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Calderon-Zygmund theory and applications

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΑΤΡΙΒΗ

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"Δηλώνω υπεύθυνα ότι η παρούσα διατριβή εκπονήθηκε κάτω από τους διεθνείς ηθικούς και ακαδημαϊκούς κανόνες δεοντολογίας και προστασίας της πνευματικής ιδιοκτησίας. Σύμφωνα με τους κανόνες αυτούς, δεν έχω προβεί σε ιδιοποίηση ξένου επιστημονικού έργου και έχω πλήρως αναφέρει τις πηγές που χρησιμοποίησα στην εργασία αυτή."

Αναστάσιος Καραμήτρος

Περιληψη

Θα δουλέψουμε κυρίως με τον Hardy-Littlewood μεγιστικό τελεστή ο οποίος ορίζεται ως εξής:
$$Mf(x) = sup \frac{1}{|Q|} \int_Q |f(y)| \, dy$$
όπου Q είναι κύβος ο οποίος περιέχει το x. Ένα απο τα κυριότερα εργαλεία μας θα είναι ο χωρισμός του χώρου $\mathbb{R}^n$ σε ένα υποσύνολο Ω που αποτελείται απο ξένους ανα δύο κύβους πάνω στους οποίους ο μέσος όρος της συνάρτησης $|f|$ είναι μεταξύ t και $2^n t$, και στο συμπλήρωμά του F όπου $|f(x)| \leq t$ σχεδόν παντού. Θα αποδείξουμε κάποιες L^p ανισότητες για αυτόν τον μεγιστικό τελεστή και θα δούμε την σχέση του με τον sharp μεγιστικό τελεστή $f^{\#}$. Μετά απο την εισαγωγή μας στην θεωρία βαρών, θα μελετήσουμε ενα πρόβλημα για δυαδικά A_1 βάρη, για το οποίο θα αποδείξουμε μια αντίστροφη Hölder L^p ανισότητα.

Abstract

We will work mostly with the Hardy-Littlewood maximal function which is defined as

$$Mf(x) = sup\frac{1}{|Q|}\int_Q |f(y)|dy$$

where Q is a cube containing x. One of the tools of constant use in our work will be the splitting of the space $\mathbb{R}^n$ into a subset Ω made up of non-overlapping cubes Q_j over each of which the average of an integrable function $|f|$ is between t and $2^n t$, and a complementary subset F where $|f(x)| \leq t$ a.e. We will obtain some L^p inequalities for this maximal function and we will see the relation with the sharp maximal function $f^{\#}$. After our introduction in weights and A_p theory we will study an interesting problem for dyadic A_1 weights from which we will get a sharp reverse Hölder type L^p-inequality.

Introduction

In 1952, A.P. Calderon and A.Zygmund invented a simple but powerful method to split the space $\mathbb{R}^n$ into a subset Ω made up of non-overlapping cubes Q_j over each of which the average of an integrable function $|f|$ is between t and $2^n t$, and a complementary subset F where $|f(x)| \leq t$ for a.e. $x \in F$. This method has become widely known as the Calderon-Zygmund decomposition. We aim to describe here this method together with some of its most immediate and interesting applications.

The first two sections of chapter one give a description of the method in connection with the (very closely related) Hardy-Littlewood maximal operator. Apart from the usual estimates for this maximal function, we also obtain some weighted inequalities which anticipate the A_p theory to be developed in chapter two, and we study some variants of the Hardy-Littlewood operator when Lebesgue measure is replaced by a more general measure. This leads us in a natural way to the definition and study of the Carleson measures.

This is not the only maximal operator to appear in the first chapter. The so-called sharp maximal function shares enough properties with the Hardy-Littlewood operator, but behaves in a different way in L^∞, which is somehow replaced by our friend, the space B.M.O, which will be further exploited in chapter two. This relation comes to light after proving the John-Nirenberg inequality for BMO functions, which is yet another application of the Calderon-Zygmund decomposition.

As we will see in chapter two, the L^p inequalities that will be obtained for several kinds of operators remain true when Lebesgue measure dx is replaced by certain measures $w(x)dx$.

We will devote chapter two to a systematic study of this type (L^p) of inequalities. We will see that for the maximal function Mf (which will be defined in chapter one), it is possible to give a very precise and satisfactory answer to the question of finding those w for which either

$$\int |Mf(x)|^p w(x)dx \leq C_p(w) \int |f(x)|^p w(x)dx \qquad (*)$$

or the corresponding weak type inequality (for which, the definition will be given in chapter one) hold. The same problem for two weights will be also considered.

Why should one be interested in inequalities like (*)? We shall briefly sketch some answers

(1) Conjugate functions, H^p spaces etc. can be defined in domains of complex plane with a "resonable" boundary ∂D. When estimating the L^p norms of operators appearing in this context, some of the problems that arise can be reduced, by change of variables, to estimates for known operators on the line or on the torus, but with respect to a measure $w(x)dx$ for certain w.

(2) Inequalities like (*) imply (as we will show), when the structure of weights satisfying them, the following

$$\int |Tf(x)|^2 u(x)dx \leq C \int |f(x)|^2 Nu(x)dx \qquad (**)$$

for arbitrary $u(x) \geq 0$, where N is (in the most desirable case) some kind of "maximal operator" which we can control. An inequality like (**) will be proved in chapter one for the Hardy-Littlewood maximal operator. Such inequalities are very easy to handle, and contain essentially all the relevant information about the boundedness properties of T.

In the end, we will determine the exact best possible range of p which depends (as we will see) on the dimension n and the corresponding A_1 constant of w, for which any dyadic A_1 weight on $\mathbb{R}^n$ satisfies a reverse Hölder inequality for p. The proof will be based on an effective linearization of the dyadic maximal operator applied to dyadic step functions.

Contents

CHAPTER 1

CALDERON-ZYGMUND THEORY

1.1 THE HARDY-LITTLEWOOD MAXIMAL FUNCTION AND THE CALDERON-ZYGMUND DECOMPOSITION

Let f be locally integrable function in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we define

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|dy$$

where the sup is taken over all cubes Q containing x (cube will always mean a compact cube with sides parallel to the axes and non empty interior), and $|Q|$ stands for the Lebesgue measure of Q.

Mf will be called (Hardy-Littlewood) maximal function of f, and the operator M sending f to Mf,(Hardy-Littlewood) maximal operator.

Observe that we obtain the same value $Mf(x)$, which can be $+\infty$, if we allow in the definition only those cubes Q for wich x is an interior point. It follows from this remark that the function Mf is lower semicontinuous,i.e, for every $t > 0$, the set $E_t = \{x \in \mathbb{R}^n : Mf(x) > t\}$ is open.

In order to study the size of Mf, we shall look at its distribution function $\lambda(t) = |E_t|$. It will be instructive to start with the case $n = 1$, which is particularly simple.Let $f \in L^1(\mathbb{R})$. The open set E_t is a disjoint union of open intervals I_j : its connected components. Let us look at one of the I_j's, and let us call it I. Take any compact set $K \subset I$. For each $x \in K$, there is (by definition of E_t) a compact interval Q_x containing x in its interior and satisfying

$$\frac{1}{|Q_x|}\int_{Q_x}|f(y)|dy > t.$$

Since $Q_x \subset E_t$, it follows that $Q_x \subset I$. Since K is compact, we can cover it with the interior of just finitely many of the Q_x's, say $\{Q_j\}$. We can even assume that this finite covering is minimal in the sence that no Q_j is superfluous. Then, no point is in more that two of the interiors of the Q_j's. It follows that:

$$|K| \leq \sum_j |Q_j| < \frac{1}{t}\sum_j \int_{Q_j}|f(y)|dy \leq \frac{2}{t}\int_{\bigcup_j Q_j}|f(y)|dy \leq \frac{2}{t}\int_I |f(y)|dy.$$

Since this is true for every compact $K \subset I$, we obtain:

$$|I| \leq \frac{2}{t}\int_I |f(y)|dy \tag{1.1}$$

This implies, in particular, that I is bounded. Let $I = (a, b)$. Then, since $b \in \overline{I}$ and $b \notin E_t$ (E_t is open and I is one of its open components), we can write:

$$\frac{1}{|I|}\int_I |f(y)|dy \leq Mf(b) \leq t$$

Finally (1.1) implies:

$$|E_t| = \sum_j |I_j| \leq \frac{2}{t}\sum_j \int_{I_j}|f(y)|dy = \frac{2}{t}\int_{E_t}|f(y)|dy$$

We have obtained the following result:

Theorem 1.1.1. *Let $f \in L^1(\mathbb{R})$. then, for every $t > 0$, the set $E_t = \{x \in \mathbb{R} : Mf(x) > t\}$ can be written as a disjoint union of bounded open intervals I_j, such that, for every $j = 1, 2, ...$*

$$\frac{t}{2} \leq \frac{1}{|I_j|}\int_{I_j}|f(y)|dy \leq t \tag{1.3}$$

and as a consequence:

$$|E_t| \leq \frac{2}{t}\int_{E_t}|f(y)|dy.$$

□

Now we seek an analogue of the previous theorem in dimension $n > 1$. The extension is not straightforward.

Let $f \in L^1(\mathbb{R}^n)$, $n > 1$, and let $t > 0$. Instead of looking at the maximal function Mf, we shall try to obtain directly a family of cubes $\{Q_j\}$ such that the average of $|f|$ over each is comparable to t in the sense that a relation like (1.3) holds. This is quite easy and it will be done most effectively by considering only dyadic cubes. For $k \in \mathbb{Z}$, we consider the lattice $\Lambda_k = 2^{-k}\mathbb{Z}^n$ formed by those points of $\mathbb{R}^n$ whose coordinates are integral multiples of 2^{-k}. Let D_k be the collection of the cubes determined by Λ_k, that is, those cubes with side lenght 2^{-k} and vertices in Λ_k. The cubes belonging to $D = \bigcup_{-\infty}^{+\infty} D_k$ are called dyadic cubes. Observe that if $Q, Q' \in D$ and $|Q'| \leq |Q|$, then either $Q' \subset Q$ or else Q and Q' do not overlap (by which we mean that their interiors are disjoint). Each $Q \in D_k$ is the union of 2^n non-overlapping cubes belonging to D_{k+1}. We shall call C_t the family formed by the cubes $Q \in D$ which satisfy the condition:

$$t < \frac{1}{|Q|} \int_Q |f(x)| dx \tag{1.4}$$

and are maximal among those which satisfy it. Every $Q \in D$ satisfying (1.4) is contained in some $Q' \in C_t$. The cubes in C_t are, by definition, non overlapping. Also, if $Q \in D_k$ is in C_t and Q' is the only cube in D_{k-1} containing Q, we shall have:

$$\frac{1}{|Q'|} \int_{Q'} |f(x)| dx \leq t$$

but, since $|Q'| = 2^n |Q|$, we get:

$$\frac{1}{|Q|} \int_Q |f(x)| dx \leq \frac{2^n}{|Q'|} \int_{Q'} |f(x)| dx \leq 2^n t$$

we have achieved our purpose by obtaining a family $C_t = \{Q_j\}$ of cubes such that, for every j:

$$t \leq \frac{1}{|Q_j|} \int_{Q_j} |f(x)| dx \leq 2^n t \tag{1.5}$$

Next, we shall investigate the relation with the maximal function Mf. Suppose $x \in \mathbb{R}^n$ is such that $Mf(x) > t$. There will be some cube R containing x in its interior and satisfying

$$t < \frac{1}{|R|} \int_R |f(x)| dx.$$

We look for a dyadic cube of comparable size over which the average of $|f|$ is comparably big. Let k be the only integer such that

$$2^{-(k+1)n} < |R| \leq 2^{-kn}.$$

For this k there is at most one point of Λ_k interior to R and there are at most 2^n cubes in D_k meeting the interior of R. Consequently, there is some cube in D_k meeting the interior of R satisfying:

$$\int_{R\cap Q} |f(y)|dy > \frac{t|R|}{2^n}$$

and that is because if we had

$$\int_{R\cap Q} |f(y)|dy \leq \frac{t|R|}{2^n}$$

for every such cube, then we get:

$$\int_R |f(y)|dy = \int_{R\cap(\cup_{i=1}^{2^n} Q_i)} |f(y)|dy = \sum_{i=1}^{2^n} \int_{(R\cap Q_i)^\circ} |f(y)|dy \leq \sum_{i=1}^{2^n} \frac{t|R|}{2^n} = t|R|$$

which is not valid (for the first equality we used that there are at most 2^n cubes in D_k meeting the interior of R) .

Now, since $|R| \leq |Q| < 2^n|R|$, we have :

$$\int_{R\cap Q} |f(y)|dy > \frac{t|R|}{2^n} > \frac{t|Q|}{4^n}$$

and therefore:

$$\frac{1}{|Q|}\int_Q |f(y)|dy > \frac{t}{4^n}$$

it follows that $Q \subset Q_j \in C_{4^{-n}t}$ for some j.

In general for any cube Q and any $a > 0$, we shall denote by Q^a the cube with the same center as Q but with side lenght a times that of Q. In our particular situation, since R and Q meet and $|R| \leq |Q|$, it follows that $R \subset Q^3 \subset Q_j^3$. We conclude that if $C_{4^{-n}t} = \{Q_j\}$, then $E_t \subset \bigcup_j Q_j^3$ and this leads to the estimate

$$|E_t| \leq \sum_j |Q_j^3| = 3^n \sum_j |Q_j| < \frac{3^n 4^n}{t} \sum_j \int_{Q_j} |f(y)|dy \leq \frac{C}{t}\int_{\mathbb{R}^n} |f(y)|dy.$$

For those Q_j's we also have that:

$$\frac{t}{4^n} \le \frac{1}{|Q_j|}\int_{Q_j}|f(y)|dy = \frac{2^n}{|Q'_j|}\int_{Q_j}|f(y)|dy \le \frac{2^n}{|Q'_j|}\int_{Q'_j}|f(y)|dy \le \frac{2^n t}{4^n} = \frac{t}{2^n}$$

where Q'_j is the unique cube in D_{k-1} (if $Q_j \in D_k$) containing Q_j, and we know that Q_j's are in $C_{4^{-n}t}$, so they are maximal. We have obtained the following result:

Theorem 1.1.2. *Let $f \in L^1(\mathbb{R}^n)$. Then ,for every $t > 0$, the set $E_t = \{x \in \mathbb{R}^n : Mf(x) > t\}$ is contained in the union of a family of cubes $\{Q_j^3\}$ which result from expanding by a factor of* 3 *the non overlapping maximal cubes $\{Q_j\}$ which satisfy:*

$$\frac{t}{4^n} < \frac{1}{|Q_j|}\int_{Q_j}|f(x)|dx \le \frac{t}{2^n} \tag{1.7}$$

it follows that :

$$|E_t| \le \frac{C}{t}\int_{\mathbb{R}^n}|f(x)|dx \tag{1.8}$$

where the constant C depends only on the dimension n. □

We shall derive some consequences of the basic inequality (1.8) which illustrate the role played by the maximal operator M. The importance of the operator M stems from the fact that it controls many operators arising naturally in Analysis. As an example, we are going to prove an extension of Lebesgue's differentiation theorem.

Theorem 1.1.3. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $r > 0$, let $Q(x;r) = \{y \in \mathbb{R}^n : |y-x|_\infty = max_j|y_j - x_j| \le r\}$. Then, for almost every $x \in \mathbb{R}^n$:*

$$\frac{1}{|Q(x;r)|}\int_{Q(x;r)}|f(y)-f(x)|dy \to 0 \quad as \quad r \to 0. \tag{1.10}$$

Proof. We may assume $f \in L^1(\mathbb{R}^n)$. It will be enough to show that, for every $t > 0$, the set

$$A_t = \{x \in \mathbb{R}^n : \limsup_{r\to 0}\frac{1}{|Q(x;r)|}\int_{Q(x;r)}|f(y)-f(x)|dy > t\}$$

has zero measure. Indeed, the set where (1.10) does not hold, is precisely $\bigcup_{j=1}^{\infty} A_{1/j}$.

Given $\epsilon > 0$, we can write $f = g + h$, where g is continuous with compact support and $\int |h| < \epsilon$ (we can do that because of the density of continuous functions in L^1). For g we clearly have:

$$\frac{1}{|Q(x;r)|}\int_{Q(x;r)} |g(y) - g(x)|dy \to 0 \quad as \quad r \to 0$$

for every $x \in \mathbb{R}^n$. Therefore, we get that:

$$\limsup_{r\to 0} \frac{1}{|Q(x;r)|}\int_{Q(x;r)} |f(y)-f(x)|dy \leq \limsup_{r\to 0} \frac{1}{|Q(x;r)|}\int_{Q(x;r)} |h(y)-h(x)|dy$$
$$\leq Mh(x) + |h(x)|$$

and

$$A_t \subset \{x \in \mathbb{R}^n : Mh(x) > t/2\} \cup \{x \in \mathbb{R}^n : |h(x)| > t/2\}.$$

But

$$(1) \qquad |\{x \in \mathbb{R}^n : Mh(x) > t/2\}| \leq C\|h\|_1/t < C\epsilon/t$$

and

$$(2) \qquad |\{x \in \mathbb{R}^n : |h(x)| > t/2\}| \leq \int_{\mathbb{R}^n} \frac{2|h(x)|}{t}dx \leq 2\epsilon/t.$$

Thus A_t is contained in a set of measure $\leq (C + 2)\frac{\epsilon}{t}$. Since this can be done for any $\epsilon > 0$, we get $|A_t| = 0$. The second inequality is obvious, so lets proof the first one:

It is enough to show that for the set $A_t = \{x \in \mathbb{R}^n : Mh(x) > t\}$, there is a constant C such that $|A_t \cap K| \leq \frac{C}{t}\|h\|_1$ for every bounded $K \subset \mathbb{R}^n$. Let $x \in A_t \cap K$, then there will be $r_x > 0$:

$$\frac{1}{|Q(x;r_x)|}\int_{Q(x;r_x)} |h(y)|dy > t$$

Now for the collection $\{Q(x;r_x)\}_{x\in A_t\cap K}$ we recall the *Besicovitch* theorem, so there is a sub collection $\{Q_k\}_{k\in\mathbb{N}}$ such that:

- $A_t \cap K \subset \cup_k Q_k$
- $\sum_k X_{Q_k}(y) \leq \theta_n$ for every $y \in \mathbb{R}^n$

It is clear that:

$$\frac{1}{|Q_k|}\int_{Q_k} |h| > t$$

so we get:

$$|A_t \cap K| \leq |\cup Q_k| \leq \sum_k |Q_k| \leq \sum_k \frac{1}{t}\int_{Q_k} |h| = \sum_k \frac{1}{t}\int_{\mathbb{R}^n} X_{Q_k}|h|$$

using the Beppo-Levi theorem we get:

$$= \frac{1}{t}\int_{\mathbb{R}^n} \sum_k |h| X_{Q_k} = \int_{\mathbb{R}^n} |h| \sum_k X_{Q_k} \leq \frac{1}{t}\int_{\mathbb{R}^n} |h|\theta_n = \frac{\theta_n}{t}\|h\|_1$$

and the proof is complete. □

The points x for which (1.10) holds are called Lebesgue points for f. We can rephrase the previous theorem by saying that almost every point $x \in \mathbb{R}^n$ is a Lebesgue point.

Proposition 1.1.1. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then, for every Lebesgue point x for f and, therefore, for a.e point $x \in \mathbb{N}^n$:*

1. $f(x) = \lim_{r\to 0} \frac{1}{|Q(x;r)|}\int_{Q(x;r)} f(y)dy$
2. $|f(x)| \leq Mf(x)$.

Proof. In order to prove 1), just note that:

$$\left|\frac{1}{|Q(x;r)|}\int_{Q(x;r)} f(y)dy - f(x)\right| \leq \frac{1}{|Q(x;r)|}\int_{Q(x;r)} |f(y) - f(x)|dy$$

whilst 2) is an immediate consequence of 1). □

Now if x is a Lebesgue point for f and we have a sequence of cubes $Q_1 \supset Q_2 \supset ....$ with $\cap_j Q_j = \{x\}$, then:

$$f(x) = \lim_{j\to\infty} \frac{1}{|Q_j|}\int_{Q_j} f(y)dy$$

Indeed if Q_j has side length r_j, we have $Q_j \subset Q(x; 2r_j)$ and $\lim_{j\to\infty} r_j^n = \lim_{j\to\infty} |Q_j| = |\cap_j Q_j| = 0$, so that $r_j \to 0$. Therefore

$$\left|\frac{1}{|Q_j|}\int_{Q_j} f(y)dy - f(x)\right| \leq \frac{1}{|Q_j|}\int_{Q_j} |f(y) - f(x)|dy \leq$$

$$\frac{2^n}{2^n|Q(x;r_j)|}\int_{Q_j}|f(y)-f(x)|dy \leq \frac{2^n}{|Q(x;2r_j)|}\int_{Q(x;2r_j)}|f(y)-f(x)|dy \to 0$$
as $j \to \infty$

Let $f \in L^1(\mathbb{R}^n)$ and let $C_t = C_t(f) = \{Q_j\}$ be the collection formed by those maximal dyadic cubes over which the average of $|f|$ is $> t$ (called Calderon-Zygmund cubes for f corresponding to t) .Let $x \notin \cup_j Q_j$. Then the average of $|f|$ over any dyadic cube will be $\leq t$. Let $\{R_k\}$ be a sequence of dyadic cubes of decreasing size such that $\cap_k R_k = \{x\}$. Then for each of them we have

$$\frac{1}{|R_k|}\int_{R_k}|f(y)|dy \leq t$$

If, besides, x is a Lebesgue point for f (and hence for $|f|$) we get, by passing to the limit :$|f(x)| \leq t$. Thus $|f(x)| \leq t$ for a.e $x \notin \cup_j Q_j$.

The splitting of the space $\mathbb{R}^n$ into a subset Ω made up of non overlapping cubes Q_j over each of which the average of $|f|$ is between t and $2^n t$ and a complementary subset F where $|f(x)| \leq t$ a.e., is the first step of the so-called Calderon-Zygmund decomposition which will be a tool of constant use here. Let us record the following:

Theorem 1.1.4. *Given $f \in L^1(\mathbb{R}^n)$ and $t > 0$, there is a family of non overlapping cubes $C_t = C_t(f)$ consisting of those maximal dyadic cubes over which the average of $|f|$ is $> t$. This family satisfies:*

1. *for every $Q \in C_t : t < \frac{1}{|Q|}\int_Q |f(x)|dx \leq 2^n t$*
2. *for a.e $x \notin \cup Q$, where Q ranges over C_t, is $|f(x)| \leq t$.*

Besides, for every $t > 0$, $E_t = \{x \in \mathbb{R}^n : Mf(x) > t\} \subset \cup Q^3$ where Q ranges over $C_{4^{-n}t}$.

□

Next we are going to study a usefull generalization of the maximal function. Let μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets and satisfying that following "doubling" condition :

$$\mu(Q^2) \leq C\mu(Q) \qquad (1.13)$$

for every cube Q, with $C > 0$ independent of Q. We shall often say simply that μ is a doubling measure. This implies, of course, that for every $\alpha > 0$, there is a constant $C = C_\alpha > 0$, depending only on α, such that $\mu(Q^\alpha) \leq C\mu(Q)$ for every cube Q (that is because there is $n_a \in \mathbb{N}$ such that $\alpha < 2^{n_a}$ so $Q^\alpha \subset Q^{2^{n_a}}$). Since we are in $\mathbb{R}^n$, the finiteness of μ on compact subsets implies that μ is regular. Notice that for every cube Q, $\mu(Q) > 0$. Indeed, if we had $\mu(Q) = 0$ for some cube Q, we would have $\mu(Q^k) \leq C_k\mu(Q) = 0$, from which $\mu(\mathbb{R}^n) = 0$, which is excluded as trivial.

Now, for μ as above, $f \in L^1_{loc}(\mu)$ and $x \in \mathbb{R}^n$, define:

$$M_\mu f(x) = \sup_{x \in Q} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y)$$

where the sup is taken over all cubes Q containing x. As before, we obtain the same value $M_\mu f(x)$ if we just take in the definition, those cubes Q containing x in their interior. This is a consequence of the regularity of μ.

Let $f \in L^1(\mu)$ and $t > 0$. We want to obtain a Calderon-Zygmund decomposition for f and t relative to the measure μ and, at the same time, we want to estimate the μ-measure of the set $E_t = \{x \in \mathbb{R}^n : M_\mu f(x) > t\}$, which is, of course, open. We are going to apply the same ideas that led to the previous theorem. We need to make two observations.

First, we are going to see that there is a constant $K > 1$ such that, every time we have dyadic cubes $Q' \subsetneq Q$, it follows that $\mu(Q) \geq K\mu(Q')$. To see this, let Q'' be a dyadic cube contained in Q, contiguous to Q' and with the same diameter. Then $Q' \subset Q''^3$ and consequently, for $C = C_3$ we have:

$$\mu(Q') \leq C\mu(Q'') \leq C(\mu(Q) - \mu(Q'))$$

This implies that $(1 + C)\mu(Q') \leq C\mu(Q)$, which gives $\mu(Q) \geq K\mu(Q')$ with $K = (1 + C)/C > 1$, and our claim is justified. As a consequence, if we have a strictly increasing sequence dyadic cubes $Q_0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq ....$, we have the inequality $\mu(Q_k) > \left(\frac{1+C_3}{C_3}\right)^k \mu(Q_0) \to \infty$ as $k \to \infty$. The conclusion is that if a chain of dyadic cubes is such that the $\mu-$measure of the cubes is bounded above, then their diameter is also bounded above or, what is the same, the chain terminates at a given cube containing all the others.

The second observation we need is the following: for every $A > 0$ there is

$B > 0$ such that every time we have cubes Q and R which meet and satisfy $|Q| < A|R|$, then they also satisfy $|\mu(Q)| < B|\mu(R)|$, lets prove that: Let $A > 0$ such that $|Q| < A|R|$, but $A|R| = A^{n/n}|R| = |A^{1/n}R| = |R^{A^{1/n}}|$, so the side length of Q is smaller than $A^{1/n}-$times the side length of R and also Q and R meet, which leads to:

$$Q \subset R^{3A^{1/n}} \Rightarrow \mu(Q) < \mu(R^{3A^{1/n}}) \leq C_{3A^{1/n}}\mu(R)$$

and for $B = C_{3A^{1/n}}$ our claim is justified.

Now we go back to our problem. Denote by $C_t = C_t(f;\mu)$ the collection formed by the maximal dyadic cubes Q satisfying the condition

$$t < \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y).$$

Since this condition forces $\mu(Q)$ to be bounded by $t^{-1}\int_{\mathbb{R}^n} |f(y)|d\mu(y) < \infty$, our first observation implies that every dyadic cube satisfying our condition is contained in some member of C_t. Take $Q \in C_t$, then $Q \in D_k$ for some k and if Q' is the only cube in D_{k-1} containing Q, we have

$$\frac{1}{\mu(Q')} \int_{Q'} |f(y)| d\mu(y) \leq t$$

But $Q' \subset Q^3$, so that $\mu(Q') \leq C\mu(Q)$. Therefore

$$\frac{1}{\mu(Q)} \int_Q |f(x)| d\mu(x) \leq \frac{C}{\mu(Q')} \int_{Q'} |f(x)| d\mu(x) \leq Ct$$

Thus, for every $Q \in C_t$:

$$\frac{1}{\mu(Q)} \int_Q |f(x)| d\mu(x) \leq Ct.$$

Let now $x \in E_t$, that is $:M_\mu f(x) > t$. Then there will be some cube containing x in its interior such that

$$\frac{1}{\mu(R)} \int_R |f(y)| d\mu(y) > t.$$

As we did for the case μ =Lebesgue measure, let Q be a dyadic cube which overlaps with R and satisfies $|R| \leq |Q| < 2^n|R|$ and

$$\int_{R\cap Q} |f|d\mu > 2^{-n}t\mu(R).$$

Let B be the constant corresponding to $A = 2^n$ in our second observation. Then

$$\int_{R\cap Q} |f|d\mu > B^{-1}2^{-n}t\mu(Q)$$

and hence

$$\frac{1}{\mu(Q)}\int_Q |f|d\mu > \frac{t}{2^nB}$$

it follows that $Q \subset Q_j$ for some $Q_j \in C_{2^{-n}B^{-1}t}$ and $R \subset Q^3 \subset Q_j^3$.

