressed as an equivalent (infinite) AR process, the condition must be satisfied is $\mid \phi \mid < 1$.

Invertibility is important because it ensures that we can model the process in a well-defined way, avoiding ambiguity in parameter estimation. If $\mid \phi \mid \geq 1$, the model becomes non-invertible, meaning multiple MA representations could produce the same data, making it unsuitable for reliable modeling and forecasting.

The MA model is particularly suited for capturing short-run dependencies in time series. It is especially useful when the data exhibit random shocks or disturbances that have only a temporary effect. Such disturbances can arise from various sources — for example, market volatility, weather events, or other random external factors. The MA model provides a way to model and forecast these transient effects, making it a valuable tool in fields such as economics, finance, and environmental science.

One of the advantages of the MA model is its simplicity and interpretability. Unlike the AR model, which is concerned with modelling the influence of past values, the MA model focuses on how past shocks or innovations contribute to the current value. This makes it a good choice when random noise or unstable behaviour is present in the data and cannot be adequately explained by past observations alone. However, the MA model is limited in its ability to capture long-term dependencies, as it is primarily designed to model short-term dynamics.

In practice, the MA model is often combined with the AR model to form the more general Autoregressive Moving Average (ARMA) model. The ARMA model includes both autoregressive and moving average components, allowing it to represent a wider range of short-run and long-run relationships in time series data. Moreover, the ARIMA model generalizes the ARMA framework by incorporating differencing, thus enabling it to handle non-stationary series as well.

In summary, the Moving Average (MA) model is a foundational technique in time series analysis that offers a straightforward and practical method for modelling random shocks in a time series. By accounting for previous error terms, it effectively captures short-term dependencies and enables reliable forecasting in the presence of noise. While it may be less suited for modelling long-term trends, it remains an essential component of more complex models like ARMA and ARIMA. When used appropriately, the MA model provides valuable insights into the underlying structure of a time series and supports robust forecasting across a wide range of applications.

2.3 Autoregressive Moving Average Model ARMA(p,q)

Autoregressive Moving Average (ARMA) (Hamilton (1994) and Box and Jenkins (1976)) is a widely used time series model that combines two integral elements: the Autoregressive (AR) and the Moving Average (MA) components. By integrating these two elements. The model is particularly valuable because it accounts for both the impact of lagged values of the series and the influence of lagged error terms. Therefore, it provides a more comprehensive framework for predicting and analyzing data affected by both trend-like behaviour and random shocks.

A model of an ARMA(p,q), where p and q are the orders of the autoregressive and moving average components respectively, can be mathematically expressed as:

$$Y_t = \mu + \rho_1 Y_{t-1} + \rho_2 Y_{t-2} + \ldots + \rho_p Y_{t-p} + \phi_1 + \varepsilon_{t-1} + \phi_2 \varepsilon_{t-2} + \ldots + \phi_q \varepsilon_{t-q} \varepsilon_t, \tag{2.5}$$

where Y_t is the value of the time series at time t, μ is the mean of the time series, $\rho_1, \rho_2, \ldots, \rho_p$ are the parameters of the AR component, and $\phi_1, \phi_2, \ldots, \phi_q$ are the parameters of the MA component. The terms $\varepsilon_t, \varepsilon_{t-1}, \ldots$ represent the error terms. The AR component captures the influence of past values of the time series, while the MA component captures the effect of past errors on the current value.

One of the main limitations of the ARMA model, despite its flexibility and effectiveness, is its reliance on the stationarity assumption. If the time series is non-stationary — for example, if it exhibits trends or seasonality — the ARMA model cannot be applied directly. In such cases, differencing is usually applied to transform the series into a stationary one.

The ARMA model is widely applied in various domains. In finance and economics, it is used to model and forecast time series such as interest rates, exchange rates, and stock prices — all of which are influenced by past values and random shocks. In engineering, ARMA models are used to predict system behaviours such as vibrations and noise. In environmental studies, they help forecast variables like temperature, rainfall, and pollution levels.

In conclusion, the ARMA model is a powerful and essential tool for modelling and forecasting stationary time series data. Overall, the ARMA model is a benchmark method in time series analysis, offering valuable insights into dynamic systems and enabling robust forecasting across a wide range of fields.

2.4 ARMA(1,1)

The ARMA(1,1) model is a special case of the ARMA(p,q) model and combines both the moving average (MA) and autoregressive (AR) components into one, unifying framework (Hamilton (1994) and Box and Jenkins (1976)). It is used extensively in time series analysis to examine data that exhibit short-run correlations and random noise. The ARMA(1,1) model is particularly effective in scenarios where the current value of a time series depends on its immediate past value and the past error term. The combination of moving average and autoregressive components in the ARMA(1,1) model enables it to capture the persistence of past observations as well as the impact of past disturbances.

The ARMA(1,1) model can be expressed algebraically as:

$$Y_t = \mu + \rho Y_{t-1} + \varepsilon_t + \phi \varepsilon_{t-1}, \tag{2.6}$$

where Y_t is the time series value at time t, μ is the mean of the series, ρ is the autoregressive coefficient, which captures the influence of the past period's value Y_{t-1} on the present value, and ε_t is the error term at time t. The parameter ϕ is the moving average coefficient that reflects the impact of the lagged error term ε_{t-1} on the level of the series in the current period.

For the ARMA(1,1) model to be valid and interpretable, certain conditions must be satisfied. Specifically, the autoregressive coefficient ρ must satisfy $\mid \rho \mid < 1$ to ensure stationarity — that is, that the series has a constant mean and variance over time. Likewise, for the model to be invertible, allowing for a unique MA representation, the moving average coefficient must satisfy $\mid \phi \mid < 1$. These constraints ensure that the model is both mathematically well-defined and statistically reliable for forecasting.

The ARMA(1,1) model assumes that the present value of the series Y_t depends not only on its recent past value but also on the shock that occurred during the previous period. This dual dependence allows the model to reflect both the structure present in the series and the effect of unexpected shocks.

The ARMA(1,1) model assumes that the underlying time series is stationary, i.e., that its statistical properties, such as mean, variance, and autocorrelation, remain constant over time. This stationarity assumption is important because it ensures that the relationships between past values and error terms remain stable, allowing the model to accurately capture the behaviour of the series. If the series is not stationary—showing trends or seasonal patterns—the data may need to be transformed (e.g., by differencing) before fitting the ARMA(1,1) model.

The ARMA(1,1) model is typically estimated using maximum likelihood estimation (MLE) or least squares, by finding the optimum values for the parameters ρ_1 and ϕ_1 that minimize the discrepancy between the model's theoretical values and the actual data. Once the parameters have been estimated, the model can be used for forecasting purposes, making predictions on future values based on both past values and past innovations. The ARMA(1,1) model is well-suited for short-run forecasting, as it leverages both types of past information, often producing more accurate forecasts than models that rely solely on lagged observations.

This model is widely used across various disciplines. In finance, it is used to model stock prices, exchange rates, and interest rates, where both historical values and random shocks play a key role in future movements. In engineering, it is used in signal processing and system control to forecast behaviour in systems subject to noise or random disturbances. In environmental science, the model is applied to forecast variables such as temperature, rainfall, or pollution levels, which show short-term variation and longer-term persistence.

Finally, the ARMA(1,1) model provides a simple but effective way of modelling and predicting stationary time series data. By encompassing both moving average and autoregressive components, it captures both short-run dependencies and the influence of lagged errors on the current value of the series. Although ARMA(1,1) applies best to stationary data and may require adjustment when applied to non-stationary series, it remains one of the most widely used and practical tools for time series forecasting and analysis. Its ability to account for both persistent patterns and the influence of random shocks makes it a general-purpose tool across numerous fields, from economics and finance to engineering and environmental science.

2.5 GLS model with ARMA(1,1) disturbances

Generalized Least Squares (GLS) model with ARMA(1,1) errors is a useful generalisation of the simple linear regression model in which there is more efficient estimation when there are autocorrelated errors. In time series, normally errors do involve serial correlation; i.e., the current error term is connected with previous error terms. This violates one of the basic assumptions of ordinary least squares (OLS) regression, namely, that errors are independent. OLS will produce inefficient (and possibly inconsistent) estimates when regressors are correlated with past error terms. To reverse this, GLS adjusts for the autocorrelation structure of the errors and provides more efficient and unbiased estimates of the regression coefficients. When terms of error follow an ARMA(1,1) process, the GLS method is even more specific in the sense that it adds both the moving average and autoregressive terms to the error process.

For ARMA(1,1) errors of the GLS model, we start with a typical linear regression equation:

$$Y_t = X_t\beta + u_t, \tag{2.7}$$

where Y_t is the dependent variable at time t, X_t is a vector of the explanatory variables, β is a vector of unknown parameters to be estimated, and u_t is the error term at time t. For ARMA(1,1) disturbances, errors u_t are characterized by an ARMA(1,1) process and can be specified as:

$$u_t = \rho u_{t-1} + \varepsilon_t + \phi\varepsilon_{t-1}, \tag{2.8}$$

where ε_t is a white noise process of zero mean and constant variance, and ρ, ϕ are the autoregressive and moving average parameters, respectively. The ARMA(1,1) process indicates that the current error term u_t is determined by both the previous error u_{t-1} as well as the previous shock ε_{t-1} and the present shock ε_t.

In case of such autocorrelated errors, the GLS method makes a change to the process of estimation by accounting for the structure of error covariance terms. GLS aims at estimating the parameters of regression β keeping in view the serial correlation of residuals. The GLS estimator is:

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y \tag{2.9}$$

Where Ω is the error covariance matrix, which in the case of ARMA(1,1) disturbances reflects the error term autocorrelation pattern. Since the error covariance matrix is not diagonal, the GLS procedure scales each observation's weight by the error correlation. Such a transformation gives more accurate and efficient parameter estimations compared to OLS.

One of the major reasons for using GLS with ARMA(1,1) errors is that it manages autocorrelated errors very well. With an accurate description of the correlation between the error terms, GLS ensures that the estimated coefficients are unbiased and more efficient than the estimated coefficients obtained by OLS, where errors are supposed to be uncorrelated. Also, by incorporating both the moving average and the autoregressive features, the model can capture both short-term relationships and the effect of past disturbances, which renders it perfectly adapted to time series data with these characteristics.

The GLS method also provides an improved estimation process by controlling for the structure of the error term. The efficiency of this process is particularly beneficial when dealing with highly autocorrelated time series data since the GLS estimator is designed to incorporate the time interdependencies between observations. This improves the parameter estimates' accuracy, which is most crucial in efficient modelling and forecasting.

To estimate the parameters of the GLS model with ARMA(1,1) errors, one has to first estimate an ARMA(1,1) model from the regression residuals. This allows for the identification and estimation of the autoregressive coefficient, ρ, and the moving average coefficient, ϕ, which describe the error correlation structure. Once these parameters are estimated, the covariance matrix Ω can be estimated and the GLS estimator can be employed to obtain the regression coefficients. The estimation is more efficient since the error structure is well accounted for.

The GLS model with ARMA(1,1) disturbances is extremely prevalent across many fields. For example, in finance and econometrics, it is used most frequently to estimate economic variables such as stock returns, and exchange or interest rates where autocorrelation of residuals is typically encountered. In engineering, GLS model with ARMA(1,1)Box and Jenkins (1976), Brockwell and Davis (1991), Hamilton (1994), Nelson (1991), Granger and Joyeux (1980), James et al. (2013) and White (1982) can be used to model the behaviour of systems over time that are affected by noise or disturbances. Similarly, in environmental science, it is found to be handy for predicting variables such as temperature, rain or pollution levels, where data are typically found to exhibit autocorrelation due to prevailing temporal mechanisms.

Generally, the GLS model with ARMA(1,1) disturbances is a useful tool in time series analysis, providing a way of dealing with autocorrelated errors and improving the efficiency of regression estimation. By specifying both the moving average and the autoregressive components in the error structure, the model is capable of modelling both data persistence and the influence of past random shocks. Despite the model's assumption of a known and appropriately specified autocorrelation structure, it remains a useful tool for the study of time series data in economics, finance, engineering and environmental science. Its ability to generate unbiased and accurate estimates even under the state of autocorrelated errors makes it a highly crucial tool for successful modelling and prediction.

Chapter 3

The generalized linear model with ARMA(1,1) disturbances

3.1 Introduction

Consider the linear regression model:

$$y = X\beta + \sigma u, \tag{3.1}$$

where y is the $T \times 1$ vector of observations on the endogenous variable, X is the $T \times n$ matrix of the exogenous variable, β is a $n \times 1$ vector of unknown parameters and σu $(\sigma > 0)$ is the $T \times 1$ vector of unobserved errors. The random vector u is distributed as $N(0, \Omega^{-1})$, where the elements of the $T \times T$ matrix Ω are known functions of the unknown $k \times 1$ parameter vector γ and, possibly, of a $T \times m$ matrix Z of observations on a set of exogenous variables, some of which may be regressors too. The vector γ belongs to the parameter space Θ, which is an open subset of the k-dimensional Euclidean space. Let $\hat{\gamma}$ is any consistent estimator of γ. For any function $f = f(\gamma)$ we write $\hat{f} = f(\hat{\gamma})$. The feasible GLS estimators of β and σ^2 are:

$$\hat{\beta} = (X'\hat{\Omega}X)^{-1}X'\hat{\Omega}y, \tag{3.2}$$

$$\hat{\sigma}^2 = (y - X\hat{\beta})'\hat{\Omega}(y - X\hat{\beta})/(T - n). \tag{3.3}$$

We write $\Omega_i = \partial\Omega/\partial\gamma_i$, $\Omega_{ij} = \partial^2\Omega/\partial\gamma_i\partial\gamma_j$ for $T \times T$ matrices of the first and second order partial derivatives of the matrix Ω with respect to the elements of the vector γ.

Let

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \rho \\ \phi \end{pmatrix} \tag{3.4}$$

We define the 3×1 vector δ with elements

$$\delta_0 = \frac{\hat{\sigma}^2 - \sigma^2}{\tau\sigma^2}, \ \delta_\rho = \frac{\hat{\gamma}_1 - \gamma_1}{\tau} = \frac{\hat{\rho} - \rho}{\tau}, \ \delta_\phi = \frac{\hat{\gamma}_2 - \gamma_2}{\tau} = \frac{\hat{\phi} - \phi}{\tau}, \tag{3.5}$$

where $\tau = 1/\sqrt{T}$, is the "asymptotic scale" of our expansions.

So δ includes δ_0, δ_ρ and δ_ϕ and is written as follows:

$$\delta = \begin{bmatrix} \delta_0 \\ \delta_\rho \\ \delta_\phi \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \delta_* \end{bmatrix} \tag{3.6}$$

where δ_* is a 2×1 vector with elements δ_ρ, δ_ϕ.

We assume that the following regularity conditions hold:

1. The elements of the matrices Ω and Ω^{-1} are bounded for all T and for all $T \in \theta$, and the matrices
$$A = X'\Omega X/T,\ F = X'X/T \tag{3.7}$$
converge to non-singular matrices as $T \to \infty$.

2. The partial derivatives, up to the fourth order, of the elements of the matrix Ω with respect to the elements of the vector γ are bounded for all T and for all $\gamma \in \theta$.

3. The estimator $\hat{\gamma}$ is an even function of u and is functionally unrelated to the parameters β, that is, it can be written as a function only of X, Z and σu.

4. The vector δ accepts a stochastic expansion of the form
$$\delta = d_1 + \tau d_2 + \omega(\tau^2), \tag{3.8}$$
and the expectations
$$\mathbb{E}(d_1 d_1'),\ \mathbb{E}(\sqrt{T} d_1 + d_2) \tag{3.9}$$
exist and have finite limits as $T \to \infty$.

The first two conditions imply that the matrices

$$A_i = X'\Omega_i X/T,\ A_{ij} = X'\Omega_{ij} X/T,\ A^*_{ij} = X'\Omega_i \Omega^{-1} \Omega_j X/T \tag{3.10}$$

are bounded and therefore the Taylor expansion of $\hat{\beta}$ is a stochastic expansion (Magdalinos (1992)). Under the condition that the parameters β and γ are functionally unrelated, assumption (3) is satisfied for a wide class of estimators of γ, including maximum likelihood estimators (ML) and the simple or iterative estimators based on regression residuals. Moreover, we can show that condition (4) is satisfied for the same classes of estimators of γ. Note that we do not assume that the estimator of γ is asymptotically efficient.

We define the scalars λ_0 and μ_0, the 2×1 vectors λ and μ, and the 2×2 matrix Λ as follows:

$$\Lambda = \begin{bmatrix} \lambda_{\rho\rho} & \lambda_{\rho\phi} \\ \lambda_{\phi\rho} & \lambda_{\phi\phi} \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_\rho \\ \lambda_\phi \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_\rho \\ \mu_\phi \end{bmatrix}, \tag{3.11}$$

$$\lim_{T\to\infty} \mathbb{E}(d_1 d_1') = \begin{bmatrix} \lambda_0 & \lambda' \\ \lambda & \Lambda \end{bmatrix} = \begin{bmatrix} \lambda_0 & \lambda_\rho & \lambda_\phi \\ \lambda_\rho & \lambda_{\rho\rho} & \lambda_{\rho\phi} \\ \lambda_\phi & \lambda_{\phi\rho} & \lambda_{\phi\phi} \end{bmatrix}, \tag{3.12}$$

$$\lim_{T\to\infty} \mathbb{E}(\sqrt{T} d_1 + d_2) = \begin{bmatrix} \mu_0 \\ \mu \end{bmatrix} = \begin{bmatrix} \mu_0 \\ \mu_\rho \\ \mu_\phi \end{bmatrix}. \tag{3.13}$$

For each $n \times m$ matrix L with elements l_{ij} we write:

$$L = [(l_{ij})_{i=1,\ldots,n;j=1,\ldots,m}] \tag{3.14}$$

with the corresponding modifications for vectors and square matrices. If L_{ij} are $n_i \times m_j$ matrices, then the notation means that the matrix L is a $(\sum_{i=1}^{n} n_i) \times (\sum_{j=1}^{m} m_j)$ partitioned matrix with submatrices L_{ij}.

The properties of the size corrections presented below have proved in Symeonides (1991).

3.2 The t Test

Let e_0 be a known scalar and let e be a known $n \times 1$ vector. In order to test the null hypothesis

$$\mathrm{e}'\beta - \mathrm{e}_0 = 0 \tag{3.15}$$

for one-sided alternative hypotheses we use the statistic

$$t = (\mathrm{e}'\hat{\beta} - \mathrm{e}_0)/[\hat{\sigma^2}\mathrm{e}'(X'\hat{\Omega}X)^{-1}\mathrm{e}]^{1/2}. \tag{3.16}$$

We define the $k \times 1$ vector l and the $k \times k$ matrix L as

$$l = \begin{bmatrix} l_\rho \\ l_\phi \end{bmatrix}, \quad L = \begin{bmatrix} L_{\rho\rho} & L_{\rho\phi} \\ L_{\phi\rho} & L_{\phi\phi} \end{bmatrix}, \tag{3.17}$$

where

$$\begin{aligned} & & l_{\rho\rho} &= \mathrm{e}'GC_{\rho\rho}G\mathrm{e}/\mathrm{e}'G\mathrm{e} \\ l_\rho &= \mathrm{e}'GA_\rho G\mathrm{e}/\mathrm{e}'G\mathrm{e} & l_{\phi\phi} &= \mathrm{e}'GC_{\phi\phi}G\mathrm{e}/\mathrm{e}'G\mathrm{e} \\ l_\phi &= \mathrm{e}'GA_\phi G\mathrm{e}/\mathrm{e}'G\mathrm{e} & l_{\rho\phi} &= \mathrm{e}'GC_{\rho\phi}G\mathrm{e}/\mathrm{e}'G\mathrm{e} \\ & & l_{\phi\rho} &= \mathrm{e}'GC_{\phi\rho}G\mathrm{e}/\mathrm{e}'G\mathrm{e} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} G &= (X'\Omega X/T)^{-1} \\ C_{\rho\rho} &= A^*_{\rho\rho} - 2A_\rho G A_\rho + A_{\rho\rho}/2 \\ C_{\phi\phi} &= A^*_{\phi\phi} - 2A_\phi G A_\phi + A_{\phi\phi}/2 \\ C_{\rho\phi} &= A^*_{\rho\phi} - 2A_\rho G A_\phi + A_{\rho\phi}/2 \\ C_{\phi\rho} &= A^*_{\phi\rho} - 2A_\phi G A_\rho + A_{\phi\rho}/2 \end{aligned} \tag{3.19}$$

and the matrices A_i, A_{ij}, and A^*_{ij} are defined in (3.10).

Lemma 3.1. *Under the null hypothesis* (3.15), *the distribution function of the statistic* (3.16) *assumes the Edgeworth expansion*

$$\Pr(t \le x) = I(x) - \frac{\tau^2}{2}\left[\left(p_1 + \frac{1}{2}\right) + \left(p_2 + \frac{1}{2}\right)x^2\right] x i(x) + O(\tau^3), \tag{3.20}$$

where

$$\begin{aligned} p_1 &= \mathrm{tr}(\Lambda L) + l'\Lambda l/4 + l'(\mu + \lambda/2) - \mu_0 + (\lambda_0 - 2)/4, \\ p_2 &= (l'\Lambda l - 2l'\lambda + \lambda_0 - 2)/4 \end{aligned} \tag{3.21}$$

and $I(\cdot)$, $i(\cdot)$ are the distribution and density functions, respectively, of the standard normal distribution.

Theorem 3.1. *Under the hypothesis* (3.15) *and if the regularity conditions are satisfied, the Cornish-Fisher corrected statistic*

$$\tilde{t} = t - \frac{\tau^2}{2}\left[\left(p_1 + \frac{1}{2}\right) + \left(p_2 + \frac{1}{2}\right)t^2\right] t, \tag{3.22}$$

is distributed, with an error of order $O(\tau^3)$, as a standard normal variable.

Lemma 3.2. *Under the null hypothesis* (3.15) *the distribution function of the statistic* (3.16) *accepts the Edgeworth-type expansion*

$$\Pr(t \leq x) = I_{T-n}(x) - \frac{\tau^2}{2}\left(p_1 + p_2 x^2\right) x i_{T-n}(x) + O(\tau^3), \tag{3.23}$$

where the quantities p_1 *and* p_2 *are defined in* (3.21) *and* $I_{T-n}(\cdot)$, $i_{T-n}(\cdot)$ *are the distribution and density functions, respectively, of a* t *variable with* $T-n$ *degrees of freedom. Moreover, the approximation is locally exact, i.e., if* γ *is known to belong to a ball of radius* θ*, then the approximation becomes exact as* $\theta \to 0$.

Theorem 3.2. *Under the null hypothesis* (3.15) *and if the regularity conditions are satisfied, the Cornish-Fisher corrected statistic*

$$\hat{t} = t - \frac{\tau^2}{2}\left(p_1 + p_2 t^2\right) t \tag{3.24}$$

is distributed, with an error of order $O(\tau^3)$*, as a* t *variable with* $T-n$ *degrees of freedom. Moreover, the approximation is locally exact, i.e., if* γ *is known to belong to a ball of radius* θ*, then the approximation becomes exact as* $\theta \to 0$.

Corollary 3.1. *The level of significance corresponding to a specific value, say* t_0*, of the* t *statistic* (3.16) *is obtained by comparing the p-value of the Cornish-Fisher corrected* t *statistic, say* $\hat{t}_0$*, with the tables of the* $Student-t$ *distribution. This means that*

$$\Pr(t \leq t_0) = I_{T-n}(\hat{t}_0) + O(\tau^3) \tag{3.25}$$

and

$$\Pr(t \leq t_0) = 1 - I_{T-n}(\hat{t}_0) + O(\tau^3), \tag{3.26}$$

where $I_{T-n}(\cdot)$*,* $i_{T-i}(\cdot)$ *are the distribution and density functions, respectively, of a* t *variable with* $T-n$ *degrees of freedom.*

In the case of the two-sided statistical significance test of the $k-th$ structural parameter β_k, it follows that e has 1 in the $k-th$ position and 0 anywhere else. Therefore, the components of l and L are estimated as

$$\hat{l}_i = \hat{g}_k' \hat{A}_i \hat{g}_k / \hat{g}_{kk}, \;\; \hat{l}_{ij} = \hat{g}_k' \hat{C}_{ij} \hat{g}_k / \hat{g}_{kk}, \tag{3.27}$$

respectively, where $\hat{g}_k$ is the $k-th$ column and $\hat{g}_{kk}$ is the $k-th$ diagonal element, respectively, of the matrix $\hat{G} = (X'\hat{\Omega}X/T)$. Moreover, the symbol "ˆ" denotes the estimates of the corresponding quantities from the data.

3.3 The F Test

Let H be an $r \times n$ known matrix of rank r and let h be a known $r \times 1$ vector. In order to test the null hypothesis

$$H\beta - h = 0 \tag{3.28}$$

we use the Wald statistic

$$\mathrm{w} = (H\hat{\beta} - h)'[H(X'\hat{\Omega}X)^{-1}H']^{-1}(H\hat{\beta} - h)/\hat{\sigma}^2. \tag{3.29}$$

We define the $k \times 1$ vector c and the $k \times k$ matrices C and D as

$$c = \begin{bmatrix} c_\rho \\ c_\phi \end{bmatrix}, \quad C = \begin{bmatrix} c_{\rho\rho} & c_{\rho\phi} \\ c_{\phi\rho} & c_{\phi\phi} \end{bmatrix}, \tag{3.30}$$

where

$$\begin{aligned} & & c_{\rho\rho} &= \mathrm{tr}(C_{\rho\rho}P) \\ c_\rho &= \mathrm{tr}(A_\rho P) & c_{\phi\phi} &= \mathrm{tr}(C_{\phi\phi}P) \\ c_\phi &= \mathrm{tr}(A_\phi P) & c_{\rho\phi} &= \mathrm{tr}(C_{\rho\phi}P) \\ & & c_{\phi\rho} &= \mathrm{tr}(C_{\phi\rho}P) \end{aligned}$$

and

$$D = \begin{bmatrix} d_{\rho\rho} & d_{\rho\phi} \\ d_{\phi\rho} & d_{\phi\phi} \end{bmatrix}, \tag{3.31}$$

where

$$\begin{aligned} d_{\rho\rho} &= [(\mathrm{tr}\, D_{\rho\rho}P) \\ d_{\phi\phi} &= [(\mathrm{tr}\, D_{\phi\phi}P) \\ d_{\rho\phi} &= [(\mathrm{tr}\, D_{\rho\phi}P) \\ d_{\phi\rho} &= [(\mathrm{tr}\, D_{\phi\rho}P) \end{aligned}$$

and the matrices A_i, C_{ij} are defined in (3.10) and (3.19), respectively, and

$$P = GQG, \quad Q = H'(HGH')^{-1}H, \quad D_{ij} = A_iPA_j/2. \tag{3.32}$$

Lemma 3.3. *Under the null hypothesis* (3.28)*, the distribution function of the statistic* (3.29) *admits the Edgeworth-type expansion*

$$\Pr(\mathrm{w} \leq x) = F_r(x) - \tau^2 \left(h_1 + h_2 \frac{x}{r+2} \right) + \frac{x}{r} f_r(x) + O(\tau^3), \tag{3.33}$$

where

$$\begin{aligned} h_1 &= \operatorname{tr}[\Lambda(C+D)] - c'\Lambda c/4 + c'\mu + r[c'\lambda/2 - \mu_0 - (r-2)\lambda_0/4], \\ h_2 &= \operatorname{tr}(\Lambda D) + [c'\Lambda c - (r+2)(2c'\lambda - r\lambda_0)]/4 \end{aligned} \tag{3.34}$$

and $F_r(\cdot)$, $f_r(\cdot)$ are the distribution and density functions, respectively, of a chi-square random variable with r degrees of freedom.

Theorem 3.3. *Under the hypothesis* (3.28) *and if the regularity conditions are satisfied, the Cornish-Fisher corrected statistic*

$$\hat{\mathrm{w}} = \mathrm{w} - \tau^2 \left[\frac{h_1}{r} + \frac{h_2}{r(r+2)} \mathrm{w} \right] \mathrm{w} \tag{3.35}$$

is distributed, with an error of order $O(\tau^3)$, as a chi-square random variable with r degrees of freedom.

The exact distribution of the statistic (3.29) has not been tabulated, even in cases where the vector γ is known. Therefore, it is preferable to adjust the statistic (3.29) by correcting the numerator degrees of freedom, thereby obtaining the modified statistic.

$$\mathrm{v} = (H\hat{\beta} - h)'[H(X'\hat{\Omega}X)^{-1}H']^{-1}(H\hat{\beta} - h)/r\hat{\sigma}^2. \tag{3.36}$$

The statistic (3.36) is the exact analogue of the well-known F statistic in the classical linear model and follows exactly an F distribution when the vector γ is known.

Lemma 3.4. *Under the null hypothesis* (3.28) *the distribution function of the statistic* (3.36) *has an Edgeworth-type expansion*

$$\Pr(\mathrm{v} \leq x) = F^r_{T-n}(x) - \tau^2 (q_1 + q_2 x) x f^r_{T-n}(x) + O(\tau^3), \tag{3.37}$$

where

$$\begin{aligned} q_1 &= h_1/r + (r-2)/2, \\ q_2 &= h_2/(r+2) - r/2 \end{aligned} \tag{3.38}$$

and $F^r_{T-n}(\cdot)$, $f^r_{T-n}(\cdot)$ are the distribution and density functions, respectively, of an F random variable with r and $T-n$ degrees of freedom. Moreover, the approximation is locally exact, that is, if γ is known to belong to a ball of radius θ, then the approximation becomes exact as $\theta \to 0$.

Theorem 3.4. *Under the hypothesis* (3.28) *and if the regularity conditions are satisfied, the Cornish-Fisher corrected statistic*

$$\hat{\mathrm{v}} = \mathrm{v} - \tau^2 \left(q_1 + q_2 \mathrm{v}\right) \mathrm{v} \tag{3.39}$$

is distributed, with an error of order $O(\tau^3)$, as an F random variable with r and $T-n$ degrees of freedom. Moreover, the approximation is locally exact, that is, if γ is known to belong to a ball of radius θ, then the approximation becomes exact as $\theta \to 0$.

Corollary 3.2. *The level of significance corresponding to a specific value, say v_0, of the F statistic is obtained by comparing the p-value of the Cornish-Fisher corrected F statistic, say $\hat{\mathrm{v}}_0$, with the tables of the F distribution. That is, if $F^r_{T-n}(\cdot)$, $f^r_{T-n}(\cdot)$ are the distribution and density functions, respectively, of the F distribution with r and $T-n$ degrees of freedom, then*

$$\Pr(\mathrm{v} > t_0) = 1 - F^r_{T-n}(\hat{\mathrm{v}}_0) + O(\tau^3). \tag{3.40}$$

3.4 Computation of the quantities involved in the correction formulas

In cases where the model exhibits ARMA(1,1) disturbances, there are no closed-form expressions available for some quantities required to implement the corrections to the t and F statistical tests.

For the t test, two approaches are considered: the Edgeworth expansion and the Cornish-Fisher expansion, both under the normal and the Student-t distribution. Similarly, for the F test, the same two approaches (Edgeworth and Cornish-Fisher) are applied, both under the χ^2 distribution and the F distribution.

3.4.1 Analysis of the quantities involved in the t test formulas

To apply corrections to t tests in models with ARMA(1,1) disturbances, several approaches are employed based on either the normal or the Student-t distribution. Each approach involves specific correction terms, the computation of which depends on the availability of closed-form expressions or the feasibility of estimation through theoretical or computational methods. The following section provides an analysis of the key terms associated with both distributions, highlighting which ones can be derived analytically and which require simulation-based techniques rooted in asymptotic expansion theory.

For the normal and Student-t distribution, the correction terms p_1 and p_2 in (3.21) are used in both the Edgeworth and Cornish-Fisher expansions.

According to Breusch (1980), the constant λ_0 takes the value of 2. Furthermore, the matrix L and the vector l are computed using the closed-form expressions in equations (3.17), (3.18), and (3.19).

The matrix Λ is a 2×2 covariance matrix that contains the second-order moments of the quantities δ_ρ and δ_ϕ. Specifically, its elements are given by:

$$\Lambda = \begin{bmatrix} \lambda_{\rho\rho} & \lambda_{\rho\phi} \\ \lambda_{\phi\rho} & \lambda_{\phi\phi} \end{bmatrix}, \tag{3.41}$$

where $\lambda_{\rho\rho} = \mathbb{E}(\delta_\rho^2)$, $\lambda_{\rho\phi} = \mathbb{E}(\delta_\rho\delta_\phi)$, $\lambda_{\phi\rho} = \mathbb{E}(\delta_\phi\delta_\rho)$, and $\lambda_{\phi\phi} = \mathbb{E}(\delta_\phi^2)$.

The quantities δ_ρ and δ_ϕ represent deviations of the estimated parameters from their true values, and are defined as:

$$\delta_\rho = \sqrt{T}(\hat{\rho} - \rho), \quad \delta_\phi = \sqrt{T}(\hat{\phi} - \phi), \tag{3.42}$$

where T denotes the sample size, $\hat{\rho}$ and $\hat{\phi}$ are the estimators of the autoregressive and moving average parameters, respectively, based on the regression residuals, and ρ, ϕ are their true values.

Since closed-form expressions for the expected values involved in the elements of Λ are not available due to the non-linearity of the ML estimators of the ARMA(1,1) parameters, these expected values are computed through simulation. By repeatedly generating synthetic data under known parameter values and estimating $\hat{\rho}$ and $\hat{\phi}$, the empirical distributions of δ_ρ and δ_ϕ can be obtained. These simulations allow for consistent approximation of the elements of Λ, which are essential for computing the correction terms in the Edgeworth and Cornish-Fisher expansions.

Similarly, the vector μ consists of the first-order moments of the scaled estimators, and is defined as:

$$\mu = \begin{bmatrix} \mu_\rho \\ \mu_\phi \end{bmatrix} \quad \text{where} \quad \mu_\rho = \mathbb{E}(\delta_\rho)/\tau, \quad \mu_\phi = \mathbb{E}(\delta_\phi)/\tau. \tag{3.43}$$

As with the elements of the matrix Λ, there are no closed-form expressions available for these expected values due to the non-linearity of the ML estimators of the ARMA(1,1) parameters. Therefore, the vector μ is also approximated via simulation. Using the same simulated data sets, the empirical means of δ_ρ and δ_ϕ are computed and used to construct estimates of μ, which are necessary for the computation of the correction terms in the Edgeworth and Cornish-Fisher expansions.

Subsequently, the vector λ consists of the elements λ_ρ and λ_ϕ, which are defined as follows:

$$\lambda_\rho = \mathbb{E}(\delta_0\delta_\rho), \quad \lambda_\phi = \mathbb{E}(\delta_0\delta_\phi). \tag{3.44}$$

The term δ_0 is computed as:

$$\delta_0 = \frac{\hat{\sigma}^2 - \sigma^2}{\tau\sigma^2}, \tag{3.45}$$

where

$$\hat{\sigma}^2 = (y - X\hat{\beta})'\hat{\Omega}(y - X\hat{\beta})/(T-n) \tag{3.46}$$

is the estimated variance, σ^2 is the true variance, and τ is the asymptotic scale of our expansions. The quantity δ_0 represents the difference between the estimated and true variances, adjusted by the asymptotic scale τ, T is the sample size and n refers to the number of parameters estimated in the model. All of these quantities are computed via simulation, given that no closed-form expressions exist for these expected values.

The computation of μ_0

In order to compute μ_0, we use the following procedure: using (3.4), (3.5), (3.6), (3.8) and (3.9), we can prove that

$$\begin{bmatrix} \mu_0 \\ \mu \end{bmatrix} = \lim_{T\to\infty} \mathbb{E} \begin{bmatrix} \sqrt{T}\sigma_0 + \sigma_1 \\ \sqrt{T}d_{1i} - d_{2i} \end{bmatrix}, \tag{3.47}$$

where

$$\mu_0 = \lim_{T\to\infty} \mathbb{E}(\sqrt{T}\sigma_0 + \sigma_1), \tag{3.48}$$

$$\mu = \lim_{T\to\infty} \mathbb{E}(\sqrt{T}d_{1i} - d_{2i}) = \lim_{T\to\infty} \mathbb{E}(\sqrt{T}\delta_i). \tag{3.49}$$

Also, we can prove that

$$\hat{\rho} \quad = \quad \rho + \tau(\rho_1 + \tau\rho_2) + \omega(\tau^3) \Rightarrow \tag{3.50}$$

$$\hat{\phi} \quad = \quad \phi + \tau(\phi_1 + \tau\phi_2) + \omega(\tau^3) \Rightarrow \tag{3.51}$$

Therefore, we calculate that

$$\mathbb{E}(\delta_\rho\delta_\rho) = \mathbb{E}(\rho_1^2) + O(\tau) = \lambda_{\rho*} \tag{3.52}$$

$$\mathbb{E}(\delta_\phi\delta_\phi) = \mathbb{E}(\phi_1^2) + O(\tau) = \lambda_{\phi*} \tag{3.53}$$

$$\mathbb{E}(\delta_\rho\delta_\phi) = \mathbb{E}(\rho_1\phi_1) + O(\tau) = \lambda_{\rho\phi*}. \tag{3.54}$$

Thus,

$$\Lambda_* = \Lambda + O(\tau) \tag{3.55}$$

where

$$\Lambda = \begin{bmatrix} \lambda_{\rho\rho*} & \lambda_{\rho\phi*} \\ \lambda_{\phi\rho*} & \lambda_{\phi\phi*} \end{bmatrix}. \tag{3.56}$$

Similarly, we can prove that

$$\mu_{\rho*} = \mathbb{E}(\sqrt{T}\rho_1 + \rho_2 + \omega(\tau)) = \mathbb{E}(\sqrt{T}\rho_1 + \rho_2) + O(\tau) = \mu_\rho + O(\tau) \tag{3.57}$$

$$\mu_{\phi*} = \mathbb{E}(\sqrt{T}\phi_1 + \phi_2 + \omega(\tau)) = \mathbb{E}(\sqrt{T}\phi_1 + \phi_2) + O(\tau) = \mu_\phi + O(\tau) \tag{3.58}$$

Then, since

$$\mu_0 = \lim_{T\to\infty} \mathbb{E}(\sqrt{T}\sigma_0 + \sigma_1) \tag{3.59}$$

where

$$\sigma_0 \quad = \quad w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi \tag{3.60}$$

$$\sigma_1 \quad = \quad w_\rho\delta_\rho + w_\phi\delta_\phi + \mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi - b'\hat{A}b + n \tag{3.61}$$

we can prove that

$$\mu_0 = \frac{1}{2}\frac{\mathrm{tr}(u'\Omega_{\rho\rho}u)}{T}\delta_\rho\delta_\rho + \frac{1}{2}\frac{\mathrm{tr}(u'\Omega_{\phi\phi}u)}{T}\delta_\phi\delta_\phi + \frac{\mathrm{tr}(u'\Omega_{\rho\phi}u)}{T}\delta_\rho\delta_\phi. \tag{3.62}$$

The proofs of the results given in this section are gathered in Appendix C.

3.4.2 Analysis of the quantities involved in the F-test formulas

To apply corrections to F tests in models with ARMA(1,1) disturbances, several approaches are employed based on either the χ^2 or the F distributions. Each approach involves specific correction terms, the computation of which depends on the availability of closed-form expressions or the feasibility of estimation through theoretical or computational methods. The following section provides an analysis of the key terms associated with both distributions, highlighting which ones can be derived analytically and which require simulation-based techniques rooted in asymptotic expansion theory.

For the χ^2 and F distributions, the correction terms h_1, h_2 in (3.34) and q_1, q_2 in (3.38) are used in both the Edgeworth and Cornish-Fisher expansions.

The parameters μ and λ, which also appear in the correction terms of the t test, retain the same definitions in the context of the F test. Due to the absence of closed-form expressions, these quantities are estimated through simulation. In contrast, the constants λ_0 and μ_0 are available in closed-form, as in the t-test case.

Finally, the quantities c, C, and D, which are critical for the computation of the correction terms, are derived analytically from equations (3.30), (3.31), and (3.32).

Chapter 4

Computational Techniques for Estimating Non-Closed Form Quantities in ARMA Models

4.1 Introduction

In this chapter, we will present the methods required to compute quantities that do not have closed-form expressions, which are essential for applying corrections to the t and F tests for models with ARMA(1,1) disturbances. Given the complexity of the expressions related to ARMA processes and the lack of direct mathematical formulas for many of these quantities, computational approaches are needed to estimate them.

Additionally, two other methods crucial for applying corrections and improving computations will be discussed: the Gradient Descent method and L2 regularization. These methods are particularly important for regularizing the coefficients, as they help address issues that arise when the coefficients approach extreme values, such as -1 or 1. The theory suggests that the ARMA(1,1) coefficients should have absolute values smaller than 1 to ensure the stability and invertibility of the model. However, in practice, when the coefficients approach these extreme values, significant problems can arise, such as model instability and overfitting. The Gradient Descent method and L2 regularization help avoid such issues by constraining the coefficients, thus improving the robustness and generalizability of the estimates.

Furthermore, the implementation of the Monte Carlo experiment will be analyzed, which will be used to simulate the necessary data and estimate the required quantities. At the end of the chapter, the results of the experiment will be presented.

4.2 Introduction to the Gradient Descent Algorithm

Gradient Descent (GD) is one of the most fundamental and widely used optimization algorithms in machine learning. It is an iterative method employed to minimize a loss function by gradually adjusting the model parameters in the direction of the steepest descent of the loss function. The effectiveness of Gradient Descent is influenced by various factors, such as the learning rate, the management of the convergence rate, and the choice of the appropriate variant (Batch, Stochastic or Mini-Batch). This algorithm has been extensively studied and applied in numerous domains of machine learning and deep learning, forming the backbone of many training procedures Box and Jenkins (1976), Brockwell and Davis (1991), Hamilton (1994), Nelson (1991), Granger and Joyeux (1980), James et al. (2013) and White (1982).

4.2.1 Basic Idea of Gradient Descent

The Gradient Descent algorithm is based on the concept of the gradient of the loss function $J(\theta)$ with respect to the model parameters θ. The core step of the algorithm is to update the parameters based on the negative gradient, which points in the direction of the maximum decrease of the loss function.

Mathematically, the process is described by the following equation:

$$\theta := \theta - \alpha \nabla J(\theta), \tag{4.1}$$

where:

- θ represents the model parameters,
- α is the learning rate,
- $\nabla J(\theta)$ is the gradient of the loss function.

This process is repeated for a predetermined number of iterations or until the loss function approaches a minimum.

4.2.2 Loss Function Calculation

The prediction error is calculated through the loss function. A common choice for the loss function is the Mean Squared Error (MSE), which calculates the average squared difference between the actual and predicted values.

The loss function $J(\theta)$ for MSE is:

$$J(\theta) = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2, \tag{4.2}$$

where:

- y_i is the actual value of the i-th sample,
- $\hat{y}_i$ is the predicted value of the i-th sample,
- n is the size of the dataset (the number of observations).

This function measures the distance between the actual and predicted values, and the goal of Gradient Descent is to minimize this value, improving the model parameters so that the predictions get as close to the actual values as possible.

4.2.3 Updating Parameters Based on the Gradient of the loss Function

Based on the gradient of the loss function with respect to the parameters θ, the model parameters are updated. For instance, for a parameter w, the update rule is:

$$w \leftarrow w - \alpha \frac{\partial J(\theta)}{\partial w}, \tag{4.3}$$

where:

- α is the learning rate,
- $\frac{\partial J(\theta)}{\partial w}$ is the derivative of the loss function with respect to the parameter w, representing the gradient.

This step is used to adjust the parameters and reduce the prediction error of the model.

4.2.4 Types of Gradient Descent

Batch Gradient Descent

In Batch Gradient Descent, the algorithm computes the gradient using the entire dataset (batch) in each iteration. The gradient computation is as follows:

$$\nabla J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \nabla J_i(\theta), \tag{4.4}$$

where m is the size of the dataset and $J_i(\theta)$ is the loss function for the i-th sample. This approach has high accuracy but can be slow when the dataset is large.

Stochastic Gradient Descent (SGD)

In Stochastic Gradient Descent, the parameter update is done for each sample individually, and the algorithm computes the gradient for a single random sample at a time:

$$\theta := \theta - \alpha \nabla J_i(\theta), \tag{4.5}$$

where $J_i(\theta)$ is the loss function for the i-th sample. This method is faster since it updates the parameters after each sample, but the process can be more unstable due to random fluctuations in the updates.

Mini-Batch Gradient Descent

The Mini-Batch Gradient Descent method combines the advantages of both Batch and Stochastic Gradient Descent. Here, the algorithm computes the gradient for small groups of data (mini-batches), offering a good compromise between speed and accuracy:

$$\nabla J(\theta) = \frac{1}{B}\sum_{i=1}^{B} \nabla J_i(\theta), \tag{4.6}$$

where B is the size of the mini-batch. This approach helps achieve better performance and speed without the instability of SGD.

4.2.5 Learning Rate

The learning rate (α) is one of the most important parameters in the Gradient Descent algorithm. If the learning rate is too small, the algorithm may require many iterations to converge, while if it is too large, it may fail to converge as the parameter updates can become too large and cause divergence.

Choosing the right learning rate is crucial for the efficiency of the algorithm. Techniques such as learning rate decay or dynamic adjustment of the learning rate during training are commonly used to optimize the convergence process.

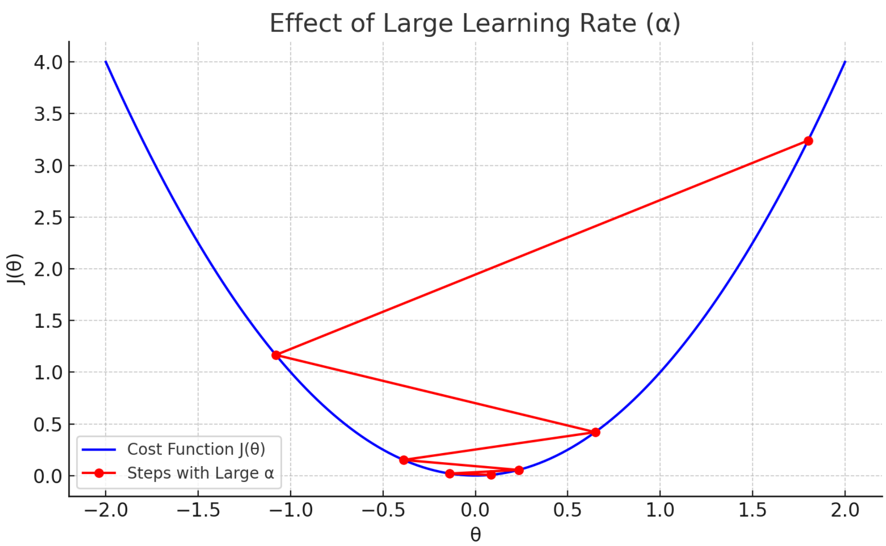

Figure 4.1 Effect of a large learning rate: the algorithm overshoots the minimum, potentially diverging.

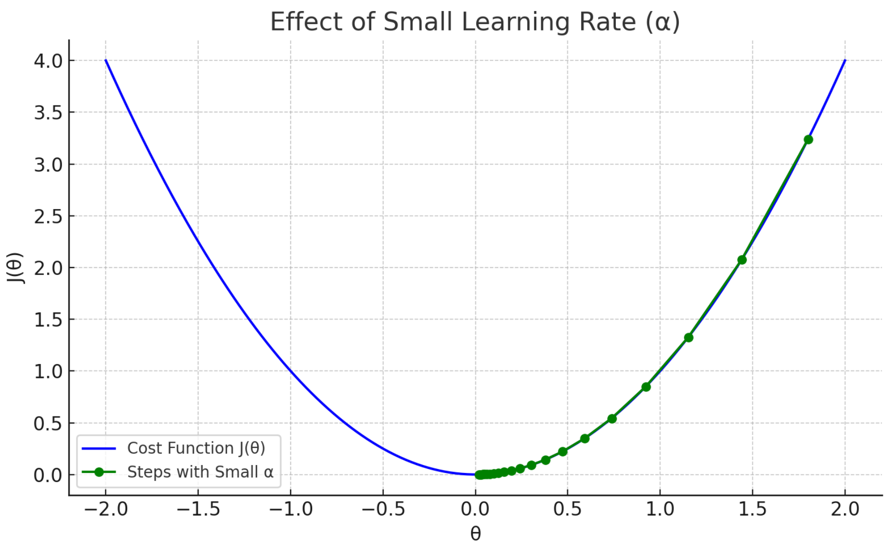

Figure 4.2 Effect of a small learning rate: convergence is very slow as the steps are tiny.

These diagrams illustrate how the learning rate affects the gradient descent trajectory. A very large α leads to oscillations or divergence, while a very small α slows down the convergence, making the process inefficient.

4.2.6 Conclusion

Gradient Descent is a fundamental optimization method in machine learning that allows finding the parameters of a model that minimize a loss function. While its basic form is simple, its successful application requires careful selection of parameters such as the learning rate, as well as the decision on the appropriate variant (Batch, Stochastic or Mini-Batch) depending on the problem characteristics and dataset.

4.3 L2 Regularization (Ridge Regularization)

L2 regularization, also known as Ridge Regularization, is one of the most widely used normalization techniques in machine learning and statistical predictive models. Its main goal is to reduce model complexity in order to limit the phenomenon of overfitting. This technique was introduced by Hoerl and Kennard (1970) and has since been applied across a wide range of problems, from linear regression to deep neural networks.

4.3.1 Mathematical Foundation of L2 Regularization

L2 regularization is based on adding a penalty term to the model's loss function, corresponding to the sum of the squares of the model parameters. For the linear regression, the classical loss function is the Mean Squared Error (MSE):

$$J(\theta) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2. \tag{4.7}$$

By adding the L2 regularization term, the revised loss function is formulated as follows:

$$J(\theta) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \psi\sum_{j=1}^{\kappa}\theta_j^2, \tag{4.8}$$

where:

- y_i is the actual value,
- $\hat{y}_i$ is the predicted value from the model,
- n is the sample size,
- θ_j are the model coefficients,
- ψ is the regularization hyperparameter that controls the magnitude of the penalty,
- κ is the number of model coefficients

The addition of the $\psi\sum_{j=1}^{m}\theta_j^2$ term shrinks the model's coefficients, preventing large values that could lead to overfitting to the training data.

The parameter ψ in L2 regularization (Ridge Regression) acts as a tuning factor, regulating the degree of coefficient shrinkage. When ψ is large, the coefficients approach zero, reducing model complexity and preventing overfitting, but simultaneously increasing bias. On the other hand, when ψ is very small, the model fits the training data more closely, increasing the risk of overfitting. Thus, ψ determines the balance between model simplicity and predictive performance, directly affecting model's generalization to new data.

4.3.2 Role in Overfitting Prevention and Generalization

L2 regularization improves the model's generalization ability—its capacity to perform well on new, unseen data—through the following mechanisms:

- Limiting model complexity: The penalty on the sum of squared coefficients discourages large parameter values, thereby reducing model variance.

- Enhancing stability: In cases of high correlation among features (multicollinearity), Ridge Regression stabilizes coefficient estimation, unlike traditional linear regression.

- Retaining all features: Unlike L1 regularization (Lasso), which zeroes out some coefficients, L2 regularization reduces the magnitude of all coefficients without eliminating any, making it ideal for problems where all features contribute to prediction.

4.3.3 Conclusion

L2 regularization is a fundamental tool in machine learning, offering an effective mechanism for improving model generalization. With the proper choice of the ψ parameter, it can reduce overfitting and lead to more reliable results. Although it does not provide feature selection like L1 regularization, its ability to keep all features active makes it valuable in problems where information is spread across many informative variables.

4.3.4 Problems Arising from Extreme Values of Coefficients

While L2 Regularization is a widely used technique for reducing overfitting and improving model generalization, it becomes particularly important in the context of time series models such as ARMA, where extreme values of the coefficients can lead to some significant issues.

Several studies have highlighted these challenges, pointing out the instability, non-invertibility, and difficulty in parameter estimation, etc., that arise when coefficients approach extreme values (e.g., Box and Jenkins (1976), Brockwell and Davis (1991), Hamilton (1994), Nelson (1991), Granger and Joyeux (1980), James et al. (2013) and White (1982)). These findings underscore the need for regularization techniques to address such problems and improve the robustness of time series models.

When the coefficients of the AR (ρ) and/or MA (ϕ) components approach extreme values — namely, close to -1 or 1 — the following problems may arise:

- Model Stability Issue
  It occurs when AR coefficients approach -1 or 1, potentially leading to non-stationary time series.

- Model Invertibility Issue
  It occurs when MA coefficients approach -1 or 1, resulting in non-invertible time series.

- Random Walk-Like Behaviour
  When AR coefficients are close to 1, the model behaviour tends to approximate that of a random walk.

- Sensitivity to Data and Numerical Accuracy
  The model may become overly responsive to data changes, leading to inaccurate predictions. Additionally, computations become numerically unstable near the boundaries.

- Overfitting Problem
  In ARMA models, overfitting can occur when the model coefficients are too large, causing the model to fit the training data too closely and reducing its ability to generalize to new data.

- Difficulty in Parameter Estimation
  Estimation algorithms such as maximum likelihood may struggle to converge when coefficients approach extreme values.

- Long Memory Effect
  It arises when past observations have a disproportionately strong impact on future predictions, potentially resulting in inaccurate forecasts if older data is no longer representative.

To mitigate these issues, L2 Regularization (Ridge Regularization) is applied during model training. By penalizing large coefficient values through an additional term in the loss function, the regularization helps to constrain the model complexity, improve numerical stability, and enhance generalization to unseen data.

4.4 ARMA(1,1) with L2 Penalty and Gradient Descent

As discussed in the previous sections, significant estimation issues arise when the coefficients of the ARMA(1,1) model approach the boundary values of -1 and 1. Although the theory (Box and Jenkins (1976) and Hamilton (1994)) states that for the model to be stationary and invertible it is sufficient that the coefficients are less than one in absolute value, in practice these estimates are adversely affected near the limits.

For this reason, it is necessary to apply techniques that constrain the values of the model's coefficients. Two such methods are:

- Gradient Descent for optimizing the loss function.

- L2 Regularization, which penalizes large parameter magnitudes.

Below we develop the mathematical framework for applying these methods to the ARMA(1,1) model, as used in this study.

Loss Function

The total loss function consists of two components:

1. The prediction error term (Mean Squared Error - MSE)
2. The L2 regularization penalty on the parameters

It is defined as follows:

$$\mathcal{L} = \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t)^2 + \psi(\rho^2 + \phi^2), \tag{4.9}$$

where

- y_t is the actual value,
- $\hat{y}_t$ is the predicted value from the model,
- n is the sample size,
- ρ, ϕ are the model coefficients (AR and MA respectively),
- ψ is the regularization hyperparameter that controls the penalty magnitude.

Gradient-Based Parameter Updates

According to the paper Park et al. (2018), the update rule for a parameter using L2 regularization is:

$$w^{(t+1)} = w^{(t)} - \alpha\left(\frac{\partial\mathcal{L}}{\partial w} + \psi\frac{\partial\Omega}{\partial w}\right), \tag{4.10}$$

where

- α is the learning rate,
- $\Omega(w) = \|w\|_2^2$ is the L2 regularizer (also known as weight decay).

For the ARMA(1,1) model, the update rules for the coefficients become:

$$\hat{\rho} = \rho - \alpha\left(\frac{\partial \text{MSE}}{\partial\rho} + 2\psi\rho\right), \tag{4.11}$$

$$\hat{\phi} = \phi - \alpha\left(\frac{\partial \text{MSE}}{\partial\phi} + 2\psi\phi\right). \tag{4.12}$$

ARMA(1,1) Model Specification

The model used is of the form:

$$y_t = \rho y_{t-1} + \varepsilon_t + \phi\varepsilon_{t-1} \tag{4.13}$$

and the corresponding prediction is:

$$\hat{y}_t = \hat{\rho} y_{t-1} + \varepsilon_t + \hat{\phi}\varepsilon_{t-1}. \tag{4.14}$$

Gradient of the MSE

Derivative with respect to ρ:

$$\frac{\partial \hat{y}_t}{\partial \hat{\rho}} = y_{t-1}, \tag{4.15}$$

$$\frac{\partial \text{MSE}}{\partial \rho} = -\frac{2}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t)y_{t-1}. \tag{4.16}$$

Derivative with respect to ϕ:

$$\frac{\partial \hat{y}_t}{\partial \hat{\phi}} = \varepsilon_{t-1}, \tag{4.17}$$

$$\frac{\partial \text{MSE}}{\partial \phi} = -\frac{2}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t).\varepsilon_{t-1} \tag{4.18}$$

Final Update Equations

The final update equations for the ARMA(1,1) coefficients using gradient descent with L2 penalty are:

$$\hat{\rho} = \rho - \alpha\left(-\frac{2}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t)y_{t-1} + 2\psi\rho\right), \tag{4.19}$$

$$\hat{\phi} = \phi - \alpha\left(-\frac{2}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t)\varepsilon_{t-1} + 2\psi\phi\right). \tag{4.20}$$

This approach allows the ARMA(1,1) model to be trained in a way that simultaneously minimizes prediction error while constraining parameter magnitudes, thus enhancing generalization and numerical stability.

4.5 Description of the Monte Carlo Experiment for the ARMA(1,1) Model

This section outlines a Monte Carlo experiment conducted within the framework of the ARMA(1,1) stochastic model, aiming to estimate important quantities required for the implementation of corrections of the t and F statistical tests.

In cases where the model exhibits ARMA(1,1) disturbances, there are no closed-form expressions available for some quantities necessary to apply the size corrections of the t and F tests. For the t test, two approaches are considered: the Edgeworth expansion and the Cornish-Fisher expansion, both under the normal and the Student-t distribution. Similarly, for the F test, the same two expansions are applied under the χ^2 and F distributions.

The lack of analytical expressions for some quantities involved in the size corrected t and F tests necessitates the use of simulation based calculation of these quantities. To address this, a Monte Carlo experiment was designed and implemented to provide reliable numerical estimates of the required quantities that underpin the correction terms in each case.

The experiment consists of 10000 repetitions for each combination of the values of the model parameters. Specifically, the values of the autoregression (AR) coefficient ρ and the moving average (MA) coefficient ϕ range from -0.9 to 0.9 in steps of 0.1, thus covering the full spectrum of parameter space. Also, four different sample sizes are considered: $T = 15$, 20, 30 and 50 observations.

This systematic approach enables the numerical estimation of the quantities μ, λ, Λ, and other derived terms, whose existence is crucial for the application of the corrections arising from asymptotic expansion theory. These estimates form the essential foundation for the analysis that follows.

For each combination of the parameters ρ and ϕ, the innovation terms ε_t are independently generated from a standard normal distribution $\mathcal{N}(0,1)$, and the standard deviation σ is set to 1. The underlying model is a generalized linear model with ARMA(1,1) residuals.

According to formula (3.4.7) from Box and Jenkins (1976), the variance of the initial value of the process u_0 is calculated as:

$$\gamma_0 = \frac{1 + \phi^2 + 2\rho\phi}{1 - \rho^2}\sigma^2. \tag{4.21}$$

Subsequently, the standard deviation is:

$$\sigma_u = \sqrt{\gamma_0}. \tag{4.22}$$

Thus, the initial value u_0 of the process is drawn from the normal distribution $\mathcal{N}(0, \sigma_u)$. Then, for $t = 1, 2, \ldots, T$ stochastic process u_t is then constructed recursively as follows:

$$u_t = \rho u_{t-1} + \varepsilon_t + \phi \varepsilon_{t-1}. \tag{4.23}$$

For $t = 1$, the observation is given by:

$$u_1 = \rho u_0 + \varepsilon_1 + \phi \varepsilon_0, \tag{4.24}$$

where $u_0 \sim \mathcal{N}(0, \sigma_u)$, and $\varepsilon_0, \varepsilon_1 \sim \mathcal{N}(0, 1)$. This specific form allows the representation of both the autoregression component through the dependence on u_{t-1} and the moving average component through the influence of ε_t and ε_{t-1}.

After generating the series u_t, the estimated parameters ρ and ϕ are estimated using the Maximum Likelihood Estimation (MLE) method. The initial estimates are further improved through the Gradient Descent method, aiming to minimize the following loss function:

$$\mathcal{L}(\rho, \phi) = \frac{1}{n} \sum_{t=1}^{n} (y_t - \hat{y}_t)^2 + \psi(\rho^2 + \phi^2). \tag{4.25}$$

This function includes the Mean Squared Error (MSE) and an L2 regularization term ψ, which penalizes large parameter values. For the purposes of the present study, we set $\psi = 1$ is set, as suggested by a relevant study Di Gangi et al. (2022), according to which this value provides a trade-off between convergence and stability.

The derivatives of the loss function with respect to the etsimated parameters are given by the following formulas:

$$\frac{\partial \mathcal{L}}{\partial \rho} = -\frac{2}{n} \sum (y_t - \hat{y}_t) y_{t-1} + 2\psi\rho, \tag{4.26}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{2}{n} \sum (y_t - \hat{y}_t) \varepsilon_{t-1} + 2\psi\phi. \tag{4.27}$$

The estimated parameters are updated using the following rules:

$$\rho \leftarrow \rho - \alpha \cdot \frac{\partial \mathcal{L}}{\partial \rho}, \quad \phi \leftarrow \phi - \alpha \cdot \frac{\partial \mathcal{L}}{\partial \phi}, \tag{4.28}$$

where α is the learning rate. Moreover, an early stopping criterion is applied if no improvement of at least 0.01 in the loss is observed over five consecutive iterations.

Only the parameter estimates that satisfy the stationarity and invertibility conditions are retained, that is:

$$|\rho| < 1 \quad \text{and} \quad |\phi| < 1. \tag{4.29}$$

For all the valid estimates, the following statistical quantities are computed:

Here, ρ and ϕ denote the true values of the respective parameters, whereas $\hat{\rho}$ and $\hat{\phi}$ denote their corresponding estimates.

the normalized differences:

$$\delta_\rho = (\hat{\rho} - \rho)\sqrt{T}, \quad \delta_\phi = (\hat{\phi} - \phi)\sqrt{T}, \tag{4.30}$$

the squared differences:

$$(\delta_\rho)^2, \quad (\delta_\phi)^2, \tag{4.31}$$

the cross-product:

$$\delta_\rho \delta_\phi. \tag{4.32}$$

The above procedure is repeated for k=10000 valid simulations to ensure statistical reliability. For the resulting quantities, the following summary statistics are calculated:

the means:

$$\tau\mu_\rho = \frac{1}{k}\sum \delta_\rho, \quad \tau\mu_\phi = \frac{1}{k}\sum \delta_\phi, \tag{4.33}$$

the variances:

$$\lambda_{\rho\rho} = \frac{1}{k}\sum (\delta_\rho)^2, \quad \lambda_{\phi\phi} = \frac{1}{k}\sum (\delta_\phi)^2, \tag{4.34}$$

the covariance:

$$\lambda_{\rho\phi} = \frac{1}{k}\sum \delta_\rho \delta_\phi. \tag{4.35}$$

These quantities are necessary for the implementation of the Edgeworth and Cornish-Fisher size corrections of the t and F tests, thereby improving the approximation of the true significance levels in small-sample distributions.

4.5.1 Conclusion

This experimental study demonstrates that the combined use of the Maximum Likelihood Estimation (MLE) method, the L2 regularization and the Gradient Descent algorithm, under appropriate stationarity and invertibility constraints, leads to consistent and reliable estimates of the asymptotic moments (expectations, variances and covariances) of the ARMA(1,1) model parameters. That are essential for the size correction of the t and F econometric tests. The estimated quantities, μ_ρ, μ_ϕ, $\lambda_{\rho\rho}$, $\lambda_{\phi\phi}$, and $\lambda_{\rho\phi}$, provide a more accurate representation of the behavior of the estimators in small samples. This enables the correction of the t and F tests using the Edgeworth and Cornish-Fisher expansions. Thus, the Monte Carlo experiment serves a dual purpose as both a parameter estimation tool and a foundation for more precise inference procedures in finite samples.

4.6 Results

This section presents the results of an extensive simulation conducted to evaluate the behavior of the model under different combinations of parameter values and sample sizes. Specifically, the experiment consists of 10000 repetitions for every possible combination of the model's parameter values.

The parameters that change are the autoregressive coefficient (ρ) and the moving average coefficient (ϕ), which take values from -0.9 to 0.9 with a step of 0.1, thus fully covering the entire parameter space. In addition, the analysis is carried out for four different sample sizes: $T = 15$, 20, 30, and 50 observations.

For each combination of values of ρ, ϕ, and T, the following statistical quantities are computed:

- The mean values of the parameter estimates μ_ρ and μ_ϕ, which reflect the accuracy of the estimators for the autoregressive and moving average parameters, respectively.
- The variances of the estimates $\lambda_{\rho\rho}$ and $\lambda_{\phi\phi}$, which represent the variability of the estimators for each parameter.
- The covariance between the estimators $\lambda_{\rho\phi}$, which indicates the interdependence of the parameter estimates.

The results are summarized in Appendix E, where the calculated quantities are presented in detail for every combination of parameter values and sample sizes. The simulation fully covers the symmetric range of the parameters, that is, both positive and negative values.

Observing the mean values and variances enables the assessment of the consistency and efficiency of the estimators as the sample size increases, while the covariance between the estimates provides additional insight into their behavior under interdependent conditions.

4.7 Sensitivity Analysis on the Number of Repetitions

In order to assess the robustness and stability of the quantities estimated via Monte Carlo simulation, a sensitivity analysis was conducted with respect to the number of repetitions. Initially, all quantities of interest—namely $\lambda_{\rho\rho}$, $\lambda_{\phi\phi}$, $\lambda_{\rho\phi}$, μ_ρ, and μ_ϕ—were computed using 10000 repetitions. Subsequently, the same experiment was repeated with reduced numbers of repetitions, specifically 1000, 2500, and 5000, in order to examine the potential impact of the number of repetition on the estimation accuracy.

The motivation behind this procedure was twofold. First, to validate whether a smaller number of repetitions could yield results of comparable quality, thus significantly reducing computational cost. Second, to ensure that the estimates of the parameters remain consistent and unbiased across different number of repetition.

This assessment was carried out across all sample sizes considered in the study, specifically for samples of 15, 20, 30, and 50 observations. However, to limit computational load, this extended analysis was restricted to selected values of ρ and ϕ. More precisely, the sensitivity checks were performed for $\rho \in \{\pm 0.1,\ \pm 0.5,\ \pm 0.9\}$ and symmetrically for $\phi \in \{\pm 0.1,\ \pm 0.5,\ \pm 0.9\}$. In the bivariate case, we focused on combinations such as $(\rho, \phi) = (0.1, \pm 0.1)$, $(-0.1, \pm 0.1)$, $(0.5, \pm 0.5)$, $(-0.5, \pm 0.5)$, $(0.9, \pm 0.9)$, and $(-0.9, \pm 0.9)$.

The results for all combinations of ρ and ϕ are presented in tabular form. A comparative inspection reveals that the differences between the estimates obtained with 10000 repetitions and those with fewer repetitions are negligible. More precisely, the values of $\lambda_{\rho\rho}$, $\lambda_{\phi\phi}$, $\lambda_{\rho\phi}$, μ_ρ, and μ_ϕ exhibit minimal variation across the different experiments, which indicates high numerical stability and convergence of the estimation procedure.

These findings justify the use of a reduced number of repetitions in practical applications where computational efficiency is critical, without compromising the reliability of the results.

Table 4.1 Monte Carlo Estimates with 1000 Repetitions

1000		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
0.1	0.1	-0.335	-0.313	0.324	0.220	0.053	-0.418	-0.347	0.440	0.264	0.075	-0.511	-0.420	0.587	0.393	0.088	-0.679	-0.503	0.867	0.587	0.115
0.1	-0.1	-0.328	0.358	0.340	0.259	-0.157	-0.401	0.437	0.458	0.352	-0.239	-0.484	0.514	0.585	0.509	-0.392	-0.639	0.643	0.886	0.846	-0.711
-0.1	0.1	0.450	-0.413	0.430	0.304	-0.224	0.503	-0.459	0.550	0.371	-0.294	0.603	-0.586	0.711	0.584	-0.495	0.777	-0.770	1.080	1.021	-0.896
-0.1	-0.1	0.459	0.256	0.419	0.196	0.089	0.512	0.296	0.506	0.244	0.088	0.582	0.334	0.630	0.359	0.074	0.668	0.405	0.799	0.568	0.045
0.5	0.5	-1.793	-1.374	3.390	1.991	2.406	-2.051	-1.636	4.391	2.772	3.288	-2.430	-2.105	6.052	4.522	5.042	-3.093	-2.845	9.682	8.173	8.733
0.5	-0.5	-1.879	1.884	3.770	3.693	-3.580	-2.172	2.206	5.017	5.041	-4.861	-2.665	2.674	7.441	7.405	-7.271	-3.459	3.443	12.453	12.276	-12.201
-0.5	0.5	2.017	-1.944	4.294	3.928	-3.956	2.296	-2.222	5.568	5.123	-5.167	2.783	-2.731	8.092	7.711	-7.736	3.607	-3.590	13.485	13.332	-13.247
-0.5	-0.5	1.691	1.375	3.073	1.984	2.270	1.924	1.655	3.934	2.835	3.118	2.302	2.105	5.489	4.530	4.775	2.987	2.839	9.064	8.153	8.419
0.9	0.9	-2.809	-2.187	9.064	5.157	6.113	-3.245	-2.686	11.643	7.616	8.590	-3.941	-3.603	16.561	13.353	13.957	-5.205	-4.986	27.862	25.261	25.625
0.9	-0.9	-3.434	3.346	12.041	11.365	-11.541	-4.013	3.890	16.404	15.340	-15.679	-4.923	4.673	24.561	22.166	-23.138	-6.372	6.071	41.008	37.359	-38.939
-0.9	0.9	3.517	-3.047	12.574	9.606	-10.722	4.068	-3.543	16.793	12.887	-14.455	4.972	-4.467	25.018	20.458	-22.316	6.380	-5.974	41.113	36.352	-38.393
-0.9	-0.9	2.683	2.271	8.246	5.590	5.936	3.113	2.761	10.858	8.088	8.382	3.857	3.667	15.956	13.893	13.795	5.095	5.038	26.841	25.898	25.218

Table 4.2 Monte Carlo Estimates with 2500 Repetitions

2500		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
0.1	0.1	-0.349	-0.312	0.341	0.218	0.060	-0.415	-0.351	0.425	0.271	0.072	-0.506	-0.435	0.552	0.418	0.086	-0.653	-0.519	0.805	0.612	0.114
0.1	-0.1	-0.329	0.352	0.341	0.254	-0.157	-0.383	0.426	0.425	0.350	-0.234	-0.505	0.521	0.585	0.531	-0.409	-0.663	0.663	0.903	0.872	-0.735
-0.1	0.1	0.449	-0.419	0.433	0.305	-0.228	0.516	-0.469	0.542	0.388	-0.314	0.583	-0.578	0.667	0.594	-0.482	0.752	-0.754	1.030	1.000	-0.861
-0.1	-0.1	0.458	0.256	0.418	0.200	0.090	0.507	0.292	0.490	0.255	0.085	0.536	0.358	0.572	0.389	0.070	0.629	0.438	0.755	0.588	0.046
0.5	0.5	-1.808	-1.366	3.441	1.967	2.412	-2.037	-1.647	4.318	2.805	3.292	-2.417	-2.121	5.980	4.586	5.056	-3.079	-2.850	9.601	8.203	8.707
0.5	-0.5	-1.889	1.877	3.808	3.668	-3.586	-2.176	2.193	5.017	4.987	-4.844	-2.690	2.679	7.556	7.440	-7.349	-3.489	3.467	12.645	12.455	-12.392
-0.5	0.5	2.010	-1.942	4.272	3.915	-3.943	2.307	-2.231	5.597	5.169	-5.219	2.773	-2.732	8.013	7.741	-7.716	3.579	-3.566	13.273	13.158	-13.058
-0.5	-0.5	1.711	1.378	3.148	1.993	2.299	1.915	1.657	3.881	2.842	3.106	2.294	2.113	5.441	4.559	4.787	2.968	2.851	8.948	8.219	8.401
0.9	0.9	-2.837	-2.167	9.189	5.068	6.116	-3.218	-2.696	11.535	7.641	8.582	-3.930	-3.608	16.465	13.404	13.942	-5.187	-4.969	27.702	25.117	25.436
0.9	-0.9	-3.456	3.351	12.207	11.395	-11.630	-4.009	3.860	16.357	15.117	-15.537	-4.929	4.685	24.598	22.284	-23.222	-6.363	6.072	40.890	37.383	-38.898
-0.9	0.9	3.527	-3.037	12.650	9.547	-10.708	4.064	-3.530	16.757	12.840	-14.382	4.948	-4.445	24.755	20.272	-22.090	6.339	-5.925	40.583	35.768	-37.831
-0.9	-0.9	2.725	2.279	8.448	5.633	6.055	3.116	2.785	10.876	8.217	8.448	3.830	3.673	15.725	13.954	13.749	5.059	5.063	26.465	26.138	25.180

Table 4.3 Monte Carlo Estimates with 5000 Repetitions

5000		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
0.1	0.1	-0.350	-0.320	0.337	0.227	0.065	-0.405	-0.365	0.409	0.293	0.073	-0.497	-0.434	0.542	0.420	0.084	-0.651	-0.514	0.797	0.601	0.111
0.1	-0.1	-0.329	0.359	0.331	0.264	-0.158	-0.387	0.419	0.415	0.357	-0.232	-0.498	0.519	0.574	0.536	-0.405	-0.682	0.676	0.933	0.869	-0.753
-0.1	0.1	0.445	-0.412	0.420	0.305	-0.223	0.510	-0.474	0.524	0.408	-0.312	0.594	-0.579	0.678	0.600	-0.490	0.731	-0.739	1.000	0.959	-0.832
-0.1	-0.1	0.445	0.261	0.400	0.206	0.087	0.487	0.301	0.471	0.275	0.087	0.536	0.353	0.576	0.391	0.071	0.647	0.433	0.794	0.573	0.049
0.5	0.5	-1.794	-1.376	3.388	1.999	2.411	-2.023	-1.648	4.258	2.809	3.270	-2.424	-2.115	6.017	4.566	5.058	-3.076	-2.844	9.578	8.167	8.677
0.5	-0.5	-1.885	1.885	3.783	3.701	-3.595	-2.180	2.188	5.024	4.981	-4.840	-2.690	2.685	7.562	7.478	-7.369	-3.509	3.484	12.780	12.554	-12.516
-0.5	0.5	2.008	-1.935	4.252	3.899	-3.924	2.307	-2.236	5.584	5.201	-5.227	2.786	-2.744	8.085	7.809	-7.786	3.560	-3.551	13.138	13.030	-12.932
-0.5	-0.5	1.698	1.388	3.105	2.020	2.303	1.905	1.662	3.843	2.858	3.105	2.286	2.117	5.402	4.572	4.778	2.972	2.848	8.970	8.206	8.407
0.9	0.9	-2.816	-2.176	9.109	5.107	6.097	-3.190	-2.697	11.403	7.662	8.483	-3.938	-3.592	16.567	13.300	13.910	-5.196	-4.965	27.794	25.083	25.441
0.9	-0.9	-3.456	3.355	12.193	11.424	-11.639	-4.013	3.857	16.387	15.113	-15.548	-4.930	4.690	24.608	22.338	-23.252	-6.364	6.077	40.898	37.431	-38.938
-0.9	0.9	3.526	-3.037	12.640	9.549	-10.709	4.059	-3.538	16.709	12.907	-14.388	4.942	-4.446	24.704	20.262	-22.077	6.334	-5.927	40.530	35.766	-37.818
-0.9	-0.9	2.702	2.286	8.378	5.664	6.033	3.096	2.792	10.749	8.252	8.419	3.821	3.677	15.679	13.989	13.722	5.062	5.073	26.506	26.224	25.242

Table 4.4 Monte Carlo Estimates with 10000 Repetitions

10000		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
0.1	0.1	-0.349	-0.315	0.337	0.222	0.062	-0.412	-0.363	0.414	0.294	0.074	-0.514	-0.421	0.549	0.402	0.086	-0.652	-0.510	0.797	0.602	0.105
0.1	-0.1	-0.326	0.363	0.334	0.268	-0.158	-0.392	0.423	0.423	0.357	-0.236	-0.504	0.520	0.584	0.526	-0.408	-0.680	0.677	0.932	0.872	-0.752
-0.1	0.1	0.450	-0.410	0.430	0.304	-0.224	0.505	-0.470	0.525	0.401	-0.308	0.589	-0.577	0.676	0.587	-0.486	0.732	-0.737	1.004	0.956	-0.830
-0.1	-0.1	0.446	0.264	0.407	0.212	0.091	0.487	0.302	0.478	0.272	0.088	0.541	0.353	0.581	0.386	0.070	0.653	0.430	0.805	0.576	0.048
0.5	0.5	-1.795	-1.371	3.397	1.987	2.403	-2.029	-1.645	4.281	2.802	3.273	-2.420	-2.109	5.995	4.537	5.036	-3.080	-2.840	9.602	8.145	8.678
0.5	-0.5	-1.877	1.888	3.757	3.713	-3.583	-2.186	2.192	5.053	4.991	-4.861	-2.698	2.690	7.610	7.498	-7.404	-3.509	3.487	12.781	12.578	-12.525
-0.5	0.5	2.007	-1.932	4.255	3.886	-3.915	2.300	-2.234	5.558	5.185	-5.206	2.783	-2.743	8.076	7.796	-7.780	3.560	-3.547	13.140	13.007	-12.920
-0.5	-0.5	1.694	1.389	3.094	2.025	2.297	1.909	1.660	3.862	2.850	3.109	2.288	2.116	5.412	4.567	4.783	2.969	2.851	8.949	8.221	8.407
0.9	0.9	-2.820	-2.158	9.129	5.053	6.057	-3.203	-2.689	11.469	7.633	8.483	-3.946	-3.584	16.654	13.253	13.897	-5.193	-4.966	27.779	25.092	25.428
0.9	-0.9	-3.452	3.362	12.174	11.471	-11.651	-4.018	3.861	16.425	15.133	-15.577	-4.939	4.700	24.701	22.426	-23.348	-6.370	6.082	40.986	37.506	-39.014
-0.9	0.9	3.529	-3.041	12.663	9.568	-10.733	4.065	-3.540	16.757	12.917	-14.422	4.932	-4.445	24.611	20.235	-22.028	6.334	-5.930	40.527	35.798	-37.833
-0.9	-0.9	2.712	2.282	8.430	5.647	6.041	3.104	2.798	10.798	8.277	8.452	3.829	3.687	15.755	14.051	13.789	5.064	5.068	26.509	26.165	25.243

Chapter 5

Implementation and Evaluation of Corrected Statistical Tests through Monte Carlo Simulation

5.1 Description of Monte Carlo Simulation

In the present chapter, we present the Monte Carlo experiment implemented with the aim of investigating the performance of various corrections of the t and F tests in the case of the linear model whose disturbance term follows the ARMA(1,1) model is presented.

The purpose of the analysis is the comparison of the classical tests with the corrected tests, based on the Cornish-Fisher and Edgeworth approximations under the assumptions of normal, Student-t, χ^2 and F distributions for significance levels 1%, 5%, and 10%, as well as for sample sizes of 15 and 30 observations.

The experiment was designed to cover combinations of the parameters of the $ARMA(1,1)$ model, namely ρ and ϕ, as well as the correlation coefficient between any two different explanatory variables, denoted by A. Each combination of the parameters ρ, ϕ and A corresponds to a point in the experimental space.

For this purpose, we selected four values for the autoregressive (AR) coefficient, $\rho = \pm 0.5, \pm 0.9$, four values for the moving average (MA) coefficient, $\phi = \pm 0.5, \pm 0.9$ and one value for the coefficient indicating the intensity of multicollinearity $A = 0.5$.

Next, we create 50 independent observations for the four independent $N(0,1)$ pseudo-random numbers z_{t1}, z_{t2}, z_{t3} and z_{t4} $(t = 1, 2, ..., 50)$. Using formula (3.1) from the McDonald and Galarneau (1975) we construct the elements x_{tj} of the matrix of explanatory variables X, using the following relations:

$$x_{t1} = 1 \quad (t = 1, 2, ..., 50) \tag{5.1}$$

$$x_{t2} = (1-\alpha^2)^{1/2} z_{t2} + \alpha z_{t1} \quad (t = 1, 2, ..., 50) \tag{5.2}$$

$$x_{t3} = (1-\alpha^2)^{1/2} z_{t3} + \alpha z_{t1} \quad (t = 1, 2, ..., 50) \tag{5.3}$$

$$x_{t4} = (1-\alpha^2)^{1/2} z_{t4} + \alpha z_{t1} \quad (t = 1, 2, ..., 50) \tag{5.4}$$

According to formula (3.4.7) in Box and Jenkins (1976), the variance of the initial value of the process u_0 is calculated as in Appendix D

$$\sigma_u^2 = \gamma_0 = \frac{\sigma_\epsilon^2(1 + \phi^2 + 2\rho\phi)}{1 - \rho^2}, \tag{5.5}$$

where σ_ε^2 is the variance of the white noise ϵ_t, which has been set equal to unity.

And the standard deviation is:

$$\sigma_u = \sqrt{\gamma_0}. \tag{5.6}$$

These relations arise from the statistical properties of the autocorrelation function of the $ARMA(1,1)$ process; the analytical derivation of these relations is given in Appendix E.

The described procedure completes the generation of the data used in the simulation.

Using the matrix of explanatory variables and 10000 different vectors of the dependent variable, which we created for each of the 32 points of the experimental space, we performed at each experimental point 10000 repetitions of the process described below.

For each of the 10000 repetitions, the generation of the time series of the disturbance term u_t, which follows the stochastic process ARMA(1,1), is implemented. This process depends on three main parameters: the autoregressive coefficient ρ, the moving average coefficient ϕ, as well as the stochastic disturbances ε_t that follow a normal distribution with zero mean and variance equal to unity.

Initially, we define the vector ε_t where $\varepsilon_t \sim N(0, \sigma^2)$, which is used to create the stochastic process. The initial value of the error ε_0 is a random number generated from the normal distribution, while the initial value of the disturbance term u_0 is generated as a random value from the normal distribution with zero mean and standard deviation σ_u, which has already been calculated from the theoretical variance of the process (see (5.5) and (5.6)).

The first value of the time series u_t, where $u \sim N(0, \sigma^2\Omega(\rho, \phi))$, i.e. u_1, is calculated based on the relation:

$$u_1 = \rho u_0 + \varepsilon_1 + \phi\epsilon_0. \tag{5.7}$$

This relation incorporates the autoregressive via the term ρu_0, as well as the influence of both the current disturbance ε_1 and the previous ε_0, through the term $\phi\varepsilon_0$.

Subsequently, the remaining values of the time series u_t are calculated recursively for each time $t = 2, 3, \ldots, T$, according to the general form of the process:

$$u_t = \rho u_{t-1} + \varepsilon_t + \phi\varepsilon_{t-1}. \tag{5.8}$$

In this way, each new value of u_t depends both on the previous value of the series itself, as well as on the current and previous values of the stochastic disturbances. This process naturally incorporates the dependence structure characterizing the ARMA(1,1) model, ensuring that the simulated series u_t has the desired stochastic properties.

Knowing the vector u, we create the vector of the dependent variable, y, with elements y_t, using the relation

$$y_t = \beta_1 + \beta_2 x_{t2} + \beta_3 x_{t3} + \beta_4 x_{t4} + \sigma u_t, \quad u_t = ARMA(1,1) \tag{5.9}$$

where $\beta_1, \beta_2, \beta_3, \beta_4$ are the parameters to be estimated. To simplify the calculations, we set $\beta_1 = 0, \beta_2 = 0, \beta_3 = 0, \beta_4 = 0$ and get the relation

$$y_t = \sigma u_t, \tag{5.10}$$

from which we calculate the elements of the vectors y of the dependent variable of the experiment given the matrix of exogenous variables.

Each vector of the dependent variable was regressed using the ordinary least squares (OLS) method on the matrix of explanatory variables and we extracted the residuals.

Subsequently, using the OLS residuals, we estimated the parameters of the ARMA(1,1) model via the Maximum Likelihood Estimation (MLE) method.

