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THE BASES OF INTERVALS AND RECTANGLES IN THE TWO DIMENSIONAL EUCLIDEAN SPACE

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ΠΑΝΕΠΙΣΤΗΜΙΟ ΙΩΑΝΝΙΝΩΝ

ΤΜΗΜΑ ΜΑΘΗΜΑΤΙΚΩΝ

Καλλιόπη Σακελλαρίου

ΟΙ ΒΑΣΕΙΣ ΤΩΝ ΔΙΑΣΤΗΜΑΤΩΝ ΚΑΙ ΤΩΝ ΟΡΘΟΓΩΝΙΩΝ ΣΤΟΝ 2-ΔΙΑΣΤΑΤΟ ΕΥΚΛΕΙΔΕΙΟ ΧΩΡΟ

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ΥΠΕΥΘΥΝΗ ΔΗΛΩΣΗ

"Δηλώνω υπεύθυνα ότι η παρούσα διατριβή εκπονήθηκε κάτω από τους διεθνείς ηθικούς και ακαδημαϊκούς κανόνες δεοντολογίας και προστασίας της πνευματικής ιδιοκτησίας. Σύμφωνα με τους κανόνες αυτούς, δεν έχω προβεί σε ιδιοποίηση ξένου επιστημονικού έργου και έχω πλήρως αναφέρει τις πηγές που χρησιμοποίησα στην εργασία αυτή."

Καλλιόπη Σακελλαρίου

ABSTRACT

We will examine the basis $\mathbb{B}_2$ of rectangles of $\mathbb{R}^2$ and $\mathbb{R}^3$ with sides parallel to the axis and their differentiation properties in relation to the corresponding maximal operators using some covering theorems.
We will present the noteworthy set called the Perron tree and use it in order to give answers to the "Needle problem" and the existence of Besicovitch sets and examine the differentiation basis $\mathbb{B}_3$ of rectangles of $\mathbb{R}^2$ and some of its subbases.
We will then work on the distribution function and the decreasing rearrangement of a given function to acquire inequalities that we will use on the last chapter, which is a generalization of what preceded, where we will examine the multidimensional analogues of what we proved previously, on the interval $S = (0,1)^n$, for $n \in \mathbb{N}$ arbitary, getting some differentiation properties of the basis $\mathbb{B}_2$ once again using the corresponding maximal operator.

ΠΕΡΙΛΗΨΗ

Θα ασχοληθούμε με την βάση $\mathbb{B}_2$ των διαστημάτων του $\mathbb{R}^2$ και του $\mathbb{R}^3$ με πλευρές παράλληλες στους άξονες και τις ιδιότητες διαφόρισής τους σε σχέση με τους αντίστοιχους μεγιστικούς τελεστές χρησιμοποιώντας κάποια θεωρήματα κάλυψης.
Θα παρουσιάσουμε το σύνολο που είναι γνωστό ως δέντρο του Perron και θα το αξιοποιήσουμε ώστε να δώσουμε απαντήσεις στο "Needle Problem" και στην ύπαρξη των συνόλων Besicovitch καθώς επίσης θα μελετήσουμε τις ιδιότητες διαφόρισης της βάσης $\mathbb{B}_3$ των ορθογωνίων του $\mathbb{R}^2$ αλλά και απο κάποιων από τις υποβάσεις της.
Έπειτα θα ορίσουμε την συνάρτηση κατανομής και την φθίνουσα αναδιάταξη μιας συνάρτησης με σκοπό να αποκτήσουμε κάποιες ανισότητες που θα χρησιμποιήσουμε στο τελαυταίο κεφάλαιο, το οποίο ειναι μια γενίκευση των όσων προηγήθηκαν, όπου θα μελετήσουμε τα ανάλογα σε πολλές μεταβλητές των όσων αποδείξαμε, στο σύνολο $S = (0,1)^n$, για $n \in \mathbb{N}$ τυχαίο, από όπου θα λάβουμε κάποιες ιδιότητες διαφόρισης για την βάση $\mathbb{B}_2$ χρησιμοποιώντας ξανά τον αντίστοιχο μεγιστικό τελεστή.

INTRODUCTION

In Chapter 1 we will present some estimates for the uncentered maximal operator $M_{\mathbb{R}^n} f$ over cubes in $\mathbb{R}^n$ as well as for the maximal operators $M_{\mathbb{R}} f$, $M_R f$ and $M_L f$ in $\mathbb{R}$, where

$$M_{\mathbb{R}^n} f(x) = \sup\{\frac{1}{|Q|}\int_Q |f(x)|dx \ , \quad where \ x \in Q \ , Q \ is \ a \ cube \ in \ \mathbb{R}^n\}$$

$$M_{\mathbb{R}} f(x) = \sup\{\frac{1}{|I|}\int_I |f(x)|dx \ , \quad x \in I \ bounded \ interval \ in \ \mathbb{R}\}$$

$$M_R f(x) = \sup\{\frac{1}{u-x}\int_x^u f dy : u \in (x, \infty)\}$$

$$M_L f(x) = \sup\{\frac{1}{x-u}\int_u^x f dy : u \in (-\infty, x)\}$$

More specifically we will prove that

$$\{M_R f(x) > t\} = \frac{1}{t}\int_{\{M_R f(x)>t\}} f \ d\lambda \ , \quad \forall f \in L^1(\mathbb{R}^n) \ , \quad \forall t > 0$$

and that the same equality holds for $M_L f$ and

$$\{M_{\mathbb{R}} f(x) > t\} \leq \frac{2}{t}\int_{\{Mf(x)>t\}} f \ d\lambda \ , \quad \forall f \in L^1(\mathbb{R}^n) \ , \quad \forall t > 0.$$

Using these we will prove the Hardy-Littlewood maximal theorem for L^p, $p > 1$ and get estimates for the norms of Mf, $M_R f$ and $M_L f$ in $L^p(\mathbb{R})$.

In Chapter 2 we shall analyze some interesting covering and differentiation properties of the basis of intervals in $\mathbb{R}^2$. For each $x \in \mathbb{R}^n$ we consider as $B_2(x)$ the family of all open bounded intervals containing x, and $\mathbb{B} = \bigcup_{x\in\mathbb{R}^2} B_2(x)$. This basis will be denoted as $\mathbb{B}_2$ and its maximal operator will be denoted by M_2 where

$$M_2 f(x) = \sup\{\frac{1}{|I|}\int_I |f(x)|dx \ , \ I \subseteq B_2(x)\} \ .$$

The basis $\mathbb{B}_2$ was the basis that allowed the expansion of the modern theory of differentiation after the Lebesgue differentiation theorem was proved.
It is known that $\mathbb{B}_2$ does not differentiate $L^1(\mathbb{R}^2)$ as for a B-F basis $\mathbb{B}$ that is invariant by homothecies, differentiation of L^1 is equivalent to the Vitali property and also equivalent to the regularity of the basis with respect to the basis of cubic intervals (Moriyon[1975])(that is there exists $0 < \delta < 1 : \forall I \in \mathbb{B}_2$ there exists a cube Q such that $I \subseteq Q$ and $\frac{|I|}{|Q|} = \delta$) and $\mathbb{B}_2$ does not satisfy the Vitali property(Banach[1924]), but it is a density basis as proved by Saks[1935].
Furthermore it differentiates $L^1(1+log^+L^1)(\mathbb{R}^2)$ and we will prove this in two ways. First by considering the basis $\mathbb{B}_2$ as the iterated Cartesian product of the interval basis of $\mathbb{R}^1$ and second by proving that if a system of intervals in $\mathbb{R}^2$ satisfies a specific covering property then we can select a finite sequence from any collection of intervals such that $|\cup R_j| < \infty$ that covers a good part of $\cup R_j$ and it has a very small overlap.
In a similar way, if we consider a system of intervals in $\mathbb{R}^3$ such that there is some reasonable constraint between their three different side-lengths, that is one of their side- length is given as a function of the other two, it is to be expected that this system will behave again like the two-dimensional basis of intervals and so the basis $\mathbb{B}_2$ of intervals of $\mathbb{R}^3$ differentiates $L^1(1 + log^+L^1)(\mathbb{R}^3)$.

In the first part of Chapter 3 we will present the construction of the Perron tree. Given a triangle ABC in $\mathbb{R}^2$ and any $\epsilon > 0$ we can obtain a new figure E, the Perron tree, that has measure $< \epsilon \cdot |ABC|$. We shall prove this by repetition of a process called the basic construction which is essentially the partitioning of the basis of the triangle ABC and the translation of those new triangles that were created parallel to the Ox axis towards each other. With every application of this process the area of this newly obtained figure will be decreased up until we get to the desired area of measure $< \epsilon \cdot |ABC|$.
The Perron tree has many applications to a number of different problems, one of them being the so called "Needle problem" proposed by Kakeya [1917]. The problem states : What is the infimum of the areas of those sets in $\mathbb{R}^2$ such that a needle of length 1 can be continuously moved within the set so that at the end it occupies the original place but in inverted position? Using the Perron tree we will prove that given $\eta > 0$ and a straight segment AB with length 1 in $\mathbb{R}^2$ we can construct a figure F of area less than η so that we can continuously move AB within F so that it finally occupies the same place but in inverted position.
From the construction of the Perron tree we can also obtain a Besicovitch set, a compact set of null measure in $\mathbb{R}^2$ that contains a segment of unit length in every

direction. To do so we will take a closed parallelogram P and partition its basis in order to get the triangles from which we will construct the Perron trees. We will then substitute each of the intersections of the parallelogram P with the small triangles of such Perron trees by parallelograms whose union has measure as small as we wish and contain a segment of unit length in every direction.
In our last application of the construction of the Perron tree we shall prove, using the Busemann-Feller criterion, that the differentiation basis $\mathbb{B}_T$ generated by all the triangles $\{T_h\}_{h=1}^{2^n}$ where T_h has basis a dyadic interval, for every $n \in \mathbb{N}$, is not a density basis and by utilizing the inequality

$$M_T f \leq c \cdot M_R f$$

we will deduce that $\mathbb{B}_R$ is not a density basis, where M_T is the maximal operator corresponding to the basis $\mathbb{B}_T$ and M_R the the maximal operator corresponding to the basis $\mathbb{B}_R$, the basis generated by the rectangles R_h, where R_h is the smallest rectangle containing T_h. This of course implies that the basis of all rectangles $\mathbb{B}_3$ is not a density basis.
In the last part of Chapter 3 we will examine the basis $\mathbb{B}_\phi$ of parallelograms in lacunary directions $\phi = \{\frac{\pi}{2^2}, \frac{\pi}{2^3}, \frac{\pi}{2^4}, ...\}$ and proving a similar covering theorem as in Chapter 2 we will obtain the weak type (2,2) inequality for the maximal operator $M_{\mathbb{B}_\phi}$ corresponding to the basis $\mathbb{B}_\phi$, which is equivalent to the fact that the basis $\mathbb{B}_\phi$ differentiates L^2.

In Chapter 4 we will define the distribution function μ_f of a given function $f \in M_o(R, \mu)$

$$\mu_f(\lambda) = \mu(\{x \in R : |f(x)| > \lambda\})$$

in order to get the definition of equimeasurability; two non negative functions f and g will be called equimeasurable if their distribution functions coincide. We will then define the decreasing rearrangement f^* of the function f as

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}$$

which is equimeasurable to f and thus we can substitute f with f^* when needed to take advantage of the extra properties of the decreasing rearrangement. We shall also prove some inequalities involving the functions f and Mf and their decreasing rearrangements that we will use in our last chapter.

In chapter 5 we will examine some results that we have already seen but in the general case. We will work on the interval $S = (0,1)^k$, $k \in \mathbb{N}$ is arbitary, and we will extract some differentiation properties for the basis $\mathbb{B}$ of the intervals of S. After

some introductory lemmas, presented in the first section, relating to $M^{(0,1)}f$ we will prove that this maximal operator is of strong type (p,p), and thus the basis $\mathbb{B}$ differentiates L^p. Then for functions $f \in L^1(log^+L^1)(S)$ we will check that the integral of f is strongly differentiable at almost every point and the derivative is equal to f so the basis $\mathbb{B}$ also differentiates $L^1(log^+L^1)(S)$.
In the third and last part of Chapter 4 we will consider functions $f \in L^1(log^+L^1)^{k-1}$ and prove, through the strong derivative of the integral of f, that the basis $\mathbb{B}$ differentiates $L^1(log^+L^1)^{k-1}(S)$. Then for intervals of a specific form, where the ratios of any two of their sides do not exceed a finite number, we will prove that the Vitali property still holds and so the basis $\mathbb{B}$ differentiates L^1 as well as $L^1(log^+L^1)^{k-r}$, for $1 \leq r \leq k$ which is the general case for the property mentioned above.

Contents

Chapter 0

Preliminaries

Definition 0.1: *For every $x \in \mathbb{R}^n$ we consider a collection $B(x)$ that consists of bounded measurable sets of positive measure that contain x and are such that for every x there exists $\{R_k(x)\}_{k\in\mathbb{N}} \subseteq B(x)$, with $x \in R_k(x) \quad \forall k \in \mathbb{N}$ and $\delta(R_k) \to 0$, where $\delta(R_k)$ is the diameter of R_k. The collection $\mathbb{B} = \bigcup_{x\in\mathbb{R}^n} B(x)$ is called differentiation basis.*

Definition 0.2 : *For every $x \in \mathbb{R}^n$ we consider the collection $B_2(x)$ that consists of bounded intervals of $\mathbb{R}^n$ that contain x such that $\forall x$ there exists $\{R_k(x)\}_{k\in\mathbb{N}} \subseteq B_2(x)$ with $\delta(R_k) \to 0$. Then the collection $\mathbb{B}_2 = \bigcup_{x\in\mathbb{R}^n} B_2(x)$ is called the differentiation basis of intervals of $\mathbb{R}^n$.*

Definition 0.3: *A differentiation basis $\mathbb{B}$ will be called a Busemann-Feller basis if:*

a) B is open, $\forall B \in \mathbb{B}$

b) $\forall x \in B \in \mathbb{B}$, B belongs in B(x)

Definition 0.4: *The differentiation basis $\mathbb{B}$ will be called invariant by homothecies and translations if $\forall B \in \mathbb{B}$ and $\forall \phi$ such that $\phi(x) = ax + b$, $a \neq 0$, $\quad b \in \mathbb{R}^n$, $\phi(B)$ belongs in $\mathbb{B}$.*

Definition 0.5: *Let $\mathbb{B}$ be a differentiation basis, then for every $f \in L^1_{loc}(\mathbb{R}^n)$ and for every $x \in \mathbb{R}^n$ we define*
a) The upper derivative of $\int f$ on x with respect to the basis $\mathbb{B}$ by

$$\overline{D}_{\mathbb{B}}(\int f, x) = \limsup_{\substack{\delta(B)\to 0 \\ B \in B(x)}} \{\frac{1}{|B|} \int_B f(y)dy\}$$

and
b) The lower derivative of $\int f$ on x with respect to the basis $\mathbb{B}$ by

$$\underline{D}_{\mathbb{B}}(\int f, x) = \liminf_{\substack{\delta(B)\to 0 \\ B \in B(x)}} \{\frac{1}{|B|} \int_B f(y)dy\}$$

If for some $x \in \mathbb{R}^n$ we have $\overline{D}_{\mathbb{B}}(\int f, x) = \underline{D}_{\mathbb{B}}(\int f, x) =: D_{\mathbb{B}}(\int f, x)$ then $D_{\mathbb{B}}(\int f, x)$ is called the derivative of $\int f$ on x for the basis $\mathbb{B}$.

Additionally if $D_{\mathbb{B}}(\int f, x)$ exists and is equal to $f(x)$ for almost every $x \in \mathbb{R}^n$ we say that the basis $\mathbb{B}$ differentiates $\int f$ and if $\mathbb{B}$ differentiates $\int f$ for every $f \in X$, for some space $X \subseteq L^1_{loc}(\mathbb{R}^n)$ then we say that $\mathbb{B}$ differentiates X.

Definition 0.6 : *The differentiation basis $\mathbb{B}$ will be called density basis if it differentiates every χ_A, for every $A \subseteq \mathbb{R}^n$ measurable, where χ_A denotes the characteristic function of A.*

Definition 0.7 : *The maximal operator corresponding to the differentiation basis*

$\mathbb{B}$ *is defined as*

$$M_{\mathbb{B}}f(x) = \sup\{\frac{1}{|R|}\int_R |f(y)|dy : \quad x \in R \in B(x)\}.$$

Definition 0.8 : *The maximal operator $M_{\mathbb{B}}$ is called of*

a) strong type (p, p), if there exists $c_p > 0$ such that

$$||M_{\mathbb{B}}f||_{L^p} \leq c_p||f||_{L^p}, \qquad \forall f \in L^p(\mathbb{R}^n)$$

b) weak type (p, p), if there exists $c_p > 0$ such that

$$|\{x \in \mathbb{R}^n : M_{\mathbb{B}}f(x) > \lambda\}| \leq \frac{{c_p}^p}{\lambda^p}||f||^p_{L^p}, \qquad \forall f \in L^p(\mathbb{R}^n) \quad \text{and} \quad \forall \lambda > 0.$$

If $M_{\mathbb{B}}$ is of strong type (p, p) then it is also of weak type (p,p).

Theorem 0.9 : *Let $\mathbb{B}$ be a differentiation basis invariant by homothecies and translations. If the corresponding maximal operator $M_{\mathbb{B}}f$ is of weak type (1,1) then $\mathbb{B}$ differentiates $L^1(\mathbb{R}^n)$.*

Theorem 0.10 (Busemann-Feller criterion): *Assume $\mathbb{B}$ is a differentiation basis invariant by homothecies and translations. The following are equivalent:*

a) $\mathbb{B}$ is a density basis

b) $\forall \lambda \in (0,1)$ there exists $c_\lambda \in (0,\infty)$ such that $\forall A \subseteq \mathbb{R}^n$ bounded and measurable the inequality

$$|\{M\chi_A > \lambda\}| \leq c_\lambda|A|$$

holds for every such A.

Theorem 0.11 : *Let $\mathbb{B}$ be a differentiation basis. The following are equivalent :*

a) $\mathbb{B}$ is a density basis

b) $\forall\lambda \in (0,1)$, $\forall\{A_k\}$ decreasing sequence of bounded measurable sets with $|A_k| \to 0$ and $\forall\{r_k\}$, r_k decreasing to 0^+ we have

$$|\{M_{r_k}\chi_{A_k} > \lambda\}| \xrightarrow{k\to\infty} 0$$

where

$$M_{r_k}\chi_{A_h}(x) = \sup_{\substack{R\in\mathbb{B}(x)\\ \delta(R)<r_k}} \{\frac{1}{|R|}\int_R |\chi_{A_h}(y)|dy\} = \sup_{\substack{R\in\mathbb{B}(x)\\ \delta(R)<r_k}} \frac{|A_h \cap R|}{|R|} \ , \quad \forall h \in \mathbb{N}.$$

Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\phi(0) = 0$ satisfying : $\exists c' > 0$, $u_o > 0$ such that

$$\phi(u) \geq c'u \ , \qquad \forall u \geq u_o.$$

We define the space

$$\phi(L^1(\mathbb{R}^n)) = \{f : \mathbb{R}^n \to \mathbb{R} \ , \ f \quad \textit{measurable with} \ \int_{\mathbb{R}^n} \phi(|f|) < \infty\}.$$

Theorem 0.12 : *a) Let $\mathbb{B}$ be a B-F differentiation basis of $\mathbb{R}^n$ invariant by homothecies and translations, $M_{\mathbb{B}}$ be the corresponding maximal operator satisfying*

$$|\{M_{\mathbb{B}}f > \lambda\}| \leq c\int_{\mathbb{R}^n} \phi(\frac{|f|}{\lambda}) \quad , \qquad \forall f \in \phi(L^1(\mathbb{R}^n)) \quad \textit{and} \quad \forall\lambda > 0,$$

where c does not depend on λ and f, ϕ is such that

$$\phi(\alpha\mu) \leq c_\mu\phi(\alpha) \ , \qquad \forall a > 0 \ , \ \forall\mu > 0 \ , \ c_\mu < \infty.$$

Then the basis $\mathbb{B}$ differentiates $\phi(L^1(\mathbb{R}^n))$.

The converse holds under weaker conditions:

b) Let $\mathbb{B}$ be a B-F differentiation basis of $\mathbb{R}^n$ invariant by homothecies and translations, differentiating $\phi(L^1(\mathbb{R}^n))$. Then there exists $c > 0$ such that $\forall \lambda > 0$ and $\forall f \in \phi(L^1(\mathbb{R}^n))$, $f \geq 0$, the corresponding maximal operator $M_{\mathbb{B}}$ satisfies

$$|\{M_{\mathbb{B}}f > \lambda\}| \leq c \int_{\mathbb{R}^n} \phi(\frac{f(x)}{\lambda})dx.$$

Definition 0.13 : *A dyadic interval of $\mathbb{R}^n$ is a bounded interval of the form*

$$(\frac{i}{2^m}, \frac{i+1}{2^m}) \times (\frac{j}{2^l}, \frac{j+1}{2^l}) \times .... \times (\frac{\zeta}{2^k}, \frac{\zeta+1}{2^k})$$

where i, j, $\zeta \in \mathbb{N}$, m, l , ..., $k \in \mathbb{Z}$, with $i = 1, 2, ..., 2^m - 1$, $j = 1, 2, ..., 2^l - 1$, ... , $\zeta = 1, 2, ..., 2^k - 1$.

Theorem 0.14 (Calderon - Zygmund decomposition): *Given $f \in L^1(\mathbb{R}^n)$ and $t > 0$ there exists an at most countable family Q_t of non overlapping cubes consisting of those maximal dyadic cubes over which the average of $|f|$ is $> t$. Then the family Q_t satisfies :*

a) for every $Q \in Q_t$: $\quad t < \frac{1}{|Q|} \int_Q |f(x)|dx \leq \ 2^n \cdot t$

b) for almost every $x \notin \cup Q_t$, $|f(x)| \leq t$.

Lemma 0.15 : *Let $f \in L^1(\mathbb{R}^n)$ and $Mf(x)$ be the uncentered maximal operator* *over cubes in* $\mathbb{R}^n$*. Then*

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq \frac{3^n \cdot 4^n}{t} \int_{\mathbb{R}^n} |f(y)| dy.$$

Lemma 0.16 (Layer cake formula): *Let* (X, A, μ) *be a measure space and* $f : X \to \mathbb{R}$*. Then,* $\forall p > 0$*,*

$$\int_X |f|^p d\mu = \int_0^\infty p\lambda^{p-1} \mu(\{|f| > \lambda\}) d\lambda.$$

Lemma 0.17 (Whitney Lemma) : *Let* G *be an open subset of* $\mathbb{R}^n$*,* $G \neq \emptyset$*,* $G \neq \mathbb{R}^n$*. Then there exists a disjoint sequence of half open cubes* $\{Q_k\}_{k \in \mathbb{N}}$ *that can be chosen as translations of dyadic cubes, such that:*

a) $G = \cup Q_k$

b) $2 \leq \dfrac{d(Q_k\ ,\ \partial G)}{\delta(Q_k)} \leq 6\ , \quad \forall k \in \mathbb{N}$

Chapter 1

The maximal function

In this chapter we will prove some estimates and weak type inequalities for the maximal function $Mf : L^p(\mathbb{R}^n) \to \mathbb{R}^+$ that we will frequently use in the next chapters.

1.1 Norm estimates for the maximal function

Theorem 1.1.1: *There exist c , c' such that:*

a) $|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq \dfrac{c}{t} \displaystyle\int_{\{x\in\mathbb{R}^n:\ |f(x)|>\frac{t}{2}\}} |f(x)|dx$

b) $|\{x \in \mathbb{R}^n : Mf(x) > t\}| \geq \dfrac{c'}{t} \displaystyle\int_{\{x\in\mathbb{R}^n:\ |f(x)|>t\}} |f(x)|dx$

$\forall t > 0$ *and* $\forall f \in L^1(\mathbb{R}^n)$ *real function, where Mf is the uncentered maximal operator over cubes in $\mathbb{R}^n$ and c , c' do not depend on f or t, but only the dimension of the space.*

Proof: We write $f = f^* + f_*$ where

$$f^*(x) = \begin{cases} f(x), & if\ |f(x)| > \frac{t}{2} \\ \\ 0, & if\ |f(x)| \leq \frac{t}{2} \end{cases}$$

and

$$f_*(x) = \begin{cases} 0, & if\ |f(x)| > \frac{t}{2} \\ f(x), & if\ |f(x)| \leq \frac{t}{2} \end{cases}$$

Then

$$Mf(x) \leq Mf^*(x) + Mf_*(x) \leq Mf^*(x) + \frac{t}{2}$$

since $Mf_*(x) \leq \dfrac{t}{2}$, $\forall x \in \mathbb{R}^n$, as $|f_*(x)| \leq \frac{t}{2}$, $\forall x \in \mathbb{R}^n$.
So

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq |\{x \in \mathbb{R}^n : Mf^*(x) > \frac{t}{2}\}| \leq$$

$$\leq \frac{3^n \cdot 4^n}{\frac{t}{2}} \int_{\mathbb{R}^n} |f^*(x)| dx =: \frac{c}{t} \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{t}{2}\}} |f(x)| dx$$

using Lemma 0.15 and (a) is proved.

Using the Calderon-Zygmund decomposition for f and t we get non-overlapping cubes Q_j such that

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f(x)| dx \leq 2^n \cdot t \ , \quad \forall j \qquad (1)$$

and

$$|f(x)| \leq t \quad for\ almost\ every\ x \notin \bigcup_j Q_j.$$

If $x \in Q_j$ we get $Mf(x) > t$, so $Q_j \subseteq \{x \in \mathbb{R}^n : Mf(x) > t\}$ for every j and thus

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \geq \sum_j |Q_j| \geq_{(1)} \frac{1}{2^n \cdot t} \int_{\{x : |f(x)| > t\}} |f(x)| dx$$

since $\bigcup_j Q_j \supseteq \{x : |f(x)| > t\}$ a.e and (b) is proved. $\square$

Theorem 1.1.2: *For every $1 < p < \infty$, there is a constant $C_p > 0$ such that for every $f \in L^p(\mathbb{R}^n)$*

$$(\int_{\mathbb{R}^n} (Mf(x))^p dx)^{\frac{1}{p}} \leq C_p(\int_{\mathbb{R}^n} |f(x)|^p dx)^{\frac{1}{p}},$$

that is $||Mf||_{L^p} \leq C_p||f||_{L^p}$,

where Mf is the uncentered maximal operator over cubes in $\mathbb{R}^n$.

Proof: Using the Layer cake formula and Theorem 1.1.1 we get

$$\int_{\mathbb{R}^n} (Mf(x))^p dx = \int_0^\infty pt^{p-1}|\{x : Mf(x) > t\}|dt \leq$$

$$\leq cp \int_0^\infty \frac{t^{p-1}}{t} \int_{\{x:|f(x)|>\frac{t}{2}\}} |f(x)|dxdt =$$

$$= cp \int_0^\infty t^{p-2} \int_{\{x:|f(x)|>\frac{t}{2}\}} |f(x)|dxdt = cp \int_{\mathbb{R}^n} (\int_0^{2|f(x)|} t^{p-2}dt)|f(x)|dx =$$

$$= \frac{cp}{p-1} 2^{p-1} \int_{\mathbb{R}^n} |f(x)|^p dx =: C_p \int_{\mathbb{R}^n} |f(x)|^p dx. \ \square$$

We have seen that the operator M is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$, since $Mf(x) \leq ||f||_\infty$, $\forall x$. However, it is not bounded in $L^1(\mathbb{R}^n)$.

Theorem 1.1.3: *Assume f is supported in a ball $B \subseteq \mathbb{R}^n$. If $|f(x)|log^+|f(x)|$ is integrable then Mf is also integrable over B , where Mf is the uncentered maximal operator over cubes in $\mathbb{R}^n$.*

Proof: We have $\int_B |f(x)|log^+|f(x)|dx < \infty$.
Then

$$\int_B Mf(x)dx = \int_0^\infty |\{x \in B : Mf(x) > t\}|dt = 2\int_0^\infty |\{x \in B : Mf(x) > 2t\}|dt \leq$$

$$\leq 2(\int_0^1 |B|dt + \int_1^\infty |\{x \in \mathbb{R}^n : Mf(x) > 2t\}|dt)$$

and using (a) from Theorem 1.1.1 we get that the previous quantity is

$$\leq 2|B| + c\int_1^\infty \frac{1}{t}\int_{\{x:|f(x)|>t\}} |f(x)|dxdt = 2|B| + c\int_{\mathbb{R}^n} |f(x)| \int_1^{|f(x)|} \frac{1}{t}dtdx =$$

$$= 2|B| + c\int_{\mathbb{R}^n} |f(x)|log^+|f(x)|dx < \infty$$

and so

$$\int_B Mf(x)dx < \infty$$

as requested. □

Now let f be a non-negative function in $L^1(\mathbb{R})$ and $M_1 f$ be the uncentered maximal operator corresponding to the differentiation basis of $\mathbb{R}$.

We define the following maximal functions:

$$M_R f(x) = \sup\{\frac{1}{u-x}\int_x^u f(y)dy \ : \ u \in (x,\infty)\}$$

$$M_L f(x) = \sup\{\frac{1}{x-u}\int_u^x f(y)dy \ : \ u \in (-\infty,x)\}$$

and then it is proved that

$$M_1 f(x) = \max\{M_R f(x), M_L f(x)\}$$

For each $t > 0$, let $M_t = \{x : Mf(x) > t\}$,

$M_t^L = \{x : M_L f(x) > t\}$ and $M_t^R = \{x : M_R f(x) > t\}$

Lemma 1.1.4: *The equalities*

$$a)\ |\{M_R f(x) > t\}| = \frac{1}{t}\int_{M_t^R} f(x)dx$$

$$b)\ |\{M_L f(x) > t\}| = \frac{1}{t}\int_{M_t^L} f(x)dx$$

and the inequality

$$c)|\{M_1 f(x) > t\}| \leq \frac{2}{t}\int_{M_t} f(x)dx,$$

hold for every $t > 0$.

Proof: We will prove (a) as the $M_L f$ case is identical.

The function $s \to \dfrac{1}{s-x}\displaystyle\int_x^s f$ is continuous on (x, ∞) so the set $M_t^R = \{x : M_R f(x) > t\}$ is open.

Therefore, M_t^R can be written as a unique union of open disjoint intervals $M_t^R = \bigcup_{k=1}^{\infty}(\beta_k, \gamma_k)$.

Consider an interval (β_k, γ_k), which may not be necessarily bounded.
For every $x \in (\beta_k, \gamma_k)$ the open set

$$N_x = \{s : \int_x^s f > t(s-x), \quad s \in (x, \gamma_k)\}$$

is nonvoid.
For $\gamma_k = \infty$ this is trivial, as for $x \in (\beta_k, \infty) \in M_t^R$ we have by definition of M_t^R that there exists $s > x$ such that

$$\frac{1}{s-x}\int_x^s f > t \quad \Rightarrow \quad \int_x^s f > t(s-x)$$

so $s \in N_x$.