If now $C_{2^{-n}B^{-1}t} = \{Q_j\}$, then $E_t \subset \bigcup_j Q_j^3$ and thus, we get the estimate:

$$\mu(E_t) \leq \sum_j \mu(Q_j^3) \leq C\sum_j \mu(Q_j) \leq \frac{C2^nB}{t}\sum_j \int_{Q_j} |f|d\mu \leq \frac{C2^nB}{t}\int_{\mathbb{R}^n} |f|d\mu$$

$$:= \frac{C}{t}\int_{\mathbb{R}^n} |f|d\mu$$

This basic estimate can be used to extend theorem (1.9) and its corollary, obtaining:

Theorem 1.1.5. *With μ as above, let $f \in L^1_{loc}(\mu)$. Then, for almost every $x \in \mathbb{R}^n$ (with respect to μ):*

1. $\lim_{r\to 0} \frac{1}{\mu(Q(x;r))}\int_{Q(x;r)} |f(y) - f(x)|d\mu(y) = 0$
2. $f(x) = \lim_{r\to 0} \frac{1}{\mu(Q(x;r))}\int_{Q(x;r)} f(y)d\mu(y)$
3. $|f(x)| \leq M_\mu f(x)$ □

In particular, if $C_t(f,\mu) = \{Q_j\}$, we have $|f(x)| \leq t$ for a.e. $x \notin \bigcup_j Q_j$ (with respect to μ). We can finally state the following:

Theorem 1.1.6. *For μ as above, letf $\in L^1(\mu)$ and $t > 0$. Then, there is a family of non overlapping cubes $C_t = C_t(f,\mu)$, consisting of those maximal dyadic cubes over which the average of $|f|$ relative to μ is $> t$, which satisfies*

1. *for every* $Q \in C_t : t < \frac{1}{\mu(Q)} \int_Q |f| d\mu \leq Ct$
2. *for a.e.* $x \notin \cup Q$ *where* Q *ranges over* C_t *(a.e is with respect to* μ*), we have :*$|f(x)| \leq t$. *Besides, for every* $t > 0$, *the set* $E_t = \{x \in \mathbb{R}^n : M_\mu f(x) > t\}$ *is contained in* $\bigcup Q^3$ *where* Q *ranges over* $C_{t/C'}$, *and we have an estimate:*
$$\mu(E_t) \leq Ct^{-1} \int_{\mathbb{R}^n} |f| d\mu.$$
Here C *represents an absolute constant, possibly different at each occurrence.*

□

1.2 NORM ESTIMATES FOR THE MAXIMAL FUNCTION

Theorem 1.2.1. *Let* f *be a measurable function on* $\mathbb{R}^n$ *and let* $t > 0$. *Then we have the following estimates for the Lebesgue measure of the set* $E_t = \{x \in \mathbb{R}^n : Mf(x) > t\}$:

$$|E_t| \leq \frac{C}{t} \int_{\{x \in \mathbb{R}^n : |f(x)| > t/2\}} |f(x)| dx \tag{2.2}$$

$$|E_t| \geq \frac{C'}{t} \int_{\{x \in \mathbb{R}^n : |f(x)| > t\}} |f(x)| dx \tag{2.3}$$

with constants C *and* C' *which do not depend on* f *or* t.

Proof. Write $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $|f(x)| > t/2$, and $f_1(x) = 0$ otherwise. Then $Mf(x) \leq Mf_1(x) + Mf_2(x) \leq Mf_1(x) + t/2$, since $|f_2| \leq t/2$ implies that $Mf_2 \leq t/2$ also. Thus

$$|E_t| \leq |\{x \in \mathbb{R}^n : Mf_1(x) > t/2\}| \leq \frac{3^n 4^n}{t/2} \int_{\mathbb{R}^n} |f_1(x)| dx$$

$$:= \frac{C}{t} \int_{\{x \in \mathbb{R}^n : |f(x)| > t/2\}} |f(x)| dx$$

which gives (2.2)

As for (2.3), we may assume that $f \in L^1(\mathbb{R}^n)$ (otherwise we truncate and apply a limiting process). Then we use the Calderon-Zygmund decomposition for f and t, so we have non overlapping cubes Q_j, such that

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f(x)|dx \leq 2^n t$$

for every j, and $|f(x)| \leq t$ for a.e. $x \notin \cup_j Q_j$. Now, since $x \in Q_j$ implies that $Mf(x) > t$, we can write:

$$|E_t| \geq \sum_j |Q_j| \geq \frac{1}{2^n t} \sum_j \int_{Q_j} |f(x)|dx \geq \frac{1}{2^n t} \int_{\{x:|f(x)|>t\}} |f(x)|dx$$

so, for $C' = 2^{-n}$ we get (2.3). □

The next result is proved in exactly the same way.

Theorem 1.2.2. *Suppose μ is a regular positive Borel measure in $\mathbb{R}^n$ satisfying a "doubling" condition like* (1.13). *Then, there are constants C, C' such that, for any Borel function f and any $t > 0$:*

$$\frac{C'}{t} \int_{\{x:|f(x)|>t\}} |f(x)|d\mu(x) \leq \mu(\{x : M_\mu f(x) > t\}) \leq$$

$$\leq \frac{C}{t} \int_{\{x\in\mathbb{R}^n:|f(x)|>t/2\}} |f(x)|d\mu(x)$$

□

From theorem (1.2.1) we easily derive several norm estimates for the maximal function.

Theorem 1.2.3. *For every p with $1 < p < \infty$, there is a constant $C_p > 0$ such that, for every $f \in L^p(\mathbb{R}^n)$:*

$$\left(\int_{\mathbb{R}^n} (Mf(x))^p dx \right)^{1/p} \leq C_p \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}$$

Proof. By Layer Cake representation we get:

$$\int_{\mathbb{R}^n} (Mf(x))^p dx = \int_0^\infty |\{x : Mf(x)^p > t\}|dt =$$

$$= \int_0^\infty |\{x : Mf(x) > t^{1/p}\}|dt = p\int_0^\infty t^{p-1}|\{x : Mf(x) > t\}|dt \le$$

$$\le C \cdot p \int_0^\infty t^{p-2} \int_{\{x:|f(x)|>t/2\}} |f(x)|dxdt = C \cdot p \int_{\mathbb{R}^n} \left(\int_0^{2|f(x)|} t^{p-2}dt \right) |f(x)|dx$$

$$= \frac{C \cdot 2^{p-1}p}{p-1} \int_{\mathbb{R}^n} |f(x)|^p dx := C_p \int_{\mathbb{R}^n} |f(x)|^p dx.$$

□

In exact same way we obtain the following

Theorem 1.2.4. *Let μ be a regular positive Borel measure in $\mathbb{R}^n$ satisfying a "doubling" condition like* (1.13)*. Then, for each p with $1 < p < \infty$, there is a constant $C_p > 0$ such that for every $f \in L^P(\mu)$:*

$$\left(\int_{\mathbb{R}^n} (M_\mu f(x))^p d\mu(x) \right)^{1/p} \le C_p \left(\int_{\mathbb{R}^n} |f(x)|^p d\mu(x) \right)^{1/p} \quad \square$$

We have seen that the operator M is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$ (since $Mf(x) \le \|f\|_\infty$ for every x). However, is not bounded in $L^1(\mathbb{R}^n)$.

Theorem 1.2.5. *Let f be integrable function supported in a ball $B \subset \mathbb{R}^n$. Then Mf is integrable over B if and only if :*

$$\int_B |f(x)|log^+|f(x)|dx < \infty. \tag{2.8}$$

Proof. If (2.8) holds, then

$$\int_B Mf(x)dx = \int_0^\infty |\{x \in B : Mf(x) > t\}|dt = 2\int_0^\infty |\{x \in B : Mf(x) > 2t\}|dt$$

$$\le 2\left(\int_0^1 |B|dt + \int_1^\infty |E_{2t}|dt \right)$$

and using (2.2), we get

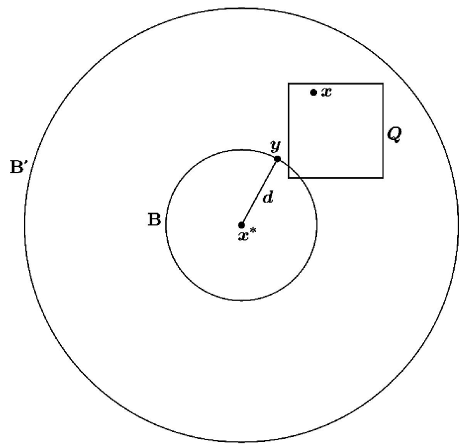
$$\le 2|B| + C\int_1^\infty \frac{1}{t} \int_{\{x:|f(x)|>t\}} |f(x)|dxdt =$$

$$2|B| + C\int_{\mathbb{R}^n} |f(x)| \int_1^{|f(x)|} \frac{1}{t} dtdx = 2|B| + C\int_{\mathbb{R}^n} |f(x)|log^+|f(x)|dx.$$

observe that for this part of the proof we do not need to use the fact that f is supported in B. Indeed, the same proof shows that if $\int_{\mathbb{R}^n} |f(x)|log^+|f(x)|dx < \infty$, which we shall indicate by saying that $f \in LlogL(\mathbb{R}^n)$, then Mf is locally integrable.

Going back to the proof of the theorem, suppose that $\int_B Mf(x)dx < \infty$. If we denote by B' the ball concentric with B but with radius $3/2$ as big, we can easily see that $\int_{B'} Mf(x)dx < \infty$ and that is because there is a constant $C > 0$ such that for $x \in B' \backslash B$ we get $Mf(x) \leq CMf(x^*)$ where x^* is the point symmetric to x with respect to the boundary of B. Lets prove that:

Let $x \in B' \backslash B$ and let Q be a cube containing x in its interior such that $\frac{1}{|Q|} \int_Q |f(y)|dy > 0$ (so $|Q \cap B| > 0$). Let $d = d(x, \partial B)$. It is obvious that the side length of Q is bigger than d. Let now $y = \overrightarrow{x^* x} \cap \partial B$. We can see that $d(y, Q) < d$.



(Let us note that the shape above stands for B' with radius twice as big compered with the one of B, but the proof is still valid.) Thus, y is in $Q' = 3Q$ and now x^* is in $Q'' = 3Q' = 9Q$, so for $C = 9^n$ we get that

$$\frac{1}{|Q|} \int_Q |f(y)|dy \leq \frac{1}{|Q|} \int_{Q''} |f(y)|dy = \frac{9^n}{|Q''|} \int_{Q''} |f(y)|dy \leq CMf(x^*)$$

so

$$Mf(x) \leq CMf(x^*)$$

Now

$$\int_{B'} Mf(x)dx = \int_{B'\setminus B} Mf(x)dx + \int_B Mf(x)dx = I_1 + I_2$$

where $I_2 < \infty$ and for I_1 we have

$$I_1 = \int_{B'\setminus B} Mf(x)dx = \int_{g(b)} Mf(x)dx$$

where $g : b \longrightarrow B'\setminus B$ with $g(y) = 2r\frac{y}{\|y\|} - y$ and $b = \{x \in B : \|x\| \geq r/2\}$ (r is the radius of B), so

$$I_1 \leq C\int_{g(b)} Mf(g^{-1}(x))dx = C\int_b Mf(y)|Jg(y)|dy$$

we can easily see that $|Jg|$ is bounded in b so:

$$I_1 \leq CK\int_b Mf(y)dy \leq CK\int_B Mf(y)dy < \infty$$

We conclude that $\int_{B''} Mf(x)dx < \infty$ where B'' is a ball with radius as big as we want. Now we see that

$$\frac{1}{|Q|}\int_Q |f(x)|dx \leq \frac{1}{|Q|}\|f\|_1$$

thus, $Mf(x) \to 0$ as $\|x\| \to \infty$, so, for any fixed $t_o > 0$, we get that

$$\{x : Mf(x) > t_o\} \subset B$$

for some ball B and thus, $\int_{\{x:Mf(x)>t_o\}} Mf(x)dx < \infty$. Now for $t_o = 1$ and using theorem (1.2.1), we get

$$\int_1^\infty |\{Mf > t\}|dt \geq \int_1^\infty \frac{C'}{t}\int_{\{|f|>t\}} |f(x)|dxdt = C'\int_{\mathbb{R}^n} |f(x)|\int_1^{|f(x)|} \frac{1}{t}dtdx$$

$$= \frac{1}{2^n}\int_{\mathbb{R}^n} |f(x)|log^+|f(x)|dx \qquad (C' = 1/2^n)$$

But

$$\int_1^\infty |\{Mf > t\}|dt = \int_{\{Mf>1\}} Mf(x)dx$$

which is $< \infty$ as we said before, so (2.8) holds and the proof is complete. □

The theorem extends clearly to M_μ for a measure μ satisfying a doubling condition. There is no need to write a new statement.

Suppose now that we have two measure spaces with respective measures μ and ν , and that T is an operator bounded from $L^p(\mu)$ to $L^q(\nu)$, that is:

$$\left(\int |Tf|^q d\nu\right)^{1/q} \leq C \left(\int |f|^p d\mu\right)^{1/p} \tag{2.9}$$

Then,

$$\nu(\{x : |Tf(x)| > t\}) \leq \int_{\{x:|Tf(x)|>t\}} (|Tf(x)|/t)^q d\nu$$

$$\leq \frac{1}{t^q} \int |Tf|^q d\nu \leq \frac{C^q}{t^q} \left(\int |f|^p d\mu\right)^{p/q}$$

and we obtain

$$\nu(\{x : |Tf(x)| > t\}) \leq \left(\frac{C\|f\|_{L^p(\mu)}}{t}\right)^q \tag{2.10}$$

When T satisfies (2.10) we say that the operator T is of weak type (p,q) with respect to the pair of measures (ν, μ). For example (1.8) is read by saying that M is of weak type (1,1)(with respect to the Lebesgue measure). However, we know that M fails to be bounded in L^1 (see proposition 3.0.1 in appendix). In general, (2.10) may hold whereas (2.9) does not hold for a given operator T. It is convenient to see (2.10) as a substitute or a weakening of (2.9). With this in mind, when (2.9) holds, we say that T is of strong type (p,q) with respect to the pair of measures (ν, μ). Sometimes it is convenient to indicate that (2.10) holds by saying that T sends $L^p(\mu)$ boundedly into $L^q_*(\nu)$ (called weak-$L^q(\nu)$).

Weak type inequalities such as (2.10) can be used to obtain strong type inequalities. This is what we have done to prove theorem (1.2.3). We are going to present a result, which is a particular case of the Marcinkiewicz interpolation theorem and is based upon the same idea as our proof of (1.2.3).

Theorem 1.2.6. *Suppose we have two measure spaces with respective measures μ and ν. Let T be an operator sending functions in $L^{p_o}(\mu) + L^{p_1}(\mu)$ to ν - measurable functions,$1 \leq p_o < p_1 \leq \infty$. Suppose that :*

1. *T is subadditive, that is, for $f_1, f_2 \in L^{P_o}(\mu) + L^{p_1}(\mu)$,*

$$|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)|, \nu - a.e.$$

2. *T is of weak type (p_o, p_o),that is:*

$$\nu(\{x : |Tf(x)| > t\}) \leq \frac{C_o \int |f|^{p_o} d\mu}{t^{p_o}}$$

with C_o independent of $f \in L^{p_o}(\mu)$ and $t > 0$.

3. *T is of weak type (p_1, p_1) which means the same as above if $p_1 < \infty$, while if $p_1 = \infty$, weak type and strong type coincide by definition:*

$$\|Tf\|_{L^\infty(\nu)} \leq C_1 \|f\|_{L^\infty(\mu)}$$

Then, for every p such that $p_o < p < p_1$, T is of strong type (p, p), that is :$\int |Tf|^p d(\nu) \leq C_p \int |f|^p d(\mu)$.

Proof. Fix p with $p_o < p < p_1$ and let $f \in L^p(\mu) \subset L^{P_o}(\mu) + L^{p_1}(\mu)$.For every $t > 0$ write $f(x) = f^t(x) + f_t(x)$ where $f^t(x) = f(x)$ if $|f(x)| > t$ and $f^t(x) = 0$ otherwise. Clearly $f^t \in L^{p_o}(\mu)$, and that is because:

$$\int (f^t)^{p_o} d\mu = \int (f^t)^{p-(p-p_o)} d\mu \leq \frac{1}{t^{p-p_o}} \int (f^t)^p d\mu < \infty$$

and since $|f_t(x)| \leq t$ we get also

$$\int |f_t|^{p_1} d\mu = \int |f_t|^{p+(p_1-p)} d\mu \leq t^{p_1-p} \int |f_t|^p d\mu < \infty$$

Suppose now $p_1 < \infty$. Then, since $|Tf(x)| \leq |T(f^t)(x)| + |T(f_t)(x)|$, we can write:

$$\nu(x : |Tf(x)| > t) \leq \nu(x : |T(f^t)(x)| > t/2) + \nu(|T(f_t)(x)| > t/2) \leq$$

$$\leq \frac{C_o \int |f^t|^p d\mu}{(t/2)^{p_o}} + \frac{C_1 \int |f_t|^{p_1} d\mu}{(t/2)^{p_1}}$$

Thus

$$\int |Tf|^P d\nu = p \int_0^\infty t^{p-1} \nu(Tf > t) dt \leq$$

$$\leq p2^{p_o}C_o\int_0^\infty t^{p-p_o-1}\int_{|f|>t}|f(x)|^{p_o}d\mu(x)dt$$
$$+\,p2^{p_1}C_1\int_0^\infty t^{p-p_1-1}\int_{|f|\leq t}|f(x)|^{p_1}d\mu(x)dt=$$

$$=p2^{p_o}C_o\int|f(x)|^{p_o}\int_0^{|f(x)|}t^{p-p_o-1}dtd\mu(x)$$
$$+\,p2^{p_1}C_1\int|f(x)|^{p_1}\int_{|f(x)|}^\infty t^{p-p_1-1}dtd\mu(x)$$

$$=\frac{p2^{p_o}C_o}{p-p_o}\int|f(x)|^pd\mu(x)\;+\;\frac{p2^{p_1}C_1}{p_1-p}\int|f(x)|^pd\mu(x):=C_p\int|f(x)|^pd\mu(x)$$

For the case $p_1=\infty$ we just have to observe (as we will see in the end of this proof), that

$$\nu(|Tf|>t)\leq\nu(|T(f^{at})|>t/2)\qquad(I)$$

where $a=1/2C_1'$ where $C_1'=C_1+\varepsilon$ and, consequently

$$\int|Tf(x)|^pd\nu(x)=p\int_0^\infty t^{p-1}\nu(|Tf|>t)dt\leq p\int_0^\infty t^{p-1}\nu(|Tf^{at}|>t/2)dt\leq$$

$\leq p\int_0^\infty t^{p-1}\frac{C_o}{(t/2)^{p_o}}\int|f^{at}(x)|^{p_o}d\mu(x)dt=$

$$=p2^{p_o}C_o\int_0^\infty t^{p-p_o-1}\int_{|f|>at}|f(x)|^{p_o}d\mu(x)dt=$$

$=C_op2^{p_o}\int|f(x)|^{p_o}\int_0^{|f(x)|/a}t^{p-p_o-1}dtd\mu(x):=$

$$:=C_p\int|f(x)|^pd\mu(x)$$

Lets prove (I): We already know that

$$|Tf|=|T(f^{at}+f_{at})|\leq|T(f^{at})|+|T(f_{at})|$$

Let now $t>0$ such that

$$\nu(\{|Tf|>t\})>0.$$

If we had

$$\nu(\{|Tf_{at}|\geq t/2\})>0$$

then, from the definition of $L^\infty(\nu)$ norm, we would have

$$\frac{t}{2} \leq \|Tf_{at}\|_{L^\infty(\nu)} \leq C_1\|f_{at}\|_{L^\infty(\nu)} < C_1'\|f_{at}\|_{L^\infty(\nu)} \leq C_1'at = \frac{t}{2}$$

which is not valid, thus

$$\nu(\{|Tf_{at} \geq t/2|\}) = 0$$

which implies that

$$\nu(|Tf| > t) \leq \nu(|T(f^at)| > t/2)$$

and the proof is complete

□

Next we shall establish a general inequality for the maximal function.This inequality involves a weight function $\phi(x)$.

Theorem 1.2.7. *For every p with $1 < p < \infty$ there is a constant C_p such that for any measurable functions on $\mathbb{R}^n$, $\phi \geq 0$ and f, we have the inequality*

$$\int_{\mathbb{R}^n} (Mf(x))^p\phi(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p(M\phi)(x)dx \tag{2.13}$$

Proof. Except when $M\phi(x) = \infty$ a.e in which (2.13) holds trivially, $M\phi$ is the density of a positive measure μ (just define $\mu(A) = \int_A M\phi(x)d\lambda(x)$ where λ is the Lebesgue measure and then $d\mu(x) = M\phi(x)dx$) and in the same way ϕ is the density of another positive measure ν ($d\nu(x) = \phi(x)dx$), consequently by this observation (2.13) means that M is bounded operator from $L^p(\mu)$ to $L^p(\nu)$. Now if $p = \infty$ then clearly M is bounded from $L^\infty(\mu)$ to $L^\infty(\nu)$, indeed if $M\phi(x) = 0$ for some x then $\phi(x) = 0$ a.e and so (2.13) holds, now if $M\phi(x) > 0$ for every x and $\|f\|_{L^\infty(\mu)} < a$ for some a (because if $\| f \|_{L^\infty(\mu)} = \infty$ then (2.13) holds again), we get that:

$$\int_{\{|f|>a\}} M\phi(x)dx = \int_{\{|f|>a\}} d\mu(x) = \mu(|f| > a) = 0$$

and consequently $|\{|f| > a\}| = 0$ or, what is the same $|f(x)| \leq a$ a.e from which we get that $Mf(x) \leq a$ a.e.Thus $\| Mf \|_{L^\infty(\nu)} \leq a$. So we have shown that if $\| f \|_{L^\infty(\mu)} < a$ then $\| Mf \|_{L^\infty(\nu)} \leq a$ which means that $\| Mf \|_{L^\infty(\nu)} \leq \| f \|_{L^\infty(\mu)}$(we can choose $a = \| f \|_{L^\infty(\mu)} + \varepsilon$ and then $\varepsilon \to 0$).

Having the (∞,∞) result, if we are able to show that M is of weak type $(1,1)$ with respect to the pair of measures (ν,μ), the previous theorem (interpolation) will give (2.13).Thus, all we need to show is that :

$$\nu(\{Mf>t\}) = \int_{\{Mf>t\}} \phi(x)dx \leq$$

$$\leq \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|d\mu(x) = \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|(M\phi)(x)dx \qquad (2.14)$$

We can obviously assume that $f \geq 0$ and we can also assume that $f \in L^1(\mathbb{R}^n)$. Indeed, we can find integrable functions f_j such that $f_1 \leq f_2 \leq \ldots \nearrow f$ a.e. and observe that

$$\{x : Mf(x) > t\} = \bigcup_j \{x : Mf_j(x) > t\}$$

So, let $f \in L^1(\mathbb{R}^n)$ and $f \geq 0$. Given $t > 0$, we know that there is a family of non-overlapping cubes $\{Q_j\}$ such that, for each j :

$$\frac{t}{4^n} < \frac{1}{|Q_j|}\int_{Q_j} f(x)dx \leq \frac{t}{2^n} \qquad (i)$$

and also

$$\{x : Mf(x) > t\} \subseteq \bigcup_j {Q_j}^3$$

Then

$$\int_{\{Mf>t\}} \phi(x)dx \leq \sum_j \int_{{Q_j}^3} \phi(x)dx = \sum_j \frac{|Q_j^3|}{|Q_j^3|}\int_{{Q_j}^3} \phi(x)dx =$$

$$= \sum_j \frac{1}{|Q_j^3|} 3^n |Q_j| \int_{Q_j^3} \phi(x)dx \leq \sum_j \frac{3^n 4^n}{|Q_j^3| t}\int_{Q_j} f(x)dx \int_{Q_j^3} \phi(x)dx =$$

For the last inequality we used (i).

$$= \frac{3^n 4^n}{t}\sum_j \int_{Q_j} \left(\frac{1}{|Q_j^3|}\int_{Q_j^3} \phi(y)dy \right) f(x)dx := a$$

But

$$\frac{1}{|Q_j^3|}\int_{Q_j^3} \phi(y)dy \leq M\phi(y)$$

for every $y \in Q_j$ and so we get

$$a \leq \frac{3^n 4^n}{t} \sum_j \int_{Q_j} f(x) M\phi(x) dx \leq \frac{3^n 4^n}{t} \int_{\mathbb{R}^n} f(x) M\phi(x) dx :=$$

$$:= \frac{C}{t} \int_{\mathbb{R}^n} f(x) M\phi(x) dx = \frac{C}{t} \int_{\mathbb{R}^n} f(x) d\mu(x)$$

The theorem we just proved identifies a whole class of weight functions ϕ for which the operator M is bounded in $L^p(\phi)$ for every $p \in (1, \infty]$ and of weak type $(1,1)$ with respect to ϕ, namely, then class, customarily denoted by A_1 , of those $\phi \geq 0$ satisfying $M\phi(x) \leq C\phi(x)$ a.e for some constant C.

□

There is an interesting extension of the previous theorem whose proof is but a repetition of the arguments which led to 1.6, 2.5 and 2.12. In order to present this result we make several definitions:

Given a function f in $\mathbb{R}^n$, we define a function Mf in $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t \geq 0\}$ by setting

$$Mf(x,t) = sup \left\{ \frac{1}{|Q|} \int_Q |f(y)| dy : x \in Q \ \ and \ \ side \ \ length \ \ of \ \ Q \geq t \right\}$$

Given a positive Borel measure μ in $\overline{\mathbb{R}^{n+1}_+}$, we define a function $N(\mu)$ in $\mathbb{R}^n$ by setting

$$N(\mu)(x) = \sup_{x \in Q} \frac{\mu(\tilde{Q})}{|Q|}$$

where the sup is taken over all cubes Q containing x and for a cube Q

$$\tilde{Q} = \{(x,t) \in \overline{\mathbb{R}^{n+1}_+} : x \in Q \ \ and \ \ 0 \leq t \leq side \ \ lenght \ \ of \ \ Q\},$$

that is,$\tilde{Q}$ is the cube in $\overline{\mathbb{R}^{n+1}_+}$ having Q as a face. With the above definitions we can state the following:

Theorem 1.2.8. *For every p with $1 < p < \infty$, there is a constant C_p such that, for every f and every μ:*

$$(2.16) \qquad \left(\int_{\overline{\mathbb{R}^{n+1}_+}} \{Mf(x,t)\}^p d\mu(x,t) \right)^{1/p} \leq C_p \left(\int_{\mathbb{R}^n} |f(x)|^p N\mu(x) dx \right)^{1/p}$$

Proof. Before we start the proof, let us note that this result includes the previous one. We know that if $\nu(A) = \int_A \phi(x)dx$ then $d\nu(x) = \phi(x)dx$ and ν is a measure in $\mathbb{R}^n$,let also δ be the unit mass consertrated at the origin in the t axis (Dirac measure on $0 \in \mathbb{R}$) which is a measure on $\mathbb{R}$, then there exist a unique measure μ in $\mathbb{R}^{n+1}$ such that $\mu(A \times B) = \nu(A) \times \delta(B)$ where $A \in \mathcal{B}(\mathbb{R}^n)$ and $B \in \mathcal{B}(\mathbb{R})$. We can see now that:

$$N\mu(x) = \sup_{x \in Q} \frac{\mu(\tilde{Q})}{|Q|} = \sup_{x \in Q} \frac{\nu(Q)}{|Q|} \delta([0, side\ lenght\ of\ Q]) = \sup_{x \in Q} \frac{\nu(Q)}{|Q|} = M\phi(x)$$

also $Mf(x,0) = Mf(x)$ and

$$\left(\int_{\overline{\mathbb{R}^{n+1}_+} = \mathbb{R}^n \times \overline{\mathbb{R}_+}} \{Mf(x,t)\}^p d\mu(x,t) \right)^{1/p} \geq \left(\int_{\mathbb{R}^n \times \{0\}} \{Mf(x,0)\}^p d\mu(x,t) \right)^{1/p} =$$

$$= \left(\int_{\{0\}} \int_{\mathbb{R}^n} Mf(x)^p d\nu(x) d\delta(x) \right)^{1/p} =$$

$$= \left(\int_{\mathbb{R}^n} Mf(x)^p d\nu(x) \int_{\{0\}} 1 d\delta(x) \right)^{1/p} = \left(\int_{\mathbb{R}^n} Mf(x)^p d\nu(x) \right)^{1/p}$$

so this observation combined with (2.16) gives (2.13).

