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Small-Sample size corrections of the t and F tests in some econometric specifications of the Generalized Linear Regression Model

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Introduction

The lack of an exact theory of statistical inference dictates the acceptance of asymptotic methods as legitimate solutions concerning inference problems of Statistics and Econometrics. About the Generalized Linear Model, the econometric bibliography suggests two alternative size corrections of the size of the t and F tests. These size corrections are based either on Edgeworth corrections of critical values or on the Cornish-Fisher corrections of testing statistics. Using the exact distributions Student-t and F instead of the corresponding asymptotic distributions (Normal and *chi*-squared) we find approximations which are "locally exact" , i.e., that they reduce to the exact distributions for a sufficient simplification of the model. In applied econometrics research most interesting economic phenomena can be described formally using the mathematical formalism of the Generalized Linear Model, whose Variance-Covariance matrix of stochastic terms is non-scalar. The econometric model which arises is estimated using the Generalized Least Squares method and its validity is the statistical significance of the its parameters tested by the t and F econometric test. In the framework of this Doctoral Thesis, a general mathematical expression of the Generalized Linear Model is given, whose regressors may be stochastic. About the special cases of the aforementioned model, Nagar, 1959 type refined asymptotic theory is used in order to derive size correction formulae of the small sample t and F econometric tests. Specifically, this doctoral thesis is concerned with the implementation of refined asymptotic size-correction techniques for the following special cases of the Generalized Linear Model:

1. The Linear Model with Heteroskedastic and Autocorrelated Disturbances, which is presented in chapters 1 and 2 (Proof are given in Appendix A). This specific model is a mixture of the heteroskedasticity and autocorrelation problems, and suggests a process for the estimation of the autocorrelation and heteroskedasticity parameters, as well as a process for the correction of these econometric problems. Moreover, an experimental procedure is presented in section 2.6 for a single-equation model with heteroskedastic and autocorrelated error terms.

2. The Generalized Model with panel data, which is presented in chapters 3 and 4 (Proof are given in Appendix B). The basic assumption of this model is that the economic behaviour parameters are the same for all economic agents, and this differentiates this model from the autocorrelated SUR model (see Parks, 1967) which studies the causes of different economic behaviours.

3. A Special Case of The Generalized Linear Model with Panel Data, which is presented in chapters 5 and 6 (Proof are given in Appendix C). This model is a special case of the Generalized Model with panel data.

Lastly, Lemmas and theorems from the existing bibliography used in all three models of this doctoral thesis are presented in section Useful Results.

Notational Convensions

Throughout this Thesis, we use the $\mathrm{tr}, \mathrm{vec}, \otimes$, and matrix differentiation notation as defined in Dhrymes, 1978, and for any two indices i,j, we denote Kronecker's delta as δ_{ij}. Moreover, any $n \times m$ matrix $\boldsymbol{L}$ with elements l_{ij} is denoted as

$$\boldsymbol{L} = [(l_{ij})_{i=1,\dots,n;j=1,\dots,m}],$$

with obvious modifications for vectors and square matrices. If l_{ij} are $n_i \times m_j$ matrices, then $\boldsymbol{L}$ is the $\sum_i n_i \times \sum_j m_j$ partioned matrix with submatrices the l_{ij}'s. The matrices

$$\begin{aligned} \boldsymbol{P}_X &= \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}', \\ \bar{\boldsymbol{P}}_X &= \boldsymbol{I}_T - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' \end{aligned}$$

denote the orthogonal projectors into the spaces spanned by the columns of the matrix $\boldsymbol{X}$ and its orthogonal complement, respectively. For any stochastic quantity (scalar, vector, or matrix) we use the symbols $\mathrm{E}(\cdot)$ and $V(\cdot)$ to denote the expectation and variance-covariance operators, respectively. Finally, we write N for the standard normal distribution function; $\mathrm{t}_{(I)}$ and $\chi_{(I)}$ for the student-t and chi-square distribution functions, respectively, with I degrees of freedom; $\mathrm{F}_{(I,J)}$ for the F-distribution function with I and J degrees of freedom. In this thesis we use the notation proposed by Abadir and Magnus, 2002 with minor modifications properly clarified mathematics.
In this thesis to denote the accuracy of our stochastic approximations we use the order $\omega(\cdot)$ as follows: "Let $(S, \|\cdot\|)$, be a finite dimensional normed linear space and J a given set of indices, which, without loss of generality, can be taken equal to the open interval (0,1). A collection x_τ $(\tau \in J)$ of random elements of S is said to be defined on the probability space (Ω, A, P) if all the mappings x_τ are measurable.

Let x_τ $(\tau \in J)$ be a collection of random elements of $(S, \|\cdot\|)$ defined on a probability space (Ω, A, P). Given a $q > 0$, we say that x_τ is of order $\omega(q)$ as $\tau \to 0$, and we write $x_\tau = \omega(q)$, if there exists $0 < \epsilon < \infty$, such that

$$P(\|x_\tau\| > (-\ln\tau)^\epsilon) = o(\tau^q) \text{ as } \tau \to 0. \tag{1}$$

If equation (1) holds for all $q > 0$, then we write $x_\tau = \omega(\infty)$." (Magdalinos, 1992)

Chapter 1

The Linear Model with Heteroskedastic and Autocorrelated Disturbances

1.1 Introduction

Most of the single-equation econometric specifications in both applied and theoretical research can be expressed in the form of the generalized normal linear regression model, provided that certain assumptions are made about the structure of the error covariance matrix. Some of the disturbance specifications, most frequently used in both applied and theoretical econometrics, are the AR(1), the heteroskedastic, and the seemingly-unrelated-regressions structures of disturbances. The volume of theoretical and applied work published in those areas can be attributed to this fact. Also, in order to cope with more complex economic phenomena, in many cases, econometricians have focused on models with random errors which are generated by a mixture of various disturbance specifications, such as models of seemingly unrelated regressions with autocorrelated errors (see, e.g., Parks, 1967), or models with mixed heteroskedastic-autoregressive disturbances, which can be estimated by using the heteroskedasticity-autocorrelation consistent (HAC) estimators of the error covariance matrix (see, inter alia, White, 1980, MacDonald and MacKinnon, 1985, Newey and West, 1987). In this chapter the normal linear regression model is presented, in which the disturbances are specified as a mixed heteroskedastic-autoregressive process. In particular, we examine the mixture of a stationary first-order autoregressive process with autocorrelation coefficient ρ, and a linear heteroskedastic specification of the form $\mathrm{var}(u_t) = z_t'\varsigma$, where ς is a vector of heteroskedasticity parameters (Amemiya, 1977). From the viewpoint of theoretical econometrics, a lot of effort has been devoted, up till now, to the construction of estimators of ς and ρ in econometric models with error terms that are either heteroskedastic or autoregressive, respectively. Thus, in the linear model with heteroskedastic variances, $\mathrm{var}(u_t) = z_t'\varsigma$, some of the most frequently used estimators of ς, described in Subsections 1.3.1 and 1.3.3, are the least squares or Goldfeld-Quandt estimator, the generelized least squares or Amemiya estimator, the iterative Amemiya estimator, and the maximum likelihood estimator. Moreover, in the linear model with AR(1) errors, some of the most frequently used estimators of ρ, described in Subsection 1.3.2, are the least squares estimator, the Durbin-Watson estimator, the generalized least squares estimator, the Prais-Winsten estimator, and the maximum likelihood estimator. However, although there are many estimators of ς and ρ in models with exclusively heteroskedastic or exclusively autoregressive disturbances, respectively, according to our knowledge, no procedure has ever been proposed for the estimation of parameters ς and ρ in order to facilitate the theoretical investigation of linear models with a mixed heteroskedastic-autoregressive specification of the disturbances. Our purpose, in this chapter, is to derive such an estimation procedure.

When a linear heteroskedastic specification is combined with a stationary first-order autoregressive process in order to generate the disturbances in a generalized normal linear regression model, the heteroskedastic variances, $\mathrm{var}(u_t) = \sigma_t^2/(1-\rho^2)$, are functions of the first-order autocorrelation coefficient, ρ. Due to this fact, the use of the standard estimators results in estimated heteroskedasticity parameters which are functions of the first-order autocorrelation coefficient. This means that, although the parameters ς and ρ are theoretically identified, they cannot be properly distinguished by any of the estimators $\hat{\varsigma}$ and $\hat{\rho}$ used in applied research. To account for this, a reparameterization of the model is being introduced, in which the heteroskedasticity parameter vector is $\varsigma_* = \varsigma(1-\rho^2)^{1/2}$. The use of this alternative parameterization results in a multi-step estimation procedure that enables us to effectively distinguish, from a theoretical viewpoint, the estimation of the heteroskedasticity parameters from the estimation of the first-order autocorrelation coefficient. Such a distinction is extremely useful whenever a researcher is interested in constructing an adjusted generalized linear model with disturbances that are exclusively heteroskedastic or exclusively autoregressive, in order to examine certain distributional properties of the estimators of ς and ρ, respectively.

1.2 The Model

Consider the linear regression model

$$\boldsymbol{y} = \boldsymbol{X\beta} + \sigma\boldsymbol{u} \tag{1.1}$$

where
$\boldsymbol{y}$ is a $T \times 1$ vector of observations on the dependent variable,
$\boldsymbol{X}$ is a $T \times n$ matrix of observations on n exogenous regressors,
$\boldsymbol{\beta}$ is a $n \times 1$ vector of unknown structural parameters, and
$\sigma\boldsymbol{u}$ (σ is a positive scalar) is a Tx1 vector of unobserved stochastic disturbances.

Assumption 1. The following assumptions hold:

1. The random vector $\boldsymbol{u}$ is distributed as $\mathrm{N}(0, \boldsymbol{\Omega}^{-1})$, where $\boldsymbol{\Omega}$ is a $T \times T$ positive definite and symmetric matrix.

2. The matrix of the regressors has full column rank, i.e.,

$$r(\boldsymbol{X}) = n. \tag{1.2}$$

3. The regressors are non-stochastic. The results of this Thesis would also be valid if the regressors were stochastic, yet uncorrelated with the errors, i.e.,

$$\mathrm{E}(\boldsymbol{X}'\boldsymbol{u}) = 0, \tag{1.3}$$

but in such a case the proofs would be a little more complicated.

1.2.1 The random vector $\boldsymbol{u}$

Let u_t be the t-th element of the $T \times 1$ random vector $\boldsymbol{u}$. The element u_t satisfies the following relationship:

$$u_t = \sigma_t u_{*t} \quad (t = 1, \dots, T), \tag{1.4}$$

where u_{*t} is the t-th element of a $T \times 1$ random vector $\boldsymbol{u}_*$ and $\sigma_t \quad (t = 1, \dots, T)$ are positive scalars, uncorrelated with elements u_{*t}.

The elements of the random vector $\boldsymbol{u}_*$ are generated by a stationary, first order autoregressive $AR(1)$ stochastic process of the form

$$u_{*t} = \rho u_{*t-1} + \varepsilon_t; \quad 0 < |\rho| < 1 \quad (t = 2, \dots, T), \tag{1.5}$$

where

$$u_{*t} \sim \mathrm{N}(0, 1/(1-\rho^2)) \tag{1.6}$$

and ε_t are independent $\mathrm{N}(0,1)$ random variables, i.e.,

$$\mathrm{E}(\varepsilon_t \ \varepsilon_{t'}) = \delta_{tt'} = \begin{cases} 1, & \text{if } t = t' \\ 0, & \text{if } t \neq t' \end{cases}, \tag{1.7}$$

where $\delta_{tt'}$ denotes Kronecker's delta.

The time-series u_{*t} $(t = 1, \dots, T)$ is a stationary $AR(1)$ stochastic process provided that

$$u_{*1} = (1-\rho^2)^{-1/2} \varepsilon_1 \quad (\text{for } t = 1). \tag{1.8}$$

It is straightforward that

$$\mathrm{E}(\boldsymbol{u}_* \boldsymbol{u}_*') = \frac{\boldsymbol{R}}{(1-\rho^2)} , \tag{1.9}$$

where

$$\boldsymbol{R} = [(\rho^{|t-t'|})_{t,t'=1,\dots,T}] \tag{1.10}$$

is a $T \times T$ positive definite and symmetric matrix.

Equations (1.4), (1.5), (1.6), (1.9) and (1.10) imply the following results:

$$\mathrm{E}(u_t) = \mathrm{E}(\sigma_t u_{*t}) = \sigma_t \, \mathrm{E}(u_{*t}) = 0, \tag{1.11a}$$

$$\mathrm{E}(u_t^2) = \mathrm{E}(\sigma_t^2 u_{*t}^2) = \sigma_t^2 \, \mathrm{E}(u_{*t}^2) = \frac{\sigma_t^2}{(1-\rho^2)}, \tag{1.11b}$$

$$\mathrm{E}(u_t \; u_{t'}) = \mathrm{E}(\sigma_t u_{*t} \; \sigma_{t'} u_{*t'}) = \sigma_t \sigma_{t'} \, \mathrm{E}(u_{*t} \; u_{*t'}) = \frac{\sigma_t \sigma_{t'} \rho^{|t-t'|}}{(1-\rho^2)}, \tag{1.11c}$$

for any $t \neq t'$. Note that if $t = t'$ then (1.11c) implies (1.11b).

1.2.2 The specification of σ_t $(t = 1, \dots, T)$

Let $\boldsymbol{x}_t'$ $(t = 1, \dots, T)$ be the rows of the $T \times n$ matrix $\boldsymbol{X}$ of the regressors in model (1.1), and let y_t, u_t be the t-th elements of the $T \times 1$ vectors $\boldsymbol{y}, \boldsymbol{u}$, respectively. Moreover, let $\boldsymbol{z}_t'$ be the rows of a $T \times m$ matrix $\boldsymbol{Z}$ of observations on a set of m exogenous variables, some of which may be regressors too, i.e., they may belong to the matrix $\boldsymbol{X}$.

Further, let

$$\boldsymbol{\varsigma} \in \mathcal{F}_s = \mathbb{R}^m \setminus \{\mathbf{0}\}, \; (\mathbf{0} \text{ is the } m \times 1 \text{ zero vector}) \tag{1.12}$$

be a $m \times 1$ vector of unknown parameters. Then, the parameters σ_t $(t = 1, \dots, T)$ in (1.4) are assumed to satisfy the linear functions

$$\sigma_t^2 = \boldsymbol{z}_t' \boldsymbol{\varsigma} \quad (t = 1, \dots, T), \tag{1.13}$$

where

$$\boldsymbol{z}_t' = (z_{t1}, z_{t2}, \dots, z_{tm}) \tag{1.14}$$

is a vector with elements the t-th observations on the m exogenous variables: $z_1 \equiv 1$ $(\forall t)$ $z_2, \dots, z_m$ and

$$\boldsymbol{\varsigma} = \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_m \end{bmatrix} \tag{1.15}$$

is a $m \times 1$ non-zero vector of unknown parameters (see Hildreth and Houck, 1968, Nonlinear Methods in Econometrics, 1972, Amemiya, 1977).

1.2.3 The specification of $\boldsymbol{\Omega}$

The elements of the $T \times T$ matrix $\boldsymbol{\Omega}$ are functions of the $(m+1) \times 1$ vector

$$\boldsymbol{\gamma} = (\rho, \boldsymbol{\varsigma}')', \tag{1.16}$$

where ρ is the autocorrelation coefficient and $\boldsymbol{\varsigma} \in \mathbb{R}^m \setminus \{0\}$.

The t-th diagonal element of $\boldsymbol{\Omega}^{-1}$ is $\sigma_t^2/(1-\rho^2)$ [see (1.11b)], and the (t,t')-th off diagonal element of $\boldsymbol{\Omega}^{-1}$ is $\sigma_t\sigma_{t'}\rho^{|t-t'|}/(1-\rho^2)$, [see (1.11c)].

Thus, the $T\times T$ matrix $\boldsymbol{\Omega}^{-1}$ can be analytically written as follows:

$$\boldsymbol{\Omega}^{-1} = \frac{1}{(1-\rho^2)}\begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho & \sigma_1\sigma_3\rho^2 & \dots & \sigma_1\sigma_T\rho^{T-1} \\ \sigma_2\sigma_1\rho & \sigma_2^2 & \sigma_2\sigma_3\rho & \dots & \sigma_2\sigma_T\rho^{T-2} \\ & & \ddots & & \\ \vdots & & & & \\ \sigma_T\sigma_1\rho^{T-1} & \sigma_T\sigma_2\rho^{T-2} & & \dots & \sigma_T^2 \end{bmatrix}. \tag{1.17}$$

Define the $T\times T$ diagonal matrix

$$\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2,\sigma_2^2,.....,\sigma_T^2) = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_T^2 \end{bmatrix}, \tag{1.18}$$

which implies that

$$\boldsymbol{\Sigma}^{1/2} = \text{diag}(\sigma_1,\sigma_2,.....,\sigma_T) = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_T \end{bmatrix}, \tag{1.19}$$

and

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2}. \tag{1.20}$$

Then by using (1.9), (1.10), (1.17) and (1.19) we can write

$$\boldsymbol{\Omega}^{-1} = \boldsymbol{\Sigma}^{1/2}[\boldsymbol{R}/(1-\rho^2)]\boldsymbol{\Sigma}^{1/2}. \tag{1.21}$$

Let $\boldsymbol{D}$ be a $T\times T$ band matrix whose (t,t')-th element is 1 if $|t-t'|=1$ and 0 elsewhere. Also, let $\boldsymbol{\Delta}$ be a $T\times T$ matrix with 1 in the $(1,1)$-st and (T,T)-th positions and 0's elsewhere. Then,

$$[\boldsymbol{R}/(1-\rho^2)]^{-1} = (1+\rho^2)\boldsymbol{I}_T - \rho\boldsymbol{D} - \rho^2\boldsymbol{\Delta} \tag{1.22}$$

$$= \begin{bmatrix} 1 & -\rho & \dots & 0 \\ -\rho & 1+\rho^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \dots & -\rho & 1+\rho^2 & -\rho \\ \dots & 0 & -\rho & 1 \end{bmatrix}. \tag{1.23}$$

Then by combining (1.21) and (1.22) we can write the $T \times T$ matrix $\boldsymbol{\Omega}$ as follows:

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1/2}[(1+\rho^2)\boldsymbol{I}_T - \rho\boldsymbol{D} - \rho^2\boldsymbol{\Delta}]\boldsymbol{\Sigma}^{-1/2} \ , \tag{1.24}$$

where

$$\boldsymbol{\Sigma}^{-1/2} = diag(1/\sigma_1, 1/\sigma_2, ..., 1/\sigma_T) = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sigma_T} \end{bmatrix}. \tag{1.25}$$

1.2.4 Identification and estimation of the parameters

Let $\hat{\boldsymbol{\gamma}} = (\hat{\rho}, \hat{\boldsymbol{\varsigma}}')'$ be any consistent estimator of the parameter vector $\boldsymbol{\gamma} = (\rho, \boldsymbol{\varsigma}')'$. For any function $f = f(\boldsymbol{\gamma})$ we can write $\hat{f} = f(\hat{\boldsymbol{\gamma}})$. The feasible GLS estimators of β and σ are

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{y} \tag{1.26}$$

and

$$\hat{\sigma} = [(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'\hat{\boldsymbol{\Omega}}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})/(T-n)]^{1/2} \,. \tag{1.27}$$

From (1.17) it is straightforward that the parameters σ and σ_t $(t = 1, \dots, T)$ cannot be distinguished, that is the parameters σ and $\boldsymbol{\varsigma}$ cannot be simultaneously identified without the restriction $\sigma = 1$, under which the estimate $\hat{\boldsymbol{\Omega}}^{-1}$ is supposed to be accurate, up to a multiplicative factor. This is not true in small samples, and a reasonable method to account for this is to use the feasible GLS estimate of $\hat{\sigma}$ from (1.27) to compute the traditional t and F test statistics. This method is meaningless from the estimation viewpoint, but its success in improving the size corrections must be the only criterion to judge its validity.

1.2.5 Regularity conditions

Let $\boldsymbol{\Omega}_i$, $\boldsymbol{\Omega}_{ij}$, etc. denote the $T \times T$ matrices of first-, second- and higher-order derivatives of the elements of $\boldsymbol{\Omega}$ with respect to the elements of the $(m+1) \times 1$ parameter vector $\boldsymbol{\gamma} = (\rho, \boldsymbol{\varsigma}')'$.

Moreover, for any estimator $\hat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$, define the $(m+2) \times 1$ vector δ with elements

$$\delta_0 = \frac{\hat{\sigma}^2 - 1}{\tau}; \quad \delta_\rho = \frac{\hat{\rho} - \rho}{\tau}; \quad \delta_{\varsigma_i} = \frac{\hat{\varsigma}_i - \varsigma_i}{\tau} \quad (i = 1, \dots, m) \tag{1.28}$$

where $\tau = \frac{1}{\sqrt{T}}$ is the asymptotic scale of our expansions.

The size corrections derived in this Doctoral Thesis are based on the following regularity conditions:

(1) The elements of $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^{-1}$ are bounded for all T, all $\rho \in (-1, 1)$, and all vectors $\boldsymbol{\varsigma} \in \mathcal{F}_s = \mathbb{R}^m \setminus \{0\}$. Moreover, the matrices

$$\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T, \quad \boldsymbol{F} = \boldsymbol{X}\boldsymbol{X}'/T, \quad \boldsymbol{\Gamma} = \boldsymbol{Z}'\boldsymbol{Z}/T, \tag{1.29}$$

converge to non-singular limits as $T \to \infty$.

(2) Up to the fourth order, the partial derivatives of the elements of $\boldsymbol{\Omega}$ with respect to the elements of $\boldsymbol{\gamma} = (\rho, \varsigma_1, \ldots, \varsigma_m)'$ are bounded for all T, all $\rho \in (-1, 1)$, and all vectors $\boldsymbol{\varsigma} \in \mathcal{F}_s = \mathbb{R}^m \setminus \{0\}$.

(3) The estimators $\hat{\rho}$ and $\hat{\boldsymbol{\varsigma}}$ are even functions of $\boldsymbol{u}$, and they are functionally unrelated to the parameter vector $\boldsymbol{\beta}$, i.e., they can be written as functions of $\boldsymbol{X}, \boldsymbol{Z}$, and $\sigma\boldsymbol{u}$ only.

(4) The vector $\boldsymbol{\delta}$ admits a stochastic expansion of the form

$$\boldsymbol{\delta} = \boldsymbol{d}_1 + \tau\boldsymbol{d}_2 + \omega(\tau^2), \tag{1.30}$$

where the order of magnitude $\omega(\cdot)$ defined in the Notational Convensions, has the same operational properties as the order $O(\cdot)$, and the expectations

$$\mathbf{E}(\boldsymbol{d}_1\boldsymbol{d}_1'), \quad \mathbf{E}(\boldsymbol{d}_1 + \sqrt{T}\boldsymbol{d}_2) \tag{1.31}$$

exist and have finite limits as $T \to \infty$.

Discussions on the Regularity Conditions:

The first two regularity conditions imply that the $n \times n$ matrices

$$\boldsymbol{A}_i = \boldsymbol{X}'\boldsymbol{\Omega}_i\boldsymbol{X}/T, \quad \boldsymbol{A}_{ij} = \boldsymbol{X}'\boldsymbol{\Omega}_{ij}\boldsymbol{X}/T, \quad \boldsymbol{A}_{ij}{}^* = \boldsymbol{X}'\boldsymbol{\Omega}_i\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_j\boldsymbol{X}/T \tag{1.32}$$

are bounded and therefore the Taylor series expansion of β is a stochastic expansion (Magdalinos, 1992). Since the parameters ρ and $\boldsymbol{\varsigma} = (\varsigma_1, \ldots, \varsigma_m)'$ are functionally unrelated to $\boldsymbol{\beta}$, regularity condition (3) is satisfied for a wide class of estimators $\hat{\rho}$ and $\hat{\boldsymbol{\varsigma}}$ including the maximum likelihood estimators and the simple and iterative estimators based on the regression residuals (Breusch, 1980, Rothenberg, 1984a). Note that we need not assume that the estimators $\hat{\rho}$ and $\hat{\boldsymbol{\varsigma}}$ are asymptotically efficient. Also, notice that the regularity conditions (1) through (4) are satisfied by all the estimators of ρ and $\boldsymbol{\varsigma}$ examined in the next section.

1.2.6 Definition of parameters

Define the scalars $\lambda_0, \kappa_0, \lambda_{0\rho}, \kappa_\rho, \lambda_{\rho\rho}$, the $m \times 1$ vectors $\boldsymbol{\lambda}_{0\varsigma}, \boldsymbol{\kappa}_\varsigma, \boldsymbol{\lambda}_{\rho\varsigma}$, and the $m \times m$ matrix $\boldsymbol{\Lambda}_{\varsigma\varsigma}$ as follows:

$$\begin{bmatrix} \lambda_0 & \lambda_{0\rho} & \boldsymbol{\lambda}_{0\varsigma}' \\ \lambda_{0\rho} & \lambda_{\rho\rho} & \boldsymbol{\lambda}_{\rho\varsigma}' \\ \boldsymbol{\lambda}_{0\varsigma} & \boldsymbol{\lambda}_{\rho\varsigma} & \boldsymbol{\Lambda}_{\varsigma\varsigma} \end{bmatrix} = \mathrm{E}(\boldsymbol{d}_1\boldsymbol{d}_1'); \quad \begin{bmatrix} \kappa_0 \\ \kappa_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix} = \mathrm{E}(\boldsymbol{d}_1 + \sqrt{T}\boldsymbol{d}_2). \tag{1.33}$$

Also define the $(m+1) \times 1$ vectors $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$ and the $(m+1) \times (m+1)$ matrix $\boldsymbol{\Lambda}$ as follows:

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_{0\rho} \\ \boldsymbol{\lambda}_{\rho\varsigma} \end{bmatrix}; \quad \boldsymbol{\kappa} = \begin{bmatrix} \kappa_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix}; \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{\rho\rho} & \boldsymbol{\lambda}_{\rho\varsigma}' \\ \boldsymbol{\lambda}_{\rho\varsigma} & \boldsymbol{\Lambda}_{\varsigma\varsigma} \end{bmatrix}. \tag{1.34}$$

1.2.7 Alternative model specification

Denote by σ_{*t}^2 the variance of u_t, i.e. [see (1.11b)],

$$\sigma_{u_t}^2 = \sigma_{*t}^2 = \mathrm{var}(u_t) = \frac{\sigma_t^2}{1-\rho^2} \ , \tag{1.35}$$

which implies that the standard deviation of u_t is

$$\sigma_{u_t} = \sigma_{*t} = \frac{\sigma_t}{(1-\rho^2)^{1/2}}. \tag{1.36}$$

Also, denote by $\sigma_{*tt'}$ the covariance of u_t and $u_{t'}$, i.e. [see (1.11c)],

$$\begin{aligned}
\sigma_{*tt'} &= \mathrm{cov}(u_t,\ u_{t'}) = \frac{\sigma_t \sigma_{t'}}{(1-\rho^2)} \rho^{|t-t'|} \\
&= \frac{\sigma_t}{(1-\rho^2)^{1/2}} \frac{\sigma_{t'}}{(1-\rho^2)^{1/2}} \rho^{|t-t'|} = [see(1.36)] \\
&= \sigma_{*t}\sigma_{*t'}\rho^{|t-t'|}.
\end{aligned} \tag{1.37}$$

Further, define the $m \times 1$ non-zero vector $\boldsymbol{\varsigma}_* = (\varsigma_{1*}, \dots, \varsigma_{m*})'$ as follows

$$\boldsymbol{\varsigma}_* = \frac{\boldsymbol{\varsigma}}{(1-\rho^2)} \implies \varsigma_{*i} = \frac{\varsigma_i}{(1-\rho^2)}, \ (i = 1, \dots, m). \tag{1.38}$$

Then, by combining (1.13), (1.35) and (1.38) we find that

$$\sigma_{*t}^2 = \frac{\sigma_t^2}{1-\rho^2} = \boldsymbol{z}_t'[\boldsymbol{\varsigma}/(1-\rho^2)] = \boldsymbol{z}_t'\boldsymbol{\varsigma}_*, \ (t = 1, \dots, T). \tag{1.39}$$

Moreover, by combining (1.13), (1.35) and (1.39) we find that

$$\sigma_t^2 = \sigma_{*t}^2(1-\rho^2) = \boldsymbol{z}_t'\boldsymbol{\varsigma}_*(1-\rho^2) \tag{1.40}$$

and

$$\boldsymbol{\varsigma} = \boldsymbol{\varsigma}_*(1-\rho^2) \implies \varsigma_i = \varsigma_{*i}(1-\rho^2), \ (i = 1, \dots, m), \tag{1.41}$$

where $\boldsymbol{\varsigma}_* \in \mathcal{F}_s = \mathbb{R}^m \setminus \{\mathbf{0}\}$.

Moreover, define $T \times 1$ random vector $\boldsymbol{u}_{**}$, the t-th element of which is

$$u_{**t} = \frac{u_t}{\sigma_{*t}} = \frac{u_t}{\frac{\sigma_t}{(1-\rho^2)^{1/2}}} = (1-\rho^2)^{1/2}\frac{u_t}{\sigma_t} = (1-\rho^2)^{1/2}u_{*t}. \tag{1.42}$$

Then, since $u_{*t} \sim \mathrm{N}(0, 1/(1-\rho^2))$ $(t = 1, \ldots, T)$, the following results hold:

$$\mathrm{E}(u_{**t}) = \mathrm{E}((1-\rho^2)^{1/2} u_{*t}) = (1-\rho^2)^{1/2}\,\mathrm{E}(u_{*t}) = 0, \tag{1.43}$$

$$\mathrm{E}(u_{**t}^2) = \mathrm{E}((1-\rho^2) u_{*t}^2) = (1-\rho^2)\,\mathrm{E}(u_{*t}^2) = (1-\rho^2)/(1-\rho^2) = 1. \tag{1.44}$$

Equation (1.9) and (1.10) imply that

$$\begin{aligned} \mathrm{E}(u_{**t} u_{**t'}) &= \mathrm{E}((1-\rho^2)^{1/2} u_{*t} (1-\rho^2)^{1/2} u_{*t'}) = (1-\rho^2)\,\mathrm{E}(u_{*t} u_{*t'}) \\ &= (1-\rho^2) \frac{\rho^{|t-t'|}}{(1-\rho^2)} = \rho^{|t-t'|}, \end{aligned} \tag{1.45}$$

i.e., we can write more compactly that

$$\mathrm{E}(u_{**}) = 0 \; \mathit{and} \; \mathrm{E}(\boldsymbol{u}_{**}\boldsymbol{u}_{**}) = \boldsymbol{R}. \tag{1.46}$$

Finally, since $u_{**t-1} = (1-\rho^2)^{1/2} u_{*t-1}$, by combining (1.5) and (1.42) we find that

$$\begin{aligned} u_{**t} &= (1-\rho^2)^{1/2} u_{*t} = (1-\rho^2)^{1/2} (\rho u_{*t-1} + \varepsilon_t) \\ &= \rho[(1-\rho^2)^{1/2} u_{*t-1}] + (1-\rho^2)^{1/2} \varepsilon_t \\ &= \rho u_{**t-1} + \varepsilon_{*t}\,, \end{aligned} \tag{1.47}$$

where the random variables

$$\varepsilon_{*t} = (1-\rho^2)^{1/2} \varepsilon_t \tag{1.48}$$

are independently distributed as $N(0, (1-\rho^2))$. Equation (1.47) implies that the elements of the random vector $\boldsymbol{u}_{**}$ are generated by a stationary, first-order autoregressive (AR(1)) stochastic process with autocorrelation coefficient ρ.

1.2.8 Alternative representation of the matrices $\boldsymbol{\Omega}^{-1}$ and $\boldsymbol{\Omega}$

By combining (1.17), (1.35), and (1.36) we find that

$$\boldsymbol{\Omega}^{-1} = \begin{bmatrix} \sigma_{*1}^2 & \sigma_{*1}\sigma_{*2}\rho & \sigma_{*1}\sigma_{*3}\rho^2 \ldots \sigma_{*1}\sigma_{*T}\rho^{T-1} \\ \sigma_{*2}\sigma_{*1}\rho & \sigma_{*2}^2 & \ldots \\ \vdots & & \\ \sigma_{*T-1}\sigma_{*1}\rho^{T-2} & \ldots\ldots & \sigma_{*T-1}\sigma_{*T}\rho \\ \sigma_{*T}\sigma_{*1}\rho^{T-1} & \ldots\ldots & \sigma_{*T}^2 \end{bmatrix}. \tag{1.49}$$

Moreover, define the $(T \times T)$ matrix

$$\begin{aligned}
\boldsymbol{\Sigma}_*^{1/2} &= \operatorname{diag}(\sigma_{*1}, \sigma_{*2}, \ldots, \sigma_{*T}) = [\textit{see } (1.36)] \\
&= \frac{1}{(1-\rho^2)^{1/2}} \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_T) = \frac{1}{(1-\rho^2)^{1/2}} \boldsymbol{\Sigma}^{1/2} \qquad (1.50) \\
&= \frac{1}{(1-\rho^2)^{1/2}} \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_T \end{bmatrix} = \begin{bmatrix} \sigma_{*1} & 0 & \ldots & 0 \\ 0 & \sigma_{*2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_{*T} \end{bmatrix}. \qquad (1.51)
\end{aligned}$$

Also, define accordingly the $(T \times T)$ matrix

$$\begin{aligned}
\boldsymbol{\Sigma}_*^{-1/2} &= \operatorname{diag}(1/\sigma_{*1}, 1/\sigma_{*2}, \ldots, 1/\sigma_{*T}) = [\textit{see } (1.36)] \\
&= (1-\rho^2)^{1/2} \operatorname{diag}(1/\sigma_1, 1/\sigma_2, \ldots, 1/\sigma_T) = (1-\rho^2)^{1/2} \boldsymbol{\Sigma}^{-1/2} \qquad (1.52) \\
&= (1-\rho^2)^{1/2} \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{\sigma_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{\sigma_T} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_{*1}} & 0 & \ldots & 0 \\ 0 & \frac{1}{\sigma_{*2}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{\sigma_{*T}} \end{bmatrix}. \qquad (1.53)
\end{aligned}$$

Note that

$$\boldsymbol{\Sigma}^{1/2} = (1-\rho^2)^{1/2} \boldsymbol{\Sigma}_*^{1/2} \qquad (1.54)$$

and

$$\boldsymbol{\Sigma}^{-1/2} = \frac{1}{(1-\rho^2)^{1/2}} \boldsymbol{\Sigma}_*^{-1/2} \qquad (1.55)$$

Then, (1.21) and (1.54) imply that

$$\begin{aligned}
\boldsymbol{\Omega}^{-1} &= \left[\frac{1}{(1-\rho^2)^{1/2}} \boldsymbol{\Sigma}^{1/2}\right] \boldsymbol{R} \left[\frac{1}{(1-\rho^2)^{1/2}} \boldsymbol{\Sigma}^{1/2}\right] \\
&= \boldsymbol{\Sigma}_*^{1/2} \boldsymbol{R} \boldsymbol{\Sigma}_*^{1/2}. \qquad (1.56)
\end{aligned}$$

Further, (1.24) and (1.55) imply that

$$\begin{aligned}
\boldsymbol{\Omega} &= \left[\frac{1}{(1-\rho^2)^{1/2}} \boldsymbol{\Sigma}_*^{-1/2}\right] [(1+\rho^2) I_T - \rho \boldsymbol{D} - \rho^2 \boldsymbol{\Delta}] \left[\frac{1}{(1-\rho^2)^{1/2}} \boldsymbol{\Sigma}_*^{-1/2}\right] \\
&= \frac{1}{(1-\rho^2)} \boldsymbol{\Sigma}_*^{-1/2} [(1+\rho^2) I_T - \rho \boldsymbol{D} - \rho^2 \boldsymbol{\Delta}] \boldsymbol{\Sigma}_*^{-1/2}. \qquad (1.57)
\end{aligned}$$

1.2.9 Estimation strategy

Denote by LS, GL, IG, ML the least squares, generalized least squares, iterative GLS and maximum likelihood estimation methods, respectively. Also, denote by $\hat{\beta}_I$ any consistent estimator of β in model (1.1), indexed by I (I=S, GL, IG, ML).

The discussion above suggests the following 7 steps of an estimation strategy:

Step 1: Estimate model (1.1) using the $\hat{\boldsymbol{\beta}}_I$ estimator. Then, the corresponding residual vector:

$$\hat{\boldsymbol{u}}_I = \boldsymbol{y} - \hat{\boldsymbol{\beta}}_I \boldsymbol{X} = \left[(\hat{u}_{t(I)})_{t=1,\ldots,T} \right] \tag{1.58}$$

is a consistent predictor of the disturbance vector $\boldsymbol{u}$.

Step 2: Use one of the consistent estimators given in Subsection 1.3.1 in order to estimate the parameter vector $\boldsymbol{\varsigma}_* = (\varsigma_{*1}, \ldots, \varsigma_{*m})'$. Then, estimate matrix $\boldsymbol{\Sigma}_*^{-1/2}$ as

$$\hat{\boldsymbol{\Sigma}}_*^{-1/2} = \mathrm{diag}(1/\hat{\sigma}_{u_1}, \ldots, 1/\hat{\sigma}_{u_T}), \tag{1.59}$$

where

$$\hat{\sigma}_{u_t} = \left(\boldsymbol{z}_{t.}^\top \hat{\boldsymbol{\varsigma}}_* \right)^{1/2} \quad \forall\, t = 1, \ldots, T. \tag{1.60}$$

Step 3: Estimate the heteroskedasticity-corrected residuals

$$\hat{\boldsymbol{u}}_{*I} = \hat{\boldsymbol{\Sigma}}_*^{-1/2} \hat{\boldsymbol{u}}_I = \left[(\hat{u}_{*t(I)})_{t=1,\ldots,T} \right], \tag{1.61}$$

where

$$\hat{u}_{*t(I)} = \frac{\hat{u}_{t(I)}}{\hat{\sigma}_{u_t}} \quad \forall\, t = 1, \ldots, T, \tag{1.62}$$

and $\hat{\boldsymbol{u}}_I$ is the predictor of $\boldsymbol{u}$ estimated by (1.58).

Step 4: Use one of the consistent estimators given in Subsection 1.3.2 in order to calculate an initial estimate $\hat{\rho}_*$ of the autocorrelation coefficient ρ.

Step 5: Use (1.41) and the consistent estimators $\hat{\boldsymbol{\varsigma}}_*$ and $\hat{\rho}_*$ in order to estimate the parameter vector $\boldsymbol{\varsigma}$ as

$$\hat{\boldsymbol{\varsigma}} = \hat{\boldsymbol{\varsigma}}_*(1 - \hat{\rho}_*^2) \implies \hat{\varsigma}_i = \hat{\varsigma}_{*i}(1 - \hat{\rho}_*^2) \quad \forall\, i = 1, \ldots, m. \tag{1.63}$$

Then, estimate matrix $\boldsymbol{\Sigma}^{-1/2}$ as

$$\hat{\boldsymbol{\Sigma}}^{-1/2} = \mathrm{diag}(1/\hat{\sigma}_1, \ldots, 1/\hat{\sigma}_T), \tag{1.64}$$

where

$$\hat{\sigma}_t = \left(\boldsymbol{z}_{t.}^\top \hat{\boldsymbol{\varsigma}} \right)^{1/2} \quad \forall\, t = 1, \ldots, T. \tag{1.65}$$

Alternatively, $\boldsymbol{\varsigma}$ can be estimated via the following asymptotically equivalent process:

(i) Use the initial estimator $\hat{\rho}_*$ in order to transform model (1.1) into the autoregression-corrected model

$$\boldsymbol{y}_H = \boldsymbol{X}_H\boldsymbol{\beta} + \boldsymbol{u}_H, \tag{1.66}$$

where the elements of vector $\boldsymbol{u}_H = \left[(u_{_H t})_{t=1,\dots,T}\right]$ are purely heteroskedastic disturbances, given by the following formulae:

$$u_{_H 1} = (1 - \hat{\rho}_*^2)^{1/2} u_1, \quad u_{_H t} = u_t - \hat{\rho}_* u_{t-1} \quad \forall\, t = 2, \dots, T. \tag{1.67}$$

(ii) Use one of the consistent estimators given in Subsection 1.3.1 in order to estimate the parameter vector $\boldsymbol{\varsigma}$, and then estimate matrix $\boldsymbol{\Sigma}^{-1/2}$ via (1.64) and (1.65).

Although from the estimation viewpoint (1.63) is perfectly adequate as a consistent estimator of $\boldsymbol{\varsigma}$, the estimator $\hat{\boldsymbol{\varsigma}}$ based on the residuals of model (1.66) enables the researcher to find the finite-sample distributional properties of any consistent estimator of $\boldsymbol{\varsigma}$ in Subsection 1.3.1.

Step 6: Premultiply model (1.1) by $\hat{\boldsymbol{\Sigma}}^{-1/2}$ given in (1.64), in order to derive heteroskedasticity-corrected model

$$\boldsymbol{y}_{AR} = \boldsymbol{X}_{AR}\boldsymbol{\beta} + \boldsymbol{u}_{AR}, \tag{1.68}$$

where the elements of vector $\boldsymbol{u}_{AR} = \left[(u_{_{AR} t})_{t=1,\dots,T}\right]$ are purely autoregressive disturbances, given by the following formula:

$$u_{_{AR} t} = u_t / \hat{\sigma}_t \quad \forall\, t = 1, \dots, T, \tag{1.69}$$

where $\hat{\sigma}_t$ are given in (1.65). Then, use one of the consistent estimators given in Subsection 1.3.2 in order to estimate the autocorrelation coefficient ρ. The estimator $\hat{\rho}$ based on the residuals of model (1.68) enables the researcher to find the finite-sample distributional properties of any consistent estimator of ρ in Subsection 1.3.2.

Step 7: Use the estimators $\hat{\boldsymbol{\Sigma}}^{-1/2}$ and $\hat{\rho}$ from Steps 5 and 6, respectively, in order to calculate the estimator

$$\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Sigma}}^{-1/2}[(1 + \hat{\rho}^2)\boldsymbol{I}_T - \hat{\rho}\boldsymbol{D} - \hat{\rho}^2\boldsymbol{\Delta}]\hat{\boldsymbol{\Sigma}}^{-1/2}, \tag{1.70}$$

which can be used for the feasible generalized least squares estimation of model (1.1).

1.3 Asymptotically efficient estimators of $\gamma = (\rho, \varsigma')'$

1.3.1 Estimators of $\varsigma_* = (\varsigma_{*1}, \dots, \varsigma_{*m})'$

Some of the most frequently used estimators of ς_* in applied econometric research are:

1. The least squares (LS) or Goldfeld and Quandt, 1965 (GQ) estimator

$$\hat{\varsigma}_{*LS} = \hat{\varsigma}_{*GQ} = \left(\sum_{t=1}^{T} z_t z_t'\right)^{-1} \left(\sum_{t=1}^{T} z_t \hat{u}^2_{(LS)t}\right), \tag{1.71}$$

where $\hat{u}_{(LS)t} = y_t - \boldsymbol{x}_t'\hat{\beta}_{LS}$ and $\hat{\beta}_{LS}$ is the least squares estimator pf β.

2. The generelized least squares (GL) or Amemiya, 1977 (A) estimator

$$\hat{\varsigma}_{*GL} = \hat{\varsigma}_{*A} = \left(\sum_{t=1}^{T} (z_t'\hat{\varsigma}_{*GQ})^{-2} z_t z_t'\right)^{-1} \sum_{t=1}^{T} \left(z_t \hat{\varsigma}_{*GQ}\right)^{-2} z_t \hat{u}^2_{(LS)t}. \tag{1.72}$$

3. The iterative generalized least squares (IG) or iterative Amemiya (IA) estimator

$$\hat{\varsigma}_{*IG} = \hat{\varsigma}_{*IA} = \left(\sum_{t=1}^{T} (z_t'\hat{\varsigma}_{*I-1})^{-2} z_t z_t'\right)^{-1} \sum_{t=1}^{T} \left(z_t'\hat{\varsigma}_{*I-1}\right)^{-2} z_t \hat{u}^2_{(i-1)t}, \tag{1.73}$$

where $\hat{u}_{(I-1)t} = y_t - x_t\hat{\boldsymbol{\beta}}_{i-1}$ and $\hat{\varsigma}_{*I-1}$ and $\hat{\beta}_{I-1}$ $(I = 2, \dots)$ denote the estimator of ς_* and the feasible GLS estimator of β taken from the previous iteration. Note that for the first iteration $\hat{\varsigma}_{*1} = \hat{\varsigma}_{*A}$.

4. The maximum likelihood (ML) estimator, $\hat{\varsigma}_{*ML}$, which can be obtained by maximising the log-likelihood function

$$\mathcal{L}(\boldsymbol{\beta}, \varsigma_*) = -{}^1\!/\!_2 \sum_{t=1}^{T} log(z_t'\varsigma_*) - {}^1\!/\!_2 \sum_{t=1}^{T} (y_t - x_t'\beta)^2 / (z_t'\varsigma_*). \tag{1.74}$$

1.3.2 Estimators of ρ

Some of the most frequently used estimators of ρ in applied econometric research are:

1. The least squares (LS) estimator

$$\hat{\rho}_{*LS} = \sum_{t=2}^{T} \hat{u}_{(LS)**t} \hat{u}_{(LS)**t-1} / \sum_{t=1}^{T} \left(\hat{u}_{(LS)**t}\right)^2, \tag{1.75}$$

where $\hat{u}_{(LS)**t} = \hat{u}^{(LS)}_t / \hat{\sigma}^{(GQ)}_{*t} = \hat{u}^{(LS)}_t / (z_t'\hat{\varsigma}_{*GQ})^{1/2}$ are the least squares residuals.

2. The Durbin and Watson, 1950, 1951 (DW) estimator, which is computed via the DW-statistic approximation as

$$\hat{\rho}_{DW} = 1 - \left(\frac{DW}{2}\right), \tag{1.76}$$

where DW is the Durbin-Watson statistic.

3. The generalized least squares (GL) estimator

$$\hat{\rho}_{*GL} = \sum_{t=2}^{T} \hat{u}_{(GL)**t}\hat{u}_{(GL)**t-1} / \sum_{t=1}^{T} \left(\hat{u}_{(GL)**t}\right)^2, \tag{1.77}$$

   where $\hat{u}_{(GL)**t} = \hat{u}_t^{(GL)}/\hat{\sigma}_{*t}^{(A)} = \hat{u}_t^{(GL)}/(z_t'\hat{\varsigma}_{*A})^{1/2}$ are the generalized least squares residuals after correcting model (1.1) for both the problems by using any asymptotically efficient estimators of ς_* and ρ.

4. The Prais and Winsten, 1954 estimator $\hat{\rho}_{*PW}$, which, together with the PW estimator $\hat{\beta}_{PW}$ minimises the sum of squared GL residuals.

5. The maximum likelihood (ML) estimator, ρ_{ML}, which satisfies a cubic equation with coefficients defined in terms of the ML residuals in the heteroskedasticity-corrected regression model (1.68) (see Beach and MacKinnon, 1978).

1.3.3 Estimators of $\varsigma = (\varsigma_1, \ldots, \varsigma_m)'$

By using (1.41) we can calculate the following estimators of ς:

$$\hat{\varsigma}_{GQ} = (1-\hat{\rho}_{LS}^2)\hat{\varsigma}_{*GQ}, \tag{1.78}$$

$$\hat{\varsigma}_{A} = (1-\hat{\rho}_{GL}^2)\hat{\varsigma}_{*A}, \tag{1.79}$$

$$\hat{\varsigma}_{IA} = (1-\hat{\rho}^2)\hat{\varsigma}_{*IA}, \tag{1.80}$$

$$\hat{\varsigma}_{ML} = (1-\hat{\rho}^2)\hat{\varsigma}_{*ML}, \tag{1.81}$$

where $\hat{\rho}$ is any asymptotically efficient estimator of ρ.

Chapter 2

Small-Sample size corrections of the t and F tests of the Linear Model with Heteroskedastic and Autocorrelated Disturbances

2.1 Introduction

In this chapter we present the analytical forms of the Edgeworth and Cornish-Fisher size corrections of the t and F tests in the Linear Model with Heteroskedastic and Autocorrelated Disturbances. The purpose of this chapter is the creation of functional formulae for the calculation of corrections using quantities already calculated during the estimation process, presented in the previous chapter. Indeed, the formulae given in Theorems (1) and (2) are a considerable improvement compared to the formulae in Rothenberg, 1984b, Rothenberg, 1988 and Magee, 1989 and they simplify the calculation of Cornish-Fisher and Edgeworth corrections in the case of the linear model with disturbance terms which are a mixture of autocorrelation and heteroskedasticity.

2.2 t-test

Let e_0 be a known scalar and $\boldsymbol{e}$ be a known $n \times 1$ vector. To test the null hypothesis

$$\boldsymbol{e}'\boldsymbol{\beta} - e_0 = 0 \tag{2.1}$$

against one-sided alternatives we use the statistic

$$t = (\boldsymbol{e}'\hat{\boldsymbol{\beta}} - e_0)/[\hat{\sigma}^2\boldsymbol{e}'(\boldsymbol{X}'\widehat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{e}]^{1/2}. \tag{2.2}$$

We define the $(m+1) \times 1$ vector $\boldsymbol{l}$ and the $(m+1) \times (m+1)$ matrix $\boldsymbol{L}$ as follows:

$$\boldsymbol{l} = [(l_i)_{i=1,\dots,m+1}], \quad \boldsymbol{L} = [(l_i)_{i,j=1,\dots,m+1}], \tag{2.3}$$

where

$$l_i = \boldsymbol{e}'\boldsymbol{G}\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e}, \;\; l_{ij} = \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{ij}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e}, \tag{2.4}$$

$$\boldsymbol{G} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}, \quad \boldsymbol{C}_{ij} = \boldsymbol{A}_{ij}{}^* - 2\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{A}_j + \boldsymbol{A}_{ij}/2, \tag{2.5}$$

and the matrices $\boldsymbol{A}_i$, $\boldsymbol{A}_{ij}$ and $\boldsymbol{A}_{ij}^*$ are defined in the equation (1.32). The corrected critical value, using the Edgeworth approximation of the t distribution is given by

$$t_\alpha{}^* = t_\alpha + \frac{\tau^2}{2}[p_1 + p_2 t_\alpha{}^2]t_\alpha, \tag{2.6}$$

(see Edgeworth, 1903). Moreover, the corrected statistic from the Cornish Fisher approximation of the t distribution is given by

$$t^* = t - \frac{\tau^2}{2}\left[p_1 + p_2 t^2\right]t, \tag{2.7}$$

(see, inter alia, Cornish and Fisher, 1937, Fisher and Cornish, 1960, Hill and Davis, 1968). In order to correct either the critical value or the t-statistic the required correction quantities p_1, p_2 are given by the following Proposition.

Proposition 1. The quantities p_1, p_2, required for the calculation of both the Edgeworth corrected critical values of the t distribution, and the Cornish-Fisher corrected t-statistic are:

$$p_1 = \operatorname{tr}\boldsymbol{\Lambda L} + \frac{\boldsymbol{l}'\boldsymbol{\Lambda l}}{4} + \boldsymbol{l}'(\boldsymbol{\kappa} + \frac{\boldsymbol{\lambda}}{2}) - \kappa_0 + \frac{\lambda_0 - 2}{4} \tag{2.8}$$

$$p_2 = \frac{\boldsymbol{l}'\boldsymbol{\Lambda l} - 2\boldsymbol{l}'\boldsymbol{\lambda} + \lambda_0 - 2}{4} \tag{2.9}$$

2.3 F-test

Let $\boldsymbol{H}$ be a $r \times n$ known matrix with $\operatorname{rank}(\boldsymbol{H}) = r$ and $\boldsymbol{h}$ be a known $r \times 1$ vector. The test of the null hypothesis

$$\boldsymbol{H\beta} - \boldsymbol{h} = 0 \tag{2.10}$$

can be based on the Wald statistic

$$w = (\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h})'[\boldsymbol{H}(\boldsymbol{X}'\widehat{\boldsymbol{\Omega}}\boldsymbol{X}/T)^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h})/\hat{\sigma}^2. \tag{2.11}$$

We define the $(m+1) \times 1$ vector $\boldsymbol{c}$ and the $(m+1) \times (m+1)$ matrices $\boldsymbol{C}$, $\boldsymbol{D}$ as follows:

$$\boldsymbol{c} = [(\operatorname{tr}\boldsymbol{A}_i\boldsymbol{P})_{i=1,\dots,m+1}], \quad \boldsymbol{C} = [(\operatorname{tr}\boldsymbol{C}_{ij}\boldsymbol{P})_{i,j=1,\dots,m+1}] \; and \; \boldsymbol{D} = [(\operatorname{tr}\boldsymbol{D}_{ij}\boldsymbol{P})_{i,j=1,\dots,m+1}] \tag{2.12}$$

where matrices $\boldsymbol{A}_i$ and $\boldsymbol{C}_{ij}$ are defined in the equations (1.32), (2.5), respectivelly, and

$$\boldsymbol{P} = \boldsymbol{GQG}, \; \boldsymbol{Q} = \boldsymbol{H}'(\boldsymbol{HGH}')^{-1}\boldsymbol{H}, \; \boldsymbol{D}_{ij} = \boldsymbol{A}_i\boldsymbol{P}\boldsymbol{A}_j/2. \tag{2.13}$$

The corrected critical value, using the Edgeworth approximation of the F distribution is given by

$$F_\alpha{}^* = F_\alpha + \tau^2\left[q_1 + q_2 F_\alpha\right]F_\alpha, \tag{2.14}$$

(see Edgeworth, 1903). Moreover, the corrected statistic from the Cornish Fisher approximation of the F distribution is given by

$$\mathcal{F} = F - \tau^2(q_1 + q_2 F)F, \tag{2.15}$$

(see, inter alia, Cornish and Fisher, 1937, Fisher and Cornish, 1960, Hill and Davis, 1968).