We define $s_x = \sup N_x$ and consider the case $\gamma_k = +\infty$. For every $x \in (\beta_k, +\infty)$ we have $s_x = +\infty$.
Indeed assume $s_x < +\infty$ for some $x \in (\beta_k, +\infty)$ then

$$\frac{1}{s_x - x}\int_x^{s_x} f \geq t$$

but $s_x \in (\beta_k, +\infty)$ so there exists $y > s_x$ such that

$$\frac{1}{y - s_x} \int_{s_x}^{y} f > t$$

and so

$$\frac{1}{y - x} \int_{x}^{y} f > t \ , \quad where\ y > s_x > x$$

which is a contradiction.
Thus for $x \in (\beta_k, +\infty)$ fixed there exists a sequence $s_n \nearrow +\infty$ such that

$$\frac{1}{s_n - x} \int_{x}^{s_n} f > t \quad and\ for\ n \to +\infty \quad we\ get \quad 0 \geq t$$

a contradiction.
So for any (β_k, γ_k) as above we have that $\gamma_k < +\infty$.

For $\gamma_k < \infty$, assume $N_x = \emptyset$ for some $x \in (\beta_k, \gamma_k)$.
Then there exists $w \geq \gamma_k$ such that

$$\int_{x}^{w} f > t(w - x).$$

Also, by $N_x = \emptyset$ we get that, $\forall s \in (x, \gamma_k)$, $\displaystyle\int_{x}^{s} f \leq t(s - x)$ and since $\displaystyle\frac{1}{s - x} \int_{x}^{s} f$ continuous on (x, γ_k) we get

$$\int_{x}^{\gamma_k} f \leq t(\gamma_k - x)$$

so that $w > \gamma_k$.
This gives us

$$\int_{\gamma_k}^{w} f = \int_{x}^{w} f - \int_{x}^{\gamma_k} f > t(w - x) - t(\gamma_k - x) = t(w - \gamma_k) \quad \Rightarrow \quad \gamma_k \in M_t^R$$

which is a contradiction.

Now let $s_x = \sup N_x$, where $x \in (\beta_k, \gamma_k)$ and $\gamma_k < \infty$. We will prove that $s_x = \gamma_k$. If $s_x < \gamma_k$ then

$$\int_x^{s_x} f = t(s_x - x) \qquad (1)$$

by the continuity of $M_R f$.
We have $\beta_k < x < s_x < \gamma_k \quad \Rightarrow \quad s_x \in M_t^R$ and since $N_{s_x} \neq \emptyset$ so there exists $y \in (s_x, \gamma_k)$ such that

$$\int_{s_x}^{y} f > t(y - s_x). \qquad (2)$$

Combining (1) and (2) we get

$$\int_{s_x}^{y} f + \int_x^{s_x} f = \int_x^y f > t(y - x),$$

which is a contradiction since $\gamma_k > y > s_x$, by the definition of s_x and N_x.
Therefore $\forall x \in (\beta_k, \gamma_k)$, $s_x = \gamma_k$ so

$$\int_x^{\gamma_k} f \geq t(\gamma_k - x).$$

Letting $x \to \beta_k$, we get

$$\int_{\beta_k}^{\gamma_k} f \geq t(\gamma_k - \beta_k) \qquad (3)$$

[from which we ensure that (β_k, γ_k) is bounded since $f \in L^1(\mathbb{R}^n)$]

and as (3) is true for every $(\beta_k, \gamma_k) \in M_t^R$ we get

$$\int_{M_t^R} f \geq t|M_t^R|. \qquad (4)$$

On the other hand $\beta_k \notin (\beta_k, \gamma_k) \subseteq M_t^R$ so

$$\int_{\beta_k}^{\gamma_k} f \leq t(\gamma_k - \beta_k)$$

and so

$$\int_{M_t^R} f \leq t|M_t^R|. \qquad (5)$$

Inequalities (4) and (5) gives us

$$|M_t^R| = |\{x : M_R f(x) > t\}| = \frac{1}{t} \int_{M_t^R} f(x)dx.$$

To prove (c) we proceed in a similar way.
We will prove that $M_1 f(x) = \max\{M_R f(x), M_L f(x)\}$, $\forall x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$ and define $M_1 f(x) = \theta$.
For every $z > x$ we have

$$\frac{1}{z-x}\int_x^z f = \lim_{\epsilon\to 0}\frac{1}{z-(x-\epsilon)}\int_{x-\epsilon}^z f \ \le M_1 f(x) = \ \theta$$

so, $M_R f(x) \le \theta$.

Similarly $M_L f(x) \le \theta$, which gives us

$$\max\{M_R f(x), M_L f(x)\} \le \theta = M_1 f(x).$$

Now assume $\max\{M_R f(x), M_L f(x)\} = \delta$ and (a,b) is an arbitary interval in $\mathbb{R}$ such that $x \in (a,b)$ we then have

$$\frac{1}{b-a}\int_a^b f = \frac{1}{b-a}\int_a^x f + \frac{1}{b-a}\int_x^b f =$$
$$\frac{x-a}{b-a}(\frac{1}{x-a}\int_a^x f) + \frac{b-x}{b-a}(\frac{1}{b-x}\int_x^b f) \le$$
$$\frac{x-a}{b-a}M_L f(x) + \frac{b-x}{b-a}M_R f(x)$$

and $M_R f(x), M_L f(x) \le \delta$ by definition of δ, so

$$\frac{1}{b-a}\int_a^b f \le \frac{x-a}{b-a}\delta + \frac{b-x}{b-a}\delta = \delta \ , \quad \forall \textit{ bounded interval } (a,b) \textit{ containing } x$$

and thus we get
$$M_1 f(x) \le \delta$$

and as a consequence

$$M_1 f(x) = \max\{M_R f(x), M_L f(x)\}$$

Also by the above equality we get

$$M_t = M_t^R \cup M_t^L$$

and using the last identity we get

$$|M_t| \leq |M_t^R| + |M_t^L| = \frac{1}{t}[\int_{M_t^R} f + \int_{M_t^L} f] \leq$$

$$\leq \frac{1}{t}[\int_{M_t} f + \int_{M_t} f] = \frac{2}{t}\int_{M_t} f. \ \square$$

1.2 The Hardy- Littlewood maximal theorem

Theorem 1.2.1: *(Hardy-Littlewood maximal theorem for L^p, $p > 1$)*
Let $f \in L^p(\mathbb{R})$ and $p > 1$ be a real number. Then:

a)

$$(\int_{\mathbb{R}} (M_R f(x))^p dx)^{\frac{1}{p}} \leq \frac{p}{p-1}[\int_{\mathbb{R}} |f|^p dx]^{\frac{1}{p}}$$

that is $\quad ||M_R f||_{L^p} \leq \dfrac{p}{p-1}||f||_{L^p}$

b)

$$(\int_{\mathbb{R}} (M_L f(x))^p dx)^{\frac{1}{p}} \leq \frac{p}{p-1}[\int_{\mathbb{R}} |f|^p dx]^{\frac{1}{p}}$$

that is $\quad ||M_L f||_{L^p} \leq \dfrac{p}{p-1}||f||_{L^p}$

and

c)

$$(\int_{\mathbb{R}} (M_1 f(x))^p dx)^{\frac{1}{p}} \leq \frac{2p}{p-1}[\int_{\mathbb{R}} |f|^p dx]^{\frac{1}{p}}$$

that is $\quad ||M_1 f||_{L^p} \leq \dfrac{2p}{p-1}||f||_{L^p}.$

Proof: We will use the equality (a) that we proved in Lemma 1.1.4, which is stated for functions $f \in L^1(\mathbb{R})$. As we know, $C_c(\mathbb{R})$, the space of continuously compacted functions, is dense in $L^1(\mathbb{R})$ and $L^p(\mathbb{R})$. Let $f \in L^p(\mathbb{R})$, we can then find a sequence $\{f_n\} \subseteq C_c(\mathbb{R})$ such that $f_n \xrightarrow{L^p} f$.
Therefore

$$||f_n - f||_{L^p} \to 0 \quad and \quad ||M_R(f_n - f)||_{L^p} \leq c_p ||f_n - f||_{L^p} \to 0$$

by Theorem 1.1.2, so

$$M_R(f_n - f) \xrightarrow{L^p} 0.$$

By the triangle inequality we get

$$|M_R f_n(x) - M_R f(x)| \leq M_R(f_n - f)(x) \ , \qquad \forall x \in \mathbb{R}$$

thus

$$M_R f_n \xrightarrow{L^p} M_R f$$

and thus it is enough to prove the theorem for functions $f \in C_c(\mathbb{R})$.

Let f be a fixed function in $C_c(\mathbb{R})$. Using

$$\int_{\mathbb{R}} |g|^p dx = \int_0^\infty pt^{p-1}|\{|g| > t\}|dt \ , \qquad (1)$$

$$|\{M_R f > t\}| = \frac{1}{t}\int_{\{M_R f>t\}} |f|dx \ , \qquad (2)$$

we get

$$\int_{\mathbb{R}} (M_R f(x))^p dx =_{(1)} \int_0^\infty pt^{p-1}|\{M_R f(x) > t\}|dt =$$

$$=_{(2)} p\int_0^\infty t^{p-2}\int_{\{M_R f>t\}} |f(x)|dxdt = p\int_{\mathbb{R}} t^{p-2}\int_0^{M_R f(x)} |f(x)|dtdx =$$

$$= p\int_{\mathbb{R}}\int_0^{M_R f(x)} (\frac{t^{p-1}}{p-1})' dt|f(x)|dx = \frac{p}{p-1}\int_{\mathbb{R}} (M_R f(x))^{p-1}|f(x)|dx \leq$$

$$\leq \frac{p}{p-1}[\int_{\mathbb{R}} |f(x)|^p dx]^{\frac{1}{p}}[\int_{\mathbb{R}} (M_R f(x))^p dx]^{\frac{p-1}{p}} < \infty$$

by Theorem 1.1.2 , where on the third equality we used Fubini's Theorem and we also used Holder's inequality to get the first inequality,

so

$$(\int_{\mathbb{R}} M_R f(x)^p dx)^{\frac{1}{p}} \leq \frac{p}{p-1}[\int_{\mathbb{R}} |f|^p dx]^{\frac{1}{p}}.$$

We will not prove (b) as the proof is identical.

To prove (c) we use

$$|\{M_1 f(x) < t\}| \leq \frac{2}{t}\int_{\{M_1 f(x)>t\}} f dx$$

from Lemma 1.1.4 and proceed similarly. □

Chapter 2

The basis $\mathbb{B}_2$ of intervals

In this chapter we will prove some covering theorems that entail differentiation properties for the basis $\mathbb{B}_2$ of intervals.

2.1 Intervals of $\mathbb{R}^2$

Assume $\mathbb{B}_2$ is the following basis: for every $x \in \mathbb{R}^2$
$B_2(x) = \{I = J \times H$, where J, H open bounded intervals of $\mathbb{R}$, $x \in I\}$.
Then $\mathbb{B}_2 = \bigcup_{x\in\mathbb{R}^2} B_2(x)$, is invariant by homothecies and is also a Busemann-Feller differentiation basis of $\mathbb{R}^2$.

Let M_2 be the corresponding maximal operator in $\mathbb{R}^2$, where

$$M_2 f(x) = \sup\{\frac{1}{|I|}\int_I |f(x)|\; dx, \quad I \in B_2(x)\}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is Lebesgue integrable and $x \in \mathbb{R}^2$.

Let also $l_1 = \{J : J$ bounded interval of $\mathbb{R}\}$ and

$$M_1 g(y) = \sup\{\frac{1}{|J|}\int_J |g(y)|\; dy, \quad y \in J \in l_1\}, \qquad y \in \mathbb{R}$$

the corresponding maximal operator in $\mathbb{R}$ where $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable on $\mathbb{R}$.

In order to prove the next theorem we will use the following lemma.

Lemma 2.1.1: *The maximal operator M_1 is of weak type (1,1), that is there exists $c > 0$ such that*

$$|\{M_1 g(y) > \lambda\}| \leq \frac{c}{\lambda} \int_{\mathbb{R}} |g(y)| dy$$

for every $g \in L^1(\mathbb{R})$ and every $\lambda > 0$.

Proof: This weak type inequality is a result of the inequality (c) proved in Lemma 1.1.4. $\square$

Theorem 2.1.2: *There exists a constant $k > 0$ such that*

$$|\{M_2 f > \lambda\}| \leq k \int_{\mathbb{R}^2} \frac{|f|}{\lambda}(1 + log^+ \frac{|f|}{\lambda}) \qquad (*)$$

$\forall \lambda > 0$ *and* $\forall f$ *measurable, for which the integral on the right side of inequality (*) is finite, and* $\forall \lambda > 0$.

Proof: Notice that for $\lambda > 0$ the following equivalence is true:

$$(1) \qquad \int_{\mathbb{R}^2} |f|(1 + log^+|f|) < \infty \Longleftrightarrow \int_{\mathbb{R}^2} \frac{|f|}{\lambda}(1 + log^+ \frac{|f|}{\lambda}) < \infty \qquad (2)$$

We will prove " $\Rightarrow$ ", with " $\Leftarrow$ " being immediate.

Assume $\int_{\mathbb{R}^2} |f|(1 + log^+|f|) < \infty$.

For $\lambda \geq 1$ we have $\frac{|f|}{\lambda} \leq |f|$ on $\mathbb{R}^2$,

so

$$\int_{\mathbb{R}^2} \frac{|f|}{\lambda}(1 + log^+(\frac{|f|}{\lambda}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^2} |f|(1 + log^+|f|) < \infty,$$

since $x \to x(1 + log^+ x)$ is an increasing function on $(0, \infty)$.

For $0 < \lambda < 1$ we have

$$\int_{\mathbb{R}^2} \frac{|f|}{\lambda}(1 + log^+ \frac{|f|}{\lambda}) = \int_{\{|f| \leq \lambda\}} \frac{|f|}{\lambda} + \int_{\{|f| > \lambda\}} \frac{|f|}{\lambda}(1 + log\frac{|f|}{\lambda}) =$$

$$= \int_{\{|f| \leq \lambda\}} \frac{|f|}{\lambda} + \int_{\{|f| > \lambda\}} \frac{|f|}{\lambda}(1 + log\frac{1}{\lambda} + log|f|) =$$

$$= \int_{\{|f| \leq \lambda\}} \frac{|f|}{\lambda} + \frac{1 + log\frac{1}{\lambda}}{\lambda} \int_{\{|f| > \lambda\}} |f| + \frac{1}{\lambda} \int_{\{|f| > \lambda\}} |f| log|f| \leq$$

$$\leq \frac{1 + log\frac{1}{\lambda}}{\lambda} \int_{\mathbb{R}^2} |f| + \frac{1}{\lambda} \int_{\mathbb{R}^2} |f| log^+|f| \leq \frac{1 + log\frac{1}{\lambda}}{\lambda} \int_{\mathbb{R}^2} |f|(1 + log^+|f|) < \infty,$$

Assume $f \in L^1_{loc}(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} |f|(1 + log^+|f|) < \infty$, $f \geq 0$ and $\lambda > 0$. Then $f \in L^1(\mathbb{R}^2)$.

For every $x = (x_1, x_2) \in \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}^+$, we define

$$T_1 f(x_1, x_2) = \sup\{\frac{1}{|J|} \int_J f(\xi_1, x_2) d\xi_1, \quad J \subseteq \mathbb{R} \quad \textit{open and bounded with } x_1 \in J\}$$

(so $T_1 f(x_1, x_2) = M_1 f(\cdot, x_2)(x_1)$, $\forall (x_1, x_2) \in \mathbb{R}^2$)

and

$$A_\lambda = A = \{(u_1, u_2) \in \mathbb{R}^2 : \ T_1 f(u_1, u_2) > \frac{\lambda}{2}\}$$

which can be easily seen that it is measurable.
We also define

$$T_2 f(x_1, x_2) = \sup\{\frac{1}{|H|} \int_H \chi_A(x_1, u_2) T_1 f(x_1, u_2) du_2 \ , \ H \subseteq \mathbb{R} \ \textit{open, bounded, } x_2 \in H\}$$

(so $T_2 f(x_1, x_2) = M_1(\chi_A(x_1, \cdot) T_1 f(x_1, \cdot))(x_2)$).
We will prove that

$$B := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : M_2(\xi_1, \xi_2) > \lambda\} \subseteq \{(\xi_1, \xi_2) \in \mathbb{R}^2 : T_2 f(\xi_1, \xi_2) > \frac{\lambda}{2}\} =: C$$

Let $(x_1, x_2) \in B$, then there exists $I = J \times H$ where $x_1 \in J,\ x_2 \in H$ and J, H open bounded intervals of $\mathbb{R}$ such that

$$\frac{1}{|I|}\int_I f > \lambda. \qquad (*)$$

Consider the following partition $\{C_1, C_2\}$ of $I = J \times H$:

a) If $\xi_2 \in H$ is such that for every $(z_1, \xi_2) \in J \times \{\xi_2\}$ we have $T_1 f(z_1, \xi_2) > \dfrac{\lambda}{2}$ we set $J \times \{\xi_2\} \subseteq C_1$.

b) If $\xi_2 \in H$ is such that there exists $(z_1, \xi_2) \in J \times \{\xi_2\}$ with $T_1 f(z_1, \xi_2) \leq \dfrac{\lambda}{2}$ we set $J \times \{\xi_2\} \subseteq C_2$.
Obviously $C_1 \cup C_2 = J \times H = I$ and $C_1 \cap C_2 = \emptyset$.

For ξ_2 such that $J \times \{\xi_2\} \subseteq C_2$, by definition of T_1 we have

$$\frac{1}{|J|}\int_J f(\xi_1, \xi_2) d\xi_1 \leq \frac{\lambda}{2} \qquad (4)$$

and integrating (4) over G, where G is the set containing every $\xi_2 \in H$ such that $J \times \{\xi_2\} \subseteq C_2$, we get

$$\int_{C_2} f = \int_{\xi_2 \in G}\int_{\xi_1 \in J} f(\xi_1, \xi_2) d\xi_1 d\xi_2 \leq_{(4)} \int_{\xi_2 \in G} \frac{\lambda}{2} \cdot |J| d\xi_2 = \frac{\lambda}{2} \cdot |C_2| \leq \frac{\lambda}{2} \cdot |I|$$

so

$$\int_I f = \int_{C_1} f + \int_{C_2} f \leq \int_{C_1} f + \frac{\lambda}{2} \cdot |I|$$

but also $\lambda \cdot |I| < \displaystyle\int_I f$ from (*) so

$$\int_{C_1} f > \frac{\lambda}{2} \cdot |I|. \qquad (**)$$

Furthermore,

$$T_2 f(x_1, x_2) \geq \frac{1}{|H|}\int_H \chi_A(x_1, u_2) T_1 f(x_1, u_2) du_2 \geq$$

$$\geq \frac{1}{|H|}\int_H \chi_A(x_1, u_2)\Big(\frac{1}{|J|}\int_J f(u_1, u_2) du_1\Big) du_2 =$$

$$= \frac{1}{|J \times H|} \int_H \chi_A(x_1, u_2) \int_J f(u_1, u_2) du_1) du_2 =$$

$$= \frac{1}{|I|} \int_H \chi_A(x_1, u_2) \int_J f(u_1, u_2) du_1) du_2. \qquad (5)$$

Moreover for $(u_1, u_2) \in (J \times H) \cap C_1$ (that is when $J \times \{u_2\} \subseteq C_1$) and since $x_1 \in J$ we have $(x_1, u_2) \in C_1$, thus $T_1 f(x_1, u_2) > \frac{\lambda}{2}$,
so $(x_1, u_2) \in A$, $\forall u_2$ such that $J \times \{u_2\} \subseteq C_1$ so $x_A(x_1, u_2) = 1$ and by (5)

$$T_2 f(x_1, x_2) \geq \frac{1}{|I|} \int_{(H \times J) \cap C_1} f(u_1, u_2) du_1 du_2 =$$

$$= \frac{1}{|I|} \int_{C_1} f(u_1, u_2) du_1 du_2 = \frac{1}{|I|} \int_{C_1} f >_{(**)} \frac{\lambda}{2}.$$

We thus get $T_2 f(x_1, x_2) > \frac{\lambda}{2}$, $\forall (x_1, x_2) \in B$ therefore $(x_1, x_2) \in C$, $\forall (x_1, x_2) \in B$ and we finally get $B \subseteq C$.

Moving on we are going to estimate $|C|$:
Using the previous lemma we have: for every fixed $x_1 \in \mathbb{R}$, fixed,

$$|\{\xi_2 \in R : T_2 f(x_1, \xi_2) > \frac{\lambda}{2}\}| \leq \frac{c}{\frac{\lambda}{2}} \int_{\mathbb{R}} \chi_A(x_1, \xi_2) T_1 f(x_1, \xi_2) d\xi_2 \ , \qquad (6)$$

as $T_2 f(x_1, \xi_2) = M_1(\chi_A(x_1, \xi_2) T_1 f(x_1, \xi_2))$.

Then

$$|C| = |\{(\xi_1, \xi_2) \in \mathbb{R}^2 : T_2 f(\xi_1, \xi_2) > \frac{\lambda}{2}\}| = \int_{\mathbb{R}} |\{\xi_2 \in \mathbb{R} : T_2 f(\xi_1, \xi_2) > \frac{\lambda}{2}\}| d\xi_1 \leq$$

$$\leq_{(6)} \frac{c}{\frac{\lambda}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(\xi_1, \xi_2) T_1 f(\xi_1, \xi_2) d\xi_1 d\xi_2 = \frac{c}{\frac{\lambda}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(\xi_1, \xi_2) T_1 f(\xi_1, \xi_2) d\xi_2 d\xi_1 =$$

$$= c \int_{\xi_2 \in \mathbb{R}} \phi(\xi_2) d\xi_2 \qquad (7)$$

where

$$\phi(\xi_2) = \int_{\mathbb{R}} \frac{\chi_A(\xi_1, \xi_2) T_1 f(\xi_1, \xi_2)}{\frac{\lambda}{2}} d\xi_1.$$

By the Layer cake representation we have that for every $\xi_2 \in \mathbb{R}$

$$\phi(\xi_2) = \int_{\sigma=0}^{\infty} |\{\xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1,\xi_2)T_1 f(\xi_1,\xi_2)}{\frac{\lambda}{2}} > \sigma\}| d\sigma =$$

$$\int_{\sigma=0}^{1} |\{\xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1,\xi_2)T_1 f(\xi_1,\xi_2)}{\frac{\lambda}{2}} > \sigma\}| d\sigma +$$
$$+ \int_{\sigma=1}^{\infty} |\{\xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1,\xi_2)T_1 f(\xi_1,\xi_2)}{\frac{\lambda}{2}} > \sigma\}| d\sigma. \qquad (8)$$

So by (7) $|C| \leq S_1 + S_2$, where

$$S_1 = c\int_{\mathbb{R}} \int_{\sigma=0}^{1} |\{\xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1,\xi_2)T_1 f(\xi_1,\xi_2)}{\frac{\lambda}{2}} > \sigma\}| d\sigma d\xi_2$$

and

$$S_2 = c\int_{\mathbb{R}} \int_{\sigma=1}^{\infty} |\{\xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1,\xi_2)T_1 f(\xi_1,\xi_2)}{\frac{\lambda}{2}} > \sigma\}| d\sigma d\xi_2.$$

Now we get an estimate for S_1:
If $(\xi_1,\xi_2) \in A$ then $T_1 f(\xi_1,\xi_2) > \dfrac{\lambda}{2}$ and for $\sigma \in (0,1)$

$$\{\xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1,\xi_2)T_1 f(\xi_1,\xi_2)}{\frac{\lambda}{2}} > \sigma\} =$$

$$=^{\sigma>0} \{\xi_1 \in \mathbb{R} : (\xi_1,\xi_2) \in A \quad and \quad T_1 f(\xi_1,\xi_2) > \frac{\lambda}{2}\sigma\} =$$

$$=_{\sigma<1} \{\xi_1 \in \mathbb{R} : (\xi_1,\xi_2) \in A\} = \{\xi_1 \in \mathbb{R} : T_1 f(\xi_1,\xi_2) > \frac{\lambda}{2}\}$$

so

$$S_1 = c\int_{\mathbb{R}} |\{\xi_1 \in \mathbb{R} : T_1 f(\xi_1,\xi_2) > \frac{\lambda}{2}\}| d\xi_2. \qquad (9)$$

Using the previous lemma again, for every $\xi_2 \in \mathbb{R}$

$$|\{\xi_1 \in \mathbb{R} : T_1 f(\xi_1,\xi_2) > \frac{\lambda}{2}\}| \leq \frac{c}{\frac{\lambda}{2}} \int_{\mathbb{R}} f(\xi_1,\xi_2) d\xi_1 \ ,$$

as $T_1 f(\xi_1,\xi_2) = M_1 f(\cdot,\xi_2)(\xi_1)$.

Thus (9) becomes

$$S_1 \le \frac{2c^2}{\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}} f(\xi_1,\xi_2)d\xi_1 d\xi_2 = \frac{2c^2}{\lambda}\int_{\mathbb{R}^2} f.$$

In order to calculate S_2 we define for every $\sigma \ge 1$

$$f_*(\xi_1,\xi_2,\sigma) = \begin{cases} f(\xi_1,\xi_2), & if\ f(\xi_1,\xi_2) \le \frac{\lambda\sigma}{4} \\ 0, & if\ f(\xi_1,\xi_2) > \frac{\lambda\sigma}{4} \end{cases}$$

and

$$f^*(\xi_1,\xi_2,\sigma) = \begin{cases} 0, & if\ f(\xi_1,\xi_2) \le \frac{\lambda\sigma}{4} \\ f(\xi_1,\xi_2), & if\ f(\xi_1,\xi_2) > \frac{\lambda\sigma}{4} \end{cases}$$

which we will simple write as $f_*(\xi_1,\xi_2)$ and $f^*(\xi_1,\xi_2)$.

Then $f = f^* + f_*$ and $T_1 f \le T_1 f^* + T_1 f_*$. (10)

But clearly $T_1 f_*(\xi_1,\xi_2) \le \frac{\lambda\sigma}{4}$, $\forall(\xi_1,\xi_2)\in\mathbb{R}^2$. (11)

So

$$|\{\xi_1\in\mathbb{R} : \frac{\chi_A(\xi_1,\xi_2)T_1 f(\xi_1,\xi_2)}{\frac{\lambda}{2}} > \sigma\}| \le |\{\xi_1\in\mathbb{R} : T_1 f(\xi_1,\xi_2) > \frac{\lambda\sigma}{2}\}| \le$$

$$\le_{(10),(11)} |\{\xi_1\in\mathbb{R} : T_1 f^*(\xi_1,\xi_2) > \frac{\lambda\sigma}{4}\}|$$

thus

$$S_2 \le c\int_{\mathbb{R}}\int_{\sigma=1}^{\infty} |\{T_1 f^*(\xi_1,\xi_2) > \frac{\lambda\sigma}{4}\}| d\sigma d\xi_2$$

and using the previous lemma we get

$$S_2 \le c\int_{\mathbb{R}}\int_{\sigma=1}^{\infty} \frac{c}{\frac{\lambda\sigma}{4}}(\int_{\mathbb{R}} f^*(\xi_1,\xi_2)d\xi_1)d\sigma d\xi_2 =$$

$$= c^2\int_{\mathbb{R}}\int_{\sigma=1}^{\infty}(\int_{\{f>\frac{\lambda\sigma}{4}\}} 4\frac{f(\xi_1,\xi_2)}{\lambda\sigma}d\xi_1)d\sigma d\xi_2 =$$

$$= c^2 \int_{\mathbb{R}} (\int_{\{f > \frac{\lambda}{4}\}} \int_{\sigma=1}^{4\frac{f(\xi_1,\xi_2)}{\lambda}} 4 \frac{f(\xi_1,\xi_2)}{\lambda\sigma} d\sigma d\xi_1) d\xi_2 =$$

$$= c^2 \int_{\mathbb{R}} (\int_{\{f > \frac{\lambda}{4}\}} \frac{f(\xi_1,\xi_2)}{\lambda}) (\int_{\sigma=1}^{4\frac{f(\xi_1,\xi_2)}{\lambda}} \frac{4}{\sigma} d\sigma d\xi_1) d\xi_2 =$$

$$= \frac{4c^2}{\lambda} \int_{\mathbb{R}} \int_{\{f > \frac{\lambda}{4}\}} f(\xi_1,\xi_2)[log4 + log(\frac{f(\xi_1,\xi_2)}{\lambda})] d\xi_1 d\xi_2 \leq$$

$$\leq \frac{4c^2}{\lambda} log4 \int_{\mathbb{R}^2} f(\xi_1,\xi_2) d\xi_1 d\xi_2 + \frac{4c^2}{\lambda} \int_{\mathbb{R}^2} f(\xi_1,\xi_2) log^+ (\frac{f(\xi_1,\xi_2)}{\lambda}) d\xi_1 d\xi_2.$$

Summarising,

$$S_1 + S_2 \leq \frac{2c^2}{\lambda} \int_{\mathbb{R}^2} f(\xi_1,\xi_2) d\xi_1 d\xi_2 + \frac{4c^2}{\lambda} log4 \int_{\mathbb{R}^2} f(\xi_1,\xi_2) d\xi_1 d\xi_2 +$$

$$+ \frac{4c^2}{\lambda} \int_{\mathbb{R}^2} f(\xi_1,\xi_2) log^+ (\frac{f(\xi_1,\xi_2)}{\lambda}) d\xi_1 d\xi_2 \leq K \int_{\mathbb{R}^2} \frac{|f|}{\lambda} (1 + log^+ (\frac{|f|}{\lambda})$$

for suitable $K > 0$

and

$$|C| \leq S_1 + S_2 \leq K \int_{\mathbb{R}^2} \frac{|f|}{\lambda} (1 + log^+ (\frac{|f|}{\lambda})$$

while $B \subseteq C$.

So finally

$$|\{(\xi_1,\xi_2) \in \mathbb{R}^2 : M_2 f(\xi_1,\xi_2) > \lambda\}| \leq K \int_{\mathbb{R}^2} \frac{|f|}{\lambda} (1 + log^+ (\frac{|f|}{\lambda})). \square$$

Now, having proved

$$|\{M_2 f > \lambda\}| \leq K \int_{\mathbb{R}^2} \frac{|f|}{\lambda}(1 + log^+ \frac{|f|}{\lambda})$$

for every f such that

$$\int_{\mathbb{R}^2} |f|(1 + log^+|f|) < \infty \ ,$$

that is $f \in L^1(1 + log^+(L^1))$, defining

$$g(x) = x(1 + log^+ x) \ , \ for \ x > 0 \quad we \ have \quad g(|f|) \in L^1(\mathbb{R}^2) \ , \quad g \geq 0.$$

Let $M_{r_k} f$ be the maximal operator with respect to the intervals of $\mathbb{R}^2$,

$$M_{r_k} f = \sup\{\frac{1}{|B|} \int_B f : \ x \in B \quad with \quad \delta(B) < r_k\}$$

with $r_k \searrow 0$ and $\{A_k\}$ a decreasing sequence of measurable bounded sets with $|A_k| \to 0$.