As we have analyzed in previous sections, although the theory states that the coefficients in absolute value should be less than unity, in practice at extreme parameter values, problems arise that affect the estimates.

For this reason, it is necessary to apply the L2 Regularization and Gradient Descent methods in order to adjust the estimated coefficient ρ and ϕ.

The initial estimates are further improved through the Gradient Descent method, aiming to minimize the following loss function:

$$\mathcal{L}(\rho, \phi) = \frac{1}{n} \sum_{t=1}^{n} (y_t - \hat{y}_t)^2 + \psi(\rho^2 + \phi^2). \tag{5.11}$$

The above function includes the Mean Squared Error (MSE) and an L2 regularization term ψ, which penalizes large parameter values. For the purposes of the present study, is set equal to unity, as suggested by a relevant study Di Gangi et al. (2022), according to which this value provides a trade-off between convergence and stability.

The derivatives of the loss function with respect to the estimated parameters are given by the following formulas:

$$\frac{\partial \mathcal{L}}{\partial \rho} = -\frac{2}{n}\sum (y_t - \hat{y}_t) y_{t-1} + 2\psi\rho, \tag{5.12}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{2}{n}\sum (y_t - \hat{y}_t) \varepsilon_{t-1} + 2\psi\phi. \tag{5.13}$$

The estimated parameters are updated using the following rules:

$$\rho \leftarrow \rho - \alpha \cdot \frac{\partial \mathcal{L}}{\partial \rho}, \quad \phi \leftarrow \phi - \alpha \cdot \frac{\partial \mathcal{L}}{\partial \phi}, \tag{5.14}$$

where α is the learning rate. Moreover, an early stopping criterion is applied if no improvement of at least 0.01 in the loss is observed over five consecutive iterations.

Only the parameter estimates that satisfy the stationarity and invertibility conditions are retained, that is:

$$|\rho| < 1 \quad \text{and} \quad |\phi| < 1. \tag{5.15}$$

In the next stage of the analysis, the matrix Ω is calculated, which expresses the variance-covariance structure of the disturbance terms of the stochastic process. This matrix constitutes a critical tool for the application of the Feasible Generalized Least Squares (FGLS) method, as it allows the adjustment of the linear model in cases of ARMA(1,1) disturbances.

The elements of Ω include both the diagonal (variances) and the non-diagonal (covariances) values, and are determined based on the formulas proposed by Tiao and Ali (1971) for ARMA(1,1) processes. The general form of these elements depends on the estimated parameters ρ and ϕ of the stochastic ARMA(1,1) model.

Specifically, the diagonal elements are calculated according to the equation

$$\begin{aligned} \omega_{tt} \quad = \quad & \frac{1}{D}\Big[(1+\rho\phi)^2(1+\rho^2+2\rho\phi) \\ & +(\phi+\rho)^2\left\{(\phi+\rho)+\rho(1+\rho\phi)\right\}\phi^{2T-1} \\ & -(\phi+\rho)^2(1+\rho\phi)^2\left\{\phi^{2(t-1)}+\phi^{2(T-t)}\right\}\Big], \end{aligned} \tag{5.16}$$

while the non-diagonal elements (for $t \neq t'$) are given by

$$\begin{aligned}\omega_{tt'} &= \frac{1}{D}\Big[-(\phi+\rho)(1+\rho\phi)^3(-\phi)^{|t-t'|-1} \\ &\quad -(\phi+\rho)^3(1+\rho\phi)(-\phi)^{2T-|t-t'|-1} \\ &\quad -(\phi+\rho)^2(1+\rho\phi)^2\left\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\right\}\Big]. \end{aligned} \tag{5.17}$$

The above formulas allow the numerical computation of all elements of the matrix Ω, which is assembled symmetrically. For a complete mathematical analysis of the above formulas, you may refer to Appendix A.

Subsequently, Ω is used for estimating the parameter β by the $FGLS$ method as

$$\hat{\beta} = \left(\frac{1}{T}X^\top \Omega X\right)^{-1} \frac{1}{T}X^\top \Omega y, \tag{5.18}$$

and the parameter σ^2 as

$$\hat{\sigma}^2 = \frac{(y-\hat{y})^\top \Omega (y-\hat{y})}{T-k}. \tag{5.19}$$

Finally, the estimated variance of the estimator $\hat{\beta}$ is

$$\widehat{\text{Var}}(\hat{\beta}) = \frac{\hat{\sigma}^2}{T}\left(\frac{1}{T}X^\top \Omega X\right)^{-1}. \tag{5.20}$$

Based on the diagonal elements of this matrix, the t statistic for each coefficient is derived as

$$t_j^{FGLS} = \frac{\hat{\beta}_j}{\sqrt{\widehat{\text{Var}}(\hat{\beta}_j)}}, \quad j = 1,2,3,4. \tag{5.21}$$

Next, the first- and second-order derivatives of the elements of the matrix Ω with respect to the estimated parameters ρ and ϕ are calculated, according to the following formulas:

We define ω_{tt} as follows:

$$\omega_{tt} = D^{-1}N, \tag{5.22}$$

where

$$D = [(1+\rho\phi)^2 - (\rho+\phi)^2\phi^{2T}][1-\phi^2] \tag{5.23}$$

and

$$\begin{aligned} N &= (1+\rho\phi)^2(1+\rho^2+2\rho\phi)+(\phi+\rho)^2\left\{(\phi+\rho)+\rho(1+\rho\phi)\right\}\phi^{2T-1} \\ &\quad -(\phi+\rho)^2(1+\rho\phi)^2\left\{\phi^{2(t-1)}+\phi^{2(T-t)}\right\}. \end{aligned} \tag{5.24}$$

We define $\omega_{tt'}$ as follows:

$$\omega_{tt'} = D^{-1}N_*, \tag{5.25}$$

where

$$\begin{aligned} N_* &= -(\phi+\rho)(1+\rho\phi)^3(-\phi)^{|t-t'|-1}-(\phi+\rho)^3(1+\rho\phi)(-\phi)^{2T-|t-t'|-1} \\ &\quad -(\phi+\rho)^2(1+\rho\phi)^2\left\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\right\}. \end{aligned} \tag{5.26}$$

Equations (B.1) and (B.4) in Appendix B imply that

$$\begin{aligned} \omega_{tt} &= \frac{N}{D}, \\ \omega_{tt'} &= \frac{N_*}{D}. \end{aligned} \tag{5.27}$$

Then,

$$\frac{\partial\omega_{tt}}{\partial\rho} = \frac{\partial}{\partial\rho}\left(\frac{N}{D}\right) = \frac{DN_\rho - ND_\rho}{D^2}, \tag{5.28}$$

$$\begin{aligned} \frac{\partial^2\omega_{tt}}{\partial\rho^2} &= \frac{\partial}{\partial\rho}\left\{\frac{DN_\rho - ND_\rho}{D^2}\right\} \\ &= \frac{D^2(D_\rho N_\rho + DN_{\rho\rho} - N_\rho D_\rho - ND_{\rho\rho}) - (DN_\rho - ND_\rho)2D_\rho D}{D^4} \\ &= \frac{D^2(DN_{\rho\rho} - ND_{\rho\rho}) - (DN_\rho D_\rho^2 - ND_\rho D_\rho^2)}{D^4}. \end{aligned} \tag{5.29}$$

Similarly,

$$\frac{\partial\omega_{tt}}{\partial\phi} = \frac{DN_\phi - ND_\phi}{D^2}, \tag{5.30}$$

$$\frac{\partial^2 \omega_{tt}}{\partial \phi^2} = \frac{D^2(DN_{\phi\phi} - ND_{\phi\phi}) - (2D^2 N_\phi D_\phi - 2DND_\phi D_\phi)}{D^4}, \tag{5.31}$$

$$\begin{aligned} \frac{\partial^2 \omega_{tt}}{\partial \rho \partial \phi} &= \frac{\partial}{\partial \phi}\left\{\frac{DN_\rho - ND_\rho}{D^2}\right\} \\ &= \frac{D^2(D_\phi N_\rho + DN_{\rho\phi} - N_\phi D_\rho - ND_{\rho\phi}) - (DN_\rho - ND_\rho)2DD_\phi}{D^4}, \end{aligned} \tag{5.32}$$

$$\begin{aligned} \frac{\partial^2 \omega_{tt}}{\partial \phi \partial \rho} &= \frac{\partial}{\partial \rho}\left\{\frac{DN_\phi - ND_\phi}{D^2}\right\} \\ &= \frac{D^2(D_\rho N_\phi + DN_{\phi\rho} - N_\rho D_\phi - ND_{\phi\rho}) - (DN_\phi - ND_\phi)2DD_\rho}{D^4}. \end{aligned} \tag{5.33}$$

Also,

$$\frac{\partial \omega_{tt'}}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{N_*}{D}\right) = \frac{DN_{*\rho} - N_* D_\rho}{D^2}, \tag{5.34}$$

$$\begin{aligned} \frac{\partial^2 \omega_{tt'}}{\partial \rho^2} &= \frac{\partial}{\partial \rho}\left\{\frac{DN_{*\rho} - N_* D_\rho}{D^2}\right\} \\ &= \frac{D^2(D_\rho N_{*\rho} + DN_{*\rho\rho} - N_{*\rho} D_\rho - N_* D_{\rho\rho}) - (DN_{*\rho} - N_* D_\rho)2DD_\rho}{D^4} \\ &= \frac{D^2(DN_{*\rho\rho} - N_* D_{\rho\rho}) - (DN_{*\rho} D_\rho^2 - 2DN_* D_\rho D_\rho^2)}{D^4}. \end{aligned} \tag{5.35}$$

Similarly,

$$\frac{\partial \omega_{tt'}}{\partial \phi} = \frac{DN_{*\phi} - N_* D_\phi}{D^2}, \tag{5.36}$$

$$\frac{\partial^2 \omega_{tt'}}{\partial \phi^2} = \frac{D^2(DN_{*\phi\phi} - N_* D_{\phi\phi}) - (2D^2 N_{*\phi} D_\phi - 2DN_* D_\phi D_\phi)}{D^4}, \tag{5.37}$$

$$\begin{aligned} \frac{\partial^2 \omega_{tt'}}{\partial \rho \partial \phi} &= \frac{\partial}{\partial \phi}\left\{\frac{DN_{*\rho} - N_* D_\rho}{D^2}\right\} \\ &= \frac{D^2(D_\phi N_{*\rho} + DN_{*\rho\phi} - N_{*\phi} D_\rho - N_* D_{\rho\phi}) - (DN_{*\rho} - N_* D_\rho)2DD_\phi}{D^4}, \end{aligned} \tag{5.38}$$

$$
\begin{aligned}
\frac{\partial^2 \omega_{tt'}}{\partial\phi\partial\rho} &= \frac{\partial}{\partial\rho}\left\{\frac{DN_{*\phi} - N_* D_\phi}{D^2}\right\} \\
&= \frac{D^2(D_\rho N_{*\phi} + DN_{*\phi\rho} - N_{*\rho}D_\phi - N_* D_{\phi\rho}) - (DN_{*\phi} - N_* D_\phi)2DD_\rho}{D^4}. \qquad (5.39)
\end{aligned}
$$

All these derivatives have been calculated analytically in Appendix D.

Using these elements, we define the matrices $\Omega_\rho, \Omega_\phi, \Omega_{\rho\rho}, \Omega_{\phi\phi}, \Omega_{\rho\phi}$ and $\Omega_{\phi\rho}$.

For the t test, we considered four hypotheses of the form (3.15), i.e.

$$\beta_1 = 0,\ \beta_2 = 0,\ \beta_3 = 0,\ \beta_4 = 0 \qquad (5.40)$$

Now, having now all the necessary elements, we calculate the following quantities which are essential for the formulas of the corrections.

From formula (3.10), we have

$$
\begin{array}{lll}
 & A_{\rho\rho} = X'\Omega_{\rho\rho}X/T & A^*_{\rho\rho} = X'\Omega_\rho\Omega^{-1}\Omega_\rho X/T \\
A_\rho = X'\Omega_\rho X/T & A_{\phi\phi} = X'\Omega_{\phi\phi}X/T & A^*_{\phi\phi} = X'\Omega_\phi\Omega^{-1}\Omega_\phi X/T \\
A_\phi = X'\Omega_\phi X/T & A_{\rho\phi} = X'\Omega_{\rho\phi}X/T & A^*_{\rho\phi} = X'\Omega_\rho\Omega^{-1}\Omega_\phi X/T \\
 & A_{\phi\rho} = X'\Omega_{\phi\rho}X/T & A^*_{\phi\rho} = X'\Omega_\phi\Omega^{-1}\Omega_\rho X/T
\end{array} \qquad (5.41)
$$

For each $\beta_i = (i = 1, 2, 3, 4)$, using formulas (3.17), (3.18) and (3.19) we calculate the elements of

$$l = \begin{bmatrix} l_\rho \\ l_\phi \end{bmatrix}, \quad L = \begin{bmatrix} L_{\rho\rho} & L_{\rho\phi} \\ L_{\phi\rho} & L_{\phi\phi} \end{bmatrix}, \qquad (5.42)$$

as follows:

$$
\begin{array}{ll}
 & l_{\rho\rho} = \mathrm{e}'GC_{\rho\rho}G\mathrm{e}/\mathrm{e}'G\mathrm{e} \\
l_\rho = \mathrm{e}'GA_\rho G\mathrm{e}/\mathrm{e}'G\mathrm{e} & l_{\phi\phi} = \mathrm{e}'GC_{\phi\phi}G\mathrm{e}/\mathrm{e}'G\mathrm{e} \\
l_\phi = \mathrm{e}'GA_\phi G\mathrm{e}/\mathrm{e}'G\mathrm{e} & l_{\rho\phi} = \mathrm{e}'GC_{\rho\phi}G\mathrm{e}/\mathrm{e}'G\mathrm{e} \\
 & l_{\phi\rho} = \mathrm{e}'GC_{\phi\rho}G\mathrm{e}/\mathrm{e}'G\mathrm{e}
\end{array} \qquad (5.43)
$$

where

$$G = (X'\Omega X/T)^{-1} \qquad (5.44)$$

and

$$C_{\rho\rho} = A^*_{\rho\rho} - 2A_\rho G A_\rho + A_{\rho\rho}/2 \tag{5.45}$$

$$C_{\phi\phi} = A^*_{\phi\phi} - 2A_\phi G A_\phi + A_{\phi\phi}/2 \tag{5.46}$$

$$C_{\rho\phi} = A^*_{\rho\phi} - 2A_\rho G A_\phi + A_{\rho\phi}/2 \tag{5.47}$$

$$C_{\phi\rho} = A^*_{\phi\rho} - 2A_\phi G A_\rho + A_{\phi\rho}/2 \tag{5.48}$$

For the F test, we considered a hypothesis of the form (3.28), i.e.

$$H\beta = h, \quad H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.49}$$

Next, using formulas (3.29) and (3.36) we compute the Wald statistic:

$$\mathrm{w} = (H\hat{\beta} - h)'[H(X'\hat{\Omega}X)^{-1}H']^{-1}(H\hat{\beta} - h)/\hat{\sigma}^2 \tag{5.50}$$

and the F statistic

$$\mathrm{v} = (H\hat{\beta} - h)'[H(X'\hat{\Omega}X)^{-1}H']^{-1}(H\hat{\beta} - h)/r\hat{\sigma}^2. \tag{5.51}$$

Then we compute all quantities necessary for the correction formulas using formulas (3.30), (3.31) and (3.32), i.e.

$$P = GQG, \quad Q = H'(HGH')^{-1}H \tag{5.52}$$

and

$$c = \begin{bmatrix} c_\rho \\ c_\phi \end{bmatrix}, \quad C = \begin{bmatrix} c_{\rho\rho} & c_{\rho\phi} \\ c_{\phi\rho} & c_{\phi\phi} \end{bmatrix}, \quad D = \begin{bmatrix} d_{\rho\rho} & d_{\rho\phi} \\ d_{\phi\rho} & d_{\phi\phi} \end{bmatrix}, \tag{5.53}$$

where

$$\begin{array}{llll} & c_{\rho\rho} = \mathrm{tr}(C_{\rho\rho}P) & d_{\rho\rho} = [(\mathrm{tr}\, D_{\rho\rho}P) & D_{\rho\rho} = A_\rho P A_\rho/2 \\ c_\rho = \mathrm{tr}(A_\rho P) & c_{\phi\phi} = \mathrm{tr}(C_{\phi\phi}P) & d_{\phi\phi} = [(\mathrm{tr}\, D_{\phi\phi}P) & D_{\phi\phi} = A_\phi P A_\phi/2 \\ c_\phi = \mathrm{tr}(A_\phi P) & c_{\rho\phi} = \mathrm{tr}(C_{\rho\phi}P) & d_{\rho\phi} = [(\mathrm{tr}\, D_{\rho\phi}P) & D_{\rho\phi} = A_\rho P A_\phi/2 \\ & c_{\phi\rho} = \mathrm{tr}(C_{\phi\rho}P) & d_{\phi\rho} = [(\mathrm{tr}\, D_{\phi\rho}P) & D_{\phi\rho} = A_\phi P A_\rho/2 \end{array} \tag{5.54}$$

According to Breusch (1980), the constant λ_0 takes the value 2. Then we compute the following:

$$\tau = 1/\sqrt{T}, \tag{5.55}$$

$$\delta_0 = \frac{\hat{\sigma}^2 - \sigma^2}{\tau\sigma^2} \quad \text{where} \quad \hat{\sigma}^2 = (y - X\hat{\beta})'\hat{\Omega}(y - X\hat{\beta})/(T-n) \quad \text{and} \quad \sigma^2 = 1. \tag{5.56}$$

Next, using all these as inputs, we conduct the internal experiment exactly as described in the previous chapter, additionally computing the quantities $\lambda_{0\rho}$ and $\lambda_{0\phi}$, which are defined by the following expressions:

$$\lambda_{0\rho} = \mathbb{E}(\delta_0\delta_\rho), \quad \lambda_{0\phi} = \mathbb{E}(\delta_0\delta_\phi). \tag{5.57}$$

This time, however, the internal experiment will be performed using 1000 repetitions instead of 10000, since we have shown in the previous chapter that the difference in the results is negligible.

Once the internal experiment has been completed and all necessary results obtained, we calculate μ_0 with the following formula (see Appendix C):

$$\mu_0 = \frac{1}{2}\frac{\text{tr}(u'\Omega_{\rho\rho}u)}{T}\delta_\rho\delta_\rho + \frac{1}{2}\frac{\text{tr}(u'\Omega_{\phi\phi}u)}{T}\delta_\phi\delta_\phi + \frac{\text{tr}(u'\Omega_{\rho\phi}u)}{T}\delta_\rho\delta_\phi. \tag{5.58}$$

Next, using the values of the statistics and the density functions of the normal, Student-t, χ^2 and F distributions, we calculated the corresponding significance levels (p-values). More specifically, we computed the significance levels of the usual t statistic under the assumption that it follows the Student-t or the normal distribution, and the significance levels of the usual χ^2 and F distributions, respectively. Additionally, we computed the significance levels of the *locally exact* Cornish-Fisher corrected t and F statistics under the assumption that they follow the Student-t and F distributions, respectively. It is worth noticing that the Cornish-Fisher corrected F statistic (3.39) can assume negative values, which indicates overcorrection, a topic that will be analyzed further below.

The procedure described in this chapter was repeated 10000 times for each of the 32 points of the experimental space.

5.2 Results

The results of the experiment are presented in Tables 5.1 through 5.18. The experiment was conducted for all combinations of parameter values $\rho = \pm 0.5$, ± 0.9 and $\phi = \pm 0.5$, ± 0.9.

For a sample of 15 observations, Tables 5.1–5.8 report the null rejection probabilities of the alternative forms of the t-statistic examined in this dissertation for testing null hypotheses of the form $\beta_j = 0$ against one-sided alternatives of the form $\beta_j > 0$ or $\beta_j < 0$, for $j = 1, 2, 3, 4$. Each table presents the null rejection probabilities at significance levels of 1%, 5%, and 10%.

For each significance level, the null rejection probabilities were computed for the following:

- the t-test based on the standard normal distribution (N),
- the Edgeworth correction of the critical values of the normal distribution (NE),
- the corresponding Cornish-Fisher corrected statistic (NCF),
- the t-test based on the Student-t distribution (T),
- the Edgeworth correction of the critical values of the Student-t distribution (TE),
- and the corresponding Cornish-Fisher corrected statistic (TCF).

Table 5.9 presents, also for a sample of 15 observations, the actual size null rejection probabilities of the χ^2 and F statistics examined in this dissertation for testing the null hypothesis $\beta_2 = \beta_3 = \beta_4 = 0$ against the alternative that at least one of them differs from zero. The table includes the null rejection probabilities for significance levels of 1%, 5%, and 10%. Specifically, it reports:

- the results of the Wald statistic based on the χ^2 distribution (X2),
- the Edgeworth correction of the critical values of the χ^2 distribution (X2E),
- the corresponding Cornish-Fisher corrected statistic (X2CF),
- the results of the F statistic based on the F distribution (F),
- the Edgeworth correction of the critical values of the F distribution (FE),
- and the corresponding Cornish-Fisher corrected statistic (FCF).

Corresponding results for a sample of 30 observations are presented in Tables 5.10–5.17 for the t-statistics and in Table 5.18 for the Wald and F statistics.

Table 5.1 Hypothesis test results for $H_0 : \beta_1 = 0$, using a sample size of 15 and showing the Positive t-statistics

POSITIVE t-STATISTICS

Sample size: T=15; H0 : $\boldsymbol{\beta_1}$

Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	1.63	2.84	5.07	0.59	2.64	4.56	8.87	5.68	7.82	6.28	4.98	7.53	16.71	11.42	12.82	14.93	10.70	12.64
	-0.5	1.82	1.78	2.50	0.60	1.59	2.17	9.50	5.77	5.92	6.68	5.08	5.67	17.49	12.86	12.36	15.44	12.16	12.40
	0.5	3.44	2.54	1.84	1.69	1.93	1.73	10.28	9.16	8.59	8.15	8.52	8.22	17.02	16.39	16.20	15.20	15.84	15.78
	0.9	2.94	2.25	1.61	1.48	1.72	1.47	8.55	7.58	7.26	6.76	7.14	7.01	14.55	13.93	13.83	13.04	13.59	13.54
-0.5	-0.9	1.36	1.31	1.79	0.61	1.11	1.66	5.43	4.61	4.81	4.07	4.18	4.73	9.77	9.87	10.39	8.72	9.50	10.04
	-0.5	1.40	1.13	1.03	0.63	0.87	0.90	5.04	4.57	4.52	3.82	4.17	4.21	9.91	9.76	9.89	8.67	9.38	9.47
	0.5	2.13	1.46	1.14	0.97	1.13	1.07	6.65	5.78	5.60	5.25	5.42	5.39	11.73	11.23	11.20	10.47	10.84	10.79
	0.9	2.16	1.54	0.83	1.14	1.12	0.95	6.60	5.79	5.44	5.43	5.35	5.21	11.44	10.79	10.61	10.16	10.38	10.30
0.5	-0.9	0.43	0.37	0.29	0.16	0.31	0.25	2.21	1.98	1.90	1.58	1.77	1.74	5.51	5.24	5.21	4.56	4.95	4.87
	-0.5	2.38	1.59	1.22	1.18	1.31	1.23	6.58	5.91	5.73	5.33	5.54	5.50	11.43	10.80	10.78	10.07	10.48	10.47
	0.5	9.81	9.18	5.37	7.02	7.92	6.07	17.38	16.50	14.28	15.42	15.99	14.84	22.93	22.48	21.26	21.78	22.19	21.43
	0.9	10.22	12.72	9.18	7.32	11.76	9.15	17.62	19.20	16.73	15.81	18.66	16.95	22.85	23.76	22.10	21.56	23.45	22.39
0.9	-0.9	15.94	13.80	9.43	12.37	11.97	10.91	23.77	22.63	20.87	21.80	21.98	21.17	28.69	28.12	27.04	27.41	27.71	27.09
	-0.5	23.31	21.21	13.85	20.06	19.77	15.86	29.68	28.50	24.99	28.15	27.99	24.77	33.47	32.93	30.71	32.59	32.69	29.82
	0.5	29.66	32.67	28.56	26.90	31.54	27.14	34.78	36.00	33.31	33.64	35.60	32.16	37.49	38.15	36.34	36.85	37.97	35.02
	0.9	30.57	35.93	33.23	27.68	35.13	31.88	35.79	38.25	36.43	34.62	37.83	35.49	38.58	39.84	38.69	37.89	39.69	37.69

Table 5.2 Hypothesis test results for $H_0 : \beta_1 = 0$, using a sample size of 15 and showing the Negative t-statistics

		NEGATIVE t-STATISTICS																	
Sample size: T=15																			
H0 :		$\boldsymbol{\beta_1}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	1.60	2.85	5.13	0.62	2.56	4.67	8.13	5.33	7.76	5.78	4.77	7.31	16.10	10.62	12.59	14.12	9.96	12.45
	-0.5	1.68	2.04	2.62	0.64	1.90	2.29	9.41	5.75	5.94	6.68	5.08	5.82	17.60	13.05	12.56	15.72	12.22	12.54
	0.5	3.29	2.38	1.67	1.60	1.81	1.60	10.13	8.88	8.33	8.01	8.25	8.02	16.59	15.82	15.50	14.75	15.23	15.07
	0.9	2.82	2.18	1.71	1.46	1.74	1.55	8.61	7.70	7.36	6.91	7.31	7.19	13.76	13.24	13.16	12.30	12.81	12.78
-0.5	-0.9	1.38	1.13	1.71	0.62	0.89	1.54	5.20	4.64	4.94	3.97	4.17	4.75	9.74	9.73	10.14	8.44	9.27	9.76
	-0.5	1.28	0.99	1.06	0.48	0.77	0.98	4.80	4.38	4.48	3.59	4.11	4.34	9.35	9.45	9.56	8.14	9.02	9.18
	0.5	2.06	1.46	1.10	0.96	1.15	1.11	6.39	5.57	5.37	5.03	5.21	5.14	11.41	10.73	10.72	10.07	10.32	10.31
	0.9	1.99	1.29	0.78	0.96	0.99	0.83	6.28	5.33	5.05	4.91	5.04	4.96	11.27	10.68	10.56	9.92	10.35	10.30
0.5	-0.9	0.53	0.40	0.39	0.15	0.30	0.30	2.56	2.22	2.21	1.72	2.08	2.06	5.83	5.57	5.59	4.88	5.32	5.33
	-0.5	2.64	1.83	1.48	1.29	1.55	1.47	6.98	6.21	6.04	5.57	5.78	5.74	12.19	11.69	11.66	11.02	11.47	11.41
	0.5	10.73	9.92	5.69	7.67	8.40	6.49	18.70	17.89	15.03	16.76	17.28	15.74	23.87	23.25	21.72	22.43	22.88	21.96
	0.9	10.45	13.13	9.23	7.57	11.97	9.17	18.20	19.50	16.95	16.21	18.97	17.19	23.44	24.07	22.48	22.17	23.77	22.54
0.9	-0.9	18.41	15.60	10.72	14.09	13.79	12.43	26.41	25.20	23.37	24.45	24.75	23.85	31.39	30.90	29.72	30.23	30.56	29.89
	-0.5	26.30	24.17	14.94	23.03	22.30	17.53	33.21	32.07	27.07	31.44	31.47	27.43	36.91	36.54	33.14	36.09	36.21	32.47
	0.5	30.94	34.17	29.41	28.18	33.20	27.78	36.42	37.73	34.24	35.30	37.31	32.94	39.41	39.92	37.15	38.71	39.77	35.72
	0.9	30.88	36.08	33.23	28.14	35.28	31.79	36.14	38.55	36.57	35.18	38.21	35.45	38.71	40.24	38.72	38.12	40.05	37.63

Table 5.3 Hypothesis test results for $H_0 : \beta_2 = 0$, using a sample size of 15 and showing the Positive t-statistics

		POSITIVE t-STATISTICS																	
Sample size: T=15																			
H0 :		β_2																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	6.26	5.99	6.78	4.29	4.46	6.43	14.18	9.84	9.96	11.63	8.39	9.90	21.47	14.39	13.75	19.56	13.35	13.82
	-0.5	6.89	4.73	4.48	4.40	3.24	4.34	15.46	10.00	8.67	13.01	8.58	8.83	22.92	16.06	13.52	21.18	15.08	14.13
	0.5	12.69	8.97	5.35	8.77	6.67	6.16	23.10	19.81	17.32	20.47	18.70	17.69	29.35	27.41	26.03	28.10	27.03	26.18
	0.9	14.12	10.59	6.76	10.02	7.79	7.34	23.41	20.62	18.15	21.03	19.74	18.40	28.80	27.13	25.52	27.58	26.79	25.67
-0.5	-0.9	1.53	2.34	1.89	0.73	2.07	1.82	5.64	5.29	4.71	4.18	4.87	4.70	10.16	9.70	9.29	8.86	9.37	9.11
	-0.5	1.90	1.74	1.19	0.93	1.31	1.12	6.59	5.33	4.85	4.95	4.95	4.80	12.24	10.66	10.33	10.73	10.28	10.16
	0.5	2.73	1.60	1.13	1.21	1.17	1.15	7.41	5.76	5.54	5.71	5.40	5.35	12.62	11.05	10.84	11.19	10.68	10.59
	0.9	1.69	0.95	0.62	0.74	0.66	0.62	5.84	4.39	4.06	4.62	4.08	3.95	10.70	8.80	8.57	9.28	8.57	8.45
0.5	-0.9	1.78	1.22	0.90	0.93	0.95	0.92	6.01	4.83	4.54	4.71	4.50	4.49	10.97	9.39	9.24	9.59	9.12	9.05
	-0.5	2.15	1.38	0.94	1.00	1.04	0.98	6.62	5.39	5.18	5.25	5.05	5.04	11.50	9.99	9.84	10.03	9.63	9.57
	0.5	1.17	0.67	0.51	0.42	0.43	0.49	4.88	3.42	2.84	3.55	2.95	2.73	9.29	7.67	7.15	8.09	7.38	7.14
	0.9	0.66	0.36	0.76	0.26	0.20	0.74	3.37	2.44	2.43	2.33	2.10	2.31	7.37	6.13	5.99	6.27	5.74	5.85
0.9	-0.9	2.45	1.44	0.92	1.08	1.00	0.95	7.48	5.73	5.46	5.70	5.30	5.28	12.41	10.76	10.62	10.99	10.54	10.48
	-0.5	2.80	1.75	1.16	1.35	1.26	1.22	8.11	6.41	5.98	6.53	5.95	5.89	13.43	11.67	11.49	12.07	11.34	11.30
	0.5	1.73	1.09	0.83	0.78	0.76	0.86	6.24	4.38	3.66	4.84	3.90	3.67	11.89	8.67	7.72	10.40	8.21	7.81
	0.9	1.49	1.39	1.78	0.74	1.14	1.67	5.23	4.07	3.81	4.02	3.57	3.72	9.91	7.50	7.07	8.60	7.03	6.94

Table 5.4 Hypothesis test results for $H_0 : \beta_2 = 0$, using a sample size of 15 and showing the Negative t-statistics

Sample size: T=15		NEGATIVE t-STATISTICS																	
H0 :		β_2																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	6.23	5.85	7.27	4.17	4.37	7.00	13.35	9.42	10.25	11.29	8.29	10.26	20.49	14.02	14.01	18.94	12.89	14.03
	-0.5	7.20	5.31	4.80	4.42	3.71	4.63	16.36	10.91	9.53	13.68	9.60	9.67	23.63	16.64	14.18	21.75	15.60	14.65
	0.5	11.92	8.28	5.19	7.88	6.03	5.72	21.36	18.08	15.96	19.04	17.20	16.27	27.18	25.10	23.86	25.82	24.68	24.07
	0.9	13.03	9.43	6.52	9.20	6.97	7.10	21.81	19.19	17.36	19.65	18.18	17.43	27.28	25.72	24.70	26.13	25.33	24.64
-0.5	-0.9	1.58	2.36	1.82	0.55	2.11	1.74	5.54	5.72	5.14	4.27	5.35	5.05	10.36	10.06	9.61	9.31	9.63	9.48
	-0.5	1.94	1.65	1.14	0.84	1.29	1.10	6.58	5.44	5.02	5.02	5.02	4.93	11.52	10.16	9.89	10.22	9.79	9.68
	0.5	2.36	1.63	1.27	1.34	1.29	1.27	6.89	5.43	5.16	5.44	5.08	5.02	12.01	10.31	10.19	10.63	9.99	9.92
	0.9	1.69	0.99	0.60	0.70	0.64	0.61	5.38	3.99	3.57	4.27	3.61	3.49	9.71	8.06	7.72	8.55	7.77	7.57
0.5	-0.9	2.15	1.27	0.88	1.04	0.99	0.94	6.48	5.16	4.96	5.14	4.81	4.78	11.32	9.63	9.45	9.87	9.30	9.27
	-0.5	2.19	1.41	1.04	1.17	1.10	1.08	6.81	5.31	5.11	5.38	4.99	4.96	11.87	10.22	9.98	10.45	9.80	9.74
	0.5	1.04	0.60	0.52	0.41	0.44	0.48	4.91	3.37	2.92	3.51	3.00	2.79	9.89	7.83	7.21	8.55	7.36	7.09
	0.9	0.60	0.39	0.86	0.20	0.23	0.84	3.06	2.24	2.19	2.21	1.93	2.11	7.21	5.83	5.64	5.96	5.45	5.49
0.9	-0.9	2.47	1.53	1.07	1.18	1.14	1.09	7.31	5.95	5.55	5.96	5.49	5.43	12.98	11.18	11.03	11.41	10.86	10.77
	-0.5	2.50	1.61	1.15	1.31	1.23	1.18	8.35	6.40	5.90	6.70	6.03	5.90	14.79	12.83	12.57	13.25	12.41	12.31
	0.5	1.81	1.10	0.71	0.79	0.69	0.71	6.49	4.35	3.47	4.93	3.94	3.47	11.85	8.68	7.53	10.46	8.23	7.55
	0.9	1.65	1.59	2.08	0.85	1.26	2.02	5.15	4.08	4.05	4.05	3.62	3.93	9.57	7.51	7.18	8.40	7.08	7.07

Table 5.5 Hypothesis test results for $H_0 : \beta_3 = 0$, using a sample size of 15 and showing the Positive t-statistics

		POSITIVE t-STATISTICS																	
Sample size: T=15																			
H0 :		$\boldsymbol{\beta_3}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	1.50	4.09	5.54	0.73	3.93	5.15	6.19	6.31	8.03	4.79	5.91	7.81	11.94	9.48	11.14	10.36	9.04	11.05
	-0.5	1.56	2.01	2.86	0.53	1.77	2.64	6.73	4.76	5.60	4.95	4.31	5.43	13.05	8.97	9.40	11.41	8.52	9.31
	0.5	3.06	2.02	1.45	1.46	1.56	1.42	9.09	7.41	6.85	7.31	6.86	6.63	15.18	13.34	12.98	13.79	12.98	12.76
	0.9	1.71	1.39	1.09	0.76	1.12	0.97	5.64	4.95	4.79	4.42	4.60	4.58	10.16	8.97	8.82	8.78	8.75	8.69
-0.5	-0.9	0.94	2.36	2.04	0.45	2.15	1.84	4.05	4.33	3.98	2.83	4.08	3.89	8.47	7.72	7.09	7.20	7.32	7.09
	-0.5	1.30	1.32	0.96	0.62	1.11	0.97	5.11	4.33	3.84	3.84	3.99	3.81	10.00	8.44	8.08	8.81	8.16	7.97
	0.5	1.92	1.59	1.23	0.90	1.28	1.16	6.26	5.52	5.35	5.07	5.16	5.14	11.16	9.97	9.90	9.75	9.74	9.73
	0.9	1.43	1.27	1.04	0.65	1.00	0.92	4.60	3.75	3.42	3.62	3.42	3.38	9.11	7.34	7.16	7.89	7.08	7.03
0.5	-0.9	1.90	1.74	1.31	0.80	1.47	1.11	6.35	5.75	5.52	4.75	5.45	5.21	10.64	10.57	10.42	9.45	10.21	10.08
	-0.5	1.81	1.51	1.24	0.96	1.28	1.12	6.20	5.43	5.24	4.89	5.14	5.04	11.00	9.84	9.85	9.41	9.59	9.65
	0.5	0.88	0.52	0.59	0.38	0.40	0.57	4.17	2.82	2.76	2.93	2.59	2.72	8.36	6.51	6.33	7.28	6.22	6.28
	0.9	0.61	0.31	1.00	0.17	0.18	0.90	3.53	2.01	2.43	2.56	1.73	2.49	7.34	5.01	5.41	6.16	4.70	5.34
0.9	-0.9	1.90	1.58	1.13	0.99	1.23	1.13	6.61	5.65	5.41	5.25	5.30	5.13	11.83	10.57	10.42	10.44	10.25	10.23
	-0.5	1.90	1.30	0.91	0.81	0.97	0.81	6.11	5.09	4.84	4.91	4.70	4.61	11.69	10.11	9.96	10.28	9.74	9.73
	0.5	0.38	0.23	0.51	0.08	0.20	0.46	2.75	1.56	1.78	1.88	1.39	1.68	7.13	4.59	4.44	5.95	4.23	4.42
	0.9	0.38	0.19	1.42	0.10	0.15	1.32	2.28	1.05	2.18	1.58	0.90	2.14	6.04	3.43	4.41	4.80	3.18	4.34

Table 5.6 Hypothesis test results for $H_0 : \beta_3 = 0$, using a sample size of 15 and showing the Negative t-statistics

		NEGATIVE t-STATISTICS																	
Sample size: T=15																			
H0 :		$\boldsymbol{\beta_3}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	1.88	3.84	5.31	1.04	3.61	4.95	6.18	6.29	7.87	4.81	5.89	7.69	11.42	9.09	10.40	10.01	8.50	10.30
	-0.5	1.88	2.35	2.79	0.84	1.94	2.70	6.86	5.22	5.67	5.46	4.76	5.55	12.91	9.41	9.29	11.31	8.75	9.22
	0.5	3.16	2.11	1.54	1.54	1.63	1.58	9.24	7.44	6.74	7.29	6.81	6.55	15.18	13.51	12.92	13.65	13.08	12.82
	0.9	1.64	1.60	1.14	0.82	1.21	1.09	5.48	4.69	4.52	4.17	4.38	4.37	9.87	9.14	9.00	8.87	8.90	8.86
-0.5	-0.9	1.21	2.57	2.28	0.60	2.47	2.15	4.45	4.78	4.49	3.28	4.56	4.39	8.73	8.32	7.86	7.59	7.96	7.75
	-0.5	1.49	1.49	1.20	0.77	1.23	1.14	5.42	4.31	3.92	3.94	3.91	3.78	10.60	8.75	8.42	9.08	8.43	8.27
	0.5	2.03	1.73	1.33	1.06	1.38	1.26	6.14	5.53	5.24	4.91	5.12	5.06	10.87	9.91	9.78	9.62	9.59	9.57
	0.9	1.23	0.95	0.71	0.52	0.74	0.62	5.00	3.69	3.33	3.70	3.42	3.34	9.24	7.47	7.25	8.09	7.27	7.19
0.5	-0.9	1.84	1.83	1.35	0.94	1.56	1.28	6.37	5.98	5.79	4.98	5.57	5.45	11.22	10.93	10.84	9.86	10.65	10.58
	-0.5	2.02	1.69	1.25	0.99	1.33	1.19	6.29	5.56	5.41	5.06	5.21	5.20	11.09	10.04	9.85	9.82	9.76	9.68
	0.5	1.00	0.50	0.38	0.35	0.31	0.36	3.96	2.70	2.50	2.98	2.54	2.47	8.26	6.22	5.99	7.09	5.96	5.92
	0.9	0.73	0.31	0.97	0.25	0.19	0.89	3.27	1.75	2.31	2.30	1.44	2.26	7.21	4.87	5.16	6.06	4.67	5.21
0.9	-0.9	2.14	1.74	1.22	1.06	1.28	1.11	6.51	5.73	5.46	5.23	5.31	5.20	11.34	10.39	10.13	9.92	9.96	9.80
	-0.5	1.73	1.20	0.86	0.77	0.84	0.83	5.94	4.77	4.47	4.66	4.38	4.31	11.32	9.75	9.50	9.81	9.39	9.26
	0.5	0.38	0.13	0.34	0.08	0.08	0.35	2.83	1.37	1.43	1.86	1.19	1.42	6.53	4.23	4.15	5.34	4.00	4.08
	0.9	0.34	0.25	1.59	0.13	0.19	1.48	2.13	1.19	2.50	1.55	1.01	2.44	5.74	3.28	4.51	4.87	3.05	4.41

Table 5.7 Hypothesis test results for $H_0 : \beta_4 = 0$, using a sample size of 15 and showing the Positive t-statistics

		POSITIVE t-STATISTICS																	
Sample size: T=15																			
H0 :		$\boldsymbol{\beta_4}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.18	2.36	3.22	0.07	2.31	2.95	1.35	3.08	4.27	0.83	2.95	4.05	4.29	4.65	6.09	3.30	4.41	5.84
	-0.5	0.19	1.03	1.31	0.08	1.02	1.26	1.26	1.66	2.14	0.82	1.58	2.04	3.77	3.31	3.83	2.91	3.06	3.63
	0.5	0.60	0.46	0.44	0.33	0.39	0.43	2.88	2.17	2.12	1.98	1.99	2.04	6.13	5.40	5.37	5.23	5.08	5.11
	0.9	1.30	0.99	0.71	0.72	0.69	0.71	4.81	3.91	3.71	3.65	3.57	3.58	8.92	7.85	7.86	7.81	7.57	7.58
-0.5	-0.9	0.81	1.85	1.50	0.32	1.65	1.36	3.72	4.20	3.94	2.61	3.86	3.71	7.57	7.79	7.64	6.44	7.53	7.50
	-0.5	0.83	1.05	0.83	0.38	0.91	0.79	3.55	3.42	3.31	2.57	3.14	3.05	7.34	6.83	6.78	6.21	6.63	6.60
	0.5	1.82	1.27	0.97	0.95	1.02	0.97	5.89	4.83	4.71	4.52	4.58	4.57	10.51	9.52	9.39	9.25	9.12	9.08
	0.9	2.17	1.39	0.95	1.10	1.07	1.02	6.50	5.37	5.15	5.05	4.98	4.94	11.13	9.92	9.77	9.76	9.66	9.60
0.5	-0.9	1.67	1.25	0.92	0.88	0.96	0.90	5.45	4.36	4.21	4.17	4.10	4.08	10.29	9.13	9.07	9.05	8.83	8.78
	-0.5	2.06	1.37	0.98	0.98	1.03	0.98	6.14	5.12	4.93	4.76	4.76	4.72	10.86	9.80	9.71	9.55	9.35	9.33
	0.5	1.49	0.90	0.66	0.72	0.62	0.67	5.73	4.01	3.89	4.21	3.74	3.84	10.33	8.57	8.53	9.05	8.23	8.37
	0.9	1.22	0.88	1.22	0.63	0.67	1.16	4.88	3.46	3.99	3.61	3.17	3.86	9.81	7.53	7.95	8.49	7.06	7.78
0.9	-0.9	1.79	1.21	0.86	0.87	0.92	0.87	5.68	4.55	4.42	4.31	4.27	4.26	10.42	9.45	9.31	9.19	9.14	9.08
	-0.5	1.77	1.17	0.81	0.86	0.87	0.83	6.01	4.82	4.59	4.56	4.36	4.36	10.61	9.40	9.32	9.29	9.04	9.02
	0.5	0.98	0.57	0.65	0.48	0.47	0.64	4.36	3.13	3.15	3.23	2.81	3.07	8.25	6.69	6.86	7.20	6.41	6.77
	0.9	0.94	0.82	1.54	0.34	0.53	1.34	3.71	2.98	3.79	2.87	2.75	3.73	7.50	5.94	6.83	6.51	5.58	6.67

Table 5.8 Hypothesis test results for $H_0 : \beta_4 = 0$, using a sample size of 15 and showing the Negative t-statistics

		NEGATIVE t-STATISTICS																	
Sample size: T=15																			
H0 :		$\boldsymbol{\beta_4}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.19	2.24	3.02	0.08	2.14	2.85	1.55	3.01	4.15	1.03	2.91	3.97	4.05	4.45	5.84	3.39	4.27	5.65
	-0.5	0.24	1.12	1.40	0.08	1.03	1.26	1.45	1.89	2.39	0.87	1.77	2.25	3.76	3.44	4.16	2.94	3.32	4.03
	0.5	0.72	0.47	0.34	0.30	0.34	0.31	2.87	2.18	2.11	2.05	1.97	1.97	6.34	5.50	5.38	5.25	5.16	5.11
	0.9	1.54	0.96	0.77	0.73	0.78	0.79	4.83	4.04	3.94	3.74	3.73	3.79	8.90	7.91	7.89	7.81	7.67	7.71
-0.5	-0.9	0.91	1.81	1.45	0.37	1.62	1.27	3.66	4.19	4.00	2.62	3.95	3.80	7.43	7.65	7.50	6.38	7.37	7.29
	-0.5	0.92	1.02	0.76	0.36	0.83	0.70	3.57	3.37	3.09	2.64	3.07	2.95	7.26	6.86	6.70	6.30	6.56	6.50
	0.5	1.80	1.38	1.10	0.98	1.09	1.05	5.90	4.94	4.72	4.57	4.58	4.53	10.57	9.51	9.47	9.38	9.26	9.24
	0.9	2.24	1.59	1.09	1.11	1.09	1.06	6.75	5.61	5.38	5.20	5.25	5.22	11.54	10.43	10.32	10.30	10.16	10.14
0.5	-0.9	1.71	1.19	0.88	0.86	0.85	0.82	5.62	4.52	4.33	4.33	4.27	4.19	9.93	8.85	8.72	8.80	8.54	8.50
	-0.5	2.02	1.42	1.10	1.01	1.10	1.07	5.94	5.11	4.96	4.84	4.75	4.73	10.52	9.47	9.38	9.15	9.15	9.14
	0.5	1.57	0.96	0.79	0.88	0.69	0.81	5.79	4.24	4.15	4.43	3.95	4.02	10.98	8.89	8.84	9.48	8.58	8.67
	0.9	1.42	0.89	1.39	0.68	0.68	1.35	5.29	3.79	4.23	4.02	3.49	4.18	10.19	7.54	8.03	8.80	7.28	7.99
0.9	-0.9	1.85	1.35	1.07	0.99	1.11	1.03	5.81	4.82	4.68	4.55	4.52	4.48	10.37	9.37	9.31	9.14	9.10	9.08
	-0.5	2.08	1.38	0.97	1.03	0.98	0.95	5.90	4.95	4.79	4.64	4.57	4.56	10.58	9.31	9.23	9.22	8.99	8.98
	0.5	1.11	0.55	0.62	0.42	0.36	0.63	4.51	2.97	2.91	3.30	2.70	2.89	8.69	6.74	6.68	7.50	6.53	6.66
	0.9	1.02	0.77	1.41	0.42	0.60	1.37	3.92	2.83	3.62	2.88	2.59	3.52	7.86	5.93	6.57	6.83	5.57	6.46

Table 5.9 Hypothesis test results for $H_0 : \beta_2, \beta_3, \beta_4 = 0$, using a sample size of 15 and showing the F-statistics

		F-STATISTICS																	
Sample size: T=15																			
H0 :		$\boldsymbol{\beta_2, \beta_3, \beta_4}$																	
Nominal size:		1%						5%						10%					
Test:		X2	X2E	X2CF	F	FE	FCF	X2	X2E	X2CF	F	FE	FCF	X2	X2E	X2CF	F	FE	FCF
ρ	ϕ	%						%						%					
-0.9	-0.9	10.97	19.12	5.77	5.50	14.68	4.17	17.17	22.07	8.97	11.72	16.85	7.40	22.29	24.55	11.66	16.82	19.17	10.45
	-0.5	11.27	12.86	2.60	5.33	8.46	2.35	18.90	16.44	4.83	12.23	11.39	5.11	25.42	19.43	7.43	18.33	14.50	7.78
	0.5	19.90	10.83	0.89	7.47	3.93	3.16	33.38	20.90	3.45	21.64	13.52	11.28	42.81	28.79	11.22	32.36	21.78	18.31
	0.9	20.94	11.37	0.92	8.32	4.36	3.57	34.01	21.52	2.73	22.62	13.31	11.47	42.18	29.94	11.31	33.15	22.76	19.16
-0.5	-0.9	3.85	3.76	1.46	0.99	2.55	1.84	8.53	5.62	1.92	4.37	4.26	3.16	12.51	7.65	3.25	8.23	6.12	4.53
	-0.5	4.67	2.80	0.75	1.25	1.49	1.29	10.48	5.74	1.71	5.36	4.28	3.91	15.50	8.71	4.44	9.98	7.18	6.54
	0.5	4.85	3.25	0.61	1.24	1.89	1.26	11.24	7.16	1.94	5.44	5.39	4.79	16.91	11.69	6.93	10.70	9.93	9.20
	0.9	2.53	1.76	0.40	0.51	1.06	0.61	7.66	3.88	1.02	3.00	2.80	2.29	12.70	6.56	3.44	7.32	5.26	4.65
0.5	-0.9	4.34	2.90	0.73	1.10	1.85	1.26	10.70	6.71	1.89	5.00	4.80	4.23	16.06	10.90	5.94	10.25	9.04	8.40
	-0.5	4.28	3.10	0.54	1.12	1.85	1.19	10.17	6.85	1.81	4.99	4.99	4.42	16.21	11.02	6.44	9.89	9.51	8.76
	0.5	1.15	0.52	0.12	0.26	0.32	0.24	3.92	1.32	0.32	1.36	0.87	0.75	6.98	2.61	1.11	3.66	1.89	1.66
	0.9	0.88	0.52	0.06	0.16	0.29	0.11	2.49	1.03	0.18	1.00	0.67	0.37	4.70	1.69	0.59	2.36	1.22	0.81
0.9	-0.9	4.59	3.03	0.59	1.00	1.71	1.13	11.05	6.98	1.75	5.13	4.95	4.50	17.04	11.33	6.59	10.61	9.55	8.93
	-0.5	3.97	2.23	0.55	0.88	1.28	0.90	10.29	5.54	1.87	4.58	4.14	3.84	16.25	9.52	5.77	9.86	7.97	7.50
	0.5	1.38	2.35	0.38	0.28	2.03	0.24	3.76	3.43	1.08	1.62	2.71	0.98	6.65	4.72	2.35	3.49	3.92	2.30
	0.9	1.50	5.64	1.59	0.39	5.07	0.63	3.70	6.74	3.02	1.76	5.94	2.24	5.63	7.82	4.48	3.56	6.91	3.78

Table 5.10 Hypothesis test results for $H_0 : \beta_1 = 0$, using a sample size of 30 and showing the Positive t-statistics

		POSITIVE t-STATISTICS																	
Sample size: T=30																			
H0 :								$\boldsymbol{\beta_1}$											
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.00	1.12	3.55	0.00	1.12	3.36	0.00	1.51	4.90	0.00	1.48	4.73	0.51	2.40	6.33	0.36	2.32	6.18
	-0.5	0.02	0.27	1.75	0.01	0.24	1.64	0.28	0.37	2.37	0.19	0.34	2.26	1.53	1.04	3.27	1.35	1.02	3.15
	0.5	0.50	0.38	0.35	0.36	0.36	0.33	2.93	2.36	2.24	2.50	2.31	2.21	6.24	5.88	5.73	5.84	5.85	5.74
	0.9	1.10	0.90	0.77	0.74	0.83	0.75	4.82	4.55	4.50	4.34	4.50	4.43	9.19	8.82	8.83	8.62	8.73	8.74
-0.5	-0.9	0.00	0.14	0.22	0.00	0.14	0.20	0.01	0.15	0.24	0.01	0.15	0.23	0.34	0.22	0.28	0.23	0.22	0.29
	-0.5	0.01	0.01	0.03	0.01	0.01	0.02	0.43	0.14	0.18	0.31	0.14	0.18	1.86	0.96	0.89	1.65	0.93	0.89
	0.5	1.39	1.05	0.94	1.00	1.00	0.93	5.55	5.11	5.00	4.97	5.02	4.92	10.04	9.77	9.73	9.49	9.70	9.66
	0.9	1.91	1.45	1.12	1.41	1.30	1.12	6.53	6.01	5.86	5.96	5.96	5.82	11.08	10.68	10.65	10.67	10.62	10.58
0.5	-0.9	0.00	0.02	0.03	0.00	0.02	0.03	0.12	0.14	0.14	0.09	0.14	0.14	0.71	0.57	0.57	0.62	0.57	0.56
	-0.5	1.52	1.25	1.03	1.13	1.18	1.06	5.68	5.31	5.25	5.20	5.24	5.20	10.31	10.02	10.01	9.71	9.97	9.95
	0.5	9.79	10.10	7.88	8.67	9.80	8.11	17.25	17.05	15.98	16.39	16.93	16.25	23.17	22.78	22.27	22.71	22.74	22.40
	0.9	9.06	11.60	9.47	8.01	11.33	9.32	16.51	17.18	15.62	15.64	16.98	15.73	22.32	22.00	21.12	21.83	21.89	21.22
0.9	-0.9	12.97	10.91	9.04	11.52	10.33	9.41	20.80	19.84	18.70	20.09	19.67	19.00	25.92	25.45	24.93	25.50	25.37	25.04
	-0.5	25.32	24.27	19.27	24.24	23.83	20.57	31.57	30.62	27.95	30.97	30.46	28.26	35.23	34.85	32.95	34.92	34.72	32.74
	0.5	30.52	36.57	34.07	29.21	36.32	33.54	35.73	38.94	37.58	35.31	38.79	37.25	38.17	40.44	39.52	37.95	40.34	39.14
	0.9	29.40	37.48	36.57	28.26	37.13	36.27	34.83	39.20	38.94	34.37	38.97	38.82	37.87	40.37	40.59	37.49	40.20	40.43

Table 5.11 Hypothesis test results for $H_0 : \beta_1 = 0$, using a sample size of 30 and showing the Negative t-statistics

		NEGATIVE t-STATISTICS																	
Sample size: T=30																			
H0 :		β_1																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.00	0.94	3.64	0.00	0.92	3.41	0.01	1.27	4.89	0.01	1.22	4.67	0.44	1.95	6.26	0.31	1.92	6.13
	-0.5	0.03	0.25	2.00	0.00	0.23	1.92	0.15	0.40	2.54	0.11	0.39	2.46	1.49	0.90	3.38	1.32	0.89	3.31
	0.5	0.39	0.26	0.22	0.25	0.24	0.21	2.68	2.06	1.92	2.26	2.01	1.90	6.16	5.81	5.68	5.75	5.75	5.63
	0.9	1.06	0.79	0.64	0.73	0.71	0.67	4.42	4.11	4.02	3.90	4.05	3.95	9.14	8.85	8.80	8.77	8.75	8.74
-0.5	-0.9	0.00	0.10	0.11	0.00	0.10	0.11	0.03	0.11	0.19	0.03	0.11	0.16	0.38	0.22	0.31	0.30	0.21	0.32
	-0.5	0.01	0.00	0.02	0.00	0.00	0.02	0.42	0.08	0.12	0.32	0.08	0.11	1.84	0.93	0.89	1.58	0.92	0.91
	0.5	1.28	0.87	0.76	0.89	0.77	0.74	5.19	4.71	4.66	4.65	4.62	4.63	10.22	9.84	9.83	9.71	9.75	9.77
	0.9	1.88	1.43	1.04	1.37	1.25	1.11	6.19	5.65	5.49	5.60	5.55	5.46	11.59	11.04	10.96	10.99	10.93	10.88
0.5	-0.9	0.00	0.01	0.01	0.00	0.01	0.01	0.18	0.13	0.14	0.11	0.13	0.14	0.90	0.79	0.78	0.80	0.76	0.75
	-0.5	1.52	1.21	0.98	1.12	1.10	0.98	5.74	5.20	5.10	5.03	5.10	5.05	10.95	10.51	10.48	10.26	10.44	10.42
	0.5	10.16	10.53	8.26	8.76	10.25	8.61	18.52	18.26	17.17	17.67	18.15	17.40	24.44	23.96	23.40	23.86	23.90	23.52
	0.9	9.52	12.36	10.01	8.16	12.14	10.01	17.70	18.25	16.67	16.76	18.11	16.86	23.58	23.41	22.19	23.04	23.36	22.33
0.9	-0.9	14.64	12.30	9.31	13.07	11.48	9.81	23.32	22.13	20.13	22.46	21.96	20.45	28.81	28.33	26.76	28.33	28.22	26.85
	-0.5	28.07	26.53	20.66	26.70	26.02	22.30	34.53	33.59	30.10	33.94	33.44	30.73	38.31	37.86	35.38	37.86	37.72	35.22
	0.5	31.02	37.93	35.60	30.01	37.76	35.23	36.48	40.27	38.78	36.06	40.14	38.52	39.59	41.94	40.99	39.31	41.92	40.77
	0.9	30.05	38.94	37.36	28.87	38.64	36.99	35.52	40.65	40.00	35.07	40.50	39.89	38.79	42.09	42.10	38.50	42.03	41.98

Table 5.12 Hypothesis test results for $H_0 : \beta_2 = 0$, using a sample size of 30 and showing the Positive t-statistics

		POSITIVE t-STATISTICS																	
Sample size: T=30																			
H0 :		$\boldsymbol{\beta_2}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.71	1.15	4.71	0.52	1.13	4.52	2.76	1.81	6.32	2.42	1.73	6.25	5.58	2.99	7.77	5.23	2.90	7.69
	-0.5	1.01	0.85	2.18	0.79	0.81	2.08	3.54	2.02	3.65	3.11	1.93	3.59	6.88	3.60	5.22	6.41	3.49	5.23
	0.5	2.43	1.71	1.41	1.72	1.55	1.40	8.58	6.87	6.34	7.87	6.74	6.40	14.90	12.92	12.59	14.17	12.82	12.59
	0.9	3.38	2.28	2.03	2.47	2.04	2.03	10.50	8.50	7.89	9.60	8.32	7.96	17.40	15.23	14.68	16.69	15.07	14.67
-0.5	-0.9	0.11	0.17	0.15	0.04	0.16	0.15	1.44	0.55	0.44	1.17	0.52	0.44	4.06	2.21	1.73	3.65	2.14	1.84
	-0.5	0.30	0.18	0.15	0.20	0.14	0.14	2.37	1.46	1.20	2.08	1.39	1.27	5.98	4.59	4.32	5.53	4.51	4.31
	0.5	1.42	1.48	1.33	1.09	1.40	1.31	5.18	4.92	4.89	4.61	4.83	4.82	10.31	9.90	9.81	9.84	9.80	9.78
	0.9	1.22	0.89	0.67	0.85	0.82	0.70	4.93	3.78	3.54	4.43	3.63	3.59	9.97	8.09	7.92	9.40	8.01	7.91
0.5	-0.9	1.28	1.51	1.18	0.98	1.43	1.14	5.23	5.32	5.10	4.50	5.24	5.05	10.11	10.47	10.36	9.67	10.38	10.29
	-0.5	1.46	1.53	1.35	1.16	1.50	1.31	5.41	5.08	4.97	4.86	4.96	4.94	10.26	9.81	9.78	9.72	9.77	9.76
	0.5	0.85	0.48	0.31	0.58	0.39	0.34	4.42	2.99	2.63	3.95	2.88	2.71	8.87	6.99	6.65	8.41	6.90	6.68
	0.9	0.62	0.28	0.18	0.39	0.22	0.18	3.32	2.07	1.72	2.86	1.99	1.72	7.27	5.32	4.65	6.77	5.20	4.73
0.9	-0.9	2.19	1.89	1.89	1.65	1.80	1.85	7.27	6.58	6.50	6.63	6.50	6.51	13.04	12.24	12.15	12.47	12.13	12.13
	-0.5	2.46	1.87	1.62	1.76	1.73	1.63	7.56	6.74	6.48	6.90	6.64	6.48	13.35	12.37	12.24	12.68	12.26	12.22
	0.5	1.03	0.54	0.54	0.70	0.46	0.56	4.92	2.50	2.06	4.37	2.35	2.13	9.89	5.74	4.88	9.24	5.57	4.98
	0.9	0.96	1.04	0.94	0.69	0.97	0.91	3.93	2.29	2.10	3.47	2.15	2.15	8.22	4.98	4.16	7.92	4.85	4.21

Table 5.13 Hypothesis test results for $H_0 : \beta_2 = 0$, using a sample size of 30 and showing the Negative t-statistics

		NEGATIVE t-STATISTICS																	
Sample size: T=30																			
H0 :		$\boldsymbol{\beta_2}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.79	1.32	4.61	0.60	1.30	4.38	2.68	2.10	6.33	2.36	1.97	6.16	5.44	3.32	7.64	5.14	3.22	7.58
	-0.5	0.85	0.63	2.10	0.65	0.62	2.04	3.24	1.81	3.41	2.93	1.73	3.32	7.06	3.72	5.23	6.60	3.69	5.22
	0.5	2.02	1.45	1.43	1.54	1.28	1.38	7.39	6.02	5.73	6.64	5.92	5.80	13.13	11.68	11.49	12.55	11.63	11.48
	0.9	2.77	2.05	1.72	2.12	1.88	1.69	8.86	7.22	6.83	8.08	7.05	6.84	15.15	13.57	13.12	14.73	13.46	13.13
-0.5	-0.9	0.12	0.19	0.23	0.03	0.19	0.23	1.36	0.67	0.58	1.19	0.66	0.56	3.81	2.19	1.88	3.44	2.16	1.96
	-0.5	0.34	0.18	0.11	0.25	0.16	0.14	2.61	1.58	1.39	2.14	1.55	1.43	5.98	4.44	4.14	5.48	4.41	4.20
	0.5	1.17	1.18	1.08	0.91	1.15	1.08	5.05	4.69	4.58	4.46	4.61	4.52	9.89	9.07	9.03	9.27	8.98	9.02
	0.9	1.07	0.71	0.60	0.70	0.66	0.60	4.74	3.56	3.36	4.16	3.49	3.40	9.20	7.53	7.30	8.72	7.46	7.30
0.5	-0.9	1.34	1.47	1.29	0.97	1.43	1.22	5.69	5.77	5.59	5.08	5.65	5.54	10.63	10.68	10.58	10.06	10.59	10.55
	-0.5	1.42	1.45	1.38	1.03	1.44	1.33	5.79	5.42	5.30	5.11	5.31	5.25	10.70	10.22	10.17	10.16	10.13	10.13
	0.5	0.87	0.40	0.31	0.50	0.35	0.32	4.56	3.09	2.82	3.91	3.00	2.87	9.27	7.44	6.97	8.75	7.36	7.03
	0.9	0.51	0.36	0.25	0.32	0.30	0.26	3.42	2.10	1.75	2.98	2.09	1.82	7.81	5.53	4.91	7.20	5.39	5.02
0.9	-0.9	2.30	1.78	1.59	1.54	1.68	1.56	8.07	7.16	6.78	7.37	7.06	6.79	13.95	12.90	12.64	13.33	12.80	12.63
	-0.5	2.56	1.84	1.52	1.89	1.61	1.55	8.46	7.34	7.02	7.63	7.23	7.01	14.53	13.47	13.17	13.84	13.33	13.11
	0.5	1.24	0.55	0.48	0.86	0.42	0.49	5.13	2.69	2.17	4.60	2.61	2.31	9.92	5.91	5.05	9.49	5.81	5.21
	0.9	0.92	1.04	1.01	0.63	0.95	1.00	3.99	2.20	2.19	3.51	2.05	2.14	8.17	4.87	4.04	7.52	4.64	4.11

Table 5.14 Hypothesis test results for $H_0 : \beta_3 = 0$, using a sample size of 30 and showing the Positive t-statistics

		POSITIVE t-STATISTICS																	
Sample size: T=30																			
H0 :		$\boldsymbol{\beta_3}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.20	0.67	3.90	0.12	0.67	3.74	1.81	1.25	5.49	1.50	1.21	5.33	4.68	2.36	7.25	4.33	2.33	7.13
	-0.5	0.35	0.39	1.83	0.24	0.35	1.70	2.97	1.38	3.16	2.36	1.35	3.09	7.32	3.85	5.54	6.88	3.77	5.54
	0.5	1.77	1.28	1.22	1.24	1.18	1.20	7.05	5.95	5.86	6.34	5.84	5.82	12.91	11.68	11.61	12.25	11.57	11.52
	0.9	0.90	0.73	0.67	0.69	0.68	0.65	4.35	3.71	3.60	3.75	3.61	3.56	8.61	8.01	7.89	8.15	7.91	7.85
-0.5	-0.9	0.53	0.35	0.33	0.32	0.33	0.36	3.09	1.44	1.28	2.71	1.37	1.32	7.27	3.68	3.09	6.67	3.65	3.15
	-0.5	0.90	0.51	0.37	0.59	0.43	0.40	5.05	3.54	3.06	4.51	3.42	3.16	9.39	7.32	6.93	8.86	7.28	7.04
	0.5	1.26	1.00	0.91	0.91	0.92	0.91	5.33	4.79	4.72	4.77	4.67	4.66	9.96	9.40	9.39	9.47	9.36	9.34
	0.9	1.10	0.80	0.75	0.62	0.72	0.70	4.55	4.09	3.97	4.07	3.99	3.96	8.99	8.70	8.67	8.52	8.63	8.60
0.5	-0.9	1.57	1.10	0.94	1.11	1.01	0.99	5.60	4.97	4.85	5.13	4.86	4.83	10.56	9.47	9.39	9.92	9.41	9.37
	-0.5	1.41	1.05	0.96	0.94	0.96	0.92	5.49	4.94	4.89	4.92	4.87	4.86	10.30	9.75	9.72	9.79	9.69	9.67
	0.5	0.35	0.19	0.16	0.17	0.18	0.16	2.90	2.20	2.08	2.44	2.16	2.07	6.64	5.73	5.67	6.17	5.70	5.67
	0.9	0.29	0.21	0.18	0.17	0.21	0.21	2.37	1.34	1.18	1.79	1.24	1.16	5.99	4.62	4.25	5.50	4.55	4.26
0.9	-0.9	1.64	1.24	1.15	1.18	1.17	1.14	6.24	5.53	5.48	5.63	5.47	5.46	11.58	10.58	10.48	10.81	10.53	10.45
	-0.5	1.03	0.71	0.68	0.66	0.66	0.66	4.99	4.30	4.19	4.45	4.23	4.18	10.16	9.38	9.31	9.57	9.27	9.23
	0.5	0.16	0.13	0.10	0.08	0.12	0.10	1.85	1.05	0.83	1.57	0.95	0.86	5.09	3.37	3.08	4.71	3.28	3.10
	0.9	0.10	0.94	0.67	0.09	0.91	0.63	1.38	1.52	1.43	1.11	1.48	1.42	4.57	3.49	3.20	4.15	3.43	3.22

Table 5.15 Hypothesis test results for $H_0 : \beta_3 = 0$, using a sample size of 30 and showing the Negative t-statistics

		NEGATIVE t-STATISTICS																	
Sample size: T=30																			
H0 :		$\boldsymbol{\beta_3}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.34	0.79	4.21	0.27	0.77	3.93	1.93	1.53	6.16	1.62	1.47	6.01	4.64	2.54	7.83	4.36	2.45	7.76
	-0.5	0.38	0.41	1.89	0.23	0.40	1.84	2.69	1.58	3.44	2.30	1.52	3.41	6.58	3.46	5.57	6.16	3.42	5.57
	0.5	1.45	1.04	0.98	1.03	0.98	0.98	6.64	5.53	5.40	5.97	5.44	5.37	12.55	11.31	11.19	11.99	11.22	11.16
	0.9	0.91	0.69	0.62	0.63	0.65	0.62	4.21	3.58	3.51	3.65	3.50	3.48	8.56	7.96	7.89	8.14	7.91	7.87
-0.5	-0.9	0.58	0.38	0.35	0.39	0.36	0.34	3.05	1.58	1.23	2.81	1.52	1.29	7.08	3.75	3.13	6.65	3.68	3.22
	-0.5	1.02	0.51	0.36	0.67	0.40	0.36	4.52	3.12	2.92	4.00	3.06	2.98	9.20	7.05	6.68	8.73	6.98	6.75
	0.5	1.21	0.87	0.80	0.81	0.79	0.79	5.20	4.59	4.53	4.60	4.53	4.52	9.89	9.34	9.31	9.34	9.30	9.28
	0.9	1.08	0.80	0.74	0.68	0.76	0.74	4.60	4.17	4.10	4.04	4.08	4.05	9.31	8.89	8.88	8.67	8.81	8.80
0.5	-0.9	1.52	1.16	1.05	1.19	1.08	1.06	5.44	4.69	4.58	4.94	4.60	4.56	10.37	9.57	9.53	9.85	9.54	9.50
	-0.5	1.33	1.03	0.92	0.94	0.93	0.91	5.29	4.70	4.61	4.72	4.60	4.56	9.83	9.30	9.29	9.30	9.22	9.23
	0.5	0.37	0.23	0.17	0.26	0.21	0.18	2.81	2.21	2.12	2.45	2.17	2.11	6.42	5.54	5.44	5.84	5.50	5.41
	0.9	0.29	0.11	0.11	0.16	0.09	0.11	2.14	1.28	1.04	1.86	1.24	1.09	5.61	4.21	3.81	5.20	4.19	3.83
0.9	-0.9	1.46	1.14	1.01	1.10	1.04	1.01	5.72	5.13	5.03	5.17	5.02	4.99	10.75	9.88	9.88	10.15	9.85	9.85
	-0.5	0.87	0.63	0.60	0.58	0.60	0.60	4.75	3.98	3.86	4.17	3.90	3.88	9.42	8.55	8.51	8.78	8.51	8.48
	0.5	0.12	0.16	0.13	0.10	0.15	0.12	1.65	0.88	0.79	1.29	0.87	0.78	4.93	3.21	3.03	4.59	3.18	3.06
	0.9	0.13	0.87	0.70	0.06	0.86	0.68	1.16	1.51	1.39	0.91	1.45	1.33	4.51	3.20	3.06	4.11	3.14	3.05

Table 5.16 Hypothesis test results for $H_0 : \beta_4 = 0$, using a sample size of 30 and showing the Positive t-statistics

		POSITIVE t-STATISTICS																	
Sample size: T=30																			
H0 :		$\boldsymbol{\beta_4}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	1.22	0.96	4.18	1.06	0.87	4.01	3.15	1.66	5.85	2.88	1.53	5.68	6.33	2.86	7.51	5.96	2.65	7.42
	-0.5	1.26	0.59	2.47	1.07	0.49	2.37	4.83	1.90	3.87	4.21	1.70	3.80	9.53	4.22	6.07	8.90	3.98	6.14
	0.5	3.76	2.09	2.12	2.75	1.85	2.15	10.88	8.31	7.89	9.99	8.06	7.95	17.49	15.24	14.55	16.70	14.97	14.54
	0.9	2.32	1.04	1.36	1.78	0.84	1.35	7.39	4.97	4.79	6.55	4.71	4.71	12.80	10.54	9.89	12.24	10.26	9.91
-0.5	-0.9	0.31	0.29	0.26	0.18	0.27	0.29	2.16	0.82	0.69	1.93	0.76	0.71	5.49	2.32	1.83	5.06	2.29	1.86
	-0.5	0.76	0.39	0.26	0.49	0.31	0.27	4.18	2.39	1.96	3.69	2.28	2.01	8.45	5.79	5.22	7.96	5.73	5.33
	0.5	1.30	1.06	0.95	0.87	0.97	0.93	5.05	4.46	4.35	4.54	4.39	4.31	9.92	9.11	9.03	9.28	9.07	9.01
	0.9	0.68	0.43	0.41	0.38	0.39	0.38	3.67	2.79	2.69	3.11	2.74	2.69	7.70	6.32	6.15	7.18	6.25	6.17
0.5	-0.9	0.91	0.73	0.65	0.61	0.71	0.64	4.32	3.76	3.70	3.70	3.69	3.65	8.68	8.03	7.88	8.12	7.96	7.89
	-0.5	1.55	1.24	1.18	1.06	1.19	1.17	5.51	4.94	4.88	4.87	4.89	4.85	10.21	9.73	9.71	9.79	9.62	9.65
	0.5	0.60	0.37	0.29	0.44	0.32	0.29	4.23	2.67	2.21	3.70	2.55	2.26	8.65	6.62	6.06	8.17	6.50	6.14
	0.9	0.29	1.03	0.37	0.17	0.95	0.37	2.18	2.13	1.39	1.85	2.01	1.40	5.78	4.92	3.95	5.44	4.80	3.99
0.9	-0.9	2.88	2.12	1.86	2.15	1.95	1.87	9.28	7.99	7.63	8.38	7.86	7.60	15.19	14.04	13.77	14.54	13.92	13.73
	-0.5	4.82	3.66	3.11	3.85	3.39	3.20	12.60	11.17	10.77	11.76	11.05	10.78	18.74	17.66	17.33	18.14	17.59	17.35
	0.5	1.79	1.06	0.92	1.39	0.92	0.93	6.23	2.78	2.16	5.58	2.58	2.15	11.31	5.88	4.29	10.64	5.73	4.43
	0.9	1.33	2.28	1.61	1.02	2.16	1.57	4.17	3.37	3.01	3.75	3.26	2.88	7.83	5.07	4.72	7.40	4.92	4.68

Table 5.17 Hypothesis test results for $H_0 : \beta_4 = 0$, using a sample size of 30 and showing the Negative t-statistics

		NEGATIVE t-STATISTICS																	
Sample size: T=30																			
H0 :		$\boldsymbol{\beta_4}$																	
Nominal size:		1%						5%						10%					
Test:		N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF	N	NE	NCF	T	TE	TCF
ρ	ϕ	%						%						%					
-0.9	-0.9	0.98	0.87	4.26	0.84	0.81	4.19	3.16	1.53	5.73	2.87	1.43	5.67	6.38	2.82	7.51	5.88	2.63	7.38
	-0.5	1.33	0.54	2.43	1.01	0.45	2.33	4.94	1.75	3.85	4.43	1.60	3.82	9.67	4.14	5.90	9.03	3.86	5.91
	0.5	3.95	2.19	2.36	2.91	1.92	2.44	10.88	8.79	8.41	9.95	8.50	8.41	17.39	15.11	14.73	16.67	14.92	14.76
	0.9	2.24	0.99	1.58	1.63	0.82	1.49	6.80	5.02	5.07	6.15	4.66	5.07	11.83	9.89	9.68	11.28	9.69	9.66
-0.5	-0.9	0.36	0.20	0.21	0.21	0.19	0.21	2.42	0.90	0.71	2.08	0.88	0.76	6.05	2.53	1.89	5.53	2.47	1.99
	-0.5	0.68	0.29	0.20	0.41	0.24	0.23	4.59	2.58	2.14	3.92	2.47	2.16	9.26	6.45	5.72	8.81	6.39	5.83
	0.5	1.27	0.97	0.91	0.83	0.92	0.90	5.35	4.84	4.79	4.74	4.76	4.71	10.41	9.63	9.56	9.82	9.49	9.47
	0.9	0.64	0.50	0.39	0.44	0.42	0.39	3.72	2.87	2.80	3.31	2.82	2.82	7.98	6.69	6.64	7.45	6.65	6.64
0.5	-0.9	0.71	0.56	0.47	0.41	0.52	0.47	4.05	3.49	3.38	3.56	3.44	3.38	8.61	8.08	7.99	8.05	7.98	7.93
	-0.5	1.46	1.24	1.06	0.95	1.18	1.05	5.51	5.12	5.06	5.02	5.06	5.02	10.50	9.73	9.70	10.00	9.69	9.67
	0.5	0.69	0.25	0.15	0.42	0.23	0.18	3.79	2.24	1.86	3.29	2.09	1.89	8.18	6.06	5.48	7.60	6.00	5.54
	0.9	0.22	0.96	0.33	0.10	0.93	0.27	2.26	2.10	1.36	1.92	1.95	1.27	5.50	4.59	3.60	5.15	4.50	3.69
0.9	-0.9	2.71	2.19	2.04	2.07	2.12	2.07	8.66	7.64	7.56	7.84	7.51	7.56	14.22	13.19	13.13	13.65	13.05	13.09
	-0.5	4.42	3.37	2.91	3.56	3.10	2.99	11.45	10.09	9.71	10.61	9.94	9.74	17.53	16.45	16.27	16.95	16.36	16.24
	0.5	1.41	1.00	0.90	1.12	0.85	0.84	5.75	2.68	2.21	5.20	2.53	2.19	11.36	5.76	4.60	10.50	5.61	4.67
	0.9	1.23	2.22	1.66	0.95	2.13	1.54	3.83	3.33	2.93	3.40	3.18	2.83	7.67	5.21	4.69	7.27	5.02	4.62

Table 5.18 Hypothesis test results for $H_0 : \beta_2, \beta_3, \beta_4 = 0$, using a sample size of 30 and showing the F-statistics

		F-STATISTICS																	
Sample size: T=30																			
H0 :		$\boldsymbol{\beta_2, \beta_3, \beta_4}$																	
Nominal size:		1%						5%						10%					
Test:		X2	X2E	X2CF	F	FE	FCF	X2	X2E	X2CF	F	FE	FCF	X2	X2E	X2CF	F	FE	FCF
ρ	ϕ	%						%						%					
-0.9	-0.9	4.27	4.22	1.95	3.35	3.60	1.64	6.88	4.89	2.73	5.91	4.03	2.48	8.66	5.43	3.19	7.75	4.40	3.03
	-0.5	4.71	3.47	0.77	3.58	2.60	0.58	8.09	4.44	1.43	6.75	3.15	1.29	10.93	5.31	2.01	9.56	3.95	2.03
	0.5	10.59	2.45	0.12	6.41	0.76	0.34	21.15	6.93	0.91	16.88	4.42	2.37	30.07	11.24	3.26	25.81	8.42	5.03
	0.9	12.82	3.33	0.22	8.70	1.21	0.65	23.01	7.86	1.33	18.90	4.68	2.81	30.88	12.16	3.81	27.46	8.51	5.58
-0.5	-0.9	0.38	0.04	0.02	0.15	0.04	0.03	1.52	0.13	0.03	0.84	0.08	0.06	3.00	0.23	0.07	2.26	0.14	0.08
	-0.5	0.89	0.15	0.01	0.39	0.06	0.03	3.50	0.36	0.06	2.17	0.23	0.11	7.02	0.69	0.12	5.30	0.44	0.19
	0.5	2.18	1.06	0.25	1.00	0.69	0.38	6.81	3.28	0.93	4.77	2.54	1.77	12.22	5.89	2.86	9.44	5.30	3.95
	0.9	1.35	0.40	0.07	0.60	0.25	0.11	4.78	1.21	0.22	3.16	0.95	0.45	8.85	2.45	0.65	6.65	1.91	1.03
0.5	-0.9	1.52	0.58	0.13	0.65	0.38	0.20	5.75	1.89	0.62	3.69	1.50	0.92	10.30	3.78	1.55	8.16	3.13	2.23
	-0.5	2.25	1.23	0.23	1.14	0.82	0.36	7.13	3.46	1.01	4.95	2.83	2.02	12.20	6.28	3.01	9.57	5.64	4.33
	0.5	0.59	0.08	0.01	0.25	0.05	0.02	2.62	0.29	0.03	1.66	0.16	0.08	5.78	0.48	0.13	4.05	0.33	0.17
	0.9	0.34	0.00	0.00	0.14	0.00	0.00	1.33	0.04	0.00	0.81	0.00	0.00	2.94	0.08	0.00	2.02	0.04	0.01
0.9	-0.9	5.05	1.85	0.18	2.66	0.98	0.41	13.04	5.15	1.18	9.74	3.95	2.52	20.11	9.43	3.80	16.45	8.27	5.77
	-0.5	7.00	2.39	0.26	3.81	1.35	0.62	16.46	6.73	1.52	12.39	4.78	2.82	24.12	11.01	4.08	20.33	9.23	5.97
	0.5	2.35	1.11	0.26	1.51	0.71	0.29	4.75	1.75	0.54	3.64	1.28	0.62	7.42	2.22	0.74	5.99	1.73	0.81
	0.9	1.95	1.27	0.50	1.29	0.91	0.39	3.75	1.63	0.66	2.97	1.23	0.61	5.64	1.84	0.78	4.63	1.41	0.73

5.2.1 Comments on the results for the t statistic

As observed in Tables 5.1–5.8 and 5.10–5.17, in almost all cases, the Edgeworth and Cornish-Fisher corrections based on the normal distribution improve the null rejection probabilities compared to the uncorrected test based on the normal distribution. This improvement lies in the fact that the null rejection probabilities of the corrected tests better approximate the nominal size (significance level) of the t tests. The same holds for the Edgeworth and Cornish-Fisher corrections based on the Student-t distribution compared to the uncorrected Student-t test.

Furthermore, the tests based on the t distribution appear to perform better than the corresponding tests based on the normal distribution in almost all regions of the experimental space.

In addition, in most areas of the experimental space, the Cornish-Fisher corrections perform better than the Edgeworth corrections, confirming the theoretical advantages of the Cornish-Fisher corrections over the Edgeworth corrections.

Finally, comparing the tables corresponding to sample size 15 with those for sample size 30, we observe that as the sample size increases (sample size 30), all t tests based on the normal and Student-t distributions exhibit improved null rejection probabilities (closer to the nominal size) compared to the corresponding tests for sample size 15.

5.2.2 Comments on the results for the Wald and F statistics

As observed in Tables 5.9 and 5.18, in almost all cases, the Edgeworth and Cornish-Fisher corrections based on the χ^2 distribution improve the null rejection probabilities compared to the uncorrected test based on the χ^2 distribution. This improvement lies in the fact that the null rejection probabilities of the corrected tests better approximate the nominal size (significance level) of the χ^2 tests. The same holds for the Edgeworth and Cornish-Fisher corrections based on the F distribution compared to the uncorrected F test.

Furthermore, the tests based on the F distribution appear to perform better than the corresponding tests based on the χ^2 distribution in almost all regions of the experimental space.

In addition, in most areas of the experimental space, the Cornish-Fisher corrections perform better than the Edgeworth corrections, confirming the theoretical advantages of the Cornish-Fisher corrections over the Edgeworth corrections.

Finally, comparing the tables corresponding to sample size 15 with those for sample size 30, we observe that as the sample size increases (sample size 30), all χ^2 and F tests exhibit improved null rejection probabilities (closer to the nominal size) compared to the corresponding tests for sample size 15.

5.2.3 Discussion of Negative Values of Cornish-Fisher Adjusted Wald and F Statistics

From formulas (3.29) and (3.36), it follows that the Cornish-Fisher adjustments may yield negative values for the adjusted Wald and F statistics. Such values can arise for three main reasons:

I. Very large values of the unadjusted Wald or F statistic.

II. Large values of the adjustment factors h_1, h_2 or q_1, q_2.

III. A combination of the above two factors.

If the occurrence of a negative adjusted statistic is due to a large value of the unadjusted Wald or F statistic, this phenomenon is not concerning, as it indicates that the unadjusted statistic is so large that the null hypothesis should be rejected with high confidence. Conversely, if the negative result is caused by large values of the adjustment factors (h_1, h_2 or q_1, q_2), the existence of a negative Cornish-Fisher adjusted Wald or F statistic cannot be considered as evidence to reject the null hypothesis, and further investigation is required.

In our experiment, several negative Cornish-Fisher adjusted Wald and F statistics were detected. For this reason, we investigated their causes as follows:

For each adjusted Wald statistic, we recorded the corresponding value of the unadjusted Wald statistic and the adjustment factors h_1 and h_2. Then, for all Cornish-Fisher adjusted Wald statistics, the following equation was estimated by OLS:

$$\mathrm{stat\_x\_2\_cf} = \alpha_0 + \alpha_1 \mathrm{w} + \alpha_2 h_1 + \alpha_3 h_2 + \varepsilon \tag{5.59}$$

where stat_x_2_cf is the Cornish-Fisher adjusted value of the Wald statistic, w is the unadjusted value of the Wald statistic, h_1 and h_2 are adjustment factors used to modify the Wald statistic, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are the regression coefficients, and ε is the error term capturing the random deviations from the model.

Subsequently, the statistical significance of the individual coefficients as well as the overall model was examined.

