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Γραμμικές απεικονήσεις γραφημάτων με διπλότοξα

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΑΤΡΙΒΗ

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Linear graph layouts with biarcs

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Αφιερώνεται στην Οικογένεια μου.

Η παρούσα Μεταπτυχιακή Διατριβή εκπονήθηκε στο πλαίσιο των σπουδών για την απόκτηση του Μεταπτυχιακού Διπλώματος Ειδίκευσης στα Εφαρμοσμένα Μαθηματικά και Πληροφορική που απονέμει το Τμήμα Μαθηματικών του Πανεπιστημίου Ιωαννίνων.

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"Δηλώνω υπεύθυνα ότι η παρούσα διατριβή εκπονήθηκε κάτω από τους διεθνείς ηθικούς και ακαδημαϊκούς κανόνες δεοντολογίας και προστασίας της πνευματικής ιδιοκτησίας. Σύμφωνα με τους κανόνες αυτούς, δεν έχω προβεί σε ιδιοποίηση ξένου επιστημονικού έργου και έχω πλήρως αναφέρει τις πηγές που χρησιμοποίησα στην εργασία αυτή."

Γεώργιος Βελισσάρης

Ευχαριστιες

Με την ολοκλήρωση της παρούσας διατριβής, θα ήθελα να εκφράσω τις ευχαριστίες μου προς τους ανθρώπους των οποίων η συμβολή ήταν καθοριστική για την επιτυχή διεκπεραίωση αυτής της μελέτης.

Αρχικά θα ήθελα να ευχαριστήσω θερμά τον επιβλέποντα καθηγητή μου κ. Μιχαήλ Μπέκο, για την αμέριστη συμπαράσταση, την καθοδήγηση, τις πολύτιμες συμβουλές του, και τον χρόνο που αφιέρωσε κατά τη διάρκεια της εκπόνησης της εργασίας.

Επιπλέον θέλω να ευχαριστήσω όλους του διδάσκοντες του Τμήματος Μαθηματικών για τις πολύτιμες γνώσεις που μου μετέδωσαν, καθώς και για την εποικοδομητική συνεργασία που είχαμε κατά τη διάρκεια των Προπτυχιακών και Μεταπτυχιακών μου σπουδών.

Το μεγαλύτερο ευχαριστώ απευθύνεται προς στους γονείς μου και την οικογένειά μου, των οποίων η αδιάκοπη στήριξη και ενθάρρυνση υπήρξαν καταλυτικές για την ολοκλήρωση των μεταπτυχιακών μου σπουδών.

Περιληψη

Στην παρούσα διατριβή εισάγουμε και μελετάμε μία νέα παραλλαγή γραμμικών απεικονίσεων γραφημάτων σύμφωνα με την οποία οι κορυφές του γραφήματος διατάσσονται κατά μήκος μιας ευθείας γραμμής, η οποία συχνά αναφέρεται ως *ράχη*, ενώ οι ακμές του γραφήματος διαμερίζονται σε σύνολα, τα οποία ονομάζονται *σελίδες*. Στην παραλλαγή που μελετάμε: (i) κάθε σελίδα αντιστοιχεί σε ένα διακριτό επίπεδο που περιέχει τη ράχη, (ii) κάθε ακμή απεικονίζεται είτε ως ημικύκλιο άνωθεν ή κάτωθεν της ράχης είτε ως δύο ημικύκλια εκατέρωθεν της ράχης τα οποία έχουν ένα κοινό σημείο που βρίσκεται στη ράχη και ανάμεσα στα δύο άκρα της ακμής και (iii) κάθε δύο ακμές της ίδιας σελίδας δεν τέμνονται. Αναφερόμαστε σε τέτοιες γραμμικές απεικονίσεις ως *μονότονες με διπλοτόξα*. Δοθέντος ενός γραφήματος G, το ενδιαφέρον μας εστιάζει στον ελάχιστο αριθμό $bn(G)$ σελίδων που απαιτούνται για την ύπαρξη μίας γραμμικής απεικόνισης με μονότονα διπλοτόξα.

Αποδεικνύουμε ότι για το πλήρες γράφημα K_n ισχύει $bn(K_n) \leq \left\lceil \frac{n}{4} \right\rceil$. Το αποτέλεσμα αυτό επιτυγχάνεται μέσω μίας γενικής κατασκευής που αποδίδει διαφορετικές γραμμικές απεικονίσεις με μονότονα διπλοτόξα, στα οποία ο αριθμός των ακμών, που απεικονίζονται ως διπλοτόξα, μπορεί να κυμαίνεται από 0 έως $\frac{n^2}{8} - \frac{n}{4} + 2$. Για το πλήρες διμερές γράφημα $K_{n,n}$ δείχνουμε ότι $bn(K_{n,n}) \leq \left\lceil \frac{n}{3} \right\rceil + 1$ στην περίπτωση που οι κορυφές του πρώτου μέρους του $K_{n,n}$ προηγούνται των κορυφών του δεύτερου του μέρους. Πέραν αυτών των αποτελεσμάτων, τα οποία είναι θεωρητικής φύσης, αναπτύξαμε και μία διατύπωση SAT για το πρόβλημα του ελέγχου εάν ένα δοθέν γράφημα επιδέχεται μιας γραμμικής απεικόνισης με διπλοτόξα σε συγκεκριμένο αριθμό σελίδων. Τέλος, ενσωματώσαμε την υλοποίησή μας σε ένα υπάρχον client-server λογισμικό, το οποίο υποστηρίζει διάφορους τύπους γραμμικών απεικονίσεων, συμπεριλαμβανομένων και αυτών της στοίβας και της ουράς.

Abstract

In this thesis, we introduce and study on a new variant of linear layouts in which the vertices are arranged along a straight line, commonly referred to as *spine*, and the edges are partitioned into a certain number of parts, called *pages*, such that: (i) each page is a distinct plane containing the spine, (ii) each edge is drawn either as a half-circle above the spine or as a half-circle below the spine, or as two half-circles on opposite sides of the spine with a single common point located on the spine and inbetween the two endvertices of the edge, and (iii) no two edges of the same page cross. We refer to such linear layouts as *monotone with biarcs*. Given a graph, our interest is on its *biarc number*, that is, the minimum number of pages that are required for a linear layout with monotone biarcs to exist.

Our contribution is as follows: We prove that the biarc number of K_n is at most $\lceil \frac{n}{4} \rceil$. This result is obtained via a general construction (of independent interest) which yields different linear layouts with monotone biarcs, in which the number of edges drawn as biarcs can be adjusted from 0 to $\frac{n^2}{8} - \frac{n}{4} + 2$. We further show that the biarc number of $K_{n,n}$ is at most $\lceil \frac{n}{3} \rceil + 1$ in the *separated setting*, namely, when all vertices of one part of $K_{n,n}$ precede those of its second part. Besides these results, which are of theoretical nature, we also developed a SAT formulation for the problem of testing whether a given graph admits a linear layout with biarcs on a certain number of pages. We integrated our implementation into an existing client-server tool, which supports various types of linear layouts, including the well-known stack and queue layouts.

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CHAPTER 1

Introduction

Linear layouts have a significant role in several aspects of Computer Science and in particular in Topological Graph Theory. For a given a graph, a linear layout of it consists of an ordering $\prec$ of its vertices and a partition of its edges into parts, called *pages*. The set of edges that are allowed to coexist in a page must avoid certain forbidden patterns and different forbidden patterns for the edges coexisting in the same page result in different types of linear layouts.

In the following, we introduce two existing types of linear layouts, which are relevant to our work; in particular, the well-studied stack and queue layouts. We then introduce and motivate the variant that we studied in this thesis.

1.1 Stack Layouts

In the literature, stack layouts are also known as book embeddings. In this type of linear layouts, the edges are partitioned into pages, called *stacks*, that avoid the following forbidden pattern: Two distinct independent edges (u, v) and (z, w), such that without loss of generality $u \prec v$ and $z \prec w$, cannot be in the same stack if and only if $u \prec z \prec v \prec w$ or $z \prec u \prec w \prec v$. In other words, one edge cannot *cross* the other. Given a graph G, the minimum number of stacks that are required in order for G to admit a stack layout is called the *stack number* of G and it is denoted as $sn(G)$. Note that in the literature the stack number of a graph is also referred to as *book thickness* and *page number*.

Since stack layouts form a deeply-studied topic in Topological Graph Theory, there exist a plethora of results proposed in the literature. Since our work focuses mainly on complete and complete bipartite graphs, we mention here two related results by Bernhart and Kainen [3] and by Enomoto et al. [5]. The former in 1979 proved that the stack number of the complete graph with

n vertices is $\lceil \frac{n}{2} \rceil$, that is, $sn(K_n) = \lceil \frac{n}{2} \rceil$. The latter proved that the stack number of the complete bipartite graph $K_{n,n}$ is at most $\lfloor \frac{2n}{3} \rfloor + 1$, that is, $sn(K_n) \leq \lfloor \frac{2n}{3} \rfloor + 1$.