Then

$$\{M_{r_k}(f\chi_{A_k}) > \lambda\} \subseteq \{M_2(f\chi_{A_k}) > \lambda\}$$

and

$$\int_{\mathbb{R}^2} g(|f|)\chi_{A_k} = \int_{A_k} g(|f|) \to 0 \ , \quad as \quad |A_k| \to 0$$

$$with \quad A_{k+1} \subseteq A_k \quad \forall k \quad and \quad g(|f|) \in L^1(\mathbb{R}^n).$$

So from (*) we get that $\forall f \in L^1(1 + log^+(L^1))$, $\forall A_k$ decreasing sequence of measurable bounded sets with $|A_k| \to 0$ and $\forall r_k \searrow 0$

$$|\{M_{r_k}(f\chi_{A_k}) > \lambda\}| \leq |\{M_2(f\chi_{A_k}) > \lambda\}| \leq \int_{A_k} g(|f|) \to 0 \quad \Rightarrow$$

$$|\{M_{r_k}(f\chi_{A_k}) > \lambda\}| \to 0.$$

Thus by an application of Theorem 0.11 the basis $\mathbb{B}_2$ differentiates $L^1(1 + log^+(L^1))$.

The same result is proved below using a covering theorem regarding the basis of intervals.

The following theorem is a special case of a more general theorem for arbitrary intervals, which can be used for the proof of the fact that the basis $\mathbb{B}_2$ differentiates $L(1+log^+(L))$.

Theorem 2.1.3: *Let $\{B_a\}_{a\in A}$ be a collection of open dyadic intervals of $\mathbb{R}^2$ with $|\bigcup B_a| < \infty$. Then we can select a finite sequence $\{R_k\} \subseteq (B_a)_{a\in A}$ such that:*

α) $|\bigcup B_a| \leq c_1 |\bigcup R_k|$

b) $\displaystyle\int_{\cup R_k} e^{\sum \chi_{R_k}} \leq c_2 |\bigcup R_k|$

where c_1, c_2 are independent of the initial collection $\{B_a\}_{a\in A}$.

Proof: Let $\{B_a\}_{a\in A}$ be a collection of dyadic intervals of $\mathbb{R}^2$ with $|\bigcup B_a| < \infty$. We take a finite sequence $\{B_k\}_{k=1}^M \subseteq \{B_a\}_{a\in A}$ with $|\bigcup B_a| \leq 2|\bigcup\limits_{k=1}^{M} B_k|$.
We can do so by using Lindelof's theorem as $\bigcup_a B_a$ can be written as a countable union $\bigcup\limits_{a\in A} B_a = \bigcup\limits_{i\in I} B_{a_i}$, where I is at most countable and $a_i \in A$, $\forall i \in I$.
Then we can define a new sequence $\{F_{a_i}\}$ where $F_{a_i} = B_{a_1} \cup B_{a_2} \cup ... \cup B_{a_i}$ from which we get $|\bigcup\limits_a B_a| = |\bigcup\limits_i B_{a_i}| = |\bigcup\limits_i F_{a_i}|$ where $\{F_{a_i}\}$ is increasing so $|\bigcup\limits_{i=1}^{\infty} F_{a_i}| = \lim\limits_{i\to\infty} \mu(F_{a_i})$ thus there exists $i_o \in \mathbb{N}$ such that $|F_{a_{i_o}}| > \frac{1}{2}|\bigcup\limits_{i=1}^{\infty} F_{a_i}|$ and we finally deduce that $|\bigcup\limits_{j=1}^{i_o} B_{a_j}| > \frac{1}{2}|\bigcup\limits_a B_a|$.
We denote the side lengths of B_k , $k = 1, 2, ..., M$ by a_k, b_k. We also may assume that $b_1 \geq b_2 \geq ... \geq b_M$ and that no B_k is contained in any other of the family

$\{B_j\}_{j=1}^M$.

We now start constructing our sequence R_k.
First we choose $R_1 = B_1$.
Assume that $R_1, R_2, ..., R_h$ have been chosen where $R_h = B_l$, for some $l \in \{1, 2, ..., M-1\}$.
Then we choose as R_{h+1} the first B_k in the sequence $B_{l+1}, B_{l+2}, ..., B_M$ such that

$$\frac{1}{|B_k|}\int_{B_k} \chi_{\bigcup_{j=1}^h R_j} e^{\sum_{j=1}^h \chi_{R_j}} \leq 1+n \qquad (1)$$

where n will be chosen later.
By this way we get a family $\{R_j\}_{j=1}^H$ of intervals with sides a_j, b_j (we continue to use the notation b_j but now for the family $\{R_j\}_{j=1}^H$ such that $b_1 \geq b_2 \geq ... \geq b_H$). We obviously then have

$$|R_j \cap (\bigcup_{k<j} R_k)| = \frac{1}{e}\int_{R_j \cap (\bigcup_{k<j} R_k)} edx.$$

Now for any $z \in \bigcup_{k<j} R_k$ there exists $k \in \{1, 2, ..., j-1\}$ such that $z \in R_k$ so $x_{R_k}(z) \geq 1$ and $\sum_{k=1}^{j-1} x_{R_k}(z) \geq 1$, thus the last quantity is

$$\leq \frac{1}{e}\int_{R_j} \chi_{\bigcup_{k<j} R_k} e^{\sum_{k=1}^{j-1} \chi_{R_k}} \leq_{(1)} \frac{1+n}{e}|R_j|. \qquad (2)$$

So, for every $j \in \{1, 2, ..., H\}$, we have

$$|R_j| = |R_j \cap (\bigcup_{k<j} R_k)| + |R_j - (\bigcup_{k<j} R_k)| \leq_{(2)} \frac{1+n}{e}|R_j| + |R_j - \bigcup_{k<j} R_k| \Rightarrow$$

$$|R_j| \leq \frac{1}{1-\frac{1+n}{e}}|R_j - \bigcup_{k<j} R_k|. \qquad (3)$$

for $0 < n < e-1$.

We shall now prove (b) for $c = 20e$ by induction.

- Firstly

$$\int_{R_1} \chi_{R_1} e^{\chi_{R_1}} = e|R_1| \leq 20e|R_1|.$$

- Next assume that

$$\int \chi_{\bigcup_{j=1}^{h} R_j} e^{\sum_{j=1}^{h} \chi_{R_j}} \leq 20e|\bigcup_{j=1}^{h} R_j|, \qquad for \quad 1 \leq h \leq H-1. \tag{4}$$

- We will prove that $\displaystyle\int \chi_{\bigcup_{j=1}^{h+1} R_j} e^{\sum_{j=1}^{h+1} \chi_{R_j}} \leq 20e|\bigcup_{j=1}^{h+1} R_j|.$

We have the following

$$\int \chi_{\bigcup_{j=1}^{h+1} R_j} e^{\sum_{j=1}^{h+1} \chi_{R_j}} = \int_{(\bigcup_{j=1}^{h} R_j)-R_{h+1}} e^{\sum_{j=1}^{h} \chi_{R_j}} + \int_{R_{h+1}\cap(\bigcup_{j=1}^{h} R_j)} e^{\sum_{j=1}^{h+1} \chi_{R_j}} \leq$$

$$\leq \int_{(\bigcup_{j=1}^{h} R_j)} e^{\sum_{j=1}^{h} \chi_{R_j}} + \int_{R_{h+1}\cap(\bigcup_{j=1}^{h} R_j)} e^{\sum_{j=1}^{h+1} \chi_{R_j}} \leq_{(4)}$$

$$\leq 20e|\bigcup_{j=1}^{h} R_j| + \int_{R_{h+1}\cap(\bigcup_{j=1}^{h} R_j)} e^{\sum_{j=1}^{h+1} \chi_{R_j}} =$$

$$= 20e|\bigcup_{j=1}^{h} R_j| + e\int_{R_{h+1}\cap(\bigcup_{j=1}^{h} R_j)} e^{\sum_{j=1}^{h} \chi_{R_j}} \leq_{(1)} 20e|\bigcup_{j=1}^{h} R_j| + e(1+n)|R_{h+1}|. \tag{5}$$

But we have from (3) that $|R_{h+1}| \leq \dfrac{1}{1-\frac{1+n}{e}}|R_{h+1} - (\bigcup_{k=1}^{h} R_k)|,$

so,

$$\int \chi_{\bigcup_{j=1}^{h+1} R_j} e^{\sum_{j=1}^{h+1} \chi_{R_j}} \leq 20e|\bigcup_{j=1}^{h} R_j| + \frac{(1+n)e}{1-\frac{1+n}{e}}|R_{h+1} - (\bigcup_{j=1}^{h} R_j)| \ ,$$

so choosing n sufficiently small such that $\dfrac{(1+n)e}{1-\frac{1+n}{e}} \leq 20e$ (which is possible since $\lim\limits_{n\to 0^+} \dfrac{(1+n)e}{1-\frac{(1+n)}{e}} = \dfrac{e}{1-\frac{1}{e}} < 20e$)

we get

$$\int \chi_{\bigcup_{j=1}^{h+1} R_j} e^{\sum_{j=1}^{h+1} \chi_{R_j}} \leq 20e|\bigcup_{j=1}^{h} R_j| + 20e|R_{h+1} - (\bigcup_{j=1}^{h} R_j)| = 20e|\bigcup_{j=1}^{h+1} R_j| \ ,$$

$\forall h = 1, 2, ..., H-1$ so the induction is completed and (b) is proved for $c_2 = 20e$ and $\{R_k\}_{k=1}^{H} \subseteq \{B_a\}_{a\in A}$.

Moving on to (a), we want to prove $|\bigcup\limits_a B_a| \leq c_1|\bigcup\limits_k R_k|$.

It is sufficient to show that

$$|(\bigcup_{k=1}^{M} B_k) - (\bigcup_{j=1}^{H} R_j)| \leq c|\bigcup_{j=1}^{H} R_j| \tag{6}$$

for a suitable constant c independent of the initial family $\{B_a\}_{a\in A}$ as we will then have

$$|\bigcup_{a\in A} B_a| \leq 2|\bigcup_{j=1}^{M} B_k| \leq 2[|(\bigcup_{k=1}^{M} B_k) - (\bigcup_{j=1}^{H} R_j)| + |\bigcup_{j=1}^{H} R_j|] \leq_{(6)}$$

$$\leq_{(6)} 2[c|\bigcup_{j=1}^{H} R_j| + |\bigcup_{j=1}^{H} R_j|] = 2(c+1)|\bigcup_{j=1}^{H} R_j|.$$

Let $B \in \{B_k\}_{k=1}^{M}$ that has not be chosen in $\{R_1, R_2, ..., R_H\}$, that is we have the finite sequence $R_1, R_2, ..., R_l, ..., B, ...,$ (where R_l is the last dyadic interval of $\{R_j\}_{j=1}^{M}$ before B in the sequence $\{B_k\}_{k=1}^{M}$) with side lengths of B a , b and $b_j \geq b$ for $1 \leq j \leq l$.

For this set B we have

$$\frac{1}{|B|}\int_B \chi_{\bigcup_{j=1}^{l} R_j} e^{\sum_{j=1}^{l} \chi_{R_j}} > 1 + n. \tag{7}$$

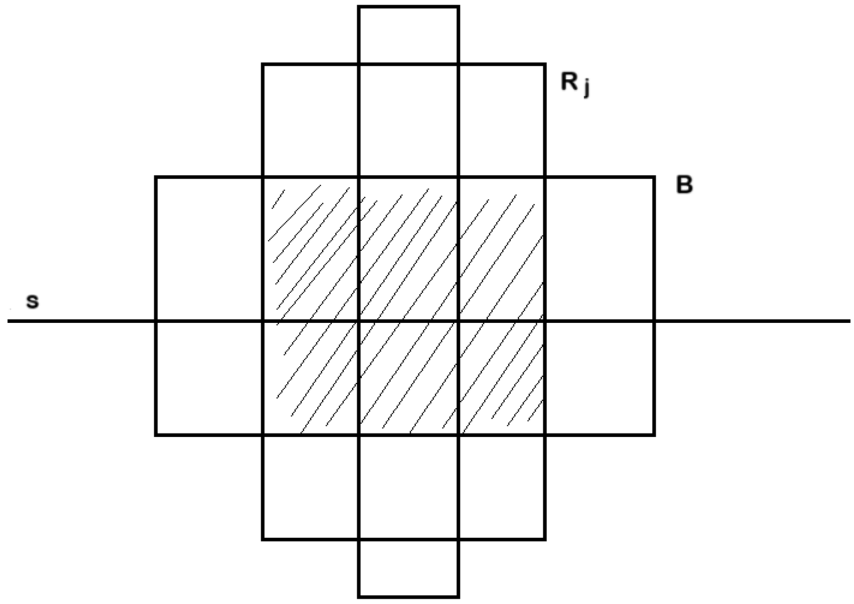


Figure 2.1.3

If we intersect $R_1, R_2, ..., R_H$ and B by a line s parallel to Ox and call the intersections $S = B \cap s$, $I_1 = R_1 \cap s, I_2 = R_2 \cap s, ..., I_H = R_H \cap s$ we get

$$\frac{1}{|S|} \int_S \chi_{\bigcup_{k=1}^l I_k} e^{\sum_{k=1}^l \chi_{I_k}} > 1 + n.$$

If this wasn't true, that is, if

$$\frac{1}{|S|} \int_S \chi_{\bigcup_{k=1}^l I_k} e^{\sum_{k=1}^l \chi_{I_k}} \leq 1 + n$$

we would have

$$\int_S \chi_{\bigcup_{k=1}^l I_{i_k}} e^{\sum_{k=1}^l \chi_{I_k}} \leq (1 + n)|S|$$

and by integrating over the projection of B on Oy we would get

$$\frac{1}{|B|} \int_B \chi_{\bigcup_{k=1}^l R_k} e^{\sum_{k=1}^l \chi_{R_k}} \leq 1 + n,$$

a contradiction.

So,
$$\frac{1}{|S|}\int_S \chi_{\bigcup_{k=1}^{l} I_k} e^{\sum_{k=1}^{l}\chi_{I_k}} \leq \frac{1}{|S|}\int_S \chi_{\bigcup_{k=1}^{H} I_k} e^{\sum_{k=1}^{H}\chi_{I_k}}$$

thus,
$$\frac{1}{|S|}\int_S \chi_{\bigcup_{k=1}^{H} I_k} e^{\sum_{k=1}^{H}\chi_{I_k}} > 1+n \quad \Rightarrow$$

$$S = B\cap s \subseteq \{x\in s : M_s(\chi_{\bigcup_{k=1}^{H} I_k} e^{\sum_{k=1}^{H}\chi_{I_k}}) > 1+n\},$$

where M_s is the 1-dimensional maximal operator with respect to intervals of s. The above arguments are given for an arbitary B in $\{B_1, B_2, ..., B_M\} - \{R_1, R_2, ..., R_H\}$ therefore we have

$$s\cap(\bigcup_{k=1}^{M} B_k - \bigcup_{j=1}^{H} R_j) \subseteq \{x\in s : M_s(\chi_{\bigcup_{k=1}^{H} I_k} e^{\sum_{k=1}^{H}\chi_{I_k}}) > 1+n\} \quad \Rightarrow$$

$$|s\cap(\bigcup_{k=1}^{M} B_k - \bigcup_{j=1}^{H} R_j)| \leq \frac{c^*}{1+n}\int \chi_{\bigcup_{k=1}^{H} I_k} e^{\sum_{k=1}^{H}\chi_{I_k}},$$

from the weak type (1,1) inequality for the one dimensional maximal operator M_s.

Integrating over the projection of $\bigcup_{k=1}^{M} B_k$ on Oy we get

$$|\bigcup_{k=1}^{M} B_k - \bigcup_{j=1}^{H} R_j| \leq \frac{c^*}{1+n}\int_{\mathbb{R}^2} \chi_{\bigcup_{j=1}^{H} R_j} e^{\sum_{j=1}^{H}\chi_{R_j}} \leq_{(b)} \frac{c^*}{1+n} 20e|\bigcup_{j=1}^{H} R_j|,$$

which proves (a) for $c_1 = 20e\frac{c^*}{1+n}$. $\square$

As we mentioned above, using Theorem 2.1.3 we can prove that the basis $\mathbb{B}_2$ differentiates $L^1(1+log^+(L^1))$.

Theorem 2.1.4: *The basis* $\mathbb{B}_2$ *differentiates* $L^1(1+log^+(L^1(\mathbb{R}^2))$*, that is*

$$|\{M_2 f > \lambda\}| \leq C \int_{\mathbb{R}^2} \frac{|f|}{\lambda}(1 + log^+(\frac{|f|}{\lambda})).$$

Proof: To prove the inequality we define the function $G : [0,\infty) \to [0,\infty)$, $G(x) = xlog^+x$.
For every $y \geq 0$ we also define

$$\Psi(y) = \begin{cases} y, & if\ 0 \leq y \leq 1 \\ e^{y-1}, & if\ y > 1 \end{cases}$$

Then $G(x) = \sup\limits_{y\geq 0}(xy - \Psi(y))$, $\forall x \geq 0$ that is for every $x, y \geq 0$ we have

$$G(x) \geq xy - \Psi(y) \quad \Rightarrow \quad xy \leq G(x) + \Psi(y), \quad \forall x, y \geq 0. \quad (1)$$

For every $\lambda > 0$ we define the set $E_\lambda = \{M_2 f > \lambda\}$ and take a compact set K such that $K \subseteq E_\lambda$.
Then $x \in K \quad \Rightarrow \quad x \in E_\lambda$ so there exists an open set $R_x \in \mathbb{B}_2$ containing x such that

$$\frac{1}{|R_x|}\int_{R_x} f > \lambda. \qquad (2)$$

So $K \subseteq \bigcup_{x\in K} R_x$, thus there exist $x_1, x_2, ..., x_n \in K$ such that $K \subseteq \bigcup_{i=1}^n R_{x_i}$.
Applying Theorem 2.1.3 to the collection $\{R_{x_i}\}_{i=1}^n =: C$ we get a finite sequence $\{R_k\} \subseteq C$ such that:

a) $|\bigcup R_k| \geq c_1 |\bigcup_{i=1}^n R_{x_i}|$ and

b) $\int_{\cup R_k} e^{\sum \chi_{R_k}} dx \leq c_2 |\bigcup_{i=1}^n R_k|$

Without loss of generality, we assume that $f \geq 0$, so we have

$$\frac{1}{|R_k|}\int_{R_k} f > \lambda \quad \Rightarrow \quad |R_k| \leq \frac{1}{\lambda}\int_{R_k} f \ , \ \forall k \qquad (*)$$

as $\{R_k\} \subseteq C$.

So

$$|K| \leq |\bigcup_{i=1}^{n} R_{x_i}| \leq_{(a)} c_3 |\cup R_k| \leq c_3 \sum_k |R_k| \leq$$

$$\leq_{(*)} \frac{c_3}{\lambda} \sum_k \int_{R_k} f = \frac{c_3}{\lambda} \sum_k \int_{R_k} f \cdot \chi_{R_k} = c_3 \int_{\cup R_k} \frac{f}{\lambda} \sum_k \chi_{R_k}$$

from which we get

$$c_3 \sum_k |R_k| \leq c_3 \int_{\cup R_k} \frac{f}{\lambda} \sum_k \chi_{R_k} = c_3 I_1 \qquad (**)$$

where $I_1 = \displaystyle\int_{\cup R_k} \frac{f}{\lambda} \sum_k \chi_{R_k}$

and continuing our calculations, we have that for any $\mu > 0$

$$c_3 I = c_3 \int_{\cup R_k} \frac{f}{\lambda} \sum_k \chi_{R_k} = c_3 \int_{\cup R_k} \frac{f}{\lambda\mu} (\mu \sum_k \chi_{R_k}) dx \leq$$

$$\leq_{(1)} c_3 \int_{\mathbb{R}^2} \frac{f}{\lambda\mu} (log^+(\frac{f}{\lambda\mu}) + c_3 \int_{\cup R_k} \Psi(\mu \sum_k \chi_{R_k}) = c_3 I_3 + c_3 I_2$$

(that is $I_1 \leq I_2 + I_3$)

where $I_2 = \displaystyle\int_{\cup R_k} \Psi(\mu \sum \chi_{R_k})$ and $I_3 = \displaystyle\int_{\mathbb{R}^2} \frac{f}{\lambda\mu} (log^+(\frac{f}{\lambda\mu})$.

Setting $\sum \chi_{R_k} =: g$ we get

$$I_2 = \int_{\cup R_k} \Psi(\mu g(x)) dx = \int_{\{g \geq \frac{1}{\mu}\} \cap (\cup R_k)} \Psi(\mu g(x)) dx + \int_{\{0 < g < \frac{1}{\mu}\} \cap (\cup R_k)} \Psi(\mu g(x)) dx =$$

$$= \int_{\{g \geq \frac{1}{\mu}\} \cap (\cup R_k)} e^{\mu g(x) - 1} dx + \int_{\{0 < g < \frac{1}{\mu}\} \cap (\cup R_k)} \mu g(x) dx \leq$$

$$\leq \frac{1}{e} \int_{\{g \geq \frac{1}{\mu}\} \cap (\cup R_k)} e^{\mu \sum \chi_{R_k}} dx + \mu \int_{\mathbb{R}^2} \sum \chi_{R_k} =$$

$$= \frac{1}{e} \int_{\{g \geq \frac{1}{\mu}\} \cap (\cup R_k)} e^{\mu \sum \chi_{R_k}} dx + \mu \sum |R_k| dx$$

$$\leq_{(**)} c_4 \mu I_1 + \frac{1}{e} \int_{\{g \geq \frac{1}{\mu}\} \cap (\cup R_k)} e^{\mu \sum \chi_{R_k}(x)} dx \leq$$

$$\leq c_4 \mu I_1 + \frac{1}{e} \int_{\{\sum \chi_{R_k} \geq \frac{1}{\mu}\}} e^{\mu \sum \chi_{R_k}}(x) dx \leq_{(***)} c_4 \cdot \mu I_1 + \frac{1}{e} \mu \int_{\cup R_k} e^{\sum \chi_{R_k}(x)} dx$$

(where we used that $e^{\mu y} \leq \mu e^{y}$, $\forall y \geq \frac{1}{\mu}$, $\quad 0 < \mu < 1, \quad (***)$)

and

$$c_4 \mu \cdot I_1 + \frac{1}{e} \mu \int_{\cup R_k} e^{\sum \chi_{R_k}(x)} dx \leq_{(b)} c_4 \mu I_1 + c_5 \mu |\cup R_k| \leq$$

$$\leq_{(**)} c_4 \mu I_1 + c_6 \mu I_1 = c_7 \mu I_1 \quad \Rightarrow$$

(for suitable constants c_5 , c_6 , $c_7 > 0$)

$$I_2 \leq c_7 \mu I_1 \qquad (****)$$

So

$$I_1 \leq I_3 + I_2 \ \leq_{(****)} \ I_3 + \mu c_7 I_1 \quad \Rightarrow$$

$$(1-\mu c_7)I_1 \ \leq \ I_3$$

and for $\mu < \dfrac{1}{c_7}$, $\mu < 1$ we get

$$I_1 \leq \frac{1}{1-\mu c_8} I_3 \quad \Rightarrow \qquad I_2 \leq c_8 \mu I_3$$

for a suitable constant c_8.

From the above conclusions we have

$$|K| \leq c_3 I_2 + c_3 I_3 \quad \Rightarrow$$

$$\Rightarrow \quad |K| \leq c_9 I_3 = c_{10} \int \frac{f}{\lambda} log^+(\frac{f}{\lambda} \cdot \frac{1}{\mu}) = c_{10} \int_{\{f > \lambda\mu\}} \frac{f}{\lambda} log(\frac{f}{\lambda} \cdot \frac{1}{\mu}) =$$

$$= c_{10} \int_{\{f>\lambda\mu\}} \frac{f}{\lambda} [log(\frac{f}{\lambda}) + log\frac{1}{\mu}] = c_{11} \int \frac{f}{\lambda} + c_{10} \int_{\{f > \frac{\lambda}{\mu}\}} \frac{f}{\lambda} log \frac{f}{\lambda} \leq$$

$$\leq c_{11} \int_{\mathbb{R}^2} \frac{f}{\lambda} + c_{10} \int_{\mathbb{R}^2} \frac{f}{\lambda} log^+(\frac{f}{\lambda}) \leq c_{12} \int_{\mathbb{R}^2} \frac{f}{\lambda}(1 + log^+(\frac{f}{\lambda})).$$

Now since $|E_\lambda| = \sup\{ \ |K| : K \subseteq E_\lambda \ , \ K \quad compact \ \}$ we have

$$|E_\lambda| = |\{M_2 f > \lambda\}| \leq c_{12} \int_{\mathbb{R}^2} \frac{f}{\lambda}(1 + log^+(\frac{f}{\lambda}))$$

and thus

$$|\{M_2 f > \lambda\}| \leq C \int_{\mathbb{R}^2} \frac{f}{\lambda}(1 + log^+(\frac{f}{\lambda}))$$

which is the required inequality.

We will now prove (****), that is

$$e^{\mu y} \leq \mu e^y, \quad \forall y \geq \frac{1}{\mu}, \mu \in (0,1).$$

Let μ be fixed and set $h(y) = e^{\mu y} - \mu e^y$, $y > 0$.

Then

$$h'(y) = \mu e^{\mu y} - \mu e^y \quad < 0, \quad for\ any\ y > 0 \quad since\ \mu < 1.$$

so $h(y)$ is strictly decreasing for $y > 0$.

Therefore for $y \geq \frac{1}{\mu}$

$$h(y) \leq h(\frac{1}{\mu}) = e - \mu e^{\frac{1}{\mu}} \quad < 0\ ,$$

as $\theta(\mu) = \mu e^{\frac{1}{\mu}}$, is also strictly decreasing for $\mu \in (0,1)$,

thus $\theta(\mu) > \theta(1) = e \quad \Rightarrow \qquad \mu e^{\frac{1}{\mu}} > e, \qquad \forall \mu \in (0,1).$

2.2 Intervals of $\mathbb{R}^3$

In a similar way as in Theorem 2.1.3, if we consider a system of intervals in $\mathbb{R}^3$ such that there is some reasonable constraint between their three side-lengths , it is to be expected that this system will behave again like the two-dimensional basis of intervals, that is its maximal operator will satisfy the same inequality as above.
We will present the theorem in the dyadic version again and in the same pattern as before the required inequality can be obtained.

Theorem 2.2.1: *Let $\{B_a\}_{a\in A}$ be a collection of dyadic intervals of $\mathbb{R}^3$ such that $|\bigcup_a B_a| < \infty$. Assume that the side-lengths of B_a in directions Ox, Oy, Oz are $\tilde{a}_a$, $\tilde{b}_a$, $\tilde{c}_a = \phi(\tilde{a}_a, \tilde{b}_a)$, respectively, where $\phi : (0,\infty)\times(0,\infty) \to (0,\infty)$ is a fixed function strictly increasing in the two variables separately.*
Then we can choose a finite sequence $\{R_k\}$ from $\{B_a\}_{a\in A}$ such that:

a) $|\bigcup_a B_a| \leq c^* |\bigcup R_k|$

b) $\int_{\bigcup R_k} e^{\sum \chi_{R_k}} \leq c^{**}|\bigcup R_k|$

where c^*, c^{**} *are absolute constants independent of* $\{B_a\}_{a\in A}$.

Proof: We will prove this Theorem on the same pattern as the proof of Theorem 2.1.3.
First we choose a finite sequence $\{B_k\}_{k=1}^M$ such that

$$|\bigcup_a B_a| \leq 2|\bigcup_{k=1}^M B_k|.$$

We can assume that $\tilde{c}_1 \geq \tilde{c}_2 \geq ... \geq \tilde{c}_M$ and that no B_k is contained in another one. By the strict monotonicity of ϕ we have either $\tilde{a}_k \geq \tilde{a}_l$ or $\tilde{b}_k \geq \tilde{b}_l$ for $1 \leq k \leq l \leq M$. We can see that this is true because if $\tilde{a}_k < \tilde{a}_l$ and $\tilde{b}_k < \tilde{b}_l$ then we would get $\phi(\tilde{a}_k, \tilde{b}_k) < \phi(\tilde{a}_l, \tilde{b}_k) < \phi(\tilde{a}_l, \tilde{b}_l)$, which gives $\tilde{c}_k < \tilde{c}_l$, a contradiction, since $\tilde{c}_k \geq \tilde{c}_l$.

Now we proceed exactly as in the previous proof for intervals in $\mathbb{R}^2$.
First we choose $R_1 = B_1$ and the R_2 as the first B_k where $k > 1$ such that

$$\frac{1}{|B_k|}\int_{B_k} \chi_{R_1} e^{\chi_{R_1}} \leq 1+n \ , \quad \textit{where } n \textit{ will be chosen later.}$$

We then choose R_3 as the first B_l, $l > k$, such that

$$\frac{1}{|B_l|}\int_{B_l} \chi_{R_1\cup R_2} e^{\sum_{i=1}^{2}\chi_{R_i}} \leq 1+n$$

and so on.