Similarly, for each Cornish-Fisher adjusted F statistic, the corresponding value of the unadjusted F statistic and the adjustment factors q_1 and q_2 were recorded. For all observations, the following equation was estimated by OLS:

$$\text{stat\_F\_cf} = \alpha_0 + \alpha_1 \text{v} + \alpha_2 q_1 + \alpha_3 q_2 + \varepsilon \qquad (5.60)$$

where stat_F_cf is the Cornish-Fisher adjusted value of the F statistic, v is the unadjusted value of the F statistic, q_1 and q_2 are adjustment factors used to modify the F statistic, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are the regression coefficients, and ε is the error term capturing the random deviations from the model.

In this case as well, the statistical significance of the coefficients and the overall model was tested.

The results of these regressions are presented in Appendix F (Tables F.1–F.64).

Additionally, for each sample size ($T = 15, 30$) and each combination of parameters ρ and ϕ, graphs were created showing all negative Cornish-Fisher adjusted Wald statistics and the values of the adjustment factors h_1, h_2 as functions of the corresponding unadjusted Wald statistic. Similarly, graphs were created for the negative adjusted F statistics and the values of q_1, q_2.

These graphs are presented in Appendix F (Figures F.1–F.64).

As an example, this chapter presents the tables and graphs for sample size $T = 15$ (Tables F.13–F.16, Figures F.13 and F.14), as well as for $T = 30$ (Tables F.45–F.48, Figures F.45 and F.46).

For instance, Table F.13 shows the estimation of the function (for the Wald statistic) for $\rho = -0.5$, $\phi = 0.5$, and $T = 15$. The table indicates that the Cornish-Fisher adjusted Wald statistic is due to the adjustment factors h_1 and h_2, whose coefficients are statistically significant at the 1% level. Similarly, Table F.15 presents the estimation of the function (for the F statistic) under the same conditions ($\rho = -0.5$, $\phi = 0.5$, $T = 15$). The results show that the Cornish-Fisher adjusted statistic is attributed to both the adjustment factors q_1, q_2 and the unadjusted F statistic, with all coefficients statistically significant at the 1% level.

The corresponding graphs (Figures F.13 and F.14) confirm that the negative adjusted Cornish-Fisher statistics are associated with large values of h_1, h_2, q_1, and q_2. Furthermore, Figure F.14 (for the F statistics) shows that the negative adjusted Cornish-Fisher statistics are almost exclusively due to large values of the unadjusted F statistic, for example when the F values exceed 7.

Table F.13: Estimated Regression Results under $\rho = -0.5$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	0.020
	(0.047)
h1	-0.051^{***}
	(0.002)
h2	-0.115^{***}
	(0.001)
Constant	3.773^{***}
	(0.262)
Observations	10,000
R^2	0.619
Adjusted R^2	0.619

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.14: Estimated Regression Results under $\rho = -0.5$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-7.163^{***}
	(1.571)
h1	0.551^{***}
	(0.023)
h2	-0.436^{***}
	(0.007)
Constant	62.640^{***}
	(7.162)
Observations	10,000
R^2	0.860
Adjusted R^2	0.860

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.15: Estimated Regression Results under $\rho = -0.5$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	0.661^{***}
	(0.047)
q1	-0.051^{***}
	(0.002)
q2	-0.192^{***}
	(0.002)
Constant	0.486^{***}
	(0.088)
Observations	10,000
R^2	0.619
Adjusted R^2	0.619

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.16: Estimated Regression Results under $\rho = -0.5$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-6.699^{***}
	(1.571)
q1	0.551^{***}
	(0.023)
q2	-0.726^{***}
	(0.011)
Constant	19.212^{***}
	(2.384)
Observations	10,000
R^2	0.860
Adjusted R^2	0.860

Note: *p<0.1; **p<0.05; ***p<0.01

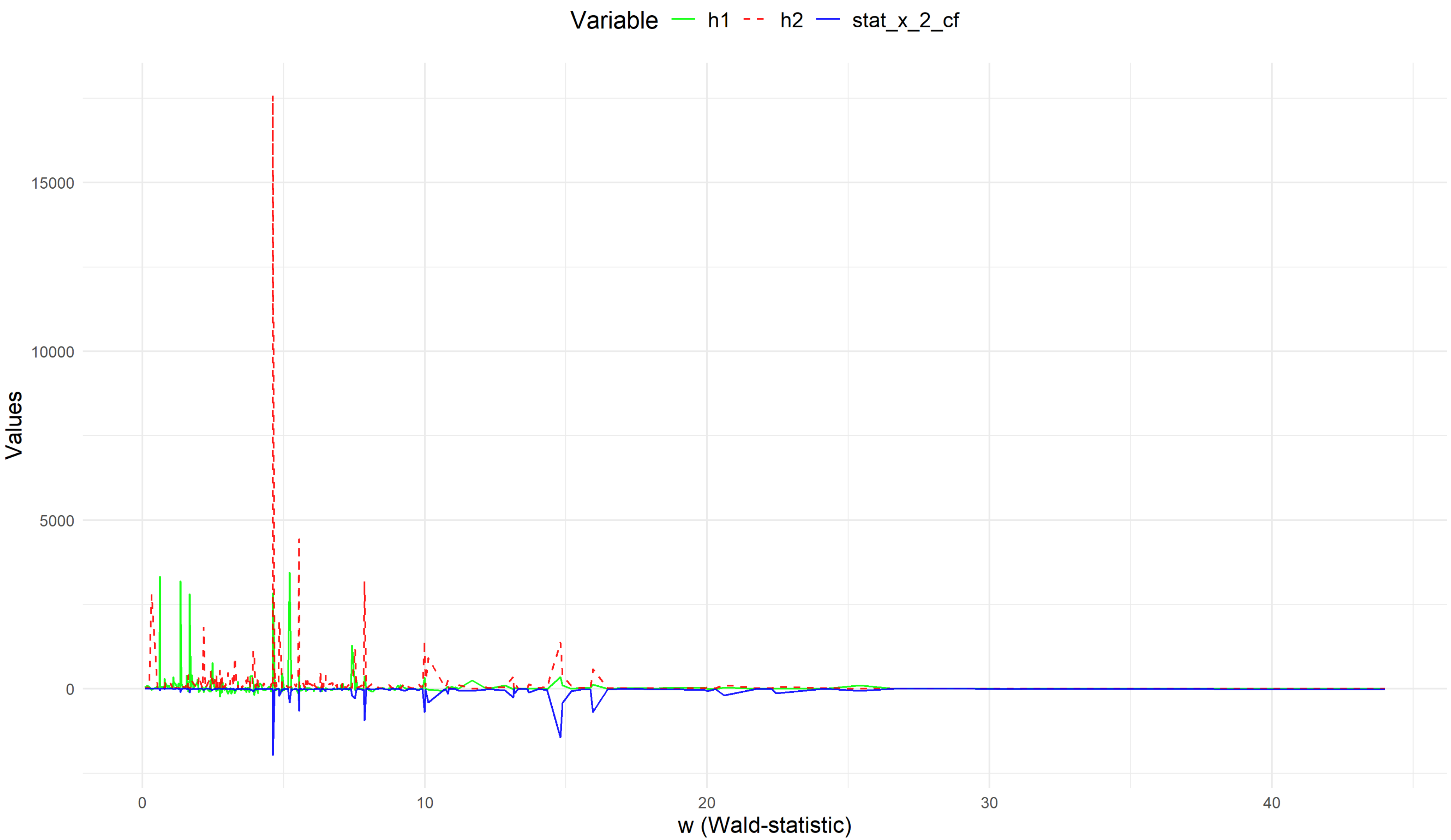

Figure F.13: Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.5$, and $T = 15$

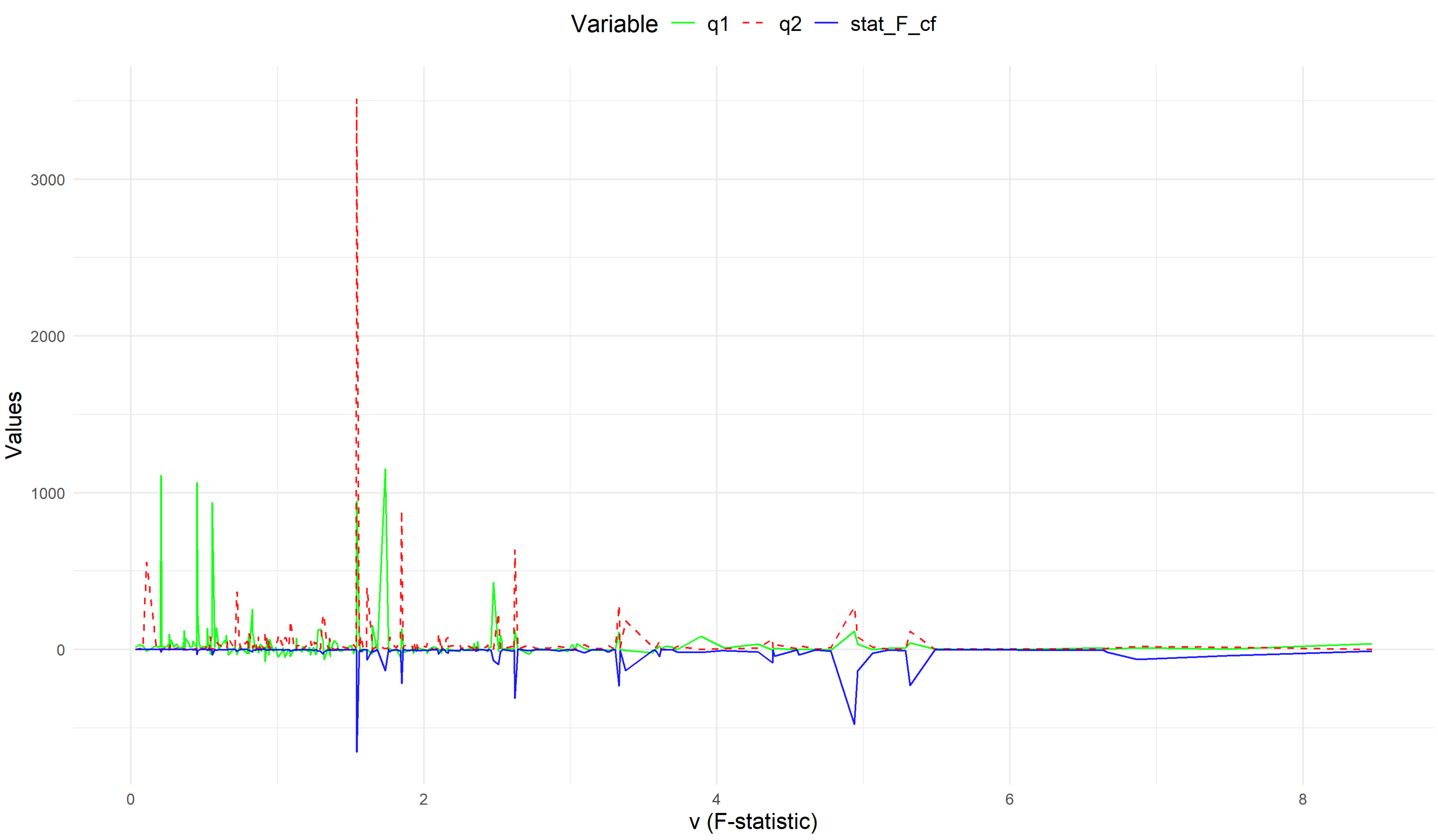

Figure F.14: Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.5$, and T=15

Table F.45: Estimated Regression Results under $\rho = -0.5$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-0.430^{***}
	(0.094)
h1	-0.039^{***}
	(0.001)
h2	-0.174^{***}
	(0.001)
Constant	9.638^{***}
	(0.411)
Observations	10,000
R^2	0.635
Adjusted R^2	0.635
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.46: Estimated Regression Results under $\rho = -0.5$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-2.839^{***}
	(0.126)
h1	-0.030^{***}
	(0.001)
h2	-0.033^{***}
	(0.0004)
Constant	11.206^{***}
	(0.479)
Observations	10,000
R^2	0.835
Adjusted R^2	0.835
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.47: Estimated Regression Results under $\rho = -0.5$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-0.219^{**}
	(0.094)
q1	-0.039^{***}
	(0.001)
q2	-0.289^{***}
	(0.002)
Constant	2.660^{***}
	(0.136)
Observations	10,000
R^2	0.635
Adjusted R^2	0.635
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.48: Estimated Regression Results under $\rho = -0.5$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-2.666^{***}
	(0.126)
q1	-0.030^{***}
	(0.001)
q2	-0.054^{***}
	(0.001)
Constant	3.571^{***}
	(0.160)
Observations	10,000
R^2	0.835
Adjusted R^2	0.835
Note:	*p<0.1; **p<0.05; ***p<0.01

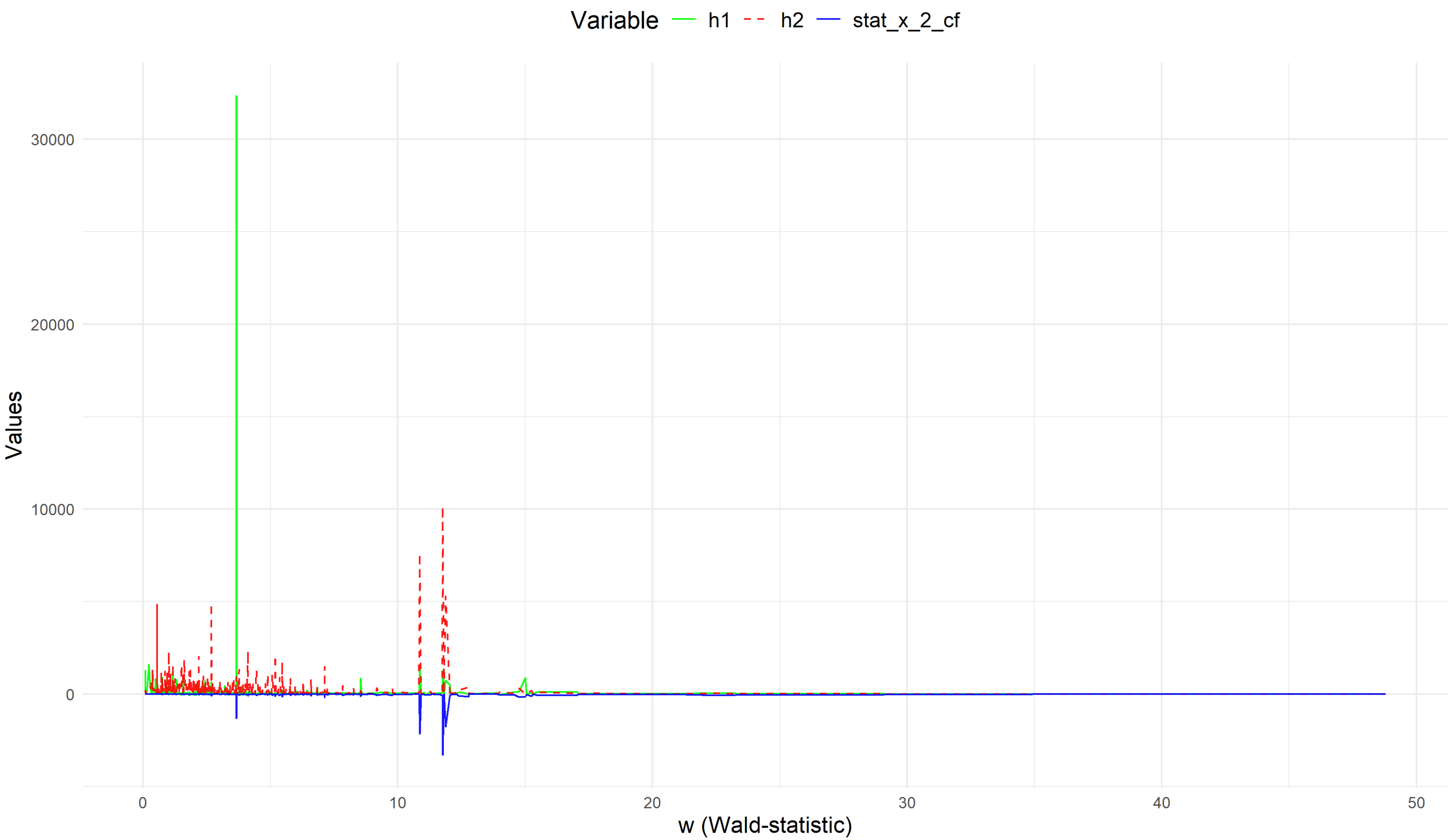

Figure F.45: Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.5$, and T=30

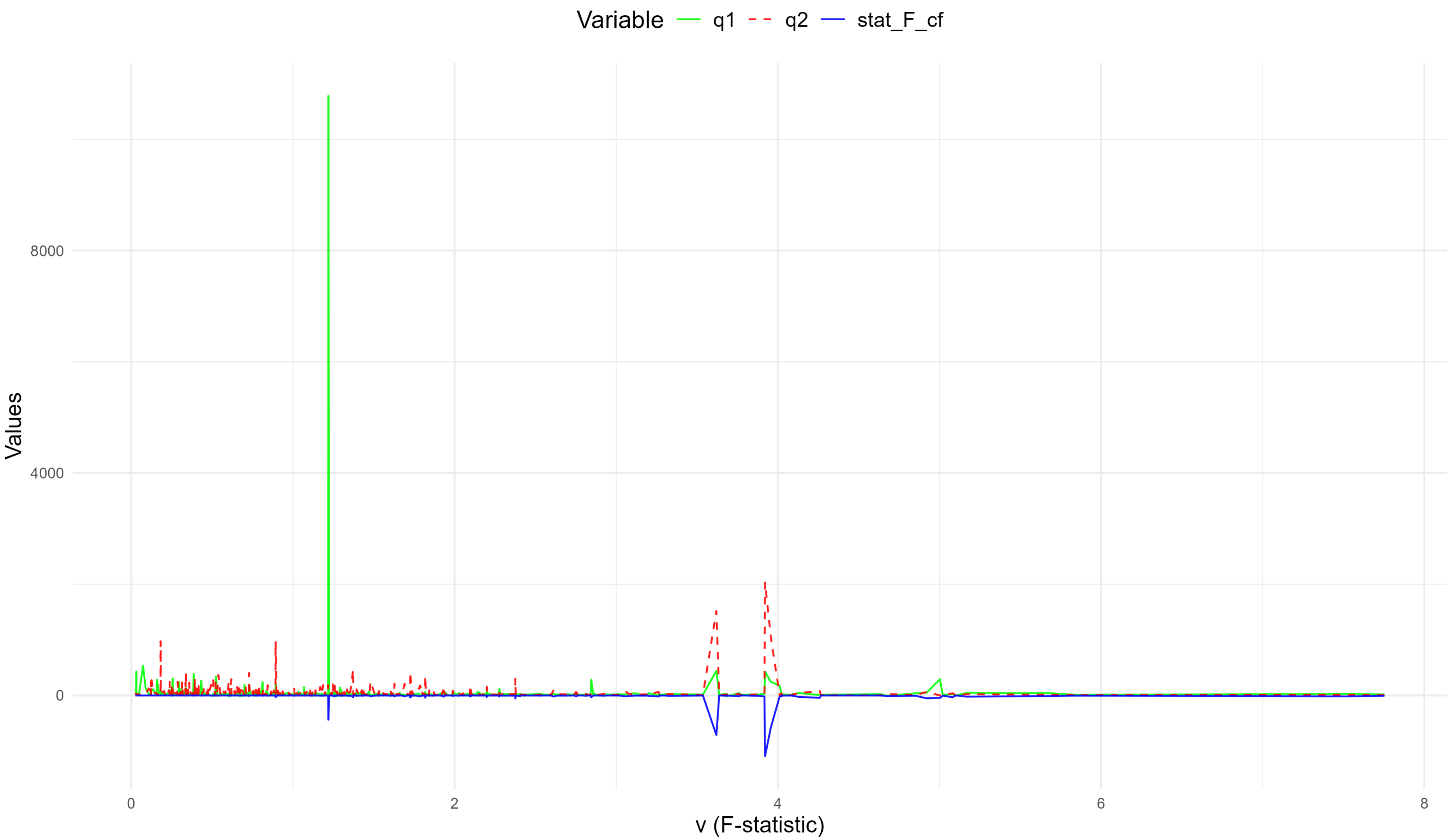

Figure F.46: Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.5$, and T=30

Chapter 6

Conclusion

In this thesis, we dealt with small-sample correction of the size of the t and F econometric tests in the generalized linear model with ARMA(1,1) disturbances. The methodology we used falls within the framework of refined asymptotic theory, according to the Nagar school approach. The corrections we propose are based on:

- Edgeworth-corrected critical values
- Cornish-Fisher-corrected test statistics

There are both theoretical and practical reasons that support the preference for Cornish-Fisher corrections. The theoretical reasons are based on the fact that:

- Edgeworth-corrected critical values are derived from Edgeworth expansions, which are not well-defined distributions (and may assign negative probabilities in the tails of the distribution)
- Cornish-Fisher-corrected statistics are well-defined random variables with well-behaved properties

Since both alternative correction methods have an error of the same order of magnitude, the comparative evaluation of their performance can only be carried out using stochastic simulation experiments (Monte Carlo).

6.1 The results of the Monte Carlo simulations

As regards the results for the t-test, in almost all cases, the Edgeworth and Cornish-Fisher corrections based on the normal distribution improve the null rejection probabilities compared to the uncorrected test based on the normal distribution. This improvement lies in the fact that the null rejection probabilities of the corrected tests better approximate the nominal size (significance level) of the t tests. The same holds for the Edgeworth and Cornish-Fisher corrections based on the Student-t distribution compared to the uncorrected Student-t test.

Furthermore, the tests based on the t distribution appear to perform better than the corresponding tests based on the normal distribution in almost all regions of the experimental space.

In addition, in most areas of the experimental space, the Cornish-Fisher corrections perform better than the Edgeworth corrections, confirming the theoretical advantages of the Cornish-Fisher corrections over the Edgeworth corrections.

Finally, comparing the tables corresponding to sample size 15 with those for sample size 30, we observe that as the sample size increases (sample size 30), all t tests based on the normal and Student-t distributions exhibit improved null rejection probabilities (closer to the nominal size) compared to the corresponding tests for sample size 15.

As regards the results for the t-test, in almost all cases, the Edgeworth and Cornish-Fisher corrections based on the χ^2 distribution improve the null rejection probabilities compared to the uncorrected test based on the χ^2 distribution. This improvement lies in the fact that the null rejection probabilities of the corrected tests better approximate the nominal size (significance level) of the χ^2 tests. The same holds for the Edgeworth and Cornish-Fisher corrections based on the F distribution compared to the uncorrected F test.

Furthermore, the tests based on the F distribution appear to perform better than the corresponding tests based on the χ^2 distribution in almost all regions of the experimental space.

In addition, in most areas of the experimental space, the Cornish-Fisher corrections perform better than the Edgeworth corrections, confirming the theoretical advantages of the Cornish-Fisher corrections over the Edgeworth corrections.

Finally, comparing the tables corresponding to sample size 15 with those for sample size 30, we observe that as the sample size increases (sample size 30), all χ^2 and F tests exhibit improved null rejection probabilities (closer to the nominal size) compared to the corresponding tests for sample size 15.

6.2 Some remarks on future research

In this thesis, we evaluated the proposed corrections of statistical tests through simulation experiments. Each repetition of the simulation corresponds to the situation faced by a researcher estimating a generalized linear model with ARMA(1,1) errors. This simulation essentially proposes a procedure for dealing with estimation and testing problems, which consists of the following steps:

1. The researcher estimates the model using the Ordinary Least Squares (OLS) method, which yields consistent estimators and residuals.

2. Using the OLS residuals, the researcher computes the estimates ρ, ϕ, and $\hat{\sigma}^2$.

3. Assuming the estimated values ρ, ϕ, and $\hat{\sigma}^2$ as the true parameters, the researcher performs simulations to estimate the following asymptotic quantities:

$$\begin{array}{lll} \mu_\rho = \mathbb{E}(\delta_\rho)/\tau & \lambda_0 = \mathbb{E}(\delta_0^2) & \lambda_{\rho\rho*} = \mathbb{E}(\delta_\rho\delta_\rho) \\ & \lambda_\rho = \mathbb{E}(\delta_\rho\delta_0) & \lambda_{\phi\phi*} = \mathbb{E}(\delta_\phi\delta_\phi) \\ \mu_\phi = \mathbb{E}(\delta_\phi)/\tau & \lambda_\phi = \mathbb{E}(\delta_\phi\delta_0) & \lambda_{\rho\phi*} = \mathbb{E}(\delta_\rho\delta_\phi) \end{array} \tag{6.1}$$

These quantities are then used to compute the proposed Edgeworth and Cornish-Fisher corrections for the t and F tests.

Alternatively, a researcher can avoid much of the simulation described in step (3) using the following approach:

1. (a) The researcher estimates the model using OLS and obtains consistent residuals.

   (b) Using the OLS residuals, the researcher calculates the estimates ρ, ϕ, and $\hat{\sigma}^2$.

   (c) Based on the estimated values of ρ and ϕ, and utilizing the simulation results from Chapter 4, the researcher can compute the quantities μ_ρ, μ_ϕ, $\lambda_{\rho\rho}$, $\lambda_{\rho\phi}$, and $\lambda_{\phi\phi}$. The remaining quantities, namely λ_0, λ_ρ, and λ_ϕ, can be obtained through the following procedure:

2. The researcher performs bootstrap sampling using the original dataset, in order to approximate the empirical distributions of the estimated $\hat{\sigma}^2$, ρ, and ϕ. From these empirical distributions, the quantities λ_0, λ_ρ, and λ_ϕ are derived.

The evaluation of this alternative approach, and its comparison with the method proposed in this thesis, can be conducted through further simulation experiments and constitutes a topic for future research.

Appendices

Appendix A

The elements of the variance-covariance matrix Ω

In this study, the computation of the elements of the variance–covariance matrix Ω are based on the theoretical framework proposed by Tiao and Ali (1971) for the ARMA(1,1) model. To verify the correct computation of the elements of this matrix, systematic checks were performed whereby, through appropriate zeroing of certain parameters, the model reduces either to a first-order moving average process (MA(1)) or to a first-order autoregressive process (AR(1)). Moreover, to simplify the mathematical manipulations, the symbol D is defined as the common term appearing in both the diagonal and non-diagonal elements of the matrix Ω, thereby facilitating the subsequent analytical treatment.

A.1 Theoretical framework and validation of the variance–covariance matrix Ω in ARMA(1,1)

According to Tiao and Ali (1971), we can calculate diagonal elements, ω_{tt}, and the non-diagonal elements, $\omega_{tt'}$, of the matrix Ω as follows:

$$\begin{aligned}
\omega_{tt} &= |\Omega|(1-\rho^2)^{-1}[1-(-\phi)^2]^{-2}[(1-\rho(-\phi))^2(1+\rho^2-2\rho(-\phi)) \\
&\quad +((-\phi)-\rho)^2\{((-\phi)-\rho)-\rho(1-\rho(-\phi))\}(-\phi)^{2T-1} \\
&\quad -((-\phi)-\rho)^2(1-\rho(-\phi))^2\{(-\phi)^{2(t-1)}+(-\phi)^{2(T-t)}\}] \\
&= |\Omega|\frac{1}{(1-\rho^2)(1-(-\phi)^2)^2}[(1+\rho\phi)^2(1+\rho^2+2\rho\phi) \\
&\quad +(\phi+\rho)^2\{-(\phi+\rho)-\rho(1+\rho\phi)\}(-\phi)^{2T-1} \\
&\quad -(\phi+\rho)^2(1+\rho\phi)^2\{\phi^{2(t-1)}+\phi^{2(T-t)}\}]. \qquad \text{(A.1)}
\end{aligned}$$

$$\begin{aligned}
\omega_{tt'} &= |\Omega|\frac{((-\phi)-\rho)(1-\rho(-\phi))}{(1-\rho^2)(1-(-\phi)^2)^2}[(1-\rho(-\phi))^2(-\phi)^{|t-t'|-1} \\
&\quad +((-\phi)-\rho)^2(-\phi)^{2T-|t-t'|-1} \\
&\quad -((-\phi)-\rho)(1-\rho(-\phi))\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\}] \\
&= |\Omega|\frac{-(\phi+\rho)(1+\rho\phi)}{(1-\rho^2)(1-\phi^2)^2}[(1+\rho\phi)^2(-\phi)^{|t-t'|-1}
\end{aligned}$$

$$+(\phi+\rho)^2(-\phi)^{2T-|t-t'|-1}$$
$$+(\phi+\rho)(1+\rho\phi)\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\}]. \quad \text{(A.2)}$$

where

$$t \neq t' \quad and \quad t,t'=1,\ldots,T. \quad \text{(A.3)}$$

The determinant of the matrix Ω^{-1} is

$$\begin{aligned} |\Omega|^{-1} &= 1+\frac{(\rho-(-\phi))^2(1-(-\phi)^{2T})}{(1-\rho^2)(1-(-\phi)^2)} \Rightarrow \\ |\Omega|^{-1} &= 1+\frac{(\rho+\phi)^2(1-\phi^{2T})}{(1-\rho^2)(1-\phi^2)}. \end{aligned} \quad \text{(A.4)}$$

A.1.1 Verification of the Reduction to a First-Order Moving Average (MA(1)) Process

When $\rho = 0$, the expressions in (A.1), (A.2), and (A.4) reduce to those corresponding to the stationary first-order moving average process. If $\rho = 0$ equation (A.4) implies that

$$\begin{aligned} |\Omega|^{-1} &= 1+\frac{(0+\phi)^2(1-\phi^{2T})}{(1-0^2)(1-\phi^2)} = 1+\frac{\phi^2(1-\phi^{2T})}{1-\phi^2} = \frac{1-\phi^2+\phi^2-\phi^{2(T+1)}}{1-\phi^2} \Rightarrow \\ |\Omega|^{-1} &= \frac{1-\phi^{2(T+1)}}{1-\phi^2} \Rightarrow \end{aligned} \quad \text{(A.5)}$$

$$|\Omega| = \frac{1-\phi^2}{1-\phi^{2(T+1)}}. \quad \text{(A.6)}$$

If $\rho = 0$ equation (A.1) implies that

$$\begin{aligned} \omega_{tt} &= \frac{(1-\phi^2)}{(1-\phi^{2(T+1)})}\frac{1}{(1-\rho 0^2)(1-(-\phi)^2)^2}[(1+0\phi)^2(1+0^2+2\phi 0) \\ &\quad +(\phi+0)^2\{-(\phi+0)-0(1+0\phi)\}(-\phi)^{2T-1} \\ &\quad -(\phi+0)^2(1+0\phi)^2\{\phi^{2(t-1)}+\phi^{2(T-t)}\}] \\ &= \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[1+\phi^2(-\phi)(-\phi)^{2T-1}-\phi^2(\phi^{2(t-1)}+\phi^{2(T-t)})] \\ &= \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[1+\phi^2(-\phi)^{2T}-\phi^{2t}-\phi^{2(T+1-t)})] \\ &= \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[1+\phi^{2(T+1)}-\phi^{2t}-\phi^{2(T+1-t)})]. \end{aligned} \quad \text{(A.7)}$$

If $\rho = 0$ equation (A.2) implies that

$$\begin{aligned} \omega_{tt'} &= \frac{(1-\phi^2)}{(1-\phi^{2(T+1)})}\frac{-(\phi+0)(1+0\phi)}{(1-0^2)(1-\phi^2)^2}[(1+0\phi)^2(-\phi)^{|t-t'|-1} \\ &\quad +(\phi+0)^2(-\phi)^{2T-|t-t'|-1} \end{aligned}$$

$$
\begin{aligned}
&+(\phi+0)(1+0\phi)\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\}] \\
= \quad & \frac{(-\phi)}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{|t-t'|-1}+(-\phi)^2(-\phi)^{2T-|t-t'|-1} \\
&+\phi\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\}] \\
= \quad & \frac{(-\phi)}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{|t-t'|-1}+(-\phi)^{2T-|t-t'|+1} \\
&+\phi\{(-\phi)^{t+t'}(-\phi)^{-2}+(-\phi)^{2T}(-\phi)^{-(t+t')}\}] \\
= \quad & \frac{(-\phi)}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{|t-t'|-1}+(-\phi)^{2T-|t-t'|+1} \\
&+\phi\{\phi^{-2}(-\phi)^{t+t'}+\phi^{2T}(-\phi)^{-(t+t')}\}] \\
= \quad & \frac{(-\phi)}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{|t-t'|-1}+(-\phi)^{2T-|t-t'|+1} \\
&+\phi^{-1}(-\phi)^{t+t'}+\phi^{2T+1}(-\phi)^{-(t+t')}] \\
= \quad & \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{|t-t'|}+(-\phi)^{2T-|t-t'|+2} \\
&-(-\phi)^{t+t'}-\phi^{2(T+1)}(-\phi)^{-(t+t')}] \\
= \quad & \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{|t-t'|}+(-\phi)^{2(T+1)-|t-t'|} \\
&-(-\phi)^{t+t'}-\phi^{2(T+1)}(-\phi)^{-(t+t')}] \\
= \quad & \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{|t-t'|}+(-\phi)^{2(T+1)-|t-t'|} \\
&-(-\phi)^{t+t'}-(-\phi)^{2(T+1)-(t+t')}]. \qquad \text{(A.8)}
\end{aligned}
$$

If $t' \geq t$ equation (A.8) implies that

$$
\begin{aligned}
\omega_{tt'} \quad = \quad & \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{t'-t}+(-\phi)^{2(T+1)-(t'-t)} \\
&-(-\phi)^{t+t'}-(-\phi)^{2(T+1)-(t+t')}]. \qquad \text{(A.9)}
\end{aligned}
$$

Then, using the results in Shaman (1969), referring to a first-order moving average process, we verify the formulae for ω_{tt} and $\omega_{tt'}$.

If $t' \geq t$ equation (A.2) implies that

$$
\begin{aligned}
\omega_s^{tt'} \quad = \quad & \frac{(-\phi)^{t'-t}}{(1-\phi^2)(1-\phi^{2(T+1)})}(1-\phi^{2t})(1-\phi^{2(T-t'+1)}) \\
= \quad & \frac{(-\phi)^{t'-t}}{(1-\phi^2)(1-\phi^{2(T+1)})}[1-\phi^{2(T+1-t')}-\phi^{2t}+\phi^{2(T+1+t-t')}] \\
= \quad & \frac{(-\phi)^{t'-t}}{(1-\phi^2)(1-\phi^{2(T+1)})}[1-(-\phi)^{2(T+1-t')}-(-\phi)^{2t}+(-\phi)^{2(T+1+t-t')}] \\
= \quad & \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{t'-t}-(-\phi)^{2(T+1)-2t'+t'-t}-(-\phi)^{2t+t'-t} \\
&+(-\phi)^{2(T+1)+2t-2t'+t'-t}]
\end{aligned}
$$

$$\begin{aligned} &= \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{t'-t} - (-\phi)^{2(T+1)-(t'+t)} - (-\phi)^{t'+t} \\ &\quad +(-\phi)^{2(T+1)-(t'-t)}] \\ &= \omega_{tt'}. \end{aligned} \tag{A.10}$$

Then, for $t = t'$ we take

$$\begin{aligned} \omega_{tt'}|t=t' &= \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{t'-t} + (-\phi)^{2(T+1)-(t'-t)} - (-\phi)^{t+t'} \\ &\quad -(-\phi)^{2(T+1)-(t+t')}] \\ &= \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[(-\phi)^{0} + (-\phi)^{2(T+1)-0} - (-\phi)^{2t} \\ &\quad -(-\phi)^{2(T+1)-2t}] \\ &= \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[1 + (-\phi)^{2(T+1)} - (-\phi)^{2t} - (-\phi)^{2(T+1-t)}] \\ &= \frac{1}{(1-\phi^2)(1-\phi^{2(T+1)})}[1 + \phi^{2(T+1)} - \phi^{2t} - \phi^{2(T+1-t)}] \\ &= \omega_{tt}. \end{aligned} \tag{A.11}$$

So, if in formulae (A.1), (A.2), and (A.4) $\rho = 0$, then we end up with MA(1).

A.1.2 Verification of the Reduction to a First-Order Autoregressive Process

When $\phi = 0$, the expressions in (A.1), (A.2), and (A.4) reduce to those corresponding to the stationary first-order autoregressive process.

If $\phi = 0$ equation (A.4) implies that

$$\begin{aligned} |\Omega|^{-1} &= 1 + \frac{(\rho+0)^2(1-0^{2T})}{(1-\rho^2)(1-0^2)} = 1 + \frac{\rho^2}{(1-\rho^2)} = \frac{1-\rho^2+\rho^2}{(1-\rho^2)} \Rightarrow \\ |\Omega|^{-1} &= \frac{1}{(1-\rho^2)} \Rightarrow \end{aligned} \tag{A.12}$$

$$|\Omega| = (1-\rho^2). \tag{A.13}$$

If $\phi = 0$, for $t \neq 1$ and $t \neq T$ equation (A.1) implies that

$$\begin{aligned}
\omega_{tt} &= (1-\rho^2)\frac{1}{(1-\rho^2)(1-0^2)^2}[(1+\rho 0)^2(1+\rho^2+2\rho 0) \\
&\quad +(0+\rho)^2\{-(0+\rho)-\rho(1+\rho 0)\}(-0)^{2T-1} \\
&\quad -(0+\rho)^2(1+\rho 0)^2\{0^{2(t-1)}+0^{2(T-t)}\}] \Rightarrow \\
\omega_{tt} &= (1+\rho^2). \qquad \text{(A.14)}
\end{aligned}$$

If $t=1$ equation (A.1) implies that

$$\begin{aligned}
\omega_{11} &= |\Omega|\frac{1}{(1-\rho^2)(1-\phi^2)^2}[(1+\rho\phi)^2(1+\rho^2+2\rho\phi) \\
&\quad +(\phi+\rho)^2\{-(\phi+\rho)-\rho(1+\rho\phi)\}(-\phi)^{2T-1} \\
&\quad -(\phi+\rho)^2(1+\rho\phi)^2\{\phi^{2(1-1)}+\phi^{2(T-1)}\}] \\
&= |\Omega|\frac{1}{(1-\rho^2)(1-\phi^2)^2}[(1+\rho\phi)^2(1+\rho^2+2\rho\phi) \\
&\quad +(\phi+\rho)^2\{-(\phi+\rho)-\rho(1+\rho\phi)\}(-\phi)^{2T-1} \\
&\quad -(\phi+\rho)^2(1+\rho\phi)^2\{1+\phi^{2(T-1)}\}], \qquad \text{(A.15)}
\end{aligned}$$

and if $t=1$ and $\phi=0$ from (A.13) and (A.15) we take

$$\begin{aligned}
\omega_{11} &= (1-\rho^2)\frac{1}{(1-\rho^2)(1-0^2)^2}[(1+\rho 0)^2(1+\rho^2+2\rho 0) \\
&\quad +(0+\rho)^2\{-(0+\rho)-\rho(1+\rho 0)\}(-0)^{2T-1} \\
&\quad -(0+\rho)^2(1+\rho 0)^2\{1+0^{2(T-1)}\}] \\
&= [1+\rho^2+0-\rho^2] \Rightarrow \\
\omega_{11} &= 1. \qquad \text{(A.16)}
\end{aligned}$$

If $t=T$ equation (A.1) implies that

$$\begin{aligned}
\omega_{TT} &= |\Omega|\frac{1}{(1-\rho^2)(1-\phi^2)^2}[(1+\rho\phi)^2(1+\rho^2+2\rho\phi) \\
&\quad +(\phi+\rho)^2\{-(\phi+\rho)-\rho(1+\rho\phi)\}(-\phi)^{2T-1} \\
&\quad -(\phi+\rho)^2(1+\rho\phi)^2\{\phi^{2(T-1)}+\phi^{2(T-T)}\}] \\
&= |\Omega|\frac{1}{(1-\rho^2)(1-\phi^2)^2}[(1+\rho\phi)^2(1+\rho^2+2\rho\phi) \\
&\quad +(\phi+\rho)^2\{-(\phi+\rho)-\rho(1+\rho\phi)\}(-\phi)^{2T-1} \\
&\quad -(\phi+\rho)^2(1+\rho\phi)^2\{\phi^{2(T-1)}+1\}], \qquad \text{(A.17)}
\end{aligned}$$

and if $t = T$ and $\phi = 0$ from (A.13) and (A.17) we take

$$\begin{aligned}
\omega_{TT} &= (1-\rho^2)\frac{1}{(1-\rho^2)(1-0^2)^2}[(1+\rho 0)^2(1+\rho^2+2\rho 0) \\
&\quad +(0+\rho)^2\{-(0+\rho)-\rho(1+\rho 0)\}(-0)^{2T-1} \\
&\quad -(0+\rho)^2(1+\rho 0)^2\{0^{2(T-1)}+1\}] \\
&= [1+\rho^2+0-\rho^2] \Rightarrow \\
\omega_{TT} &= 1. \qquad \text{(A.18)}
\end{aligned}$$

If $|t-t'| = 1$ equation (A.2) implies that

$$\begin{aligned}
\omega_{tt'} &= |\Omega|\frac{-(\phi+\rho)(1+\rho\phi)}{(1-\rho^2)(1-\phi^2)^2}[(1+\rho\phi)^2(-\phi)^{1-1} \\
&\quad +(\phi+\rho)^2(-\phi)^{2T-1-1} \\
&\quad +(\phi+\rho)(1+\rho\phi)\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\}] \\
&= |\Omega|\frac{-(\phi+\rho)(1+\rho\phi)}{(1-\rho^2)(1-\phi^2)^2}[(1+\rho\phi)^2 1 \\
&\quad +(\phi+\rho)^2(-\phi)^{2(T-1)} \\
&\quad +(\phi+\rho)(1+\rho\phi)\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\}]. \qquad \text{(A.19)}
\end{aligned}$$

If $|t-t'| = 1$ and $\phi = 0$ from (A.13) and (A.19) we take

$$\begin{aligned}
\omega_{tt'} &= (1-\rho^2)\frac{-(0+\rho)(1+\rho 0)}{(1-\rho^2)(1-0^2)^2}[(1+\rho 0)^2 1 \\
&\quad +(0+\rho)^2(-0)^{2(T-1)} \\
&\quad +(0+\rho)(1+\rho 0)\{(-0)^{t+t'-2}+(-0)^{2T-(t+t')}\}] \Rightarrow \\
\omega_{tt'} &= -\rho. \qquad \text{(A.20)}
\end{aligned}$$

If $|t-t'| \neq 1$ and $\phi = 0$ we find

$$\begin{aligned}
\omega_{tt'} &= (1-\rho^2)\frac{-(0+\rho)(1+\rho 0)}{(1-\rho^2)(1-0^2)^2}[(1+\rho 0)^2(-0)^{|t-t'|-1} \\
&\quad +(0+\rho)^2(-0)^{2T-|t-t'|-1} \\
&\quad +(0+\rho)(1+\rho 0)\{(-0)^{t+t'-2}+(-0)^{2T-(t+t')}\}] \Rightarrow \\
\omega_{tt'} &= 0. \qquad \text{(A.21)}
\end{aligned}$$

A.1.3 Definition and Role of the Common Term D in Simplifying Matrix Expressions

From equation (A.4) we take

$$
\begin{aligned}
|\Omega|^{-1} &= 1 + \frac{(\rho+\phi)^2(1-\phi^{2T})}{(1-\rho^2)(1-\phi^2)} \\
&= \frac{(1-\rho^2)(1-\phi^2) + (\rho+\phi)^2(1-\phi^{2T})}{(1-\rho^2)(1-\phi^2)}
\end{aligned} \tag{A.22}
$$

$$
|\Omega| = \frac{(1-\rho^2)(1-\phi^2)}{(1-\rho^2)(1-\phi^2) + (\rho+\phi)^2(1-\phi^{2T})}. \tag{A.23}
$$

Then we define

$$
\frac{1}{D} = |\Omega| \frac{1}{(1-\rho^2)(1-\phi^2)^2}. \tag{A.24}
$$

Using equations (A.23) and (A.24) we find

$$
\begin{aligned}
\frac{1}{D} &= \frac{(1-\rho^2)(1-\phi^2)}{[(1-\rho^2)(1-\phi^2) + (\rho+\phi)^2(1-\phi^{2T})](1-\rho^2)(1-\phi^2)^2} \\
&= \frac{(1-\rho^2)(1-\phi^2)}{[1-\rho^2-\phi^2+\rho^2\phi^2+(\rho+\phi)^2-(\rho+\phi)^2\phi^{2T}](1-\rho^2)(1-\phi^2)^2} \\
&= \frac{(1-\rho^2)(1-\phi^2)}{[1-\rho^2-\phi^2+\rho^2\phi^2+\rho^2+2\rho\phi+\phi^2-(\rho+\phi)^2\phi^{2T}](1-\rho^2)(1-\phi^2)^2} \\
&= \frac{(1-\rho^2)(1-\phi^2)}{[1+\rho^2\phi^2+2\rho\phi-(\rho+\phi)^2\phi^{2T}](1-\rho^2)(1-\phi^2)^2} \\
&= \frac{(1-\rho^2)(1-\phi^2)}{[(1+\rho\phi)^2-(\rho+\phi)^2\phi^{2T}](1-\rho^2)(1-\phi^2)^2} \Rightarrow \\
\frac{1}{D} &= \frac{1}{[(1+\rho\phi)^2-(\rho+\phi)^2\phi^{2T}](1-\phi^2)}.
\end{aligned} \tag{A.25}
$$

From equation (A.25) we take

$$
\begin{aligned}
D &= [(1+\rho\phi)^2-(\rho+\phi)^2\phi^{2T}](1-\phi^2) \\
&= (1+\rho\phi)^2-(\rho+\phi)^2\phi^{2T}-(1+\rho\phi)^2\phi^2+(\rho+\phi)^2\phi^{2T+2}.
\end{aligned} \tag{A.26}
$$

Using equations (A.1), (A.24) we find

$$\begin{aligned}\omega_{tt} &= \frac{1}{D}[(1+\rho\phi)^2(1+\rho^2+2\rho\phi) \\ &+(\phi+\rho)^2\{(\phi+\rho)+\rho(1+\rho\phi)\}\phi^{2T-1} \\ &-(\phi+\rho)^2(1+\rho\phi)^2\{\phi^{2(t-1)}+\phi^{2(T-t)}\}].\end{aligned} \tag{A.27}$$

Using equations (A.2), (A.24) we find

$$\begin{aligned}\omega_{tt'} &= \frac{1}{D}[-(\phi+\rho)(1+\rho\phi)^3(-\phi)^{|t-t'|-1} \\ &-(\phi+\rho)^3(1+\rho\phi)(-\phi)^{2T-|t-t'|-1} \\ &-(\phi+\rho)^2(1+\rho\phi)^2\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\}].\end{aligned} \tag{A.28}$$

Appendix B

First- and second-order derivatives of the elements of Ω with respect to ρ and ϕ

Initially, all the necessary individual components and their corresponding derivatives are computed in order to subsequently assemble the first- and second-order derivatives of the elements of the matrix Ω with respect to ρ and ϕ. This approach follows a modular strategy, given the extensive and complex nature of the calculations, which necessitate their division into manageable and distinct stages.

B.1 Derivatives

We define ω_{tt} as follows:

$$\omega_{tt} = D^{-1}N, \tag{B.1}$$

where

$$D = [(1+\rho\phi)^2 - (\rho+\phi)^2\phi^{2T}][1-\phi^2] \tag{B.2}$$

and

$$\begin{aligned} N &= (1+\rho\phi)^2(1+\rho^2+2\rho\phi) + (\phi+\rho)^2\left\{(\phi+\rho)+\rho(1+\rho\phi)\right\}\phi^{2T-1} \\ &\quad -(\phi+\rho)^2(1+\rho\phi)^2\left\{\phi^{2(t-1)}+\phi^{2(T-t)}\right\}. \end{aligned} \tag{B.3}$$

Also, we define $\omega_{tt'}$ as follows:

$$\omega_{tt'} = D^{-1}N_*, \tag{B.4}$$

where

$$\begin{aligned} N_* &= -(\phi+\rho)(1+\rho\phi)^3(-\phi)^{|t-t'|-1} - (\phi+\rho)^3(1+\rho\phi)(-\phi)^{2T-|t-t'|-1} \\ &\quad -(\phi+\rho)^2(1+\rho\phi)^2\left\{(-\phi)^{t+t'-2}+(-\phi)^{2T-(t+t')}\right\}. \end{aligned} \tag{B.5}$$

Using equation (B.2) we can write D as follows:

$$\begin{aligned} D &= (1+\rho\phi)^2 - (\rho+\phi)^2\phi^{2T} - (1+\rho\phi)^2\phi^2 + (\rho+\phi)^2\phi^{2T+2} \\ &= D_1 - D_2 - D_3 + D_4, \end{aligned} \tag{B.6}$$

where

$$\begin{aligned} D_1 &= (1+\rho\phi)^2, \\ D_2 &= (\rho+\phi)^2\phi^{2T}, \\ D_3 &= (1+\rho\phi)^2\phi^2, \\ D_4 &= (\rho+\phi)^2\phi^{2T+2}. \end{aligned} \tag{B.7}$$

Using equation (B.3) we can write N as follows:

$$\begin{aligned} N &= (1+\rho\phi)^2(1+\rho^2+2\rho\phi)+(\phi+\rho)^3\phi^{2T-1}+(\phi+\rho)^2\rho(1+\rho\phi)\phi^{2T-1} \\ &\quad -(\phi+\rho)^2(1+\rho\phi)^2\phi^{2(t-1)}-(\phi+\rho)^2(1+\rho\phi)^2\phi^{2(T-t)} \\ &= N_1+N_2+N_3-N_4-N_5, \end{aligned} \tag{B.8}$$

where

$$\begin{aligned} N_1 &= (1+\rho\phi)^2(1+\rho^2+2\rho\phi), \\ N_2 &= (\phi+\rho)^3\phi^{2T-1}, \\ N_3 &= (\phi+\rho)^2\rho(1+\rho\phi)\phi^{2T-1}, \\ N_4 &= (\phi+\rho)^2(1+\rho\phi)^2\phi^{2(t-1)}, \\ N_5 &= (\phi+\rho)^2(1+\rho\phi)^2\phi^{2(T-t)}. \end{aligned} \tag{B.9}$$

Using equation (B.5) we can write N_* as follows:

$$\begin{aligned} N_* &= -(\phi+\rho)(1+\rho\phi)^3(-\phi)^{|t-t'|-1}-(\phi+\rho)^3(1+\rho\phi)(-\phi)^{2T-|t-t'|-1} \\ &\quad -(\phi+\rho)^2(1+\rho\phi)^2(-\phi)^{t+t'-2}-(\phi+\rho)^2(1+\rho\phi)^2(-\phi)^{2T-(t+t')} \\ &= -N_{1*}-N_{2*}-N_{3*}-N_{4*}, \end{aligned} \tag{B.10}$$

where

$$\begin{aligned} N_{1*} &= (\phi+\rho)(1+\rho\phi)^3(-\phi)^{|t-t'|-1}, \\ N_{2*} &= (\phi+\rho)^3(1+\rho\phi)(-\phi)^{2T-|t-t'|-1}, \\ N_{3*} &= (\phi+\rho)^2(1+\rho\phi)^2(-\phi)^{t+t'-2}, \\ N_{4*} &= (\phi+\rho)^2(1+\rho\phi)^2(-\phi)^{2T-(t+t')}. \end{aligned} \tag{B.11}$$

B.1.1 Derivatives of D

Derivatives of D_1

Equation (B.7) implies that

$$D_1 = (1 + \rho\phi)^2. \tag{B.12}$$

Then,

$$\frac{\partial D_1}{\partial \rho} = D_{1\rho} = 2(1 + \rho\phi)\phi, \tag{B.13}$$

$$\frac{\partial^2 D_1}{\partial \rho^2} = D_{1\rho\rho} = 2\phi\phi = 2\phi^2 \tag{B.14}$$

$$\frac{\partial D_1}{\partial \phi} = D_{1\phi} = 2(1 + \rho\phi)\rho, \tag{B.15}$$

$$\frac{\partial^2 D_1}{\partial \phi^2} = D_{1\phi\phi} = 2\rho\rho = 2\rho^2, \tag{B.16}$$

$$\begin{aligned} \frac{\partial^2 D_1}{\partial \rho \partial \phi} = D_{1\rho\phi} &= \frac{\partial}{\partial \phi}\{2(1 + \rho\phi)\phi\} = 2(1 + \rho\phi)'\phi + 2(1 + \rho\phi)\phi' \\ &= 2\rho\phi + 2(1 + \rho\phi) = 2\rho\phi + 2 + 2\rho\phi = 2 + 4\rho\phi \\ &= 2(1 + 2\rho\phi), \end{aligned} \tag{B.17}$$

and

$$\begin{aligned} \frac{\partial^2 D_1}{\partial \phi \partial \rho} = D_{1\phi\rho} &= \frac{\partial}{\partial \rho}\{2(1 + \rho\phi)\rho\} = 2(1 + \rho\phi)'\rho + 2(1 + \rho\phi)\rho' \\ &= 2\rho\phi + 2(1 + \rho\phi) = 2\rho\phi + 2 + 2\rho\phi = 2 + 4\rho\phi \\ &= 2(1 + 2\rho\phi). \end{aligned} \tag{B.18}$$

Derivatives of D_2

Equation (B.7) implies that

$$D_2 = (\rho + \phi)^2\phi^{2T}. \tag{B.19}$$

Then,

$$\frac{\partial D_2}{\partial \rho} = D_{2\rho} = 2(\rho + \phi)\phi^{2T}, \tag{B.20}$$

$$\frac{\partial^2 D_2}{\partial \rho^2} = D_{2\rho\rho} = 2\phi^{2T}, \tag{B.21}$$

$$\begin{aligned}\frac{\partial D_2}{\partial \phi} = D_{2\phi} &= 2(\rho+\phi)\phi^{2T} + (\rho+\phi)^2 2T\phi^{2T-1} \\ &= 2(\rho+\phi)\phi^{2T} + 2T(\rho+\phi)^2\phi^{2T-1} \\ &= 2(\rho+\phi)\phi^{2T-1}[\phi + T(\rho+\phi)] \\ &= 2(\rho+\phi)\phi^{2T} + 2T(\rho+\phi)^2\phi^{2T-1},\end{aligned} \tag{B.22}$$

$$\begin{aligned}\frac{\partial^2 D_2}{\partial \phi^2} = D_{2\phi\phi} &= 2\phi^{2T} + 2(\rho+\phi)2T\phi^{2T-1} \\ &\quad +2T2(\rho+\phi)\phi^{2T-1} + 2T(\rho+\phi)^2(2T-1)\phi^{2T-2} \\ &= 2\phi^{2T} + 4T(\rho+\phi)\phi^{2T-1} + 4T(\rho+\phi)\phi^{2T-1} \\ &\quad +2T(2T-1)(\rho+\phi)^2\phi^{2(T-1)} \\ &= 2\phi^{2T} + 8T(\rho+\phi)\phi^{2T-1} + 2T(2T-1)(\rho+\phi)^2\phi^{2(T-1)},\end{aligned} \tag{B.23}$$

$$\begin{aligned}\frac{\partial^2 D_2}{\partial \rho \partial \phi} = D_{2\rho\phi} &= \frac{\partial}{\partial \phi}\left\{2(\rho+\phi)\phi^{2T}\right\} = 2\phi^{2T} + 2(\rho+\phi)2T\phi^{2T-1} \\ &= 2\phi^{2T} + 4T(\rho+\phi)\phi^{2T-1},\end{aligned} \tag{B.24}$$

and

$$\begin{aligned}\frac{\partial^2 D_2}{\partial \phi \partial \rho} = D_{2\phi\rho} &= \frac{\partial}{\partial \rho}\left\{2(\rho+\phi)\phi^{2T} + 2T(\rho+\phi)^2\phi^{2T-1}\right\} \\ &= 2\phi^{2T} + 2T2(\rho+\phi)\phi^{2T-1} \\ &= 2\phi^{2T} + 4T(\rho+\phi)\phi^{2T-1}.\end{aligned} \tag{B.25}$$

Derivatives of D_3

Equation (B.7) implies that

$$D_3 = (1+\rho\phi)^2\phi^2. \tag{B.26}$$

Then,

$$\frac{\partial D_3}{\partial \rho} = D_{3\rho} = 2(1+\rho\phi)\phi\phi^2 = 2\phi^3(1+\rho\phi), \tag{B.27}$$

$$\frac{\partial^2 D_3}{\partial \rho^2} = D_{3\rho\rho} = 2\phi^3\phi = 2\phi^4, \tag{B.28}$$

$$\begin{aligned}\frac{\partial D_3}{\partial \phi} = D_{3\phi} &= 2(1+\rho\phi)\rho\phi^2 + (1+\rho\phi)^2 2\phi \\ &= 2(1+\rho\phi)\phi[\rho\phi + (1+\rho\phi)],\end{aligned} \tag{B.29}$$

$$\begin{aligned}\frac{\partial^2 D_3}{\partial \phi^2} = D_{3\phi\phi} &= 2\rho^2\phi^2 + 2(1+\rho\phi)2\phi\rho + 2(1+\rho\phi)2\rho\phi + (1+\rho\phi)^2 2 \\ &= 2\rho^2\phi^2 + 4\phi\rho(1+\rho\phi) + 4\rho\phi(1+\rho\phi) + 2(1+\rho\phi)^2 \\ &= 2\rho^2\phi^2 + 8\phi\rho(1+\rho\phi) + 2(1+\rho\phi)^2,\end{aligned} \tag{B.30}$$

$$\begin{aligned}\frac{\partial^2 D_3}{\partial \rho \partial \phi} = D_{3\rho\phi} &= \frac{\partial}{\partial \phi}\left\{2\phi^3(1+\rho\phi)\right\} = 6\phi^2(1+\rho\phi) + 2\phi^3\rho \\ &= 6\phi^2 + 6\rho\phi^3 + 2\phi^3\rho \\ &= 6\phi^2 + 8\rho\phi^3,\end{aligned} \tag{B.31}$$

and

$$\begin{aligned}\frac{\partial^2 D_3}{\partial \phi \partial \rho} = D_{3\phi\rho} &= \frac{\partial}{\partial \rho}\left\{2(1+\rho\phi)\rho\phi^2 + (1+\rho\phi)^2 2\phi\right\} \\ &= 2\phi\rho\phi^2 + 2(1+\rho\phi)\phi^2 + 2(1+\rho\phi)\phi 2\phi \\ &= 2\rho\phi^3 + 2\phi^2(1+\rho\phi) + 4\phi^2(1+\rho\phi) \\ &= 2\rho\phi^3 + 6\phi^2(1+\rho\phi) \\ &= 2\rho\phi^3 + 6\phi^2 + 6\rho\phi^3 \\ &= 6\phi^2 + 8\rho\phi^3.\end{aligned} \tag{B.32}$$

Derivatives of D_4

Equation (B.7) implies that

$$D_4 = (\rho + \phi)^2\phi^{2T+2}. \tag{B.33}$$

Then,

$$\frac{\partial D_4}{\partial \rho} = D_{4\rho} = 2(\rho + \phi)\phi^{2T+2}, \tag{B.34}$$

$$\frac{\partial^2 D_4}{\partial \rho^2} = D_{4\rho\rho} = 2\phi^{2T+2}, \tag{B.35}$$

$$\frac{\partial D_4}{\partial \phi} = D_{4\phi} = 2(\rho+\phi)\phi^{2T+2} + (\rho+\phi)^2(2T+2)\phi^{2T+1}, \tag{B.36}$$

$$\begin{aligned}
\frac{\partial^2 D_4}{\partial \phi^2} = D_{4\phi\phi} &= 2\phi^{2T+2} + 2(\rho+\phi)(2T+2)\phi^{2T+1} \\
&\quad +(2T+2)2(\rho+\phi)\phi^{2T+1} + (2T+2)(\rho+\phi)^2(2T+1)\phi^{2T} \\
&= 2\phi^{2T+2} + 4(\rho+\phi)(T+1)\phi^{2T+1} \\
&\quad +4(T+1)(\rho+\phi)\phi^{2T+1} + 2(T+1)(\rho+\phi)^2(2T+1)\phi^{2T} \\
&= 2\phi^{2T+2} + 8(T+1)(\rho+\phi)\phi^{2T+1} \\
&\quad +2(T+1)(\rho+\phi)^2(2T+1)\phi^{2T},
\end{aligned} \tag{B.37}$$

$$\begin{aligned}
\frac{\partial^2 D_4}{\partial \rho \partial \phi} = D_{4\rho\phi} &= \frac{\partial}{\partial \phi}\left\{2(\rho+\phi)\phi^{2T+2}\right\} \\
&= 2\phi^{2T+2} + 2(\rho+\phi)(2T+2)\phi^{2T+1} \\
&= 2\phi^{2T+2} + 4(\rho+\phi)(T+1)\phi^{2T+1} \\
&= 2\phi^{2(T+1)} + 4(\rho+\phi)(T+1)\phi^{2T+1},
\end{aligned} \tag{B.38}$$

and

$$\begin{aligned}
\frac{\partial^2 D_4}{\partial \phi \partial \rho} = D_{4\phi\rho} &= \frac{\partial}{\partial \rho}\left\{2(\rho+\phi)\phi^{2T+2} + (\rho+\phi)^2(2T+2)\phi^{2T+1}\right\} \\
&= 2\phi^{2T+2} + 2(\rho+\phi)2(T+1)\phi^{2T+1} \\
&= 2\phi^{2T+2} + 4(\rho+\phi)(T+1)\phi^{2T+1}.
\end{aligned} \tag{B.39}$$

B.1.2 Derivatives of N

Derivatives of N_1

Equation (B.9) implies that

$$N_1 = (1+\rho\phi)^2(1+\rho^2+2\rho\phi). \tag{B.40}$$

Then,

$$\begin{aligned}
\frac{\partial N_1}{\partial \rho} = N_{1\rho} &= 2(1+\rho\phi)\phi(1+\rho^2+2\rho\phi)+(1+\rho\phi)^2(2\rho+2\phi) \\
&= (1+\rho\phi)[2\phi(1+\rho^2+2\rho\phi)+(1+\rho\phi)(2\rho+2\phi)] \\
&= (1+\rho\phi)[2\phi+2\rho^2\phi+4\rho\phi^2+2\rho+2\phi+2\rho^2\phi+2\rho\phi^2] \\
&= (1+\rho\phi)[2\rho+4\phi+4\rho^2\phi+6\rho\phi^2],
\end{aligned} \tag{B.41}$$

$$\frac{\partial^2 N_1}{\partial \rho^2} = N_{1\rho\rho} = \phi[2\rho+4\phi+4\rho^2\phi+6\rho\phi^2]+(1+\rho\phi)[2+8\rho\phi+6\phi^2], \tag{B.42}$$

$$\begin{aligned}
\frac{\partial N_1}{\partial \phi} = N_{1\phi} &= 2(1+\rho\phi)\rho(1+\rho^2+2\rho\phi)+(1+\rho\phi)^2(2\rho) \\
&= (1+\rho\phi)[2\rho+2\rho^3+4\rho^2\phi+2\rho+2\rho^2\phi] \\
&= (1+\rho\phi)[4\rho+2\rho^3+6\rho^2\phi],
\end{aligned} \tag{B.43}$$

$$\frac{\partial^2 N_1}{\partial \phi^2} = N_{1\phi\phi} = \rho[4\rho+2\rho^3+6\rho^2\phi]+(1+\rho\phi)6\rho^2, \tag{B.44}$$

$$\begin{aligned}
\frac{\partial^2 N_1}{\partial \rho\partial \phi} = N_{1\rho\phi} &= \frac{\partial}{\partial \phi}\left\{(1+\rho\phi)[2\rho+4\phi+4\rho^2\phi+6\rho\phi^2]\right\} \\
&= \rho[2\rho+4\phi+4\rho^2\phi+6\rho\phi^2]+(1+\rho\phi)[4+4\rho^2+12\rho\phi] \\
&= 2\rho^2+4\rho\phi+4\rho^3\phi+6\rho^2\phi^2+4+4\rho^2+12\rho\phi+4\rho\phi \\
&\quad +4\rho^3\phi+12\rho^2\phi^2 \\
&= 6\rho^2+20\rho\phi+8\rho^3\phi+18\rho^2\phi^2+4,
\end{aligned} \tag{B.45}$$

and

$$\begin{aligned}
\frac{\partial^2 N_1}{\partial \phi\partial \rho} = N_{1\phi\rho} &= \frac{\partial}{\partial \rho}\left\{(1+\rho\phi)[4\rho+2\rho^3+6\rho^2\phi]\right\} \\
&= \phi[4\rho+2\rho^3+6\rho^2\phi]+(1+\rho\phi)[4+6\rho^2+12\rho\phi] \\
&= 4\rho\phi+2\rho^3\phi+6\rho^2\phi^2+(1+\rho\phi)[4+6\rho^2+12\rho\phi] \\
&= 4\rho\phi+2\rho^3\phi+6\rho^2\phi^2+4+6\rho^2+12\rho\phi+4\rho\phi \\
&\quad +6\rho^3\phi+12\rho^2\phi^2 \\
&= 6\rho^2+20\rho\phi+8\rho^3\phi+18\rho^2\phi^2+4.
\end{aligned} \tag{B.46}$$

Derivatives of N_2

Equation (B.9) implies that

$$N_2 = (\phi+\rho)^3\phi^{2T-1}. \tag{B.47}$$

Then,

$$\frac{\partial N_2}{\partial \rho} = N_{2\rho} = 3(\phi+\rho)^2\phi^{2T-1}, \tag{B.48}$$

$$\frac{\partial^2 N_2}{\partial \rho^2} = N_{2\rho\rho} = 6(\phi+\rho)\phi^{2T-1}, \tag{B.49}$$

$$\begin{aligned}\frac{\partial N_2}{\partial \phi} = N_{2\phi} &= 3(\phi+\rho)^2\phi^{2T-1} + (\phi+\rho)^3(2T-1)\phi^{2T-2} \\ &= (\phi+\rho)^2[3\phi^{2T-1} + (\phi+\rho)(2T-1)\phi^{2T-2}],\end{aligned} \tag{B.50}$$

$$\begin{aligned}\frac{\partial^2 N_2}{\partial \phi^2} = N_{2\phi\phi} &= 2(\phi+\rho)[3\phi^{2T-1} + (\phi+\rho)(2T-1)\phi^{2T-2}] \\ &\quad +(\phi+\rho)^2[3(2T+1)\phi^{2T} + (2T-1)\phi^{2T-2} \\ &\quad +(\phi+\rho)(2T-1)(2T-2)\phi^{2T-3}],\end{aligned} \tag{B.51}$$

$$\begin{aligned}\frac{\partial^2 N_2}{\partial \rho \partial \phi} = N_{2\rho\phi} &= \frac{\partial}{\partial \phi}\left\{3(\phi+\rho)^2\phi^{2T-1}\right\} \\ &= 6(\phi+\rho)\phi^{2T-1} + 3(\phi+\rho)^2(2T-1)\phi^{2T-2},\end{aligned} \tag{B.52}$$

and

$$\begin{aligned}\frac{\partial^2 N_2}{\partial \phi \partial \rho} = N_{2\phi\rho} &= \frac{\partial}{\partial \rho}\left\{(\phi+\rho)^2[3\phi^{2T-1} + (\phi+\rho)(2T-1)\phi^{2T-2}]\right\} \\ &= 2(\phi+\rho)[3\phi^{2T-1} + (\phi+\rho)(2T-1)\phi^{2T-2}] \\ &\quad +(\phi+\rho)^2(2T-1)\phi^{2T-2} \\ &= 6(\phi+\rho)\phi^{2T-1} + 2(\phi+\rho)^2(2T-1)\phi^{2T-2} \\ &\quad +(\phi+\rho)^2(2T-1)\phi^{2T-2} \\ &= 6(\phi+\rho)\phi^{2T-1} + 3(\phi+\rho)^2(2T-1)\phi^{2T-2}.\end{aligned} \tag{B.53}$$

Derivatives of N_3

Equation (B.9) implies that

$$N_3 = (\phi + \rho)^2(\rho\phi^{2T-1} + \rho^2\phi^{2T}). \tag{B.54}$$

Then,

$$\frac{\partial N_3}{\partial \rho} = N_{3\rho} = 2(\phi + \rho)(\rho\phi^{2T-1} + \rho^2\phi^{2T}) + (\phi + \rho)^2(\phi^{2T-1} + 2\rho\phi^{2T}), \tag{B.55}$$

$$\begin{aligned}\frac{\partial^2 N_3}{\partial \rho^2} = N_{3\rho\rho} \quad &= \quad 2(\rho\phi^{2T-1} + \rho^2\phi^{2T}) + 2(\phi + \rho)(\phi^{2T-1} + 2\rho\phi^{2T}) \\ &\quad +2(\phi + \rho)(\phi^{2T-1} + 2\rho\phi^{2T}) + (\phi + \rho)^2(2\phi^{2T}),\end{aligned} \tag{B.56}$$

$$\begin{aligned}\frac{\partial N_3}{\partial \phi} = N_{3\phi} \quad &= \quad 2(\phi + \rho)(\rho\phi^{2T-1} + \rho^2\phi^{2T}) \\ &\quad +(\phi + \rho)^2[\rho(2T-1)\phi^{2T-2} + \rho^2 2T\phi^{2T-1}],\end{aligned} \tag{B.57}$$

$$\begin{aligned}\frac{\partial^2 N_3}{\partial \phi^2} = N_{3\phi\phi} \quad &= \quad 2(\rho\phi^{2T-1} + \rho^2\phi^{2T}) \\ &\quad +2(\phi + \rho)[\rho(2T-1)\phi^{2T-2} + \rho^2 2T\phi^{2T-1}] \\ &\quad +2[\rho(2T-1)\phi^{2T-2} + \rho^2 2T\phi^{2T-1}](\phi + \rho) \\ &\quad +(\phi + \rho)^2[\rho(2T-1)(2T-2)\phi^{2T-3} \\ &\quad +\rho^2 2T(2T-1)\phi^{2T-2}],\end{aligned} \tag{B.58}$$

$$\begin{aligned}\frac{\partial^2 N_3}{\partial \rho \partial \phi} = N_{3\rho\phi} \quad &= \quad \frac{\partial}{\partial \phi}\left\{2(\phi + \rho)(\rho\phi^{2T-1} + \rho^2\phi^{2T}) + (\phi + \rho)^2(\phi^{2T-1} + 2\rho\phi^{2T})\right\} \\ &= \quad 2(\rho\phi^{2T-1} + \rho^2\phi^{2T}) \\ &\quad +2(\phi + \rho)[\rho(2T-1)\phi^{2T-2} + \rho^2 2T\phi^{2T-1}] \\ &\quad +2(\phi + \rho)(\phi^{2T-1} + 2\rho\phi^{2T}) \\ &\quad +(\phi + \rho)^2[(2T-1)\phi^{2T-2} + 4T\rho\phi^{2T-1}],\end{aligned} \tag{B.59}$$

and

$$\begin{aligned}
\frac{\partial^2 N_3}{\partial\phi\partial\rho} = N_{3\phi\rho} &= \frac{\partial}{\partial\rho}\{2(\phi+\rho)(\rho\phi^{2T-1}+\rho^2\phi^{2T}) \\
&\quad +(\phi+\rho)^2[\rho(2T-1)\phi^{2T-2}+\rho^2 2T\phi^{2T-1}]\} \\
&= 2(\rho\phi^{2T-1}+\rho^2\phi^{2T})+2(\phi+\rho)(\phi^{2T-1}+2\rho\phi^{2T}) \\
&\quad +2(\phi+\rho)[\rho(2T-1)\phi^{2T-2}+\rho^2 2T\phi^{2T-1}] \\
&\quad +(\phi+\rho)^2[(2T-1)\phi^{2T-2}+4T\rho\phi^{2T-1}]. \qquad \text{(B.60)}
\end{aligned}$$

Derivatives of N_4

Equation (B.9) implies that

$$N_4 = (\phi+\rho)^2(1+\rho\phi)^2\phi^{2(t-1)}. \qquad \text{(B.61)}$$

Then,

$$\begin{aligned}
\frac{\partial N_4}{\partial\rho} = N_{4\rho} &= [2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\phi]\phi^{2(t-1)} \\
&= 2(\phi+\rho)(1+\rho\phi)^2\phi^{2t-2}+(\phi+\rho)^2(2\phi+2\rho\phi^2)\phi^{2t-2}, \qquad \text{(B.62)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 N_4}{\partial\rho^2} = N_{4\rho\rho} &= 2\phi^{2t-2}[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\phi] \\
&\quad +\phi^{2t-2}[2(\phi+\rho)(2\phi+2\rho\phi^2)+(\phi+\rho)^2(2\phi^2)], \qquad \text{(B.63)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial N_4}{\partial\phi} = N_{4\phi} &= [2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\rho]\phi^{2(t-1)} \\
&\quad +(\phi+\rho)^2(1+\rho\phi)^2 2(t-1)\phi^{2t-3} \\
&= 2(\phi+\rho)(1+\rho\phi)^2\phi^{2(t-1)}+2(\phi+\rho)^2(1+\rho\phi)\rho\phi^{2t-2} \\
&\quad +2(t-1)(\phi+\rho)^2(1+\rho\phi)^2\phi^{2t-3}, \qquad \text{(B.64)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 N_4}{\partial\phi^2} = N_{4\phi\phi} &= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\rho]\phi^{2(t-1)} \\
&\quad +2(\phi+\rho)(1+\rho\phi)^2 2(t-1)\phi^{2t-3} \\
&\quad +2[2(\phi+\rho)(1+\rho\phi)+(\phi+\rho)^2\rho]\phi^{2t-2}\rho \\
&\quad +2(\phi+\rho)^2(1+\rho\phi)(2t-2)\phi^{2t-3}\rho \\
&\quad +2(t-1)[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\rho]\phi^{2t-3} \\
&\quad +2(t-1)(\phi+\rho)^2(1+\rho\phi)^2(2t-3)\phi^{2t-4}, \qquad \text{(B.65)}
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 N_4}{\partial\rho\partial\phi} = N_{4\rho\phi} &= \frac{\partial}{\partial\phi}\left\{2(\phi+\rho)(1+\rho\phi)^2\phi^{2t-2} + (\phi+\rho)^2(2\phi+2\rho\phi^2)\phi^{2t-2}\right\} \\ &= 2[(1+\rho\phi)^2 + (\phi+\rho)2(1+\rho\phi)\rho]\phi^{2t-2} \\ &\quad +2(\phi+\rho)(1+\rho\phi)^2(2t-2)\phi^{2t-3} \\ &\quad +[2(\phi+\rho)(2\phi+2\rho\phi^2) + (\phi+\rho)^2(2+4\rho\phi)]\phi^{2t-2} \\ &\quad +(\phi+\rho)^2(2\phi+2\rho\phi^2)(2t-2)\phi^{2t-3} \\ &= 2(1+\rho\phi)^2\phi^{2t-2} + 2(\phi+\rho)2(1+\rho\phi)\rho\phi^{2t-2} \\ &\quad +2(\phi+\rho)(1+\rho\phi)^2(2t-2)\phi^{2t-3} \\ &\quad +2\phi^{2t-2}(\phi+\rho)(2\phi+2\rho\phi^2) + (\phi+\rho)^2(2+4\rho\phi)\phi^{2t-2} \\ &\quad +(\phi+\rho)^2(2\phi+2\rho\phi^2)(2t-2)\phi^{2t-3}, \end{aligned} \tag{B.66}$$

and

$$\begin{aligned}\frac{\partial^2 N_4}{\partial\phi\partial\rho} = N_{4\phi\rho} &= \frac{\partial}{\partial\rho}\{2(\phi+\rho)(1+\rho\phi)^2\phi^{2(t-1)} + 2(\phi+\rho)^2(\rho+\rho^2\phi)\phi^{2t-2}] \\ &\quad +2(t-1)(\phi+\rho)^2(1+\rho\phi)^2\phi^{2t-3}\} \\ &= 2\phi^{2(t-1)}[(1+\rho\phi)^2 + (\phi+\rho)2(1+\rho\phi)\phi \\ &\quad +2\phi^{2t-2}[2(\phi+\rho)(\rho+\rho^2\phi) + (\phi+\rho)^2(1+2\rho\phi)] \\ &\quad +2(t-1)\phi^{2t-3}[2(\phi+\rho)(1+\rho\phi)^2 + (\phi+\rho)^2 2(1+\rho\phi)\phi] \\ &= 2\phi^{2t-2}(1+\rho\phi)^2 + 2\phi^{2t-2}(\phi+\rho)(2\phi+2\rho\phi^2) \\ &\quad +2\phi^{2t-2}2(\phi+\rho)\rho(1+\rho\phi) + \phi^{2t-2}(\phi+\rho)^2(2+4\rho\phi) \\ &\quad +(2t-2)\phi^{2t-3}2(\phi+\rho)(1+\rho\phi)^2 \\ &\quad +(2t-2)\phi^{2t-3}(\phi+\rho)^2 2(1+\rho\phi)\phi. \end{aligned} \tag{B.67}$$

Derivatives of N_5

Equation (B.9) implies that

$$N_5 = (\phi+\rho)^2(1+\rho\phi)^2\phi^{2(T-t)}. \tag{B.68}$$

Then,

$$\begin{aligned}\frac{\partial N_5}{\partial\rho} = N_{5\rho} &= [2(\phi+\rho)(1+\rho\phi)^2 + (\phi+\rho)^2 2(1+\rho\phi)\phi]\phi^{2(T-t)} \\ &= 2(\phi+\rho)(1+\rho\phi)^2\phi^{2T-2t} + (\phi+\rho)^2(2\phi+2\rho\phi^2)\phi^{2T-2t}, \end{aligned} \tag{B.69}$$

$$\begin{aligned}\frac{\partial^2 N_5}{\partial\rho^2} = N_{5\rho\rho} &= 2\phi^{2T-2t}[(1+\rho\phi)^2 + (\phi+\rho)2(1+\rho\phi)\phi] \\ &\quad +\phi^{2T-2t}[2(\phi+\rho)(2\phi+2\rho\phi^2) + (\phi+\rho)^2(2\phi^2)], \end{aligned} \tag{B.70}$$

$$\begin{aligned}\frac{\partial N_5}{\partial\phi}=N_{5\phi} &= [2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\rho]\phi^{2(T-t)}\\ &\quad +(\phi+\rho)^2(1+\rho\phi)^2 2(T-t)\phi^{2T-2t-1}\\ &= 2(\phi+\rho)(1+\rho\phi)^2\phi^{2(T-t)}+2(\phi+\rho)^2(1+\rho\phi)\rho\phi^{2(T-t)}\\ &\quad +2(T-t)(\phi+\rho)^2(1+\rho\phi)^2\phi^{2T-2t-1}, \end{aligned} \tag{B.71}$$

$$\begin{aligned}\frac{\partial^2 N_5}{\partial\phi^2}=N_{5\phi\phi} &= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\rho]\phi^{2(T-t)}\\ &\quad +2(\phi+\rho)(1+\rho\phi)^2 2(T-t)\phi^{2(T-t)-1}\\ &\quad +2[2(\phi+\rho)(1+\rho\phi)+(\phi+\rho)^2\rho]\phi^{2T-2t}\rho\\ &\quad +2(\phi+\rho)^2(1+\rho\phi)(2T-2t)\phi^{2T-2t-1}\rho\\ &\quad +2(T-t)[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\rho]\phi^{2T-2t-1}\\ &\quad +2(T-t)(\phi+\rho)^2(1+\rho\phi)^2(2T-2t-1)\phi^{2T-2t-2}, \end{aligned} \tag{B.72}$$

$$\begin{aligned}\frac{\partial^2 N_5}{\partial\rho\partial\phi}=N_{5\rho\phi} &= \frac{\partial}{\partial\phi}\left\{2(\phi+\rho)(1+\rho\phi)^2\phi^{2T-2t}+(\phi+\rho)^2(2\phi+2\rho\phi^2)\phi^{2T-2t}\right\}\\ &= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\rho]\phi^{2T-2t}\\ &\quad +2(\phi+\rho)(1+\rho\phi)^2(2T-2t)\phi^{2T-2t-1}\\ &\quad +[2(\phi+\rho)(2\phi+2\rho\phi^2)+(\phi+\rho)^2(2+4\rho\phi)]\phi^{2T-2t}\\ &\quad +(\phi+\rho)^2(2\phi+2\rho\phi^2)(2T-2t)\phi^{2T-2t-1}\\ &= 2\phi^{2T-2t}(1+\rho\phi)^2+2(\phi+\rho)2(1+\rho\phi)\rho\phi^{2T-2t}\\ &\quad +2(\phi+\rho)(1+\rho\phi)^2(2T-2t)\phi^{2T-2t-1}\\ &\quad +2\phi^{2T-2t}(\phi+\rho)(2\phi+2\rho\phi^2)+(\phi+\rho)^2(2+4\rho\phi)\phi^{2T-2t}\\ &\quad +(\phi+\rho)^2(2\phi+2\rho\phi^2)(2T-2t)\phi^{2T-2t-1}, \end{aligned} \tag{B.73}$$

and

$$\begin{aligned}\frac{\partial^2 N_5}{\partial\phi\partial\rho}=N_{5\phi\rho} &= \frac{\partial}{\partial\rho}\{2(\phi+\rho)(1+\rho\phi)^2\phi^{2(T-t)}+2(\phi+\rho)^2(\rho+\rho^2\phi)\phi^{2(T-t)}\\ &\quad +2(T-t)(\phi+\rho)^2(1+\rho\phi)^2\phi^{2T-2t-1}\}\\ &= 2\phi^{2(T-t)}[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\phi]\\ &\quad +2\phi^{2(T-t)}[2(\phi+\rho)(\rho+\rho^2\phi)+(\phi+\rho)^2(1+2\rho\phi)]\\ &\quad +2(T-t)\phi^{2T-2t-1}[2(\phi+\rho)(1+\rho\phi)^2 \end{aligned}$$

$$
\begin{aligned}
& +(\phi+\rho)^2 2(1+\rho\phi)\phi] \\
= \; & 2\phi^{2T-2t}(1+\rho\phi)^2 + 2\phi^{2T-2t}(\phi+\rho)2(1+\rho\phi)\phi \\
& +2\phi^{2T-2t}2(\phi+\rho)(\rho+\rho^2\phi) + 2\phi^{2T-2t}(\phi+\rho)^2(1+2\rho\phi) \\
& +(2T-2t)\phi^{2T-2t-1}2(\phi+\rho)(1+\rho\phi)^2 \\
& +(2T-2t)\phi^{2T-2t-1}(\phi+\rho)^2 2(1+\rho\phi)\phi. \qquad \text{(B.74)}
\end{aligned}
$$