Now we prove the theorem. As in the proof of the preceding result, if we exclude the trivial case when $N\mu(x) = \infty$ a.e, we have in the same way that (for the case $p = \infty$) M is bounded operator from $L^\infty(\mathbb{R}^n, \nu)$ to $L^\infty(\overline{\mathbb{R}^{n+1}_+}, \mu)$ where ν is defined as before. So all we need to prove is that M is of weak type $(1,1)$ and then use interpolation (theorem(2.11)). So if we call $E_a = \{(x,t) \in \overline{\mathbb{R}^{n+1}_+} : Mf(x,t) > a\}$ we have to show that there is a constant C such that for every $a > 0$:

$$\mu(E_a) \leq \frac{C}{a} \int_{\mathbb{R}^n} |f(x)| d\nu(x) = \frac{C}{a} \int_{\mathbb{R}^n} |f(x)| N\mu(x) dx.$$

Fix $a > 0$ and suppose that $(x,t) \in E_a$, then there is a cube R containing x with side length $R \geq t$ and such that:

$$\frac{1}{|R|} \int_R |f(y)| dy \geq a$$

Let now k be the only integer such that : $2^{-(k+1)n} < |R| \leq 2^{-kn}$. As in the proof of theorem (1.6) there is some $Q \in D_k$ which meets the interior of R and satisfies

$$\int_{R\cap Q} |f(y)|dy > \frac{a|R|}{2^n} > \frac{a|Q|}{4^n}$$

so that

$$\frac{1}{|Q|}\int_Q |f(y)|dy > \frac{a}{4^n} \qquad .$$

It follows that $Q \subset Q_j \in C_{a4^{-n}}$ for some j and $x \in R \subset Q^3 \subset Q_j^3$. On the other hand $t \leq$ side length of $R \leq$ side length of Q_j^3, so that $(x,t) \in \tilde{Q_j^3}$. Thus we have seen that:

$$E_a \subset \bigcup_j \tilde{Q_j^3}$$

where $\quad C_{a4^{-n}} = \{Q_j\}$. Then

$$\mu(E_a) \leq \sum_j \mu(\tilde{Q_j^3}) \leq \sum_j \mu(\tilde{Q_j^3}) \cdot \frac{\frac{1}{|Q_j|}\int_{Q_j}|f(y)|dy}{a4^{-n}} =$$

$$= \sum_j \frac{\mu(\tilde{Q_j^3})}{|Q_j^3|}\frac{3^n 4^n}{a}\int_{Q_j}|f(y)|dy \leq \frac{C}{a}\sum_j \int_{Q_j}|f(y)|N\mu(y)dy \leq \frac{C}{a}\int_{\mathbb{R}^n}|f(x)|N\mu(x)dx.$$

For the 3rd inequality we used the definition of $N\mu$.

□

In particular, if the measure μ is such that:

$$(2.17) \qquad \mu(\tilde{Q}) \leq C|Q|$$

for every cube $Q \subset \mathbb{R}^n$ with C independent of Q, then $N\mu(x) \leq C$ and (2.16) implies that $f \to Mf$ is an operator bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^{n+1}_+, \mu)$ for every p with $1 < p < \infty$. Actually, given any p with $1 < p < \infty$, (2.17) is not only sufficient but also necessary for M to be bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^{n+1}_+, \mu)$. Indeed, since $M(X_Q)(x,t) \geq 1$ for every $(x,t) \in \tilde{Q}$, the boundedness of M implies that:

$$\mu(\tilde{Q}) = \int_{\tilde{Q}} d\mu \leq \int_{\tilde{Q}} M(X_Q)(x,t)^p d\mu(x,t) \leq C\|X_Q\|_{L^p(\mathbb{R}^n)} = C|Q|$$

The importance of M stems from the fact that Mf controls the Poisson integral of f, $P(f)$, defined by:

$$P(f)(x,t) = C_n \int_{\mathbb{R}^n} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} f(y)dy$$

for $x \in \mathbb{R}^n$ and $t > 0$, where

$$C_n = \left(\int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^{\frac{n+1}{2}}} dx \right)^{-1}$$

Indeed:

$$|P(f)(x,t)| \leq$$

$$\leq C_n \int_{|x-y|\leq t} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} |f(y)|dy +$$

$$+ C_n \sum_{k=0}^{\infty} \int_{2^k t < |y-x| \leq 2^{k+1}t} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} |f(y)|dy \leq$$

$$\leq C_n \left\{ \frac{1}{t^n} \int_{|x-y|\leq t} |f(y)|dy + \sum_{k=0}^{\infty} \frac{t}{(2^k t)^{n+1}} \int_{|x-y|\leq 2^{k+1}t} |f(y)|dy \right\} \leq$$

$$\leq C_n \left\{ \frac{1}{t^n} \int_{Q(x,2t)} |f(y)|dy + \sum_{k=0}^{\infty} \frac{t}{(2^k t)^{n+1}} \int_{Q(x,2^{k+2}t)} |f(y)|dy \right\} =$$

$$= C_n \left\{ \frac{2^n}{(2t)^n} \int_{Q(x,2t)} |f(y)|dy + \sum_{k=0}^{\infty} \frac{t(2^2)^{n+1}}{(2^{k+2}t)^{n+1}} \int_{Q(x,2^{k+2}t)} |f(y)|dy \right\}$$

$$\leq C_n \left\{ 2^n Mf(x,t) + t(2^2)^{n+1} Mf(x,t) \sum_{k=0}^{\infty} \frac{1}{2^{k+2}t} \right\}$$

$$= C_n \left\{ 2^n Mf(x,t) + 4^n Mf(x,t) \cdot 1 \right\} := CMf(x,t)$$

Actually, for $f \geq 0$

$$P(f)(x,t) = C_n \int_{\mathbb{R}^n} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} f(y)dy \geq \frac{C_n}{t^n} \int_{|x-y|\leq t} f(y)dy \quad ,(a)$$

In particular if $f = X_Q$ and $(x,t) \in \tilde{Q}$, we get $P(X_Q)(x,t) \geq a_n > 0$, where a_n depends only on the dimension n , and that is because by (a) we get that:

$$P(f)(x,t) \geq \frac{C_n}{t^n} \int_{|x-y|\leq t} X_Q(y)dy \geq \frac{C_n}{t^n} \int_{Q'} dy = \frac{C_n}{(\sqrt{2})^n} := a_n$$

where Q' is a cube with side lenght equal to $\frac{t}{\sqrt{2}}$. And so for $(x,t) \in \tilde{Q}$, we get that $\frac{P(X_Q)(x,t)}{a_n} \geq 1$.
Consequently, using the same argument that we used for M $(M(X_Q)(x,t) \geq 1)$ shows that if the operator $f \to P(f)$ is bounded from $L^P(\mathbb{R}^n)$ to $L^p(\overline{\mathbb{R}^{n+1}_+}, \mu)$, then μ satisfies (2.17). The measures μ satisfying (2.17) are called Carleson measures. We can state the following:

Theorem 1.2.9. *Let μ be a positive Borel measure on $\overline{\mathbb{R}^{n+1}_+}$ and let $1 < p < \infty$.Then $f \to P(f)$ is bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\overline{\mathbb{R}^{n+1}_+}, \mu)$ if and only if μ is a Carleson measure, that is, if and only if* (2.17) *holds for some constant C.* □

Observe that the condition obtained does not depend on p and is also equivalent to the fact that $f \to P(f)$ sends $L^1(\mathbb{R}^n)$ boundedly into $L^1_*(\overline{\mathbb{R}^{n+1}_+}, \mu)$.

1.3 THE SHARP MAXIMAL FUNCTION AND THE SPACE OF BOUNDED MEAN OSCILATION

For a real locally integrable function f in $\mathbb{R}^n$, the sharp maximal function $f^{\#}$ is defined at $x \in \mathbb{R}^n$ by setting

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy$$

where f_Q stands for the average of f over Q, that is:

$$f_Q = \frac{1}{|Q|}\int_Q f(x)dx.$$

The sharp maximal operator $f \to f^{\#}$ is an analogue of the Hardy-Littlewood maximal operator M, but it has certain advantages over it which we shall presently see. Of course, $f^{\#}(x) \leq 2Mf(x)$. It is also clear that in the definition of $f^{\#}(x)$ one can take only those cubes Q containing x in its interior.Actually

$$\text{(3.1)} \qquad f^{\#}(x) \cong \sup_{x\in Q}\inf_{a\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(y)-a|dy$$

where $\cong$ is used to indicate that each side is bounded by the other times an absolute constant. It is clear that the right hand side of (3.1) is $\leq f^{\#}(x)$. For the opposite inequality we see that

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq \frac{1}{|Q|}\int_Q |f(x)-a|dx + |f_Q - a| \leq$$

$$\leq \frac{2}{|Q|}\int_Q |f(x)-a|dx$$

for every $a \in \mathbb{R}$. It follows that $f^{\#}(x)$ is bounded by twice the right hand side of (3.1). We also note that:

$$\text{(3.2)} \qquad (|f|)^{\#}(x) \leq 2f^{\#}(x)$$

Indeed by (3.1) we get that

$$|f|^{\#}(x) \leq 2\sup_{x\in Q}\inf_{a\in\mathbb{R}} \frac{1}{|Q|}\int_Q ||f(y)|-a|dy \leq 2\sup \frac{1}{|Q|}\int_Q ||f(y)|-|f_Q||dy \leq$$

$$\leq 2\sup \frac{1}{|Q|}\int_Q |f(y)-f_Q|dy = 2f^{\#}(x).$$

If f is such that $f^{\#}$ is bounded, we say that f is a function of bounded mean oscillation, and we denote by the initials B.M.O. the space formed by these functions. Thus

$$B.M.O. = \{f \in L^1_{loc}(\mathbb{R}^n) : f^{\#} \in L^\infty\}$$

We write $B.M.O.(\mathbb{R}^n)$ when we need to specify the underlying space. For $f \in B.M.O$ we write

$$\|f\|_* = \|f^{\#}\|_\infty = \sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|dx.$$

Of course we get after (3.1):

$$\frac{1}{2}\|f\|_* \leq \sup_Q . \inf_{a\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(x)-a|dx \leq \|f\|_*$$

Thus, in order to be able to say that $f \in B.M.O.$, it suffices to make sure that there exists $C < \infty$ and, for each Q, a constant a_Q such that

$$\frac{1}{|Q|}\int_Q |f(x)-a_Q|dx \leq C.$$

Then $\|f\|_* \leq 2C$. This is the usual way to see that a certain $f \in B.M.O.$

Clearly, $f \to \|f\|_*$ is a seminorm and $\|f\|_* = 0$ if and only if f is constant. It is natural to consider the quotient space of B.M.O. modulo constants, which is a normed space and, actually a Banach space. This space of equivalence classes modulo constants will also be called B.M.O. The ambiguity does not cause any problem.

Of course $L^\infty \subset B.M.O.$ (because $\|f\|_* \leq 2\|f\|_\infty$). However, there are unbounded B.M.O. functions as we shall soon see. We shall give two results which provide many examples of B.M.O functions.

Theorem 1.3.1. *If w is an A_1 weight, that is, if $Mw(x) \leq Cw(x)$ a.e., then $logw$ is in B.M.O. with a norm depending only on the A_1 constant for w i.e. the smallest C such that the above inequality is true.*

Proof. Call $logw = \phi$, that is $w = e^\phi$. We have for every cube Q

$$\frac{1}{|Q|}\int_Q e^{\phi(x)}dx \leq Ce^{\phi(x)}$$

for a.e.$x \in Q$ or, equivalently

$$\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}dx\right)\cdot ess\sup_{x\in Q}(e^{-\phi(x)}) \leq C$$

But $ess\sup_{x\in Q}(e^{-\phi(x)}) = \exp(-ess\inf_{x\in Q}\phi(x))$, and Jensen's inequality implies

$$\frac{1}{|Q|}\int_Q e^{\phi(x)}dx \geq \exp(\phi_Q)$$

Thus, $\exp(\phi_Q - ess\inf_{x\in Q}\phi(x)) \leq C$ and consequently, ϕ satisfies, for some other constant C independent of Q, the property

$$\phi_Q - ess\inf_{x\in Q}\phi(x) \leq C$$

We express this by saying that ϕ is of bounded lower oscillation, and denote by $B.L.O.$ the class formed by all the functions of bounded lower oscilation. Now, we see that $B.L.O. \subset B.M.O.$ Indeed

$$|\phi(x) - \phi_Q| \leq (\phi(x) - ess\inf_{x\in Q}\phi(x)) + (\phi_Q - ess\inf_{x\in Q}\phi(x))$$

for almost every x. Averaging over Q we obtain

$$\frac{1}{|Q|}\int_Q |\phi(x) - \phi_Q|dx \leq 2(\phi_Q - ess\inf_{x\in Q}\phi(x)).$$

and the inclusion $B.L.O. \subset B.M.O.$ follows readily □

Observe that the class $B.L.O.$ introduced above fails to be a vector space even though it is stable under the sum and the product by a non negative number. Actually $B.L.O. \cap (-B.L.O.) = L^\infty$. Indeed, if both ϕ and $-\phi$ are in $B.L.O.$, we have at the same time

$$\phi_Q - ess\inf_{x\in Q}\phi(x) \leq C \quad and \quad -\phi_Q + ess\sup_{x\in Q}\phi(x) \leq C$$

Adding up we get: $ess\sup_{x\in Q}\phi(x) - ess\inf_{x\in Q}\phi(x) \leq 2C$ with C independent of the cube Q. This is only possible if ϕ is essentially bounded.

The previous theorem gives us $B.M.O.$ functions from A_1 weights. We shall presently see a nice way to produse A_1 weights by using the Hardy-Littlewood maximal operator M.

Let μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets, and hence regular. It makes sense to consider the maximal function

$$M\mu(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q d\mu$$

where the sup is taken over all cubes containing x. Exactly as in the case of integrable functions, one obtains for measures the fundamental estimate

$$|\{x \in \mathbb{R}^n : M\mu(x) > t\}| \leq \frac{C}{t}\int_{\mathbb{R}^n} d\mu$$

with C depending only on the dimension. We can state the following:

Theorem 1.3.2. *Let μ be a positive Borel measure such that $M\mu(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, and let $0 < \gamma < 1$. Then the function $w(x) = (M\mu(x))^\gamma$ is an A_1 weight with a constant depending only on γ and the dimension n.*

Proof. Let Q be a fixed cube, we shall see that

$$\frac{1}{|Q|}\int_Q w(x)dx \le Cw(x)$$

for a.e. $x \in Q$ with C independent of Q. Let $\tilde{Q} = Q^3$, we write $\mu = \mu_1 + \mu_2$ with $\mu_1 = X_{\tilde{Q}}\mu$, the restriction of μ to $\tilde{Q}$. Then $M\mu(x) \le M\mu_1(x) + M\mu_2(x)$ and, since $0 < \gamma < 1$, also $(M\mu(x))^\gamma \le (M\mu_1(x))^\gamma + (M\mu_2(x))^\gamma$. Therefore, it will be enough to see that the averages of $(M\mu_1(x))^\gamma$ and $(M\mu_2(x))^\gamma$ over Q are both bounded by $Cw(x)$ for any $x \in Q$ with C depending only on γ and the dimension. We carry out these two estimates separately:

$$\frac{1}{|Q|}\int_Q (M\mu_1(x))^\gamma dx = \frac{1}{|Q|}\int_0^\infty \gamma t^{\gamma-1}|\{x \in Q : M\mu_1(x) > t\}|dt =$$

$$= \frac{1}{|Q|}\left(\int_0^R + \int_R^\infty\right)$$

(we split the integral by using an arbitrary R). Near to 0 we use the trivial estimate for the distribution function, which is, clearly, $\le |Q|$. From R to ∞ we use the weak type estimate

$$|\{x \in Q : M\mu_1(x) > t\}| \le \frac{C}{t}\mu_1(\mathbb{R}^n) := \frac{C}{t}\|\mu_1\|$$

Thus

$$\frac{1}{|Q|}\int_Q (M\mu_1(x))^\gamma dx \le \frac{1}{|Q|}\left(|Q|R^\gamma + C\int_R^\infty \gamma t^{\gamma-2}dt\|\mu_1\|\right)$$

$$= R^\gamma\left(1 + \frac{C\gamma}{1-\gamma}\frac{\|\mu_1\|}{R|Q|}\right)$$

Taking $R = \|\mu_1\||Q|^{-1}$ we get:

$$\frac{1}{|Q|}\int_Q (M\mu_1(x))^\gamma dx \le \left(\frac{\|\mu_1\|}{|Q|}\right)^\gamma\left(1 + \frac{C\gamma}{1-\gamma}\right) = \left(\frac{\mu(\tilde{Q})3^n}{|\tilde{Q}|}\right)^\gamma\left(1 + \frac{C\gamma}{1-\gamma}\right)$$

$$:= C' \left(\frac{\mu(\tilde{Q})}{|\tilde{Q}|} \right)^{\gamma} \leq C' w(x)$$

for every $x \in Q \subset \tilde{Q}$.

Comment: What we have just done is to realize that every operator of weak type (1,1) in a finite measure space, actually takes L^1 boundedly into L^p, if $p < 1$. This fact is known as Kolmogorov's inequality.

Let us explain this comment before we continue with the proof : Let (X, μ) be a finite measure space and let T be an operator of weak type (1,1) (with respect to μ) then :

$$\int_X |Tf|^{\gamma} d\mu = \int_0^{\infty} \gamma t^{\gamma-1} \mu(\{|Tf| > t\}) dt := \int_0^R + \int_R^{\infty} \leq$$

Since T is of weak type (1,1)

$$\leq R^{\gamma}\mu(X) + C \int_R^{\infty} \gamma t^{\gamma-2} dt \|f\|_1 =$$

$$= R^{\gamma}\mu(X) + \frac{C\gamma R^{\gamma-1}}{1-\gamma}\|f\|_1 =$$

$$= R^{\gamma}\left(\mu(X) + \frac{C\gamma}{(1-\gamma)R}\|f\|_1\right)$$

Thus, for $R = \|f\|_1$ we get that

$$\left(\int_X |Tf|^{\gamma} d\mu\right)^{1/\gamma} \leq C'\|f\|_1$$

Lets continue with the proof:

To deal now with $M\mu_2$ is even simpler. It is enough to see that, because of the fact that μ_2 lives far from Q (outside of $\tilde{Q}$), for any two points $x, y \in Q$, we have $M\mu_2(x) \leq CM\mu_2(y)$, with C an absolute constant. Indeed if Q' is a cube containing x and meeting $\mathbb{R}^n \backslash \tilde{Q}$, then $Q \subset Q'^3$, so we get:

$$\frac{1}{|Q'|}\int_{Q'} d\mu_2 = \frac{3^n}{|Q'^3|}\int_{Q'} d\mu_2 \leq \frac{3^n}{|Q'^3|}\int_{Q'^3} d\mu_2 \leq 3^n M\mu_2(y)$$

which leads to $M\mu_2(x) \leq 3^n M\mu_2(y)$ for any $x, y \in Q$. Thus

$$\frac{1}{|Q|}\int_Q (M\mu_2(y))^\gamma dy \leq \frac{|Q|}{|Q|}3^{n\gamma}(M\mu_2(x))^\gamma \leq 3^{n\gamma}w(x)$$

for every $x \in Q$ and the proof is complete □

For example,take $\mu = \delta_0$, the Dirac delta or unit mass at the origin in $\mathbb{R}^n$. Then $M\delta(x) = \|x\|_\infty^{-n}$ where $\|x\|_\infty = max|x_j|$ and that is because, in order to have $M\delta(x) > 0$ for some x, we need to look at the cubes containing both zero and x, and the smallest cubes which include those points is of side length equal to $\|x\|_\infty$. Thus $M\delta(x) \cong |x|^{-n}$ (because $|\ |_\infty$ and $|\ |$ are equivalent)

It follows that for any γ with $0 \leq \gamma < 1$, $|x|^{-n\gamma}$ is an A_1 weight, or, in other words $|x|^a$ is an A_1 weight for any a with $-n < a \leq 0$ and only for these a's actually, since w has to be locally integrable, let as explain this :

$$\int_{B(0,1)} \frac{1}{|x|^a}dx = \int_0^1 \int_{\partial B(0,t)} \frac{1}{|x|^a}dxdt =$$

$$= \int_0^1 \frac{1}{t^a}|\partial B(0,t)|dt = \int_0^1 \frac{1}{t^a}nt^{n-1}|B(0,1)|dt =$$

$$= n|B(0,1)|\left|\frac{t^{n-a}}{n-a}\right|_0^1$$

which is $< \infty$ if $a < n$. However, our main concern here is the fact that $\log|x|$ is an example of an unbounded $B.M.O.$ function. Note that (we will see it later) $\log\frac{1}{|x|}$ is actually in $B.L.O.$ In general we have:

Corollary 1.3.1. *1. For any positive Borel measure μ such that $M\mu(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, $\log M\mu(x)$ is a B.M.O. function with norm depending only on the dimension.*

2. $\log|x|$ *is in B.M.O.*

Proof. It is clear from the definition of B.M.O. functions, that, $f \in B.M.O.$ implies that $cf \in B.M.O.$ for any constant c.

□

Since $(|f|)^{\#} \leq 2f^{\#}(x)$, we know that $f \in B.M.O.$ implies $|f| \in B.M.O.$ Consequently B.M.O. is a lattice (if $f, g \in B.M.O.$, then the functions $max(f,g) = (|f-g|+f+g)/2$ and $min(f,g) = (f+g-|f-g|)/2$ will also be in B.M.O.). However, we may have $|f| \in B.M.O.$ without having $f \in B.M.O.$ For example if :

$$f(x) = \begin{cases} 0 & \text{for} \quad |x| > 1 \\ -\log|x| & \text{if} \quad 0 < x < 1 \\ \log|x| & \text{if} \quad -1 < x < 0 \end{cases}$$

It is clear that $|f(x)| = max(\log \frac{1}{|x|}, 0)$ is in $B.M.O.$ However, f is not in $B.M.O.$ Since f is odd and, consequently, has average 0 on every interval $[-a,a]$, we just need to observe that

$$\frac{1}{2a}\int_{-a}^{a} |f(x)|dx = \frac{1}{a}\int_{0}^{a} \log\frac{1}{x}dx = 1 - \log a \longrightarrow \infty$$

for $a \to 0$.

There is an intimate relation between the operator $f \to f^{\#}$ and the Hardy-Littlewood maximal operator M. It is contained in the following statement:

Theorem 1.3.3. *If f is such that $Mf \in L^{p_o}$ for some p_o with $0 < p_o < \infty$, then for every p such that $p_o \leq p < \infty$, we have:*

$$\int_{\mathbb{R}^n} (Mf(x))^p dx \leq C \int_{\mathbb{R}^n} (f^{\#}(x))^p dx$$

with C independent of f.

Proof. We may assume that $f \geq 0$ since $Mf = M(|f|)$ and $(|f|)^{\#} \leq 2f^{\#}$.

The proof is based upon the Calderon-Zygmund decomposition. First we see that this decomposition can be carried our for oun function f. Let $t > 0$ and suppose that Q is a cube such that $f_Q > t$. Then, for every x in Q

$$t < \frac{1}{|Q|}\int_{Q} f(y)dy \leq Mf(x)$$

and thus $\quad t^{p_o} < (Mf(x))^{p_o}$ so,

$$t^{p_o} \leq \frac{1}{|Q|}\int_{Q} (Mf(x))^{p_o} dx \leq \frac{1}{|Q|}\int_{\mathbb{R}^n} (Mf(x))^{p_o} dx := \frac{C}{|Q|}$$

It follows that if $Q_1 \subsetneq Q_2 \subsetneq ....$ is an increasing family of dyadic cubes for each of which is

$$\frac{1}{|Q_k|}\int_{Q_k} f(y)dy > t,$$

then the family is necessarily finite since $|Q_k|$ is bounded independently of k. Thus, each dyadic cube Q satisfying $f_Q > t$ will be contained in a maximal one. If $\{Q_j\}$ is the family consisting of these maximal dyadic cubes, for each of them will be

$$t < \frac{1}{|Q_j|}\int_{Q_j} f(y)dy \leq 2^n t$$

In order to indicate the dependence on t, we shall denote this family by $\{Q_{j,t}\}$. For a.e. $x \notin \cup_j Q_{t,j}$ is $f(x) \leq t$.