In that way we obtain $\{R_j\}_{j=1}^H$ from $\{B_k\}_{k=1}^M$ satisfying

i)$\tilde{c}_1 \geq \tilde{c}_2 \geq ... \geq \tilde{c}_H$

ii) If $\tilde{c}_k \geq \tilde{c}_l$ then either $\tilde{a}_k \geq \tilde{a}_l$ or $\tilde{b}_k \geq \tilde{b}_l$ (where $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j$ are the side lengths of R_j, j=1,2, ...,H)

iii) $\dfrac{1}{|R_{j+1}|}\displaystyle\int_{R_{j+1}} \chi_{\bigcup_{i\leq j} R_j} e^{\sum_{i=1}^{j}\chi_{R_j}} \leq 1+n$

iv) If $B \in \{B_1, B_2, ..., B_M\} - \{R_1, R_2, ..., R_H\}$ and if $R_1, R_2, ..., R_l$ are intervals with $\tilde{c}_k \geq \tilde{c}$, where $\tilde{c}$ is the side-length of B parallel to Oz, then

$$\frac{1}{|B|}\int_B \chi_{\bigcup_{j=1}^{l} R_j} e^{\sum_{j=1}^{l}\chi_{R_j}} > 1+n.$$

As in the previous proof, for every $j \geq 2$ (for $j=1$ it's trivial),

$$|R_j \cap (\bigcup_{k<j} R_k)| = \frac{1}{e}\int_{R_j \cap (\bigcup_{k<j} R_k)} e \leq \frac{1}{e}\int_{R_j} \chi_{\bigcup_{k<j} R_k} e^{\sum_{k=1}^{j-1}\chi_{R_k}} \leq_{(iii)} \frac{1+n}{e}|R_j|.$$

Therefore,

$$|R_j| = |R_j \cap (\bigcup_{k<j} R_k)| + |R_j - (\bigcup_{k<j} R_k)| \leq \frac{1+n}{e}|R_j| + |R_j - (\bigcup_{k=1}^{j-1} R_k)|$$

$$\Rightarrow (1 - \frac{1+n}{e})|R_j| \leq |R_j - (\bigcup_{k<j} R_k)|$$

and for n small enough, we get

$$|R_j| < \frac{1}{1-\frac{1+n}{e}}|R_j - (\bigcup_{k<j} R_k)|.$$

We are now going to prove (b) by induction: We have

$$\int \chi_{R_1} e^{\chi_{R_1}} = e|R_1| < 20e|R_1|.$$

We assume that

$$\int \chi_{\bigcup_{j=1}^{h} R_j} e^{\sum_{j=1}^{h}\chi_{R_j}} \leq 20e|\bigcup_{j=1}^{h} R_j|,$$

where $h < H$.
Then

$$\int \chi_{\bigcup_{j=1}^{h+1} R_j} e^{\sum_{j=1}^{h+1}\chi_{R_j}} = \int \chi_{(\bigcup_{j=1}^{h} R_j) - R_{h+1}} e^{\sum_{j=1}^{h}\chi_{R_j}} + \int_{R_{h+1} \cap (\bigcup_{j=1}^{h} R_j)} e^{\sum_{j=1}^{h+1}\chi_{R_j}} \leq$$

$$\leq \int_{\bigcup_{j=1}^{h} R_j} \chi_{\bigcup_{j=1}^{h} R_j} e^{\sum_{j=1}^{h} \chi_{R_j}} + e \int_{R_{h+1}} \chi_{\bigcup_{j=1}^{h} R_j} e^{\sum_{j=1}^{h} \chi_{R_j}} \leq$$

$$\leq_{(iii)} 20e|\bigcup_{j=1}^{h} R_j| + e|R_{j+1}|(1+n).$$

Continuing as before we conclude that

$$\int \chi_{\bigcup_{j=1}^{h+1} R_j} e^{\sum_{j=1}^{h+1} \chi_{R_j}} \leq 20e|\bigcup_{j=1}^{h+1} R_j|.$$

so we have obtained $\{R_j\}_{j=1}^{H}$ satisfying

$$\int \chi_{\bigcup_{j=1}^{H} R_j} e^{\sum_{j=1}^{H} \chi_{R_j}} \leq c^{**}|\bigcup_{j=1}^{H} R_j|.$$

We shall now prove (a) using (i)-(iv) and (b).
In order to do so we will prove that

$$|\bigcup_{k=1}^{M} B_k - \bigcup_{j=1}^{H} R_j| \leq c|\bigcup_{j=1}^{H} R_j|.$$

If B is an interval of $\{B_k\}_{k=1}^{M}$ which has not been chosen, between R_l and R_{l+1}, for $l < H$, with side-lengths $a, b, c = \phi(a, b)$ then

$$\frac{1}{|B|} \int_B \chi_{\bigcup_{j=1}^{l} R_j} e^{\sum_{j=1}^{l} \chi_{R_j}} > 1 + n. \ (1)$$

We intersect B and $R_1, R_2, ..., R_l$ by a plane σ orthogonal to Oz obtaining $S, I_1, I_2, ..., I_l$ where $S = B \cap \sigma$, $I_1 = R_1 \cap \sigma$ and so on, as in the figure bellow.

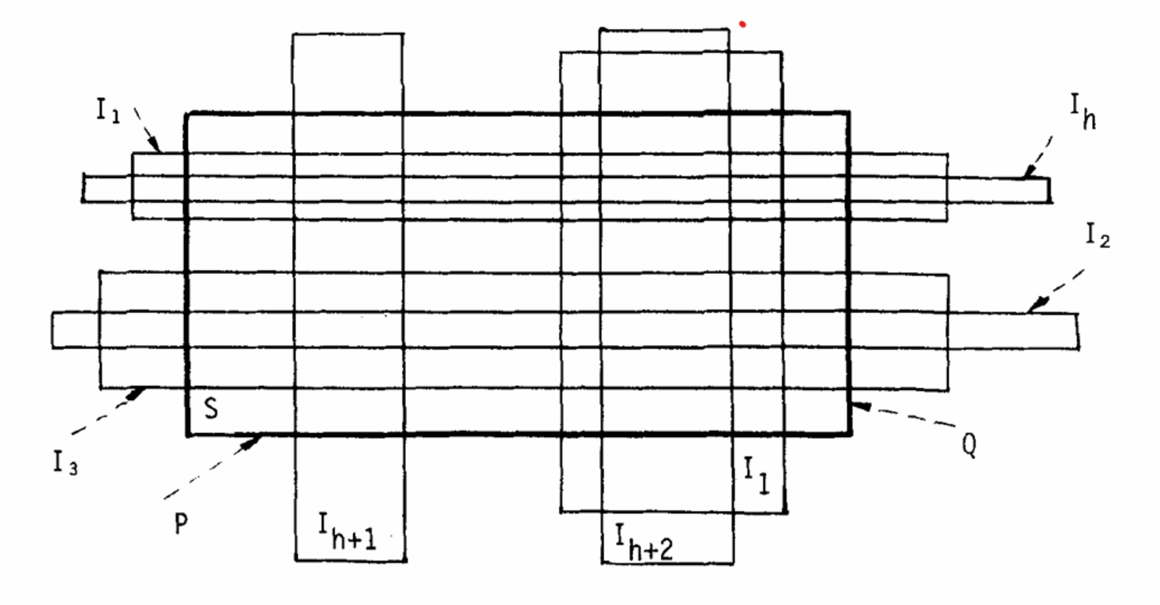


Figure 2.2.1(a)

The 3-dimensional inequality (1) is transformed into the 2-dimensional one

$$\frac{1}{|S|}\int\int_S \chi_{\bigcup_{j=1}^l I_j} e^{\sum_{j=1}^l \chi_{I_j}} > 1+n. \qquad (2)$$

If this was not true, that is if

$$\int\int_S \chi_{\bigcup_{j=1}^l I_j} e^{\sum_{j=1}^l \chi_{I_j}} \leq (1+n)|S|$$

then by integrating over the projection of B, on Oz we would get

$$\int_B \chi_{\bigcup_{j=1}^l R_j} e^{\sum_{j=1}^l \chi_{R_j}} < (1+n)|B|,$$

which is a contradiction to (1).
As we stated before, for every $j = 1, 2, ..., l$ we have $c \leq c_j$ so either $a_j \geq a$ or $b_j \geq b$.
Let's assume that $a_j > a$ and $b_j \leq b$ for $i = 1, 2, ..., h$ and $a_j \leq a$ and $b_j > b$ for $i = h+1, h+2, ..., l$, forgetting about the order of the sides c_j.
Let us call P, Q the projections of S and J_j, K_j the projection of I_j over the axes Ox, Oy.
We can now write $I_j = J_j \times K_j$ and $S = P \times Q$, where $J_j = (\alpha_j, \beta_j)$, $K_j = (\gamma_j, \delta_j)$.

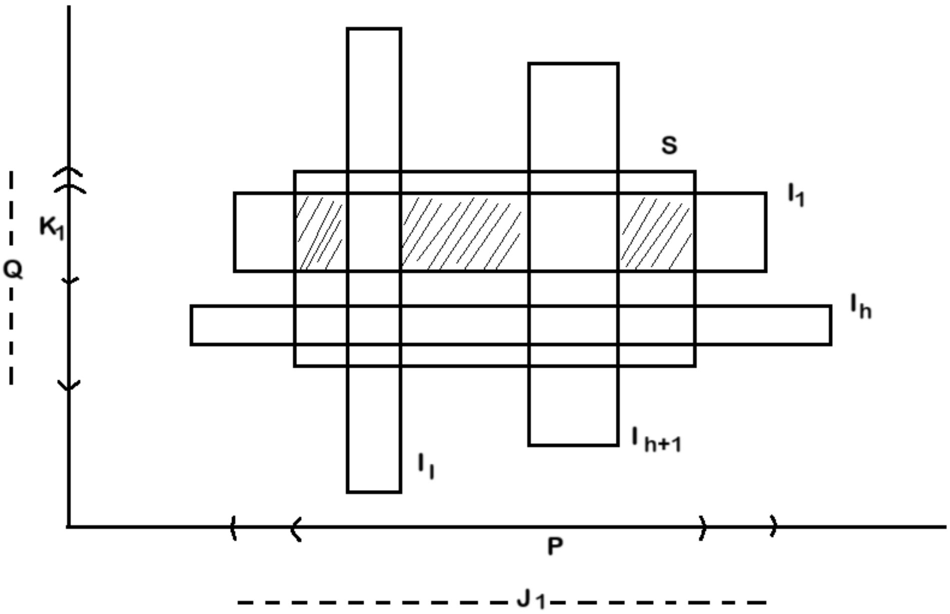


Figure 2.2.1 (b)

Without loss of generality assume that $I_j \cap I_i = \emptyset$ for $i \neq j$, $i, j = 1, 2, .., h$ or $i,\ j = h+1, h+2, ..., l$. Then

$$\frac{1}{|S|}\int\int_{(x,y)\in S} \chi_{\bigcup_{j=1}^{l} I_j}(x,y) e^{\sum_{j=1}^{l}\chi_{I_j}(x,y)} dxdy = A_1 + A_2 + A_3$$

where

$$A_1 = \frac{1}{|S|}\int\int_{S\cap(\bigcup_{j=1}^{h} I_j - \bigcup_{j=h+1}^{l} I_j)} \chi_{\bigcup_{j=1}^{h} I_j}(x,y) e^{\sum_{j=1}^{h}\chi_{I_j}(x,y)} dxdy$$

$$A_2 = \frac{1}{|S|}\int\int_{S\cap(\bigcup_{j=h+1}^{l} I_j - \bigcup_{j=1}^{h} I_j)} \chi_{\bigcup_{j=h+1}^{l} I_j}(x,y) e^{\sum_{j=h+1}^{l}\chi_{I_j}(x,y)} dxdy$$

and

$$A_3 = \frac{1}{|S|}\int\int_{S\cap[(\bigcup_{j=1}^{h} I_j)\cap(\bigcup_{j=h+1}^{l} I_j)]} \chi_{\bigcup_{j=1}^{h} I_j}\chi_{\bigcup_{j=h+1}^{l} I_j} e^{\sum_{j=1}^{h}\chi_{I_j}} e^{\sum_{j=h+1}^{l}\chi_{I_j}} dxdy$$

Now

$$A_1 = x_1 + x_2 + ... + x_h$$

where

$$x_1 = \frac{1}{|S|}\int\int_{T_1} e^{\sum_{j=1}^{h}\chi_{I_j}},\ x_2 = \frac{1}{|S|}\int\int_{T_2} e^{\sum_{j=1}^{h}\chi_{I_j}},\ ...,\ x_h = \frac{1}{|S|}\int\int_{T_h} e^{\sum_{j=1}^{h}\chi_{I_j}}\ ,$$

$$T_1 = \{P - (\bigcup_{m=j+1}^{l} J_j)\} \times \{Q \cap K_1\},\ ...,\ T_h = \{P - (\bigcup_{j=h+1}^{l} J_j)\} \times \{Q \cap K_h\}$$

and T_1 the shaded part of the figure above.

Now,

$$x_1 \leq \frac{1}{|S|}\int\int_{P\times[\gamma_1,\delta_1]} e^{\sum_{j=1}^{h}\chi_{I_j}} dxdy$$

as $T_1 \subseteq P \times K_1 = P \times [\gamma_1, \delta_1]$,

$$= \frac{1}{|P||Q|}\int\int_{P\times[\gamma_1,\delta_1]} e^{\sum_{j=1}^{h}\chi_{K_j(y)}} dxdy$$

(as $I_j = J_j \times K_j$ and $x \in P$ implies $x \in J_j\ \forall j = 1, 2, ..., h$)

$$= \frac{1}{|P||Q|}\int_{y=\gamma_1}^{\delta_1}\int_{x\in P} e^{\sum_{j=1}^{h}\chi_{K_j}(y)} dxdy = \frac{1}{|Q|}\int_{\gamma_1}^{\delta_1} e^{\sum_{j=1}^{h}\chi_{K_j}(y)} dy$$

Similarly we get

$$x_h \leq \frac{1}{|Q|}\int_{[\gamma_h,\delta_h]} e^{\sum_{j=1}^{h}\chi_{K_j}(y)} dy$$

So

$$A_1 = x_1 + x_2 + ... + x_h \leq \frac{1}{|Q|}\int_{(\bigcup_{j=1}^{k} K_j)\cap Q} e^{\sum_{k=1}^{h}\chi_{K_j}(y)} dy =: a_1$$

as $K_j = [\gamma_j, \delta_j]$ and $K_j \subseteq Q$.

Analogously we get

$$A_2 \leq \frac{1}{|P|} \int_P \chi_{P \cap (\bigcup_{j=h+1}^{l} J_j)}(x) e^{\sum_{j=h+1}^{l} \chi_{J_j}(x)} dx =: a_2$$

Furthermore,

$$A_3|P||Q| = \int_{y \in Q \cap (\bigcup_{j=1}^{h} K_j)} \int_{x \in P \cap (\bigcup_{j=h+1}^{l} J_j)} e^{\sum_{j=1}^{l} \chi_{J_j \times K_j}} \chi_{\bigcup_{j=1}^{h}(J_j \times K_j)} \chi_{\bigcup_{j=h+1}^{l}(J_j \times K_j)} dxdy$$

We define

$$F(x,y) := e^{\sum_{j=1}^{h} \chi_{J_j \times K_j}(x,y)} e^{\sum_{j=h+1}^{l} \chi_{J_j \times K_j}(x,y)}$$

and for $x \in P \cap (\bigcup_{j=h+1}^{l} J_j)$, $y \in Q \cap (\bigcup_{j=1}^{h} K_j)$, we have

$$F(x,y) = e^{\sum_{j=1}^{h} K_j(y)} e^{\sum_{j=h+1}^{l} J_j(x)} =: G(x,y)$$

as for $j = 1, 2, ..., h$, we have $P \subseteq J_j$ and for $j = h+1, h+2, ..., l$, $Q \subseteq K_j$.

As a consequence we have

$$A_3|P||Q| = \int_{y \in Q \cap (\bigcup_{j=1}^{h} K_j)} \int_{x \in P \cap (\bigcup_{j=h+1}^{l} J_j)} G(x,y) dxdy.$$

So

$$A_3 = \frac{1}{|P|} \int_{P \cap (\bigcup_{j=h+1}^{l} J_j)} e^{\sum_{j=h+1}^{l} J_j(x)} \frac{1}{|Q|} \int_{Q \cap (\bigcup_{j=1}^{h} K_j)} e^{\sum_{j=1}^{h} K_j(y)},$$

that is $A_3 = a_1 a_2$.
From (2) we get $A_1 + A_2 + A_3 > 1 + n$, that is $a_1 + a_2 + a_1 a_2 > 1 + n$ so either $a_1 > \rho$ or $a_2 > \rho$ where $\rho := \min(\frac{1+n}{3}, \sqrt{\frac{1+n}{3}})$.
We can see this is true because if $a_1 \leq \rho$ and $a_2 \leq \rho$ then $a_1 \leq \frac{1+n}{3}$, $a_2 \leq \frac{1+n}{3}$ and $a_1 a_2 \leq \frac{1+n}{3}$.
This entails that $a_1 + a_2 + a_1 a_2 \leq 1 + n$, a contradiction.

So for the set $B \in \{B_1, B_2, ..., B_M\} - \{R_1, R_2, ..., R_H\}$, if $(x, y, z) \in B$ and $a_2 > \rho$ then

$$\frac{1}{|P|}\int_P \chi_{P\cap(\bigcup_{j=h+1}^{l} J_j)}(t)e^{\sum_{j=h+1}^{l}\chi_{J_j(t)}}dt > \rho$$

and since $x \in P$

$$\frac{1}{|P|}\int_P \chi_{\bigcup_{j=h+1}^{l} R_j}(t, y, z)e^{\sum_{j=h+1}^{l}\chi_{R_j}(t,y,z)}dt > \rho$$

and if $a_1 > \rho$ then similarly we get

$$\frac{1}{|Q|}\int_Q \chi_{\bigcup_{j=1}^{h} R_j}(x, s, z)e^{\sum_{j=1}^{h}\chi_{\bigcup_{j=1}^{h} R_j}(x,s,z)}ds > \rho.$$

Therefore either

$$B \subseteq \{(x, y, z) : \sup_{\substack{x\in I \\ I\ interval}} \{\frac{1}{|I|}\int_I \chi_{\bigcup_{j=h+1}^{l} R_j}(t, y, z)e^{\sum_{j=h+1}^{l}\chi_{R_j}(t,y,z)}dt > \rho\} =: E$$

or

$$B \subseteq \{(x, y, z) : \sup_{\substack{y\in J \\ J\ interval}} \{\frac{1}{|J|}\int_J \chi_{\bigcup_{j=1}^{h} x_{R_j}}(x, s, z)e^{\sum_{j=1}^{H}\chi_{\bigcup_{j=1}^{H} R_j}(x,s,z)}ds > \rho\} =: F.$$

That is $B \subseteq E \cup F$.

For fixed y, z we have $E = \cup_{y,z}E(y, z)$ where

$$E(y, z) = \{(x, y, z) : M_1[\chi_{\bigcup R_j}(\cdot, y, z)e^{\sum \chi_{R_j}(\cdot,y,z)}] > \rho\}.$$

Using the weak type (1,1) for the one-dimensional Hardy-Littlewood maximal operator we get

$$|E(y, z)|_1 \leq \frac{c}{\rho}\int_{\mathbb{R}} \chi_{\bigcup_{j=1}^{H} R_j}(t, y, z)e^{\sum_{j=1}^{H}\chi_{R_j}(t,y,z)}dt$$

and integrating we obtain

$$|E|_3 \leq \frac{c}{\rho} \int_{\mathbb{R}^3} \chi_{\bigcup_{j=1}^H R_j} e^{\sum_{j=1}^H \chi_{R_j}} \leq_{(b)} \frac{c}{\rho} c^{**} |\bigcup R_j|.$$

Similarly we get

$$|F|_3 \leq \frac{c}{\rho} \int_{\mathbb{R}^3} \chi_{\bigcup_{j=1}^H R_j} e^{\sum_{j=1}^H \chi_{R_j}} \leq_{(b)} \frac{c}{\rho} c^{**} |\bigcup R_j|$$

concluding

$$|\bigcup B_a| \leq 2\frac{c}{\rho} c^{**} |\bigcup R_j| = c^* |\bigcup R_j|. \ \square$$

Chapter 3

The basis of rectangles $\mathbb{B}_3$

In this chapter we will present the construction of an important set called the "Perron tree" and its contribution in the solution of problems like the "needle problem" and the construction of other notable sets such as the "Besicovitch set". We will also use the Perron tree to prove that the basis of rectangles $\mathbb{B}_3$ does not have good differentiation properties.

3.1 The Perron tree

Theorem 3.1.1: *Let $\{A_h\}_{h=1}^{2^n}$ be the 2^n open triangles in $\mathbb{R}^2$ obtained by joining the point (0,1) with the points (0,0), (1,0), ... ,(2^n,0). Let A_h be the triangle with vertices (0,1), (h-1,0), (h,0). Then given α, where $1/2 < \alpha < 1$, we can make a parallel translation of each A_h along the Ox axis to a new position $\tilde{A}_h$ so that we have*

$$|\bigcup_{h=1}^{2^n} \tilde{A}_h| \leq (a^{2n} + 2(1-a))|\bigcup_{h=1}^{2^n} A_h|.$$

Proof: We will prove the theorem by repetition of a process called basic construction.

Basic construction: Consider two adjacent triangles T_1, T_2, with basis on the Ox axis, the same basis length b and height length h, as shown in Figure 1.1(a) below.

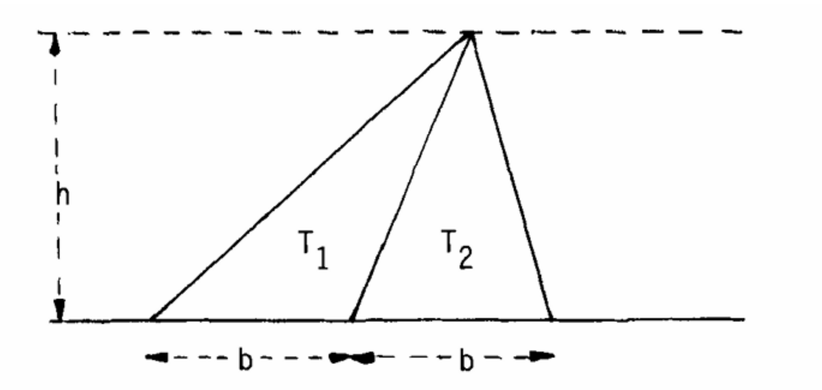


Figure 3.1.1(a)

With T_1 fixed, we transfer T_2 towards T_1, so that the sides of the triangles that are not parallel meet at a point with distance ah from Ox as in Figure 3.1.1(b).

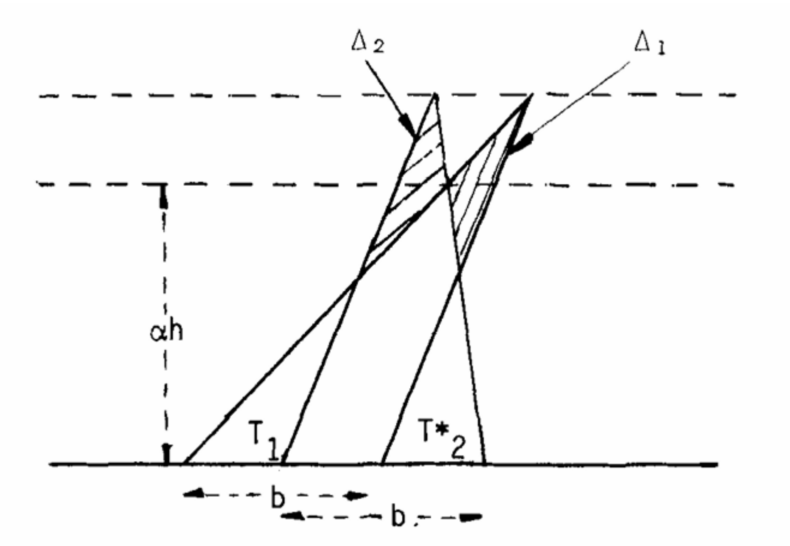


Figure 3.1.1(b)

The union of T_1 and T_2^* consists of a triangle S (not shaded) homothetic to $T_1 \cup T_2$ and two "excess triangles" Δ_1, Δ_2 (shaded).

We will prove that S is homothetic to $T_1 \cup T_2$.
Their sides are parallel and, without loss of generality, we can assume $A = (0, 0)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$ and $E = (z, w)$, as in figure below, where

triangle EAD is $T_1 \cup T_2$ and triangle ABC is S. We also denote by B'=(x,y) any point in the segment BC.

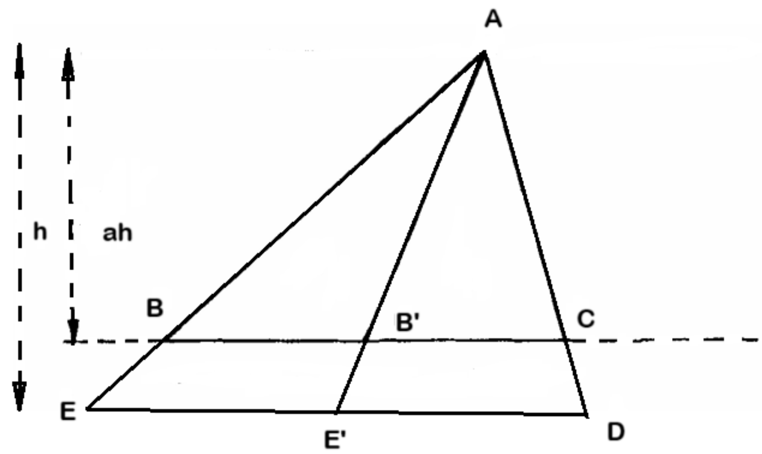


Figure 3.1.1 (c)

Then from Thales' theorem we get

$$\frac{|AB|}{|AE|} = \frac{|AB'|}{|AE'|} = \frac{|AC|}{|AD|} = \lambda$$

where $\lambda \in (0, 1)$, with

$$|AE'| = \frac{1}{\lambda}|AB'|.$$

Additionally $B' \in BC$ and $E' \in ED$ so $(z, w) = \frac{1}{\lambda}(x, y)$.

Considering the homothecy $P : \mathbb{R}^2 \to \mathbb{R}^2$, where $P(u, v) = \frac{1}{\lambda}(u, v)$, with homothecy center the point A= (0,0), we have :
since $B'(x, y) \in BC$ then $P(B') = \frac{1}{\lambda}(x, y) = (z, w) = E' \in ED$, so $P(AB') = AE'$ by the continuity of P.

From that we get $P(ABC) = AED$, that is S is homothetic to $T_1 \cup T_2$.

Again from Thales' theorem and looking at the previous figure we have

$$\frac{|AC|}{|AD|} = \frac{ah}{h} = \alpha \ (= \lambda)$$

so

$$P(ABC) = P(S) = \frac{1}{\alpha}S = AED = T_1 \cup T_2 \quad \Rightarrow$$

$$S = \alpha \cdot T_1 \cup T_2 \quad \Rightarrow \qquad |S| = \alpha^2 |T_1 \cup T_2|. \qquad (1)$$

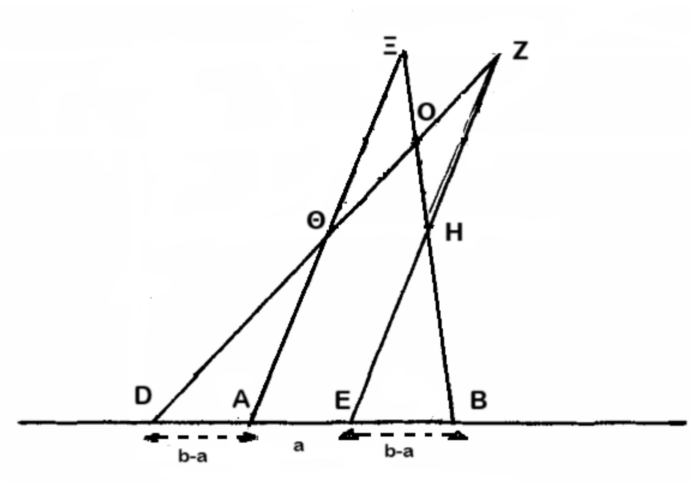


Figure 3.1.1 (d)

As we can see on the figure above, the triangles ABΞ and EHB are similar and so are the triangles $DA\Theta$ and DZE as their sides are parallel so

$$\frac{EB}{AB} = \frac{EH}{EZ} \quad and \quad \frac{DA}{DE} = \frac{A\Theta}{A\Xi}$$

and

$$\frac{EB}{AB} = \frac{\beta - a}{\beta} = \frac{DA}{DE}$$

entailing that $\dfrac{A\Theta}{A\Xi} = \dfrac{EH}{EZ}$ so by Thales' theorem once again we get that

$$\Theta H // AB \quad \Rightarrow \qquad \Xi\Theta = ZH.$$

Having proved that the two excess triangles Δ_1, Δ_2 are equal and we are moving on to prove that

$$|\Delta_1| + |\Delta_2| = 2(1-\alpha)^2 |T_1 \cup T_2|.$$

We draw the parallel to ΞZ that goes through the point O and separates Δ_1 into the two triangles KOZ and KOH and Δ_2 into the triangles MOΞ and MOΘ for Δ_2.

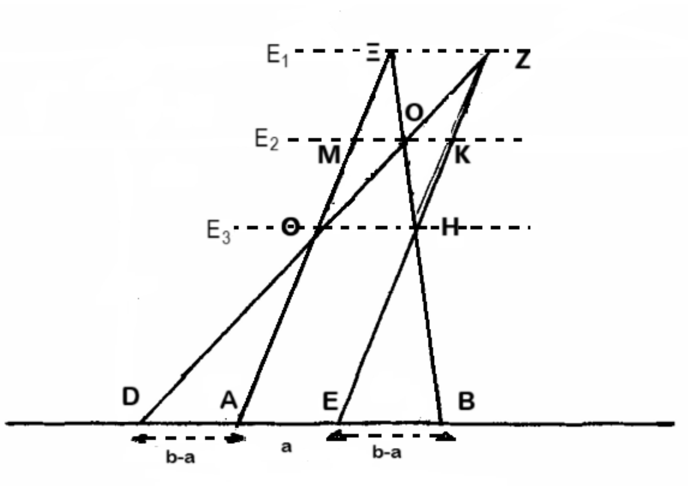


Figure 3.1.1 (e)

We know from the basic construction that $d(E_1, E_2) = (1-\alpha)h$ and we define $d(E_2, E_3) = \epsilon \cdot h$, where $0 < \epsilon < \alpha$.
Now

$$|\Delta_1| = |KOZ| + |KOH| = \frac{1}{2}|OK| \cdot (1-a)h + \frac{1}{2}|OK| \cdot \epsilon h$$

and

$$|\Delta_2| = |MO\Xi| + |MO\Theta| = \frac{1}{2}(1-\alpha)h|OM| + \frac{1}{2}\epsilon h|OM|$$

but $KOH = MO\Xi$ as O is the point where the diagonals intersect each other and also the center of the parallelogram, thus $OK = OM$ and $O\Xi = OH$ and $\angle HOK = \angle MO\Xi$ as opposite angles.

So the triangles KOH *and* $MO\Xi$ are equal, therefore $\epsilon = (1-a)$,

$$|\Delta_1| = |KOZ| + |MO\Xi| = \frac{1}{2}(1-a)h|OK| + \frac{1}{2}(1-\alpha)h|OM|$$
$$= |OK|(1-a)h$$

and

$$|\Delta_1| + |\Delta_2| = 2|\Delta_1| = 2(1-a)h|OK| = (1-a)h|\Theta H|$$

where $\dfrac{|H\Theta|}{|BD|} = \dfrac{|O\Theta|}{|OD|} = \dfrac{(1-a)h}{ah} = \dfrac{1-a}{a} \quad \Rightarrow \quad |H\Theta| = \dfrac{1-a}{a}|BD|$
and so

$$|\Delta_1| + |\Delta_2| = (1-a)h\frac{1-a}{a}|BD| =$$
$$= (1-a)^2\frac{h}{a}a \cdot 2b = 2(1-a)^2\frac{1}{2}h \cdot 2b = 2(1-a)^2|T_1 \cup T_2|$$

as $|BD| = a \cdot (2b)$ and 2b is the length of the basis of the triangle $T_1 \cup T_2$.

Thus, from (1) and (2) we get

$$|T_1 \cup T_2^*| = (a^2 + 2(1-a)^2)|T_1 \cup T_2|. \qquad (*)$$

We are now ready to apply this basic construction to our theorem. Consider the 2^{n-1} pairs of adjacent triangles (A_1, A_2), ..., (A_{2^n-1},A_{2^n}). Applying the basic construction to each pair we obtain the triangles S_1, S_2, ..., S_{2^n-1} and the excess triangles Δ_1^1, Δ_2^1 corresponding to S_1, Δ_1^2, Δ_2^2 corresponding to S_2 and so on. We shift S_2 along the Ox towards S_1 so that it is adjacent to S_1 and call it $\tilde{S}_2$. Then we shift S_3 along Ox so that it is adjacent to $S_1 \cup \tilde{S}_2$ and so on.