B.1.3 Derivatives of N_*

Derivatives of N_{1*}

Equation (B.11) implies that

$$N_{1*} = (\phi+\rho)(1+\rho\phi)^3(-\phi)^{|t-t'|-1}. \qquad \text{(B.75)}$$

For

$$t' > t, \text{ we have } t - t' < 0, \; t' - t > 0, \qquad \text{(B.76)}$$

and

$$N_{1*} = (\phi+\rho)(1+\rho\phi)^3(-\phi)^{(t'-t)-1}. \qquad \text{(B.77)}$$

Then,

$$
\begin{aligned}
\frac{\partial N_{1*}}{\partial\rho} = N_{1*\rho} \; = \; & [(1+\rho\phi)^3 + (\phi+\rho)3(1+\rho\phi)^2\phi](-\phi)^{(t'-t)-1} \\
= \; & [(1+\rho\phi)^3 + (3\phi^2+3\rho\phi)(1+\rho\phi)^2](-\phi)^{(t'-t)-1}, \qquad \text{(B.78)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 N_{1*}}{\partial\rho^2} = N_{1*\rho\rho} \; = \; & 6\phi^3(-\phi)^{t'-t-1} + 6\rho\phi^4(-\phi)^{t'-t-1} + 6\phi(-\phi)^{t'-t-1} \\
& +18\rho\phi^2(-\phi)^{t'-t-1} + 12\rho^2\phi^3(-\phi)^{t'-t-1}, \qquad \text{(B.79)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial N_{1*}}{\partial\phi} = N_{1*\phi} \; = \; & [(1+\rho\phi)^3 + (\phi+\rho)3(1+\rho\phi)^2\rho](-\phi)^{(t'-t)-1} \\
& +(\phi+\rho)(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2}(-1) \\
= \; & [(1+\rho\phi)^3 + (\phi+\rho)3(1+\rho\phi)^2\rho](-\phi)^{(t'-t)-1} \\
& -(\phi+\rho)(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2}, \qquad \text{(B.80)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 N_{1*}}{\partial \phi^2} = N_{1*\phi\phi} &= 3(1+\rho\phi)^2\rho(-\phi)^{(t'-t)-1} \\
&+(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2}(-1) \\
&+[3\rho(1+\rho\phi)^2+3\rho(\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{(t'-t)-1} \\
&+(\phi+\rho)3\rho(1+\rho\phi)^2[(t'-t)-1](-\phi)^{(t'-t)-2}(-1) \\
&+[(t'-t)-1][(1+\rho\phi)^3(-\phi)^{(t'-t)-2}(-1) \\
&+(\phi+\rho)3(1+\rho\phi)^2\rho(-\phi)^{(t'-t)-2}(-1)] \\
&+(-1)(\phi+\rho)(1+\rho\phi)^3[(t'-t)-1] \\
&\times[(t'-t)-2](-\phi)^{(t'-t)-3}(-1) \\
&= 3(1+\rho\phi)^2\rho(-\phi)^{(t'-t)-1} \\
&-(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2} \\
&+[3\rho(1+\rho\phi)^2+3\rho(\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{(t'-t)-1} \\
&-(\phi+\rho)3\rho(1+\rho\phi)^2[(t'-t)-1](-\phi)^{(t'-t)-2} \\
&+[(t'-t)-1][-(1+\rho\phi)^3(-\phi)^{(t'-t)-2} \\
&-(\phi+\rho)3(1+\rho\phi)^2\rho(-\phi)^{(t'-t)-2}] \\
&+(\phi+\rho)(1+\rho\phi)^3[(t'-t)-1] \\
&\times[(t'-t)-2](-\phi)^{(t'-t)-3}, \qquad \text{(B.81)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 N_{1*}}{\partial \rho \partial \phi} = N_{1*\rho\phi} &= \frac{\partial}{\partial \phi}\{(1+\rho\phi)^3(-\phi)^{(t'-t)-1} \\
&+(3\phi^2+3\rho\phi)(1+\rho\phi)^2(-\phi)^{(t'-t)-1}\} \\
&= 3(1+\rho\phi)^2\rho(-\phi)^{(t'-t)-1} \\
&+(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2}(-1) \\
&+(6\phi+3\rho)(1+\rho\phi)^2(-\phi)^{(t'-t)-1} \\
&+(3\phi^2+3\rho\phi)2(1+\rho\phi)\rho(-\phi)^{(t'-t)-1} \\
&+(3\phi^2+3\rho\phi)(1+\rho\phi)^2[(t'-t)-1](-\phi)^{(t'-t)-2}(-1) \\
&= 3(1+\rho\phi)^2\rho(-\phi)^{(t'-t)-1} \\
&-(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2} \\
&+(6\phi+3\rho)(1+\rho\phi)^2(-\phi)^{(t'-t)-1} \\
&+(3\phi^2+3\rho\phi)2(1+\rho\phi)\rho(-\phi)^{(t'-t)-1} \\
&-(3\phi^2+3\rho\phi)(1+\rho\phi)^2[(t'-t)-1](-\phi)^{(t'-t)-2}, \qquad \text{(B.82)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 N_{1*}}{\partial\phi\partial\rho} = N_{1*\phi\rho} \quad &= \quad \frac{\partial}{\partial\rho}\{[(1+\rho\phi)^3 + (3\rho\phi+3\rho^2)(1+\rho\phi)^2](-\phi)^{(t'-t)-1} \\
&\quad +(\phi+\rho)(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2}(-1)\} \\
&= \quad 3(1+\rho\phi)^2\phi(-\phi)^{(t'-t)-1} \\
&\quad +(3\phi+6\rho)(1+\rho\phi)^2(-\phi)^{(t'-t)-1} \\
&\quad +(3\rho\phi+3\rho^2)2(1+\rho\phi)\phi(-\phi)^{(t'-t)-1} \\
&\quad +[(1+\rho\phi)^3+(\phi+\rho)3(1+\rho\phi)^2\phi] \\
&\quad \times[(t'-t)-1](-\phi)^{(t'-t)-2}(-1) \\
&= \quad 3(1+\rho\phi)^2\phi(-\phi)^{(t'-t)-1} \\
&\quad +(3\phi+6\rho)(1+\rho\phi)^2(-\phi)^{(t'-t)-1} \\
&\quad +(3\rho\phi+3\rho^2)2(1+\rho\phi)\phi(-\phi)^{(t'-t)-1} \\
&\quad +(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2}(-1) \\
&\quad +(\phi+\rho)3(1+\rho\phi)^2\phi[(t'-t)-1](-\phi)^{(t'-t)-2}(-1) \\
&= \quad 3(1+\rho\phi)^2\phi(-\phi)^{(t'-t)-1} \\
&\quad +(3\phi+6\rho)(1+\rho\phi)^2(-\phi)^{(t'-t)-1} \\
&\quad +(3\rho\phi+3\rho^2)2(1+\rho\phi)\phi(-\phi)^{(t'-t)-1} \\
&\quad -(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2} \\
&\quad -(\phi+\rho)3(1+\rho\phi)^2\phi[(t'-t)-1](-\phi)^{(t'-t)-2} \\
&= \quad 3(1+\rho\phi)^2\phi(-\phi)^{(t'-t)-1} \\
&\quad +3\phi(1+\rho\phi)^2(-\phi)^{(t'-t)-1} \\
&\quad +6\rho(1+\rho\phi)^2(-\phi)^{(t'-t)-1} \\
&\quad +\rho(3\phi^2+3\rho\phi)2(1+\rho\phi)(-\phi)^{(t'-t)-1} \\
&\quad -(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2} \\
&\quad -(3\phi^2+\rho\phi)(1+\rho\phi)^2[(t'-t)-1](-\phi)^{(t'-t)-2} \\
&= \quad 3(1+\rho\phi)^2\rho(-\phi)^{(t'-t)-1} \\
&\quad -(1+\rho\phi)^3[(t'-t)-1](-\phi)^{(t'-t)-2} \\
&\quad +(6\phi+3\rho)(1+\rho\phi)^2(-\phi)^{(t'-t)-1} \\
&\quad +(3\phi^2+3\rho\phi)2(1+\rho\phi)\rho(-\phi)^{(t'-t)-1} \\
&\quad -(3\phi^2+3\rho\phi)(1+\rho\phi)^2[(t'-t)-1](-\phi)^{(t'-t)-2}. \qquad \text{(B.83)}
\end{aligned}
$$

Derivatives of N_{2*}

Equation (B.11) implies that

$$N_{2*} = (\phi+\rho)^3(1+\rho\phi)(-\phi)^{2T-|t-t'|-1}. \tag{B.84}$$

For

$$t' > t,\ \ t-t' < 0,\ \ t'-t > 0 \tag{B.85}$$

and

$$N_{2*} = (\phi+\rho)^3(1+\rho\phi)(-\phi)^{2T-(t'-t)-1}. \tag{B.86}$$

Then,

$$\frac{\partial N_{2*}}{\partial \rho} = N_{2*\rho} = [3(\phi+\rho)^2(1+\rho\phi)+(\phi+\rho)^3\phi](-\phi)^{2T-(t'-t)-1}, \tag{B.87}$$

$$\frac{\partial^2 N_{2*}}{\partial \rho^2} = N_{2*\rho\rho} = [6(\phi+\rho)(1+\rho\phi)+3(\phi+\rho)^2\phi+3(\phi+\rho)^2\phi](-\phi)^{2T-(t'-t)-1}, \tag{B.88}$$

$$\begin{aligned}\frac{\partial N_{2*}}{\partial \phi} = N_{2*\phi} &= [3(\phi+\rho)^2(1+\rho\phi)+(\phi+\rho)^3\rho](-\phi)^{2T-(t'-t)-1}\\ &\quad +(\phi+\rho)^3(1+\rho\phi)[2T-(t'-t)-1](-\phi)^{2T-(t'-t)-2}(-1)\\ &= [3(\phi+\rho)^2(1+\rho\phi)+(\phi+\rho)^3\rho](-\phi)^{2T-(t'-t)-1}\\ &\quad -(\phi+\rho)^3(1+\rho\phi)[2T-(t'-t)-1](-\phi)^{2T-(t'-t)-2},\end{aligned} \tag{B.89}$$

$$\begin{aligned}\frac{\partial^2 N_{2*}}{\partial \phi^2} = N_{2*\phi\phi} &= 6\phi(-\phi)^{2T-t'+t-1} - 6\phi^2(2T-t'+t-1)(-\phi)^{2T-t'+t-2}\\ &\quad +\phi^3(2T-t'+t-1)(2T-t'+t-2)(-\phi)^{2T-t'+t-3}\\ &\quad +6\rho(-\phi)^{2T-(t'-t)-1} - 12\rho\phi(2T-t'+t-1)(-\phi)^{2T-t'+t-2}\\ &\quad +3\rho\phi^2(2T-t'+t-1)(2T-t'+t-2)(-\phi)^{2T-t'+t-3}\\ &\quad -6\rho^2(2T-t'+t-1)(-\phi)^{2T-t'+t-2}\\ &\quad +3\rho^2\phi(2T-t'+t-1)(2T-t'+t-2)(-\phi)^{2T-t'+t-3}\\ &\quad +\rho^3(2T-t'+t-1)(2T-t'+t-2)(-\phi)^{2T-t'+t-3}\\ &\quad +12\rho\phi^2(-\phi)^{2T-t'+t-1} - 8\rho\phi^3(2T-t'+t-1)(-\phi)^{2T-t'+t-2}\\ &\quad +\rho\phi^4(2T-t'+t-1)(2T-t'+t-2)(-\phi)^{2T-t'+t-3}\\ &\quad +18\rho^2\phi(-\phi)^{2T-t'+t-1} - 18\rho^2\phi^2(2T-t'+t-1)(-\phi)^{2T-t'+t-2}\\ &\quad +3\rho^2\phi^3(2T-t'+t-1)(2T-t'+t-2)(-\phi)^{2T-t'+t-3}\end{aligned}$$

$$
\begin{aligned}
&+6\rho^3(-\phi)^{2T-t'+t-1} - 12\rho^3\phi(2T-t'+t-1)(-\phi)^{2T-t'+t-2} \\
&+3\rho^3\phi^2(2T-t'+t-1)(2T-t'+t-2)(-\phi)^{2T-t'+t-3} \\
&-\rho^4(2T-t'+t-1)(-\phi)^{2T-t'+t-2} \\
&+\rho^4\phi(2T-t'+t-1)(2T-t'+t-2)(-\phi)^{2T-t'+t-3}, \qquad \text{(B.90)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 N_{2*}}{\partial\rho\partial\phi} = N_{2*\rho\phi} &= \frac{\partial}{\partial\phi}\{3(\phi+\rho)^2(1+\rho\phi)(-\phi)^{2T-(t'-t)-1} \\
&\quad +(\phi+\rho)^3\phi(-\phi)^{2T-(t'-t)-1}\} \\
&= [6(\phi+\rho)(1+\rho\phi)+3(\phi+\rho)^2\rho](-\phi)^{2T-(t'-t)-1} \\
&\quad +3(\phi+\rho)^2(1+\rho\phi)[2T-(t'-t)-1](-\phi)^{2T-(t'-t)-2}(-1) \\
&\quad +[3(\phi+\rho)^2\phi+(\phi+\rho)^3](-\phi)^{2T-(t'-t)-1} \\
&\quad +(\phi+\rho)^3\phi[2T-(t'-t)-1](-\phi)^{2T-(t'-t)-2}(-1) \\
&= [6(\phi+\rho)(1+\rho\phi)+3(\phi+\rho)^2\rho](-\phi)^{2T-(t'-t)-1} \\
&\quad -3(\phi+\rho)^2(1+\rho\phi)[2T-(t'-t)-1](-\phi)^{2T-(t'-t)-2} \\
&\quad +[3(\phi+\rho)^2\phi+(\phi+\rho)^3](-\phi)^{2T-(t'-t)-1} \\
&\quad -(\phi+\rho)^3\phi[2T-(t'-t)-1](-\phi)^{2T-(t'-t)-2}, \qquad \text{(B.91)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 N_{2*}}{\partial\phi\partial\rho} = N_{2*\phi\rho} &= \frac{\partial}{\partial\rho}\{3(\phi+\rho)^2(1+\rho\phi)(-\phi)^{2T-(t'-t)-1} \\
&\quad +(\phi+\rho)^3\rho](-\phi)^{2T-(t'-t)-1} \\
&\quad +(\phi+\rho)^3(1+\rho\phi)[2T-(t'-t)-1] \\
&\quad \times(-\phi)^{2T-(t'-t)-2}(-1)\} \\
&= [6(\phi+\rho)(1+\rho\phi)+3(\phi+\rho)^2\phi](-\phi)^{2T-(t'-t)-1} \\
&\quad +3(\phi+\rho)^2\rho(-\phi)^{2T-(t'-t)-1}+(\phi+\rho)^3(-\phi)^{2T-(t'-t)-1} \\
&\quad +[3(\phi+\rho)^2(1+\rho\phi)+(\phi+\rho)^3\phi][2T-(t'-t)-1] \\
&\quad \times(-\phi)^{2T-(t'-t)-2}(-1) \\
&= [6(\phi+\rho)(1+\rho\phi)+3(\phi+\rho)^2\phi](-\phi)^{2T-(t'-t)-1} \\
&\quad +3(\phi+\rho)^2\rho(-\phi)^{2T-(t'-t)-1}+(\phi+\rho)^3(-\phi)^{2T-(t'-t)-1} \\
&\quad -[3(\phi+\rho)^2(1+\rho\phi)+(\phi+\rho)^3\phi][2T-(t'-t)-1] \\
&\quad \times(-\phi)^{2T-(t'-t)-2}. \qquad \text{(B.92)}
\end{aligned}
$$

Derivatives of N_{3*}

Equation (B.11) implies that

$$N_{3*} = (\phi+\rho)^2(1+\rho\phi)^2(-\phi)^{t+t'-2}. \tag{B.93}$$

Then,

$$\frac{\partial N_{3*}}{\partial \rho} = N_{3*\rho} = [2(\phi+\rho)(1+\rho\phi)^2 + (\phi+\rho)^2 2(1+\rho\phi)\phi](-\phi)^{t+t'-2}, \tag{B.94}$$

$$\begin{aligned}\frac{\partial^2 N_{3*}}{\partial \rho^2} = N_{3*\rho\rho} &= [2(1+\rho\phi)^2 + 2(\phi+\rho)2(1+\rho\phi)\phi \\ &\quad +2(\phi+\rho)2\phi(1+\rho\phi) + (\phi+\rho)^2 2\phi^2](-\phi)^{t+t'-2},\end{aligned} \tag{B.95}$$

$$\begin{aligned}\frac{\partial N_{3*}}{\partial \phi} = N_{3*\phi} &= [2(\phi+\rho)(1+\rho\phi)^2 + (\phi+\rho)^2 2(1+\rho\phi)\rho](-\phi)^{t+t'-2} \\ &\quad +(\phi+\rho)^2(1+\rho\phi)^2(t+t'-2)(-\phi)^{t+t'-3}(-1) \\ &= 2(\phi+\rho)(1+\rho\phi)^2(-\phi)^{t+t'-2} + (\phi+\rho)^2 2(1+\rho\phi)\rho(-\phi)^{t+t'-2} \\ &\quad +(\phi+\rho)^2(1+\rho\phi)^2(t+t'-2)(-\phi)^{t+t'-3}(-1) \\ &= 2(\phi+\rho)(1+\rho\phi)^2(-\phi)^{t+t'-2} + (\phi+\rho)^2 2(1+\rho\phi)\rho(-\phi)^{t+t'-2} \\ &\quad -(\phi+\rho)^2(1+\rho\phi)^2(t+t'-2)(-\phi)^{t+t'-3},\end{aligned} \tag{B.96}$$

$$\begin{aligned}\frac{\partial^2 N_{3*}}{\partial \phi^2} = N_{3*\phi\phi} &= 2[(1+\rho\phi)^2 + (\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{t+t'-2} \\ &\quad +2(\phi+\rho)(1+\rho\phi)^2(t+t'-2)(-\phi)^{t+t'-3}(-1) \\ &\quad +2\rho[2(\phi+\rho)(1+\rho\phi) + (\phi+\rho)^2\rho](-\phi)^{t+t'-2} \\ &\quad +2\rho(\phi+\rho)^2(1+\rho\phi)(t+t'-2)(-\phi)^{t+t'-3}(-1) \\ &\quad +(t+t'-2)(-1)[2(\phi+\rho)(1+\rho\phi)^2 + (\phi+\rho)^2 2(1+\rho\phi)\rho] \\ &\quad \times(-\phi)^{t+t'-3} \\ &\quad +(\phi+\rho)^2(1+\rho\phi)^2(t+t'-2)(-1)(t+t'-3) \\ &\quad \times(-\phi)^{t+t'-4}(-1) \\ &= 2[(1+\rho\phi)^2 + (\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{t+t'-2}\end{aligned}$$

$$-2(\phi+\rho)(1+\rho\phi)^2(t+t'-2)(-\phi)^{t+t'-3}$$
$$+2\rho[2(\phi+\rho)(1+\rho\phi)+(\phi+\rho)^2\rho](-\phi)^{t+t'-2}$$
$$-2\rho(\phi+\rho)^2(1+\rho\phi)(t+t'-2)(-\phi)^{t+t'-3}$$
$$-(t+t'-2)[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\rho]$$
$$\times(-\phi)^{t+t'-3}$$
$$+(\phi+\rho)^2(1+\rho\phi)^2(t+t'-2)(t+t'-3)(-\phi)^{t+t'-4}, \quad \text{(B.97)}$$

$$\begin{aligned}
\frac{\partial^2 N_{3*}}{\partial\rho\partial\phi} = N_{3*\rho\phi} &= \frac{\partial}{\partial\phi}\{[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2(2\phi+2\rho\phi^2)](-\phi)^{t+t'-2}\} \\
&= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{t+t'-2} \\
&\quad +2(\phi+\rho)(1+\rho\phi)^2(t+t'-2)(-\phi)^{t+t'-3}(-1) \\
&\quad +[2(\phi+\rho)(2\phi+2\rho\phi^2)+(\phi+\rho)^2(2+4\rho\phi)](-\phi)^{t+t'-2} \\
&\quad +(\phi+\rho)^2(2\phi+2\rho\phi^2)(t+t'-2)(-\phi)^{t+t'-3}(-1) \\
&= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{t+t'-2} \\
&\quad -2(\phi+\rho)(1+\rho\phi)^2(t+t'-2)(-\phi)^{t+t'-3} \\
&\quad +[2(\phi+\rho)(2\phi+2\rho\phi^2)+(\phi+\rho)^2(2+4\rho\phi)](-\phi)^{t+t'-2} \\
&\quad -(\phi+\rho)^2(2\phi+2\rho\phi^2)(t+t'-2)(-\phi)^{t+t'-3}, \qquad \text{(B.98)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 N_{3*}}{\partial\phi\partial\rho} = N_{3*\phi\rho} &= \frac{\partial}{\partial\rho}\{2(\phi+\rho)(1+\rho\phi)^2(-\phi)^{t+t'-2}+(\phi+\rho)^2(2\rho+2\rho^2\phi) \\
&\quad \times(-\phi)^{t+t'-2}+(\phi+\rho)^2(1+\rho\phi)^2(t+t'-2)(-\phi)^{t+t'-3}(-1)\} \\
&= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\phi](-\phi)^{t+t'-2} \\
&\quad +[2(\phi+\rho)(2\rho+2\rho^2\phi)+(\phi+\rho)^2(2+4\rho\phi)](-\phi)^{t+t'-2} \\
&\quad +[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\phi](t+t'-2) \\
&\quad \times(-\phi)^{t+t'-3}(-1) \\
&= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\phi](-\phi)^{t+t'-2} \\
&\quad +[2(\phi+\rho)(2\rho+2\rho^2\phi)+(\phi+\rho)^2(2+4\rho\phi)](-\phi)^{t+t'-2} \\
&\quad -[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\phi](t+t'-2) \\
&\quad \times(-\phi)^{t+t'-3}. \qquad \text{(B.99)}
\end{aligned}$$

Derivatives of N_{4*}

Equation (B.11) implies that

$$N_{4*} = (\phi+\rho)^2(1+\rho\phi)^2(-\phi)^{2T-(t+t')}. \tag{B.100}$$

Then,

$$\frac{\partial N_{4*}}{\partial \rho} = N_{4*\rho} = [2(\phi+\rho)(1+\rho\phi)^2 + (\phi+\rho)^2 2(1+\rho\phi)\phi](-\phi)^{2T-(t+t')}, \tag{B.101}$$

$$\begin{aligned}\frac{\partial^2 N_{4*}}{\partial \rho^2} = N_{4*\rho\rho} \quad = \quad & [2(1+\rho\phi)^2 + 2(\phi+\rho)2(1+\rho\phi)\phi \\ & +2(\phi+\rho)2\phi(1+\rho\phi) + (\phi+\rho)^2 2\phi^2](-\phi)^{2T-(t+t')},\end{aligned} \tag{B.102}$$

$$\begin{aligned}\frac{\partial N_{4*}}{\partial \phi} = N_{4*\phi} \quad = \quad & [2(\phi+\rho)(1+\rho\phi)^2 + (\phi+\rho)^2 2(1+\rho\phi)\rho](-\phi)^{2T-(t+t')} \\ & +(\phi+\rho)^2(1+\rho\phi)^2[2T-(t+t')](-\phi)^{2T-(t+t')-1}(-1) \\ = \quad & 2(\phi+\rho)(1+\rho\phi)^2(-\phi)^{2T-(t+t')} + (\phi+\rho)^2 2(1+\rho\phi)\rho(-\phi)^{2T-(t+t')} \\ & +(\phi+\rho)^2(1+\rho\phi)^2[2T-(t+t')](-\phi)^{2T-(t+t')-1}(-1) \\ = \quad & 2(\phi+\rho)(1+\rho\phi)^2(-\phi)^{2T-(t+t')} + (\phi+\rho)^2 2(1+\rho\phi)\rho(-\phi)^{2T-(t+t')} \\ & -(\phi+\rho)^2(1+\rho\phi)^2[2T-(t+t')](-\phi)^{2T-(t+t')-1},\end{aligned} \tag{B.103}$$

$$\begin{aligned}\frac{\partial^2 N_{4*}}{\partial \phi^2} = N_{4*\phi\phi} \quad = \quad & 2[(1+\rho\phi)^2 + (\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{2T-(t+t')} \\ & +2(\phi+\rho)(1+\rho\phi)^2[2T-(t+t')](-\phi)^{2T-(t+t')-1}(-1) \\ & +2\rho[2(\phi+\rho)(1+\rho\phi) + (\phi+\rho)^2\rho](-\phi)^{2T-(t+t')} \\ & +2\rho(\phi+\rho)^2(1+\rho\phi)[2T-(t+t')](-\phi)^{2T-(t+t')-1}(-1) \\ & +[2T-(t+t')](-1)[2(\phi+\rho)(1+\rho\phi)^2 + (\phi+\rho)^2 2(1+\rho\phi)\rho] \\ & \times(-\phi)^{2T-(t+t')-1} \\ & +(\phi+\rho)^2(1+\rho\phi)^2[2T-(t+t')](-1)[2T-(t+t')-1] \\ & \times(-\phi)^{2T-(t+t')-2}(-1) \\ = \quad & 2[(1+\rho\phi)^2 + (\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{2T-(t+t')} \\ & -2(\phi+\rho)(1+\rho\phi)^2[2T-(t+t')](-\phi)^{2T-(t+t')-1}\end{aligned}$$

$$
\begin{aligned}
&+2\rho[2(\phi+\rho)(1+\rho\phi)+(\phi+\rho)^2\rho](-\phi)^{2T-(t+t')} \\
&-2\rho(\phi+\rho)^2(1+\rho\phi)[2T-(t+t')](-\phi)^{2T-(t+t')-1} \\
&-[2T-(t+t')][2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\rho] \\
&\times(-\phi)^{2T-(t+t')-1} \\
&+(\phi+\rho)^2(1+\rho\phi)^2[2T-(t+t')][2T-(t+t')-1] \\
&\times(-\phi)^{2T-(t+t')-2},
\end{aligned}
\tag{B.104}
$$

$$
\begin{aligned}
\frac{\partial^2 N_{4*}}{\partial\rho\partial\phi}=N_{4*\rho\phi} &= \frac{\partial}{\partial\phi}\{[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2(2\phi+2\rho\phi^2)](-\phi)^{2T-(t+t')}\} \\
&= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{2T-(t+t')} \\
&\quad+2(\phi+\rho)(1+\rho\phi)^2[2T-(t+t')](-\phi)^{2T-(t+t')-1}(-1) \\
&\quad+[2(\phi+\rho)(2\phi+2\rho\phi^2)+(\phi+\rho)^2(2+4\rho\phi)](-\phi)^{2T-(t+t')} \\
&\quad+(\phi+\rho)^2(2\phi+2\rho\phi^2)[2T-(t+t')](-\phi)^{2T-(t+t')-1}(-1) \\
&= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\rho](-\phi)^{2T-(t+t')} \\
&\quad-2(\phi+\rho)(1+\rho\phi)^2[2T-(t+t')](-\phi)^{2T-(t+t')-1} \\
&\quad+[2(\phi+\rho)(2\phi+2\rho\phi^2)+(\phi+\rho)^2(2+4\rho\phi)](-\phi)^{2T-(t+t')} \\
&\quad-(\phi+\rho)^2(2\phi+2\rho\phi^2)[2T-(t+t')](-\phi)^{2T-(t+t')-1},
\end{aligned}
\tag{B.105}
$$

and

$$
\begin{aligned}
\frac{\partial^2 N_{4*}}{\partial\phi\partial\rho}=N_{4*\phi\rho} &= \frac{\partial}{\partial\rho}\{2(\phi+\rho)(1+\rho\phi)^2(-\phi)^{2T-(t+t')} \\
&\quad+(\phi+\rho)^2(2\rho+2\rho^2\phi)(-\phi)^{2T-(t+t')} \\
&\quad+(\phi+\rho)^2(1+\rho\phi)^2[2T-(t+t')](-\phi)^{2T-(t+t')-1}(-1)\} \\
&= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\phi](-\phi)^{2T-(t+t')} \\
&\quad+[2(\phi+\rho)(2\rho+2\rho^2\phi)+(\phi+\rho)^2(2+4\rho\phi)](-\phi)^{2T-(t+t')} \\
&\quad+[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\phi][2T-(t+t')] \\
&\quad\times(-\phi)^{2T-(t+t')-1}(-1) \\
&= 2[(1+\rho\phi)^2+(\phi+\rho)2(1+\rho\phi)\phi](-\phi)^{2T-(t+t')} \\
&\quad+[2(\phi+\rho)(2\rho+2\rho^2\phi)+(\phi+\rho)^2(2+4\rho\phi)](-\phi)^{2T-(t+t')} \\
&\quad-[2(\phi+\rho)(1+\rho\phi)^2+(\phi+\rho)^2 2(1+\rho\phi)\phi][2T-(t+t')] \\
&\quad\times(-\phi)^{2T-(t+t')-1}.
\end{aligned}
\tag{B.106}
$$

B.2 Derivatives of the Elements of Ω (ω_{tt} and $\omega_{tt'}$)

In this section, we present the explicit expressions for the first- and second-order derivatives of both the diagonal and non-diagonal elements of the matrix Ω with respect to ρ and ϕ. These expressions are obtained by systematically combining all intermediate quantities and derivative components computed in the preceding sections. The final formulas encapsulate the complete derivative structure of Ω and serve as the culmination of the analytical process developed above.

Equations (B.1) and (B.4) imply that

$$\begin{aligned} \omega_{tt} &= \frac{N}{D}, \\ \omega_{tt'} &= \frac{N_*}{D}. \end{aligned} \tag{B.107}$$

Then,

$$\frac{\partial \omega_{tt}}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{N}{D}\right) = \frac{DN_\rho - ND_\rho}{D^2}, \tag{B.108}$$

$$\begin{aligned} \frac{\partial^2 \omega_{tt}}{\partial \rho^2} &= \frac{\partial}{\partial \rho}\left\{\frac{DN_\rho - ND_\rho}{D^2}\right\} \\ &= \frac{D^2(D_\rho N_\rho + DN_{\rho\rho} - N_\rho D_\rho - ND_{\rho\rho}) - (DN_\rho - ND_\rho)2D_\rho D}{D^4} \\ &= \frac{D^2(DN_{\rho\rho} - ND_{\rho\rho}) - (DN_\rho D_\rho^2 - ND_\rho D_\rho^2)}{D^4}. \end{aligned} \tag{B.109}$$

Similarly,

$$\frac{\partial \omega_{tt}}{\partial \phi} = \frac{DN_\phi - ND_\phi}{D^2}, \tag{B.110}$$

$$\frac{\partial^2 \omega_{tt}}{\partial \phi^2} = \frac{D^2(DN_{\phi\phi} - ND_{\phi\phi}) - (2D^2N_\phi D_\phi - 2DND_\phi D_\phi)}{D^4}, \tag{B.111}$$

$$\begin{aligned} \frac{\partial^2 \omega_{tt}}{\partial \rho \partial \phi} &= \frac{\partial}{\partial \phi}\left\{\frac{DN_\rho - ND_\rho}{D^2}\right\} \\ &= \frac{D^2(D_\phi N_\rho + DN_{\rho\phi} - N_\phi D_\rho - ND_{\rho\phi}) - (DN_\rho - ND_\rho)2DD_\phi}{D^4}, \end{aligned} \tag{B.112}$$

$$\begin{aligned}\frac{\partial^2\omega_{tt}}{\partial\phi\partial\rho} &= \frac{\partial}{\partial\rho}\left\{\frac{DN_\phi - ND_\phi}{D^2}\right\}\\ &= \frac{D^2(D_\rho N_\phi + DN_{\phi\rho} - N_\rho D_\phi - ND_{\phi\rho}) - (DN_\phi - ND_\phi)2DD_\rho}{D^4}. \qquad \text{(B.113)}\end{aligned}$$

Also,

$$\frac{\partial\omega_{tt'}}{\partial\rho} = \frac{\partial}{\partial\rho}\left(\frac{N_*}{D}\right) = \frac{DN_{*\rho} - N_*D_\rho}{D^2}, \qquad \text{(B.114)}$$

$$\begin{aligned}\frac{\partial^2\omega_{tt'}}{\partial\rho^2} &= \frac{\partial}{\partial\rho}\left\{\frac{DN_{*\rho} - N_*D_\rho}{D^2}\right\}\\ &= \frac{D^2(D_\rho N_{*\rho} + DN_{*\rho\rho} - N_{*\rho}D_\rho - N_*D_{\rho\rho}) - (DN_{*\rho} - N_*D_\rho)2DD_\rho}{D^4}\\ &= \frac{D^2(DN_{*\rho\rho} - N_*D_{\rho\rho}) - (DN_{*\rho}D_\rho^2 - 2DN_*D_\rho D_\rho^2)}{D^4}. \qquad \text{(B.115)}\end{aligned}$$

Similarly,

$$\frac{\partial\omega_{tt'}}{\partial\phi} = \frac{DN_{*\phi} - N_*D_\phi}{D^2}, \qquad \text{(B.116)}$$

$$\frac{\partial^2\omega_{tt'}}{\partial\phi^2} = \frac{D^2(DN_{*\phi\phi} - N_*D_{\phi\phi}) - (2D^2N_{*\phi}D_\phi - 2DN_*D_\phi D_\phi)}{D^4}, \qquad \text{(B.117)}$$

$$\begin{aligned}\frac{\partial^2\omega_{tt'}}{\partial\rho\partial\phi} &= \frac{\partial}{\partial\phi}\left\{\frac{DN_{*\rho} - N_*D_\rho}{D^2}\right\}\\ &= \frac{D^2(D_\phi N_{*\rho} + DN_{*\rho\phi} - N_{*\phi}D_\rho - N_*D_{\rho\phi}) - (DN_{*\rho} - N_*D_\rho)2DD_\phi}{D^4}, \qquad \text{(B.118)}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\omega_{tt'}}{\partial\phi\partial\rho} &= \frac{\partial}{\partial\rho}\left\{\frac{DN_{*\phi} - N_*D_\phi}{D^2}\right\}\\ &= \frac{D^2(D_\rho N_{*\phi} + DN_{*\phi\rho} - N_{*\rho}D_\phi - N_*D_{\phi\rho}) - (DN_{*\phi} - N_*D_\phi)2DD_\rho}{D^4}. \qquad \text{(B.119)}\end{aligned}$$

Appendix C

Proofs of the computation of μ_0

In the case of the Generalized Linear Model with stochastic errors following an ARMA(1,1) process, we cannot use theoretical formulas to approximate the expected values of δ_ρ, δ_ϕ, δ_ρ^2, δ_ϕ^2 and $\delta_\rho\delta_\phi$. For this reason, we can use a simulation experiment to empirically compute these expected values.

Obviously, the approximation of the quantities of interest via simulation introduces an error whose order of magnitude, as we will see below, is $O(\tau)$. Due to the structure of our expansions, the total error will be of the order $O(\tau)^3$, which is acceptable based on the accuracy of the method.

In this Appendix, we provide the proofs of these results and the formulas with which we can approximate the quantities μ_ρ, μ_ϕ, $\lambda_{\rho\rho}$, $\lambda_{\phi\phi}$, and $\lambda_{\rho\phi}$. The quantities

$$\lambda_\rho = \mathbb{E}(\delta_\rho\delta_0) \tag{C.1}$$

$$\lambda_\phi = \mathbb{E}(\delta_\phi\delta_0) \tag{C.2}$$

are obtained from the internal simulation experiment conducted in each iteration of the overall experiment.

C.1 Proofs of the computation

$$\delta = \begin{bmatrix} \delta_0 \\ \delta_\rho \\ \delta_\phi \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \delta_* \end{bmatrix}, \tag{C.3}$$

where δ_* is a 2×1 vector with elements δ_ρ and δ_ϕ, and

$$\delta_0 = \frac{\hat{\sigma}^2 - \sigma^2}{\tau\sigma^2}, \tag{C.4}$$

$$\delta_\rho = \frac{\hat{\rho} - \rho}{\tau}, \tag{C.5}$$

$$\delta_\phi = \frac{\hat{\phi} - \phi}{\tau}. \tag{C.6}$$

Also, δ admits a stochastic expansion of the form

$$\delta = \begin{bmatrix} \delta_0 \\ \delta_\rho \\ \delta_\phi \end{bmatrix} = d_1 + \tau d_2 + \omega(\tau^2). \tag{C.7}$$

Let $\Omega = P'P$. Equation (3.1) can be transformed as follows:

$$Py = PX\beta + P\sigma u. \tag{C.8}$$

Since the matrix Ω is unknown, we must use $\hat{\Omega}$ instead of Ω, and by setting $\hat{\Omega} = \hat{P}'\hat{P}$, we write the transformed equation as follows:

$$\hat{P}y = \hat{P}X\beta + \hat{P}\sigma u. \tag{C.9}$$

Let $\hat{\beta}$ be the feasible GLS estimator of β.

$$\hat{\beta} = \beta + \tau\sigma(b + \tau b_*) \Rightarrow \tag{C.10}$$

$$\hat{\beta} - \beta = \tau\sigma(b + \tau b_*) \Rightarrow \tag{C.11}$$

$$\sqrt{T}(\hat{\beta} - \beta) = \sigma(b + \tau b_*) = \sigma b + \omega(\tau) \Rightarrow \tag{C.12}$$

$$\sigma b = \sqrt{T}(\hat{\beta} - \beta) + \omega(\tau) \Rightarrow \tag{C.13}$$

$$b = \frac{\sqrt{T}(\hat{\beta} - \beta)}{\sigma} + \omega(\tau) \Rightarrow \tag{C.14}$$

We define the $n \times 1$ vector

$$k = \frac{\sqrt{T}(\hat{\beta} - \beta)}{\sigma} = b + \omega(\tau) \tag{C.15}$$

$$k = \frac{\sqrt{T}(\hat{\beta} - \beta)}{\sigma} = \sqrt{T}(X'\hat{\Omega}X)^{-1}X'\hat{\Omega}u = (X'\hat{\Omega}X/T)^{-1}X'\hat{\Omega}u/\sqrt{T} \tag{C.16}$$

$$(X'\hat{\Omega}X/T)k = (X'\hat{\Omega}X/T)(X'\hat{\Omega}X/T)^{-1}X'\hat{\Omega}u/\sqrt{T} = X'\hat{\Omega}u/\sqrt{T} \tag{C.17}$$

From (3.1), (C.15), and (C.9), it follows that

$$\begin{aligned} \hat{u} &= \hat{P}y - \hat{P}X\hat{\beta} \\ &= \hat{P}(y - X\hat{\beta}) \\ &= \hat{P}(\sigma u + X\beta - X\hat{\beta}) \\ &= \hat{P}\sigma[u - \tau X(\hat{\beta} - \beta)/\tau\sigma] \\ &= \hat{P}\sigma(u - \tau X k) \end{aligned} \tag{C.18}$$

Therefore, using (C.17) and (C.18), we obtain

$$\begin{aligned}\hat{u}'\hat{u} &= \sigma^2[u'\hat{\Omega}u - k'(X'\hat{\Omega}X/T)k] \\ &= \sigma^2[u'\hat{\Omega}u - (b+\omega(\tau))'(X'\hat{\Omega}X/T)(b+\omega(\tau))] \\ &= \sigma^2[u'\hat{\Omega}u - b'(X'\hat{\Omega}X/T)b] + \omega(\tau) \qquad \text{(C.19)}\end{aligned}$$

$$\hat{\sigma}^2 = \hat{u}'\hat{u}/T = \sigma^2\{[u'\hat{\Omega}u - b'(X'\hat{\Omega}X/T)b] + \omega(\tau)\}/T \qquad \text{(C.20)}$$

$$\begin{aligned}\frac{\hat{\sigma}^2}{\sigma^2} &= [u'\hat{\Omega}u - b'(X'\hat{\Omega}X/T)b]/T + \omega(\tau^3) \\ &= (u'\hat{\Omega}u/T) - b'(X'\hat{\Omega}X/T)b/T + \omega(\tau^3) \Rightarrow \\ \frac{\hat{\sigma}^2}{\sigma^2} &= (u'\hat{\Omega}u/T) - b'Ab/T + \omega(\tau^3) \qquad \text{(C.21)}\end{aligned}$$

Since $\partial\Omega/\partial\rho = \Omega_\rho$, $\partial\Omega/\partial\phi = \Omega_\phi$, $\partial^2\Omega/\partial\rho^2 = \Omega_{\rho\rho}$, $\partial^2\Omega/\partial\phi^2 = \Omega_{\phi\phi}$ and $\partial^2\Omega/\partial\rho\partial\phi = \Omega_{\rho\phi}$, by performing a Taylor expansion of the quantity $u'\hat{\Omega}u/T$ around $u'\Omega u/T$, we obtain

$$\begin{aligned}u'\hat{\Omega}u/T &= u'\Omega u/T + \frac{\partial u'\Omega u}{\partial\rho}(\hat{\rho}-\rho)/T + \frac{\partial u'\Omega u}{\partial\phi}(\hat{\phi}-\phi)/T \\ &\quad + \frac{1}{2}\frac{\partial^2 u'\Omega u}{\partial\rho\partial\rho}(\hat{\rho}-\rho)^2/T + \frac{1}{2}\frac{\partial^2 u'\Omega u}{\partial\phi\partial\phi}(\hat{\phi}-\phi)^2/T \\ &\quad + \frac{\partial^2 u'\Omega u}{\partial\rho\partial\phi}(\hat{\rho}-\rho)(\hat{\phi}-\phi)/T \\ &= u'\Omega u/T + (u'\Omega_\rho u/T)\tau\left(\frac{\hat{\rho}-\rho}{\tau}\right) + (u'\Omega_\phi u/T)\tau\left(\frac{\hat{\phi}-\phi}{\tau}\right) \\ &\quad + \frac{1}{2}(u'\Omega_{\rho\rho}u/T)\tau^2\left(\frac{\hat{\rho}-\rho}{\tau}\right)^2 + \frac{1}{2}(u'\Omega_{\phi\phi}u/T)\tau^2\left(\frac{\hat{\phi}-\phi}{\tau}\right)^2 \\ &\quad + (u'\Omega_{\rho\phi}u/T)\tau^2\left(\frac{\hat{\rho}-\rho}{\tau}\right)\left(\frac{\hat{\phi}-\phi}{\tau}\right) \\ &= u'\Omega u/T + \tau(u'\Omega_\rho u/T)\delta_\rho + \tau(u'\Omega_\phi u/T)\delta_\phi + \frac{\tau^2}{2}(u'\Omega_{\rho\rho}u/T)\delta_\rho\delta_\rho \\ &\quad + \frac{\tau^2}{2}(u'\Omega_{\phi\phi}u/T)\delta_\phi\delta_\phi + \tau^2(u'\Omega_{\rho\phi}u/T)\delta_\rho\delta_\phi + \omega(\tau^3). \qquad \text{(C.22)}\end{aligned}$$

Analogously to the stochastic expansion (3.8), δ_0 has a stochastic expansion of the form

$$\delta_0 = \sigma_0 + \tau\sigma_1 + \omega(\tau^2) \qquad \text{(C.23)}$$

where σ_0 and σ_1 are the first elements of the vectors d_1 and d_2, respectively, from equation (3.8). We will now compute the quantities σ_0 and σ_1.

$$\mathsf{a}_i = -E(u'\Omega_i u/T) \tag{C.24}$$

$$\mathsf{a}_{ij} = \frac{1}{2}E(u'\Omega_{ij} u/T) \tag{C.25}$$

We also define the scalars

$$w_0 = \sqrt{T}(u'\Omega u/T - 1) \tag{C.26}$$

$$w_i = \sqrt{T}(u'\Omega_i u/T + \mathsf{a}_i) \tag{C.27}$$

$$w_{ij} = \sqrt{T}(u'\Omega_{ij} u/T - 2\mathsf{a}_{ij}), \tag{C.28}$$

and

$$\tau = \frac{1}{\sqrt{T}} \tag{C.29}$$

Using (C.22), (C.23), (C.24), (C.25), (C.26), (C.27), (C.28), and (C.29), it follows that

$$\begin{aligned}
u'\hat{\Omega}u/T &= 1 - \tau\sqrt{T} + \tau\sqrt{T}(u'\Omega u/T) + \tau^2\sqrt{T}(u'\Omega_\rho u/T + \mathsf{a})\delta_\rho - \tau\mathsf{a}_\rho\delta_\rho \\
&\quad + \tau^2\sqrt{T}(u'\Omega_\phi u/T + \mathsf{a})\delta_\phi - \tau\mathsf{a}_\phi\delta_\phi \\
&\quad + \frac{\tau^2}{2}2\mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \frac{\tau^3}{2}\sqrt{T}(u'\Omega_{\rho\rho}u/T - 2\mathsf{a}_{\rho\rho})\delta_\rho\delta_\rho \\
&\quad + \frac{\tau^2}{2}2\mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + \frac{\tau^3}{2}\sqrt{T}(u'\Omega_{\phi\phi}u/T - 2\mathsf{a}_{\phi\phi})\delta_\phi\delta_\phi \\
&\quad + \tau^2 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi + \tau^3\sqrt{T}(u'\Omega_{\rho\phi}u/T - 2\mathsf{a}_{\rho\phi})\delta_\rho\delta_\phi + \omega(\tau^3) \\
&= 1 + \tau w_0 + \tau^2 w_\rho\delta_\rho - \tau\mathsf{a}_\rho\delta_\rho + \tau^2 w_\phi\delta_\phi - \tau\mathsf{a}_\phi\delta_\phi \\
&\quad + \tau^2\mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \frac{\tau^3}{2}w_{\rho\rho}\delta_\rho\delta_\rho + \tau^2\mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + \frac{\tau^3}{2}w_{\phi\phi}\delta_\phi\delta_\phi \\
&\quad + \tau^2 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi + \tau^3 w_{\rho\phi}\delta_\rho\delta_\phi + \omega(\tau^3) \\
&= 1 + \tau(w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi) \\
&\quad + \tau^2(w_\rho\delta_\rho + w_\phi\delta_\phi + \mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi) + \omega(\tau^3)
\end{aligned} \tag{C.30}$$

From (C.21), it follows that

$$\hat{\sigma}^2 = \hat{u}'\hat{u}/(T-n) \quad \Leftrightarrow \quad (T-n)\hat{\sigma}^2 = \hat{u}'\hat{u} \Rightarrow \tag{C.31}$$

$$\hat{u}'\hat{u}/T = (T-n)\hat{\sigma}^2/T = \hat{\sigma}^2 - \hat{\sigma}^2 n/T = \hat{\sigma}^2 - \hat{\sigma}^2 n\tau^2. \tag{C.32}$$

Provided that

$$\hat{\sigma}^2 = \sigma^2 + \omega(\tau) \tag{C.33}$$

it follows that

$$\hat{\sigma}^2 n\tau^2 = (\sigma^2 + \omega(\tau))n\tau^2 \quad \Rightarrow \tag{C.34}$$

$$\hat{\sigma}^2 = \sigma^2 n\tau^2 + \omega(\tau^3). \tag{C.35}$$

Therefore,

$$\hat{u}'\hat{u}/T = \hat{\sigma}^2 - \hat{\sigma}^2 n\tau^2. \tag{C.36}$$

From relations (C.20), (C.23), (C.34), (C.35), and (C.36), and using the definition (3.5), we find that

$$\begin{aligned}
\hat{\sigma}^2 &= \hat{u}'\hat{u}/T + \hat{\sigma}^2 n\tau^2 \\
&= \sigma^2[u'\hat{\Omega}u/T - b'\hat{A}b/T + \omega(\tau^3)] + \hat{\sigma}^2 n\tau^2 \\
&= \sigma^2[1 + \tau(w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi) \\
&\quad + \tau^2(w_\rho\delta_\rho + w_\phi\delta_\phi + \mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi)] \\
&\quad + \tau^2\sigma^2(-b'\hat{A}b + n) + \omega(\tau^3) &\text{(C.37)} \\
\hat{\sigma}^2 &= \sigma^2[1 + \tau(w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi) \\
&\quad + \tau^2(w_\rho\delta_\rho + w_\phi\delta_\phi + \mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi - b'\hat{A}b + n)] \\
&\quad + \omega(\tau^3), &\text{(C.38)}
\end{aligned}$$

where

$$\delta_0 = \frac{\hat{\sigma}^2 - \sigma^2}{\tau\sigma^2}. \tag{C.39}$$

From (C.38) it follows that

$$\begin{aligned}
\delta_0 &= (w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi) \\
&\quad + \tau^2(w_\rho\delta_\rho + w_\phi\delta_\phi + \mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi - b'\hat{A}b + n) \\
&\quad + \omega(\tau^2) &\text{(C.40)}
\end{aligned}$$

and

$$\begin{aligned}
\sigma_0 &= w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi &\text{(C.41)} \\
\sigma_1 &= w_\rho\delta_\rho + w_\phi\delta_\phi + \mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi - b'\hat{A}b + n. &\text{(C.42)}
\end{aligned}$$

Following (Breusch 1980) and relations (3.8), (3.9), and (C.38) it follows that

$$\begin{bmatrix} \lambda_0 & \lambda' \\ \lambda & \Lambda \end{bmatrix} = \lim_{T\to\infty} \mathbb{E}(d_1 d_1') = \lim_{T\to\infty} \mathbb{E} \begin{bmatrix} \sigma_0^2 & \sigma_0 d_{1i}' \\ \sigma_0 d_{1i} & d_{1i} d_{1i}' \end{bmatrix} \tag{C.43}$$

where

$$\lambda_0 = \lim_{T\to\infty} \mathbb{E}(\sigma_0^2) = \lim_{T\to\infty} \mathbb{E}[(w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi)^2] \tag{C.44}$$

and also that

$$\delta = \begin{bmatrix}\delta_0\\ \delta_*\end{bmatrix} = \begin{bmatrix}\sigma_0 + \tau\sigma_1\\ d_{1i} - \tau d_{2i}\end{bmatrix} + \omega(\tau^2) = \begin{bmatrix}\sigma_0\\ d_{1i}\end{bmatrix} + \tau\begin{bmatrix}\sigma_1\\ -d_{2i}\end{bmatrix} + \omega(\tau^2) = d_1 + \tau d_2 + \omega(\tau^2) \tag{C.45}$$

$$\begin{bmatrix}\mu_0\\ \mu\end{bmatrix} = \lim_{T\to\infty} \mathbb{E}\begin{bmatrix}\sqrt{T}\sigma_0 + \sigma_1\\ \sqrt{T}d_{1i} - d_{2i}\end{bmatrix}. \tag{C.46}$$

Therefore,

$$\delta_i = d_{1i} - \tau d_{2i} + \omega(\tau^2) \Rightarrow \tag{C.47}$$

$$\sqrt{T}\delta_i = \sqrt{T}d_{1i} - d_{2i} + \omega(\tau) \Rightarrow \tag{C.48}$$

$$\lim_{T\to\infty} \sqrt{T}\delta_i = \lim_{T\to\infty} \sqrt{T}d_{1i} - d_{2i}. \tag{C.49}$$

Therefore, for the parameter ρ we have:

$$\delta_\rho = \frac{\hat{\rho} - \rho}{\tau} \Rightarrow \tag{C.50}$$

$$\sqrt{T}\delta_\rho = \frac{\hat{\rho} - \rho}{\tau^2}. \tag{C.51}$$

$$\hat{\rho} = \rho + \tau(\rho_1 + \tau\rho_2) + \omega(\tau^3) \Rightarrow \tag{C.52}$$

$$\hat{\rho} - \rho = \tau(\rho_1 + \tau\rho_2) + \omega(\tau^3) \Rightarrow \tag{C.53}$$

$$\delta_\rho = \frac{\hat{\rho} - \rho}{\tau} = (\rho_1 + \tau\rho_2) + \omega(\tau^2) \tag{C.54}$$

and

$$\begin{aligned}\delta_\rho\delta_\rho &= [(\rho_1 + \tau\rho_2) + \omega(\tau^2)][(\rho_1 + \tau\rho_2) + \omega(\tau^2)]\\ &= \rho_1^2 + \omega(\tau)\end{aligned} \tag{C.55}$$

and taking expected values we find:

$$\mathbb{E}(\delta_\rho \delta_\rho) = \mathbb{E}(\rho_1^2) + O(\tau) = \lambda_{\rho\rho*} \tag{C.56}$$

From (C.55) and (C.56), it is shown that theoretically the expected value of $\delta_\rho \delta_\rho$ is $\mathbb{E}(\rho_1^2) + O(\tau)$.

In the case where the stochastic terms follow an ARMA(1,1) process, we cannot use theoretical formulas to compute the expected value of $\delta_\rho \delta_\rho$. Instead, we use a simulation experiment from which the expected value $\mathbb{E}(\delta_\rho \delta_\rho)$ is computed "empirically". We denote by $\lambda_{\rho\rho}$ the theoretical value of $\mathbb{E}(\delta_\rho \delta_\rho)$. Correspondingly, we denote by $\lambda_{\rho\rho*}$ the empirical estimate of $\mathbb{E}(\delta_\rho \delta_\rho)$ from the Monte Carlo experiment. Therefore, from (C.56) it follows that using the empirical estimate $\lambda_{\rho\rho*}$ instead of the theoretical estimate $\lambda_{\rho\rho}$ introduces an error of order $O(\tau)$.

Since the term $\lambda_{\rho\rho*}$ in our expansions is multiplied by a coefficient τ^2, the total error in the expansion is of order $O(\tau^3)$, which is acceptable based on the accuracy of the method.

Therefore, for the parameter ϕ we have:

$$\delta_\phi = \frac{\hat{\phi} - \phi}{\tau} \Rightarrow \tag{C.57}$$

$$\sqrt{T}\delta_\phi = \frac{\hat{\phi} - \phi}{\tau^2} \tag{C.58}$$

Provided that

$$\hat{\phi} = \phi + \tau(\phi_1 + \tau\phi_2) + \omega(\tau^3) \Rightarrow \tag{C.59}$$

$$\hat{\phi} - \phi = \tau(\phi_1 + \tau\phi_2) + \omega(\tau^3) \Rightarrow \tag{C.60}$$

$$\delta_\phi = \frac{\hat{\phi} - \phi}{\tau} = (\phi_1 + \tau\phi_2) + \omega(\tau^2). \tag{C.61}$$

and

$$\begin{aligned} \delta_\phi \delta_\phi &= [(\phi_1 + \tau\phi_2) + \omega(\tau^2)][(\phi_1 + \tau\phi_2) + \omega(\tau^2)] \\ &= \phi_1^2 + \omega(\tau) \end{aligned} \tag{C.62}$$

and taking expected values we find:

$$\mathbb{E}(\delta_\phi \delta_\phi) = \mathbb{E}(\phi_1^2) + O(\tau) = \lambda_{\phi\phi*} \tag{C.63}$$

From (C.62) and (C.63), it is shown that theoretically the expected value of $\delta_\phi\delta_\phi$ is $\mathbb{E}(\phi_1^2) + O(\tau)$.

In the case where the stochastic terms follow an ARMA(1,1) process, we cannot use theoretical formulas to compute the expected value of $\delta_\phi\delta_\phi$. Instead, we use a simulation experiment from which the expected value $\mathbb{E}(\delta_\phi\delta_\phi)$ is computed "empirically". We denote by $\lambda_{\phi\phi}$ the theoretical value of $\mathbb{E}(\delta_\phi\delta_\phi)$. Correspondingly, we denote by $\lambda_{\phi\phi*}$ the empirical estimate of $\mathbb{E}(\delta_\phi\delta_\phi)$ from the Monte Carlo experiment. Therefore, from (C.63) it follows that using the empirical estimate $\lambda_{\phi\phi*}$ instead of the theoretical estimate $\lambda_{\phi\phi}$ introduces an error of order $O(\tau)$.

Since the term $\lambda_{\phi\phi*}$ in our expansions is multiplied by a coefficient τ^2, the total error in the expansion is of order $O(\tau^3)$, which is acceptable based on the accuracy of the method.

Using (C.54) and (C.61) we find that

$$\delta_\rho\delta_\phi = [(\rho_1 + \tau\rho_2) + \omega(\tau^2)][(\phi_1 + \tau\phi_2) + \omega(\tau^2)] \tag{C.64}$$

and taking expected values we find:

$$\mathbb{E}(\delta_\rho\delta_\phi) = \mathbb{E}(\rho_1\phi_1) + O(\tau) = \lambda_{\rho\phi*} \tag{C.65}$$

From (C.65), it is shown that theoretically the expected value of $\delta_\rho\delta_\phi$ is $\mathbb{E}(\rho_1\phi_1) + O(\tau)$.

In the case where the stochastic terms follow an ARMA(1,1) process, we cannot use theoretical formulas to compute the expected value of $\delta_\rho\delta_\phi$. Instead, we use a simulation experiment from which the expected value $\mathbb{E}(\delta_\rho\delta_\phi)$ is computed "empirically". We denote by $\lambda_{\rho\phi}$ the theoretical value of $\mathbb{E}(\delta_\rho\delta_\phi)$. Correspondingly, we denote by $\lambda_{\rho\phi*}$ the empirical estimate of $\mathbb{E}(\delta_\rho\delta_\phi)$ from the Monte Carlo experiment. Therefore, from (C.65) it follows that using the empirical estimate $\lambda_{\rho\phi*}$ instead of the theoretical estimate $\lambda_{\rho\phi}$ introduces an error of order $O(\tau)$.

Since the term $\lambda_{\rho\phi*}$ in our expansions is multiplied by a coefficient τ^2, the total error in the expansion is of order $O(\tau^3)$, which is acceptable based on the accuracy of the method.

Based on (3.11), (C.56), (C.62), and (C.65), it follows that instead of the matrix Λ we will use the matrix Λ_*.

$$\Lambda_* = \begin{bmatrix} \lambda_{\rho\rho*} & \lambda_{\rho\phi*} \\ \lambda_{\phi\rho*} & \lambda_{\phi\phi*} \end{bmatrix} \tag{C.66}$$

$$\Lambda_* = \Lambda + O(\tau) \tag{C.67}$$

$$\mu_i = \lim_{T\to\infty} \mathbb{E}(\sqrt{T}d_{1i} - d_{2i}) = \lim_{T\to\infty} \mathbb{E}(\sqrt{T}\delta_i) \tag{C.68}$$

$$\begin{aligned}\mu_\rho &= \lim_{T\to\infty}\mathbb{E}(\sqrt{T}d_{1\rho}-d_{2\rho}) = \lim_{T\to\infty}\mathbb{E}(\sqrt{T}\delta_\rho)\\ &= \lim_{T\to\infty}\mathbb{E}(\frac{\hat{\rho}-\rho}{\tau^2})\end{aligned} \tag{C.69}$$

$$\frac{\hat{\rho}-\rho}{\tau^2} = \frac{\tau(\rho_1+\tau\rho_2)+\omega(\tau^3)}{\tau^2} = \frac{(\rho_1+\tau\rho_2)+\omega(\tau^2)}{\tau} = \sqrt{T}\rho_1+\rho_2+\omega(\tau) \tag{C.70}$$

and taking expected value we find:

$$\mu_{\rho*} = \mathbb{E}(\sqrt{T}\rho_1+\rho_2+\omega(\tau)) = \mathbb{E}(\sqrt{T}\rho_1+\rho_2)+O(\tau) = \mu_\rho+O(\tau) \tag{C.71}$$

From (C.69), (C.70), and (C.71), it follows that theoretically $\mu_\rho = \sqrt{T}\delta_\rho$. In the case where the stochastic terms follow an ARMA(1,1) process, we cannot use theoretical formulas to compute the expected value of δ_ρ. Instead, we use a simulation experiment from which $\mathbb{E}(\delta_\rho)$ is computed "empirically". We denote by μ_ρ the theoretical $\mathbb{E}(\delta_\rho)$. Correspondingly, we denote by $\mu_{\rho*}$ the empirical estimate of $\mathbb{E}(\delta_\rho)$ from the Monte Carlo experiment. Therefore, from (C.71) it follows that using the empirical estimate $\mu_{\rho*}$ instead of the theoretical estimate μ_ρ introduces an error of order $O(\tau)$.

Since the term $\mu_{\rho*}$ in our expansions is multiplied by a coefficient $\tau^2/2$, the total error in the expansion is of order $O(\tau^3)$, which is acceptable based on the accuracy of the method.

$$\begin{aligned}\mu_\phi &= \lim_{T\to\infty}\mathbb{E}(\sqrt{T}d_{1\phi}-d_{2\phi}) = \lim_{T\to\infty}\mathbb{E}(\sqrt{T}\delta_\phi)\\ &= \lim_{T\to\infty}\mathbb{E}(\frac{\hat{\phi}-\phi}{\tau^2})\end{aligned} \tag{C.72}$$

$$\frac{\hat{\phi}-\phi}{\tau^2} = \frac{\tau(\phi_1+\tau\phi_2)+\omega(\tau^3)}{\tau^2} = \frac{(\phi_1+\tau\phi_2)+\omega(\tau^2)}{\tau} = \sqrt{T}\phi_1+\phi_2+\omega(\tau) \tag{C.73}$$

and taking expected value we find:

$$\mu_{\phi*} = \mathbb{E}(\sqrt{T}\phi_1+\phi_2+\omega(\tau)) = \mathbb{E}(\sqrt{T}\phi_1+\phi_2)+O(\tau) = \mu_\phi+O(\tau) \tag{C.74}$$

From (C.72), (C.73), and (C.74), it follows that theoretically $\mu_\phi = \sqrt{T}\delta_\phi$. In the case where the stochastic terms follow an ARMA(1,1) process, we cannot use theoretical formulas to compute the expected value of δ_ϕ. Instead, we use a simulation experiment from which $\mathbb{E}(\delta_\phi)$ is computed "empirically". We denote by μ_ϕ the theoretical $\mathbb{E}(\delta_\phi)$. Correspondingly, we denote by $\mu_{\phi*}$ the empirical estimate of $\mathbb{E}(\delta_\phi)$ from the Monte Carlo experiment. Therefore, from (C.74) it follows that using the empirical estimate $\mu_{\phi*}$ instead of the theoretical estimate μ_ϕ introduces an error of order $O(\tau)$.

Since the term $\mu_{\phi*}$ in our expansions is multiplied by a coefficient $\tau^2/2$, the total error in the expansion is of order $O(\tau^3)$, which is acceptable based on the accuracy of the method.

For μ_0 we can use theoretical formulas according to the following method. Theoretically, μ_0 is given by the formula:

$$\mu_0 = \lim_{T\to\infty} \mathbb{E}(\sqrt{T}\sigma_0 + \sigma_1). \tag{C.75}$$

Provided that

$$\sigma_0 = w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi, \tag{C.76}$$

$$\sigma_1 = w_\rho\delta_\rho + w_\phi\delta_\phi + \mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi - b'\hat{A}b + n, \tag{C.77}$$

it follows that

$$\begin{aligned} \mathbb{E}(\sigma_0) &= E(w_0 - \mathsf{a}_\rho\delta_\rho - \mathsf{a}_\phi\delta_\phi) && \text{(C.78)}\\ &= \mathbb{E}(w_0) - \mathsf{a}_\rho\mathbb{E}(\delta_\rho) - \mathsf{a}_\phi\mathbb{E}(\delta_\phi) && \text{(C.79)}\\ &= 0 && \text{(C.80)} \end{aligned}$$

Using (C.54), (C.55), (C.61), (C.62), (C.24) and (C.25) we find that

$$\begin{aligned} \sigma_1 &= \sqrt{T}\left(\frac{u'\Omega_\rho u}{T} + \mathsf{a}_\rho\right)\delta_\rho + \sqrt{T}\left(\frac{u'\Omega_\phi u}{T} + \mathsf{a}_\phi\right)\delta_\phi \\ &\quad + \mathsf{a}_{\rho\rho}\delta_\rho\delta_\rho + \mathsf{a}_{\phi\phi}\delta_\phi\delta_\phi + 2\mathsf{a}_{\rho\phi}\delta_\rho\delta_\phi - b'\hat{A}b + n \\ &= \sqrt{T}\left[\left(\frac{u'\Omega_\rho u}{T}\right) - E\left(\frac{u'\Omega_\rho u}{T}\right)\right]\delta_\rho + \sqrt{T}\left[\left(\frac{u'\Omega_\phi u}{T}\right) - \mathbb{E}\left(\frac{u'\Omega_\phi u}{T}\right)\right]\delta_\phi \\ &\quad + \frac{1}{2}\mathbb{E}\left(\frac{u'\Omega_{\rho\rho} u}{T}\right)\delta_\rho\delta_\rho + \frac{1}{2}\mathbb{E}\left(\frac{u'\Omega_{\phi\phi} u}{T}\right)\delta_\phi\delta_\phi \\ &\quad + 2\frac{1}{2}\mathbb{E}\left(\frac{u'\Omega_{\rho\phi} u}{T}\right)\delta_\rho\delta_\phi - b'\hat{A}b + n. \end{aligned} \tag{C.81}$$

Provided that

$$\mathbb{E}(b'Ab) = \operatorname{tr} AG = \operatorname{tr} I_n = n \tag{C.82}$$

taking expected values we find:

$$\mu_0 = \mathbb{E}(\sqrt{T}\sigma_0 + \sigma_1) = \sqrt{T}\mathbb{E}(\sigma_0) + \mathbb{E}(\sigma_1) = \mathbb{E}(\sigma_1). \tag{C.83}$$

Therefore from (C.81), (C.82) and (C.83) imply that

$$\mu_0 = \frac{1}{2}\frac{\operatorname{tr}(u'\Omega_{\rho\rho}u)}{T}\delta_\rho\delta_\rho + \frac{1}{2}\frac{\operatorname{tr}(u'\Omega_{\phi\phi}u)}{T}\delta_\phi\delta_\phi + \frac{\operatorname{tr}(u'\Omega_{\rho\phi}u)}{T}\delta_\rho\delta_\phi \tag{C.84}$$

Appendix D

Initial value

D.1 Analysis of the ARMA(1,1) Model

The ARMA(1,1) model (Autoregressive Moving Average) is one of the fundamental structures in time series analysis, combining the characteristics of the autoregressive process (AR) and the moving average process (MA). The ARMA(1,1) models allows the representation and forecasting of time series data that exhibit dependencies with errors.

D.1.1 General Form of the Model

The general form of the ARMA(1,1) model is as follows:

$$Y_t = \mu + \rho Y_{t-1} + \epsilon_t + \phi \epsilon_{t-1}, \tag{D.1}$$

where:

- Y_t is the value of the time series at time t,
- μ is a constant term,
- ρ is the coefficient of the autoregressive term (AR),
- ϕ is the coefficient of the moving average term (MA),
- ϵ_t is the stochastic error (white noise) with mean 0 and variance σ^2.

The same equation can be expressed with the lag operator L, as follows:

$$(1 - \rho L)Y_t = \delta + (1 - \phi L)\epsilon_t. \tag{D.2}$$

This formulation facilitates the algebraic manipulation of the model and helps in the calculation of important statistical quantities such as the mean, variance, and autocovariance of the series.

D.1.2 Stationarity and Invertibility

For the ARMA(1,1) model to be used statistically in a valid manner, it is necessary to meet two basic conditions: stationarity and invertibility.

- The model is stationary if $|\rho| < 1$, meaning the process does not diverge over the long term.
- It is invertible if $|\phi| < 1$, which ensures that the errors can be determined from the observed values of the series.

D.1.3 Mean of the Process

The mean $\mu = E(Y_t)$ is derived as follows:

First, solving the model for Y_t:

$$Y_t = \frac{\delta + (1 - \phi L)\epsilon_t}{(1 - \rho L)} \tag{D.3}$$

$$Y_t = \frac{\delta}{(1 - \rho L)} + \frac{(1 - \phi L)\epsilon_t}{(1 - \rho L)} \tag{D.4}$$

$$Y_t = (1 - \rho L)^{-1}\delta + (1 - \rho L)^{-1}(1 - \phi L)\epsilon_t. \tag{D.5}$$

Since the error term ϵ_t is white noise with zero expected value, the second term of the equation vanishes when taking the expected value, i.e.

$$\mathbb{E}(Y_t) = (1 - \rho L)^{-1}\delta = \frac{\delta}{1 - \rho}. \tag{D.6}$$

Thus, the mean of the process is:

$$\mu = \frac{\delta}{1 - \rho}. \tag{D.7}$$

D.1.4 Variance and Covariance

The variance of the series is crucial for understanding the dispersion around the mean. If we define $y_t = Y_t - \mu$, then the series of deviations from the mean satisfies the relation

$$y_t = \rho y_{t-1} + \epsilon_t + \phi\epsilon_{t-1}. \tag{D.8}$$

Thus,

$$\mathbb{E}(y_t) = \rho\mathbb{E}(y_{t-1}) \Rightarrow$$

$$\mu = \rho\mu \Rightarrow$$

$$\mu(1-\rho) = 0 \Rightarrow$$

$$\mu = 0.$$

The variance $\gamma_0 = E(y_t)^2$ is calculated as:

$$\begin{aligned}
\gamma_0 = \mathbb{E}(y_t)^2 &= E(\rho y_{t-1} + \epsilon_t + \phi\epsilon_{t-1})^2 \\
&= \mathbb{E}(\rho^2 y_{t-1}^2 + \epsilon_t^2 + \phi^2\epsilon_{t-1}^2 + 2\rho y_{t-1}\epsilon_t + 2\rho\phi y_{t-1}\epsilon_{t-1} + 2\phi\epsilon_t\epsilon_{t-1}) \\
&= \rho^2\mathbb{E}(y_{t-1})^2 + \mathbb{E}(\epsilon_t)^2 + \phi^2\mathbb{E}(\epsilon_{t-1})^2 + 2\rho\mathbb{E}(y_{t-1}\epsilon_t) \\
&\quad +2\rho\phi\mathbb{E}(y_{t-1}\epsilon_{t-1}) + 2\phi\mathbb{E}(\epsilon_t\epsilon_{t-1}), \qquad \text{(D.9)}
\end{aligned}$$

which implies that

$$\begin{aligned}
\gamma_0 &= \rho^2\gamma_0 + \sigma_\epsilon^2 + \phi^2\sigma_\epsilon^2 + 2\rho\phi\mathbb{E}[(\rho y_{t-2} + \epsilon_{t-1} + \phi\epsilon_{t-2})\epsilon_{t-1}] \\
&= \rho^2\gamma_0 + \sigma_\epsilon^2 + \phi^2\sigma_\epsilon^2 + 2\rho\phi\mathbb{E}(\epsilon_{t-1})^2 \\
&= \rho^2\gamma_0 + \sigma_\epsilon^2 + \phi^2\sigma_\epsilon^2 + 2\rho\phi\sigma_\epsilon^2. \qquad \text{(D.10)}
\end{aligned}$$

Therefore,

$$(1-\rho^2)\gamma_0 = \sigma_\epsilon^2(1 + \phi^2 + 2\rho\phi) \Rightarrow \qquad \text{(D.11)}$$

$$\gamma_0 = \frac{\sigma_\epsilon^2(1 + \phi^2 + 2\rho\phi)}{1-\rho^2}. \qquad \text{(D.12)}$$

Similarly, the first- and second-order autocovariances are calculated as:

$$\begin{aligned}
\gamma_1 = \mathbb{E}(y_{t-1}y_t) &= \mathbb{E}[y_{t-1}(\rho y_{t-1} + \epsilon_t + \phi\epsilon_{t-1})] \\
&= \mathbb{E}(\rho y_{t-1}^2 + y_{t-1}\epsilon_t + \phi y_{t-1}\epsilon_{t-1}) \\
&= \rho\mathbb{E}(y_{t-1})^2 + \phi\mathbb{E}(y_{t-1}\epsilon_{t-1}) \\
&= \rho\gamma_0 + \phi\sigma_\epsilon^2, \qquad \text{(D.13)}
\end{aligned}$$

$$\begin{aligned}
\gamma_2 = \mathbb{E}(y_{t-2}y_t) &= E[y_{t-2}(\rho y_{t-1} + \epsilon_t + \phi\epsilon_{t-1})] \\
&= \mathbb{E}(\rho y_{t-1}y_{t-2} + y_{t-2}\epsilon_t + \phi\epsilon_{t-1})y_{t-2} \\
&= \rho\mathbb{E}(y_{t-1}y_{t-2}) \\
&= \rho\gamma_1, \qquad \text{(D.14)}
\end{aligned}$$

and generally

$$\gamma_k = \rho\gamma_{k-1}. \qquad \text{(D.15)}$$

These quantities are crucial for understanding the temporal dependence of the series.

D.1.5 Calculation of the Initial Value u_0

The value u_0 refers to the initial value of u_t and is necessary for proper simulation or forecasting of the model.

Once γ_0 is calculated, we calculate σ_u from the following formula:

$$\sigma_u = \sqrt{\gamma_0}, \qquad \text{(D.16)}$$

and then u_0, which can be expressed as a number generated from a distribution with mean 0 and standard deviation σ_u.

Appendix E

Monte Carlo estimation of the expectation variance and covariance of ρ and ϕ

This appendix includes the complete set of results obtained from the extensive Monte Carlo simulation conducted to study the behavior of the estimators of the ARMA(1,1) model in small sample sizes. The purpose of the simulation was to quantify the behavior of the maximum likelihood estimates (MLE) under different combinations of parameters and sample sizes, as well as to compute the asymptotic moments — namely the means, variances, and covariances — of the parameter estimators.

The results are systematically presented, covering a wide range of values for the autoregressive coefficient (ρ) and the moving average coefficient (ϕ), which take values from -0.9 to 0.9 in increments of 0.1. The analysis is performed for four different sample sizes: $T = 15$, 20, 30, and 50 observations. For each parameter combination, 10000 replications were conducted to ensure the statistical reliability of the calculations. The estimated quantities include the means μ_ρ and μ_ϕ, the variances $\lambda_{\rho\rho}$ and $\lambda_{\phi\phi}$, and the covariance $\lambda_{\rho\phi}$.

These quantities are critical for adjusting the size of econometric t and F tests, especially in small samples, through the use of asymptotic expansions such as the Edgeworth and Cornish-Fisher expansions. Therefore, the simulation results provide a statistical basis for more accurate inference in empirical data with limited sample sizes.

Subsequently, the results are presented in tables with a uniform format to facilitate reading and analysis. Each table is structured to display the values of the above quantities for all combinations of parameters and sample sizes.

More specifically, in the tables, columns are grouped according to the sample size T, while rows correspond to the combinations of parameters ρ and ϕ. For each case, the estimated means and moments described above are presented, allowing the reader to accurately evaluate the effect of parameters and sample size on the performance of the estimators.