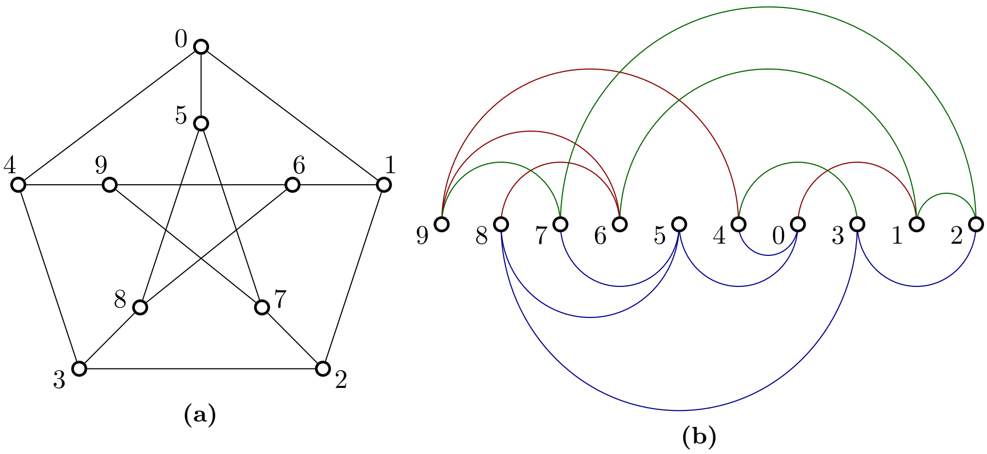


Figure 1.1: (a) The Petersen's graph and (b) one of its possible stack layouts with three stacks where edges of the same color are in the same stack.

Since the subgraphs induced by the edges of each page of a stack layout form an outerplanar graph, it follows that the stack number of a graph is lower bounded by its *thickness*, where the thickness of a graph is defined as the minimum number of planar subgraphs in which the graph can be decomposed.

1.2 Queue Layouts

In a queue layout, the partition of the edges is done into pages, called *queues*, that avoid the following forbidden pattern: Two distinct independent edges (u, v) and (z, w) such that without loss of generality $u \prec v$ and $z \prec w$ cannot be on the same queue if and only if $u \prec z \prec w \prec v$ or $z \prec u \prec v \prec w$. In other words, one edge cannot *nest* the other. Given a graph G, the minimum number of queues that are required in order for G to admit a queue layout is called the *queue number* of G and it is denoted as $qn(G)$.

The queue number of the complete and the complete bipartite graphs have been studied by Heath and Rosenberg [7]. More precisely, the queue number of K_n is $\lfloor \frac{n}{2} \rfloor$, while the queue number of $K_{n,n}$ is $\lceil \frac{n}{2} \rceil$, that is, $qn(K_n) = \lfloor \frac{n}{2} \rfloor$ and $qn(K_{n,n}) = \lceil \frac{n}{2} \rceil$.

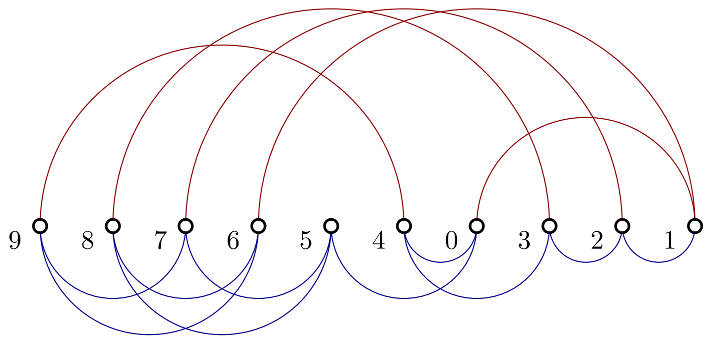


Figure 1.2: A queue layout of Petersen's graph with two queues, where edges of the same color are in the same queue.

1.3 Linear Layouts with Biarcs

A common approach to visualize a linear layout of a graph is by laying out its vertices along a straight line, commonly refer to as *spine*, from left to right according to $\prec$ and then draw its edges as half-circles connecting their endpoints. Usually, the edges of each page (stack or queue) are drawn with the same color in one of the two half planes bounded by the spine as seen in Fig. 1.1b and Fig. 1.2. By definition, in a stack (queue) layout, the half-circles corresponding to two edges of the same stack (queue) do not cross (nest).

In the model that we introduce and study in this thesis, each *page* in the layout is a whole plane containing the spine (as opposed to stack and queue layouts, in which their pages are half-planes). Under this assumption, an edge in the layout can be drawn either as a half-circle above or as a half-circle below the spine, or as two half-circles on opposite sides of the spine with a single common point located on the spine and inbetween the two endvertices of the edge. The later way of representing an edge is referred to as *monotone biarcs* in the literature; see, e.g., [4]. Furthermore, representations of planar graphs in which the edges are allowed to cross the spine are referred to as *topological book embeddings* in the literature; see, e.g., [9]. In a *linear layout with monotone biarcs* of a graph the task is to determine a linear order of its vertices and a partition of its edges into pages such that the edges belonging to the same page admit a planar representation under the restrictions described above. The *biarc number* of a graph, denoted by $bn(G)$, is the minimum number of pages that are required for a linear layout with monotone biarcs to exist. The next theorem provides a trivial upper bound on the biarc number of a graph.

Theorem 1.3.1. *The biarc number of a graph G is at most half of its stack number, that is, $bn(G) \leq \lceil sn(G)/2 \rceil$.*

Proof. Let $\mathcal{L}$ be a stack layout of graph G with $sn(G)$ stacks. By definition, any two stacks s and s' of G form a biarc page, since one can draw as half-circles all edges of s above the spine and all edges of s' below the spine. It follows that no two edges of the resulting layout will cross with each other, which by definition yields a biarc page without monotone biarcs. This implies that $bn(G) \leq \lceil sn(G)/2 \rceil$, as desired. □

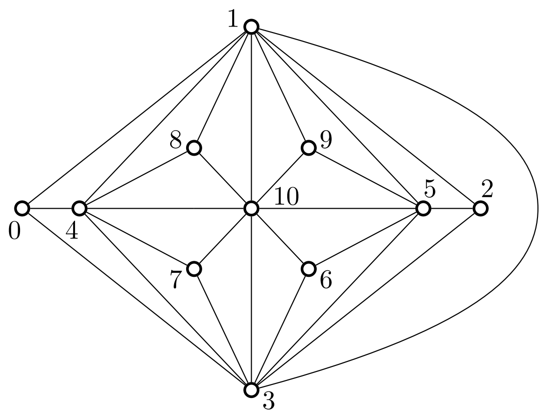
Theorem 1.3.1 provides an upper bound on the biarc number of a graph. A lower bound can be derived by leveraging the *thickness* of the graph, which is formally defined as the minimum number of planar graphs a given graph can be decomposed to. Given a graph G, the thickness of G is commonly denoted by $t(G)$. The next theorem follows by the definitions of biarc number and thickness.

Theorem 1.3.2. *The biarc number of a graph G is lower bounded by its thickness, that is, $t(G) \leq bn(G)$.*

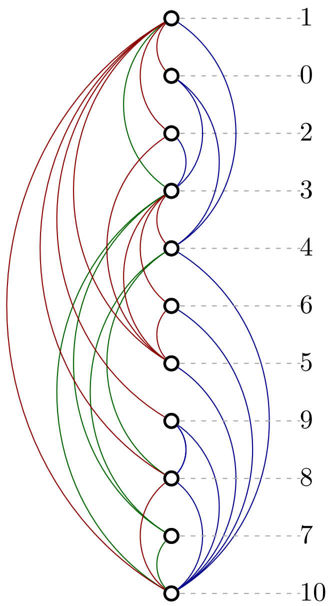
Proof. Since each page of a biarc layout $\mathcal{L}$ of graph G is a planar graph, it follows that $\mathcal{L}$ cannot have less that $t(G)$ pages, as otherwise one can decompose G into less that $t(G)$ planar graphs; a contradiction to the definition of $t(G)$. This implies that $t(G) \leq bn(G)$, as desired. □

Our motivation for studying linear graph layouts with monotone biarcs stems from the following observations that can be made for planar graphs. More presicely, since the stack number of the class of planar graphs is 4 [11, 2] (that is, every planar graph admits a stack layout with four stacks [11], while there exist planar graphs that require four stacks [2]), Theorem 1.3.1 implies that every planar graph admits a linear layout with biarcs on at most two pages. On the other hand, it is known that each planar graph admits a linear layout with biarcs on a single page with at most $\frac{15n}{16}$ edges drawn as biarcs [4] with n being the number of vertices of the graph.