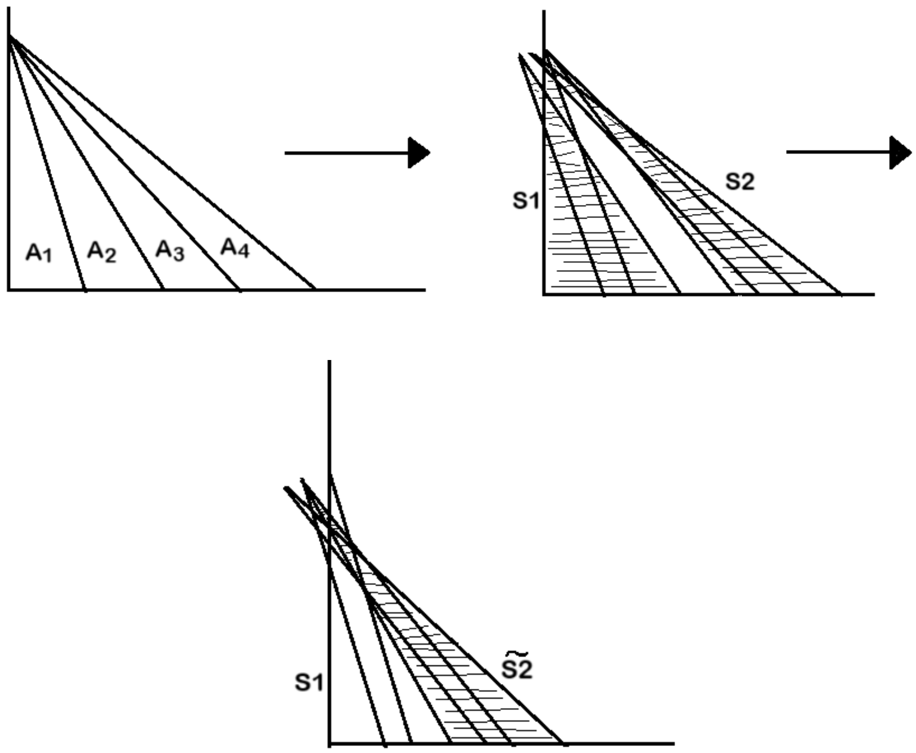


Figure 3.1.1(f)

In these motions every S_h carries with it the two excess triangles Δ_1^h, Δ_2^h, so we are in fact shifting the triangles A_1, A_2, ..., A_{2^n} to new positions $\tilde{A}_1$, $\tilde{A}_2$, ..., $\tilde{A}_{2^n}$. So $\tilde{A}_1 \cup \tilde{A}_2 \cup ... \cup \tilde{A}_{2^n}$ consists of S_1, $\tilde{S}_2$, ..., $\tilde{S}_{2^{n-1}}$ where

$$|S_1 \cup \tilde{S}_2 \cup ... \cup \tilde{S}_{2^{n-1}}| = \alpha^2 |A_1 \cup A_2 \cup ... \cup A_{2^n}| \qquad (3)$$

and the excess triangles $\Delta_1^1, \Delta_2^1, ..., \Delta_1^{2^{n-1}}, \Delta_2^{2^{n-1}}$ whose union has area

$$\leq 2(1-a)^2 |A_1 \cup A_2 \cup ... \cup A_{2^n}|. \qquad (4)$$

We then apply the basic construction to S_1, $\tilde{S}_2$, ..., $\tilde{S}_{2^{n-1}}$ and then applying the basic construction again, after n-times, we obtain a figure $A_1 \cup \bar{A}_2 \cup ... \cup \bar{A}_{2^n}$ which consists of a triangle H homothetic to $A_1 \cup A_2 \cup ... \cup A_{2^n}$ of area $\alpha^{2n}|A_1 \cup A_2 \cup ... \cup A_{2^n}|$ and the excess triangles of area

$$\leq |\bigcup_{h=1}^{2^n} A_h|[2(1-\alpha)^2 + 2\alpha^2(1-\alpha)^2 + ... + 2\alpha^{2(n-1)}(1-\alpha)^2].$$

Setting $A_1 = \bar{A}_1$ we finally get

$$|\bigcup_{h=1}^{2^n} \bar{A}_h| \leq (\alpha^{2n} + 2(1-\alpha)^2 \frac{1}{1-\alpha^2})|\bigcup_{h=1}^{2^n} A_h| < [\alpha^{2n} + 2(1-\alpha)]|\bigcup_{h=1}^{2^n} A_h|.$$

Theorem 3.1.2: *Let ABC be a triangle with area H. Given any $\epsilon > 0$ we can partition the basis BC into 2^n parts I_1, I_2, ..., I_{2^n}, where n depends on ϵ, and shift the triangles T_1, T_2, ..., T_{2^n} with basis I_1, I_2, ..., I_{2^n} and common vertex A along BC to new positions $\tilde{T}_1$, $\tilde{T}_2$, ..., $\tilde{T}_{2^n}$ so that*

$$|\tilde{T}_1 \cup \tilde{T}_2 \cup ... \cup \tilde{T}_{2^n}| < \epsilon H.$$

<u>Proof:</u> This theorem is proved using the results of Theorem 3.1.1 working with a suitable affine transformation. $\square$

The set $\bigcup_{h=1}^{2^n} \tilde{T}_h$ is called a *Perron tree* and is noteworthy thanks to its additional features that we present below.

<u>Remark 1:</u> The triangles $\tilde{T}_1$, $\tilde{T}_2$, ..., $\tilde{T}_{2^n}$ of the construction of Theorem 3.1.2 have their upper vertices in reversed order with respect to their basis as shown in the figure below.

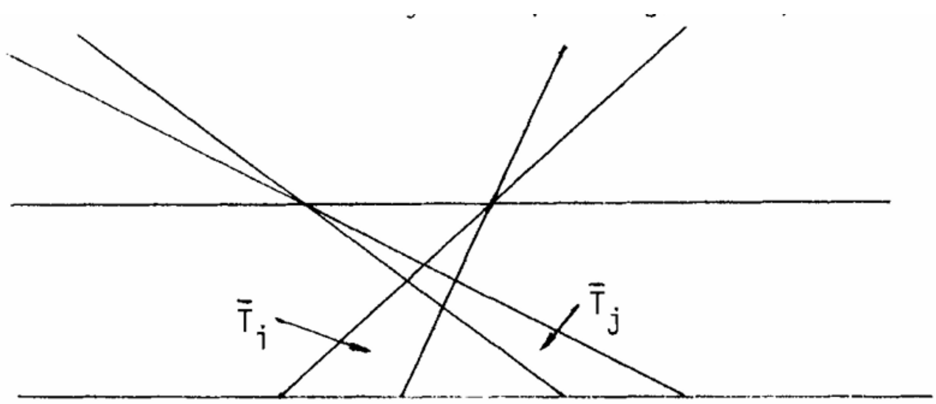


Figure R.1

As a result, if we extend them above their upper vertices the extensions are disjoint.

Remark 2: If we now extend the triangles $\tilde{T}_h$ below their basis these extensions cover at least a triangle equal to the original one ABC, on the strip parallel to the basis of width h_α as shown in the figure below, no matter of a and n in the construction of the Perron tree of Theorem 3.1.2.

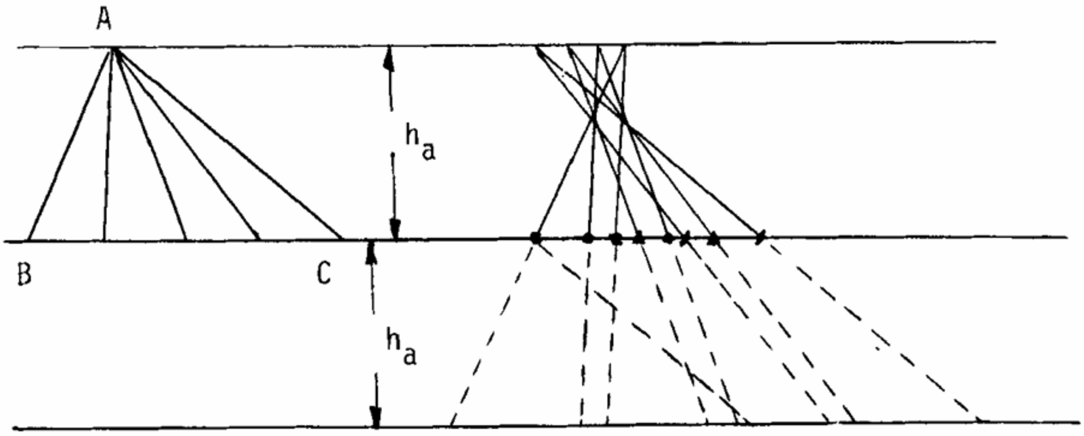


Figure R.2

Remark 3: In their final positions the upper vertices of $\tilde{T}_h$ never get further to the left of that of $\tilde{T}_1$ by more than the length of the basis of ABC.

Lemma 3.1.3 (Fefferman): *Let $\eta > 0$ be a small fixed number. There exists a measurable set $E \subseteq \mathbb{R}^2$ and a finite collection of rectangles $\{R_h\}$, which is pairwise disjoint, such that:*

a) $|E \cap \tilde{R}_h| \geq \frac{1}{100}|\tilde{R}_h|$

b)$|E| \leq \eta \sum |R_h|$

where $\tilde{R}_h$ denotes the shaded portion of the figure below.

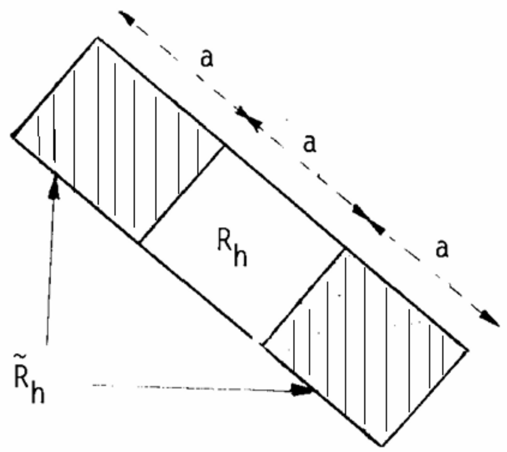


Figure 3.1.3 (a)

Proof: To prove this lemma we will combine Theorem 3.1.2. and Remarks 1 and 2. For each triangle $\tilde{T}_h$ as in Theorem 3.1.2 we perform the construction indicated in Figure 3.1.3(a) below taking as R_h the rectangle indicated and as E the Perron tree $\cup\tilde{T}_h$.

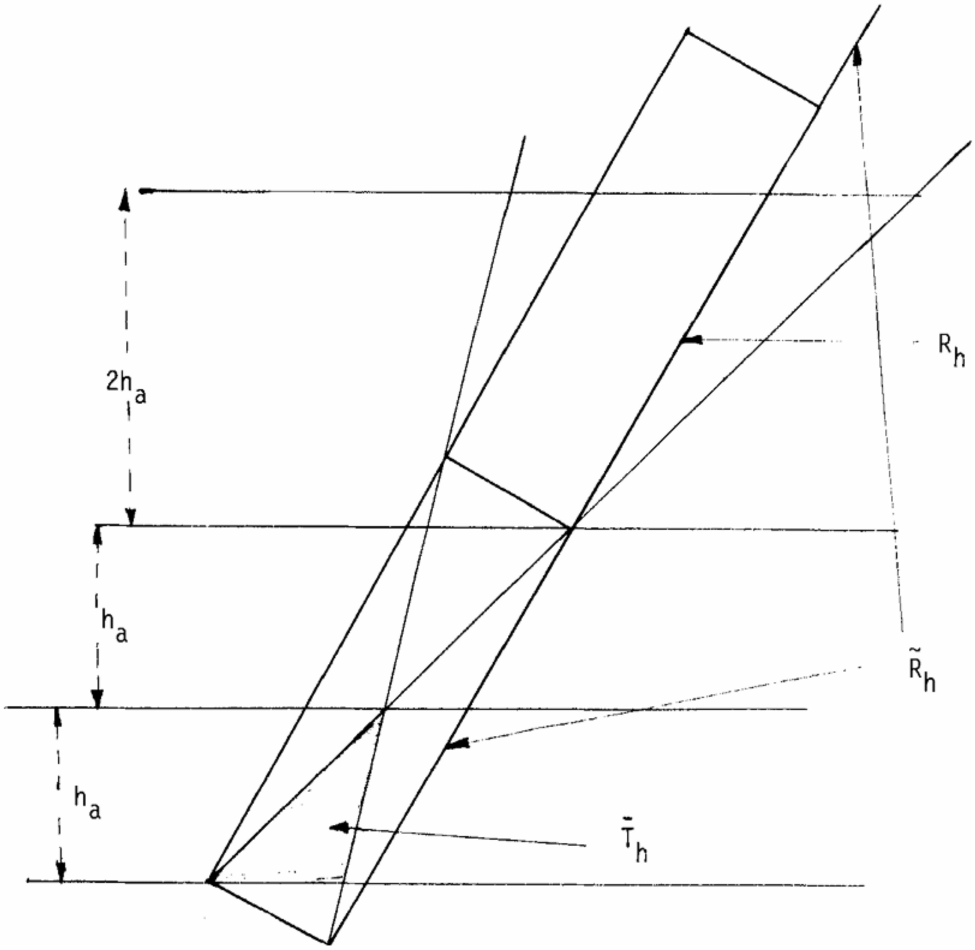


Figure 3.1.3(b)

From Remark 1 the sets R_h are disjoint as they are contained in the extension above the upper vertices of $\tilde{T}_h$.
From Remark 2 we have $\sum |R_h| \geq |ABC|$ and $|E| < \epsilon |ABC|$, $\forall \epsilon > 0$ and thus for $\epsilon = \eta$ we get $|E| \leq \eta \sum |R_h|$.

As shown on the figure below, considering a traingle $\tilde{T}_h$ such that ϕ is a acute angle, we have the following

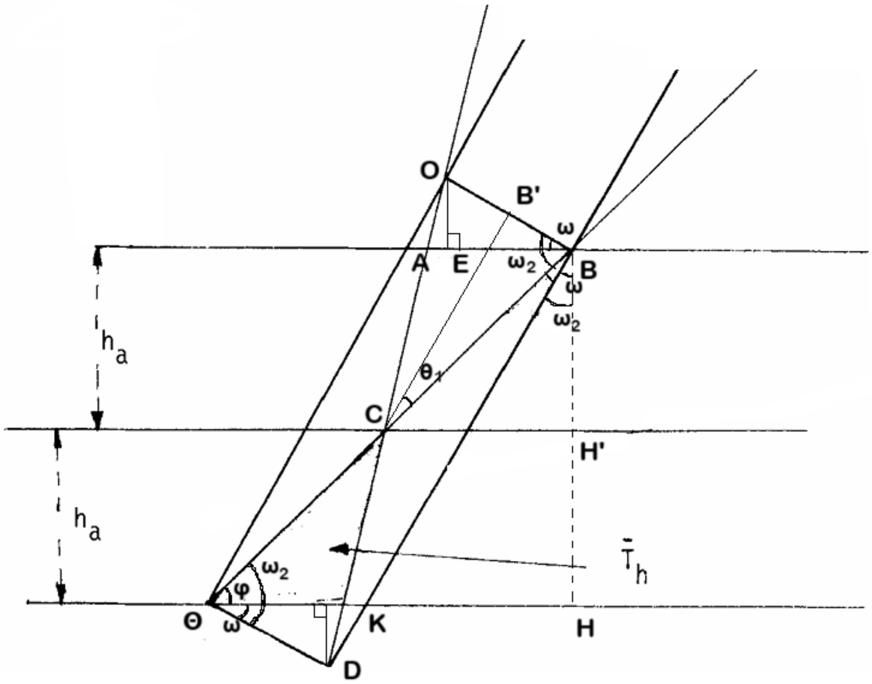


Figure 3.1.3(c)

$$|OAB| = \frac{1}{2}|OE| \cdot |AB| \qquad (1)$$

$$\frac{1}{8}|\tilde{R}_h| = |\tilde{T}_h| + |OAB| \qquad (2)$$

$$|OE| = \sin\omega |OB|, \qquad where \quad \angle OBA = \angle HBD = \omega$$

$$and \qquad \omega = \angle HBD < \omega_2 = \angle HB\Theta.$$

So

$$|OE| = \sin\omega|OB| < \sin\omega_2|OB| = \frac{|CH'|}{|CB|}2|BB'| =$$

$$= 2|CH'|\frac{|BB'|}{|CB|} = 2|CH'|\sin\theta_1 < 2|CH'| = 2|BH'|\tan(\omega_2) =$$

$$= 2h_a tan\omega_2 = 2h_a\cot\phi = 2h_a\frac{\cos\phi}{\sin\phi} \leq 2\frac{h_a}{\sin\phi} \qquad (3)$$

as

$$\tan(\omega_2) = \frac{|CH'|}{|BH'|} \quad \Rightarrow \quad |CH'| = \tan(\omega_2)|BH'| = \cot(\phi)|BH'|$$

since ω_2 , ϕ are complementary.
But ϕ is the acute angle of T_h so there exists an angle ϕ_o for every angle ϕ and for every T_h such that

$$\phi \geq \phi_o \quad \Rightarrow \quad \sin\phi \geq \sin\phi_o.$$

So from (3) we get

$$|OE| < 2\frac{h_a}{\sin\phi_o} = c_1 h_a, \qquad where \quad c_1 > 1$$

and

$$\frac{1}{8}|\tilde{R}_h| = |OAB| + |\tilde{T}_h| = \frac{1}{2}|OE|\cdot|AB| + \frac{1}{2}|AB|\cdot h_a.$$

But

$$|OE| < c_1 h_a \quad \Rightarrow \quad \frac{1}{2}|OE|\cdot|AB| < \frac{c_1}{2}|AB|h_a = c_1|\tilde{T}_h| \Rightarrow$$

$$|OAB| < c_1|\tilde{T}_h| \quad \Rightarrow \quad c_1|\tilde{T}_h| + |\tilde{T}_h| > \frac{1}{8}|\tilde{R}_h| \quad \Rightarrow$$

$$|\tilde{T}_h| > \frac{1}{8}(1+c_1)|\tilde{R}_h|. \qquad (4)$$

Furthermore $\tilde{R}_h \cap E \supseteq \tilde{T}_h$ and by (4) $\tilde{T}h$ is a good portion of $\tilde{R}_h$,

so $\forall h$

$$|\tilde{R}_h \cap E| > |\tilde{T}h| > \frac{1}{4(c_1+1)}|\tilde{R}_h| > \frac{1}{100}|\tilde{R}_h|. \;\Box$$

3.2 The needle problem

In 1917 Kakeya proposed a version of what is now called "the Kakeya problem" or "the needle problem": What is the infimum of the areas of those sets in $\mathbb{R}^2$ such that a needle of length 1 can be continuously moved within the set so that at the end it occupies the original place but in inverted position? The solution of the needle problem is also immediate by using the Perron tree.

Theorem 3.2.1: *Given $\eta > 0$ and a straight segment AB with length 1 in $\mathbb{R}^2$ we can construct a figure F of area less than η so that we can continuously move AB within F so that it finally occupies the same place but in inverted position.*

Proof: We will first show that we can continuously move a segment from one straight line to another one parallel to it sweeping out an area as small as one wishes.

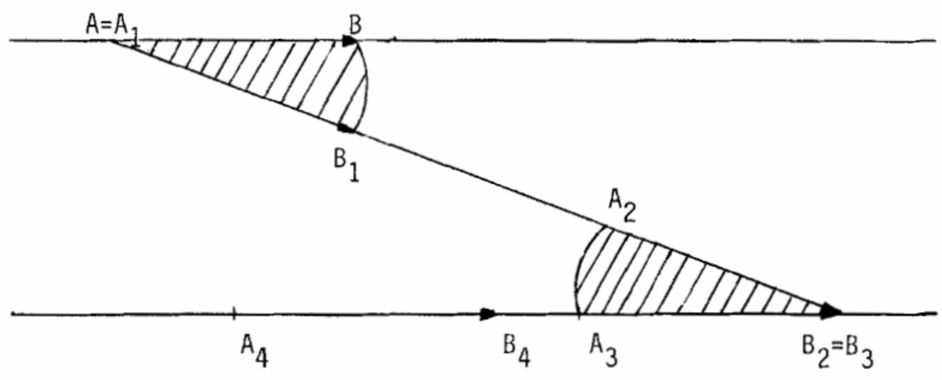


Figure 3.2.1 (a)

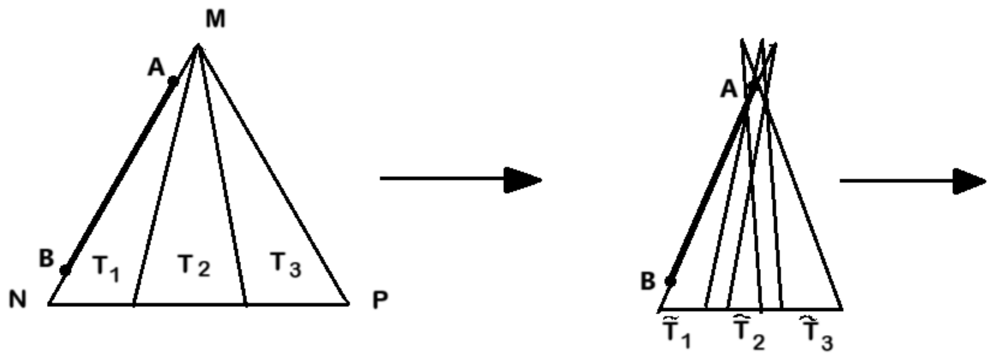
It is sufficient to observe in the figure above that we can move AB to A_4B_4 sweeping out the shaded area which can be made as small as we wish by taking AB_3 sufficiently large.

We will now show that AB can be moved to a straight line forming an angle of 60^o with its original position within a figure of area less than $\eta < \frac{1}{6}$. Repeating the process six times should get us the figure F.
Let MNP be an equilateral triangle with area 10, placed so that AB is in the interior of MN. As a result the height of MNP is bigger than 1.
We apply Theorem 3.1.2 to MNP taking as basis NP and $\epsilon > 0$ such that $10\epsilon < \dfrac{\eta}{12}$ and we obtain the triangles $\tilde{T}_1$, $\tilde{T}_2$, ..., $\tilde{T}_{2^n}$, so the area of the respective Perron tree is $< \dfrac{\eta}{12}$.

We can continuously move AB within $\tilde{T}_1$ from MN to the opposite side of $\tilde{T}_1$. From there we can move AB to the side of $\tilde{T}_2$ parallel to it sweeping an area $< \dfrac{\eta}{12 \cdot 2^n}$.

Now we move it again within $\tilde{T}_2$ and so on.

The whole process can be seen in the figure below.



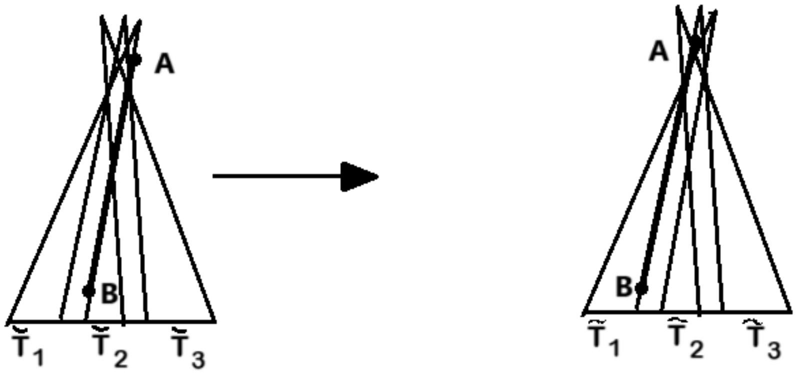


Figure 3.2.1(b)

The area swept out in this process is less than $\dfrac{\eta}{6}$ and the needle is at MP, that is at the end of a line forming an angle of 60^o with its original position.

As we mentioned before, repeating the same process six times we will have swept out an area less than η and the needle will be in the same place but in inverted position. $\square$

3.3 The Besicovitch set

In 1918 Besicovitch, in his attempt to answer a question concerning the Riemann integral, constructed a compact set in $\mathbb{R}^2$ of null measure containing a segment of length one in each direction. Such type of sets are called Besicovitch sets.

We now present a lemma that is needed to prove the existence of Besicovitch sets.

Lemma 3.3.1: *Given a closed parallelogram P of sides a, b, c, d and $\eta > 0$ there is a finite collection of closed parallelograms $\Omega = \{\omega_1, \omega_2, ..., \omega_H\}$ with one side on a and another on c such that:*

1. $|\cup \omega_j| \leq \eta$
2. *Each segment joining a point of a to another point of c admits a parallel translation that carries it to $\cup\omega_j$.*

Proof: We start by taking two stripes $\omega_1 = ASTD$ and $\omega_2 = DLBT$ such that $|\omega_1 \cup \omega_2| \leq \frac{\eta}{4}$, as shown on Figure 3.3.1(a).

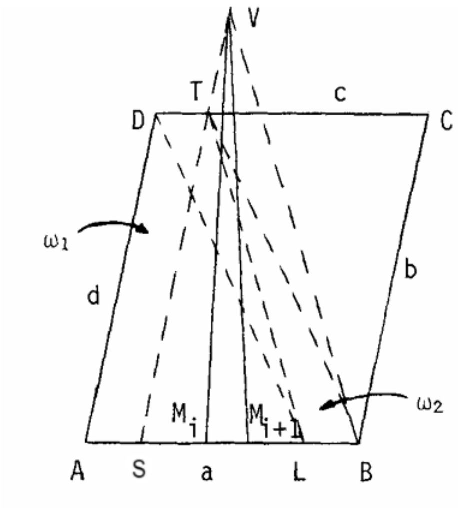


Figure 3.3.1(a)

We take the point V as the point where the extension of ST intersects with the parallel to LT from B. We divide SB into a finite number of equal segments with length smaller than AS and we then join V with these dividing points M_i and consider the triangles VM_iM_{i+1}.
To each of them we apply the construction of the Perron tree of Theorem 3.1.2 with ϵ such that the area of the union of all the Perron trees obtained is $< \dfrac{\eta}{4}$.
By Remark 3 we know that the upper vertices of the small triangles obtained in these Perron trees never go to the left of d.
We now repeat the same process starting from the side BC taking two stripes ω_3 and ω_4 and so on.
Finally we substitute each one of the intersections of P with the small triangles of such Perron tress by the corresponding parallelograms that are contained in these small triangles, as required in the statement of the theorem and as shown in Figure 3.3.1(b).

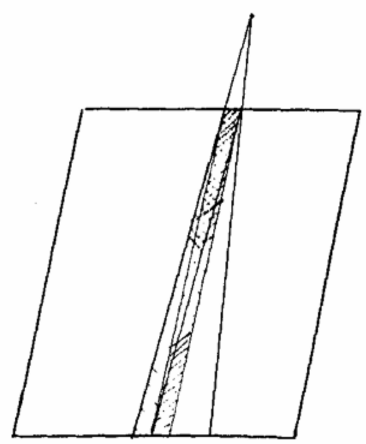


Figure 3.3.1 (b)

Each union of the Perron trees has area $< \dfrac{\eta}{4}$ and

$$|\omega_1 \cup \omega_2| \leq \frac{\eta}{4}\ , \quad |\omega_3 \cup \omega_4| \leq \frac{\eta}{4}$$

so $| \cup \omega_j| \leq \eta$, as required. $\Box$

The next theorem shows how we can obtain a Besicovitch set from the Perron tree of Theorem 3.1.2.

Theorem 3.3.2: *There exists a compact set F in $\mathbb{R}^2$ of null measure containing a segment of unit length in each direction.*

Proof: To prove this it is sufficient to construct a compact null set F that contains a segment of unit length in each direction of an angle of 45^o.
We apply the previous lemma to the closed unit square $Q = ABCD$ with $\eta_1 = \frac{1}{2}$ obtaining $\{\omega_1, \omega_2, ..., \omega_{H_1^1}\}$. The set $L_1 = \bigcup_{j=1}^{H_1^1} \omega_j$ is compact and its area is $\leq \frac{1}{2}$, is contained on Q and containing segments of unit length in each direction of the angle $\angle ACB$ of 45^o.

Now we apply the lemma to each ω_j with η_2 so small that $H_1^1 \eta_2 \leq \dfrac{1}{2^2}$ and we obtain $\{\omega(j,1), \omega(j,2), ..., \omega(j, H_j^2)\}$.

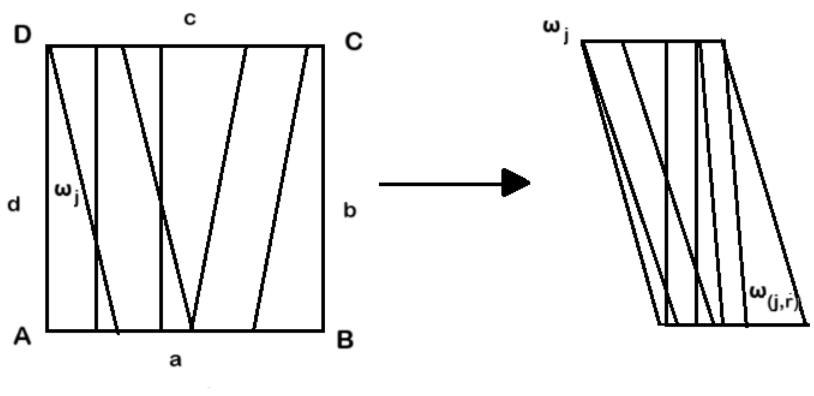


Figure 3.3.2(a)

The set $L_2 = \bigcup_{j=1}^{H_1^1} \bigcup_{r=1}^{H_j^2} \omega(j,r)$ is compact, contained in L_1, it's area is $\leq \frac{1}{2^2}$ and contains segments of unit length in each direction of the angle of 45^o.
Continuing in this way we get the set $F = \bigcap_{j=1}^{\infty} L_j$ which is a compact null set containing segments in each direction of $\angle ACB$ as required. Indeed :

Fix $E \in AB$.
From the construction of L_j, $\forall j$, there exists $[a_j, c_j] \subseteq L_j$ with $c_j \in CD$ and $a_j \in AB$, such that $[a_j, c_j] \parallel CE$.

By using the Bolzano theorem we can assume without loss of generality that there exist $a' \in AB$ and $c' \in CD$ so that $a_j \to a'$ and $c_j \to c'$, that is $[a_j, c_j] \to [a', c'] \parallel CE$ from before.

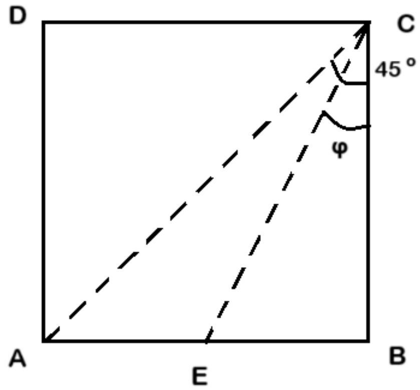


Figure 3.3.2(b)

Let $j_o \in N$, $j_o > 1$, be a fixed number. Then $\forall k \geq j_o$ we have $[a_k, c_k] \subseteq L_k \subseteq L_{j_0}$ and $([a_k, c_k]) \to [a', c']$ and since L_{j_o} is compact we get $[a', c'] \subseteq L_{j_o}$.

So $[a', c'] \subseteq \bigcap_{j_o=1}^{\infty} L_{j_o} = F$ where $[a', c'] \parallel CE$ $\square$.