Table E.1 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	0.1	-0.349	-0.315	0.337	0.222	0.062	-0.412	-0.363	0.414	0.294	0.074	-0.514	-0.421	0.549	0.402	0.086	-0.652	-0.510	0.797	0.602	0.105
	0.2	-0.364	-0.650	0.343	0.540	0.189	-0.432	-0.736	0.415	0.689	0.248	-0.522	-0.865	0.527	0.941	0.342	-0.625	-1.069	0.697	1.410	0.496
	0.3	-0.396	-0.969	0.360	1.045	0.338	-0.454	-1.099	0.420	1.332	0.439	-0.531	-1.302	0.504	1.841	0.608	-0.623	-1.637	0.640	2.855	0.900
	0.4	-0.424	-1.282	0.377	1.730	0.503	-0.483	-1.462	0.438	2.231	0.659	-0.547	-1.753	0.500	3.173	0.899	-0.631	-2.240	0.610	5.120	1.338
0.1	0.5	-0.458	-1.593	0.409	2.606	0.694	-0.516	-1.829	0.468	3.410	0.906	-0.570	-2.222	0.519	5.001	1.227	-0.659	-2.865	0.624	8.267	1.843
	0.6	-0.496	-1.902	0.455	3.670	0.911	-0.552	-2.199	0.517	4.883	1.184	-0.608	-2.698	0.572	7.316	1.612	-0.698	-3.503	0.683	12.307	2.416
	0.7	-0.539	-2.204	0.520	4.902	1.152	-0.599	-2.565	0.593	6.613	1.508	-0.651	-3.169	0.655	10.067	2.036	-0.739	-4.138	0.784	17.144	3.036
	0.8	-0.585	-2.479	0.605	6.191	1.407	-0.647	-2.906	0.694	8.479	1.846	-0.701	-3.619	0.778	13.123	2.508	-0.783	-4.745	0.928	22.537	3.685
	0.9	-0.627	-2.706	0.699	7.378	1.637	-0.696	-3.193	0.817	10.233	2.173	-0.758	-4.015	0.942	16.149	3.003	-0.840	-5.306	1.134	28.176	4.417
	0.1	-0.735	-0.269	0.738	0.199	0.150	-0.854	-0.303	0.943	0.252	0.187	-1.025	-0.349	1.287	0.336	0.246	-1.257	-0.428	1.867	0.482	0.358
	0.2	-0.753	-0.592	0.754	0.469	0.398	-0.858	-0.674	0.931	0.595	0.514	-1.010	-0.794	1.224	0.802	0.710	-1.220	-1.006	1.721	1.227	1.092
	0.3	-0.766	-0.909	0.765	0.930	0.650	-0.870	-1.037	0.942	1.189	0.848	-1.006	-1.246	1.195	1.677	1.182	-1.211	-1.601	1.658	2.701	1.842
	0.4	-0.784	-1.222	0.793	1.578	0.916	-0.885	-1.406	0.963	2.062	1.197	-1.015	-1.712	1.200	3.017	1.681	-1.228	-2.218	1.674	5.004	2.656
0.2	0.5	-0.811	-1.534	0.841	2.420	1.204	-0.910	-1.775	1.010	3.211	1.572	-1.039	-2.184	1.250	4.826	2.223	-1.260	-2.849	1.749	8.169	3.541
	0.6	-0.846	-1.841	0.911	3.444	1.515	-0.942	-2.144	1.085	4.645	1.980	-1.074	-2.659	1.341	7.108	2.813	-1.301	-3.485	1.872	12.181	4.490
	0.7	-0.884	-2.136	1.000	4.617	1.839	-0.984	-2.506	1.193	6.320	2.418	-1.115	-3.125	1.467	9.798	3.438	-1.342	-4.110	2.030	16.926	5.468
	0.8	-0.924	-2.405	1.107	5.841	2.162	-1.030	-2.840	1.329	8.109	2.864	-1.164	-3.566	1.638	12.750	4.092	-1.384	-4.707	2.229	22.186	6.450
	0.9	-0.961	-2.624	1.221	6.960	2.441	-1.077	-3.118	1.487	9.776	3.280	-1.220	-3.953	1.854	15.669	4.745	-1.440	-5.258	2.502	27.676	7.484
	0.1	-1.111	-0.221	1.409	0.182	0.199	-1.264	-0.253	1.780	0.225	0.255	-1.495	-0.296	2.423	0.285	0.347	-1.829	-0.390	3.554	0.402	0.573
	0.2	-1.113	-0.540	1.403	0.413	0.555	-1.263	-0.620	1.763	0.520	0.725	-1.473	-0.751	2.333	0.717	1.028	-1.801	-0.986	3.408	1.146	1.669
	0.3	-1.118	-0.855	1.411	0.836	0.910	-1.261	-0.988	1.750	1.084	1.193	-1.465	-1.214	2.297	1.587	1.714	-1.804	-1.594	3.395	2.655	2.795
	0.4	-1.131	-1.167	1.444	1.449	1.277	-1.272	-1.358	1.779	1.926	1.677	-1.478	-1.682	2.330	2.908	2.430	-1.828	-2.214	3.475	4.978	3.985
0.3	0.5	-1.154	-1.477	1.503	2.255	1.657	-1.293	-1.728	1.840	3.050	2.184	-1.502	-2.155	2.408	4.704	3.184	-1.865	-2.843	3.618	8.138	5.248
	0.6	-1.182	-1.781	1.584	3.237	2.051	-1.323	-2.093	1.939	4.438	2.716	-1.536	-2.625	2.537	6.940	3.977	-1.905	-3.471	3.797	12.093	6.553
	0.7	-1.215	-2.070	1.691	4.354	2.452	-1.361	-2.448	2.072	6.047	3.266	-1.576	-3.082	2.702	9.546	4.788	-1.943	-4.084	4.001	16.723	7.862
	0.8	-1.249	-2.331	1.810	5.513	2.832	-1.403	-2.774	2.237	7.761	3.811	-1.622	-3.513	2.915	12.393	5.612	-1.981	-4.667	4.239	21.836	9.148
	0.9	-1.282	-2.544	1.938	6.571	3.162	-1.446	-3.045	2.420	9.351	4.300	-1.677	-3.892	3.180	15.212	6.414	-2.037	-5.209	4.578	27.194	10.480

Table E.2 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	0.1	-1.464	-0.175	2.298	0.170	0.211	-1.661	-0.210	2.914	0.201	0.291	-1.949	-0.270	3.944	0.253	0.448	-2.406	-0.390	5.928	0.353	0.837
	0.2	-1.455	-0.493	2.266	0.369	0.672	-1.645	-0.581	2.853	0.470	0.901	-1.926	-0.731	3.839	0.675	1.342	-2.394	-0.992	5.844	1.126	2.294
	0.3	-1.455	-0.806	2.268	0.760	1.128	-1.640	-0.950	2.836	1.010	1.506	-1.925	-1.195	3.831	1.535	2.243	-2.407	-1.602	5.898	2.668	3.791
	0.4	-1.464	-1.116	2.298	1.341	1.586	-1.649	-1.318	2.867	1.827	2.121	-1.939	-1.664	3.886	2.851	3.169	-2.437	-2.221	6.045	5.008	5.353
0.4	0.5	-1.479	-1.423	2.357	2.112	2.052	-1.667	-1.685	2.941	2.915	2.751	-1.964	-2.132	3.999	4.615	4.126	-2.473	-2.842	6.240	8.143	6.967
	0.6	-1.504	-1.722	2.449	3.050	2.526	-1.693	-2.045	3.052	4.256	3.396	-1.995	-2.592	4.151	6.786	5.100	-2.508	-3.457	6.452	12.017	8.597
	0.7	-1.532	-2.004	2.563	4.109	2.993	-1.727	-2.392	3.203	5.800	4.046	-2.031	-3.039	4.344	9.308	6.082	-2.540	-4.055	6.674	16.514	10.201
	0.8	-1.564	-2.259	2.699	5.208	3.437	-1.768	-2.710	3.395	7.436	4.686	-2.075	-3.461	4.591	12.056	7.066	-2.575	-4.628	6.943	21.499	11.784
	0.9	-1.593	-2.467	2.833	6.218	3.811	-1.808	-2.974	3.598	8.959	5.247	-2.129	-3.832	4.902	14.780	8.011	-2.629	-5.163	7.346	26.745	13.400
	0.1	-1.800	-0.138	3.380	0.167	0.207	-2.036	-0.184	4.280	0.192	0.323	-2.400	-0.264	5.874	0.231	0.573	-3.005	-0.408	9.118	0.328	1.160
	0.2	-1.784	-0.453	3.324	0.339	0.766	-2.018	-0.551	4.205	0.438	1.061	-2.385	-0.725	5.794	0.654	1.674	-3.005	-1.012	9.108	1.144	2.983
	0.3	-1.779	-0.762	3.313	0.700	1.310	-2.012	-0.919	4.185	0.959	1.797	-2.386	-1.188	5.799	1.516	2.779	-3.022	-1.619	9.213	2.717	4.839
	0.4	-1.781	-1.069	3.331	1.253	1.855	-2.017	-1.283	4.213	1.749	2.532	-2.399	-1.650	5.874	2.814	3.900	-3.049	-2.230	9.391	5.055	6.742
0.5	0.5	-1.795	-1.371	3.397	1.987	2.403	-2.029	-1.645	4.281	2.802	3.273	-2.420	-2.109	5.995	4.537	5.036	-3.080	-2.840	9.602	8.145	8.678
	0.6	-1.813	-1.664	3.485	2.881	2.946	-2.052	-1.998	4.403	4.091	4.022	-2.447	-2.559	6.158	6.643	6.176	-3.106	-3.441	9.807	11.928	10.597
	0.7	-1.838	-1.941	3.604	3.892	3.478	-2.084	-2.337	4.572	5.571	4.770	-2.478	-2.996	6.366	9.080	7.315	-3.130	-4.025	10.019	16.304	12.475
	0.8	-1.865	-2.190	3.743	4.940	3.975	-2.122	-2.647	4.782	7.137	5.494	-2.521	-3.409	6.645	11.742	8.452	-3.162	-4.590	10.313	21.189	14.345
	0.9	-1.889	-2.395	3.877	5.907	4.391	-2.159	-2.907	5.000	8.607	6.122	-2.571	-3.775	6.989	14.385	9.524	-3.216	-5.118	10.777	26.328	16.238
	0.1	-2.118	-0.107	4.629	0.170	0.189	-2.400	-0.165	5.888	0.187	0.352	-2.855	-0.269	8.247	0.220	0.724	-3.622	-0.436	13.175	0.321	1.538
	0.2	-2.099	-0.417	4.554	0.320	0.836	-2.381	-0.529	5.798	0.420	1.215	-2.844	-0.725	8.180	0.651	2.017	-3.625	-1.034	13.204	1.178	3.709
	0.3	-2.088	-0.723	4.520	0.658	1.466	-2.372	-0.892	5.765	0.922	2.065	-2.844	-1.182	8.192	1.510	3.313	-3.640	-1.636	13.316	2.771	5.906
	0.4	-2.086	-1.025	4.529	1.183	2.089	-2.372	-1.252	5.779	1.691	2.913	-2.852	-1.637	8.252	2.788	4.608	-3.658	-2.236	13.472	5.097	8.124
0.6	0.5	-2.093	-1.323	4.575	1.886	2.706	-2.381	-1.607	5.844	2.707	3.756	-2.867	-2.086	8.365	4.467	5.906	-3.676	-2.832	13.630	8.124	10.334
	0.6	-2.107	-1.610	4.662	2.737	3.314	-2.400	-1.953	5.966	3.948	4.597	-2.886	-2.527	8.514	6.512	7.194	-3.692	-3.420	13.793	11.814	12.521
	0.7	-2.128	-1.882	4.781	3.703	3.905	-2.428	-2.284	6.142	5.367	5.432	-2.913	-2.955	8.723	8.875	8.477	-3.709	-3.995	13.986	16.101	14.670
	0.8	-2.150	-2.126	4.920	4.706	4.451	-2.462	-2.588	6.359	6.873	6.229	-2.953	-3.361	9.021	11.460	9.753	-3.738	-4.554	14.298	20.903	16.822
	0.9	-2.169	-2.326	5.045	5.631	4.904	-2.494	-2.843	6.582	8.289	6.918	-3.001	-3.720	9.399	14.030	10.950	-3.791	-5.074	14.814	25.946	18.976

Table E.3 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	0.1	-2.420	-0.079	6.020	0.177	0.163	-2.752	-0.151	7.713	0.187	0.382	-3.309	-0.276	11.040	0.217	0.882	-4.243	-0.460	18.061	0.324	1.926
	0.2	-2.393	-0.386	5.905	0.310	0.890	-2.727	-0.512	7.587	0.414	1.359	-3.293	-0.727	10.941	0.657	2.356	-4.240	-1.052	18.040	1.211	4.430
	0.3	-2.375	-0.688	5.843	0.631	1.596	-2.712	-0.870	7.524	0.904	2.315	-3.288	-1.176	10.924	1.509	3.821	-4.243	-1.645	18.080	2.811	6.938
	0.4	-2.366	-0.987	5.825	1.134	2.286	-2.706	-1.223	7.510	1.646	3.254	-3.286	-1.622	10.934	2.763	5.267	-4.247	-2.234	18.132	5.107	9.430
0.7	0.5	-2.366	-1.279	5.854	1.805	2.964	-2.710	-1.571	7.560	2.628	4.186	-3.292	-2.062	11.007	4.401	6.707	-4.251	-2.818	18.196	8.075	11.896
	0.6	-2.375	-1.560	5.929	2.619	3.626	-2.723	-1.909	7.668	3.822	5.106	-3.305	-2.495	11.132	6.390	8.133	-4.258	-3.396	18.304	11.692	14.341
	0.7	-2.390	-1.826	6.038	3.545	4.264	-2.745	-2.235	7.835	5.196	6.013	-3.326	-2.916	11.327	8.693	9.548	-4.269	-3.966	18.466	15.918	16.760
	0.8	-2.406	-2.067	6.163	4.508	4.852	-2.774	-2.534	8.049	6.648	6.874	-3.361	-3.315	11.633	11.210	10.949	-4.294	-4.520	18.779	20.656	19.176
	0.9	-2.418	-2.264	6.277	5.400	5.334	-2.802	-2.784	8.267	8.016	7.613	-3.407	-3.669	12.030	13.712	12.253	-4.345	-5.034	19.336	25.607	21.570
	0.1	-2.683	-0.059	7.450	0.189	0.137	-3.069	-0.145	9.622	0.195	0.420	-3.733	-0.281	14.067	0.219	1.030	-4.836	-0.475	23.466	0.331	2.285
	0.2	-2.645	-0.362	7.286	0.313	0.931	-3.035	-0.501	9.439	0.420	1.489	-3.709	-0.726	13.906	0.667	2.665	-4.819	-1.061	23.317	1.235	5.091
	0.3	-2.619	-0.661	7.186	0.622	1.699	-3.010	-0.852	9.322	0.898	2.524	-3.690	-1.168	13.789	1.511	4.268	-4.805	-1.645	23.199	2.827	7.867
	0.4	-2.602	-0.954	7.141	1.105	2.441	-2.996	-1.198	9.272	1.620	3.539	-3.676	-1.606	13.721	2.741	5.843	-4.791	-2.225	23.088	5.092	10.602
0.8	0.5	-2.594	-1.239	7.150	1.745	3.163	-2.991	-1.540	9.287	2.572	4.537	-3.672	-2.039	13.730	4.346	7.403	-4.783	-2.801	23.042	8.017	13.311
	0.6	-2.594	-1.515	7.204	2.530	3.864	-2.996	-1.873	9.365	3.733	5.518	-3.675	-2.466	13.800	6.295	8.942	-4.780	-3.374	23.069	11.589	15.998
	0.7	-2.602	-1.776	7.302	3.417	4.535	-3.010	-2.192	9.508	5.061	6.475	-3.689	-2.881	13.962	8.550	10.466	-4.785	-3.939	23.185	15.765	18.661
	0.8	-2.612	-2.013	7.415	4.346	5.156	-3.030	-2.487	9.697	6.474	7.377	-3.719	-3.275	14.256	11.013	11.967	-4.804	-4.489	23.469	20.450	21.307
	0.9	-2.614	-2.212	7.500	5.221	5.664	-3.048	-2.732	9.891	7.796	8.141	-3.758	-3.622	14.640	13.445	13.344	-4.851	-4.997	24.032	25.315	23.899
	0.1	-2.868	-0.049	8.823	0.213	0.133	-3.287	-0.146	11.332	0.211	0.469	-4.056	-0.291	16.829	0.234	1.174	-5.335	-0.487	28.707	0.348	2.600
	0.2	-2.822	-0.342	8.634	0.332	0.965	-3.236	-0.493	11.081	0.438	1.587	-4.007	-0.728	16.504	0.688	2.904	-5.286	-1.063	28.214	1.254	5.610
	0.3	-2.791	-0.630	8.539	0.622	1.767	-3.196	-0.837	10.903	0.906	2.662	-3.966	-1.163	16.241	1.526	4.584	-5.244	-1.641	27.805	2.836	8.576
	0.4	-2.772	-0.914	8.509	1.077	2.549	-3.168	-1.176	10.809	1.609	3.710	-3.933	-1.595	16.061	2.744	6.227	-5.208	-2.216	27.466	5.087	11.488
0.9	0.5	-2.765	-1.193	8.543	1.686	3.310	-3.157	-1.510	10.812	2.531	4.740	-3.912	-2.022	15.972	4.320	7.838	-5.182	-2.789	27.249	7.990	14.362
	0.6	-2.771	-1.465	8.647	2.436	4.064	-3.155	-1.838	10.887	3.659	5.752	-3.899	-2.444	15.957	6.245	9.423	-5.163	-3.358	27.129	11.534	17.199
	0.7	-2.784	-1.723	8.791	3.293	4.790	-3.167	-2.152	11.048	4.951	6.745	-3.902	-2.855	16.079	8.459	10.991	-5.154	-3.919	27.125	15.673	20.002
	0.8	-2.805	-1.960	8.976	4.195	5.476	-3.185	-2.443	11.256	6.331	7.681	-3.921	-3.243	16.326	10.872	12.518	-5.159	-4.465	27.298	20.313	22.760
	0.9	-2.820	-2.158	9.129	5.053	6.057	-3.203	-2.689	11.469	7.633	8.483	-3.946	-3.584	16.654	13.253	13.897	-5.193	-4.966	27.779	25.092	25.428

Table E.4 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	0.1	0.450	-0.410	0.430	0.304	-0.224	0.505	-0.470	0.525	0.401	-0.308	0.589	-0.577	0.676	0.587	-0.486	0.732	-0.737	1.004	0.956	-0.830
	0.2	0.441	-0.755	0.426	0.695	-0.376	0.489	-0.868	0.515	0.918	-0.497	0.577	-1.046	0.666	1.324	-0.745	0.715	-1.322	0.964	2.119	-1.214
	0.3	0.415	-1.086	0.412	1.292	-0.496	0.461	-1.250	0.485	1.706	-0.649	0.537	-1.491	0.611	2.424	-0.927	0.661	-1.858	0.847	3.751	-1.443
	0.4	0.378	-1.413	0.390	2.097	-0.577	0.411	-1.611	0.441	2.716	-0.723	0.472	-1.912	0.519	3.812	-0.996	0.609	-2.386	0.711	5.889	-1.587
-0.1	0.5	0.317	-1.724	0.351	3.056	-0.583	0.346	-1.965	0.383	3.953	-0.724	0.411	-2.343	0.435	5.592	-1.014	0.558	-2.956	0.594	8.851	-1.711
	0.6	0.253	-2.033	0.318	4.195	-0.540	0.283	-2.325	0.343	5.471	-0.680	0.357	-2.799	0.378	7.897	-1.016	0.514	-3.569	0.515	12.794	-1.842
	0.7	0.195	-2.335	0.307	5.500	-0.473	0.218	-2.688	0.323	7.266	-0.592	0.302	-3.268	0.353	10.719	-0.978	0.471	-4.200	0.484	17.678	-1.956
	0.8	0.141	-2.623	0.318	6.922	-0.385	0.159	-3.038	0.333	9.264	-0.481	0.246	-3.727	0.366	13.917	-0.896	0.423	-4.823	0.501	23.285	-2.005
	0.9	0.095	-2.867	0.354	8.270	-0.293	0.101	-3.341	0.371	11.198	-0.336	0.185	-4.140	0.411	17.163	-0.742	0.366	-5.404	0.563	29.227	-1.932
	0.1	0.832	-0.450	0.906	0.351	-0.409	0.945	-0.520	1.144	0.463	-0.559	1.116	-0.638	1.555	0.684	-0.849	1.389	-0.824	2.358	1.116	-1.420
	0.2	0.838	-0.797	0.930	0.772	-0.708	0.955	-0.916	1.181	1.020	-0.946	1.137	-1.123	1.624	1.517	-1.423	1.440	-1.444	2.545	2.499	-2.371
	0.3	0.830	-1.142	0.927	1.427	-0.991	0.937	-1.314	1.162	1.888	-1.302	1.120	-1.591	1.603	2.758	-1.924	1.408	-2.024	2.451	4.459	-3.120
	0.4	0.798	-1.472	0.893	2.274	-1.218	0.896	-1.691	1.094	2.995	-1.585	1.057	-2.027	1.463	4.297	-2.262	1.314	-2.539	2.162	6.724	-3.543
-0.2	0.5	0.749	-1.792	0.835	3.304	-1.381	0.832	-2.049	0.992	4.309	-1.759	0.964	-2.442	1.250	6.099	-2.432	1.222	-3.059	1.855	9.530	-3.847
	0.6	0.673	-2.099	0.735	4.482	-1.439	0.744	-2.399	0.852	5.835	-1.815	0.879	-2.871	1.069	8.332	-2.551	1.145	-3.629	1.614	13.265	-4.178
	0.7	0.601	-2.400	0.658	5.817	-1.457	0.671	-2.753	0.760	7.638	-1.852	0.809	-3.326	0.949	11.119	-2.680	1.083	-4.244	1.459	18.068	-4.562
	0.8	0.533	-2.689	0.610	7.284	-1.441	0.597	-3.103	0.696	9.672	-1.841	0.740	-3.783	0.874	14.349	-2.761	1.033	-4.865	1.401	23.706	-4.961
	0.9	0.481	-2.941	0.603	8.709	-1.421	0.533	-3.412	0.673	11.689	-1.797	0.673	-4.201	0.844	17.682	-2.774	0.975	-5.451	1.384	29.754	-5.231
	0.1	1.192	-0.492	1.616	0.401	-0.615	1.344	-0.562	2.031	0.520	-0.817	1.579	-0.684	2.761	0.759	-1.205	1.963	-0.871	4.206	1.189	-1.947
	0.2	1.214	-0.831	1.682	0.845	-1.043	1.384	-0.956	2.160	1.116	-1.390	1.651	-1.170	3.030	1.658	-2.071	2.074	-1.501	4.720	2.702	-3.390
	0.3	1.227	-1.180	1.734	1.533	-1.487	1.402	-1.360	2.236	2.035	-1.979	1.685	-1.667	3.166	3.039	-2.954	2.149	-2.149	5.086	5.033	-4.909
	0.4	1.220	-1.525	1.734	2.449	-1.904	1.382	-1.758	2.204	3.251	-2.500	1.662	-2.134	3.120	4.775	-3.686	2.103	-2.725	4.913	7.780	-6.003
-0.3	0.5	1.181	-1.852	1.667	3.536	-2.229	1.331	-2.130	2.084	4.667	-2.902	1.580	-2.560	2.870	6.731	-4.160	1.966	-3.222	4.327	10.637	-6.526
	0.6	1.118	-2.165	1.550	4.778	-2.454	1.247	-2.479	1.886	6.250	-3.137	1.460	-2.971	2.489	8.952	-4.402	1.831	-3.733	3.735	14.087	-6.909
	0.7	1.038	-2.465	1.401	6.148	-2.578	1.151	-2.826	1.672	8.062	-3.269	1.350	-3.398	2.167	11.623	-4.593	1.736	-4.306	3.358	18.631	-7.458
	0.8	0.963	-2.753	1.283	7.641	-2.656	1.067	-3.167	1.513	10.088	-3.367	1.263	-3.843	1.960	14.825	-4.814	1.658	-4.912	3.105	24.193	-8.063
	0.9	0.904	-3.011	1.223	9.131	-2.723	0.999	-3.479	1.429	12.165	-3.450	1.191	-4.259	1.843	18.196	-5.003	1.590	-5.498	2.968	30.291	-8.622

Table E.5 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	0.1	1.518	-0.524	2.485	0.441	-0.818	1.714	-0.596	3.134	0.568	-1.072	2.019	-0.715	4.298	0.793	-1.545	2.496	-0.873	6.475	1.145	-2.357
	0.2	1.565	-0.863	2.641	0.915	-1.379	1.773	-0.986	3.359	1.193	-1.809	2.100	-1.196	4.670	1.735	-2.637	2.623	-1.509	7.214	2.723	-4.191
	0.3	1.598	-1.207	2.760	1.618	-1.963	1.825	-1.390	3.574	2.143	-2.604	2.192	-1.700	5.103	3.189	-3.866	2.760	-2.172	8.023	5.181	-6.270
	0.4	1.617	-1.560	2.844	2.579	-2.561	1.848	-1.799	3.684	3.424	-3.395	2.234	-2.208	5.321	5.136	-5.078	2.854	-2.851	8.616	8.547	-8.429
-0.4	0.5	1.610	-1.905	2.848	3.756	-3.111	1.830	-2.198	3.652	4.989	-4.091	2.206	-2.676	5.236	7.380	-6.042	2.794	-3.423	8.318	12.064	-9.837
	0.6	1.571	-2.225	2.761	5.056	-3.537	1.769	-2.562	3.469	6.691	-4.596	2.100	-3.092	4.813	9.732	-6.602	2.623	-3.903	7.381	15.479	-10.421
	0.7	1.500	-2.528	2.584	6.478	-3.821	1.675	-2.903	3.177	8.528	-4.901	1.965	-3.496	4.265	12.337	-6.915	2.446	-4.409	6.411	19.581	-10.828
	0.8	1.417	-2.815	2.385	8.001	-4.002	1.571	-3.236	2.879	10.548	-5.089	1.842	-3.910	3.813	15.375	-7.185	2.333	-4.972	5.856	24.820	-11.540
	0.9	1.351	-3.074	2.256	9.532	-4.157	1.492	-3.545	2.697	12.640	-5.270	1.747	-4.317	3.523	18.709	-7.473	2.255	-5.544	5.569	30.822	-12.369
	0.1	1.825	-0.550	3.497	0.475	-1.017	2.063	-0.610	4.428	0.585	-1.296	2.436	-0.710	6.102	0.757	-1.802	3.037	-0.837	9.372	1.004	-2.648
	0.2	1.880	-0.883	3.711	0.965	-1.681	2.133	-1.002	4.742	1.236	-2.184	2.520	-1.191	6.558	1.717	-3.096	3.139	-1.473	10.072	2.564	-4.783
	0.3	1.938	-1.225	3.942	1.686	-2.400	2.203	-1.404	5.064	2.207	-3.150	2.623	-1.697	7.130	3.203	-4.573	3.287	-2.135	11.115	5.017	-7.239
	0.4	1.982	-1.574	4.130	2.652	-3.154	2.266	-1.815	5.370	3.520	-4.177	2.729	-2.220	7.734	5.242	-6.196	3.450	-2.838	12.297	8.535	-10.065
-0.5	0.5	2.007	-1.932	4.255	3.886	-3.915	2.300	-2.234	5.558	5.185	-5.206	2.783	-2.743	8.076	7.796	-7.780	3.560	-3.547	13.140	13.007	-12.920
	0.6	2.002	-2.273	4.272	5.300	-4.595	2.280	-2.629	5.512	7.072	-6.063	2.749	-3.211	7.941	10.528	-8.965	3.491	-4.123	12.721	17.334	-14.664
	0.7	1.960	-2.586	4.153	6.797	-5.109	2.219	-2.984	5.287	9.032	-6.679	2.629	-3.615	7.351	13.236	-9.605	3.286	-4.581	11.347	21.221	-15.226
	0.8	1.894	-2.874	3.957	8.356	-5.473	2.120	-3.307	4.915	11.042	-7.044	2.486	-4.002	6.657	16.138	-9.984	3.081	-5.070	10.003	25.851	-15.638
	0.9	1.831	-3.132	3.789	9.908	-5.755	2.039	-3.608	4.649	13.113	-7.362	2.369	-4.381	6.149	19.292	-10.348	2.954	-5.602	9.265	31.503	-16.449
	0.1	2.113	-0.568	4.635	0.494	-1.203	2.393	-0.614	5.885	0.576	-1.490	2.859	-0.684	8.299	0.681	-1.994	3.625	-0.781	13.225	0.820	-2.878
	0.2	2.171	-0.896	4.882	0.993	-1.953	2.467	-0.995	6.252	1.223	-2.483	2.931	-1.163	8.747	1.615	-3.466	3.682	-1.405	13.682	2.262	-5.249
	0.3	2.241	-1.232	5.194	1.722	-2.775	2.547	-1.394	6.675	2.197	-3.590	3.026	-1.655	9.352	3.052	-5.092	3.787	-2.055	14.538	4.614	-7.914
	0.4	2.312	-1.578	5.528	2.694	-3.671	2.632	-1.807	7.133	3.521	-4.810	3.143	-2.179	10.117	5.089	-6.962	3.953	-2.751	15.916	8.045	-11.078
-0.6	0.5	2.368	-1.934	5.804	3.930	-4.608	2.705	-2.229	7.545	5.214	-6.092	3.266	-2.725	10.948	7.759	-9.036	4.137	-3.494	17.494	12.706	-14.722
	0.6	2.397	-2.292	5.970	5.423	-5.532	2.750	-2.652	7.829	7.244	-7.362	3.330	-3.263	11.414	10.931	-11.008	4.268	-4.234	18.676	18.360	-18.359
	0.7	2.396	-2.624	6.008	7.028	-6.329	2.732	-3.047	7.782	9.454	-8.392	3.301	-3.734	11.293	14.162	-12.462	4.192	-4.808	18.134	23.458	-20.431
	0.8	2.365	-2.929	5.919	8.697	-6.967	2.679	-3.383	7.563	11.581	-9.121	3.178	-4.119	10.576	17.134	-13.185	3.963	-5.242	16.321	27.716	-20.945
	0.9	2.323	-3.190	5.790	10.290	-7.448	2.599	-3.674	7.222	13.617	-9.591	3.053	-4.465	9.882	20.075	-13.673	3.775	-5.692	14.903	32.574	-21.510

Table E.6 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	0.1	2.373	-0.569	5.832	0.494	-1.343	2.715	-0.601	7.542	0.548	-1.631	3.285	-0.649	10.915	0.596	-2.137	4.237	-0.732	18.015	0.679	-3.112
	0.2	2.442	-0.894	6.152	0.989	-2.175	2.782	-0.982	7.914	1.181	-2.733	3.346	-1.119	11.325	1.467	-3.760	4.273	-1.343	18.337	1.993	-5.754
	0.3	2.519	-1.226	6.524	1.712	-3.082	2.866	-1.368	8.390	2.122	-3.931	3.427	-1.599	11.898	2.822	-5.514	4.335	-1.965	18.903	4.123	-8.558
	0.4	2.602	-1.563	6.946	2.673	-4.068	2.962	-1.767	8.960	3.397	-5.258	3.532	-2.099	12.665	4.728	-7.473	4.436	-2.615	19.854	7.211	-11.692
-0.7	0.5	2.682	-1.911	7.375	3.880	-5.139	3.058	-2.188	9.556	5.073	-6.734	3.661	-2.639	13.633	7.329	-9.761	4.609	-3.333	21.516	11.594	-15.541
	0.6	2.745	-2.269	7.730	5.366	-6.253	3.144	-2.616	10.112	7.120	-8.283	3.796	-3.206	14.684	10.645	-12.301	4.819	-4.120	23.592	17.505	-20.116
	0.7	2.789	-2.627	7.998	7.096	-7.364	3.196	-3.042	10.474	9.496	-9.788	3.881	-3.764	15.384	14.471	-14.748	4.972	-4.906	25.171	24.517	-24.678
	0.8	2.798	-2.946	8.089	8.847	-8.287	3.203	-3.432	10.571	11.969	-11.061	3.866	-4.232	15.342	18.136	-16.493	4.910	-5.476	24.684	30.330	-27.164
	0.9	2.785	-3.226	8.078	10.558	-9.040	3.166	-3.743	10.415	14.173	-11.924	3.773	-4.582	14.737	21.184	-17.405	4.704	-5.871	22.828	34.733	-27.820
	0.1	2.599	-0.548	7.048	0.470	-1.413	2.993	-0.572	9.227	0.502	-1.691	3.689	-0.616	13.812	0.530	-2.243	4.834	-0.703	23.457	0.606	-3.383
	0.2	2.669	-0.873	7.392	0.948	-2.307	3.068	-0.951	9.654	1.101	-2.888	3.753	-1.074	14.258	1.327	-4.006	4.870	-1.299	23.801	1.820	-6.310
	0.3	2.752	-1.202	7.811	1.651	-3.281	3.159	-1.328	10.198	1.994	-4.169	3.826	-1.539	14.797	2.583	-5.868	4.915	-1.902	24.250	3.790	-9.329
	0.4	2.849	-1.530	8.329	2.571	-4.336	3.256	-1.717	10.817	3.214	-5.571	3.917	-2.016	15.515	4.327	-7.894	4.982	-2.514	24.939	6.560	-12.522
-0.8	0.5	2.949	-1.867	8.890	3.738	-5.489	3.364	-2.114	11.525	4.767	-7.112	4.031	-2.512	16.447	6.638	-10.151	5.082	-3.157	26.002	10.312	-16.081
	0.6	3.048	-2.215	9.476	5.170	-6.747	3.478	-2.531	12.307	6.730	-8.824	4.171	-3.057	17.627	9.737	-12.821	5.252	-3.877	27.857	15.510	-20.495
	0.7	3.123	-2.560	9.942	6.820	-8.009	3.577	-2.957	13.012	9.071	-10.621	4.322	-3.636	18.944	13.633	-15.826	5.486	-4.693	30.453	22.614	-25.993
	0.8	3.177	-2.900	10.304	8.655	-9.246	3.645	-3.375	13.533	11.682	-12.363	4.425	-4.206	19.888	18.052	-18.749	5.667	-5.517	32.545	30.934	-31.548
	0.9	3.205	-3.200	10.518	10.451	-10.307	3.676	-3.743	13.798	14.248	-13.834	4.448	-4.664	20.156	22.023	-20.889	5.666	-6.094	32.661	37.498	-34.811
	0.1	2.741	-0.479	8.055	0.413	-1.313	3.161	-0.508	10.599	0.433	-1.577	3.974	-0.567	16.333	0.465	-2.204	5.329	-0.690	28.683	0.589	-3.627
	0.2	2.812	-0.806	8.407	0.836	-2.243	3.245	-0.885	11.077	0.971	-2.818	4.055	-1.029	16.917	1.215	-4.101	5.393	-1.280	29.307	1.757	-6.846
	0.3	2.898	-1.135	8.858	1.487	-3.247	3.340	-1.265	11.645	1.807	-4.149	4.151	-1.485	17.613	2.386	-6.086	5.451	-1.870	29.905	3.634	-10.134
	0.4	3.005	-1.464	9.430	2.364	-4.347	3.456	-1.643	12.371	2.940	-5.592	4.249	-1.945	18.395	4.006	-8.171	5.514	-2.464	30.578	6.236	-13.520
-0.9	0.5	3.119	-1.794	10.070	3.462	-5.537	3.588	-2.021	13.227	4.365	-7.174	4.365	-2.407	19.333	6.060	-10.433	5.588	-3.062	31.412	9.597	-17.035
	0.6	3.243	-2.128	10.802	4.804	-6.853	3.722	-2.413	14.151	6.139	-8.926	4.499	-2.886	20.514	8.656	-12.934	5.693	-3.675	32.634	13.815	-20.866
	0.7	3.363	-2.457	11.552	6.339	-8.224	3.861	-2.812	15.168	8.261	-10.831	4.648	-3.399	21.870	11.955	-15.794	5.853	-4.344	34.544	19.316	-25.440
	0.8	3.460	-2.771	12.185	8.005	-9.568	3.980	-3.199	16.079	10.622	-12.734	4.813	-3.940	23.440	15.996	-19.012	6.102	-5.120	37.599	26.837	-31.406
	0.9	3.529	-3.041	12.663	9.568	-10.733	4.065	-3.540	16.757	12.917	-14.422	4.932	-4.445	24.611	20.235	-22.028	6.334	-5.930	40.527	35.798	-37.833

Table E.7 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	-0.1	-0.326	0.363	0.334	0.268	-0.158	-0.392	0.423	0.423	0.357	-0.236	-0.504	0.520	0.584	0.526	-0.408	-0.680	0.677	0.932	0.872	-0.752
	-0.2	-0.314	0.703	0.327	0.632	-0.252	-0.375	0.807	0.410	0.826	-0.367	-0.493	0.976	0.574	1.208	-0.615	-0.673	1.239	0.915	1.943	-1.108
	-0.3	-0.298	1.027	0.316	1.184	-0.329	-0.363	1.177	0.394	1.548	-0.479	-0.484	1.408	0.550	2.212	-0.791	-0.647	1.758	0.827	3.437	-1.350
	-0.4	-0.281	1.348	0.307	1.929	-0.390	-0.343	1.538	0.376	2.500	-0.558	-0.460	1.832	0.506	3.537	-0.911	-0.635	2.286	0.738	5.476	-1.577
0.1	-0.5	-0.260	1.671	0.299	2.880	-0.436	-0.318	1.903	0.356	3.723	-0.611	-0.440	2.267	0.467	5.267	-1.018	-0.621	2.858	0.664	8.324	-1.823
	-0.6	-0.232	1.988	0.294	4.012	-0.451	-0.293	2.278	0.347	5.257	-0.655	-0.416	2.731	0.442	7.539	-1.121	-0.606	3.469	0.624	12.131	-2.094
	-0.7	-0.206	2.301	0.304	5.340	-0.462	-0.257	2.652	0.349	7.082	-0.658	-0.379	3.214	0.434	10.383	-1.183	-0.587	4.105	0.628	16.917	-2.369
	-0.8	-0.174	2.593	0.323	6.760	-0.441	-0.215	3.012	0.366	9.107	-0.625	-0.325	3.696	0.446	13.695	-1.158	-0.545	4.754	0.657	22.641	-2.538
	-0.9	-0.141	2.852	0.357	8.179	-0.401	-0.173	3.326	0.402	11.099	-0.559	-0.264	4.126	0.479	17.045	-1.047	-0.461	5.379	0.680	28.957	-2.421
	-0.1	-0.725	0.406	0.742	0.298	-0.339	-0.852	0.474	0.980	0.401	-0.476	-1.056	0.588	1.419	0.595	-0.761	-1.368	0.770	2.287	0.993	-1.323
	-0.2	-0.714	0.748	0.739	0.695	-0.573	-0.838	0.867	0.973	0.931	-0.797	-1.051	1.066	1.432	1.392	-1.265	-1.388	1.383	2.397	2.328	-2.211
	-0.3	-0.694	1.085	0.716	1.313	-0.784	-0.814	1.251	0.939	1.736	-1.081	-1.027	1.516	1.397	2.548	-1.691	-1.357	1.930	2.319	4.126	-2.891
	-0.4	-0.669	1.408	0.686	2.105	-0.961	-0.789	1.618	0.901	2.772	-1.324	-0.996	1.939	1.325	3.977	-2.033	-1.295	2.429	2.101	6.223	-3.340
0.2	-0.5	-0.643	1.728	0.658	3.087	-1.118	-0.750	1.974	0.843	4.019	-1.502	-0.951	2.358	1.221	5.726	-2.294	-1.250	2.950	1.914	8.922	-3.784
	-0.6	-0.607	2.047	0.623	4.271	-1.236	-0.710	2.337	0.787	5.552	-1.653	-0.905	2.796	1.123	7.930	-2.525	-1.210	3.522	1.766	12.539	-4.266
	-0.7	-0.571	2.357	0.597	5.612	-1.329	-0.665	2.709	0.741	7.401	-1.774	-0.859	3.266	1.049	10.737	-2.762	-1.186	4.139	1.710	17.217	-4.855
	-0.8	-0.536	2.654	0.588	7.090	-1.401	-0.624	3.069	0.719	9.463	-1.877	-0.805	3.742	1.002	14.052	-2.946	-1.146	4.784	1.680	22.942	-5.395
	-0.9	-0.504	2.920	0.598	8.574	-1.453	-0.581	3.389	0.715	11.532	-1.931	-0.744	4.178	0.972	17.498	-3.036	-1.065	5.417	1.604	29.388	-5.666
	-0.1	-1.115	0.450	1.440	0.343	-0.549	-1.292	0.517	1.892	0.448	-0.742	-1.573	0.636	2.732	0.659	-1.130	-1.991	0.821	4.298	1.054	-1.859
	-0.2	-1.115	0.788	1.455	0.761	-0.923	-1.299	0.916	1.936	1.023	-1.263	-1.602	1.128	2.863	1.531	-1.948	-2.069	1.462	4.682	2.547	-3.293
	-0.3	-1.102	1.132	1.443	1.419	-1.285	-1.287	1.312	1.928	1.902	-1.760	-1.600	1.610	2.888	2.849	-2.721	-2.093	2.087	4.850	4.774	-4.660
	-0.4	-1.074	1.466	1.395	2.283	-1.604	-1.254	1.694	1.859	3.038	-2.185	-1.564	2.057	2.799	4.472	-3.348	-2.041	2.622	4.659	7.267	-5.621
0.3	-0.5	-1.044	1.788	1.341	3.315	-1.884	-1.217	2.057	1.778	4.377	-2.545	-1.510	2.471	2.634	6.312	-3.825	-1.946	3.107	4.232	9.960	-6.224
	-0.6	-1.007	2.103	1.279	4.521	-2.121	-1.162	2.408	1.653	5.909	-2.807	-1.444	2.888	2.431	8.491	-4.202	-1.861	3.618	3.836	13.286	-6.787
	-0.7	-0.965	2.416	1.215	5.911	-2.320	-1.109	2.767	1.548	7.740	-3.048	-1.369	3.331	2.219	11.199	-4.530	-1.800	4.194	3.581	17.716	-7.504
	-0.8	-0.923	2.711	1.161	7.412	-2.478	-1.053	3.126	1.451	9.834	-3.249	-1.302	3.796	2.071	14.478	-4.870	-1.755	4.822	3.464	23.343	-8.359
	-0.9	-0.885	2.984	1.137	8.962	-2.614	-1.006	3.450	1.402	11.961	-3.421	-1.241	4.232	1.975	17.970	-5.157	-1.677	5.457	3.288	29.848	-9.012

Table E.8 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	-0.1	-1.489	0.483	2.394	0.385	-0.763	-1.715	0.551	3.130	0.493	-1.014	-2.048	0.662	4.403	0.690	-1.464	-2.541	0.826	6.694	1.030	-2.269
	-0.2	-1.501	0.826	2.442	0.833	-1.286	-1.737	0.953	3.228	1.100	-1.727	-2.105	1.155	4.676	1.603	-2.559	-2.661	1.472	7.396	2.566	-4.135
	-0.3	-1.503	1.170	2.467	1.516	-1.801	-1.747	1.357	3.296	2.032	-2.444	-2.149	1.666	4.903	3.046	-3.720	-2.767	2.149	8.043	5.037	-6.208
	-0.4	-1.489	1.512	2.448	2.430	-2.289	-1.735	1.753	3.282	3.257	-3.113	-2.148	2.153	4.943	4.891	-4.769	-2.802	2.788	8.320	8.191	-8.102
0.4	-0.5	-1.460	1.844	2.381	3.533	-2.722	-1.696	2.133	3.174	4.718	-3.675	-2.103	2.594	4.785	6.966	-5.584	-2.726	3.316	7.943	11.377	-9.307
	-0.6	-1.419	2.163	2.284	4.795	-3.086	-1.644	2.490	3.021	6.338	-4.131	-2.028	3.005	4.496	9.227	-6.180	-2.592	3.787	7.197	14.631	-9.978
	-0.7	-1.379	2.470	2.191	6.198	-3.406	-1.578	2.836	2.824	8.150	-4.477	-1.943	3.418	4.161	11.824	-6.657	-2.476	4.299	6.548	18.665	-10.664
	-0.8	-1.337	2.769	2.107	7.750	-3.687	-1.518	3.185	2.668	10.229	-4.807	-1.852	3.858	3.836	14.984	-7.093	-2.391	4.881	6.135	23.946	-11.579
	-0.9	-1.300	3.045	2.054	9.349	-3.933	-1.469	3.510	2.569	12.399	-5.112	-1.782	4.286	3.638	18.455	-7.546	-2.311	5.498	5.839	30.327	-12.546
	-0.1	-1.846	0.511	3.566	0.427	-0.987	-2.098	0.572	4.562	0.526	-1.260	-2.492	0.669	6.367	0.687	-1.750	-3.093	0.803	9.719	0.934	-2.593
	-0.2	-1.868	0.851	3.659	0.891	-1.631	-2.146	0.971	4.786	1.150	-2.150	-2.568	1.158	6.791	1.610	-3.075	-3.197	1.436	10.437	2.424	-4.743
	-0.3	-1.886	1.197	3.741	1.595	-2.301	-2.184	1.386	4.975	2.127	-3.097	-2.643	1.671	7.221	3.076	-4.538	-3.338	2.114	11.436	4.890	-7.268
	-0.4	-1.896	1.544	3.799	2.541	-2.971	-2.198	1.793	5.068	3.420	-4.013	-2.694	2.196	7.538	5.103	-6.055	-3.466	2.827	12.390	8.425	-10.057
0.5	-0.5	-1.877	1.888	3.757	3.713	-3.583	-2.186	2.192	5.053	4.991	-4.861	-2.698	2.690	7.610	7.498	-7.404	-3.509	3.487	12.781	12.578	-12.525
	-0.6	-1.851	2.215	3.683	5.039	-4.129	-2.145	2.567	4.909	6.753	-5.562	-2.647	3.131	7.386	10.038	-8.415	-3.412	4.006	12.175	16.422	-13.932
	-0.7	-1.808	2.529	3.556	6.512	-4.590	-2.080	2.915	4.664	8.633	-6.095	-2.553	3.532	6.927	12.661	-9.094	-3.249	4.466	11.072	20.217	-14.652
	-0.8	-1.764	2.826	3.429	8.087	-4.985	-2.016	3.251	4.439	10.678	-6.553	-2.451	3.940	6.449	15.662	-9.660	-3.104	4.975	10.124	24.930	-15.426
	-0.9	-1.730	3.104	3.360	9.732	-5.359	-1.958	3.570	4.255	12.840	-6.961	-2.372	4.348	6.122	19.012	-10.256	-3.002	5.549	9.558	30.924	-16.532
	-0.1	-2.179	0.534	4.893	0.465	-1.199	-2.475	0.580	6.266	0.539	-1.479	-2.933	0.655	8.721	0.647	-1.975	-3.674	0.766	13.582	0.807	-2.868
	-0.2	-2.219	0.867	5.076	0.935	-1.962	-2.525	0.973	6.530	1.167	-2.509	-3.004	1.136	9.169	1.547	-3.483	-3.745	1.381	14.149	2.198	-5.249
	-0.3	-2.251	1.212	5.230	1.653	-2.768	-2.584	1.380	6.848	2.130	-3.626	-3.090	1.639	9.732	2.975	-5.157	-3.863	2.034	15.116	4.511	-7.988
	-0.4	-2.272	1.564	5.345	2.622	-3.596	-2.627	1.803	7.097	3.475	-4.803	-3.183	2.172	10.356	5.025	-7.031	-4.015	2.741	16.401	7.952	-11.206
0.6	-0.5	-2.284	1.912	5.418	3.824	-4.409	-2.650	2.222	7.252	5.154	-5.958	-3.242	2.718	10.784	7.684	-8.950	-4.168	3.493	17.740	12.659	-14.820
	-0.6	-2.269	2.252	5.379	5.224	-5.148	-2.641	2.620	7.248	7.063	-6.990	-3.250	3.219	10.892	10.630	-10.607	-4.220	4.179	18.278	17.891	-17.926
	-0.7	-2.244	2.573	5.302	6.759	-5.806	-2.597	2.987	7.061	9.091	-7.815	-3.199	3.660	10.625	13.624	-11.836	-4.108	4.693	17.431	22.398	-19.544
	-0.8	-2.206	2.881	5.172	8.424	-6.375	-2.540	3.330	6.817	11.230	-8.495	-3.092	4.048	10.004	16.569	-12.591	-3.928	5.145	16.016	26.737	-20.349
	-0.9	-2.169	3.162	5.056	10.119	-6.867	-2.482	3.639	6.581	13.364	-9.042	-3.004	4.425	9.531	19.726	-13.307	-3.779	5.636	14.906	31.957	-21.288

Table E.9 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	-0.1	-2.497	0.550	6.385	0.494	-1.400	-2.834	0.581	8.166	0.541	-1.678	-3.378	0.636	11.510	0.596	-2.180	-4.285	0.727	18.416	0.693	-3.137
	-0.2	-2.544	0.879	6.621	0.972	-2.265	-2.892	0.963	8.506	1.152	-2.821	-3.438	1.102	11.934	1.445	-3.824	-4.333	1.329	18.848	1.980	-5.785
	-0.3	-2.594	1.217	6.883	1.684	-3.190	-2.955	1.363	8.886	2.099	-4.071	-3.521	1.589	12.541	2.796	-5.648	-4.410	1.948	19.558	4.080	-8.635
	-0.4	-2.637	1.567	7.118	2.658	-4.167	-3.026	1.781	9.327	3.419	-5.441	-3.622	2.106	13.294	4.746	-7.706	-4.532	2.614	20.720	7.213	-11.949
0.7	-0.5	-2.665	1.922	7.278	3.892	-5.162	-3.076	2.212	9.655	5.135	-6.864	-3.723	2.657	14.081	7.393	-10.003	-4.694	3.348	22.304	11.670	-15.901
	-0.6	-2.677	2.273	7.364	5.348	-6.126	-3.101	2.638	9.843	7.189	-8.247	-3.792	3.226	14.643	10.725	-12.365	-4.863	4.141	24.005	17.637	-20.394
	-0.7	-2.666	2.610	7.339	6.980	-6.998	-3.097	3.035	9.864	9.419	-9.466	-3.806	3.735	14.808	14.230	-14.358	-4.931	4.857	24.767	24.029	-24.234
	-0.8	-2.644	2.924	7.265	8.702	-7.767	-3.061	3.392	9.696	11.686	-10.444	-3.764	4.170	14.567	17.623	-15.830	-4.829	5.374	23.891	29.241	-26.214
	-0.9	-2.623	3.213	7.195	10.467	-8.453	-3.029	3.716	9.555	13.964	-11.302	-3.691	4.536	14.101	20.776	-16.843	-4.667	5.798	22.449	33.906	-27.232
	-0.1	-2.785	0.560	7.958	0.516	-1.578	-3.170	0.578	10.217	0.538	-1.852	-3.814	0.615	14.663	0.550	-2.357	-4.890	0.698	23.981	0.615	-3.417
	-0.2	-2.846	0.885	8.283	0.993	-2.539	-3.235	0.950	10.621	1.128	-3.093	-3.875	1.070	15.131	1.348	-4.154	-4.934	1.288	24.412	1.816	-6.354
	-0.3	-2.910	1.219	8.638	1.704	-3.569	-3.312	1.339	11.123	2.038	-4.459	-3.953	1.537	15.756	2.606	-6.094	-4.994	1.885	25.021	3.755	-9.412
	-0.4	-2.978	1.570	9.030	2.686	-4.701	-3.393	1.745	11.673	3.307	-5.953	-4.047	2.030	16.523	4.410	-8.249	-5.078	2.502	25.897	6.530	-12.714
0.8	-0.5	-3.026	1.920	9.327	3.908	-5.840	-3.469	2.174	12.207	4.998	-7.584	-4.158	2.554	17.463	6.855	-10.679	-5.206	3.170	27.276	10.430	-16.569
	-0.6	-3.060	2.280	9.542	5.413	-7.013	-3.525	2.611	12.617	7.084	-9.260	-4.268	3.120	18.428	10.100	-13.407	-5.374	3.919	29.138	15.850	-21.217
	-0.7	-3.069	2.635	9.623	7.137	-8.127	-3.560	3.043	12.897	9.514	-10.895	-4.342	3.705	19.111	14.078	-16.211	-5.557	4.753	31.217	23.123	-26.655
	-0.8	-3.064	2.968	9.625	8.988	-9.135	-3.559	3.436	12.939	12.033	-12.296	-4.372	4.226	19.428	18.155	-18.614	-5.651	5.506	32.379	30.780	-31.397
	-0.9	-3.048	3.268	9.568	10.845	-9.999	-3.544	3.779	12.883	14.482	-13.461	-4.352	4.647	19.325	21.856	-20.365	-5.588	6.023	31.788	36.648	-33.931
	-0.1	-2.999	0.559	9.436	0.530	-1.691	-3.425	0.565	12.120	0.524	-1.946	-4.178	0.592	17.731	0.509	-2.467	-5.428	0.674	29.639	0.568	-3.641
	-0.2	-3.082	0.878	9.862	0.995	-2.723	-3.513	0.931	12.668	1.089	-3.280	-4.253	1.037	18.322	1.257	-4.402	-5.482	1.258	30.201	1.710	-6.878
	-0.3	-3.174	1.215	10.369	1.705	-3.877	-3.611	1.310	13.311	1.959	-4.742	-4.342	1.492	19.064	2.438	-6.469	-5.549	1.846	30.923	3.561	-10.213
	-0.4	-3.264	1.559	10.899	2.666	-5.110	-3.717	1.709	14.042	3.188	-6.374	-4.451	1.962	20.010	4.107	-8.726	-5.637	2.435	31.918	6.139	-13.690
0.9	-0.5	-3.348	1.923	11.412	3.933	-6.463	-3.826	2.132	14.846	4.833	-8.190	-4.571	2.463	21.080	6.372	-11.270	-5.740	3.044	33.109	9.547	-17.442
	-0.6	-3.412	2.297	11.834	5.505	-7.866	-3.922	2.576	15.577	6.932	-10.145	-4.700	3.005	22.294	9.386	-14.168	-5.879	3.705	34.769	14.119	-21.792
	-0.7	-3.454	2.675	12.122	7.369	-9.276	-3.985	3.036	16.088	9.504	-12.155	-4.821	3.587	23.466	13.261	-17.371	-6.060	4.460	36.998	20.409	-27.141
	-0.8	-3.462	3.036	12.204	9.404	-10.552	-4.015	3.473	16.358	12.322	-14.005	-4.903	4.183	24.296	17.878	-20.616	-6.246	5.312	39.345	28.809	-33.396
	-0.9	-3.452	3.362	12.174	11.471	-11.651	-4.018	3.861	16.425	15.133	-15.577	-4.939	4.700	24.701	22.426	-23.348	-6.370	6.082	40.986	37.506	-39.014

Table E.10 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	-0.1	0.446	0.264	0.407	0.212	0.091	0.487	0.302	0.478	0.272	0.088	0.541	0.353	0.581	0.386	0.070	0.653	0.430	0.805	0.576	0.048
	-0.2	0.437	0.592	0.392	0.480	0.239	0.475	0.669	0.453	0.611	0.273	0.521	0.790	0.535	0.848	0.319	0.606	0.988	0.675	1.285	0.426
	-0.3	0.432	0.916	0.385	0.948	0.383	0.463	1.039	0.431	1.214	0.452	0.499	1.233	0.481	1.692	0.552	0.571	1.568	0.567	2.677	0.781
	-0.4	0.432	1.236	0.387	1.612	0.528	0.459	1.414	0.424	2.097	0.632	0.489	1.690	0.450	2.974	0.790	0.563	2.175	0.520	4.872	1.159
-0.1	-0.5	0.442	1.560	0.404	2.496	0.686	0.467	1.793	0.436	3.285	0.831	0.493	2.166	0.453	4.770	1.053	0.571	2.802	0.516	7.939	1.571
	-0.6	0.459	1.881	0.437	3.584	0.858	0.486	2.175	0.474	4.777	1.052	0.510	2.650	0.487	7.075	1.347	0.585	3.438	0.553	11.883	2.003
	-0.7	0.485	2.187	0.488	4.820	1.046	0.518	2.551	0.538	6.542	1.311	0.545	3.138	0.565	9.882	1.706	0.605	4.074	0.630	16.641	2.457
	-0.8	0.513	2.467	0.551	6.125	1.237	0.559	2.899	0.626	8.434	1.594	0.602	3.609	0.694	13.051	2.155	0.647	4.705	0.774	22.165	3.029
	-0.9	0.540	2.711	0.628	7.400	1.419	0.600	3.197	0.731	10.257	1.874	0.667	4.017	0.853	16.161	2.644	0.739	5.301	1.018	28.119	3.881
	-0.1	0.796	0.214	0.824	0.189	0.149	0.884	0.242	0.997	0.237	0.164	1.017	0.281	1.284	0.322	0.187	1.229	0.354	1.802	0.467	0.252
	-0.2	0.776	0.541	0.789	0.415	0.403	0.859	0.615	0.939	0.531	0.491	0.976	0.731	1.166	0.728	0.637	1.173	0.945	1.599	1.146	0.978
	-0.3	0.765	0.863	0.773	0.844	0.649	0.840	0.989	0.898	1.098	0.803	0.951	1.189	1.094	1.558	1.079	1.154	1.551	1.511	2.579	1.705
	-0.4	0.763	1.188	0.774	1.487	0.898	0.834	1.368	0.888	1.961	1.122	0.941	1.658	1.065	2.852	1.528	1.154	2.172	1.493	4.833	2.455
-0.2	-0.5	0.770	1.512	0.798	2.343	1.152	0.839	1.750	0.911	3.127	1.452	0.946	2.140	1.083	4.649	2.002	1.166	2.804	1.525	7.941	3.239
	-0.6	0.786	1.830	0.843	3.394	1.418	0.859	2.132	0.969	4.593	1.810	0.964	2.626	1.146	6.947	2.512	1.182	3.440	1.596	11.887	4.039
	-0.7	0.811	2.130	0.913	4.582	1.693	0.890	2.503	1.063	6.303	2.194	1.000	3.108	1.264	9.697	3.079	1.201	4.066	1.702	16.578	4.848
	-0.8	0.838	2.403	0.995	5.824	1.961	0.932	2.842	1.185	8.120	2.594	1.058	3.567	1.453	12.759	3.723	1.243	4.682	1.898	21.960	5.763
	-0.9	0.863	2.641	1.086	7.039	2.205	0.972	3.132	1.318	9.861	2.967	1.125	3.963	1.678	15.745	4.382	1.337	5.262	2.260	27.720	6.950
	-0.1	1.124	0.170	1.440	0.168	0.173	1.256	0.198	1.768	0.213	0.207	1.458	0.237	2.323	0.272	0.263	1.784	0.337	3.380	0.397	0.466
	-0.2	1.099	0.494	1.384	0.359	0.528	1.223	0.572	1.674	0.467	0.666	1.420	0.700	2.185	0.657	0.932	1.752	0.944	3.225	1.091	1.560
	-0.3	1.085	0.820	1.357	0.766	0.877	1.204	0.951	1.625	1.014	1.119	1.399	1.169	2.111	1.491	1.590	1.745	1.563	3.178	2.582	2.665
	-0.4	1.082	1.144	1.358	1.382	1.222	1.198	1.332	1.615	1.858	1.571	1.392	1.644	2.092	2.794	2.253	1.754	2.189	3.204	4.891	3.794
-0.3	-0.5	1.088	1.467	1.388	2.213	1.572	1.205	1.715	1.650	3.007	2.037	1.398	2.126	2.127	4.589	2.940	1.768	2.819	3.271	8.019	4.944
	-0.6	1.104	1.781	1.448	3.227	1.927	1.224	2.093	1.730	4.435	2.523	1.416	2.607	2.215	6.851	3.651	1.779	3.444	3.357	11.924	6.085
	-0.7	1.128	2.074	1.532	4.361	2.283	1.257	2.457	1.854	6.089	3.027	1.450	3.079	2.369	9.529	4.407	1.795	4.059	3.481	16.530	7.222
	-0.8	1.155	2.340	1.633	5.546	2.623	1.299	2.788	2.009	7.831	3.534	1.510	3.526	2.618	12.485	5.238	1.836	4.660	3.725	21.768	8.460
	-0.9	1.178	2.572	1.734	6.703	2.926	1.338	3.069	2.170	9.493	3.994	1.579	3.911	2.909	15.359	6.054	1.931	5.224	4.205	27.350	9.955

Table E.11 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	-0.1	1.436	0.131	2.232	0.150	0.174	1.613	0.165	2.767	0.187	0.232	1.895	0.225	3.742	0.240	0.364	2.362	0.356	5.708	0.347	0.754
	-0.2	1.411	0.456	2.161	0.320	0.629	1.580	0.542	2.658	0.425	0.829	1.863	0.696	3.604	0.629	1.248	2.346	0.972	5.607	1.102	2.216
	-0.3	1.394	0.781	2.122	0.703	1.072	1.561	0.923	2.602	0.956	1.413	1.847	1.167	3.544	1.474	2.118	2.349	1.592	5.618	2.648	3.690
	-0.4	1.390	1.106	2.122	1.302	1.511	1.556	1.306	2.599	1.792	2.001	1.843	1.643	3.538	2.788	2.991	2.359	2.214	5.674	4.991	5.180
-0.4	-0.5	1.397	1.426	2.161	2.108	1.954	1.562	1.685	2.643	2.913	2.588	1.847	2.119	3.580	4.565	3.869	2.369	2.836	5.750	8.118	6.670
	-0.6	1.411	1.733	2.232	3.078	2.388	1.583	2.057	2.743	4.304	3.195	1.863	2.592	3.682	6.787	4.768	2.376	3.450	5.832	11.975	8.132
	-0.7	1.436	2.021	2.334	4.168	2.821	1.617	2.413	2.896	5.901	3.814	1.898	3.052	3.870	9.390	5.704	2.385	4.051	5.950	16.487	9.570
	-0.8	1.461	2.279	2.445	5.296	3.222	1.657	2.735	3.079	7.570	4.411	1.957	3.487	4.174	12.241	6.698	2.424	4.639	6.235	21.599	11.112
	-0.9	1.481	2.505	2.551	6.399	3.578	1.696	3.008	3.267	9.160	4.952	2.027	3.861	4.531	15.002	7.662	2.522	5.188	6.840	27.005	12.901
	-0.1	1.735	0.102	3.180	0.141	0.164	1.960	0.148	3.994	0.173	0.264	2.333	0.236	5.563	0.217	0.508	2.962	0.397	8.854	0.322	1.126
	-0.2	1.708	0.428	3.092	0.296	0.714	1.929	0.526	3.878	0.403	0.989	2.310	0.706	5.449	0.626	1.595	2.957	1.011	8.818	1.146	2.949
	-0.3	1.691	0.752	3.045	0.665	1.247	1.909	0.904	3.811	0.925	1.695	2.295	1.175	5.388	1.487	2.662	2.961	1.626	8.851	2.743	4.777
	-0.4	1.686	1.074	3.045	1.248	1.773	1.903	1.285	3.810	1.749	2.404	2.288	1.647	5.378	2.806	3.725	2.966	2.241	8.901	5.110	6.603
-0.5	-0.5	1.694	1.389	3.094	2.025	2.297	1.909	1.660	3.862	2.850	3.109	2.288	2.116	5.412	4.567	4.783	2.969	2.851	8.949	8.221	8.407
	-0.6	1.709	1.690	3.178	2.959	2.809	1.931	2.026	3.982	4.207	3.827	2.302	2.578	5.520	6.742	5.854	2.965	3.452	8.985	12.013	10.153
	-0.7	1.733	1.970	3.293	3.998	3.307	1.965	2.373	4.161	5.743	4.543	2.335	3.029	5.734	9.281	6.953	2.968	4.042	9.076	16.441	11.873
	-0.8	1.758	2.221	3.418	5.072	3.772	2.005	2.685	4.372	7.345	5.228	2.395	3.452	6.093	12.037	8.103	3.007	4.620	9.403	21.466	13.715
	-0.9	1.773	2.442	3.522	6.134	4.172	2.043	2.950	4.583	8.863	5.844	2.466	3.813	6.514	14.686	9.198	3.105	5.154	10.127	26.704	15.769
	-0.1	2.016	0.084	4.251	0.141	0.160	2.297	0.144	5.434	0.166	0.314	2.776	0.256	7.814	0.205	0.685	3.582	0.441	12.889	0.317	1.553
	-0.2	1.988	0.409	4.148	0.288	0.793	2.264	0.520	5.299	0.397	1.153	2.750	0.722	7.683	0.640	1.961	3.574	1.048	12.844	1.204	3.720
	-0.3	1.973	0.731	4.101	0.648	1.408	2.240	0.895	5.210	0.919	1.969	2.733	1.189	7.606	1.522	3.214	3.572	1.654	12.841	2.835	5.877
	-0.4	1.969	1.050	4.107	1.218	2.014	2.234	1.272	5.207	1.734	2.786	2.719	1.653	7.559	2.839	4.443	3.567	2.261	12.829	5.212	8.017
-0.6	-0.5	1.976	1.358	4.162	1.970	2.610	2.242	1.641	5.271	2.816	3.598	2.714	2.115	7.570	4.588	5.664	3.552	2.859	12.767	8.287	10.086
	-0.6	1.993	1.653	4.260	2.873	3.193	2.263	2.000	5.403	4.137	4.412	2.724	2.569	7.667	6.732	6.890	3.540	3.451	12.735	12.038	12.109
	-0.7	2.017	1.924	4.389	3.866	3.750	2.298	2.338	5.610	5.626	5.220	2.758	3.009	7.912	9.205	8.147	3.540	4.035	12.811	16.428	14.129
	-0.8	2.038	2.168	4.517	4.891	4.262	2.340	2.639	5.857	7.156	5.990	2.820	3.420	8.326	11.873	9.440	3.577	4.603	13.167	21.364	16.247
	-0.9	2.049	2.384	4.613	5.911	4.705	2.375	2.898	6.078	8.613	6.662	2.891	3.770	8.807	14.419	10.650	3.678	5.122	14.018	26.437	18.552

Table E.12 Full Monte Carlo Estimates with 10000 Repetitions

		T=15					T=20					T=30					T=50				
ρ	ϕ	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$	μ_ρ	μ_ϕ	$\lambda_{\rho\rho}$	$\lambda_{\phi\phi}$	$\lambda_{\rho\phi}$
	-0.1	2.278	0.081	5.420	0.147	0.171	2.610	0.151	7.015	0.168	0.383	3.204	0.278	10.408	0.205	0.883	4.194	0.469	17.660	0.327	1.961
	-0.2	2.250	0.403	5.307	0.298	0.879	2.570	0.525	6.837	0.413	1.323	3.172	0.741	10.225	0.668	2.330	4.179	1.069	17.546	1.250	4.454
	-0.3	2.234	0.724	5.257	0.663	1.570	2.546	0.899	6.739	0.947	2.241	3.146	1.201	10.084	1.565	3.740	4.162	1.667	17.424	2.892	6.911
	-0.4	2.228	1.036	5.257	1.223	2.239	2.540	1.269	6.734	1.759	3.151	3.121	1.663	9.968	2.899	5.125	4.137	2.266	17.255	5.264	9.325
-0.7	-0.5	2.238	1.337	5.324	1.956	2.898	2.546	1.633	6.794	2.832	4.050	3.115	2.119	9.955	4.642	6.502	4.113	2.863	17.093	8.346	11.693
	-0.6	2.254	1.623	5.431	2.825	3.535	2.567	1.985	6.941	4.130	4.948	3.123	2.566	10.044	6.764	7.873	4.091	3.450	16.973	12.078	13.986
	-0.7	2.276	1.885	5.567	3.775	4.139	2.606	2.310	7.179	5.557	5.835	3.156	2.997	10.308	9.191	9.267	4.089	4.031	17.026	16.451	16.289
	-0.8	2.293	2.122	5.690	4.759	4.690	2.648	2.601	7.450	7.021	6.668	3.219	3.395	10.778	11.768	10.683	4.121	4.593	17.392	21.335	18.664
	-0.9	2.298	2.333	5.769	5.731	5.168	2.678	2.851	7.671	8.416	7.387	3.289	3.733	11.308	14.209	11.986	4.224	5.096	18.364	26.258	21.189
	-0.1	2.500	0.098	6.589	0.168	0.225	2.873	0.177	8.589	0.185	0.494	3.587	0.304	13.130	0.219	1.093	4.766	0.479	22.834	0.343	2.290
	-0.2	2.471	0.418	6.466	0.338	0.991	2.835	0.549	8.399	0.460	1.517	3.536	0.766	12.806	0.719	2.680	4.732	1.076	22.545	1.281	5.083
	-0.3	2.457	0.733	6.420	0.714	1.737	2.807	0.919	8.272	1.017	2.513	3.502	1.226	12.585	1.651	4.235	4.696	1.674	22.234	2.944	7.830
	-0.4	2.453	1.036	6.431	1.272	2.456	2.799	1.284	8.253	1.842	3.496	3.475	1.683	12.420	3.004	5.756	4.657	2.272	21.906	5.324	10.517
-0.8	-0.5	2.462	1.328	6.507	1.989	3.159	2.805	1.638	8.320	2.904	4.463	3.460	2.135	12.347	4.763	7.254	4.616	2.870	21.575	8.430	13.137
	-0.6	2.477	1.604	6.619	2.828	3.836	2.831	1.977	8.495	4.162	5.428	3.468	2.574	12.437	6.870	8.747	4.589	3.459	21.377	12.191	15.704
	-0.7	2.494	1.856	6.754	3.736	4.471	2.866	2.291	8.748	5.545	6.362	3.505	2.994	12.733	9.246	10.264	4.583	4.036	21.387	16.564	18.260
	-0.8	2.506	2.085	6.873	4.672	5.051	2.905	2.571	9.027	6.941	7.231	3.567	3.379	13.240	11.742	11.765	4.617	4.591	21.787	21.401	20.879
	-0.9	2.506	2.292	6.938	5.616	5.558	2.925	2.813	9.223	8.280	7.969	3.634	3.701	13.796	14.061	13.125	4.723	5.076	22.878	26.146	23.586
	-0.1	2.656	0.161	7.655	0.221	0.386	3.044	0.241	9.953	0.241	0.703	3.842	0.355	15.387	0.274	1.346	5.206	0.495	27.504	0.377	2.590
	-0.2	2.637	0.470	7.593	0.433	1.182	3.005	0.608	9.763	0.570	1.766	3.791	0.815	15.027	0.831	3.039	5.153	1.097	26.981	1.350	5.632
	-0.3	2.624	0.773	7.566	0.830	1.953	2.985	0.970	9.665	1.165	2.812	3.747	1.271	14.719	1.808	4.679	5.088	1.698	26.394	3.056	8.586
	-0.4	2.624	1.067	7.620	1.400	2.708	2.979	1.325	9.666	2.010	3.832	3.713	1.719	14.489	3.182	6.271	5.029	2.299	25.837	5.495	11.465
-0.9	-0.5	2.637	1.349	7.739	2.114	3.446	2.990	1.667	9.777	3.068	4.844	3.692	2.166	14.381	4.958	7.845	4.982	2.897	25.387	8.638	14.284
	-0.6	2.660	1.611	7.919	2.930	4.158	3.014	1.993	9.977	4.306	5.843	3.696	2.597	14.449	7.062	9.407	4.944	3.485	25.057	12.450	17.020
	-0.7	2.682	1.854	8.116	3.809	4.833	3.051	2.294	10.267	5.640	6.804	3.721	3.007	14.723	9.410	10.949	4.931	4.056	24.984	16.814	19.729
	-0.8	2.705	2.076	8.308	4.717	5.470	3.087	2.563	10.577	6.989	7.692	3.776	3.377	15.228	11.824	12.462	4.962	4.602	25.370	21.602	22.479
	-0.9	2.712	2.282	8.430	5.647	6.041	3.104	2.798	10.798	8.277	8.452	3.829	3.687	15.755	14.051	13.789	5.064	5.068	26.509	26.165	25.243

Appendix F

Regression Outputs and Visualizations

This appendix presents the detailed results of the estimated regressions concerning the Cornish-Fisher adjusted statistics, as well as the corresponding graphs illustrating the relationships between the adjusted statistics and the adjustment factors. Specifically, for each observed value of the adjusted Wald statistic, the corresponding values of the unadjusted Wald statistic and the two adjustment factors h_1 and h_2 were recorded. Subsequently, the relationship connecting them was estimated using ordinary least squares. Similarly, for the adjusted F statistics, the values of the unadjusted F statistic and the adjustment factors q_1 and q_2 were recorded, and a corresponding regression was estimated.

This part includes detailed tables with the estimation results for different combinations of parameters (ρ, ϕ) and sample sizes (T=15, 30). Each table presents the coefficient estimates and the assessment of their statistical significance, thus providing a comprehensive picture of the impact of the adjustment factors on the adjusted statistics.

Additionally, the appendix contains graphs depicting the relationship between the negative adjusted statistics and the respective adjustment factors, as well as their relationship with the unadjusted statistics. These visualizations contribute to the intuitive understanding of the factors influencing the values of the adjusted statistics, highlighting characteristic patterns and extreme values associated with the adjustments.

The graphs included in the appendix illustrate the relationships for the same parameter point and sample size, first showing the values and adjustment factors related to the Wald statistic, followed by those related to the F statistic. Correspondingly, the regression tables are organized so that the estimates for the Wald statistic appear directly above the respective estimates for the F statistic, facilitating comparative analysis and interpretation of the results.

Table F.1 Estimated Regression Results under $\rho = -0.9$, $\phi = -0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	444,473***
	(72,958.370)
h1	$-$0.109***
	(0.003)
h2	$-$3,640.535***
	(23.900)
Constant	13,072,341***
	(1,077,562)
Observations	10,000
R^2	0.876
Adjusted R^2	0.875
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.2 Estimated Regression Results under $\rho = -0.9$, $\phi = -0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	579,345.400***
	(57,264.750)
h1	$-$0.029***
	(0.0004)
h2	$-$3,305.705***
	(18.001)
Constant	3,687,546***
	(917,046.100)
Observations	10,000
R^2	0.893
Adjusted R^2	0.893
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.3 Estimated Regression Results under $\rho = -0.9$, $\phi = -0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	444,479.500***
	(72,958.280)
q1	$-$0.109***
	(0.003)
q2	$-$6,067.548***
	(39.833)
Constant	4,348,327***
	(359,200.800)
Observations	10,000
R^2	0.876
Adjusted R^2	0.875
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.4 Estimated Regression Results under $\rho = -0.9$, $\phi = -0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	579,353.800***
	(57,264.680)
q1	$-$0.029***
	(0.0004)
q2	$-$5,509.500***
	(30.002)
Constant	1,220,900***
	(305,692.900)
Observations	10,000
R^2	0.893
Adjusted R^2	0.893
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.5 Estimated Regression Results under $\rho = -0.9$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-$55,939.970***
	(2,033.581)
h1	$-$7,314.562***
	(6.429)
h2	1,583.949***
	(2.849)
Constant	194,813.800***
	(23,792.250)
Observations	10,000
R^2	0.999
Adjusted R^2	0.999
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.6 Estimated Regression Results under $\rho = -0.9$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-$3,190.856
	(2,203.769)
h1	$-$822.632***
	(4.258)
h2	60.310***
	(1.907)
Constant	34,272.520
	(27,058.570)
Observations	10,000
R^2	0.940
Adjusted R^2	0.940
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.7 Estimated Regression Results under $\rho = -0.9$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-$55,937.910***
	(2,033.576)
q1	$-$7,314.551***
	(6.429)
q2	2,639.914***
	(4.748)
Constant	72,551.340***
	(7,932.138)
Observations	10,000
R^2	0.999
Adjusted R^2	0.999
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.8 Estimated Regression Results under $\rho = -0.9$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-$3,188.699
	(2,203.764)
q1	$-$822.628***
	(4.258)
q2	100.515***
	(3.179)
Constant	11,982.260
	(9,019.638)
Observations	10,000
R^2	0.940
Adjusted R^2	0.940
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.9 Estimated Regression Results under $\rho = -0.5$, $\phi = -0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-$410.835
	(389.252)
h1	$-$0.044***
	(0.00001)
h2	0.017
	(0.348)
Constant	3,695.191*
	(2,012.954)
Observations	10,000
R^2	1.000
Adjusted R^2	1.000
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.10 Estimated Regression Results under $\rho = -0.5$, $\phi = -0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-$622.765***
	(137.507)
h1	$-$0.090***
	(0.00000)
h2	$-$0.249
	(0.249)
Constant	1,735.809**
	(763.693)
Observations	10,000
R^2	1.000
Adjusted R^2	1.000
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.11 Estimated Regression Results under $\rho = -0.5$, $\phi = -0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-$410.151
	(389.252)
q1	$-$0.044***
	(0.00001)
q2	0.029
	(0.580)
Constant	1,231.311*
	(670.766)
Observations	10,000
R^2	1.000
Adjusted R^2	1.000
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.12 Estimated Regression Results under $\rho = -0.5$, $\phi = -0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-$622.013***
	(137.507)
q1	$-$0.090***
	(0.00000)
q2	$-$0.415
	(0.415)
Constant	577.425**
	(254.484)
Observations	10,000
R^2	1.000
Adjusted R^2	1.000
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.13 Estimated Regression Results under $\rho = -0.5$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	0.020
	(0.047)
h1	-0.051^{***}
	(0.002)
h2	-0.115^{***}
	(0.001)
Constant	3.773^{***}
	(0.262)
Observations	10,000
R^2	0.619
Adjusted R^2	0.619

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.14 Estimated Regression Results under $\rho = -0.5$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-7.163^{***}
	(1.571)
h1	0.551^{***}
	(0.023)
h2	-0.436^{***}
	(0.007)
Constant	62.640^{***}
	(7.162)
Observations	10,000
R^2	0.860
Adjusted R^2	0.860

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.15 Estimated Regression Results under $\rho = -0.5$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	0.661^{***}
	(0.047)
q1	-0.051^{***}
	(0.002)
q2	-0.192^{***}
	(0.002)
Constant	0.486^{***}
	(0.088)
Observations	10,000
R^2	0.619
Adjusted R^2	0.619

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.16 Estimated Regression Results under $\rho = -0.5$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-6.699^{***}
	(1.571)
q1	0.551^{***}
	(0.023)
q2	-0.726^{***}
	(0.011)
Constant	19.212^{***}
	(2.384)
Observations	10,000
R^2	0.860
Adjusted R^2	0.860

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.17 Estimated Regression Results under $\rho = 0.5$, $\phi = -0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	0.205***
	(0.025)
h1	−0.033***
	(0.0003)
h2	−0.075***
	(0.001)
Constant	3.020***
	(0.140)
Observations	10,000
R^2	0.644
Adjusted R^2	0.644
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.18 Estimated Regression Results under $\rho = 0.5$, $\phi = -0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	0.177***
	(0.024)
h1	−0.065***
	(0.001)
h2	−0.106***
	(0.001)
Constant	3.284***
	(0.129)
Observations	10,000
R^2	0.837
Adjusted R^2	0.837
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.19 Estimated Regression Results under $\rho = 0.5$, $\phi = -0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	0.795***
	(0.025)
q1	−0.033***
	(0.0003)
q2	−0.124***
	(0.002)
Constant	0.392***
	(0.046)
Observations	10,000
R^2	0.653
Adjusted R^2	0.653
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.20 Estimated Regression Results under $\rho = 0.5$, $\phi = -0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	0.754***
	(0.024)
q1	−0.065***
	(0.001)
q2	−0.177***
	(0.001)
Constant	0.436***
	(0.043)
Observations	10,000
R^2	0.840
Adjusted R^2	0.840
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.21 Estimated Regression Results under $\rho = 0.5$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-938.788^{***}
	(150.044)
h1	0.084^{***}
	(0.010)
h2	-0.269^{***}
	(0.007)
Constant	$2{,}675.689^{***}$
	(535.587)
Observations	10,000
R^2	0.179
Adjusted R^2	0.179

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.22 Estimated Regression Results under $\rho = 0.5$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-551.627^{*}
	(301.606)
h1	-0.083^{***}
	(0.0001)
h2	0.009^{***}
	(0.003)
Constant	$1{,}686.222^{*}$
	(915.847)
Observations	10,000
R^2	0.975
Adjusted R^2	0.975

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.23 Estimated Regression Results under $\rho = 0.5$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-938.369^{***}
	(150.044)
q1	0.084^{***}
	(0.010)
q2	-0.448^{***}
	(0.011)
Constant	890.952^{***}
	(178.529)
Observations	10,000
R^2	0.179
Adjusted R^2	0.179

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.24 Estimated Regression Results under $\rho = 0.5$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-551.261^{*}
	(301.606)
q1	-0.083^{***}
	(0.0001)
q2	0.014^{***}
	(0.005)
Constant	561.969^{*}
	(305.282)
Observations	10,000
R^2	0.975
Adjusted R^2	0.975

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.25 Estimated Regression Results under $\rho = 0.9$, $\phi = -0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-0.229^{**}
	(0.102)
h1	-0.092^{***}
	(0.006)
h2	-0.205^{***}
	(0.002)
Constant	5.954^{***}
	(0.558)
Observations	10,000
R^2	0.561
Adjusted R^2	0.561

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.26 Estimated Regression Results under $\rho = 0.9$, $\phi = -0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-0.873^{***}
	(0.098)
h1	-0.020^{***}
	(0.008)
h2	-0.084^{***}
	(0.002)
Constant	6.088^{***}
	(0.520)
Observations	10,000
R^2	0.740
Adjusted R^2	0.740

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.27 Estimated Regression Results under $\rho = 0.9$, $\phi = -0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	0.382^{***}
	(0.102)
q1	-0.092^{***}
	(0.006)
q2	-0.341^{***}
	(0.003)
Constant	1.041^{***}
	(0.185)
Observations	10,000
R^2	0.561
Adjusted R^2	0.561

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.28 Estimated Regression Results under $\rho = 0.9$, $\phi = -0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-0.282^{***}
	(0.098)
q1	-0.020^{***}
	(0.008)
q2	-0.140^{***}
	(0.003)
Constant	1.367^{***}
	(0.172)
Observations	10,000
R^2	0.740
Adjusted R^2	0.740

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.29 Estimated Regression Results under $\rho = 0.9$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-597.656^{***}
	(96.971)
h1	-0.041^{***}
	(0.002)
h2	-0.092^{***}
	(0.002)
Constant	$1,909.597^{***}$
	(339.207)
Observations	10,000
R^2	0.480
Adjusted R^2	0.480

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table F.30 Estimated Regression Results under $\rho = 0.9$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-626.015
	(423.731)
h1	-0.019^{***}
	(0.00002)
h2	-0.017^{***}
	(0.003)
Constant	715.948
	(1,488.198)
Observations	10,000
R^2	0.990
Adjusted R^2	0.990

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table F.31 Estimated Regression Results under $\rho = 0.9$, $\phi = 0.5$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-597.263^{***}
	(96.971)
q1	-0.041^{***}
	(0.002)
q2	-0.153^{***}
	(0.003)
Constant	636.120^{***}
	(113.069)
Observations	10,000
R^2	0.480
Adjusted R^2	0.480

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table F.32 Estimated Regression Results under $\rho = 0.9$, $\phi = 0.9$, and $T = 15$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-625.438
	(423.731)
q1	-0.019^{***}
	(0.00002)
q2	-0.028^{***}
	(0.004)
Constant	238.321
	(496.065)
Observations	10,000
R^2	0.990
Adjusted R^2	0.990

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table F.33 Estimated Regression Results under $\rho = -0.9$, $\phi = -0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-16{,}917.510^{***}$
	(564.531)
h1	0.183^{***}
	(0.013)
h2	-0.055^{***}
	(0.003)
Constant	$35{,}345.830^{***}$
	(3,210.674)
Observations	10,000
R^2	0.110
Adjusted R^2	0.110
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.34 Estimated Regression Results under $\rho = -0.9$, $\phi = -0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-16{,}902.700^{***}$
	(3,693.997)
h1	50.074^{***}
	(0.439)
h2	-15.796^{***}
	(0.132)
Constant	$270{,}023.700^{***}$
	(22,356.580)
Observations	10,000
R^2	0.658
Adjusted R^2	0.658
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.35 Estimated Regression Results under $\rho = -0.9$, $\phi = -0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-16{,}916.980^{***}$
	(564.530)
q1	0.183^{***}
	(0.013)
q2	-0.091^{***}
	(0.005)
Constant	$11{,}781.430^{***}$
	(1,070.222)
Observations	10,000
R^2	0.110
Adjusted R^2	0.110
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.36 Estimated Regression Results under $\rho = -0.9$, $\phi = -0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-16{,}901.990^{***}$
	(3,693.991)
q1	50.074^{***}
	(0.439)
q2	-26.327^{***}
	(0.220)
Constant	$89{,}942.750^{***}$
	(7,452.264)
Observations	10,000
R^2	0.658
Adjusted R^2	0.658
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.37 Estimated Regression Results under $\rho = -0.9$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	1,640.171***
	(158.240)
h1	$-$85.811***
	(0.403)
h2	$-$15.249***
	(0.064)
Constant	$-$3,669.433***
	(1,178.259)
Observations	10,000
R^2	0.953
Adjusted R^2	0.953
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.38 Estimated Regression Results under $\rho = -0.9$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	10,072.960***
	(498.356)
h1	$-$90.624***
	(2.219)
h2	$-$3.729***
	(0.468)
Constant	$-$41,060.320***
	(4,056.903)
Observations	10,000
R^2	0.869
Adjusted R^2	0.869
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.39 Estimated Regression Results under $\rho = -0.9$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	1,640.567***
	(158.240)
q1	$-$85.811***
	(0.403)
q2	$-$25.414***
	(0.106)
Constant	$-$1,218.803***
	(392.793)
Observations	10,000
R^2	0.953
Adjusted R^2	0.953
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.40 Estimated Regression Results under $\rho = -0.9$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	10,073.470***
	(498.355)
q1	$-$90.623***
	(2.219)
q2	$-$6.215***
	(0.780)
Constant	$-$13,651.410***
	(1,352.682)
Observations	10,000
R^2	0.869
Adjusted R^2	0.869
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.41 Estimated Regression Results under $\rho = -0.5$, $\phi = -0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-$324.082***
	(67.147)
h1	$-$0.012***
	(0.0001)
h2	$-$0.017***
	(0.0001)
Constant	895.454***
	(174.135)
Observations	10,000
R^2	0.803
Adjusted R^2	0.803
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.42 Estimated Regression Results under $\rho = -0.5$, $\phi = -0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	93.724
	(137.620)
h1	$-$41.914***
	(0.421)
h2	11.518***
	(0.128)
Constant	$-$7,146.587***
	(486.629)
Observations	10,000
R^2	0.995
Adjusted R^2	0.995
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.43 Estimated Regression Results under $\rho = -0.5$, $\phi = -0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-$323.947***
	(67.147)
q1	$-$0.012***
	(0.0001)
q2	$-$0.028***
	(0.0002)
Constant	298.395***
	(58.045)
Observations	10,000
R^2	0.803
Adjusted R^2	0.803
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.44 Estimated Regression Results under $\rho = -0.5$, $\phi = -0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	93.900
	(137.620)
q1	$-$41.914***
	(0.421)
q2	19.197***
	(0.214)
Constant	$-$2,332.546***
	(162.096)
Observations	10,000
R^2	0.995
Adjusted R^2	0.995
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.45 Estimated Regression Results under $\rho = -0.5$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-0.430^{***}
	(0.094)
h1	-0.039^{***}
	(0.001)
h2	-0.174^{***}
	(0.001)
Constant	9.638^{***}
	(0.411)
Observations	10,000
R^2	0.635
Adjusted R^2	0.635
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.46 Estimated Regression Results under $\rho = -0.5$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-2.839^{***}
	(0.126)
h1	-0.030^{***}
	(0.001)
h2	-0.033^{***}
	(0.0004)
Constant	11.206^{***}
	(0.479)
Observations	10,000
R^2	0.835
Adjusted R^2	0.835
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.47 Estimated Regression Results under $\rho = -0.5$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-0.219^{**}
	(0.094)
q1	-0.039^{***}
	(0.001)
q2	-0.289^{***}
	(0.002)
Constant	2.660^{***}
	(0.136)
Observations	10,000
R^2	0.635
Adjusted R^2	0.635
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.48 Estimated Regression Results under $\rho = -0.5$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-2.666^{***}
	(0.126)
q1	-0.030^{***}
	(0.001)
q2	-0.054^{***}
	(0.001)
Constant	3.571^{***}
	(0.160)
Observations	10,000
R^2	0.835
Adjusted R^2	0.835
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.49 Estimated Regression Results under $\rho = 0.5$, $\phi = -0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-$1.844***
	(0.111)
h1	$-$0.007***
	(0.0001)
h2	$-$0.036***
	(0.001)
Constant	7.938***
	(0.448)
Observations	10,000
R^2	0.476
Adjusted R^2	0.476

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.50 Estimated Regression Results under $\rho = 0.5$, $\phi = -0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	0.112***
	(0.028)
h1	$-$0.013***
	(0.0002)
h2	$-$0.040***
	(0.001)
Constant	2.650***
	(0.125)
Observations	10,000
R^2	0.394
Adjusted R^2	0.394

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.51 Estimated Regression Results under $\rho = 0.5$, $\phi = -0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-$1.661***
	(0.111)
q1	$-$0.007***
	(0.0001)
q2	$-$0.060***
	(0.001)
Constant	2.452***
	(0.149)
Observations	10,000
R^2	0.475
Adjusted R^2	0.475

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.52 Estimated Regression Results under $\rho = 0.5$, $\phi = -0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	0.328***
	(0.028)
q1	$-$0.013***
	(0.0002)
q2	$-$0.066***
	(0.001)
Constant	0.647***
	(0.041)
Observations	10,000
R^2	0.400
Adjusted R^2	0.400

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.53 Estimated Regression Results under $\rho = 0.5$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-60.420^{***}
	(2.585)
h1	0.061^{***}
	(0.011)
h2	-0.038^{***}
	(0.003)
Constant	138.167^{***}
	(8.814)
Observations	10,000
R^2	0.560
Adjusted R^2	0.560

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.54 Estimated Regression Results under $\rho = 0.5$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-709.540^{***}
	(83.419)
h1	-0.018^{***}
	(0.00002)
h2	-0.013^{***}
	(0.0002)
Constant	$1,327.280^{***}$
	(205.677)
Observations	10,000
R^2	0.989
Adjusted R^2	0.989

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.55 Estimated Regression Results under $\rho = 0.5$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-60.280^{***}
	(2.585)
q1	0.061^{***}
	(0.011)
q2	-0.064^{***}
	(0.005)
Constant	45.864^{***}
	(2.933)
Observations	10,000
R^2	0.560
Adjusted R^2	0.560

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.56 Estimated Regression Results under $\rho = 0.5$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-709.422^{***}
	(83.419)
q1	-0.018^{***}
	(0.00002)
q2	-0.022^{***}
	(0.0003)
Constant	442.359^{***}
	(68.559)
Observations	10,000
R^2	0.989
Adjusted R^2	0.989

Note: *p<0.1; **p<0.05; ***p<0.01

Table F.57 Estimated Regression Results under $\rho = 0.9$, $\phi = -0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	27.360***
	(3.386)
h1	−3.443***
	(0.090)
h2	−1.935***
	(0.013)
Constant	107.802***
	(18.841)
Observations	10,000
R^2	0.898
Adjusted R^2	0.898
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.58 Estimated Regression Results under $\rho = 0.9$, $\phi = -0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	15.513***
	(5.805)
h1	0.624***
	(0.071)
h2	−3.966***
	(0.014)
Constant	354.751***
	(36.467)
Observations	10,000
R^2	0.888
Adjusted R^2	0.888
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.59 Estimated Regression Results under $\rho = 0.9$, $\phi = -0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	27.638***
	(3.385)
q1	−3.443***
	(0.090)
q2	−3.224***
	(0.022)
Constant	32.587***
	(6.290)
Observations	10,000
R^2	0.898
Adjusted R^2	0.898
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.60 Estimated Regression Results under $\rho = 0.9$, $\phi = -0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	15.828***
	(5.804)
q1	0.623***
	(0.071)
q2	−6.610***
	(0.024)
Constant	107.731***
	(12.159)
Observations	10,000
R^2	0.888
Adjusted R^2	0.888
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.61 Estimated Regression Results under $\rho = 0.9$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	-366.489^{***}
	(9.527)
h1	0.063^{***}
	(0.011)
h2	-0.035^{***}
	(0.003)
Constant	$1{,}098.936^{***}$
	(46.018)
Observations	10,000
R^2	0.232
Adjusted R^2	0.232
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.62 Estimated Regression Results under $\rho = 0.9$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	X^2 statistic Cornish Fisher
w (Wald-stat)	$-2{,}393.622^{***}$
	(99.369)
h1	-0.025^{***}
	(0.001)
h2	0.005^{***}
	(0.0005)
Constant	$3{,}663.431^{***}$
	(384.328)
Observations	10,000
R^2	0.322
Adjusted R^2	0.321
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.63 Estimated Regression Results under $\rho = 0.9$, $\phi = 0.5$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	-366.112^{***}
	(9.527)
q1	0.063^{***}
	(0.011)
q2	-0.058^{***}
	(0.005)
Constant	365.934^{***}
	(15.332)
Observations	10,000
R^2	0.232
Adjusted R^2	0.231
Note:	*p<0.1; **p<0.05; ***p<0.01

Table F.64 Estimated Regression Results under $\rho = 0.9$, $\phi = 0.9$, and $T = 30$

	Dependent variable:
	F statistic Cornish Fisher
v (F-statistic)	$-2{,}393.217^{***}$
	(99.367)
q1	-0.025^{***}
	(0.001)
q2	0.008^{***}
	(0.001)
Constant	$1{,}220.973^{***}$
	(128.107)
Observations	10,000
R^2	0.322
Adjusted R^2	0.321
Note:	*p<0.1; **p<0.05; ***p<0.01