In order to correct either the critical value or the F-statistic the required correction quantities q_1, q_2 are given by the following Proposition.

Proposition 2. The quantities q_1, q_2, required for the calculation of both the Edgeworth corrected critical values of the F distribution and the Cornish-Fisher corrected F-statistic are:

$$q_1 = \xi_1/r + (r-2)/2,\ q_2 = \xi_2/(r+2) - r/2, \tag{2.16}$$

where

$$\xi_1 = \mathrm{tr}[\boldsymbol{\Lambda}(\boldsymbol{C}+\boldsymbol{D})] - \boldsymbol{c}'\boldsymbol{\Lambda}\boldsymbol{c}/4 + \boldsymbol{c}'\boldsymbol{\kappa} + r[\boldsymbol{c}'\boldsymbol{\lambda}/2 - \kappa_0 - (r-2)\lambda_0/4] \tag{2.17}$$

$$\xi_2 = \mathrm{tr}\,(\boldsymbol{\Lambda}\boldsymbol{D}) + [\boldsymbol{c}'\boldsymbol{\Lambda}\boldsymbol{c} - (r+2)(2\boldsymbol{c}'\boldsymbol{\lambda} - r\lambda_0)]/4. \tag{2.18}$$

2.4 Comparison of the t and F tests

We have that

$$\boldsymbol{H} = \boldsymbol{e}',\ \boldsymbol{h} = \boldsymbol{e}_0,\ r = 1. \tag{2.19}$$

Let

$$\boldsymbol{k} = \boldsymbol{e}/(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}, \tag{2.20}$$

Equations (2.13), (2.19) and (2.20) we find

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{H}'(\boldsymbol{H}\boldsymbol{G}\boldsymbol{H}')^{-1}\boldsymbol{H} = \boldsymbol{e}(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{-1}\boldsymbol{e}' = \boldsymbol{k}\boldsymbol{k}' \\ &\text{and} \\ \boldsymbol{P} &= \boldsymbol{G}\boldsymbol{Q}\boldsymbol{G} = \boldsymbol{G}\boldsymbol{k}\boldsymbol{k}'\boldsymbol{G}. \end{aligned} \tag{2.21}$$

From equations (2.3), (2.4), (2.5), (2.12), (2.20) and (2.21) we get the following results:

$$l_i = \boldsymbol{e}'\boldsymbol{G}\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} = \boldsymbol{k}\boldsymbol{G}\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{k} = \mathrm{tr}\,\boldsymbol{k}\boldsymbol{G}\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{k} = \mathrm{tr}\,\boldsymbol{A}_i\boldsymbol{k}\boldsymbol{G}\boldsymbol{G}\boldsymbol{k} = \mathrm{tr}\,\boldsymbol{A}_i\boldsymbol{P} \tag{2.22a}$$

$$l_{ij} = \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{ij}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} = \boldsymbol{k}\boldsymbol{G}\boldsymbol{C}_{ij}\boldsymbol{G}\boldsymbol{k} = \mathrm{tr}\,\boldsymbol{k}\boldsymbol{G}\boldsymbol{C}_{ij}\boldsymbol{G}\boldsymbol{k} = \mathrm{tr}\,\boldsymbol{C}_{ij}\boldsymbol{k}\boldsymbol{G}\boldsymbol{G}\boldsymbol{k} = \mathrm{tr}\,\boldsymbol{C}_{ij}\boldsymbol{P} \tag{2.22b}$$

Using equations

$$p_1 = \sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\lambda_{ij}(l_{ij}+\frac{1}{4}l_il_j)+\sum_{i=1}^{m+1}l_i(\kappa_i+\frac{1}{2}\lambda_{i0})+\frac{1}{4}\lambda_0-\kappa_0-\frac{1}{2}, \tag{2.23}$$

$$p_2 = \frac{1}{4}(\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\lambda_{ij}l_il_j - 2\sum_{i=1}^{m+1}l_i\lambda_{i0}+\lambda_0)-\frac{1}{2} \tag{2.24}$$

and

$$\begin{aligned} h_1 &= \sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\lambda_{ij}(\operatorname{tr}\boldsymbol{C}_{ij}\boldsymbol{P})-\frac{1}{4}\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\lambda_{ij}[(\operatorname{tr}\boldsymbol{A}_i\boldsymbol{P})(\operatorname{tr}\boldsymbol{A}_j\boldsymbol{P})-2(\operatorname{tr}\boldsymbol{A}_i\boldsymbol{P}\boldsymbol{A}_j\boldsymbol{P})]+ \\ &\quad +\sum_{i=1}^{m+1}(\kappa_i+\frac{r}{2}\lambda_{i0})(\operatorname{tr}\boldsymbol{A}_i\boldsymbol{P})-r(\kappa_0+\frac{r-2}{4}\lambda_0) \\ &= \sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\lambda_{ij}(l_{ij}+\frac{1}{4}l_il_j)+\sum_{i=1}^{m+1}l_i(\kappa_i+\frac{1}{2}\lambda_{i0})+\frac{1}{4}\lambda_0-\kappa_0, \end{aligned} \tag{2.25}$$

$$\begin{aligned} h_2 &= \frac{1}{4}\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\lambda_{ij}(\operatorname{tr}\boldsymbol{A}_i\boldsymbol{P})(\operatorname{tr}\boldsymbol{A}_j\boldsymbol{P})+\frac{1}{2}\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\lambda_{ij}(\operatorname{tr}\boldsymbol{A}_i\boldsymbol{P}\boldsymbol{A}_j\boldsymbol{P}) \\ &\quad -\frac{r+2}{2}\sum_{i=1}^{m+1}\lambda_{i0}(\operatorname{tr}\boldsymbol{A}_i\boldsymbol{P})+\frac{r(r+2)}{4}\lambda_0) \\ &= \frac{1}{4}(\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\lambda_{ij}l_il_j-2\sum_{i=1}^{m+1}l_i\lambda_{i0}+\lambda_0), \end{aligned} \tag{2.26}$$

We can prove that

$$q_1 = h_1-\frac{1}{2}=p_1,\;\; q_2=\frac{h_2}{3}-\frac{1}{2}=p_2, \tag{2.27}$$

(see Symeonides, 1991). Therefore, the corrected critical value, using the Edgeworth approximation of the t distribution is

$$t^*{}_{\alpha/2} = t_{\alpha/2}+\frac{\tau^2}{2}(p_1+p_2t^2{}_{\alpha/2})t_{\alpha/2}, \tag{2.28}$$

and the corrected critical value, using the Edgeworth approximation of the F distribution is

$$F_\alpha{}^* = F_\alpha+\tau^2(q_1+q_2F_\alpha)F_\alpha, \tag{2.29}$$

Using equations (2.27) (2.28), (2.29), and given that $t^2{}_{\alpha/2}=F_\alpha$ we have that

$$\begin{aligned} (t^*{}_{\alpha/2})^2 &= [t_{\alpha/2}+\frac{\tau^2}{2}(p_1+p_2t^2{}_{\alpha/2})t_{\alpha/2}]^2 \\ &= t^2{}_{\alpha/2}+2\frac{\tau^2}{2}(p_1+p_2t^2{}_{\alpha/2})t^2{}_{\alpha/2}+O(\tau^4) \\ &= F_\alpha+\tau^2(q_1+q_2F_\alpha)F_\alpha+O(\tau^4)=F_\alpha{}^*+O(\tau^4). \end{aligned} \tag{2.30}$$

2.5 Theorems

Theorem 1. Vectors $\boldsymbol{l}$, $\boldsymbol{c}$ and matrices $\boldsymbol{L}$, $\boldsymbol{C}$, $\boldsymbol{D}$, in equations (2.3) and (2.12) can be calculated by the following formulae:

$$\boldsymbol{l} = \begin{bmatrix} \left[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\rho} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \\ \left[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'1} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \\ \vdots \\ \left[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'm} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \end{bmatrix}. \tag{2.31}$$

$$\begin{aligned} l_{ij} \quad = \quad & \left[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{r=1}^{T} \sum_{m=1}^{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'i} \omega^{t'm} \omega_{mrj} x_{r\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \\ & -2 \sum_{\kappa_1=1}^{n} \left[\sum_{\kappa_2=1}^{n} \sum_{d_2=1}^{n} \sum_{d_1=1}^{n} \left[\left(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{\kappa_1 t} \omega_{tt'i} x_{t'd_1} \right) g_{d_1 d_2} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{d_2 t} \omega_{tt'j} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \\ & + \frac{1}{2} \sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'ij} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right], \end{aligned} \tag{2.32}$$

where

$$\boldsymbol{L} = [(l_{ij})_{i,j=(\rho,1,\ldots,m)}]. \tag{2.33}$$

$$\boldsymbol{c} = \begin{bmatrix} \left[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\rho} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \\ \left[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'1} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \\ \vdots \\ \left[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'm} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \end{bmatrix}. \tag{2.34}$$

$$\begin{aligned} c_{ij} \quad = \quad & \left[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{r=1}^{T} \sum_{m=1}^{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'i} \omega^{t'm} \omega_{mrj} x_{r\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \\ & -2 \sum_{\kappa_1=1}^{n} \left[\sum_{\kappa_2=1}^{n} \sum_{d_2=1}^{n} \sum_{d_1=1}^{n} \left[\left(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{\kappa_1 t} \omega_{tt'i} x_{t'd_1} \right) g_{d_1 d_2} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{d_2 t} \omega_{tt'j} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right] \right] \\ & + \frac{1}{2} \sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \left[\frac{1}{T} \left(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'ij} x_{t'\kappa_2} \right) p_{\kappa_2 \kappa_1} \right], \end{aligned} \tag{2.35}$$

where

$$\boldsymbol{C} = [(c_{ij})_{i,j=(\rho,1,\ldots,m)}]. \tag{2.36}$$

$$d_{ij} = \frac{1}{2}\sum_{\kappa_1=1}^{n}\Big[\sum_{\kappa_2=1}^{n}\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\Big[\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Big)p_{d_1d_2}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2t}\omega_{tt'j}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\Big]\Big], \tag{2.37}$$

where

$$\boldsymbol{D} = [(d_{ij})_{i,j=(\rho,1,\ldots,m)}]. \tag{2.38}$$

Theorem 2. Given the hypotheses of model (1.1) and for each asymptotically efficient estimator of ρ and ς, the parameters (1.33) are:

$$\begin{aligned}\lambda_0 = {} & 2 - 2\boldsymbol{a}'\lim_{T\to\infty} 2\varsigma\Big[\Big(\frac{1+\rho^2}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2} - \frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\Big)\\ & -\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) - \sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\ldots,T}\Big]\\ & -2O(\tau^2)\boldsymbol{a}'\lim_{T\to\infty}\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2}) - \sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\ldots,T}\Big]\\ & +\boldsymbol{a}'\boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a} + O(\tau^4).\end{aligned} \tag{2.39}$$

$$\begin{aligned}\kappa_0 = {} & -1 + \lim_{T\to\infty}\Big(-\operatorname{tr}\varsigma\Big[\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Big)_{l=1,\ldots,T,\ i=1,\ldots,m}\Big]\Big) - \boldsymbol{a}'\kappa_\varsigma + \operatorname{tr}\bar{\boldsymbol{A}}\boldsymbol{\Lambda}_{\varsigma\varsigma}\\ & -O(\tau^4)\lim_{T\to\infty}\Big[-(\alpha/2\rho\alpha)[2(\rho^2-n\alpha)+\alpha\operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}+\operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}]] + O(\tau^2)\Big]\\ & +\boldsymbol{a}'_{\rho\varsigma}\lim_{T\to\infty}\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2}) - \sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\ldots,T}\Big].\end{aligned} \tag{2.40}$$

$$\lambda_{0\rho} = -\boldsymbol{a}'\lim_{T\to\infty}\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2}) - \sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\ldots,T}\Big] + O(\tau^2). \tag{2.41}$$

$$\begin{aligned}\lambda_{0\varsigma} = {} & \lim_{T\to\infty} 2\varsigma\Big[\Big(\frac{1+\rho^2}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2} - \frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\Big) - \frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\\ & -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\ldots,T}\Big]\\ & -O(\tau^2)\lim_{T\to\infty}\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2}) - \sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\ldots,T}\Big]\\ & -\boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a}.\end{aligned} \tag{2.42}$$

$$\lambda_{\rho\rho} = \lim_{T\to\infty}\mathrm{E}(\rho_1{}^2) = \alpha. \tag{2.43}$$

$$\lambda_{\rho\varsigma} = \lim_{T\to\infty}\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2}) - \sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\ldots,T}\Big]. \tag{2.44}$$

For the GQ estimator of $\boldsymbol{\varsigma}$, matrix $\boldsymbol{\Lambda}_{\varsigma\varsigma}$ can be estimated as

$$\boldsymbol{\Lambda}_{\varsigma\varsigma} = 2\bar{\boldsymbol{B}}\bar{\boldsymbol{\Gamma}}_H\bar{\boldsymbol{B}}. \tag{2.45}$$

For the A, IA and ML estimators of $\boldsymbol{\varsigma}$ matrix $\boldsymbol{\Lambda}_{\varsigma\varsigma}$ can be estimated as

$$\boldsymbol{\Lambda}_{\varsigma\varsigma}{}^{A} = 2\bar{\boldsymbol{G}}_H. \tag{2.46}$$

Also, depending on the estimator of ρ being used we get:

$$\kappa_{LS} = -[(n+3)\rho + (c_1 - 2n)/2\rho], \tag{2.47}$$

where $c_1 = \alpha \operatorname{tr} \boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \operatorname{tr} \boldsymbol{A}_{AR} \operatorname{tr} \boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} \operatorname{tr} \boldsymbol{B}_{AR}$.

$$\kappa_{GL} = \kappa_{PW} = \kappa_{LS} - \alpha c_2/2\rho + (c_1 - \alpha n)/2\rho, \tag{2.48}$$

where $c_2 = \alpha \operatorname{tr} \boldsymbol{F}_{AR}\boldsymbol{G}_{AR}$.

$$\kappa_{ML} = \kappa_{PW} + \rho = \kappa_{GL} + \rho. \tag{2.49}$$

$$\kappa_{DW} = \kappa_{LS} + 1. \tag{2.50}$$

Also, depending on the estimator of $\boldsymbol{\varsigma}$ being used we get:

For the GQ estimator of $\boldsymbol{\varsigma}$, κ_{ς} expressed as

$$\boldsymbol{\kappa}_{\varsigma} = -\bar{\boldsymbol{B}}\boldsymbol{\xi}_H, \tag{2.51}$$

where $\bar{\boldsymbol{B}} = (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}$. For the A estimator of $\boldsymbol{\varsigma}$, $\boldsymbol{\kappa}$ can be estimated as

$$\boldsymbol{\kappa}_{\varsigma} = -\bar{\boldsymbol{G}}_H\xi_{H1} - 4\bar{\boldsymbol{G}}_H\sum_{i=1}^{m}[\bar{\boldsymbol{A}}_{H\varsigma_i}g_{Hi} - (\boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T)\bar{b}_i \tag{2.52}$$

where $\bar{\boldsymbol{A}}_{H\varsigma_i} = \boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T, \bar{\boldsymbol{g}}_i$ is the i-th column of matrix $\bar{\boldsymbol{G}}_H$ and $\bar{b}_i$ is the i-th column of matrix $\bar{\boldsymbol{B}}_H$.

For IA and ML estimators of $\boldsymbol{\varsigma}$ we have that

$$\boldsymbol{\kappa}_{\varsigma} = -\bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H2}. \tag{2.53}$$

2.6 Experimental Procedure of The Linear Model with Heteroskedastic and Autocorrelated Disturbances

In this section we will theoretically describe an experimental procedure which could be used in order to investigate the performance of various size corrections of the t- and F-tests in the case of the linear model with heteroskedasticity and autocorrelation in the disturbances. The performance of various size corrections of t- and F-tests can be measured as the difference between the true and the nominal size of the corrected tests.

For the simulation we consider a four-parameter linear model as follows:

$$y_t = \beta_1 x_{t1} + \beta_2 x_{t2} + \beta_3 x_{t3} + \beta_4 x_{t4} + \sigma u_t, \;\; (t = 1, \ldots, T), \tag{2.54}$$

where β_j the parameters to be estimated and $x_{t1} = 1 \; \forall t$. We considered sample sizes of $T(15, 20, 30)$ observations.

For the error term we assume that

$$\mathrm{E}(u_t) = 0, \; {\sigma_{u_t}}^2 = \mathrm{var}(u_t) = \frac{{\sigma_t}^2}{1-\rho^2} = z_t' \varsigma_* = z_t'[\varsigma/(1-\rho^2)] \tag{2.55}$$

where

$$z_t' = (1, x_{t2}, x_{t3}, z_{t4}), \;\; \varsigma = (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4). \tag{2.56}$$

It is clear that, given the vectors z_t' $(t = 1, \ldots, T)$ the variances ${\sigma_{u_t}}^2$, and consequently the intensity of the considered mixture of heteroskedasticity and autocorrelation, depend on the values of the coordinates of the vector ς and the parameter ρ. Multicollinearity describes a situation in which different variables reflect related variation, where the A is the coefficient which states the intensity of multicollinearity between any two interpretative variables except the constant.

Each combination of the values of the parameters ρ, ς, and A constitutes a point of the experimental space which we try to make representative of the parameter space defined by the sets of possible values of the parameters ρ, ς, and A.

For this purpose we considered six values of the vector ς

$$\begin{aligned} \varsigma_{(1)}' &= (\varsigma_1, 0, 0, 0), \; \varsigma_{(2)}' = (\varsigma_1, 1, 0, 0), \; \varsigma_{(3)}' = (\varsigma_1, 0, 0, 1) \\ \varsigma_{(4)}' &= (\varsigma_1, 1, 1, 0), \; \varsigma_{(5)}' = (\varsigma_1, 1, 0, 1), \; \varsigma_{(6)}' = (\varsigma_1, 1, 1, 1), \end{aligned} \tag{2.57}$$

six values of the parameter ρ

$$\rho = \pm 0.1, \;\; \rho = \pm 0.5, \;\; \rho = \pm 0.9, \tag{2.58}$$

and four values of the coefficient which states the intensity of multicollinearity between any two interpretative variables except the constant, i.e., (A = 0.0, 0.1, 0.5, 0.9). At each experimental point the value of the vector of the parameters ς is determined in a manner described in more detail below. Combining the values of the parameters ρ, ς and A, we can create our experimental space, which consists of 144 points. The experimental space we use is representative of all the combinations of heteroscedasticity, autocorrelation, and multicollinearity that can be encountered in applied econometric research. It contains points showing high, moderate, low, or no multicollinearity, and autocorrelation combined with heteroscedasticity. The cases $\rho = 0$ and $\rho = 1$ will not be studied experimentally because if $\rho = 0$ there is no autocorrelation to be examined and if $\rho = 1$ the AR(1) process is not stationary. The cases $\boldsymbol{\varsigma}' = (\varsigma_1, 0, 0, 0)$ will be studied experimentally because we are interested in investigating the consequences of the Edgeworth and Cornish-Fisher corrections of the t and F tests in the case where the error term is homoscedastic. The cases $A = 0$ are very rare in applied research but will be studied experimentally because they give us information on the behavior of the Edgeworth and Cornish-Fisher corrections of the t and F tests in the "ideal" case in which there is no multicollinearity and the regressors are linearly independent.

For each combination of the values of the parameters ρ, ς and A, a matrix of explanatory variables can be created as follows: Using some random number generator, we can generate T independent observations for the four independent $N(0, 1)$ pseudorandom numbers $\zeta_{t1}, \zeta_{t2}, \zeta_{t3}, \zeta_{t4}$ $(t = 1, \ldots, T)$. Following McDonald and Galarneau, 1975 (p.409) we can construct the elements x_{tj}, of the matrix of explanatory variables, $\boldsymbol{X}$, using the following relations:

$$\begin{aligned} x_{tj} &= 1 \;\; (t = 1, \ldots, T \text{ and } j = 1) \\ \text{and} & \\ x_{tj} &= (1 - A)^{1/2} \zeta_{tj} + \sqrt{A} \zeta_{t1} \;\; (t = 1, \ldots, T \text{ and } j = 2, 3, 4), \end{aligned} \tag{2.59}$$

from which it follows that the correlation coefficient between any two explanatory variables, excluding the constant, is A. We must note that the matrix $\boldsymbol{X}$ can be accepted and used by the experiment under the assumption that the matrix $(\boldsymbol{X}'\boldsymbol{X})$ can be inverted. If the matrix $\boldsymbol{X}$ is rejected, the procedure must be repeated until we obtain a matrix $\boldsymbol{X}$ such that the matrix $(\boldsymbol{X}'\boldsymbol{X})$ is invertible. Since the variance ${\sigma_t}^2$ for each observation of the stochastic term u_t is given by equation (2.55), it follows that the calculation of all ${\sigma_t}^2$ $(t = 1, \ldots, T)$ requires the knowledge of the matrix Z with $z_t' = (z_{t1}, z_{t2}, z_{t3}, z_{t4})$ rows. From equation (2.56) it is clear that the first three columns of matrices $\boldsymbol{Z}$ and $\boldsymbol{X}$ are identical. To construct the fourth column of the $\boldsymbol{Z}$ matrix we generate T independent $N(0, 1)$ pseudorandom observations. Consequently, for every $\boldsymbol{X}$ matrix we also made a $\boldsymbol{Z}$ one. We must note that the matrix $\boldsymbol{Z}$ is accepted and used by the experiment under the assumption that the matrix $(\boldsymbol{Z}'\boldsymbol{Z})$ can be inverted. If the matrix $\boldsymbol{Z}$ is rejected the procedure is repeated until we obtain a matrix $\boldsymbol{Z}$ such that the matrix $(\boldsymbol{Z}'\boldsymbol{Z})$ is invertible.

Each pair of matrices $\boldsymbol{X}$ and $\boldsymbol{Z}$, created for a given combination of the values of the parameters ρ, $\boldsymbol{\varsigma}$, and A, can be used in 10.000 replications of the experiment. For each of these replications we construct a vector $\boldsymbol{y}$. Next we will describe the construction of each of these 10.000 vectors.

Without loss of generality, our interest will be limited to the study of the case with ${\sigma_t}^2 \geq 1$ $(t = 1, \ldots, T)$. (Cases with $0 < {\sigma_t}^2 < 1$ are handled by using the inverse of ${\sigma_t}^2$ instead of ${\sigma_t}^2$ for all t). For this purpose, for each replication of the experiment, we must create an error term vector $\boldsymbol{u}$ with elements u_t, such that

$$\text{var}(u_t) = {\sigma_t}^2 \geq 1 \ \ (t = 1, \ldots, T). \tag{2.60}$$

However, from equation (2.55) it is understood that, given the vector $\boldsymbol{z}_t'$, the relation $\boldsymbol{z}_t'\boldsymbol{\varsigma} \geq 1$ is not satisfied for every vector $\boldsymbol{\varsigma}$. This problem can be solved as follows: First, we assumed that the vector $\boldsymbol{\varsigma}$ is of the form $\boldsymbol{\varsigma}' = (0, \varsigma_2, \varsigma_3, \varsigma_4)$, therefore

$${\sigma_t}^* = \boldsymbol{z}_t'\boldsymbol{\varsigma} = \sum_{j=2}^{4} z_{tj}\varsigma_j \ \ (t = 1, \ldots, T). \tag{2.61}$$

Then we set $\sigma_{min} = \min {\sigma_t}^* \ (t = 1, \ldots, T)$ and calculated the first coordinate of the vector $\boldsymbol{\varsigma}$ as $\varsigma_1 = 1 - \sigma_{min}$, getting $\boldsymbol{\varsigma}' = (1 - \sigma_{min}, \varsigma_2, \varsigma_3, \varsigma_4)$. Since $z_{t1} = 1 \ \ (t = 1, \ldots, T)$, from (2.61) we get:

$${\sigma_t}^2 = \boldsymbol{z}_t'\boldsymbol{\varsigma} = \sum_{j=1}^{4} z_{tj}\varsigma_j = 1 - \sigma_{min} + {\sigma_t}^* \geq 1 \ \ (t = 1, \ldots, T). \tag{2.62}$$

The calculation of the first coordinate of the vector $\boldsymbol{\varsigma}$ as $\varsigma_1 = 1 - \sigma_{min}$ ensures us that all the variances ${\sigma_t}^2$ will be greater than or equal to 1. However, it creates serious problems by increasing the effect of the constant z_{t1} in shaping the value of ${\sigma_t}^2$ and consequently minimazing the intensity of the problem of heteroskedasticity. Consequently, in order to be able to combine the existence of significant heteroskedasticity with variances ${\sigma_t}^2 \geq 1$, we set an upper limit to the value of the first coordinate of the vector $\boldsymbol{\varsigma}$. Specifically, since the coordinates ς_2, ς_3 and ς_4 take values of 0 or 1, we decided to discard each vector $\boldsymbol{\varsigma}$ for which the coordinate ς_1 is greater than or equal to 4 and to repeat the entire process initiating from the creation of the matrix $\boldsymbol{X}$ until the calculation of vector $\boldsymbol{\varsigma}$ which satisfies equation (2.62). Having calculated the variances ${\sigma_t}^2 \geq 1$ from equation (2.62) it is very easy to construct a vector of heteroskedastic and autocorrelated error terms, $\boldsymbol{u}$.

Using random numbers we can construct T for $N(0, 1/(1 - \rho^2))$ numbers u_{*t} $(t = 1, \ldots, T)$. The elements u_t of the error vector, $\boldsymbol{u}$ can be constructed using the relation:

$$u_t = \sigma_t u_{*t}, \ \sigma_t = \sqrt{{\sigma_t}^2} \ \ (t = 1, \ldots, T), \tag{2.63}$$

from which it follows that the variance of each u_t is equal to ${\sigma_t}^2 \geq 1$.

Knowing the vector $\boldsymbol{u}$ we can create the vector of the dependent variable, $\boldsymbol{y}$, with elements, y_t, using equation (2.54) where β_j are the parameters of the model to be estimated. From Theorem 5 of Breusch, 1980 p. 336, and taking into account that the t and F statistics arise as special cases of the Wald statistic, it follows that the distributions of the t and F statistics for testing hypotheses (2.1) and (2.10) do not depend on the true values of the parameters β_j $(j = 1, \ldots, 4)$ of model (2.54), when the null hypothesis is true. Since the study of the actual size of a test is done under the assumption that the null hypothesis is true, it is clear that the results of the experiment do not depend on the values of the parameters

β_j $(j = 1, \ldots, 4)$. So, we can set $\beta_j = 0$ $(j = 1, \ldots, 4)$. Thus, we simplified the computational procedure of the experiment, while our results did not lose their generality. Setting $\beta_j = 0$ $(j = 1, \ldots, 4)$ in equation (2.54) we get

$$y_t = \sigma u_t \;\; (t = 1, \ldots, 20), \tag{2.64}$$

from which we calculated the elements of the vectors $\boldsymbol{y}$ of the dependent variable for each of the 10.000 replications of the experiment, given the matrix of exogenous variables.

Since $\beta_j = 0$ $(j = 1, \ldots, 4)$ the null hypotheses of the tests are:

$$\beta_1 = 0, \; \beta_2 = 0, \; \beta_3 = 0, \; \beta_4 = 0, \tag{2.65}$$

and the null hypothesis of the F test is:

$$\boldsymbol{H\beta} = \boldsymbol{h}, \text{ where } \boldsymbol{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{h} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.66}$$

Using the matrix of regressors, $\boldsymbol{X}$, the matrix $\boldsymbol{Z}$ and the 10.000 different vectors of the dependent variable, $\boldsymbol{y}$, created for each of the 144 points of the experimental space, at each we can construct 10.000 replications of the procedure that is described below .

1. We estimate model (2.54) using the OLS estimator:

$$\hat{\boldsymbol{\beta}}_{OLS} = \left[\sum_{t=1}^{20} \boldsymbol{x}_t \boldsymbol{x}_t' \right]^{-1} \sum_{t=1}^{20} \boldsymbol{x}_t y_t. \tag{2.67}$$

   Then, we calculate the OLS residuals:

$$\hat{\boldsymbol{u}}_{OLS} = \boldsymbol{y} - \hat{\boldsymbol{\beta}}_{OLS} \boldsymbol{X} = \left[(\hat{u}_{t(OLS)})_{t=1,\ldots,T} \right]. \tag{2.68}$$

2. We can use one of the consistent estimators given in Subsection 1.3.1 in order to estimate the parameter vector $\boldsymbol{\varsigma}_* = (\varsigma_{*1}, \ldots, \varsigma_{*m})'$. Then, we can estimate matrix $\boldsymbol{\Sigma}_*^{-1/2}$ as

$$\hat{\boldsymbol{\Sigma}}_*^{-1/2} = \mathrm{diag}(1/\hat{\sigma}_{u_1}, \ldots, 1/\hat{\sigma}_{u_T}), \tag{2.69}$$

   where

$$\hat{\sigma}_{u_t} = \left(\boldsymbol{z}_{t.}^{\top} \hat{\boldsymbol{\varsigma}}_* \right)^{1/2} \quad \forall\, t = 1, \ldots, T. \tag{2.70}$$

3. We can estimate the heteroskedasticity-corrected residuals

$$\hat{\boldsymbol{u}}_{*OLS} = \hat{\boldsymbol{\Sigma}}_*^{-1/2} \hat{\boldsymbol{u}}_{OLS} = \left[(\hat{u}_{*t(OLS)})_{t=1,\ldots,T} \right], \tag{2.71}$$

where

$$\hat{u}_{*t(OLS)} = \frac{\hat{u}_{t(OLS)}}{\hat{\sigma}_{u_t}} \quad \forall\, t = 1, \ldots, T, \tag{2.72}$$

and $\hat{\boldsymbol{u}}_{OLS}$ is the predictor of $\boldsymbol{u}$ estimated by (6.4).

4. We can use one of the consistent estimators given in Subsection 1.3.2 in order to calculate an initial estimate $\hat{\rho}_*$ of the autocorrelation coefficient ρ.

5. We can use equation (1.41) and the consistent estimators $\hat{\boldsymbol{\varsigma}}_*$ and $\hat{\rho}_*$ in order to estimate the parameter vector $\boldsymbol{\varsigma}$ as

$$\hat{\boldsymbol{\varsigma}} = \hat{\boldsymbol{\varsigma}}_*(1 - \hat{\rho}_*^2) \implies \hat{\varsigma}_i = \hat{\varsigma}_{*i}(1 - \hat{\rho}_*^2) \quad \forall\, i = 1, \ldots, m. \tag{2.73}$$

Then, we can estimate matrix $\boldsymbol{\Sigma}^{-1/2}$ as

$$\hat{\boldsymbol{\Sigma}}^{-1/2} = \mathrm{diag}(1/\hat{\sigma}_1, \ldots, 1/\hat{\sigma}_T), \tag{2.74}$$

where

$$\hat{\sigma}_t = \left(\boldsymbol{z}_{t.}^\top \hat{\boldsymbol{\varsigma}}\right)^{1/2} \quad \forall\, t = 1, \ldots, T. \tag{2.75}$$

Alternatively, $\boldsymbol{\varsigma}$ can be estimated via the following asymptotically equivalent process:

(i) We can use the initial estimator $\hat{\rho}_*$ in order to transform model (2.54) into the autoregression-corrected model

$$\boldsymbol{y}_H = \boldsymbol{X}_H\boldsymbol{\beta} + \boldsymbol{u}_H, \tag{2.76}$$

where the elements of vector $\boldsymbol{u}_H = \left[(u_{_Ht})_{t=1,\ldots,T}\right]$ are purely heteroskedastic disturbances, given by the following formulae:

$$u_{_H1} = (1 - \hat{\rho}_*^2)^{1/2} u_1, \quad u_{_Ht} = u_t - \hat{\rho}_* u_{t-1} \quad \forall\, t = 2, \ldots, T. \tag{2.77}$$

(ii) Then, we can use one of the consistent estimators given in Subsection 1.3.1 in order to estimate the parameter vector $\boldsymbol{\varsigma}$, and the matrix $\boldsymbol{\Sigma}^{-1/2}$ via (2.74) and (2.75).

Although from the estimation viewpoint the estimator (2.73) is perfectly adequate as a consistent estimator of $\boldsymbol{\varsigma}$, the estimator $\hat{\boldsymbol{\varsigma}}$ based on the residuals of model (2.76) enables the researcher to find the finite-sample distributional properties of any consistent estimator of $\boldsymbol{\varsigma}$ in Subsection 1.3.1.

6. We can premultiply model (2.54) by $\hat{\boldsymbol{\Sigma}}^{-1/2}$ given in (2.74), in order to derive heteroskedasticity-corrected model

$$\boldsymbol{y}_{AR} = \boldsymbol{X}_{AR}\boldsymbol{\beta} + \boldsymbol{u}_{AR}, \tag{2.78}$$

where the elements of vector $\boldsymbol{u}_{AR} = \left[(u_{_{AR}t})_{t=1,\ldots,T}\right]$ are purely autoregressive disturbances, given by the following formula:

$$u_{_{AR}t} = u_t/\hat{\sigma}_t \quad \forall\, t = 1, \ldots, T, \tag{2.79}$$

where $\hat{\sigma}_t$ are given in (2.75). Then, we can use one of the consistent estimators given in Subsection 1.3.2 in order to estimate the autocorrelation coefficient ρ. The estimator $\hat{\rho}$ based on the residuals of model (2.78) enables the researcher to find the finite-sample distributional properties of any consistent estimator of ρ in Subsection 1.3.2.

7. We can use the estimators $\hat{\boldsymbol{\Sigma}}^{-1/2}$ and $\hat{\rho}$ from Steps 5 and 6, respectively, in order to calculate the estimator

$$\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Sigma}}^{-1/2}[(1+\hat{\rho}^2)\boldsymbol{I}_T - \hat{\rho}\boldsymbol{D} - \hat{\rho}^2\boldsymbol{\Delta}]\hat{\boldsymbol{\Sigma}}^{-1/2}, \tag{2.80}$$

which can be used for the feasible generalized least squares estimation of model (2.54).

From this estimation strategy we calculate residuals that are exclusively autocorrelated and residuals that are exclusively heteroscedastic in order to calculate the estimate of the parameters ρ and $\boldsymbol{\varsigma}$, respectively. Then we can calculate the feasible GLS estimator

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{y} \tag{2.81}$$

of the parameter vector $\boldsymbol{\beta}$ of model (2.54). Then, using Cornish-Fisher corrected t statistic $t^* = t - \frac{\tau^2}{2}\left[p_1 + p_2 t^2\right]t$ and Cornish-Fisher corrected F statistic $\hat{\mathcal{F}} = F - \tau^2(q_1 + q_2 F)F$ we can test the null hypotheses

Having at our disposal the estimators of the parameter ρ and the vector $\boldsymbol{\varsigma}$ but also the GLS estimators of the parameters of the model (2.54) and using the corrected statistic from the Cornish Fisher approximation of the normal distribution that givens by and the corrected statistic from the Cornish Fisher approximation of the F distribution that givens by we calculate the values of the t and F statistics as well as the values of locally exact according to Cornish -Fisher corrected t and F statistics for testing hypotheses (2.65) and (2.66) against the alternative hypotheses

$$\beta_j > 0 \text{ or } \beta_j < 0 \ \ (j = 1, \ldots, 4) \tag{2.82}$$

and

$$\boldsymbol{H\beta} \neq \boldsymbol{h}, \tag{2.83}$$

respectively, where the matrix H and the vector h are defined in equation (2.66). Let $I_{T-n}(\cdot)$, $i_{T-n}(\cdot)$ be the distribution and density functions, respectively, of a t-random variable with $T-n$ d.o.f. Also, let t_a be the α% critical value of the t-distribution. Then, under the null hypothesis $\boldsymbol{e}'\boldsymbol{\beta} - e_0 = 0$, the distribution function of the t-statistic admits an Edgeworth expansion of the form:

$$\Pr(t \leq \xi) = I_{T-n}(\xi) - \frac{\tau^2}{2}(p_1 + p_2\xi^2)\xi i_{t-n}(\xi) + O(\tau^3). \tag{2.84}$$

Moreover, let $F^r_{T-n}(\cdot)$, $f^r_{T-n}(\cdot)$ be the distribution and density functions, respectively, of a F-random variable with r and $T-n$ d.o.f. Also, let F_a be the upper α% critical value of the F-distribution. Then, under the null hypothesis $\boldsymbol{H\beta} - \boldsymbol{h} = 0$, the distribution function of the F-statistic admits an Edgeworth expansion of the form:

$$\Pr(F \le \xi) = F^r_{T-n}(\xi) - \tau^2(q_1 + q_2\xi^2)\xi f^r_{T-n}(\xi) + O(\tau^3). \tag{2.85}$$

Concluding our reference to the method of calculating the various statistics and the corresponding significance levels, we consider it appropriate to emphasize that we use one-sided alternative hypotheses (2.82) for two reasons: First, because the t-test for each of the hypotheses (2.65) against two-sided alternative hypotheses is a special case of F test, and secondly, because the Edgeworth expansions of the t-Student density functions are not symmetric about zero and therefore the level of significance corresponding to the corrected critical value of the usual t statistic for $t = t_0$ is generally different from the corresponding significance level for $t = -t_0$. The procedure we have just described can be replicated 10000 times at each of the 144 points of the experimental space. By using the values of these statistics and the density functions of the t-Student and F distribution respectivelly, we can calculate the corresponding p-values.More specifically, we can calculate the significance level of the t statistic (see (2.7)) under the assumption that it is distributed according to the t-Student and the significance level of the F statistic under the assumption that it is distributed according to the F distribution. Furthermore, the significance levels of the locally exact Cornish-Fisher corrected t and F statistics can be calculated under the assumption that they follow the t-Student and F distributions, respectively. At this point it should be noted that the Cornish-Fisher corrected F statistic (see (2.15)) may admit negative values, and in such a case we have a major problem given that the Cornish-Fisher corrected F statistic (see (2.15)) is assumed to be distributed as an F variable.

All that remains is the calculation of the significance levels corresponding to the Edgeworth corrected critical values of the t and F statistics. First, we will calculate the values of the Edgeworth expansions of the distribution functions of t and F statistics in terms of t-Student (see (2.84)) and of the F (see (2.85)) distribution, respectively, for the specific values of these statistics. The required significance levels for the F, and positive t statistics are equal to the values of the Edgeworth expansions of the distribution functions of these statistics.

From the performance of random experiments concerning the case of linear regression model with autocorrelation AR(1) as well as the case of the linear regression model with heteroskedasticity we deduced the following: The performance of various t and F test forms is affected either from the specializations of vector ς or from the theorized values of parameter A. About parameter ρ, locally exact Edgeworth size corrections of t and F test are preferable for the t and F tests for small and intermediary values of ρ(e.g. $\rho = \pm.1$ or$\rho = \pm.5$). In both experiments, locally exact Cornish-Fisher size corrections of t and F tests are preferable to the respective locally exact Edgeworth corrections in almost every point of the parameter space. Finally, for the t and F tests with a mixture of autocorrelation and heteroskedasticity, we expect that the locally exact Edgeworth corrections to be preferable for A, ς as well as small and intermediary values of ρ. Also, we expect Cornish-Fisher corrections to verify their theoretical advantages over Edgeworth corrections on average.

Chapter 3

The Generalized Linear Model with Panel Data

3.1 The Model

Seemingly Unrelated Regressions (S.U.R.) model is a special case of the Generalized Least Squares (GLS) model and refers to the case in which the disturbances of a system of equations are contemporaneously correlated. In this case, regression coefficients in all equations are better estimated simultaneously, because these estimators are at least asymptotically more efficient than those obtained by an equation-by-equation application of least squares. Zellner, 1962 proposed a method of estimating seemingly unrelated regressions. He assumed that the disturbances in each equation are not autocorrelated but the disturbances of two different equations are contemporaneously correlated. Using the theory proposed by Zellner, 1962 about S.U.R., this chapter is concerned with the Generalized Model with Panel Data, i.e., a combination of correlated cross-sectional data with autoregressive time-series, which describe the individual behavior both across time and across individuals and is described by a system of M regression equations, of form (3.1), examined bellow.

3.1.1 Generalized Linear Model with Panel Data

Consider a Panel system of M contemporaneously regression equations of the form:

$$\boldsymbol{y}_\mu = \boldsymbol{X}_\mu \boldsymbol{\beta} + \sigma \boldsymbol{u}_\mu, \tag{3.1}$$

where

$\boldsymbol{y}_\mu$ is a $T \times 1$ vector of observations on the μ-th dependent variable;

$\boldsymbol{X}_\mu$ is a $T \times n$ matrix of observations on n exogenous variables of μ-th unit ;

$\boldsymbol{\beta}$ is a $n \times 1$ vector of unknown structural parameters;

and

$\sigma \boldsymbol{u}_\mu$ ($\sigma > 0$) is a Tx1 vector of unobserved stochastic disturbances.

The model can be written as

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_M \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_M \end{bmatrix} \boldsymbol{\beta} + \sigma \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_M \end{bmatrix}. \tag{3.2}$$

More compactly, the model can be written as

$$y = X\beta + \sigma u, \tag{3.3}$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix}, \; X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_M \end{bmatrix}, \; u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix} \tag{3.4}$$

and

$$\mathrm{E}(uu') = \Omega^{-1}. \tag{3.5}$$

Assumption 2. The following assumptions hold:

1. The random vector u is distributed as a $\mathrm{N}(0, \Omega^{-1})$, where Ω is $MT \times MT$ positive definite and symmetric partitioned matrix;

2. The matrix X_μ of the regressors has full column rank, i.e.

$$r(X_\mu) = n; \tag{3.6}$$

3. The regressors are non-stochastic. The results of this thesis would also be valid if the regressors were stochastic, yet uncorrelated with the errors, i.e.,

$$\mathrm{E}(X'_\mu u) = 0, \tag{3.7}$$

but in such a case the proofs would be a little more complicated.

3.1.2 Autoregressive extension of the Generalized Linear Model with Panel Data

Let $u_{t\mu}$ be the t-th observation of the random vector u_μ of the μ-th equation. Then, we assume the autoregressive scheme:

$$u_{t\mu} = \rho u_{(t-1)\mu} + \varepsilon_{t\mu}\,; \;\; 0 < |\rho_\mu| < 1 \;\; (t = 2, \ldots, T; \; \mu = 1, \ldots, M), \tag{3.8}$$

where the random variables $\varepsilon_{t\mu}$ satisfy the conditions:
For $t \neq 1$ or $t' \neq 1$,

$$\mathrm{E}(\varepsilon_{t\mu}) = 0 \;\; (t = 1, \ldots, T; \; \mu = 1, \ldots, M), \tag{3.9}$$

$$\mathrm{E}(\varepsilon_{t\mu}\varepsilon_{t'\mu'}) = \delta_{tt'}\sigma_{\mu\mu'} = \begin{cases} \sigma_{\mu\mu'} & \text{if } t{=}t';\, \mu, \mu' = 1, \ldots, M, \\ 0 & \text{if } t{\neq}t';\, \mu, \mu' = 1, \ldots, M, \end{cases}, \tag{3.10}$$

where $\delta_{tt'}$ is Kronecker's delta. For $t' = t = 1$ and $\mu, \mu' = 1, \dots, M$, $\mathrm{E}(\varepsilon_{t\mu}\varepsilon_{t'\mu'})$ becomes

$$\mathrm{E}(\varepsilon_{1\mu}\varepsilon_{1\mu'}) = \sigma_{\mu\mu'}(1-\rho_\mu{}^2)^{1/2}(1-\rho_{\mu'}{}^2)^{1/2}/(1-\rho_\mu\rho_{\mu'}) \tag{3.11}$$

(see Parks, 1967).

The time series $u_{t\mu}$,$(t = 1, \dots, T;\ \mu = 1, \dots, M)$ is stationary provided that

$$u_{1\mu} = (1-\rho_\mu{}^2)^{1/2}\varepsilon_{1\mu}, \text{ for } t = 1. \tag{3.12}$$

Equations (3.8) and (3.12) imply that, for all $t = 1, \dots, T$ and $\mu, \mu' = 1, \dots, M$, the disturbances $u_{t\mu}$ satisfy the following conditions

$$\mathrm{E}(u_{t\mu}) = 0, \tag{3.13a}$$

$$\mathrm{E}(u_{t\mu}^2) = \sigma_{\mu\mu}/(1-\rho_\mu^2), \tag{3.13b}$$

$$\mathrm{E}(u_{t\mu}\ u_{t\mu'}) = \sigma_{\mu\mu'}/(1-\rho_\mu\rho_{\mu'}). \tag{3.13c}$$

Note that if $\mu = \mu'$ then (3.13c) implies (3.13b).

Let $\boldsymbol{\varepsilon}_t'$ $(t = 1, \dots, T)$ be the rows of the $T \times M$ matrix $\boldsymbol{E}$ (i.e. ε_t are the columns of $\boldsymbol{E}'$). Also, let $\boldsymbol{\varepsilon}_\mu$ $(\mu = 1, \dots, M)$ be the columns of $\boldsymbol{E}$ (i.e. $\boldsymbol{\varepsilon}_\mu'$ are the rows of $\boldsymbol{E}'$). So,

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{\varepsilon}_1' \\ \vdots \\ \boldsymbol{\varepsilon}_T' \end{bmatrix} = [(\boldsymbol{\varepsilon}_t')_{t=1,\dots,T}];\quad \boldsymbol{E} = [\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_M] = [(\boldsymbol{\varepsilon}_\mu)_{\mu=1,\dots,M}]. \tag{3.14}$$

Then, equations (3.9) and (3.10) imply that

$$\begin{aligned} \mathrm{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') &= \begin{bmatrix} \mathrm{E}(\varepsilon_{t1}\varepsilon_{t1}) & \dots & \mathrm{E}(\varepsilon_{t1}\varepsilon_{tM}) \\ \vdots & & \vdots \\ \mathrm{E}(\varepsilon_{tM}\varepsilon_{t1}) & \dots & \mathrm{E}(\varepsilon_{tM}\varepsilon_{tM}) \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \dots & \sigma_{1M} \\ \vdots & & \vdots \\ \sigma_{M1} & \dots & \sigma_{MM} \end{bmatrix} && (3.15) \\ &= [(\sigma_{\mu\mu'})_{\mu,\mu'=1,\dots,M}] = \boldsymbol{\Sigma}, && (3.16) \end{aligned}$$

which is a $(M \times M)$ matrix of contemporaneous covariances between the t-th elements of any two random variables ε_μ and ε_μ'.

Similarly for any random vector ε_μ it holds that

$$\mathrm{E}(\boldsymbol{\varepsilon}_\mu) = 0, \tag{3.17a}$$

$$\mathrm{E}(\boldsymbol{\varepsilon}_\mu \boldsymbol{\varepsilon}_\mu{}') = \begin{bmatrix} \mathrm{E}(\varepsilon_{1\mu}\varepsilon_{1\mu}) & \dots & \mathrm{E}(\varepsilon_{1\mu}\varepsilon_{T\mu}) \\ \vdots & & \vdots \\ \mathrm{E}(\varepsilon_{T\mu}\varepsilon_{1\mu}) & \dots & \mathrm{E}(\varepsilon_{T\mu}\varepsilon_{T\mu}) \end{bmatrix} = [(\delta_{tt'}\sigma_{\mu\mu})_{t,t'=1,\dots,T}] = \sigma_{\mu\mu}\boldsymbol{I}_T. \tag{3.17b}$$

Moreover, for any two random vectors ε_μ, $\varepsilon_{\mu'}$ $(\mu \neq \mu',\ \mu, \mu' = 1, \dots, M)$

$$\mathrm{E}(\boldsymbol{\varepsilon}_\mu\ \boldsymbol{\varepsilon}_{\mu'}{}') = [(\delta_{tt'}\sigma_{\mu\mu'})_{t,t'=1,\dots,T}] = \sigma_{\mu\mu'}\boldsymbol{I}_T. \tag{3.17c}$$

Define the $(TM \times 1)$ vector

$$\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{E}) = \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_M \end{bmatrix}. \tag{3.18}$$

Then,

$$\mathrm{E}(\boldsymbol{\varepsilon}) = 0 \tag{3.19a}$$

and

$$\mathrm{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \begin{bmatrix} \sigma_{11}\boldsymbol{I}_T & \dots & \sigma_{1M}\boldsymbol{I}_T \\ \vdots & & \vdots \\ \sigma_{M1}\boldsymbol{I}_T & \dots & \sigma_{MM}\boldsymbol{I}_T \end{bmatrix} = [(\sigma_{\mu\mu'}\boldsymbol{I}_T)_{\mu,\mu'=1,\dots,M}] = \boldsymbol{\Sigma} \otimes \boldsymbol{I}_T. \tag{3.19b}$$

3.1.3 Representation of the Generalized Linear Model with Panel Data

Define the $(T \times T)$ matrix (see Parks, 1967)

$$\boldsymbol{P}_\mu = \begin{bmatrix} (1-\rho_\mu{}^2)^{-1/2} & 0 & \dots & 0 \\ (1-\rho_\mu{}^2)^{-1/2}\rho_\mu & 1 & \dots & 0 \\ \vdots & & & \\ (1-\rho_\mu{}^2)^{-1/2}\rho_\mu{}^{T-1} & \dots & & 1 \end{bmatrix}. \tag{3.20}$$

The inverse of $\boldsymbol{P}_\mu$ is

$$\boldsymbol{P}_\mu{}^{-1} = \begin{bmatrix} (1-\rho_\mu{}^2)^{1/2} & 0 & & \dots & 0 \\ -\rho_\mu & 1 & 0 & \dots & 0 \\ 0 & -\rho_\mu & 1 & \dots & 0 \\ \vdots & & \vdots & \ddots & \\ 0 & 0 & \dots & -\rho_\mu & 1 \end{bmatrix}. \tag{3.21}$$

Then, equation (3.8) implies that

$$\boldsymbol{u}_\mu = \boldsymbol{P}_\mu \varepsilon_\mu. \tag{3.22}$$

By using equation (3.22), model (3.1) can be written as

$$\boldsymbol{y}_\mu = \boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{P}_\mu\boldsymbol{\varepsilon}_\mu. \tag{3.23}$$

Define the $(TM \times TM)$ block diagonal matrix $\boldsymbol{P}$ as

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_1 & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M \end{bmatrix}. \tag{3.24}$$

The inverse of matrix $\boldsymbol{P}$ is

$$\boldsymbol{P}^{-1} = \begin{bmatrix} \boldsymbol{P}_1{}^{-1} & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M{}^{-1} \end{bmatrix}. \tag{3.25}$$

Then, since

$$\boldsymbol{u} = \boldsymbol{P}\boldsymbol{\varepsilon}, \tag{3.26}$$

model (3.3) can be written as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{P}\boldsymbol{\varepsilon}. \tag{3.27}$$

Obviously,

$$\mathrm{E}(\boldsymbol{u}) = \mathrm{E}(\boldsymbol{P}\boldsymbol{\varepsilon}) = \boldsymbol{P}\,\mathrm{E}(\boldsymbol{\varepsilon}) = 0 \tag{3.28a}$$

and

$$\mathrm{E}(\boldsymbol{u}\boldsymbol{u}') = \boldsymbol{\Omega}^{-1} = \mathrm{E}(\boldsymbol{P}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\boldsymbol{P}') = \boldsymbol{P}\,\mathrm{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\boldsymbol{P}' = \boldsymbol{P}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{P}' \tag{3.28b}$$

$$= \begin{bmatrix} \sigma_{11}\boldsymbol{P}_1\boldsymbol{P}_1' & \dots & \sigma_{1M}\boldsymbol{P}_1\boldsymbol{P}_M' \\ \vdots & & \vdots \\ \sigma_{M1}\boldsymbol{P}_M\boldsymbol{P}_1' & \dots & \sigma_{MM}\boldsymbol{P}_M\boldsymbol{P}_M' \end{bmatrix}. \tag{3.28c}$$

The $TM \times TM$ block diagonal matrix

$$\boldsymbol{P} = [(\delta_{\mu\mu'}\boldsymbol{P}_\mu)_{\mu,\mu'=1,\dots,M}] \tag{3.29}$$

and the $T \times T$ matrix

$$\boldsymbol{R}_{\mu\mu'} = \frac{1}{1-\rho_\mu\rho_{\mu'}} \begin{bmatrix} 1 & \rho_{\mu'} & \dots & \rho_{\mu'}{}^{T-1} \\ \rho_\mu & \ddots & & \vdots \\ \vdots & & & \\ \rho_\mu{}^{T-1} & \dots & & 1 \end{bmatrix}. \tag{3.30}$$

As in equation (3.22) consider the $T \times 1$ vectors $\boldsymbol{y}_{\mu*}$ and the $T \times n$ matrices $\boldsymbol{X}_{\mu*}$ with non-autocorrelated elements, satisfying the following relations:

$$\boldsymbol{y}_{\mu*} = \boldsymbol{P}_\mu{}^{-1}\boldsymbol{y}_\mu,\ \boldsymbol{X}_{\mu*} = \boldsymbol{P}_\mu{}^{-1}\boldsymbol{X}_\mu, \tag{3.31}$$

and define the $MT \times 1$ vector $\boldsymbol{y}_*$ and $MT \times n$ matrix $\boldsymbol{X}_*$ as follows:

$$\boldsymbol{y}_* = \begin{bmatrix} \boldsymbol{y}_{1*} \\ \vdots \\ \boldsymbol{y}_{M*} \end{bmatrix}, \ \boldsymbol{X}_* = \begin{bmatrix} \boldsymbol{X}_{1*} \\ \vdots \\ \boldsymbol{X}_{M*} \end{bmatrix}. \tag{3.32}$$

Then, premultiplying each regression in equation (3.1) by $\boldsymbol{P}_\mu{}^{-1}$ we can derive the following model with non-autocorrelated error terms:

$$\begin{aligned} \boldsymbol{P}_\mu{}^{-1}\boldsymbol{y}_\mu &= \boldsymbol{P}_\mu{}^{-1}\boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{P}_\mu{}^{-1}\boldsymbol{u}_\mu \Rightarrow \\ \boldsymbol{y}_{\mu*} &= \boldsymbol{X}_{\mu*}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_\mu \end{aligned} \tag{3.33}$$

(see Zellner, 1962, Zellner, 1963 Zellner and Huang, 1962, Zellner and Theil, 1962). Alternatively, by premultiplying (3.3) by the matrix $\boldsymbol{P}^{-1}$ defined in (3.25) we take

$$\begin{aligned} \boldsymbol{P}^{-1}\boldsymbol{y} &= \boldsymbol{P}^{-1}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{P}^{-1}\boldsymbol{u} \Rightarrow \\ \boldsymbol{y}_* &= \boldsymbol{X}_*\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \end{aligned} \tag{3.34}$$

where $\boldsymbol{y}_* = \boldsymbol{P}^{-1}\boldsymbol{y},\ \boldsymbol{\varepsilon} = \boldsymbol{P}^{-1}\boldsymbol{u},\ \boldsymbol{X}_* = \boldsymbol{P}^{-1}\boldsymbol{X}$.