3.4 Differentiation properties of the basis of rectangles

In the next theorems we will present some differentiation properties of some bases of rectangles. Considering what we have proved so far we will now see some differentiation properties of $\mathbb{B}_3$ and some of its subbases.

Theorem 3.4.1 *Consider the B-F differentiation basis $\mathbb{B}_T$ invariant by homothecies generated by all the triangles $\{T_i\}_{i=0}^{2^n}$, where T_i has basis the dyadic interval $(\frac{i}{2^n}, \frac{i+1}{2^n})$, $i = 0, 1, ..., 2^n - 1$, $\forall n \in \mathbb{N}$. Then $\mathbb{B}_T$ is not a density basis.*

Proof: Assume M_T is the maximal operator associated to $\mathbb{B}_T$ and $E = \cup T_i$ is a Perron tree constructed from $\{T_i\}_{i=0}^{2^n}$ as in Theorem 3.1.1 and let $\tilde{T_i}$ be the extension of T_i below the basis up to y=-1.

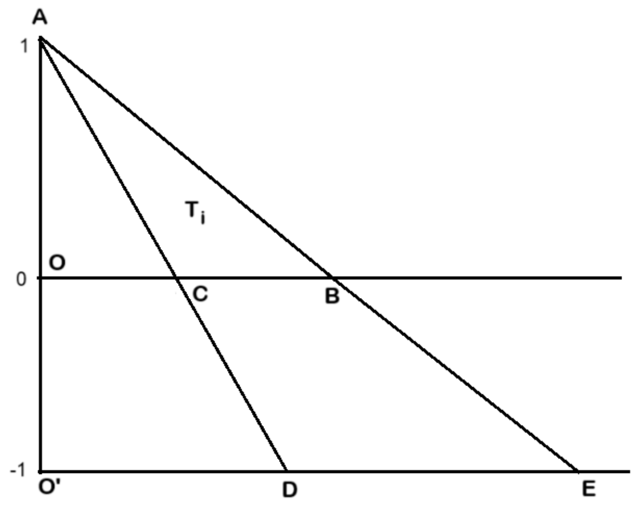


Figure 3.4.1(a)

For $\tilde{T}_i \in \mathbb{B}_T$, we have

$$\frac{|\tilde{T}_i \cap E|}{|\tilde{T}_i|} = \frac{|T_i|}{|\tilde{T}_i|} > \frac{1}{8} = c,$$

as

$$\frac{|T_i|}{|\tilde{T}_i|} = \frac{\frac{BC}{2}}{\frac{2 \cdot DE}{2}} = \frac{1}{2}\frac{BC}{DE} = \frac{1}{2}\frac{AC}{AD}$$

since T_i similar to $\tilde{T}_i$ and

$$\frac{1}{2}\frac{AC}{AD} = \frac{1}{2}\frac{AO}{AO'} = \frac{1}{2}\frac{1}{2} = \frac{1}{4} > \frac{1}{8}.$$

That gives us

$$\cup \tilde{T}_i \subseteq \{M_T(\chi_E) > \frac{1}{8}\}.$$

By Remark 2, $|\cup \tilde{T}_i| \geq |ABC|$ so

$$|\{x : M_T(\chi_E)(x) > \frac{1}{8}\}| \geq |ABC| = \frac{|ABC|}{|E|}|E|$$

and by the Busemann-Feller criterion for density bases, $\mathbb{B}_T$ cannot be a density basis, since $\dfrac{|ABC|}{|E|}$ can be arbitary large, for suitable E. $\Box$

Now for every T_i of the previous theorem consider R_i the rectangle shown in the figure below and let $\mathbb{B}_R$ be the B-F basis invariant by homothecies, generated by $\{R_i\}$.

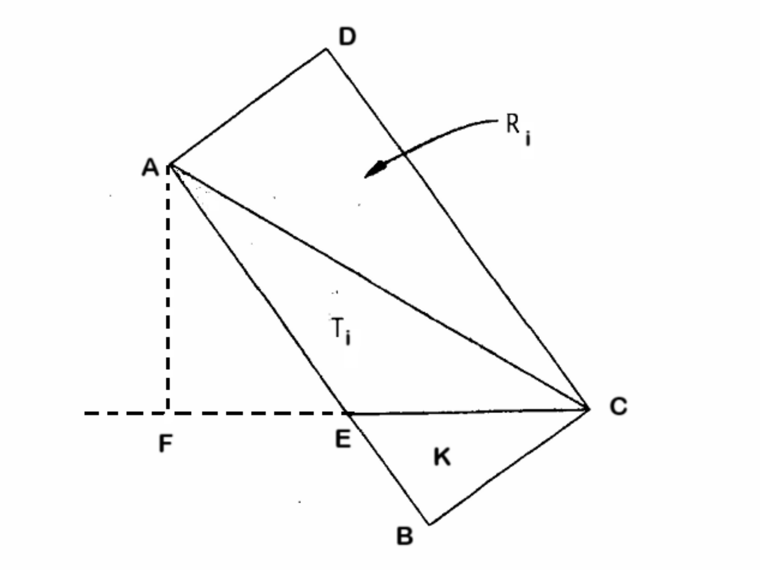


Figure 3.4.1 (b)

If M_R is the corresponding maximal operator then, for $x \in T_i \subseteq R_i$

$$M_T f(x) = \frac{1}{|T_i|}\int_{T_i}|f| \leq \frac{1}{|T_i|}\int_{R_i}|f| = \frac{|R_i|}{|T_i|}(\frac{1}{|R_i|}\int_{R_i}|f|) \leq \frac{|R_i|}{|T_i|}M_R f(x)$$

and

$$|T_i| + |K| = \frac{1}{2}|R_i|$$

where

$$|K| = \frac{1}{2}|BC||EB| \quad and \quad |T_i| = \frac{1}{2}|AE||BC|$$

so

$$\frac{|K|}{|T_i|} = \frac{|EB|}{|AE|} < 1$$

as $|AE| > 1$ and $|EC| < 1$, so $|EB| < 1$ by the Pythagorian theorem.
Thus $|R_i| < 4|T_i|$ so finally $\dfrac{|R_i|}{|T_i|} < c' = 4.$

Therefore we have $M_T f(x) \leq c' \cdot M_R f(x)$ and using the result proved in Theorem 3.4.1 we get

$$|\{x : M_R\chi_E > \frac{1}{8}c'\}| \geq |\{x : M_T\chi_E > \frac{1}{8}c' \cdot \frac{1}{c'}\}| = |\{x : M_T\chi_E > \frac{1}{8}\}| \geq \frac{|ABC|}{|E|}|E|$$

concluding that $\mathbb{B}_3$ is not a density basis.

We will now see some results concerning bases of rectangles in lacunary directions.

Let ϕ be the set of directions $\phi = \{\frac{\pi}{2^2}, \frac{\pi}{2^3}, \frac{\pi}{2^4}, ...\}$ and consider the basis $\mathbb{B}_\phi$ of rectangles with one side in one of those directions. We want to examine the differentiation properties of the basis $\mathbb{B}_\phi$.
Stromberg [1976] proved that B_ϕ differentiates $L^2(log^+L)^{4+\epsilon}(\mathbb{R}^2)$ for every $\epsilon > 0$ and Cordoba and R. Fefferman [1977] proved that $\mathbb{B}_\phi$ differentiates L^2, which is equivalent by Theorem 0.12 (as $\mathbb{B}_\phi$ is invariant by homothecies) to the fact that the maximal operator M_ϕ associated to $\mathbb{B}_\phi$ is of weak type (2,2).

In this chapter we will examine the method of Cordoba and R. Fefferman with a modified version of $\mathbb{B}_\phi$ which is easier to handle. We can obtain the same covering theorem and the weak type (2,2) for the corresponding maximal operator for the above basis from this result.

Let $\mathbb{B}_\phi$ be the basis of all parallelograms R such that:

i) Two of their sides are parallel to Oy.

ii) The other pair of sides have one of the directions $\phi = \{\frac{\pi}{2^2}, \frac{\pi}{2^3}, \frac{\pi}{2^4}, ...\}$.

iii)The projection p(R) of R over Ox is a dyadic interval.

iv) Each R is so thin that if $\tilde{R}$ is the minimal interval containing R then $\frac{|R|}{|\tilde{R}|} \leq \frac{1}{8}$.

Theorem 3.4.2: *Let $\{B_\alpha\}_{\alpha\in A}$ be any collection of open parallelograms of the basis $\mathbb{B}_\phi$ defined above with $|\bigcup_\alpha B_\alpha| < \infty$. Then we can derive a finite sequence $\{R_1, R_2, ..., R_H\}$ from $\{B_\alpha\}_{\alpha\in A}$ such that*

1. $|\bigcup_{\alpha\in A} B_a| \leq c|\bigcup_{j=1}^H R_j|$
2. $\int_{\mathbb{R}^2} (\sum \chi_{R_j})^2 \leq c|\bigcup_{j=1}^H R_j|$

where the constant c is a positive absolute constant not depending on the collection $\{B_\alpha\}_{\alpha\in A}$.

Proof: First, we select a finite sequence $\{B_1, B_2, ..., B_N\}$ from $\{B_\alpha\}_{\alpha\in A}$ so that $|\cup B_\alpha| \leq 2|\bigcup\limits_{k=1}^{N} B_k|$.
By Lindelof's theorem the set $\bigcup_{\alpha\in A} B_\alpha$ can be written as a countable union $\bigcup\limits_{\alpha\in A} B_\alpha = \bigcup\limits_{i\in I} B_{\alpha_i}$, $I \subseteq A$, countable. Then we can define a new sequence $\{F_{\alpha_i}\}$ where $F_{\alpha_i} = B_{\alpha_1} \cup B_{\alpha_2} \cup ... \cup B_{\alpha_i}$ from which we get $|\bigcup\limits_{\alpha\in A} B_\alpha| = |\bigcup\limits_{i\in I} B_{\alpha_i}| = |\bigcup\limits_{i\in I} F_{\alpha_i}|$ where $\{F_{\alpha_i}\}$ is increasing and $|\bigcup\limits_{i=1}^{\infty} F_{\alpha_i}| = \lim\limits_{i\to\infty} \mu(F_{\alpha_i})$ so there exists an $i_o \in N$ such that $|F_{\alpha_{i_o}}| > \frac{1}{2}|\bigcup\limits_{i=1}^{\infty} F_{\alpha_i}|$ and we finally deduce that $|\bigcup\limits_{j=1}^{i_o} B_{\alpha_j}| > \frac{1}{2}|\bigcup\limits_{\alpha} B_\alpha|$.
We assume that $B_1, B_2, ..., B_N$ have been ordered so that $b(B_j)$ = length of projection of B_j over Ox $\geq$ length of projection of B_{j+1} over Ox $= b(B_{j+1})$
and also no B_j is contained in another one.

We begin the construction of $\{R_j\}_{j=1}^H$ by setting $R_1 = B_1$.
If

$$|B_2 \cap R_1| = \int \chi_{B_2}\chi_{R_1} \leq \frac{1}{2}|B_2|$$

we set $R_2 = B_2$, else we leave B_2 aside.

Assume $R_1 = B_1$, $R_2 = B_2$. If

$$\sum_{i=j}^{2} |B_3 \cap R_j| = \int \chi_{B_3} \sum_{j=1}^{2} \chi_{R_j} \leq \frac{1}{2}|B_3|$$

we set $R_3 = B_3$, otherwise we leave B_3 aside.
Assume $R_1 = B_1$, $R_2 = B_2$ and B_3 has not been chosen.
We examine B_4. If

$$\int \chi_{B_4}(\chi_{R_1} + \chi_{R_2}) \leq \frac{1}{2}|B_4|$$

then we set $R_3 = B_4$, otherwise we leave B_4 aside.
And so on.

By this way we can finish our construction in a finite number of steps obtaining $\{R_j\}_{j=1}^{H}$ that satisfies:

a) $b(R_j) \geq b(R_{j+1})$

b) $\int \chi_{R_{h+1}} \sum_{j=1}^{h} \chi_{R_j} \leq \frac{1}{2}|R_{h+1}|$, $\forall h = 1, 2, ..., H-1$

c) If B_i has not been chosen in the selection process, then

$$\int \chi_{B_i}\Big(\sum_{b(R_j)\geq b(B_i)} \chi_{R_j}\Big) > \frac{1}{2}|B_i|$$

Using (b) we get

$$\int \Big(\sum_{j=1}^{H} \chi_{R_j}\Big)^2 = \int \sum_{j=1}^{H} \chi_{R_j} + 2\sum_{k=1}^{H} \int \Big(\sum_{j<k} \chi_{R_j}\Big)\chi_{R_k} \leq$$

$$\leq \sum_{j=1}^{H} |R_j| + 2\,\frac{1}{2} \sum_{j=1}^{H} |R_j| = 2\sum_{j=1}^{H} |R_j|.$$

We have

$$\sum_{j=1}^{H} |R_j| = |R_1| + |R_2| + ... + |R_H| =$$

$$= |R_1| + |R_2 - R_1| + |R_2 \cap R_1| + |R_3 - (R_1 \cup R_2)| + |R_3 \cap (R_1 \cup R_2)| + ... =$$

$$= |\cup R_j| + |R_2 \cap R_1| + |R_3 \cap (R_2 \cup R_1)| + \ldots =$$

$$= |\cup R_j| + \sum_{k=2}^{H} \int \chi_{R_k} (\sum_{j=1}^{k-1} \chi_{R_j}) \leq |\cup R_j| + \frac{1}{2} \sum |R_j|$$

from (b) again.

Therefore

$$\sum |R_j| \leq 2 |\cup R_j| \qquad (*)$$

and finally

$$\int (\sum_{j=1}^{H} \chi_{R_j})^2 \leq 4 |\cup_{j=1}^{H} R_j|$$

getting us (2) for a suitable absolute constant C.

We will now try to prove (1) using (c), (iv) and the lacunarity of $\{B_\alpha\}_{\alpha \in A}$.
Let (a,b) be the projection of B_i over Ox.
If B_i has not been chosen we get from (c)

$$\sum_{b(R_j) \geq b(B_i)} \frac{|R_j \cap B_i|}{|B_i|} > \frac{1}{2}.$$

Therefore we obviously have either

$$(A) \qquad \sum_{\substack{b(R_j) \geq b(B_i) \\ d(R_j) = d(B_i)}} \frac{|R_j \cap B_i|}{|B_i|} > \frac{1}{6}\ , \qquad \textit{is true}$$

or

$$(B) \qquad \sum_{\substack{b(R_j) \geq b(B_i) \\ d(R_j) > d(B_i)}} \frac{|R_j \cap B_i|}{|B_i|} > \frac{1}{6}\ , \qquad \textit{is true}$$

or

$$(C) \qquad \sum_{\substack{b(R_j) \geq b(B_i) \\ d(R_j) < d(B_i)}} \frac{|R_j \cap B_i|}{|B_i|} > \frac{1}{6}\ , \qquad \textit{is true}\ ,$$

where $d(R_j)$, $d(B_i)$ are the directions of R_j and B_i respectively.

The sets B_i for which (A) is true are in the situation of the figure

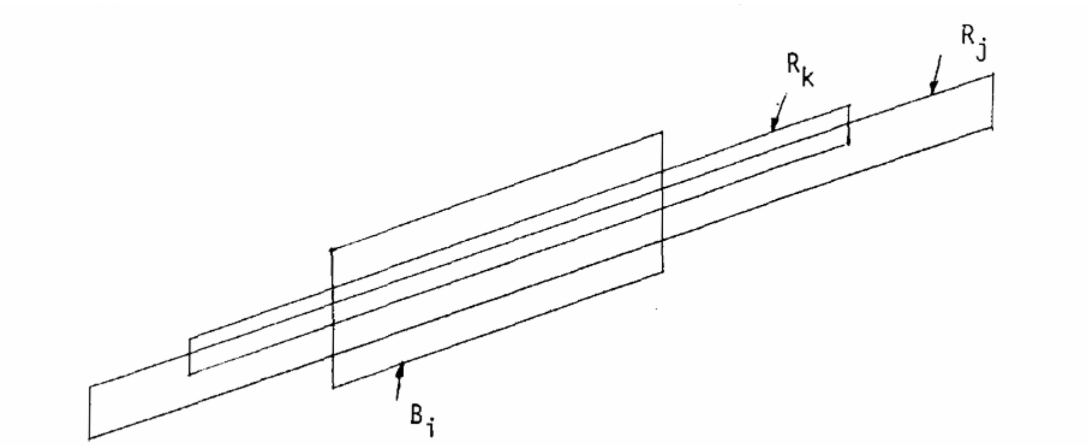


Figure 3.4.2 (a)

By intersecting by a vertical line l, $x = \lambda$, and calling M_l the unidimensional maximal operator with respect to intervals of l=l(λ) we get

$$l(\lambda) \cap B_i \subseteq \{(\lambda, y) : M_l(\sum \chi_{R_j})(\lambda, y) > \frac{1}{6}\} \qquad (**)$$

where the sum is over those R_j that are described in case (A).

Suppose this is not true, so there exists a $(\lambda, x) \in l \cap B_i$ for which

$$M_l(\sum \chi_{R_j}(\lambda, x)) \leq \frac{1}{6}.$$

Then

$$\frac{1}{|l \cap B_i|} \int_{l \cap B_i} \sum \chi_{R_j}(\lambda, t) dt \leq \frac{1}{6} \ ,$$

where λ ranges over the projection of B_i to the Ox axis, say $x \in (a,b)$. Thus by integrating the previous inequality over (α, b) we get

$$\int_{\lambda=a}^{b} \int_{B_i \cap l(\lambda)} \sum \chi_{R_j}(\lambda,t) dt d\lambda \leq \frac{1}{6} \int_a^b |B_i \cap l(\lambda)| d\lambda$$

so

$$\int_{B_i} \sum_j \chi_{R_j} d\lambda_2 \leq \frac{1}{6}|B_i| \Rightarrow$$

$$\sum_j |R_j \cap B_i| \leq \frac{1}{6}|B_i|$$

a contradiction to (A).

By (**) we have

$$l(\lambda) \cap (\cup B_i) \subseteq \{(\lambda, y) : M_l(\sum_{j=1}^{H} \chi_{R_j})(\lambda, y) > \frac{1}{6}\} =: \Delta_\lambda$$

where the union is over those B_i for which (A) is true and

$$|\Delta_\lambda| \leq 6c \int \sum_{j=1}^{H} \chi_{R_j}(\lambda, t) dt$$

$\forall \lambda$ such that the line $l(\lambda)$ intersects the set $\cup B_i$, so

$$|\cup B_i| \leq \int_\lambda |\Delta_\lambda| d\lambda \leq 6c \int_\lambda \int_t \sum_{j=1}^{H} \chi_{R_j}(\lambda,t) dt d\lambda \leq 6c \sum |R_j| \leq_{(*)} c|\cup R_j|.$$

Consider now a set for which (B) is true.
We choose a R_j such that $d(R_j) > d(B_i)$ and $b(R_j) \geq b(B_i)$ and draw the minimal closed interval $\tilde{B}_i$ containing B_i.
We shall prove

$$\frac{|\tilde{B}_i \cap R_j|}{|\tilde{B}_i|} \geq c \frac{|B_i \cap R_j|}{|B_i|} \qquad (***)$$

so we will then get

$$\sum_j \frac{|\tilde{B}_i \cap R_j|}{|\tilde{B}_i|} > \frac{c}{6}$$

and therefore similarly as before

$$B_i \subseteq \tilde{B}_i \subseteq \{x : M_2(\sum \chi_{R_j})(x) > \frac{c}{6}\}$$

from which entails that the union of those B_i satisfying (B) has area less than

$$|\{x : M_2(\sum \chi_{R_j})(x) > \frac{c}{6}\}| \leq \frac{1}{6} \int (\sum \chi_{R_j})^2 \leq c|\cup R_j|$$

where for the first inequality we used the weak (2,2) inequality for M_2, as we have proved in the previous chapter that the basis $\mathbb{B}_2$ differentiates $L^1(1 + log^+ L^1)(\mathbb{R}^2)$, and $L^2(\mathbb{R}^2) \subseteq L^1(1 + log^+ L^1(\mathbb{R}^2)$ and for the second we used (2).

To prove (***) we consider the following figure

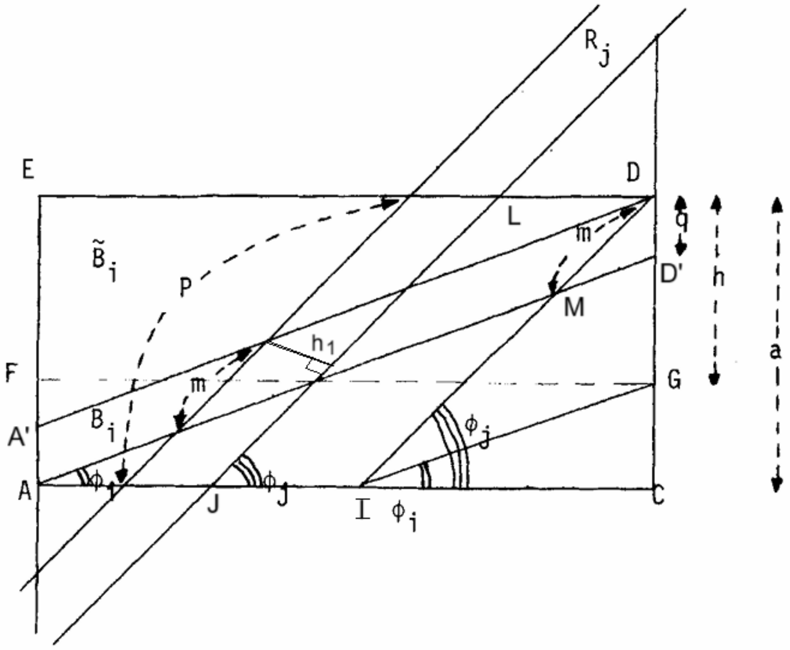


Figure 3.4.2(b)

where $ACDE = \tilde{B}_i$, $ID \parallel JL$, $IG \parallel AD'$.

For every fixed R_j as in (B) , we have

$$\frac{|R_j \cap B_i|}{|R_j \cap \tilde{B}_i|} =_{(1)} \frac{m}{p} =_{(2)} \frac{q}{h} =_{(3)} \frac{|B_i|}{|EDGF|} = \frac{|B_i|}{|\tilde{B}_i|}\frac{|\tilde{B}_i|}{|EDGF|} = \frac{|B_i|}{|\tilde{B}_i|}\frac{a}{h}$$

where (1) entails from $|R_j \cap B_i| = mh_1$ and $|R_j \cap \tilde{B}_i| = ph_1$

(2) from IDG and MDD' being similar triangles

and (3) from $\dfrac{|B_i|}{|EDGF|} = \dfrac{q \cdot |AC|}{|EF| \cdot |AC|} = \dfrac{q}{h}$

and because of the lacunarity of ϕ we have $\frac{a}{h} \leq c$ as

$$\phi_j \geq 2\phi_i \Rightarrow sin(\phi_j) \geq sin(2\phi_i) = 2sin(\phi_i) \cdot cos(\phi_i) \Rightarrow$$
$$\frac{a}{|DI|} \geq 2\frac{a-h}{|IG|}cos(\phi_i) \Rightarrow$$
$$\frac{a}{|DI|} \cdot 2\frac{a-h}{|IG|}cos(\phi_i) \Rightarrow$$
$$\frac{a}{h} \geq \frac{|DI|}{|IG|} \geq 2 \cdot (\frac{a}{h} - 1) \cdot cos(\phi_i) \geq \sqrt{2} \cdot (\frac{a}{h} - 1)$$

where $\phi_i \leq \dfrac{\pi}{4}$ so $cos(\phi_i) \geq \frac{\sqrt{2}}{2}$ and $DI \geq IG$ thus $\dfrac{a}{h} < \dfrac{\sqrt{2}}{\sqrt{2}-1} = c.$

That proves (***) $\dfrac{|R_j \cap B_i|}{|B_i|} \leq c \cdot \dfrac{|R_j \cap \tilde{B}_i|}{|\tilde{B}_i|}.$

For a set that satisfies (C) a similar consideration holds.

Therefore we finally obtain $| \cup B_a| \leq c| \cup R_j|$. $\square$

Colloraly 3.4.3 : *The maximal operator $M_{\mathbb{B}_\phi}$ corresponding to $\mathbb{B}_\phi$ is of weak type (2,2).*

<u>Proof:</u> Let $f \in L^2(\mathbb{R}^2)$ and $A = \{M_{\mathbb{B}_\phi} f > \lambda\} > 0$.
Let K be any compact subset of A and $x \in K$. Then there exists $R_x \in \mathbb{B}_\phi$, $x \in R_x$ such that

$$\frac{1}{|R_x|}\int_{R_x} |f| > \lambda. \qquad (*)$$

Since $K \subseteq \cup_{x\in K} R_x$ is compact there exist $x_1, x_2, ..., x_n$ such that $K \subseteq R_{x_1} \cup R_{x_2} \cup ... \cup R_{x_n}$. Then we apply Theorem 3.4.2 to $\{R_x\}_{x\in K}$ and obtain $\{R_j\}_{j=1}^{H}$ such that
$|\bigcup_{x\in K} R_x| \leq c|\bigcup_j R_j|$.

Then $|K| \leq |\bigcup_{i=1}^{n} R_{x_i}| \leq c|\bigcup_j R_j|$ using (1) and so

$$|K| \leq c|\bigcup R_j| \leq c\sum |R_j| \leq_{(*)} \frac{c}{\lambda}\int |f|\sum \chi_{R_j}$$

which using Holder's inequality gives

$$c|\cup R_j| \leq \frac{c}{\lambda}||f||_{L^2}(\int (\sum \chi_{R_j})^2)^{\frac{1}{2}} \leq_{(2)} \frac{c}{\lambda}||f||_{L^2}|\bigcup R_j|^{\frac{1}{2}}.$$

Therefore

$$|\bigcup R_j| \leq \frac{1}{\lambda}||f||_{L^2}|\bigcup R_j|^{\frac{1}{2}} \Rightarrow |\bigcup R_j| \leq \frac{1}{\lambda^2}||f||^2_{L^2}$$

and so

$$|K| \leq \frac{c}{\lambda^2}||f||^2_{L^2}\ , \quad \forall K \subseteq A \quad \textit{compact}.$$

Thus

$$|A| = |\{M_{\mathbb{B}_\phi} f > \lambda\}| \leq \frac{c}{\lambda^2}||f||^2_{L^2}$$

which gives us the weak (2,2) type inequality. $\Box$

Chapter 4

The Decreasing Rearrangement

In our 4^{th} chapter we will examine the decreasing rearrangement of a given function f proving some useful results that we will utilize in the 5^{th} and last chapter.

We give the following definition: The non-negative measurable functions f and g will be rearrangement of one another or *equimeasurable* if their distribution function coincide.

Note that this notion allows equimeasurability to be defined for functions defined in different measure spaces.
For every measurable function f we can construct a decreasing right continuous f^* on $(0, \infty)$ that is equimeasurable with f.
The function f^* is called the *decreasing rearrangement* of f and constructing f^* from f is analogous to rearranging the terms of a finite sequence in decreasing order.

From now on (R, μ) denotes a totally σ- finite measure space and M_o is the class of finite μ - almost everywhere functions.

4.1 The distribution function

Definition 4.1.1 : The distribution function μ_f of a function f in $M_o = M_o(R, \mu)$ is given by

$$\mu_f(\lambda) = \mu(\{x \in R : |f(x)| > \lambda\}), \quad \lambda \geq 0 \ .$$

Note that μ_f depends only on the absolute value $|f|$ and may be ∞.

Definition 4.1.2: Two functions $f \in M_o(R, \mu)$ and $g \in M_o(S, \nu)$ are called equimeasurable if they have the same distribution function, that is if

$$\mu_f(\lambda) = \nu_g(\lambda), \qquad \forall \lambda \geq 0.$$

Proposition 4.1.3: *Let $f, g, f_n \in M_o(R, \mu)$, $n = 1, 2, ...$ and $a \neq 0$. The distribution function μ_f is non-negative, decreasing and right continuous on* $[0, \infty)$. *Also,*

$$a)\ |g| \leq |f| \ a.e \ \Rightarrow \mu_g \leq \mu_f$$

$$b)\ \mu_{af}(\lambda) = \mu_f(\frac{\lambda}{|a|}), \qquad \lambda \geq 0$$

$$c)\ \mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2)$$

$$d)\ |f| \leq \liminf_{n\to\infty} |f_n| \quad \mu - a.e \quad \Rightarrow \quad \mu_f \leq \liminf_{n\to\infty} \mu_{f_n}$$

more specifically,

$$|f_n| \nearrow |f| \quad \mu - a.e \quad \Rightarrow \quad \mu_{f_n} \nearrow \mu_f$$

Proof: It is obvious that μ_f is non-negative and decreasing.
To prove the right continuity we define $E(\lambda) = \{x : |f(x)| > \lambda\}$, $\lambda \geq 0$, and fix $\lambda_o \geq 0$.
For $\lambda_1 > \lambda_2$ we have $E(\lambda_1) \subseteq E(\lambda_2)$ and

$$E(\lambda_o) = \bigcup_{\lambda > \lambda_o} E(\lambda) = \bigcup_{n=1}^{\infty} E(\lambda_n), \quad \forall (\lambda_n)_n \quad satisfying\ \lambda_n > \lambda_o\ and\ \lambda_n \searrow \lambda_o.$$

Thus from the monotone convergence theorem we get

$$\mu_f(\lambda_n) = \mu(E(\lambda_n)) \to \mu(E(\lambda_o)) = \mu_f(\lambda_o).$$

To prove (a) assume $|g| \leq |f|$. Then $\{|g(x)| > \lambda\} \subseteq \{|f(x)| > \lambda\}$
$\Rightarrow \mu_g(\lambda) \leq \mu_f(\lambda)$.