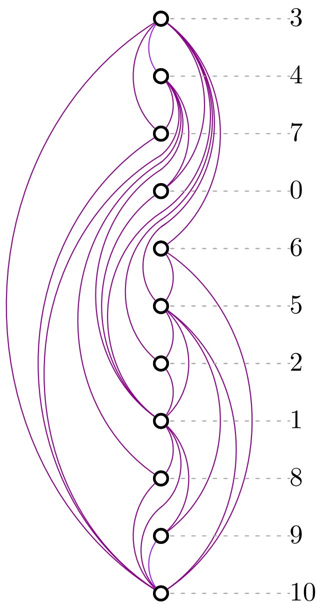
A concrete example is the Goldner Harary graph [6], which is a maximum planar graph consisting of 11 vertices and 27 edges; see Fig. 1.3a. Even though this graph has stack number of 3 [6] (for a linear layout with three stacks, see Fig. 1.3b), it requires only a single page for its biarc layout, as depicted in Fig. 1.3c.



(a) The Goldner Harary graph.



(b) A stack layout with three stacks.



(c) A single-page linear layout with monotone biarcs.

Figure 1.3: (a) The Goldner Harary graph and (b)-(c) different linear layouts of it, in which edges with the same col are of the same page.

The aforementioned observations imply that the upper bound of Theorem Theorem 1.3.1 is not tight. Hence, it is tempting to study other graph classes for which improved upper bounds can be obtained.

1.4 Thesis Contribution

Our research focused on bounds on the biarc number of the complete graph K_n and of the complete bipartite graph $K_{n,n}$ (for definitions refer to Chapter 2).

- For the former, we present a construction that yields biarc layouts with monotone biarcs on $\lceil \frac{n}{4} \rceil$ pages. Since the stack number of K_n is $\lceil \frac{n}{2} \rceil$, our result does not improve the upper bound that one would obtain by an application of Theorem 1.3.1. However, it forms a general construction of independent interest, as it yields different layouts in which the number of biarcs ranges from 0 to $\frac{n^2}{8} - \frac{n}{4} + 2$.

- For the latter, since the best-known upper bound on the stack number of $K_{n,n}$ is $\lfloor \frac{2n}{3} \rfloor + 1$ [5], the upper bound that one obtains on the biac number of $K_{n,n}$ by an application of Theorem 1.3.1 is $\lceil \frac{\lfloor \frac{2n}{3} \rfloor + 1}{2} \rceil$. In this thesis, we show that the biarc number of $K_{n,n}$ is $\lceil \frac{n}{3} \rceil + 1$ in the *separated setting*, namely, when all vertices of one part of $K_{n,n}$ precede those of its second part. It is worth noting that the two bounds are equal when n is a multiple of 3; otherwise ours is by one worse than the one derived by Theorem 1.3.1.

The upper bounds that we introduced above can be coupled by corresponding lower bounds that one can derived from Theorem 1.3.2 and the thickness of K_n and $K_{n,n}$. For the former, Mutzel et al. [10] proved that $t(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor$, for $n \neq 9, 10$ and $t(K_9) = t(K_{10}) = 3$, while for the latter Hu and Chen [8] proved that $t(K_{n,n}) = \lceil \frac{n}{4} \rceil$. We summarize these results in the following two theorems:

Theorem 1.4.1. *For $n > 10$, the biarc number of the complete graph K_n is at least $\left\lfloor \frac{n+7}{6} \right\rfloor$ and at most $\lceil \frac{n}{4} \rceil$.*

Theorem 1.4.2. *The biarc number of the complete bipartite graph $K_{n,n}$ (in the separated setting) is at least $\left\lfloor \frac{n}{4} \right\rfloor$ and at most $\lceil \frac{n}{3} \rceil + 1$.*

Besides the aforementioned results, which are of theoretical nature, we also developed a SAT formulation for the problem of testing whether a given graph admits a linear layout with biarcs on a certain number of pages. We integrated our formulation into an existing client-server tool [1], which supports different types of linear layouts (including stack and queue layouts).

1.5 Thesis Structure

The rest of this thesis is structured as follows:

- Chapter 2 summarizes preliminary definitions and notions that are used in the remaining parts of the thesis.
- In Chapter 3, we present our constructions for obtaining the linear layouts that we described above for K_n and $K_{n,n}$.
- In Chapter 4, we present our SAT formulation for the problem of testing whether a given graph admits a linear layout with biarcs on a certain number of pages.
- The thesis is concluded in Chapter 5 by listing our future plans, applications and open problems raised by our work.

CHAPTER 2

Preliminaries

In this section, we present preliminary definitions and notions that are used in the remaining parts of the thesis.

2.1 Complete and Complete Bipartite Graphs

Definition 2.1.1. *A graph is called* complete *if and only if every two distinct vertices of it are connected by an edge.*

The complete graph on n vertices is commonly denoted by K_n. It is well known that the number of edges of K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$.

Definition 2.1.2. *A graph is called* complete bipartite *if its vertex set can be partitioned into two independent sets A and B such that every vertex in A is connected with every vertex in B.*

If, in the aforementioned definition, $|A| = a$ and $|B| = b$ holds, then the obtained complete bipartite graph is denoted by $K_{a,b}$. In this thesis, we focus on complete bipartite graphs in which $|A| = |B|$.

2.2 A Method For Computing Stack Layouts for Complete Graphs.

There exist several methods for obtaining a stack layout of the compete graph K_n with $\lceil \frac{n}{2} \rceil$ stacks. In this section, we present the method by Bernhart and Kainen [3], which is the earliest method yielding a stack layout $\mathcal{L}$ of K_n with $\lceil \frac{n}{2} \rceil$ stacks. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of K_n and assume without loss of generality $v_0 \prec v_1 \prec \ldots \prec v_{n-1}$ holds in $\mathcal{L}$. Assuming that the indices are taken $\texttt{mod}\ n$, for $i \in [0, \lceil \frac{n}{2} \rceil - 1]$, the edges assigned to the i-stack p_i of $\mathcal{L}$ are the following:

$$\left(v_{\lceil \frac{n}{2} \rceil + i - 1 - \lfloor \frac{j}{2} \rfloor}, v_{\lceil \frac{n}{2} \rceil + i - 1 + \lceil \frac{j}{2} \rceil}\right), \quad 1 \leq j \leq n-1$$

It is not difficult to see that each page in $\mathcal{L}$ contains exactly $n-1$ edges which induce a path. Also, no edge of K_n is assigned to two distinct pages of $\mathcal{L}$, which implies that $\mathcal{L}$ is a valid stack layout of K_n, since the total number of edges in $\mathcal{L}$ is $\frac{n(n-1)}{2}$.

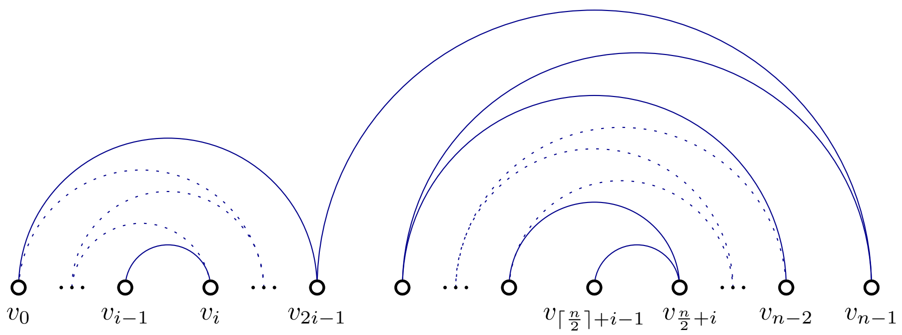


Figure 2.1: Illustration of the i-th stack p_i of a stack layout of K_n with $\lceil \frac{n}{2} \rceil$ stacks computed using the method by Bernhart and Kainen [3].

2.3 A General SAT Formulation for Different Types of Linear Layouts.

Let X a set of n boolean variables $x_1, x_2, \ldots, x_n$. A *term* over X is either x_i or its negation $\neg x_i$ and a clause is a disjunction of distinct terms e.g., $(x_2 \vee \neg x_6 \vee x_1)$. A *truth assignment* for X is an assignment of either "true" or "false" to each variable x_i. A clause C is *satisfied* if at least on of the terms in it has received the value "true". For a collection of clauses $C_1, C_2, \ldots, C_k$, the assignment satisfies the collection if the conjunction of all the clauses $\Phi = C_1 \wedge C_2 \wedge \ldots \wedge C_k$ evaluates to "true". This problem is known as Boolean satisfiability problem more commonly refereed as SAT.