3.1.4 The specification of $\boldsymbol{\Omega}$

The elements of the $T \times T$ matrix $\boldsymbol{\Omega}$ are functions of the $(M + M^2) \times 1$ vector

$$\gamma = (\rho', \varsigma')' , \tag{3.35}$$

where $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_M)'$ is the $T \times 1$ vector of autocorrelation coefficients and $\boldsymbol{\varsigma} = \text{vec}(\boldsymbol{\Sigma}^{-1}) \in \mathbb{R}^{M^2} - \mho$ where $\mho$ is the subspace of $\mathbb{R}^{M^2}$ in which $\boldsymbol{\Sigma}$ is not positive definite. $\boldsymbol{\Omega}$ can be written as

$$\boldsymbol{\Omega} = \boldsymbol{P}'^{-1}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}. \tag{3.36}$$

Define, for any two indexes $\mu, \mu' = 1, \ldots, M$, the composite index

$$((\mu\mu') = \mu + M(\mu' - 1))_{(\mu\mu')=1,\ldots,M^2}, \tag{3.37}$$

It can be easily seen that the $(\mu\mu')$-th element of vector $\boldsymbol{\varsigma}$ denoted as $\varsigma(\mu\mu')$, is actually the $((\mu, \mu')$-th element of matrix $\boldsymbol{\Sigma}^{-1}$, denoted as $\sigma^{\mu\mu'}$.

3.1.5 Vectorization of the Model

The system of equations (3.31) or (3.32) can be seen as the outcome of vectorizing the following model:

$$\boldsymbol{Y}_* = \boldsymbol{Z}\boldsymbol{B} + \boldsymbol{E}, \tag{3.38}$$

which can be defined as in the S.U.R. model.

In the generalized linear model with panel data, the columns $\boldsymbol{b}_\mu$ ($\mu = 1, \ldots, M$) of the $(k \times M)$ parameter matrix $\boldsymbol{B}$ obey the restrictions:

$$\boldsymbol{b}_\mu = \boldsymbol{\Psi}_\mu \boldsymbol{\beta}, \tag{3.39}$$

where $\boldsymbol{\Psi}_\mu$ are $(k \times n)$ known matrices and $\boldsymbol{\beta}$ is a $(n \times 1)$ vector of unknown parameters to be estimated. Define the $(Mk \times n)$ matrix $\boldsymbol{\Psi}$ as

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \\ \vdots \\ \boldsymbol{\Psi}_M \end{bmatrix} \tag{3.40}$$

By vectorizing model (3.38) we take

$$\boldsymbol{y}_* = \boldsymbol{X}_* \boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{3.41}$$

where

$$\begin{aligned} \boldsymbol{y}_* &= \text{vec}(\boldsymbol{Y}_*),\ \boldsymbol{\varepsilon} = \text{vec}(\boldsymbol{E}) \\ &\text{and} \\ \boldsymbol{X}_* &= (\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi} = [(\delta_{\mu\mu'}\boldsymbol{Z})_{\mu\mu'}] \cdot [(\boldsymbol{\Psi}_\mu)_\mu] = \left[\left(\sum_{\mu=1}^{M} \delta_{\mu\mu'} \boldsymbol{Z}\boldsymbol{\Psi}_\mu\right)_\mu\right] = [(\boldsymbol{Z}\boldsymbol{\Psi}_\mu)_\mu] \\ &= \begin{bmatrix} \boldsymbol{Z}\boldsymbol{\Psi}_1 \\ \vdots \\ \boldsymbol{Z}\boldsymbol{\Psi}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_{1*} \\ \vdots \\ \boldsymbol{X}_{M*} \end{bmatrix}. \end{aligned} \tag{3.42}$$

By partitioning $\boldsymbol{y}_*$ and $\boldsymbol{\varepsilon}$ according to $\boldsymbol{X}_*$ in (3.42), model (3.41) can be decomposed as follows:

$$\begin{bmatrix} \boldsymbol{y}_{1*} \\ \vdots \\ \boldsymbol{y}_{M*} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_{1*} \\ \vdots \\ \boldsymbol{X}_{M*} \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_M \end{bmatrix}, \tag{3.43}$$

where $\boldsymbol{X}_{\mu*}$ ($\mu = 1, \ldots, M$) are $(T \times n)$ matrices.

Note that:

$\boldsymbol{Y}_*$ is a $(T \times M)$ matrix, $\boldsymbol{X}_*$ is a $(TM \times n)$ matrix, $\boldsymbol{X}_*'\boldsymbol{X}_*$ is a $(n \times n)$ matrix, $\boldsymbol{\Psi}$ is a $(Mk \times n)$ matrix and $\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'$ is a $(Mk \times MT)$ matrix.

3.1.6 Identification and estimation of the parameters

Let $\hat{\boldsymbol{\gamma}} = (\hat{\rho}, \hat{\varsigma}')'$ be any consistent estimator of the parameter vector $\boldsymbol{\gamma}$. For any function $f = f(\boldsymbol{\gamma})$ we can write $\hat{f} = f(\hat{\boldsymbol{\gamma}})$. The feasible GLS estimator σ is

$$\hat{\sigma} = [(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'(\hat{\boldsymbol{P}}'_{GL}{}^{-1}(\hat{\boldsymbol{\Sigma}}_{GL}^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}_{GL}^{-1})(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})/(MT - n)]^{1/2}. \tag{3.44}$$

It is straightforward that the parameters σ and $\boldsymbol{\gamma}$ cannot be simultaneously identified without the restriction $\sigma = 1$, under which the estimate $\hat{\boldsymbol{\Sigma}}$ is supposed to be accurate, up to a multiplicative factor. This is not true in small samples, and a reasonable method to account for this is to use the feasible GLS estimate of $\hat{\sigma}$ from (3.44) in order to compute the traditional t and F test statistics. This method is meaningless from the estimation viewpoint, but its success in improving the size corrections must be the only criterion to judge its validity.

3.1.7 Regularity conditions

Denote as Ω_i, Ω_{ij}, etc., the $MT \times MT$ matrices of first-, second-, and higher-order derivatives of the elements of Ω with respect to the elements of the $(M + M^2) \times 1$ vector of nuisance parameters $\boldsymbol{\gamma} = (\boldsymbol{\rho}', \boldsymbol{\varsigma}')'$. Moreover, for any estimator $\hat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$, define the $(1 + M + M^2) \times 1$ vector $\boldsymbol{\delta}$ with elements

$$\delta_0 = \frac{\hat{\sigma}^2 - 1}{\tau}; \quad \delta_{\rho_\mu} = \frac{\hat{\rho}_\mu - \rho_\mu}{\tau}; \quad \delta_{\varsigma_{(\mu,\mu')}} = \frac{\hat{\varsigma}_{(\mu\mu')} - \varsigma_{(\mu\mu')}}{\tau} \tag{3.45}$$

where $\mu = 1, \ldots, M$, $(\mu\mu') = 1, \ldots, M^2$ and $\tau = \frac{1}{\sqrt{T}}$ is the "asymptotic scale" of our expansions.

The suggested size corrections are based on the following

Regularity Conditions:

(1) The elements of matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^{-1}$ are bounded for all T, for all vectors $\boldsymbol{\rho}$ with elements $\rho_\mu \in (-1, 1)$, and for all vectors $\boldsymbol{\varsigma} \in \mathcal{F}_s = \mathbb{R}^m \setminus \{0\}$. Moreover, the matrices

$$\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T, \quad \boldsymbol{F} = \boldsymbol{X}\boldsymbol{X}'/T, \quad \boldsymbol{\Gamma} = \boldsymbol{Z}'\boldsymbol{Z}/T \tag{3.46}$$

converge to non-singular limits as $T \to \infty$.

(2) Up to the fourth order, the partial derivatives of the elements of $\boldsymbol{\Omega}$ with respect to the elements of $\boldsymbol{\rho}$ and $\boldsymbol{\varsigma}$, are bounded for all T, for all vectors $\boldsymbol{\rho}$ with elements in interval $(-1, 1)$, and for all vectors $\boldsymbol{\varsigma} \in \mathcal{F}_s$.

(3) The estimators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varsigma}}$ are even functions of $\boldsymbol{u}$, and they are functionally unrelated to the parameter vector $\boldsymbol{\beta}$, i.e., they can be written as functions of $\boldsymbol{X}$, $\boldsymbol{Z}$ and $\boldsymbol{u}$ only.

(4) The vector $\boldsymbol{\delta}$ admits a stochastic expansion of the form

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_0 \\ [(\delta_{\rho_\mu})_{\mu=1,\ldots,M}]' \\ [(\delta_{\varsigma_{(\mu\mu')}})_{(\mu\mu')=1,\ldots,M^2}]' \end{bmatrix} \tag{3.47}$$

$$= \boldsymbol{d}_1 + \tau \boldsymbol{d}_2 + \omega(\tau^2), \tag{3.48}$$

where the order of magnitude $\omega(\cdot)$, defined in Notational Convensions, has the same operational properties as the order $O(\cdot)$, and the expectations

$$\mathbf{E}(\boldsymbol{d}_1 \boldsymbol{d}_1'), \quad \mathbf{E}(\boldsymbol{d}_1 + \sqrt{T}\boldsymbol{d}_2) \tag{3.49}$$

exist and have finite limits as $T \to \infty$.

Discussions on the Regularity Conditions:
The first two regularity conditions imply that the $n \times n$ matrices

$$\boldsymbol{A}_i = \boldsymbol{X}'\boldsymbol{\Omega}_i\boldsymbol{X}/T, \quad \boldsymbol{A}_{ij} = \boldsymbol{X}'\boldsymbol{\Omega}_{ij}\boldsymbol{X}/T, \quad \boldsymbol{A}_{ij}{}^* = \boldsymbol{X}'\boldsymbol{\Omega}_i\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_j\boldsymbol{X}/T \tag{3.50}$$

are bounded, and therefore the Taylor series expansion of β is a stochastic expansion (see Magdalinos, 1992). Since the parameters $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_\mu)'$ and $\boldsymbol{\varsigma} = (\varsigma_1, \ldots, \varsigma_m)'$ are functionally unrelated to $\boldsymbol{\beta}$, regularity condition (3) is satisfied for a wide class of estimators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varsigma}}$ including the maximum likelihood estimators and the simple and iterative estimators based on the regression residuals (see Breusch, 1980, Rothenberg, 1984a). Note that we need not assume that the estimators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varsigma}}$ are asymptotically efficient. Also, notice that the regularity conditions (1) through (4) are satisfied by all the estimators of ρ and $\boldsymbol{\varsigma}$ examined in the next section. Some of the estimators of the elements $\rho_{\mu_{(\mu=1,\ldots,M)}}$ of the vector $\boldsymbol{\rho}$, are the least squares (LS), Durbin-Watson (DW), generalized least squares (GL), Prais-Winsten (PW) and maximum likelihood (ML) estimators. The elements of vector $\boldsymbol{\varsigma} = \text{vec}(\boldsymbol{\Sigma}^{-1})$ can be estimated by

$$\hat{\boldsymbol{\varsigma}} = \text{vec}[(\boldsymbol{Y}_* - \boldsymbol{Z}\hat{\boldsymbol{B}})'(\boldsymbol{Y}_* - \boldsymbol{Z}\hat{\boldsymbol{B}})/T]^{-1}, \tag{3.51}$$

where $\hat{\boldsymbol{B}}$ is any consistent estimator of the parameter matrix $\boldsymbol{B}$ in the regression model (3.38).

3.1.8 Definition of parameters

Finally, define the scalars λ_0, κ_0, the $M \times 1$ vectors $\boldsymbol{\lambda}_\rho$, $\boldsymbol{\kappa}_\rho$, the $M^2 \times 1$ vectors $\boldsymbol{\lambda}_\varsigma$, the $M \times M$ matrix $\boldsymbol{\Lambda}_\rho$, the $M^2 \times M$ matrix $\boldsymbol{\Lambda}_{\rho\varsigma}$, and the $M^2 \times M^2$ matrix $\boldsymbol{\Lambda}_{\varsigma\varsigma}$, as follows:

$$\boldsymbol{\Lambda}_* = \begin{bmatrix} \lambda_0 & \boldsymbol{\lambda}_\rho{}' & \boldsymbol{\lambda}_\varsigma{}' \\ \boldsymbol{\lambda}_\rho & \boldsymbol{\Lambda}_\rho & \boldsymbol{\Lambda}_{\rho\varsigma}{}' \\ \boldsymbol{\lambda}_\varsigma & \boldsymbol{\Lambda}_{\rho\varsigma} & \boldsymbol{\Lambda}_\varsigma \end{bmatrix} = \mathrm{E}(\boldsymbol{d}_1\boldsymbol{d}_1'); \;\; \boldsymbol{\kappa}_* = \begin{bmatrix} \kappa_0 \\ \boldsymbol{\kappa}_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix} = \mathrm{E}(\boldsymbol{d}_1 + \sqrt{T}\boldsymbol{d}_2). \tag{3.52}$$

We partition matrix $\boldsymbol{\Lambda}_*$ and vector $\boldsymbol{\kappa}_*$ as follows:

$$\begin{bmatrix} \lambda_0 & \boldsymbol{\lambda}' \\ \boldsymbol{\lambda} & \boldsymbol{\Lambda} \end{bmatrix} \text{ and } \begin{bmatrix} \kappa_0 \\ \boldsymbol{\kappa} \end{bmatrix}. \tag{3.53}$$

Equation (3.52) and (3.53) imply that

$$\boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_\rho \\ \boldsymbol{\lambda}_{\rho\varsigma} \end{bmatrix}, \; \boldsymbol{\kappa} = \begin{bmatrix} \boldsymbol{\kappa}_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix}, \; \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_\rho & \boldsymbol{\Lambda}_{\rho\varsigma}' \\ \boldsymbol{\Lambda}_{\rho\varsigma} & \boldsymbol{\Lambda}_\varsigma \end{bmatrix}, \tag{3.54}$$

and $\boldsymbol{\Lambda}$ is a $(M \times M^2) \times (M \times M^2)$ matrix and $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$ are $((M \times M^2) \times 1)$ vectors. The elements of $\boldsymbol{\Lambda}_*$ and $\boldsymbol{\kappa}_*$ in equations (3.52), (3.53), and (3.54) can be interpreted as "measures" of the accuracy of the expansions of $\hat{\sigma}^2$, $\hat{\rho}_\mu$ and $\hat{\varsigma}_{(\mu\mu')}$ around the true values of the corresponding parameters.

3.1.9 A 3-step Estimation Process

Denote by LS, GL, IG, ML the least squares, generalized least squares, iterative GLS, and maximum likelihood estimation methods, respectively. Also, denote by $\hat{\beta}_I$ any consistent estimator of β in the model (3.1), indexed by I (I=LS, GL, IG, ML). The discussion above suggests the following 3 steps of an estimation strategy:

- Step 1: Single equation estimation of autoregressive parameters ρ_μ

$$\begin{aligned} \hat{u}_{\mu_{(I)}} &= \boldsymbol{y}_\mu - \boldsymbol{X}_\mu\hat{\boldsymbol{\beta}}_{(I)} \\ \hat{\rho}_{\mu_{(I)}} &= \frac{\sum_{t=2}^{T} \hat{u}_{t\mu_{(I)}}\hat{u}_{(t-1)\mu_{(I)}}}{\sum_{t=2}^{T} \hat{u}^2_{(t-1)\mu_{(I)}}} \end{aligned} \tag{3.55}$$

- Step 2: Transform model (3.1) to obtain estimations of contemporaneous covariances $\sigma_{\mu\mu'}$

  i. Transorm the model in order to cancel out first-order autoregression

$$\begin{aligned} \hat{\boldsymbol{P}}_\mu^{-1}\boldsymbol{y}_\mu &= \hat{\boldsymbol{P}}_\mu^{-1}\boldsymbol{X}_\mu\boldsymbol{\beta} + \hat{\boldsymbol{P}}_\mu^{-1}\boldsymbol{P}_\mu\boldsymbol{\varepsilon}_\mu \\ &\text{or} \\ \boldsymbol{y}_{\mu*} &= \boldsymbol{X}_{\mu*}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_{\mu*}\,. \end{aligned} \tag{3.56}$$

ii. Estimate (3.56) via (I) to obtain the estimators $\hat{\boldsymbol{\beta}}_I^*$ and the residuals

$$\hat{\boldsymbol{\varepsilon}}_{\mu*} = \boldsymbol{y}_{\mu*} - \boldsymbol{X}_{\mu*}\hat{\boldsymbol{\beta}}_I^* \,. \tag{3.57}$$

iii. Estimate covariances by

$$\hat{\sigma}_{\mu\mu'} = \frac{\hat{\varepsilon}_{\mu}^{*\,\prime}\hat{\varepsilon}_{\mu}^{*}}{T} \tag{3.58}$$

to obtain $\hat{\boldsymbol{\Sigma}}_{(I)}$.

- Step 3: Aitken estimation of (3.3) by using $\hat{\boldsymbol{\Omega}}$.

  Since,

$$\begin{aligned}
\boldsymbol{\Omega}^{-1} &= \boldsymbol{P}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{P}' \Rightarrow \\
\hat{\boldsymbol{\Omega}} &= \hat{\boldsymbol{P}}'^{-1}(\hat{\boldsymbol{\Sigma}}_I^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1} \,.
\end{aligned} \tag{3.59}$$

  and

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_{GLS} &= (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{y} \\
&= [\boldsymbol{X}'(\hat{\boldsymbol{P}}^{-1})'(\hat{\boldsymbol{\Sigma}}_{(I)}^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1}\boldsymbol{X}]^{-1}\boldsymbol{X}'[(\hat{\boldsymbol{P}}^{-1})'(\hat{\boldsymbol{\Sigma}}_{(I)}^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1}]\boldsymbol{y} \\
&= [\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}_{(I)}^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1}\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}_{(I)}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{y}_* \,.
\end{aligned} \tag{3.60}$$

3.2 Asymptotically efficient estimators of $\boldsymbol{\rho}$ and $\boldsymbol{B}$

3.2.1 Estimators of ρ

Some of the most frequently used estimators of ρ in applied econometric research are:

1. The least squares (LS) estimator

$$\tilde{\rho}_\mu = \sum_{t=2}^{T} \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu} / \sum_{t=1}^{T} \left(\tilde{u}_{t\mu}\right)^2, \tag{3.61}$$

   where $\tilde{u}_{t\mu}$ are the LS residuals in the regression model (3.1).

2. The Durbin-Watson (DW) estimator, which is computed via the DW-statistic approximation as

$$\hat{\rho}^{DW} = 1 - \left(\frac{DW}{2}\right) \tag{3.62}$$

3. The generalized least squares (GL) estimator

$$\hat{\rho}_\mu = \sum_{t=2}^{T} \hat{u}_{t\mu}\hat{u}_{(t-1)\mu} / \sum_{t=1}^{T} \left(\hat{u}_{t\mu}\right)^2, \tag{3.63}$$

   where $\hat{u}_{t\mu}$ are the GL residuals in the regression model (3.1).

4. The Prais and Winsten, 1954 estimator $\hat{\rho}_{\mu}^{PW}$, which, together with the PW estimator $\hat{\beta}_{\mu}^{PW}$ minimises the sum of squared GL residuals.

5. The maximum likelihood (ML) estimator, $\hat{\rho}_{\mu}^{ML}$, which satisfies a cubic equation with coefficients defined in terms of the ML residuals in the regression model (3.1) (see Beach and MacKinnon, 1978).

3.2.2 Estimators of $\boldsymbol{B}$

Some of the most frequently used estimators of $\boldsymbol{B}$ in applied econometric research are:

1. The unrestricted least squares (UL) estimator

$$\hat{\boldsymbol{B}}_{(UL)} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{Y}_{*}. \tag{3.64}$$

2. The restricted least squares (RL) estimator

$$\text{vec}\left(\hat{\boldsymbol{B}}_{(RL)}\right) = \boldsymbol{\Psi}(\boldsymbol{X}_{*}'\boldsymbol{X}_{*})^{-1}\boldsymbol{X}_{*}'\boldsymbol{y}_{*}. \tag{3.65}$$

3. The The generalized least squares (GL) estimator

$$\text{vec}\left(\hat{\boldsymbol{B}}_{(GL)}\right) = \boldsymbol{\Psi}[\boldsymbol{X}_{*}'(\hat{\boldsymbol{\Sigma}}_{I}^{-1} \otimes \boldsymbol{I}_{T})\boldsymbol{X}_{*}]^{-1}\boldsymbol{X}_{*}'(\hat{\boldsymbol{\Sigma}}_{I}^{-1} \otimes \boldsymbol{I}_{T})\boldsymbol{y}_{*}, \tag{3.66}$$

   where $\hat{\boldsymbol{\Sigma}}_{I}^{-1}$ is the UL or RL estimator of $\boldsymbol{\Sigma}^{-1}$.

4. The iterative generalized least squares (IG) estimator $\hat{\boldsymbol{B}}_{(IG)}$ which is computed by the iterative implementation of GL estimator.

5. The maximum likelyhood (ML) estimator $\hat{\boldsymbol{B}}_{(ML)}$ which can be computed by iterating the GL estimation process up to convergence (Dhrymes, 1971).

Chapter 4

Size Corrected Test Statistics

4.1 Introduction

This chapter specifies the analytical forms of the Edgeworth and Cornish-Fisher size corrections of the t and F tests in the Generalized Linear Model with Panel Data. For this purpose, we calculate some useful quantities.

4.2 t-test

Let e_o be a known scalar and let $\boldsymbol{e}$ be a known $n \times 1$ vector. To test the null hypothesis

$$H_0 : \boldsymbol{e}'\boldsymbol{\beta} - e_0 = 0 \tag{4.1}$$

for one-sided alternative hypotheses we use the statistic

$$t = (\boldsymbol{e}'\hat{\boldsymbol{\beta}} - e_0)/[\hat{\sigma}^2\boldsymbol{e}'(\boldsymbol{X}'\widehat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{e}]^{1/2}. \tag{4.2}$$

We define the $((M+M^2)\times 1)$ vector $\boldsymbol{l}$ and the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ as follows:

$$\boldsymbol{l} = \left[[(l_{\rho_\mu})_{\mu=1,\dots,M}]',\ [(l_{\varsigma_{(\mu\mu')}})_{(\mu\mu')=1,\dots,M^2}]'\right]', \tag{4.3}$$

$$\boldsymbol{L} = \begin{bmatrix} [(l_{\rho_\mu\rho_{\mu'}})_{\mu,\mu'=1,\dots,M}] & \left[(l_{\rho_\mu\varsigma_{(\nu\nu')}})_{\mu=1,\dots,M;\ (\nu\nu')=1,\dots,M^2}\right] \\ \left[(l_{\varsigma_{(\nu\nu')}\rho_\mu})_{(\nu\nu')=1,\dots,M^2;\ \mu=1,\dots,M}\right] & \left[(l_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}})_{(\mu\mu')=1,\dots,M^2;\ (\nu\nu')=1,\dots,M^2}\right] \end{bmatrix}, \tag{4.4}$$

where the elements of vector $\boldsymbol{l}$ and matrix $\boldsymbol{L}$ are defined as follows:

$$\begin{aligned} l_{\rho_\mu} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{h}, \\ l_{\varsigma_{(\mu\mu')}} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G}\boldsymbol{h}, \\ l_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G}\boldsymbol{h}, \\ l_{\rho_\mu\varsigma_{(\nu\nu')}} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{G}\boldsymbol{h}, \\ l_{\varsigma_{(\nu\nu')}\rho_\mu} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\varsigma_{(\nu\nu')}\rho_\mu}\boldsymbol{G}\boldsymbol{h}, \\ l_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{G}\boldsymbol{h}, \end{aligned} \tag{4.5}$$

where $\boldsymbol{G} = \boldsymbol{A}^{-1} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}$ is a $(n \times n)$ matrix, $\boldsymbol{h} = \boldsymbol{e}/(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}$ is a $(n \times 1)$ vector and

$$\begin{aligned} \boldsymbol{C}_{\rho_\mu \rho_{\mu'}} &= \boldsymbol{A}_{\rho_\mu \rho_{\mu'}}{}^{*} - 2\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}} + \boldsymbol{A}_{\rho_\mu \rho_{\mu'}}/2, \\ \boldsymbol{C}_{\rho_\mu \varsigma_{(\nu\nu')}} &= \boldsymbol{A}_{\rho_\mu \varsigma_{(\nu\nu')}}{}^{*} - 2\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}} + \boldsymbol{A}_{\rho_\mu \varsigma_{(\nu\nu')}}/2, \\ \boldsymbol{C}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \boldsymbol{A}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}{}^{*} - 2\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}} + \boldsymbol{A}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}/2, \end{aligned} \tag{4.6}$$

with the obvious adjustments for $\boldsymbol{C}_{\varsigma_{(\nu\nu')}\rho_\mu}$. Matrices $\boldsymbol{A}_i$, $\boldsymbol{A}_{ij}$ and $\boldsymbol{A}_{ij}{}^{*}$ are defined in the equation (3.50).

The corrected critical value, using the Edgeworth approximation of the t distribution is given by

$$t_\alpha{}^{*} = t_\alpha + \frac{\tau^2}{2}[p_1 + p_2 t_\alpha{}^2]t_\alpha, \tag{4.7}$$

(see Edgeworth, 1903). Moreover, the corrected statistic from the Cornish Fisher approximation of the t distribution is given by

$$t^{*} = t - \frac{\tau^2}{2}\left[p_1 + p_2 t^2\right]t, \tag{4.8}$$

(see, inter alia, Cornish and Fisher, 1937, Fisher and Cornish, 1960, Hill and Davis, 1968). In order to correct either the critical value or the t-statistic the required correction quantities p_1, p_2 are given by the following Proposition.

Proposition 3. The quantities p_1, p_2, required for the calculation of both the Edgeworth corrected critical values of the t distribution, and the Cornish-Fisher corrected t-statistic are:

$$p_1 = \operatorname{tr}\boldsymbol{\Lambda}\boldsymbol{L} + \frac{\boldsymbol{l}'\boldsymbol{\Lambda}\boldsymbol{l}}{4} + \boldsymbol{l}'(\boldsymbol{\kappa} + \frac{\boldsymbol{\lambda}}{2}) - \kappa_0 + \frac{\lambda_0 - 2}{4} \tag{4.9}$$

$$p_2 = \frac{\boldsymbol{l}'\boldsymbol{\Lambda}\boldsymbol{l} - 2\boldsymbol{l}'\boldsymbol{\lambda} + \lambda_0 - 2}{4} \tag{4.10}$$

4.3 The Wald and F Tests

Let $\boldsymbol{H}$ be a $r \times n$ known matrix with $\mathrm{rank}(\boldsymbol{H}) = r$ and let $\boldsymbol{h}_0$ be a known $r \times 1$ vector. The test of the null hypothesis

$$H_0 : \boldsymbol{H}\boldsymbol{\beta} - \boldsymbol{h}_0 = 0 \tag{4.11}$$

is based in Wald statistic

$$w = (\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h}_0)'[\boldsymbol{H}(\boldsymbol{X}'\widehat{\boldsymbol{\Omega}}\boldsymbol{X}/T)^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h}_0)/\hat{\sigma}^2, \tag{4.12}$$

or on the degrees-of-freedom-adjusted F statistic

$$F = (\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h}_0)'[\boldsymbol{H}(\boldsymbol{X}'\widehat{\boldsymbol{\Omega}}\boldsymbol{X}/T)^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h}_0)/r\hat{\sigma}^2. \tag{4.13}$$

Define the $(n \times n)$ matrix $\boldsymbol{G}$ and the $(n \times n)$ matrix $\boldsymbol{\Xi}$ as follows:

$$\boldsymbol{G} = \boldsymbol{A}^{-1} \text{ and } \boldsymbol{\Xi} = \boldsymbol{GQG}, \tag{4.14}$$

where

$$\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T \text{ and } \boldsymbol{Q} = \boldsymbol{H}'(\boldsymbol{HGH}')^{-1}\boldsymbol{H}. \tag{4.15}$$

Next, define the $(M+M^2)\times 1$ vector $\boldsymbol{c}$ and the $(M+M^2)\times(M+M^2)$ matrices $\boldsymbol{C}$, $\boldsymbol{D}$ as follows:

$$\boldsymbol{c} = \left[[(c_{\rho_\mu})_{\mu=1,\ldots,M}]', [(\boldsymbol{c}_{\varsigma_{(\mu\mu')}})_{(\mu\mu')=1,\ldots,M^2}]'\right]', \tag{4.16}$$

$$\boldsymbol{C} = \begin{bmatrix} [(c_{\rho_\mu\rho_{\mu'}})_{\mu,\mu'=1,\ldots,M}] & \left[(c_{\rho_\mu\varsigma_{(\nu\nu')}})_{\mu=1,\ldots,M;\ (\nu\nu')=1,\ldots,M^2}\right] \\ \left[(c_{\varsigma_{(\nu\nu')}\rho_\mu})_{(\nu\nu')=1,\ldots,M^2;\ \mu=1,\ldots,M}\right] & \left[(c_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}})_{(\mu\mu')=1,\ldots,M^2;\ (\nu\nu')=1,\ldots,M^2}\right] \end{bmatrix} \tag{4.17}$$

and

$$\boldsymbol{D} = \begin{bmatrix} [(d_{\rho_\mu\rho_{\mu'}})_{\mu,\mu'=1,\ldots,M}] & \left[(d_{\rho_\mu\varsigma_{(\nu\nu')}})_{\mu=1,\ldots,M;\ (\nu\nu')=1,\ldots,M^2}\right] \\ \left[(d_{\varsigma_{(\nu\nu')}\rho_\mu})_{(\nu\nu')=1,\ldots,M^2;\ \mu=1,\ldots,M}\right] & \left[(d_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}})_{(\mu\mu')=1,\ldots,M^2;\ (\nu\nu')=1,\ldots,M^2}\right] \end{bmatrix}, \tag{4.18}$$

where the elements of the vector $\boldsymbol{c}$ and of the matrices $\boldsymbol{C}$, $\boldsymbol{D}$ are defined as follows:

$$\begin{aligned} c_{\rho_\mu} &= \operatorname{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}), \\ c_{\rho_\mu\rho_{\mu'}} &= \operatorname{tr}(\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}), \\ c_{\rho_\mu\varsigma_{(\nu\nu')}} &= \operatorname{tr}(\boldsymbol{C}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}), \\ c_{\varsigma_{(\mu\mu')}} &= \operatorname{tr}(\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{\Xi}), \\ c_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \operatorname{tr}(\boldsymbol{C}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}), \\ d_{\rho_\mu\rho_{\mu'}} &= \operatorname{tr}(\boldsymbol{D}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}), \\ d_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \operatorname{tr}(\boldsymbol{D}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}), \\ d_{\rho_\mu\varsigma_{(\nu\nu')}} &= \operatorname{tr}(\boldsymbol{D}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}), \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} \boldsymbol{D}_{\rho_\mu\rho_{\mu'}} &= \frac{\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_{\mu'}}}{2}, \\ \boldsymbol{D}_{\rho_\mu\varsigma_{(\nu\nu')}} &= \frac{\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}}{2}, \\ \boldsymbol{D}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \frac{\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}}{2}, \end{aligned} \tag{4.20}$$

with the obvious adjustments for $c_{\varsigma_{(\nu\nu')}\rho_\mu}$, $d_{\varsigma_{(\nu\nu')}\rho_\mu}$ and $\boldsymbol{D}_{\varsigma_{(\nu\nu')}\rho_\mu}$.

The corrected critical value, using the Edgeworth approximation of the F distribution is given by

$$F_\alpha{}^* = F_\alpha + \tau^2\left[q_1 + q_2F_\alpha\right]F_\alpha, \tag{4.21}$$

(see Edgeworth, 1903). Moreover, the corrected statistic from the Cornish Fisher approximation of the F distribution is given by

$$\mathcal{F} = F - \tau^2(q_1 + q_2 F)F, \tag{4.22}$$

(see, inter alia, Cornish and Fisher, 1937, Fisher and Cornish, 1960, Hill and Davis, 1968). In order to correct either the critical value or the F-statistic the required correction quantities q_1, q_2 are given by the following Proposition.

Proposition 4. The quantities q_1, q_2, required for the calculation of both the Edgeworth corrected critical values of the F distribution and the Cornish-Fisher corrected F statistic are:

$$q_1 = \xi_1/r + (r-2)/2,\ q_2 = \xi_2/(r+2) - r/2, \tag{4.23}$$

where

$$\xi_1 = \mathrm{tr}[\boldsymbol{\Lambda}(\boldsymbol{C} + \boldsymbol{D})] - \boldsymbol{c}'\boldsymbol{\Lambda}\boldsymbol{c}/4 + \boldsymbol{c}'\boldsymbol{\kappa} + r[\boldsymbol{c}'\boldsymbol{\lambda}/2 - \kappa_0 - (r-2)\lambda_0/4] \tag{4.24}$$

$$\xi_2 = \mathrm{tr}\,(\boldsymbol{\Lambda}\boldsymbol{D}) + [\boldsymbol{c}'\boldsymbol{\Lambda}\boldsymbol{c} - (r+2)(2\boldsymbol{c}'\boldsymbol{\lambda} - r\lambda_0)]/4. \tag{4.25}$$

4.4 Theorems

Theorem 3. Vectors $\boldsymbol{l}$, $\boldsymbol{c}$ and matrices $\boldsymbol{L}, \boldsymbol{C}, \boldsymbol{D}$, in equations (4.3), (4.4), (4.5), (4.6), (4.16),(4.17),(4.18), (4.19) and (4.20) can be calculated as follows:

(i) The $\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}$ matrix

$$\begin{aligned}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}} &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa l}\boldsymbol{R}_{\kappa l} - 2\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/T \\ &+ \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j/2T.\end{aligned} \tag{4.26}$$

(ii) The $\boldsymbol{D}_{\rho_\mu\rho_{\mu'}}$ matrix

$$\boldsymbol{D}_{\rho_\mu\rho_{\mu'}} = \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{X}_\kappa\boldsymbol{\Xi}\boldsymbol{X}_l'\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/2T^2. \tag{4.27}$$

(iii) The $\boldsymbol{C}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}$ matrix

$$\boldsymbol{C}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} = \sigma_{\mu'\nu}\boldsymbol{B}_{\mu\nu'} - 2\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}. \tag{4.28}$$

(iv) The $\boldsymbol{D}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}$ matrix

$$\boldsymbol{D}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} = \boldsymbol{B}_{\mu\mu'}\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu'}/2. \tag{4.29}$$

(v) The $\boldsymbol{C}_{\rho_\mu \varsigma_{(\nu\nu')}}$ matrix

$$\begin{aligned}\boldsymbol{C}_{\rho_\mu \varsigma_{(\nu\nu')}} &= \sum_{i=1}^{M}\sum_{\kappa=1}^{M} \sigma^{i\kappa}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa\nu}\boldsymbol{R}_{\kappa\nu} - 2\boldsymbol{X}_\kappa \boldsymbol{G}\boldsymbol{X}_\nu'/T]\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}/T \\ &\quad +\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}/2T. \end{aligned} \tag{4.30}$$

(vi)

(vii) The $\boldsymbol{D}_{\rho_\mu \varsigma_{(\nu\nu')}}$ matrix

$$\begin{aligned}\boldsymbol{D}_{\rho_\mu \varsigma_{(\nu\nu')}} &= \boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}/2 \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}/2T^2. \end{aligned} \tag{4.31}$$

(viii) The $\boldsymbol{C}_{\varsigma_{(\nu\nu')}\rho_\mu}$ matrix

$$\begin{aligned}\boldsymbol{C}_{\varsigma_{(\nu\nu')}\rho_\mu} &= \sum_{l=1}^{M}\sum_{j=1}^{M} \sigma^{lj}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}[\sigma_{\nu' l}\boldsymbol{R}_{\nu' l} - 2\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j/T \\ &\quad +\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}/2T. \end{aligned} \tag{4.32}$$

(ix) The $\boldsymbol{D}_{\varsigma_{(\nu\nu')}\rho_\mu}$ matrix

$$\boldsymbol{D}_{\varsigma_{(\nu\nu')}\rho_\mu} = \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/2T^2 \tag{4.33}$$

(x) The μ-th element of the $((M+M^2)\times 1)$ vector $\boldsymbol{l}$ is

$$l_{\rho_\mu} = \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{h}/T, \tag{4.34}$$

where

$$\boldsymbol{h} = \frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}. \tag{4.35}$$

(xi) Similarly the (μ,μ')-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ is

$$\begin{aligned} l_{\rho_\mu\rho_{\mu'}} &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M} \sigma^{i\kappa}\sigma^{lj}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa l}\boldsymbol{R}_{\kappa l} - 2\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{h}/T \\ &\quad + \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{h}/2T. \end{aligned} \tag{4.36}$$

(xii) The μ-th element of the $((M+M^2)\times 1)$ vector $\boldsymbol{c}$ is

$$c_{\rho_\mu} = \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T). \quad (4.37)$$

(xiii) The (μ,μ')-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned} c_{\rho_\mu\rho_{\mu'}} = & \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi})/T \\ & -2\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_l'\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi})/T^2 \\ & +\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi})/2T. \end{aligned} \quad (4.38)$$

(xiv) The (μ,μ')-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{D}$ is

$$d_{\rho_\mu\rho_{\mu'}} = \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{X}_\kappa\boldsymbol{\Xi}\boldsymbol{X}_l'\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi})/2T^2. \quad (4.39)$$

(xv) The $(\mu\mu')$-th element of the $((M+M^2)\times 1)$ vector $\boldsymbol{l}$ is

$$l_{\varsigma_{(\mu\mu')}} = \boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{h}/T. \quad (4.40)$$

(xvi) Similarly the $((\mu\mu'),(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ is

$$l_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} = \sigma_{\mu'\nu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/T - 2\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/T^2. \quad (4.41)$$

(xvii) The $(\mu\mu')$-th element of the $((M+M^2)\times 1)$ vector $\boldsymbol{c}$ is

$$c_{\varsigma_{(\mu\mu')}} = \mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi})/T. \quad (4.42)$$

(xviii) The $((\mu\mu'),(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{C}$ is

$$c_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} = \sigma_{\mu'\nu}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/T - 2(\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/T^2. \quad (4.43)$$

(xix) The $((\mu\mu'),(\nu\nu')$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{D}$ is

$$d_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} = \mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/2T^2. \quad (4.44)$$

(xx) Similarly the $(\mu,(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ is

$$l_{\rho_\mu\varsigma_{(\nu\nu')}} = \sum_{i=1}^{M}\sum_{\kappa=1}^{M}\sigma^{i\kappa}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa\nu}\boldsymbol{R}_{\kappa\nu}-2\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_\nu'/T]\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/T + \boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/2T. \quad (4.45)$$

(xxi) The $(\mu,(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{C}$ is

$$c_{\rho_\mu\varsigma_{(\nu\nu')}} = \sum_{i=1}^{M}\sum_{\kappa=1}^{M}\sigma^{i\kappa}\sigma_{\kappa\nu}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}^{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/T - 2\sum_{i=1}^{M}\sum_{\kappa=1}^{M}\sigma^{i\kappa}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/T^2 + \mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/2T. \quad (4.46)$$

(xxii) The $(\mu,(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{D}$ is

$$d_{\rho_\mu\varsigma_{(\nu\nu')}} = \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/2T^2. \quad (4.47)$$

(xxiii) The $((\nu\nu'),\mu)$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ is

$$l_{\varsigma_{(\nu\nu')}\rho_\mu} = \sum_{l=1}^{M}\sum_{j=1}^{M}\sigma^{lj}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}[\sigma_{\nu'l}\boldsymbol{R}_{\nu'l}-2\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{h}/T + \boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/2T. \quad (4.48)$$

(xxiv) The $((\nu\nu'),\mu)$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{C}$ is

$$c_{\varsigma_{(\nu\nu')}\rho_\mu} = \sum_{l=1}^{M}\sum_{j=1}^{M}\sigma^{lj}\sigma_{\nu'l}\,\mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu'l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi})/T - 2\sum_{l=1}^{M}\sum_{j=1}^{M}\sigma^{lj}\,\mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_l'\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi})/T^2 + \mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/2T. \quad (4.49)$$

(xxv) The $((\nu\nu'),\mu)$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{D}$ is

$$d_{\varsigma_{(\nu\nu')}\rho_\mu} = \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\,\mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi})/2T^2. \quad (4.50)$$

Theorem 4. Given the assumptions of model (3.1), for each asymptotically efficient estimator of ρ and $\boldsymbol{\varsigma}$, the parameters (3.52) are:

(i)

$$\lambda_0 = \lim_{T\to\infty} \mathrm{E}(\sigma_0{}^2) = 0. \tag{4.51}$$

(ii)

$$\boldsymbol{\lambda}_\rho = \lim_{T\to\infty} \mathrm{E}(\sigma_0 \boldsymbol{d}_{1\rho}) = 0. \tag{4.52}$$

(iii)

$$\boldsymbol{\lambda}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\sigma_0 \boldsymbol{d}_{1\varsigma}) = 0. \tag{4.53}$$

(iv)

$$\boldsymbol{\Lambda}_\varsigma = (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\boldsymbol{N}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}). \tag{4.54}$$

(v)

$$\kappa_0 = \mathrm{tr}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL} - \boldsymbol{\Delta}_I)]/M + n/M \text{ (I=UL, RL, GL, IG, ML)}. \tag{4.55}$$

(vi)

$$\boldsymbol{\kappa}_\varsigma = \mathrm{vec}[(M + K + 1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1}]. \tag{4.56}$$

(vii)

$$\kappa_{\rho_\mu} = -[\rho_\mu(3 + n) + (2n - c_1)/2\rho_\mu]. \tag{4.57}$$

(viii)

$$\kappa_{\rho_\mu}{}^{GL} = \kappa_{\rho_\mu}{}^{LS} - (1 - \rho_\mu{}^2)c_2/2\rho_\mu + [c_1 - (1 - \rho_\mu{}^2)n]/2\rho_\mu. \tag{4.58}$$

(ix)

$$\kappa_{\rho_\mu}{}^{DW} = \kappa_{\rho_\mu}{}^{LS} + 1. \tag{4.59}$$

(x)

$$\boldsymbol{\Lambda}_{\varsigma\rho} = \boldsymbol{\Lambda}'_{\rho\varsigma} = 0. \tag{4.60}$$

Chapter 5

A Special Case of The Generalized Linear Model with Panel Data

5.1 The Model

The Generalized Model with data that are cross-sectional heteroskedastic and AR(1) time series is a special case of the Generalized Linear Model with Panel Data (3.1).

5.1.1 The Model

Consider a system of M regression equations, of which the typical **μ**-th (**μ** = 1, . . . , M) equation is

$$\boldsymbol{y}_\mu = \boldsymbol{X}_\mu \boldsymbol{\beta} + \sigma \boldsymbol{u}_\mu, \tag{5.1}$$

where

$\boldsymbol{y}_\mu$ is a $T \times 1$ vector of observations on the μ-th dependent variable;

$\boldsymbol{X}_\mu$ is a $T \times n$ matrix of observations on κ exogenous variables of μ-th unit ;

$\boldsymbol{\beta}$ is a $n \times 1$ vector of unknown structural parameters;

and

$\sigma \boldsymbol{u}_\mu$ $(\sigma > 0)$ is a $T \times 1$ vector of unobserved stochastic disturbances.

The model can be written as

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_M \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_M \end{bmatrix} \boldsymbol{\beta} + \sigma \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_M \end{bmatrix}. \tag{5.2}$$

More compactly, the model can be written as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \sigma \boldsymbol{u}, \tag{5.3}$$

where

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_M \end{bmatrix}, \boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_M \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_M \end{bmatrix} \tag{5.4}$$

and

$$E(\boldsymbol{u}\boldsymbol{u}') = [(\delta_{\mu\mu'}\boldsymbol{\Omega}_\mu{}^{-1})_{\mu,\mu'=1,\dots,M}]. \tag{5.5}$$

Assumption 3. The following assumptions hold:

1. The random vector $\boldsymbol{u}$ is distributed as a $N(0, \boldsymbol{\Omega}^{-1})$ random variable, where $\boldsymbol{\Omega}$ is $MT \times MT$ positive definite and symmetric partitioned matrix;
2. The matrix $\boldsymbol{X}_\mu$ of the regressors has full column rank, i.e.

$$r(\boldsymbol{X}_\mu) = n; \tag{5.6}$$

3. The regressors are non-stochastic. The results of this thesis would also be valid if the regressors were stochastic, yet uncorrelated with the errors, i.e.,

$$E(\boldsymbol{X}_\mu'\boldsymbol{u}) = 0, \tag{5.7}$$

but in such a case the proofs would be a little more complicated.

5.1.2 Autoregressive extension of the Special Case

Let $u_{t\mu}$ be the t-th observation of the random vector $\boldsymbol{u}_\mu$ of the μ-th equation. Then, we assume the autoregressive scheme:

$$u_{t\mu} = \rho u_{(t-1)\mu} + \varepsilon_{t\mu}; \;\; -1 < \rho_\mu < 1 \;\; (t = 2, \dots, T; \mu = 1, \dots, M), \tag{5.8}$$

where the random variables $\varepsilon_{t\mu}$ satisfy the conditions:

$$E(\varepsilon_{t\mu}) = 0 \;\; (t = 1, \dots, T; \mu = 1, \dots, M), \tag{5.9}$$

$$E(\varepsilon_{t\mu}\varepsilon_{t'\mu'}) = \delta_{tt'}\delta_{\mu\mu'}\sigma_{\mu\mu'} = \begin{cases} \sigma_{\mu\mu} & \text{if } t = t'; \mu = \mu', \\ 0 & \text{if } t \neq t' \text{ or } \;\mu \neq \mu', \end{cases} \tag{5.10}$$

where $\delta_{tt'}$ and $\delta_{\mu\mu'}$ are Kronecker's delta. (see Parks, 1967).

In addition to assumption $\rho_\mu \in (˘1, 1)$, stationarity of AR(1) processes (5.8) implies the following relationships on the initial conditions of the disturbances

$$u_{1\mu} = (1 - \rho_\mu{}^2)^{-1/2}\varepsilon_{1\mu} \tag{5.11}$$

These relationships imply that, for all $t = 1, \dots, T$ and $\mu, \mu' = 1, \dots, M$, the disturbances $u_{t\mu}$ satisfy the following conditions

$$E(u_{t\mu}) = 0 \tag{5.12a}$$

$$\mathrm{E}(u_{t\mu}^2) = \sigma_{\mu\mu}/(1-\rho_\mu{}^2) = \sigma_{u_\mu}{}^2 \tag{5.12b}$$

$$\mathrm{E}(u_{t\mu}\ u_{t\mu'}) = Cov(u_{t\mu}\ u_{t\mu'}) = 0 \text{ for } \mu' \neq \mu \tag{5.12c}$$

$$\mathrm{E}(u_{t\mu}\ u_{t'\mu}) = \rho_\mu^{|t-t'|}\sigma_{u_\mu}{}^2 \text{ for } t \neq t' \tag{5.12d}$$

$$\mathrm{E}(u_{t\mu}\ u_{t'\mu'}) = 0 \text{ for } \mu' \neq \mu \tag{5.12e}$$

Let $\varepsilon_t^{'}$ $(t = 1, \ldots, T)$ be the rows of the $T \times M$ matrix $\boldsymbol{E}$ (i.e. ε_t are the columns of $\boldsymbol{E}'$). Also, let $\varepsilon_\mu^{'}$ $(\mu = 1, \ldots, M)$ be the columns of $\boldsymbol{E}$ (i.e. ε_μ are the rows of $\boldsymbol{E}'$). So,

$$\boldsymbol{E} = \begin{bmatrix} \varepsilon_1^{'} \\ \vdots \\ \varepsilon_T^{'} \end{bmatrix} = [(\varepsilon_t^{'})_{t=1,\ldots,T}]; \quad \boldsymbol{E} = [\varepsilon_1, \ldots, \varepsilon_M] = [(\varepsilon_\mu)_{\mu=1,\ldots,M}]. \tag{5.13}$$

Then, (5.9) and (5.10) imply that

$$\mathrm{E}(\varepsilon_t\varepsilon_t^{'}) = \begin{bmatrix} \mathrm{E}(\varepsilon_{t1}\varepsilon_{t1}) & \ldots & \mathrm{E}(\varepsilon_{t1}\varepsilon_{tM}) \\ \vdots & & \vdots \\ \mathrm{E}(\varepsilon_{tM}\varepsilon_{t1}) & \ldots & \mathrm{E}(\varepsilon_{tM}\varepsilon_{tM}) \end{bmatrix} = \begin{bmatrix} \delta_{tt}\delta_{11}\sigma_{11} & \ldots & \delta_{tt}\delta_{1M}\sigma_{1M} \\ \vdots & & \vdots \\ \delta_{tt}\delta_{M1}\sigma_{M1} & \ldots & \delta_{tt}\delta_{MM}\sigma_{MM} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & \sigma_{MM} \end{bmatrix} = \boldsymbol{\Sigma}, \tag{5.14}$$

which is a $(M \times M)$ matrix of contemporaneous covariances between the t-th elements of any two random variables ε_μ and $\varepsilon_\mu^{'}$.