For (b) : $\mu_{af}(\lambda) = \mu(\{x : |af(x)| > \lambda\}) = \mu(\{x : |f(x)| > \frac{\lambda}{|a|}\}) = \mu_f(\frac{\lambda}{|a|})$

In order to prove (c) assume $|f(x) + g(x)| > \lambda_1 + \lambda_2$. Then either $|f(x)| > \lambda_1$ or $|g(x)| > \lambda_2$ so

$$\mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2).$$

Moving on to (d), assume $|f| \leq \liminf_{n\to\infty} |f_n| \quad \mu - a.e$ and let $\lambda \geq 0$ and $E = \{x : |f(x)| > \lambda\}$, $E_n = \{x : |f_n(x)| > \lambda\}$, $n = 1, 2, ...$

Clearly $E \subseteq \bigcup\limits_{m=1}^{\infty} \bigcap\limits_{n>m} E_n$,
so

$$\mu(\bigcap_{n>m} E_n) \leq \inf_{n>m} \mu(E_n) \leq \liminf_{n\to\infty} \mu(E_n), \quad \forall m = 1, 2, ... \ (*)$$

Moreover $\bigcap\limits_{n>m} E_n$ is increasing with m so using the monotone convergence theorem we get

$$\mu(E) \leq \mu(\bigcup_{m=1}^{\infty} \bigcap_{n>m} E_n) = \lim_{m\to\infty} \mu(\bigcap_{n>m} E_n) \leq_{(*)} \liminf_{n\to\infty} \mu(E_n),$$

that is

$$\mu_f \leq \liminf_{n\to\infty} \mu_{f_n}.$$

If $|f_n| \nearrow |f| \quad \mu - a.e.$, we have

$$\liminf_{n\to\infty} |f_n| = \lim_{n\to\infty} |f_n| = |f|$$

and by (d) we get

$$\mu_f \leq \liminf_{n\to\infty} \mu_{f_n} \leq \limsup_{n\to\infty} \mu_{f_n}. \qquad (1)$$

On the other hand, since $|f_n| \nearrow |f|$

$$|f_n| \leq |f| \quad \Rightarrow \qquad \{|f_n| > \lambda\} \subseteq \{|f| > \lambda\} \quad \Rightarrow$$

$$E_n(\lambda) \leq E(\lambda) \quad \Rightarrow \quad \mu_{f_n}(\lambda) \leq \mu_f(\lambda) \quad \Rightarrow \quad \limsup \mu_{f_n} \leq \mu_f. \qquad (2)$$

Thus from (1), (2)

$$\mu_{f_n} \to \mu_f. \ \square$$

Example 4.1.4: The distribution function μ_f of a non-negative simple function f. Let

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$$

where E_j are pairwise disjoint subsets of R with finite μ-measure and $a_1 > a_2 > ... > a_n > 0$.
If $\lambda \geq a_1$ then $\mu_f(\lambda) = \mu(\{x \in R : |f(x)| > \lambda\}) = 0$.
If $a_2 \leq \lambda < a_1$ then $f(x) \geq \lambda$ for $x \in E_1$ and $\mu_f(\lambda) = \mu(E_1)$.
If $a_3 \leq \lambda < a_2$ the $\mu_f(\lambda) = \mu(E_1 \cup E_2)$ and so on.
In general we have

$$\mu_f(\lambda) = \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}(\lambda), \quad \lambda \geq 0,$$

where $m_j = \sum_{i=1}^{j} \mu(E_i)$, $j = 1, 2, ..., n$ and $a_{n+1} = 0$.

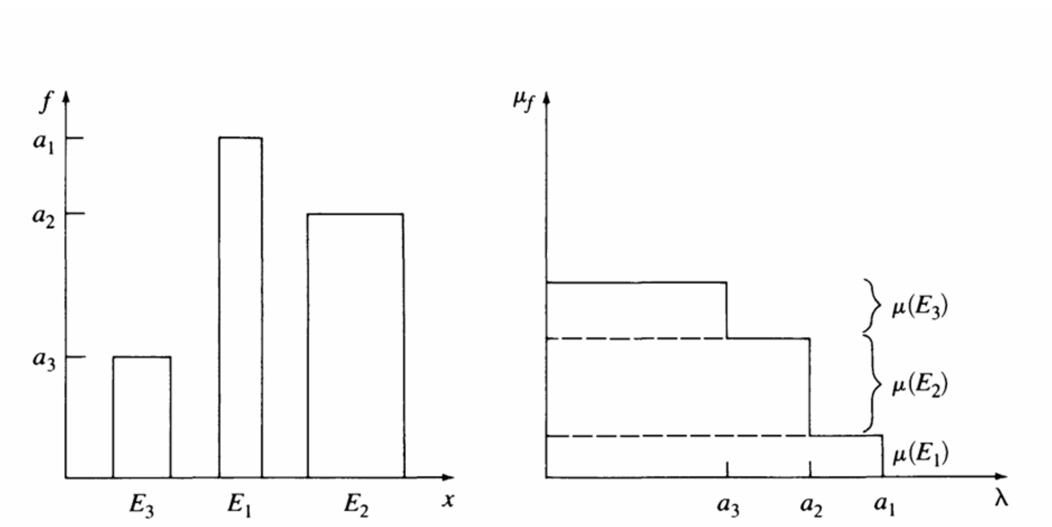


Figure 4.1.4

4.2 The decreasing rearrangement

Definition 4.2.1: Suppose $f \in M_o(R,\mu)$. The decreasing rearrangement of f is the function f^* defined in $[0,\infty)$ by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \le t\}, \quad t \ge 0.$$

Here we assume $\inf \emptyset = \infty$ so if $\mu_f(\lambda) > t$ for all $\lambda \ge 0$ the $f^*(t) = \infty$.
If (R,μ) is a finite measure space then μ_f is bounded by $\mu(R)$ so $f^*(t) = 0$ for every $t \ge \mu(R)$ and we can regard f^* as a function defined on the interval $[0,\mu(R))$.

Example 4.2.2 (a): We will now compute the decreasing rearrangement of the simple function $f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$, as seen on Example 4.1.4.

By Definition 4.2.1, $f^*(t) = \inf\{\lambda : \mu_f(\lambda) \le t\}$ we can see that $f^*(t) = 0$ for $t \ge m_n = \sum_{j=1}^{n} \mu(E_j)$.

While for $m_n > t \ge m_{n-1}$ we have $f^*(t) = a_n$ and so on.

More generally we see that

$$f^*(t) = \sum_{j=1}^{n} a_j \chi_{[m_{j-1},m_j)}(t), \quad t \ge 0,$$

where $m_o = 0$, when f is given as in Example 4.1.4.

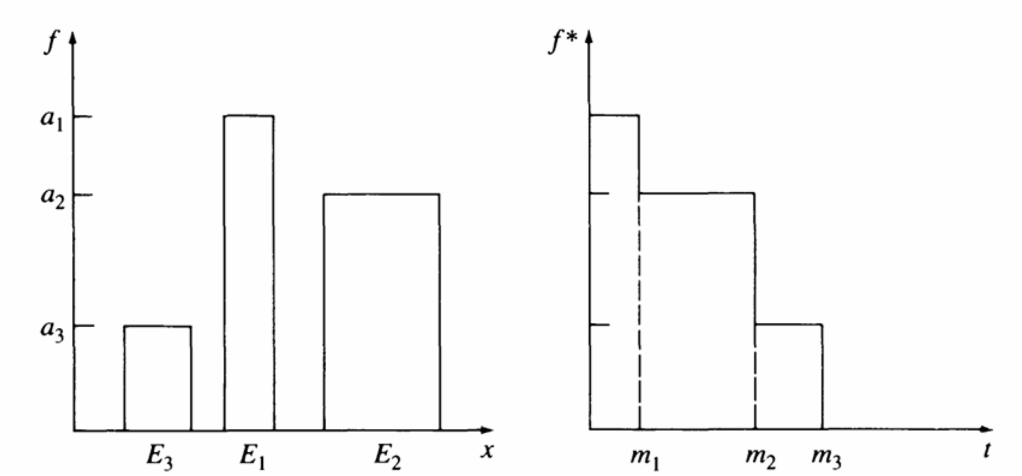


Figure 4.2.2 (a)

Geometrically, what we are actually doing is we rearrange the vertical blocks in the graph of f in decreasing order and in this way we obtain f^*. The values of f^* at the jumps are determined by the right continuity, as we will prove in a moment.

Example 4.2.2 (b): Sometimes it is more useful to section functions into horizontal blocks rather than vertical ones. By doing so the simple function f of Example 4.1.4

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

can also be written

$$f(x) = \sum_{k=1}^{n} b_k \chi_{F_k}$$

where $b_k = a_k - a_{k+1} > 0$, and $a_{n+1} = 0$, $F_k = \bigcup_{j=1}^{k} E_j, k = 1, 2, ..., n$ with $\mu(F_k) < \infty$ and $F_1 \subseteq F_2 \subseteq .... \subseteq F_k$.

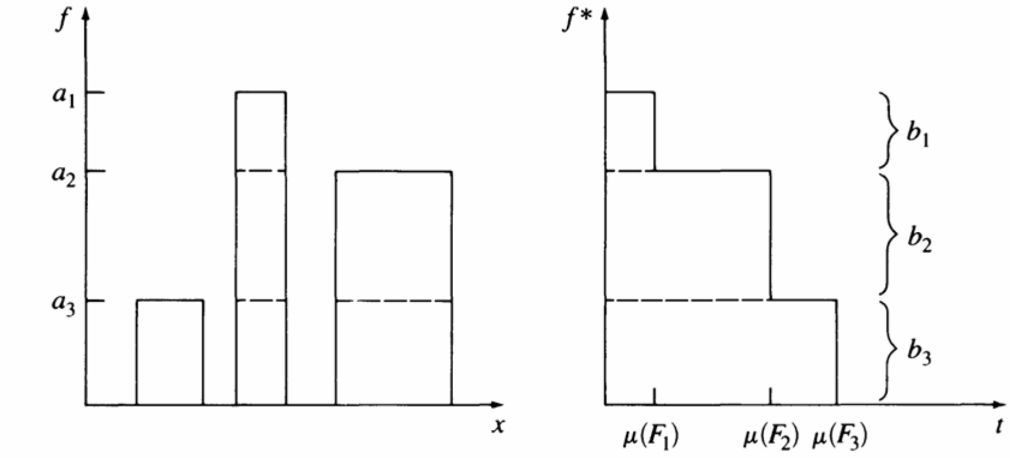


Figure 4.2.2(b)

So now

$$f^*(t) = \sum_{k=1}^{n} b_k \chi_{[0,\mu(F_k))}.$$

Corollary 4.2.3: *Let $f, g, f_n \in M_o(R, \mu)$, $n = 1, 2, ...$ and a be any scalar. The decreasing rearrangement f^* is a non-negative, decreasing right continuous function on $[0, \infty)$. Additionally*

a) $|g| \leq |f|\ \mu - a.e \Rightarrow g^* \leq f^*$

b) $(af)^* = |a| f^*$

c) $(f+g)^*(t_1 + t_2) \leq f^*(t_1) + f^*(t_2), \quad t_1, t_2 \geq 0$

d) $|f| \leq \liminf_{n\to\infty} |f_n| \quad a.e \quad \Rightarrow \quad f^* \leq \liminf_{n\to\infty} f_n^*$

specifically, if $|f_n| \nearrow |f| \quad \mu - a.e \quad \Rightarrow \quad f_n^* \nearrow f^*.$

e) $f^*(\mu_f(\lambda)) \leq \lambda$ *when* $\mu_f(\lambda) < \infty$ *and* $\mu_f(f^*(t)) \leq t$ *when* $f^*(t) < \infty$

f) f *and* f^* *are equimeasurable*

g) $(|f|^p)^* = (f^*)^p$, *for* $0 < p < \infty$

4.3 Some results

The next proposition gives alternative descriptions of the L^p-norm in terms of the distribution function and the decreasing rearrangement.

Proposition 4.3.1 : *Assume* $f \in M_o$*. For* $0 < p < \infty$ *we have*

$$\int_R |f|^p d\mu = p \int_0^\infty \lambda^{p-1} \mu_f(\lambda) d\lambda = \int_0^\infty f^*(t) dt$$

and for $p = \infty$ *we have*

$$ess \sup_{x \in R} |f(x)| = inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0).$$

Proof: Having proved Proposition 4.1.3 (d) and mentioning Corollary 4.2.3 (d) and by the Monotone Convergence Theorem it is sufficient to prove the first two equalities for an arbitary non-negative simple function f.
For

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$$

where E_j are disjoint with finite measure and $a_1 > a_2 > ... > a_n > 0$ we obtained, as seen on Example 4.2.2(a), the decreasing rearrangement

$$f^*(t) = \sum_{j=1}^{n} a_j \chi_{[m_{j-1}, m_j)}(t)$$

where $m_j = \sum_{i=1}^{j} \mu(E_i)$.

Now,

$$\int_R |f|^p d\mu = \sum_{j=1}^{n} {a_j}^p \mu(E_j) =$$

$$= \sum_{j=1}^{n} {a_j}^p |[m_{j-1}, m_j)| = \int_0^\infty (f^*)^p dm.$$

We also obtained $\mu_f(\lambda) = \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}(\lambda)$,

so

$$p \int_0^\infty \lambda^{p-1} \mu_f(\lambda) d\lambda = p \sum_{j=1}^{n} m_j \int_{a_{j+1}}^{a_j} \lambda^{p-1} d\lambda = \sum_{j=1}^{n} ({a_j}^p - a_{j+1}^p) m_j =$$

$$= \sum_{j=1}^{n} {a_j}^p \mu(E_j) = \int |f|^p d\mu.$$

For the case $p = \infty$ we have

$$esssup_{x \in \mathbb{R}} |f(x)| = inf\{c \in \mathbb{R} : |f(x)| \leq c \ \mu - a.e\} =$$

$$= inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0)$$

by the definition of f^*. $\square$

Unfortunately the decreasing rearrangement does not preserve products of functions but we will state a basic inequality that is true for products of functions.

Lemma 4.3.2: *Let g be a non-negative simple function on (R, μ) and E be an arbitary μ-measurable subset of R. Then*

$$\int_E g \ d\mu \leq \int_0^{\mu(E)} g^*(s) ds.$$

Proof: We write g in the form of Example 4.2.2 (b)

$$g(x) = \sum_{j=1}^{n} b_j \chi_{F_j}(x)$$

where $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$, $b_j \geq 0$, $j = 1, 2, ..., n.$

Then the decreasing rearrangement g^* is given by

$$g^*(t) = \sum_{j=1}^{n} b_j \chi_{[0,\mu(F_j))}(t)$$

so,

$$\int_E g d\mu = \sum_{j=1}^{n} b_j \mu(E \cap F_j) \leq \sum_{j=1}^{n} b_j \min(\mu(E), \mu(F_j)).$$

If $\mu(E) \leq \mu(F_j)$ then

$$\int_0^{\mu(E)} \chi_{[0,\mu(F_j))}(s) ds = \mu(E)$$

if $\mu(F_j) < \mu(E)$ then

$$\int_0^{\mu(E)} \chi_{[0,\mu(F_j))}(s) ds = \mu(F_j) \ , \quad \forall j$$

so we obtain $\min(\mu(E), \mu(F_j)) = \int_0^{\mu(E)} \chi_{[0,\mu(F_j))}(s) ds$
and thus

$$\sum_{j=1}^{n} b_j \min(\mu(E), \mu(F_j)) = \sum_{j=1}^{n} b_j \int_0^{\mu(E)} \chi_{[0,\mu(F_j))}(s) ds = \int_0^{\mu(E)} g^*(s) ds. \ \square$$

Theorem 4.3.3: (G. H Hardy and J. E Littlewood) *Assume* $f, g \in M_o(\mathbb{R}, \mu)$. *Then*

$$\int_R |fg| d\mu \leq \int_0^{\infty} f^*(s) g^*(s) ds.$$

Proof: As we stated before f^*, g^* depend only on the absolute values of f and g, so it is enough to prove the theorem for non-negative functions f and g.
By Proposition's 4.2.3 (d) and the Monotone Convergence Theorem we can suppose f and g to be simple.

We can write $f(x) = \sum_{j=1}^{m} a_j \chi_{E_j}(x)$ where $E_1 \subseteq E_2 \subseteq ... \subseteq E_m$ and $a_j > 0$ as in Example 4.2.2 (b).

So by the lemma we just proved we have

$$\int_R |fg| d\mu = \sum_{j=1}^{m} a_j \int_{E_j} g d\mu \leq \sum_{0}^{m} a_j \int_0^{\mu(E_j)} g^*(s) ds =$$

$$= \int_0^{\infty} \sum_{j=1}^{m} a_j \chi_{[0,\mu(E_j))}(s) g^*(s) ds = \int_0^{\infty} f^*(s) g^*(s) ds. \quad \square$$

Applying Lemma 1.1.4 we get the following.

Theorem 4.3.4: *Let $f : (0,1) \to \mathbb{R}^+$, $f \in L^1((0,1))$. The decreasing rearrangement $f^* : (0,1) \to \mathbb{R}^+$ satisfies*

$$(M_R f)^*(t) \leq \frac{1}{t} \int_0^t f^*(u) du \ , \quad \forall t \in (0,1)$$

where $(M_R f)^*$ is the decreasing rearrangement of $M_R f$

and $M_R f : (0,1) \to \mathbb{R}^+ \cup 0$ is defined

$$M_R f(x) = sup\{\frac{1}{u-x} \int_x^u |f(t)| dt : x < u \leq 1\}.$$

The same inequality holds for the maximal operator $M_L f : (0,1) \to \mathbb{R}^+ \cup \{0\}$, where

$$M_L f(x) = sup\{\frac{1}{x-u} \int_u^x |f(t)| dt : 0 \leq u < x\}.$$

Chapter 5

Differentiability of multiple integrals

Our last chapter is a generalization of some of the differentiation properties we have proved so far as we will work in the k-dimensional space.

Assume $f(x_1, x_2, ..., x_k) = f(P)$ is an integrable function defined in the interval $S = \{(x_1, x_2, ..., x_k) : 0 < x_i < 1,\ i = 1, 2, ..., k\}$.

We will say that the integral of the function f is *strongly differentiable* at the point P_o , if

$$\lim_{\delta(I)\to 0} \frac{1}{|I|} \int_I f(P)dP \qquad (1)$$

exists and is finite, where I is any interval with sides parallel to the axes, $I \subseteq S$, $P_o \in I$, $|I|$ is the measure of I and $\delta(I)$ the diameter of I.

The limit (1) is called the *strong derivative* of the integral of f at the point P_o.

Theorem A: *There is a function $f(P) \in L^1$ such that its integral is nowhere strongly differentiable.*

Theorem B: *If $f(P) \in L^p(S)$, $p > 1$, the strong derivative of the integral of $f(P)$ exists almost everywhere and is equal to f(P).*

Given a function $f(P) \in L^1(S)$, we write

$$f^*(P_o) = \sup_{P_o \in I} \frac{1}{|I|} \int_I |f(P)| dP \qquad (2)$$

and

$$f_*(P_o) = \limsup_{\substack{\delta(I) \to 0 \\ P_o \in I}} \frac{1}{|I|} \int_I |f(P)| dP. \qquad (3)$$

Note: Definition (2) should not be confused with the definition of the decreasing rearrangement of the function f.

5.1 Introductory results

Let f be a function $f : S = (0,1) \to \mathbb{R}$, $f \in L^1(S)$ and we define its extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$

$$\tilde{f}(x) = \begin{cases} f(x), & if \ x \in S \\ \\ 0, & if \ x \notin S \end{cases}$$

and the maximal operators $M_R^{(0,1)}$, $M_L^{(0,1)}$ on $L^1(S)$ by :

$$M_R^{(0,1)} f(x) = sup\{\frac{1}{u-x} \int_x^u |f(t)| dt : x < u \leq 1\}.$$
$$M_L^{(0,1)} f(x) = sup\{\frac{1}{x-u} \int_u^x |f(t)| dt : 0 \leq u < x\}.$$

By Lemma 1.1.4 we have that

$$|\{M_R \tilde{f}(x) > \lambda\}| = \frac{1}{\lambda} \int_{\{M_R \tilde{f}(x) > \lambda\}} |\tilde{f}(x)| dx \ , \qquad \forall \lambda > 0$$

and the same equality holds for $M_L\tilde{f}$.
In this chapter we will work on the interval $S = (0,1)$ for a function $f : (0,1) \to \mathbb{R}$ using the maximal operator $M^{(0,1)}$, which is defined by

$$M^{(0,1)}f(x) = \sup_{\substack{\xi_1<x<\xi_2 \\ \xi_1,\xi_2\in(0,1)}} \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} f(x)dx.$$

Theorem 5.1.1: *The maximal operators $M_R f$, $M_L f$ satisfy*

$$|\{M_R^{(0,1)}f(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{M_R^{(0,1)}f(x)>\lambda\}} f$$

$$|\{M_L^{[0,1]}f(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{M_L^{(0,1)}f(x)>\lambda\}} |f|$$

$\forall\lambda > 0$, $\forall f : S \to \mathbb{R}$, $f \in L^1(S)$.

Proof: The above inequalities can be obtained easily by applying the identities

$$|\{M_R g(x) > t\}| = \frac{1}{t}\int_{M_t^R} |g|dx$$

and

$$|\{M_L g(x) > t\}| = \frac{1}{t}\int_{M_t^L} |g|dx$$

of Lemma 1.1.4 that we proved for $g : \mathbb{R} \to \mathbb{R}$ for the extension $\tilde{f}$ of $f : (0,1) \to \mathbb{R}$ and the corresponding maximal operators $M_R\tilde{f}$, $M_L\tilde{f}$. $\Box$

Lemma 5.1.2: *Let $f(x) \in L^p(S)$, $p > 1$. Then*

$$f^*(x) = \sup_{\xi_1<x<\xi_2} \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} |f(u)|du \ , \ \forall x \in (0,1) \qquad (4)$$

belongs to L^p also, where ξ_1, ξ_2 range over $(0,1)$ under the condition $\xi_1 < x < \xi_2$. Moreover

$$\int_0^1 (f^*(x))^p dx \leq c_p \int_0^1 |f(x)|^p dx$$

where $c_p = 2^p(\frac{p}{p-1})^p$.

<u>Proof:</u> The proof is analogous to the one of Theorem 1.2.1. We now use Theorem 5.1.1 instead of Lemma 1.1.4 and we obtain the result. □

Lemma 5.1.3: *If* $f(x)log^+|f(x)|$ *is integrable over* S *then* $f^*(x)$ *is integrable and*

$$\int_0^1 f^*(x)dx \leq A\int_0^1 |f(x)|log^+|f(x)|dx + B$$

where A *and* B *are absolute constants.*

<u>Proof:</u> We will prove this lemma as in Theorem 1.1.3.

Assume $\int_0^1 f(x)log^+|f(x)| < \infty$.
We define the extension of f, $\tilde{f} : \mathbb{R} \to \mathbb{R}$

$$\tilde{f}(x) = \begin{cases} f(x), & if\ x \in S \\ \\ 0, & if\ x \notin S \end{cases}$$

Then

$$\int_0^1 f^*(x)dx \leq \int_0^1 M_1\tilde{f}(x)dx = \int_0^\infty |\{x \in (0,1) : M_1\tilde{f}(x) > \lambda\}|d\lambda =$$

$$= 2\int_0^\infty |\{x \in (0,1) : M_1\tilde{f}(x) > 2\lambda\}|d\lambda$$

$$\leq \int_0^1 2|S| + 2\int_1^\infty |\{x : M_1\tilde{f}(x) > \lambda\}|d\lambda \leq 2 + 2c\int_1^\infty \frac{1}{\lambda}\int_{\{x:|\tilde{f}(x)|>\lambda\}} |\tilde{f}(x)|dxd\lambda$$

where we used Theorem 1.1.1

$$= 2 + 2c \int_{\{|\tilde{f}(x)|>1\}} |\tilde{f}(x)| \int_1^{|\tilde{f}(x)|} \frac{1}{\lambda} d\lambda dx = 2 + 2c \int_{\{|\tilde{f}(x)|>1\}} |\tilde{f}(x)| [log\lambda]_1^{|\tilde{f}(x)|} dx =$$

$$= 2 + 2c \int_{\{|f(x)|>1\}} |f(x)| log|f(x)| dx = B + A \int_0^1 |f(x)| log^+ |f(x)| dx$$

for A=2c and B=2 as requested. □

5.2 The case k=2

We will now consider the case $k = 2$ and we will write x, y for x_1 and x_2. We denote $S = \{(x, y) \in \mathbb{R}^2 : \ 0 < x < 1 \ , \ 0 < y < 1\}$.

From Lemma 5.1.2 we get the following theorem.

Theorem 5.2.1: *Assume $f(P) \in L^p$, $p > 1$. Then $f^*(P) \in L^p$ and*

$$(5) \qquad \int_S (f^*(P))^p dP \leq A_p \int_S |f(P)|^p dP$$

where $A_p = {c_p}^2$ and c_p the constant from Lemma 5.1.2.

Proof: We will prove this theorem using Lemma 5.1.2

We define

$$(6) \qquad g(x, y) = \sup_{v_1 < y < v_2} \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} |f(x, v)| dv$$

and

$$(7) \qquad h(x,y) = \sup_{\xi_1 < x < \xi_2} \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} g(u,y)du.$$

We will prove that g is L^p- integrable on S so h is finite at almost every point of S.
By Lemma 5.1.2 we have

$$\int_S g^p(P)dP = \int_0^1 dx \int_0^1 g^p(x,y)dy \leq$$

$$\leq \int_0^1 dx c_p \int_0^1 |f(x,y)|^p dy = c_p \int_S |f(P)|^p dP < \infty$$

and so

$$(*) \int_S g^p(P)dP \leq c_p \int_S |f(P)|^p dP < \infty \; and \; g \in L^p$$

Similarly,

$$\int_S h^p(P)dP \leq c_p \int_S g^p(P)dP \leq_{(*)} {c_p}^2 \int_S |f(P)|dP \qquad (8)$$

For $0 < \xi_1 < x < \xi_2 < 1$ and $0 < v_1 < y < v_2 < 1$ we notice that

$$\frac{1}{(\xi_2 - \xi_1)(v_2 - v_1)} \int_{\xi_1}^{\xi_2} \int_{v_1}^{v_2} |f(u,v)|dudv \leq$$

$$\leq_{(6)} \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} g(u,y)du \leq_{(7)} h(x,y) \qquad (9)$$

hence

$$f^*(P) \leq h(P). \qquad (10)$$

Using (8) we get

$$\int_S (f^*(P))^p \leq \int_S h^P(P) \leq {c_p}^2 \int_S |f(P)|^p dP$$

as required. □

Lemma 5.2.2: If $f(P)log^{+}|f(P)|$ is integrable over S, then the function

$$f_*(P) = \limsup_{\delta(I)\to 0} \frac{1}{|I|}\int_I |f(P)|dP$$

is integrable and

$$(11)\quad \int_S f_*(P)dP \le A\int_S |f(P)|log^{+}|f(P)|dP + B$$

where A and B are the constants of Lemma 5.1.3.

Proof: By Lemma 5.1.3 we have

$$\int_0^1 g(x,y)dy \le A\int_0^1 |f(x,y)|log^{+}|f(x,y)|dy + B\ ,$$

where g is defined by equation (6) $\forall x \in (0,1)$.

Integrating with respect to x we get

$$(12)\quad \int_S g(P)dP \le A\int_S |f(P)|log^{+}|f(P)|dP + B < \infty$$

So for almost every y, g(x,y) is integrable as a function of x.

Then by definition of integrability we have

$$\limsup_{\substack{\xi_1<x<\xi_2\\ \xi_2-\xi_1\to 0}} \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} g(u,y)du = g(x,y),\ \textit{for almost every}\ (x,y)$$

and (9) gives us $f_*(P) \le g(P)$ so

$$\int_S f_*(P) \le \int_S g(P) \le_{(12)} A\int_S |f(P)|log^{+}|f(P)|dP + B.\ \square$$

Theorem 5.2.3: *If $f(P)log^+|f(P)|$ is integrable over S, then at almost every point P, the integral of f is strongly differentiable and the derivative is equal to f(P).*

Proof: We apply inequality (11) to the function λf, where $\lambda > 0$ is a constant and obtain

$$(13)\quad \int_S f_*(P)dP \leq A\int_S |f(P)|log^+|\lambda f(P)|dP + \frac{B}{\lambda}.$$

Given $\epsilon > 0$ we take λ large enough so that $\frac{B}{\lambda} < \frac{\epsilon}{2}$ and put $f(P) = \phi(P) + \psi(P)$, where ϕ is continuous and

$$(14)\quad \int_S |\psi(P)|dP < \epsilon$$

and

$$(15)\ A\int_S |\psi(P)|log^+|\lambda\psi(P)|dP + \frac{B}{\lambda} < \epsilon.$$

Define

$$E(\epsilon) = \{|\psi| > \sqrt{\epsilon}\} \cup \{\psi_* > \sqrt{\epsilon}\}\ ,\quad E_j = E(\frac{1}{j^4}).$$

Now

$$|\{|\psi(P)| > \sqrt{\epsilon}\}| \leq \frac{1}{\sqrt{\epsilon}}\int_{\{|\psi(P)|>\sqrt{\epsilon}\}} |\psi(P)| \leq \frac{1}{\sqrt{\epsilon}}\int_S |\psi(P)| <_{(14)} \sqrt{\epsilon}$$

so $|\{P : |\psi(P)| > \sqrt{\epsilon}\}| \leq \sqrt{\epsilon}$ and similarly $|\{P : \psi_*(P) > \sqrt{\epsilon}\}| \leq \sqrt{\epsilon}$. Therefore $|E(\epsilon)| \leq 2\sqrt{\epsilon}$ and for $\epsilon = \dfrac{1}{j^4}$ we get $|E_j| \leq \dfrac{2}{j^2}$.

We will prove that

$$\lim_{\substack{\delta(I)\to 0\\ P_o\in I}} \frac{1}{|I|}\int_I f(P)dP = f(P_o)\ ,\quad \textit{for almost all } P_o \in S \qquad (*)$$

Indeed, we have $E_j = E(\frac{1}{j^4})$ with $|E_j| \leq 2\sqrt{\frac{1}{j^4}} = \frac{2}{j^2}$.

We set $S_o = \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} E_j = \{x \in S : x \text{ is in infinite } E_j\ 's\}$

$\bigcup_{j \geq k} E_j$ is decreasing with respect to k and

$$|\bigcup_{j \geq 1} E_j| \leq \sum_{j=1}^{\infty} |E_j| < \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.$$

Then

$$|\bigcap_{k=1}^{\infty} \bigcup_{j \geq k} E_j| = \lim_{k \to \infty} |\bigcup_{j \geq k} E_j| \leq \lim_{k \to \infty} \sum_{j \geq k} |E_j| \leq \lim_{k \to \infty} \sum_{j \geq k} \frac{1}{j^2} = 0.$$

If $P_o \notin \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} E_j = S_o$, there exists $j_o \in \mathbb{N}$ such that $P_o \notin E_j$, $\forall j \geq j_o$, so there exists a j_o such that $P_o \notin E(\frac{1}{j^4})$, $\forall j \geq j_o$.

Thus, since for every $j \geq j_o \quad P_o \notin E_j \ \Rightarrow \ |\psi(P_o)| < \frac{1}{j^4}$, so we have

$$\limsup_{\substack{\delta(I) \to 0 \\ P_o \in I}} |\frac{1}{|I|} \int_I f(P) dP - f(P_o)| = \limsup_{\substack{\delta(I) \to 0 \\ P_o \in I}} |\frac{1}{|I|} \int_I \psi(P) dP - \psi(P_o)| \leq$$

$$\leq \psi_*(P_o) + |\psi(P_o)| < \sqrt{\frac{1}{j^4}} + \sqrt{\frac{1}{j^4}}$$

since $\forall j \geq j_o$

$$= 2\sqrt{\frac{1}{j^4}} = \frac{2}{j^2} \xrightarrow{j \to \infty} 0. \ \square$$

5.3 The case of arbitary k

We now examine the case of arbitary k. Theorem 5.2.1 still holds for $A_p = {c_p}^k$ and so also Theorem B is still true.

Theorem 5.3.1: *Assume $|f|(log^+|f|)^{k-1}$ is integrable over S. Then the integral of $f(P)$ is strongly differentiable at almost every point of S to the value f(P), where $S = (0,1)^k$.*

Proof: The proof is analogous to the proof of Theorem 5.2.3 and is omitted. We would need the following lemma.