As part of this thesis, we expanded a preexisting project introduced by Bekos et al. [1], which formulates various types of linear layouts as SAT instances. In the following, we recall the most important aspects of this formulation. Given a graph G, for every pair of distinct vertices u and v of G, the formulation has a variable $\sigma(u, v)$, which is true if and only if vertex u has precedes v in the order of the layout, namely, $u \prec v$. Further, for every edge e of G, the formulation has a variable $\phi_p(e)$, which is true if and only if e is assigned to page p. Finally, for every pair of two distinct edges e and e' of G, the formulation has a variable $\chi(e, e')$, which is true if and only if e and e' are both assigned in the same page. If n and m are the number of vertices and edges of G, respectively, a set of $O(n^3 + m^2)$ clauses ensures that the underlying order is indeed linear, and that no two edges of the same page form a forbidden pattern (that is, cross in a stack or nest in a queue).

CHAPTER 3

Bounds on the Biarc-Number of Complete and Compete Bipartite Graphs

This section is devoted on upper bounds on the biarc-number of the complete graph K_n (Section 3.1) and of the compete bipartite graph $K_{n,n}$ (Section 3.2).

3.1 The Upper Bound on the Biarc Number of K_n

In the following, we assume that n is a multiple of 4 and we prove that K_n admits a linear layout with monotone biarcs in $\frac{n}{4}$ pages. This implies that the biarc number of K_n is at most $\lceil \frac{n}{4} \rceil$ and proves Theorem 1.4.1. In our construction, we assume that there exist integers x and y, such that:

$$n = x + y, \quad x \bmod 2 = 0, \quad n \bmod \frac{x}{2} = 0, \quad n \bmod \frac{y}{2} = 0, \quad x \leq \frac{n}{2} \qquad (3.1)$$

Since n is a multiple of 4, a feasible solution of Eq. (3.1) can be derived by, e.g., setting $x = y = \frac{n}{2}$. Another feasible solution can be derived by setting $x = 0$ and $y = n$. To simplify the presentation, we assume the vertices of K_n are denoted by $v_0, \ldots, v_{x-1}, u_0, \ldots, u_{y-1}$ (colored white and gray in Fig. 3.1, respectively) and their order in the constructed layout $\mathcal{L}$ is:

$$v_0 \prec \ldots \prec v_{\frac{x}{2}-1} \prec u_0 \prec \ldots \prec u_{y-1} \prec v_{\frac{x}{2}} \prec \ldots \prec v_{x-1}$$

We further assume in the following that all indices at $u-$vertices are taken $\bmod\ y$, while all indices at $v-$vertices are taken $\bmod\ x$. Under these assumptions, we next describe how to assigned the edges of K_n to the $\frac{n}{4}$ pages

$p_0, \ldots, p_{\frac{n}{4}-1}$ of $\mathcal{L}$. We partition the available pages into two sets. The first set consists of the pages $p_0, \ldots, p_{\frac{x}{2}-1}$, while the second set consists of the remaining available pages, namely, $p_{\frac{x}{2}}, \ldots, p_{\frac{n}{4}-1}$. In the following, we first describe the edges in the first set of the partition. In particular, for each $i \in [0, \frac{x}{2}-1]$, we describe which edges are assigned to page p_i of $\mathcal{L}$ and their type (arc above or arc below the spine or biarc).

Page p_i, with $i \in [0, \frac{x}{2}-1]$, of $\mathcal{L}$ contains the following $y+2i+3$ edges, drawn as arcs above the spine:

- $(u_{\frac{y}{2}+i-1-\lfloor\frac{j}{2}\rfloor}, u_{\frac{y}{2}+i-1+\lceil\frac{j}{2}\rceil})$, $\frac{y}{2}-2i \le j \le \frac{y}{2}-1$
- (v_i, u_j) $\quad 2i-1 \le j \le \frac{y}{2}+i-1$
- $(u_{\frac{y}{2}-1-j}, v_{x-1-i})$ $\quad i \le j \le \frac{y}{2}$
- (v_i, v_{x-1-j}) $\quad 0 \le j \le i$
- (v_j, v_i) $\quad 0 \le j \le i-1$

Page p_i, with $i \in [0, \frac{x}{2}-1]$, of $\mathcal{L}$ contains the following $\frac{y}{2}+2i$ edges, drawn as arcs below the spine:

- $(u_{\frac{y}{2}+i-1-\lfloor\frac{j}{2}\rfloor}, u_{\frac{y}{2}+i-1+\lceil\frac{j}{2}\rceil})$, $\quad 1 \le j \le y-2i-1$
- $(u_j, v_{\frac{x}{2}-i-1})$ $\quad i-1 \le j \le 2i-1$
- (v_i, u_j) $\quad 0 \le j \le i-1$
- (v_j, v_{x-i-1}) $\quad 0 \le j \le i-1$
- (v_{x-i-1}, v_{x-j}) $\quad 1 \le j \le i$

Page p_i, with $i \in [0, \frac{x}{2}-1]$, of $\mathcal{L}$ contains the following $\frac{y}{2}-1$ edges, drawn as biarcs starting above the spine and ending below the spine; in the following, the former intersect the spine between the vertices $u_{\frac{y}{2}+2i-2+j}$ and $u_{\frac{y}{2}+2i-2+j}$, while the latter between the vertices u_{2i-2-j} and u_{2i-1-j}.

- $(v_i, u_{\frac{y}{2}+j})$ $\quad i \le j \le \frac{y}{2}-1$
- (u_j, v_{x-i-1}) $\quad 0 \le j \le i-2$

Page p_i, with $i \in [0, \frac{x}{2}-1]$, of $\mathcal{L}$ contains the following $\frac{y}{2}-2$ edges, drawn as biarcs starting below the spine and ending above the spine; in the following, the former intersect the spine between the vertices u_{i-2+j} and u_{i-1+j}, while the latter between the vertices $u_{y+2i-2-j}$ and $u_{y+2i-1-j}$.

- (v_i, u_{i+j}) $\quad 0 \leq j \leq i-2$
- (u_j, v_{x-i-1}) $\quad 2i \leq j \leq \frac{y}{2}+i-2$

The schemization of Fig. 3.1 shows that the edges assigned to page p_i, with $i \in [0, \frac{x}{2}-1]$, (as described above) do not cross in $\prec$. Furthermore, the total number of edges that have been assigned to pages $p_0, \ldots, p_{\frac{x}{2}-1}$ of $\mathcal{L}$ is:

$$\sum_{i=0}^{\frac{x}{2}-1} \Big((y+2i+3)+(y+2i)+(\frac{y}{2}-1)+(\frac{y}{2}-2)\Big) = \sum_{i=0}^{\frac{x}{2}-1}(3y+4i) = \frac{1}{2}x(x+3y-2)$$

To complete the proof, we now turn our attention to the remaining pages in $\mathcal{L}$, namely, $p_{\frac{x}{2}}, \ldots, p_{\frac{n}{4}-1}$. More precisely, the edges of K_n that are assigned to page p_i, with $i \in [\frac{x}{2}, \frac{n}{4}-1]$ are the following $2y-2$ ones; in the following, the former are drawn above the spine, while the latter are drawn below the spine.

- $(u_{\frac{y}{2}+2i-\frac{x}{2}-1-\lfloor\frac{j}{2}\rfloor}, u_{\frac{y}{2}+2i-\frac{x}{2}-1+\lceil\frac{j}{2}\rceil})$, $\quad 1 \leq j \leq y-1$
- $(u_{\frac{y}{2}+2i-\frac{x}{2}-\lfloor\frac{j}{2}\rfloor}, u_{\frac{y}{2}+2i-\frac{x}{2}+\lceil\frac{j}{2}\rceil})$, $\quad 1 \leq j \leq y-1$

These edges do not cross paiwise, since the ones above the spine as well as the ones below the spine in page p_i, with $i \in [\frac{x}{2}, \frac{n}{4}-1]$ follow the scheme described in Section 2.2. In total, the edges assigned to pages $p_{\frac{x}{2}}, \ldots, p_{\frac{n}{4}-1}$ are $\left(\frac{n}{4}-\frac{x}{2}\right)\cdot(2y-2)$. Summing up with the number of edges assigned to pages $p_0, \ldots, p_{\frac{n}{4}-1}$, we obtain that the total number of edges in $\mathcal{L}$ is $\frac{1}{2}(n(y-1)+x(x+y))$. By setting $x+y=n$ (refer to Eq. (3.1)), we obtain that the total number of edges in $\mathcal{L}$ is $\frac{1}{2}(n(n-1))$, which equals the number of edges of K_n, as desired. Since no edge is assigned to two distinct pages in $\mathcal{L}$, the proof of Theorem 1.4.1 is completed.