Similarly for any random vector ε_μ it holds that

$$\mathrm{E}(\varepsilon_\mu) = 0 \tag{5.15a}$$

$$\mathrm{E}(\varepsilon_\mu\varepsilon_\mu^{'}) = \begin{bmatrix} \mathrm{E}(\varepsilon_{1\mu}\varepsilon_{1\mu}) & \ldots & \mathrm{E}(\varepsilon_{1\mu}\varepsilon_{T\mu}) \\ \vdots & & \vdots \\ \mathrm{E}(\varepsilon_{T\mu}\varepsilon_{1\mu}) & \ldots & \mathrm{E}(\varepsilon_{T\mu}\varepsilon_{T\mu}) \end{bmatrix} = \begin{bmatrix} \delta_{\mu\mu}\delta_{11}\sigma_{\mu\mu} & \ldots & \delta_{\mu\mu}\delta_{1T}\sigma_{\mu\mu} \\ \vdots & & \vdots \\ \delta_{\mu\mu}\delta_{T1}\sigma_{\mu\mu} & \ldots & \delta_{\mu\mu}\delta_{TT}\sigma_{\mu\mu} \end{bmatrix} = [(\delta_{tt'}\sigma_{\mu\mu})_{t,t'=1,\ldots,T}] = \sigma_{\mu\mu}\boldsymbol{I}_T. \tag{5.15b}$$

Moreover, for any two random vectors ε_μ, $\varepsilon_{\mu'}^{'}$ $(\mu \neq \mu',\ \mu, \mu' = 1, \ldots, M)$

$$\mathrm{E}(\varepsilon_\mu\ \varepsilon_{\mu'}^{'}) = [(\delta_{tt'}\delta_{\mu\mu'}\sigma_{\mu\mu'})_{t,t'=1,\ldots,T}] = 0. \tag{5.15c}$$

Define the $(TM \times 1)$ vector

$$\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{E}) = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_M \end{bmatrix}. \tag{5.16}$$

Then,

$$\mathrm{E}(\varepsilon) = 0 \tag{5.17a}$$

and

$$\mathrm{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \begin{bmatrix} \sigma_{11}\boldsymbol{I}_T & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \sigma_{MM}\boldsymbol{I}_T \end{bmatrix} = [(\delta_{\mu\mu'}\sigma_{\mu\mu'}\boldsymbol{I}_T)_{\mu,\mu'=1,\dots,M}] = \boldsymbol{\Sigma} \otimes \boldsymbol{I}_T. \tag{5.17b}$$

5.1.3 Representation of the Special Case

Define the $(T \times T)$ matrix (see Parks, 1967)

$$\boldsymbol{P}_\mu = \begin{bmatrix} (1-\rho_\mu{}^2)^{-1/2} & 0 & \dots & 0 \\ (1-\rho_\mu{}^2)^{-1/2}\rho_\mu & 1 & \dots & 0 \\ \vdots & & & \\ (1-\rho_\mu{}^2)^{-1/2}\rho_\mu{}^{T-1} & \dots & & 1 \end{bmatrix}. \tag{5.18}$$

The inverse of $\boldsymbol{P}_\mu$ is

$$\boldsymbol{P}_\mu{}^{-1} = \begin{bmatrix} (1-\rho_\mu{}^2)^{1/2} & 0 & & \dots & 0 \\ -\rho_\mu & 1 & 0 & \dots & 0 \\ 0 & -\rho_\mu & 1 & \dots & 0 \\ \vdots & & \vdots & \ddots & \\ 0 & 0 & \dots & -\rho_\mu & 1 \end{bmatrix}. \tag{5.19}$$

Then, equation (5.8) implies that

$$\boldsymbol{u}_\mu = \boldsymbol{P}_\mu \varepsilon_\mu. \tag{5.20}$$

By using equation (5.20), model (5.1) can be written as

$$\boldsymbol{y}_\mu = \boldsymbol{X}_\mu \boldsymbol{\beta} + \boldsymbol{P}_\mu \boldsymbol{\varepsilon}_\mu. \tag{5.21}$$

Define the $(TM \times TM)$ block diagonal matrix $\boldsymbol{P}$ as

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_1 & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M \end{bmatrix}. \tag{5.22}$$

The inverse of matrix $\boldsymbol{P}$ is

$$\boldsymbol{P}^{-1} = \begin{bmatrix} \boldsymbol{P}_1{}^{-1} & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M{}^{-1} \end{bmatrix}. \tag{5.23}$$

Then, since

$$\boldsymbol{u} = \boldsymbol{P}\boldsymbol{\varepsilon}, \tag{5.24}$$

model (5.3) can be written as

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{P\varepsilon} \tag{5.25}$$

Obviously,

$$\mathrm{E}(\boldsymbol{u}) = \mathrm{E}(\boldsymbol{P\varepsilon}) = \boldsymbol{P}\,\mathrm{E}(\boldsymbol{\varepsilon}) = 0 \tag{5.26a}$$

and

$$\mathrm{E}(\boldsymbol{uu}') = \boldsymbol{\Omega}^{-1} = \mathrm{E}(\boldsymbol{P\varepsilon\varepsilon}'\boldsymbol{P}') = \boldsymbol{P}\,\mathrm{E}(\boldsymbol{\varepsilon\varepsilon}')\boldsymbol{P}' = \boldsymbol{P}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{P}' \tag{5.26b}$$

$$= \begin{bmatrix} \sigma_{11}\boldsymbol{P}_1\boldsymbol{P}_1' & \dots & \mathbf{O} \\ \vdots & & \vdots \\ \mathbf{O} & \dots & \sigma_{MM}\boldsymbol{P}_M\boldsymbol{P}_M' \end{bmatrix} \tag{5.26c}$$

The $TM \times TM$ block diagonal matrix $\boldsymbol{P} = [(\delta_{\mu\mu'}\boldsymbol{P}_\mu)_{\mu,\mu'=1,\dots,M}]$ and the $T \times T$

$$\boldsymbol{R}_{\mu\mu} = \frac{1}{1-\rho_\mu{}^2}\begin{bmatrix} 1 & \rho_\mu & \dots & \rho_\mu{}^{T-1} \\ \rho_\mu & \ddots & & \vdots \\ \vdots & & & \\ \rho_\mu{}^{T-1} & \dots & & 1 \end{bmatrix}. \tag{5.27}$$

As in equation (5.20) consider the $T \times 1$ vectors $\boldsymbol{y}_{\mu*}$ and the $T \times n$ matrices $\boldsymbol{X}_{\mu*}$ with non-autocorrelated elements, satisfying the following relations:

$$\boldsymbol{y}_{\mu*} = \boldsymbol{P}_\mu{}^{-1}\boldsymbol{y}_\mu,\ \boldsymbol{X}_{\mu*} = \boldsymbol{P}_\mu{}^{-1}\boldsymbol{X}_\mu, \tag{5.28}$$

and define the $MT \times 1$ vector $\boldsymbol{y}_*$ and $MT \times n$ matrix $\boldsymbol{X}_*$ as follows:

$$\boldsymbol{y}_* = \begin{bmatrix} \boldsymbol{y}_{1*} \\ \vdots \\ \boldsymbol{y}_{M*} \end{bmatrix},\ \boldsymbol{X}_* = \begin{bmatrix} \boldsymbol{X}_{1*} \\ \vdots \\ \boldsymbol{X}_{M*} \end{bmatrix}. \tag{5.29}$$

Then, premultiplying each regression equation of the form (5.1) by $\boldsymbol{P}_\mu{}^{-1}$ we can derive the following model with non-autocorrelated error terms:

$$\begin{aligned} \boldsymbol{P}_\mu{}^{-1}\boldsymbol{y}_\mu &= \boldsymbol{P}_\mu{}^{-1}\boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{P}_\mu{}^{-1}\boldsymbol{u}_\mu \Rightarrow \\ \boldsymbol{y}_{\mu*} &= \boldsymbol{X}_{\mu*}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_\mu \end{aligned} \tag{5.30}$$

(see Zellner, 1962, Zellner, 1963 Zellner and Huang, 1962, Zellner and Theil, 1962). Alternatively, by premultiplying (5.3) by the matrix $\boldsymbol{P}^{-1}$ defined in (5.23) we take

$$\begin{aligned} \boldsymbol{P}^{-1}\boldsymbol{y} &= \boldsymbol{P}^{-1}\boldsymbol{X\beta} + \boldsymbol{P}^{-1}\boldsymbol{u} \Rightarrow \\ \boldsymbol{y}_* &= \boldsymbol{X}_*\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \end{aligned} \tag{5.31}$$

where $\boldsymbol{y}_* = \boldsymbol{P}^{-1}\boldsymbol{y},\ \boldsymbol{\varepsilon} = \boldsymbol{P}^{-1}\boldsymbol{u},\ \boldsymbol{X}_* = \boldsymbol{P}^{-1}\boldsymbol{X}$.

5.1.4 The specification of $\boldsymbol{\Omega}$

The elements of the $T \times T$ matrix $\boldsymbol{\Omega}$ are functions of the $2M \times 1$ vector

$$\gamma = (\rho', \varsigma')' \tag{5.32}$$

where $\boldsymbol{\rho} = [(\rho_\mu)_{\mu=1....,M}]$ is the $T \times 1$ vector of autocorrelation coefficients and $\boldsymbol{\varsigma} = [(\sigma^{\mu\mu})_{\mu\mu=1,...,M}] = [({\sigma_\mu}^{-2})_{\mu=1,...,M}]$. It can be easily seen that the (μ)-th element of vector $\boldsymbol{\varsigma}$ denoted, as $\sigma^{\mu\mu}$, is actually the (μ,μ)-th element of matrix $\boldsymbol{\Sigma}^{-1}$. $\boldsymbol{\Omega}$ can be written as

$$\boldsymbol{\Omega} = \boldsymbol{P'}^{-1}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}, \tag{5.33}$$

5.1.5 Vectorization of the Model

The system of equations (5.28) or (5.29) can be seen as the outcome of vectorizing the following model:

$$\boldsymbol{Y}_* = \boldsymbol{ZB} + \boldsymbol{E}, \tag{5.34}$$

which can be defined as in the S.U.R. model. In the generalized linear model with panel data, the columns $\boldsymbol{b}_\mu$ $(\mu = 1, \ldots, M)$ of the $(k \times M)$ parameter matrix $\boldsymbol{B}$ obey the restrictions:

$$\boldsymbol{b}_\mu = \boldsymbol{\Psi}_\mu \boldsymbol{\beta}, \tag{5.35}$$

where $\boldsymbol{\Psi}_\mu$ are $(k \times n)$ known matrices and $\boldsymbol{\beta}$ is a $(n \times 1)$ vector of unknown parameters to be estimated. Define the $(Mk \times n)$ matrix $\boldsymbol{\Psi}$ as follows:

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \\ \vdots \\ \boldsymbol{\Psi}_M \end{bmatrix} \tag{5.36}$$

By vectorizing model (5.34) we take

$$\boldsymbol{y}_* = \boldsymbol{X}_*\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{5.37}$$

where

$$\boldsymbol{y}_* = \mathrm{vec}(\boldsymbol{Y}_*),\ \boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{E}), \text{ and } \boldsymbol{X}_* = (\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}. \tag{5.38}$$

In the special case, the columns $\boldsymbol{b}_{\mu_{(\mu=1,\ldots,M)}}$ of the $(k \times M)$ parameter matrix $\boldsymbol{B}$ obey the restrictions:

$$\boldsymbol{b}_\mu = \boldsymbol{\Psi}_\mu \boldsymbol{\beta}, \tag{5.39}$$

where $\boldsymbol{\Psi}_\mu$ are $(k \times n)$ known matrices and $\boldsymbol{\beta}$ is a $(n \times 1)$ vector of unknown parameters to be estimated. Define the $(Mk \times n)$ matrix Ψ as

and

$$\begin{aligned} \boldsymbol{X}_* &= (\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi} = [(\delta_{\mu\mu'}\boldsymbol{Z})\ \mu\mu'] \cdot [(\boldsymbol{\Psi}_{\mu'})\ \mu'] \\ &= \begin{bmatrix} \boldsymbol{Z}\boldsymbol{\Psi}_1 \\ \vdots \\ \boldsymbol{Z}\boldsymbol{\Psi}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_{1*} \\ \vdots \\ \boldsymbol{X}_{M*} \end{bmatrix}. \end{aligned} \tag{5.40}$$

By partitioning $\boldsymbol{y}_*$ and $\boldsymbol{\varepsilon}$ according to $\boldsymbol{X}_*$ in (5.38), model (5.34) can be decomposed as follows:

$$\begin{bmatrix} \boldsymbol{y}_{1*} \\ \vdots \\ \boldsymbol{y}_{M*} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_{1*} \\ \vdots \\ \boldsymbol{X}_{M*} \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_M \end{bmatrix}, \tag{5.41}$$

where $\boldsymbol{X}_{\mu*}$ $(\mu = 1, \ldots, M)$ are $(T \times n)$ matrices.

Note that:

$\boldsymbol{Y}_*$ is a $(T \times M)$ matrix, $\boldsymbol{X}_*$ is a $(MT \times n)$ matrix, $\boldsymbol{X}_*'\boldsymbol{X}_*$ is a $(n \times n)$ matrix, $\boldsymbol{\Psi}$ is a $(Mk \times n)$ matrix and $\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'$ is a $(Mk \times MT)$ matrix.

5.1.6 Identification and estimation of the parameters

Let $\hat{\boldsymbol{\gamma}} = (\hat{\rho}, \hat{\varsigma}')'$ be any consistent estimator of the parameter $\boldsymbol{\gamma}$. For any function $f = f(\gamma)$ we can write $\hat{f} = f(\hat{\gamma})$. The feasible GLS estimator σ is

$$\hat{\sigma} = [(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'(\hat{\boldsymbol{P}}'^{-1}_{GL}(\hat{\boldsymbol{\Sigma}}^{-1}_{GL} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1}_{GL})(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})/(MT - n)]^{1/2}, \tag{5.42}$$

From (5.20) it is straightforward that the parameters σ and σ_t $(t = 1, \ldots, T)$ cannot be distinguished, that is the parameters σ and s cannot be simultaneously identified without the restriction $\sigma = 1$, under which the estimate $\hat{\boldsymbol{\Omega}}^{-1}$ is supposed to be accurate, up to a multiplicative factor. This is not true in small samples, and a reasonable method to account for this is to use the feasible GLS estimate of $\hat{\sigma}$ from (5.33) in order to the traditional t and F test statistics. This method is meaningless from the estimation viewpoint, but its success in improving the size corrections must be the only criterion to judge its validity.

5.1.7 Regularity conditions

Denote as Ω_i, Ω_{ij}, etc., the $MT \times MT$ matrices of first-, second- and higher-order derivatives of the elements of Ω with respect to the elements of the $(M+M) \times 1$ vector of nuisance parameters $\boldsymbol{\gamma} = (\boldsymbol{\rho}', \boldsymbol{\varsigma}')'$. Moreover, for any estimator $\hat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$, define the $(1+M+M) \times 1$ vector $\boldsymbol{\delta}$ with elements

$$\delta_0 = \frac{\hat{\sigma}^2 - 1}{\tau}; \quad \delta_{\rho_\mu} = \frac{\hat{\rho}_\mu - \rho_\mu}{\tau}; \quad \delta_{\sigma^{\mu\mu}} = \frac{\hat{\sigma}^{\mu\mu} - \sigma^{\mu\mu}}{\tau} \tag{5.43}$$

where $\mu = 1, \ldots, M$ and $\tau = \frac{1}{\sqrt{T}}$ is the "asymptotic scale" of our expansions.

The suggested size corrections are based on the following
Regularity Conditions:

(1) The elements of matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^{-1}$ are bounded for all T, all vectors $\boldsymbol{\rho}$ with elements $\rho_\mu \in (-1, 1)$, and all vectors $\boldsymbol{\varsigma} \in \mathcal{F}_s = \mathbb{R}^m \setminus \{0\}$. Moreover, the matrices

$$\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T, \quad \boldsymbol{F} = \boldsymbol{X}\boldsymbol{X}'/T, \quad \boldsymbol{\Gamma} = \boldsymbol{Z}'\boldsymbol{Z}/T \tag{5.44}$$

converge to non-singular limits as $T \to \infty$.

(2) Up to the fourth order, the partial derivatives of the elements of $\boldsymbol{\Omega}$ with respect to the elements of $\boldsymbol{\rho}$ and $\boldsymbol{\varsigma}$, are bounded for all T, all vectors $\boldsymbol{\rho}$ with elements in interval $(-1, 1)$ and all vectors $\boldsymbol{\varsigma} \in \mathcal{F}_s$.

(3) The estimators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varsigma}}$ are even functions of $\boldsymbol{u}$, and they are functionally unrelated to the parameter vector $\boldsymbol{\beta}$, i.e., they can be written as functions of $\boldsymbol{X}$, $\boldsymbol{Z}$ and $\boldsymbol{u}$ only.

(4) The vector $\boldsymbol{\delta}$ admits a stochastic expansion of the form

$$\begin{aligned}\boldsymbol{\delta} &= \begin{bmatrix} \delta_0 \\ [(\delta_{\rho_\mu})_{\mu=1,\ldots,M}]' \\ [(\delta_{\sigma^{\mu\mu}})_{\mu\mu=1,\ldots,M}]' \end{bmatrix} \\ &= \boldsymbol{d}_1 + \tau \boldsymbol{d}_2 + \omega(\tau^2) \end{aligned} \tag{5.45}$$

where the order of magnitude $\omega(\cdot)$ defined in Notational Convensions, has the same operational properties as the order $O(\cdot)$, and the expectations

$$\mathbf{E}(\boldsymbol{d}_1 \boldsymbol{d}_1'), \quad \mathbf{E}(\boldsymbol{d}_1 + \sqrt{T}\boldsymbol{d}_2) \tag{5.46}$$

exist and have finite limits as $T \to \infty$.

Discussions on the Regularity Conditions:

The first two regularity conditions imply that the $n \times n$ matrices

$$\boldsymbol{A}_i = \boldsymbol{X}'\boldsymbol{\Omega}_i\boldsymbol{X}/T, \quad \boldsymbol{A}_{ij} = \boldsymbol{X}'\boldsymbol{\Omega}_{ij}\boldsymbol{X}/T, \quad \boldsymbol{A}_{ij}{}^{*} = \boldsymbol{X}'\boldsymbol{\Omega}_i\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_j\boldsymbol{X}/T \tag{5.47}$$

are bounded, and therefore the Taylor series expansion of β is a stochastic expansion (see Magdalinos, 1992). Since the parameters $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_\mu)'$ and $\boldsymbol{\varsigma} = [(\sigma^{\mu\mu})_{\mu\mu=1,\ldots,M}]'$ are functionally unrelated to $\boldsymbol{\beta}$, regularity condition (3) is satisfied for a wide class of estimators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varsigma}}$ including the maximum likelihood estimators and the simple and iterative estimators based on the regression residuals [see Breush (1980); Rothenberg (1984a)]. Note that we need not assume that the estimators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varsigma}}$ are asymptotically efficient. Also, notice that the regularity conditions (1) through (4) are satisfied by all the estimators of $\boldsymbol{\rho}$ and $\boldsymbol{\varsigma}$ examined in the next section.

5.1.8 Definition of parameters

Finally, define

the scalars $\lambda_0, \ \kappa_0$, the $M \times 1$ vectors $\boldsymbol{\lambda}_\rho, \ \boldsymbol{\kappa}_\rho$, the $M^2 \times 1$ vectors $\boldsymbol{\lambda}_\varsigma$, the $M \times M$ matrix $\boldsymbol{\Lambda}_\rho$;

the $M^2 \times M$ matrix $\boldsymbol{\Lambda}_{\rho\varsigma}$, the $M^2 \times M^2$ matrix $\boldsymbol{\Lambda}_{\varsigma\varsigma}$, as follows:

$$\boldsymbol{\Lambda}_* = \begin{bmatrix} \lambda_0 & \boldsymbol{\lambda}_\rho{}' & \boldsymbol{\lambda}_\varsigma{}' \\ \boldsymbol{\lambda}_\rho & \boldsymbol{\Lambda}_\rho & \boldsymbol{\Lambda}_{\rho\varsigma}{}' \\ \boldsymbol{\lambda}_\varsigma & \boldsymbol{\Lambda}_{\rho\varsigma} & \boldsymbol{\Lambda}_\varsigma \end{bmatrix} = \mathrm{E}(\boldsymbol{d}_1\boldsymbol{d}_1'); \ \boldsymbol{\kappa}_* = \begin{bmatrix} \kappa_0 \\ \boldsymbol{\kappa}_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix} = \mathrm{E}(\boldsymbol{d}_1 + \sqrt{T}\boldsymbol{d}_2) \tag{5.48}$$

We partition matrix $\boldsymbol{\Lambda}_*$ and vector $\boldsymbol{\kappa}_*$ as follows:

$$\begin{bmatrix} \lambda_0 & \boldsymbol{\lambda}' \\ \boldsymbol{\lambda} & \boldsymbol{\Lambda} \end{bmatrix} \text{ and } \begin{bmatrix} \kappa_0 \\ \boldsymbol{\kappa} \end{bmatrix} \tag{5.49}$$

where

$$\boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_\rho \\ \boldsymbol{\lambda}_{\rho\varsigma} \end{bmatrix}, \ \boldsymbol{\kappa} = \begin{bmatrix} \boldsymbol{\kappa}_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix}, \ \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_\rho & \boldsymbol{\Lambda}'_{\rho\varsigma} \\ \boldsymbol{\Lambda}_{\rho\varsigma} & \boldsymbol{\Lambda}_\varsigma \end{bmatrix}, \tag{5.50}$$

and $\boldsymbol{\Lambda}$ is a $(M \times M^2) \times (M \times M^2)$ matrix and $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$ are $((M \times M^2) \times 1)$ vectors. The elements of $\boldsymbol{\Lambda}_*$ and $\boldsymbol{\kappa}_*$ in equations (5.47), (5.48) and (5.49) can be interpreted as "measures" of the accuracy of the expansions of $\hat{\sigma}^2, \hat{\rho}_\mu$ and $\hat{\sigma}_{(\mu\mu)}$ around the true values of the corresponding parameters.

5.1.9 3-step Estimation

Denote by LS, GL, IG, ML the least squares, generalized least squares, iterative GLS and maximum likelihood estimation methods, respectively. Also, denote by $\hat{\beta}_I$ any consistent estimator of β in model (5.1), indexed by I (I=S, GL, IG, ML).

The discussion above suggests the following 3 steps of an estimation strategy:

- Step 1: Single equation estimation of autoregressive parameters ρ_μ

$$\begin{aligned} \hat{u}_{\mu_{(I)}} &= \boldsymbol{y}_\mu - \boldsymbol{X}_\mu \hat{\boldsymbol{\beta}}_{(I)} \\ \hat{\rho}_{\mu_{(I)}} &= \frac{\sum_{t=2}^T \hat{u}_{t\mu_{(I)}} \hat{u}_{(t-1)\mu_{(I)}}}{\sum_{t=2}^T \hat{u}^2_{(t-1)\mu_{(I)}}} \end{aligned} \tag{5.51}$$

- Step 2: Transform model (5.1) to obtain estimations of contemporaneous covariances $\sigma_{\mu\mu'}$

  i Transform the model in order to cancel out first-order autoregression

$$\begin{aligned} \hat{\boldsymbol{P}}_\mu^{-1} \boldsymbol{y}_\mu &= \hat{\boldsymbol{P}}_\mu^{-1} \boldsymbol{X}_\mu \boldsymbol{\beta} + \hat{\boldsymbol{P}}_\mu^{-1} \boldsymbol{P}_\mu \boldsymbol{\varepsilon}_\mu \\ &\text{or} \\ \boldsymbol{y}_{\mu*} &= \boldsymbol{X}_{\mu*} \boldsymbol{\beta} + \boldsymbol{\varepsilon}_{\mu*} \,. \end{aligned} \tag{5.52}$$

  ii Estimate (5.52) via (I) to obtain the estimators $\hat{\boldsymbol{\beta}}_I^*$ and the residuals

$$\hat{\varepsilon}_\mu^* = \boldsymbol{y}_{\mu*} - \boldsymbol{X}_{\mu*} \hat{\boldsymbol{\beta}}_{(I)} \,. \tag{5.53}$$

  iii Estimate covariances by

$$\hat{\sigma}_{\mu\mu} = \frac{\hat{\varepsilon}_\mu^{*\prime} \hat{\varepsilon}_\mu^*}{T-n} \,. \tag{5.54}$$

  to obtain $\hat{\boldsymbol{\Sigma}}_{(I)}$.

- Step 3: Aitken estimation of (5.3) by using $\hat{\boldsymbol{\Omega}}$.

  Since,

$$\begin{aligned} \boldsymbol{\Omega}^{-1} &= \boldsymbol{P}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{P}' \Rightarrow \\ \hat{\boldsymbol{\Omega}} &= \hat{\boldsymbol{P}}'^{-1}(\hat{\boldsymbol{\Sigma}}_I^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1} \,. \end{aligned} \tag{5.55}$$

  and

$$\begin{aligned} \hat{\boldsymbol{\beta}}_{GLS} &= (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{y} \\ &= [\boldsymbol{X}'\hat{\boldsymbol{P}}'^{-1}(\hat{\boldsymbol{\Sigma}}_{(I)}^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1}\boldsymbol{X}]^{-1}\boldsymbol{X}'[\hat{\boldsymbol{P}}'^{-1}(\hat{\boldsymbol{\Sigma}}_{(I)}^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1}]\boldsymbol{y} \\ &= [\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}_{(I)}^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{P}}^{-1}\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}_{(I)}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{y}_* \,. \end{aligned} \tag{5.56}$$

5.2 Asymptotically efficient estimators of $\boldsymbol{\rho}$ and $\boldsymbol{B}$

5.2.1 Estimators of ρ

Some of the most frequently used estimators of ρ in applied econometric research are:

1. The least squares (LS) estimator

$$\tilde{\rho}_{\mu} = \sum_{t=2}^{T} \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu} / \sum_{t=1}^{T} \left(\tilde{u}_{t\mu}\right)^2, \tag{5.57}$$

   where $\tilde{u}_{t\mu}$ are the LS residuals of regression model (5.1).

2. The Durbin-Watson (DW) estimator, which is computed via the DW-statistic approximation as

$$\hat{\rho}^{DW} = 1 - \left(\frac{DW}{2}\right). \tag{5.58}$$

3. The generalized least squares (GL) estimator

$$\hat{\rho}_{\mu} = \sum_{t=2}^{T} \hat{u}_{t\mu}\hat{u}_{(t-1)\mu} / \sum_{t=1}^{T} \left(\hat{u}_{t\mu}\right)^2, \tag{5.59}$$

   where $\hat{u}_{t\mu}$ are the GL residuals after correcting model (5.1).

4. The Prais-Winston (1954) estimator $\hat{\rho}_{\mu}^{PW}$, which, together with the PW estimator $\hat{\beta}_{\mu}^{PW}$ minimises the sum of squared GL residuals.

5. The maximum likelihood (ML) estimator, $\hat{\rho}_{\mu}^{ML}$, which satisfies a cubic equation with coefficients defined in terms of the (heteroskedasticity corrected) ML residuals in the (heteroskedasticity corrected) regression model (5.1) [see Beach and Mac Kinnon (1978)].

5.2.2 Estimators of $\boldsymbol{B}$

Some of the most frequently used estimators of $\boldsymbol{B}$ in applied econometric research are (Symeonides et al., 2016)

1. The unrestricted least squares (UL) estimator

$$\hat{\boldsymbol{B}}_{(UL)} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{Y}_{*}. \tag{5.60}$$

2. The restricted least squares (RL) estimator

$$\text{vec}(\hat{\boldsymbol{B}}_{(RL)}) = \boldsymbol{\Psi}(\boldsymbol{X}_{*}'\boldsymbol{X}_{*})^{-1}\boldsymbol{X}_{*}'\boldsymbol{y}_{*}. \tag{5.61}$$

3. The The generalized least squares (GL) estimator

$$\mathrm{vec}(\hat{\boldsymbol{B}}_{(GL)}) = \boldsymbol{\Psi}[\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}_I^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}_I^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{y}_*, \tag{5.62}$$

   where $\hat{\boldsymbol{\Sigma}}_I^{-1}$ is the UL or RL estimator of $\boldsymbol{\Sigma}^{-1}$.

4. The iterative generalized least squares (IG) estimator $\hat{\boldsymbol{B}}_{(IG)}$ is computed by the iterative implementation of GL estimator.

5. The maximum likelyhood (ML) estimator $\hat{\boldsymbol{B}}_{(ML)}$ can be computed computed by the iterating the GL estimation process up to convergence (Dhrymes, 1971).

Chapter 6

Size Corrected Test Statistics

6.1 Introduction

This chapter specifies the analytical forms of the Edgeworth and Cornish-Fisher size corrections of the t and F tests in the Special Case of The Generalized Linear Model with Panel Data. For this purpose, we calculate some useful quantities.

6.2 t-test

Let e_o be a known scalar and let $\boldsymbol{e}$ be a known $n \times 1$ vector.To test the null hypothesis

$$H_0 : \boldsymbol{e}'\boldsymbol{\beta} - e_0 = 0 \tag{6.1}$$

for one-sided alternative hypotheses we use the statistic

$$t = (\boldsymbol{e}'\hat{\boldsymbol{\beta}} - e_0)/[\hat{\sigma}^2 \boldsymbol{e}'(\boldsymbol{X}'\widehat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{e}]^{1/2}. \tag{6.2}$$

We define the $((M+M) \times 1)$ vector $\boldsymbol{l}$ and the $((M+M) \times (M+M))$ matrix $\boldsymbol{L}$ as follows:

$$\boldsymbol{l} = \begin{bmatrix} [(l_{\rho_\mu})_{\mu=1,\ldots,M}]' \\ [(l_{(\mu\mu)})_{(\mu\mu)=1,\ldots,M}]' \end{bmatrix} \tag{6.3}$$

and

$$\boldsymbol{L} = \begin{bmatrix} [(l_{\rho_\mu \rho_{\mu'}})_{\mu,\mu'=1,\ldots,M}] & \left[(l_{\rho_\mu (\nu\nu)})_{\mu=1,\ldots,M;\ (\nu\nu)=1,\ldots,M}\right] \\ \left[(l_{(\nu\nu)\rho_\mu})_{(\nu\nu)=1,\ldots,M;\ \mu=1,\ldots,M}\right] & \left[(l_{(\mu\mu)(\nu\nu)})_{(\mu\mu)=1,\ldots,M;\ (\nu\nu)=1,\ldots,M}\right] \end{bmatrix} \tag{6.4}$$

where the elements of vector $\boldsymbol{l}$ and matrix $\boldsymbol{L}$ are defined as follows:

$$\begin{aligned} l_{\rho_\mu} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{h}, \\ l_{(\mu\mu)} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{A}_{(\mu\mu)}\boldsymbol{G}\boldsymbol{h}, \\ l_{\rho_\mu \rho_{\mu'}} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu \rho_{\mu'}}\boldsymbol{G}\boldsymbol{h}, \\ l_{\rho_\mu (\nu\nu)} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu (\nu\nu)}\boldsymbol{G}\boldsymbol{h}, \\ l_{(\nu\nu)\rho_\mu} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{(\nu\nu)\rho_\mu}\boldsymbol{G}\boldsymbol{h}, \\ l_{(\mu\mu)(\nu\nu)} &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{(\mu\mu)(\nu\nu)}\boldsymbol{G}\boldsymbol{h}, \end{aligned} \tag{6.5}$$

and $\boldsymbol{G} = \boldsymbol{A}^{-1} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}$ is a $(n \times n)$ matrix, $\boldsymbol{h} = \boldsymbol{e}/(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}$ is a $(n \times 1)$ vector, and

$$\begin{aligned} \boldsymbol{C}_{\rho_\mu \rho_{\mu'}} &= \boldsymbol{A}_{\rho_\mu \rho_{\mu'}}{}^{*} - 2\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}} + \boldsymbol{A}_{\rho_\mu \rho_{\mu'}}/2, \\ \boldsymbol{C}_{\rho_\mu(\nu\nu)} &= \boldsymbol{A}_{\rho_\mu(\nu\nu)}{}^{*} - 2\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)} + \boldsymbol{A}_{\rho_\mu(\nu\nu)}/2, \\ \boldsymbol{C}_{(\mu\mu)(\nu\nu)} &= \boldsymbol{A}_{(\mu\mu)(\nu\nu)}{}^{*} - 2\boldsymbol{A}_{(\mu\mu)}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)} + \boldsymbol{A}_{(\mu\mu)(\nu\nu)}/2, \end{aligned} \tag{6.6}$$

with the obvious adjustments for $\boldsymbol{C}_{\varsigma_\nu \rho_\mu}$. Matrices $\boldsymbol{A}_i$, $\boldsymbol{A}_{ij}$ and $\boldsymbol{A}_{ij}{}^{*}$ are defined in the equation (5.47). The corrected critical value, using the Edgeworth approximation of the t distribution is given by

$$t_\alpha{}^{*} = t_\alpha + \frac{\tau^2}{2}[p_1 + p_2 t_\alpha{}^2]t_\alpha, \tag{6.7}$$

(see Edgeworth, 1903). Moreover, the corrected statistic from the Cornish Fisher approximation of the t distribution is given by

$$t^{*} = t - \frac{\tau^2}{2}\left[p_1 + p_2 t^2\right]t, \tag{6.8}$$

(see, inter alia, Cornish and Fisher, 1937, Fisher and Cornish, 1960, Hill and Davis, 1968). In order to correct either the critical value or the t-statistic the required correction quantities p_1, p_2 are given by the following Proposition.

Proposition 5. The quantities p_1, p_2, required for the calculation of both the Edgeworth corrected critical values of the t distribution, and the Cornish-Fisher corrected t-statistic are:

$$p_1 = \operatorname{tr}\boldsymbol{\Lambda}\boldsymbol{L} + \frac{\boldsymbol{l}'\boldsymbol{\Lambda}\boldsymbol{l}}{4} + \boldsymbol{l}'(\boldsymbol{\kappa} + \frac{\boldsymbol{\lambda}}{2}) - \kappa_0 + \frac{\lambda_0 - 2}{4} \tag{6.9}$$

$$p_2 = \frac{\boldsymbol{l}'\boldsymbol{\Lambda}\boldsymbol{l} - 2\boldsymbol{l}'\boldsymbol{\lambda} + \lambda_0 - 2}{4} \tag{6.10}$$

6.3 The Wald and F Tests

Let $\boldsymbol{H}$ be a $r \times n$ known matrix of $\operatorname{rank}(\boldsymbol{H}) = r$ and let $\boldsymbol{h}_0$ be a known $r \times 1$ vector. The test of the null hypothesis

$$H_0 : \boldsymbol{H}\boldsymbol{\beta} - \boldsymbol{h}_0 = 0 \tag{6.11}$$

we use the Wald statistic

$$w = (\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h}_0)'[\boldsymbol{H}(\boldsymbol{X}'\widehat{\boldsymbol{\Omega}}\boldsymbol{X}/T)^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h}_0)/\hat{\sigma}^2. \tag{6.12}$$

or the degrees-of-freedom-adjusted F statistic

$$F = (\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h}_0)'[\boldsymbol{H}(\boldsymbol{X}'\widehat{\boldsymbol{\Omega}}\boldsymbol{X}/T)^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}} - \boldsymbol{h}_0)/r\hat{\sigma}^2. \tag{6.13}$$

Define the $(n \times n)$ matrix $\boldsymbol{G}$ and the $(n \times n)$ matrix $\boldsymbol{\Xi}$ as follows

$$\boldsymbol{G} = \boldsymbol{A}^{-1} \text{ and } \boldsymbol{\Xi} = \boldsymbol{GQG}, \tag{6.14}$$

where

$$\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T \text{ and } \boldsymbol{Q} = \boldsymbol{H}'(\boldsymbol{HGH}')^{-1}\boldsymbol{H}. \tag{6.15}$$

Next, define the $(M+M)\times 1$ vector $\boldsymbol{c}$ and the $(M+M)\times(M+M)$ matrices $\boldsymbol{C}$, $\boldsymbol{D}$ as follows:

$$\boldsymbol{c} = \begin{bmatrix} [(c_{\rho_\mu})_{\mu=1,\dots,M}]' \\ [(\boldsymbol{c}_{(\mu\mu)})_{(\mu\mu)=1,\dots,M}]' \end{bmatrix}, \tag{6.16}$$

$$\boldsymbol{C} = \begin{bmatrix} [(c_{\rho_\mu\rho_{\mu'}})_{\mu,\mu'=1,\dots,M}] & \left[(c_{\rho_\mu(\nu\nu)})_{\mu=1,\dots,M;\ (\nu\nu)=1,\dots,M}\right] \\ \left[(c_{(\nu\nu)\rho_\mu})_{(\nu\nu)=1,\dots,M;\ \mu=1,\dots,M}\right] & \left[(c_{(\mu\mu)(\nu\nu)})_{(\mu\mu)=1,\dots,M;\ (\nu\nu)=1,\dots,M}\right] \end{bmatrix} \tag{6.17}$$

and

$$\boldsymbol{D} = \begin{bmatrix} [(d_{\rho_\mu\rho_{\mu'}})_{\mu,\mu'=1,\dots,M}] & \left[(d_{\rho_\mu(\nu\nu)})_{\mu=1,\dots,M;\ (\nu\nu)=1,\dots,M}\right] \\ \left[(d_{(\nu\nu)\rho_\mu})_{(\nu\nu)=1,\dots,M;\ \mu=1,\dots,M}\right] & \left[(d_{(\mu\mu)(\nu\nu)})_{(\mu\mu)=1,\dots,M;\ (\nu\nu)=1,\dots,M}\right] \end{bmatrix} \tag{6.18}$$

where the elements of matrices $\boldsymbol{C}$, $\boldsymbol{D}$ and vector $\boldsymbol{c}$ are defined as follows:

$$\begin{aligned} c_{\rho_\mu} &= \text{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}), \\ c_{\rho_\mu\rho_{\mu'}} &= \text{tr}(\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}) \\ c_{\rho_\mu(\nu\nu)} &= \text{tr}(\boldsymbol{C}_{\rho_\mu(\nu\nu)}\boldsymbol{\Xi}) \\ c_{(\mu\mu)} &= \text{tr}(\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi}), \\ c_{(\mu\mu)(\nu\nu)} &= \text{tr}(\boldsymbol{C}_{(\mu\mu)(\nu\nu)}\boldsymbol{\Xi}) \\ d_{\rho_\mu\rho_{\mu'}} &= \text{tr}(\boldsymbol{D}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}), \\ d_{(\mu\mu)(\nu\nu)} &= \text{tr}(\boldsymbol{D}_{(\mu\mu)(\nu\nu)}\boldsymbol{\Xi}), \\ d_{\rho_\mu(\nu\nu)} &= \text{tr}(\boldsymbol{D}_{\rho_\mu(\nu\nu)},\boldsymbol{\Xi}) \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} \boldsymbol{D}_{\rho_\mu\rho_{\mu'}} &= \frac{\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_{\mu'}}}{2}, \\ \boldsymbol{D}_{\rho_\mu(\nu\nu)} &= \frac{\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{(\nu\nu)}}{2}, \\ \boldsymbol{D}_{(\mu\mu)(\nu\nu)} &= \frac{\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi}\boldsymbol{A}_{(\nu\nu)}}{2}, \end{aligned} \tag{6.20}$$

with the obvious adjustments for $c_{(\nu\nu)\rho_\mu}$, $d_{(\nu\nu)\rho_\mu}$ and $\boldsymbol{D}_{(\nu\nu)\rho_\mu}$.

The corrected critical value, using the Edgeworth approximation of the F distribution is given by

$$F_\alpha{}^* = F_\alpha + \tau^2\left[q_1 + q_2 F_\alpha\right] F_\alpha, \tag{6.21}$$

(see Edgeworth, 1903). Moreover, the corrected statistic from the Cornish Fisher approximation of the F distribution is given by

$$\mathcal{F} = F - \tau^2(q_1 + q_2 F)F, \tag{6.22}$$

(see, inter alia, Cornish and Fisher, 1937, Fisher and Cornish, 1960, Hill and Davis, 1968). In order to correct either the critical value or the F-statistic the required correction quantities q_1, q_2 are given by the following Proposition.

Proposition 6. The quantities q_1, q_2, required for the calculation of both the Edgeworth corrected critical values of the F distribution and the Cornish-Fisher corrected F statistic are:

$$q_1 = \xi_1/r + (r-2)/2,\ q_2 = \xi_2/(r+2) - r/2, \tag{6.23}$$

where

$$\xi_1 \;=\; \mathrm{tr}[\boldsymbol{\Lambda}(\boldsymbol{C}+\boldsymbol{D})] - \boldsymbol{c}'\boldsymbol{\Lambda}\boldsymbol{c}/4 + \boldsymbol{c}'\boldsymbol{\kappa} + r[\boldsymbol{c}'\boldsymbol{\lambda}/2 - \kappa_0 - (r-2)\lambda_0/4] \tag{6.24}$$

$$\xi_2 \;=\; \mathrm{tr}\,(\boldsymbol{\Lambda}\boldsymbol{D}) + [\boldsymbol{c}'\boldsymbol{\Lambda}\boldsymbol{c} - (r+2)(2\boldsymbol{c}'\boldsymbol{\lambda} - r\lambda_0)]/4. \tag{6.25}$$

6.4 Theorems

Theorem 5. The vectors $\boldsymbol{l}$, $\boldsymbol{c}$ and the matrices $\boldsymbol{L}, \boldsymbol{C}, \boldsymbol{D}$, can be calculated as follows:

(i) The $\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}$ matrix is

$$\begin{aligned}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}} \;=\;& \delta_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/T - 2\sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}_{\mu'}'\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/T^2\\ &+\delta_{\mu\mu'}\sigma^{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/2T.\end{aligned} \tag{6.26}$$

(ii) The $\boldsymbol{D}_{\rho_\mu\rho_{\mu'}}$ matrix is

$$\boldsymbol{D}_{\rho_\mu\rho_{\mu'}} \;=\; \sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}_{\mu'}'\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/2T^2. \tag{6.27}$$

(iii) The $\boldsymbol{C}_{(\mu\mu)(\nu\nu)}$ matrix is

$$\boldsymbol{C}_{(\mu\mu)(\nu\nu)} = \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{B}_{\mu\mu} - 2\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu}. \tag{6.28}$$

(iv) The $\boldsymbol{D}_{(\mu\mu)(\nu\nu)}$ matrix is

$$\boldsymbol{D}_{(\mu\mu)(\nu\nu)} = \boldsymbol{B}_{\mu\mu}\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu}/2. \tag{6.29}$$

(v) The $\boldsymbol{C}_{\rho_\mu(\nu\nu)}$ matrix is

$$\begin{aligned}\boldsymbol{C}_{\rho_\mu(\nu\nu)} \;=\;& \delta_{\mu\nu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T - 2\sigma^{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu/T^2\\ &+\delta_{\mu\nu}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/2T.\end{aligned} \tag{6.30}$$

(vi) The $\boldsymbol{D}_{\rho_\mu(\nu\nu)}$ matrix is

$$\boldsymbol{D}_{\rho_\mu(\nu\nu)} = \sigma^{\mu\mu}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{\Xi}\boldsymbol{X}'_\nu \boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu/2T^2. \tag{6.31}$$

(vii) The $\boldsymbol{C}_{(\nu\nu)\rho_\mu}$ matrix is

$$\begin{aligned}\boldsymbol{C}_{(\nu\nu)\rho_\mu} &= \delta_{\mu\nu}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T - 2\sigma^{\mu\mu}\boldsymbol{X}'_\nu \boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu \boldsymbol{G}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T^2 \\ &+\delta_{\mu\nu}\boldsymbol{X}'_\nu \boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_\nu/2T. \end{aligned}\tag{6.32}$$

(viii) The $\boldsymbol{D}_{(\nu\nu)\rho_\mu}$ matrix is

$$\boldsymbol{D}_{(\nu\nu)\rho_\mu} = \sigma^{\mu\mu}\boldsymbol{X}'_\nu \boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu \boldsymbol{\Xi}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/2T^2. \tag{6.33}$$

(ix) The μ-th element of the $((M+M)\times 1)$ vector $\boldsymbol{l}$ is

$$l_{\rho_\mu} = \sigma^{\mu\mu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{G}\boldsymbol{h}/T, \tag{6.34}$$

where

$$\boldsymbol{h} = \frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}. \tag{6.35}$$

(x) Similarly, the $(\mu\mu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{L}$ is

$$\begin{aligned} l_{\rho_\mu\rho_{\mu'}} &= \delta_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{h}/T - 2\sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{G}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{h}/T^2 \\ &+\delta_{\mu\mu'}\sigma^{\mu\mu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{G}\boldsymbol{h}/2T. \end{aligned}\tag{6.36}$$

(xi) The μ-th element of the $((M+M)\times 1)$ vector $\boldsymbol{c}$ is

$$c_{\rho_\mu} = \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{\Xi}/T). \tag{6.37}$$

(xii) The $(\mu\mu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned} c_{\rho_\mu\rho_{\mu'}} &= \delta_{\mu\mu'}\sigma^{\mu'\mu'}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T) - 2\sigma^{\mu\mu}\sigma^{\mu'\mu'}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{G}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T^2) \\ &+\delta_{\mu\mu'}\sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{\Xi}/2T). \end{aligned}\tag{6.38}$$

(xiii) The (μ,μ')-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{D}$ is

$$d_{\rho_\mu\rho_{\mu'}} = \sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{\Xi}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/2T^2. \tag{6.39}$$

(xiv) The $(\mu\mu)$-th element of the $((M+M)\times 1)$ vector $\boldsymbol{l}$ is

$$l_{(\mu\mu)} = \boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu \boldsymbol{G}\boldsymbol{h}/T. \tag{6.40}$$

(xv) Similarly, the $((\mu\mu),(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{L}$ is

$$l_{(\mu\mu)(\nu\nu)} = \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_{\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{G}\boldsymbol{h}/T - 2\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_{\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{G}\boldsymbol{X}'_{\nu}\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_{\nu}\boldsymbol{G}\boldsymbol{h}/T^2. \tag{6.41}$$

(xvi) The $(\mu\mu')$-th element of the $((M+M)\times 1)$ vector $\boldsymbol{c}$ is

$$c_{(\mu\mu)} = \mathrm{tr}(\boldsymbol{X}'_{\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{\Xi}/T). \tag{6.42}$$

(xvii) The $((\mu\mu),(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{C}$ is

$$c_{(\mu\mu)(\nu\nu)} = \delta_{\mu\nu}\sigma_{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_{\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{\Xi})/T - 2\,\mathrm{tr}(\boldsymbol{X}'_{\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{G}\boldsymbol{X}'_{\nu}\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_{\nu}\boldsymbol{\Xi}/T^2). \tag{6.43}$$

(xviii) The $((\mu\mu),(\nu\nu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{D}$ is

$$d_{(\mu\mu)(\nu\nu)} = \mathrm{tr}(\boldsymbol{X}'_{\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{\Xi}\boldsymbol{X}'_{\nu}\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_{\nu}\boldsymbol{\Xi}/2T^2). \tag{6.44}$$

(xix) Similarly, the $(\mu,(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{L}$ is

$$l_{\rho_\mu(\nu\nu)} = \boldsymbol{h}'\boldsymbol{G}\left[\delta_{\mu\nu}\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}/T - 2\sigma^{\mu\mu}\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{G}\boldsymbol{X}'_{\nu}\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_{\nu}/T^2 + \delta_{\mu\nu}\boldsymbol{X}'_{\nu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_{\nu}/T\right]\boldsymbol{G}\boldsymbol{h}. \tag{6.45}$$

(xx) The $(\mu,(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned} c_{\rho_\mu(\nu\nu)} = {} & \delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{\Xi}/T) \\ & -2\sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{\Xi}\boldsymbol{X}'_{\nu}\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_{\nu}\boldsymbol{\Xi}/2T^2) \\ & +\delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}'_{\nu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_{\nu}\boldsymbol{\Xi}/2T). \end{aligned} \tag{6.46}$$

(xxi) The $(\mu,(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{D}$ is

$$d_{\rho_\mu(\nu\nu)} = \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{\Xi}\boldsymbol{X}'_{\nu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_{\nu}\boldsymbol{\Xi}/2T^2). \tag{6.47}$$

(xxii) The $((\nu\nu),\mu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{L}$ is

$$l_{(\nu\nu)\rho_\mu} = \boldsymbol{h}'\boldsymbol{G}\left[\delta_{\mu\nu}\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}/T - 2\sigma^{\mu\mu}\boldsymbol{X}'_{\nu}\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_{\nu}\boldsymbol{G}\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}/T^2 + \delta_{\mu\nu}\boldsymbol{X}'_{\nu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu}/2T\right]\boldsymbol{G}\boldsymbol{h}. \tag{6.48}$$

(xxiii) The $((\nu\nu),\mu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned} c_{(\nu\nu)\rho_\mu} = {} & \delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{\Xi}/T) \\ & -2\sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_{\nu}\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_{\nu}\boldsymbol{G}\boldsymbol{X}'_{\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_{\mu}\boldsymbol{\Xi}/T^2) \\ & +\delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}'_{\nu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu}\boldsymbol{\Xi}/2T). \end{aligned} \tag{6.49}$$

(xxiv) The $((\nu\nu),\mu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{D}$ is

$$d_{(\nu\nu)\rho_\mu} \quad = \quad \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/2T^2). \tag{6.50}$$

Theorem 6. Given the asssumptions of model (5.1), for each asymptotically efficient estimator of ρ and $\boldsymbol{\varsigma}$, the parameters (5.48) are:

(i)

$$\lambda_0 = \lim_{T\to\infty} \mathrm{E}(\sigma_0{}^2) = 0. \tag{6.51}$$

(ii)

$$\boldsymbol{\lambda}_\rho = \lim_{T\to\infty} \mathrm{E}(\sigma_0\boldsymbol{d}_{1\rho}) = 0. \tag{6.52}$$

(iii)

$$\boldsymbol{\lambda}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\sigma_0\boldsymbol{d}_{1\varsigma}) = 0. \tag{6.53}$$

(iv)

$$\boldsymbol{\Lambda}_\varsigma = \boldsymbol{\Sigma}^{-2}. \tag{6.54}$$

(v)

$$\kappa_0 = \mathrm{tr}[\boldsymbol{\Sigma}^{-1}(\Delta_{GL} - \Delta_I)]/M + n/M \text{ (I=UL, RL, GL, IG, ML)}. \tag{6.55}$$

(vi)

$$\boldsymbol{\kappa}_\varsigma = [((M+K+1)\sigma^{ii} - \sigma^{ii}\mathrm{d}_{ii}{}^I\sigma^{ii})_{i=1,\ldots,M}]. \tag{6.56}$$

(vii)

$$\kappa_{\rho_\mu} = -[\rho_\mu(3+n) + (2n - c_1)/2\rho_\mu]. \tag{6.57}$$

(viii)

$$\kappa_{\rho_\mu}{}^{GL} = \kappa_{\rho_\mu}{}^{LS} - (1-\rho_\mu{}^2)c_2/2\rho_\mu + [c_1 - (1-\rho_\mu{}^2)n]/2\rho_\mu. \tag{6.58}$$

(ix)

$$\kappa_{\rho_\mu}{}^{DW} = \kappa_{\rho_\mu}{}^{LS} + 1. \tag{6.59}$$

(x)

$$\boldsymbol{\Lambda}_{\varsigma\rho} = \boldsymbol{\Lambda}_{\rho\varsigma}' = 0. \tag{6.60}$$

Useful Results

Lemmas

Lemma UR.1. I. Let $\boldsymbol{X}_\tau$, $\boldsymbol{Y}_\tau$, $(\tau \in J)$ be two conformable collections of square random matrices. If $\boldsymbol{X}_\tau^{-1}, \boldsymbol{Y}_\tau$ are of order $\omega(q)$ for some positive integer q, then outside a set of probability $o(p)$,

$$(\boldsymbol{X}_\tau + \tau \boldsymbol{Y}_\tau)^{-1} = \sum_{i=0}^{p} (-\tau)^i \boldsymbol{D}^i \boldsymbol{X}_\tau^{-1} + \tau^{p+1}\omega(p), \tag{UR.1}$$

When the quantity of interest is a more complicated function of the data, stochastic expansions can be based on the Taylor expansion of the function. Let Γ, Λ be subsets of some finite-dimensional vector spaces and consider the collection of random elements

$$z_\tau = \gamma + \tau^p \omega(q) \in \Gamma \; (p, q > 0) \tag{UR.2}$$

and the collection of nonrandom elements

$$\lambda_\tau = \lambda + o(1) \in \Lambda \tag{UR.3}$$

Given any function $f : \Gamma \times \Lambda \to S$, we write $f_\kappa(x - \gamma, \lambda_\tau)$ for the κ-order term of the Taylor expansion of the function $f(x, \lambda_\tau)$ around the point (γ, λ_τ).

II. Consider a measurable function

$$f : \Gamma \times \Lambda \to S \tag{UR.4}$$

and assume that, for some integer $s \leq 2$, all the partial derivatives (with respect to Γ) of orders s and less exist and are continuous in a neighborhood of $(\gamma, \lambda) \in \Gamma \times \Lambda$. Then, given the collections (UR.2) and (UR.3) we have

$$f(z_\tau, \lambda_\tau) = \sum_{\kappa=0}^{m-1} f_\kappa(z_\tau - \gamma, \lambda_\tau) + \tau^{pm}\omega(q) \tag{UR.5}$$

for all $m \geq s - 1$, (see Magdalinos, 1992, Corollary 1, Corollary 2).

Lemma UR.2. If $\boldsymbol{x}$ is a $\mathcal{N}(0, \boldsymbol{\Sigma})$ vector, and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are symmetric constant matrices, then

$$\begin{aligned} \mathrm{E}(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) &= \operatorname{tr} \boldsymbol{A\Sigma}, \\ \mathrm{E}(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x}) &= \operatorname{tr} \boldsymbol{A\Sigma} \operatorname{tr} \boldsymbol{B\Sigma} + 2(\operatorname{tr} \boldsymbol{A\Sigma B\Sigma}), \end{aligned} \tag{UR.6}$$

$$\begin{aligned} \mathrm{E}(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x}\boldsymbol{x}'\boldsymbol{C}\boldsymbol{x}) &= \operatorname{tr} \boldsymbol{A\Sigma} \operatorname{tr} \boldsymbol{B\Sigma} \operatorname{tr} \boldsymbol{C\Sigma} + 2 \operatorname{tr} \boldsymbol{A\Sigma}(\operatorname{tr} \boldsymbol{B\Sigma C\Sigma}) \\ &+ 2 \operatorname{tr} \boldsymbol{B\Sigma}(\operatorname{tr} \boldsymbol{A\Sigma C\Sigma}) + 2 \operatorname{tr} \boldsymbol{C\Sigma}(\operatorname{tr} \boldsymbol{A\Sigma B\Sigma}) + 8(\operatorname{tr} \boldsymbol{A\Sigma B\Sigma C\Sigma}), \end{aligned} \tag{UR.7}$$

(see Magnus and Neudecker, 1979).

Lemma UR.3. If $\boldsymbol{V}$ is a $T \times 1$ matrix, the rows of which are independent $\mathcal{N}(0, \boldsymbol{C})$ vectors and $\boldsymbol{A}$ is a confomable matrix, then we have:

$$\mathrm{E}(\boldsymbol{V}'\boldsymbol{A}\boldsymbol{V}) = (\mathrm{tr}\,\boldsymbol{A})\boldsymbol{C},\ \mathrm{E}(\boldsymbol{V}\boldsymbol{A}\boldsymbol{V}') = (\mathrm{tr}\,\boldsymbol{C}\boldsymbol{A})\boldsymbol{I}_T,$$

$$\mathrm{E}(\boldsymbol{V}\boldsymbol{A}\boldsymbol{V}) = \boldsymbol{A}'\boldsymbol{C},\ \mathrm{E}(\boldsymbol{V}'\boldsymbol{A}\boldsymbol{V}') = \boldsymbol{C}\boldsymbol{A}', \tag{UR.8}$$

$$\mathrm{E}(\boldsymbol{V}'\boldsymbol{V}\boldsymbol{A}\boldsymbol{V}'\boldsymbol{V}) = T(\mathrm{tr}\,\boldsymbol{C}\boldsymbol{A})\boldsymbol{C} + T(T+1)\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}, \tag{UR.9}$$

(Magdalinos, 1983, page 263 Lemma E.1).