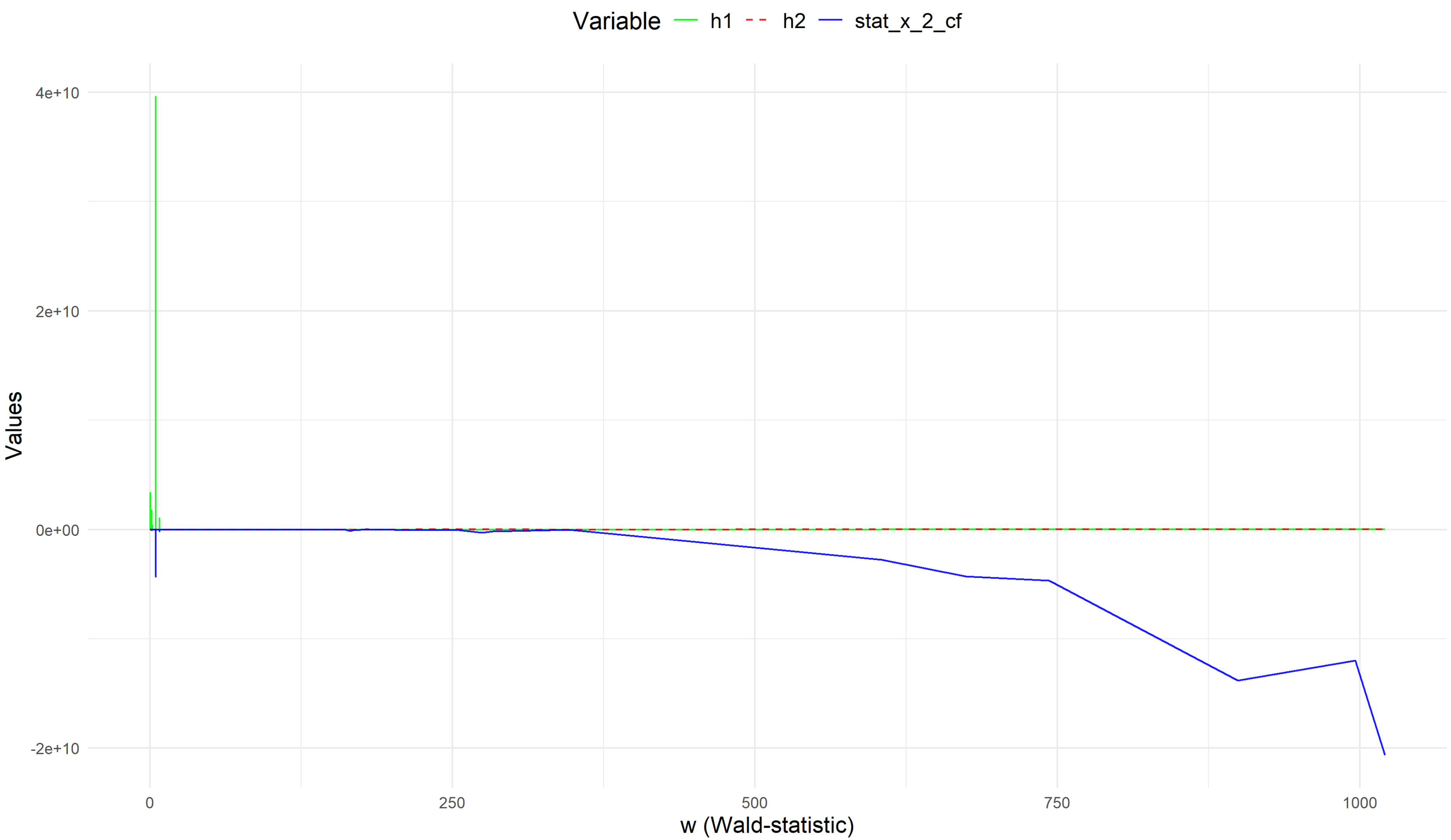

Figure F.1 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = -0.9$, and T=15

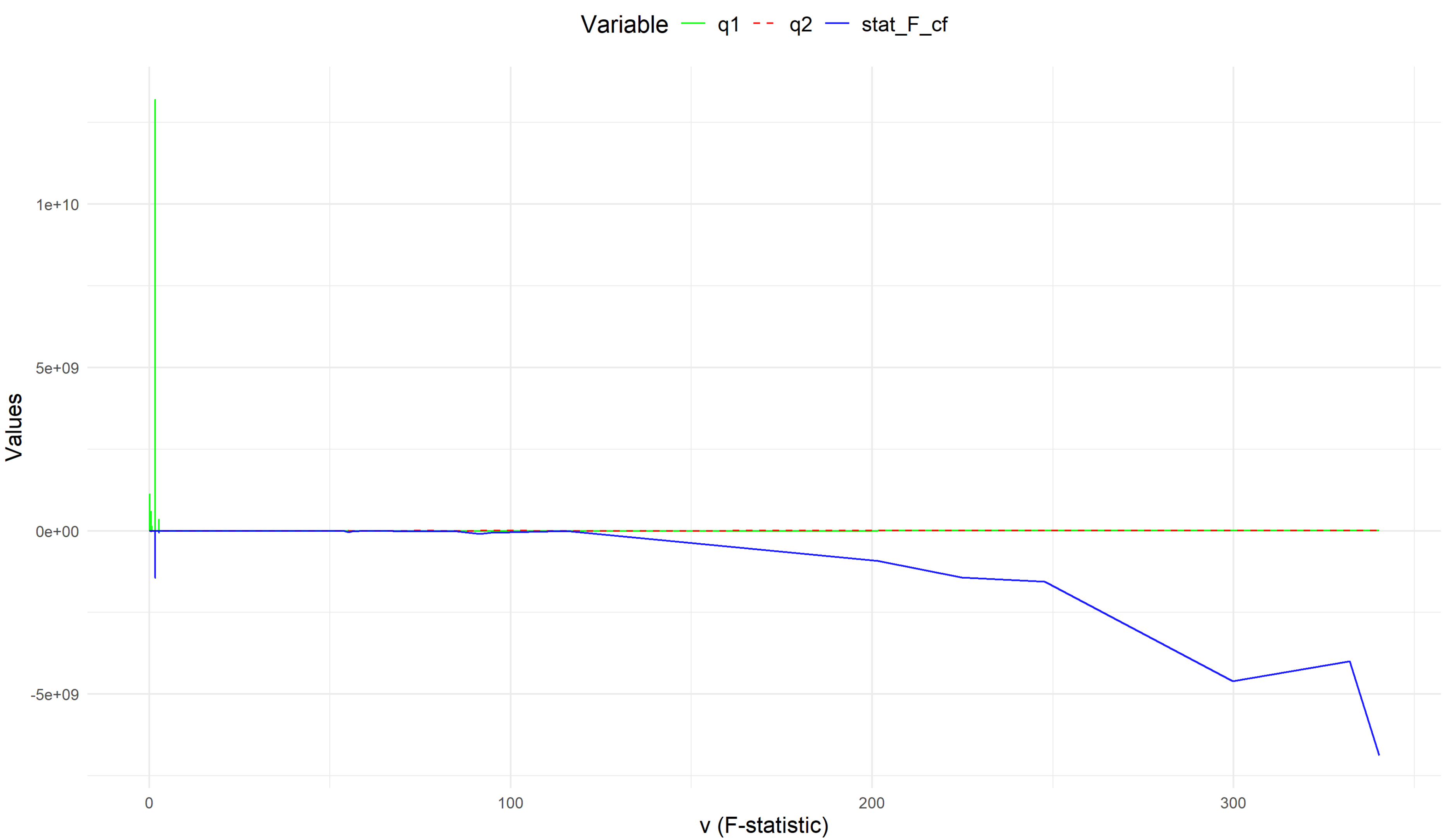

Figure F.2 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = -0.9$, and T=15

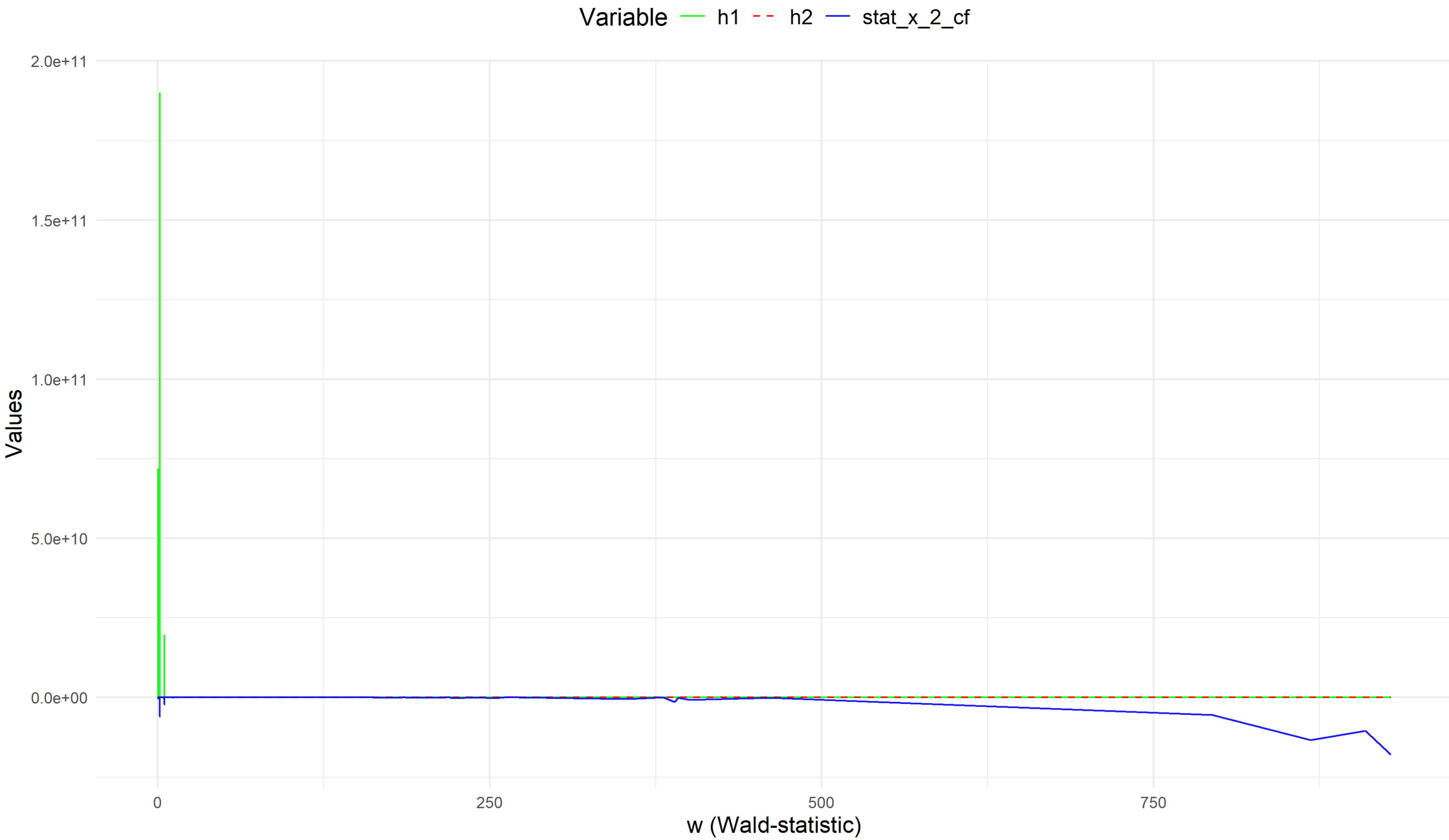

Figure F.3 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = -0.5$, and T=15

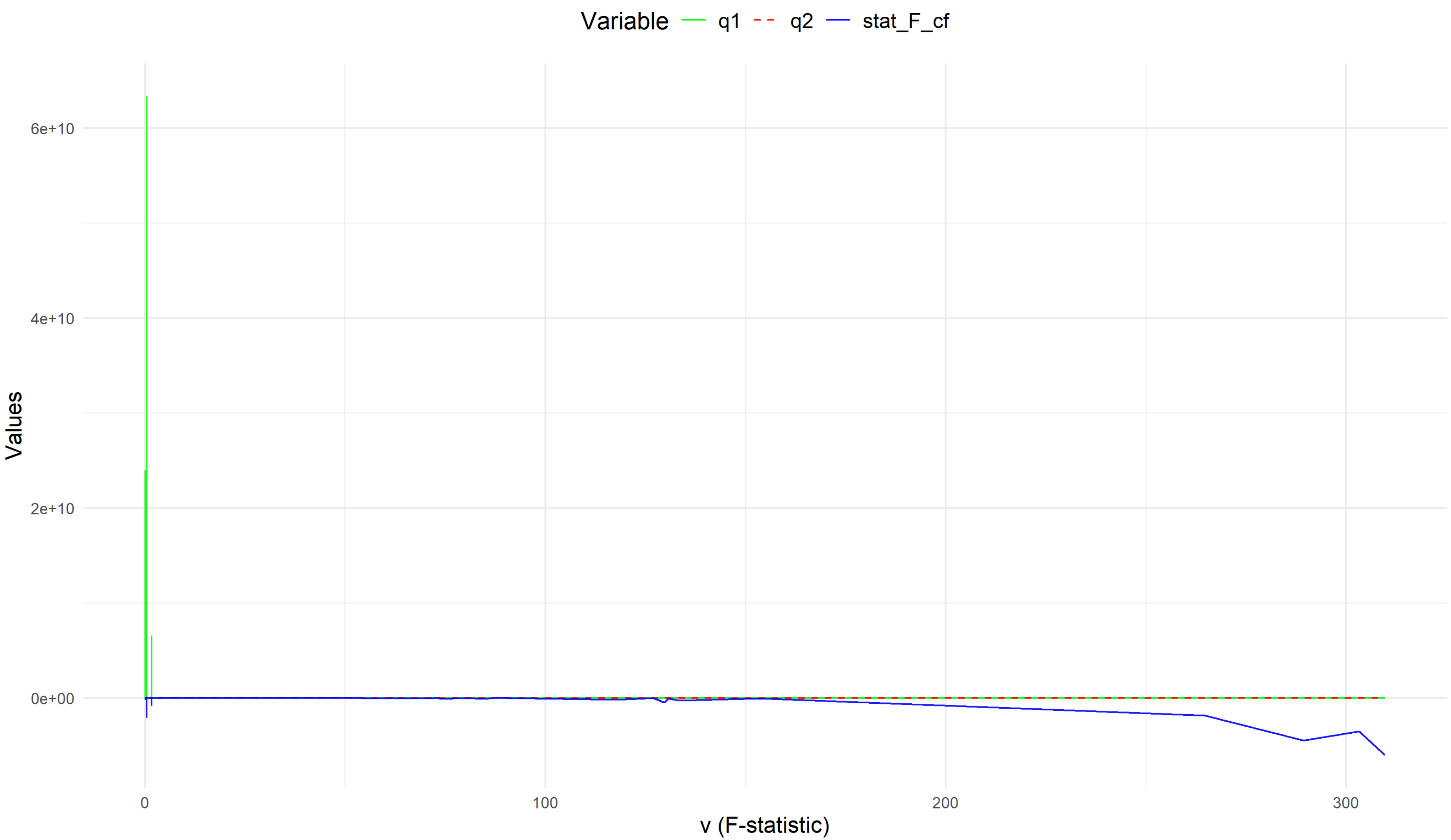

Figure F.4 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = -0.5$, and T=15

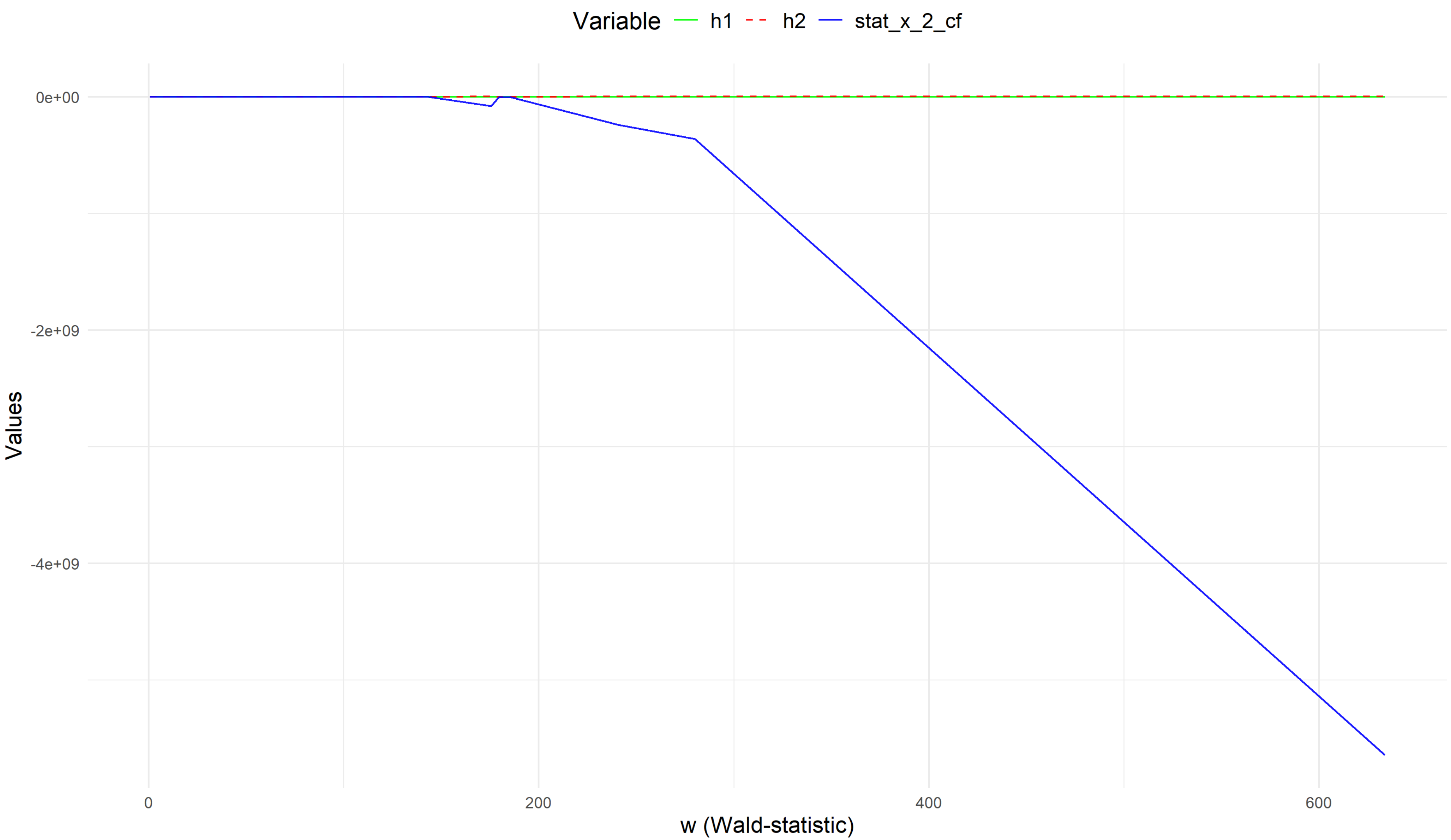

Figure F.5 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = 0.5$, and T=15

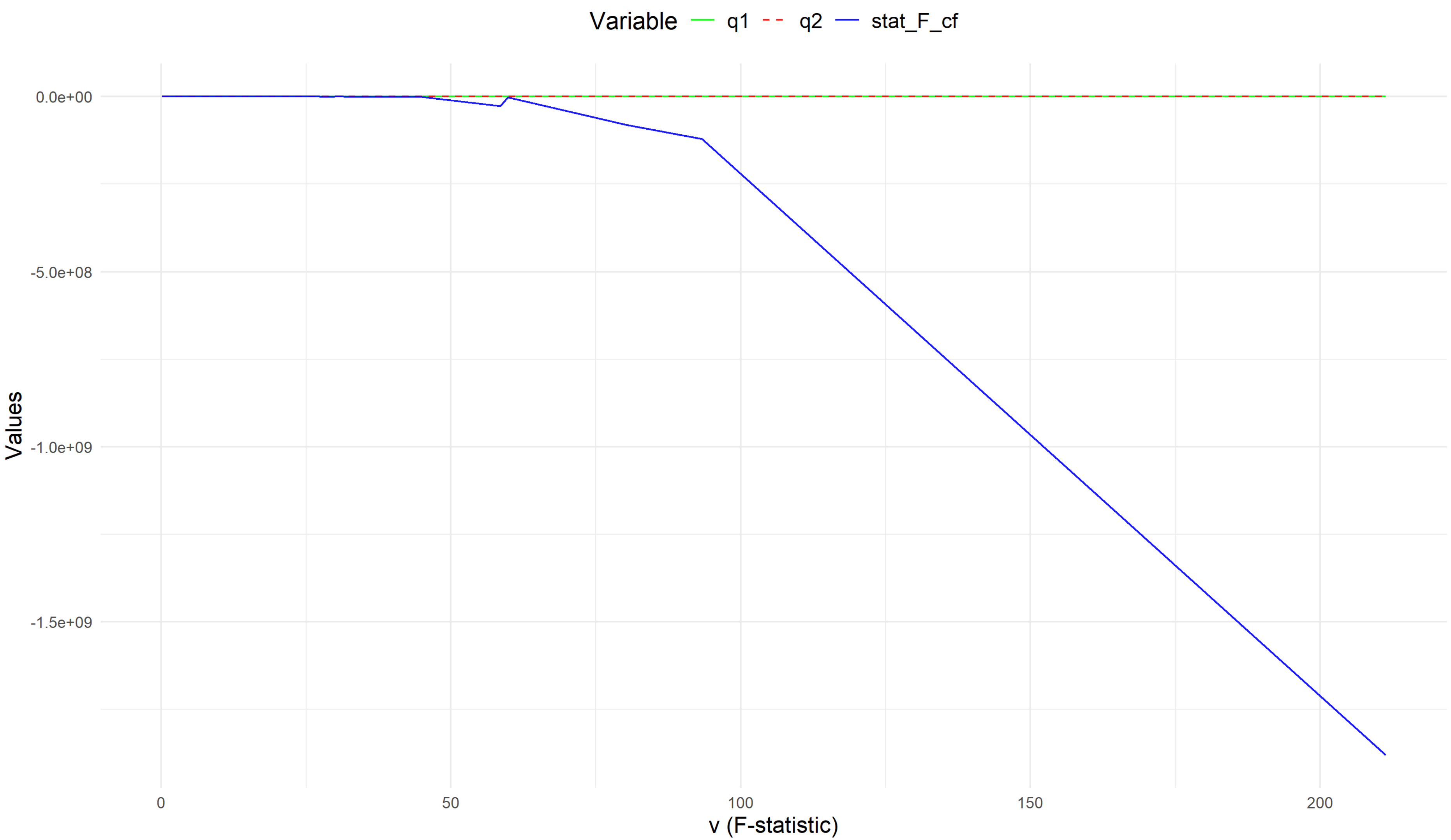

Figure F.6 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = 0.5$, and T=15

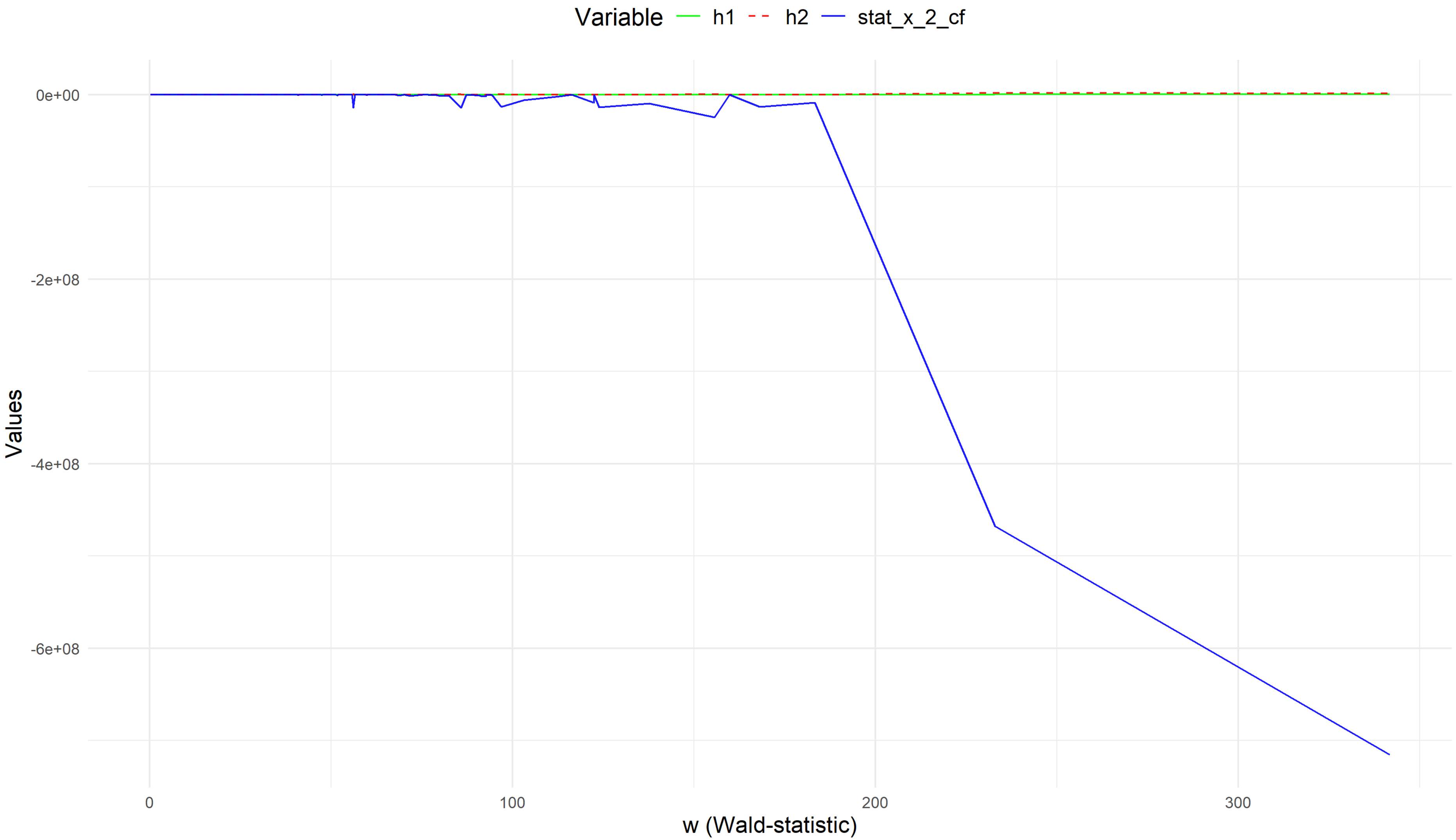

Figure F.7 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = 0.9$, and T=15

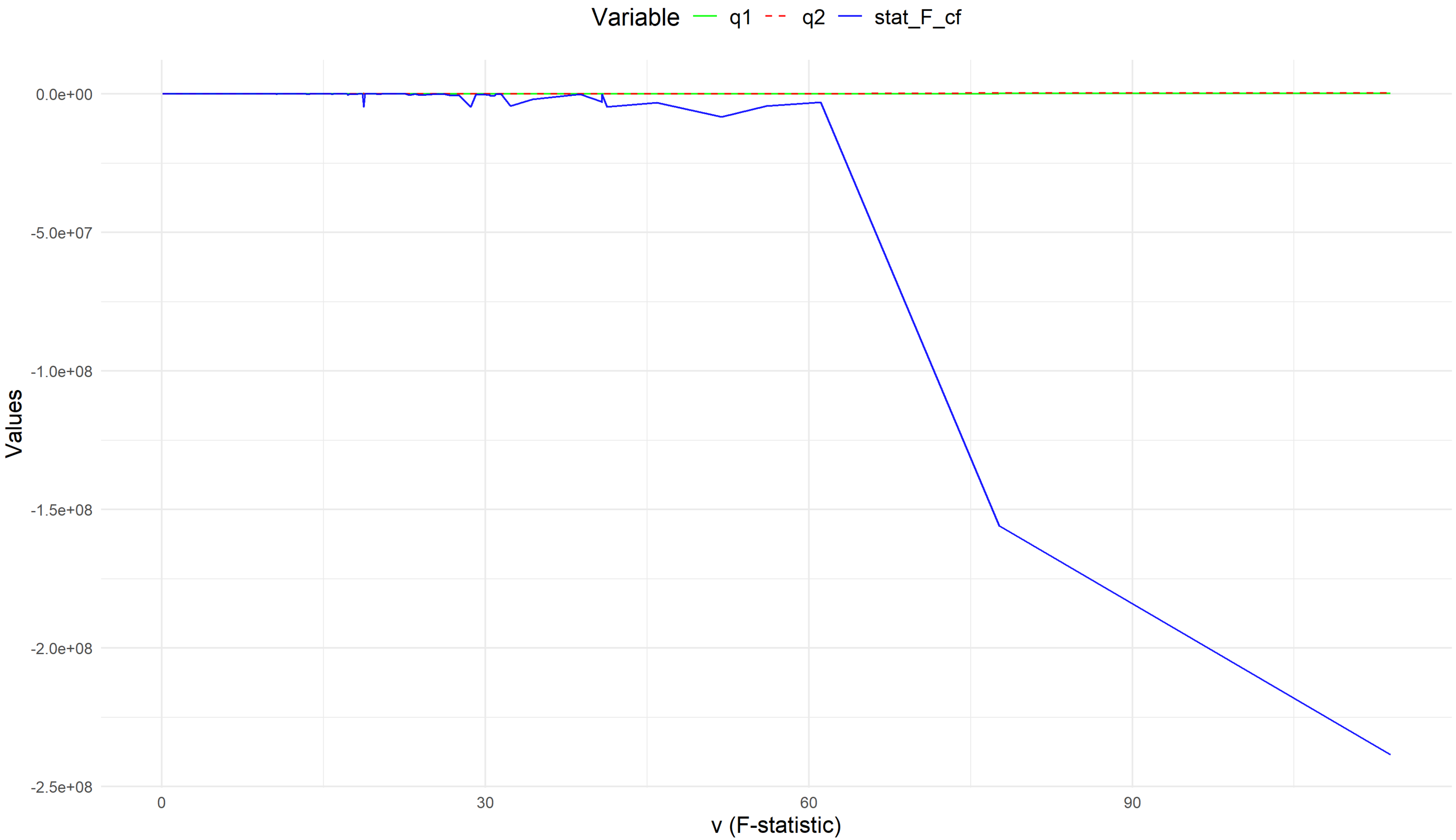

Figure F.8 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = 0.9$, and T=15

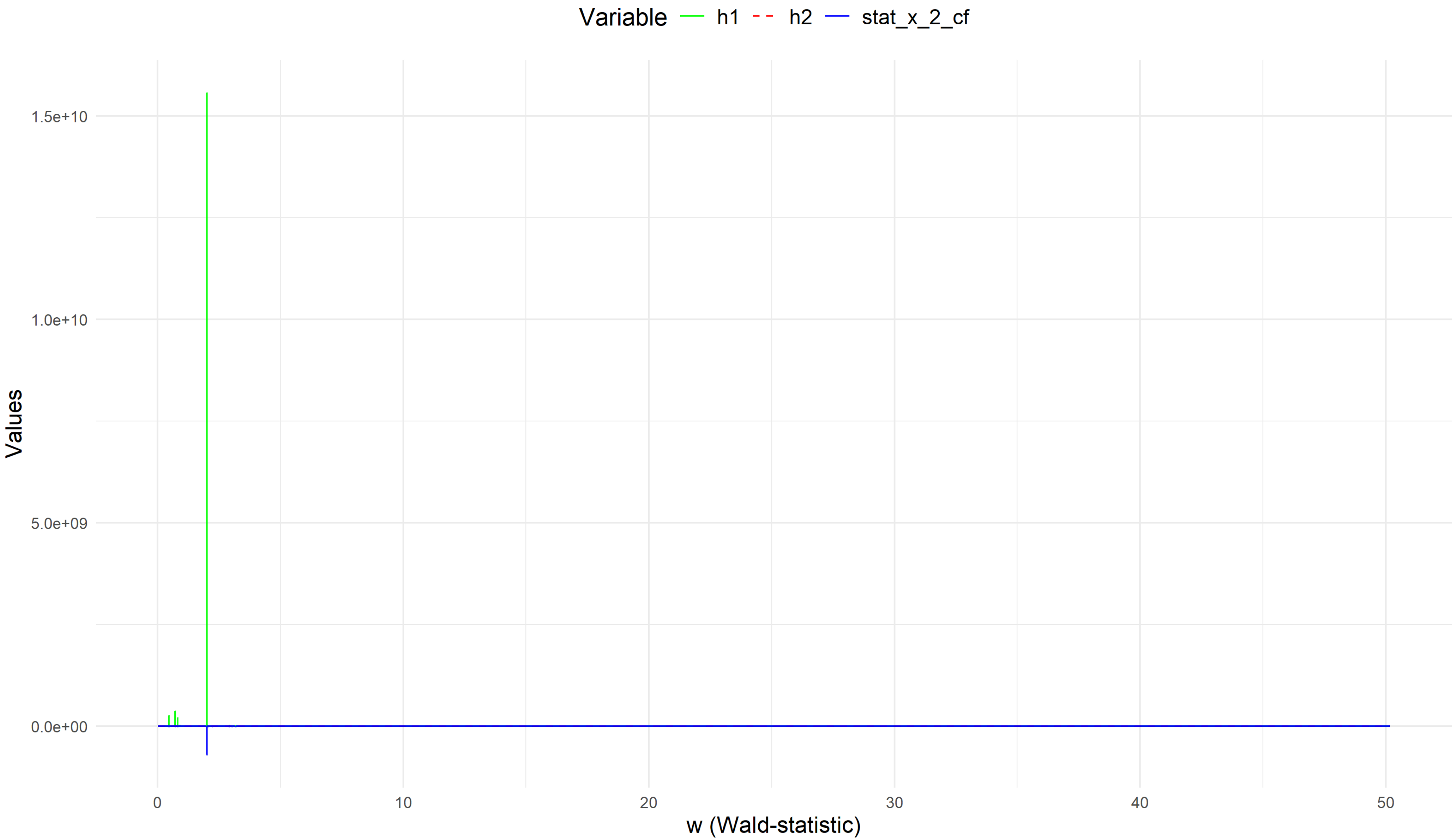

Figure F.9 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = -0.9$, and T=15

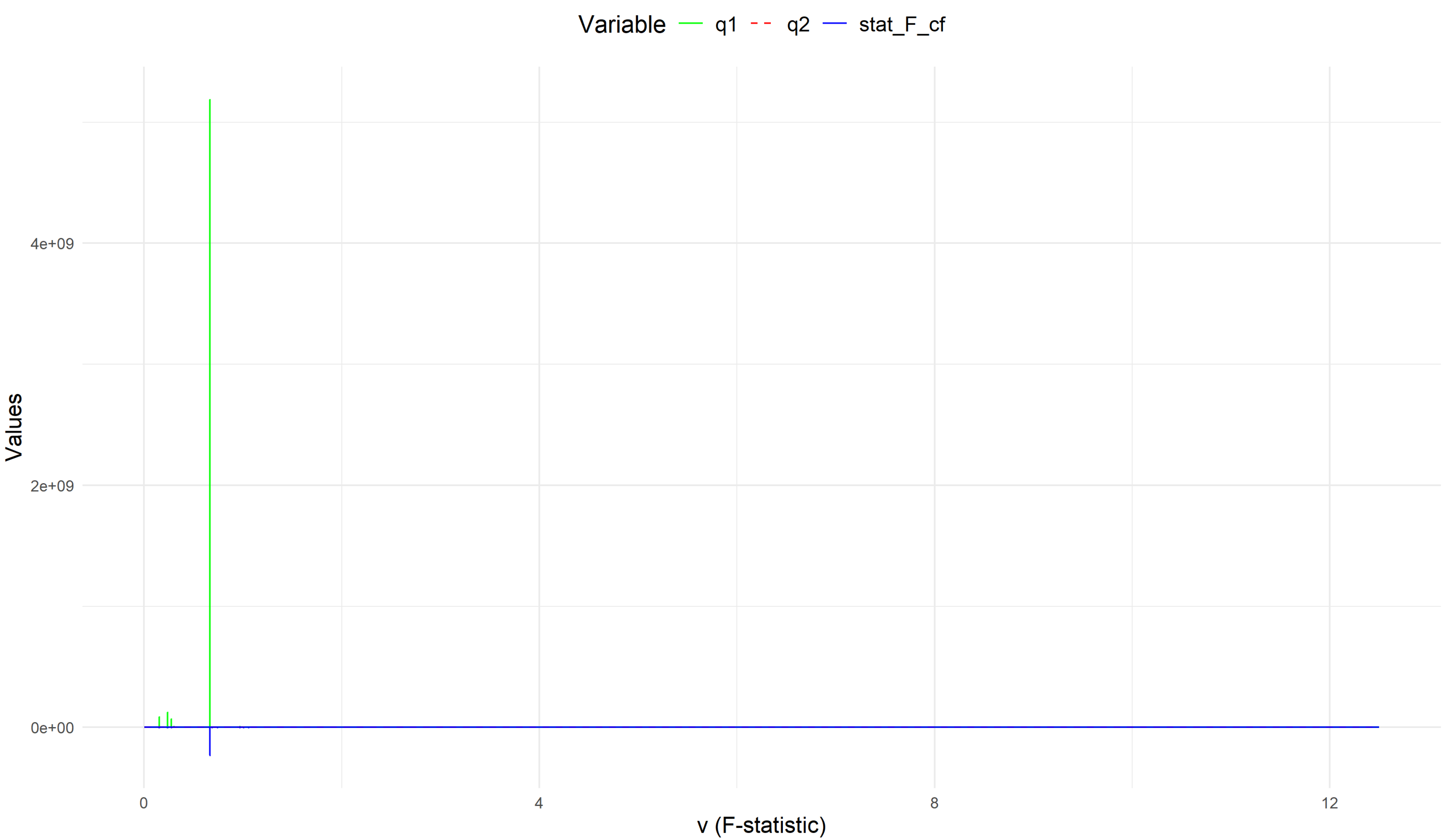

Figure F.10 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = -0.9$, and T=15

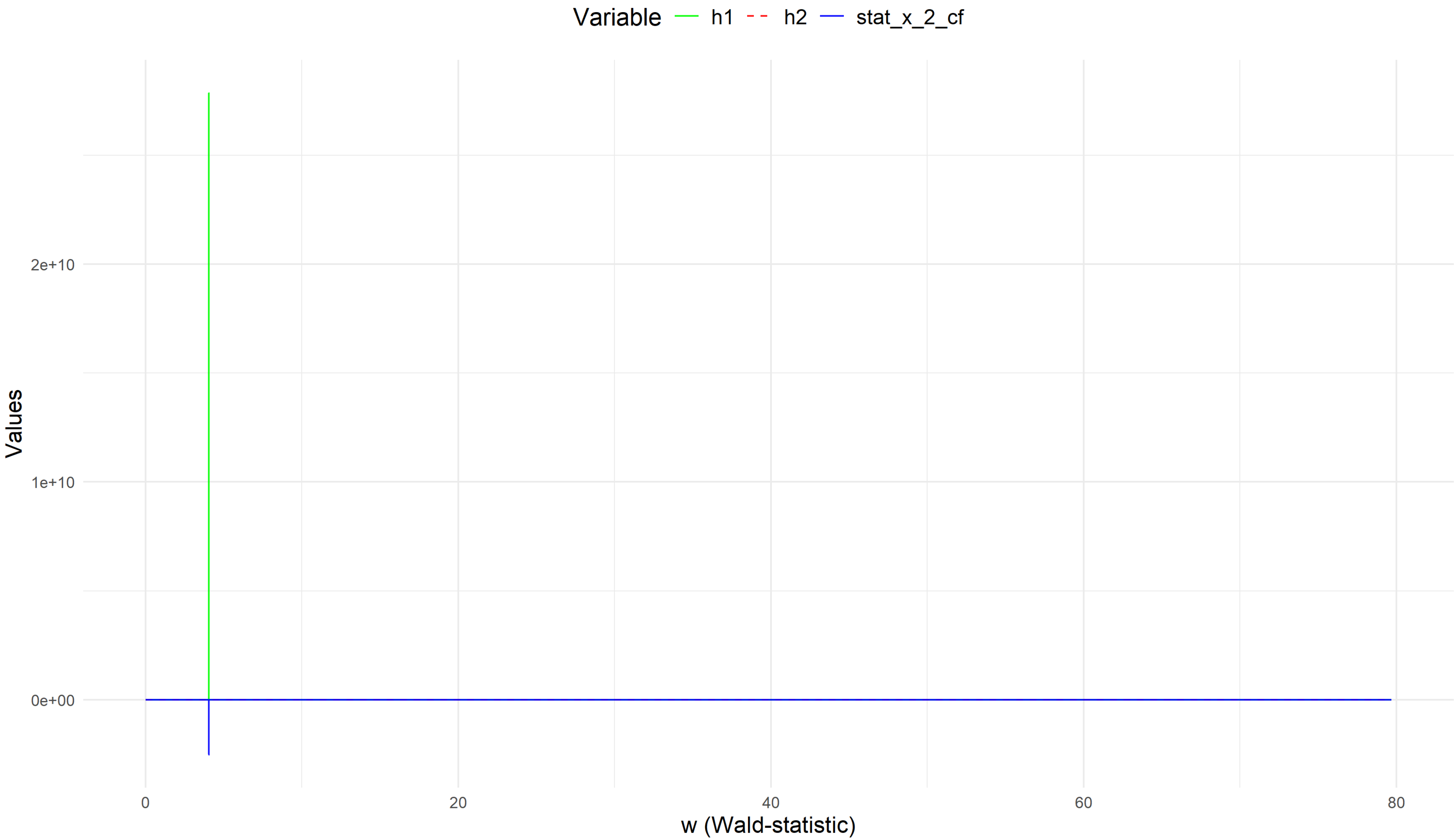

Figure F.11 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = -0.5$, and T=15

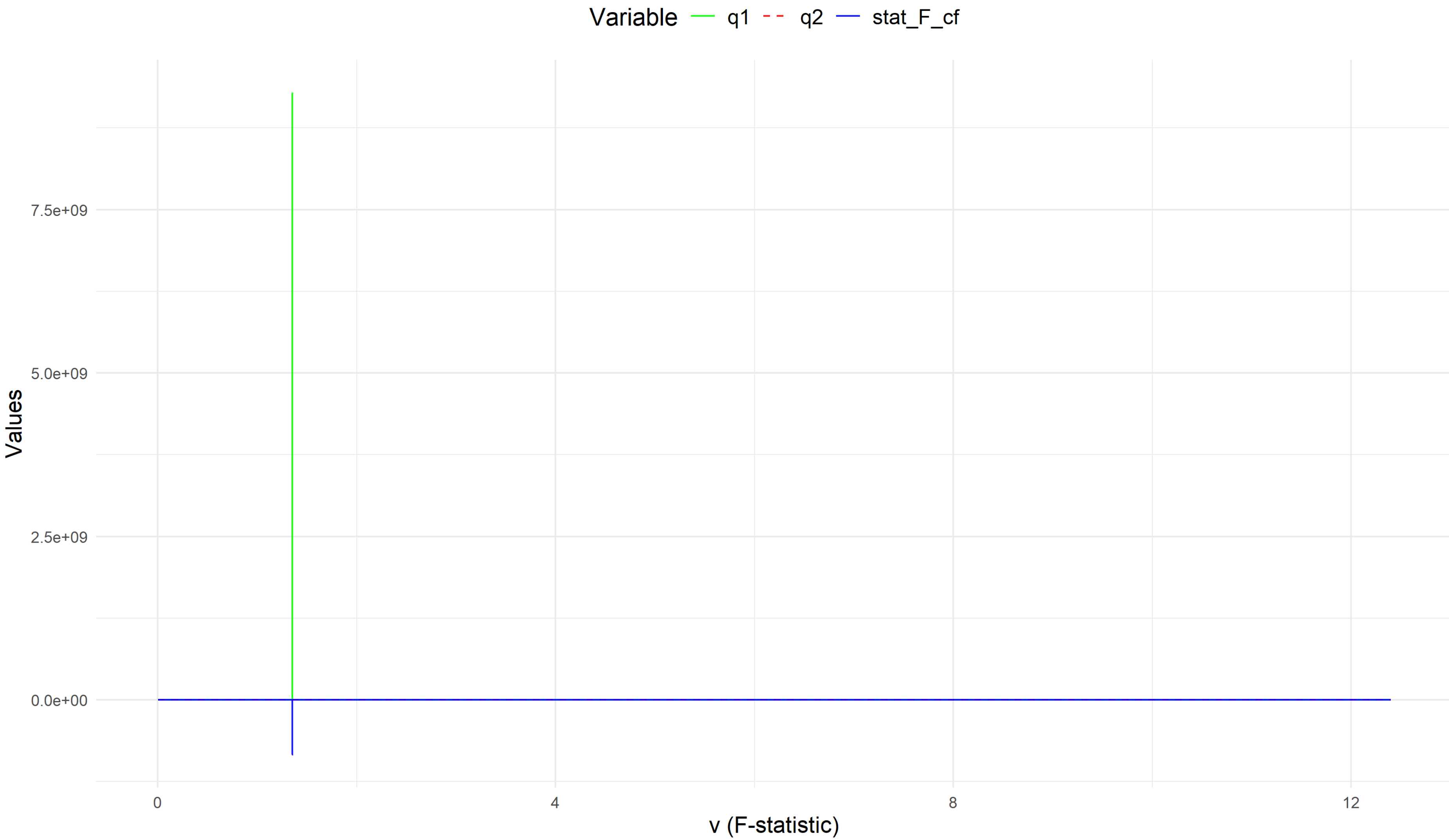

Figure F.12 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = -0.5$, and T=15

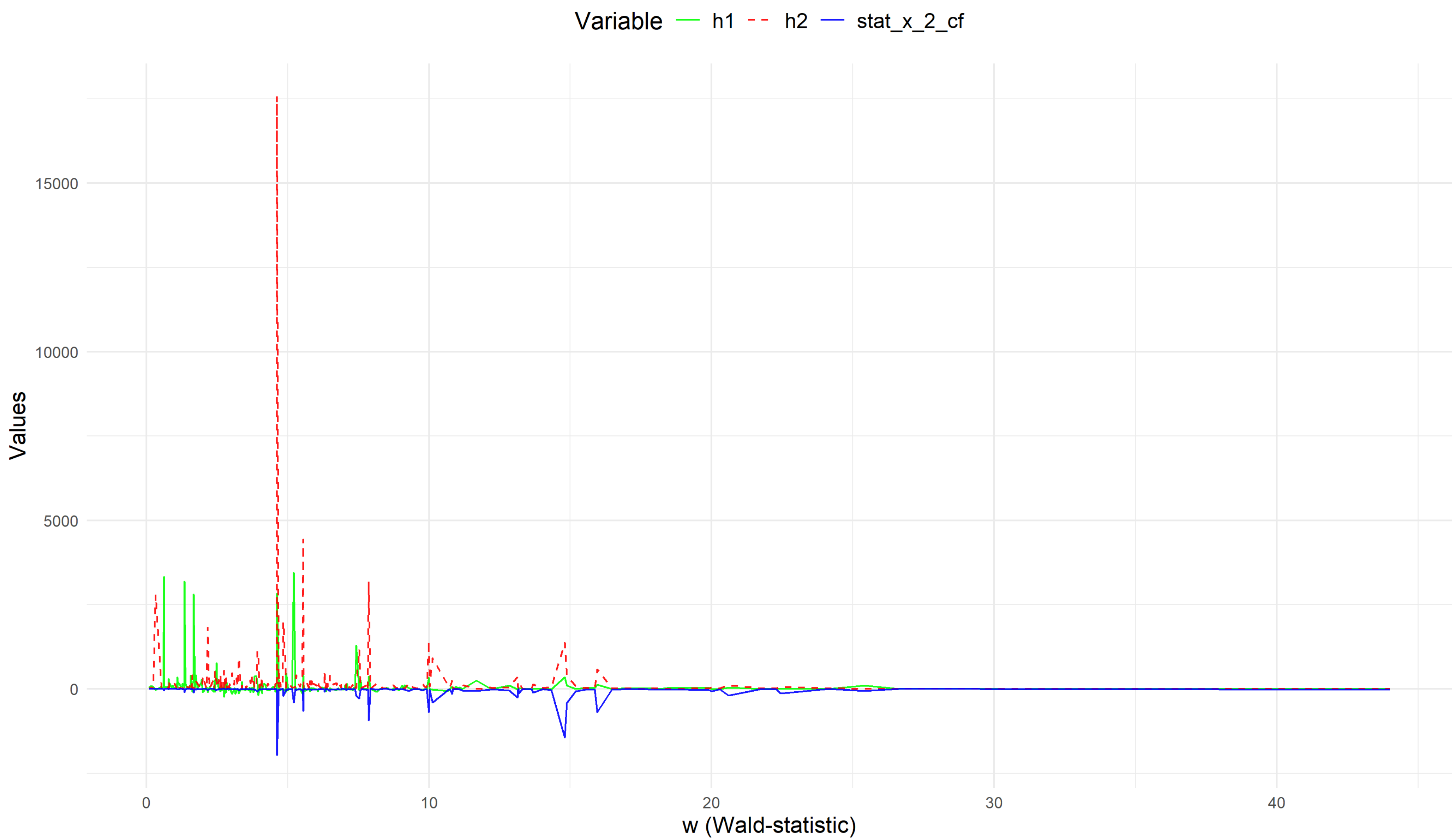

Figure F.13 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.5$, and T=15

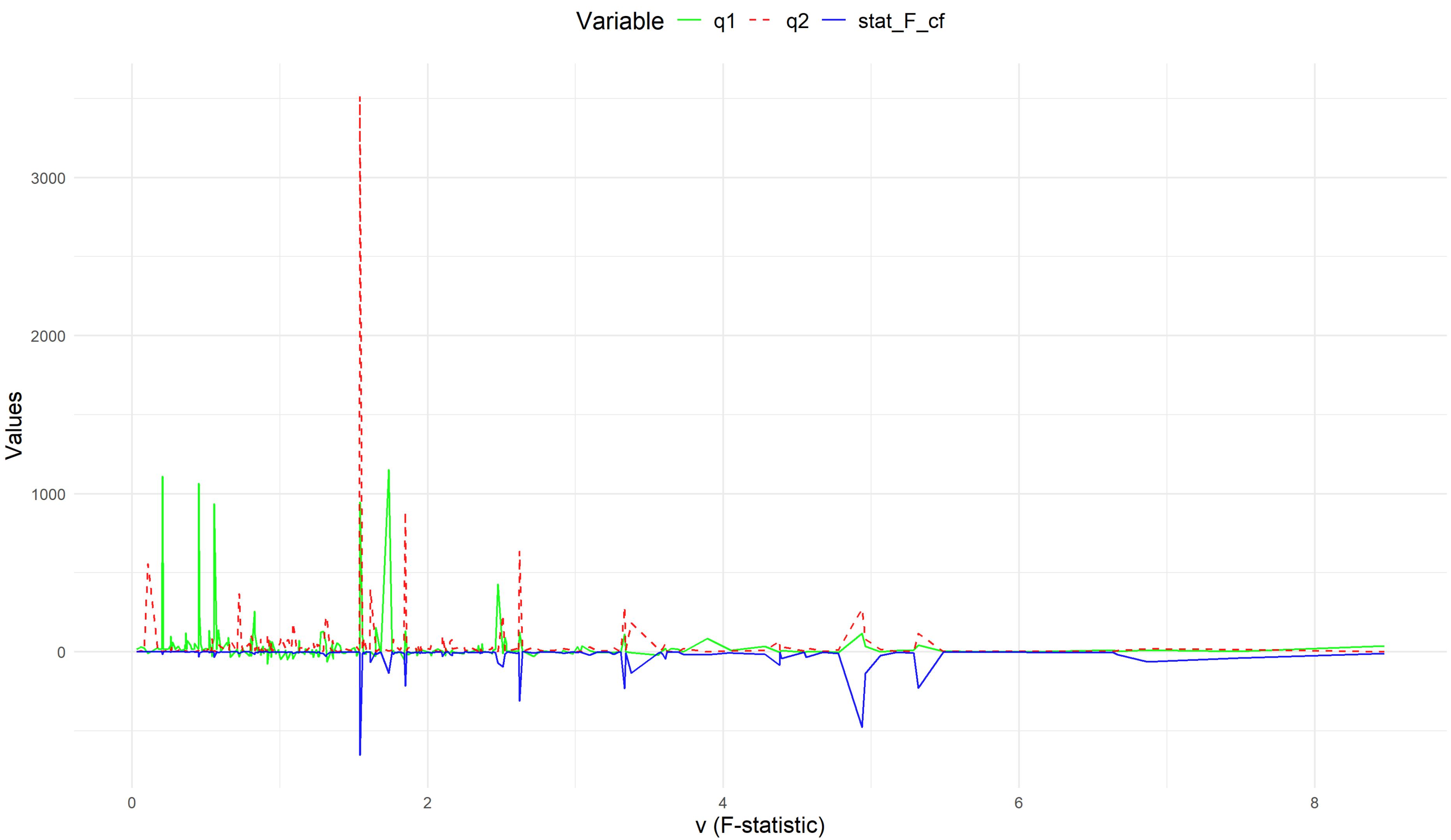

Figure F.14 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.5$, and T=15

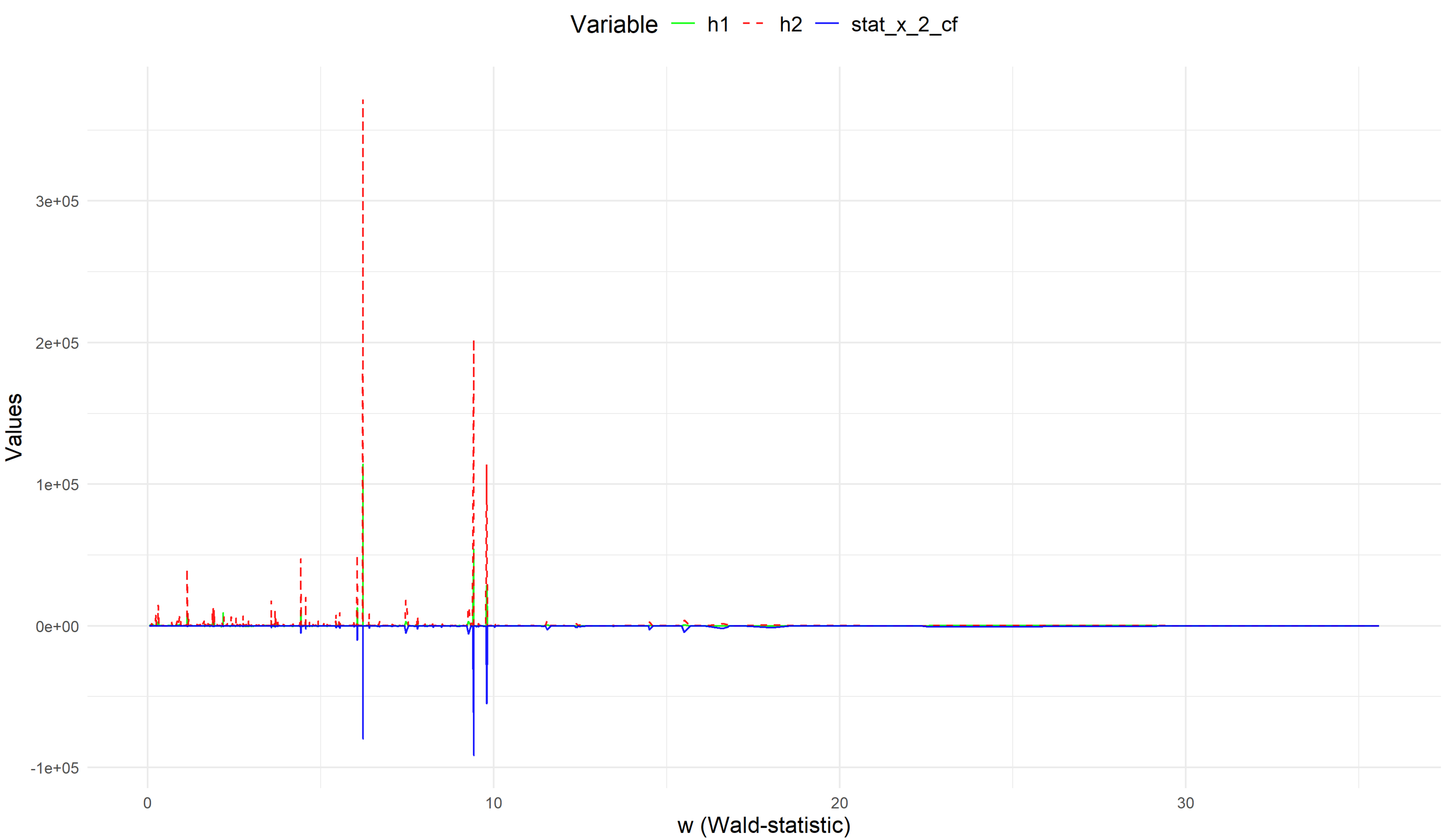

Figure F.15 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.9$, and T=15

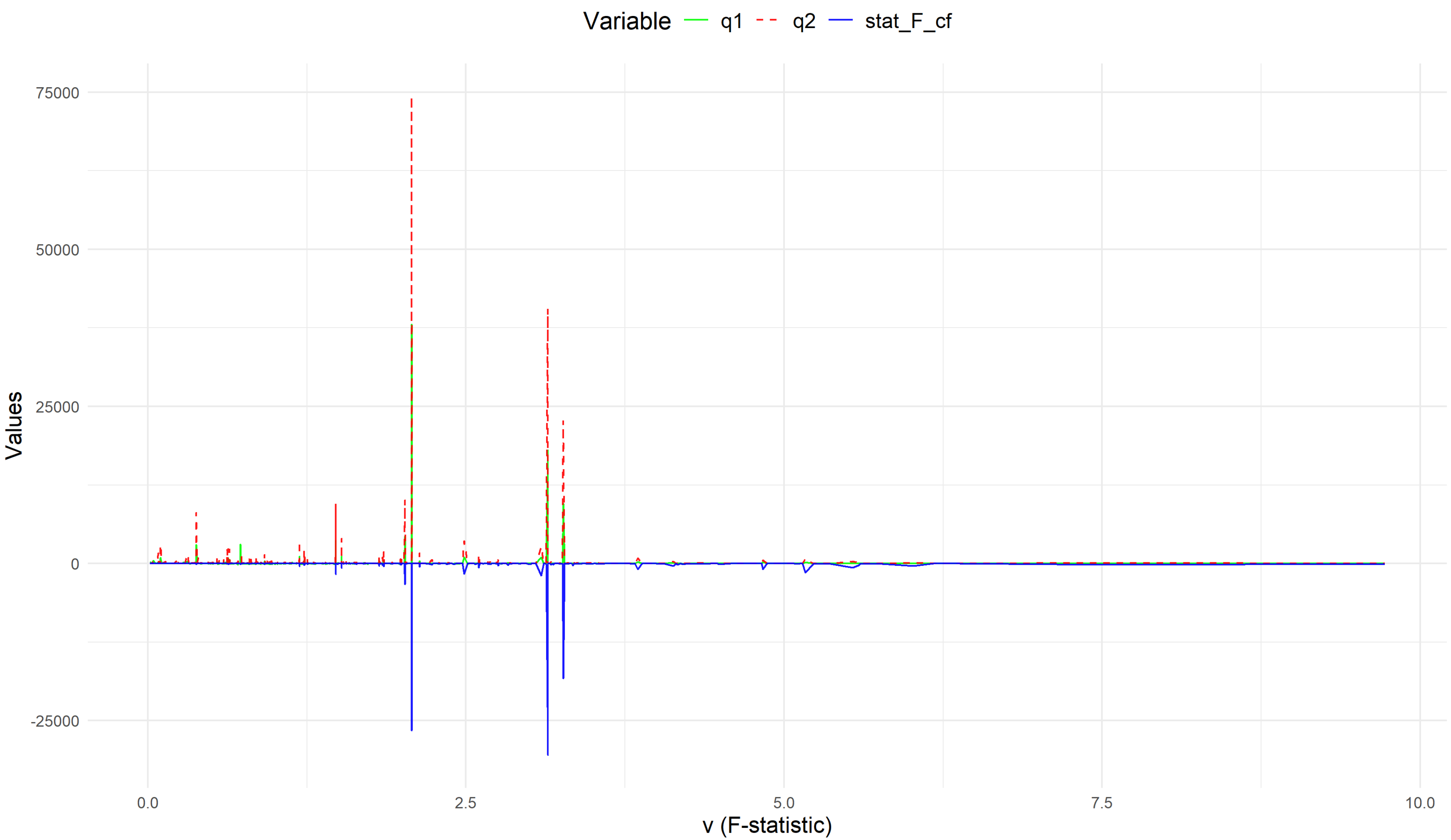

Figure F.16 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.9$, and T=15

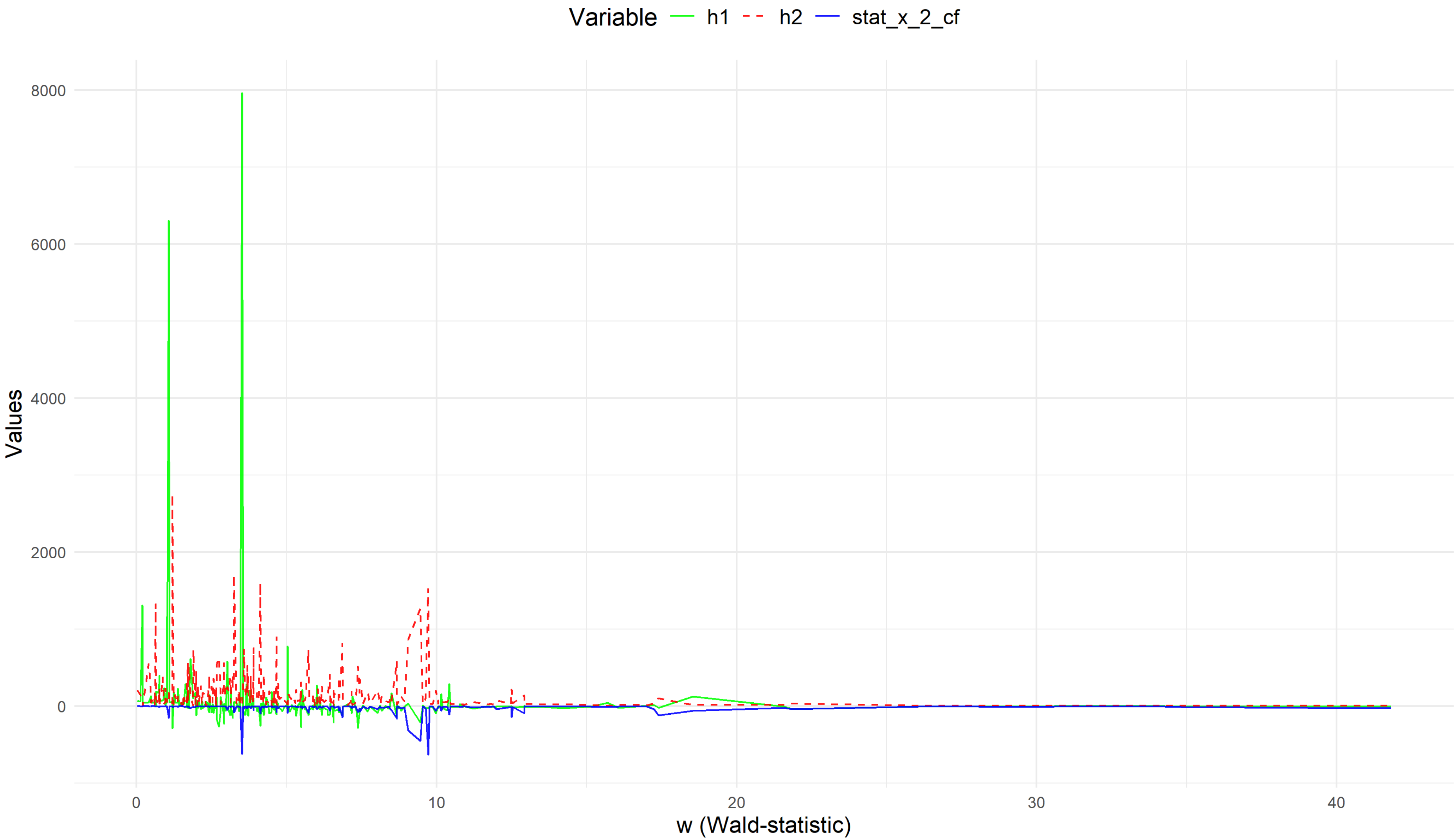

Figure F.17 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = -0.9$, and T=15

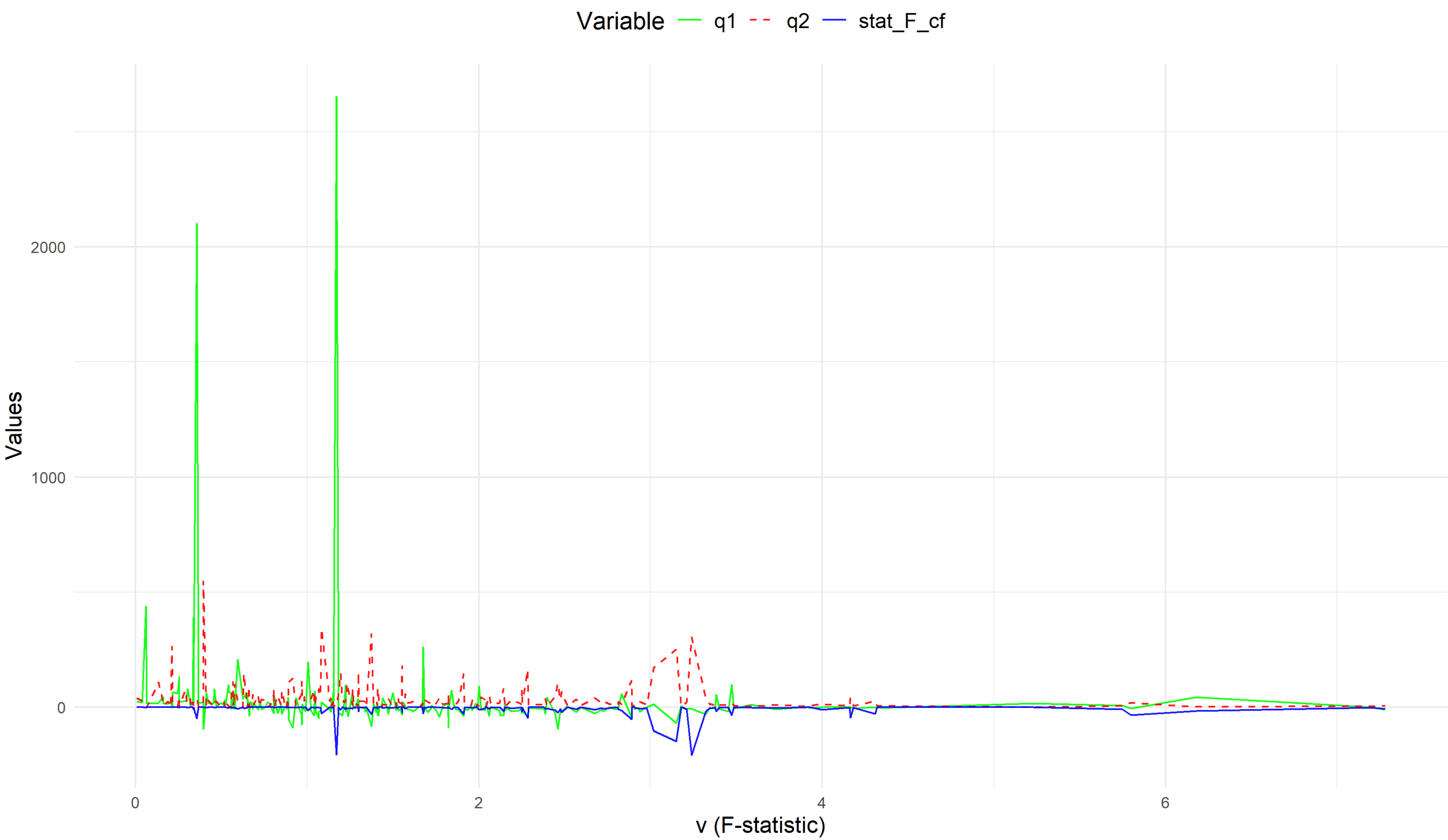

Figure F.18 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = -0.9$, and T=15

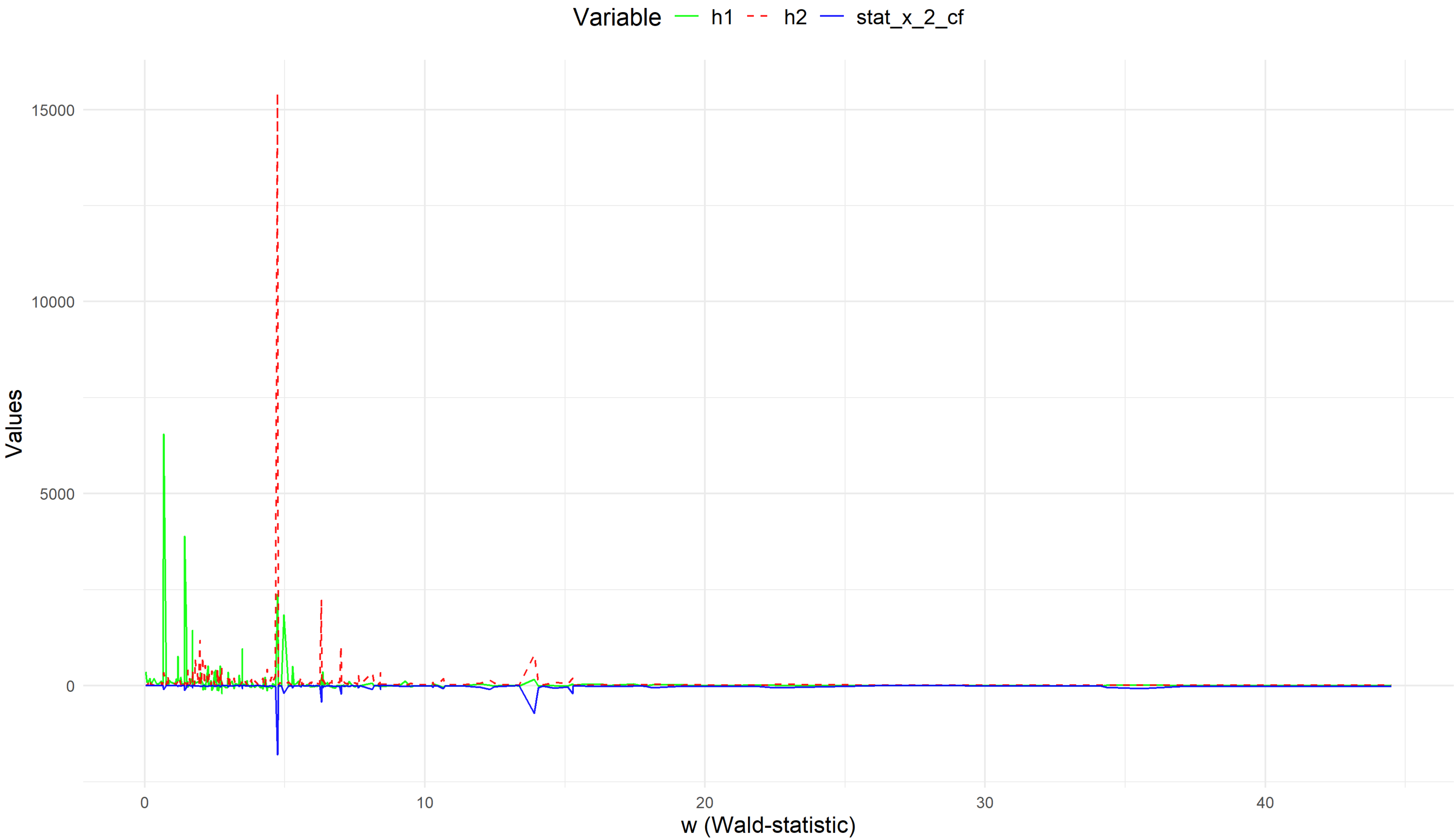

Figure F.19 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = -0.5$, and T=15

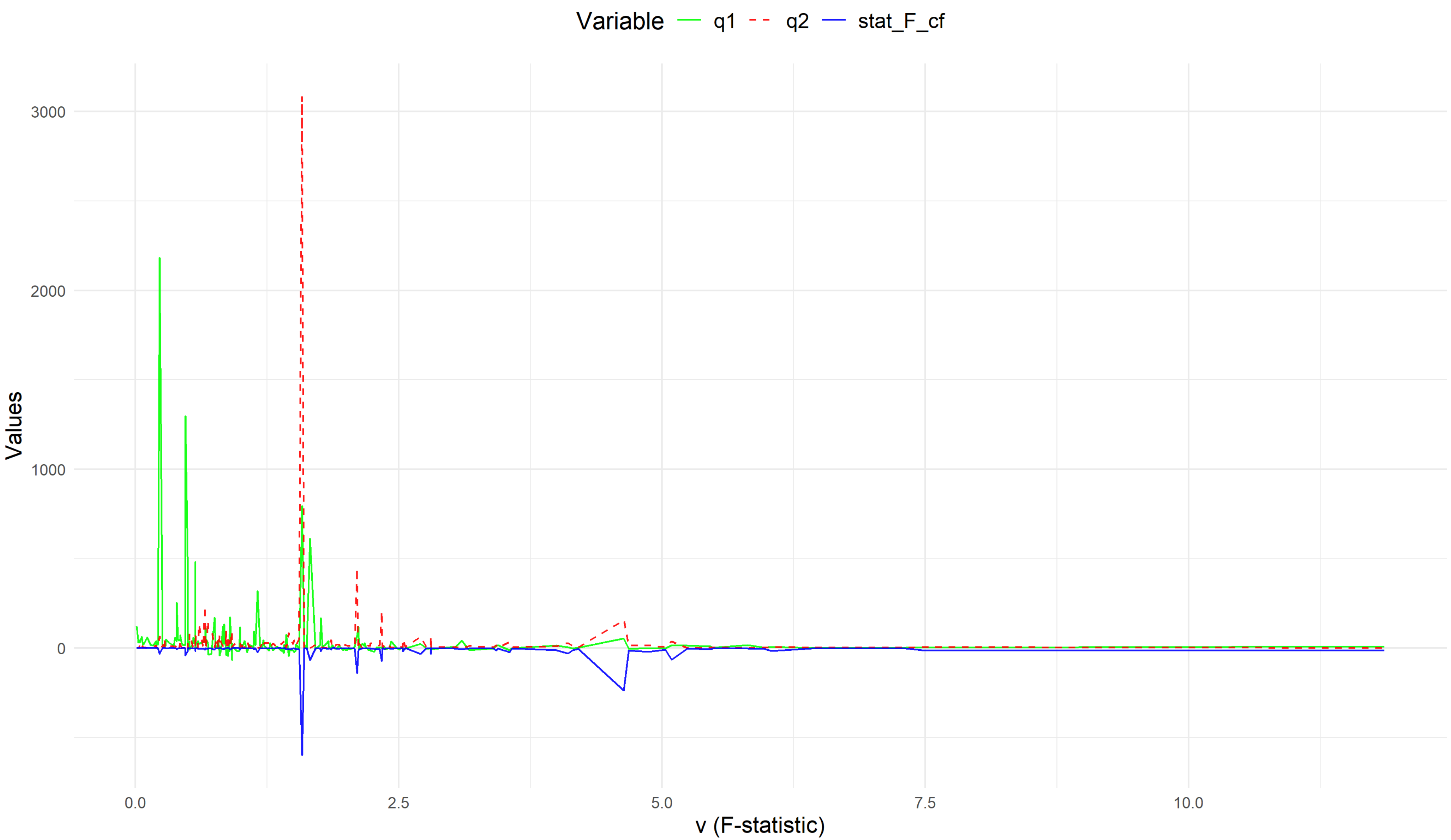

Figure F.20 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = -0.5$, and T=15

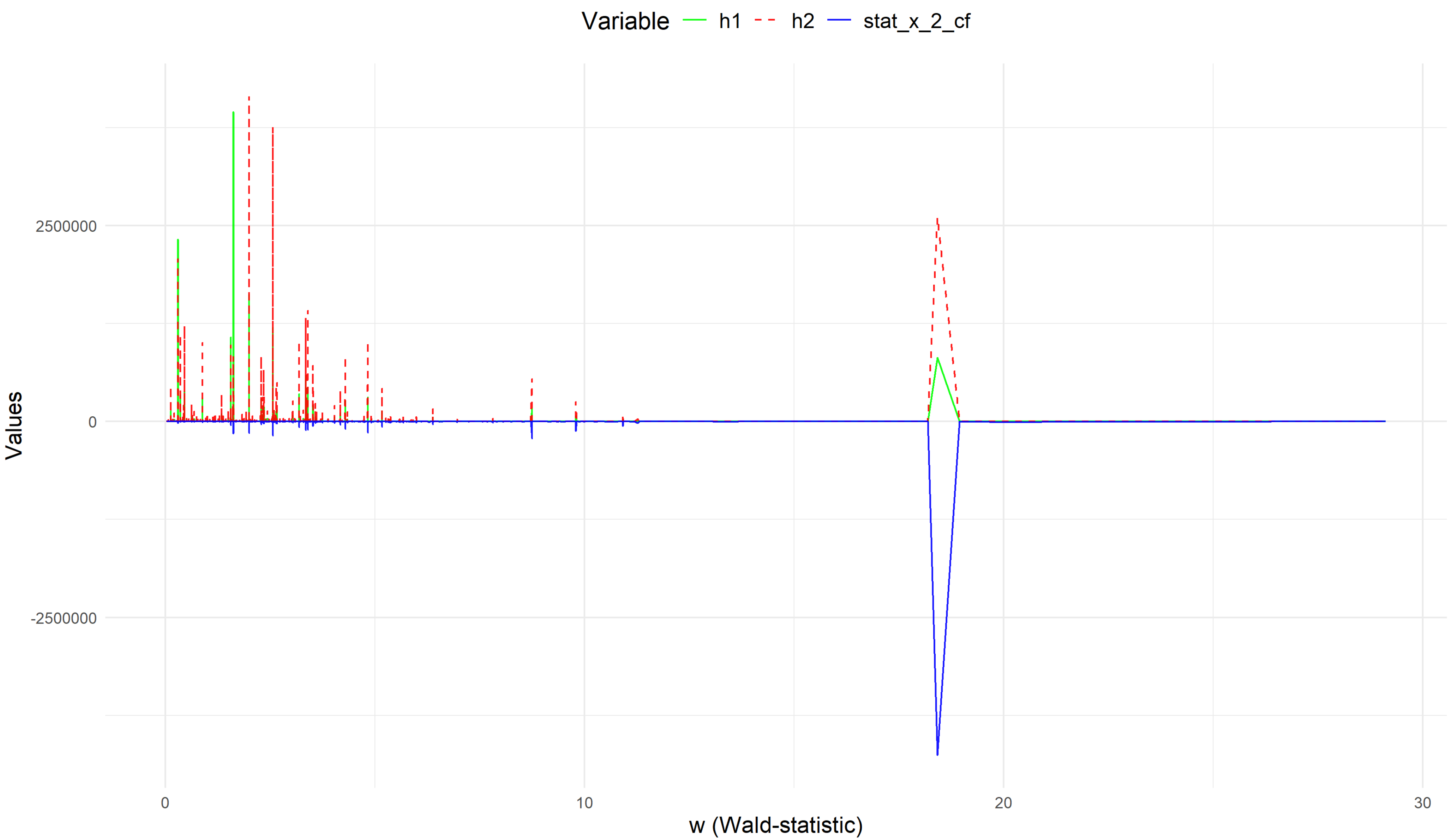

Figure F.21 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = 0.5$, and T=15

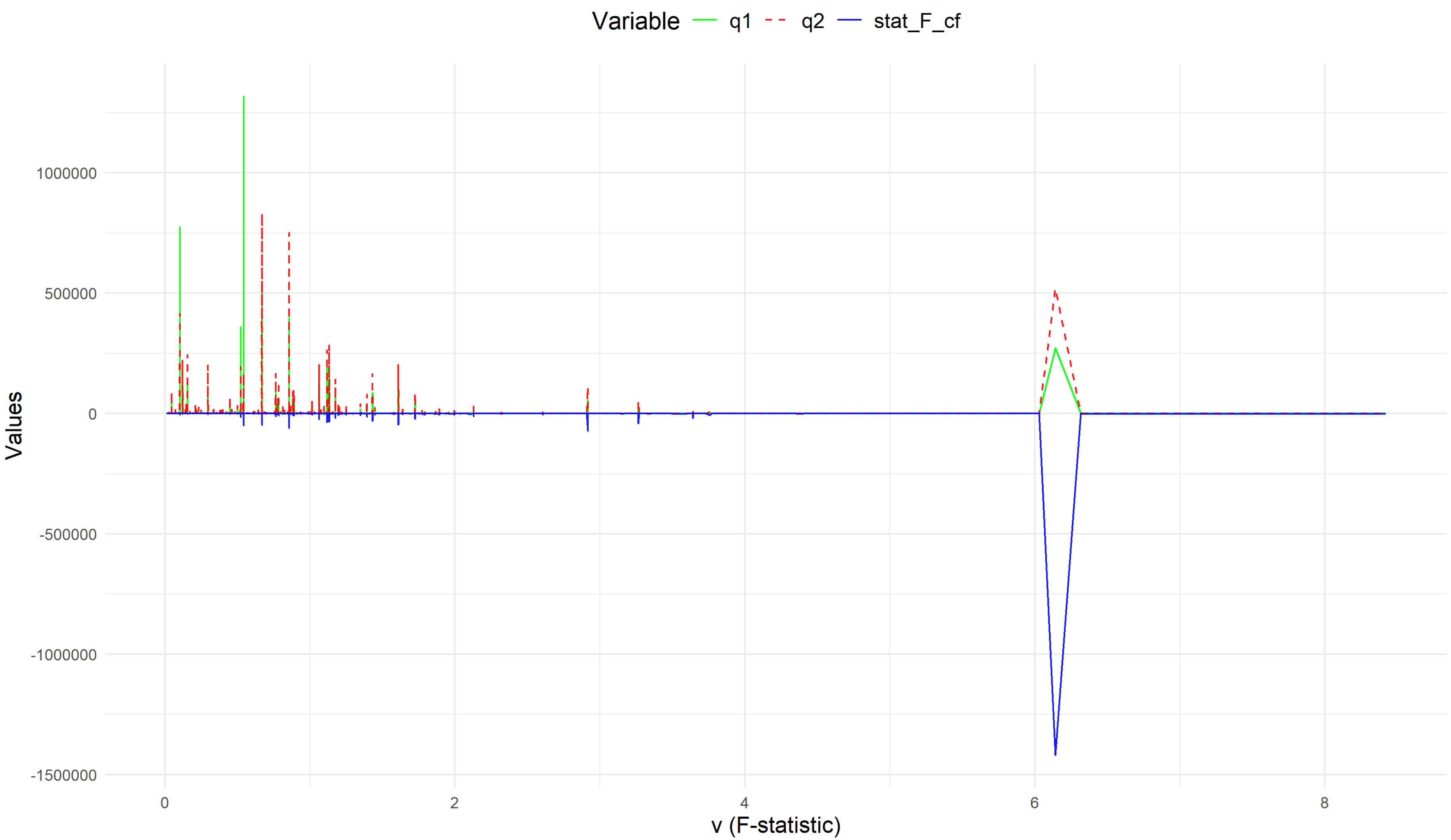

Figure F.22 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = 0.5$, and T=15

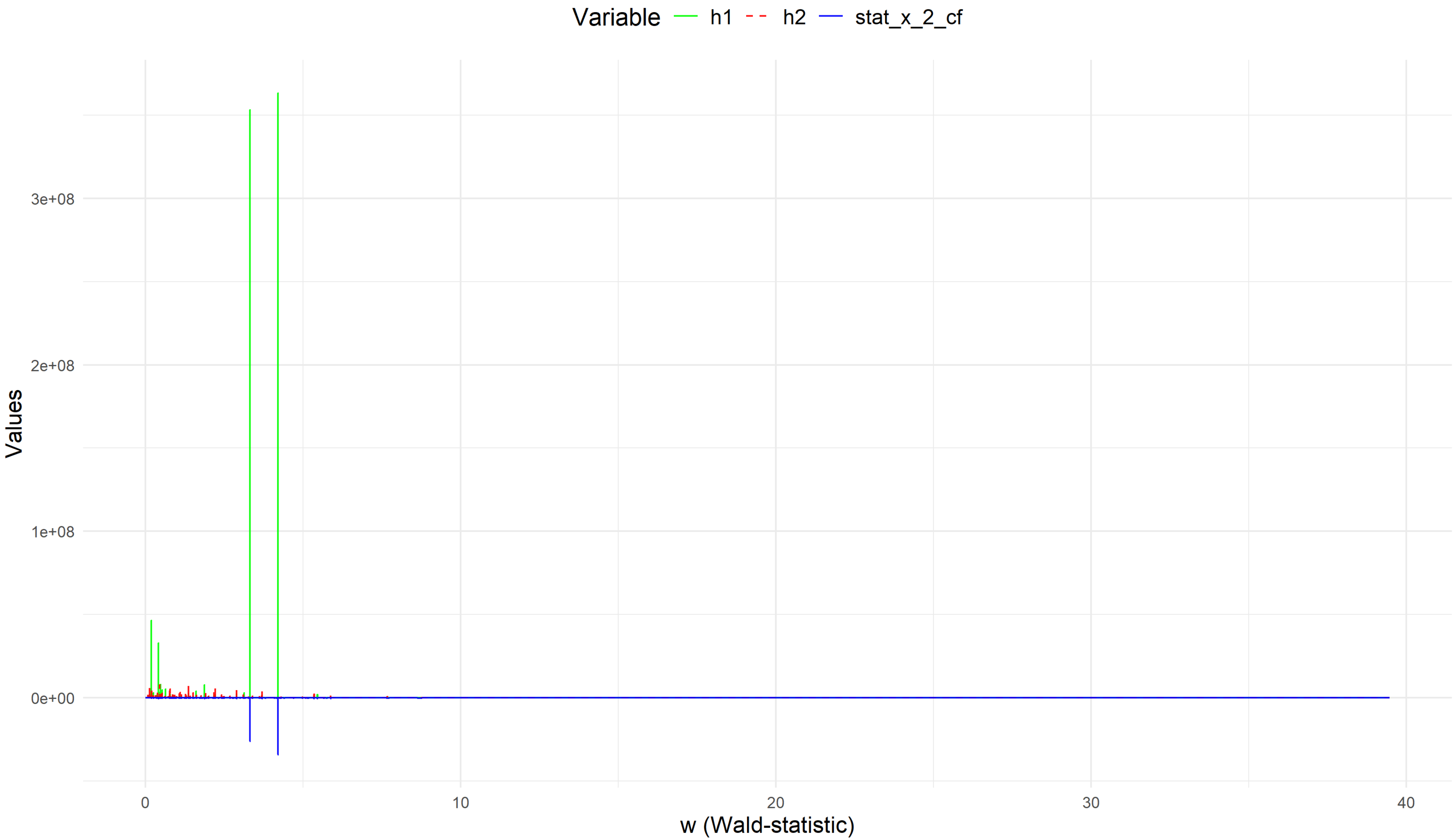

Figure F.23 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = 0.9$, and T=15