Remark 1. *The solution that one obtains combining Theorem 1.3.1 with the construction provided in Section 2.2 which yields stack layouts of K_n with $\frac{n}{2}$ pages is derived from our scheme by setting $x = 0$ and $y = n$.*

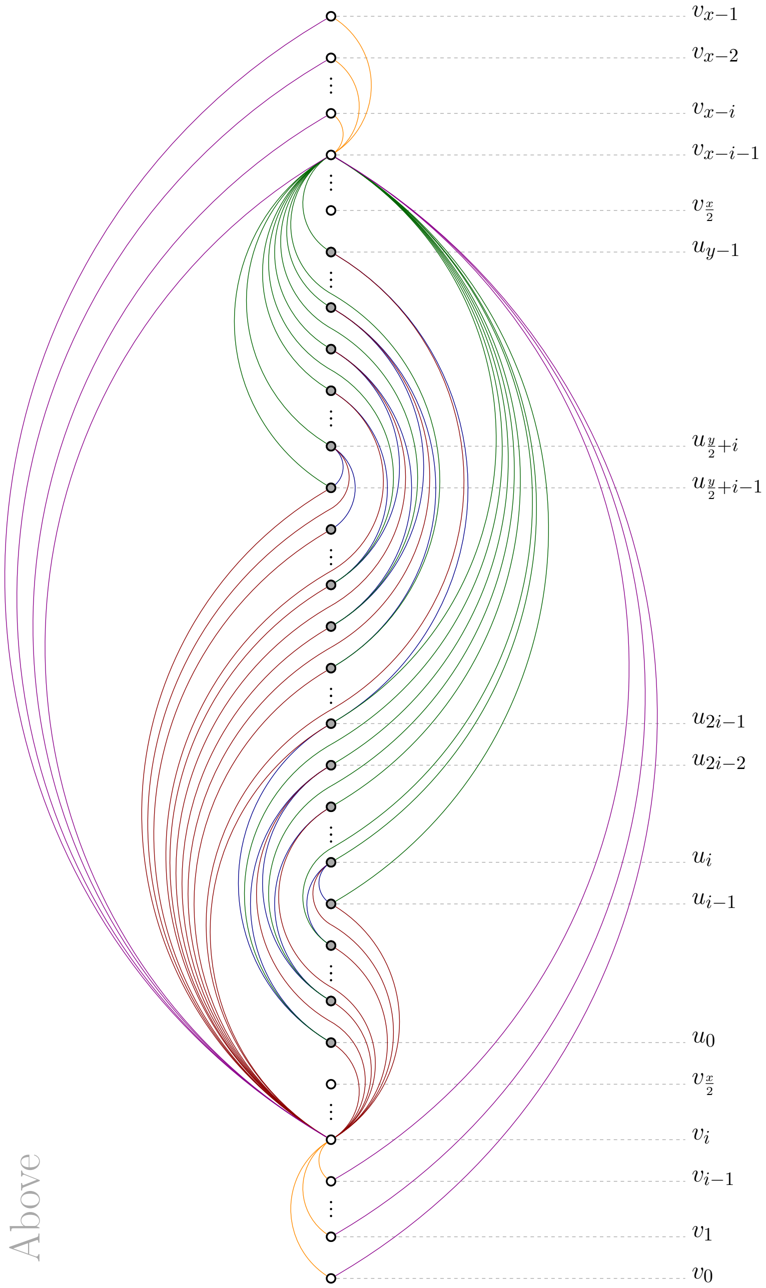


Figure 3.1: Page p_i, with $i \in [0, \frac{x}{2} - 1]$ of the linear layout of K_n with monotone biarcs provided in Section 3.1.

3.2 The Upper Bound on the Biarc Number of $K_{n,n}$

In this section, we turn our attention to the complete bipartite graph $K_{n,n}$ and, assuming that n is a multiple of 3, we prove that $K_{n,n}$ admits a linear layout $\mathcal{L}$ with monotone biarcs in $\frac{n}{3}+1$ pages. This implies that the biarc-number of $K_{n,n}$ is at most $\lceil \frac{n}{3} \rceil + 1$, proving Theorem 1.4.2. We denote by $u_0, \ldots, u_{n-1}$ and by $v_0, \ldots, v_{n-1}$ the vertices of the two parts of $K_{n,n}$. In the constructed linear layout $\mathcal{L}$, the order of the vertices is as follows:

$$u_0 \prec u_1 \prec \ldots \prec u_{n-1} \prec v_0 \prec v_1 \prec \ldots \prec v_{n-1}$$

This immediately implies that $\mathcal{L}$ is separated, as desired. We next describe how to assign the edges of $K_{n,n}$ to the pages of $\mathcal{L}$.

Page p_i, with $i \in [0, \frac{n}{3}-1]$, of $\mathcal{L}$ contains the following edges that are drawn as arcs above the spine. In particular, if $i \in [0, \frac{n}{6}-1]$, then the total number of these edges is $n-i-1$; otherwise (that is, $i \in [\frac{n}{6}, \frac{n}{3}-1]$), their total number is $\frac{n}{3}+3i+2$.

- $(u_{\frac{n}{3}-i-1}, v_{\frac{2n}{3}-1-j})$ $0 \leq j \leq \frac{n}{3}+i$
- $(u_{\frac{n}{3}-i+j}, v_{\frac{n}{3}-i-1})$ $0 \leq j \leq 2i$
- $(u_{\frac{2n}{3}+i+2+j}, v_i)$ $0 \leq j \leq \frac{n}{3}-2i-3$
- $(u_{\frac{2n}{3}+i+1}, v_{i+j})$ $0 \leq j \leq \frac{n}{3}-2i-2$

Page p_i, with $i \in [0, \frac{n}{3}-1]$, of $\mathcal{L}$ contains the following edges that are drawn as arcs below the spine. In particular, if $i \in [0, \frac{n}{6}-1]$, then the total number of these edges is $\frac{5n}{3}-3i-2$; otherwise (that is, $i \in [\frac{n}{6}, \frac{n}{3}-1]$), their total number is $n+i+1$.

- $(u_i, v_{\frac{2n}{3}+j})$ $i \leq j \leq \frac{n}{3}-1$
- $(u_{i+1+j}, v_{\frac{2n}{3}+i})$ $0 \leq j \leq \frac{2n}{3}-1$
- $(u_{\frac{2n}{3}+i}, v_{\frac{2n}{3}+i-1-j})$ $0 \leq j \leq 2i$
- $(u_{\frac{2n}{3}+i+1+j}, v_{\frac{n}{3}-i-1})$ $0 \leq j \leq \frac{n}{3}-2i-2$
- $(u_{n-i-1}, v_{\frac{n}{3}-i-2-j})$ $0 \leq j \leq \frac{n}{3}-2i-3$

Page p_i, with $i \in [0, \frac{n}{3}-1]$, of $\mathcal{L}$ contains the following $\frac{2n}{3}-1$ edges, drawn as biarcs starting above the spine and ending below the spine; these edges intersect the spine between the vertices $u_{\frac{2n}{3}+i}$ and $u_{\frac{2n}{3}+i+1}$

- $(u_{\frac{n}{3}+i}, v_{\frac{n}{3}-i+j})$, $0 \leq j \leq \frac{n}{3}-1$
- $(u_{\frac{n}{3}+i+1+j}, v_{\frac{2n}{3}-i-1})$, $0 \leq j \leq \frac{n}{3}-2$

We conclude the description of the edge-to-page assignment by describing the edges of the last page $p_{\frac{n}{3}}$ of $\mathcal{L}$. More precisely, in this page, there exist $\frac{n}{3}$ edges drawn as simple arcs above the spine:

- $(u_{\frac{2n}{3}-1}, v_j)$, $0 \leq j \leq \frac{n}{3}-1$

The schemizations of Figs. 3.2 and 3.3 show that the edges assigned to page p_i, with $i \in [0, \frac{n}{3}-1]$, (as described above) do not cross in $\prec$. The same holds in page $p_{\frac{n}{3}}$, since the edges assigned to this page form a star routed at $u_{\frac{2n}{3}-1}$. Furthermore, the total number of edges that have been assigned to $\mathcal{L}$ is:

$$\sum_{i=0}^{\frac{n}{6}-1}\Big((n-i-1)+(\frac{5n}{3}-3i-2)+(\frac{2n}{3}-1)\Big)+$$
$$\sum_{i=\frac{n}{6}}^{\frac{n}{3}-1}\Big((\frac{n}{3}+3i+2)+(n+i+1)+(\frac{2n}{3}-1)\Big)+\frac{n}{3}$$
$$=\sum_{i=0}^{\frac{n}{6}-1}\Big(\frac{10n}{3}-4i-4\Big)+\sum_{i=\frac{n}{6}}^{\frac{n}{3}-1}(2n+4i+2)+\frac{n}{3}=n^2$$

It follows that that the total number of edges in $\mathcal{L}$ equals the number of edges of $K_{n,n}$, as desired. Since no edge is assigned to two distinct pages in $\mathcal{L}$, the proof of Theorem 1.4.2 is completed.