We define the GLS estimator of β when the matrix $\boldsymbol{\Omega}$ is known.

$$\bar{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{y} \tag{UR.10}$$

By using the Theorem of Basu (Rothenberg, 1984a, Rothenberg, 1984b) and the definitions (1.26) and (UR.10), we can show that $\hat{\gamma}$ and $\hat{\beta} - \bar{\beta}$ the distribute independently from the $\bar{\beta}$

Lemma UR.4. Applies that

$$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + \tau\sigma(\boldsymbol{b} + \tau\boldsymbol{b}_*) \tag{UR.11}$$

where

$$\boldsymbol{b} = \sqrt{T}(\bar{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sigma,\ \boldsymbol{b}_* = T(\hat{\boldsymbol{\beta}} - \bar{\boldsymbol{\beta}})/\sigma. \tag{UR.12}$$

In addition, the following apply:

$$\begin{aligned} \boldsymbol{b} &= \boldsymbol{G}\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{u}/\sqrt{T}, \boldsymbol{b} \sim \mathcal{N}(0, \boldsymbol{G}),\ \text{where } \boldsymbol{G} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1} \\ &\text{and} \\ \boldsymbol{b}_* &= \hat{\boldsymbol{G}}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{M}\boldsymbol{u},\ \text{where } \boldsymbol{M} = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Omega}. \end{aligned} \tag{UR.13}$$

Proof of Lemma UR.4. substituting equations (1.25) and (1.26) in the equation (UR.10) we find

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}(\boldsymbol{X}\boldsymbol{\beta} + \sigma\boldsymbol{u}) = \boldsymbol{\beta} + (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\sigma\boldsymbol{u} \tag{UR.14}$$

and

$$\bar{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Omega}(\boldsymbol{X}\boldsymbol{\beta} + \sigma\boldsymbol{u}) = \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{X}'\boldsymbol{\Omega}\sigma\boldsymbol{u}. \tag{UR.15}$$

Using (1.29) and substituting (UR.14) and (UR.15) in definitions (UR.13), we find:

i.

$$\begin{aligned} \boldsymbol{b} &= \sqrt{T}(\bar{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sigma = \sqrt{T}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{X}'\boldsymbol{\Omega}\sigma\boldsymbol{u}/\sigma = \sqrt{T}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{u}/T \\ &= (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{u}/\sqrt{T} = \boldsymbol{G}\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{u}/\sqrt{T}, \end{aligned} \tag{UR.16}$$

where

$$G = (X'\Omega X/T)^{-1} = A^{-1}. \tag{UR.17}$$

From the assumptions of model (1.1) we have that

$$u \sim \mathcal{N}(0, \Omega^{-1}). \tag{UR.18}$$

From equation (UR.16), and since b and b^3 are odd functions of u we have

$$\mathrm{E}(b) = GX'\Omega\,\mathrm{E}(u)/\sqrt{T} = 0 \text{ hence } \mathrm{E}(b^3) = 0. \tag{UR.19}$$

Furthermore, by using equations (UR.16), (UR.17) and (UR.18) we find

$$\begin{aligned} Cov(b) &= \mathrm{E}(bb') = \mathrm{E}(GX'\Omega uu'\Omega XG/T) = GX'\Omega\,\mathrm{E}(uu')\Omega XG/T = GX'\Omega\Omega^{-1}\Omega XG/T \\ &= GX'\Omega XG/T = G(X'\Omega X/T)G = GG^{-1}G = G = A^{-1}. \end{aligned} \tag{UR.20}$$

It follows that

$$b \sim \mathcal{N}(0, G), \text{ where } G = A^{-1} = (X'\Omega X/T)^{-1} \tag{UR.21}$$

ii.

$$\begin{aligned} b_* &= T(\hat{\beta} - \bar{\beta})/\sigma = T[\beta + (X'\hat{\Omega}X)^{-1}X'\hat{\Omega}\sigma u - \beta - (X'\Omega X)^{-1}X'\Omega\sigma u]/\sigma \\ &= T[(X'\hat{\Omega}X)^{-1}X'\hat{\Omega}u - (X'\hat{\Omega}X)^{-1}(X'\hat{\Omega}X)(X'\Omega X)^{-1}X'\Omega u] \\ &= T[(X'\hat{\Omega}X)^{-1}X'\hat{\Omega}[I - X(X'\Omega X)^{-1}X'\Omega]u] = T(X'\hat{\Omega}X)^{-1}X'\hat{\Omega}Mu \\ &= (X'\hat{\Omega}X/T)^{-1}X'\hat{\Omega}Mu = \hat{G}X'\hat{\Omega}Mu, \end{aligned} \tag{UR.22}$$

where

$$\hat{G} = (X'\hat{\Omega}X/T)^{-1} = \hat{A}^{-1},\ \hat{A} = X'\hat{\Omega}X/T \text{ and } M = I - X(X'\Omega X)^{-1}X'\Omega. \tag{UR.23}$$

From the definitions of b and b_* and the definition of τ we find that $\tau\sigma b = \bar{\beta} - \beta$ and $\tau^2\sigma b_* = \hat{\beta} - \bar{\beta}$. Therefore we have

$$\tau\sigma b + \tau^2\sigma b_* = \tau\sigma(b + \tau b_*) = \bar{\beta} - \beta + \hat{\beta} - \bar{\beta} = \hat{\beta} - \beta \implies \hat{\beta} = \beta + \tau\sigma(b + \tau b_*), \tag{UR.24}$$

(see Symeonides, 1991, lemma B.1).

□

The following matrices are defined:

$$
\begin{aligned}
r &= \rho^2 \text{ where } |r| < 1, \\
\operatorname{tr} \boldsymbol{R} &= T \implies \operatorname{tr} \boldsymbol{R}/T = 1, \\
\operatorname{tr} \boldsymbol{R}^2/T &= \frac{1+\rho^2}{1-\rho^2} + o(T^{-1}), \\
\operatorname{tr} \boldsymbol{R}^3/T &= \frac{1+\rho^4}{(1-\rho^2)^2} + o(T^{-1}), \qquad \text{(UR.25)}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{E} &= \boldsymbol{\Delta R} \implies \operatorname{tr} \boldsymbol{E} = 2, \\
\boldsymbol{Z} &= \boldsymbol{R\Delta R} \implies \operatorname{tr} \boldsymbol{Z} = 2\frac{1-\rho^{2T}}{1-\rho^2}, \\
\boldsymbol{X} &= (\boldsymbol{\Delta R})^2 = \boldsymbol{E}^2 \implies \operatorname{tr} \boldsymbol{X} = 2(1+\rho^{2(T-1)}), \\
\operatorname{tr} \boldsymbol{\Theta} &= 2\Big[T\rho^{2(T-1)} + \frac{1-\rho^{2T}}{1-\rho^2}\Big], \\
\boldsymbol{\Phi} &= (\boldsymbol{\Delta R})^3 = \boldsymbol{\Delta R}(\boldsymbol{\Delta R})^2 = \boldsymbol{EX} \implies \operatorname{tr} \boldsymbol{\Phi} = 2(1+3\rho^{2(T-1)}), \\
\boldsymbol{\Psi} &= \boldsymbol{\Delta R}^3 \implies \operatorname{tr} \boldsymbol{\Psi} = \frac{2}{1-\rho^2} + o(T^{-1}), \qquad \text{(UR.26)}
\end{aligned}
$$

(see Symeonides, 1991, lemmas Γ.1, Γ.2 and Γ.5).

Theorems

Theorem UR.1. Isserlis' Theorem or Wick's probability Theorem is a formula that allows one to compute higher-order moments of the multivariate normal distribution in terms of its covariance matrix.

$$
\mathrm{E}[\, X_1X_2X_3X_4 \,] = \mathrm{E}[X_1X_2]\ \mathrm{E}[X_3X_4] + \mathrm{E}[X_1X_3]\ \mathrm{E}[X_2X_4] + \mathrm{E}[X_1X_4]\ \mathrm{E}[X_2X_3]. \qquad \text{(UR.27)}
$$

Appendix A

Notational Conventions

The Model

Lemma A.1. Define the $T \times T$ matrices $\boldsymbol{R}_*$, $\boldsymbol{R}_{*\rho}$, $\boldsymbol{R}_{*\rho\rho}$ as follows:

$$\boldsymbol{R}_* = [\boldsymbol{R}/(1-\rho^2)]^{-1},\ \boldsymbol{R}_{*\rho} = \frac{\partial \boldsymbol{R}_*}{\partial \rho},\ \boldsymbol{R}_{*\rho\rho} = \frac{\partial^2 \boldsymbol{R}_*}{\partial \rho^2}. \tag{A.1}$$

Then,

$$\boldsymbol{R}_{*\rho} = 2\rho \boldsymbol{I}_T - \boldsymbol{D} - 2\rho \boldsymbol{\Delta}, \tag{A.2}$$

and

$$\boldsymbol{R}_{*\rho\rho} = 2(\boldsymbol{I}_T - \boldsymbol{\Delta}). \tag{A.3}$$

Proof of Lemma A.1. Equations (1.22) and (A.1) imply that

$$\boldsymbol{R}_* = (1+\rho^2)\boldsymbol{I}_T - \rho \boldsymbol{D} - \rho^2 \boldsymbol{\Delta}, \tag{A.4}$$

where $\boldsymbol{I}_T$ is the identity matrix, $\boldsymbol{D}$ is a matrix with elements 1 if $|i-j| = 1$ and 0 elsewhere, and $\boldsymbol{\Delta}$ is a matrix with elements 1 in (1,1)-st and (T,T)-th position and 0 elsewhere.

The following results hold:

i. Using equation (A.4), the first order derivative of $\boldsymbol{R}_*$ is

$$\begin{aligned} \boldsymbol{R}_{*\rho} &= \frac{\partial \boldsymbol{R}_*}{\partial \rho} = \frac{\partial}{\partial \rho}[(1+\rho^2)\boldsymbol{I}_T - \rho \boldsymbol{D} - \rho^2 \boldsymbol{\Delta}] \\ &= 2\rho \boldsymbol{I}_T - \boldsymbol{D} - 2\rho \boldsymbol{\Delta}. \end{aligned} \tag{A.5}$$

ii. Using equation (A.5), the second order derivative of $\boldsymbol{R}_*$ is

$$\begin{aligned} \boldsymbol{R}_{*\rho\rho} &= \frac{\partial^2 \boldsymbol{R}_*}{\partial \rho^2} = \frac{\partial}{\partial \rho}\left(\frac{\partial \boldsymbol{R}_*}{\partial \rho}\right) \\ &= \frac{\partial}{\partial \rho}(2\rho \boldsymbol{I}_T - \boldsymbol{D} - 2\rho \boldsymbol{\Delta}) \\ &= 2\boldsymbol{I}_T - 2\boldsymbol{\Delta} \\ &= 2(\boldsymbol{I}_T - \boldsymbol{\Delta}). \end{aligned} \tag{A.6}$$

□

Lemma A.2. Define the $T \times T$ matrices

$$\boldsymbol{R}_{*j} = \boldsymbol{R}_{*\rho} + j\rho\boldsymbol{\Delta},\ \boldsymbol{R}_{*jj} = \boldsymbol{R}_{*\rho\rho} + j\boldsymbol{\Delta}\ (j = 1, 2). \tag{A.7}$$

Then,

$$\boldsymbol{R}_{*\rho} = \boldsymbol{R}_{*1} - \rho\boldsymbol{\Delta} = \boldsymbol{R}_{*2} - 2\rho\boldsymbol{\Delta}, \tag{A.8}$$

and

$$\boldsymbol{R}_{*\rho\rho} = \boldsymbol{R}_{*11} - \boldsymbol{\Delta} = \boldsymbol{R}_{*22} - 2\boldsymbol{\Delta}. \tag{A.9}$$

Proof of Lemma A.2. Equation (A.7) implies that

$$\boldsymbol{R}_{*\rho} = \boldsymbol{R}_{*j} - j\rho\boldsymbol{\Delta} \tag{A.10}$$

and

$$\boldsymbol{R}_{*\rho\rho} = \boldsymbol{R}_{*jj} - j\boldsymbol{\Delta}. \tag{A.11}$$

For $j = 1$, the following results hold:

i.

$$\boldsymbol{R}_{*\rho} = \boldsymbol{R}_{*1} - \rho\boldsymbol{\Delta}, \tag{A.12}$$

ii.

$$\boldsymbol{R}_{*\rho\rho} = \boldsymbol{R}_{*11} - \boldsymbol{\Delta}. \tag{A.13}$$

For $j = 2$, the following results hold:

i.

$$\boldsymbol{R}_{*\rho} = \boldsymbol{R}_{*2} - 2\rho\boldsymbol{\Delta}, \tag{A.14}$$

ii.

$$\boldsymbol{R}_{*\rho\rho} = \boldsymbol{R}_{*22} - 2\boldsymbol{\Delta}. \tag{A.15}$$

□

Lemma A.3. The (t, t')-th element of the $T \times T$ matrix $\boldsymbol{R}_*$ is

$$r_{*tt'} = \delta_{tt'} + \rho^2\delta_{tt'}(1 - \delta_{1t} - \delta_{tT}) - \rho(\delta_{t(t'+1)} + \delta_{(t+1)t'}). \tag{A.16}$$

Proof of Lemma A.3. The following results hold:

i. The (t, t')-th element of matrix $\boldsymbol{I}_T$ is $\delta_{tt'}$, i.e., it is Kronecker's delta.

ii. The (t,t')-th element of the $T \times T$ band matrix $\boldsymbol{D}$ equals 1 if $|t-t'|=1$ and it equals zero otherwise. Therefore, the (t,t')-th element of matrix $\boldsymbol{D}$ is

$$\delta_{t(t'+1)} + \delta_{(t+1)t'}. \tag{A.17}$$

iii. The $T \times T$ matrix $\boldsymbol{\Delta}$ has 1 in the $(1,1)$-st and (T,T)-th position and zero's elsewhere. Therefore, the (t,t')-th element of matrix $\boldsymbol{\Delta}$ is

$$\delta_{1t}\delta_{tt'} + \delta_{Tt}\delta_{tt'}. \tag{A.18}$$

Equation (A.4) implies that

$$\boldsymbol{R}_* = (1+\rho^2)\boldsymbol{I}_T - \rho\boldsymbol{D} - \rho^2\boldsymbol{\Delta}. \tag{A.19}$$

By using the results (i.), (ii.) and (iii.), we can write the (t,t')-th element $r_{*tt'}$ of the $T \times T$ matrix $\boldsymbol{R}_*$ as follows:

$$\begin{aligned} r_{*tt'} &= (1+\rho^2)\delta_{tt'} - \rho(\delta_{t(t'+1)} + \delta_{(t+1)t'}) - \rho^2(\delta_{1t}\delta_{tt'} + \delta_{Tt}\delta_{tt'}) \\ &= \delta_{tt'} + \rho^2(\delta_{tt'} - \delta_{1t}\delta_{tt'} - \delta_{Tt}\delta_{tt'}) - \rho(\delta_{t(t'+1)} + \delta_{(t+1)t'}) \\ &= \delta_{tt'} + \rho^2\delta_{tt'}(1-\delta_{1t}-\delta_{Tt}) - \rho(\delta_{t(t'+1)} + \delta_{(t+1)t'}). \end{aligned} \tag{A.20}$$

□

Lemma A.4. Confirmation of equation (1.23)

Proof of Lemma A.4. Lemma A.3 implies the following results:

i. Elements on the principal diagonal:$t = t'$. equation (A.20) implies that

$$\begin{aligned} r_{*tt} &= \delta_{tt} + \rho^2\delta_{tt}(1-\delta_{1t}-\delta_{Tt}) - \rho(\delta_{t(t+1)} + \delta_{(t+1)t}) \\ &= 1 + \rho^2(1-\delta_{1t}-\delta_{Tt}) \end{aligned} \tag{A.21}$$

(1) For $t = 2,\ldots,T-1$, $\delta_{1t} = 0$ and $\delta_{Tt} = 0$, and equation (A.21) implies that

$$r_{*tt} = 1+\rho^2 \quad (t = 2,\ldots,T-1) \tag{A.22}$$

(2) For $t = 1$, $\delta_{1t} = \delta_{11} = 1$ and $\delta_{Tt} = \delta_{T1} = 0$, and equation (A.21) implies that

$$r_{*11} = 1+\rho^2(1-1-0) = 1 \tag{A.23}$$

(3) For $t = T$, $\delta_{1t} = \delta_{1T} = 0$ and $\delta_{Tt} = \delta_{TT} = 1$, and equation (A.21) implies that

$$r_{*TT} = 1+\rho^2(1-0-1) = 1 \tag{A.24}$$

ii. Elements on the lower secondary diagonal: $t = t' + 1$. Equation (A.16) implies that

$$\begin{aligned} r_{*tt'} &= r_{*(t'+1)t'} \\ &= \delta_{(t'+1)t'} + \rho^2\delta_{(t'+1)t'}(1 - \delta_{1(t'+1)} - \delta_{T(t'+1)}) - \rho(\delta_{(t'+1)(t'+1)} + \delta_{(t'+1+1)t'}) \\ &= -\rho. \end{aligned} \tag{A.25}$$

iii. Elements on the upper secondary diagonal: $t = t' - 1$. Equation (A.16) implies that

$$\begin{aligned} r_{*tt'} &= r_{*(t'-1)t'} \\ &= \delta_{(t'-1)t'} + \rho^2\delta_{(t'-1)t'}(1 - \delta_{1(t'-1)} - \delta_{T(t'-1)}) - \rho(\delta_{(t'-1)(t'+1)} + \delta_{(t'-1+1)t'}) \\ &= -\rho. \end{aligned} \tag{A.26}$$

iv. Lower off-diagonal elements: $t = t' + j$ $(j \geq 2)$. Equation (A.16) implies that, for $j \geq 2$,

$$\begin{aligned} r_{*tt'} &= r_{*(t'+j)t'} \\ &= \delta_{(t'+j)t'} + \rho^2\delta_{(t'+j)t'}(1 - \delta_{1(t'+j)} - \delta_{T(t'+j)}) - \rho(\delta_{(t'+j)(t'+1)} + \delta_{(t'+j+1)t'}) \\ &= 0. \end{aligned} \tag{A.27}$$

v. Upper off-diagonal elements: $t = t' - j$ $(j \geq 2)$. Equation (A.16) implies that, for $j \geq 2$,

$$\begin{aligned} r_{*tt'} &= r_{*(t'-j)t'} \\ &= \delta_{(t'-j)t'} + \rho^2\delta_{(t'-j)t'}(1 - \delta_{1(t'-j)} - \delta_{T(t'-j)}) - \rho(\delta_{(t'-j)(t'+1)} + \delta_{(t'-j+1)t'}) \\ &= 0. \end{aligned} \tag{A.28}$$

□

Lemma A.5. The (t, t')-th element of the $T \times T$ matrix Ω is

$$\omega_{tt'} = r_{*tt'}\sigma_t{}^{-1}\sigma_{t'}{}^{-1}, \tag{A.29}$$

where $r_{*tt'}$ is defined in equation (A.16).

Proof of Lemma A.5. Equations (1.22), (1.24) and (A.1)—or (1.24), (A.19)—imply that

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{R}_*\boldsymbol{\Sigma}^{-1/2}. \tag{A.30}$$

Let $\sigma_{tt'}$ be the (t, t')-th element of matrix $\boldsymbol{\Sigma}$. Further, let $\sigma_*{}^{tt'}$ be the (t, t')-th element of matrix $\boldsymbol{\Sigma}^{-1}$, and $\sigma_*{}^{tt'} = (\sigma^{tt'})^{1/2}$ be the (t, t')-th element of matrix $\boldsymbol{\Sigma}^{-1/2}$.

Equation (1.25) implies that the (t,t')-th element of the $T \times T$ diagonal matrix $\boldsymbol{\Sigma}^{-1/2}$ is

$$\sigma_*{}^{tt'} = \delta_{tt'}\frac{1}{\sigma_t} = \delta_{tt'}\sigma_t{}^{-1}, \tag{A.31}$$

which implies that

$$\boldsymbol{\Sigma}^{-1/2} = [(\sigma_*{}^{tt'})_{t,t'=1,\ldots,T}] \tag{A.32a}$$

$$= [(\delta_{tt'}\sigma_t{}^{-1})_{t,t'=1,\ldots,T}]. \tag{A.32b}$$

Let $\omega_{tt'}$ be the (t,t')-th element of the $T \times T$ matrix $\boldsymbol{\Omega}$, i.e.,

$$\boldsymbol{\Omega} = [(\omega_{tt'})_{t,t'=1,\ldots,T}]. \tag{A.33}$$

Since $r_{*tt'}$ is the (t,t')-th element of the $T \times T$ matrix $\boldsymbol{R}_*$, equations (A.30), (A.32a) and (A.33) imply that

$$\begin{aligned}
\boldsymbol{\Omega} &= [(\omega_{tt'})_{t,t'=1,\ldots,T}] \\
&= [(\delta_{tt_*}\sigma_t{}^{-1})_{t,t_*=1,\ldots,T}][(r_{*t_*t'_*})_{t_*,t'_*=1,\ldots,T}][(\delta_{t'_*t'}\sigma_{t'_*}{}^{-1})_{t'_*,t'=1,\ldots,T}] \\
&= \left[\left(\sum_{t_*=1}^{T}\sum_{t'_*=1}^{T} \underbrace{\delta_{tt_*}}_{t_*=t} \underbrace{\delta_{t'_*t'}}_{t'_*=t'} r_{*t_*t'_*}\sigma_t{}^{-1}\sigma_{t'}{}^{-1}\right)_{t,t'=1,\ldots,T}\right] \\
&= [(r_{*tt'}\sigma_t{}^{-1}\sigma_{t'}{}^{-1})_{t,t'=1,\ldots,T}],
\end{aligned} \tag{A.34}$$

which implies that

$$\omega_{tt'} = r_{*tt'}\sigma_t{}^{-1}\sigma_{t'}{}^{-1}. \tag{A.35}$$

□

Lemma A.6. The following results hold:

$$\begin{aligned}
\omega_{11} = \frac{1}{\sigma_1{}^2},\quad \omega_{TT} = \frac{1}{\sigma_T{}^2},\quad \omega_{tt} &= (1+\rho^2)\frac{1}{\sigma_t{}^2}\quad (t = 2,\ldots,T-1) \\
\omega_{tt'} &= -\rho\frac{1}{\sigma_t\sigma_{t'}} \text{ for } t = t'+1 \text{ and } t = t'-1, \\
\omega_{tt'} &= 0 \text{ for } \underbrace{t = t'+j \text{ and } t = t'-j \quad (j \geq 2)}_{(t'<t=3,\ldots,T \text{ or } t<t'=3,\ldots,T)}.
\end{aligned} \tag{A.36}$$

Proof of Lemma A.6. Lemmas A.4 and A.5 imply the following results:

i. (a) For $t = 2,\ldots,T-1$, equations (A.22) and (A.35) imply that

$$\omega_{tt} = (1+\rho^2)\frac{1}{\sigma_t{}^2}. \tag{A.37}$$

(b) For $t = 1$, equations (A.23) and (A.35) imply that

$$\omega_{11} = \frac{1}{{\sigma_1}^2}. \tag{A.38}$$

(c) For $t = T$, equations (A.24) and (A.35) imply that

$$\omega_{TT} = \frac{1}{{\sigma_T}^2}. \tag{A.39}$$

ii. For the lower secondary diagonal: $t = t' + 1$. Equations (A.25) and (A.35) imply that

$$\omega_{tt'} = -\rho\frac{1}{\sigma_t\sigma_{t'}}. \tag{A.40}$$

iii. For the upper secondary diagonal: $t = t' - 1$. Equations (A.26) and (A.35) imply that

$$\omega_{tt'} = -\rho\frac{1}{\sigma_t\sigma_{t'}}. \tag{A.41}$$

iv. Lower off-diagonal elements of $\boldsymbol{\Omega} : t = t' + j$ $(j \geq 2)$. Equations (A.27) and (A.35) imply that

$$\omega_{tt'} = 0 \quad (t' < t = 3, \ldots, T). \tag{A.42}$$

v. Upper off-diagonal elements of $\boldsymbol{\Omega} : t = t' - j$ $(j \geq 2)$. Equations (A.28) and (A.35) imply that

$$\omega_{tt'} = 0 \quad (t < t' = 3, \ldots, T). \tag{A.43}$$

□

Lemma A.7. The $T \times T$ matrix $\boldsymbol{\Omega}$ can be written as follows:

$$\boldsymbol{\Omega} = \begin{bmatrix} \frac{1}{{\sigma_1}^2} & -\rho\frac{1}{\sigma_1\sigma_2} & \cdots & 0 \\ -\rho\frac{1}{\sigma_1\sigma_2} & (1+\rho^2)\frac{1}{{\sigma_2}^2} & -\rho\frac{1}{\sigma_2\sigma_3} & \\ & \ddots & \ddots & -\rho\frac{1}{\sigma_{T-1}\sigma_T} \\ 0 & \cdots & -\rho\frac{1}{\sigma_{T-1}\sigma_T} & \frac{1}{{\sigma_T}^2} \end{bmatrix}. \tag{A.44}$$

Proof of Lemma A.7. The proof follows by using Lemma A.6. □

Lemma A.8. Let $r_{*tt'\rho}$ and $r_{*tt'\rho\rho}$ be the first- and second- order derivatives of $r_{*tt'}$ with respect to the parameter ρ, i.e.,

$$r_{*tt'\rho} = \frac{\partial r_{*tt'}}{\partial \rho}, \quad r_{*tt'\rho\rho} = \frac{\partial^2 r_{*tt'}}{\partial \rho^2}. \tag{A.45}$$

The following results hold:

$$r_{*tt'\rho} = 2\rho\delta_{tt'}(1 - \delta_{1t} - \delta_{Tt}) - (\delta_{t(t'+1)} + \delta_{(t+1)t'}),$$

$$r_{*tt'\rho\rho} = 2\delta_{tt'}(1 - \delta_{1t} - \delta_{Tt}). \tag{A.46}$$

Proof of Lemma A.8. Equation (A.16) implies that

$$\begin{aligned} r_{*tt'\rho} &= \frac{\partial r_{*tt'}}{\partial \rho} \\ &= 2\rho\delta_{tt'}(1 - \delta_{1t} - \delta_{Tt}) - (\delta_{t(t'+1)} + \delta_{(t+1)t'}). \end{aligned} \tag{A.47}$$

Equation (A.47) implies that

$$\begin{aligned} r_{*tt'\rho\rho} &= \frac{\partial^2 r_{*tt'}}{\partial \rho^2} = \frac{\partial}{\partial \rho}\left(\frac{\partial r_{*tt'}}{\partial \rho}\right) = \frac{\partial}{\partial \rho}(r_{*tt'\rho}) \\ &= 2\delta_{tt'}(1 - \delta_{1t} - \delta_{Tt}). \end{aligned} \tag{A.48}$$

□

Lemma A.9. Let $\boldsymbol{R}_{*\rho}$ be the first-order derivative of the $T \times T$ matrix $\boldsymbol{R}_*$ with respect to ρ. Then, $\boldsymbol{R}_{*\rho}$ can be analytically written as follows:

$$\boldsymbol{R}_{*\rho} = \begin{bmatrix} 0 & -1 & \dots & & 0 \\ -1 & 2\rho & -1 & & \\ & -1 & 2\rho & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & \dots & -1 & 0 \end{bmatrix}. \tag{A.49}$$

Proof of Lemma A.9. Lemma A.8 implies the following results:

i. Elements on the principal diagonal $t = t'$. Equation (A.47) implies that

$$\begin{aligned} r_{*tt\rho} &= 2\rho\delta_{tt}(1 - \delta_{1t} - \delta_{Tt}) - (\delta_{t(t+1)} + \delta_{(t+1)t}) \\ &= 2\rho(1 - \delta_{1t} - \delta_{Tt}). \end{aligned} \tag{A.50}$$

(a) For $t = 2, \dots, T-1$, $\delta_{1t} = 0$ and $\delta_{Tt} = 0$ and equation (A.50) implies that

$$r_{*tt\rho} = 2\rho, \ (t = 2, \dots, T-1). \tag{A.51}$$

(b) For $t = 1$, $\delta_{1t} = \delta_{11} = 1$ and $\delta_{Tt} = \delta_{T1} = 0$, and equation (A.50) implies that

$$r_{*11\rho} = 2\rho(1 - 1 - 0) = 0. \tag{A.52}$$

(c) For $t = T$, $\delta_{1t} = \delta_{1T} = 0$ and $\delta_{Tt} = \delta_{TT} = 1$, and equation (A.50) implies that

$$r_{*TT\rho} = 2\rho(1 - 0 - 1) = 0. \tag{A.53}$$

ii. Elements on the lower secondary diagonal $t = t' + 1$. Equation (A.47) implies that

$$\begin{aligned} r_{*tt'\rho} &= r_{*(t'+1)t'\rho} \\ &= 2\rho\delta_{(t'+1)t'}(1 - \delta_{1(t'+1)} - \delta_{T(t'+1)}) - (\delta_{(t'+1)(t'+1)} + \delta_{(t'+1+1)t'}) \\ &= -1. \end{aligned} \tag{A.54}$$

iii. Elements on the upper secondary diagonal $t = t' - 1$. Equation (A.47) implies that

$$\begin{aligned} r_{*tt'\rho} &= r_{*(t'-1)t'\rho} \\ &= 2\rho\delta_{(t'-1)t'}(1 - \delta_{1(t'-1)} - \delta_{T(t'-1)}) - (\delta_{(t'-1)(t'+1)} + \delta_{(t'-1+1)t'}) \\ &= -1. \end{aligned} \tag{A.55}$$

iv. Lower off-diagonal elements: $t = t' + j$ $(j \geq 2)$. Equation (A.47) implies that, for $(j \geq 2)$,

$$\begin{aligned} r_{*tt'\rho} &= r_{*(t'+j)t'\rho} \\ &= 2\rho\delta_{(t'+j)t'}(1 - \delta_{1(t'+j)} - \delta_{T(t'+j)}) - (\delta_{(t'+j)(t'+1)} + \delta_{(t'+j+1)t'}) \\ &= 0. \end{aligned} \tag{A.56}$$

v. Upper off-diagonal elements: $t = t' - j$ $(j \geq 2)$. Equation (A.47) implies that, for $(j \geq 2)$,

$$\begin{aligned} r_{*tt'\rho} &= r_{*(t'-j)t'\rho} \\ &= 2\rho\delta_{(t'-j)t'}(1 - \delta_{1(t'-j)} - \delta_{T(t'-j)}) - (\delta_{(t'-j)(t'+1)} + \delta_{(t'-j+1)t'}) \\ &= 0. \end{aligned} \tag{A.57}$$

Equation (A.41) follows immediately from the results (i.) through (v.). □

Lemma A.10. Let $\boldsymbol{R}_{*\rho\rho}$ be the second-order derivative of the $T \times T$ matrix $\boldsymbol{R}_*$ with respect to ρ. Then, $\boldsymbol{R}_{*\rho\rho}$ can be analytically written as follows:

$$\boldsymbol{R}_{*\rho\rho} = \begin{bmatrix} 0 & 0 & \dots & & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 2 & \ddots & \\ & & \ddots & \ddots & \\ 0 & & \dots & 0 & 0 \end{bmatrix}. \tag{A.58}$$

Proof of Lemma A.10. Lemma A.8 implies the following results:

i. Elements on the principal diagonal $t = t'$. Equation (A.48) implies that

$$\begin{aligned} r_{*tt\rho\rho} &= 2\delta_{tt}(1 - \delta_{1t} - \delta_{Tt}) \\ &= 2(1 - \delta_{1t} - \delta_{Tt}). \end{aligned} \tag{A.59}$$

(a) For $t = 2, \ldots, T-1$, $\delta_{1t} = 0$ and $\delta_{Tt} = 0$ and equation (A.59) implies that

$$r_{*tt\rho\rho} = 2, \ (t = 2, \ldots, T-1). \tag{A.60}$$

(b) For $t = 1$, $\delta_{1t} = \delta_{11} = 1$ and $\delta_{Tt} = \delta_{T1} = 0$, and equation (A.59) implies that

$$r_{*11\rho\rho} = 2(1 - 1 - 0) = 0. \tag{A.61}$$

(c) For $t = T$, $\delta_{1t} = \delta_{1T} = 0$ and $\delta_{Tt} = \delta_{TT} = 1$, and equation (A.59) implies that

$$r_{*TT\rho\rho} = 2(1 - 0 - 1) = 0. \tag{A.62}$$

ii. Elements on the lower secondary diagonal $t = t' + 1$. Equation (A.48) implies that

$$\begin{aligned} r_{*tt'\rho\rho} &= r_{*(t'+1)t'\rho\rho} \\ &= 2\delta_{(t'+1)t'}(1 - \delta_{1(t'+1)} - \delta_{T(t'+1)}) \\ &= 0. \end{aligned} \tag{A.63}$$

iii. Elements on the upper secondary diagonal $t = t' - 1$. Equation (A.48) implies that

$$\begin{aligned} r_{*tt'\rho\rho} &= r_{*(t'-1)t'\rho\rho} \\ &= 2\delta_{(t'-1)t'}(1 - \delta_{1(t'-1)} - \delta_{T(t'-1)}) \\ &= 0. \end{aligned} \tag{A.64}$$

iv. Lower off-diagonal elements: $t = t' + j$ $(j \geq 2)$. Equation (A.48) implies that, for $(j \geq 2)$,

$$\begin{aligned} r_{*tt'\rho\rho} &= r_{*(t'+j)t'\rho\rho} \\ &= 2\delta_{(t'+j)t'}(1 - \delta_{1(t'+j)} - \delta_{T(t'+j)}) \\ &= 0. \end{aligned} \tag{A.65}$$

v. Upper off-diagonal elements: $t = t' - j$ $(j \geq 2)$. Equation (A.48) implies that, for $(j \geq 2)$,

$$\begin{aligned} r_{*tt'\rho\rho} &= r_{*(t'-j)t'\rho\rho} \\ &= 2\delta_{(t'-j)t'}(1 - \delta_{1(t'-j)} - \delta_{T(t'-j)}) = 0. \end{aligned} \tag{A.66}$$

Equation (A.58) follows immediately from the results (i.) through (v.). □

Lemma A.11. Let $\omega_{tt'\rho}$, $\omega_{tt'\rho\rho}$ be the first- and second-order derivatives of $\omega_{tt'}$ with respect to the parameter ρ, i.e.,

$$\omega_{tt'\rho} = \frac{\partial \omega_{tt'}}{\partial \rho}, \quad \omega_{tt'\rho\rho} = \frac{\partial^2 \omega_{tt'}}{\partial \rho^2}. \tag{A.67}$$

The following results hold:

$$\omega_{tt'\rho} = r_{*tt'\rho}\sigma_t^{-1}\sigma_{t'}^{-1}, \tag{A.68a}$$

$$\omega_{tt'\rho\rho} = r_{*tt'\rho\rho}\sigma_t^{-1}\sigma_{t'}^{-1}, \tag{A.68b}$$

where $r_{*tt'\rho}$ and $r_{*tt'\rho\rho}$ are defined in equation (A.46).

Proof of Lemma A.11. Equation (1.13) implies that σ_t is functionally unrelated to the parameter ρ. Therefore, Lemma A.5 and equation (A.45) imply the following results:

$$\begin{aligned} \omega_{tt'\rho} &= \frac{\partial \omega_{tt'}}{\partial \rho} = \frac{\partial}{\partial \rho}(r_{*tt'}\sigma_t^{-1}\sigma_{t'}^{-1}) = \left(\frac{\partial r_{*tt'}}{\partial \rho}\right)\sigma_t^{-1}\sigma_{t'}^{-1} \\ &= r_{*tt'\rho}\sigma_t^{-1}\sigma_{t'}^{-1}. \end{aligned} \tag{A.69}$$

$$\begin{aligned} \omega_{tt'\rho\rho} &= \frac{\partial^2 \omega_{tt'}}{\partial \rho^2} = \frac{\partial^2}{\partial \rho^2}(r_{*tt'}\sigma_t^{-1}\sigma_{t'}^{-1}) = \left(\frac{\partial^2 r_{*tt'}}{\partial \rho^2}\right)\sigma_t^{-1}\sigma_{t'}^{-1} \\ &= r_{*tt'\rho\rho}\sigma_t^{-1}\sigma_{t'}^{-1}. \end{aligned} \tag{A.70}$$

□

Lemma A.12. Let $\boldsymbol{\Omega}_\rho$, $\boldsymbol{\Omega}_{\rho\rho}$ be the first-and second order derivatives of the $T \times T$ matrix $\boldsymbol{\Omega}$ with respect to ρ, i.e.,

$$\boldsymbol{\Omega}_\rho = \frac{\partial \boldsymbol{\Omega}}{\partial \rho}, \ \boldsymbol{\Omega}_{\rho\rho} = \frac{\partial^2 \boldsymbol{\Omega}}{\partial \rho^2}. \tag{A.71}$$

The following results hold:

$$\begin{aligned} \boldsymbol{\Omega}_\rho &= \boldsymbol{\Sigma}^{-1/2}\boldsymbol{R}_{*\rho}\boldsymbol{\Sigma}^{-1/2}, \\ \boldsymbol{\Omega}_{\rho\rho} &= \boldsymbol{\Sigma}^{-1/2}\boldsymbol{R}_{*\rho\rho}\boldsymbol{\Sigma}^{-1/2}, \\ \text{where } \boldsymbol{R}_{*\rho} &= \frac{\partial \boldsymbol{R}_*}{\partial \rho}, \ \text{ and } \boldsymbol{R}_{*\rho\rho} = \frac{\partial^2 \boldsymbol{R}_*}{\partial \rho^2}. \end{aligned} \tag{A.72}$$

Proof of Lemma A.12. Equation (1.13) implies that the elements $\sigma_*^{tt'} = \delta_{tt'}\sigma_t^{-1}$ of the $T \times T$ matrix $\boldsymbol{\Sigma}^{-1/2}$ are functionally unrelated to the parameter ρ. Therefore, equation (A.30) implies the following results:

i. The first order derivative of the $T \times T$ matrix $\boldsymbol{\Omega}$ with respect to ρ is

$$\boldsymbol{\Omega}_\rho = \frac{\partial \boldsymbol{\Omega}}{\partial \rho} = \frac{\partial}{\partial \rho}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{R}_*\boldsymbol{\Sigma}^{-1/2}) = \boldsymbol{\Sigma}^{-1/2}\left(\frac{\partial \boldsymbol{R}_*}{\partial \rho}\right)\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{R}_{*\rho}\boldsymbol{\Sigma}^{-1/2}. \tag{A.73}$$

ii. The second order derivative of the $T \times T$ matrix $\boldsymbol{\Omega}$ with respect to ρ is

$$\boldsymbol{\Omega}_{\rho\rho} = \frac{\partial^2 \boldsymbol{\Omega}}{\partial \rho^2} = \frac{\partial^2}{\partial \rho^2}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{R}_*\boldsymbol{\Sigma}^{-1/2}) = \boldsymbol{\Sigma}^{-1/2}\left(\frac{\partial^2 \boldsymbol{R}_*}{\partial \rho^2}\right)\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{R}_{*\rho\rho}\boldsymbol{\Sigma}^{-1/2}. \tag{A.74}$$

□

Lemma A.13. The $T \times T$ matrices $\boldsymbol{\Omega}_\rho$ and $\boldsymbol{\Omega}_{\rho\rho}$ can be analytically written as follows:

$$\boldsymbol{\Omega}_\rho = \begin{bmatrix} 0 & -\frac{1}{\sigma_1\sigma_2} & & & 0 \\ -\frac{1}{\sigma_1\sigma_2} & 2\rho\frac{1}{\sigma_2{}^2} & -\frac{1}{\sigma_2\sigma_3} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{\sigma_{T-2}\sigma_{T-1}} & 2\rho\frac{1}{\sigma_{T-1}{}^2} & -\frac{1}{\sigma_{T-1}\sigma_T} \\ 0 & & & -\frac{1}{\sigma_{T-1}\sigma_T} & 0 \end{bmatrix}. \tag{A.75}$$

$$\boldsymbol{\Omega}_{\rho\rho} = \begin{bmatrix} 0 & 0 & & \dots & 0 \\ 0 & 2\frac{1}{\sigma_2{}^2} & 0 & & \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 2\frac{1}{\sigma_{T-1}{}^2} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{A.76}$$

Proof of Lemma A.13. Since,

$$\boldsymbol{\Omega}_\rho = [(\omega_{tt'\rho})_{t,t'=1,\dots,T}], \;\; \boldsymbol{\Omega}_{\rho\rho} = [(\omega_{tt'\rho\rho})_{t,t'=1,\dots,T}]. \tag{A.77}$$

the proof of equations (A.75) and (A.76) follows by combining Lemma A.11 with Lemmas A.9 and A.10, respectively. □

Lemma A.14. Let x_t' and z_t' be the t-th rows of the $T \times n$ matrix $\boldsymbol{X}$ and the $T \times m$ matrix $\boldsymbol{Z}$, respectively. The following results hold:

$$\boldsymbol{X}'\boldsymbol{X} = \sum_{t=1}^{T} x_t x_t', \;\; \boldsymbol{Z}'\boldsymbol{Z} = \sum_{t=1}^{T} z_t z_t'. \tag{A.78}$$

Proof of Lemma A.14. Since x_t' and z_t' be the t-th rows of the $T \times n$ matrix $\boldsymbol{X}$ and the $T \times m$ matrix $\boldsymbol{Z}$, respectively, x_t and z_t are the t-th columns of the matrices $\boldsymbol{X}'$ and $\boldsymbol{Z}'$, respectively, i.e., we can write that

$$\boldsymbol{X} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_T' \end{bmatrix}, \;\; \boldsymbol{Z} = \begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_T' \end{bmatrix}, \;\; \boldsymbol{X}' = \begin{bmatrix} x_1, & x_2, & \cdots, & x_T \end{bmatrix}, \;\; \boldsymbol{Z} = \begin{bmatrix} z_1, & z_2, & \cdots, & z_T \end{bmatrix}. \tag{A.79}$$

Therefore,

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} x_1, & x_2, & \cdots, & x_T \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_T' \end{bmatrix} = \sum_{t=1}^{T} x_t x_t'. \tag{A.80}$$

Similarly,

$$\boldsymbol{Z}'\boldsymbol{Z} = \begin{bmatrix} z_1, & z_2, & \cdots & , z_T \end{bmatrix} \begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_T' \end{bmatrix} = \sum_{t=1}^{T} z_t z_t'. \tag{A.81}$$

□

Lemma A.15. Let σ^{tt} be the (t,t)-th diagonal element of the $T \times T$ diagonal matrix $\boldsymbol{\Sigma}^{-1}$, i.e., by using equation (1.18) we write that

$$\sigma^{tt} = {\sigma_t}^{-2} = \frac{1}{{\sigma_t}^2}. \tag{A.82}$$

Moreover, let ${\sigma_i}^{tt}$, ${\sigma_{ij}}^{tt}$ be the first-and second-order derivatives of σ^{tt} with respect to the element of the $m \times 1$ non-zero vector $\boldsymbol{\varsigma} = (\varsigma_1, \ldots, \varsigma_m)'$, i.e.,

$${\sigma_i}^{tt} = \frac{\partial \sigma^{tt}}{\partial \varsigma_i}, \; {\sigma_{ij}}^{tt} = \frac{\partial \sigma^{tt}}{\partial \varsigma_i \varsigma_j}. \tag{A.83}$$

The following results hold:

$$\begin{aligned} {\sigma_i}^{tt} &= -\frac{z_{ti}}{{\sigma_t}^4}, \\ {\sigma_{ij}}^{tt} &= \frac{2 z_{ti} z_{tj}}{{\sigma_t}^6}. \end{aligned} \tag{A.84}$$

Proof of Lemma A.15. Equation (1.13) and (A.82) imply that

$$\sigma^{tt} = (z_t' \boldsymbol{\varsigma})^{-1}. \tag{A.85}$$

Further, since $\boldsymbol{\Sigma}^{-1} = \mathsf{diag}(\sigma^{tt})$, equation (A.85) implies that

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} (z_1'\boldsymbol{\varsigma})^{-1} & \ldots & & 0 \\ 0 & (z_2'\boldsymbol{\varsigma})^{-1} & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & (z_T'\boldsymbol{\varsigma})^{-1} \end{bmatrix}. \tag{A.86}$$

Let z_{ti} be the i-th element of the $1 \times m$ row vector z_t' i.e., z_{ti} is the (t,i)-th element of the $T \times m$ matrix $\boldsymbol{Z}$.

Equations (A.83) and (A.85) imply the following results:

i. The first order derivative of σ^{tt} with respect to the element of the $m \times 1$ non-zero vector $\boldsymbol{\varsigma}$ is

$$\begin{aligned} \sigma_i{}^{tt} &= \frac{\partial (z_t'\boldsymbol{\varsigma})^{-1}}{\partial \varsigma_i} = \frac{\partial}{\partial \varsigma_i}\Big(\frac{1}{(z_t'\boldsymbol{\varsigma})}\Big) = -\frac{z_{ti}}{(z_t'\boldsymbol{\varsigma})^2} = [\text{see (1.13)}] \\ &= -\frac{z_{ti}}{\sigma_t{}^4}. \end{aligned} \tag{A.87}$$

ii. The second order derivative of σ^{tt} with respect to the element of the $m \times 1$ non-zero vector $\boldsymbol{\varsigma}$ is

$$\begin{aligned} \sigma_{ij}{}^{tt} &= \frac{\partial^2 (z_t'\boldsymbol{\varsigma})^{-1}}{\partial \varsigma_i \partial \varsigma_j} = \frac{\partial}{\partial \varsigma_j}\Big(\frac{\partial (z_t'\boldsymbol{\varsigma})^{-1}}{\partial \varsigma_i}\Big) = [\text{see (A.87)}] \\ &= \frac{\partial}{\partial \varsigma_j}\Big(-\frac{z_{ti}}{(z_t'\boldsymbol{\varsigma})^2}\Big) = -z_{ti}\frac{\partial}{\partial \varsigma_j}\Big(\frac{1}{(z_t'\boldsymbol{\varsigma})^2}\Big) \\ &= -z_{ti}\Big[-\frac{2z_{tj}(z_t'\boldsymbol{\varsigma})}{(z_t'\boldsymbol{\varsigma})^4}\Big] = \frac{2z_{ti}z_{tj}}{(z_t'\boldsymbol{\varsigma})^3} \\ &= \frac{2z_{ti}z_{tj}}{\sigma_t{}^6}. \end{aligned} \tag{A.88}$$

□

Lemma A.16. Define the scalars

$$\sigma_{ti} = \frac{\partial \sigma_t}{\partial \varsigma_i}, \quad \sigma_{tij} = \frac{\partial^2 \sigma_t}{\partial \varsigma_i \partial \varsigma_j}. \tag{A.89}$$

The following results hold:

$$\sigma_{ti} = \frac{z_{ti}}{2\sigma_t}, \quad \sigma_{tij} = -\frac{z_{ti}z_{tj}}{4\sigma_t^3}. \tag{A.90}$$

Proof of Lemma A.16. Equation (1.13) implies that

$$\sigma_t = (z_t'\boldsymbol{\varsigma})^{1/2}. \tag{A.91}$$

By combining equations (A.89) and (A.91), we find the following results:

$$\begin{aligned} \sigma_{ti} &= \frac{\partial}{\partial \varsigma_i}[(z_t'\boldsymbol{\varsigma})^{1/2}] = \frac{1}{2}(z_t'\boldsymbol{\varsigma})^{-1/2}\frac{\partial}{\partial \varsigma_i}(z_t'\boldsymbol{\varsigma}) = \frac{1}{2}\frac{1}{(z_t'\boldsymbol{\varsigma})^{1/2}}z_{ti} \\ &= \frac{z_{ti}}{2\sigma_t}. \end{aligned} \tag{A.92}$$

$$\begin{aligned}
\sigma_{tij} &= \frac{\partial^2}{\partial\varsigma_i\partial\varsigma_j}[(z_t'\varsigma)^{1/2}] = \frac{\partial}{\partial\varsigma_j}\left[\frac{\partial}{\partial\varsigma_i}(z_t'\varsigma)^{1/2}\right] = [\mathit{see}\ (\text{A.92})] \\
&= \frac{\partial}{\partial\varsigma_j}\left(\frac{z_{ti}}{2\sigma_t}\right) = \frac{z_{ti}}{2}\frac{\partial}{\partial\varsigma_j}\left(\frac{1}{\sigma_t}\right) = [\mathit{see}\ (\text{A.91})] \\
&= \frac{z_{ti}}{2}\frac{\partial}{\partial\varsigma_j}[(z_t'\varsigma)^{-1/2}] = \frac{z_{ti}}{2}\left(-\frac{1}{2}\right)(z_t'\varsigma)^{-3/2}\frac{\partial}{\partial\varsigma_j}(z_t'\varsigma) \\
&= -\frac{z_{ti}}{4}\frac{1}{(z_t'\varsigma)^{3/2}}z_{tj} \\
&= -\frac{z_{ti}z_{tj}}{4\sigma_t^3}.
\end{aligned} \tag{A.93}$$

□

Lemma A.17. Define the scalars

$$(\sigma_t{}^3)_i = \frac{\partial\sigma_t^3}{\partial\varsigma_i}, \quad (\sigma_t{}^3)_{ij} = \frac{\partial^2\sigma_t^3}{\partial\varsigma_i\partial\varsigma_j}. \tag{A.94}$$

The following results hold:

$$(\sigma_t{}^3)_i = \frac{3}{2}\sigma_t z_{ti}, \quad (\sigma_t{}^3)_{ij} = \frac{3z_{ti}z_{tj}}{4\sigma_t}. \tag{A.95}$$

Proof of Lemma A.17. Equation (A.91) implies that

$$\sigma_t^3 = (z_t'\varsigma)^{3/2}. \tag{A.96}$$

By combining equations (A.94) and (A.96) we find the following results:

$$\begin{aligned}
(\sigma_t{}^3)_i &= \frac{\partial}{\partial\varsigma_i}[(z_t'\varsigma)^{3/2}] = \frac{3}{2}(z_t'\varsigma)^{1/2}\frac{\partial}{\partial\varsigma_i}(z_t'\varsigma) = [\text{see (A.91)}] \\
&= \frac{3}{2}\sigma_t z_{ti}.
\end{aligned} \tag{A.97}$$

$$\begin{aligned}
(\sigma_t{}^3)_{ij} &= \frac{\partial^2}{\partial\varsigma_i\partial\varsigma_j}[(z_t'\varsigma)^{3/2}] = \frac{\partial}{\partial\varsigma_j}\left[\frac{\partial}{\partial\varsigma_i}[(z_t'\varsigma)^{3/2}]\right] = [\text{see (A.97)}] \\
&= \frac{\partial}{\partial\varsigma_j}(\frac{3}{2}\sigma_t z_{ti}) = \frac{3z_{ti}}{2}\frac{\partial}{\partial\varsigma_j}(\sigma_t) = [\text{see (A.89)}] \\
&= \frac{3z_{ti}}{2}\sigma_{tj} = [\text{see (A.92)}] \\
&= \frac{3z_{ti}}{2}\frac{z_{tj}}{2\sigma_t} \\
&= \frac{3z_{ti}z_{tj}}{4\sigma_t}.
\end{aligned} \tag{A.98}$$

□

Lemma A.18. Define the scalar

$$\sigma^{tt'} = \frac{1}{\sigma_t \sigma_{t'}}, \tag{A.99}$$

and let ${\sigma_i}^{tt'}$, ${\sigma_{ij}}^{tt'}$ be the first- and second-order derivatives of $\sigma^{tt'}$ with respect to the elements of the $m \times 1$ non-zero vector $\boldsymbol{\varsigma} = (\varsigma_1, \ldots, \varsigma_m)'$, i.e.,

$${\sigma_i}^{tt'} = \frac{\partial \sigma^{tt'}}{\partial \varsigma_i}, \quad {\sigma_{ij}}^{tt'} = \frac{\partial^2 \sigma^{tt'}}{\partial \varsigma_i \partial \varsigma_j}. \tag{A.100}$$

The following results hold:

$${\sigma_i}^{tt'} = -\frac{1}{2\sigma_t \sigma_{t'}} \left[\frac{z_{ti}}{{\sigma_t}^2} + \frac{z_{t'i}}{{\sigma_{t'}}^2} \right], \quad {\sigma_{ij}}^{tt'} = \frac{1}{4\sigma_t \sigma_{t'}} \left[\frac{3z_{ti} z_{tj}}{{\sigma_t}^4} + \frac{3z_{t'i} z_{t'j}}{{\sigma_{t'}}^4} + \frac{z_{ti} z_{t'j} + z_{t'i} z_{tj}}{{\sigma_t}^2 {\sigma_{t'}}^2} \right]. \tag{A.101}$$

Proof of Lemma A.18. Equations (A.99) and (A.100) imply the following results:

i. First order derivative of $\sigma^{tt'}$ with respect to the elements of the $m \times 1$ non-zero vector $\boldsymbol{\varsigma}$ is

$$\begin{aligned}
{\sigma_i}^{tt'} &= \frac{\partial}{\partial \varsigma_i} \left(\frac{1}{\sigma_t \sigma_{t'}} \right) = -\frac{1}{(\sigma_t \sigma_{t'})^2} \frac{\partial}{\partial \varsigma_i} (\sigma_t \sigma_{t'}) \\
&= -\frac{1}{{\sigma_t}^2 {\sigma_{t'}}^2} \left[\sigma_t \frac{\partial}{\partial \varsigma_i} (\sigma_{t'}) + \sigma_{t'} \frac{\partial}{\partial \varsigma_i} (\sigma_t) \right] = [\text{see (A.89)}] \\
&= -\frac{1}{{\sigma_t}^2 {\sigma_{t'}}^2} \left[\sigma_t \sigma_{t'i} + \sigma_{t'} \sigma_{ti} \right] [\text{see Lemma (A.16), and equation (A.89)}] \\
&= -\frac{1}{{\sigma_t}^2 {\sigma_{t'}}^2} \left[\sigma_t \frac{z_{t'i}}{2\sigma_{t'}} + \sigma_{t'} \frac{z_{ti}}{2\sigma_t} \right] \\
&= -\frac{z_{t'i}}{2{\sigma_{t'}}^3 \sigma_t} - \frac{z_{ti}}{2{\sigma_t}^3 \sigma_{t'}} \\
&= -\frac{1}{2\sigma_t \sigma_{t'}} \left[\frac{z_{ti}}{{\sigma_t}^2} + \frac{z_{t'i}}{{\sigma_{t'}}^2} \right].
\end{aligned} \tag{A.102}$$

ii. Second order derivative of $\sigma^{tt'}$ with respect to the elements of the $m \times 1$ non-zero vector $\boldsymbol{\varsigma}$ is

$$\begin{aligned}
{\sigma_{ij}}^{tt'} &= \frac{\partial^2}{\partial \varsigma_i \partial \varsigma_j} \left(\frac{1}{\sigma_t \sigma_{t'}} \right) = \frac{\partial}{\partial \varsigma_j} \left[\frac{\partial}{\partial \varsigma_i} \left(\frac{1}{\sigma_t \sigma_{t'}} \right) \right] \\
&= \frac{\partial}{\partial \varsigma_j} \left[-\frac{z_{t'i}}{2{\sigma_{t'}}^3 \sigma_t} - \frac{z_{ti}}{2{\sigma_t}^3 \sigma_{t'}} \right] \\
&= -\frac{z_{ti}}{2} \frac{\partial}{\partial \varsigma_j} \left(\frac{1}{\sigma_t^3 \sigma_{t'}} \right) - \frac{z_{t'i}}{2} \frac{\partial}{\partial \varsigma_j} \left(\frac{1}{\sigma_{t'}^3 \sigma_t} \right).
\end{aligned} \tag{A.103}$$

To calculate (A.103), we must compute the following intermediate results:

(a) First we calculate the following quantity:

$$\begin{aligned}
\frac{\partial}{\partial\varsigma_j}\left(\frac{1}{\sigma_t^3\sigma_{t'}}\right) &= -\frac{1}{(\sigma_t{}^3\sigma_{t'})^2}\frac{\partial}{\partial\varsigma_j}(\sigma_t{}^3\sigma_{t'}) \\
&= -\frac{1}{\sigma_t{}^6\sigma_{t'}^2}\left[\sigma_t{}^3\frac{\partial}{\partial\varsigma_j}(\sigma_{t'}) + \sigma_{t'}\frac{\partial}{\partial\varsigma_j}(\sigma_t{}^3)\right] \\
&= [\text{see (A.89) and (A.94)}] \\
&= -\frac{1}{\sigma_t{}^6\sigma_{t'}^2}[\sigma_t{}^3\sigma_{t'j} + \sigma_{t'}(\sigma_t{}^3)_j] \\
&= [\text{see Lemmas (A.16) and (A.17), Equations (A.92) and (A.97), respectively.}] \\
&= -\frac{1}{\sigma_t{}^6\sigma_{t'}^2}\left[\sigma_t{}^3\frac{z_{t'j}}{2\sigma_{t'}} + \sigma_{t'}\frac{3}{2}\sigma_t z_{tj}\right] \\
&= -\frac{3z_{tj}}{2\sigma_t{}^5\sigma_{t'}} - \frac{z_{t'j}}{2\sigma_t{}^3\sigma_{t'}{}^3}. \qquad \text{(A.104)}
\end{aligned}$$

(b) Similarly, we find that [by interchanging indices t, t'].

$$\frac{\partial}{\partial\varsigma_j}\left(\frac{1}{\sigma_t\sigma_{t'}{}^3}\right) = -\frac{3z_{t'j}}{2\sigma_{t'}{}^5\sigma_t} - \frac{z_{tj}}{2\sigma_t{}^3\sigma_{t'}{}^3}. \qquad \text{(A.105)}$$

Equations (A.103) (A.104) and (A.105) imply that

$$\begin{aligned}
\sigma_{ij}{}^{tt'} &= -\frac{z_{ti}}{2}\left[-\frac{3z_{tj}}{2\sigma_t{}^5\sigma_{t'}} - \frac{z_{t'j}}{2\sigma_t{}^3\sigma_{t'}{}^3}\right] \\
&\quad -\frac{z_{t'i}}{2}\left[-\frac{3z_{t'j}}{2\sigma_{t'}{}^5\sigma_t} - \frac{z_{tj}}{2\sigma_t{}^3\sigma_{t'}{}^3}\right] \\
&= \frac{3z_{ti}z_{tj}}{4\sigma_t{}^5\sigma_{t'}} + \frac{z_{ti}z_{t'j}}{4\sigma_{t'}{}^3\sigma_t{}^3} + \frac{3z_{t'i}z_{t'j}}{4\sigma_t\sigma_{t'}{}^5} + \frac{z_{t'i}z_{tj}}{4\sigma_t{}^3\sigma_{t'}{}^3} \\
&= \frac{1}{4\sigma_t\sigma_{t'}}\left[\frac{3z_{ti}z_{tj}}{\sigma_t{}^4} + \frac{3z_{t'i}z_{t'j}}{\sigma_{t'}{}^4} + \frac{z_{ti}z_{t'j} + z_{t'i}z_{tj}}{\sigma_t{}^2\sigma_{t'}{}^2}\right]. \qquad \text{(A.106)}
\end{aligned}$$

□

Lemma A.19. Confirmation of the results in Lemma A.18.