Observe that if $t < s$, then each $Q_{s,j}$ is $\subset Q_{t,k}$ for some k. Given $t > 0$ we fix $Q_o = Q_{2^{-n-1}t,j_o}$ and take $A > 0$. There are two possibilities: either

$$Q_o \subset \{x : f^{\#}(x) > t/A\} \quad or \quad Q_o \not\subset \{x : f^{\#}(x) > t/A\}.$$

In the first case

$$\sum_{\{j:Q_{t,j}\subset Q_o\}} |Q_{t,j}| \leq |\{x : f^{\#}(x) > t/A\}|$$

In the second case

$$\frac{1}{|Q_o|}\int_{Q_o} |f(y) - f_{Q_o}|dy \leq t/A$$

(that is because there is x in Q_o such that $f^{\#}(x) \leq t/A$). Now taking into account that $f_{Q_o} \leq 2^n 2^{-n-1} t = t/2$, we can write:

$$\sum_{\{j:Q_{t,j}\subset Q_o\}} (t - \frac{t}{2})|Q_{t,j}| \leq \sum_{\{j:Q_{t,j}\subset Q_o\}} (f_{Q_{t,j}} - f_{Q_o})|Q_{t,j}| =$$

$$= \sum_{\{j:Q_{t,j}\subset Q_o\}} \int_{Q_{t,j}} (f(y) - f_{Q_o})dy \leq \sum_{\{j:Q_{t,j}\subset Q_o\}} \int_{Q_{t,j}} |f(y) - f_{Q_o}|dy \leq$$

$$\leq \int_{Q_o} |f(y) - f_{Q_o}|dy \leq \frac{t|Q_o|}{A}.$$

and so

$$\sum_{\{j:Q_{t,j}\subset Q_o\}} |Q_{t,j}| \leq \frac{2|Q_o|}{A} \qquad (I)$$

Let us note that the sum $\sum_{\{j:Q_{t,j}\subset Q_o\}}$ has meaning because of the fact that each $Q_{t,j}$ is subset of some $Q_{t/2^{n+1},k}$ for some k due to the observation we made earlier. Adding up now in all the possoble Q_o's, we get

$$\sum_j |Q_{t,j}| =$$

$$= \sum_{\{j:Q_{t,j}\subset Q_{t2^{-n-1},k}\ \ for\ \ some\ \ k\ \ and\ \ Q_{t2^{-n-1},k}\subset\{x:f^{\#}(x)>t/A\}\}} |Q_{t,j}| +$$

$$+ \sum_{\{j:Q_{t,j}\subset Q_{t2^{-n-1},k}\ \ for\ \ some\ \ k\ \ and\ \ Q_{t2^{-n-1},k}\not\subset\{x:f^{\#}(x)>t/A\}\}} |Q_{t,j}| \leq$$

and by (I) and because of the fact that $Q_{t,j}$ are non overlapping, we get

$$\leq |\{x : f^{\#}(x) > t/A\}| + \sum_k \frac{2}{A}|Q_{t2^{-n-1},k}|$$

Call

$$\alpha(t) = \sum_j |Q_{t,j}|$$

and

$$\beta(t) = |\{x : Mf(x) > t\}|$$

We know that $\alpha(t) \leq \beta(t)$, and using theorem (1.1.2) we get

$$\beta(t) \leq \sum_j |Q^3_{4^{-n}t,j}| = 3^n \sum_j |Q_{t/4^n,j}| := C_1\alpha(t/C_2)$$

where $C_1 = 3^n$ and $C_2 = 4^n$. In terms of α we have got the following inequality:

$$\alpha(t) \leq |\{x : f^{\#}(x) > t/A\}| + 2A^{-1}\alpha(2^{-n-1}t) \qquad (II)$$

Now, for $N > 0$ we consider

$$I_N = \int_0^N pt^{p-1}\alpha(t)dt \leq \int_0^N pt^{p-1}\beta(t)dt = \int_0^N p\frac{p_o}{p_o}t^{p-p_o}t^{p_o-1}\beta(t)dt \leq$$

$$\leq p(p_o)^{-1}N^{p-p_o}\int_0^N p_o t^{p_o-1}\beta(t)dt \leq p(p_o)^{-1}N^{p-p_o}\int_0^\infty p_o t^{p_o-1}\beta(t)dt =$$

$$= p(p_o)^{-1}N^{p-p_o}\int_{\mathbb{R}^n}(Mf(x))^{p_o}dx < \infty$$

since we are assuming that $Mf \in L^{p_o}$. Also, using (II) we get:

$$I_N \leq \int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt + \frac{2}{A}\int_0^N pt^{p-1}\alpha(t/2^{n+1})dt =$$

we set $t/2^{n+1} = y$ in the second integral an we get

$$= \int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt + \frac{2}{A}2^{n+1}2^{(n+1)(p-1)}\int_0^{N/2^{n+1}} pt^{p-1}\alpha(t)dt$$

$$:= \int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt + \frac{C}{A}\int_0^{N/2^{n+1}} pt^{p-1}\alpha(t)dt$$

from which:

$$I_N \leq \int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt + \frac{C}{A}I_N$$

with C depending only on n and p. Take now $A = 2C$ and obtain:

$$I_N \leq 2\int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt.$$

Letting $N \to \infty$, we arrive at

$$\int_0^\infty pt^{p-1}\alpha(t)dt \leq 2\int_0^\infty pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt \qquad (III)$$

and then

$$\int_{\mathbb{R}^n}(Mf(x))^p dx = \int_0^\infty pt^{p-1}\beta(t)dt \leq C_1\int_0^\infty pt^{p-1}\alpha(t/C_2)dt =$$

$$C_1C_2C_2^{p-1}\int_0^\infty pt^{p-1}\alpha(t)dt := C\int_0^\infty pt^{p-1}\alpha(t)dt \leq$$

and using (III) we get

$$\leq 2C\int_0^\infty pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt := C\int_{\mathbb{R}^n}(f^{\#}(x))^p dx$$

and the proof is complete □

We have seen that the maximal function Mf and $f^{\#}$ are closely related. We have the trivial pointwise estimate $f^{\#}(x) \leq 2Mf(x)$, but we also have an estimate going in the opposite direction, this time an L^p estimate.

Theorem 1.3.4. *Let T be a linear operator bounded in L^{p_o} for some p_o with $1 < p_o < \infty$. Assume also that T carries L^∞ to B.M.O. boundedly. Then, for every p with $p_o < p < \infty$, T is bounded in L^p.*

Proof. We consider the operator $f \to (Tf)^\#$, which is a sublinear operator (subadditive), bounded in L^{p_o}, and that is because:

$$\|(Tf)^\#\|_{L^{p_o}} \leq 2\|M(Tf)\|_{L^{p_o}}$$

also $(Tf) \in L^{p_o}$ for $f \in L^{p_o}$, so there is a constant C_{p_o} (from theorem (1.2.3)) such that:

$$2\|M(Tf)\|_{L^{p_o}} \leq 2C_{p_o}\|Tf\|_{L^{p_o}} \leq 2C_{p_o}C'\|f\|_{L^{p_o}}$$

where C' comes from the boundness of T. Thus :

$$\|(Tf)^\#\|_{L^{p_o}} \leq 2C_{P_o}C'\|f\|_{L^{p_o}} := C\|f\|_{L^{p_o}}$$

Also this operator is bounded in L^∞: for every $f \in L^\infty$, we have, $\|(Tf)^\#\|_\infty = \|Tf\|_* \leq C\|f\|_\infty$.(where $\|\cdot\|_*$ is by definition, the norm in the $B.M.O.$ space). With other words, the new operator is of strong type (p_o, p_o) and (∞, ∞), consequently, is of weak type (p_o, p_o) and (∞, ∞). By Marcinkiewicz's interpolation theorem, it will also be bounded in L^p(of strong type (p,p)) for every $p \geq p_o$. Let $f \in L^p \cap L^{p_o}$. Then $Tf \in L^{p_o}$ (because T is bounded in L^{p_o}), and consequently $M(Tf) \in L^{p_o}$ (since $p_o > 1$ and we know that M is of strong type (p,p) for $p > 1$). On the other hand $(Tf)^\# \in L^p$ and

$$\int ((Tf)^\#)^p \leq C \int |f|^p.$$

The preceding theorem gives:

$$\int (M(Tf))^p \leq C' \int ((Tf)^\#)^p \leq C'C \int |f|^p$$

Thus (since $Mf \geq f$ a.e.), we get that

$$\int |Tf|^p \leq C'C \int |f|^p := C \int |f|^p$$

and this inequality extends to every $f \in L^p$ because of the fact that the set S of all simple functions with finite support is dense in L^p for $1 \leq p < \infty$ and clearly $S \subset L^q$ for all $q > 0$, so we can conclude that $L^P \cap L^q$ is dense in L^p simply because S is dense in L^p and $S \subset L^p \cap L^q$. □

The most important result regarding B.M.O. is the following theorem of F. John and L. Nirenberg.

Theorem 1.3.5. *There exist constants C_1, C_2 depending only on the dimension n, such that for every $f \in B.M.O. = B.M.O.(\mathbb{R}^n)$, every cube Q and every $t > 0$:*

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq C_1 e^{-(C_2/\|f\|_*)t}|Q| \tag{3.5}$$

Proof. It is again an application of the Calderon-Zygmund decomposition. Observe, first of all, that we can assume $\|f\|_* = 1$, because the inequality (3.5) does not change if we replace f by a constant times it. Fix Q and take $\alpha > 1$. We know that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq 1 < \alpha.$$

We make the C-Z decomposition of Q for the function $f - f_Q$ relative to α, obtaining cubes $Q_{1,j}$ (dyadic subcubes of Q) for each of which:

$$\alpha < \frac{1}{|Q_{1,j}|}\int_{Q_{1,j}} |f(x) - f_Q|dx \leq 2^n\alpha \qquad (I)$$

Besides, for a.e.$x \notin \bigcup_j Q_{1,j}$ is $|f(x) - f_Q| \leq \alpha$. So by (I) we get:

$$|f_{Q_{1,j}} - f_Q| = \left|\frac{1}{|Q_{1,j}|}\int_{Q_{1,j}} (f(y) - f_Q)dy\right| \leq 2^n\alpha$$

Also:

$$\sum_j |Q_{1,j}| \leq \frac{1}{\alpha}\sum_j \int_{Q_{1,j}} |f(x) - f_Q|dx \leq$$
$$\leq \frac{1}{\alpha}\int_Q |f(x) - f_Q|dx \leq \frac{|Q|}{\alpha} \qquad (II)$$

We make now the Calderon-Zygmund decomposition on each $Q_{1,j}$ for the function $f - f_{Q_{1,j}}$ relative to α. Thus we obtain for each j, a family $\{Q_{1,j,k}\}_k$ of dyadic subcubes of $Q_{1,j}$ for each of which, (like earlier):

$$|f_{Q_{1,j,k}} - f_{Q_{1,j}}| \leq 2^n\alpha$$

and also for a.e. $x \in Q_{1,j} \backslash (\cup_k Q_{1,j,k})$ is $|f(x) - f_{Q_{1,j}}| \leq \alpha$. Also with the same way we got (II), we have:

$$\sum_k |Q_{1,j,k}| \leq \frac{1}{\alpha}|Q_{1,j}|.$$

Now we put together all the families $\{Q_{1,j,k}\}_k$ corresponding to different $Q_{1,j}$'s and call the resulting family $\{Q_{2,k}\}_k =: \{Q_{1,j,k}\}_{j,k}$. Then, outside of the union of the $Q_{2,k}$'s, we have:

$$|f(x) - f_Q| \leq |f(x) - f_{Q_{1,j}}| + |f_{Q_{1,j}} - f_Q| \leq$$

$$\leq \alpha + 2^n\alpha \leq 2 \cdot 2^n\alpha$$

and also

$$\sum_k |Q_{2,k}| = \sum_j \sum_k |Q_{1,j,k}| \leq \sum_j \frac{1}{\alpha}|Q_{1,j}| \leq \left(\frac{1}{\alpha}\right)^2 |Q|$$

Subsequently, we obtain for each natural number N, a family on non overlapping cubes $\{Q_{N,j}\}_j$ in such a way that outside of their union is $|f(x) - f_Q| \leq N \cdot 2^n\alpha$ and such that:

$$\sum_j |Q_{N,j}| \leq \alpha^{-N}|Q|.$$

Now if $\quad N \cdot 2^n\alpha \leq t < (N+1) \cdot 2^n\alpha \quad$ with $N = 1, 2, ...$, then

$$\{x \in Q : |f(x) - f_Q| > t\} \subset \left(\bigcup_j Q_{N,j}\right) \cup A$$

where $|A| = 0$, and that is because, for a.e.$x \in Q' \backslash (\cup_j Q_{N,j})$, we have:

$$|f(x) - f_Q| \leq N2^n\alpha$$

where Q' is some subcube of Q produced in the $N-1$ step of the process, so we get:

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq \sum_j |Q_{N,j}| + |A| = \sum_j |Q_{N,j}| \leq$$

$$\leq \left(\frac{1}{\alpha}\right)^N |Q| = e^{-N\log\alpha}|Q|$$

But

$$-2^n\alpha(N+1) < -t \leq -N2^n\alpha \implies -2^n\alpha(N+1)N\log\alpha < -tN\log\alpha$$

$$\implies -N\log\alpha < \frac{-N\log\alpha}{2^n\alpha(N+1)}t := -C_2 t$$

so we get:

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq e^{-C_2 t}|Q| \qquad (III)$$

which is (3.5) since $\|f\|_* = 1$. On the other hand, if $t < 2^n\alpha$, then $C_2 t < C_2 2^n \alpha$ and we use the trivial majorization

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq |Q| < e^{(C_2 2^n\alpha - C_2 t)}|Q| =$$

$$= e^{C_2 2^n \alpha} e^{-C_2 t}|Q|$$

we can also (since $C_2 2^n\alpha > 0$, so $e^{-C_2 t} < e^{C_2 2^n\alpha}e^{-C_2 t}$) bring (III) into the same form as above. Thus, we get (3.5) for every t by choosing C_2 as above and $C_1 = e^{C_2 2^n \alpha}$. Finally, α can be chosen in order to get an optimal value of the constant C_2 ($\alpha = e$). □

Corollary 1.3.2. *If $f \in B.M.O.$ then:*

1. *For every p with $0 < p < \infty$:*

$$\|f\|_{*,p} \equiv \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p} \leq C_p \|f\|_*$$

with C_p independent of f, in such a way that, for $1 < p < \infty$, $f \to \|f\|_{*,p}$ is a norm equivalent to $f \to \|f\|_*$ on $B.M.O.$

2. *For every λ such that $0 < \lambda < C_2/\|f\|_*$, where C_2 is the same constant appearing in (3.5), we have:*

$$\sup_Q \frac{1}{|Q|} \int_Q e^{\lambda |f(x) - f_Q|} dx < \infty$$

Proof. $\int_Q |f(x) - f_Q|^p dx = \int_0^\infty p t^{p-1} |\{x \in Q : |f(x) - f_Q| > t\}| dt \leq$

$$\leq C_1 \int_0^\infty p t^{p-1} e^{-C_2/\|f\|_* t} dt |Q|$$

After a change of variables ($\frac{C_2}{\|f\|_*} t = s$) we get:

$$\int_Q |f(x) - f_Q|^p dx \leq C_1 p \left(\frac{\|f\|_*}{C_2} \right)^p \int_0^\infty s^{p-1} e^{-s} ds =$$

$$= C_1 p \Gamma(p) C_2^{-p} \|f\|_*^p = C_p^p \|f\|_*^p$$

which gives (1) with $C_p = (C_1 p\Gamma(p) C_2^{-p})^{1/p}$.

If $1 < p$, using Hölder's inequality, we get

$$\int_Q |f(x) - f_Q| dx \leq \left(\int_Q |f(x) - f_Q|^p dx \right)^{1/p} \cdot |Q|^{1/q} \Rightarrow$$

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq \left(\int_Q |f(x) - f_Q|^p dx \right)^{1/p} |Q|^{\frac{1}{q}-1} =$$

$$= \left(\int_Q |f(x) - f_Q|^p dx \right)^{1/p} |Q|^{-1/p} = \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p}$$

Thus

$$f^{\#}(x) \leq \|f\|_{*,p} \Rightarrow \|f\|_* = \|f^{\#}\|_\infty \leq \|f\|_{*,p} \leq C_p \|f\|_*$$

so that the norms $\| \cdot \|_*$ and $\| \cdot \|_{*,p}$ are equivalent over $B.M.O.$

(2):

$$\int_Q e^{\lambda |f(x) - f_Q|} dx = \int_0^\infty \lambda e^{\lambda t} |\{x \in Q : |f(x) - f_Q| > t\}| dt \leq$$

$$\leq \int_0^\infty \lambda e^{\lambda t} C_1 e^{-(C_2/\|f\|_*)t} dt |Q| = C_1 \lambda \int_0^\infty e^{(\lambda - C_2/\|f\|_*)t} dt |Q| =$$

$$= C_1 \lambda (C_2/\|f\|_* - \lambda)^{-1} |Q|$$

if $0 < \lambda < C_2/\|f\|_*$, and the proof is complete □

CHAPTER 2

WEIGHTED NORM INEQUALITIES

2.1 THE CONDITION A_p

By a weight on a given measure space, we shall always mean a measurable function w with values in $[0, \infty]$. Our main problem is going to be the following :

PROBLEM 1. Given p, $1 < p < \infty$, determine those weights w on $\mathbb{R}^n$ for which the maximal operator M is of strong type (p, p) with respect to the measure $w(x)dx$, that is, for which we have an inequality:

$$\left(\int_{\mathbb{R}^n} (Mf(x))^p w(x) dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} \qquad (1.1)$$

We can also pose this more general

PROBLEM 2. Given p, $1 < p < \infty$, determine those pairs of weights (u,w) on $\mathbb{R}^n$, for which M is of strong type (p, p) with respect to the pair of measures $(u(x)dx, w(x)dx)$, that is, for which we have an inequality:

$$\left(\int_{\mathbb{R}^n} (Mf(x))^p u(x) dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} \qquad (1.2)$$

We can pose similar problems substituting weak type for strong type in the two problems above. For example:

PROBLEM 3. Given p, $1 \leq p < \infty$ determine those pairs of weights (u, w) on $\mathbb{R}^n$, for which M is of weak type (p, p) with respect to the pair of measures

$(u(x)dx, w(x)dx)$, that is, for which we have the inequality:

$$u(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq Ct^{-p} \int_{\mathbb{R}^n} |f(x)|^p w(x)dx \quad (1.3)$$

For a set E, $u(E)$ stands for $\int_E u(x)dx$. This notation has been used in (1.3) and it will be used systematically.

We shall keep the usual conventions for multiplication in $[0, \infty]$, namely $\infty \cdot t = t \cdot \infty = \infty$ for $0 < t \leq \infty$ and $0 \cdot \infty = \infty \cdot 0 = 0$. Also $\infty^{-1} = 0$ and $0^{-1} = \infty$ when we consider w^{-1} for a weight w.

Let us start by analyzing problem 3, Suppose that the pair of weights (u, w) is such that (1.3) holds for a given p, $1 \leq p < \infty$, every function f and every $t > 0$. Let f be a function ≥ 0. Let Q be a cube such that the average $f_Q = \frac{1}{|Q|} \int_Q f(x)dx > 0$.

Observe that $f_Q \leq M(fX_Q)(x)$ for every $x \in Q$ (since$f_Q = \frac{1}{|Q|} \int_Q f(x)dx = \frac{1}{|Q|} \int_Q fX_Q(x)dx \leq M(fX_Q)(x)$). Then, for every t with $0 < t < f_Q$, $Q \subset E_t = \{x \in \mathbb{R}^n : M(fX_Q)(x) > t\}$ so that, by (1.3):

$$u(Q) \leq Ct^{-p} \int_Q f(x)^p w(x)dx$$

It follows that: (let $t \to f_Q$)

$$(f_Q)^p u(Q) \leq C \int_Q f(x)^p w(x)dx \quad (1.4)$$

We can actually write this inequality in seemingly stronger form. If S is a measurable subset of Q, we can replace f in (1.4) by fX_S, obtaining

$$\left(\frac{1}{|Q|} \int_S f(x)dx \right)^p u(Q) \leq C \int_S f(x)^p w(x)dx \quad (1.5)$$

Of course (1.5) is just equivalent to (1.4), but (1.5) is more readily applicable sometimes. For $f \equiv 1$, (1.5) yields:

$$(|S|/|Q|)^p u(Q) \leq Cw(S) \qquad (1.6)$$

From (1.6) we draw some relevant information about the pair (u, w):

a) $w(x) > 0$ for a.e. $x \in \mathbb{R}^n$ (unless $u(x) = 0$ for a.e. $x \in \mathbb{R}^n$, trivial case which we shall exclude).
b) u is locally integrable (unless $w(x) = \infty$ for a.e. $x \in \mathbb{R}^n$, again trivial case which we shall also exclude).

Let as proof a) and b): If $w(x) = 0$ on a set S with $|S| > 0$, a set which we could assume to be bounded, (1.6) would imply that $u(Q) = 0$ for every cube Q containing S, and consequently $u(x) = 0$ for a.e. $x \in \mathbb{R}^n$. Now if u is not locally integrable, then $u(Q) = \infty$ for some cube Q and, consequently, for any cube containing Q, this implies that $w(S) = \infty$ for any set $S \subset Q$ with $|S| > 0$, which implies $w(x) = \infty$ for a.e.$x \in \mathbb{R}^n$.

We are about to derive a necessary condition on the pair (u, w) for (1.3) to hold for every f and t. If $p = 1$, (1.6) can be written in the form:

$$\frac{1}{|Q|}\int_Q u(x)dx \leq C\frac{1}{|S|}\int_S w(x)dx \qquad (1.7)$$

the inequality being valid for every cube Q and every set $S \subset Q$ with $|S| > 0$. Fix Q and let $a > ess._Q inf.(w)$, the essential infimum of w over Q, which is defined as the

$$\inf\{t > 0 : |\{x \in Q : w(x) < t\}| > 0\}$$

Then, the set $S_a = \{x \in Q : w(x) < a\}$ has $|S_a| > 0$ and (1.7) holds for $S = S_a$, from which we get:

$$\frac{u(Q)}{|Q|} \leq Ca.$$

Since this is true for every $a > ess.inf.(w)$, we arrive finally at:

$$\frac{1}{|Q|}\int_Q u(x)dx \leq Cess._Q inf.(w) \leq Cw(x) \qquad (1.8)$$

for a.e.$x \in Q$.

Observe that the fact that (1.8) holds for every Q is equivalent to :

$$M(u)(x) \leq Cw(x) \qquad (1.9)$$

for a.e.$x \in \mathbb{R}^n$. Indeed, it is clear that (1.9) implies (1.8) for every cube Q. Conversely if (1.8) holds for every Q, let us show (1.9) holds, that is, the set $\{x \in \mathbb{R}^n : M(u)(x) > Cw(x)\}$ has measure equal to zero. If $M(u)(x) > Cw(x)$, it will be

$$\frac{1}{|Q|}\int_Q u(x)dx > Cw(x) \qquad (I)$$

for some cube Q containing x, and we can assume that Q has vertices with all coordinates rational, the cardinality of those cubes is at most Q^n which is countable, so, we can denote all those cubes as $\{Q_n\}_{n\in K\subset\mathbb{N}}$. Now, by (I) and having in mind that (1.8) holds for every cube Q, we can see that x belongs to a subset N_Q of Q with $|N_Q| = 0$. Thus we get:

$$\{x \in \mathbb{R}^n : M(u)(x) > Cw(x)\} \subset \cup_{n\in K} N_{Q_n}$$

and we get the equivalence between (1.8) and (1.9).

Condition (1.9) is known as condition A_1 for the pair (u, w). When it holds, we also say that the pair (u, w) belongs to the class A_1, viewing A_1 as a collection of pairs of weights (u, w). We often speak of the A_1 constant for the pair (u, w) which is the smallest C for which (1.8), or equivalently (1.9), holds.

We have just seen that $(u, w) \in A_1$ is a necessary condition for M to be of weak type $(1, 1)$ with respect to the pair (u, w). It is very satisfactory to realize that this condition is actually sufficient. Indeed, let $(u, w) \in A_1$, so that (1.9) holds. Then, using the inequality (2.14) in chapter I, we get:

$$u(\{x \in \mathbb{R}^n : Mf(x) > t\}) = \int_{\{x\in\mathbb{R}^n : Mf(x)>t\}} u(x)dx \le$$

$$\le Ct^{-1}\int_{\mathbb{R}^n} |f(x)|Mu(x)dx \le Ct^{-1}\int_{\mathbb{R}^n} |f(x)|w(x)dx.$$

Now we shall treat the case $1 < p < \infty$. We start by looking for a necessary condition. So far we know that if M is of weak type (p, p) with respect to (u, w), then (1.5) holds for every function $f \ge 0$, every cube Q and every measurable set $S \subset Q$. Let us choose f such that $f(x) = f(x)^p w(x)$. This gives $f(x) = w(x)^{-1/(p-1)}$. A priori this function needs not to be locally integrable. Fix a cube Q and take $S = S_j = \{x \in Q : w(x) > j^{-1}\}$ for $j = 1, 2, ...$ On every S_j our f is bounded, so that $\int_{S_j} f < \infty$. With our choice for f, (1.5) gives:

$$\left(\frac{1}{|Q|}\int_{S_j} w(x)^{-1/(p-1)}dx\right)^p u(Q) \le C\int_{S_j} w(x)^{-1/(p-1)}dx \implies$$

$$\left(\frac{1}{|Q|}\int_{S_j} w(x)^{-1/(p-1)}dx\right)^p \frac{u(Q)}{|Q|} \leq C\frac{1}{|Q|}\int_{S_j} w(x)^{-1/(p-1)}dx$$

or, since the integrals are finite,

$$(\frac{1}{|Q|}\int_Q u(x)dx)\left(\frac{1}{|Q|}\int_{S_j} w(x)^{-1/(p-1)}\right)^{p-1} \leq C$$

Now $S_1 \subset S_2 \subset ....$ and $\cup_{j=1}^{\infty} S_j = \{x \in Q : w(x) > 0\}$, whose complement in Q has measure zero, as we previously observed ($w > 0$ a.e.). Thus, letting $j \to \infty$, we get finally

$$(\frac{1}{|Q|}\int_Q u(x)dx)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\right)^{p-1} \leq C \qquad (1.10)$$

We shall say that the pair (u, w) satisfies the condition A_p, if and only if there is a constant C such that (1.10) holds for every cube Q. The smallest such constant will be called the A_p constant for the pair (u, w). We have proved that $(u, w) \in A_p$ is necessary for M to be of weak type $(p.p)$ with respect to the pair (u, w). Observe that $(u, w) \in A_p$ implies that both u and $w^{-1/(p-1)}$ are locally integrable. Indeed if one of the integrals in (1.10) were ∞, the same would happen for any cube containing Q, and that would force the other factor to be zero. This would imply either $u(x) = 0$ for a.e.$x \in \mathbb{R}^n$ or $w(x) = \infty$ for a.e.$x \in \mathbb{R}^n$. Both trivial situations that have been excluded beforehand. Another observation that has to be made is that the condition A_1 can be viewed as a limit case of the condition A_p for $p \downarrow 1$. Indeed (1.8) can be written as

$$(\frac{1}{|Q|}\int_Q u(x)dx) ess._Q sup.(w^{-1}) \leq C \qquad (1.11)$$

while

$$(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx)^{p-1} = \|w^{-1}\|_{L^{1/(p-1)}(Q,|Q|^{-1}dx)} \to$$

$$\to \|w^{-1}\|_{L^\infty(Q,|Q|^{-1}dx:=d\mu)} = \|w^{-1}\|_{L^\infty(Q,dx)} := \|w^{-1}\|_{L^\infty(Q)}$$

as $p \to 1$ where $\mu(A) = \int_A |Q|^{-1}dx = \frac{|A|}{|Q|}$, that is why we have

$$\|w^{-1}\|_{L^\infty(Q,|Q|^{-1}dx:=d\mu)} = \|w^{-1}\|_{L^\infty(Q,dx)}$$

($\mu(A) > 0$ if and only if $|A| > 0$).

Let us also explain the convergence above: Let

$$g = \frac{f}{ess.sup(f)}$$

then $ess.sup(g) = 1$ and :

$$\left(\int_Q |g|^p d\mu\right)^{1/p} = \left(\int_{\{|g|>1-\varepsilon\}} |g|^p + \int_{\{|g|\le 1-\varepsilon\}} |g|^p\right)^{1/p} \ge$$

$$\ge ((1-\varepsilon)^p \mu(\{|g| > 1-\varepsilon\}) + 0)^{1/p} =$$
$$= (1-\varepsilon)\mu(\{|g| > 1-\varepsilon\})^{1/p} \longrightarrow (1-\varepsilon)$$

as $p \to \infty$. On the other hand

$$\left(\int_Q |g|^p\right)^{1/p} \le \mu(Q)^{1/p} \longrightarrow 1 < 1+\varepsilon$$

Thus $\|g\|_{L^p(\mu)} \longrightarrow 1$ as $p \to \infty$, which implies that

$$\|f\|_{L^p(\mu)} \longrightarrow ess.sup(f) = \|f\|_{L^\infty(\mu)}$$

Thus (1.11) is the right companion for (1.10) when $p = 1$. Also note that $(u, w) \in A_1$ (that is (u,w) satisfy (1.9)) implies that u is locally integrable and w^{-1} is locally bounded.