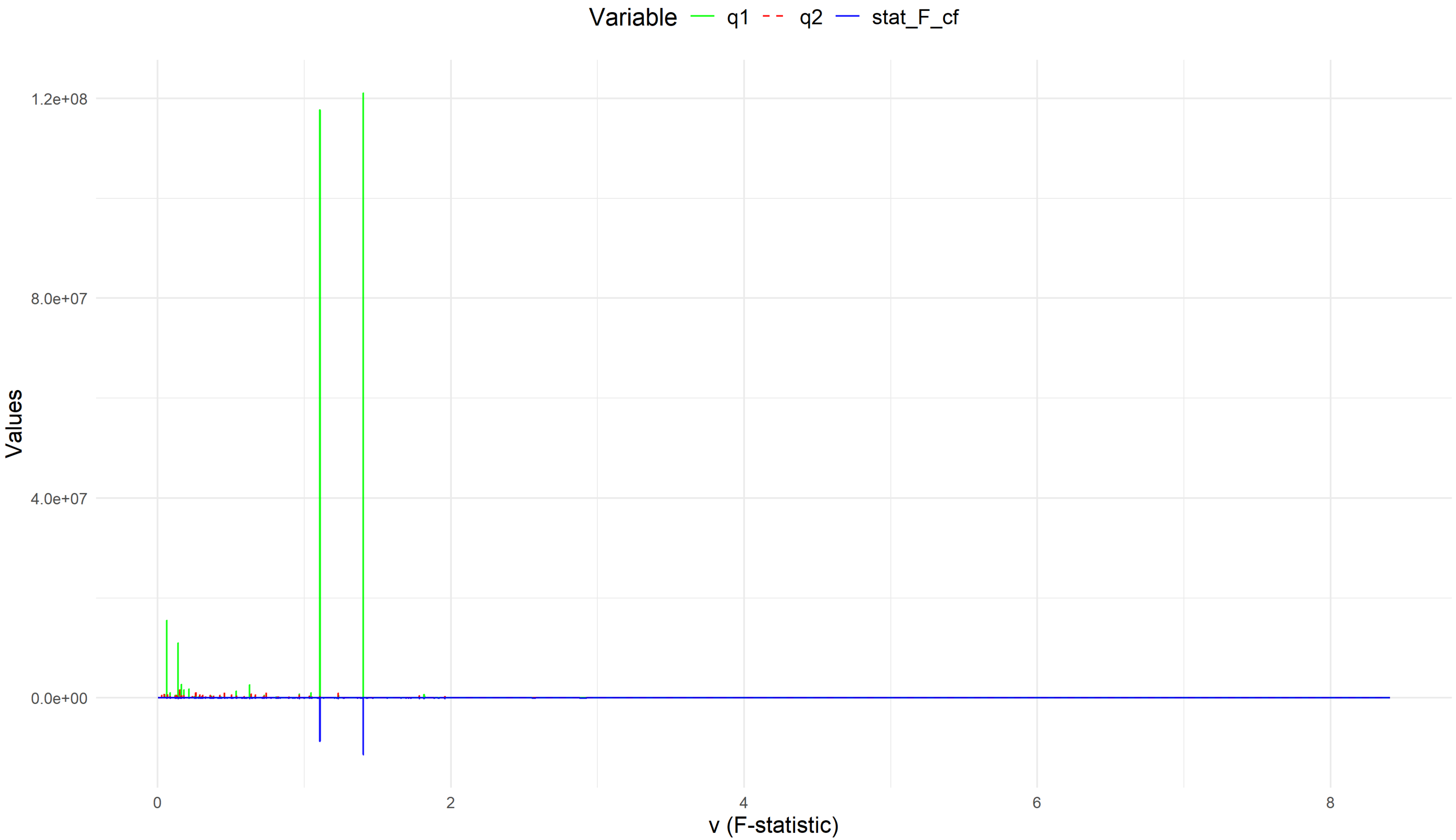

Figure F.24 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = 0.9$, and T=15

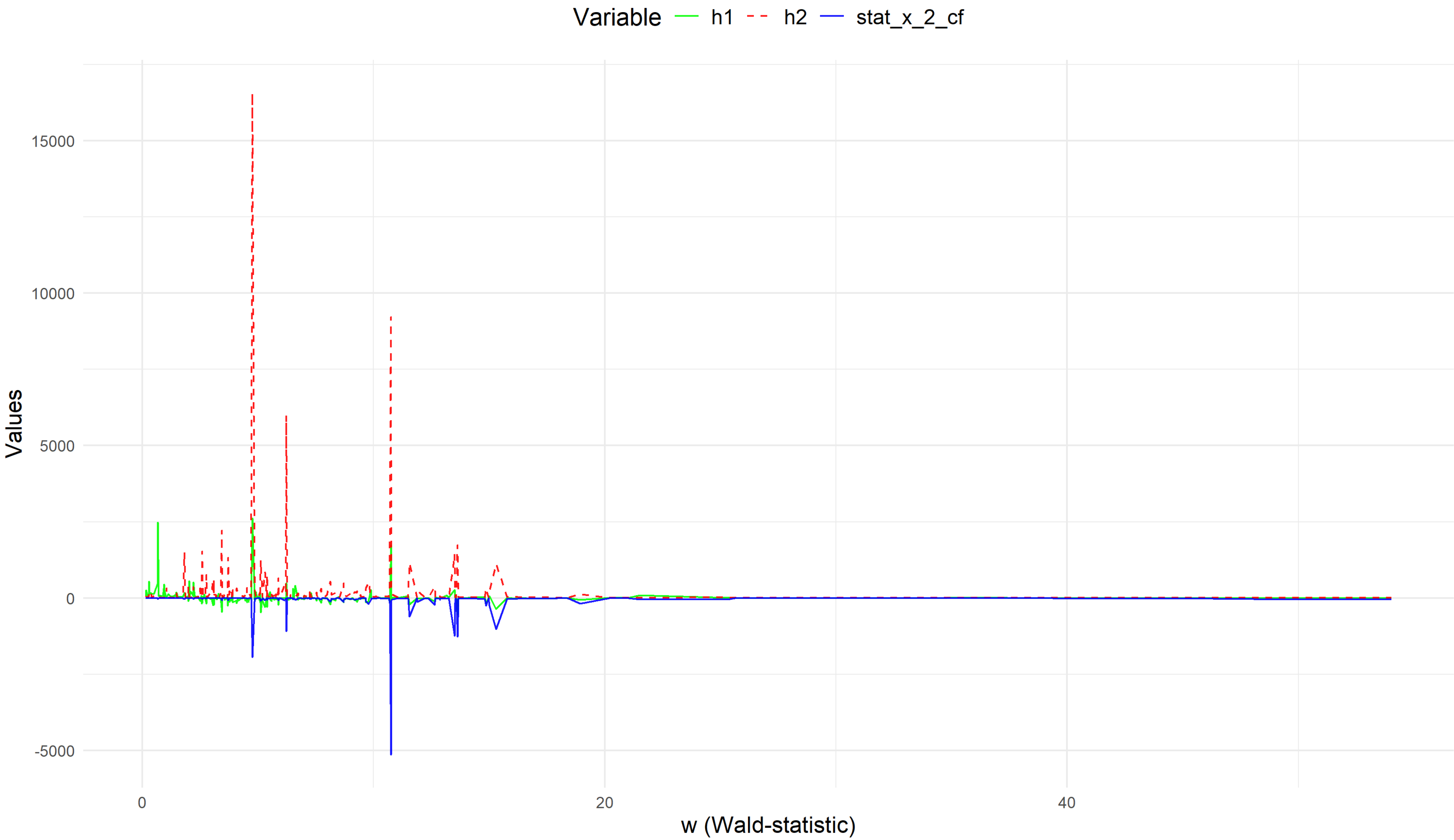

Figure F.25 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = -0.9$, and T=15

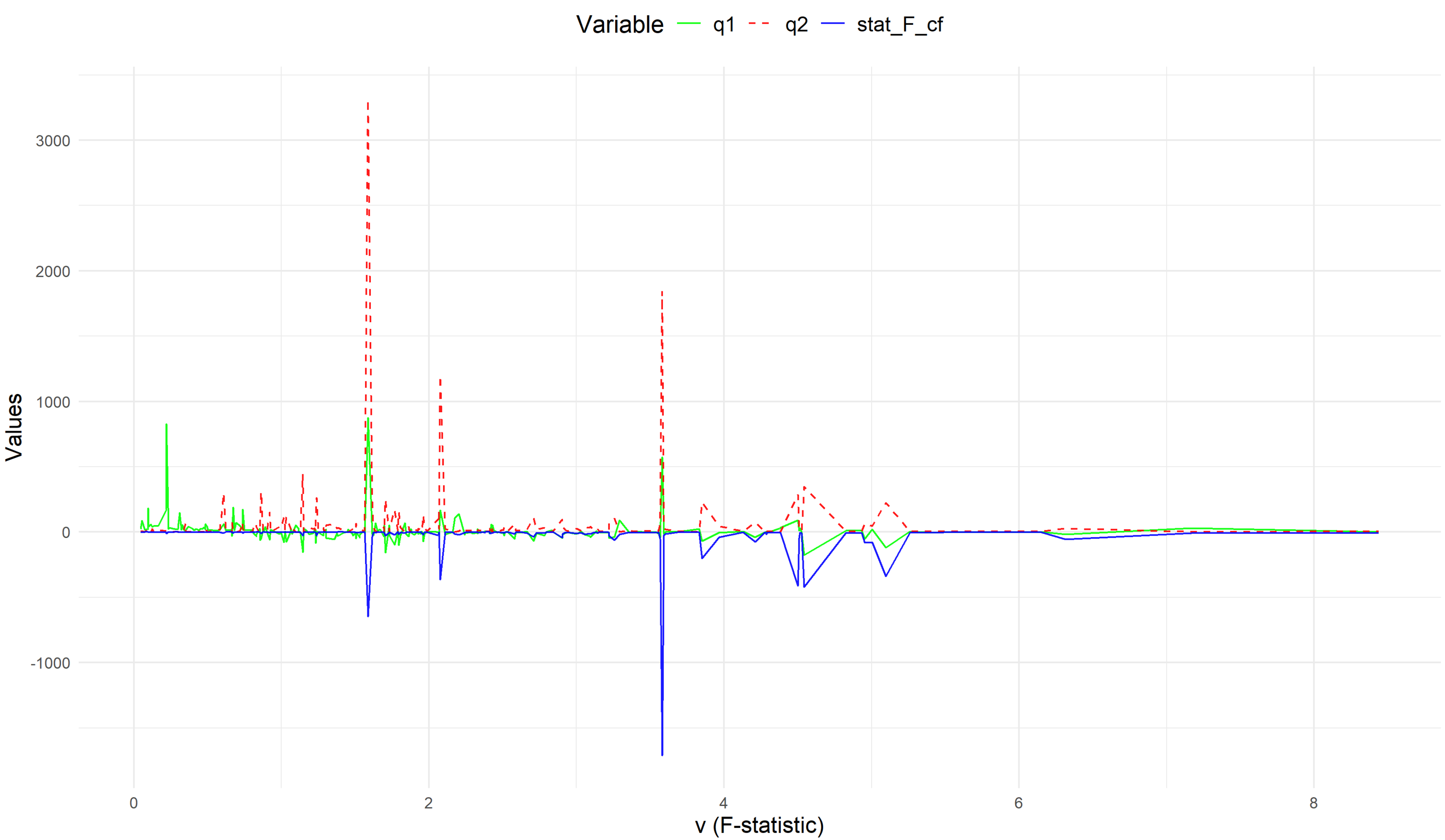

Figure F.26 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = -0.9$, and T=15

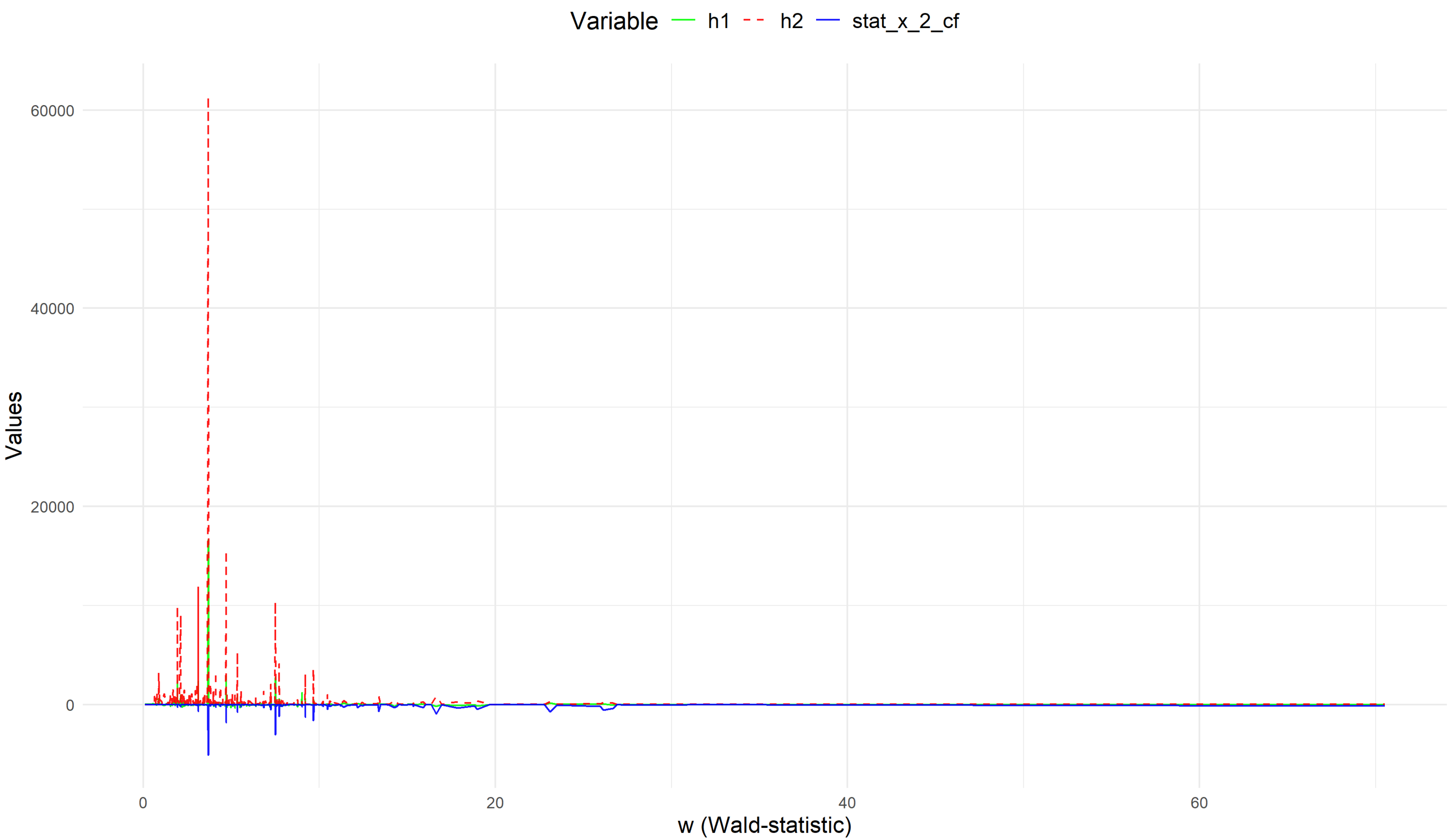

Figure F.27 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = -0.5$, and T=15

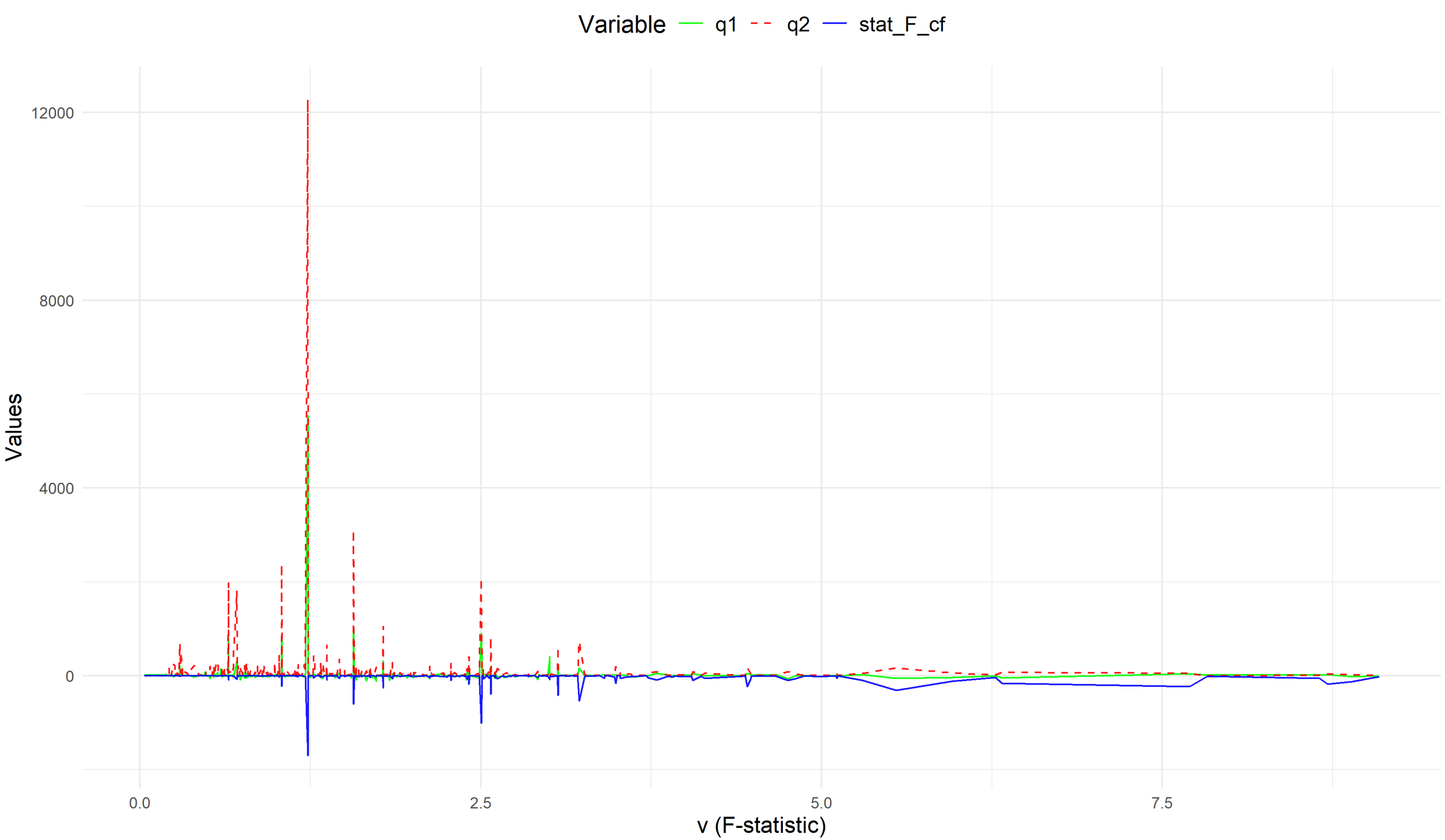

Figure F.28 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = -0.5$, and T=15

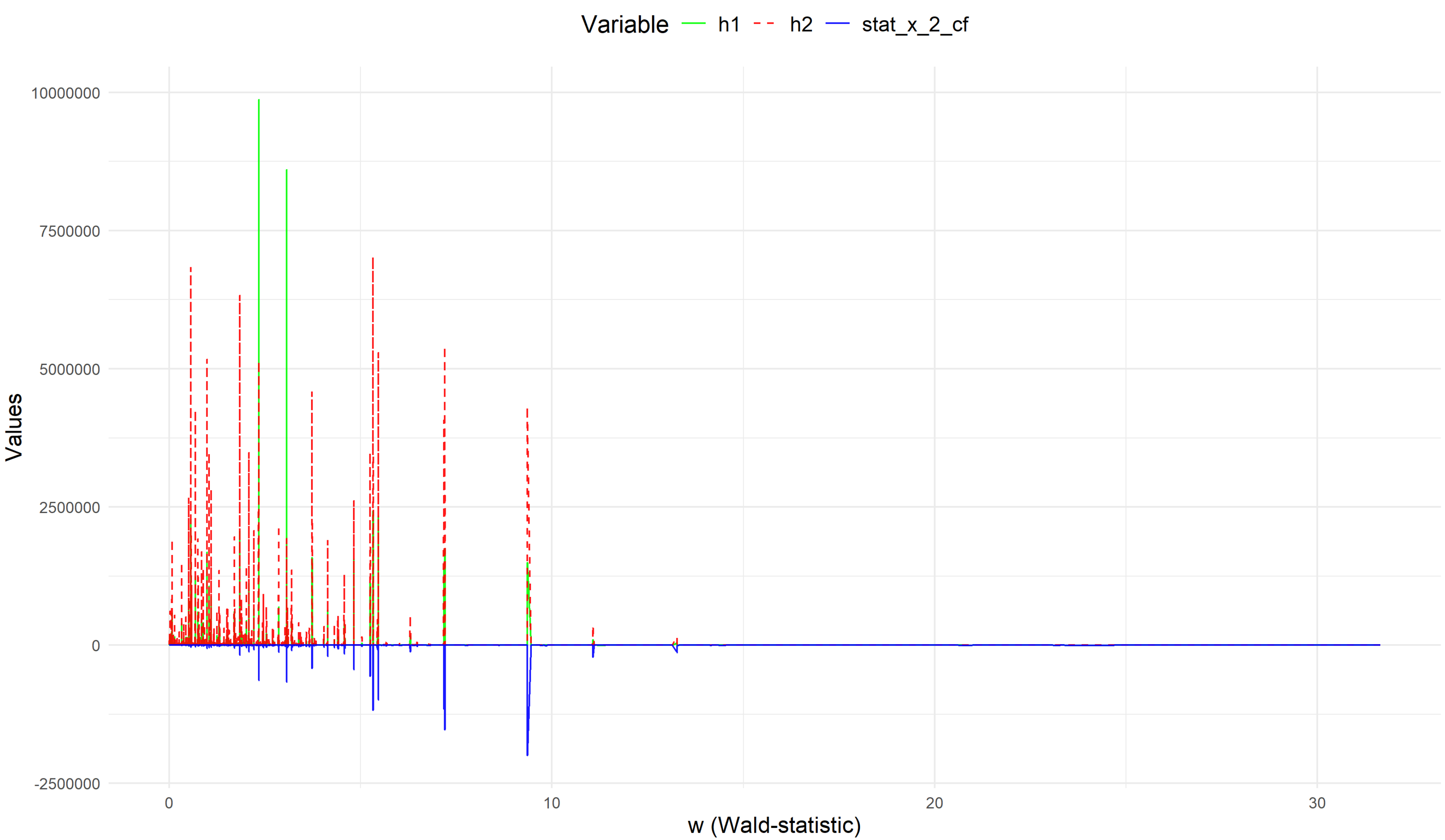

Figure F.29 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = 0.5$, and T=15

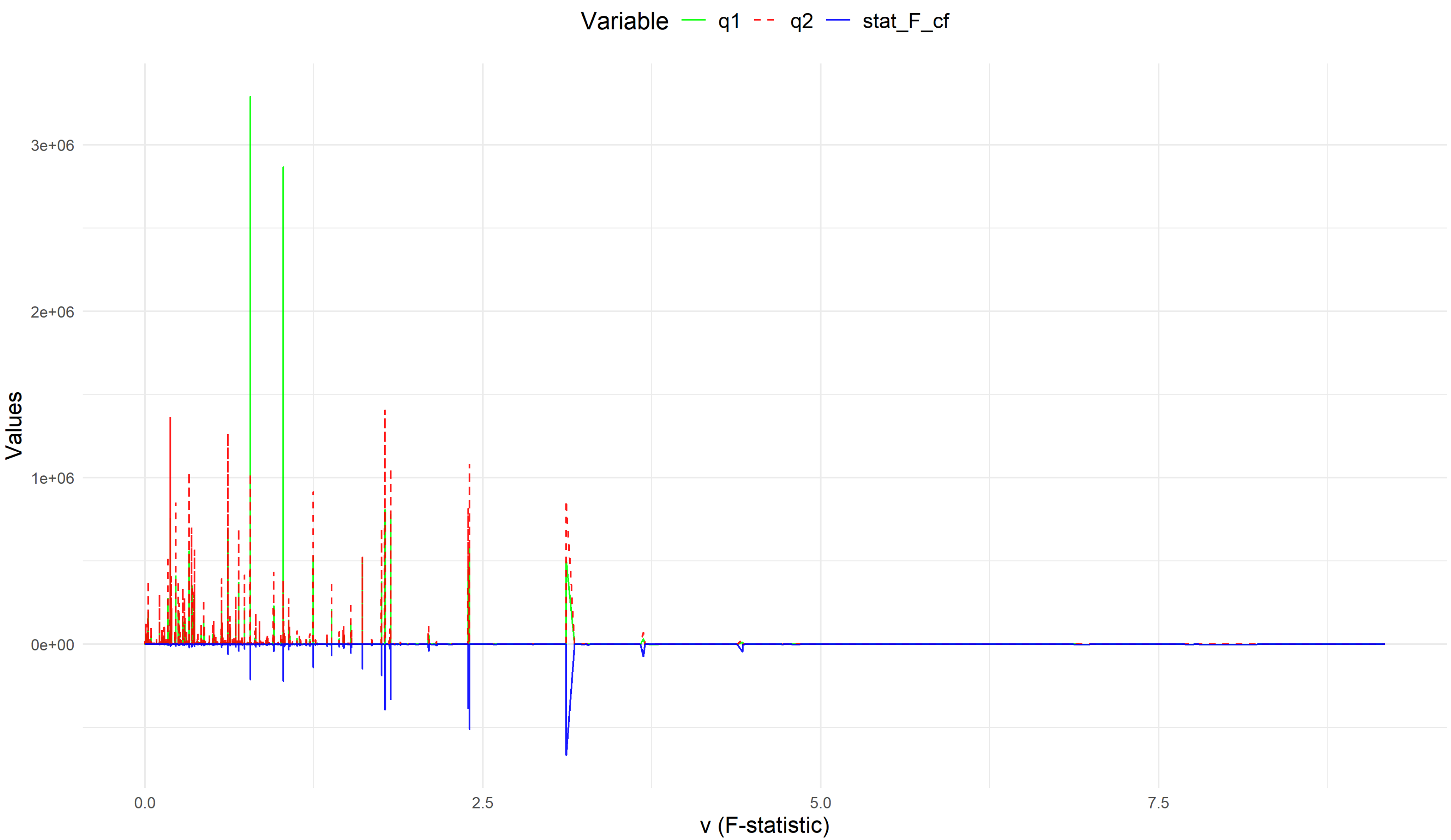

Figure F.30 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = 0.5$, and T=15

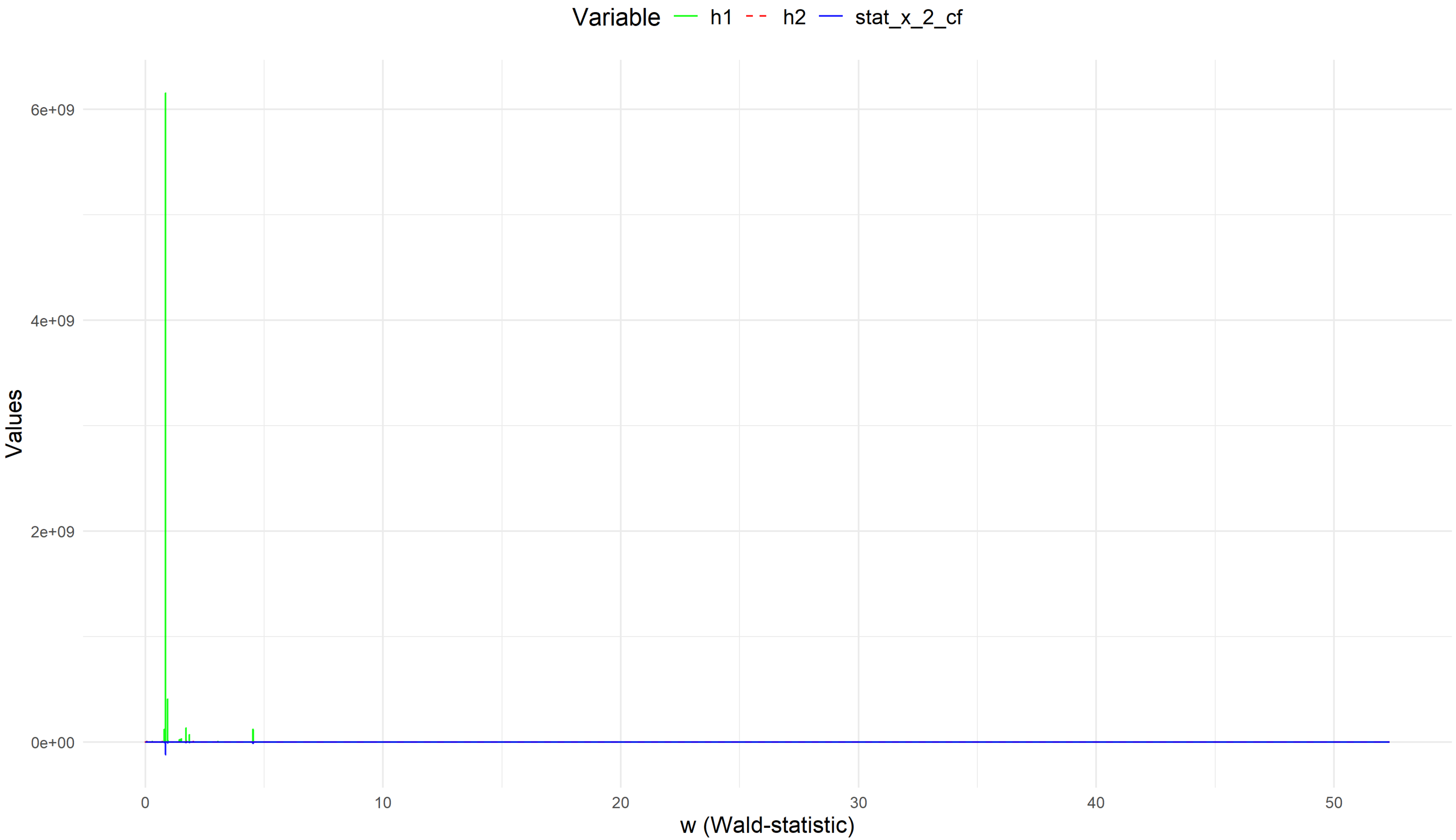

Figure F.31 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = 0.9$, and T=15

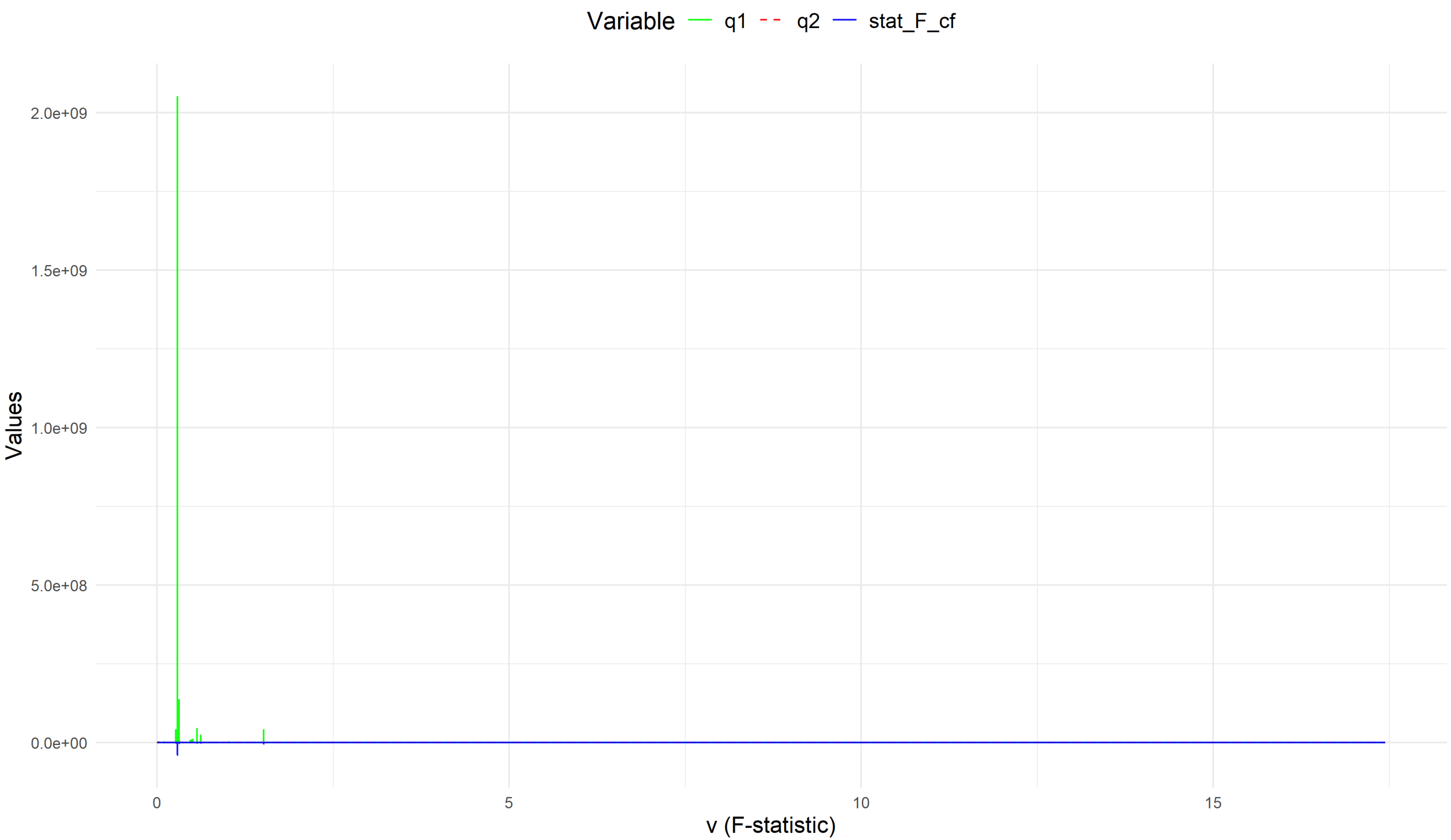

Figure F.32 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = 0.9$, and T=15

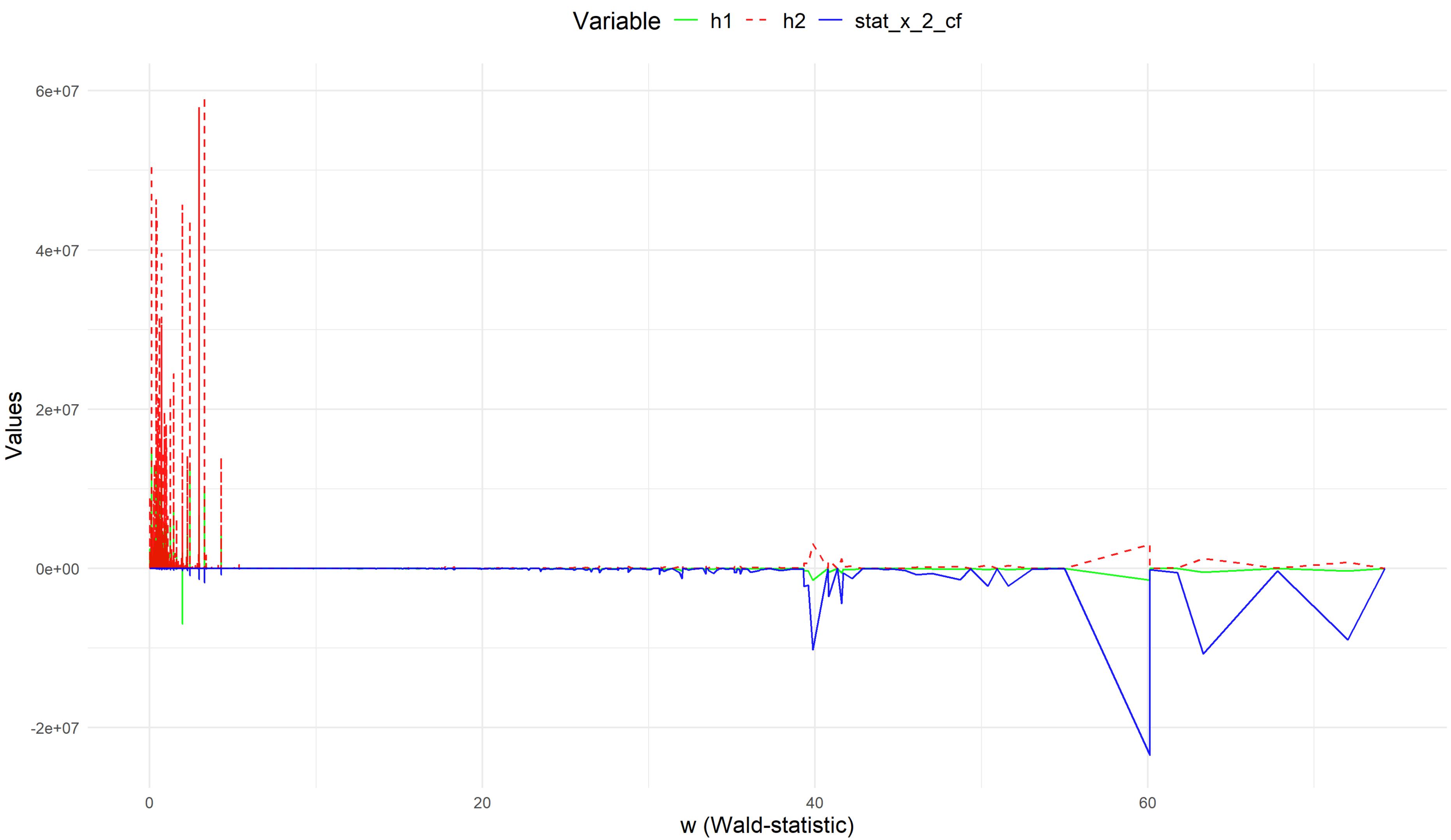

Figure F.33 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = -0.9$, and T=30

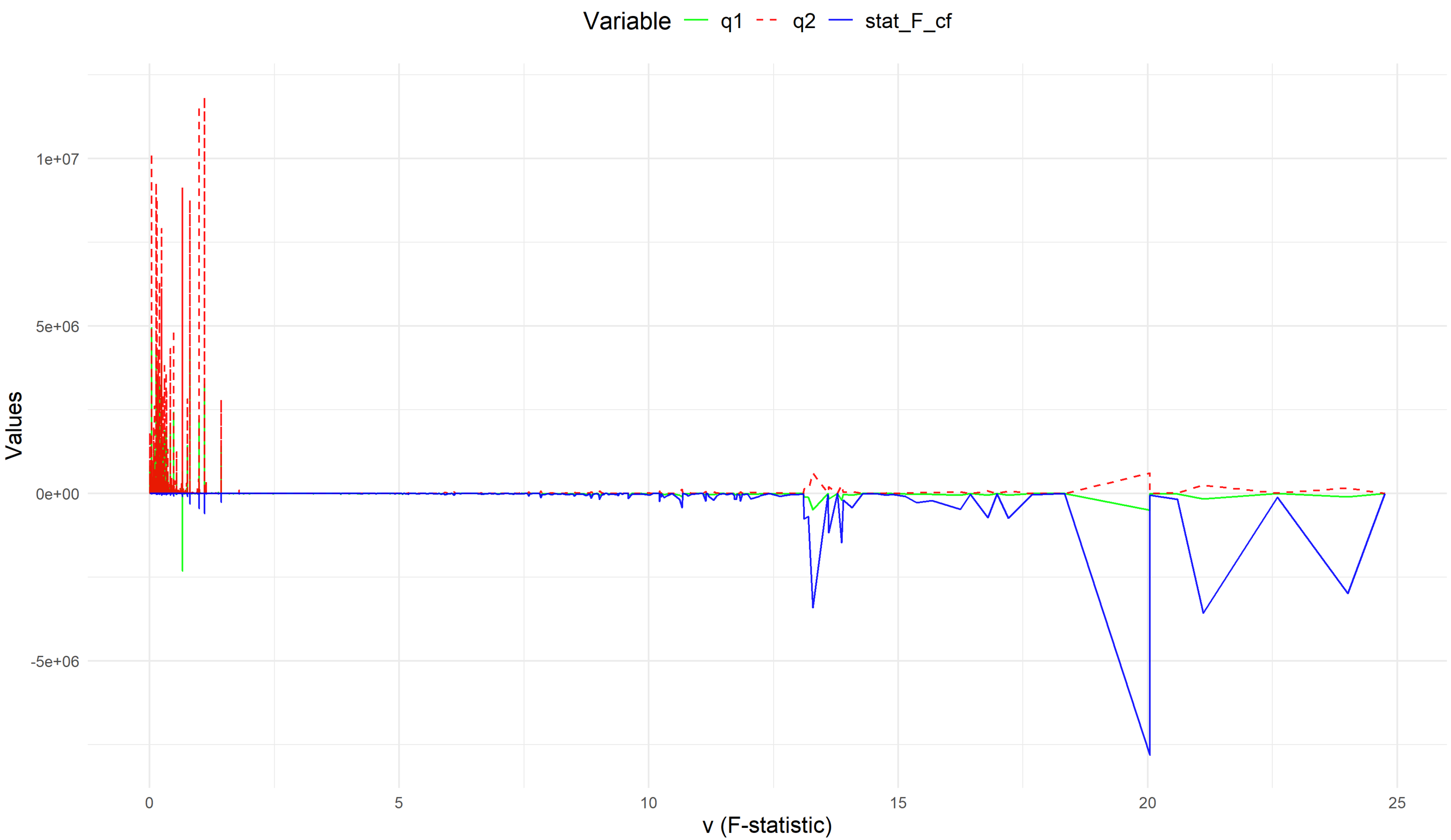

Figure F.34 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = -0.9$, and T=30

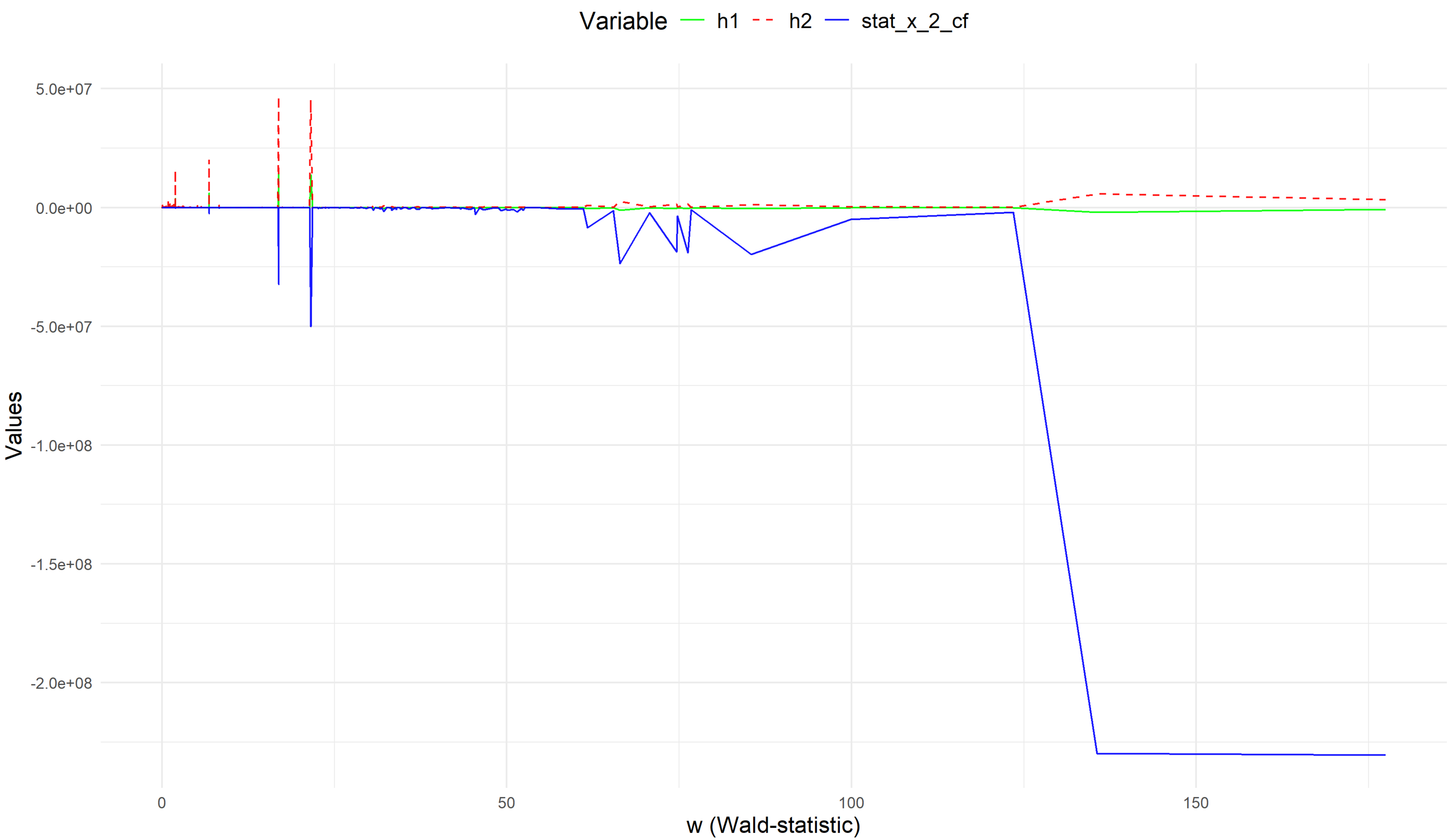

Figure F.35 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = -0.5$, and T=30

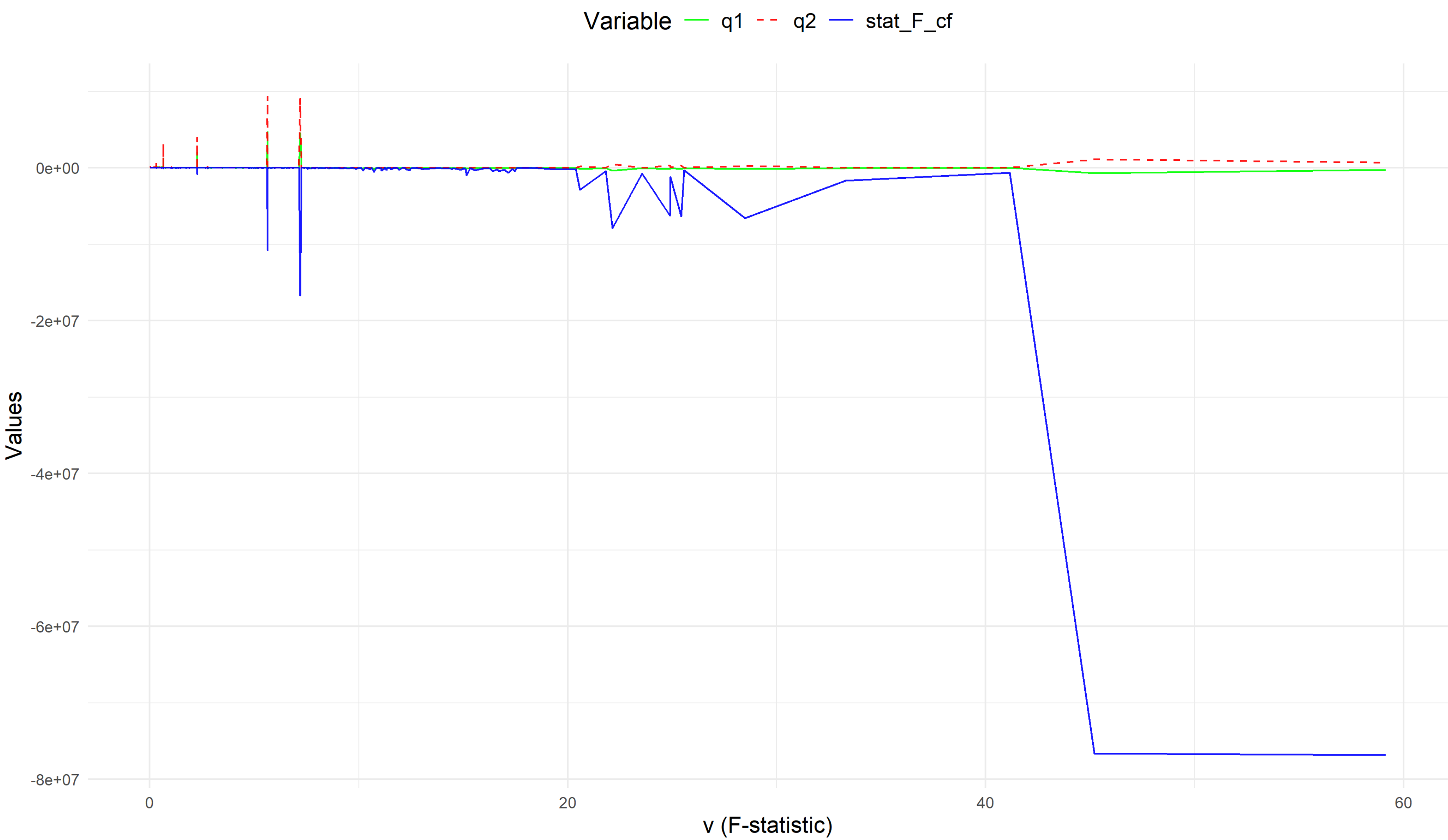

Figure F.36 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = -0.5$, and T=30

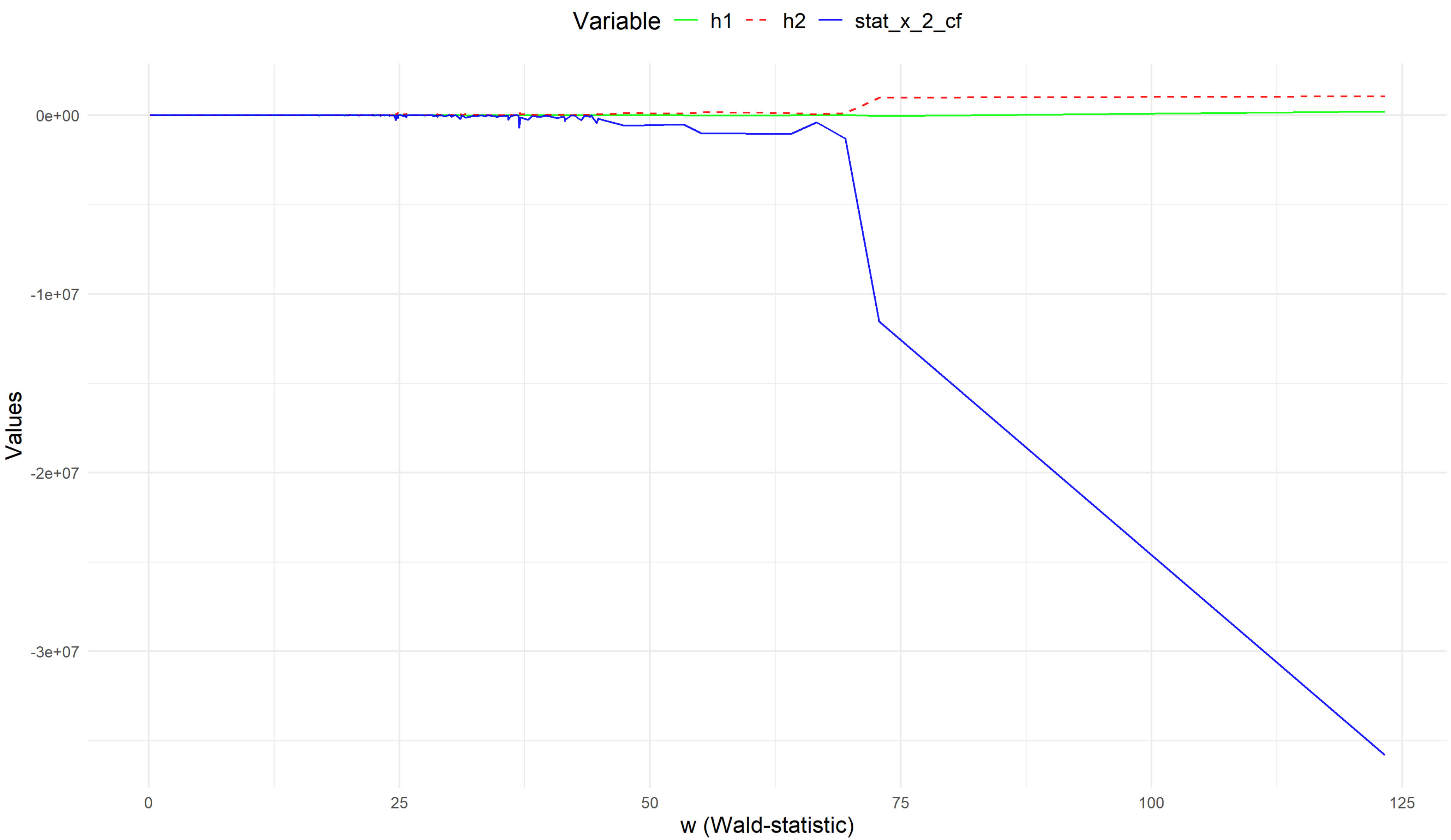

Figure F.37 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = 0.5$, and T=30

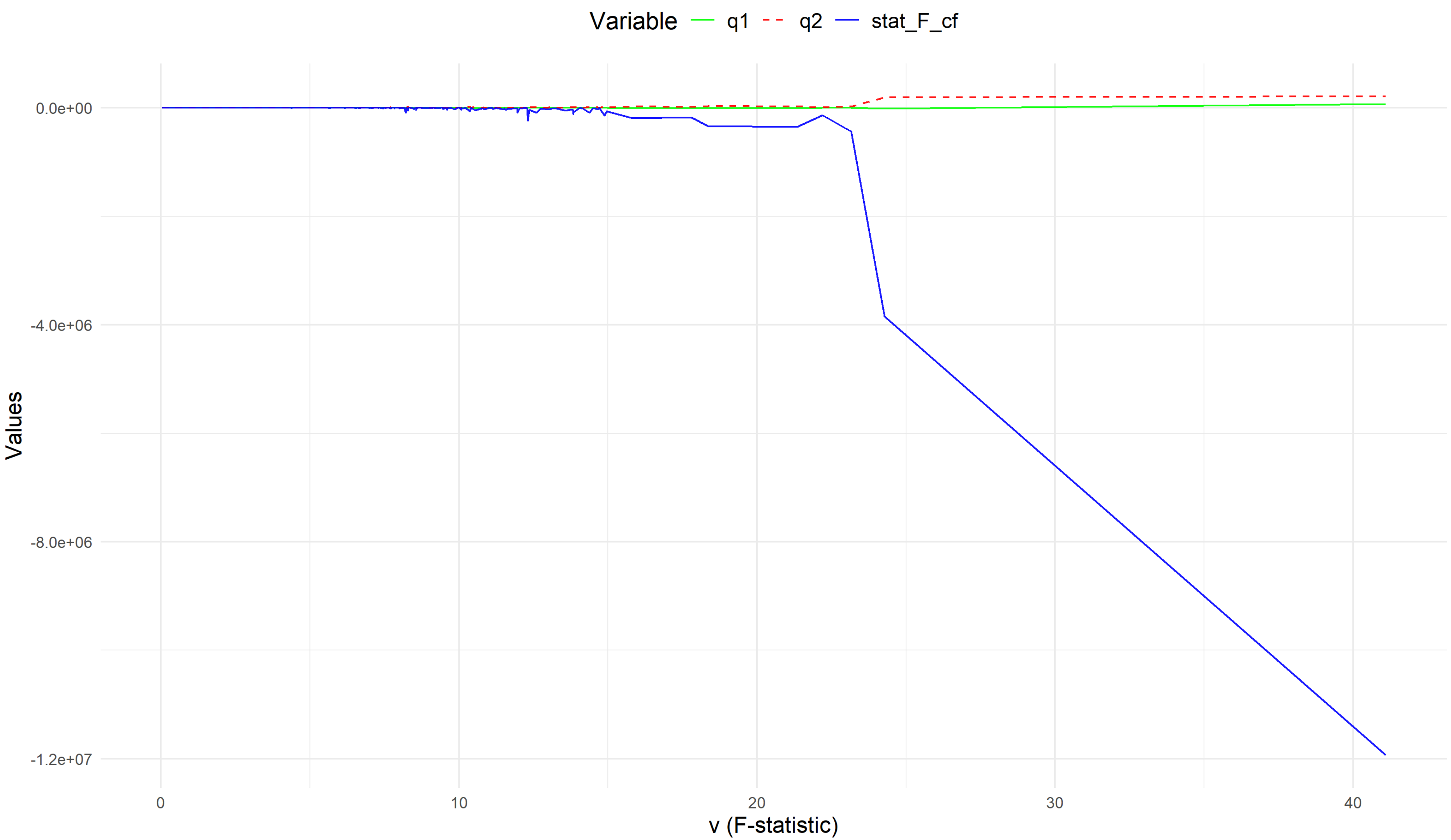

Figure F.38 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = 0.5$, and T=30

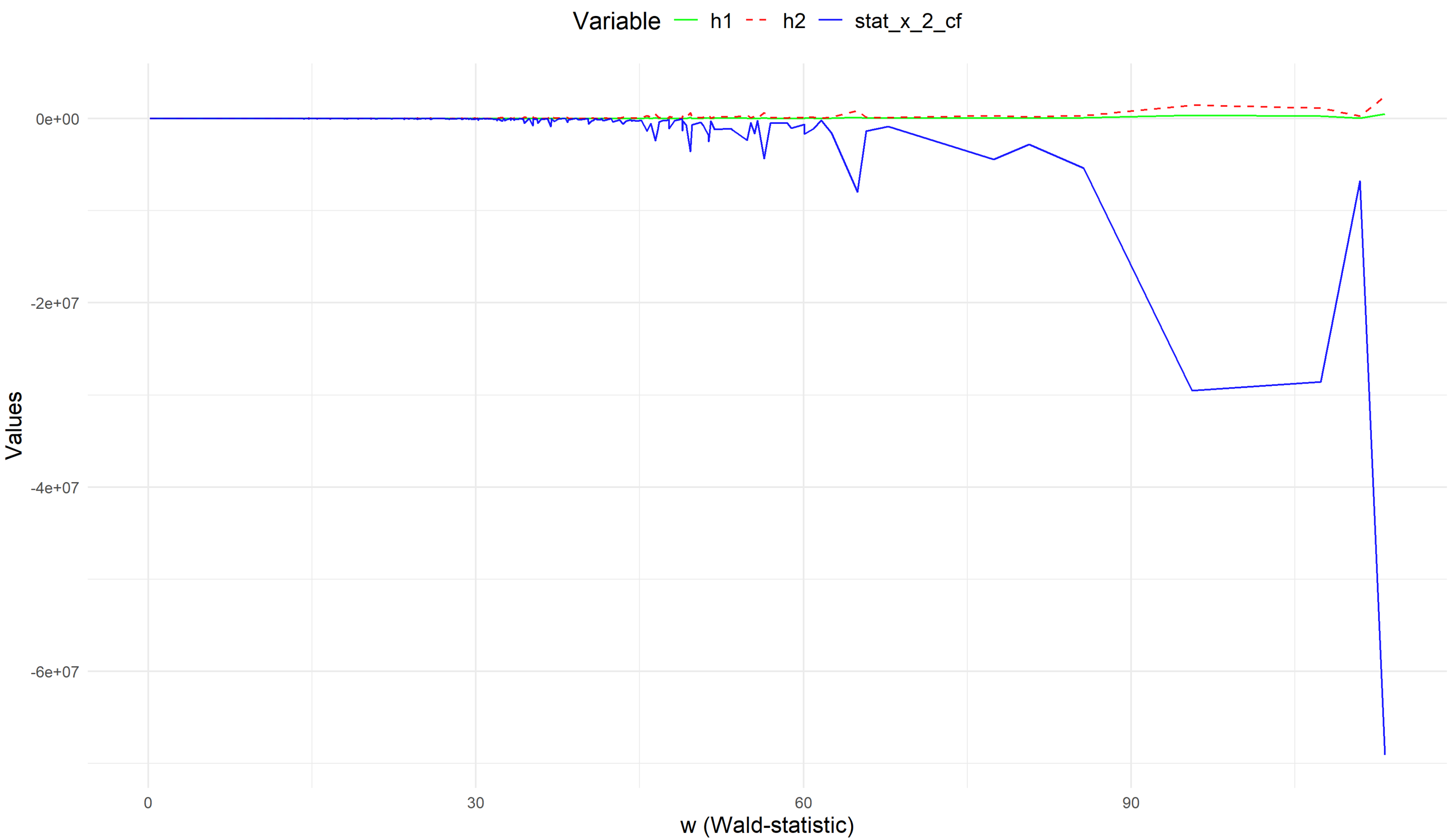

Figure F.39 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = 0.9$, and T=30

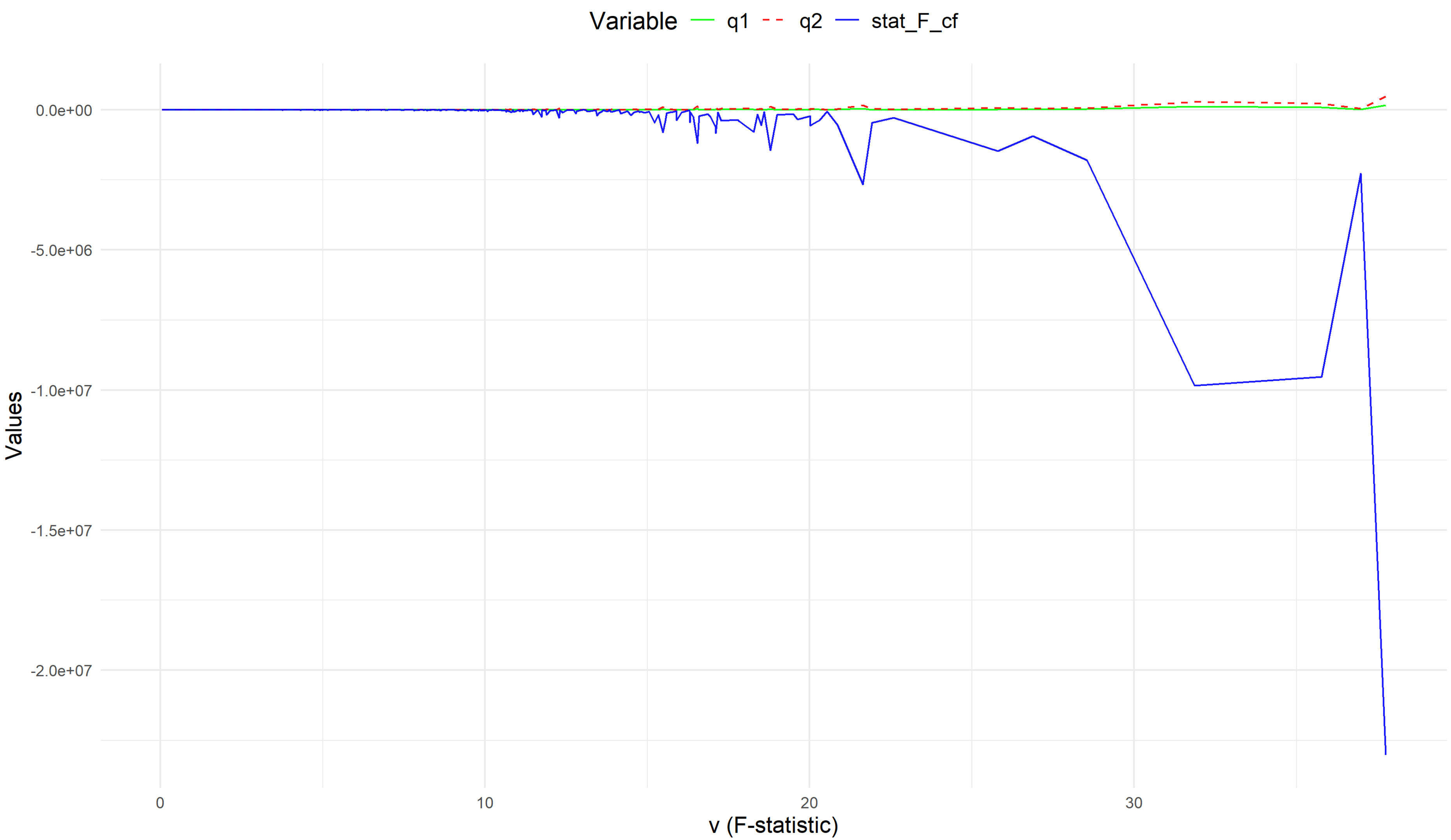

Figure F.40 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.9$, $\phi = 0.9$, and T=30

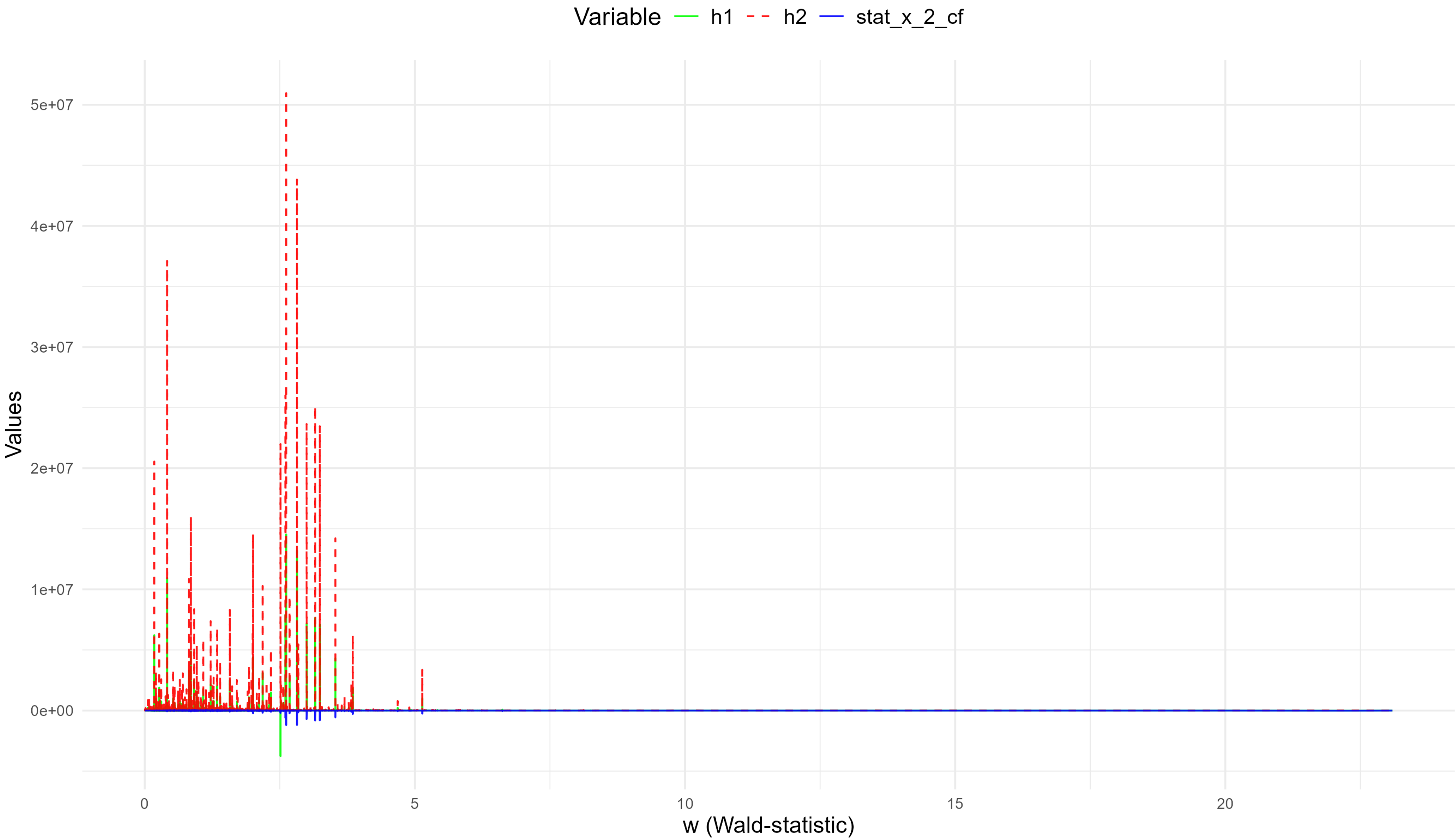

Figure F.41 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = -0.9$, and T=30

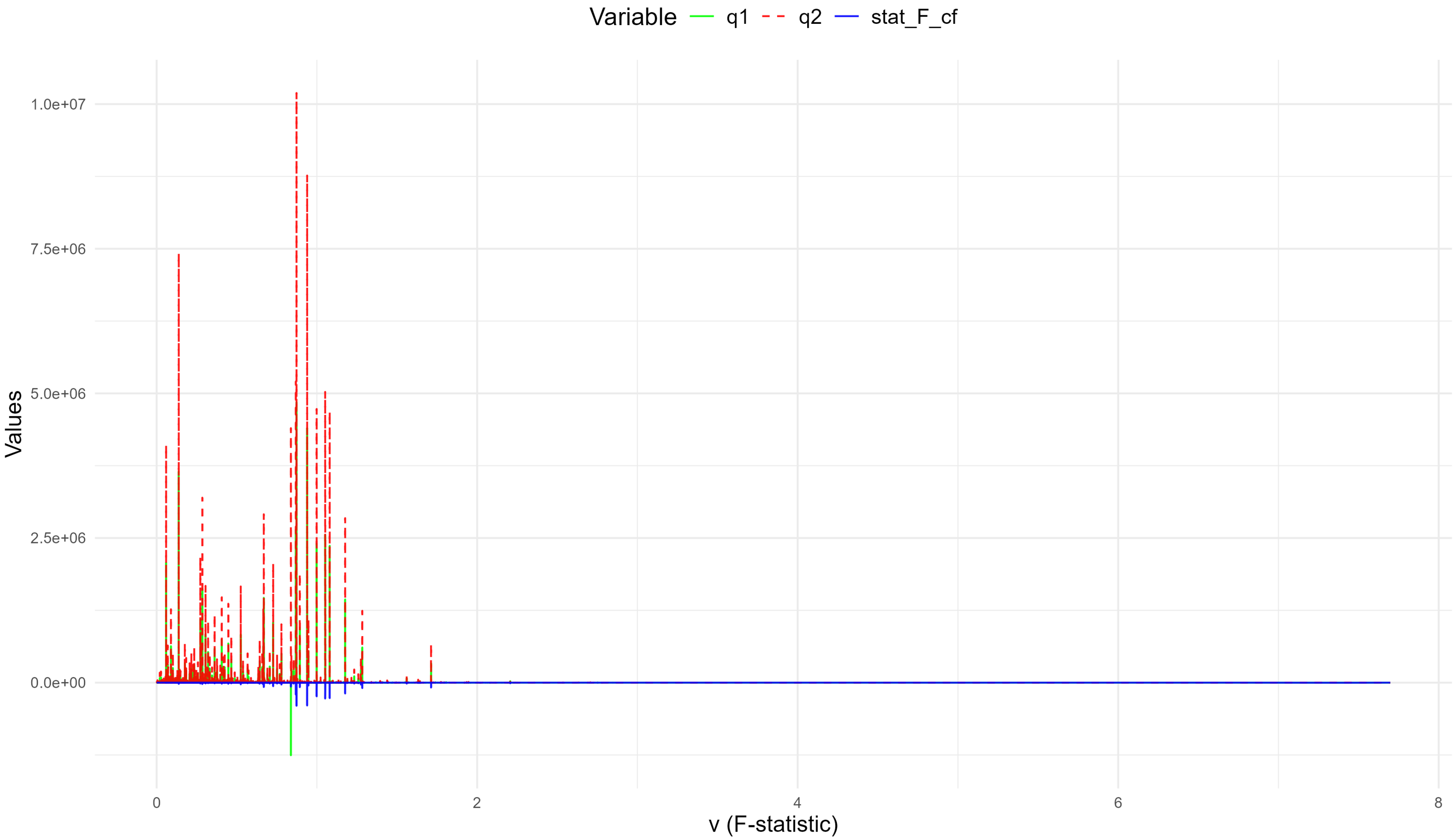

Figure F.42 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = -0.9$, and T=30

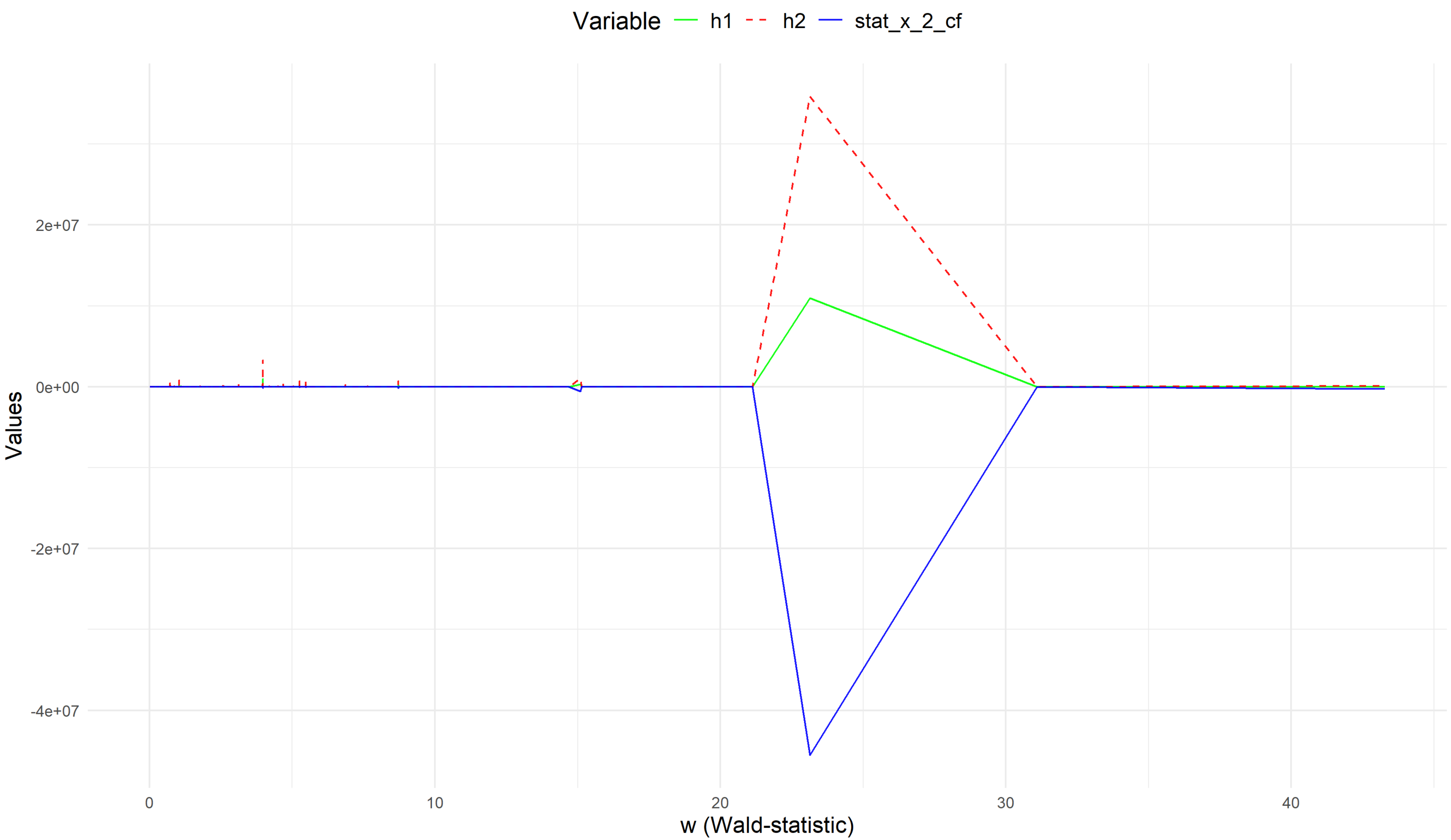

Figure F.43 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = -0.5$, and T=30

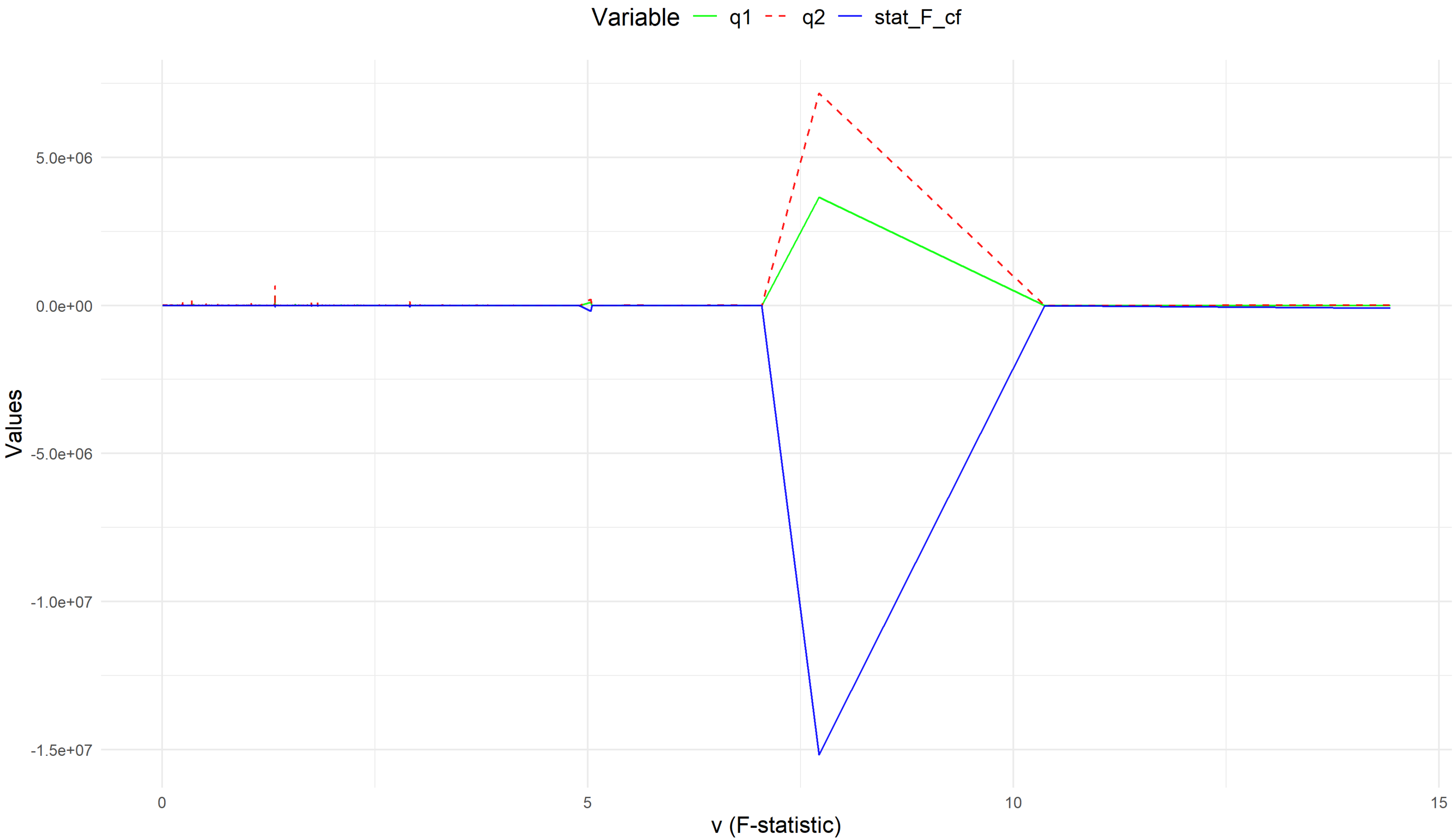

Figure F.44 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = -0.5$, and T=30