Proof of Lemma A.19. For $t = t'$, Lemma A.18 implies the following results:

$$\begin{aligned}
\sigma_i{}^{tt'} &= -\frac{1}{2\sigma_t{}^2}\left[\frac{z_{ti}}{\sigma_t{}^2} + \frac{z_{ti}}{\sigma_t{}^2}\right] = -\frac{1}{2\sigma_t{}^2}\left[\frac{2z_{ti}}{\sigma_t{}^2}\right] \\
&= -\frac{z_{ti}}{\sigma_t{}^4} = [\text{see Lemma (A.15)}] \\
&= \sigma_i{}^{tt}. \qquad \text{(A.107)}
\end{aligned}$$

$$\begin{aligned}
{\sigma_{ij}}^{tt'} &= \frac{1}{4{\sigma_t}^2}\Big[\frac{3z_{ti}z_{tj}}{{\sigma_t}^4} + \frac{3z_{ti}z_{tj}}{{\sigma_t}^4} + \frac{z_{ti}z_{tj} + z_{ti}z_{tj}}{{\sigma_t}^4}\Big] \\
&= \frac{1}{4{\sigma_t}^2}\Big[\frac{8z_{ti}z_{tj}}{{\sigma_t}^4}\Big] \\
&= \frac{2z_{ti}z_{tj}}{{\sigma_t}^6} = \text{[see Lemma (A.15)]} \\
&= {\sigma_{ij}}^{tt}. \qquad \text{(A.108)}
\end{aligned}$$

□

Lemma A.20. The (t, t')-th element of the $T \times T$ matrix $\boldsymbol{\Omega}$ can be written as

$$\omega_{tt'} = r_{*tt'}\sigma^{tt'}, \qquad \text{(A.109)}$$

where $r_{*tt'}$ and $\sigma^{tt'}$ are defined in equations (A.16) and (A.99), respectively.

Proof of Lemma A.20. The proof follows by combining Lemma A.5 and (A.99). □

Lemma A.21. Let $\omega_{tt'i}$, $\omega_{tt'ij}$ be the first- and second-order derivatives of $\omega_{tt'}$ with respect to the elements of the $m \times 1$ non-zero vector $\boldsymbol{\varsigma} = (\varsigma_1, \ldots, \varsigma_m)'$, i.e.,

$$\omega_{tt'i} = \frac{\partial \omega_{tt'}}{\partial \varsigma_i}, \; \omega_{tt'ij} = \frac{\partial^2 \omega_{tt'}}{\partial \varsigma_i \partial \varsigma_j}. \qquad \text{(A.110)}$$

The following results hold:

$$\omega_{tt'i} = r_{*tt'}{\sigma_i}^{tt'}, \quad \omega_{tt'ij} = r_{*tt'}{\sigma_{ij}}^{tt'}. \qquad \text{(A.111)}$$

where ${\sigma_i}^{tt'}$ and ${\sigma_{ij}}^{tt'}$ are defined in equation (A.101).

Proof of Lemma A.21. Equation (1.13) implies that σ_t is functionally unrelated to the parameter ρ. Therefore, Lemma A.5 and equations (A.99) and (A.100) imply the following results:

$$\begin{aligned}
\omega_{tt'i} &= \frac{\partial \omega_{tt'}}{\partial \varsigma_i} = \frac{\partial}{\partial \varsigma_i}(r_{*tt'}{\sigma_t}^{-1}{\sigma_{t'}}^{-1}) = r_{*tt'}\frac{\partial}{\partial \varsigma_i}\Big(\frac{1}{\sigma_t \sigma_{t'}}\Big) \\
&= r_{*tt'}\frac{\partial \sigma^{tt'}}{\partial \varsigma_i} \\
&= r_{*tt'}{\sigma_i}^{tt'}. \qquad \text{(A.112)}
\end{aligned}$$

$$\begin{aligned}
\omega_{tt'ij} &= \frac{\partial^2 \omega_{tt'}}{\partial \varsigma_i \partial \varsigma_j} = \frac{\partial^2}{\partial \varsigma_i \partial \varsigma_j}(r_{*tt'}{\sigma_t}^{-1}{\sigma_{t'}}^{-1}) = r_{*tt'}\frac{\partial^2}{\partial \varsigma_i \partial \varsigma_j}\Big(\frac{1}{\sigma_t \sigma_{t'}}\Big) \\
&= r_{*tt'}\frac{\partial^2 \sigma^{tt'}}{\partial \varsigma_i \partial \varsigma_j} = r_{*tt'}{\sigma_{ij}}^{tt'}. \qquad \text{(A.113)}
\end{aligned}$$

□

Lemma A.22. Let $\boldsymbol{\Omega}_{\varsigma_i}$, $\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}$ be the first- and second-order derivatives of the $T \times T$ matrix $\boldsymbol{\Omega}$ with respect to the elements of the $m \times 1$ non-zero vector $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_m)'$, i.e.,

$$\boldsymbol{\Omega}_{\varsigma_i} = \frac{\partial \boldsymbol{\Omega}}{\partial \varsigma_i}, \ \boldsymbol{\Omega}_{\varsigma_i\varsigma_j} = \frac{\partial^2 \boldsymbol{\Omega}}{\partial \varsigma_i \partial \varsigma_j}. \tag{A.114a}$$

The matrices $\boldsymbol{\Omega}_{\varsigma_i}$ and $\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}$ can be analytically written as follows:

$$\boldsymbol{\Omega}_{\varsigma_i} = \begin{bmatrix} -\frac{z_{1i}}{\sigma_1{}^4} & -\rho\sigma_i{}^{12} & & & 0 \\ -\rho\sigma_i{}^{21} & -(1+\rho^2)\frac{z_{2i}}{\sigma_2{}^4} & -\rho\sigma_i{}^{23} & & \\ & \ddots & \ddots & \ddots & \\ & & -\rho\sigma_i^{(T-1)(T-2)} & -(1+\rho^2)\frac{z_{(T-1)i}}{\sigma^4_{(T-1)}} & -\rho\sigma_i{}^{(T)(T-1)} \\ 0 & & & -\rho\sigma_i^{(T)(T-1)} & -\frac{z_{Ti}}{\sigma_T^4} \end{bmatrix}, \tag{A.114b}$$

where

$$\sigma_i^{t(t\pm1)} = -\frac{1}{2\sigma_t\sigma_{t\pm1}}\left(\frac{z_{ti}}{\sigma_t^2} + \frac{z_{(t\pm1)i}}{\sigma_{t\pm1}^2}\right) \tag{A.114c}$$

and

$$\boldsymbol{\Omega}_{\varsigma_i\varsigma_j} = \begin{bmatrix} \frac{2z_{1i}z_{1j}}{\sigma_1^6} & -\rho\sigma_{ij}^{12} & & & 0 \\ -\rho\sigma_{ij}^{21} & (1+\rho^2)\frac{2z_{2i}z_{2j}}{\sigma_2^6} & -\rho\sigma_{ij}^{23} & & \\ & \ddots & \ddots & \ddots & \\ & & -\rho\sigma_{ij}^{(T-1)(T-2)} & (1+\rho^2)\frac{2z_{(T-1)i}z_{(T-1)j}}{\sigma^6_{(T-1)}} & -\rho\sigma_{ij}^{(T)(T-1)} \\ 0 & & & -\rho\sigma_{ij}^{(T)(T-1)} & \frac{2z_{Ti}z_{Tj}}{\sigma_T^6} \end{bmatrix}, \tag{A.114d}$$

where

$$\sigma_{ij}^{t(t\pm1)} = -\frac{1}{4\sigma_t\sigma_{(t\pm1)}}\left[\frac{3z_{ti}z_{tj}}{\sigma_t^4} + \frac{3z_{(t\pm1)i}z_{(t\pm1)j}}{\sigma_{t\pm1}^4} + \frac{z_{ti}z_{(t\pm1)j} + z_{tj}z_{(t\pm1)i}}{\sigma_t^2\sigma_{t\pm1}^2}\right]. \tag{A.114e}$$

Proof of Lemma A.22. Lemmas A.4, A.6, A.15, A.18, A.20, and A.21 imply the following results:

I. First-order derivatives

i. 1. For $t = 2, \dots, T-1$, equations (A.22), (A.37), (A.84), (A.109), (A.111) imply that

$$\omega_{tti} = (1+\rho^2)\sigma_i^{tt} \tag{A.115a}$$

$$= -(1+\rho^2)\frac{z_{ti}}{\sigma_t^4}. \tag{A.115b}$$

2. For $t = 1$, equations (A.23), (A.38), (A.84), (A.109), (A.111) imply that

$$\omega_{11i} = 1 \cdot \sigma_i^{11} \tag{A.116a}$$

$$= -\frac{z_{1i}}{\sigma_1^4}. \quad (A.116b)$$

3. For $t = T$, equations (A.24), (A.39), (A.84), (A.109), (A.111) imply that

$$\omega_{TTi} = 1 \cdot \sigma_i^{TT} \quad (A.117a)$$

$$= -\frac{z_{Ti}}{\sigma_T^4}. \quad (A.117b)$$

ii. For the lower secondary diagonal $t = t' + 1$. Equations (A.25), (A.40), (A.101), (A.109), and (A.111) imply that

$$\omega_{tt'i} = -\rho\sigma_i^{tt'} = -\rho\sigma_i^{t(t-1)} = -\rho\Big[-\frac{1}{2\sigma_t\sigma_{t'}}\Big(\frac{z_{ti}}{\sigma_t^2} + \frac{z_{t'i}}{\sigma_{t'}^2}\Big)\Big] \quad (A.118a)$$

$$= [\text{and since } t' = t-1] = \frac{\rho}{2\sigma_t\sigma_{t-1}}\Big(\frac{z_{ti}}{\sigma_t^2} + \frac{z_{t-1i}}{\sigma_{t-1}^2}\Big). \quad (A.118b)$$

iii. For the upper secondary diagonal $t = t' - 1$. Equations (A.26), (A.41), (A.101), (A.109), and (A.111) imply that

$$\omega_{tt'i} = -\rho\sigma_i^{tt'} = -\rho\sigma_i^{t(t+1)} = -\rho\Big[-\frac{1}{2\sigma_t\sigma_{t'}}\Big(\frac{z_{ti}}{\sigma_t^2} + \frac{z_{t'i}}{\sigma_{t'}^2}\Big)\Big] \quad (A.119a)$$

$$= [\text{and since } t' = t+1] = \frac{\rho}{2\sigma_t\sigma_{t+1}}\Big(\frac{z_{ti}}{\sigma_t^2} + \frac{z_{t+1i}}{\sigma_{t+1}^2}\Big). \quad (A.119b)$$

iv. Lower off-diagonal elements: $t = t' + j \ \ (j \geq 2)$. Equations (A.27) and (A.111) imply that

$$\omega_{tt'i} = 0 \ \ (t' < t = 3, \dots, T). \quad (A.120)$$

v. Upper off-diagonal elements: $t = t' - j \ \ (j \geq 2)$. Equations (A.28) and (A.111) imply that

$$\omega_{tt'i} = 0 \ \ (t < t' = 3, \dots, T). \quad (A.121)$$

II. Second-order derivatives

i. 1. For $t = 2, \dots, T-1$, equations (A.22), (A.37), (A.84), (A.109), (A.111) imply that

$$\omega_{ttij} = (1+\rho^2)\sigma_{ij}^{tt} \quad (A.122a)$$

$$= (1+\rho^2)\frac{2z_{ti}z_{tj}}{\sigma_t^6}. \quad (A.122b)$$

2. For $t = 1$, equations (A.23), (A.38), (A.84), (A.109), (A.111) imply that

$$\omega_{11ij} = 1 \cdot \sigma_{ij}^{11} \quad (A.123a)$$

$$= \frac{2z_{1i}z_{1j}}{\sigma_1^6}. \tag{A.123b}$$

3. For $t = T$, equations (A.24), (A.39), (A.84), (A.109), (A.111) imply that

$$\omega_{TTij} = 1 \cdot \sigma_{ij}^{TT} \tag{A.124a}$$

$$= \frac{2z_{Ti}z_{Tj}}{\sigma_T^6}. \tag{A.124b}$$

ii. For the lower secondary diagonal $t = t' + 1$ (and since $t' = t - 1$). Equations (A.25), (A.40), (A.101), (A.109), and (A.111) imply that

$$\omega_{tt'ij} = -\rho\sigma_{ij}^{tt'} = -\rho\sigma_{ij}^{t(t-1)} = -\rho\left[\frac{1}{4\sigma_t\sigma_{t'}}\left(\frac{3z_{ti}z_{tj}}{\sigma_t^4} + \frac{3z_{t'i}z_{t'j}}{\sigma_{t'}^4} + \frac{z_{ti}z_{t'j} + z_{tj}z_{t'i}}{\sigma_t^2\sigma_{t'}^2}\right)\right] \tag{A.125a}$$

$$= -\frac{\rho}{4\sigma_t\sigma_{t-1}}\left[\frac{3z_{ti}z_{tj}}{\sigma_t^4} + \frac{3z_{(t-1)i}z_{(t-1)j}}{\sigma_{t-1}^4} + \frac{z_{ti}z_{(t-1)j} + z_{tj}z_{(t-1)i}}{\sigma_t^2\sigma_{t-1}^2}\right]. \tag{A.125b}$$

iii. For the upper secondary diagonal $t = t' - 1$ (and since $t' = t + 1$). Equations (A.26), (A.41), (A.101), (A.109), and (A.111) imply that

$$\omega_{tt'ij} = -\rho\sigma_{ij}{}^{tt'} = -\rho\sigma_{ij}{}^{t(t+1)} = -\rho\left[\frac{1}{4\sigma_t\sigma_{t'}}\left(\frac{3z_{ti}z_{tj}}{\sigma_t{}^4} + \frac{3z_{t'i}z_{t'j}}{\sigma_{t'}{}^4} + \frac{z_{ti}z_{t'j} + z_{tj}z_{t'i}}{\sigma_t{}^2\sigma_{t'}{}^2}\right)\right] \tag{A.126a}$$

$$= -\frac{\rho}{4\sigma_t\sigma_{t+1}}\left[\frac{3z_{ti}z_{tj}}{\sigma_t{}^4} + \frac{3z_{(t+1)i}z_{(t+1)j}}{\sigma_{t+1}{}^4} + \frac{z_{ti}z_{(t+1)j} + z_{tj}z_{(t+1)i}}{\sigma_t{}^2\sigma_{t+1}{}^2}\right]. \tag{A.126b}$$

iv. Lower off-diagonal elements:$t = t' + j \;\; (j \geq 2)$. Equations (A.27) and (A.111) imply that

$$\omega_{tt'ij} = 0 \;\; (t' < t = 3, \ldots, T). \tag{A.127}$$

v. Upper off-diagonal elements:$t = t' - j \;\; (j \geq 2)$. Equations (A.28) and (A.111) imply that

$$\omega_{tt'ij} = 0 \;\; (t < t' = 3, \ldots, T). \tag{A.128}$$

□

Lemma A.23. Let $\boldsymbol{\Omega}_{\rho\varsigma_i}$ be the second-order derivatives of the $T \times T$ matrix $\boldsymbol{\Omega}$ with respect to ρ and the elements of $m \times 1$ non-zero vector $\boldsymbol{\varsigma} = (\varsigma_1, \ldots, \varsigma_m)'$, i.e.,

$$\boldsymbol{\Omega}_{\rho\varsigma_i} = \frac{\partial^2}{\partial\rho\partial\varsigma_i}. \tag{A.129}$$

The $T \times T$ matrix $\boldsymbol{\Omega}_{\rho\varsigma_i}$ can be analytically written as

$$\boldsymbol{\Omega}_{\rho\varsigma_i} = \begin{bmatrix} 0 & -\sigma_i{}^{12} & & & & 0 \\ -\sigma_i{}^{21} & -2\rho\frac{z_{2i}}{\sigma_2{}^4} & -\sigma_i{}^{23} & & & \\ & -\sigma_i{}^{32} & \ddots & & & \\ & \ddots & & \ddots & & \\ & & -\sigma_i{}^{(T-1)(T-2)} & -2\rho\frac{z_{(T-1)i}}{\sigma_{(T-1)}{}^4} & -\sigma_i{}^{(T)(T-1)} \\ 0 & & & -\sigma_i{}^{(T)(T-1)} & 0 \end{bmatrix}, \tag{A.130a}$$

where

$$\sigma_i{}^{t(t\pm1)} = -\frac{1}{2\sigma_t\sigma_{t\pm1}}\left(\frac{z_{ti}}{\sigma_t^2} + \frac{z_{(t\pm1)i}}{\sigma_{t\pm1}^2}\right). \tag{A.130b}$$

Proof of Lemma A.23. Equations (A.75) and (A.99)

i. 1. For $t = 2, \ldots, T-1$, the t-th diagonal element of the $T \times T$ matrix $\boldsymbol{\Omega}_\rho$ is

$$\begin{aligned} \omega_{tt\rho} &= 2\rho\frac{1}{\sigma_t{}^2} \\ &= 2\rho\sigma^{tt}. \end{aligned} \tag{A.131}$$

2. For $t = 1$, the diagonal element of matrix $\boldsymbol{\Omega}_\rho$ is

$$\omega_{11\rho} = 0. \tag{A.132}$$

3. For $t = T$, the diagonal element of matrix $\boldsymbol{\Omega}_\rho$ is

$$\omega_{TT\rho} = 0. \tag{A.133}$$

ii. For the lower secondary diagonal $t = t' + 1$ (and since $t' = t - 1$). The (t,t')-th element of $\boldsymbol{\Omega}_\rho$ is

$$\omega_{tt'\rho} = -\frac{1}{\sigma_t\sigma_{t'}} \tag{A.134a}$$

$$= -\sigma^{tt'} \tag{A.134b}$$

$$-\sigma^{t(t-1)} \tag{A.134c}$$

$$= \omega_{t(t-1)\rho}. \tag{A.134d}$$

iii. For the upper secondary diagonal $t = t' - 1$ (and since $t' = t + 1$). The (t,t')-th element of $\boldsymbol{\Omega}_\rho$ is

$$\omega_{tt'\rho} = -\frac{1}{\sigma_t \sigma_{t'}} \tag{A.135a}$$

$$= -\sigma^{tt'} \tag{A.135b}$$

$$-\sigma^{t(t+1)} \tag{A.135c}$$

$$= \omega_{t(t+1)\rho}. \tag{A.135d}$$

iv. Lower off-diagonal elements:$t = t' + j \;\; (j \geq 2)$. The (t,t')-th element of $\boldsymbol{\Omega}_\rho$ is

$$\omega_{tt'\rho} = 0 \;\; (t' < t = 3, \ldots, T). \tag{A.136}$$

v. Upper off-diagonal elements:$t = t' - j \;\; (j \geq 2)$. The (t,t')-th element of $\boldsymbol{\Omega}_\rho$ is

$$\omega_{tt'\rho} = 0 \;\; (t < t' = 3, \ldots, T). \tag{A.137}$$

I. Derivatives of the diagonal elements of $\boldsymbol{\Omega}_\rho$ with respect to the elements of the $m \times 1$ non-zero vector $\varsigma = (\varsigma_1, \ldots, \varsigma_m)'$. Equation (1.13) implies that σ_t is functionally unrelated to the parameter ρ. Therefore, Lemma A.15 implies the following results:

i. 1. For $t = 2, \ldots, T - 1$, equations (A.83), (A.84) and (A.131) imply that

$$\omega_{tt\rho_i} = \frac{\partial^2 \omega_{tt}}{\partial \rho \partial \varsigma_i} = \frac{\partial}{\partial \varsigma_i}\left(\frac{\partial \omega_{tt}}{\partial \rho}\right) = \frac{\partial}{\partial \varsigma_i}\omega_{tt\rho} = \frac{\partial}{\partial \varsigma_i}(2\rho\sigma^{tt}) = 2\rho\frac{\partial \sigma^{tt}}{\partial \varsigma_i} \tag{A.138a}$$

$$= 2\rho{\sigma_i}^{tt} \tag{A.138b}$$

$$= -2\rho\frac{z_{ti}}{{\sigma_t}^4}. \tag{A.138c}$$

2. For t=1, equation (A.132) implies that

$$\omega_{11\rho_i} = \frac{\partial^2 \omega_{11}}{\partial \rho \partial \varsigma_i} = 0. \tag{A.139}$$

3. For t=T, equation (A.133) implies that

$$\omega_{TT\rho_i} = \frac{\partial^2 \omega_{TT}}{\partial \rho \partial \varsigma_i} = 0. \tag{A.140}$$

Moreover, Lemma A.18 implies the following results:

ii. For the lower secondary diagonal $t = t' + 1$. Equations (A.100), (A.101), (A.134a) and (A.134b) imply that, since $t' = t - 1$,

$$\omega_{tt'\rho i} = \frac{\partial^2 \omega_{tt'}}{\partial \rho \partial \varsigma_i} = \frac{\partial}{\partial \varsigma_i}\left(\frac{\partial \omega_{tt'}}{\partial \rho}\right) = \frac{\partial}{\partial \varsigma_i}\omega_{tt'\rho} = \frac{\partial}{\partial \varsigma_i}(-\sigma^{tt'}) = \frac{\partial}{\partial \varsigma_i}(-\sigma^{t(t-1)}) \tag{A.141a}$$

$$= -\sigma_i^{t(t-1)} \quad (A.141b)$$

$$= \frac{1}{2\sigma_t\sigma_{(t-1)}}\Big[\frac{z_{ti}}{\sigma_t{}^2} + \frac{z_{(t-1)i}}{\sigma_{t-1}^2}\Big]. \quad (A.141c)$$

iii. For the upper secondary diagonal $t = t' - 1$. Equations (A.100), (A.101), (A.134a), and (A.134b), imply that, since $t' = t + 1$,

$$\omega_{tt'\rho i} = \frac{\partial^2\omega_{tt'}}{\partial\rho\partial\varsigma_i} = \frac{\partial}{\partial\varsigma_i}\Big(\frac{\partial\omega_{tt'}}{\partial\rho}\Big) = \frac{\partial}{\partial\varsigma_i}\omega_{tt'\rho} = \frac{\partial}{\partial\varsigma_i}(-\sigma^{tt'}) = \frac{\partial}{\partial\varsigma_i}(-\sigma^{t(t+1)}) \quad (A.142a)$$

$$= -\sigma_i^{t(t+1)} \quad (A.142b)$$

$$= \frac{1}{2\sigma_t\sigma_{(t+1)}}\Big[\frac{z_{ti}}{\sigma_t{}^2} + \frac{z_{(t+1)i}}{\sigma_{t+1}{}^2}\Big]. \quad (A.142c)$$

iv. Lower off-diagonal elements:$t = t' + j \ \ (j \geq 2)$. Equation (A.136) implies that

$$\omega_{tt'\rho i} = 0. \quad (A.143)$$

v. Upper off-diagonal elements:$t = t' - j \ \ (j \geq 2)$. Equation (A.137) implies that

$$\omega_{tt'\rho i} = 0. \quad (A.144)$$

□

Lemma A.24. Let $\boldsymbol{A}_i$, $\boldsymbol{A}_{ij}$ be the first and second-order derivatives of the $T \times T$ matrix $\boldsymbol{A}$ with respect to ρ and the elements of $m \times 1$ non-zero vector $\boldsymbol{\varsigma} = (\varsigma_1, \ldots, \varsigma_m)$' ,i.e.,

$$\boldsymbol{A}_i = \boldsymbol{X'\Omega}_i\boldsymbol{X}/T = [(a_{\kappa_1\kappa_2 i})_{\kappa_1,\kappa_2=1,\ldots,n}] = \Bigg[\Bigg(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}x_{t'\kappa_2}\Bigg)_{\kappa_1,\kappa_2=1,\ldots,n}\Bigg]. \quad (A.145)$$

$$\boldsymbol{A}_{ij} = \boldsymbol{X'\Omega}_{ij}\boldsymbol{X}/T = [(a_{\kappa_1\kappa_2 ij})_{\kappa_1,\kappa_2=1,\ldots,n}] = \Bigg[\Bigg(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'ij}x_{t'\kappa_2}\Bigg)_{\kappa_1,\kappa_2=1,\ldots,n}\Bigg]. \quad (A.146)$$

Proof of Lemma A.24.

$$\begin{aligned}\boldsymbol{A}_i &= \boldsymbol{X'\Omega}_i\boldsymbol{X}/T = \tfrac{1}{T}[(x_{\kappa_1 t})_{\kappa_1=1,\ldots,n,\ t=1\ldots T}]\cdot[(\omega_{tt'i})_{t,t'=1,\ldots,T}]\cdot[(x_{t'\kappa_2})_{t'=1,\ldots,T,\ \kappa_2=1,\ldots,n}] \\ &= \Bigg[\Bigg(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}x_{t'\kappa_2}\Bigg)_{\kappa_1,\kappa_2=1,\ldots,n}\Bigg]. \quad (A.147)\end{aligned}$$

$$\begin{aligned}\boldsymbol{A}_{ij} &= \boldsymbol{X'\Omega}_{ij}\boldsymbol{X}/T = \tfrac{1}{T}[(x_{\kappa_1 t})_{\kappa_1=1,\ldots,n,\ t=1\ldots T}]\cdot[(\omega_{tt'ij})_{t,t'=1\ldots T}]\cdot[(x_{t'\kappa_2})_{t'=1,\ldots,T,\ \kappa_2=1,\ldots,n}] \\ &= \Bigg[\Bigg(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'ij}x_{t'\kappa_2}\Bigg)_{\kappa_1,\kappa_2=1,\ldots,n}\Bigg]. \quad (A.148)\end{aligned}$$

□

Lemma A.25. By combining (1.21) and (A.114b) we have that

$$\boldsymbol{A^*}_{ij} = \boldsymbol{X}'\boldsymbol{\Omega}_i\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_j\boldsymbol{X}/T = [(a^*{}_{\kappa_1\kappa_2 ij})_{\kappa_1,\kappa_2=1,\ldots,n}] = \left[\left(\frac{1}{T}\sum_{r=1}^{T}\sum_{m=1}^{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}\omega^{t'm}\omega_{mrj}x_{r\kappa_2}\right)_{\kappa_1,\kappa_2=1,\ldots,n}\right]. \tag{A.149}$$

Proof of Lemma A.25.

$$\begin{aligned}
\boldsymbol{A^*}_{ij} &= \boldsymbol{X}'\boldsymbol{\Omega}_i\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_j\boldsymbol{X}/T = \tfrac{1}{T}[(x_{\kappa_1 t})_{\kappa_1=1,\ldots,n,\ t=1\ldots T}]\cdot[(\omega_{tt'i})_{t,t'=1\ldots T}] \\
&\cdot\ [(\omega^{t'm})_{t',m=1\ldots T}]\cdot[(\omega_{mrj})_{m,r=1\ldots T}]\cdot[(x_{r\kappa_2})_{r=1,\ldots,T,\ \kappa_2=1,\ldots,n}] \\
&= \left[\left(\frac{1}{T}\sum_{r=1}^{T}\sum_{m=1}^{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}\omega^{t'm}\omega_{mrj}x_{r\kappa_2}\right)_{\kappa_1,\kappa_2=1,\ldots,n}\right].
\end{aligned} \tag{A.150}$$

□

Lemma A.26. By combining (A.145), (A.146),(A.149) and (2.5) we have that

$$\begin{aligned}
\boldsymbol{C}_{ij} &= \boldsymbol{A^*}_{ij} - 2\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{A}_j + \boldsymbol{A}_{ij}/2 \\
&= \left[\left(\frac{1}{T}\sum_{r=1}^{T}\sum_{m=1}^{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}\omega^{t'm}\omega_{mrj}x_{r\kappa_2}\right)_{\kappa_1,\kappa_2=1,\ldots,n}\right] \\
&\quad -2\left[\left[\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\left(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\right)g_{d_1d_2}\left(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\right)\right]_{\kappa_1,\kappa_2=1,\ldots,n}\right] \\
&\quad +\frac{1}{2}\left[\left(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'ij}x_{t'\kappa_2}\right)_{\kappa_1,\kappa_2=1,\ldots,n}\right].
\end{aligned} \tag{A.151}$$

Proof of Lemma A.26.

$$\begin{aligned}
\boldsymbol{C}_{ij} &= \boldsymbol{A^*}_{ij} - 2\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{A}_j + \boldsymbol{A}_{ij}/2 \\
&= [(a^*{}_{\kappa_1\kappa_2})_{\kappa_1,\kappa_2=1,\ldots,n}] - 2[(a_{\kappa_1 d_1 i})\ \kappa_1, d_1 = 1,\ldots,n][(g_{d_1d_2})_{d_1,d_2=1,\ldots,n}][(a_{d_2\kappa_2 j})_{d_2,\kappa_2=1,\ldots,n}] \\
&\quad +\tfrac{1}{2}[(a_{\kappa_1\kappa_2 ij})_{\kappa_1,\kappa_2=1,\ldots,n}] \\
&= \left[\left(\frac{1}{T}\sum_{r=1}^{T}\sum_{m=1}^{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}\omega^{t'm}\omega_{mrj}x_{r\kappa_2}\right)_{\kappa_1,\kappa_2=1,\ldots,n}\right] \\
&\quad -2\left[\left(\sum_{d_2=1}^{n}\sum_{d_1=1}^{n} a_{\kappa_1 d_1}g_{d_1d_2}a_{d_2\kappa_2}\right)_{\kappa_1,\kappa_2=1,\ldots,n}\right] \\
&\quad +\frac{1}{2}\left[\left(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'ij}x_{t'\kappa_2}\right)_{\kappa_1,\kappa_2=1,\ldots,n}\right] \\
&= \left[\left(\frac{1}{T}\sum_{r=1}^{T}\sum_{m=1}^{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}\omega^{t'm}\omega_{mrj}x_{r\kappa_2}\right)_{\kappa_1,\kappa_2=1,\ldots,n}\right] \\
&\quad -2\left[\left[\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\left(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\right)g_{d_1d_2}\left(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\right)\right]_{\kappa_1,\kappa_2=1,\ldots,n}\right]
\end{aligned}$$

$$+\frac{1}{2}\Big[\Big(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'ij}x_{t'\kappa_2}\Big)_{\kappa_1,\kappa_2=1,\dots,n}\Big]. \tag{A.152}$$

□

Lemma A.27. Using (A.145) and (2.13) we have

$$\boldsymbol{D}_{ij}=\boldsymbol{A}_i\boldsymbol{P}\boldsymbol{A}_j/2=\frac{1}{2}\Big[\Big(\sum_{d_1=1}^{n}\sum_{d_2=1}^{n}\Big(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Big)p_{d_1d_2}\Big(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\Big)\Big)_{\kappa_1,\kappa_2=1,\dots,n}\Big]. \tag{A.153}$$

Proof of Lemma A.27.

$$\begin{aligned}\boldsymbol{D}_{ij} &= \boldsymbol{A}_i\boldsymbol{P}\boldsymbol{A}_j/2=\frac{1}{2}\Big[\Big(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Big)_{\kappa_1,d_1=1,\dots,n}\Big][(p_{d_1d_2})_{d_1,d_2=1,\dots,n}]\Big[\Big(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\Big)_{\kappa_1,\kappa_2=1,\dots,n}\Big]\\ &= \frac{1}{2}\Big[\Big(\sum_{d_1=1}^{n}\sum_{d_2=1}^{n}\Big(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Big)p_{d_1d_2}\Big(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\Big)\Big)_{\kappa_1,\kappa_2=1,\dots,n}\Big].\end{aligned} \tag{A.154}$$

□

Lemma A.28. By using equations (A.145), (A.152) and (A.153) we have that

$$\operatorname{tr}\boldsymbol{A}_i\boldsymbol{P}=\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Big[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\Big], \tag{A.155}$$

$$\begin{aligned}\operatorname{tr}\boldsymbol{C}_{ij}\boldsymbol{P} &= \Big[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Big[\frac{1}{T}\Big(\sum_{r=1}^{T}\sum_{m=1}^{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'i}\omega^{t'm}\omega_{mrj}x_{r\kappa_2}\Big)p_{\kappa_2\kappa_1}\Big]\Big]\\ &\quad -2\sum_{\kappa_1=1}^{n}\Big[\sum_{\kappa_2=1}^{n}\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\Big[\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Big)g_{d_1d_2}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2 t}\omega_{tt'j}x_{t'd_3}\Big)p_{\kappa_2\kappa_1}\Big]\Big]\\ &\quad +\frac{1}{2}\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Big[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'ij}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\Big].\end{aligned} \tag{A.156}$$

$$\operatorname{tr}\boldsymbol{D}_{ij}\boldsymbol{P}=\frac{1}{2}\sum_{\kappa_1=1}^{n}\Big[\sum_{\kappa_2=1}^{n}\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\Big[\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Big)p_{d_1d_2}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\Big]\Big]. \tag{A.157}$$

Proof of Lemma A.28.

$$\begin{aligned}\boldsymbol{A}_i\boldsymbol{P} &= \tfrac{1}{T}[(x_{\kappa_1 t})_{\kappa_1=1,\dots,n,\ t=1,\dots,T}]\cdot[(\omega_{tt'i})_{t,t'=1,\dots,T}]\\ &\cdot\ [(x_{t'\kappa_2})_{\kappa_2=1,\dots,n,\ t'=1,\dots,T}][(p_{\kappa_2 r})_{\kappa_2,r=1,\dots,n}]\\ &= \Big[\Big(\sum_{\kappa_2=1}^{n}\Big[\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'\kappa_2}\Big]p_{\kappa_2 r}\Big)_{\kappa_1,r=1,\dots,n}\Big]\Rightarrow\end{aligned} \tag{A.158}$$

$$\operatorname{tr} \boldsymbol{A}_i \boldsymbol{P} = \Big[\sum_{\kappa_1=1}^{n} \sum_{\kappa_2=1}^{n} \Big[\frac{1}{T} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,i} x_{t'\kappa_2} \Big) p_{\kappa_2 \kappa_1} \Big] \Big]. \tag{A.159}$$

$$\begin{aligned}
\boldsymbol{C}_{ij}\boldsymbol{P} &= \Big[\Big(\frac{1}{T} \sum_{r=1}^{T} \sum_{m=1}^{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,i} \omega^{t'm} \omega_{mr\,j} x_{r\kappa_2} \Big)_{\kappa_1,\kappa_2=1,\ldots,n} \Big] \cdot [(p_{\kappa_2 l})_{\kappa_2, l=1,\ldots,n}] \\
&\quad -2 \Big[\Big[\sum_{d_2=1}^{n} \sum_{d_1=1}^{n} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{\kappa_1 t} \omega_{tt'\,i} x_{t' d_1} \Big) g_{d_1 d_2} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{d_2 t} \omega_{tt'\,j} x_{t'\kappa_2} \Big) \Big]_{\kappa_1,\kappa_2=1,\ldots,n} \Big] \cdot [(p_{\kappa_2 l})_{\kappa_2, l=1,\ldots,n}] \\
&\quad + \frac{1}{2} \Big[\Big(\frac{1}{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,ij} x_{t'\kappa_2} \Big)_{\kappa_1,\kappa_2=1,\ldots,n} \Big] \cdot [(p_{\kappa_2 l})_{\kappa_2, l=1,\ldots,n}] \\
&= \Big[\Big(\frac{1}{T} \sum_{\kappa_2=1}^{n} \Big(\sum_{r=1}^{T} \sum_{m=1}^{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,i} \omega^{t'm} \omega_{mr\,j} x_{r\kappa_2} \Big) p_{\kappa_2 l} \Big)_{\kappa_1, l=1,\ldots,n} \Big] \\
&\quad -2 \Big[\Big(\sum_{\kappa_2=1}^{n} \Big[\sum_{d_2=1}^{n} \sum_{d_1=1}^{n} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{\kappa_1 t} \omega_{tt'\,i} x_{t' d_1} \Big) g_{d_1 d_2} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{d_2 t} \omega_{tt'\,j} x_{t'\kappa_2} \Big) \Big] p_{\kappa_2 l} \Big)_{\kappa_1, l=1,\ldots,n} \Big] \\
&\quad + \frac{1}{2} \Big[\Big(\sum_{\kappa_2=1}^{n} \Big(\frac{1}{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,ij} x_{t'\kappa_2} \Big) p_{\kappa_2 l} \Big)_{\kappa_1, l=1,\ldots,n} \Big].
\end{aligned} \tag{A.160}$$

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{C}_{ij}\boldsymbol{P}) &= \operatorname{tr} \Big[\Big(\frac{1}{T} \sum_{\kappa_2=1}^{n} \Big(\sum_{r=1}^{T} \sum_{m=1}^{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,i} \omega^{t'm} \omega_{mr\,j} x_{r\kappa_2} \Big) p_{\kappa_2 l} \Big)_{\kappa_1, l=1,\ldots,n} \Big] \\
&\quad -2 \operatorname{tr} \Big[\Big(\sum_{\kappa_2=1}^{n} \Big[\sum_{d_2=1}^{n} \sum_{d_1=1}^{n} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{\kappa_1 t} \omega_{tt'\,i} x_{t' d_1} \Big) g_{d_1 d_2} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{d_2 t} \omega_{tt'\,j} x_{t'\kappa_2} \Big) \Big] p_{\kappa_2 l} \Big)_{\kappa_1, l=1,\ldots,n} \Big] \\
&\quad + \operatorname{tr} \frac{1}{2} \Big[\Big(\sum_{\kappa_2=1}^{n} \Big(\frac{1}{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,ij} x_{t'\kappa_2} \Big) p_{\kappa_2 l} \Big)_{\kappa_1, l=1,\ldots,n} \Big] \\
&= \sum_{\kappa_1=1}^{n} \Big(\frac{1}{T} \sum_{\kappa_2=1}^{n} \Big(\sum_{r=1}^{T} \sum_{m=1}^{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,i} \omega^{t'm} \omega_{mr\,j} x_{r\kappa_2} \Big) p_{\kappa_2 \kappa_1} \Big) \\
&\quad -2 \sum_{\kappa_1=1}^{n} \Big(\sum_{\kappa_2=1}^{n} \Big[\sum_{d_2=1}^{n} \sum_{d_1=1}^{n} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{\kappa_1 t} \omega_{tt'\,i} x_{t' d_1} \Big) g_{d_1 d_2} \Big(\sum_{t'=1}^{T} \sum_{t=1}^{T} \frac{1}{T} x_{d_2 t} \omega_{tt'\,j} x_{t'\kappa_2} \Big) \Big] p_{\kappa_2 \kappa_1} \Big) \\
&\quad + \frac{1}{2} \sum_{\kappa_1=1}^{n} \Big(\sum_{\kappa_2=1}^{n} \Big(\frac{1}{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,ij} x_{t'\kappa_2} \Big) p_{\kappa_2 \kappa_1} \Big).
\end{aligned} \tag{A.161}$$

$$\begin{aligned}
\boldsymbol{D}_{ij}\boldsymbol{P} &= \boldsymbol{A}_i \boldsymbol{P} \boldsymbol{A}_j \boldsymbol{P}/2 = \frac{1}{2} \Big[\Big(\sum_{d_1=1}^{n} \sum_{d_2=1}^{n} \Big(\frac{1}{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,i} x_{t' d_1} \Big) p_{d_1 d_2} \Big(\frac{1}{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{d_2 t} \omega_{tt'\,j} x_{t'\kappa_2} \Big) \Big)_{\kappa_1,\kappa_2=1,\ldots,n} \Big] \cdot [(p_{\kappa_2 l})_{\kappa_2, l=1,\ldots,n}] \\
&= \frac{1}{2} \Big[\Big(\sum_{\kappa_2=1}^{n} \Big(\sum_{d_1=1}^{n} \sum_{d_2=1}^{n} \Big(\frac{1}{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{\kappa_1 t} \omega_{tt'\,i} x_{t' d_1} \Big) p_{d_1 d_2} \Big(\frac{1}{T} \sum_{t'=1}^{T} \sum_{t=1}^{T} x_{d_2 t} \omega_{tt'\,j} x_{t'\kappa_2} \Big) \Big) p_{\kappa_2 l} \Big)_{\kappa_1, l=1,\ldots,n} \Big].
\end{aligned} \tag{A.162}$$

$$\begin{aligned}\operatorname{tr}(\boldsymbol{D}_{ij}\boldsymbol{P}) &= \frac{1}{2}\operatorname{tr}\Bigg[\Bigg(\sum_{\kappa_2=1}^{n}\Bigg(\sum_{d_1=1}^{n}\sum_{d_2=1}^{n}\Bigg(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Bigg)p_{d_1d_2}\Bigg(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{d_2t}\omega_{tt'j}x_{t'\kappa_2}\Bigg)\Bigg)p_{\kappa_2 l}\Bigg)_{\kappa_1,l=1,\ldots,n}\Bigg] \\ &= \frac{1}{2}\sum_{\kappa_1=1}^{n}\Bigg[\sum_{\kappa_2=1}^{n}\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Bigg)p_{d_1d_2}\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2t}\omega_{tt'j}x_{t'\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]\Bigg]. \qquad (A.163)\end{aligned}$$

□

Proof of Theorem 1. I. For t-test it holds that $\boldsymbol{H} = \boldsymbol{e}'$ where $\boldsymbol{e}$ is a $n \times 1$ known vector. So by setting $\boldsymbol{H} = \boldsymbol{e}'$ we have

$$\boldsymbol{Q} = \boldsymbol{Q}_t = \boldsymbol{H}'(\boldsymbol{H}\boldsymbol{G}\boldsymbol{H}')^{-1}\boldsymbol{H} = \boldsymbol{e}(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{-1}\boldsymbol{e}' = \boldsymbol{k}\boldsymbol{k}' \qquad (A.164)$$

where $\boldsymbol{k} = \boldsymbol{e}/(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}$.

Therefore,

$$\boldsymbol{P} = \boldsymbol{P}_t = \boldsymbol{G}\boldsymbol{Q}\boldsymbol{G} = \boldsymbol{G}\boldsymbol{e}\boldsymbol{e}'\boldsymbol{G}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} = \boldsymbol{G}\boldsymbol{k}\boldsymbol{k}'\boldsymbol{G}. \qquad (A.165)$$

Using definition (2.4) and Lemma A.28 we have that

i. For $i = (\rho, 1, \ldots, m)$ the following results hold:

$$\begin{aligned}l_i &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} = \boldsymbol{k}\boldsymbol{G}\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{k} = \operatorname{tr}\boldsymbol{k}\boldsymbol{G}\boldsymbol{A}_i\boldsymbol{G}\boldsymbol{k} = \operatorname{tr}\boldsymbol{A}_i\boldsymbol{k}\boldsymbol{G}\boldsymbol{G}\boldsymbol{k} = \operatorname{tr}\boldsymbol{A}_i\boldsymbol{P} \Rightarrow \\ l_i &= \operatorname{tr}\boldsymbol{A}_i\boldsymbol{P} \Rightarrow \\ l_i &= \Bigg[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Bigg[\frac{1}{T}\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]\Bigg]. \qquad (A.166)\end{aligned}$$

$$\begin{aligned}\boldsymbol{l} &= [(l_i)_{i=(\rho,1,\ldots,m)}] = \begin{bmatrix} l_\rho \\ l_1 \\ \vdots \\ l_m \end{bmatrix} \\ &= \begin{bmatrix} \Big[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Big[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'\rho}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\Big]\Big] \\ \Big[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Big[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'1}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\Big]\Big] \\ \vdots \\ \Big[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Big[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'm}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\Big]\Big] \end{bmatrix}. \qquad (A.167)\end{aligned}$$

ii. For $i, j = (\rho, 1, \ldots, m)$ the following results hold:

$$
\begin{aligned}
l_{ij} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{ij}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} = \boldsymbol{k}\boldsymbol{G}\boldsymbol{C}_{ij}\boldsymbol{G}\boldsymbol{k} = \operatorname{tr}\boldsymbol{k}\boldsymbol{G}\boldsymbol{C}_{ij}\boldsymbol{G}\boldsymbol{k} = \operatorname{tr}\boldsymbol{C}_{ij}\boldsymbol{k}\boldsymbol{G}\boldsymbol{G}\boldsymbol{k} = \operatorname{tr}\boldsymbol{C}_{ij}\boldsymbol{P} \Rightarrow \\
l_{ij} &= \operatorname{tr}\boldsymbol{C}_{ij}\boldsymbol{P} \Rightarrow \\
l_{ij} &= \Bigg[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Bigg[\frac{1}{T}\Bigg(\sum_{r=1}^{T}\sum_{m=1}^{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}\omega^{t'm}\omega_{mrj}x_{r\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]\Bigg] \\
&\quad -2\sum_{\kappa_1=1}^{n}\Bigg[\sum_{\kappa_2=1}^{n}\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Bigg)g_{d_1 d_2}\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]\Bigg] \\
&\quad +\frac{1}{2}\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Bigg[\frac{1}{T}\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}x_{\kappa_1 t}\omega_{tt'ij}x_{t'\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]. \qquad \text{(A.168)}
\end{aligned}
$$

$$\boldsymbol{L} = [(l_{ij})_{i,j=(\rho,1,\ldots,m)}]. \qquad \text{(A.169)}$$

II. For F-test, $\boldsymbol{H} \neq \boldsymbol{e}'$ therefore $\boldsymbol{Q}_F \neq \boldsymbol{Q}_t \Rightarrow \boldsymbol{P}_F \neq \boldsymbol{P}_t$.