Our task will be now to show that, exactly as in the case $p = 1$, when $1 < p < \infty$, $(u, w) \in A_p$ is not only necessary, but also sufficient for M to be of weak type (p,p) with respect to the pair (u, w). We have obtained condition A_p from (1.4). The first step will be to show that, conversely, if $(u, w) \in A_p$, then (1.4) holds for every $f \ge 0$ and every cube Q. This is actually true for $1 \le p < \infty$. If $p = 1$ and $(u, w) \in A_1$, we have, for every cube Q and every $f \ge 0$:

$$\left(\frac{1}{|Q|}\int_Q f(x)dx\right) u(Q) = \int_Q f(x)dx \frac{u(Q)}{|Q|} \le$$

$$\le \int_Q f(x)M(u)(x)dx \le C\int_Q f(x)w(x)dx$$

which is (1.4) for $p = 1$. If now $1 < p < \infty$ and $(u, w) \in A_p$, we have, for every cube Q and every $f \geq 0$, using Hölder's inequality with p and its conjugate exponent $p' = p/(p-1)$,

$$f_Q = \frac{1}{|Q|}\int_Q f(x)w(x)^{1/p}w(x)^{-1/p}dx \leq$$

$$\leq \left(\frac{1}{|Q|}\int_Q f(x)^p w(x)\right)^{1/p} \cdot \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{\frac{p-1}{p}}.$$

Thus

$$(f_Q)^p u(Q) \leq \frac{u(Q)}{|Q|}\int_Q f(x)^p w(x)dx \cdot \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}$$

and using (1.10) we get

$$\leq C\int_Q f(x)^p w(x)dx$$

so that (1.4) holds. We have established the equivalence between (1.4) and A_p. Now, suppose that (1.4) holds for every cube Q and every $f \geq 0$. We shall obtain (1.3) with a possibly bigger C. Of course, we have (1.5) for every $f \geq 0$, every cube Q and every set $S \subset Q$. Let $f \in L^p(w)$. We can obviously assume that $f \geq 0$. Observe that $L^p_{loc}(w) \subset L^1_{loc}(\mathbb{R}^n)$ as follows from (1.4) using Q such that $u(Q) > 0$. Now we can also assume that $f \in L^1(\mathbb{R}^n)$. Indeed, we can always write $f = \lim_{k\to\infty} f_k$ where $f_k = f \cdot X_{Q(0,k)}$ and if now, we have (1.3) for every f_k in place of f, passing to the limit we obtain (1.3) for f. Thus, assuming f integrable, we want to estimate $u(E_t)$ where $E_t = \{x \in \mathbb{R}^n : Mf(x) > t\}$. We use theorem (1.1.2) from chapter I, to write $E_t \subset \cup_j Q_j^3$, where the Q_j's are non overlapping cubes for which

$$\frac{t}{4^n} < \frac{1}{|Q_j|}\int_{Q_j} f(x)dx \leq \frac{t}{2^n} \qquad (a)$$

Then,

$$u(E_t) \leq \sum_j u(Q_j^3) \leq$$

and applying (1.5) with $Q = Q_j^3$ and $S = Q_j$, we get

$$\leq C\sum_j \left(\frac{1}{|Q_j^3|}\int_{Q_j} f(x)dx\right)^{-p}\int_{Q_j} f(x)^p w(x)dx =$$

$$= C \sum_j \left(\frac{1}{3^n |Q_j|} \int_{Q_j} f(x)dx \right)^{-p} \int_{Q_j} f(x)^p w(x)dx \leq$$

and using (a) we get

$$\leq C 3^{np} 4^{np} t^{-p} \sum_j \int_{Q_j} f(x)^p w(x)dx \leq C' t^{-p} \int_{\mathbb{R}^n} f(x)^p w(x)dx.$$

where $C' = C3^{np}4^{np}$. We have a complete proof of the fact that the solution to Problem 3 is precisely the class A_p of pairs of weights. We can collect our findings in the following

Theorem 2.1.1. *Let u and w be weights on $\mathbb{R}^n$ and let $1 \leq p < \infty$. Then, the following conditions are equivalent:*

1. *M is of weak type (p,p) with respect to (u,w), that is:M takes $L^p(w)$ to $L^p_*(u)$ boundedly or, in other words, there is a constant C such that for every function $f \in L^1_{loc}(\mathbb{R}^n)$ and every $t > 0$*

$$u(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq Ct^{-p} \int_{\mathbb{R}^n} |f(x)|^p w(x)dx.$$

2. *There is a constant C such that for every function $f \geq 0$ in $\mathbb{R}^n$ and for every cube Q*

$$\left(\frac{1}{|Q|} \int_Q f(x)dx \right)^p u(Q) \leq C \int_Q f(x)^p w(x)dx$$

3. *$(u,w) \in A_p$, that is, there is a constant C such that for every cube Q we have, in case $1 < p < \infty$,*

$$\left(\frac{1}{|Q|} \int_Q u(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

and in case $p = 1$,

$$(\frac{1}{|Q|} \int_Q u(x)dx) ess._Q sup.(w^{-1}) \leq C$$

Besides, the constant C appearing in 1), 2) and 3) are of the same order.

Corollary 2.1.1. *Let $(u,w) \in A_p$. Then, for every q with $p < q < \infty$, the maximal operator M is bounded from $L^q(w)$ to $L^q(u)$, that is, there is a constant C such that for every $f \in L^1_{loc}(\mathbb{R}^n)$:*

$$\int_{\mathbb{R}^n} |Mf(x)|^q u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^q w(x)dx$$

Proof. We already know that M is of weak type (p,p) with respect to (u,w), that is, M takes $L^p(w)$ boundedly to $L^p_*(u)$. We shall see presently that M is also bounded from $L^\infty(w)$ to $L^\infty(u)$. Lets prove that: Since

$$\|Mf\|_{\infty,u} = sup\{a \geq 0 : u(\{x \in \mathbb{R}^n : Mf(x) > a\}) > 0\}$$

let $a > 0$ such that : $u(\{x \in \mathbb{R}^n : Mf(x) > a\}) > 0$, then, there exists $x \in \mathbb{R}^n : Mf(x) > a$, and so, there is cube Q containing x such that (we can assume $f \geq 0$):

$$\frac{1}{|Q|}\int_Q f(x)dx > a \Rightarrow f(x) > a \;\; for \;\; a.e. \;\; x \in Q \Rightarrow$$

$$\Rightarrow |\{x \in \mathbb{R}^n : f(x) > a\}| > 0$$

and since $w(x) > 0$ for a.e. $x \in \mathbb{R}^n$, we get:

$$w(\{x \in \mathbb{R}^n : f(x) > a\}) > 0 \Rightarrow \|f\|_{\infty,w} \geq a$$

for each such a, which leads to:

$$\|Mf\|_{\infty,u} \leq \|f\|_{\infty,w}$$

so, indeed M is bounded from $L^\infty(w)$ to $L^\infty(u)$. Once we know that M is bounded from $L^p(w)$ to $L^p_*(u)$ and from $L^\infty(w)$ to $L^\infty(u)$, we use Marcinkiewicz interpolation theorem to conclude that M is bounded from $L^q(w)$ to $L^q(u)$ provided $p < q < \infty$. □

A particular instance of the previous corollary is the inequality

$$\int_{\mathbb{R}^n} |Mf(x)|^p u(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p Mu(x)dx$$

valid for $1 < p < \infty$, which appeared in chapter I,(2.13). It is contained in our corollary because $(u, Mu) \in A_1$ and $p > 1$. The following theorem contains some simple basic facts about the classes A_p.

Theorem 2.1.2. *1. Let $1 < p < q < \infty$. Then $A_1 \subset A_p \subset A_q$*

2. Let $1 \leq p < \infty$, $0 < \varepsilon < 1$ and $(u, w) \in A_p$. Then $(u^\varepsilon, w^\varepsilon) \in A_{\varepsilon p+1-\varepsilon}$.

3. Let $1 < p < \infty$. Then $(u, w) \in A_p$ if and only if

$$(w^{-1/(p-1)}, u^{-1/(p-1)}) \in A_{p'}$$

where p' is, as usual, the exponent conjugate to p, that is $p' = p/(p-1)$.

Proof. 1) We just need to observe that

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}dx\right)^{q-1} \leq \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq$$

$$\leq ess._Q sup.(w^{-1})$$

lets prove the first inequality (the second is obvious): Since $(q-1)/(p-1) > 1$ we can use Jensen's inequality getting

$$\left(\int_Q \frac{w(x)^{-1/(q-1)}}{|Q|}dx\right)^{(q-1)/(p-1)} \leq \int_Q \frac{w(x)^{-1/(p-1)}}{|Q|} \Rightarrow$$

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}dx\right)^{q-1} \leq \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}$$

2) For $r = \varepsilon p + 1 - \varepsilon$, we have $r - 1 = \varepsilon(p-1)$. Then

$$\frac{1}{|Q|}\int_Q u(x)^\varepsilon dx \left(\frac{1}{|Q|}\int_Q (w(x)^\varepsilon)^{-1/(r-1)}dx\right)^{r-1} \leq$$

again we use, just for the first integral, the Jensen's inequality for $1/\varepsilon > 1$ and we get:

$$\leq \left(\frac{1}{|Q|}\int_Q u(x)dx\right)^\varepsilon \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{\varepsilon(p-1)} \leq C^\varepsilon$$

where C is the A_p constant for the pair (u,w). For the case $p = 1$ we have

$$\frac{1}{|Q|}\int_Q u(x)^\varepsilon dx \leq \left(\frac{1}{|Q|}\int_Q u(x)dx\right)^\varepsilon \Rightarrow$$

$$\Rightarrow M(u^\varepsilon)(x) \leq (M(u)(x))^\varepsilon \leq (Cw(x))^\varepsilon = C^\varepsilon w(x)^\varepsilon$$

Thus, $(u^\varepsilon, w^\varepsilon) \in A_1 = A_{1\varepsilon+1-\varepsilon}$

3) Suppose that $(u, w) \in A_p$. Thus:

$$\frac{1}{|Q|}\int_Q u(x)dx\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C$$

Since $(p-1)(p'-1) = 1$, we can write the previous inequality as:

$$\frac{1}{|Q|}\int_Q (u(x)^{-1/(p-1)})^{-1/(p'-1)}dx\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{1/(p'-1)} \leq C$$

$$\Rightarrow$$

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)\left(\frac{1}{|Q|}\int_Q (u(x)^{-1/(p-1)})^{-1/(p'-1)}dx\right)^{p'-1} \leq C^{p'-1}$$

and we conclude that $(w^{-1/(p-1)}, u^{-1/(p-1)}) \in A_{p'}$. Actually we see that $(u, w) \in A_p$ is equivalent to $(w^{-1/(p-1)}, u^{-1/(p-1)}) \in A_{p'}$. □

EXAMPLE: We shall give here an example which shows that corollary 2.1.1. can not be improved so as to include also the case $q = p$. We shall give weights u,w such that $(u, w) \in A_p$ and, however, M is not bounded from $L^p(w)$ to $L^p(u)$. If $p = 1$, we just need to take $u = w \equiv 1$ because we know that M is not bounded in $L^1(\mathbb{R}^n) = L^1(u) = L^1(w)$. We already know that if $g \geq 0$ is integrable and is not 0 at a.e. x, then Mg is never integrable (see the remark after theorem 1.2.4. in chapter I). This same fact leads to an example for $p > 1$. Let $g \geq 0$, integrable and non trivial, in such a way that $Mg \notin L^1$. Take g bounded so that you can guarantee that $Mg(x)$ is always finite. Then $(g, Mg) \in A_1 \subset A_{p'}$, and hence (from the previous theorem 3),

$$((Mg)^{-1/(p'-1)}, g^{-1/(p'-1)}) = ((Mg)^{1-p}, g^{1-p}) \in A_p.$$

If we take $u = (Mg)^{1-p}$ and $w = g^{1-p}$, we have a pair $(u, w) \in A_p$ for which the inequality

$$\int |Mf(x)|^p u(x)dx \leq C\int |f(x)|^p w(x)dx$$

can not hold, since for $f = g$ we have : $\int |Mf|^p u = \int Mg = \infty$ and $\int |f|^p w = \int g < \infty$.

In this way, we have seen that the condition $(u, w) \in A_p$ does not solve problem 2. It is though, necessary conditions for (1.2) to hold, since (1.2) implies (1.3) which implies the condition A_p i.e. (1.10). However, it is not sufficient.

2.2 THE REVERSE HÖLDER'S INEQUALITY & THE CONDITION A_∞

The theory developed in section 1 becomes particularly interesting for the case $u = w$. First of all, theorem 2.1.1. reads as follows in this situation:

Theorem 2.2.1. *Let w be a weight on $\mathbb{R}^n$, and le $1 \le p < \infty$. Then, the following conditions are equivalent:*

1. *M is of weak type (p,p) with respect to w, i.e.*
$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \le Ct^{-p} \int_{\mathbb{R}^n} |f(x)|^p w(x)dx$$
2. *There is a constant C such that, for every function $f \ge 0$ and for every cube Q*
$$(f_Q)^p w(Q) \le C \int_Q f(x)^p w(x)dx$$
3. *$(w, w) \in A_p$, that is, in case $1 < p < \infty$*
$$\frac{1}{|Q|} \int_Q w(x)dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \le C$$
for every cube Q, and, in case $p = 1$, $Mw(x) \le Cw(x)$ a.e.

The constants C appearing in 1), 2) and 3) are of the same order.

When w satisfies 3), we say that w satisfies the condition A_p, and write $w \in A_p$. We also speak of the A_p constant for w, with the natural meaning. Notice that that the class A_1 is the same which appeared in chapter I.

We saw in the example we gave earlier, that a pair of weights (u,w) may be in A_p and yet M may not be bounded from $L^p(w)$ to $L^p(u)$. In contrast to this situation, for $p > 1$, it suffices that $w \in A_p$ for M to be bounded in $L^p(w)$.

This fact depends on a basic property enjoyed by the A_p weights: the reverse Hölder's inequality (R.H.I.) appearing in the third lemma below. First we present a couple of simple properties of the A_p weights.

We start with an estimate for the w-measure of the dilated Q^λ of a cube Q.

Lemma 2.1. *Let w be an A_p weight in $\mathbb{R}^n$. Then, for every cube Q and every $\lambda > 1$*

$$w(Q^\lambda) \le C\lambda^{np} w(Q)$$

where C is of the same order as the A_p constant for w.

Proof. In 2) of the previous theorem, take $f = X_S$ with $S \subset Q$, and Q a cube. Then

$$(|S|/|Q|)^p w(Q) \le Cw(S) \qquad (1.1)$$

using (1.1) with Q in place of S and Q^λ in place of Q we get:

$$w(Q^\lambda) \le C\lambda^{np} w(Q).$$

□

In particular the lemma implies that for an A_p weight w, the measure μ given by $d\mu(x) = w(x)dx$ is a doubling measure.

Actually, what we have shown is that the second property in theorem 2.2.1. implies that μ is a doubling measure. Observe that the same property (property 2) implies that

$$f_Q \le C^{1/p} \left(\frac{1}{w(Q)} \int_Q f(x)^p w(x) dx \right)^{1/p}$$

for every cube Q, which implies

$$Mf(x) \le C^{1/p} \left(M_\mu(f^p)(x)\right)^{1/p} \quad a.e. \Rightarrow$$

$$\Rightarrow Mf(x) \le C^{1/p} \left(M_\mu(|f|^p)(x)\right)^{1/p} \quad a.e.$$

where the operator M_μ is the one introduced in the previous chapter. We showed there that, for μ doubling, M_μ is of weak type (1,1) with respect to μ. We can rely upon this fact to prove that 2) implies 1) in theorem 2.2.1. Indeed

$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \le w(\{x \in \mathbb{R}^n : CM_\mu(|f|^p)(x) > t^p\}) =$$

$$= w(\{x \in \mathbb{R}^n : M_\mu(C|f|^p)(x) > t^p\}) \le C'Ct^{-p} \int_{\mathbb{R}^n} |f(x)|^p w(x) dx$$

where C' is the doubling constant, therefore 2) implies 1).

The next lemma is a comparison between the measure $w(x)dx$ and Lebesgue measure

Lemma 2.2. *Let $w \in A_p$. Then, for every positive $a < 1$, there exists $\beta < 1$ depending on a such that, whenever A is a measurable set contained in a cube Q and satisfying $|A| \leq a|Q|$, it follows that $w(A) \leq \beta w(Q)$.*

Proof. We start from (1.1) where, of course, it is always $C \geq 1$ (set S=Q). If we use in (1.1) $S = Q \backslash A$ where $|A| \leq a|Q|$, we get :

$$(1-a)^p w(Q) \leq (1 - |A|/|Q|)^p w(Q) =$$

$$= \left(\frac{|Q \backslash A|}{|Q|} \right)^p w(Q) \leq C w(Q \backslash A) = C(w(Q) - w(A))$$

Thus

$$w(A) \leq C^{-1}(C - (1-a)^p) w(Q) := \beta w(Q)$$

□

We shall use the previous lemma to establish our basic inequality

Lemma 2.3. *Let $w \in A_p$. Then, there exists $\varepsilon > 0$, depending only on p and on the A_p constant for w, such that, for every cube Q*

$$\left(\frac{1}{|Q|} \int_Q w(x)^{1+\varepsilon} dx \right)^{1/(1+\varepsilon)} \leq C \frac{1}{|Q|} \int_Q w(x) dx$$

with a constant C not depending on Q.

The opposite inequality holds, with $C = 1$, for every function w and is a particular case of Hölder's inequality. This is why the lemma is called the reverse Hölder's inequality (R.H.I.).

Proof. We shall fix cube Q and we shall get the inequality with ε and C independent of Q. We take an increasing sequence $\lambda_o < \lambda_1 < ..... < \lambda_k < ..$ with $\lambda_o = w_Q = \frac{1}{|Q|} \int_Q w(x) dx$ and, for each λ_k, we make the Calderón-Zygmund decomposition of Q for the function w and the value λ_k; that is, we consider those maximal dyadic subcubes of Q over which the average of w is $> \lambda_k$ (the dyadic subcubes of Q are the cubes resulting from dividing each side of Q in 2^N equal parts $N = 0, 1, 2, ...$). Let them be $\{Q_{k,j}\}_{j=1,2,..}$. It follows that, for each j is

$$\lambda_k < w_{Q_{k,j}} = \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(x) dx \leq 2^n \lambda_k$$

while for a.e. x not belonging to $\cup_j Q_{k,j} = D_k$ is $w(x) \leq \lambda_k$. Since $\lambda_{k+1} > \lambda_k$, each $Q_{k+1,j}$ is contained in $Q_{k,i}$ for some i, in such a way that $D_{k+1} \subset D_k$. Let us see what portion of $Q_{k,i}$ can be covered by D_{k+1}. We know that:

$$2^n \lambda_k \geq \frac{1}{|Q_{k,i}|} \int_{Q_{k,i} \cap D_{k+1}} w(x)dx =$$

$$= \frac{1}{|Q_{k,i}|} \sum_{Q_{k+1,j} \subset Q_{k,i}} \int_{Q_{k+1,j}} w(x)dx =$$

$$= \frac{1}{|Q_{k,i}|} \sum_{Q_{k+1,j} \subset Q_{k,i}} |Q_{k+1,j}| \cdot \frac{1}{|Q_{k+1,j}|} \int_{Q_{k+1,j}} w(x)dx >$$

$$> \frac{\lambda_{k+1}}{|Q_{k,i}|} \sum_{Q_{k+1,j} \subset Q_{k,i}} |Q_{k+1,j}| = \lambda_{k+1} \frac{|Q_{k,i} \cap D_{k+1}|}{|Q_{k+1}|}$$

Thus

$$\frac{|Q_{k,i} \cap D_{k+1}|}{|Q_{k+1}|} < \frac{2^n \lambda_k}{\lambda_{k+1}}.$$

Let as take this ratio equal to $a < 1$ ($\frac{2^n \lambda_k}{\lambda_{k+1}} = a$), that is $\lambda_{k+1} = 2^n a^{-1} \lambda_k$, $\lambda_k = (2^n a^{-1})^k \lambda_o$. If we consider the β associated to a according to the previous lemma, we shall have

$$w(Q_{k,i} \cap D_{k+1}) \leq \beta w(Q_{k,i})$$

and, summing over i, we get : $w(D_{k+1}) \leq \beta w(D_k)$, which leads to $w(D_k) \leq \beta^k w(D_o)$. Of course, we also have $|D_{k+1}| \leq a|D_k|$ (see that $|Q_{k,i} \cap D_{k+1}| < a|Q_{k,i}|$) and $|D_k| \leq a^k |D_o|$, which implies that

$$|\cap_{k=0}^{\infty} D_k| = \lim_{k \to \infty} |D_k| = 0.$$

Then:

$$\int_Q w(x)^{1+\varepsilon} dx =$$

$$= \int_{Q \setminus D_o} w(x)^{1+\varepsilon} dx + \sum_{k=0}^{\infty} \int_{D_k \setminus D_{k+1}} w(x)^{1+\varepsilon} dx$$

$$= \int_{Q \setminus D_o} w(x) w(x)^{\varepsilon} dx + \sum_{k=0}^{\infty} \int_{D_k \setminus D_{k+1}} w(x) w(x)^{\varepsilon} dx \leq$$

$$\leq \lambda_o^\varepsilon \int_{Q\backslash D_o} w(x)dx + \sum_{k=0}^{\infty} \lambda_{k+1}^\varepsilon \int_{D_k\backslash D_{k+1}} w(x)dx =$$

$$= \lambda_o^\varepsilon w(Q\backslash D_o) + \sum_{k=0}^{\infty} \lambda_{k+1}^\varepsilon w(D_k\backslash D_{k+1}) \leq$$

and since $w \geq 0$

$$\leq \lambda_o^\varepsilon w(Q\backslash D_o) + \sum_{k=0}^{\infty} \lambda_{k+1}^\varepsilon w(D_k) \leq$$

$$\leq \lambda_o^\varepsilon w(Q\backslash D_o) + \sum_{k=0}^{\infty} (2^n a^{-1})^{(k+1)\varepsilon} \lambda_o^\varepsilon \beta^k w(D_o) =$$

$$= \lambda_o^\varepsilon \left\{ w(Q\backslash D_o) + (2^n a^{-1})^\varepsilon \sum_{k=0}^{\infty} ((2^n a^{-1})^\varepsilon \beta)^k w(D_o) \right\}$$

If we take ε small enough to have $(2^n a^{-1})^\varepsilon \beta < 1$, the series will have a finite sum and we shall get :

$$\int_Q w(x)^{1+\varepsilon} dx \leq \lambda_o^\varepsilon \{ w(Q\backslash D_o) + (2^n a^{-1})^\varepsilon C' w(D_o) \} :=$$

$$:= C'' \lambda_o^\varepsilon (w(Q\backslash D_o) + w(D_o)) = C'' \lambda_o^\varepsilon w(Q) = C'' w_Q^\varepsilon w(Q)$$

Thus

$$\frac{1}{|Q|} \int_Q w(x)^{1+\varepsilon} dx \leq C'' w_Q^{1+\varepsilon} := C \left(\frac{1}{|Q|} \int_Q w(x)dx \right)^{1+\varepsilon}.$$

□

Lemma 3 has far reaching consequences which we shall presently see

Theorem 2.2.2. *Let $w \in A_p$ with $1 < p < \infty$, then there is some $q < p$ such that $w \in A_q$, that is, for every p, $1 < p < \infty$, we have*

$$A_p = \cup_{q<p} A_q.$$

Proof. Theorem 2.1.2. (3) for the special case $u = w$ tells us that $w \in A_p$ implies $w^{-1/(p-1)} \in A_{p'}$. On the other hand, from lemma 3 for the weight $w^{-1/(p-1)}$ we know that there exist $\varepsilon > 0$, $C > 0$ such that, for every cube Q:

$$\left(\frac{1}{|Q|} \int_Q w(x)^{-(1+\varepsilon)/(p-1)} dx \right)^{1/(1+\varepsilon)} \leq \frac{C}{|Q|} \int_Q w(x)^{-1/(p-1)} dx$$

But $\frac{1+\varepsilon}{p-1} > \frac{1}{p-1}$ implies $\frac{1+\varepsilon}{p-1} = \frac{1}{q-1}$ for some $1 < q < p$. Then

$$\frac{1}{|Q|}\int_Q w(x)dx \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}dx\right)^{q-1} =$$

$$= \frac{1}{|Q|}\int_Q w(x)dx \left(\frac{1}{|Q|}\int_Q w(x)^{-(1+\varepsilon)/(p-1)}dx\right)^{(p-1)/(1+\varepsilon)} \leq$$

$$C^{p-1}\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{(p-1)} \leq$$

and since $w \in A_p$

$$\leq C^{p-1}C' := C$$

□

Actually, since w itself satisfies a R.H.I., we obtain the following stronger result.

Theorem 2.2.3. *If $w \in A_p$ with $1 \leq p < \infty$, then, there exists $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_p$.*

Proof. If $p = 1$, from lemma 3, there exists $\varepsilon > 0$ such that:

$$\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx \leq \left(C\frac{1}{|Q|}\int_Q w(x)dx\right)^{1+\varepsilon} \leq$$

$$\leq (CC_1 w(x))^{1+\varepsilon} := Cw(x)^{1+\varepsilon}$$

where C_1 is the A_1 constant for the $w \in A_1$, so, $w^{1+\varepsilon} \in A_1$. If now $p > 1$, it suffices to take $\varepsilon > 0$ small enough to have, at the same time

$$\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx \leq \left(C_1\frac{1}{|Q|}\int_Q w(x)dx\right)^{1+\varepsilon}$$

and

$$\frac{1}{|Q|}\int_Q w(x)^{-(1+\varepsilon)/(p-1)}dx \leq \left(C_2\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{1+\varepsilon}$$

and then

$$\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx \left(\frac{1}{|Q|}\int_Q w(x)^{-(1+\varepsilon)/(p-1)}dx\right)^{p-1} \leq$$

$$\leq \left(C_1 C_2 \frac{1}{|Q|} \int_Q w(x)dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \right)^{1+\varepsilon} \leq$$

$$\leq C_1^{1+\varepsilon} C_2^{(1+\varepsilon)(p-1)} C^{1+\varepsilon}$$

where C is the A_p constant for w. □

Of course theorem 2.2.3. combined with part 2 of theorem 2.1.2., gives theorem 2.2.2.

Now with the help of theorem 2.2.2.. we can improve theorem 2.2.1. as anticipated, obtaining

Theorem 2.2.4. *Let w be weight on $\mathbb{R}^n$ and let $1 < p < \infty$. Then, the following conditions are equivalent:*

1. *M is of weak type (p,p) with respect to w, that is, there is a constant C such that for every function $f \in L^1_{loc}(\mathbb{R}^n)$ and every $t > 0$*

$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq Ct^{-p} \int_{\mathbb{R}^n} |f(x)|^p w(x)dx$$

2. *There is a constant C such that for every function $f \geq 0$ in $\mathbb{R}^n$ and every cube Q*

$$\left(\frac{1}{|Q|} \int_Q f(x)dx \right)^p w(Q) \leq C \int_Q f(x)^p w(x)dx$$

3. *$w \in A_p$, that is, there is a constant C such that for every cube Q*

$$\frac{1}{|Q|} \int_Q w(x)dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

4. *M is bounded in $L^P(w)$, that is, there is a constant C such that for every $f \in L^p(w)$:*

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)dx.$$

Proof. All that remains to be proved is that 3) implies 4). Here is the proof. We have $w \in A_p$. Since $1 < p < \infty$, theorem 2.2.2. tells us that $w \in A_q$ for some $q < p$. Then M is of weak type (q,q) with respect to w and, since M is also bounded in $L^\infty(w) = L^\infty$ (this inequality follows from the fact that $0 < w(x) < \infty$ for a.e. x), the Marcinkiewicz interpolation theorem allows us to conclude that M is bounded in $L^p(w)$. □

Also, the reverse Hölder's inequality allows us to give a more precise version of lemma 2.