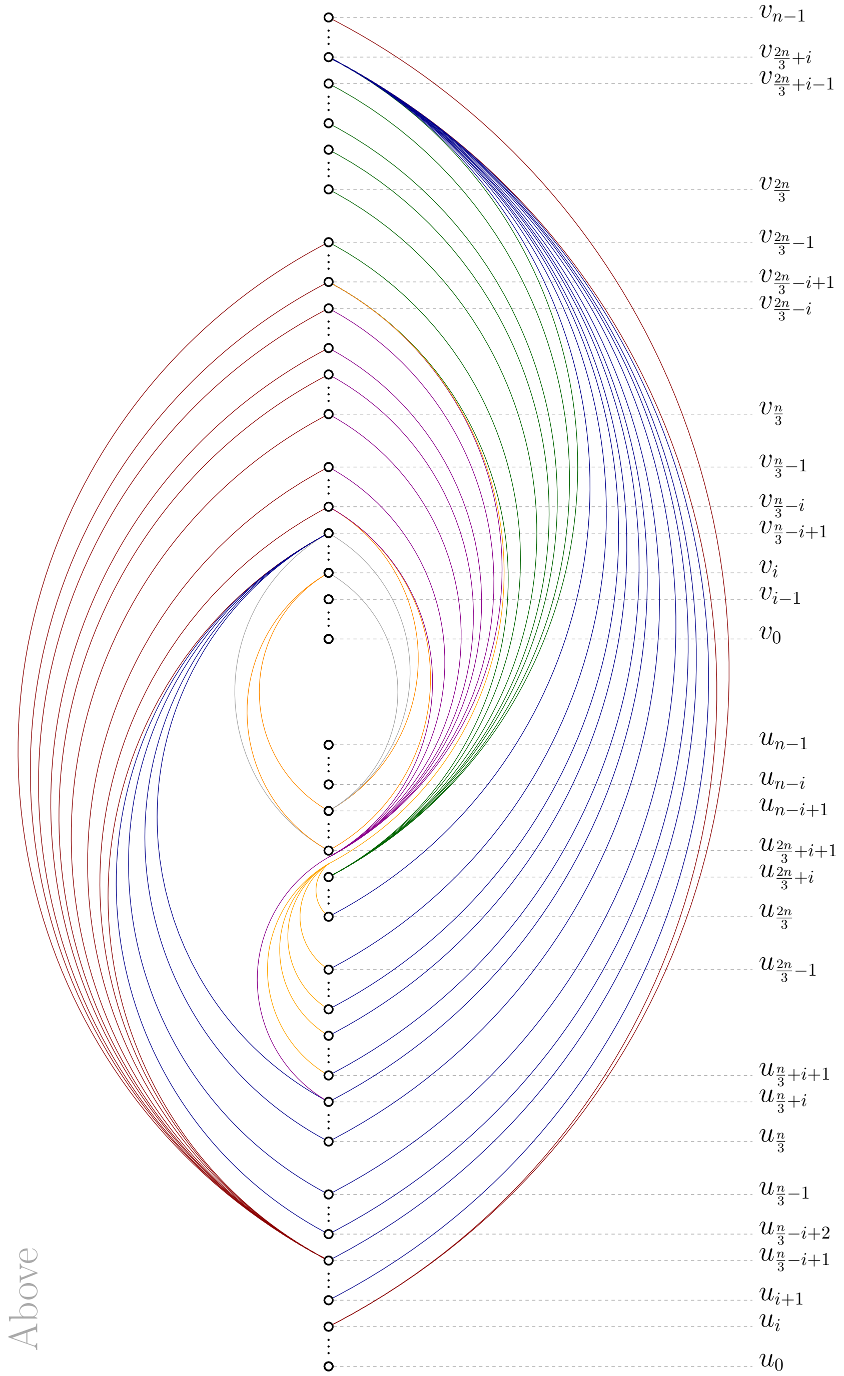


Figure 3.2: Page p_i, with $i \in [0, \frac{n}{6} - 1]$, of the linear layout of $K_{n,n}$ with monotone biarcs provided in Section 3.2.

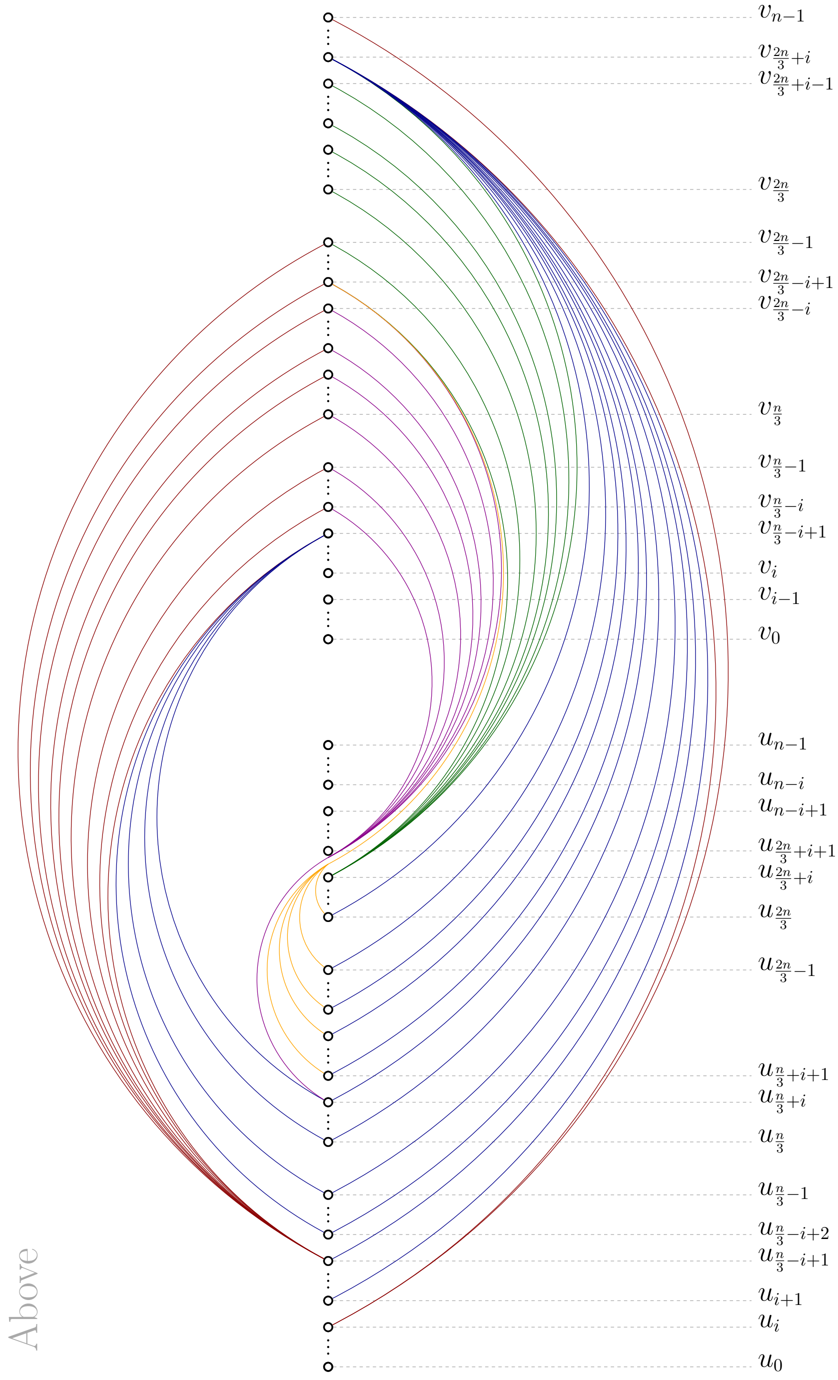


Figure 3.3: Page p_i, with $i \in [\frac{n}{6}, \frac{n}{3} - 1]$, of the linear layout of $K_{n,n}$ with monotone biarcs provided in Section 3.2.

CHAPTER 4

SAT Formulation

In this section, we present a SAT formulation for the problem of testing whether a given graph admits a linear layout with monotone biarcs on a certain number of pages. It is worth noting that our implementation has been integrated into an existing client-server tool [1], which supports various types of linear layouts, including the well-known stack and queue layouts.

4.1 The Variables of the SAT Formulation

Consider a graph G with n vertices and m edges and let p be a positive integer. For testing whether G admits a linear layout $\mathcal{L}$ with monotone biarcs on p pages, we extend the SAT formulation that was described in Section 2.3 by introducing $O(nmp)$ new variables. More precisely, for each vertex v of G, for each edge e of G and for each page q with $q \in [0, p-1]$ of $\mathcal{L}$, our formulation contains the following two variables:

- Variable $t_q(e, v)$ is `true` if and only if edge e is above the spine at vertex v in page q.
- Variable $b_q(e, v)$ is `true` if and only if edge e is below the spine at vertex v in page q.