Let $\boldsymbol{H}$ a known $r \times n$ matrix with rank $r < n$. Matrix $\boldsymbol{G}$ is a $n \times n$ positive definite and symmetric. Consequently, matrix $\boldsymbol{HGH}'$ is a $r \times r$ positive definite and symmetric i.e.,

$$\boldsymbol{HGH}' = (\boldsymbol{HGH}')' \overset{d}{>} 0. \qquad \text{(A.170)}$$

Equation (A.170) implies that matrix $(\boldsymbol{HGH}')^{-1}$ is positive define and symmetric matrix i.e., $(\boldsymbol{HGH}')^{-1} \overset{d}{>} 0$ which implies that matrix $(\boldsymbol{HGH}')^{-1}$ is positively semi-defined matrix i.e.,

$$(\boldsymbol{HGH}')^{-1} \overset{d}{\geq} 0. \qquad \text{(A.171)}$$

Equation (A.171) implies that for the $n \times n$ matrix $\boldsymbol{Q}$ we have

$$\boldsymbol{Q}_F = \boldsymbol{H}'(\boldsymbol{HGH}')^{-1}\boldsymbol{H} \overset{d}{\geq} 0, \qquad \text{(A.172)}$$

and for the matrix $\boldsymbol{P}$ we have

$$\boldsymbol{P}_F = \boldsymbol{GQG} = \boldsymbol{G}'\boldsymbol{QG} \overset{d}{\geq} 0. \qquad \text{(A.173)}$$

Let λ_i the eigenvalues of the matrix $\boldsymbol{P}$ by the equation (A.173) we have that $\lambda_i \geq 0$ $(i = 1, \ldots, n)$. We set the $n \times r$ matrix $\boldsymbol{V} = \boldsymbol{G}^{1/2}\boldsymbol{H}'$ where $G^{1/2}$ is a positive definite and symmetric matrix like $\boldsymbol{G}$. The projector in the space created by the columns of the matrix $\boldsymbol{V}$ is

$$\boldsymbol{P_V} = \boldsymbol{V}(\boldsymbol{V}'\boldsymbol{V})^{-1}\boldsymbol{V}' = \boldsymbol{G}^{1/2}\boldsymbol{H}'(\boldsymbol{HG}^{1/2}\boldsymbol{G}^{1/2}\boldsymbol{H}')^{-1}\boldsymbol{HG}^{1/2} = \boldsymbol{G}^{1/2}\boldsymbol{H}'(\boldsymbol{HGH}')^{-1}\boldsymbol{HG}^{1/2} = \boldsymbol{G}^{1/2}\boldsymbol{QG}^{1/2}. \qquad \text{(A.174)}$$

Since, $\boldsymbol{P_V V} = \boldsymbol{V}(\boldsymbol{V'V})^{-1}\boldsymbol{V'V} = \boldsymbol{V}$, the columns of $\boldsymbol{V}$ are the eigenvectors of $\boldsymbol{P_V}$. Hence, by using the definition (2.13), we find that

$$\boldsymbol{P} = \boldsymbol{GQG} = \boldsymbol{G}^{1/2}(\boldsymbol{G}^{1/2}\boldsymbol{Q}\boldsymbol{G}^{1/2})\boldsymbol{G}^{1/2} = \boldsymbol{G}^{1/2}\boldsymbol{P_V}\boldsymbol{G}^{1/2}. \tag{A.175}$$

Since $\boldsymbol{G}$ is a symmetric, positive definite, and non-singular matrix the same holds for $\boldsymbol{G}^{1/2}$. Therefore, $\operatorname{rank}(\boldsymbol{P}) = \operatorname{rank}(\boldsymbol{P}_V)$ i.e.,

$$\operatorname{rank}(\boldsymbol{P}) = \operatorname{rank}(\boldsymbol{G}^{1/2}\boldsymbol{P}_V\boldsymbol{G}^{1/2}) = \operatorname{rank}(\boldsymbol{P}_V). \tag{A.176}$$

However, matrix $\boldsymbol{P_V}$ is idempotent matrix and its rank is equal to its trace.

$$\begin{aligned}
\operatorname{rank}\boldsymbol{P}_V &= \operatorname{tr}\boldsymbol{P}_V = \operatorname{tr}\boldsymbol{V}(\boldsymbol{V'V})^{-1}\boldsymbol{V'} = \operatorname{tr} I_r = r \Rightarrow & \text{(A.177)}\\
\operatorname{rank}\boldsymbol{P} &= r. & \text{(A.178)}
\end{aligned}$$

By using the relations $\lambda_i \geq 0$ $(i = 1, \ldots, n)$ and $\operatorname{rank}\boldsymbol{P} = r$ we conclude that r of eigenvalues of the matrix $\boldsymbol{P}$ are positive and the remaining $n - r$ are equal to 0. Let $\boldsymbol{\mathcal{L}}$ a $n \times n$ diagonal matrix and let eigenvalues λ_i be the elements of matrix $\boldsymbol{P}$. Also, let $\boldsymbol{\mathcal{W}}$ be a $n \times n$ diagonal matrix whose columns are the normalized eigenvectors w_i of matrix $\boldsymbol{P}$. By using the Theorem of spectral analysis, matrix $\boldsymbol{P}$ can be write as follows:

$$\boldsymbol{P} = \boldsymbol{\mathcal{W}\mathcal{L}\mathcal{W}'} = \sum_{i=1}^{n} \lambda_i w_i w_i' = \sum_{i=1}^{r} \lambda_i w_i w_i'. \tag{A.179}$$

Using definition (2.12) and Lemma A.28 we have

i. The i-th element of vector $\boldsymbol{c}$ is

$$\begin{aligned}
c_i &= = \operatorname{tr}\boldsymbol{A}_i\boldsymbol{P} \Rightarrow\\
c_i &= \left[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\left[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\right]\right]. & \text{(A.180)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{c} &= [(c_i)\ i = (\rho, 1, \ldots, m)] = \begin{bmatrix} c_\rho \\ c_1 \\ \vdots \\ c_m \end{bmatrix}\\
&= \begin{bmatrix}
\left[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\left[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'\rho}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\right]\right]\\
\left[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\left[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'1}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\right]\right]\\
\vdots\\
\left[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\left[\frac{1}{T}\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'm}x_{t'\kappa_2}\Big)p_{\kappa_2\kappa_1}\right]\right]
\end{bmatrix}. & \text{(A.181)}
\end{aligned}$$

ii. The (i,j)-th element of matrix $\boldsymbol{C}$ is

$$\begin{aligned}
c_{ij} &= \operatorname{tr} \boldsymbol{C}_{ij}\boldsymbol{P} \Rightarrow \\
c_{ij} &= \Bigg[\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Bigg[\frac{1}{T}\Bigg(\sum_{r=1}^{T}\sum_{m=1}^{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'i}\omega^{t'm}\omega_{mrj}x_{r\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]\Bigg] \\
&\quad -2\sum_{\kappa_1=1}^{n}\Bigg[\sum_{\kappa_2=1}^{n}\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Bigg)g_{d_1 d_2}\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]\Bigg] \\
&\quad +\frac{1}{2}\sum_{\kappa_1=1}^{n}\sum_{\kappa_2=1}^{n}\Bigg[\frac{1}{T}\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T} x_{\kappa_1 t}\omega_{tt'ij}x_{t'\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]. \qquad (A.182)
\end{aligned}$$

$$\boldsymbol{C} = [(c_{ij})_{i,j=(\rho,1,\ldots,m)}]. \qquad (A.183)$$

iii. The (i,j)-th element of matrix $\boldsymbol{D}$ is

$$\begin{aligned}
d_{ij} &= \operatorname{tr} \boldsymbol{D}_{ij}\boldsymbol{P} \Rightarrow \\
d_{ij} &= \frac{1}{2}\sum_{\kappa_1=1}^{n}\Bigg[\sum_{\kappa_2=1}^{n}\sum_{d_2=1}^{n}\sum_{d_1=1}^{n}\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{\kappa_1 t}\omega_{tt'i}x_{t'd_1}\Bigg)p_{d_1 d_2}\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{T}x_{d_2 t}\omega_{tt'j}x_{t'\kappa_2}\Bigg)p_{\kappa_2\kappa_1}\Bigg]\Bigg]. \qquad (A.184)
\end{aligned}$$

$$\boldsymbol{D} = [(d_{ij})_{i,j=(\rho,1,\ldots,m)}]. \qquad (A.185)$$

□

We define the following matrices for the Linear Regression Models (1.1), (A.221) and (A.251)

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T,\ \boldsymbol{A}_{AR} = \boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR}/T,\ \boldsymbol{A}_{H} = \boldsymbol{X}'_{H}\boldsymbol{\Omega}_{H}\boldsymbol{X}_{H}/T, \\
\boldsymbol{G} &= \boldsymbol{A}^{-1},\ \boldsymbol{G}_{AR} = \boldsymbol{A}_{AR}{}^{-1},\ \boldsymbol{G}_{H} = \boldsymbol{A}_{H}{}^{-1}, \\
\bar{\boldsymbol{A}} &= \boldsymbol{Z}'\boldsymbol{\Omega}^{2}\boldsymbol{Z}/T,\ \bar{\boldsymbol{A}}_{AR} = \boldsymbol{Z}'\boldsymbol{\Omega}_{AR}{}^{2}\boldsymbol{Z}/T,\ \bar{\boldsymbol{A}}_{H} = \boldsymbol{Z}'\boldsymbol{\Omega}_{H}^{2}\boldsymbol{Z}/T, \\
\bar{\boldsymbol{G}} &= \bar{\boldsymbol{A}}^{-1},\ \bar{\boldsymbol{G}}_{AR} = \bar{\boldsymbol{A}}_{AR}^{-1},\ \bar{\boldsymbol{G}}_{H} = \bar{\boldsymbol{A}}_{H}^{-1}, \\
\boldsymbol{F} &= \boldsymbol{X}'\boldsymbol{X}/T,\ \boldsymbol{F}_{AR} = \boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T,\ \boldsymbol{F}_{H} = \boldsymbol{X}'_{H}\boldsymbol{X}_{H}/T, \\
\boldsymbol{B} &= \boldsymbol{F}^{-1},\ \boldsymbol{B}_{AR} = \boldsymbol{F}_{AR}{}^{-1},\ \boldsymbol{B}_{H} = \boldsymbol{F}_{H}{}^{-1}, \qquad (A.186) \\
\bar{\boldsymbol{F}} &= \bar{\boldsymbol{F}}_{AR} = \bar{\boldsymbol{F}}_{H} = \boldsymbol{Z}'\boldsymbol{Z}/T, \\
\bar{\boldsymbol{B}} &= \bar{\boldsymbol{B}}_{AR} = \bar{\boldsymbol{B}}_{H} = \bar{\boldsymbol{F}}^{-1}, \\
\boldsymbol{\Gamma} &= \boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}/T,\ \boldsymbol{\Gamma}_{AR} = \boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{X}_{AR}/T,\ \boldsymbol{\Gamma}_{H} = \boldsymbol{X}'_{H}\boldsymbol{\Omega}_{H}{}^{-1}\boldsymbol{X}_{H}/T, \\
\bar{\boldsymbol{\Gamma}} &= \boldsymbol{Z}'\boldsymbol{\Omega}^{-2}\boldsymbol{Z}/T,\ \bar{\boldsymbol{\Gamma}}_{AR} = \boldsymbol{Z}'_{AR}\boldsymbol{\Omega}_{AR}{}^{-2}\boldsymbol{Z}_{AR}/T,\ \bar{\boldsymbol{\Gamma}}_{H} = \boldsymbol{Z}'_{H}\boldsymbol{\Omega}_{H}{}^{-2}\boldsymbol{Z}_{H}/T,
\end{aligned}$$

According to Lemmas Γ.1, Γ.2 and Γ.5, (see Symeonides, 1991), the following quantities are defined:

$$\begin{aligned}
\operatorname{tr} \boldsymbol{\Omega}_{AR1}\boldsymbol{\Omega}_{AR}{}^{-1} &= \operatorname{tr}(\boldsymbol{I}-\boldsymbol{R})/\rho = 0,\ \operatorname{tr}\boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Delta} = 2/\alpha,\ \operatorname{tr}\boldsymbol{\Omega}_{AR2}\boldsymbol{\Omega}_{AR}{}^{-1} = 2\rho/\alpha. \\
\text{For } i = 1,2,\rho, & \\
\operatorname{tr}\boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Omega}_{ARi}\boldsymbol{\Omega}_{AR}{}^{-1} &= -2\rho T/\alpha^2 + O(1), \\
\operatorname{tr}(\boldsymbol{\Omega}_{ARi}\boldsymbol{\Omega}_{AR}{}^{-1})^2 &= 2T/\alpha + O(1), \\
\operatorname{tr}\boldsymbol{\Omega}_{AR}{}^{-1}(\boldsymbol{\Omega}_{ARi}\boldsymbol{\Omega}_{AR}{}^{-1})^2 &= 2(2\rho^{-1}-1)T/\alpha^3 + O(1), \\
\operatorname{tr}(\boldsymbol{\Omega}_{ARi}\boldsymbol{\Omega}_{AR}{}^{-1})^3 &= 2(2-3\rho^{-1})T/\rho\alpha^2 + O(1) \\
\operatorname{tr}\boldsymbol{P}_{ARX}\boldsymbol{\Omega}_{ARi} &= (\operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR} - n\alpha)/\rho + O(T^2), \\
\operatorname{tr}\boldsymbol{P}_{ARX}\boldsymbol{\Omega}_{ARi}\boldsymbol{\Omega}_{AR}^{-1} &= (n - \operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho + O(T^2), \qquad \text{(A.187)}\\
\operatorname{tr}\boldsymbol{P}_{ARX}\boldsymbol{\Omega}_{ARi}\boldsymbol{P}_{ARX}\boldsymbol{\Omega}_{AR}^{-1} &= (\operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha - \operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho + O(T^2), \\
\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T &= O(\tau^2),\ \boldsymbol{X}'_{AR}\boldsymbol{\Delta}\boldsymbol{X}_{AR}/T = O(\tau^2), \boldsymbol{X}'_{AR}\boldsymbol{R}\boldsymbol{X}_{AR}/T = O(\tau^2), \\
\boldsymbol{X}'_{AR}\boldsymbol{\Delta}\boldsymbol{R}\boldsymbol{X}_{AR}/T &= O(\tau^2),\ \boldsymbol{X}'_{AR}\boldsymbol{R}\boldsymbol{\Delta}\boldsymbol{X}_{AR}/T = O(\tau^2), \boldsymbol{X}'_{AR}\boldsymbol{R}\boldsymbol{\Delta}\boldsymbol{R}\boldsymbol{X}_{AR}/T = O(\tau^2).
\end{aligned}$$

Lemma A.29. We consider the $T \times T$ matrix

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1/2}[(1+\rho^2)\boldsymbol{I}_T - \rho\boldsymbol{D} - \rho^2\boldsymbol{\Delta}]\boldsymbol{\Sigma}^{-1/2}, \qquad \text{(A.188)}$$

where the $T \times T$ matrices $\boldsymbol{I}_T$, $\boldsymbol{D}$ and $\boldsymbol{\Delta}$ are defined in Lemma A.1. From (1.21) we know that

$$\boldsymbol{\Omega}^{-1} = \frac{1}{1-\rho^2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{R}\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{R}/\alpha\boldsymbol{\Sigma}^{1/2},\ \textit{where } \alpha = 1-\rho^2. \qquad \text{(A.189)}$$

We define the

$$\alpha_* = \frac{\rho^2}{1-\rho^2}. \qquad \text{(A.190)}$$

Let

$$\begin{aligned}
\boldsymbol{\Omega}_\rho &= \partial\boldsymbol{\Omega}/\partial\rho = \boldsymbol{\Sigma}^{-1/2}[2\rho\boldsymbol{I}_T - \boldsymbol{D} - 2\rho\boldsymbol{\Delta}]\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Omega}_1 - \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}, \\
\text{and} & \qquad \text{(A.191)}\\
\boldsymbol{\Omega}_{\rho\rho} &= \partial^2\boldsymbol{\Omega}/\partial\rho^2 = \boldsymbol{\Sigma}^{-1/2}[2\boldsymbol{I}_T - 2\boldsymbol{\Delta}]\boldsymbol{\Sigma}^{-1/2},
\end{aligned}$$

where

$$\boldsymbol{\Omega}_i = \boldsymbol{\Omega}_\rho + i\rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2},\ \boldsymbol{\Omega}_{ii} = \boldsymbol{\Omega}_{\rho\rho} + i\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2},\ (i = 1,2). \qquad \text{(A.192)}$$

Using (A.191), The following results apply:

$$\begin{aligned}
\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1} &= O(T^{-1}),\\
\operatorname{tr}(\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1})^2 &= 2/(1-\rho^2)+O(T^{-1}),\\
\operatorname{tr}(\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1})^3 &= 2(2-3\rho^2)/\rho(1-\rho^2)^2+O(T^{-1}),
\end{aligned} \tag{A.193}$$

and

$$\operatorname{tr}\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{\Omega}^{-1} = \frac{2}{\alpha}-\frac{4}{\alpha T}. \tag{A.194}$$

Proof of Lemma A.29. From (A.192) we have

$$\begin{aligned}
\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} &= \boldsymbol{\Sigma}^{-1/2}[2\rho\boldsymbol{I}_T-\boldsymbol{D}-2\rho\boldsymbol{\Delta}+\rho\boldsymbol{\Delta}]\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}[\boldsymbol{R}/(1-\rho^2)]\boldsymbol{\Sigma}^{1/2}\\
&= \boldsymbol{\Sigma}^{-1/2}[2\rho\boldsymbol{I}_T-\boldsymbol{D}-\rho\boldsymbol{\Delta}][\boldsymbol{R}/(1-\rho^2)]\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[2\rho^2\boldsymbol{I}_T-\rho\boldsymbol{D}-\rho^2\boldsymbol{\Delta}][\boldsymbol{R}/(1-\rho^2)]\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[(1+\rho^2)\boldsymbol{I}_T-\rho\boldsymbol{D}-\rho^2\boldsymbol{\Delta}-(1-\rho^2)\boldsymbol{I}_T][\boldsymbol{R}/(1-\rho^2)]\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{R}^*-\alpha\boldsymbol{I}_T][\boldsymbol{R}/\alpha]\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-\boldsymbol{R}]\boldsymbol{\Sigma}^{1/2}.
\end{aligned} \tag{A.195}$$

By using (UR.25), (A.191), (A.195) we find

$$\begin{aligned}
\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1} &= (\boldsymbol{\Omega}_1-\rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2})\boldsymbol{\Omega}^{-1} = \boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1}-\frac{\rho}{\alpha}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{R}\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-\boldsymbol{R}]\boldsymbol{\Sigma}^{1/2}-\frac{\rho}{\alpha}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{R}\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-\boldsymbol{R}-\frac{\rho^2}{\alpha}\boldsymbol{\Delta}\boldsymbol{R}]\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-\boldsymbol{R}-\frac{\rho^2}{\alpha}\boldsymbol{\Delta}\boldsymbol{R}]\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-\boldsymbol{R}-\alpha_*\boldsymbol{E}]\boldsymbol{\Sigma}^{1/2}.
\end{aligned} \tag{A.196}$$

From equation (UR.25) and (A.196) we have

$$\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}/T = \operatorname{tr}\frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-\boldsymbol{R}-\alpha_*\boldsymbol{E}]\boldsymbol{\Sigma}^{1/2}/T = \frac{1}{\rho}(T-T-2\alpha_*)/T = \frac{2\alpha_*}{\rho T} = O(T^{-1}). \tag{A.197}$$

By using (A.196) we have

$$\begin{aligned}
(\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1})^2 &= \frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-\boldsymbol{R}-\alpha_*\boldsymbol{E}]\boldsymbol{\Sigma}^{1/2}\frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-\boldsymbol{R}-\alpha_*\boldsymbol{E}]\boldsymbol{\Sigma}^{1/2}\\
&= \frac{1}{\rho^2}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T-2\boldsymbol{R}+\boldsymbol{R}^2+\alpha_*(\boldsymbol{E}\boldsymbol{R}+\boldsymbol{R}\boldsymbol{E}-2\boldsymbol{E})+a_*{}^2\boldsymbol{E}^2]\boldsymbol{\Sigma}^{1/2}.
\end{aligned} \tag{A.198}$$

From equations (UR.25) and (A.198) we have

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1})^2/T &= \frac{1}{\rho^2}\operatorname{tr}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - 2\boldsymbol{R} + \boldsymbol{R}^2 + \alpha_*(\boldsymbol{ER}+\boldsymbol{RE}-2\boldsymbol{E}) + a_*{}^2\boldsymbol{E}^2]\boldsymbol{\Sigma}^{1/2}/T \\
&= \frac{1}{\rho^2}\operatorname{tr}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - 2\boldsymbol{R} + \boldsymbol{R}^2 + \alpha_*(\boldsymbol{ER}+\boldsymbol{RE}-2\boldsymbol{E}) + a_*{}^2\boldsymbol{E}^2]/T \\
&= \frac{1}{\rho^2}\operatorname{tr}[\boldsymbol{I}_T - 2\boldsymbol{R} + \boldsymbol{R}^2 + \alpha_*(\boldsymbol{ER}+\boldsymbol{RE}-2\boldsymbol{E}) + a_*{}^2\boldsymbol{E}^2]/T \\
&= \frac{1}{\rho^2}[\operatorname{tr}\boldsymbol{I}_T - 2\operatorname{tr}\boldsymbol{R} + \operatorname{tr}\boldsymbol{R}^2 + \alpha_*(\operatorname{tr}\boldsymbol{ER}+\operatorname{tr}\boldsymbol{RE}-2\operatorname{tr}\boldsymbol{E}) + a_*{}^2\operatorname{tr}\boldsymbol{E}^2]/T \\
&= \frac{1}{\rho^2}[\operatorname{tr}\boldsymbol{I}_T - 2\operatorname{tr}\boldsymbol{R} + \operatorname{tr}\boldsymbol{R}^2 + \alpha_*(2\operatorname{tr}\boldsymbol{RE}-2\operatorname{tr}\boldsymbol{E}) + a_*{}^2\operatorname{tr}\boldsymbol{E}^2]/T \\
&= \frac{1}{\rho^2}[\operatorname{tr}\boldsymbol{I}_T - 2\operatorname{tr}\boldsymbol{R} + \operatorname{tr}\boldsymbol{R}^2 + 2\alpha_*(\operatorname{tr}\boldsymbol{Z}-\operatorname{tr}\boldsymbol{E}) + a_*{}^2\operatorname{tr}\boldsymbol{X}]/T \\
&= \frac{1}{\rho^2}[\operatorname{tr}\boldsymbol{I}_T/T - 2\operatorname{tr}\boldsymbol{R}/T + \operatorname{tr}\boldsymbol{R}^2/T + O(T^{-1})] \\
&= \frac{1}{\rho^2}[1 - 2 + (1+\rho^2)/(1-\rho^2)] + O(T^{-1}) \\
&= \frac{2\rho^2}{\rho^2(1-\rho^2)} + O(T^{-1}) = \frac{2}{1-\rho^2} + O(T^{-1}).
\end{aligned} \tag{A.199}
$$

By using (A.196) and (A.198) we have

$$
\begin{aligned}
(\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1})^3 &= \frac{1}{\rho^2}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - 2\boldsymbol{R} + \boldsymbol{R}^2 + \alpha_*(\boldsymbol{ER}+\boldsymbol{RE}-2\boldsymbol{E}) + a_*{}^2\boldsymbol{E}^2]\boldsymbol{\Sigma}^{1/2}\frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - \boldsymbol{R} - \alpha_*\boldsymbol{E}]\boldsymbol{\Sigma}^{1/2} \\
&= \frac{1}{\rho^3}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - 2\boldsymbol{R} + \boldsymbol{R}^2 + \alpha_*(\boldsymbol{ER}+\boldsymbol{RE}-2\boldsymbol{E}) + a_*{}^2\boldsymbol{E}^2][\boldsymbol{I}_T - \boldsymbol{R} - \alpha_*\boldsymbol{E}]\boldsymbol{\Sigma}^{1/2} \\
&= \frac{1}{\rho^3}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - 3\boldsymbol{R} + 3\boldsymbol{R}^2 - \boldsymbol{R}^3 + \alpha_*(3\boldsymbol{ER}+3\boldsymbol{RE}-3\boldsymbol{E}-\boldsymbol{RER}-\boldsymbol{ER}^2-\boldsymbol{R}^2\boldsymbol{E}) \\
&\quad -a_*{}^2(\boldsymbol{ERE}+\boldsymbol{RE}^2+\boldsymbol{E}^2\boldsymbol{R}-3\boldsymbol{E}^2) - \alpha_*{}^3\boldsymbol{E}^3]\boldsymbol{\Sigma}^{1/2}.
\end{aligned} \tag{A.200}
$$

From equations (UR.25) and (A.200) we have

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1})^3/T &= \frac{1}{\rho^3}\operatorname{tr}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - 3\boldsymbol{R} + 3\boldsymbol{R}^2 - \boldsymbol{R}^3 + \alpha_*(3\boldsymbol{ER}+3\boldsymbol{RE}-3\boldsymbol{E}-\boldsymbol{RER}-\boldsymbol{ER}^2-\boldsymbol{R}^2\boldsymbol{E}) \\
&\quad -a_*{}^2(\boldsymbol{ERE}+\boldsymbol{RE}^2+\boldsymbol{E}^2\boldsymbol{R}-3\boldsymbol{E}^2) - \alpha_*{}^3\boldsymbol{E}^3]\boldsymbol{\Sigma}^{1/2}/T \\
&= \frac{1}{\rho^3}\operatorname{tr}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - 3\boldsymbol{R} + 3\boldsymbol{R}^2 - \boldsymbol{R}^3 + \alpha_*(3\boldsymbol{ER}+3\boldsymbol{RE}-3\boldsymbol{E}-\boldsymbol{RER}-\boldsymbol{ER}^2-\boldsymbol{R}^2\boldsymbol{E}) \\
&\quad -a_*{}^2(\boldsymbol{ERE}+\boldsymbol{RE}^2+\boldsymbol{E}^2\boldsymbol{R}-3\boldsymbol{E}^2) - \alpha_*{}^3\boldsymbol{E}^3]/T \\
&= \frac{1}{\rho^3}\operatorname{tr}[\boldsymbol{I}_T - 3\boldsymbol{R} + 3\boldsymbol{R}^2 - \boldsymbol{R}^3 + \alpha_*(3\boldsymbol{ER}+3\boldsymbol{RE}-3\boldsymbol{E}-\boldsymbol{RER}-\boldsymbol{ER}^2-\boldsymbol{R}^2\boldsymbol{E}) \\
&\quad -a_*{}^2(\boldsymbol{ERE}+\boldsymbol{RE}^2+\boldsymbol{E}^2\boldsymbol{R}-3\boldsymbol{E}^2) - \alpha_*{}^3\boldsymbol{E}^3]/T \\
&= \frac{1}{\rho^3}[\operatorname{tr}\boldsymbol{I}_T - 3\operatorname{tr}\boldsymbol{R} + 3\operatorname{tr}\boldsymbol{R}^2 - \operatorname{tr}\boldsymbol{R}^3 + \alpha_*(3\operatorname{tr}\boldsymbol{ER}+3\operatorname{tr}\boldsymbol{RE}-3\operatorname{tr}\boldsymbol{E}-\operatorname{tr}\boldsymbol{RER}-\operatorname{tr}\boldsymbol{ER}^2-\operatorname{tr}\boldsymbol{R}^2\boldsymbol{E}) \\
&\quad -a_*{}^2(\operatorname{tr}\boldsymbol{ERE}+\operatorname{tr}\boldsymbol{RE}^2+\operatorname{tr}\boldsymbol{E}^2\boldsymbol{R}-3\operatorname{tr}\boldsymbol{E}^2) - \alpha_*{}^3\operatorname{tr}\boldsymbol{E}^3]/T \\
&= \frac{1}{\rho^3}[\operatorname{tr}\boldsymbol{I}_T - 3\operatorname{tr}\boldsymbol{R} + 3\operatorname{tr}\boldsymbol{R}^2 - \operatorname{tr}\boldsymbol{R}^3 + \alpha_*(6\operatorname{tr}\boldsymbol{RE}-3\operatorname{tr}\boldsymbol{E}-3\operatorname{tr}\boldsymbol{ER}^2) \\
&\quad -a_*{}^2(3\operatorname{tr}\boldsymbol{RE}^2-3\operatorname{tr}\boldsymbol{E}^2) - \alpha_*{}^3\operatorname{tr}\boldsymbol{E}^3]/T
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\rho^3}[\operatorname{tr} \boldsymbol{I}_T - 3 \operatorname{tr} \boldsymbol{R} + 3 \operatorname{tr} \boldsymbol{R}^2 - \operatorname{tr} \boldsymbol{R}^3 + \alpha_*(6 \operatorname{tr} \boldsymbol{Z} - 3 \operatorname{tr} \boldsymbol{E} - 3 \operatorname{tr} \boldsymbol{\Psi}) \\
&\quad -a_*{}^2(3 \operatorname{tr} \boldsymbol{\Theta} - 3 \operatorname{tr} \boldsymbol{X}) - \alpha_*{}^3 \operatorname{tr} \boldsymbol{\Phi}]/T \\
&= \frac{1}{\rho^3}[\operatorname{tr} \boldsymbol{I}_T/T - 3 \operatorname{tr} \boldsymbol{R}/T + 3 \operatorname{tr} \boldsymbol{R}^2/T - \operatorname{tr} \boldsymbol{R}^3/T + O(T^{-1})] \\
&= \frac{1}{\rho^3}[1 - 3 + 3(1+\rho^2)/(1-\rho^2) - (1+\rho^4)/(1-\rho^2)^2] + O(T^{-1}) \\
&= 2\rho^2(2-3\rho^2)/\rho^3(1-\rho^2)^2 + O(T^{-1}) = 2(2-3\rho^2)/\rho(1-\rho^2)^2 + O(T^{-1}). \qquad \text{(A.201)}
\end{aligned}
$$

From (A.189), (A.191)

$$
\begin{aligned}
\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{\Omega}^{-1} &= \boldsymbol{\Sigma}^{-1/2}[2\boldsymbol{I}_T - 2\boldsymbol{\Delta}]\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{R}/\alpha\boldsymbol{\Sigma}^{1/2} \\
&= \boldsymbol{\Sigma}^{-1/2}[2\boldsymbol{I}_T - 2\boldsymbol{\Delta}]\boldsymbol{R}/\alpha\boldsymbol{\Sigma}^{1/2} \\
&= 2\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{R}/\alpha - \boldsymbol{\Delta}\boldsymbol{R}/\alpha]\boldsymbol{\Sigma}^{1/2} \\
&= \frac{2}{\alpha}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{R} - \boldsymbol{E}]\boldsymbol{\Sigma}^{1/2}. \qquad \text{(A.202)}
\end{aligned}
$$

From equations (A.202) and (UR.25) we have

$$
\begin{aligned}
\operatorname{tr} \boldsymbol{\Omega}_{\rho\rho}\boldsymbol{\Omega}^{-1}/T &= \frac{2}{\alpha} \operatorname{tr} \boldsymbol{\Sigma}^{-1/2}[\boldsymbol{R} - \boldsymbol{E}]\boldsymbol{\Sigma}^{1/2}/T \\
&= \frac{2}{\alpha} \operatorname{tr} \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{R} - \boldsymbol{E}]/T \\
&= \frac{2}{\alpha} \operatorname{tr}[\boldsymbol{R} - \boldsymbol{E}] = \frac{2}{\alpha T}[T-2] = \frac{2}{\alpha} - \frac{4}{\alpha T}. \qquad \text{(A.203)}
\end{aligned}
$$

□

Lemma A.30. By following equation (A.192) we know that

$$
\begin{aligned}
\boldsymbol{\Omega}_1 &= \boldsymbol{\Omega}_\rho + \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}, \\
\boldsymbol{\Omega}_2 &= \boldsymbol{\Omega}_\rho + 2\rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Omega}_1 + \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}. \qquad \text{(A.204)}
\end{aligned}
$$

The following results hold:

$$\operatorname{tr} \boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} = 0, \qquad \text{(A.205)}$$

and

$$\operatorname{tr} \boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1} = \frac{2\rho}{\alpha}. \qquad \text{(A.206)}$$

Proof of Lemma A.30. By using (UR.25) and Lemma A.31 we have

$$\operatorname{tr}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} = \operatorname{tr}\frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - \boldsymbol{R}]\boldsymbol{\Sigma}^{1/2} = \frac{1}{\rho}\operatorname{tr}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}_T - \boldsymbol{R}] = \frac{1}{\rho}][T - T] = 0. \tag{A.207}$$

and

$$\begin{aligned}
\operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1} &= \operatorname{tr}[\boldsymbol{\Omega}_1 + \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}]\boldsymbol{\Omega}^{-1} = \operatorname{tr}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \rho\operatorname{tr}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Omega}^{-1} \\
&= \operatorname{tr}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \rho\operatorname{tr}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{R}/\alpha\boldsymbol{\Sigma}^{1/2} = \operatorname{tr}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \frac{\rho}{\alpha}\operatorname{tr}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{R} \\
&= \operatorname{tr}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \frac{\rho}{\alpha}\operatorname{tr}\boldsymbol{\Delta}\boldsymbol{R} = \operatorname{tr}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \frac{\rho}{\alpha}\operatorname{tr}\boldsymbol{E} = 0 + 2\frac{\rho}{\alpha} = \frac{2\rho}{\alpha}.
\end{aligned} \tag{A.208}$$

□

Lemma A.31. We define the quantities

$$\begin{aligned}
a_i &= -\operatorname{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T), \\
a_\rho &= -\operatorname{E}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T), \\
a_{ij} &= \frac{1}{2}\operatorname{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}\boldsymbol{u}/T), \\
a_{\rho\rho} &= \frac{1}{2}\operatorname{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{u}/T), \\
a_{\rho j} &= \operatorname{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\varsigma_j}\boldsymbol{u}/T).
\end{aligned} \tag{A.209}$$

Also, we define the $m \times 1$ vectors

$$\begin{aligned}
\boldsymbol{a} &= [(a_i)\ \ i = \rho, 1, \ldots, m], \\
\boldsymbol{a}_{\rho\varsigma} &= [(a_{\rho l})\ \ l = 1, \ldots, m],
\end{aligned} \tag{A.210}$$

and the $m \times m$ matrix

$$\bar{\boldsymbol{A}} = [(a_{ij})\ \ i, j = \rho, 1, \ldots, m]. \tag{A.211}$$

In addition we define the scalars

$$\begin{aligned}
w_0 &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T - 1), \\
w_i &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T + a_i), \\
w_\rho &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T + a_\rho), \\
w_{ij} &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}\boldsymbol{u}/T - 2a_{ij}), \\
w_{\rho\rho} &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{u}/T - 2a_{\rho\rho}), \\
w_{\rho j} &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\varsigma_j}\boldsymbol{u}/T - a_{\rho j}).
\end{aligned} \tag{A.212}$$

The following results are proved

$$\begin{aligned}
\mathrm{E}(w_0) &= 0,\\
\mathrm{E}(w_i) &= 0,\\
\mathrm{E}(w_\rho) &= 0,\\
\mathrm{E}(w_{ij}) &= 0,\\
\mathrm{E}(w_{\rho\rho}) &= 0,\\
\mathrm{E}(w_{\rho j}) &= 0,\\
\mathrm{E}(w_0^2) &= 2.
\end{aligned} \tag{A.213}$$

Proof of Lemma A.31.

$$\mathrm{E}(w_0) = \mathrm{E}[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T-1)] = \sqrt{T}[\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T)-1] = \sqrt{T}(1-1)=0, \tag{A.214}$$

$$\mathrm{E}(w_i) = \mathrm{E}[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T+a_i)] = \sqrt{T}[\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T)+a_i] = \sqrt{T}(-a_i+a_i)=0, \tag{A.215}$$

$$\mathrm{E}(w_\rho) = \mathrm{E}[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho}\boldsymbol{u}/T+a_\rho)] = \sqrt{T}[\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho}\boldsymbol{u}/T)+a_\rho] = \sqrt{T}(-a_\rho+a_\rho)=0, \tag{A.216}$$

$$\mathrm{E}(w_{ij}) = \mathrm{E}[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}\boldsymbol{u}/T-2a_{ij})] = \sqrt{T}[\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}\boldsymbol{u}/T)-2a_{ij}] = \sqrt{T}(2a_{ij}-2a_{ij})=0, \tag{A.217}$$

$$\mathrm{E}(w_{\rho\rho}) = \mathrm{E}[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{u}/T-2a_{\rho\rho})] = \sqrt{T}[\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{u}/T)-2a_{\rho\rho}] = \sqrt{T}(2a_{\rho\rho}-2a_{\rho\rho})=0, \tag{A.218}$$

$$\mathrm{E}(w_{\rho j}) = \mathrm{E}[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\varsigma_j}\boldsymbol{u}/T-a_{\rho j})] = \sqrt{T}[\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\varsigma_j}\boldsymbol{u}/T)-a_{\rho j}] = \sqrt{T}(a_{\rho j}-a_{\rho j})=0 \tag{A.219}$$

.

$$\begin{aligned}
\mathrm{E}(w_0^2) &= \mathrm{E}[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T-1)\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T-1)]\\
&= \mathrm{E}[\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T-2\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}+T]\\
&= \mathrm{E}[\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T]-2\,\mathrm{E}[\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}]+T\\
&= \frac{1}{T}[\mathrm{tr}\,\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\,\mathrm{tr}\,\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}+2\,\mathrm{tr}\,\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}]-2\,\mathrm{tr}\,\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}+T\\
&= \frac{1}{T}[T^2+2T]-2T+T = T+2-2T+T = 2.
\end{aligned} \tag{A.220}$$

□

Two Discrete Models

Due to the estimation strategy, the model with heteroskedastic and autocorrelated disturbances can be split into two discrete models,one concerning heteroskedastic disturbances and another concerning autoregressive disturbances. The linear regression model with heteroskedastic disturbances and the linear regression model with autocorrelated disturbances are estimated by Generalized Least Squares (GLS). Conventional F and t-testing procedures of any linear hypotheses on the parameters for these model are justified under the implicit assumption that the sample size is large enough to permit inference on the parameters estimates based on the chi-square or normal distributions. However, in finite samples there is a considerable discrepancy between the true and the nominal size of the test, and this may results in erroneous inferences and to incorrect structural specification. Also, the well-known conflict among the classical testing procedures is mainly due to the fact that the Wald, likelihood ratio, and Lagrange multiplier tests have different sizes. Given that the differences between the true and nominal size are large, compared with the differences in power (e.g., Rothenberg, 1983, p. 529), the size correction should eliminate most of the probability of conflict. Thus, once a size correction has been made, little may be lost by using the F (or t) test, even in cases where there exists a second-order more efficient test. In particular, Rothenberg, 1984b, 1988 derived general formulae giving the Edgeworth-corrected critical values for the Wald and t-test statistics based on Edgeworth expansions of their corresponding asymptotic, chi-square and normal distributions, respectively. This is done for a wide class of regression models used in practice. Instead of using the asymptotic form of the tests, Magdalinos and Symeonides, 1995, 1996 recommended to use the degrees of freedom adjusted forms of the above statistics and derived expansions in terms of the F and t distributions, respectively.(Symeonides et al., 2007).

Linear Model with Heteroskedastic Disturbances

The Linear Model with Heteroskedastic Disturbances is

$$\boldsymbol{y}_H = \boldsymbol{X}_H\boldsymbol{\beta} + \sigma\boldsymbol{u}_H \tag{A.221}$$

where

$$\begin{aligned} \boldsymbol{y}_H &= (1-\rho^2)^{1/2}\boldsymbol{y} \\ \boldsymbol{X}_H &= (1-\rho^2)^{1/2}\boldsymbol{X} \\ \boldsymbol{u}_H &= (1-\rho^2)^{1/2}\boldsymbol{u} \end{aligned} \tag{A.222}$$

We note that $\boldsymbol{x}'_{Ht}$, z'_t are the rows of the $T \times n$ matrices $\boldsymbol{X}_H$, $\boldsymbol{Z}$ respectively. Thus, they can be analytically written as follows

$$\boldsymbol{X}_H = \begin{bmatrix} \boldsymbol{x}'_{H1} \\ \vdots \\ \boldsymbol{x}'_{HT} \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} z'_1 \\ \vdots \\ z'_T \end{bmatrix}, \quad \boldsymbol{X}'_H\boldsymbol{X}_H = (\boldsymbol{x}_{H1}, \dots, \boldsymbol{x}_{HT}) \begin{bmatrix} \boldsymbol{x}'_{H1} \\ \vdots \\ \boldsymbol{x}'_{HT} \end{bmatrix} = \sum_{t=1}^{T} \boldsymbol{x}_{Ht}\boldsymbol{x}'_{Ht}, \quad \boldsymbol{Z}'\boldsymbol{Z} = \sum_{t=1}^{T} z_t z'_t \tag{A.223}$$

Lemma A.32. According to Lemma Δ.1, (see Symeonides, 1991) we have:

The T×T matrices $\boldsymbol{\Omega}_H$ and $\boldsymbol{\Omega}_H^{-1}$ can be written as follows:

and

$$\boldsymbol{\Omega}_H = \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1}^2 & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sigma_T}^2 \end{bmatrix} = \operatorname{diag}(\sigma_t^{-2}) = \operatorname{diag}(\omega_{Ht}) \tag{A.224}$$

and

$$\boldsymbol{\Omega}_H^{-1} = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_T^2 \end{bmatrix}, \tag{A.225}$$

where

$$\sigma_t^2 = z_t'\varsigma, \tag{A.226}$$

and

$$z_t' = (1, z_{t2}, \dots, z_{tm}). \tag{A.227}$$

We define the matrices:

$$\begin{aligned} \boldsymbol{\Omega}_{H\varsigma_i} &= \frac{\partial \boldsymbol{\Omega}_H}{\partial \varsigma_i} = \operatorname{diag}(\omega_{Hti}), \quad \text{where} \quad \omega_{Hti} = \frac{\partial \omega_{Ht}}{\partial \varsigma_i}, \\ \boldsymbol{\Omega}_{H\varsigma_i\varsigma_j} &= \frac{\partial^2 \boldsymbol{\Omega}_H}{\partial \varsigma_i \partial \varsigma_j} = \operatorname{diag}(\omega_{Htij}), \quad \text{where} \quad \omega_{Htij} = \frac{\partial^2 \omega_{Ht}}{\partial \varsigma_i \partial \varsigma_j}. \end{aligned} \tag{A.228}$$

The following will be proven later on

$$\begin{aligned} \omega_{Hti} &= \frac{-z_{ti}}{\sigma_t^4} = -\omega_{Ht}^2 z_{ti}, \quad \boldsymbol{\Omega}_{H\varsigma_i} = -\operatorname{diag}(z_{ti})\boldsymbol{\Omega}_H^2, \\ \omega_{Htij} &= \frac{2z_{ti}z_{tj}}{\sigma_t^6} = 2\omega_{Ht}^3 z_{ti}z_{tj}, \quad \boldsymbol{\Omega}_{H\varsigma_i\varsigma_j} = 2\operatorname{diag}(z_{ti}z_{tj})\boldsymbol{\Omega}_H^3. \end{aligned} \tag{A.229}$$

Proof of Lemma A.32. Using the fact that the matrices $\boldsymbol{\Omega}_H$, $\boldsymbol{\Omega}_{H\varsigma_i}$ and $\boldsymbol{\Omega}_{H\varsigma_i\varsigma_j}$ are diagonals we find the following results:

i.

$$\omega_{Hti} = \frac{\partial \omega_{Ht}}{\partial \varsigma_i} = \frac{\partial (z_t'\varsigma)^{-1}}{\partial \varsigma_i} = \frac{\partial}{\partial \varsigma_i}(1/z_t'\varsigma) = -z_{ti}/(z_t'\varsigma)^2 = -z_{ti}/\sigma_t^4 = -\omega_{Ht}^2 z_{ti} \implies \tag{A.230}$$

$$\begin{aligned} \implies \boldsymbol{\Omega}_{H\varsigma_i} &= \operatorname{diag}(\omega_{Hti}) = \operatorname{diag}(-\omega_{Ht}^2 z_{ti}) = -\operatorname{diag}(\omega_{Ht}^2)\operatorname{diag}(z_{ti}) \\ &= -\boldsymbol{\Omega}_H^2 \operatorname{diag}(z_{ti}) = -\operatorname{diag}(z_{ti})\boldsymbol{\Omega}_H^2. \end{aligned} \tag{A.231}$$

ii.

$$\begin{aligned}\omega_{Htij} &= \frac{\partial^2\omega_{Ht}}{\partial\varsigma_i\partial\varsigma_j} = \frac{\partial}{\partial\varsigma_j}\Big(\frac{\partial\omega_{Ht}}{\partial\varsigma_i}\Big) = \frac{\partial}{\partial\varsigma_j}[-z_{ti}/(z_t'\varsigma)^2] = -z_{ti}\frac{\partial}{\partial\varsigma_j}[1/(z_t'\varsigma)^2] \\ &= -z_{ti}[-2z_{tj}(z_t'\varsigma)/(z_t'\varsigma)^4] = 2z_{ti}z_{tj}/\sigma_t^6 = 2\omega_{Ht}^3 z_{ti}z_{tj} \qquad \text{(A.232)}\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}\boldsymbol{\Omega}_{H\varsigma_i\varsigma_j} &= \operatorname{diag}(\omega_{Htij}) = \operatorname{diag}(2\omega_{Ht}^3 z_{ti}z_{tj}) = 2\operatorname{diag}(\omega_{Ht}^3)\operatorname{diag}(z_{ti}z_{tj}) \\ &= 2\boldsymbol{\Omega}_H^3\operatorname{diag}(z_{ti}z_{tj}) = 2\operatorname{diag}(z_{ti}z_{tj})\boldsymbol{\Omega}_H^3. \qquad \text{(A.233)}\end{aligned}$$

□

Lemma A.33. We define the $T\times 1$ vector $\boldsymbol{v}$ with elements

$$v_t = 2\sigma_t^2\boldsymbol{x}_{Ht}'\boldsymbol{B}_H\boldsymbol{x}_{Ht} - \boldsymbol{x}_{Ht}'\boldsymbol{B}_H\boldsymbol{\Gamma}_H\boldsymbol{B}_H\boldsymbol{x}_{Ht}, \qquad \text{(A.234)}$$

and the $T\times 1$ vectors $\bar{\boldsymbol{u}}$,$\boldsymbol{\varepsilon}$ and $\bar{\boldsymbol{\varepsilon}}$ with elements

$$\begin{aligned}\bar{u}_t &= u_{Ht}^2 - \sigma_t^2, \\ \varepsilon_t &= 2u_{Ht}e_t - \tau e_t^2, \quad e_t = \boldsymbol{x}_{Ht}'\boldsymbol{B}_H\boldsymbol{X}_H'\boldsymbol{u}_H/\sqrt{T}, \qquad \text{(A.235)} \\ \bar{\varepsilon}_t &= 2u_{Ht}\bar{e}_t - \tau\bar{e}_t^2, \quad \bar{e}_t = \boldsymbol{x}_{Ht}'\boldsymbol{G}_H\boldsymbol{X}_H'\boldsymbol{\Omega}_H\boldsymbol{u}_H/\sqrt{T},\end{aligned}$$

respectively. The following will be proven:

$$\begin{aligned}\mathrm{E}(\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}') &= 2\boldsymbol{\Omega}_H^{-2}, \\ \mathrm{E}(\varepsilon_t) = v_t/\sqrt{T}, &\quad \mathrm{E}(\boldsymbol{\varepsilon}) = \boldsymbol{v}/\sqrt{T}, \qquad \text{(A.236)} \\ \mathrm{E}(\bar{\varepsilon}_t) = \boldsymbol{x}_{Ht}'\boldsymbol{G}_H\boldsymbol{x}_{Ht}/\sqrt{T}, &\quad \mathrm{E}(\bar{\boldsymbol{\varepsilon}}) = \boldsymbol{X}\boldsymbol{G}\boldsymbol{x}_{Ht}/\sqrt{T},\end{aligned}$$

(see Symeonides, 1991, Lemma Δ.3)

Proof of Lemma A.33. From the definition of the Linear Model with Heteroskedastic Disturbances we know that $\boldsymbol{u}_H \sim \mathrm{N}(0,\boldsymbol{\Omega}_H^{-1})$. By using (A.224), (A.226) and (A.227) we find

$$\boldsymbol{\Omega}_H^{-1} = \begin{bmatrix}\sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_T^2\end{bmatrix} = [(\delta_{ts}\sigma_t^2)_{t,s=1,\dots,T}] \qquad \text{(A.237)}$$

and

$$\boldsymbol{\Omega}_H^{-2} = \begin{bmatrix} \sigma_1^4 & 0 & \dots & 0 \\ 0 & \sigma_2^4 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_T^4 \end{bmatrix} = [(\delta_{ts}\sigma_t^4)_{t,s=1,\dots,T}] \tag{A.238}$$

where δ_{ts} is Kronecker's delta. Therefore, for the $T \times 1$ vector $\boldsymbol{u}_H$

$$\begin{aligned} \boldsymbol{u}_H &= [(u_{Ht})\ t = 1, \dots, T], \\ \mathrm{E}(\boldsymbol{u}_H) = 0, &\quad \mathrm{E}(\boldsymbol{u}_H\boldsymbol{u}_H') = [(\delta_{ts}\sigma_t^2)_{t,s=1,\dots,T}] \\ \mathrm{E}(u_{Ht}^2) = \sigma_t^2, &\quad \mathrm{E}(u_{Ht}u_{Hs}) = \delta_{ts}{\sigma_t}^2 = \delta_{ts}\,\mathrm{E}(u_{Ht}^2) \end{aligned} \tag{A.239}$$

$$u_{Ht} \sim \mathrm{N}(0, \sigma_t^2) \tag{A.240}$$

We define the variable

$$\psi_{Ht} = u_{Ht}/\sigma_t \tag{A.241}$$

for which apply

$$\psi_{Ht} = u_{Ht}/\sigma_t \sim \mathrm{N}(0,1), \quad \psi_{Ht}^2 = u_{Ht}^2/\sigma_t^2 \sim \chi_1^2 \tag{A.242}$$

where χ_1^2 is chi-square distribution with 1 degree of freedom. From equation (A.242) we have that

$$\mathrm{E}(\psi_{Ht}^2) = \mathrm{E}(u_{Ht}^2/\sigma_t^2) = 1, \quad \mathrm{E}[(\psi_{Ht}^2 - 1)^2] = \mathrm{E}[(u_{Ht}^2/\sigma_t^2 - 1)^2] = 2, \tag{A.243}$$

□

Lemma A.34. For $\boldsymbol{\Omega}_H^2$ and $\hat{\boldsymbol{\Omega}}_H^2$ holds that

$$\hat{\boldsymbol{\Omega}}_H^2 = \boldsymbol{\Omega}_H^2 + 2\tau \sum_{i=1}^{m} \boldsymbol{\Omega}_H \boldsymbol{\Omega}_{H\varsigma_i} d_{1\varsigma_i} + \omega(\tau^2), \tag{A.244}$$

where $\hat{\boldsymbol{\Omega}}_H^2$ is an estimator of matrix $\boldsymbol{\Omega}_H^2$ and $d_{1\varsigma_i}$ is the i-element of $\boldsymbol{d}_{1\varsigma}'$ vector, which is a sub-vector of $d_1 = (\sigma_0, \rho_1, \boldsymbol{d}_{1\varsigma}')$.

Proof of Lemma A.34. Using equations (1.13), (1.14), (A.223) and (A.224) we find that

$$\boldsymbol{\Omega}_H^2 = \mathrm{diag}(\omega_{Ht}^2), \;\; \omega_{Ht} = (z_t'\varsigma)^{-1}. \tag{A.245}$$

Therefore, using Lemma A.32 we find that the derivative of $\boldsymbol{\Omega}_H^2$ with respect to the elements ς_i is

$$\begin{aligned}\frac{\partial \boldsymbol{\Omega}_H^2}{\partial \boldsymbol{\varsigma}_i} &= \operatorname{diag}(\frac{\partial \omega_{_Ht}^2}{\partial \boldsymbol{\varsigma}_i}) = \operatorname{diag}(2\omega_{_Ht}\frac{\partial \omega_{_Ht}}{\partial \boldsymbol{\varsigma}_i}) \\ &= \operatorname{diag}(2\omega_{_Ht}\omega_{_Hti}) = \operatorname{diag}(\omega_{_Ht})\operatorname{diag}(\omega_{_Hti}) = 2\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{_H\varsigma_i}. \end{aligned} \quad \text{(A.246)}$$

Doing Taylor expansion of $\hat{\boldsymbol{\Omega}}^2$ around $\boldsymbol{\Omega}_H^2$ we have

$$\begin{aligned}\hat{\boldsymbol{\Omega}}_H^2 &= \boldsymbol{\Omega}_H^2 + \sum_{i=1}^m \frac{\partial \boldsymbol{\Omega}_H^2}{\partial \varsigma_i}(\hat{\varsigma}_i - \varsigma_i) + \ldots = \boldsymbol{\Omega}_H^2 + \sum_{i=1}^m \frac{\partial \boldsymbol{\Omega}_H^2}{\partial \varsigma_i}\tau\frac{(\hat{\varsigma}_i - \varsigma_i)}{\tau} + \ldots \\ &= \boldsymbol{\Omega}_H^2 + \tau\sum_{i=1}^m \frac{\partial \boldsymbol{\Omega}_H^2}{\varsigma_i}\delta_{\varsigma_i} + \omega(\tau^2), \end{aligned} \quad \text{(A.247)}$$

where $\delta_{\varsigma_i} = \frac{(\hat{\varsigma}_i - \varsigma_i)}{\tau}$. Letting $\sigma = 1$ we have that for the $\boldsymbol{\delta}$ vector applies that

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_0 \\ \delta_\rho \\ [(\delta_{\varsigma_i})_{i=1,\ldots,m}] \end{bmatrix} \quad \text{(A.248)}$$

The δ_{ς_i} admits a stochastic expansion of the form:

$$\delta_{\varsigma_i} = d_{1\varsigma_i} - \tau d_{2\varsigma_i} + \omega(\tau^2) \quad \text{(A.249)}$$

Using (A.246), (A.247) and (A.249)

$$\begin{aligned}\hat{\boldsymbol{\Omega}}_H^2 &= \boldsymbol{\Omega}_H^2 + \tau\sum_{i=1}^m 2\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{_H\varsigma_i}(d_{1\varsigma_i} - \tau d_{2\varsigma_i} + \omega(\tau^2)) + \omega(\tau^2) \\ &= \boldsymbol{\Omega}_H^2 + 2\tau\sum_{i=1}^m \boldsymbol{\Omega}_H\boldsymbol{\Omega}_{_H\varsigma_i}d_{1\varsigma_i} + \omega(\tau^2). \end{aligned} \quad \text{(A.250)}$$

□

The Linear Model with Autocorrelated Disturbances

According to Appendix Γ (Symeonides, 1991) we have:

The Linear Model with Autocorrelated Disturbances is

$$\boldsymbol{y}_{AR} = \boldsymbol{X}_{AR}\boldsymbol{\beta} + \sigma\boldsymbol{u}_{AR}, \tag{A.251}$$

where

$$\begin{aligned} \boldsymbol{y}_{AR} &= \boldsymbol{\Sigma}^{-1/2}\boldsymbol{y}, \\ \boldsymbol{X}_{AR} &= \boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}, \\ \boldsymbol{u}_{AR} &= \boldsymbol{\Sigma}^{-1/2}\boldsymbol{u}. \end{aligned} \tag{A.252}$$

Lemma A.35. We consider the $T \times T$ matrix

$$\boldsymbol{\Omega}_{AR} = [\boldsymbol{R}/(1-\rho^2)]^{-1} = (1+\rho^2)\boldsymbol{I}_T - \rho\boldsymbol{D} - \rho^2\boldsymbol{\Delta}, \tag{A.253}$$

where $\boldsymbol{I}_T$ is the identity matrix, $\boldsymbol{D}$ is a matrix with elements 1 if $|i-j| = 1$ and 0 elsewhere, and $\boldsymbol{\Delta}$ is a matrix with elements 1 in (1,1)-st and (T,T)-th position and 0 elsewhere.

We know that $\boldsymbol{\Omega}_{AR}^{-1}$ can be written as follows:

$$\boldsymbol{\Omega}_{AR}^{-1} = [\boldsymbol{R}/(1-\rho^2)]. \tag{A.254}$$

Let

$$\begin{aligned} \boldsymbol{\Omega}_{AR\rho} &= \frac{\partial \boldsymbol{\Omega}_{AR}}{\partial \rho} = 2\rho\boldsymbol{I} - \boldsymbol{D} - 2\rho\boldsymbol{\Delta} = \Omega_{AR1} - \rho\boldsymbol{\Delta}, \\ &\text{and} \\ \boldsymbol{\Omega}_{AR\rho\rho} &= \frac{\partial^2 \boldsymbol{\Omega}_{AR}}{\partial \rho^2} = 2\boldsymbol{I} - 2\boldsymbol{\Delta} = 2(\boldsymbol{I} - \boldsymbol{\Delta}), \end{aligned} \tag{A.255}$$

where

$$\begin{aligned} \boldsymbol{\Omega}_{ARi} &= \boldsymbol{\Omega}_{AR\rho} + i\rho\boldsymbol{\Delta} \\ &\text{and} \\ \boldsymbol{\Omega}_{ARii} &= \boldsymbol{\Omega}_{AR\rho\rho} + i\boldsymbol{\Delta}. \end{aligned} \tag{A.256}$$

Then, the following results apply

$$\begin{aligned} \operatorname{tr} \boldsymbol{\Omega}_{AR\rho}{\boldsymbol{\Omega}_{AR}}^{-1}/T &= O(T^{-1}),\ \operatorname{tr}(\boldsymbol{\Omega}_{AR\rho}{\boldsymbol{\Omega}_{AR}}^{-1})^2/T = 2/(1-\rho^2) + O(T^{-1}), \\ \operatorname{tr}(\boldsymbol{\Omega}_{AR\rho}{\boldsymbol{\Omega}_{AR}}^{-1})^3/T &= 2(2-3\rho^2)/\rho(1-\rho^2)^2 + O(T^{-1}). \end{aligned} \tag{A.257}$$

Proof of Lemma A.35. This equation's proof is morphologically tautological with Lemma's Γ.3 proof, (Symeonides, 1991, (App.Γ)) if matrix $\boldsymbol{\Omega}$ replaced by matrix $\boldsymbol{\Omega}_{AR}$. □

Estimator of σ

Since, estimators $\hat{\varsigma}$ and $\hat{\rho}$ have been calculated we can find estimator $\hat{\boldsymbol{\Omega}}$ of the $T \times T$ matrix $\boldsymbol{\Omega}$ as follows:

$$\begin{aligned} \hat{\boldsymbol{\Omega}}^{-1} &= \hat{\boldsymbol{\Sigma}}^{1/2}[\hat{\boldsymbol{R}}/(1-\hat{\rho}^2)]\hat{\boldsymbol{\Sigma}}^{1/2} \implies \\ \hat{\boldsymbol{\Omega}} &= \hat{\boldsymbol{\Sigma}}^{-1/2}[(1-\hat{\rho}^2)\boldsymbol{I}_T - \hat{\rho}\boldsymbol{D} - \hat{\rho}^2\boldsymbol{\Delta}]\hat{\boldsymbol{\Sigma}}^{-1/2}. \end{aligned} \tag{A.258}$$

Having calculated $\hat{\boldsymbol{\Omega}}$ we can find the feasible GLS estimators of $\boldsymbol{\beta}$ and σ as follows:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{y} \tag{A.259}$$

and

$$\hat{\sigma} = [(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'\hat{\boldsymbol{\Omega}}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})/(T-n)]^{1/2}. \tag{A.260}$$

Let

$$\boldsymbol{\Omega} = \boldsymbol{P}'\boldsymbol{P} \text{ and } \sigma = 1. \tag{A.261}$$

Equation (1.1) can be transformed as follows:

$$\boldsymbol{P}\boldsymbol{y} = \boldsymbol{P}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{P}\boldsymbol{u} \tag{A.262}$$

Since $\boldsymbol{\Omega}$ is unknown we must use $\hat{\boldsymbol{\Omega}}$ instead of $\boldsymbol{\Omega}$ and by letting $\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{P}}'\hat{\boldsymbol{P}}$ we can write the transformed equation (A.262) as follows

$$\hat{\boldsymbol{P}}\boldsymbol{y} = \hat{\boldsymbol{P}}\boldsymbol{X}\boldsymbol{\beta} + \hat{\boldsymbol{P}}\boldsymbol{u}. \tag{A.263}$$

Let $\hat{\hat{\boldsymbol{\beta}}}$ be the feasible GLS estimator of $\boldsymbol{\beta}$ and $\hat{\boldsymbol{u}}$ the GLS residuals of (A.263).