Theorem 2.2.5. *If $w \in A_p$ for some $p \in [1, \infty)$, then there exist $\delta > 0$, $C > 0$ such that, every time we have a measurable set A contained in a cube Q, the following inequality holds:*

$$\frac{w(A)}{w(Q)} \leq C \left(\frac{|A|}{|Q|} \right)^\delta \qquad (2.10)$$

Proof. The key fact is that w satisfies an inequality like the one appearing in lemma 3 for some $\varepsilon > 0$ (R.H.I.). We start by using Hölder's inequality with exponents $1+\varepsilon$ and its conjugate $(1+\varepsilon)/\varepsilon$, and then we apply the R.H.I.. We get:

$$w(A) = \int_A w(x)dx = \int_A X_A(x)w(x)dx \leq$$

$$\leq \left(\int_A w(x)^{1+\varepsilon} dx \right)^{1/(1+\varepsilon)} \left(\int_A X_A(x)^{(1+\varepsilon)/\varepsilon} dx \right)^{\varepsilon/(1+\varepsilon)} =$$

$$= \left(\int_A w(x)^{1+\varepsilon} dx \right)^{1/(1+\varepsilon)} |A|^{\varepsilon/(1+\varepsilon)} =$$

$$= \left(\frac{1}{|Q|} \int_A w(x)^{1+\varepsilon} dx \right)^{1/(1+\varepsilon)} |Q|^{1/(1+\varepsilon)} |A|^{1/(1+\varepsilon)} \leq$$

$$\leq \frac{C}{|Q|} \int_Q w(x)dx |Q|^{1/(1+\varepsilon)} |A|^{1/(1+\varepsilon)} = Cw(Q) \left(\frac{|A|}{|Q|} \right)^{\varepsilon/(1+\varepsilon)}$$

which is (2.10) with $\delta = \varepsilon/(1+\varepsilon)$.

□

Condition (2.10) is known as A_∞ for reasons which will appear very soon. We also speak of the class A_∞ which is, naturally, the class formed by those locally integrable weights w satisfying the A_∞ condition.

For the next result, μ_1 and μ_2 are going to be doubling measures, that is, both satisfy a doubling condition like (1.13) in chapter I . For these measures, we give the following definition:

Definition : μ_1 is comparable to μ_2 when there exist $a, \beta < 1$ such that, every time we have a measurable subset A of a cube Q with $\mu_2(A)/\mu_2(Q) \leq a$, it follows that $\mu_1(A)/\mu_1(Q) \leq \beta$.

With this definition we can write

Theorem 2.2.6. *The following conditions are equivalent*

1. *There exist* $\delta > 0$, $C > 0$ *such that for every measurable set* A *contained in a cube* Q
$$\frac{\mu_2(A)}{\mu_2(Q)} \leq C \left(\frac{\mu_1(A)}{\mu_1(Q)}\right)^\delta$$
2. μ_2 *is comparable to* μ_1
3. μ_1 *is comparable to* μ_2
4. $d\mu_2(x) = w(x)d\mu_1(x)$ *with:*
$$\left(\frac{1}{\mu_1(Q)}\int_Q w(x)^{1+\varepsilon}d\mu_1(x)\right)^{1/(1+\varepsilon)} \leq C\frac{1}{\mu_1(Q)}\int_Q w(x)d\mu_1(x) < \infty$$
for some $\varepsilon > 0$

Proof. 1) $\Rightarrow$ 2) is clear. Indeed, if $\mu_1(A)/\mu_1(Q) \leq a$, it will be $\mu_2(A)/\mu_2(Q) \leq Ca^\delta$. It suffices to start with some $a > 0$ such that $Ca^\delta < 1$ and we obtain μ_2 comparable to μ_1 with constants a and $\beta = Ca^\delta$.

2) $\Rightarrow$ 3). To say that $\mu_1(A)/\mu_1(Q) \leq a$ implies $\mu_2(A)/\mu_2(Q) \leq \beta$ is equivalent to saying that $\mu_2(A)/\mu_2(Q) > \beta$ implies that $\mu_1(A)/\mu_1(Q) > a$. Then if $\mu_2(A)/\mu_2(Q) \leq a'$, where $a' = (1-\beta)/2 < 1-\beta$, we get

$$\mu_2(A)/\mu_2(Q) < 1-\beta \Rightarrow \mu_2(Q) - \mu_2(A) > \beta\mu_2(Q) \Rightarrow$$
$$\Rightarrow \frac{\mu_2(Q\backslash A)}{\mu_2(Q)} > \beta$$

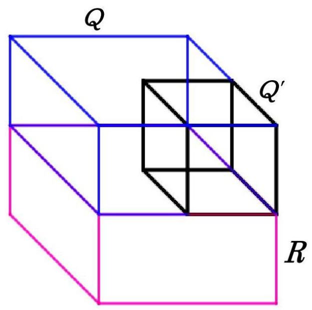
which implies
$$\frac{\mu_1(Q\backslash A)}{\mu_1(Q)} > a$$

and consequently

$$\frac{\mu_1(A)}{\mu_1(Q)} < 1 - a.$$

Thus, we have seen that μ_1 is comparable to μ_2 with constants $a' = (1-\beta)/2$ and $\beta' = 1 - a$. It becomes clear that 2) and 3) are equivalent.

Let us see now that 2) $\Rightarrow$ 4). We start from the fact that μ_2 is comparable to μ_1 with constants a and β. We see, first of all, that μ_2 is absolutely continuous with respect to μ_1, that is : $\mu_1(E) = 0 \Rightarrow \mu_2(E) = 0$. Once this is proved, the Radon-Nikodym theorem guarantees that $d\mu_2(x) = w(x)d\mu_1(x)$ with w locally integrable with respect to μ_1. Let $\mu_1(E) = 0$ and suppose that $\mu_2(E) > 0$. Since the measure is regular, there will be an open set Ω such that $\Omega \supset E$ and $\mu_2(\Omega) < \beta^{-1}\mu_2(E)$. Let $\Omega = \cup_j Q_j$ where the Q_j's are non overlapping cubes. Since for each j is $0 = \mu_1(Q_j \cap E) \leq a\mu_1(Q_j)$, we shall have $\mu_2(Q_j \cap E) \leq \beta\mu_2(Q_j)$ and, adding in j, we get : $\mu_2(E) \leq \beta\mu_2(\Omega)$, which contradicts the election of Ω. Let us note, that for this part of the proof we used the fact that the faces or edges of the cubes have measure μ_2 (or μ_1 for that matter) equal to 0. This follows easily from the doubling condition. Indeed, if μ is doubling, there is a constant $K < 1$ such that if Q is a cube and R is a half of Q, that is, if $Q = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]$, the half of the Q (one of the many half's) is $R = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, (a_n + b_n)/2]$, then (as will be shown) $\mu(R) \leq K\mu(Q)$. Let's prove that: let Q' be a dyadic subcube of Q with side length equal the half the side length of Q, and contiguous to R.



Then: $$R \subset Q'^3 \Rightarrow$$

$$\mu(R) \leq C_3\mu(Q') \leq C_3\mu(Q \backslash R) = C_3(\mu(Q) - \mu(R)) \Rightarrow$$

$$\mu(R)(1 + C) \leq C\mu(Q) \Rightarrow \mu(R) \leq \frac{C}{1 + C}\mu(Q) := K\mu(Q)$$

Now if R_1 is the half of R, with the same argument, we get that:

$$\mu(R_1) \leq K\mu(R) \leq K^2\mu(Q).$$

Viewing now,the face of Q as an intersection of the R_j's resulting from repeatedly dividing by 2 a side of Q, we see that a face has μ measure equal to zero for μ doubling. So, let

$$d\mu_2(x) = w(x)d\mu_1(x).$$

It remains to see that the inequality in 4) holds. All we have to do is to repeat the proof of lemma 3 with μ_1 in place of Lebesgue measure. Observe that in the proof of lemma 3 we just used these two facts: $w(x)dx$ is comparable to Lebesgue measure and Lebesgue measure is doubling. These hypotheses still hold for $d\mu_2(x) = w(x)d\mu_1(x)$ and $d\mu_1(x)$. Thus, we obtain the inequality in 4).

Finally we have to see that 4) implies 1). But this is done exactly as in the proof of the previous theorem 2.2.5. □

Corollary 2.2.1. *The comparability of measures ia an equivalence relation.*

Proof. The equivalence between 2) and 3) in the previous theorem tells us that comparability is a symmetric relation. Transitivity is proved very simply by using the characterization given by 1) in theorem 2.2.6., lets see that: Let μ_1 be comparable to μ_2 and μ_2 comparable to μ_3. First of all there exist $a, b < 1$ such that:

$$\frac{\mu_2(A)}{\mu_2(Q)} \leq a \Rightarrow \frac{\mu_1(A)}{\mu_1(Q)} \leq b$$

we know also (previous theorem) that μ_3 is also comparable to μ_2 which implies the existence of $a', b' < 1$ such that

$$\frac{\mu_2(A)}{\mu_2(Q)} \leq a' \Rightarrow \frac{\mu_3(A)}{\mu_3(Q)} \leq b'$$

There exists also $\delta > 0, C > 0$ such that

$$\frac{\mu_2(A)}{\mu_2(Q)} \leq C\left(\frac{\mu_3(A)}{\mu_3(Q)}\right)^{\delta}$$

So, if

$$\frac{\mu_3(A)}{\mu_3(Q)} \leq \left(\frac{a}{C}\right)^{1/\delta}$$

we get that

$$\frac{\mu_2(A)}{\mu_2(Q)} \leq a \Rightarrow \frac{\mu_1(A)}{\mu_1(Q)} \leq b$$

Thus, μ_1 is comparable to μ_3 with constants $(a/C)^{1/\delta}$ and b □

Corollary 2.2.2. *Let $w(x) \geq 0$ be locally integrable in $\mathbb{R}^n$. The following conditions are equivalent*

1. *$w \in A_p$ for some $p \in [1, \infty)$*
2. *There exist $a, \beta < 1$ such that $|E| \leq a|Q|$ implies $w(E) \leq \beta w(Q)$ whenever E is measurable subset of the cube Q*
3. *There exist $\varepsilon > 0$ and $C > 0$ such that for every cube Q*

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \leq \frac{C}{|Q|}\int_Q w(x)dx$$

4. *$w \in A_\infty$*

Proof. All the implications 1) ⇒ 2) ⇒ 3) ⇒ 4) have already been proved. Observe that the proof of lemma 3 actually yields the fact that 2) ⇒ 3)). It only remains to see that 4) ⇒ 1)). Let us see it. We know from theorem 2.2.6. that $w \in A_\infty$ is equivalent to saying that the measures dx and $w(x)dx$ are comparable and, taking into account that

$$d\mu_2(x) =: dx = w(x)^{-1}w(x)dx := w(x)^{-1}d\mu_1(x)$$

Thus $(\mu_1(Q) = w(Q))$, the following R.H.I. must hold:

$$\left(\frac{1}{w(Q)}\int_Q w(x)^{-(1+\varepsilon)}w(x)dx\right)^{1/(1+\varepsilon)} \leq \frac{C}{w(Q)}\int_Q w(x)^{-1}w(x)dx = C\frac{|Q|}{w(Q)}$$

$$\Rightarrow$$

$$\left(\frac{1}{w(Q)}\int_Q w(x)^{-\varepsilon}dx\right)^{1/(1+\varepsilon)} \leq C\frac{|Q|}{w(Q)}$$

Hence, setting $\varepsilon = 1/(p-1)$ for some $p > 1$, we have that $1/(1+\varepsilon) = (p-1)/p$ and the inequality above comes to the form

$$\left(\frac{1}{w(Q)}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C^p\left(\frac{|Q|}{w(Q)}\right)^p \Rightarrow$$

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C^p \frac{|Q|}{w(Q)} \Rightarrow$$

$$w_Q\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C^p := C$$

which means that $w \in A_p$ □

Thus, we have shown that $A_\infty = \cup_{1\leq p<\infty} A_p$, which explains the name A_∞ given to condition (2.10).

Actually, the name A_∞ is just perfect, since, as we shall presently show, A_∞ coincides with the formal limit of condition A_p as p tends to ∞

$$\lim_{p\to\infty}\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} = \lim_{q\to 0}\|w^{-1}\|_{L^q(|Q|^{-1}dx)} =$$

$$= \exp\left(\frac{1}{|Q|}\int_Q \log(w(x)^{-1})dx\right)$$

where the last identity is a simple exercise in measure theory.

Thus, the condition obtained by passing to the limit as p tends to ∞ in condition A_p is:

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\exp\left(\frac{1}{|Q|}\int_Q \log(w(x)^{-1})dx\right) \leq C$$

or, equivalently

$$\frac{1}{|Q|}\int_Q w(x)dx \leq C\exp\left(\frac{1}{|Q|}\int_Q \log w(x)dx\right) \qquad (2.14)$$

The exponential in the right hand side of (2.14) is the geometric mean of w on Q, which is, of course, dominated by the arithmetic mean w_Q (Jensen's inequality). Thus (2.14) implies that the arithmetic and the geometric means of w on every cube, are equivalent. The equivalence between this condition and A_∞ is contained in the following

Theorem 2.2.7. *Let $w \geq 0$ be locally integrable in $\mathbb{R}^n$. Then, the following conditions are equivalent:*

1. *There exist $a, \beta \in (0,1)$ such that, for every cube Q:*

$$|\{x \in Q : w(x) \leq a w_Q\}| \leq \beta |Q|$$

2. $w \in A_\infty$

3. *There exists C, such that, for every cube Q:*

$$\frac{1}{|Q|}\int_Q w(x)dx \leq C \exp\left(\frac{1}{|Q|}\int_Q \log w(x)dx\right)$$

Proof. Suppose 1) holds. Let us prove 2). After the proof of theorem 2.2.6., especially from corollary 2.2.2 and 2.2.1., it will be enough to see that, for appropriately chosen $\gamma, \delta \in (0,1)$, the following property holds: If E is a subset of a cube Q such that $w(E)/w(Q) \leq \gamma$, then $|E|/|Q| \leq \delta$. To prove this property, assume $w(E)/w(Q) \leq \gamma$, to be chosen later. Then we split $E = E_1 \cup E_2$, where

$$E_1 = \{x \in E : w(x) > a w_Q\} \text{ and } E_2 = \{x \in E : w(x) \leq a w_Q\}$$

For E_2, 1) gives the estimate $|E_2| \leq \beta|Q|$. For E_1 we use Chebichev's inequality to get:

$$|E_1| \leq \frac{1}{a w_Q}\int_{E_1} w(x)dx \leq \frac{1}{a w_Q}\int_E w(x)dx =$$

$$= \frac{|Q|}{a}\frac{w(E)}{w(Q)} \leq \frac{\gamma}{a}|Q|$$

Adding up the two estimates, we have

$$|E| \leq (\beta + \frac{\gamma}{a})|Q|$$

If we choose γ so small that $\beta + \gamma/a < 1$, we get what we wanted with $\delta = \beta + (\gamma/a)$.

To see now that 2) implies 3) is quite easy. Indeed, if $w \in A_\infty$, it follows from corollary 2.2.2. that $w \in A_p$ for some $1 \leq p < \infty$, which in turn, implies that there is a constant C such that

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\right)^{p-1} \leq C$$

But from the proof of theorem 2.1.2. 1), we can see that for every $q > p$ the conditions A_q holds with the A_p constant C. Thus. for every $q > p$ we have

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}\right)^{q-1} \leq C$$

Letting q tend to ∞ we obtain 3).

Finally, assuming 3), we are going to see that 1) holds. Take a cube Q. Dividing w by an appropriate constant (we can do that because it does not affect us on what we want to prove), we can assume that $\int_Q \log w(x)dx < \varepsilon$ for ε as close to zero as we want, so, without loss of generality we cam assume that $\int_Q \log w(x)dx = 0$ and, consequently, $w_Q \leq C$. Then, with $\lambda > 0$ still undetermined, we have:

$$|\{x \in Q : w(x) \leq \lambda\}| = |\{x \in Q : \log(1 + w(x)^{-1}) \geq \log(1 + \lambda^{-1})\}| \leq$$

$$\leq \frac{1}{\log(1+\lambda^{-1})}\int_Q \log(1 + w(x)^{-1})dx = \frac{1}{\log(1+\lambda^{-1})}\int_Q \log\frac{1+w(x)}{w(x)}dx =$$

$$= \frac{1}{\log(1+\lambda^{-1})}\int_Q \log(1 + w(x))dx$$

since by assumption $\int_Q \log w(x)dx = 0$. By using the simple inequality $\log(1+w) \leq w$ and the hypothesis $w_Q \leq C$, we get:

$$|\{x \in Q : w(x) \leq \lambda\}| \leq \frac{1}{\log(1+\lambda^{-1})}\int_Q w(x)dx \leq$$

$$\leq \frac{C}{\log(1+\lambda^{-1})}|Q| \leq \frac{1}{2}|Q|$$

if λ is small enough. In particular

$$|\{x \in Q : w(x) \leq aw_Q\}| \leq |\{x \in Q : w(x) \leq Ca\}| \leq (1/2)|Q|$$

if a is small enough. We have obtained 1) with $\beta = 1/2$. □

In chapter I we gave examples of A_1 weights, namely those those functions w of the form $w(x) = (M_\mu(x))^\gamma$ where μ is a positive Borel measure such that $M_\mu(x) < \infty$ for a.e. $x \in \mathbb{R}^n$ and $0 < \gamma < 1$. We used this result to show that $|x|^a$ is an A_1 weight in $\mathbb{R}^n$ if and only if $-n < a \leq 0$.

Starting with A_1 weights one can easily generate A_p weights for $1 < p < \infty$. Let $w_o, w_1 \in A_1$ in $\mathbb{R}^n$, and let $1 < p < \infty$. Then $w(x) = w_o(x)w_1(x)^{1-p}$ is an A_p weight. Indeed, since $w_1 \in A_1$ we have for every $x \in Q$ for some cube Q, that

$$\frac{1}{|Q|}\int_Q w_1(x)dx \le Mw_1(x) \le Cw_1(x)$$

and since $1-p<0$ we get that

$$w_1(x)^{1-p} \le C^{p-1}\left(\frac{1}{|Q|}\int_Q w_1(x)dx\right)^{1-p} := C\left(\frac{1}{|Q|}\int_Q w_1(x)dx\right)^{1-p}$$

using the same argument for w_o we get:

$$\left(\frac{1}{|Q|}\int_Q w_o(x)w_1(x)^{1-p}dx\right)\left(\frac{1}{|Q|}\int_Q (w_o(x)w_1(x)^{1-p})^{-1/(p-1)}dx\right)^{p-1} \le$$

$$\le C\left(\frac{1}{|Q|}\int_Q w_1(x)dx\right)^{1-p}\left(\frac{1}{|Q|}\int_Q w_o(x)dx\right)\cdot$$

$$\cdot\left(\frac{1}{|Q|}\int_Q w_o(x)dx\right)^{-1}\left(\frac{1}{|Q|}\int_Q w_1(x)dx\right)^{p-1} = C.$$

We shall show in the next section that every A_p weight w is actually of the form $w(x) = w_o(x)w_1(x)^{1-p}$ for some $w_o, w_1 \in A_1$ (factorization theorem). For the time being, we shall content ourselves with giving examples of A_p weights. If $-n < a \le 0$ and $-n < \beta \le 0$, $|x|^a|x|^{\beta(1-p)}$ is an A_p weight in $\mathbb{R}^n$. Thus, for $a = 0$ we get that $|x|^{\beta(1-p)}$ is an A_p weight with $-n < \beta \le 0$, which implies that $|x|^{\beta(p-1)}$ is an A_p weight, but now, with $0 \le \beta < n$. Hence, $|x|^a$ is an A_p weight in $\mathbb{R}^n$ if and only if $-n < a < n(p-1)$ since $|x|^a$ and $(|x|^a)^{-1/(p-1)}$ have to be locally integrable.

By using the R.H.I. we get a converse of theorem 1.3.2. in chapter I, giving the following characterization of A_1 weights:

Theorem 2.2.8. *Let w(x) be ≥ 0 and finite a.e. Then, $w \in A_1$ if and only if*

$$w(x) = k(x)(Mf(x))^\gamma$$

where $k(x) \ge 0$ is such that $k, k^{-1} \in L^\infty$, f is locally integrable and $0 < \gamma < 1$.

Proof. Theorem 1.3.2. of chapter I implies that every function of the given form is an A_1 weight. Lets see this: Since $k, k^{-1} \in L^\infty$, there exist $C_1, C_1 \geq 0$ such that $C_1 \leq k(x) \leq C_2$ for a.e. $x \in \mathbb{R}^n$. Thus:

$$Mw(x) \leq C_2 M((Mf(x))^\gamma) \leq C_2 C (Mf(x))^\gamma \leq$$

$$\leq C_2 C \frac{k(x)}{C_1} (Mf(x))^\gamma = C \frac{C_2}{C_1} w(x) := Cw(x)$$

Conversely, let $w \in A_1$. We know that w satisfies a R.H.I. :

$$\left(\frac{1}{|Q|} \int_Q w(x)^{1+\varepsilon} \right)^{1/(1+\varepsilon)} \leq C \frac{1}{|Q|} \int_Q w(x) dx \leq Cw(x) \;\; a.e.$$

Thus

$$w(x)^{1+\varepsilon} \leq (Mw(x))^{1+\varepsilon} \leq$$

using Jensen's inequality

$$\leq M(w^{1+\varepsilon})(x) \Rightarrow$$

$$\Rightarrow w(x) \leq (M(w^{1+\varepsilon})(x))^{1/(1+\varepsilon)} \leq Cw(x).$$

We can write now, $w(x) = k(x)(M(w^{1+\varepsilon})(x))^{1/(1+\varepsilon)}$ with $C^{-1} \leq k(x) \leq 1$ and we obtain the representation required with $f(x) = w(x)^{1+\varepsilon}$ and $\gamma = 1/(1+\varepsilon)$ □

There is a relation between weights and B.M.O. functions. We have already seen in chapter I that the logarithm of an A_1 weight is a B.M.O. function. We shall see presently that the same is true for any A_∞ weight. Of course this follows trivially after the factorization theorem, but a simple proof can be given without appealing to that result which we have not proved yet. First of all, we give a characterization of A_p weights in terms of of their logarithms.

Theorem 2.2.9. *1. Let ϕ be a real locally integrable function on $\mathbb{R}^n$ and let $1 < p < \infty$. Then $e^\phi \in A_p$ if and only if the following conditions are satisfied:*

(a) $\frac{1}{|Q|} \int_Q e^{(\phi(x) - \phi_Q) dx} \leq C$, with C independent of the cube Q

(b) $\frac{1}{|Q|} \int_Q e^{-(\phi(x) - \phi_Q)/(p-1)} dx \leq C$, with C independent of the cube Q

2. For ϕ as in 1), $e^\phi \in A_\infty$ if and only if (a) holds. Note that for $p = \infty$, condition (b) becomes empty, so that 2) is just an extension of 1) to $p = \infty$

3. It follows from 1) and 2) that w is in A_p if and only if both w and $w^{-1/(p-1)}$ are in A_∞.

Proof. It is clear the the two conditions (a) and (b) imply together that $e^\phi \in A_p$, since

$$\frac{1}{|Q|}\int_Q e^{\phi(x)}dx\left(\frac{1}{|Q|}\int_Q (e^\phi(x))^{-1/(p-1)}dx\right)^{p-1} =$$

$$e^{\phi_Q-\phi_Q}\frac{1}{|Q|}\int_Q e^{\phi(x)}dx\left(\frac{1}{|Q|}\int_Q (e^\phi(x))^{-1/(p-1)}dx\right)^{p-1} =$$

$$= \frac{1}{|Q|}\int_Q e^{\phi(x)-\phi_Q}dx\left(\frac{1}{|Q|}\int_Q e^{-(\phi(x)-\phi_Q)/(p-1)}dx\right)^{p-1}.$$

Conversely, suppose that $e^\phi \in A_p$. Then

$$\frac{1}{|Q|}\int_Q e^{\phi(x)-\phi_Q}dx = e^{-\phi_Q}\frac{1}{|Q|}\int_Q e^{\phi(x)}dx =$$

$$= \left(e^{-\phi_Q/(p-1)}\right)^{p-1}\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}dx\right) \le$$

using jensen's inequality

$$\le \left(\frac{1}{|Q|}\int_Q e^{-\phi(x)/(p-1)}\right)^{p-1}\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}dx\right) \le C$$

Also

$$\frac{1}{|Q|}\int_Q e^{-(\phi(x)-\phi_Q)/(p-1)}dx = \left(\frac{1}{|Q|}\int_Q e^{-\phi(x)/(p-1)}dx\right)(e^{\phi_Q})^{1/(p-1)} \le$$

again using Jensen's inequality

$$\le \left(\frac{1}{|Q|}\int_Q e^{-\phi(x)/(p-1)}dx\right)\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}dx\right)^{1/(p-1)} \le C^{1/(p-1)}$$

2): theorem 2.2.7. implies that $e^\phi \in A_\infty$ if and only if

$$\frac{1}{|Q|}\int_Q e^{\phi(x)}dx \le Ce^{\phi_Q}$$

which is equivalent to condition a).

3): It follows from 2) that, for $w = e^\phi$, condition a) is equivalent to saying that $w \in A_\infty$ and condition b) is equivalent to saying that $w^{-1/(p-1)} \in A_\infty$. □

In case $p = 2$, conditions a) and b) become:

$$\frac{1}{|Q|}\int_Q e^{\phi(x)-\phi_Q}dx \le C \quad \textit{and} \quad \frac{1}{|Q|}\int_Q e^{-(\phi(x)-\phi_Q)}dx \le C$$

These two inequalities together are equivalent to

$$\frac{1}{|Q|}\int_Q e^{|\phi(x)-\phi_Q|}dx \le C$$

We can write:

Corollary 2.2.3. *Let ϕ be a real locally integrable function on $\mathbb{R}^n$. Then $e^\phi \in A_2$ if and only if there is a constant C such that for every cube $Q \subset \mathbb{R}^n$*

$$\frac{1}{|Q|}\int_Q e^{|\phi(x)-\phi_Q|}dx \le C$$

The relation between weights and $B.M.O.$ functions is now clear.

Corollary 2.2.4. $w \in A_\infty \Rightarrow \log w \in B.M.O.$

Proof. Let $w \in A_\infty$ and write $w = e^\phi$ that is: $\phi = \log w$. If $w \in A_2$, we know from the previous corollary that

$$\|\phi\|_* = \sup_Q \frac{1}{|Q|}\int_Q |\phi(x) - \phi_Q|dx \le \sup_Q \frac{1}{|Q|}\int_Q e^{|\phi(x)-\phi_Q|}dx \le C$$

so that $\phi = \log w \in B.M.O.$

In general $w \in A_\infty \Rightarrow w \in A_p$ fro some $p \in [1, \infty)$. Thus, if $p \le 2$, we have $w \in A_p \subset A_2$ and, as we have just seen, $\log w \in B.M.O.$. If $p > 2$, we look at $w^{-1/(p-1)} \in A_{p'} \subset A_2$. It follows that $\log(w^{-1/(p-1)}) = -\frac{1}{p-1}\log w \in B.M.O.$ Thus, in any case, $\log w \in B.M.O$ □

Observe that, if $w \in A_p$, $\|\log w\|_*$ depends only on p and on the A_p constant for w.