4.2 The Clauses of the SAT formulation

In this section, we describe the clauses that our SAT formulation has in order to guarantee that the solution of the constructed SAT instance corresponds to a valid linear layout with monotone biarcs.

For each edge $e = (u, v)$ of G, and for each page q, we first need to ensure that if e is assigned to page q, then for each endpoint w of the edge e (i.e., $w \in \{u, v\}$), variables $t_q(e, w)$ and $b_q(e, w)$ cannot be simultaneously false. In other words, the edge must leave each of its endpoints either above or below the spine in the layout $\mathcal{L}$. This is guaranteed by the following clauses:

$$\phi_q(e) \rightarrow t_q(e, u) \vee b_q(e, u)$$
$$\phi_q(e) \rightarrow t_q(e, v) \vee b_q(e, v)$$

For each edge $e = (u, v)$ of G, for each vertex x with $x \notin \{u, v\}$ of G, and for each page q, we need to ensure that variables $t_q(e, x)$ and $b_q(e, x)$ are defined *consistently*, that is, (i) $t_q(e, x)$ and $b_q(e, x)$ cannot be simultaneously `true`, (ii) if e is assigned to page q and vertex x precedes or follows both endpoints of e in the order, then both variables $t_q(e, x)$ and $b_q(e, x)$ must be `false`, and (iii) if e is assigned to page q and vertex x appears between the endpoints of e in the order, then variables $t_q(e, x)$ and $b_q(e, x)$ cannot be simultaneously `false`.

The first (i.e., point (i) above) is guaranteed by the following clause:

$$\neg t_q(e, x) \vee \neg b_q(e, x)$$

The second (i.e., point (ii) above) is guaranteed by the following clauses:

$$\phi_q(e) \wedge \sigma(u, x) \wedge \sigma(v, x) \rightarrow \neg t_q(e, x) \wedge \neg b_q(e, x)$$
$$\phi_q(e) \wedge \sigma(v, x) \wedge \sigma(u, x) \rightarrow \neg t_q(e, x) \wedge \neg b_q(e, x)$$

The third (i.e., point (iii) above) is guaranteed by the following clauses:

$$\phi_q(e) \wedge \sigma(u, x) \wedge \sigma(x, v) \rightarrow t_q(e, x) \vee b_q(e, x)$$
$$\phi_q(e) \wedge \sigma(v, x) \wedge \sigma(x, u) \rightarrow t_q(e, x) \vee b_q(e, x)$$

Note that point (i) above further guarantees that all biarcs are drawn monotone. To see this observe that for a non-monotone biarc, say e, to exist, e.g., at page q, there has to exist a vertex, say x, for which both variables $t_q(e, x)$ and $b_q(e, x)$ have been assigned the value `true`, which is prevented by the clause of point (i). So, in the following we turn our attention to the clauses that are needed in order to guarantee that no edge crosses the spine more than once.

For each edge $e = (u, v)$ of G, for each vertex x with $x \notin \{u, v\}$ of G, and for each page q, if edge e is assigned to page q and vertex x appears between u and v in the order, then edge e cannot leave its endpoints from the same side of the spine, while for vertex x edge e resides on the other side of the spine, as this would imply that e is crossing the spine at least twice. We avoid this scenario by introducing the following clauses.

$$\phi_q(e) \wedge \sigma(u, x) \wedge \sigma(x, v) \rightarrow \neg(t_q(e, u) \wedge b_q(e, x) \wedge t_q(e, v))$$
$$\phi_q(e) \wedge \sigma(u, x) \wedge \sigma(x, v) \rightarrow \neg(b_q(e, u) \wedge t_q(e, x) \wedge b_q(e, v))$$
$$\phi_q(e) \wedge \sigma(v, x) \wedge \sigma(x, u) \rightarrow \neg(t_q(e, v) \wedge b_q(e, x) \wedge t_q(e, u))$$
$$\phi_q(e) \wedge \sigma(v, x) \wedge \sigma(x, u) \rightarrow \neg(b_q(e, v) \wedge t_q(e, x) \wedge b_q(e, u))$$

For each edge $e = (u, v)$ of G, for each pair of distinct vertices x and y with $x, y \notin \{u, v\}$ of G, and for each page q, if edge e is assigned to page q and vertices x and y appear between u and v in the order, then edge e cannot cross the spine between u and x, between x and y and also between y and v, as this would imply that e is crossing the spine at least twice. Since the case in which e leaves its endpoints from the same side of the spine is covered above, we avoid the scenario that we just discussed by introducing the following clauses.

$$\phi_q(e) \wedge \sigma(u, x) \wedge \sigma(x, y) \wedge \sigma(y, v) \rightarrow \neg(t_q(e, u) \wedge b_q(e, x) \wedge t_q(e, y) \wedge b_q(e, v))$$
$$\phi_q(e) \wedge \sigma(u, x) \wedge \sigma(x, y) \wedge \sigma(y, v) \rightarrow \neg(b_q(e, u) \wedge t_q(e, x) \wedge b_q(e, y) \wedge t_q(e, v))$$
$$\phi_q(e) \wedge \sigma(v, x) \wedge \sigma(x, y) \wedge \sigma(y, u) \rightarrow \neg(t_q(e, v) \wedge b_q(e, x) \wedge t_q(e, y) \wedge b_q(e, u))$$
$$\phi_q(e) \wedge \sigma(v, x) \wedge \sigma(x, y) \wedge \sigma(y, u) \rightarrow \neg(b_q(e, v) \wedge t_q(e, x) \wedge b_q(e, y) \wedge t_q(e, u))$$

For each edge $e = (u, v)$ of G, for each triplet of distinct vertices x, y and z with $x, y, z \notin \{u, v\}$ of G, and for each page q, if edge e is assigned to page q and vertices x, y and z appear between u and v in the order, then edge e cannot cross the spine between x and y and also between y and z, as this would imply that e is crossing the spine at least twice. We avoid this scenario that we just discussed by introducing the following clauses.

$$\phi_q(e) \wedge \sigma(u, x) \wedge \sigma(x, y) \wedge \sigma(y, z) \wedge \sigma(z, v) \rightarrow \neg(t_q(e, x) \wedge b_q(e, y) \wedge t_q(e, z))$$
$$\phi_q(e) \wedge \sigma(u, x) \wedge \sigma(x, y) \wedge \sigma(y, z) \wedge \sigma(z, v) \rightarrow \neg(b_q(e, x) \wedge t_q(e, y) \wedge b_q(e, z))$$
$$\phi_q(e) \wedge \sigma(v, x) \wedge \sigma(x, y) \wedge \sigma(y, z) \wedge \sigma(z, u) \rightarrow \neg(t_q(e, x) \wedge b_q(e, y) \wedge t_q(e, z))$$
$$\phi_q(e) \wedge \sigma(v, x) \wedge \sigma(x, y) \wedge \sigma(y, z) \wedge \sigma(z, u) \rightarrow \neg(b_q(e, x) \wedge t_q(e, y) \wedge b_q(e, z))$$

So far, the clauses that we have introduced guarantee that no edge is crossing the spine more than once; in particular, the three cases that we distinguished above suffice, as the edge $e = (u, v)$ may cross the spine either between u and a non-incident vertex, or between v and a non-incident vertex or between two non-incident vertices.

It remains to guarantee that no two edges assigned to the same page of the layout cross. We describe the case in which the two edges do not share an endvertex; the case in which the two edges share an endvertex is handled similarly. Under this assumption, consider a pair of distinct independent edges $e = (u, v)$ and $e' = (z, w)$ of G. Then, for each page q, if edges e and e' are assigned to page q and their endpoints alternate in the order, then e and e' cannot reside on the same side of the spine at the two endvertices of e and e' that are neither the first or the last among these endvertices in the order. This is guaranteed by the following clauses:

$$\begin{aligned}
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,z) \wedge \sigma(z,v) \wedge \sigma(v,w) &\rightarrow \neg(t_q(e,z) \wedge t_q(e',v))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,z) \wedge \sigma(z,u) \wedge \sigma(u,w) &\rightarrow \neg(t_q(e,z) \wedge t_q(e',u))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,w) \wedge \sigma(w,v) \wedge \sigma(v,z) &\rightarrow \neg(t_q(e,w) \wedge t_q(e',v))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,w) \wedge \sigma(w,u) \wedge \sigma(u,z) &\rightarrow \neg(t_q(e,w) \wedge t_q(e',u))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,u) \wedge \sigma(u,w) \wedge \sigma(w,v) &\rightarrow \neg(t_q(e,u) \wedge t_q(e',w))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,u) \wedge \sigma(u,z) \wedge \sigma(z,v) &\rightarrow \neg(t_q(e,u) \wedge t_q(e',z))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,v) \wedge \sigma(v,w) \wedge \sigma(w,u) &\rightarrow \neg(t_q(e,v) \wedge t_q(e',w))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,v) \wedge \sigma(v,z) \wedge \sigma(z,u) &\rightarrow \neg(t_q(e,v) \wedge t_q(e',z))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,z) \wedge \sigma(z,v) \wedge \sigma(v,w) &\rightarrow \neg(b_q(e,z) \wedge b_q(e',v))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,z) \wedge \sigma(z,u) \wedge \sigma(u,w) &\rightarrow \neg(b_q(e,z) \wedge b_q(e',u))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,w) \wedge \sigma(w,v) \wedge \sigma(v,z) &\rightarrow \neg(b_q(e,w) \wedge b_q(e',v))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,w) \wedge \sigma(w,u) \wedge \sigma(u,z) &\rightarrow \neg(b_q(e,w) \wedge b_q(e',u))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,u) \wedge \sigma(u,w) \wedge \sigma(w,v) &\rightarrow \neg(b_q(e,u) \wedge b_q(e',w))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,u) \wedge \sigma(u,z) \wedge \sigma(z,v) &\rightarrow \neg(b_q(e,u) \wedge b_q(e',z))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,v) \wedge \sigma(v,w) \wedge \sigma(w,u) &\rightarrow \neg(b_q(e,v) \wedge b_q(e',w))\\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,v) \wedge \sigma(v,z) \wedge \sigma(z,u) &\rightarrow \neg(b_q(e,v) \wedge b_q(e',z))
\end{aligned}$$

In addition to the aforementioned case, when the endpoints of e and e' alternate in the order, a crossing may occur also in the presence of a vertex, say x, that is inbetween the endpoints of each of the edges e and e', such that the edge incident to the first endvertex (among the endvertices of e and e') crosses the spine between u and x, while the other edge is on the opposite side of the spine at vertex x. We avoid this case, by introducing the following clauses for each page q and each vertex x with $x \notin \{u, v, z, w\}$:

$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,z) \wedge \sigma(z,x) \wedge \sigma(x,v) \wedge \sigma(v,w) \rightarrow \neg(t_q(e,z) \wedge t_q(e',x) \wedge b_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,z) \wedge \sigma(z,x) \wedge \sigma(x,u) \wedge \sigma(u,w) \rightarrow \neg(t_q(e,z) \wedge t_q(e',x) \wedge b_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,w) \wedge \sigma(w,x) \wedge \sigma(x,v) \wedge \sigma(v,z) \rightarrow \neg(t_q(e,w) \wedge t_q(e',x) \wedge b_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,w) \wedge \sigma(w,x) \wedge \sigma(x,u) \wedge \sigma(u,z) \rightarrow \neg(t_q(e,w) \wedge t_q(e',x) \wedge b_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,u) \wedge \sigma(u,x) \wedge \sigma(x,w) \wedge \sigma(w,v) \rightarrow \neg(t_q(e,u) \wedge t_q(e',x) \wedge b_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,u) \wedge \sigma(u,x) \wedge \sigma(x,z) \wedge \sigma(z,v) \rightarrow \neg(t_q(e,u) \wedge t_q(e',x) \wedge b_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,v) \wedge \sigma(v,x) \wedge \sigma(x,w) \wedge \sigma(w,u) \rightarrow \neg(t_q(e,v) \wedge t_q(e',x) \wedge b_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,v) \wedge \sigma(v,x) \wedge \sigma(x,z) \wedge \sigma(z,u) \rightarrow \neg(t_q(e,v) \wedge t_q(e',x) \wedge b_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,z) \wedge \sigma(z,x) \wedge \sigma(x,v) \wedge \sigma(v,w) \rightarrow \neg(b_q(e,z) \wedge b_q(e',x) \wedge t_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,z) \wedge \sigma(z,x) \wedge \sigma(x,u) \wedge \sigma(u,w) \rightarrow \neg(b_q(e,z) \wedge b_q(e',x) \wedge t_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,w) \wedge \sigma(w,x) \wedge \sigma(x,v) \wedge \sigma(v,z) \rightarrow \neg(b_q(e,w) \wedge b_q(e',x) \wedge t_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,w) \wedge \sigma(w,x) \wedge \sigma(x,u) \wedge \sigma(u,z) \rightarrow \neg(b_q(e,w) \wedge b_q(e',x) \wedge t_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,u) \wedge \sigma(u,x) \wedge \sigma(x,w) \wedge \sigma(w,v) \rightarrow \neg(b_q(e,u) \wedge b_q(e',x) \wedge t_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,u) \wedge \sigma(u,x) \wedge \sigma(x,z) \wedge \sigma(z,v) \rightarrow \neg(b_q(e,u) \wedge b_q(e',x) \wedge t_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,v) \wedge \sigma(v,x) \wedge \sigma(x,w) \wedge \sigma(w,u) \rightarrow \neg(b_q(e,v) \wedge b_q(e',x) \wedge t_q(e,x))$$
$$\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,v) \wedge \sigma(v,x) \wedge \sigma(x,z) \wedge \sigma(z,u) \rightarrow \neg(b_q(e,v) \wedge b_q(e',x) \wedge t_q(e,x))$$

Note that edges e and e' may cross, even if their endpoints do not alternate in the order. To avoid this, we introduce the following clause for each page q:

$$
\begin{aligned}
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,z) \wedge \sigma(z,w) \wedge \sigma(w,v) &\rightarrow \neg(t_q(e,z) \wedge b_q(e,w)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,z) \wedge \sigma(z,w) \wedge \sigma(w,u) &\rightarrow \neg(t_q(e,z) \wedge b_q(e,w)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,w) \wedge \sigma(w,z) \wedge \sigma(z,v) &\rightarrow \neg(t_q(e,w) \wedge b_q(e,z)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,w) \wedge \sigma(w,z) \wedge \sigma(z,u) &\rightarrow \neg(t_q(e,w) \wedge b_q(e,z)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,u) \wedge \sigma(u,v) \wedge \sigma(v,w) &\rightarrow \neg(t_q(e,u) \wedge b_q(e,v)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,u) \wedge \sigma(u,v) \wedge \sigma(v,z) &\rightarrow \neg(t_q(e,u) \wedge b_q(e,v)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,v) \wedge \sigma(v,u) \wedge \sigma(u,w) &\rightarrow \neg(t_q(e,v) \wedge b_q(e,u)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,v) \wedge \sigma(v,u) \wedge \sigma(u,z) &\rightarrow \neg(t_q(e,v) \wedge b_q(e,u)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,z) \wedge \sigma(z,w) \wedge \sigma(w,v) &\rightarrow \neg(b_q(e,z) \wedge t_q(e,w)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,z) \wedge \sigma(z,w) \wedge \sigma(w,u) &\rightarrow \neg(b_q(e,z) \wedge t_q(e,w)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(u,w) \wedge \sigma(w,z) \wedge \sigma(z,v) &\rightarrow \neg(b_q(e,w) \wedge t_q(e,z)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(v,w) \wedge \sigma(w,z) \wedge \sigma(z,u) &\rightarrow \neg(b_q(e,w) \wedge t_q(e,z)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,u) \wedge \sigma(u,v) \wedge \sigma(v,w) &\rightarrow \neg(b_q(e,u) \wedge t_q(e,v)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,u) \wedge \sigma(u,v) \wedge \sigma(v,z) &\rightarrow \neg(b_q(e,u) \wedge t_q(e,v)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(z,v) \wedge \sigma(v,u) \wedge \sigma(u,w) &\rightarrow \neg(b_q(e,v) \wedge t_q(e,u)) \\
\phi_q(e) \wedge \phi_q(e') \wedge \sigma(w,v) \wedge \sigma(v,u) \wedge \sigma(u,z) &\rightarrow \neg(b_q(e,v) \wedge t_q(e,u))
\end{aligned}
$$

This completes the description of our SAT formulation. Since each edge cannot cross the spine more than once and since no two edges of the same page can cross, the correctness of our SAT formulation follows. We conclude this section by mentioning that the additional clauses that we introduced to the original formulation as described in Section 2.3 is $O(n^5 m^2 p)$.

CHAPTER 5

Conclusions

In this thesis, we introduced and studied a new variant of linear graph layouts. The following research question directly follow from our findings.

- As we strongly believe that the upper bound on the biarc number of K_n that we provided in Theorem 1.4.1 can be improved, the first question that we pose is whether it is possible to develope a construction yielding a linear layout with monotone biarcs of K_n using strictly less than $\lceil \frac{n}{4} \rceil$.
- We studied the biarc number of $K_{n,n}$ in the separated setting. It would be interesting to study whether it is possible to improve the upper bound of Theorem 1.4.2 by relaxing the constraint of having the vertices of one part of $K_{n,n}$ to precede the ones of the second part, that is, by allowing vertex orders in which the vertices of both parts mix?
- Another interesting direction is to extend the study to other meaningful classes of graphs.

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