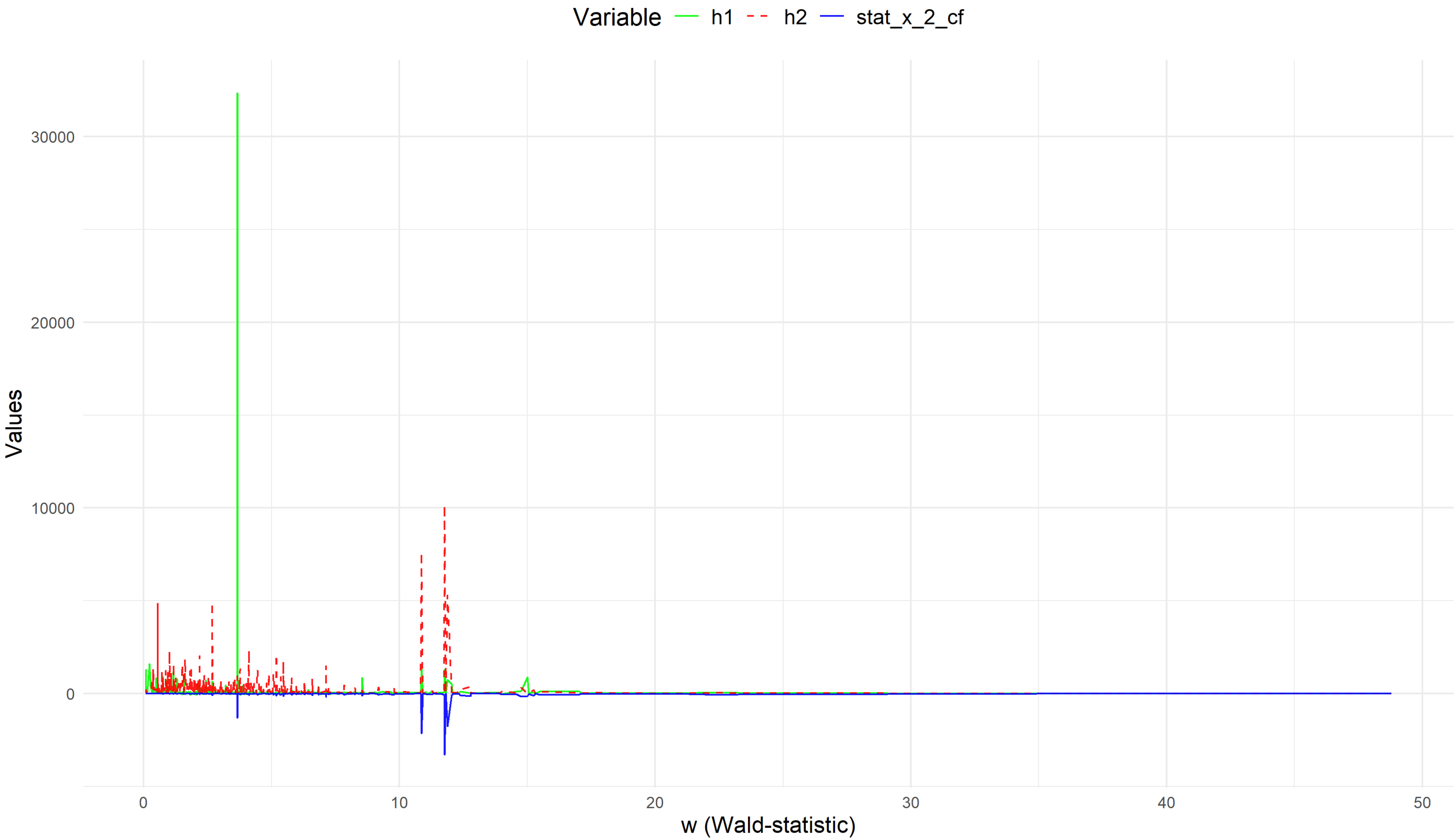

Figure F.45 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.5$, and T=30

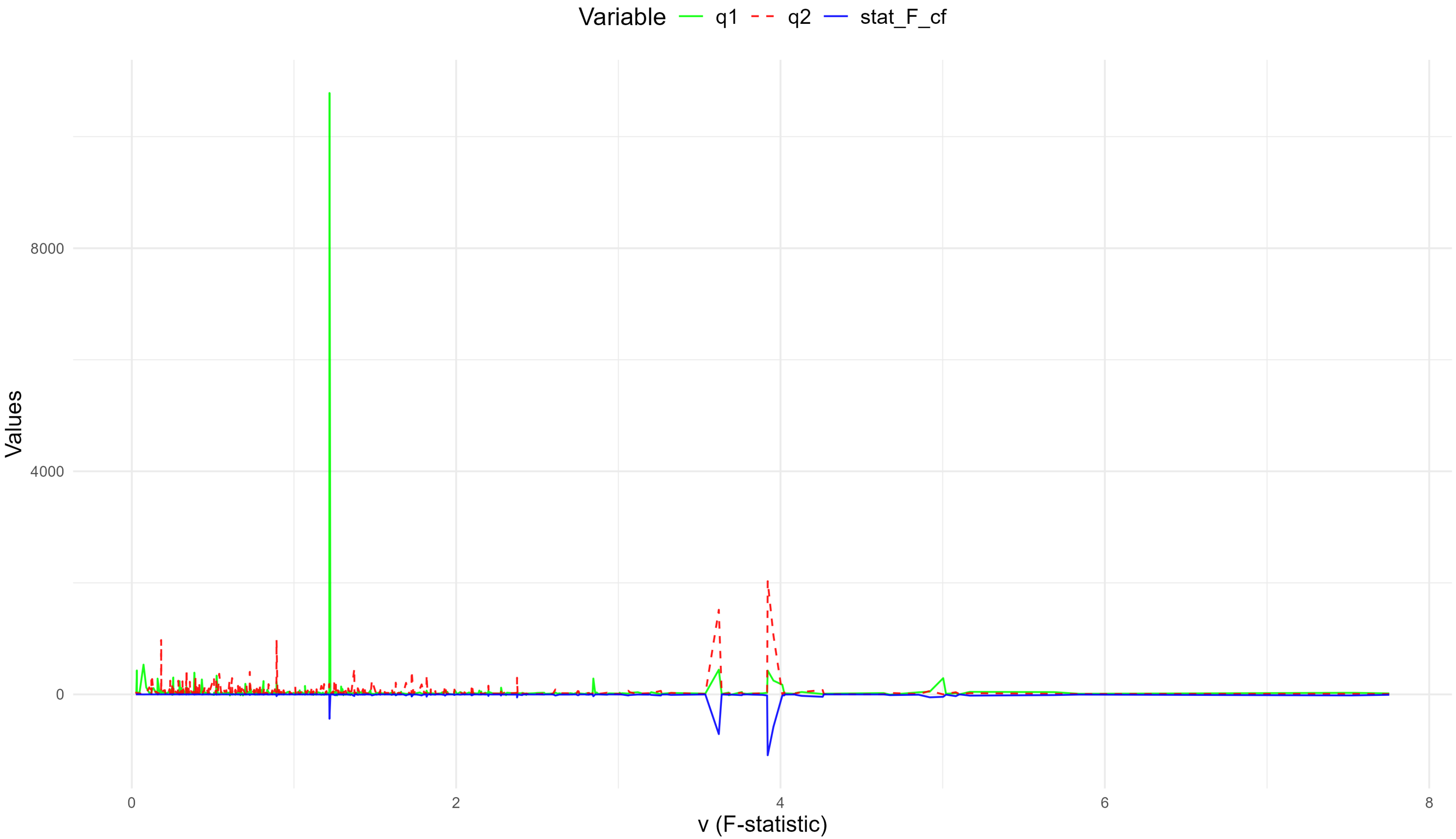

Figure F.46 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.5$, and T=30

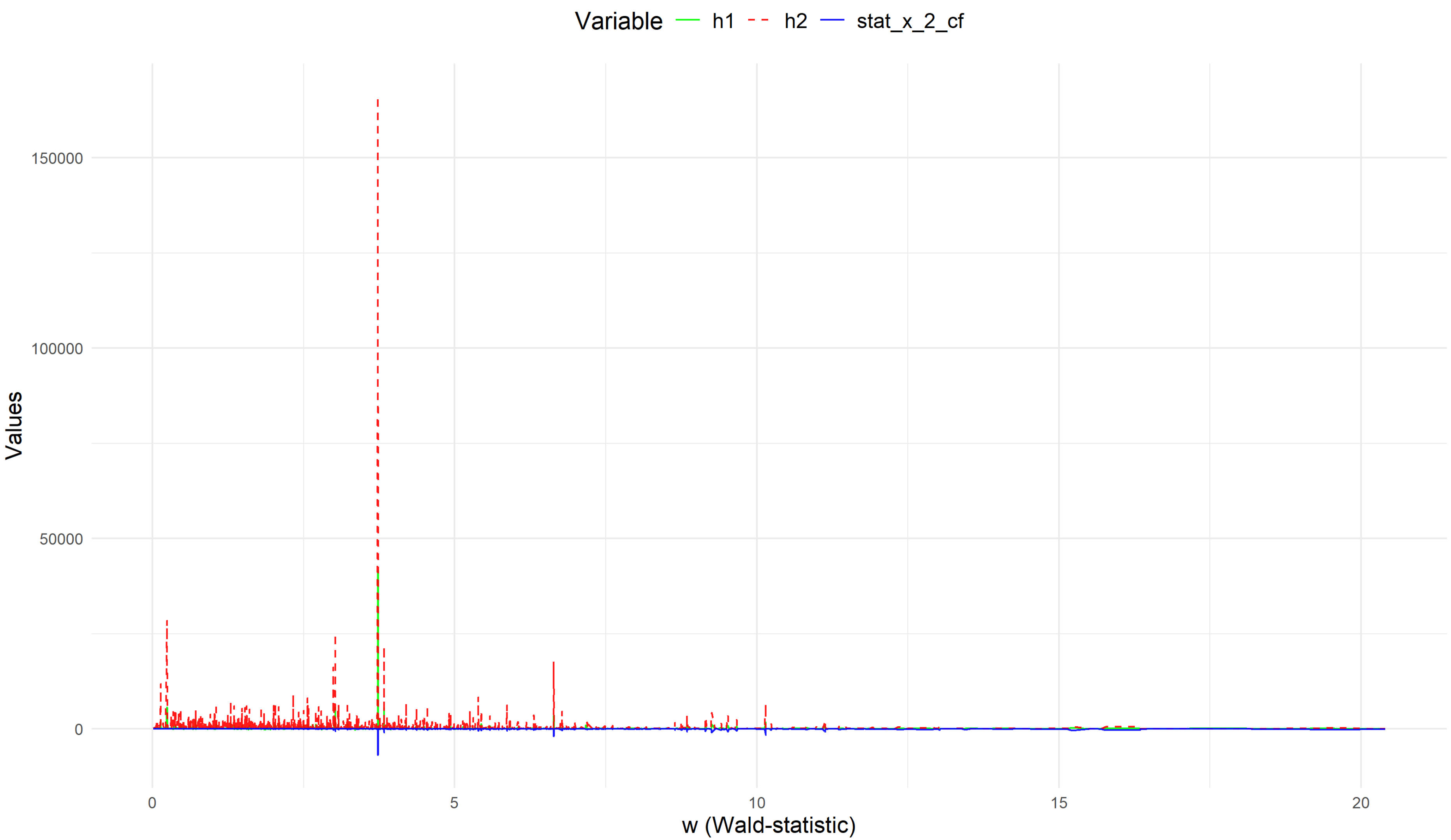

Figure F.47 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.9$, and T=30

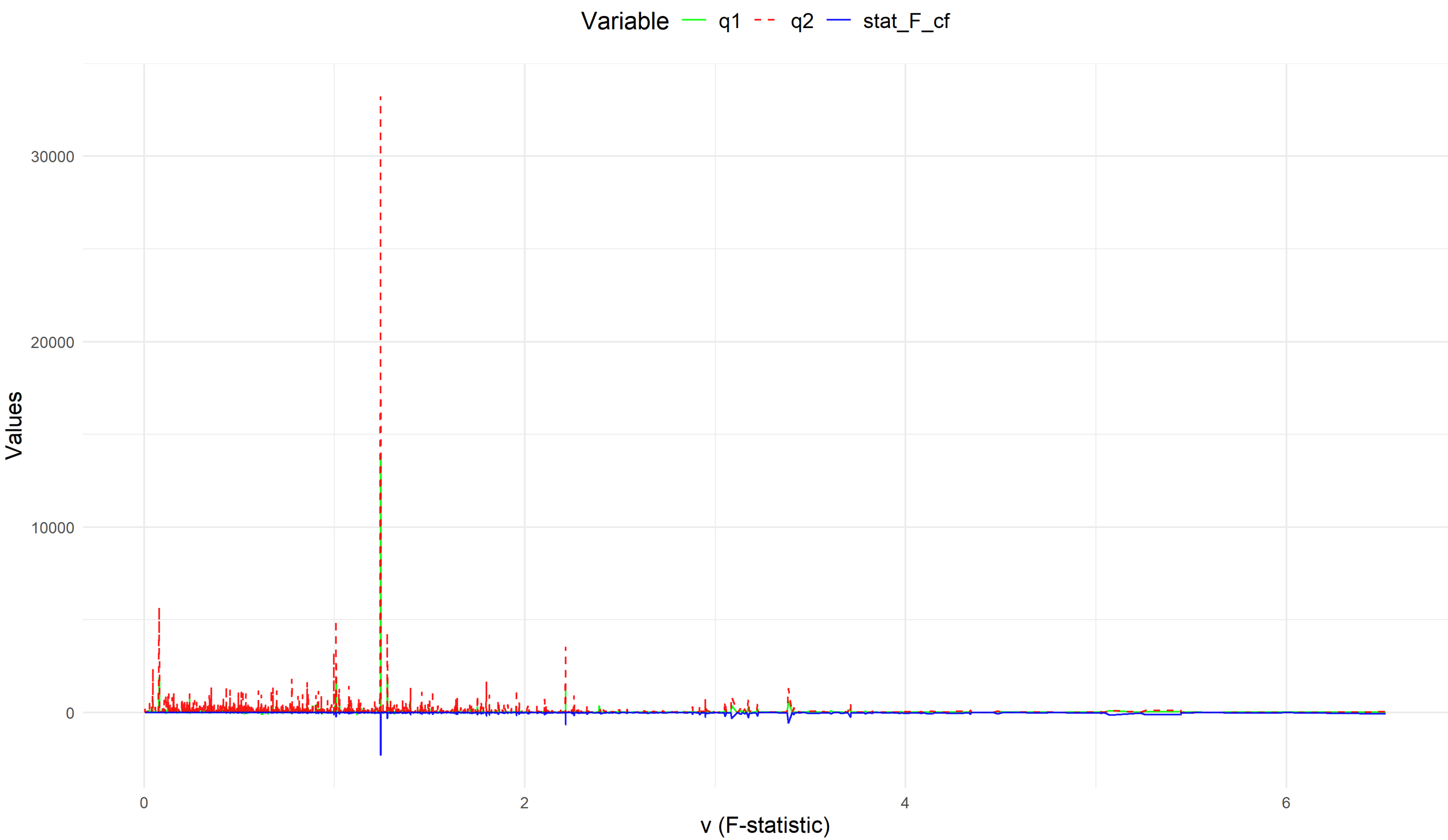

Figure F.48 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = -0.5$, $\phi = 0.9$, and T=30

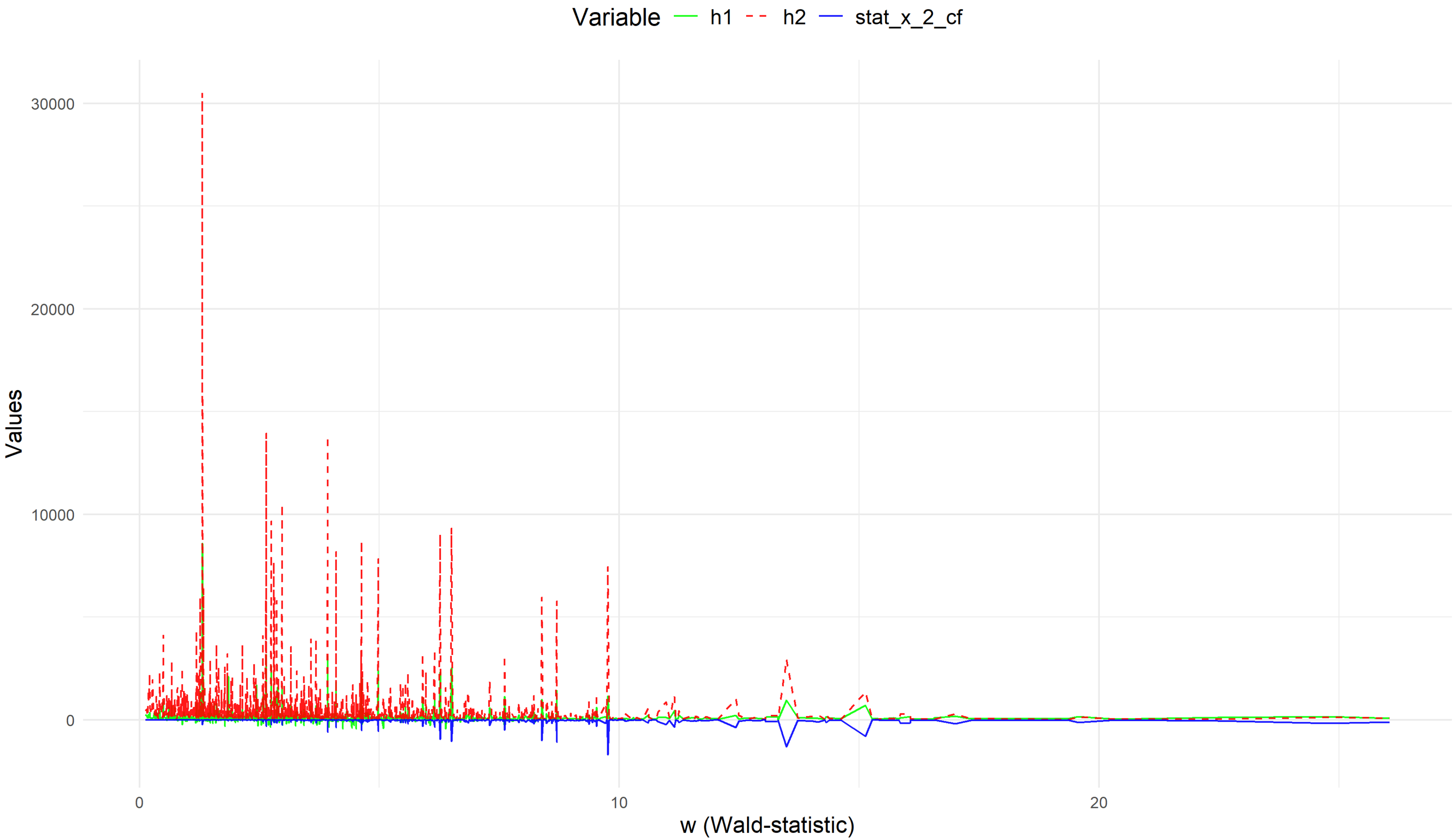

Figure F.49 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = -0.9$, and T=30

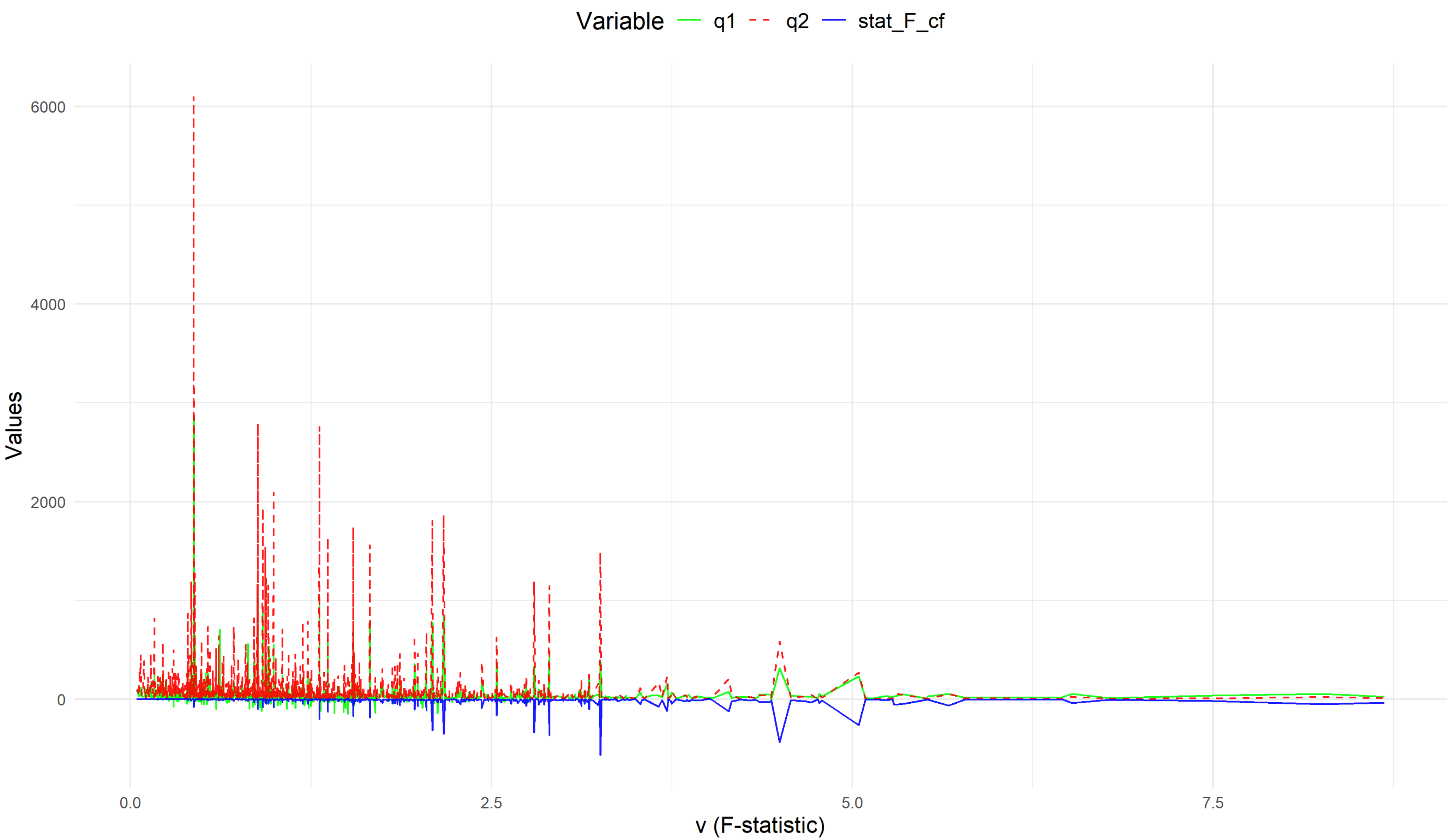

Figure F.50 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = -0.9$, and T=30

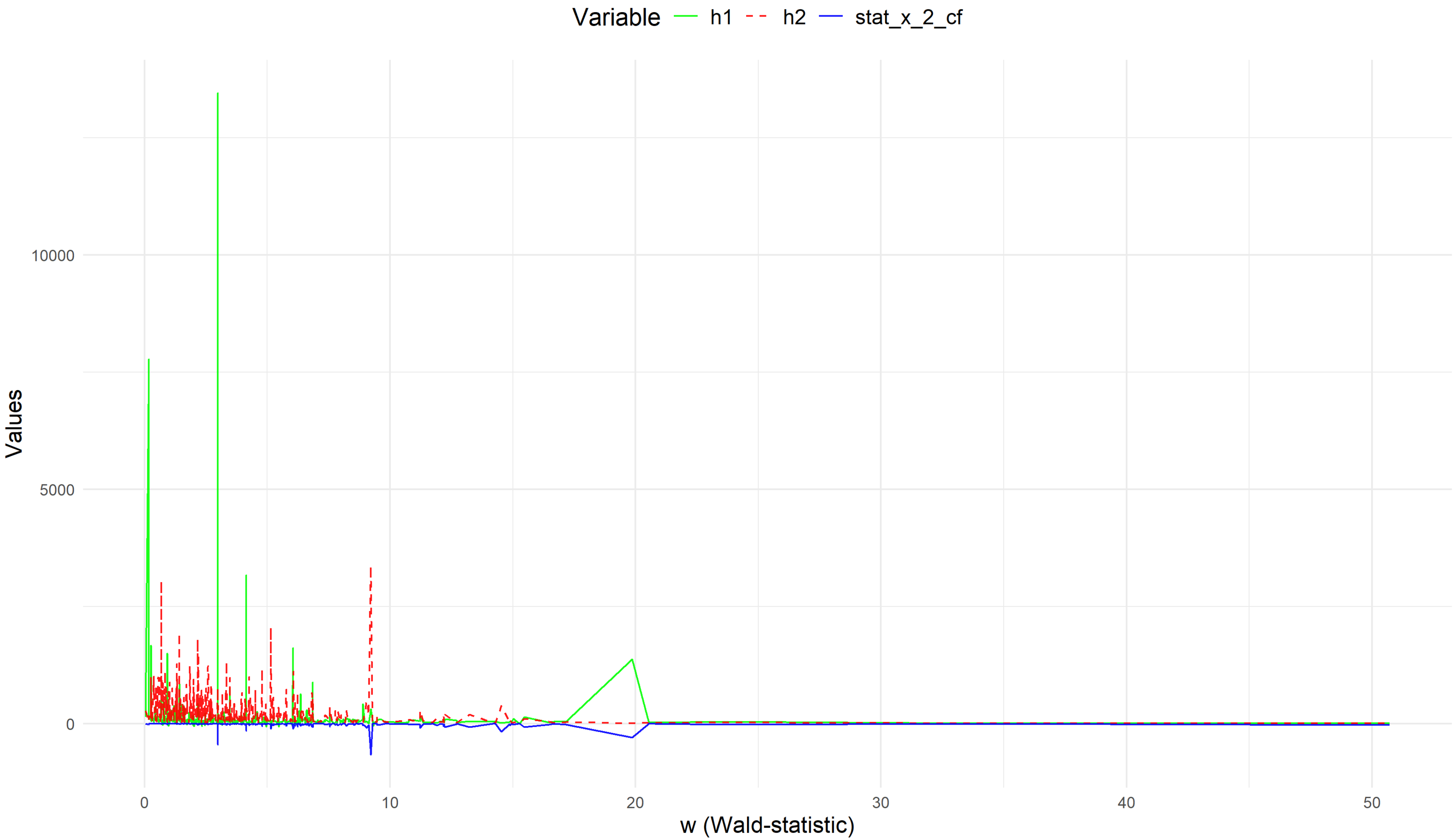

Figure F.51 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = -0.5$, and T=30

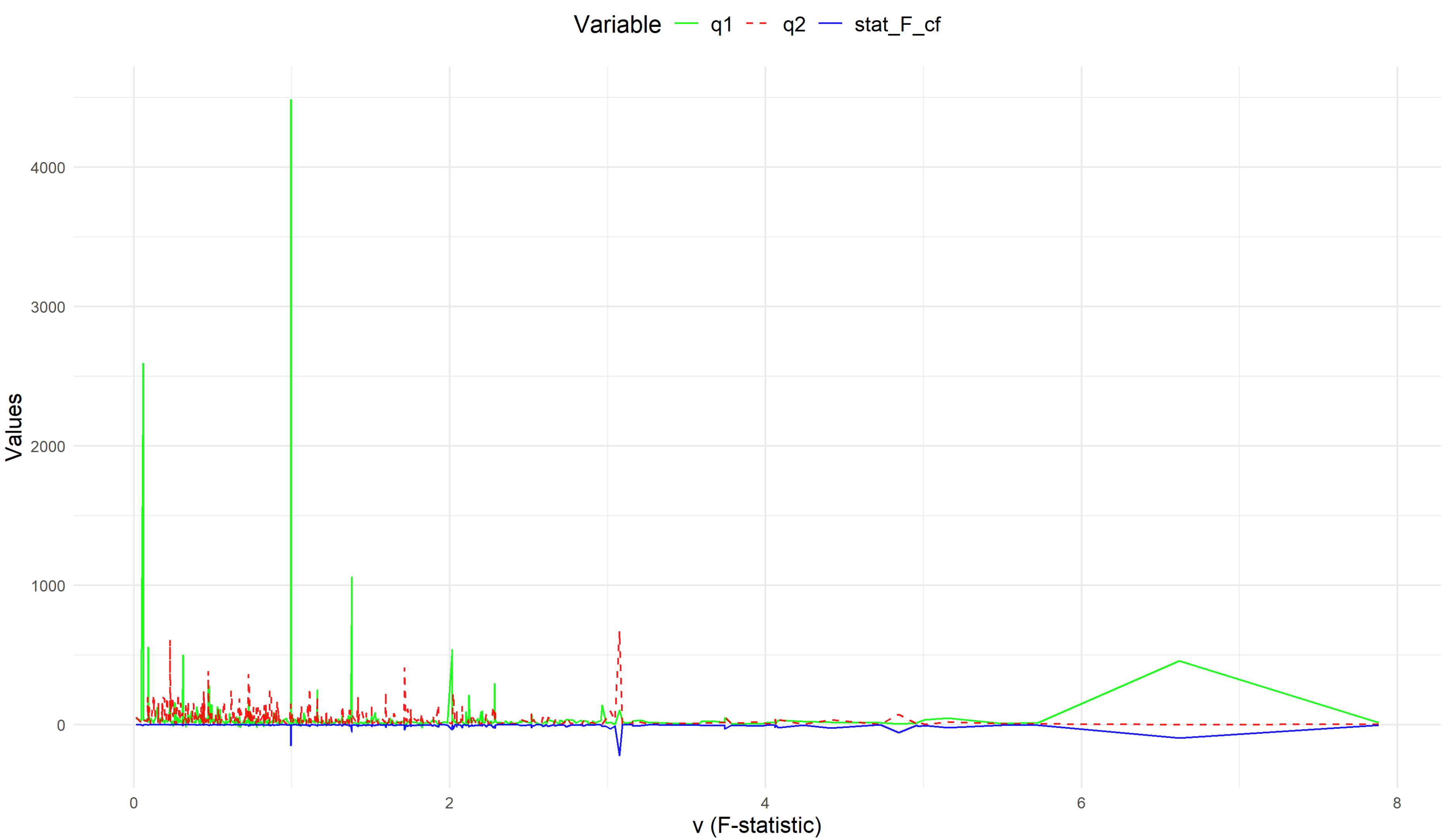

Figure F.52 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = -0.5$, and T=30

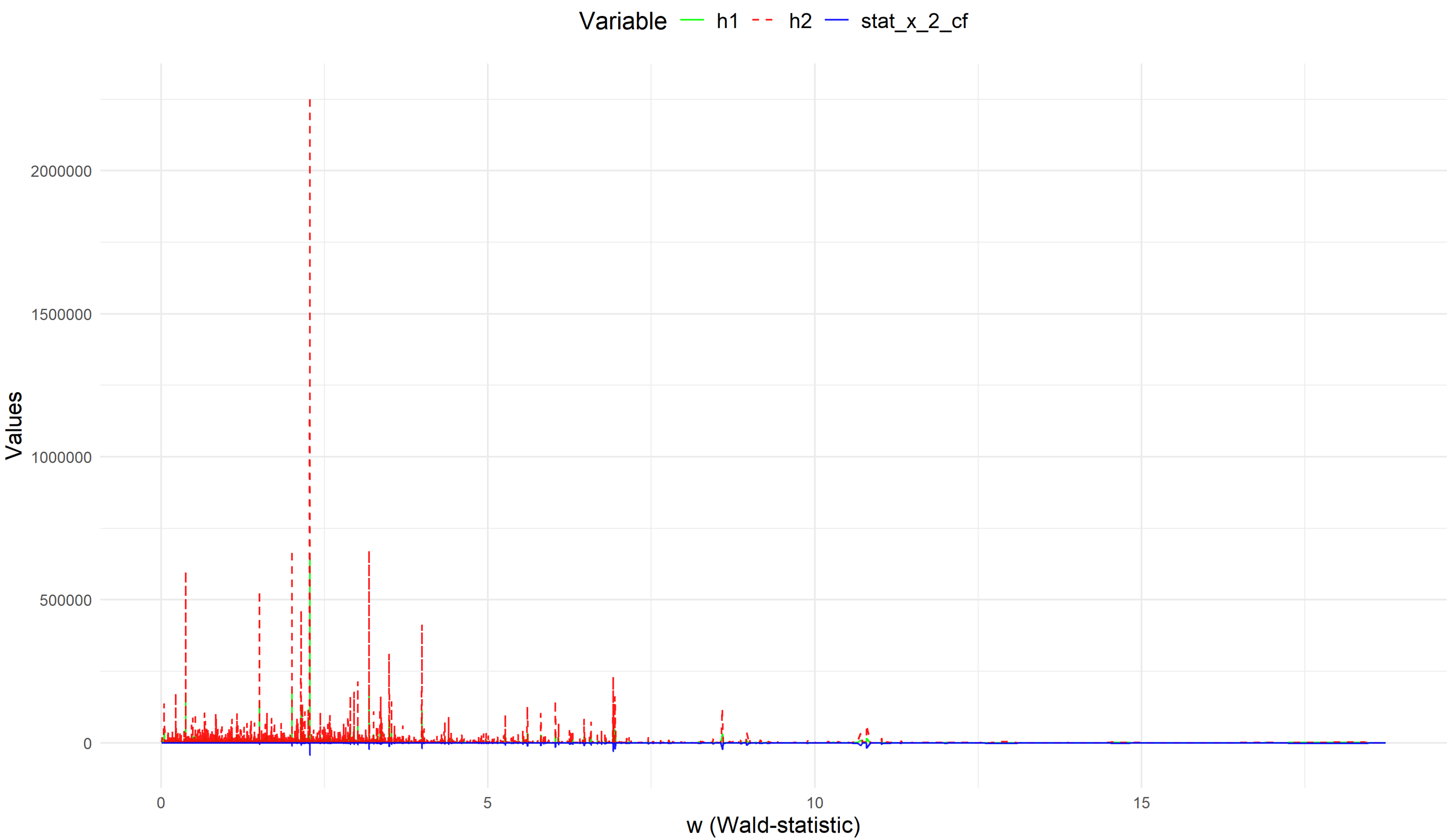

Figure F.53 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = 0.5$, and T=30

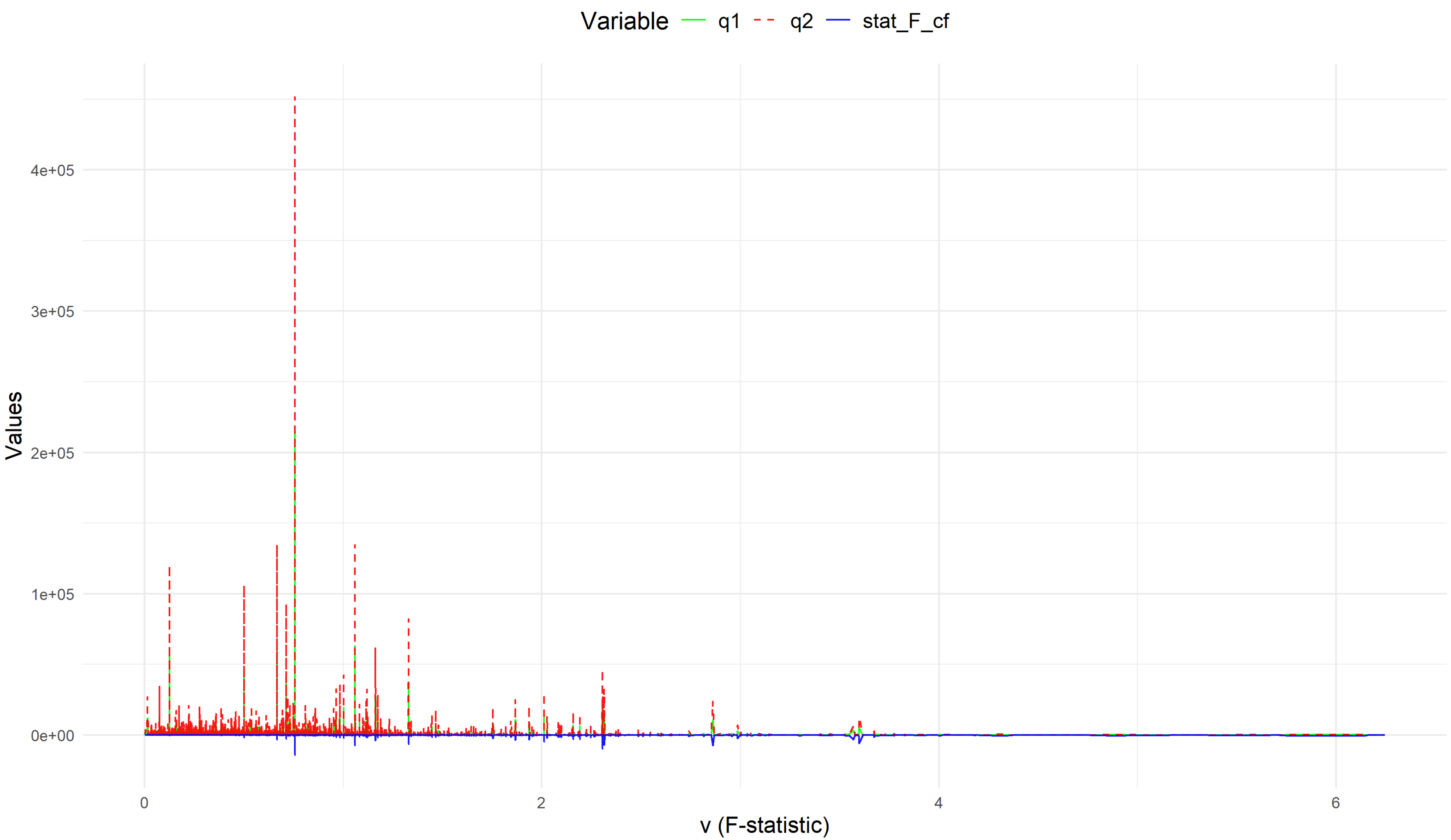

Figure F.54 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = 0.5$, and T=30

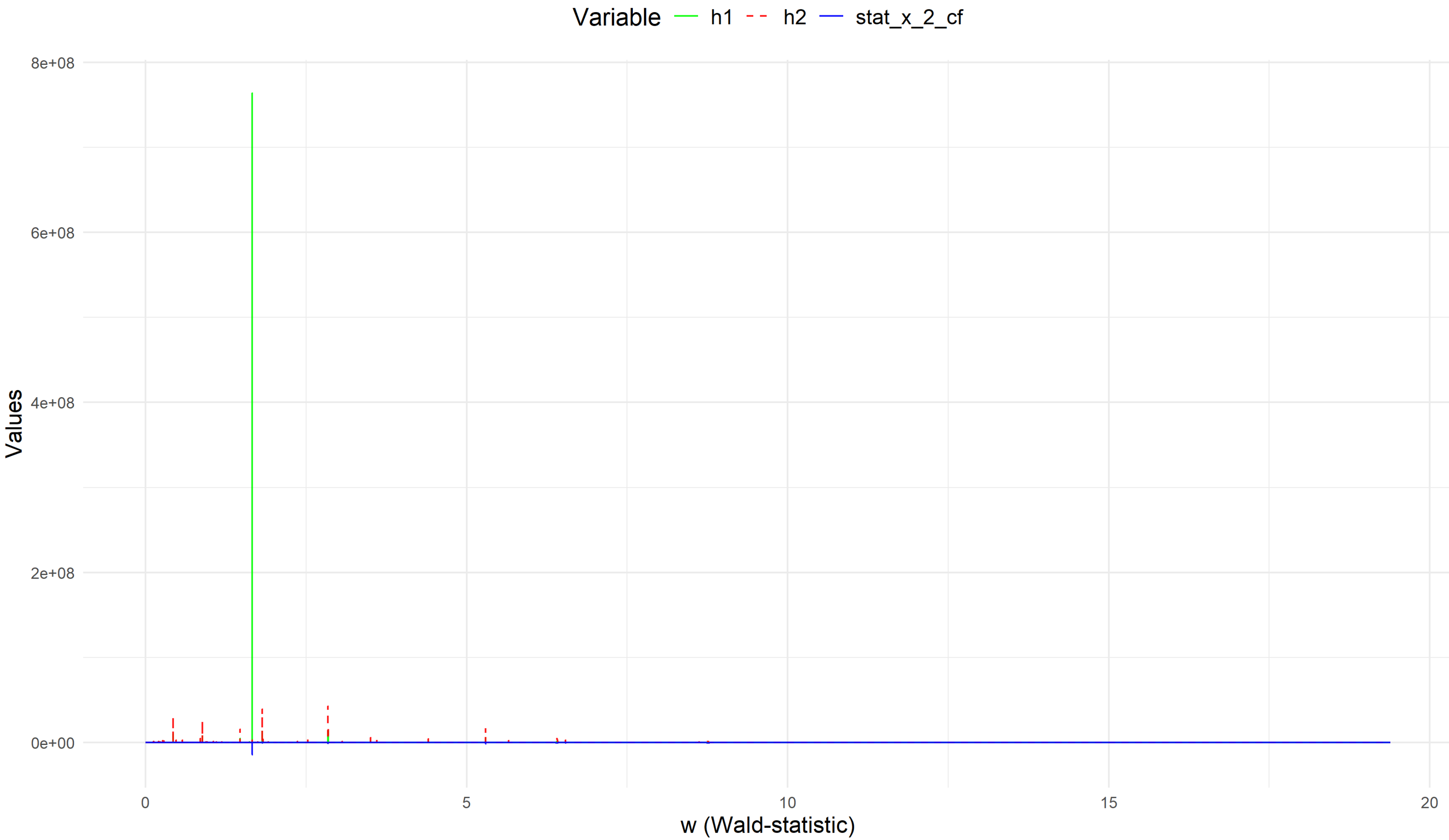

Figure F.55 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = 0.9$, and T=30

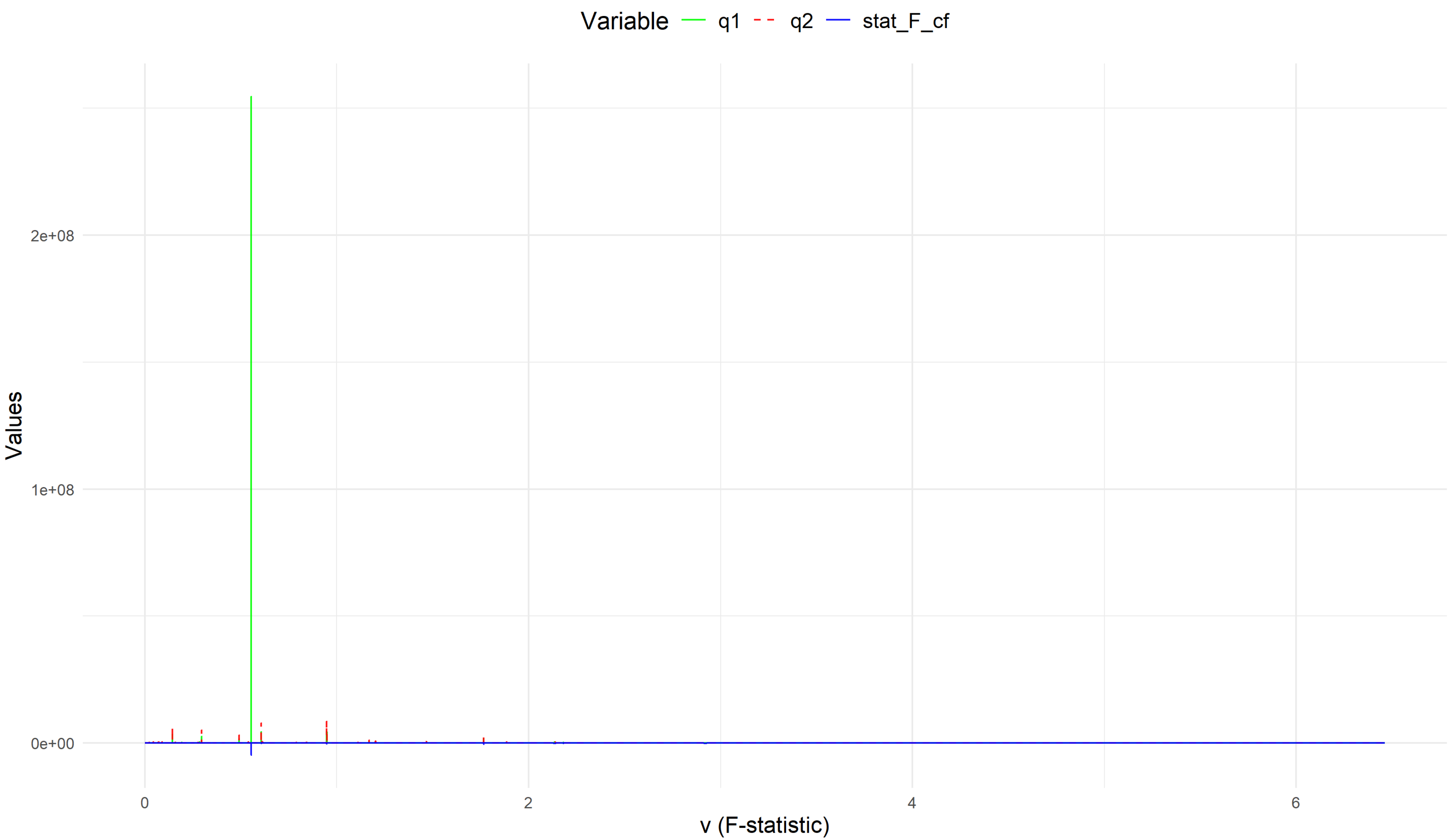

Figure F.56 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.5$, $\phi = 0.9$, and T=30

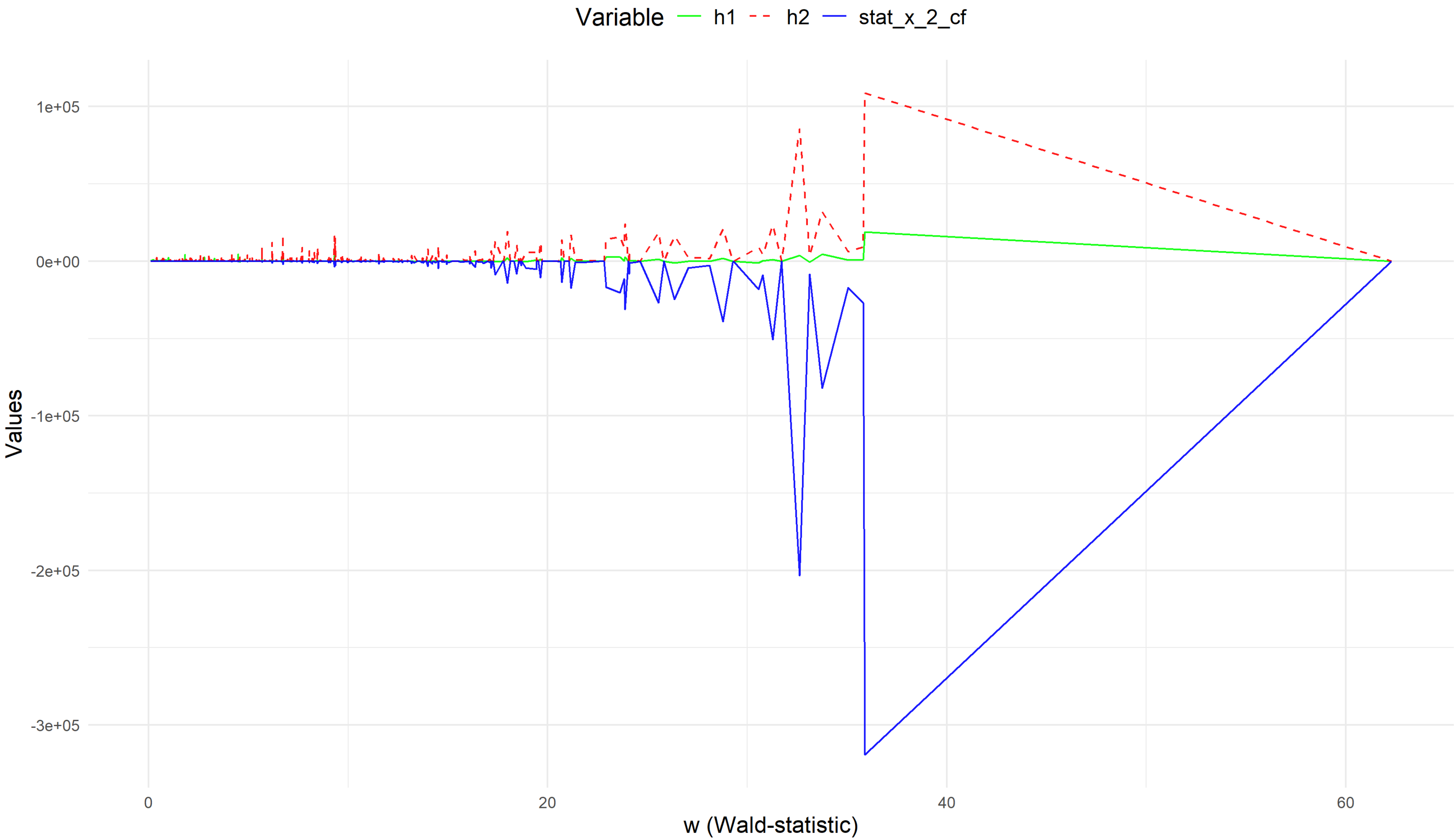

Figure F.57 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = -0.9$, and T=30

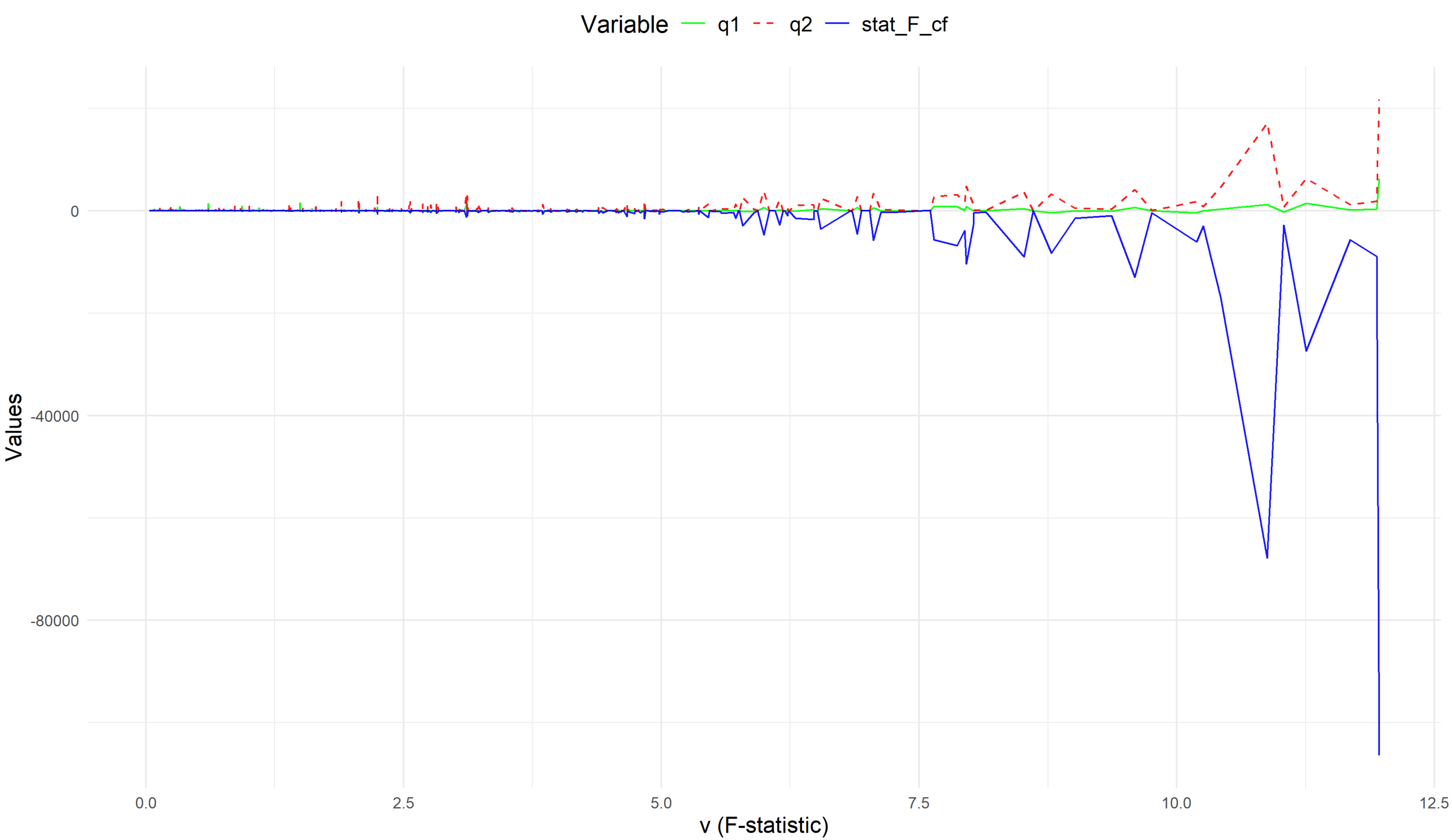

Figure F.58 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = -0.9$, and T=30

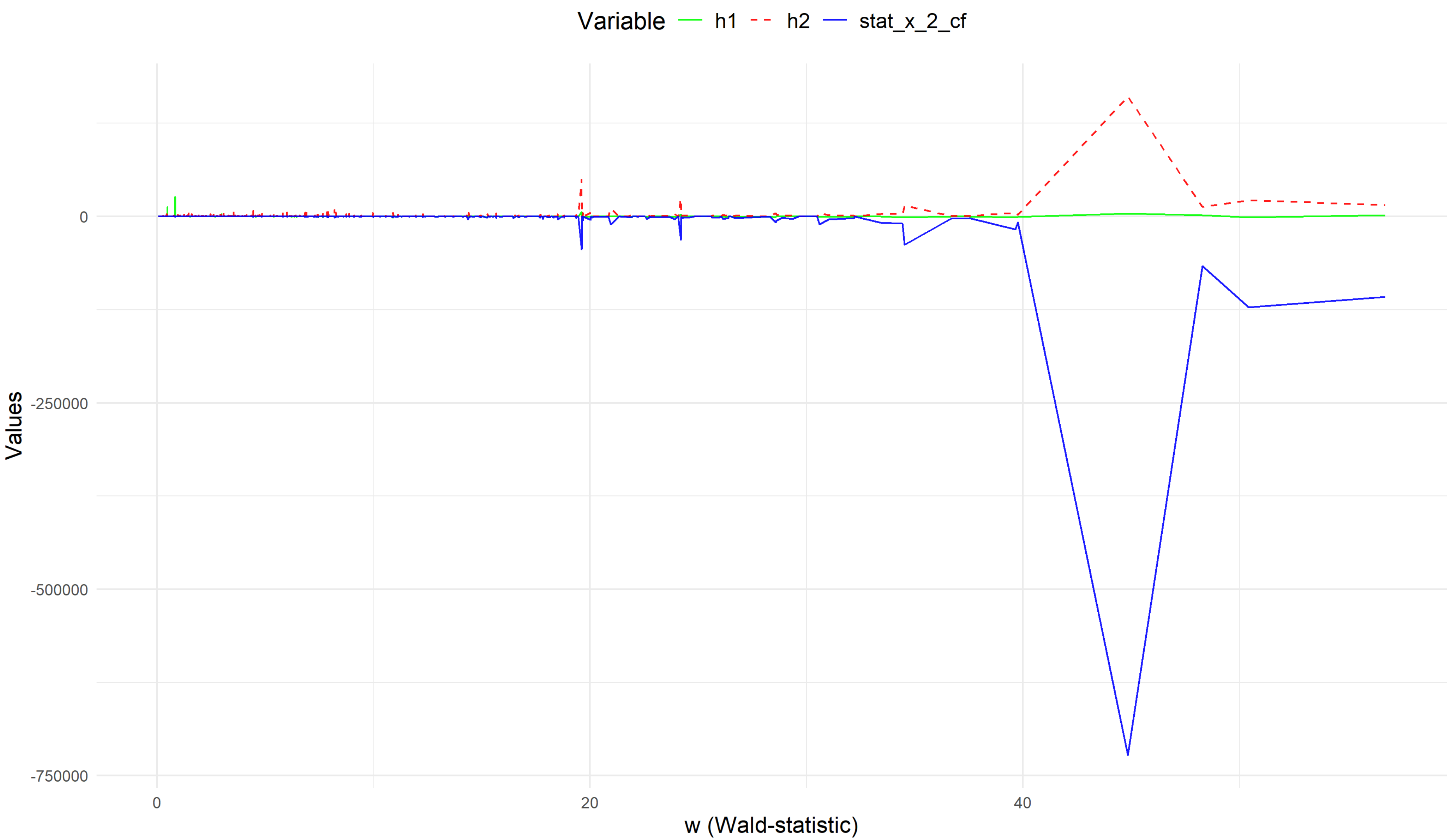

Figure F.59 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = -0.5$, and T=30

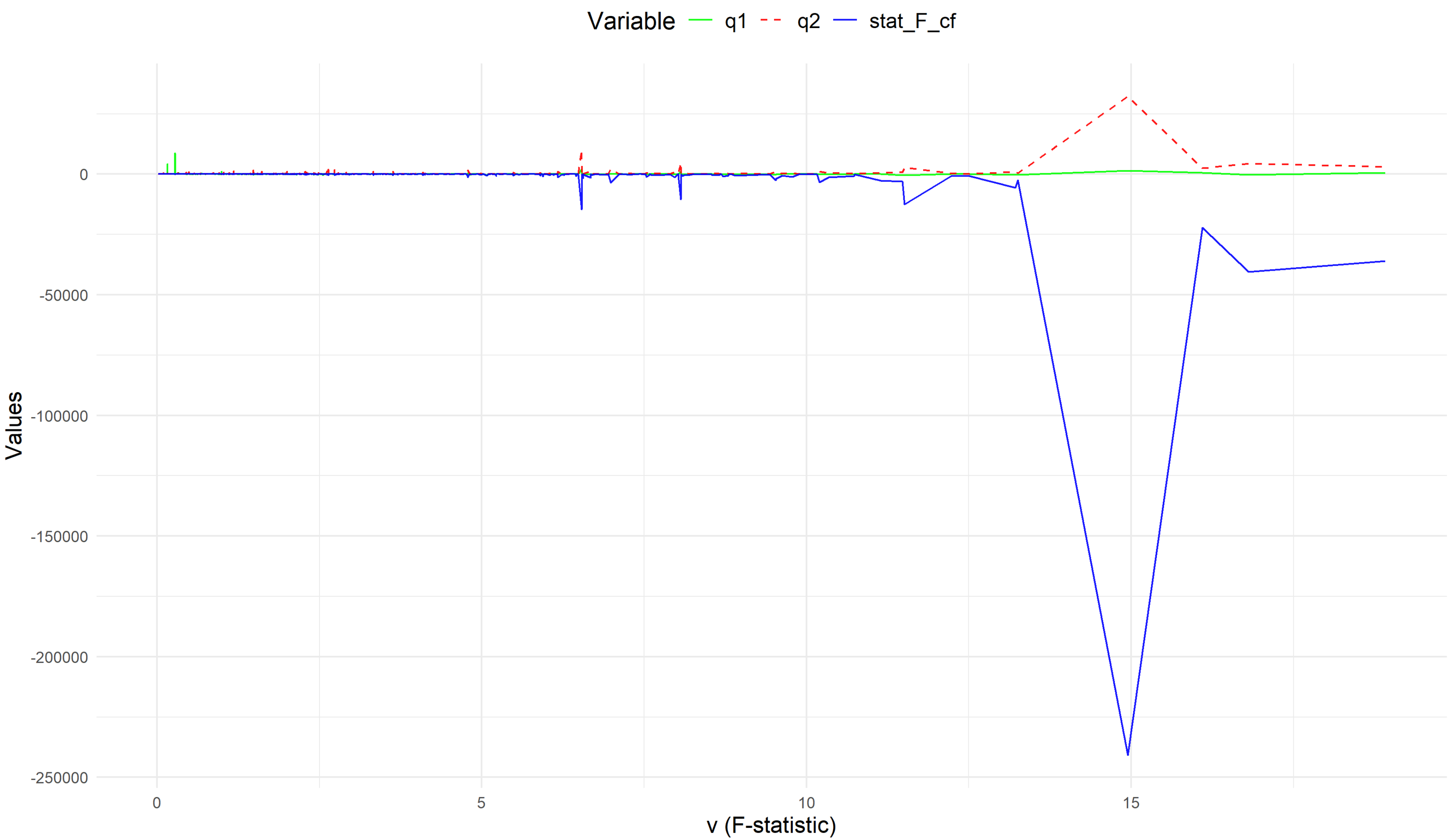

Figure F.60 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = -0.5$, and T=30

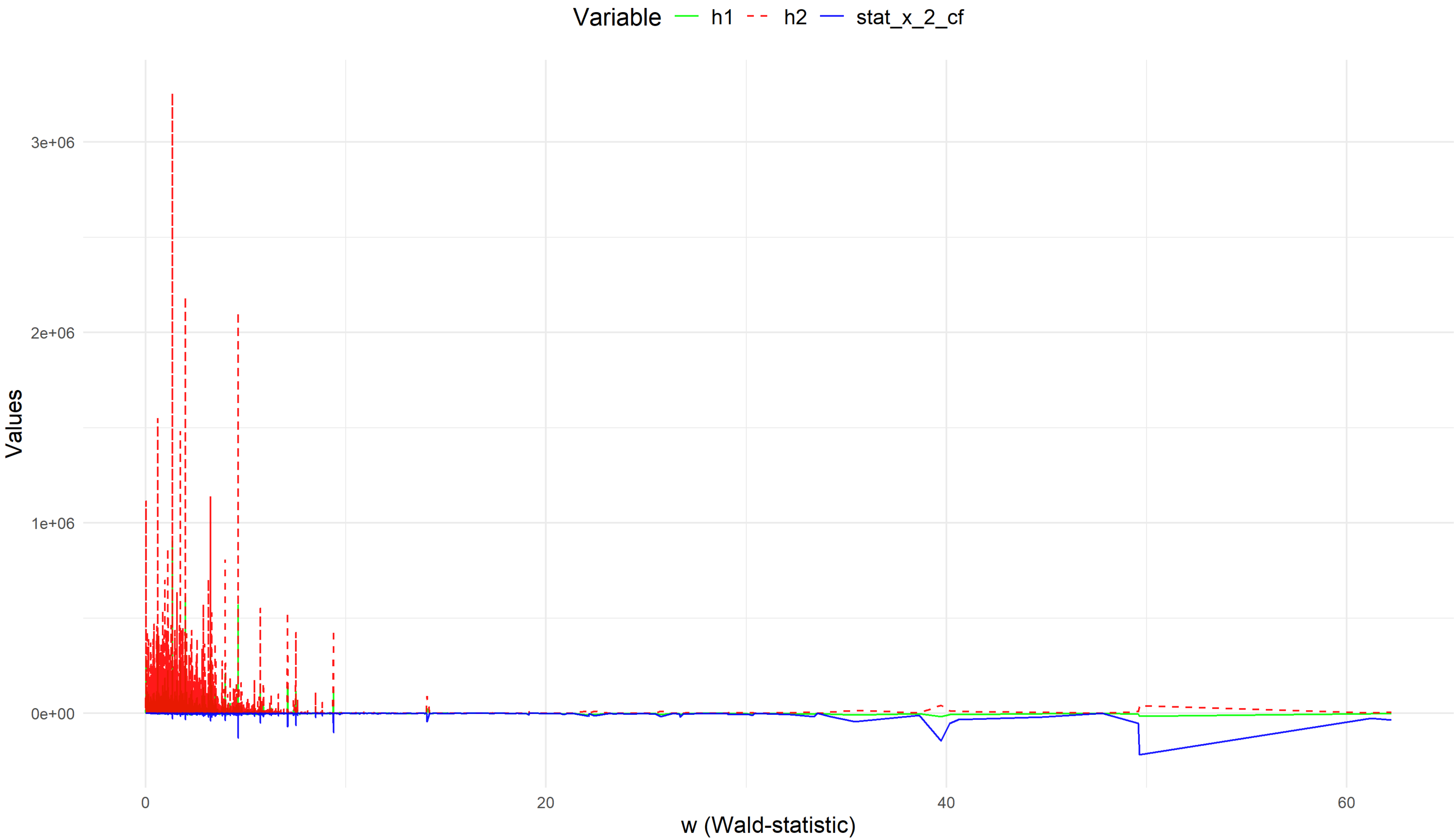

Figure F.61 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = 0.5$, and T=30

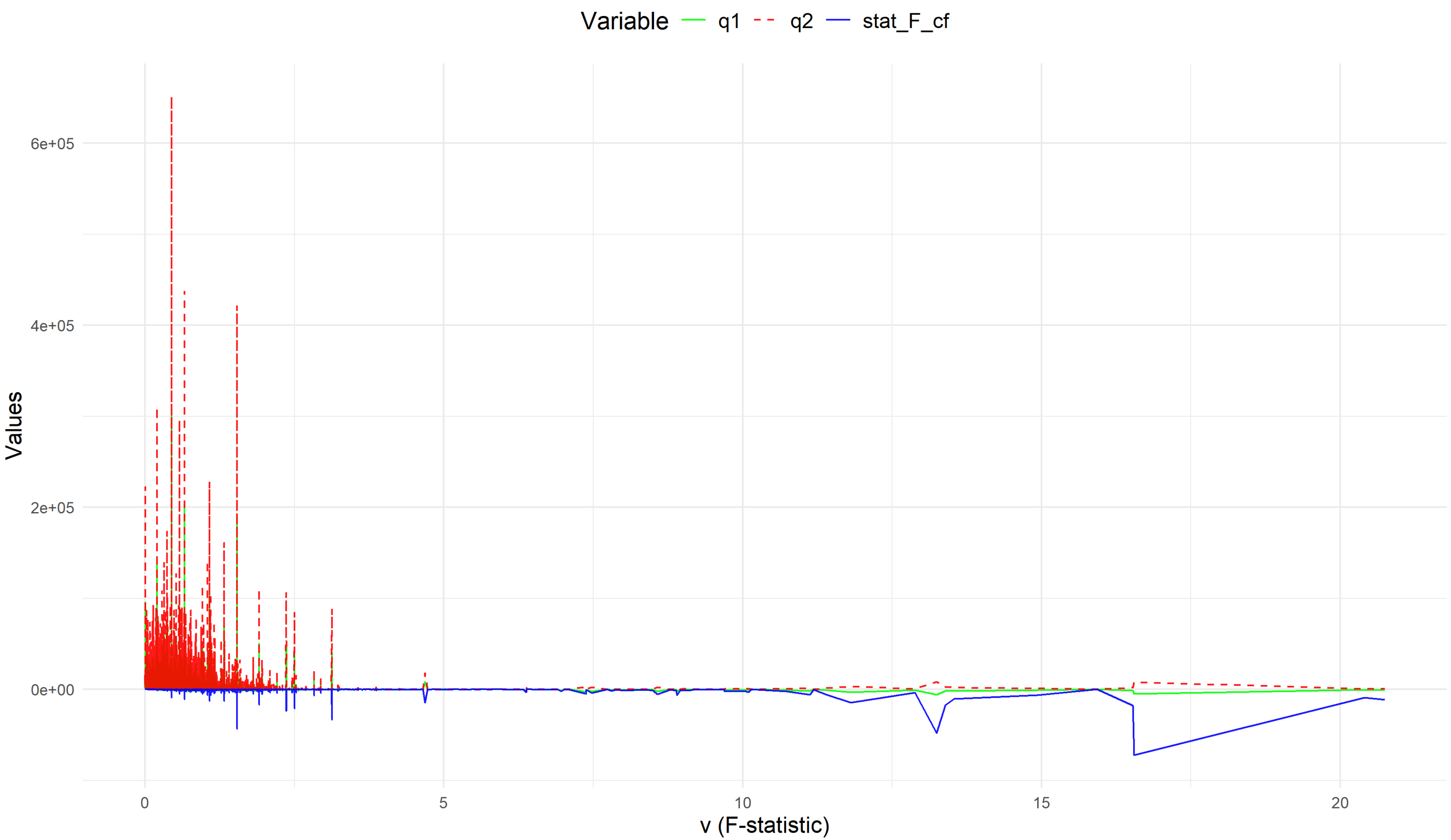

Figure F.62 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = 0.5$, and T=30

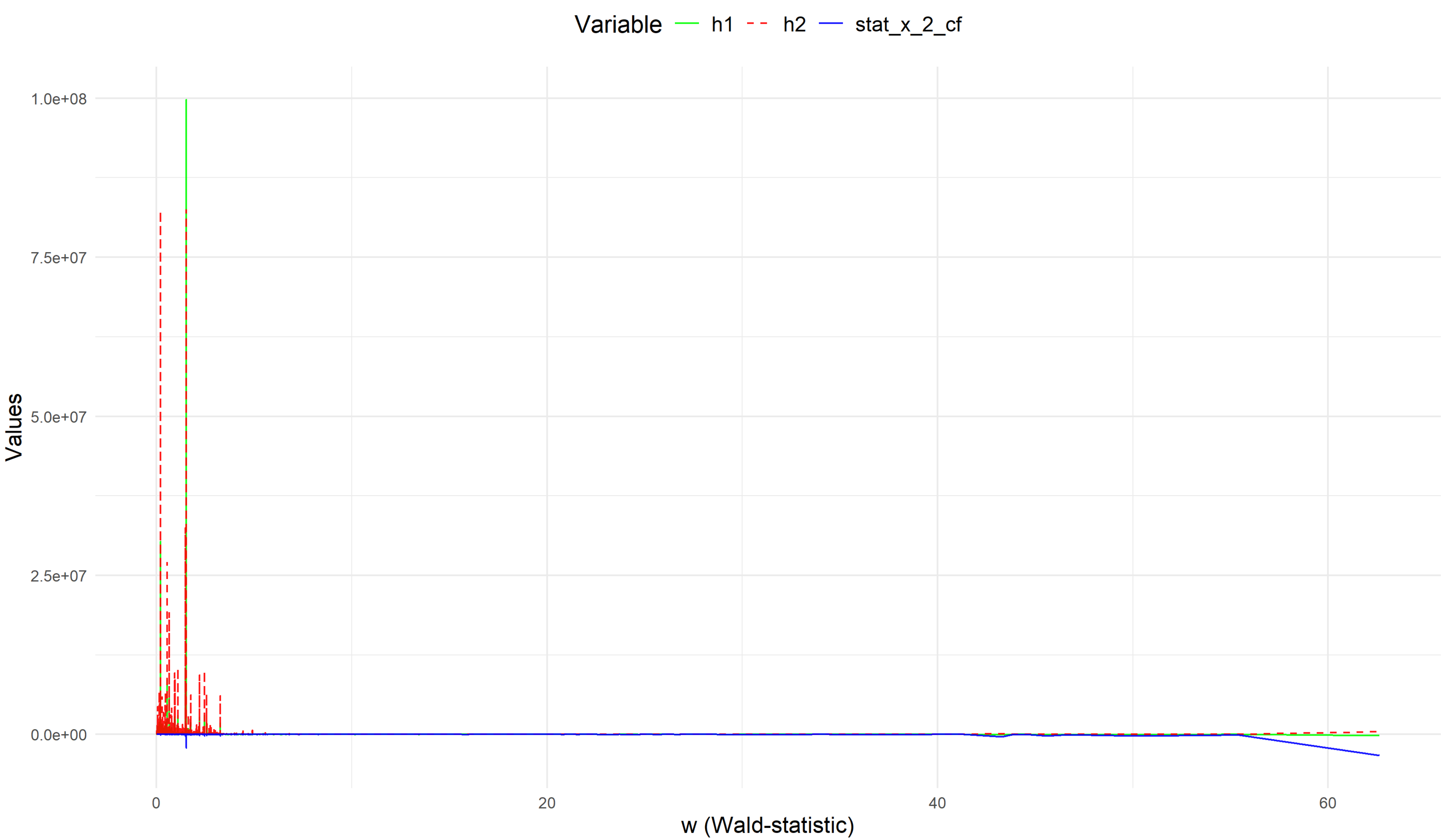

Figure F.63 Statistical relationship between h1, h2, and the Wald-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = 0.9$, and T=30

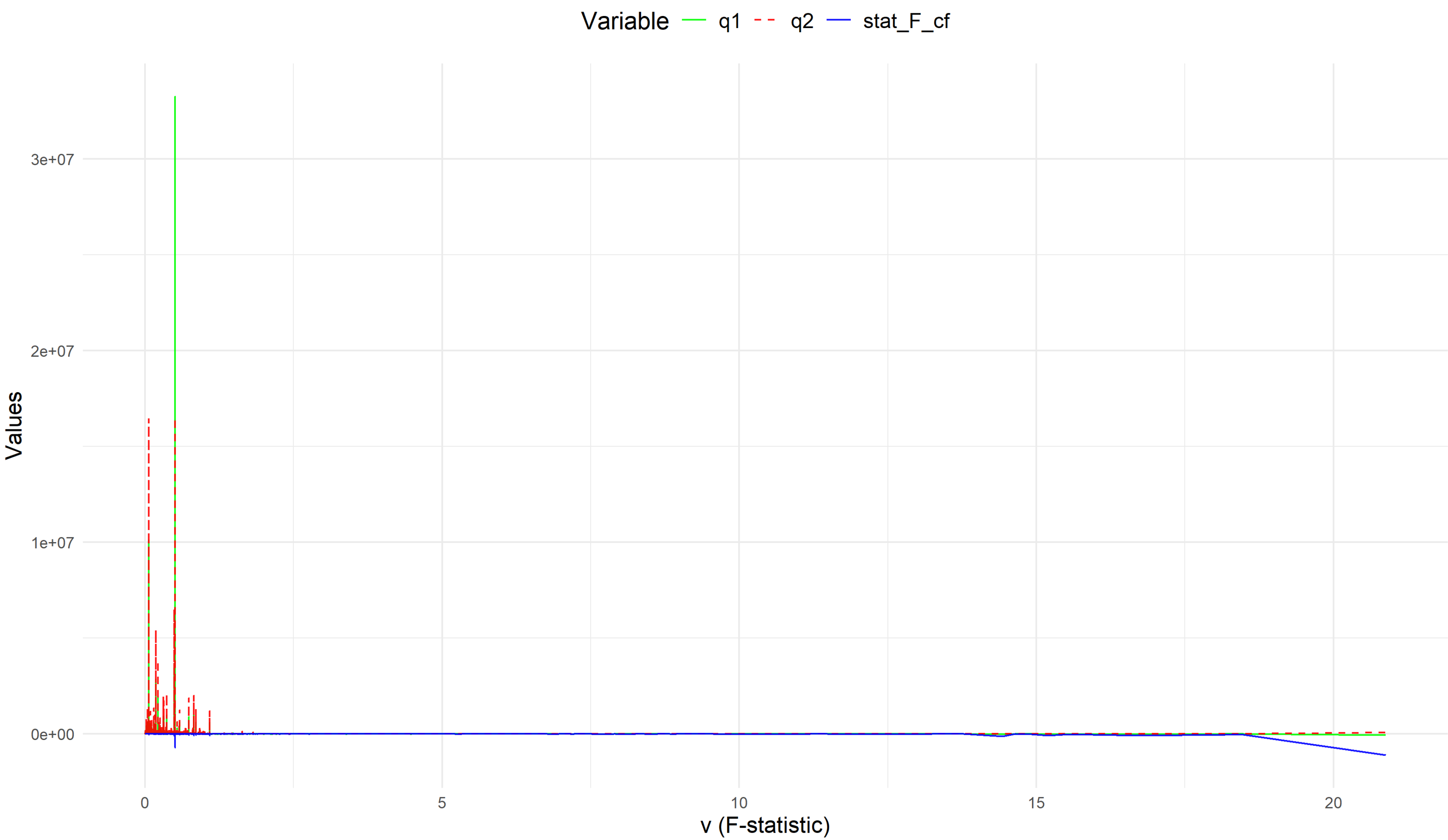

Figure F.64 Statistical relationship between q1, q2, and the F-Cornish-Fisher statistic under $\rho = 0.9$, $\phi = 0.9$, and T=30

Bibliography

Amemiya, T. (1966), 'The moments of the two-stage least-squares estimator', Journal of Econometrics **4**(3), 291–295.

Ansley, C. F. (1979), 'An algorithm for the exact likelihood of a mixed autoregressive-moving average process', Biometrika **66**(1), 59–65.

Basmann, R. (1961), 'A note on the moments of two-stage least squares', Econometrica **29**(1), 130–136.

Bhattacharya, R. and Ghosh, J. (1978), 'Edgeworth expansions in statistics', Annals of Statistics **6**, 434–475.

Box, G. E. P. and Jenkins, G. M. (1976), Time Series Analysis: Forecasting and Control, Holden-Day.

Breusch, T. (1980), 'Useful Invariance Results for Generalized Regression Models', Journal of Econometrics **13**, 327–340.

Brockwell, P. J. and Davis, R. A. (1991), Time Series: Theory and Methods, Springer Series in Statistics, Springer.

Brown, R., Smith, P. and Jones, M. (1974), 'Edgeworth approximations for econometric estimators', Journal of the American Statistical Association **69**(348), 668–673.

Chambers, M. (1967), 'Edgeworth expansions in the theory of econometric estimators and tests', Unpublished manuscript .

Chandra, T. and Ghosh, J. (1979), 'Expansions for distributions using laguerre polynomials', Sankhyā: The Indian Journal of Statistics **41**, 15–34.

Cornish, E. A. and Fisher, R. A. (1938), 'Moments and cumulants in the specification of distributions', Revue de l'Institut International de Statistique **5**(4), 307–320.

Dent, W. (1977), 'Computation of the exact likelihood function of an arima process', Journal of Statistical Computation and Simulation **6**(2), 105–115.

Di Gangi, L., Lapucci, M., Schoen, F. and Sortino, A. (2022), 'Improved maximum likelihood estimation of arma models', Lobachevskii Journal of Mathematics .

Elashmawi, H. (2014), Optimization of mathematical functions using gradient descent based algorithms, Master's thesis, Governors State University.

Fisher, R. (1925), 'Expansion of Student's Integral in Powers of n^{-1}', Metron **5**, 109–120.

Fisher, R. and Cornish, E. (1960), 'The Percentile Points of Distributions Having Known Cumulants', Technometrics **2**(2), 209–225.

Gardner, G., Harvey, A. C. and Phillips, G. D. A. (1980), 'Algorithm AS 154: An Algorithm for Exact Maximum Likelihood Estimation of Autoregressive-Moving Average Models by Means of Kalman Filtering', Journal of the Royal Statistical Society. Series C (Applied Statistics) **29**(3), 311–322.

Granger, C. W. J. and Joyeux, R. (1980), 'An introduction to long-memory time series models and fractional differencing', Journal of Time Series Analysis **1**(1), 15–29.

Hamilton, J. D. (1994), Time Series Analysis, Princeton University Press.

Harvey, A. C. and Phillips, G. D. A. (1979), 'Maximum likelihood estimation of regression models with autoregressive-moving average disturbances', Biometrika **66**(1), 49–58.

Hastie, T., Tibshirani, R. and Friedman, J. (2009), The Elements of Statistical Learning: Data Mining, Inference, and Prediction, 2nd edn, Springer.

Hoerl, A. E. and Kennard, R. W. (1970), 'Ridge regression: Biased estimation for nonorthogonal problems', Technometrics **12**(1), 55–67.

James, G., Witten, D., Hastie, T. and Tibshirani, R. (2013), An Introduction to Statistical Learning, Springer.

James, G., Witten, D., Hastie, T. and Tibshirani, R. (2021), An Introduction to Statistical Learning: with Applications in R, 2 edn, Springer.

J.G. de Gooijer, J. (1978), 'On the Inverse of the Autocovariance Matrix for a General Mixed Autoregressive Moving Average Process', Statistische Hefte **19**, 114–123.

Jones, M. C. (1987), 'Randomly choosing parameters from the stationarity and invertibility region of autoregressive-moving average models', Journal of the Royal Statistical Society. Series C (Applied Statistics) **36**(2), 134–138.

Koenecke, A. (2019), 'Applying deep neural networks to financial time series forecasting', Institute for Computational and Mathematical Engineering **14**(1).

Liu, Y. and Tajbakhsh, S. D. (2024), Fitting arma time series models without identification: A proximal approach, in 'Proceedings of Machine Learning Research', Vol. 238, pp. 1–15.

Magdalinos, M. (1992), 'Stochastic Expansions and Asymptotic Approximations', Econometric Theory **8**(3), 343–367.

Magdalinos, M. and Symeonides, S. (1995), 'Alternative Size Corrections for Some GLS Test Statistics: The Case of the AR(1) Model', Journal of Econometrics **66**(1–2), 35–59.

Magdalinos, T. (1983), 'Asymptotic expansions in the theory of econometric estimators and tests', Unpublished manuscript . London School of Economics.

McDonald, G. C. and Galarneau, D. I. (1975), 'A monte carlo evaluation of some ridge-type estimators', Journal of the American Statistical Association **70**(350), 407–416.

Nagar, A. (1959), 'The Bias and Moment Matrix of the General k-class Estimator of the Parameters in Structural Equations', Econometrica **27**, 575–595.

Nagar, A. (1961), 'A Note on the Residual Variance Estimation in Simultaneous Equations', Econometrica **29**, 238–243.

Nagar, A. (1962), 'A note on the moments of the two-stage least squares estimator', Econometrica **30**(3), 575–578.

Nagar, A. (1970), 'Distribution of simultaneous equation estimators: The role of higher moments', International Economic Review **11**(3), 460–471.

Nagar, A. and Gupta, Y. (1968), 'The Bias of Liviatan's Consistent Estimator in a Distributed Lag Model', Econometrica **36**, 337–342.

Nagar, A. and Gupta, Y. (1970), 'The Moment Matrix of the Two-Stage Least-Squares Estimator of Coefficients in Different Equations of a Complete System of Simultaneous Equations', Econometrica **38**(1), 39–49.

Nagar, A. and Ullah, A. (1973), 'Note on the Approximate Skewness and Kurtosis of the Two-Stage Least-Square Estimator', Indian Economic Review **8**(1), 69–80.

Nelson, D. B. (1991), 'Conditional heteroskedasticity in asset returns: A new approach', Econometrica **59**(2), 347–370.

Newbold, P. (1974), 'The exact likelihood function for a mixed autoregressive-moving average process', Biometrika **61**(3), 423–426.

Park, D. H., Ho, C. M., Chang, Y. and Zhang, H. (2018), 'Gradient-coherent strong regularization for deep neural networks with stochastic gradient descent', arXiv preprint arXiv:1811.08056 .

Phillips, P. (1977a), 'Approximations in the theory of econometrics', Unpublished manuscript .

Phillips, P. (1977b), 'Second order approximations in econometrics', Unpublished manuscript .

Phillips, P. (1978), 'Expansions for distributions of econometric estimators and test statistics', Unpublished manuscript .

Pierce, D. (1971), 'Least Squares Estimation in the Regression Model with Autoregressive Moving Average Errors', Biometrika **58**(2), 299–312.

Ramage, J. (1971), 'An expansion for the distribution of econometric estimators', Econometrica **39**(1), 41–46.

Richards, D., Dobriban, E. and Rebeschini, P. (2021), 'Comparing classes of estimators: When does gradient descent beat ridge regression in linear models?', arXiv preprint arXiv:2108.11872 .

Rothenberg, T. (1984a), 'Approximate Normality of Generalized Least Squares Estimates', Econometrica **52**(4), 811–825.

Rothenberg, T. (1984b), 'Hypothesis Testing in Linear Models when the Error Covariance Matrix is Nonscalar', Econometrica **52**(4), 827–842.

Rothenberg, T. (1988), 'Approximate Power Functions for some Robust Tests of Regression Coefficients', Econometrica **56**(5), 997–1019.

Rothenberg, T. J. (1982), 'Wald, likelihood ratio, and lagrange multiplier tests in econometrics', **2**, 775–826.

Ruder, S. (2016), 'An overview of gradient descent optimization algorithms', arXiv preprint arXiv:1609.04747 .

Sargan, J. (1975), 'Gram-Charlier Approximations Applied to t Ratios of k-class Estimators', Econometrica **43**(2), 327–346.

Sargan, J. (1976), 'Econometric Estimators and the Edgeworth Approximation', Econometrica **44**(3), 421–448.

Sargan, J. (1980), 'Some Approximations to the Distribution of Econometric Criteria which are Asymptotically Distributed as Chi-Squared', Econometrica **48**(5), 1107–1138.

Shaman, P. (1969), 'On the inverse of the covariance matrix of a first order moving average', Biometrika **56**(3), 595.

Shaman, P. (1973), 'On the Inverse of the Covariance Matrix for an Autoregressive-Moving Average Process', Biometrika **60**(1), 193–196.

Symeonides, S. D. (1991), Applications of Refined Asymptotic Theory in Econometric Testing, Athens University of Economics and Business, Athens. PhD dissertation.

Symeonides, S. D., Karavias, Y. and Tzavalis, E. (2017), 'Size corrected significance tests in seemingly unrelated regressions with autocorrelated errors', Journal of Time Series Econometrics **9**(1).

Tapkir, A. (2023), 'A comprehensive overview of gradient descent and its optimization algorithms', International Advanced Research Journal in Science, Engineering and Technology **10**(11), 37–42.

Tian, Y., Zhang, Y. and Zhang, H. (2023), 'Recent advances in stochastic gradient descent in deep learning', Mathematics **11**(3), 682.

Tiao, G. C. and Ali, M. M. (1971), 'Analysis of Correlated Random Effects: Linear Model with Two Random Components', Biometrika **58**(1), 37–51.

Tibshirani, R. (1996), 'Regression shrinkage and selection via the lasso', Journal of the Royal Statistical Society: Series B (Methodological) **58**(1), 267–288.

White, H. (1982), 'Maximum likelihood estimation of misspecified models', Econometrica **50**(1), 1–25.

Zhang, Y. and McLeod, A. I. (2006), 'Fitting ma(q) models in the closed invertible region', Statistics and Probability Letters **76**(13), 1331–1334.

Zhu, Y. and Liu, R. (2021), 'An algorithmic view of l2 regularization and some path-following algorithms', Journal of Machine Learning Research **22**, 1–38.

Zou, H. and Hastie, T. (2005), 'Regularization and variable selection via the elastic net', Journal of the Royal Statistical Society: Series B (Statistical Methodology) **67**(2), 301–320.