If $\phi \in B.M.O.$, we know from corollary 1.3.1 (2) in chapter I, that

$$\frac{1}{|Q|}\int_Q e^{\lambda|\phi(x)-\phi_Q|}dx \le C$$

for every cube Q, thus, using corollary 2.2.3., we see that $e^{\lambda\phi} \in A_2$ for λ small enough ($0 < \lambda < C_2/\|\phi\|_*$ with the notation used in corollary 1.3.1.). If we set $e^{\lambda\phi} = w$, we get $\phi = \lambda^{-1}\log w$. Thus

$$B.M.O. = \{a \log w : a \geq 0, w \in A_2\}$$

and the reason why we have $a \geq 0$ is that $f \in B.M.O. \Rightarrow Cf \in B.M.O.$

Actually, the same is true for any p with $1 < p \leq \infty$, i.e.

$$B.M.O. = \{a \log w : a \geq 0, w \in A_p\}$$

We already know that this is true for $p \geq 2$ since $A_2 \subset A_p$. For $1 < p < 2$, if $\phi \in B.M.O.$, we can write $\phi = a \log w$ with $a \geq 0$ and $w \in A_2$. But $\sigma = w^{p-1} \in A_p$ since $2(p-1)+1-(p-1) = p$ (see theorem 2.1.2 part 2). Therefore,

$$\phi = a \log w = a \log(\sigma^{1/(p-1)}) = (a/(p-1)) \log \sigma.$$

In contrast to this situation, we have (as we will prove), that:

$$\{a \log w : a \geq 0, \;\; w \in A_1\} = B.L.O. \subsetneqq B.M.O.$$

let's prove it: In fact, we already know that $a \log w \in B.L.O.$ when $a \geq 0$ and $w \in A_1$ (see the proof of theorem 1.3.1). Conversely, let $\phi \in B.L.O.$ Then, according to corollary 1.3.1. (2), we have for $\varepsilon > 0$ small enough, every cube Q and given C, that:

$$C \geq \frac{1}{|Q|}\int_Q e^{\varepsilon|\phi(x)-\phi_Q|}dx \geq \frac{1}{|Q|}\int_Q e^{\varepsilon(\phi(x)-\phi_Q)}dx$$

which implies

$$\frac{1}{|Q|}\int_Q e^{\varepsilon\phi(x)}dx \leq C\exp(\varepsilon\phi_Q) \leq$$

we use that $\phi \in B.L.O.$ (i.e. $\phi_Q - ess_Q inf\phi \leq C'$, for some C')

$$\leq C\exp(\varepsilon(C' + ess_Q inf\phi)) = Ce^{\varepsilon C'}\exp(\varepsilon \cdot ess_Q inf\phi) =$$

$$= Ce^{\varepsilon C'} ess_Q inf(e^{\varepsilon\phi}) := C ess_Q inf(e^{\varepsilon\phi}) \Rightarrow$$

$$\Rightarrow M(e^{\varepsilon\phi(x)}) \leq Ce^{\varepsilon\phi(x)} \quad for \;\; a.e. x \in \mathbb{R}^n$$

It follows that $e^{\varepsilon\phi} \in A_1$. Thus $\phi = \varepsilon^{-1}\log w$ with $w = e^{\varepsilon\phi} \in A_1$.

Combining this with theorem 2.2.8., which tells us that every $w \in A_1$ can be written as $w(x) = k(x)(Mf(x))^\gamma$, with $k(x) \geq 0$ such that $\log k \in L^\infty$ and $0 < \gamma < 1$, we are led to:

$$B.L.O. = \{h + \beta \log(Mf) : h \in L^\infty, \;\; f \in L^1_{loc}, \;\; \beta \geq 0\}$$

We finish this section by observing that the L^p inequality established in chapter I (theorem 1.3.3.) between the Hardy-Littlewood maximal function Mf and the sharp maximal function $f^\#$, also holds when Lebesgue measure dx is replaced by the measure $w(x)dx$, where w is any A_∞ weight.

The concrete statement without proof is as follows

Theorem 2.2.10. *Let $w \in A_\infty$ in $\mathbb{R}^n$ and let f be such that $Mf \in L^{p_o}(w)$ for some p_o with $0 < p_o < \infty$. Then, for every p such that $p_o \leq p < \infty$*

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x) dx \leq C \int_{\mathbb{R}^n} (f^\#(x))^p w(x) dx.$$

□

2.3 FACTORIZATION THEOREM

We have already seen that if we have two A_1 weights w_o and w_1 and if $1 < p < \infty$, then $w(x) = w_o(x)w_1(x)^{1-p}$ is an A_p weight. Now we are going to show that, conversely, every A_p weight w can be written in this form for certain $w_o, w_1 \in A_1$. This factorization theorem will have important consequences. The proof will be based on a single lemma, which, as we shall see, provides a strikingly powerful method to deal with several problems about weights.

Lemma 2.4. *Let S be a sublinear operator bounded in $L^p(\mu)$, where $p \geq 1$ and μ is an arbitrary positive measure on some measurable space. Suppose that $Sf \geq 0$ for every $f \in L^p(\mu)$. Then, for every $u \geq 0$ in $L^p(\mu)$ there is $v \geq 0$ in $L^p(\mu)$ such that:*

1. *$u(x) \leq v(x)$ for a.e. x*
2. *$\|v\|_p \leq 2\|u\|_p$*
3. *$Sv(x) \leq Cv(x)$ for a.e. x ($C = 2\|S\|$ is enough).*

Proof. It suffices to take

$$v = \sum_{j=0}^{\infty} (2\|S\|)^{-j} S^j(u).$$

where $S^j = S \circ S \circ ... \circ S$ j-times. Indeed, since, (we start with 2)

$$\|S\| = inf\{C > 0 : \|Sv\|_p \leq C\|v\|_p, \ \ for \ \ all \ \ v \in L^P(\mu)\}$$

we get that

$$\|v\|_p \leq \sum_{j=0}^{\infty} (2\|S\|)^{-j} \|S\|^j \|u\|_p =$$

$$= \|u\|_p \sum_{j=0}^{\infty} 2^{-j} = 2\|u\|_p$$

On the other hand, since $S^o(u) = u$ and since $Sf \geq 0$ for every $f \in L^p(\mu)$, we get that, all the partial sums in the definition of v, are $\geq u$, thus $u \leq v$ a.e. (actually everywhere)

Finally, since S is sublinear, we have:

$$Sv \le \sum_{j=0}^{\infty}(2\|S\|)^{-j}S^{j+1}(u) =$$

$$= 2\|S\|\sum_{j=0}^{\infty}(2\|S\|)^{-(j+1)}S^{j+1}(u) = 2\|S\|(v - (2\|S\|)^0S^0(u)) =$$

$$= 2\|S\|(v-u) \le 2\|S\|v.$$

□

Actually with the help of this lemma, we can give a general factorization theorem which includes the one we were seeking for A_p weights

Theorem 2.3.1. *Let T be a positive symmetric sublinear operator acting on measurable functions on some measure space (X, dx) (this means that $|T(f+g)| \le |T(f)|+|T(g)|$ and also that $|f| \le g$ implies $|Tf| \le Tg$). For $1 < p < \infty$, let us call*

$$W_p = \{w : 0 \le w(x) < \infty \text{ a.e. and } T \text{ is bounded in } L^p(w) = L^p(w(x)dx)\}$$

Also, we call

$$W_1 = \{w : 0 \le w(x) < \infty \text{ a.e. and } Tw(x) \le Cw(x), \text{a.e}\}$$

for some C independent of x

Then, for every $1 < p < \infty$, we have:

$$W_p \cap W_{p'}^{1-p} \subset W_1W_1^{1-p}$$

that is: If $w \in W_p$ and also $w^{-1/(p-1)} \in W_{p'}$, then, there exist $w_o, w_1 \in W_1$ such that $w = w_ow_1^{1-p}$. Besides, the constants C for w_o and w_1 in the class W_1 depend only upon the constants for w and $w^{-1/(p-1)}$ in W_p and $W_{p'}$ respectively, that is, on the respective norms of T on $L^p(w)$ and $L^{p'}(w^{-1/(p-1)})$.

Proof. We just need to consider the case $1 < p \le 2$ since :

$$W_p \cap W_{p'}^{1-p} \subset W_1W_1^{1-p} \iff (W_p \cap W_{p'}^{1-p})^{1-p'} \subset (W_1W_1^{1-p})^{1-p'}$$

which implies that:

$$W_p \cap W_{p'}^{1-p} \subset W_1 W_1^{1-p} \Longleftrightarrow W_{p'} \cap W_p^{1-p'} \subset W_1 W_1^{1-p'}$$

and also

$$p \geq 2 \Longleftrightarrow p' = p/(p-1) \leq 2$$

So, let $1 < p \leq 2$, and suppose that $w \in W_p \cap W_{p'}^{1-p}$, i.e. $w \in W_p$ and $w^{-1/(p-1)} \in W_{p'}$. We want to see that $w = w_o w_1^{1-p}$ with $w_o, w_1 \in W_1$. After writing $v^{-1} = w_1^{1-p}$, we see that this is equivalent to finding v such that:

1. $vw(= w_o) \in W_1$, that is: $T(vw) \leq Cvw$ and also
2. $v^{1/(p-1)} \in W_1$, that is $T(v^{1/(p-1)}) \leq Cv^{1/(p-1)}$, or equivalently

$$(T(v^{1/(p-1)}))^{p-1} \leq Cv$$

Suppose now that for every u in some L^q space we can find Su so that:

$$|T(uw)| \leq S(u)w$$

and

$$\left(T(|u|^{1/(p-1)})\right)^{p-1} \leq S(u)$$

If the operator S satisfies the hypotheses of lemma 4, we shall be able to find $v \geq 0$ such that $S(v) \leq Cv$. This would be suffice, because then we should have:

$$T(vw) \leq S(v)w \leq Cvw$$

and

$$\left(T(v^{1/(p-1)})\right)^{p-1} \leq S(v) \leq Cv$$

All we have to do is to look for S and make sure that it satisfies the hypotheses of the lemma. The natural candidate for S is the operator sending the function u into Su given by

$$S(u) = |T(uw)|w^{-1} + \left(T(|u|^{1/(p-1)})\right)^{p-1}$$

First of all, we observe that S is sublinear : For the first term of the sum, sublinearity is clear, lets prove it and for the second term. Let f,g be measurable functions, we write

$$f = (1-\lambda)F, \quad g = \lambda G$$

then

$$|(1-\lambda)F+\lambda G|^{1/(p-1)} \le (1-\lambda)|F|^{1/(p-1)}+\lambda|G|^{1/(p-1)}$$

since $|x|^a$ is convex for $a \ge 1$, and being $1 < p \le 2$, we have $1/(p-1) \ge 1$. Now, combining that T is a positive sublinear operator with the comment in the statement of the theorem, we get that:

$$\left(T(|f+g|^{1/(p-1)})\right)^{p-1} \le \left(T((1-\lambda)|F|^{1/(p-1)}+\lambda|G|^{1/(p-1)})\right)^{p-1} \le$$

$$\le \left((1-\lambda)T(|F|^{1/(p-1)})+\lambda T(|G|^{1/(p-1)})\right)^{p-1} \le$$

since $p-1 \le 1$

$$\le (1-\lambda)^{p-1}\left(T(|F|^{1/(p-1)})\right)^{p-1}+\lambda^{p-1}\left(T(|G|^{1/(p-1)})\right)^{p-1} =$$

$$= \left(T(|f|^{1/(p-1)})\right)^{p-1} + \quad \left(T(|g|^{1/(p-1)})\right)^{p-1}$$

$\Rightarrow$ S is sublinear. Besides, S is bounded in $L^{p'}(w)$. Indeed:

$$\int_{\mathbb{R}^n} |T(uw)w^{-1}|^{p'}w = \int_{\mathbb{R}^n} |T(uw)|^{p'}w^{1-p'}$$

But $w^{1-p'} = w^{-1/(p-1)} \in W_{p'}$, thus, T is bounded in $L^{p'}(w^{1-p'})$, which implies that:

$$\int_{\mathbb{R}^n} |T(uw)|^{p'}w^{1-p'} \le C\int_{\mathbb{R}^n} |uw|^{p'}w^{1-p'} = C\int_{\mathbb{R}^n} |u|^{p'}w \qquad (I)$$

and also

$$\int_{\mathbb{R}^n} |T(|u|^{1/(p-1)})|^{(p-1)p'}w = \int_{\mathbb{R}^n} |T(|u|^{1/(p-1)})|^{p}w \le$$

$w \in W_p$

$$\le C\int_{\mathbb{R}^n} \left(|u|^{1/(p-1)}\right)^{p} w = C\int_{\mathbb{R}^n} |u|^{p'}w. \qquad (II)$$

Using now (I),(II) and Minkowski's inequality, we get that S is bounded in $L^{p'}(w)$.

From the definition of S, it is clear that $Su \ge 0$ for every $u \in L^{p'}(w)$. Thus, S satisfies all the conditions required in lemma 4. Note that C in lemma 4 (iii) depends only on the norm of S in $L^{p'}(w)$, and the norm of S in $L^{p'}(w)$ depends only on the norms for T in $L^p(w)$ and in $L^{p'}(w^{-1/(p-1)})$. This finishes the proof.

□

Corollary 2.3.1. *(P.Jones′ factorization theorem) For $1 < p < \infty$,*

$$A_p = A_1 A_1^{1-p}$$

that is : $w \in A_p$ if and only if there exist $w_o, w_1 \in A_1$ such that $w = w_o w_1^{1-p}$

Proof. If we take $T = M =$ the Hardy-Littlewood maximal operator in theorem 2.3.1. (previous theorem), we know that $W_p = A_p$ and $W_1 = A_1$. Besides $W_{p'}^{1-p} = A_{p'}^{1-p} = A_p$ because $w \in A_p$ if and only if $w^{-1/(p-1)} \in A_{p'}$. Therefore, applying the previous theorem we get that

$$A_p \subset A_1 A_1^{1-p}$$

The inclusion $A_1 A_1^{1-p} \subset A_p$ has been already established in section 2. □

By combining the factorization theorem with the characterization of A_1 weights given by theorem 2.2.8., we obtain a general expression for A_p weights in terms of maximal functions. This is the natural extension to $p > 1$ of theorem 2.2.8. Then, by using the John-Nirenberg theorem, this yields an expression for $B.M.O.$ functions in terms of maximal functions:

Corollary 2.3.2.

1. *Let w be a weight in $\mathbb{R}^n$ such that $w(x) < \infty$ a.e. Then, $w \in A_p$ if and only if, it can be written as*

   $$w(x) = k(x)(Mf(x))^a (Mg(x))^{\beta(1-p)}$$

   with $f, g \in L^1_{loc}(\mathbb{R}^n)$, k bounded away from zero and ∞, and $0 < a$ and $\beta < 1$. In this representation, k can be taken between two positive bounds which depend only on the A_p constant for w.

2. *There are constants C_1 and C_2 depending only on the dimension n, such that every $\phi \in B.M.O.$ in $\mathbb{R}^n$ can be written as :*

   $$\phi(x) = b(x) + \gamma \log Mf(x) - h \log Mg(x)$$

   with $f, g \in L^1$, $\gamma, h \geq 0$ and

   $$\|b\|_\infty + \gamma + h \leq C_1 \|\phi\|_*$$

   Conversely, every ϕ which can be written as above, belongs to $B.M.O.$ with

   $$\|\phi\|_* \leq C_2(\|b\|_\infty + \gamma + h)$$

3. *We can write a statement like (2) with B.L.O. in place of B.M.O. and* $h = 0$

4. *As a consequence of 2) and 3), every B.M.O. function can be written as a difference of two B.L.O. functions. In short:*

$$B.M.O. \subset B.L.O. - B.L.O.$$

Proof. 1):($\Longleftarrow$) It follows from theorem 2.2.8. that both $(Mf(x))^a$ and $(Mg(x))^\beta$ are A_1 weights, thus $(Mf(x))^a(Mg(x))^{\beta(1-p)} \in A_1 A_1^{1-p} = A_p$, which implies that $w \in A_p$ since $kA_p = A_p$ for any such k.

Conversely if $w \in A_p$, the previous corollary implies that $w = w_o w_1^{1-p}$ with $w_o, w_1 \in A_1$. Then we just need to apply theorem 2.2.8. to obtain the desired representation:

$$w(x) = k_o(x)(Mf(x))^a k_1(x)^{(1-p)}(Mg(x))^{\beta(1-p)} =$$
$$= k_o(x)k_1(x)^{(1-p)}(Mf(x))^a(Mg(x))^{\beta(1-p)} :=$$
$$:= k(x)(Mf(x))^a(Mg(x))^{\beta(1-p)}.$$

Observe that, in the proof of theorem 2.2.8., the lower bound obtained for the function k depends only upon the constant C in the reverse Hölder's inequality for the A_1 weight, and this, in turn, depends only upon its A_1 constant. The upper bound obtained for k in the proof of theorem 2.2.8. is just 1. In our present situation, the factorization theorem tells us that the A_1 constants for w_o and w_1 depend only upon the A_p constant for w. Therefore in our representation for the A_p weight w, the function $k = k_o k_1^{1-p}$ is bounded away from zero and ∞ with bounds depending only upon the A_p constant for w.

(2):($\Longleftarrow$)We have $f, g \in L^1$ which implies that $Mf(x)$ and $Mg(x)$ are $< \infty$ a.e. Then, according to corollary in chapter I, $\log Mf(x)$ and $\log Mg(x)$ are both in $B.M.O.$ with norms independent of f and g respectively. Consequently, if ϕ has the representation exhibited in 2), we have $\phi \in B.M.O.$ with

$$\|\phi\|_* \leq C_2(\|b\|_\infty + \gamma + n)$$

for some absolute constant C_2. Indeed, since $b(x)^\# \leq 2Mb(x) \leq 2\|b\|_\infty$, we get that:

$$\|\phi\|_* \leq \|b\|_* + \gamma\|\log Mf\|_* + n\|\log Mg\|_* \leq$$
$$\leq 2\|b\|_\infty + \gamma C' + nC'' \leq C_2(\|b\|_\infty + \gamma + n)$$

where $C_2 = max\{2, C', C''\}$

($\Longrightarrow$) Conversely, if $\phi \in B.M.O.$, it follows from corollary 1.3.1. that, taking $\lambda = C_2/2\|\phi\|_*$, where C_2 is the constant appearing in corollary 1.3.1., and using the proof of the second part of the same corollary, we get that:

$$\frac{1}{|Q|}\int_Q e^{\lambda|\phi(x)-\phi_Q|}dx \leq C_1\lambda(C_2/\|\phi\|_* - \lambda)^{-1} = C_1$$

where C_1 is again, the one appearing in the end of the proof of corollary 1.3.1., which in turn, is the same one appearing in theorem 1.3.5.(from where we can see that C_1 depends only on the dimension n). Consequently, from corollary 2.2.3., we get that the function $w(x) = e^{\lambda\phi(x)}$ is in A_2 with an A_2 constant $(= C_1^2)$ independent of ϕ. Applying part 1) to our w, we obtain:

$$\log w(x) = \log k(x) + a\log(Mf(x)) - \beta(p-1)\log(Mg(x)) \Longrightarrow$$

$$\phi(x) = \lambda^{-1}\log k(x) + \lambda^{-1}a\log(Mf(x)) - \lambda^{-1}\beta(p-1)\log(Mg(x))$$

and we get the desired decomposition with

$$b = \lambda^{-1}\log k,\ \ \gamma = \lambda^{-1}a,\ \ h = \lambda^{-1}\beta(p-1) = \lambda^{-1}\beta \quad since\ \ p = 2$$

Observe that the L^∞ norm of $\log k$ does not depend on ϕ. Then, since $\lambda^{-1} = C^{-1}\|\phi\|_*$ $(C := C_2/2)$ and $0 \leq a < 1$, $0 \leq \beta < 1$, we have

$$\|b\|_\infty + \gamma + h = \|\phi\|_*(\|\log k\|_\infty + a + \beta(p-1))C^{-1} \leq$$

$$\leq C'\|\phi\|_*$$

for some constant C' since, as we said before, the L^∞ norm of k is independent of ϕ.

3) As we observed in the proof of theorem 1.3.1., $\log Mf(x)$ is actually in $B.L.O.$, so that any $\phi = b + \gamma\log Mf$ with $\|b\|_\infty < \infty$ and γ a real number ≥ 0, will also belong to $B.L.O.$

For the converse, the proof is very much like the one in part 2). The difference is that, as we noted in the second remark following corollary 2.2.4., if $\phi \in B.L.O.$, the weight $w(x) = e^{\lambda\phi(x)}$ is actually in A_1, not in A_2. Then we can use part 1) as before, but now $p = 1$. so that we obtain the representation with $h = 0$.

Finally 4) follows obviously from 2) and 3). □

2.4 A SHARP L^P INEQUALITY FOR DYADIC A_1 WEIGHTS IN $\mathbb{R}^n$

Lets remind some definitions. A locally non-negative function w on $\mathbb{R}^n$ is called a dyadic A_1 weight if it satisfies the condition

$$\frac{1}{|Q|}\int_Q w(x)dx \leq C essinf_{x\in Q} w(x)$$

for any dyadic cube Q in $\mathbb{R}^n$, which is equivalent to the inequality

$$M_d w(x) \leq C w(x)$$

for almost every $x \in \mathbb{R}^n$. Here M_d is the dyadic maximal operator defined by

$$M_d w(x) = sup\{\frac{1}{|Q|}\int_Q w(x)dx : x \in Q, Q \subseteq \mathbb{R}^n \ \ is \ \ a \ \ dyadic \ \ cube\}.$$

The smallest $C \geq 1$ for which the above inequalities hold is called the dyadic A_1 constant of w and is denoted by $[w]_1$.

It is well known that such weights satisfy reverse Hölder inequalities for certain real numbers p greater that 1 depending on the dimension n and the A_1 constant $[w]_1$. The purpose of this section is to determine the exact best possible range of p for which the reverse Hölder inequalities hold. Our main result is the following.

Theorem 2.4.1. *Let w be a dyadic A_1 weight on $\mathbb{R}^n$. Then for every p such that*

$$1 \leq p < \frac{log(2^n)}{log\left(2^n - \frac{2^n-1}{[w]_1}\right)} = p(n,[w]_1) \qquad (a)$$

and for every dyadic cube Q we have

$$\frac{1}{|Q|}\int_Q (M_d w(x))^p dx \leq \frac{2^n-1}{\left(2^n - \frac{2^n-1}{[w]_1}\right)^p - 2^n}\left(\frac{1}{|Q|}\int_Q w(x)dx\right)^p \qquad (b)$$

Moreover both the range of p and the corresponding constants in (b) are best possible.

Clearly for such weights the inequality (b) is equivalent to a reverse Hölder inequality for w (with different sharp constant) so it gives the best possible range of p for such an equality to hold. Note that for any fixed n we have $p(n,\lambda) \to \infty$ as $\lambda \to 1^+$ as expected. Moreover for fixed $\lambda > 1$ we have $p(n,\lambda) \to 1$ as $n \to \infty$ which implies that the range of p shrinks to $\{1\}$ as the dimension increases. In proving that the range is best possible we will produce for any $\lambda > 1$ a dyadic A_1 weight w on $[0,1]^n$ such that $[w]_1 = \lambda$ and $\int_{[0,1]^n} w(x)^{p(n,\lambda)} dx = \infty$.

We remark that by using a standard dilation and approximation argument it suffices to prove (b) for $Q = [0,1]^n$ and for all functions w defined only on $[0,1]^n$ and satisfying the A_1 condition only for dyadic cubes contained in $[0,1]^n$. Actually we will work on more general non-atomic probability spaces (X,μ) equipped with a structure T similar to the dyadic one.

The analogous question of finding the best range of good p for the full A_1 condition,that is, for w satisfying

$$\frac{1}{|Q|}\int_Q w(x)dx \le Cessinf_{x\in Q}w(x)$$

for all cubes, has been studied for dimension $n = 1$ and it was proved that in this case the best possible range of p is $1 \le p < [w]_1/([w]_1 - 1)$ where $[w]_1$ denotes the corresponding full A_1 constant. It is easy to see that $p(1,\lambda) < \lambda/(\lambda-1)$ for any $\lambda > 1$ and this reflects the fact that the dyadic A_1 condition is much weaker than the full one.

Lets start now by giving the precise structure of the family T we will work on: We fix a non atomic probability space (X,μ) and a positive integer $k \ge 2$. We also suppose that we are given a family T of measurable subsets of X satisfying the following properties

1. For every $I \in T$ there corresponds a subset $C(I) \subset T$ containing exactly k pairwise disjoint subsets of I such that

$$I = \cup C(I)$$

   and each element of $C(I)$ has measure $(1/k)\mu(I)$.

2. $T = \bigcup_{m\ge 0} T_{(m)}$ where $T_0 = \{X\}$ and $T_{(m+1)} = \bigcup_{I\in T_{(m)}} C(I)$.

<u>*EXAMPLE*</u> : If Q_o is the unit cube in $\mathbb{R}^n$ we let E be the union of all the boundaries of all dyadic cubes in Q_o. Let $X = Q_o \backslash E$ and let T be the set of

all open dyadic cubes $Q \subset Q_o$. Here $k = 2^n$ and each $C(Q)$ is the set of 2^n subcubes of Q obtained by bisecting its sides. More generally for any integer $m > 1$ we may consider all m-adic cubes $Q \subset Q_o$ with $C(Q)$ being the set of the m^n open subcubes of Q obtained by dividing each side of it into m equal parts.

It is clear that each $T(m)$ consists of k^m pairwise disjoint sets each having measure k^{-m} whose union is X; moreover, if $I, J \in T$ and $I \cap J$ is non empty then $I \subset J$ or $J \subset I$.

For this family T we define the corresponding maximal operator M_T as

$$M_T(f)(x) = sup\left\{\frac{1}{\mu(I)}\int_I |f|d\mu : x \in I \in T\right\} \qquad (4.1)$$

for any $f \in L^1(X, \mu)$ and we will say that a non negative integrable function w is an A_1 weight with respect to T if

$$M_T(w)(x) \leq Cw(x) \qquad (4.2)$$

for almost every $x \in X$. The smallest constant C for which (4.2) holds will be called the A_1 constant of w and will be denoted by $[w]_1$.

Now we will describe an effective linearization for the operator M_T valid for certain good functions w. This will be important for proving the theorem 4.1. Let w be a positive non-constant T-step function; that is, there exist an integer $m > 0$ and positive λ_P for each $P \in T(m)$ such that

$$w = \sum_{P \in T(m)} \lambda_P X_P \qquad (4.3)$$

(where X_P denotes the characteristic function of P). It is clear that w is an A_1 weight (with respect to T) since, for each $I \in T$ we have

$$\frac{1}{\mu(I)}\int_I |w|d\mu = \frac{1}{\mu(I)}\sum_{P \in T(m), P \subset I} \lambda_P \mu(P) \leq$$

$$\leq \max_{P \in T(m)} \lambda_P \leq \frac{\max_{P \in T(m)} \lambda_P}{\min_{P \in T(m)} \lambda_P} \cdot w := Cw \Rightarrow$$

$$\Rightarrow M_T(w) \leq Cw$$

Let $\delta = 1/[w]_1$, $0 < \delta < 1$ and for any $I \in T$ write

$$Av_I(w) = \frac{1}{\mu(I)}\int_I w d\mu.$$

Now for every $x \in X$ let $I_w(x)$ be the largest element of the set

$$\{I \in T : x \in I, M_T w(x) = Av_I(w)\}$$

(which is non-empty since $Av_J(w) = Av_P(w)$ whenever $P \in T(m)$ and $J \subset P$). Next for any $I \in T$ we define the set

$$A_I = A(w, I) = \{x \in X : I_w(x) = I\}$$

and we let $S = S_w$ be the set of all $I \in T$ such that A_I is non-empty.