Lemma 5.3.2: *Assume $f(x)$ is an integrable function defined over the interval $0 < x < 1$ and*

$$f^*(x) = \sup_{\xi_1 < x < \xi_2} \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} |f(u)| du.$$

If $|f|(log^+|f|)^r$, $r = 1, 2, ...$ is integrable over $(0,1)$, then $f^*(log^+ f^*)^{r-1}$ is integrable too and

$$\int_0^1 f^*(log^+ f^*)^{r-1} dx \leq A_r \int_0^1 |f|(log^+|f|)^r dx + B$$

where A_r and B depend only on r.

Proof: We will use Theorem 4.3.4 and denote the decreasing rearrangements of f and $M_R f$ respectively as $\bar{f}$ and $\overline{M_R} f$ at this moment.
Let $\phi(x) = x(log^+ x)^{r-1}$, $x \in (0, \infty)$.
Let f be an arbitary integrable function on (0,1), we then have :

$$\int_0^1 f^*(log^+ f^*)^{r-1} dx = \int_0^1 M_1^{(0,1)} f(x)(log^+(M_1^{(0,1)} f(x)))^{r-1} \leq$$

$$\leq \int_{\{M_R^{(0,1)} f \geq M_L^{(0,1)} f\}} M_R^{(0,1)} f(x)(log^+(M_R^{(0,1)} f(x)))^{r-1} +$$
$$+ \int_{\{M_L f \geq M_R f\}} M_L^{(0,1)} f(x)(log^+(M_L^{(0,1)} f(x)))^{r-1} dx \leq$$

$$\leq \int_0^1 M_R f(x)(log^+(M_R f(x)))^{r-1} + \int_0^1 M_L f(x)(log^+(M_L f(x)))^{r-1} = I_1 + I_2$$

where

$$I_1 = \int_0^1 \phi(M_R f(x))dx = \int_{t=0}^{\infty} |\{x : \phi(M_R f(x)) > t\}|dt =$$

$$= \int_{t=0}^{\infty} |\{x : \phi(\overline{M_R f}(x)) > t\}|dt = \int_0^1 \phi(\overline{M_R f})(x)dx =$$

$$= \int_0^1 \overline{M_R f}(x)(log^+ \overline{M_R f})^{r-1}(x)dx = I_1'.$$

and similarly

$$I_2 = \int_0^1 \overline{M_L f}(x)(log^+ \overline{M_L f})^{r-1}(x)dx = I_2'$$

and by Theorem 4.3.4 we get

$$I_1' \leq \int_0^1 \phi(\frac{1}{x}\int_0^x \bar{f}(t)dt)dx \quad and \quad I_2' \leq \int_0^1 \phi(\frac{1}{x}\int_0^x \bar{f}(t)dt)dx$$

and

$$\int_0^1 \phi(\frac{1}{x}\int_0^x \bar{f}(t)dt)dx = \int_0^1 (\bar{f})^*(log^+(\bar{f})^*)^{r-1}$$

since

$$(\bar{f})^*(x) = \frac{1}{x}\int_0^x \bar{f}(t)dt \ . \quad (*)$$

We will prove (*). To do so we will first prove that if $t_1 < t_2$ then

$$\frac{1}{t_2}\int_0^{t_2} \bar{f} \leq \frac{1}{t_1}\int_0^{t_1} \bar{f}.$$

We will use the fact that $\bar{f}$ is decreasing on (0,1).
Indeed this is true if and only if

$$t_1 \int_0^{t_2} \bar{f} \leq t_2 \int_0^{t_1} \bar{f} \quad \Leftrightarrow \quad t_1 \int_0^{t_1} \bar{f} + t_1 \int_{t_1}^{t_2} \bar{f} \leq (t_1 + \delta) \int_0^{t_1} \bar{f}$$

(where $t_2 = t_1 + \delta$, $\delta > 0$)

$$\Leftrightarrow t_1 \int_{t_1}^{t_2} \bar{f} \leq \delta \int_0^{t_1} \bar{f} \Leftrightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \bar{f} \leq \frac{1}{t_1} \int_0^{t_1} \bar{f}$$

which is true as

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \bar{f}(u) du \leq \bar{f}(t_1)$$

and

$$\frac{1}{t_1} \int_0^{t_1} \bar{f}(u) du \geq \bar{f}(t_1) \ ,$$

since $\bar{f}$ is decreasing, so

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \bar{f}(u) du \leq \bar{f}(t_1) \leq \frac{1}{t_1} \int_0^{t_1} \bar{f}(u) du$$

and thus

$$\frac{1}{t_2} \int_0^{t_2} \bar{f} \leq \frac{1}{t_1} \int_0^{t_1} \bar{f} \ .$$

Therefore $(\bar{f})^*(x) = M_1 \bar{f}(x) = \frac{1}{x} \int_0^x \bar{f}(t) dt$.

So

$$\int f^*(x)(log^+(f^*(x)))^{r-1} dx \leq 2 \int_0^1 \phi(\frac{1}{x} \int_0^x \bar{f}(t) dt) dx$$

and the proof of the lemma will be completed once we have provided the right upper bound for the right side of the inequality which involves the decreasing rearrangement $\bar{f}$ of f.
Using Jensen's inequality as $\phi(x)$ is convex we get

$$\int_0^1 \phi(\frac{1}{x} \int_0^x \bar{f}(t) dt) dx \leq \int_0^1 (\frac{1}{x} \int_0^x \phi(\bar{f}(t)) dt) dx. \qquad (18)$$

Now since $\phi(\bar{f}(t))$ is a non-increasing function of t we have

$$\int_0^1 \frac{1}{x} \int_0^x \phi(\bar{f}(t))dtdx = \int_0^1 (\phi(\bar{f}))^* \leq_{L.5.1.3} A \int_0^1 \phi(\bar{f}) log^+ \phi(\bar{f}) dx + B =$$

$$= A \int_0^1 \bar{f}(log^+ \bar{f})^{r-1} log^+ [\bar{f}(log^+ \bar{f})^{r-1}]dx + B \leq A_r \int_0^1 \bar{f}(log^+ \bar{f})^r dx + B_r$$

$$= A_r \int_0^1 |f|(log^+ |f|)^r dx + B \ ,$$

since the functions $|f|(log^+ |f|)^r$ and $|\bar{f}|(log^+ |\bar{f}|)^r$ are equimeasurable.
The latter inequality can be proved by using simple calculus arguments. □

Assume now that $f(P)$ is an integrable function defined in the k-dimensional interval $S = (0,1)^k$. By the fundamental theorem of the Lebesgue theory of integration we deduce that the integral of f is differentiable, in the ordinary sense, at almost every point P_o and the value of the derivative is $f(P_o)$. That is,

$$\lim_{\delta(I)\to 0} \frac{1}{|I|} \int_I f(P)dP = f(P_o)$$

as in (1), where the ratios of any two sides of the interval I containing P_o do not exceed a finite constant number, for every $P_o \in S = (0,1)^k$.

Theorem 5.3.3: *Let $a_1(t), a_2(t), ..., a_k(t)$ be arbitary non-decreasing functions defined to the right of $t = 0$, vanishing and continuous for $t = 0$ and positive for $t > 0$. When the intervals I, where $P_o \in I$, are of the form*

$$\begin{aligned} &\xi_i' \leq x_i \leq \xi_i'', \\ &\xi_i'' - \xi_i' = a_i(t), \qquad\qquad i=1,2,\ ...,\ k \quad (19) \end{aligned}$$

the limit

$$\lim_{\delta(I)\to 0} \frac{1}{|I|} \int_I f(P)dP$$

exists and is equal to $f(P_o)$ at almost every point P_o.

For the proof of Theorem 5.3.3 we just need to prove that the Vitali covering lemma remains valid for intervals I of the form (19).

Theorem 5.3.4: *Let $A \subseteq \mathbb{R}^n$. Assume that for every $x \in A$ there is a sequence of intervals $\{K_k(x)\}_{k\in\mathbb{N}}$, containing x such that $\delta(K_k(x)) \xrightarrow{k\to\infty} 0$.*
Let the collection $V = \{K_k(x)\}_{\substack{x\in A \\ k\in\mathbb{N}}}$ satisfies: if $T_1, T_2 \in V$ then there exists a translation of one of them that puts it inside the other. Then we can choose from V a disjoint sequence $\{S_k\} \subseteq V$ such that $|A - \cup S_k| = 0$.

Proof: To construct the sequence $\{S_k\}$ we choose $S_1 \in V$ such that

$$|S_1| \geq \frac{1}{2}\sup\{|T| : T \in V\}.$$

Without loss of generality assume $A \subseteq (0,1)^n$ and $\{K_k(x)\} \subseteq (0,1)^n$, $\forall x \in A$, $\forall k \in \mathbb{N}$.
If $A \subseteq S_1$ we stop.
If $A \nsubseteq S_1$ then the exists $x \in A - S_1$ and as S_1 is compact, there exists a set $T \in \{K_k(x)\}_{k\in\mathbb{N}} \subseteq V$ such that $T \cap S_1 = \emptyset$.

We then choose $S_2 \in V$ such that

$$|S_2| \geq \frac{1}{2}\sup\{|T| : T \in V, \quad T \cap S_1 = \emptyset\}.$$

If $A \subseteq S_1 \cup S_2$ we stop.
Is $A \nsubseteq S_1 \cup S_2$ then there exists $x \in A - (S_1 \cup S_2)$ and as $S_1 \cup S_2$ is compact there exists a set $T \in \{K_k(x)\}_{k\in\mathbb{N}} \subseteq V$ such that $T \cap (S_1 \cup S_2) = \emptyset$.

We then choose $S_3 \in V$ such that

$$|S_3| \geq \frac{1}{2}\sup\{|T| : T \in V, \quad T \cap (S_1 \cup S_2) = \emptyset\}$$

and so on.

If the sequence we constructed is finite obviously $A \subseteq \cup S_k$.
If not, we will show that $T \cap (\bigcup_{k=1}^{\infty} S_k) \neq \emptyset$, $\quad for\ any \quad T \in V.$ (1)

Assume there exists a set $T \in V$ such that $T \cap (\bigcup_{k=1}^{\infty} S_k) = \emptyset$,
that is $\{T \in V : T \cap (\bigcup_{k=1}^{\infty} S_k) = \emptyset\} \neq \emptyset$.
We choose $S \in V$ such that

$$|S| \geq \frac{1}{2}\sup\{|T| : T \in V, \quad T \cap (\bigcup_{k=1}^{\infty} S_k) = \emptyset\}.$$

We have $S_k \subseteq (0,1)^n$, $\forall k$ and $S_k \cap S_j = \emptyset$, $\forall k \neq j$, so

$$\sum_{k=1}^{\infty} |S_k| < \infty \qquad \Rightarrow \qquad |S_k| \to 0 \qquad \Rightarrow$$

$$\exists j_o \in \mathbb{N} \quad such \quad that \quad |S_{j_o}| < \frac{1}{2}|S|$$

but

$$|S_{j_o}| \geq \frac{1}{2}\sup\{|T| : T \in V, \quad T \cap (\bigcup_{k=1}^{j_o-1} S_k) = \emptyset\}$$

and

$$S \cap (\bigcup_{k=1}^{\infty} S_k) = \emptyset \quad \Rightarrow \quad S \cap (\bigcup_{k=1}^{j_o-1} S_k) = \emptyset$$

so combining these two results we get

$$|S_{j_o}| \geq \frac{1}{2}|S|,$$

a contradiction.

Now we need to prove that $|A - (\bigcup_{k=1}^{\infty} S_k)| = 0$ and to do so we will prove that

$$\forall \epsilon > 0 \quad \exists h \in \mathbb{N} \quad such \quad that \quad |A - (\bigcup_{k=1}^{h} S_k)| \leq \epsilon$$

so we will then get

$$|A - (\bigcup_{k=1}^{\infty} S_k)| \leq |A - (\bigcup_{k=1}^{h} S_k)| \leq \epsilon \ , \quad for\ any\ \epsilon > 0.$$

Let $\eta > 0$ and $h \in \mathbb{N}$ such that

$$\sum_{k=h+1}^{\infty} |S_k| < \eta$$

where η will be chosen later.
It is true that

$$A - (\bigcup_{k=1}^{h} S_k) \subseteq \bigcup\{T : T \in V, \quad T \cap (\bigcup_{k=1}^{h} S_k) = \emptyset\} \qquad (2)$$

and for $x \in A - (\bigcup_{k=1}^{h} S_k) \quad \Rightarrow \quad \exists k \in \mathbb{N}$ such that $K_k(x) \cap (\bigcup_{k=1}^{h} S_k) = \emptyset$, as $\bigcup_{k=1}^{h} S_k$ is compact and $\delta(K_k(x)) \to 0$.

Also

$$\bigcup\{T : T \in V, \quad T \cap (\bigcup_{k=1}^{h} S_k) = \emptyset\} =$$

$$= \bigcup\{T : T \in V, \quad T \cap (\bigcup_{k=1}^{h} S_k) = \emptyset \ , \quad T \cap (\bigcup_{k=1}^{\infty} S_k) \neq \emptyset\} \qquad (3)$$

since $\forall T \in V$ we have $T \cap (\bigcup\limits_{k=1}^{\infty} S_k) \neq \emptyset$, by (1)

and

$$\bigcup\{T : T \in V, \quad T \cap (\bigcup_{k=1}^{h} S_k) = \emptyset \ , \quad T \cap (\bigcup_{k=1}^{\infty} S_k) \neq \emptyset\} =$$

$$\bigcup_{j=h}^{\infty}\{\bigcup\{T : T \in V, \quad T \cap (\bigcup_{k=1}^{j} S_k) = \emptyset, \quad T \cap S_{j+1} \neq \emptyset\}$$

$$\subseteq \bigcup_{j=h}^{\infty}\{\cup\{T : T \in V, \quad |T| \leq 2|S_{j+1}|, \quad T \cap S_{j+1} \neq \emptyset\} \qquad (4)$$

as for $T \in V$ such that $T \cap (\bigcup_{k=1}^{j} S_k) = \emptyset$ and $T \cap S_{j+1} \neq \emptyset$, $j \geq h$, by our construction, we have

$$|S_{j+1}| \geq \frac{1}{2}\sup\{|T| : T \in V, \quad T \cap (\bigcup_{k=1}^{j} S_k) = \emptyset\} \geq \frac{1}{2}|T| \quad \Rightarrow$$

$$|T| \leq 2|S_{j+1}| \quad and \quad T \cap S_{j+1} \neq \emptyset.$$

So by (2), (3), (4) we get

$$|A - \bigcup_{k=1}^{h} S_k| \leq \sum_{j=h}^{\infty} |\bigcup\{T : T \in V, \quad |T| \leq 2|S_{j+1}|, \quad T \cap S_{j+1} \neq \emptyset\}|$$

Now we observe that for $T_1, T_2 \in V$ such that $|T_1| > \dfrac{|T_2|}{2}$, the union $\mathbf{U}$ of all of the sets obtained by translating T_2 and having non empty intersection with T_1 has area

$\leq 9^n|T_1|$.

This is true as by the existence of a translation that puts one interval into the other we get that either

$$a_1^1 \leq a_2^1 \ , \ a_1^2 \leq a_2^2 \ , \ \ldots \ , \ a_1^n \leq a_2^n \ , \quad if \quad |T_1| \leq |T_2|$$

or

$$a_1^1 \geq a_2^1 \ , \ a_1^2 \geq a_2^2 \ , \ \ldots \ , \ a_1^n \geq a_2^n \ , \quad if \quad |T_2| \leq |T_1|$$

where a_1^i are the sides of T_1 and a_2^i are the sides of T_2, $i = 1, 2, ..., n$.

Assuming $|T_1| \leq |T_2|$ we get

$$|\mathbf{U}| \leq (a_1^1 + 2a_2^1) \cdot (a_1^2 + 2a_2^2) \cdot \ldots \cdot (a_1^n + 2a_2^n) < 3a_2^1 \cdot 3a_2^2 \cdot \ldots \cdot 3a_2^n =$$
$$= 3^n|T_2| < 2 \cdot 3^n|T_1| < 9^n|T_1|$$

and the same inequality holds assuming $|T_2| \geq |T_1|$.

So the previous quantity is

$$\leq \sum_{j=h}^{\infty} 9^n|S_{j+1}| = 9^n \sum_{j=h+1}^{\infty} |S_j| \leq 9^n \cdot \eta$$

and choosing $\eta < \dfrac{\epsilon}{9^n}$ we get the required inequality. $\square$

Combining Theorems 5.3.1 and 5.3.3 we get the following.

Theorem 5.3.5: *Assume $f(P)$ is a function defined in the k-dimensional interval $S = (0,1)^k$ and $a_1(t), a_2(t), ..., a_r(t)$, $2 \leq r \leq k$, are r functions satisfying the properties of Theorem 5.3.3. If $f(log^+|f|)^{k-r}$ is integrable over S, the limit (1) exists, at almost every point P_o and is equal to $f(P_o)$, provided the intervals I containing P_o are of the form (20) :*

$$\begin{aligned} &\xi_i' \leq x_i \leq \xi_i'', && i = 1, 2, ...k \\ &\xi_j'' - \xi_j' = a_j(t), && j = 1, 2, ..., r \end{aligned}$$

Proof: We will prove this for $k = 3$ and $r = 2$. The general case is similar. We will write x, y, z for x_1, x_2, x_3 and $a(t), b(t)$ for $a_1(t), a_2(t)$.
Let

$$f_*(x_o, y_o, z_o) = f_*(P_o) = \limsup_{\substack{\delta(I)\to 0 \\ P_o \in I}} \frac{1}{|I|} \int_I |f(P)|dP$$

where I is of the form (20).
We just need to prove that

$$\int_S f_*(P)dP \leq A \int_S |f(P)|log^+|f(P)|dP + B \qquad (21)$$

where A , B are independent of f
and then using an argument similar to the one we used for Theorem 5.3.1 the proof will be complete.

Let

$$g(x, y, z) = \sup_{\zeta' < z < \zeta''} \frac{1}{\zeta'' - \zeta'} \int_{\zeta'}^{\zeta''} |f(x, y, w)|dw.$$

From Lemma 5.1.3 we obtain

$$\int_0^1 g(x, y, z)dz \leq A \int_0^1 |f(x, y, z)|log^+|f(x, y, z)|dz + B$$

from which we can see that g is integrable over S and

$$\int_S g(P)dP \leq A \int_S |f(P)|log^+|f(P)|dP + B. \qquad (22)$$

Now, define

$$g_*(x, y, z) = \limsup_{\substack{\xi' < x < \xi'' \\ u' < y < u'' \\ \xi'' - \xi \to 0 \;,\; u'' - u' \to 0}} \frac{1}{(\xi'' - \xi')(u'' - u')} \int_{\xi'}^{\xi''} \int_{u'}^{u''} g(u, v, z)dudv$$

Then obviously

$$g(x, y, z) = \sup_{\zeta' < z < \zeta''} \frac{1}{\zeta'' - \zeta'} \int_{\zeta'}^{\zeta''} |f(x, y, w)|dw \geq \frac{1}{\zeta'' - \zeta'} \int_{\zeta'}^{\zeta''} |f(x, y, w)|dw$$

$\forall\zeta'$, ζ'' such that $\zeta'' < z < \zeta'$

$$\Rightarrow \qquad \frac{1}{\xi''-\xi'}\cdot\frac{1}{u''-u'}\int_{\xi'}^{\xi''}\int_{u'}^{u''} g(u,v,z)dudv \geq$$

$$\geq \frac{1}{\xi''-\xi'}\cdot\frac{1}{u''-u'}\cdot\frac{1}{\zeta''-\zeta'}\int_{\xi'}^{\xi''}\int_{u'}^{u''}\int_{\zeta'}^{\zeta''} |f(u,v,w)|dwdudv$$

$\forall\zeta'$, ζ'' such that $\zeta'' < z < \zeta'$, so

$$g_*(x,y,z) = \limsup_{\substack{\xi'<x<\xi'' \\ u'<y<u'' \\ \xi''-\xi\to 0\ ,\ u''-u'\to 0}} \frac{1}{\xi''-\xi'}\cdot\frac{1}{u''-u'}\int_{\xi'}^{\xi''}\int_{u'}^{u''} g(u,v,z)dudv \geq$$

$$\geq \limsup_{\substack{\xi'<x<\xi'' \\ u'<y<u'' \\ \zeta'<z<\zeta'' \\ \xi''-\xi\to 0\ ,\ u''-u'\to 0\ ,\ \zeta''-\zeta'\to 0}} \frac{1}{\xi''-\xi'}\cdot\frac{1}{u''-u'}\cdot\frac{1}{\zeta''-\zeta'}\int_{\xi'}^{\xi''}\int_{u'}^{u''}\int_{\zeta'}^{\zeta''} |f(u,v,w)|dwdudv$$

so $g_*(x,y,z) \geq f_*(x,y,z)$. (23)

Since $a(t), b(t)$ satisfy the properties of Theorem 5.3.3 we have

$$g_*(x,y,z) = g(x,y,z) \text{ at almost every point } (x,y,z) \ \ of\ S$$

and by the inequalities (22), (23) we get

$$f_*(x,y,z) \leq g(x,y,z) \Rightarrow \int_S f_*(P)dP \leq \int_S g(P)dP \leq A\int_S |f|log^+|f| + B. \quad \square$$

We will now prove that Theorem 5.3.1 cannot be strengthened.

Assume $\phi(t)$, $0 \leq t < \infty$, is a strictly increasing function satisfying

$$\phi(0) = 0, \quad \liminf_{t \to \infty} \frac{\phi(t)}{t} > 0 \qquad (24)$$

and L_ϕ denotes the class of functions f such that $\phi(|f|)$ is integrable over S.

Therefore by (24), $L_\phi(S) \subseteq L^1(S)$.

Lemma 5.3.6: *Assume E is an arbitary bounded and measurable set and $\sigma_a(E)$, $0 < a < 1$, is the union of all intervals I for which*

$$|E \cap I| > a|I|. \qquad (25)$$

If the differentiability theorem holds for all the functions of the class L_ϕ then the inequality

$$|\sigma_a(E)| \leq c\phi(\frac{1}{a})|E| \qquad (26)$$

is true for all E and all a, where C is a constant independent of a and E.

Proof: We will prove this lemma by contradiction.
Suppose (26) is not true. We will prove that there is a function f in L_ϕ for which the differentiation theorem is false.

Let c_n be positive numbers chosen such that

$$\sum_n \frac{1}{c_n} < \frac{1}{2}\phi(1). \qquad (*)$$

By our assumption there exists, for every n, a bounded and measurable set E_n and a number a_n, $0 < a_n < 1$ such that

$$|\sigma_a(E_n)| > c_n\phi(\frac{1}{a_n})|E_n|.$$

For every I_n such that $\dfrac{|E_n \cap I_n|}{|I_n|} > a_n$ there exists c'_n with $\delta(I_n) < c'_n \qquad (**)$

We write $\sigma_{a_n}(E_n) = H_n$ and for every n, through a homothecy $\psi_{n,k}$ with ratio $\lambda_{\psi_{n,k}}$ such that $\lambda_{\psi_{n,k}} \cdot c'_n \xrightarrow{n\to\infty} 0$, we choose a sequence of sets H_n^k, which are homothetic to H_n, cover S except for a null set and satisfy the condition

$$\sum_k |H_n^k| < 2|S| = 2. \qquad (***)$$

We can do so by fixing n and writing H for H_n. We then let K be a closed subset of H such that $|H| < 2|K|$ and let I be a square containing K. We divide S into an enumerable number of disjoint squares $\{I_i\}_{i\in\mathbb{N}}$.
We write

$$k = \frac{|K|}{|I|} \quad \textit{and so} \quad |\bigcup_{i=1}^{\infty} K^i| = \sum_{i=1}^{\infty} |K^i| = k \sum_{i=1}^{\infty} |I^i| = k|S|.$$

Therefore there exists $p_1 \in \mathbb{N}$ such that

$$\sum_{i=1}^{p_1} |K^i| > k'|S| \ , \quad \textit{where } k' < k \ ,$$

and thus we can get a finite number of squares I^{p_1} and consider, for each p_1, the sets H^{p_1} and K^{p_1} derived from H and K by the same homothetical application by which I is carried over in I^{p_1}.
We then have

$$S_1 = S - \sum_{p_1} K^{p_1} \quad \textit{and so} \quad |S_1| = 1 - k'.$$

Using Whitney's Lemma we can divide S_1, except for a null set, into a finite or enumerable number of squares and proceed with each of these squares in exactly the same manner as we proceeded with S. This way we get the sets H^{p_2} and K^{p_2} so that by writing

$$S_2 = S_1 - \sum_{p_2} K^{p_2}$$

we have

$$|S_2| = (1-k')|S_1| = (1-k')^2.$$

Continuing this process and denoting by H^k and K^k the sets $H^{p_1}, H^{p_2}, ...$ and $K^{p_1}, K^{p_2}, ...$ respectively, the sets H^k will satisfy the conditions, since already the sets K^k will cover S except for a null set,

$$\sum_k |K^k| = 1 \quad \textit{and} \quad |H^k| < 2|K^k| \quad \textit{for each } k.$$

Now let E_n^k be the set derived by the same homothetical application on E_n by which H_n is carried over in H_n^k.
We assumed before $|H_n| > c_n\phi(\frac{1}{a_n})|E_n|$ and so

$$|H_n^k| > c_n\phi(\frac{1}{a_n})|E_n^k|. \qquad (****)$$

Set $f_n(P) = \frac{1}{a_n}$ in the set $S \cap (\bigcup_k E_n^k) \qquad (*****)$

and $f_n(P) = 0$ in the remaining points of S and $f(P) = \sup_n f_n(P)$ so we have

$$\int_S \phi(f(P))dP \leq \sum_n \int_S \phi(f_n(P))dP \leq_{(*****)} \sum_n \sum_k \phi(\frac{1}{a_n})|E_n^k| \leq$$

$$\leq_{(****)} \sum_n \sum_k \phi(\frac{1}{a_n})\frac{|H_n^k|}{c_n\phi(\frac{1}{a_n})} \leq_{(***)} 2|S|\sum_n \frac{1}{c_n} <_{(*)} \phi(1). \qquad (28)$$

so

$$\int_S \phi(f(P))dP < \phi(1)$$

and so f belongs in L_ϕ.

Also, for every n, almost every point P of S belongs to at least one $H_n^k = \psi_{n,k}(H_n)$ so there exists $P' \in H_n = \sigma_{a_n}(E_n)$ with $\psi_{n,k}(P') = P$ and as $P' \in \sigma_{a_n}(E_n)$ there exists I_n such that $\dfrac{|E_n \cap I_n|}{|I_n|} > a_n$ and by (**) we get $\delta(I_n) < c'_n$.
For $I_n^k = \psi_{n,k}(I_n)$ we get $\delta(I_n^k) = \lambda_{\psi_{n,k}} \cdot \delta(I_n) < \lambda_{\psi_{n,k}} c'_n \to 0$.

Therefore P belongs in an interval I_n^k such that

$$|E_n^k \cap I_n^k| > a_n|I_n^k| \qquad (*****)$$

where $\delta(I_n^k) \xrightarrow{n\to\infty} 0$.

As $f(P) \geq \frac{1}{a_n}$ for $P \in S \cap E_n^k$, we obtain

$$\frac{1}{|I_n^k|}\int_{I_n^k} f(P)dP \geq \frac{1}{|I_n^k|}\frac{1}{a_n}|I_n^k \cap E_n^k| \geq_{(*****)} \frac{1}{|I_n^k|}\frac{1}{a_n} \cdot a_n|I_n^k| = 1.$$

Supposing that the differentiability theorem holds for f, we get $f(P) \geq 1$ for almost every point and so

$$\int_S \phi(f(P))dP \geq \phi(1)$$

a contradiction to (28) and the lemma is proved. $\Box$

Theorem 5.3.7: *If for every f in L_ϕ the integral of f is strongly differentiable almost everywhere, then*

$$\phi(t) > c \cdot t(log^+ t)^{k-1}$$

where $c > 0$ a constant.
In other words, $f(log^+|f|)^{k-1}$ is integrable over S.

Proof: For simplicity we give the proof for $k = 2$.
We will use the lemma we just proved.
We take S as the set E, $S = \{(x, y) : 0 < x < 1\ ,\ 0 < y < 1\}$ and then $\sigma_a(E)$ contains the subset $\{(x, y) : 1 \leq x \leq \frac{1}{a}, \quad 0 \leq xy \leq \frac{1}{a}\}$.

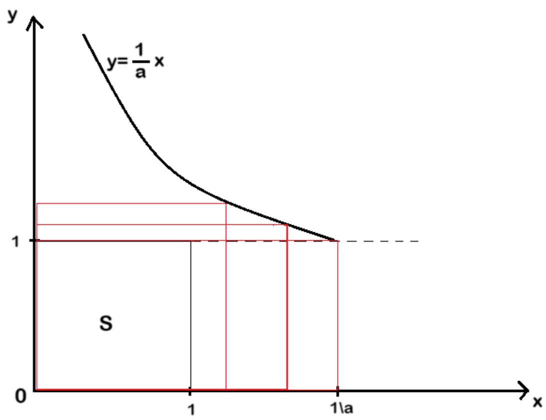


Figure 5.3.5

Hence using Lemma 5.3.6

$$\sigma_a(E) > \int_1^{\frac{1}{a}} \frac{1}{ax} dx = \frac{1}{a} log \frac{1}{a} |E| \quad \Rightarrow \frac{1}{a} log \frac{1}{a} < c\phi(\frac{1}{a})$$

for all $0 < a < 1$ and so $\phi(t) > ct \cdot log^{+} t$ which completes the proof. $\square$

Chapter 6

Bibliography

1. Calderon-Zygmund Theory and Applications - Master's Thesis Anastasios Karamitros 2024

2. Equivalence between the regularity property and the differentiation of L^1 for a homothecy invariant basis, R. Moriyon [1975] - Appendix III in: M. De Guzman, Differentiation of integrals in $\mathbb{R}^n$- Springer, Berlin, 1975

3. Interpolation of Operators- C. Bennet, R. Sharpley- Academic Press INC 1988

4. Lecture Notes - M. De Guzman - Springer-Verlag 1975

5. Maximal functions for rectangles with given directions (Thesis,Mittag-Leffler Institute, Djursholm, Sweden)- J. Stromberg [1976]

6. Note on differentiability of multiple integrals -B. Jessen, J. Marcinkiewizc, A. Zygmund

7. On differentiation of integrals, A. Cordoba and R. Fefferman [1977], Proc. Natl. Acad. USA 74

8. On the strong derivatives of functions of an interval -Saks,S. [1935] - Fund. Math. 25 1935

9. Real and Abstract Analysis - Edwin Hewitt, Karl Stromberg- Springer-Verlag

10. Real Variable Methods in Fourier Analysis- M. De Guzman- North Holland 1981

11. Sur un theoreme de M. Vitali, S. Banach [1924] , Fund. Math. 5 1924

This thesis contains some figures from "Real Variable Methods in Fourier Analysis- M. De Guzman- North Holland 1981" and "Interpolation of Operators- C. Bennet, R. Sharpley- Academic Press INC 1988".