Let $x \in A_I$, then $M_T w(x) = Av_I(w)$ and $x \in P$ for some $P \in T(m)$ with $P \subset I$. Now for any other $y \in P$ with $y \neq x$ we get that

$$M_T w(y) = Av_{I_w(y)}$$

but $I_w(y) \supset P$ for every such y, which implies (since $x \in P$) that: $M_T w(x) \geq M_T w(y)$. On the other hand each $y \in P$ belongs also in I, which implies in turn, that $M_T w(x) = Av_I(w) \leq M_T w(y)$ for every such y. Consequently, we get that:

$$M_T w(x) = Av_I(w) = M_T w(y)$$

for every $y \in P$, and since $x \in A_I$ we get that $I_w(y) = I_w(x) = I \Rightarrow P \subset A_I$. It is now clear that each A_I is a union of certain P from $T(m)$.

It is also clear that each $x \in X$ belongs also in $A_{I_w(x)}$ and that is because

$$M_T w(x) = Av_{I_w(x)}(w)$$

for every x. Thus, we can conclude that

$$X = \bigcup_{I \in S = S_w} A_I.$$

Now if there is $x \in A_I \cap A_J$ for some $I, J \in S$, then

$$M_T w(x) = Av_I(w) = Av_J(w)$$

and since I, J are the biggest elements for which the average of w on each of them respectively is equal to $M_T w(x)$, we get that $I = J$. Thus for $I, J \in S$ with $I \neq J$, we get that A_I and A_J are disjoint. We can write now $M_T w$ in the following form :

$$M_T w = \sum_{I \in S} Av_I(w) X_{A_I}$$

We also define the correspondence $I \to I^*$ with respect to S as follows: I^* is the smallest element of $\{J \in S_w : I \subsetneq J\}$. This is defined for every I in S that is not maximal with respect to $\subseteq$.

The main properties of these sets are given in the following two lemmas which can be viewed as a version of Calderon-Zygmund decomposition in a more general setting

Lemma 2.5. *1. For every $I \in S$ we have*

$$I = \bigcup_{S \ni J \subseteq I} A_J$$

2. For every $I \in S$ we have

$$A_I = I \setminus \cup_{J \in S : J^* = I} J$$

and so

$$\mu(A_I) = \mu(I) - \sum_{J \in S : J^* = I} \mu(J). \qquad (5.1)$$

3. For all $I \in T$ we have $I \in S$ if and only if $Av_Q(w) < Av_I(w)$ whenever $I \subset Q \in T$, $I \neq Q$. In particular $X \in S$ and so $I \to I^*$ is defined for all $I \in S$ such that $I \neq X$.

Proof. (1) clearly we have

$$\cup_{S \ni J \subset I} A_J \subset I$$

Let now $x \in I$. Since $I \in S$ we have that $A_I \neq \varnothing$, so there will be $y \in X$ such that $I = I_w(y)$ which means that $M_T w(y) = Av_I(w)$

Suppose now that $I_w(x) \neq J$ for each $J \subset I$ (which is equivalent to $x \notin \cup_{S \ni J \subset I} A_J$), then, it will be $I_w(x) = I'$ for some $I' \in S$ with $I \subsetneq I'$, but $y \in I'$ since $I_w(y) = I$, thus, we get that

$$Av_{I'}(w) \leq M_T w(y) = Av_I(w)$$

also $x \in I$ which implies in turn that

$$Av_I(w) \leq M_T w(x) = Av_{I'}(w).$$

Consequently we get that $Av_I(w) = Av_{I'}(w) = M_T w(x)$ with $I' \supsetneq I$, thus, $I \notin S$ which is contradiction to our assumption.

(2) Let $x \in A_I$, then $I_w(x) = I$ which implies that $x \notin \cup_{S \ni J \subsetneq I} A_J$. Thus,

$$x \notin \cup_{S \ni J' \subset J} A_{J'}$$

for each J such that $J^* = I$ which is equivalent (using (1)) to $x \notin J$ for each J such that $J^* = I$, consequently:

$$x \in I \setminus \cup_{S \ni J : J^* = I} J$$

For the opposite direction let $x \in I \setminus \cup_{S \ni J : J^* = I} J$, then it will clearly be

$$I_w(x) \supseteq I$$

but $I = I_w(y)$ for some y (since $I \in S$), thus ($y \in I \subseteq I_w(x)$), we get that

$$M_T w(y) = Av_I(w) \geq Av_{I_w(x)}(w) = M_T w(x)$$

we also have (since $x \in I$) that

$$M_T w(y) = Av_I(w) \leq Av_{I_w(x)}(w) = M_T w(x)$$

and since $I \in S$ we get that $I_w(x) = I$ which implies that $x \in A_I$

(3) ($\Longrightarrow$) Let $I \in S$ then $I_w(x) = I$ for some $x \in X$. Let also $Q \in T$ such that $I \subsetneqq Q$. Then, since $x \in I \subset Q$, we get that

$$Av_Q(w) < M_T w(x) = Av_{I_w(x)}(w) = Av_I(w)$$

actually it is $Av_Q(w) < M_T w(x)$ (otherwise it would be $I \supset Q$ since $I_w(x) = I$, which is not valid)

($\Longleftarrow$) Suppose now that $Av_Q(w) < Av_I(w)$ whenever $I \subsetneqq Q$.

Clearly $I \in T(m-k)$ where $k \geq 0$ because if we had $I \in T(m+k')$ for some $k' > 0$ then $Av_Q(w) = Av_I(w)$ for every $Q \supsetneqq I$ with $Q \in T(m+k'-1)$ which is contradiction.

Now, since every $Av_J(w)$ can be written in the form:

$$Av_J(w) = \frac{\sum_{F \in C(J)} \mu(F) Av_F(w)}{\sum_{F \in C(J)} \mu(F)}$$

we conclude that for each $J \in T$ there exists $F \in C(J)$ such that $Av_F(w) \leq Av_J(w)$. Starting from I and applying the above k times, we get a chain $I_O = I \supseteq I_1 \supseteq .... \supseteq I_k$ such that $I_r \in T(m-k+r)$ for each r and moreover

$$Av_{I_k}(w) \leq Av_{I_{k+1}}(w) \leq ... \leq Av_{I_o}(w) = Av_I(w).$$

Now from this and the assumption on I and also from the fact that for every $J \in T(n)$ there is a unique $J' \in T(n-1)$ such that $J' \supset J$, it is clear that $I_w(x) = I$ for every $x \in I_k$ and therefore $I \in S$.

□

Next we write $y_I = Av_I(w)$ for every $I \in S$ and with $\delta = 1/[w]_1$ we have the following

Lemma 2.6. *Let $I \in S$. Then:*

1. *If $J \in S$ is such that $J^* = I$ then*

$$y_I < y_J \leq (k-(k-1)\delta)y_I \qquad (6.1)$$

2. *we have*

$$\sum_{J \in S: J^*=I} y_J \mu(J) \leq \left((1-\delta)\mu(I) + \delta \sum_{J \in S: J^*=I} \mu(J) \right) y_I. \qquad (6.2)$$

Proof. (1) The inequality $y_I < y_J$ follows from the third result of lemma 2.5. Now consider the unique $F \in T$ such that $J \in C(F)$. Clearly (since $J^* = I$) $J \subsetneqq F \subseteq I$. We claim that:

$$Av_F(w) \leq y_I = Av_I(w)$$

Indeed, if $F = I$ then $Av_F(w) = Av_I(w)$. Let now F be $\subsetneqq I$. Of course $I \in T(s)$ for some s and $F \in T(s+m)$. If $m = 1$ (that is $F \in T(s+1)$) and if we had

$$Av_F(w) > y_I \qquad (I)$$

Let $Q \in T$ such that $F \subset Q$ and $F \neq Q$. Since $F \in T(s+1)$ and $F \subset I \in T(s)$, we get that

$$I \subseteq Q$$

If $Q = I$, then using (I) we get :

$$Av_F(w) > Av_Q(w) \qquad (i)$$

If $Q \neq I$ then $I \subsetneqq Q$ and since $I \in S$ we get that:

$$Av_Q(w) < y_I \qquad (ii)$$

and

$$Av_F(w) > Av_Q(w) \qquad (iii)$$

For (ii) we used the previous lemma and for (iii) if there was $Av_Q(w) \geq Av_F(w)$, then, combining (ii) and (I) we lead ourselves in contradiction. Therefore we get that

$$Av_F(w) > Av_Q(w)$$

whenever $Q \supseteq F$ with $F \neq Q$ and according to the previous lemma, this implies that $F \in S$ which is again contradiction to our assumption $J^* = I$ (because $J \subsetneqq F \subsetneqq I$). Thus, in case $m = 1$ we get that

$$Av_F(w) \leq y_I$$

which is what we want.

If now $F \in T(s+m)$ with $m > 1$, then, suppose again that

$$Av_F(w) > y_I \qquad (II)$$

There will be unique F_i such that $F_o = F \subset F_1 \subset F_2 \subset ... \subset F_{m-1} \subset I$ where $F_i \in T(s+m-i)$ for each $i \in \{0, 1, 2, ..., m-1\}$. Consider now

$$I_M = max\{i : Av_{F_i}(w) > y_I\}$$

(I_M is well defined since we have assumed (II)).

Let now Q in T such that $F_{I_M} \subset Q$ with $Q \neq F_{I_M}$.

If $Q \in \{F_{I_M+1}, ..., F_{m-1}\}$, then, using the definition of I_M, we get that

$$Av_{F_{I_M}} > y_I \geq Av_Q(w).$$

If $Q = I$ then using again the definition of I_M, we get that

$$Av_{F_{I_M}} > y_Q = y_I.$$

If $Q \supsetneqq I$, then, using the fact that $I \in S$ and the definition of I_M, we get that

$$Av_{F_{I_M}} > y_I > Av_Q(w).$$

Consequently $F_{I_M} \in S$ which is not valid (contradiction) since $J \subsetneqq F \subseteq F_{I_M} \subsetneqq I$ and $J^* = I$. Thus, our claim is justified. Let us note that the case $m > 1$ covers the case $m = 1$ but the case $m = 1$ is the first and easier thought that someone does in order to prove this claim.

Now note that for every $x \in F \backslash J \subset I$ we have

$$[w]_1 w(x) \geq M_T w(x) \geq y_I \Rightarrow$$

$$\Rightarrow w(x) \geq y_I/[w]_1 \Rightarrow y_{F\backslash J} \geq \frac{y_I}{[w]_1}$$

hence using the claim we get

$$y_I \geq Av_F(w) = \frac{\mu(J)}{\mu(F)} y_J + \frac{\mu(F\backslash J)}{\mu(F)} y_{F\backslash J} \geq$$

$$\frac{\mu(J)}{\mu(F)} y_J + \frac{\mu(F\backslash J)}{\mu(F)} \frac{y_I}{[w]_1} =$$

$$\frac{\mu(J)}{\mu(F)} y_J + \frac{\mu(F) - \mu(J)}{\mu(F)} \frac{y_I}{[w]_1} =$$

$$= \frac{1}{k} y_J + (\delta - \frac{1}{k}\delta) y_I$$

which implies that

$$\frac{1}{k} y_J \leq \frac{k - (k-1)\delta}{k} y_I$$

and this proves (1).

(2) Note that for every $x \in A_I$ we have

$$[w]_1 w(x) \geq M_T w(x) = y_I$$

hence, integrating this over A_I we get :

$$\int_{A_I} [w]_1 w(x) d\mu(x) \geq \int_{A_I} y_I d\mu(x) = y_I \mu(A_I) =$$

we use lemma 5 (ii)

$$= \left(\mu(I) - \sum_{J \in S: J^* = I} \mu(J) \right) y_I \Longrightarrow$$

$$\int_{A_I} w(x)d\mu(x) \geq \delta \left(\mu(I) - \sum_{J \in S : J^* = I} \mu(J) \right) y_I$$

But

$$A_I = I \setminus \cup_{J \in S : J^* = I} J$$

So

$$\int_I w d\mu - \sum_{J \in S : J^* = I} \int_J w d\mu \geq \delta \left(\mu(I) - \sum_{J \in S : J^* = I} \mu(J) \right) y_I \Longrightarrow$$

$$\mu(I) y_I - \sum_{J \in S : J^* = I} \mu(J) y_J \geq \delta y_I \left(\mu(I) - \sum_{J \in S : J^* = I} \mu(J) \right) \Longrightarrow$$

$$\sum_{J \in S : J^* = I} \mu(J) y_J \leq \left(\mu(I)(1 - \delta) + \delta \sum_{J \in S : J^* = I} \mu(J) \right) y_I$$

and the proof is complete.

□

Then defining the function

$$P_k(\lambda) = \frac{logk}{log(k - (k-1)\lambda)} > 1$$

for $0 < \lambda < 1$, we have the following.

Lemma 2.7. *Let w be a T-step function as above. Then*

$$\int_X (M_T w)^p d\mu \leq \frac{k-1}{k - (k - (k-1)\delta)^p} \left(\int_X w d\mu \right)^p$$

whenever $1 \leq p < P_k(\delta)$.

Proof. Fix $p > 1$ and use the previous lemma and the convexity of the function $F(t) = t^p$ to get

$$\frac{y_J^p - y_I^p}{y_J - y_I} \leq \frac{((k - (k-1)\delta) y_I)^p - y_I^p}{(k - (k-1)\delta) y_I - y_I} \Longrightarrow$$

$$y_J^p - y_I^p \leq \frac{(k - (k-1)\delta)^p - 1}{(k-1)(1-\delta)} (y_J - y_I) y_I^{p-1} \qquad (7.1)$$

whenever $I, J \in S$ are such that $J^* = I$.

Now, using (5.1) on (6.2) we get

$$\sum_{J\in S:J^*=I} y_J\mu(J) \leq \left\{(1-\delta)\left(\mu(A_I) + \sum_{J\in S:J^*=I}\mu(J)\right) + \delta\sum_{J\in S:J^*=I}\mu(J)\right\} y_I \Longrightarrow$$

$$\sum_{J\in S:J^*=I}(y_J - y_I)\mu(J) \leq (1-\delta)\mu(A_I)y_I. \qquad (7.2)$$

Multiplying (7.1) by $\mu(J)$ and, with I fixed, adding for all J with $J^* = I$ we get using (7.2) that :

$$\sum_{J\in S:J^*=I}(y_I^p - y_J^p)\mu(J) \leq \frac{(k-(k-1)\delta)^p - 1}{k-1}\mu(A_I)y_I^p \qquad (7.3)$$

for every $I \in S$ that is not minimal with respect to $\subseteq$ (otherwise we do not sum anything)

Let us before we continue, remind that

$$M_T w = \sum_{I\in S} y_I X_{A_I} = \sum_{I\in S} Av_I(w)X_{A_I}$$

so that

$$(M_T w)^p = \sum_{I\in S} y_I^p X_{A_I}$$

and therefore

$$\int_X (M_T w)^p d\mu = \sum_{I\in S} y_I^p \mu(A_I)$$

Next we sum all the inequalities (7.3) for all $I \in S'$ where S' consists of all elements of S that are not minimal. On the right hand side we have the estimate

$$\sum_{I\in S'} \mu(A_I)y_I^p \leq \int_X (M_T w)^p d\mu \qquad (7.4)$$

On the other hand , using that

$$\mu(A_I) = \mu(I) - \sum_{J\in S:J^*=I}\mu(J)$$

and the fact that X is the only $I \in S$ for which I^* is not defined, we have

$$\sum_{I\in S'} \sum_{J\in S: J^*=I} (y_J^p - y_I^p)\mu(J) =$$

$$\sum_{I\in S'} \sum_{J\in S: J^*=I} y_J^p \mu(J) - \sum_{I\in S'} \sum_{J\in S: J^*=I} y_I^p \mu(J) =$$

$$\sum_{I\in S, I\neq X} y_I^p \mu(I) - \sum_{I\in S'} y_I^p \sum_{J\in S: J^*=I} \mu(J) =$$

$$\sum_{I\in S, I\neq X} y_I^p \mu(I) - \sum_{I\in S'} y_I^p (\mu(I) - \mu(A_I)) =$$

($\mu(I) = \mu(A_I)$ for I minimal)

$$= \sum_{I\in S, I\neq X} y_I^p \mu(I) - \sum_{I\in S} y_I^p (\mu(I) - \mu(A_I)) =$$

$$\sum_{I\in S, I\neq X} y_I^p \mu(I) - \sum_{I\in S} y_I^p \mu(I) + \sum_{I\in S} y_I^p \mu(A_I) =$$

$$\sum_{I\in S} y_I^p \mu(A_I) - y_X^p =$$

$$= \int_X (M_T w)^p d\mu - \left(\int_X w d\mu \right)^p .$$

Hence, assuming that $1 < p < P_k(\delta)$ which gives $(k-(k-1)\delta)^p < k$ and consequently $(k-(k-1)\delta)^p - 1 < k-1$ and since $\int_X (M_T w)^p d\mu$ is obviously finite, we get

$$\int_X (M_T w)^p d\mu = \left(\int_X w d\mu \right)^p + \sum_{I\in S'} \sum_{J\in S: J^*=I} (y_J^p - y_I^p)\mu(J) \leq$$

use (7.3)

$$\leq \left(\int_X w d\mu \right)^p + \sum_{I\in S'} \frac{(k-(k-1)\delta)^p - 1}{k-1} \mu(A_I) y_I^p =$$

$$= \left(\int_X w d\mu \right)^p + \frac{(k-(k-1)\delta)^p - 1}{k-1} \sum_{I\in S'} \mu(A_I) y_I^p \leq$$

use (7.4)

$$\leq \left(\int_X w d\mu \right)^p + \frac{(k-(k-1)\delta)^p - 1}{k-1} \int_X (M_T w)^p d\mu$$

the above implies that

$$\int_X (M_T w)^p d\mu \leq \frac{k-1}{k-(k-(k-1)\delta)^p} \left(\int_X w d\mu \right)^p$$

which is what we want.

□

Next we show that the previous result holds for general w and that it is actually best possible.

Theorem 2.4.2. *For any A_1 weight (with respect to T) w and any p such that $1 \leq p < P_k(1/[w]_1)$ we have*

$$\int_X (M_T w)^p d\mu \leq \frac{k-1}{k-(k-(k-1)\delta)^p} \left(\int_X w d\mu \right)^p \tag{7.5}$$

and both the range of p and the constant in (7.5) are sharp (best possible).

Proof. For the general non-negative A_1 weight w we consider the sequence (w_n) where

$$w_m = \sum_{P \in T(m)} Av_P(w) X_P$$

and set

$$\phi_m = \sum_{P \in T(m)} max\{Av_I(w) : P \subseteq I \in T\} X_P = M_T w_m$$

(since $Av_I(w) = Av_I(w_m)$ whenever $P \in T(m)$ and $P \subseteq I \in T$)

Then

$$\int_X w_m d\mu = \int_X w d\mu$$

for all m and ϕ_m converges monotonically to $M_T w$. Since each w_m is a positive T- step function, from the previous lemma we get that:

$$\int_X \phi_m^p d\mu \leq \frac{k-1}{k-(k-(k-1)\delta)^p} \left(\int_X w d\mu \right)^p$$

and so letting $m \to \infty$ we get (7.5) for the general w.

Now to complete the proof of the theorem we choose an infinite chain $X =$

$I_o \supseteq I_1 \supseteq ... \supseteq I_s \supseteq I_{s+1} \supseteq$ ...such that $I_s \in T(s)$ for all $s \geq 0$ (and so $\mu(I_s) = k^{-s}$) and for $\gamma > 1$ consider the function

$$w = \sum_{s=0}^{\infty} \gamma^s X_{I_s \backslash I_{s+1}} \qquad (7.6)$$

Then it is easy to see that for all $s \geq 0$

$$Av_{I_s}(w) = \frac{k-1}{k-\gamma}\gamma^s \qquad (7.7)$$

provided $\gamma < k$. Indeed:

$$Av_{I_s}(w) = \frac{1}{\mu(I_s)} \int_{I_s} w d\mu =$$

$$\frac{1}{\mu(I_s)} \sum_{r \geq s} \gamma^r \mu(I_r \backslash I_{r+1}) = k^s \sum_{r \geq s} \gamma^r \left(\frac{1}{k^r} - \frac{1}{k^{r+1}} \right) = k^s \sum_{r \geq s} \gamma^r \frac{k-1}{k^{r+1}} =$$

$$= k^{s-1}(k-1) \sum_{r \geq s} \left(\frac{\gamma}{k}\right)^r = k^{s-1}(k-1) \left(\sum_{r \geq 0} \left(\frac{\gamma}{k}\right)^r - \sum_{r=0}^{s-1} \left(\frac{\gamma}{k}\right)^r \right) =$$

$$= k^{s-1}(k-1) \left(\frac{1}{1-\left(\frac{\gamma}{k}\right)} - \frac{\left(\frac{\gamma}{k}\right)^s - 1}{\frac{\gamma}{k} - 1} \right) =$$

$$= \frac{k^{s-1}(k-1)\left(\frac{\gamma}{k}\right)^s}{1-\frac{\gamma}{k}} = \frac{k-1}{k-\gamma}\gamma^s.$$

We next claim that

$$M_T w(x) = Av_{I_s}(w)$$

whenever $x \in I_s \backslash I_{s+1}$ and $s \geq 0$. Indeed suppose that $x \in I_s \backslash I_{s+1}$ and let J be the unique element of $T(s+1)$ such that $x \in J$ (clearly $J \in C(I_s)$ and $J \neq I_s$). Then the set of all I in T containing x consists of $I_o, I_1, ..., I_s$ and J and certain subintervals of it (of J), but since $\gamma > 1$, (7.7) implies that $Av_{I_s}(w) > Av_{I_r}(w)$ for all $0 \leq r < s$ and since w is constant on J (and every sub interval of it) equal to $\gamma^s < \frac{k-1}{k-\gamma}\gamma^s = Av_{I_s}(w)$, we get

$$M_T w(x) = Av_{I_s}(w)$$

for every $x \in I_s \backslash I_{s+1}$. This combined with (7.6) and (7.7) implies that

$$M_T w(x) = \frac{k-1}{k-\gamma} w(x)$$

so that w is an A_1 weight with $[w]_1 = \frac{k-1}{k-\gamma}$ and so

$$\gamma = k - \frac{k-1}{[w]_1} \qquad (7.8)$$

Now for any $p > 1$ we have

$$\int_X (M_T w)^p d\mu = \left(\frac{k-1}{k-\gamma}\right)^p \int_X w^p d\mu =$$

$$= \left(\frac{k-1}{k-\gamma}\right)^p \sum_{s=0}^{\infty} \gamma^{sp}(\mu(I_s) - \mu(I_{s+1})) = \left(\frac{k-1}{k-\gamma}\right)^p \sum_{s=0}^{\infty} \gamma^{sp} \left(\frac{1}{k^s} - \frac{1}{k^{s+1}}\right) =$$

$$\left(\frac{k-1}{k-\gamma}\right)^p \left(\sum_{s=0}^{\infty} \left(\frac{\gamma^p}{k}\right)^s - \frac{1}{k}\sum_{s=0}^{\infty} \left(\frac{\gamma^p}{k}\right)^s\right) =$$

$$\left(\frac{k-1}{k-\gamma}\right)^p \frac{k-1}{k} \sum_{s=0}^{\infty} \left(\frac{\gamma^p}{k}\right)^s =$$

$$\left(\frac{k-1}{k-\gamma}\right)^p \frac{k-1}{k} \frac{k}{k-\gamma^p} = \frac{k-1}{k-\gamma^p} (Av_{I_o}(w))^p \Longrightarrow$$

$$\int_X (M_T w)^p d\mu = \frac{k-1}{k-(k-(k-1)\delta)^p} \left(\int_X w d\mu\right)^p$$

and it is finite if and only if $\gamma^p < k$

The above gives us the sharpness and the proof is complete.

□

Now theorem 2.4.2. applied to the special case of dyadic cubes given in example before and combined with standard dilation and approximation arguments completes the proof of theorem 2.4.1.

CHAPTER 3

APPENDIX

Here we will not present that much of info, just a couple of lemmas that will help us solve an exercise in measure theory that we used somewhere in the previous chapters. To be precise we will show that if we have a finite positive Borel measure μ on a space X with $\mu(X) = 1$ and a function f for which $\|f\|_q < \infty$ for at least one $q > 0$ then :

$$\|f\|_p \longrightarrow exp\left(\int_X log|f(x)|d\mu(x)\right)$$

as p tends to zero.

Lemma 3.1. *If* $0 < r < s(< 1)$, *then*

$$\|f\|_r \leq \|f\|_s$$

which implies of course that $L^s(X) \subset L^r(X)$

Proof. Since the function $\phi(x) = x^{s/r}$ is convex, we can apply Jensen's inequality to $\int_X |f|^r d\mu$ to get

$$\left\{\int_X |f|^r d\mu\right\}^{s/r} \leq \int_X |f|^s d\mu.$$

Hence $\|f\|_r \leq \|f\|_s$. □

Lemma 3.2. *If* $0 < p < 1$, *then*

$$\int_X log|f|d\mu \leq log\|f\|_p \qquad (I)$$

Proof. We know that log(x) is a concave function so we can use again Jensen's inequality to $\int_X |f|^p d\mu$ to obtain

$$log\left(\int_X |f|^p d\mu\right) \geq p\int_X log|f|d\mu$$

which is what we want. □

From lemmas 8 and 9, it follows that the sequence $\log \|f\|_{1/n}$ is decreasing and bounded from below. Therefore, it converges as $n \to \infty$.

To find the limit, use the inequality $\log x \leq x - 1$ or equivalently the inequality $\log a \leq n(a^{1/n} - 1)$ with $a = \left(\int_X |f|^{1/n} d\mu\right)^n$ to get (since $\mu(X) = 1$) :

$$\log \|f\|_{1/n} \leq \int_X \frac{|f|^{1/n} - 1}{1/n} d\mu$$

The sequence $a_n = \frac{|f|^{1/n}-1}{1/n}$ is increasing. Thus, we can apply the the monotone convergence theorem in the integral above to get

$$\lim_{n\to\infty} \log \|f\|_{1/n} \leq \int_X \lim_{n\to\infty} \frac{|f|^{1/n} - 1}{1/n} d\mu$$

$$= \int_X \log |f| d\mu \qquad (II)$$

since

$$\lim_{n\to\infty} \frac{|f|^{1/n} - 1}{1/n} = \log |f|$$

From (I) and (II) we get that

$$\lim_{n\to\infty} \log \|f\|_{1/n} = \int_X \log |f| d\mu$$

and since the logarithm is continuous function we get

$$\lim_{p\to 0} \|f\|_p = exp\left(\int_X log|f(x)|d\mu(x)\right)$$

Proposition 3.0.1. *For $f \in L^1$, Mf is not bounded in L^1. Actually Mf is never integrable in L^1 unless f is almost everywhere equal to zero.*

Proof. Suppose $f \in L^1$ with $|f| > 0$ in a set of a positive measure, then, we choose cube $Q' = [-\|y\|, \|y\|] \times ... \times [-\|y\|, \|y\|]$ for some fixed $y \in \mathbb{R}^n$ such that:

$$0 < C = \int_{Q'} |f(x)|dx < \infty.$$

Consequently for every $x \in \mathbb{R}^n$ with $\|x\| > \|y\| := M$, and for

$$Q = [-\|x\|, \|x\|] \times ... \times [-\|x\|, \|x\|]$$

we get that:

$$Mf(x) \geq \frac{1}{|Q|} \int_Q |f(z)|dz = \frac{1}{2^n \|x\|^n} \int_Q |f(z)|dz \geq \frac{C'}{\|x\|^n}$$

where $C' = \frac{C}{2^n}$, and we know that

$$\int_{\{x:\|x\|>M\}} \frac{1}{\|x\|^n} dx = \infty.$$

□

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