From Lemma UR.4 we have

$$\hat{\hat{\boldsymbol{\beta}}} = \boldsymbol{\beta} + (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{u} \tag{A.264}$$

and

$$\hat{\hat{\boldsymbol{\beta}}} - \boldsymbol{\beta} = \tau\boldsymbol{b} + \tau^2\boldsymbol{b}_* = \tau\boldsymbol{b} + \omega(\tau^2), \tag{A.265}$$

where $\boldsymbol{b}$ and $\boldsymbol{b}_*$ have been defined in Lemma UR.4.

We define the $n \times 1$

$$\boldsymbol{\kappa} = \sqrt{T}(\hat{\hat{\boldsymbol{\beta}}} - \boldsymbol{\beta}) = (\hat{\hat{\boldsymbol{\beta}}} - \boldsymbol{\beta})/\tau = \boldsymbol{b} + \omega(\tau). \tag{A.266}$$

Also, combining equations (A.264) and (A.266) we find that

$$\boldsymbol{\kappa} = \sqrt{T}(\hat{\hat{\boldsymbol{\beta}}} - \boldsymbol{\beta}) = \sqrt{T}(\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{u} = (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X}/T)^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{u}/\sqrt{T} \implies \tag{A.267}$$

$$(\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X}/T)\boldsymbol{\kappa} = (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X}/T)(\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X}/T)^{-1}\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{u}/\sqrt{T} = \boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{u}/\sqrt{T}. \tag{A.268}$$

From equations (1.1), (A.263) and (A.266) derives that

$$\begin{aligned}\hat{\boldsymbol{u}} &= \hat{\boldsymbol{P}}\boldsymbol{y} - \hat{\boldsymbol{P}}\boldsymbol{X}\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{P}}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) = \hat{\boldsymbol{P}}(\boldsymbol{u} + \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) = \\ &= \hat{\boldsymbol{P}}[(\boldsymbol{u} - \tau\boldsymbol{X}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})/\tau] = \hat{\boldsymbol{P}}(\boldsymbol{u} - \tau\boldsymbol{X}\boldsymbol{\kappa}).\end{aligned} \tag{A.269}$$

Thus using equations (A.268) and (A.269) we find

$$\begin{aligned}\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}} &= (\boldsymbol{u} - \tau\boldsymbol{X}\boldsymbol{\kappa})'\hat{\boldsymbol{P}}'\hat{\boldsymbol{P}}(\boldsymbol{u} - \tau\boldsymbol{X}\boldsymbol{\kappa}) = (\boldsymbol{u}' - \tau\boldsymbol{\kappa}'\boldsymbol{X}')\hat{\boldsymbol{\Omega}}(\boldsymbol{u} - \tau\boldsymbol{X}\boldsymbol{\kappa}) = \\ &= (\boldsymbol{u}' - \boldsymbol{\kappa}'\boldsymbol{X}'/\sqrt{T})\hat{\boldsymbol{\Omega}}(\boldsymbol{u} - \boldsymbol{X}\boldsymbol{\kappa}/\sqrt{T}) = \boldsymbol{u}'\hat{\boldsymbol{\Omega}}\boldsymbol{u} - 2\boldsymbol{\kappa}'\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{u}/\sqrt{T} + \boldsymbol{\kappa}'(\boldsymbol{X}\hat{\boldsymbol{\Omega}}\boldsymbol{X}/T)\boldsymbol{\kappa} \\ &= \boldsymbol{u}'\hat{\boldsymbol{\Omega}}\boldsymbol{u} - 2\boldsymbol{\kappa}'(\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X}/T)\boldsymbol{\kappa} + \boldsymbol{\kappa}'(\boldsymbol{X}\hat{\boldsymbol{\Omega}}\boldsymbol{X}/T)\boldsymbol{\kappa} \\ &= \boldsymbol{u}'\hat{\boldsymbol{\Omega}}\boldsymbol{u} - \boldsymbol{\kappa}'(\boldsymbol{X}'\hat{\boldsymbol{\Omega}}\boldsymbol{X}/T)\boldsymbol{\kappa} = \boldsymbol{u}'\hat{\boldsymbol{\Omega}}\boldsymbol{u} - \boldsymbol{\kappa}'\hat{\boldsymbol{A}}\boldsymbol{\kappa}.\end{aligned} \tag{A.270}$$

Doing Taylor expansion of $\boldsymbol{u}'\hat{\boldsymbol{\Omega}}\boldsymbol{u}$ around $\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}$ and using Lemma A.31 and equation (1.28) we have

$$\begin{aligned}\boldsymbol{u}'\hat{\boldsymbol{\Omega}}\boldsymbol{u}/T &= \boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T + \sum_{i=1}^{m+1}(\boldsymbol{u}'\frac{\partial\boldsymbol{\Omega}}{\partial\gamma_i}\boldsymbol{u}/T)(\hat{\gamma}_i - \gamma_i) + \frac{1}{2}\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}(\boldsymbol{u}'\frac{\partial^2\boldsymbol{\Omega}}{\partial\gamma_i\partial\gamma_j}\boldsymbol{u}/T)(\hat{\gamma}_i - \gamma_i)(\hat{\gamma}_j - \gamma_j) + \ldots = \\ &= \boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T + \tau(\boldsymbol{u}'\frac{\partial\boldsymbol{\Omega}}{\partial\gamma_{m+1}}\boldsymbol{u}/T)\frac{(\hat{\gamma}_{m+1} - \gamma_{m+1})}{\tau} + \tau\sum_{i=1}^{m}(\boldsymbol{u}'\frac{\partial\boldsymbol{\Omega}}{\partial\gamma_i}\boldsymbol{u}/T)\frac{(\hat{\gamma}_i - \gamma_i)}{\tau} \\ &+ \frac{\tau^2}{2}\Big[(\boldsymbol{u}'\frac{\partial^2\boldsymbol{\Omega}}{\partial\gamma_{m+1}\partial\gamma_{m+1}}\boldsymbol{u}/T)\frac{(\hat{\gamma}_{m+1} - \gamma_{m+1})^2}{\tau^2} + \sum_{i=1}^{m}\sum_{j=1}^{m}(\boldsymbol{u}'\frac{\partial^2\boldsymbol{\Omega}}{\partial\gamma_i\partial\gamma_j}\boldsymbol{u}/T)\frac{(\hat{\gamma}_i - \gamma_i)}{\tau}\frac{(\hat{\gamma}_j - \gamma_j)}{\tau} \\ &+ 2\sum_{j=1}^{m}(\boldsymbol{u}'\frac{\partial^2\boldsymbol{\Omega}}{\partial\gamma_{m+1}\partial\gamma_j}\boldsymbol{u}/T)\frac{(\hat{\gamma}_{m+1} - \gamma_{m+1})}{\tau}\frac{(\hat{\gamma}_j - \gamma_j)}{\tau}\Big] + \omega(\tau^3) = \\ &= \boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T + \tau(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T)\delta_\rho + \tau\sum_{i=1}^{m}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T)\delta_{\varsigma_i} + \frac{\tau^2}{2}\Big[(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{u}/T)\delta_\rho^2 \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}\boldsymbol{u}/T)\delta_{\varsigma_i}\delta_{\varsigma_j} + 2\sum_{j=1}^{m}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\varsigma_j}\boldsymbol{u}/T)\delta_\rho\delta_{\varsigma_j}\Big] + \omega(\tau^3) = \\ &= 1 - \tau\sqrt{T} + \tau\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T) + \tau[\tau\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T) + \tau\sqrt{T}a_\rho - a_\rho]\delta_\rho \\ &+ \tau[\tau\sqrt{T}\sum_{i=1}^{m}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T) + \tau\sqrt{T}a_i - a_i]\delta_{\varsigma_i} + \frac{\tau^2}{2}[\tau\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{u}/T) - 2\tau\sqrt{T}a_{\rho\rho} + 2a_{\rho\rho}]\delta_\rho^2 \\ &+ \frac{\tau^2}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}[\tau\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}\boldsymbol{u}/T) - 2\tau\sqrt{T}a_{ij} + 2a_{ij}]\delta_{\varsigma_i}\delta_{\varsigma_j} \\ &+ \tau^2\sum_{j=1}^{m}\Big[\tau\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\varsigma_j}\boldsymbol{u}/T) - \tau\sqrt{T}a_{\rho j} + a_{\rho j}\Big]\delta_\rho\delta_{\varsigma_j} + \omega(\tau^3) = \\ &= 1 + \tau\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T - 1) + \tau^2[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T) + a_\rho)]\delta_\rho - \tau a_\rho\delta_\rho \\ &+ \tau^2\sum_{i=1}^{m}\sqrt{T}[(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T) + a_i]\delta_{\varsigma_i} - \tau\sum_{i=1}^{m}a_i\delta_{\varsigma_i}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\tau^3}{2}\sqrt{T}[(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{u}/T)-2a_{\rho\rho})]\delta_\rho{}^2+\tau^2 a_{\rho\rho}\delta_\rho{}^2\\
&+\frac{\tau^3}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sqrt{T}[(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i\varsigma_j}\boldsymbol{u}/T)-2a_{ij}]\delta_{\varsigma_i}\delta_{\varsigma_j}+\tau^2\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}\delta_{\varsigma_i}\delta_{\varsigma_j}\\
&+\tau^3\sum_{j=1}^{m}\sqrt{T}[(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\varsigma_j}\boldsymbol{u}/T)-a_{\rho j}]\delta_\rho\delta_{\varsigma_j}+\tau^2\sum_{j=1}^{m}a_{\rho j}\delta_\rho\delta_{\varsigma_j}+\omega(\tau^3)\\
=\;& 1+\tau[w_0-a_\rho\delta_\rho-\sum_{i=1}^{m}a_i\delta_{\varsigma_i}]\\
&+\tau^2[w_\rho\delta_\rho+\sum_{i=1}^{m}w_i\delta_{\varsigma_i}+\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}\delta_{\varsigma_i}\delta_{\varsigma_j}+a_{\rho\rho}\delta_\rho{}^2+\sum_{i=1}^{m}\sum_{j=1}^{m}a_{\rho j}\delta_\rho\delta_{\varsigma_j}]+\omega(\tau^3)\\
=\;& 1+\tau[w_0-a_\rho\delta_\rho-\boldsymbol{a}'\boldsymbol{\delta}_\varsigma]\\
&+\tau^2[w_\rho\delta_\rho+\boldsymbol{w}'\boldsymbol{\delta}_\varsigma+\boldsymbol{\delta}_\varsigma'\bar{\boldsymbol{A}}\boldsymbol{\delta}_\varsigma+a_{\rho\rho}\delta_\rho{}^2+\delta_\rho\boldsymbol{a}_{\rho\varsigma}'\boldsymbol{\delta}_\varsigma]+\omega(\tau^3)
\end{aligned}
\tag{A.271}
$$

By using equation (1.30) we have that

$$
\boldsymbol{\delta}_\varsigma=\boldsymbol{d}_{1\varsigma}-\tau\boldsymbol{d}_{2\varsigma}+\omega(\tau^2)
\tag{A.272}
$$

and

$$
\delta_\rho=\rho_1+\tau\rho_2+\omega(\tau^2)
\tag{A.273}
$$

substituting equations (A.272) and (A.273) in the equation (A.271) we have

$$
\begin{aligned}
\boldsymbol{u}'\hat{\boldsymbol{\Omega}}\boldsymbol{u}/T \;=\;& 1+\tau[w_0-a_\rho(\rho_1+\tau\rho_2+\omega(\tau^2))-\boldsymbol{a}'(\boldsymbol{d}_{1\varsigma}-\tau\boldsymbol{d}_{2\varsigma}+\omega(\tau^2))]\\
&+\tau^2[w_\rho(\rho_1+\tau\rho_2+\omega(\tau^2))+\boldsymbol{w}'(\boldsymbol{d}_{1\varsigma}-\tau\boldsymbol{d}_{2\varsigma}+\omega(\tau^2))\\
&+(\boldsymbol{d}_{1\varsigma}-\tau\boldsymbol{d}_{2\varsigma}+\omega(\tau^2))'\bar{\boldsymbol{A}}(\boldsymbol{d}_{1\varsigma}-\tau\boldsymbol{d}_{2\varsigma}+\omega(\tau^2))+a_{\rho\rho}(\rho_1+\tau\rho_2+\omega(\tau^2))^2\\
&+(\rho_1+\tau\rho_2+\omega(\tau^2))\boldsymbol{a}_{\rho\varsigma}'(\boldsymbol{d}_{1\varsigma}-\tau\boldsymbol{d}_{2\varsigma}+\omega(\tau^2))]+\omega(\tau^3)\\
=\;& 1+\tau[w_0-a_\rho\rho_1-\boldsymbol{a}'\boldsymbol{d}_{1\varsigma}]+\tau^2[w_\rho\rho_1+\boldsymbol{w}'\boldsymbol{d}_{1\varsigma}-a_\rho\rho_2+\boldsymbol{a}'\boldsymbol{d}_{2\varsigma}\\
&+\boldsymbol{d}_{1\varsigma}'\bar{\boldsymbol{A}}\boldsymbol{d}_{1\varsigma}+a_{\rho\rho}\rho_1{}^2+\rho_1\boldsymbol{a}_{\rho\varsigma}'\boldsymbol{d}_{1\varsigma}]+\omega(\tau^3).
\end{aligned}
\tag{A.274}
$$

Using Lemma UR.4 and equation (A.266) we get

$$
\begin{aligned}
\boldsymbol{\kappa}'\hat{\boldsymbol{A}}\boldsymbol{\kappa} &= (\boldsymbol{b}+\omega(\tau))'(\boldsymbol{A}+\omega(\tau))(\boldsymbol{b}+\omega(\tau))=\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b}+\omega(\tau)\implies\\
\boldsymbol{\kappa}'\hat{\boldsymbol{A}}\boldsymbol{\kappa}/T &= \boldsymbol{b}'\boldsymbol{A}\boldsymbol{b}/T+\omega(\tau^3).
\end{aligned}
\tag{A.275}
$$

Using equations (A.270), (A.274) and (A.275) we find

$$
\begin{aligned}
\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}}/T \;=\;& \boldsymbol{u}'\hat{\boldsymbol{\Omega}}\boldsymbol{u}/T-\boldsymbol{\kappa}'\hat{\boldsymbol{A}}\boldsymbol{\kappa}/T=\\
=\;& 1+\tau[w_0-a_\rho\rho_1-\boldsymbol{a}'\boldsymbol{d}_{1\varsigma}]+\tau^2[w_\rho\rho_1+\boldsymbol{w}'\boldsymbol{d}_{1\varsigma}-a_\rho\rho_2+\boldsymbol{a}'\boldsymbol{d}_{2\varsigma}\\
&+\boldsymbol{d}_{1\varsigma}'\bar{\boldsymbol{A}}\boldsymbol{d}_{1\varsigma}+a_{\rho\rho}\rho_1{}^2+\rho_1\boldsymbol{a}_{\rho\varsigma}'\boldsymbol{d}_{1\varsigma}-\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b}]+\omega(\tau^3)
\end{aligned}
\tag{A.276}
$$

Also, from the equation (A.269), the definitions of model (1.1) and since $\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{P}}'\hat{\boldsymbol{P}}$ we get

$$\begin{aligned} \hat{\sigma}^2 &= \hat{\boldsymbol{u}}'\hat{\boldsymbol{u}}/(T-n) \implies (T-n)\hat{\sigma}^2 = \hat{\boldsymbol{u}}'\hat{\boldsymbol{u}} \implies \\ \hat{\boldsymbol{u}}'\hat{\boldsymbol{u}}/T &= \frac{(T-n)}{T}\hat{\sigma}^2 = \hat{\sigma}^2 - \hat{\sigma}^2 n\tau^2 \end{aligned} \tag{A.277}$$

and

$$\begin{aligned} \hat{\sigma}^2 &= 1 + \omega(\tau) \implies \\ \hat{\sigma}^2 n\tau^2 &= (1+\omega(\tau))n\tau^2 = n\tau^2 + \omega(\tau^3) \end{aligned} \tag{A.278}$$

Using equation (A.276) we have

$$\begin{aligned} \hat{\sigma}^2 &= \hat{\boldsymbol{u}}'\hat{\boldsymbol{u}}/T + n\tau^2 + \omega(\tau^3) \\ &= 1 + \tau[w_0 - a_\rho\rho_1 - \boldsymbol{a}'\boldsymbol{d}_{1\varsigma}] + \tau^2[w_\rho\rho_1 + \boldsymbol{w}'\boldsymbol{d}_{1\varsigma} - a_\rho\rho_2 + \boldsymbol{a}'\boldsymbol{d}_{2\varsigma} \\ &\quad + \boldsymbol{d}'_{1\varsigma}\bar{\boldsymbol{A}}\boldsymbol{d}_{1\varsigma} + a_{\rho\rho}{\rho_1}^2 + \rho_1\boldsymbol{a}'_{\rho\varsigma}\boldsymbol{d}_{1\varsigma} - \boldsymbol{b}'\boldsymbol{A}\boldsymbol{b} + n] + \omega(\tau^3). \end{aligned} \tag{A.279}$$

Using equations (1.28) and (A.279) we have

$$\begin{aligned} \delta_0 &= \frac{\hat{\sigma}^2 - 1}{\tau} = [w_0 - a_\rho\rho_1 - \boldsymbol{a}'\boldsymbol{d}_{1\varsigma}] + \tau[w_\rho\rho_1 + \boldsymbol{w}'\boldsymbol{d}_{1\varsigma} - a_\rho\rho_2 + \boldsymbol{a}'\boldsymbol{d}_{2\varsigma} \\ &\quad + \boldsymbol{d}'_{1\varsigma}\bar{\boldsymbol{A}}\boldsymbol{d}_{1\varsigma} + a_{\rho\rho}{\rho_1}^2 + \rho_1\boldsymbol{a}'_{\rho\varsigma}\boldsymbol{d}_{1\varsigma} - \boldsymbol{b}'\boldsymbol{A}\boldsymbol{b} + n] + \omega(\tau^2) \\ &= \sigma_0 + \tau\sigma_1 + \omega(\tau^2), \end{aligned} \tag{A.280}$$

where

$$\begin{aligned} \sigma_0 &= w_0 - a_\rho\rho_1 - \boldsymbol{a}'\boldsymbol{d}_{1\varsigma} \\ &\text{and} \\ \sigma_1 &= w_\rho\rho_1 + \boldsymbol{w}'\boldsymbol{d}_{1\varsigma} - a_\rho\rho_2 + \boldsymbol{a}'\boldsymbol{d}_{2\varsigma} \\ &\quad + \boldsymbol{d}'_{1\varsigma}\bar{\boldsymbol{A}}\boldsymbol{d}_{1\varsigma} + a_{\rho\rho}{\rho_1}^2 + \rho_1\boldsymbol{a}'_{\rho\varsigma}\boldsymbol{d}_{1\varsigma} - \boldsymbol{b}'\boldsymbol{A}\boldsymbol{b} + n. \end{aligned} \tag{A.281}$$

Estimators of ρ

OLS estimator of ρ

Lemma A.36. Following Symeonides, 1991 the OLS estimator $\tilde{\rho}_{LS}$ of ρ admits a stochastic expansion of the form:

$$\tilde{\rho}_{LS} = \rho + \tau(\rho_1 + \tau\rho_2) + \omega(\tau^3), \tag{A.282}$$

where

$$\begin{aligned} \rho_1 &= -\alpha \boldsymbol{u}'_{AR} \boldsymbol{\Omega}_{_{AR}2} \boldsymbol{u}_{AR}/2\sqrt{T} \\ &\text{and} \\ \rho_2 &= -(\alpha \boldsymbol{u}'_{AR} \bar{\boldsymbol{P}}_{X_{AR}} \boldsymbol{\Omega}_{_{AR}2} \bar{\boldsymbol{P}}_{X_{AR}} \boldsymbol{u}_{AR}/2 - \alpha^2 \boldsymbol{u}'_{AR} \boldsymbol{u}_{AR} \boldsymbol{u}'_{AR} \boldsymbol{\Omega}_{_{AR}2} \boldsymbol{u}_{AR}/2T). \end{aligned} \tag{A.283}$$

Proof of Lemma A.36.

$$\tilde{\rho}_{LS} = \sum_{t=2}^{T} \tilde{u}_{_{AR}t} \tilde{u}_{_{AR}t-1} \Big/ \sum_{t=1}^{T} \tilde{u}^2_{_{AR}t} = \sum_{t=1}^{T} \tilde{u}_{_{AR}t} \tilde{u}_{_{AR}t+1} \Big/ \sum_{t=1}^{T} \tilde{u}^2_{_{AR}t} = N/\mathcal{D}, \tag{A.284}$$

where $\tilde{u}_{_{AR}t}$ are the OLS residuals of (A.251) equation. From (A.284) it follows that

$$\begin{aligned} N &= \frac{1}{2} \tilde{\boldsymbol{u}}'_{AR} \boldsymbol{D} \tilde{\boldsymbol{u}}_{AR}/T\sigma^2\sigma^2_{u_{AR}} \\ &\text{and} \\ \mathcal{D} &= \tilde{\boldsymbol{u}}'_{AR} \tilde{\boldsymbol{u}}_{AR}/T\sigma^2\sigma^2_{u_{AR}}. \end{aligned} \tag{A.285}$$

Let $\tilde{\boldsymbol{\beta}}$ be the OLS estimator of β. Since

$$\boldsymbol{y}_{AR} = \boldsymbol{X}_{AR}\boldsymbol{\beta} + \sigma \boldsymbol{u}_{AR}, \tag{A.286}$$

we have that

$$\tilde{\boldsymbol{u}}_{AR} = \boldsymbol{y}_{AR} - \boldsymbol{X}_{AR}\tilde{\boldsymbol{\beta}} = \sigma\boldsymbol{u}_{AR} + \boldsymbol{X}_{AR}\boldsymbol{\beta} - \boldsymbol{X}_{AR}\tilde{\boldsymbol{\beta}} = \sigma[\boldsymbol{u}_{AR} - \tau\sqrt{T}\boldsymbol{X}_{AR}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sigma] = \sigma(\boldsymbol{u}_{AR} - \tau\boldsymbol{X}_{AR}m), \tag{A.287}$$

where

$$\begin{aligned} m &= \sqrt{T}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sigma = \sqrt{T}[(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{y}_{AR} - \boldsymbol{\beta}]/\sigma \\ &= \sqrt{T}[(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}(\boldsymbol{X}_{AR}\boldsymbol{\beta} + \sigma\boldsymbol{u}_{AR}) - \boldsymbol{\beta}]/\sigma \\ &= \sqrt{T}(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{u}_{AR} = (\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)^{-1}\boldsymbol{X}'_{AR}\boldsymbol{u}_{AR}/\sqrt{T}. \end{aligned} \tag{A.288}$$

By using (A.288) we have

$$\boldsymbol{X}'_{AR}\boldsymbol{u}_{AR}/\sqrt{T} = (\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)m. \tag{A.289}$$

From (A.287) we have

$$\begin{aligned}\tilde{\boldsymbol{u}}'_{AR}\boldsymbol{D}\tilde{\boldsymbol{u}}_{AR}/\sigma^2 &= \sigma^2(\boldsymbol{u}_{AR}-\tau\boldsymbol{X}_{AR}m)'\boldsymbol{D}(\boldsymbol{u}_{AR}-\tau\boldsymbol{X}_{AR}m)/\sigma^2\\ &= \boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}-2m'(\boldsymbol{X}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}/\sqrt{T})+m'(\boldsymbol{X}'_{AR}\boldsymbol{D}\boldsymbol{X}_{AR}/T)m. \qquad (A.290)\end{aligned}$$

From (A.285), (A.288), and (A.290) we have

$$\begin{aligned}N &= \frac{1}{2}\tilde{\boldsymbol{u}}'_{AR}\boldsymbol{D}\tilde{\boldsymbol{u}}_{AR}/T\sigma^2\sigma^2_{u_{AR}} = (\tilde{\boldsymbol{u}}'_{AR}\boldsymbol{D}\tilde{\boldsymbol{u}}_{AR}/\sigma^2)/2T\sigma^2_{u_{AR}}\\ &= \boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}/2T\sigma^2_{u_{AR}} - 2[\boldsymbol{u}'_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)^{-1}/\sqrt{T}](\boldsymbol{X}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}/\sqrt{T})/2T\sigma^2_{u_{AR}}\\ &\quad +[\boldsymbol{u}'_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)^{-1}/\sqrt{T}](\boldsymbol{X}'_{AR}\boldsymbol{D}\boldsymbol{X}_{AR}/T)[(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)^{-1}\boldsymbol{X}'_{AR}\boldsymbol{u}_{AR}/\sqrt{T}]/2T\sigma^2_{u_{AR}}\\ &= \boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}/2T\sigma^2_{u_{AR}} + \tau^2(\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/2\sigma^2_{u_{AR}} - \boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{u}_{AR}/\sigma^2_{u_{AR}})\\ &= \rho-\rho+\boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}/2T\sigma^2_{u_{AR}} + \tau^2(\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/2\sigma^2_{u_{AR}} - \boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{u}_{AR}/\sigma^2_{u_{AR}})\\ &= \rho+\tau[\sqrt{T}(\boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}/2T\sigma^2_{u_{AR}}-\rho)] + \tau^2(\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/2 - \boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{u}_{AR})/\sigma^2_{u_{AR}}\\ &= \rho+\tau N_1+\tau^2 N_2, \qquad (A.291)\end{aligned}$$

where

$$\begin{aligned}N_1 &= \sqrt{T}(\boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}/2T\sigma^2_{u_{AR}}-\rho) = \sqrt{T}(\sum_{t=1}^{T-1} u_{_{AR}t}u_{_{AR}t+1}/T\sigma^2_{u_{AR}}-\rho)\\ &\text{and} \qquad (A.292)\\ N_2 &= (\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/2 - \boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{u}_{AR})/\sigma^2_{u_{AR}}.\end{aligned}$$

Similarly by using the equations (A.287), (A.288) and (A.289) we have

$$\begin{aligned}\tilde{\boldsymbol{u}}'_{AR}\tilde{\boldsymbol{u}}_{AR}/\sigma^2 &= \sigma^2(\boldsymbol{u}_{AR}-\tau\boldsymbol{X}_{AR}m)'(\boldsymbol{u}_{AR}-\tau\boldsymbol{X}_{AR}m)/\sigma^2 = \boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}-2m'(\boldsymbol{X}'_{AR}\boldsymbol{u}_{AR}/\sqrt{T})+m'(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)m\\ &= \boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}-2m'(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)m+m'(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)m = \boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}-m'(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)m\\ &= \boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}-\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}. \qquad (A.293)\end{aligned}$$

From the equations (A.285) and (A.293) we have

$$\begin{aligned}\mathcal{D} &= \tilde{\boldsymbol{u}}'_{AR}\tilde{\boldsymbol{u}}_{AR}/T\sigma^2\sigma^2_{u_{AR}} = (\tilde{\boldsymbol{u}}'_{AR}\tilde{\boldsymbol{u}}_{AR}/\sigma^2)/T\sigma^2_{u_{AR}}\\ &= \boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}/T\sigma^2_{u_{AR}} - \tau^2\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/\sigma^2_{u_{AR}}\\ &= 1-1+\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}/T\sigma^2_{u_{AR}} - \tau^2\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/\sigma^2_{u_{AR}}\\ &= 1+\tau[\sqrt{T}(\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}/T\sigma^2_{u_{AR}}-1)] - \tau^2\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/\sigma^2_{u_{AR}}\\ &= 1+\tau\mathcal{D}_1-\tau^2\mathcal{D}_2, \qquad (A.294)\end{aligned}$$

where

$$\mathcal{D}_1 = \sqrt{T}(\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}/T\sigma^2_{u_{AR}}-1),\ \mathcal{D}_2 = \boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/\sigma^2_{u_{AR}}. \qquad (A.295)$$

By using Lemma (UR.1) and equation (A.294) we have

$$\begin{aligned}
\mathcal{D} &= 1+\tau\mathcal{D}_1-\tau^2\mathcal{D}_2 \implies \\
\mathcal{D}^{-1} &= [1+\tau\mathcal{D}_1-\tau^2\mathcal{D}_2]^{-1} = 1-\tau(\mathcal{D}_1-\tau\mathcal{D}_2)+\tau^2(\mathcal{D}_1-\tau\mathcal{D}_2)^2+\tau^3\omega(\tau^2) \\
&= 1-\tau\mathcal{D}_1+\tau^2({\mathcal{D}_1}^2+\mathcal{D}_2)+\omega(\tau^3). \qquad \text{(A.296)}
\end{aligned}$$

From the equations (A.284), (A.291) and (A.296) we have

$$\begin{aligned}
\tilde{\rho}_{LS} &= N\mathcal{D}^{-1} = (\rho+\tau N_1+\tau^2N_2)[1-\tau\mathcal{D}_1+\tau^2({\mathcal{D}_1}^2+\mathcal{D}_2)+\omega(\tau^3)] \\
&= \rho-\tau\rho\mathcal{D}_1+\tau^2\rho({\mathcal{D}_1}^2+\mathcal{D}_2)+\tau N_1-\tau^2N_1\mathcal{D}_1+\tau^3N_1({\mathcal{D}_1}^2+\mathcal{D}_2)+\tau^2N_2-\tau^3N_2\mathcal{D}_1 \\
&\quad +\tau^4N_2({\mathcal{D}_1}^2+\mathcal{D}_2)+\omega(\tau^3) \\
&= \rho-\tau(\rho\mathcal{D}_1-N_1)+\tau^2[N_2-N_1\mathcal{D}_1+\rho({\mathcal{D}_1}^2+\mathcal{D}_2)]+\omega(\tau^3) \\
&= \rho+\tau(\rho_1+\tau\rho_2)+\omega(\tau^3), \qquad \text{(A.297)}
\end{aligned}$$

where

$$\begin{aligned}
\rho_1 &= -(\rho\mathcal{D}_1-N_1) \\
&\text{and} \qquad \text{(A.298)} \\
\rho_2 &= N_2-N_1\mathcal{D}_1+\rho({\mathcal{D}_1}^2+\mathcal{D}_2).
\end{aligned}$$

We know that $\boldsymbol{\Omega}_{AR\rho} = \frac{\partial\boldsymbol{\Omega}_{AR}}{\partial\rho}$. By using equations (A.253) and (A.256) we have

$$\boldsymbol{\Omega}_{AR2} = \boldsymbol{\Omega}_{AR\rho}+2\rho\boldsymbol{\Delta} = 2\rho\boldsymbol{I}-\boldsymbol{D}-2\rho\Delta+2\rho\boldsymbol{\Delta} = 2\rho\boldsymbol{I}-\boldsymbol{D}. \qquad \text{(A.299)}$$

We will then express the quantities ρ_1 and ρ_2 as a function of $\boldsymbol{\Omega}_{AR2}$. From the equations (A.292), (A.295), (A.298) and (A.299) we find:

$$\begin{aligned}
\rho_1 &= -(\rho\mathcal{D}_1-N_1) = -[\rho\sqrt{T}(\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}/T\sigma^2_{u_{AR}}-1)-\sqrt{T}(\boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}/2T\sigma^2_{u_{AR}}-\rho)] \\
&= -\sqrt{T}(2\rho\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}-\boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR})/2T\sigma^2_{u_{AR}} = -\boldsymbol{u}'_{AR}(2\rho\boldsymbol{I}-\boldsymbol{D})\boldsymbol{u}_{AR}/2\sqrt{T}\sigma^2_{u_{AR}} \\
&= -\alpha\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR2}\boldsymbol{u}_{AR}/2\sqrt{T}. \qquad \text{(A.300)}
\end{aligned}$$

$$\begin{aligned}
\rho_2 &= N_2-N_1\mathcal{D}_1+\rho({\mathcal{D}_1}^2+\mathcal{D}_2) = N_2-N_1\mathcal{D}_1+\rho{\mathcal{D}_1}^2+\rho\mathcal{D}_2 = N_2+\rho\mathcal{D}_2+\mathcal{D}_1(\rho\mathcal{D}_1-N_1) \\
&= N_2+\rho\mathcal{D}_2-\mathcal{D}_1[-(\rho\mathcal{D}_1-N_1)] = N_2+\rho\mathcal{D}_2-\mathcal{D}_1\rho_1. \qquad \text{(A.301)}
\end{aligned}$$

From the equations (A.291), (A.295) and (A.299) we have

$$\begin{aligned}
2\sigma^2_{u_{AR}}(N_2+\rho\mathcal{D}_2) &= 2\sigma^2_{u_{AR}}[(\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/2-\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{u}_{AR})/\sigma^2_{u_{AR}}+\rho\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}/\sigma^2_{u_{AR}}] \\
&= \boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}-2\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{u}_{AR}+2\rho\boldsymbol{u}'_{AR}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR} \\
&= \boldsymbol{u}'_{AR}(\boldsymbol{I}-\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}})\boldsymbol{D}(\boldsymbol{I}-\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}})\boldsymbol{u}_{AR}-2\boldsymbol{u}'_{AR}(\boldsymbol{I}-\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}})\boldsymbol{D}\boldsymbol{u}_{AR}+2\rho\boldsymbol{u}'_{AR}(\boldsymbol{I}-\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}})\boldsymbol{u}_{AR} \\
&= \boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}+\boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}-2\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{u}_{AR}-2\boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR}+2\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\boldsymbol{u}_{AR} \\
&\quad +2\rho\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}-2\rho\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR} \\
&= \boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{D}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}-2\rho\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}+2\rho\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}-\boldsymbol{u}'_{AR}\boldsymbol{D}\boldsymbol{u}_{AR} \\
&= \boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}(\boldsymbol{D}-2\rho\boldsymbol{I})\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}+\boldsymbol{u}'_{AR}(\boldsymbol{D}-2\rho\boldsymbol{I})\boldsymbol{u}_{AR} \\
&= -\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{u}_{AR}+\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR},
\end{aligned} \tag{A.302}$$

due to matrix $\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}$ being idempotent. From the equations (A.295), (A.300), (A.301) and (A.302) we have

$$\begin{aligned}
2\sigma^2_{u_{AR}}\rho_2 &= 2\sigma^2_{u_{AR}}[(N_2+\rho\mathcal{D}_2)-\mathcal{D}_1\rho_1] \\
&= -\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{u}_{AR}+\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}+2\sigma^2_{u_{AR}}\sqrt{T}(\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}/T\sigma^2_{u_{AR}}-1)\alpha\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}/2\sqrt{T} \\
&= -\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{u}_{AR}+\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}+\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}/T\sigma^2_{u_{AR}}-\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR} \implies \\
\rho_2 &= -\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{u}_{AR}/2\sigma^2_{u_{AR}}+\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}/2T\sigma^2_{u_{AR}} \\
&= -(\alpha\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{u}_{AR}/2-\alpha^2\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}/2T).
\end{aligned} \tag{A.303}$$

□

By using equations (1.28) and (A.297) we find that the sampling error of $\tilde{\rho}_{LS}$:

$$\begin{aligned}
\delta_{\rho}{}^{LS} &= \sqrt{T}(\tilde{\rho}_{LS}-\rho)=(\tilde{\rho}_{LS}-\rho)/\tau=[\rho+\tau(\rho_1+\tau\rho_2)+\omega(\tau^3)-\rho]/\tau \\
&= \rho_1+\tau\rho_2+\omega(\tau^2).
\end{aligned} \tag{A.304}$$

P-W estimator

The Prais-Winston (1954) estimator is

$$\hat{\rho}_{PW}=\tilde{\rho}_{LS}-\tau^2\alpha[\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{X_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{X_{AR}}\boldsymbol{\Sigma}_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{u}_{AR}+(1/2)\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma}_{AR}\boldsymbol{P}_{X_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{X_{AR}}\boldsymbol{\Sigma}_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{u}_{AR}]+\omega(\tau^3), \tag{A.305}$$

where

$$\boldsymbol{\Sigma}_{AR}=\boldsymbol{\Omega}_{AR}{}^{-1}-\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}=[\boldsymbol{I}_T-\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}]\boldsymbol{\Omega}_{AR}{}^{-1}=\boldsymbol{M}\boldsymbol{\Omega}_{AR}{}^{-1}.$$

By using equations (1.28) and (A.305) we find that the sampling error of $\hat{\rho}_{PW}$:

$$\begin{aligned}
\delta_{\rho}{}^{GL} &= \delta_{\rho}{}^{PW} = \sqrt{T}(\hat{\rho}_{PW} - \rho) \\
&= [(\tilde{\rho}_{LS} - \rho) - \tau^2[\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{u}_{AR} + \frac{1}{2}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{u}_{AR}] + \omega(\tau^3)]/\tau \\
&= \delta_{\rho}{}^{LS} - \tau\alpha[\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{u}_{AR} + \frac{1}{2}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{u}_{AR}] + \omega(\tau^2). \qquad (A.306)
\end{aligned}$$

ML estimator

The maximum likelihood (ML) estimator, ρ_{ML}, which satisfies a cubic equation with coefficients defined in terms of the (heteroskedasticity corrected) ML residuals in the (heteroskedasticity corrected) regression model (A.251) (see Beach and MacKinnon, 1978, Magee, 1985) is

$$\hat{\rho}_{ML} = \hat{\rho}_{PW} + \tau^2[\rho\alpha(u_{_{AR}1}{}^2 + u_{_{AR}T}{}^2) - \rho] + \omega(\tau^3). \qquad (A.307)$$

By using equations (1.28) and (A.307) we find that the sampling error of $\hat{\rho}_{ML}$:

$$\begin{aligned}
\delta_{\rho}{}^{ML} &= \sqrt{T}(\hat{\rho}_{ML} - \rho) \\
&= [(\hat{\rho}_{PW} - \rho) + \tau^2[\rho\alpha(u_{_{AR}1}{}^2 + u_{_{AR}T}{}^2) - \rho] + \omega(\tau^3)]/\tau \\
&= \delta_{\rho}{}^{PW} + \tau[\rho\alpha(u_{_{AR}1}{}^2 + u_{_{AR}T}{}^2) - \rho] + \omega(\tau^2). \qquad (A.308)
\end{aligned}$$

DW estimator

The Durbin-Watson (DW) estimator is

$$\hat{\rho}_{DW} = 1 - d/2, \qquad (A.309)$$

where d is the Durbin-Watson statistic. We know that

$$\begin{aligned}
d &= \frac{\sum_{t=2}^{T}(\tilde{u}_{_{AR}t} - \tilde{u}_{_{AR}t-1})^2}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}} = \frac{\sum_{t=2}^{T}(\tilde{u}^2_{_{AR}t} - 2\tilde{u}_{_{AR}t}\tilde{u}_{_{AR}t-1} + \tilde{u}^2_{_{AR}t-1})}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}} \\
&= \frac{\sum_{t=2}^{T}\tilde{u}^2_{_{AR}t} - 2\sum_{t=2}^{T}\tilde{u}_{_{AR}t}\tilde{u}_{_{AR}t-1} + \sum_{t=2}^{T}\tilde{u}^2_{_{AR}t-1}}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}} \\
&= \frac{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t} - \tilde{u}^2_{_{AR}1} - 2\sum_{t=2}^{T}\tilde{u}_{_{AR}t}\tilde{u}_{_{AR}t-1} + \sum_{t=1}^{T}\tilde{u}^2_{_{AR}t} - \tilde{u}^2_{_{AR}T}}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}}, \qquad (A.310)
\end{aligned}$$

wherefore

$$\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t} = (\tilde{u}^2_{_{AR}1} + \tilde{u}^2_{_{AR}2} + \cdots + \tilde{u}^2_{_{AR}T-1}) + \tilde{u}^2_{_{AR}T} = \sum_{t=1}^{T-1}\tilde{u}^2_{_{AR}t} + \tilde{u}^2_{_{AR}T} = \sum_{t=2}^{T}\tilde{u}^2_{_{AR}t-1} + \tilde{u}^2_{_{AR}T}. \qquad (A.311)$$

From equations (A.309) and (A.310) we have that

$$
\begin{aligned}
d &= \frac{2\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t} - (2\sum_{t=2}^{T}\tilde{u}_{_{AR}t}\tilde{u}_{_{AR}t-1} + \tilde{u}^2_{_{AR}1} + \tilde{u}^2_{_{AR}T})}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}} = 2 - \frac{2\sum_{t=2}^{T}\tilde{u}_{_{AR}t}\tilde{u}_{_{AR}t-1} + \tilde{u}^2_{_{AR}1} + \tilde{u}^2_{_{AR}T}}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}} \implies \\
\hat{\rho}_{DW} &= 1 - d/2 = 1 - \left[1 - \frac{\sum_{t=2}^{T}\tilde{u}_{_{AR}t}\tilde{u}_{_{AR}t-1} + (\tilde{u}^2_{_{AR}1} + \tilde{u}^2_{_{AR}T})/2}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}}\right] \\
&= \frac{\sum_{t=2}^{T}\tilde{u}_{_{AR}t}\tilde{u}_{_{AR}t-1}}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}} + \frac{(\tilde{u}^2_{_{AR}1} + \tilde{u}^2_{_{AR}T})/2}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}} = \tilde{\rho}_{LS} + \frac{(\tilde{u}^2_{_{AR}1} + \tilde{u}^2_{_{AR}T})/2T\sigma^2\sigma^2{}_{u_{AR}}}{\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}/T\sigma^2\sigma^2{}_{u_{AR}}} \\
&= \tilde{\rho}_{LS} + \frac{1}{2T}\frac{(\tilde{u}^2_{_{AR}1} + \tilde{u}^2_{_{AR}T})(1/\sigma^2\sigma^2{}_{u_{AR}})}{\sum_{t=1}^{T}(\tilde{u}^2_{_{AR}t}/T)(1/\sigma^2\sigma^2{}_{u_{AR}})} = \tilde{\rho}_{LS} + \frac{1}{2T}\frac{(\tilde{u}^2_{_{AR}1} + \tilde{u}^2_{_{AR}T})/\sigma^2}{\sigma^2{}_{u_{AR}}} + \omega(\tau^3) \\
&= \tilde{\rho}_{LS} + \tau^2\alpha(u^2{}_{_{AR}1} + u^2{}_{_{AR}T})/2 + \omega(\tau^3), \qquad \text{(A.312)}
\end{aligned}
$$

where $\tilde{u}_{_{AR}t}$ is consistent predictor of $\sigma u_{_{AR}t}$ and $\sum_{t=1}^{T}\tilde{u}^2_{_{AR}t}/T$ is a consistent predictor of $\sigma^2\sigma^2{}_{u_{AR}}$ with an error of order $\omega(\tau^3)$.

By using equations (1.28) and (A.312) we find that

$$
\begin{aligned}
\delta_\rho{}^{DW} &= \sqrt{T}(\hat{\rho}_{DW} - \rho) \\
&= [(\tilde{\rho}_{LS} - \rho) + \tau^2\alpha(u^2{}_{_{AR}1} + u^2{}_{_{AR}T})/2 + \omega(\tau^3)]/\tau \\
&= \delta_\rho{}^{LS} + \tau\alpha(u^2{}_{_{AR}1} + u^2{}_{_{AR}T})/2 + \omega(\tau^2). \qquad \text{(A.313)}
\end{aligned}
$$

Estimators of ς

Since,

$$\boldsymbol{y}_H = \boldsymbol{X}_H\boldsymbol{\beta} + \sigma\boldsymbol{u}_H, \qquad \text{(A.314)}$$

let $\tilde{\boldsymbol{u}}_H$ be the vector of OLS residuals, we have that

$$\tilde{\boldsymbol{u}}_H = \boldsymbol{u}_H - \boldsymbol{X}_H(\boldsymbol{X}'_H\boldsymbol{X}_H)^{-1}\boldsymbol{X}'_H\boldsymbol{u}_H \qquad \text{(A.315)}$$

Let $\tilde{u}_{_Ht}$ be the t-th element of vector $\tilde{u}_H$. From equations (A.186), (A.235) and (A.315) we have that

$$
\begin{aligned}
\tilde{u}_{_Ht} &= u_{_Ht} - \boldsymbol{x}'_{_Ht}(\boldsymbol{X}'_H\boldsymbol{X}_H)^{-1}\boldsymbol{X}'_H\boldsymbol{u}_H = u_{_Ht} - \boldsymbol{x}'_{_Ht}(\boldsymbol{X}'_H\boldsymbol{X}_H/T)^{-1}\boldsymbol{X}'_H\boldsymbol{u}_H/T \\
&= u_{_Ht} - \tau\boldsymbol{x}'_{_Ht}\boldsymbol{B}_H\boldsymbol{X}'_H\boldsymbol{u}_H/\sqrt{T} = u_{_Ht} - \tau e_t, \qquad \text{(A.316)}
\end{aligned}
$$

where $e_t = \boldsymbol{x}'_{_Ht}\boldsymbol{B}_H\boldsymbol{X}'_H\boldsymbol{u}_H/\sqrt{T}$.
According to our assumptions we can deduce that the $T \times 1$ vector

$$\boldsymbol{e} = [(e_t)_{t=1,\dots,T}] = O(1). \qquad \text{(A.317)}$$

Thus,

$$\tilde{u}_{Ht}^2 = (u_{Ht} - \tau e_t)^2 = {u_{Ht}}^2 - 2\tau u_{Ht} e_t + \tau^2 {e_t}^2 = {u_{Ht}}^2 - \tau(2u_{Ht} e_t - \tau {e_t}^2) = {u_{Ht}}^2 - \tau \varepsilon_t, \tag{A.318}$$

where $\varepsilon_t = 2u_{Ht} e_t - \tau {e_t}^2$.

Let $\hat{u}_H$ be the $T \times 1$ vector of the GLS residuals of equation (A.314), when the matrix $\boldsymbol{\Omega}_H$ is known. Then,

$$\hat{\boldsymbol{u}}_H = \boldsymbol{u}_H - \boldsymbol{X}_H (\boldsymbol{X}_H' \boldsymbol{\Omega}_H \boldsymbol{X}_H)^{-1} \boldsymbol{X}_H' \boldsymbol{\Omega}_H \boldsymbol{u}_H. \tag{A.319}$$

Also, let $\hat{u}_{Ht}$ be the th-element of vector $\hat{\boldsymbol{u}}_H$. From equations (A.186), (A.235) and (A.319), it follows that

$$\begin{aligned} \hat{u}_{Ht} &= u_{Ht} - \boldsymbol{x}_{Ht}' (\boldsymbol{X}_H' \boldsymbol{\Omega}_H \boldsymbol{X}_H)^{-1} \boldsymbol{X}_H' \boldsymbol{\Omega}_H \boldsymbol{u}_H \\ &= u_{Ht} - \boldsymbol{x}_{Ht}' (\boldsymbol{X}_H' \boldsymbol{\Omega}_H \boldsymbol{X}_H / T)^{-1} \boldsymbol{X}_H' \boldsymbol{\Omega}_H \boldsymbol{u}_H / T \\ &= u_{Ht} - \tau \boldsymbol{x}_{Ht}' \boldsymbol{G}_H \boldsymbol{X}_H' \boldsymbol{\Omega}_H \boldsymbol{u}_H / \sqrt{T} = u_{Ht} - \tau \bar{e}_t, \end{aligned} \tag{A.320}$$

where $\bar{e}_t = \boldsymbol{x}_{Ht}' \boldsymbol{G}_H \boldsymbol{X}_H' \boldsymbol{\Omega}_H \boldsymbol{u}_H / \sqrt{T}$. It is straightforward that the $T \times 1$ vector

$$\bar{\boldsymbol{e}} = [(\bar{e}_t)_{t=1,\dots,T}] = O(1). \tag{A.321}$$

Thus,

$$\hat{u}_{Ht}^2 = (u_{Ht} - \tau \bar{e}_t)^2 = {u_{Ht}}^2 - 2\tau u_{Ht} \bar{e}_t + \tau^2 \bar{e}_t^2 = {u_{Ht}}^2 - \tau(2u_{Ht} \bar{e}_t + \tau \bar{e}_t^2) = {u_{Ht}}^2 - \tau \bar{\varepsilon}_t, \tag{A.322}$$

where $\bar{\varepsilon}_t = 2u_{Ht} \bar{e}_t + \tau \bar{e}_t^2$. The most frequently used estimators of vector $\boldsymbol{\varsigma}$ are:

GQ estimator of ς

$$\hat{\varsigma}_{GQ} = \left[\sum_{t=1}^{T} z_t z_t' \right]^{-1} \sum_{t=1}^{T} z_t (y_{Ht} - \boldsymbol{x}_{Ht} \tilde{\boldsymbol{\beta}})^2, \tag{A.323}$$

where $\tilde{\boldsymbol{\beta}}$ is the OLS estimator of $\boldsymbol{\beta}$ and $y_{Ht} - \boldsymbol{x}_{Ht} \tilde{\boldsymbol{\beta}} = \tilde{u}_{Ht}$ are the OLS residuals from equation (A.314). We define the $T \times 1$ vector $\tilde{U}$ as follows:

$$\tilde{U} = [(\tilde{u}_{Ht}^2)_{t=1,\dots,T}]. \tag{A.324}$$

The GQ estimator of $\boldsymbol{\varsigma}$ is the OLS estimator of $\boldsymbol{\varsigma}$ from the equation

$$\tilde{U} = \boldsymbol{Z} \boldsymbol{\varsigma} + \boldsymbol{v}, \tag{A.325}$$

where

$$\boldsymbol{v} \sim \mathrm{N}(0, \boldsymbol{\Omega}_H^{-2}). \tag{A.326}$$

This result is implied by equations (A.223), (A.324), (A.325) and (A.326) since

$$\hat{\varsigma}_{OLS} = (\boldsymbol{Z}' \boldsymbol{Z})^{-1} \boldsymbol{Z}' \tilde{U} = \left[\sum_{t=1}^{T} z_t z_t' \right]^{-1} \sum_{t=1}^{T} z_t (y_{Ht} - \boldsymbol{x}_{Ht} \tilde{\boldsymbol{\beta}})^2 = \hat{\varsigma}_{GQ}. \tag{A.327}$$

From equations (1.13), (1.14), (A.318), (A.325) and (A.326) we find that the t-th element of the $T \times 1$ vector $\boldsymbol{\upsilon} = [(\upsilon_t)_{t=1,...,T}]$ is

$$\upsilon_t = \tilde{u}^2_{Ht} - \boldsymbol{z}'_t\boldsymbol{\varsigma} = u^2_{Ht} - \tau\varepsilon_t - \sigma^2_t = (\tilde{u}^2_{Ht} - \sigma^2_t) - \tau\varepsilon_t = \bar{u}_t - \tau\varepsilon_t, \tag{A.328}$$

where $\bar{u}_t = \tilde{u}^2_{Ht} - \sigma^2_t$. Thus,

$$\boldsymbol{\upsilon} = \bar{\boldsymbol{u}} - \tau\boldsymbol{\varepsilon}, \;\; \bar{\boldsymbol{u}} = [(\bar{u}_t)_{t=1,...,T}]. \tag{A.329}$$

From equations (A.223), (1.28), (1.30), (A.325), (A.327) and (A.329) we have

$$\begin{aligned}
\hat{\varsigma}_{GQ} &= (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\tilde{U} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{Z}\boldsymbol{\varsigma} + \boldsymbol{\upsilon}) \\
&= (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\varsigma} + (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{\upsilon} = \boldsymbol{\varsigma} + (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{\upsilon} \Rightarrow \\
\boldsymbol{\delta}^{GQ}_{\varsigma} &= \sqrt{T}(\hat{\varsigma}_{GQ} - \varsigma) = \sqrt{T}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{\upsilon} = \sqrt{T}(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{\upsilon}/T \\
&= \bar{\boldsymbol{B}}\boldsymbol{Z}'(\bar{\boldsymbol{u}} - \tau\boldsymbol{\varepsilon})/\sqrt{T} = \bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T} - \tau\bar{\boldsymbol{B}}\boldsymbol{Z}'\boldsymbol{\varepsilon}/\sqrt{T} \\
&= d_{1\varsigma} - \tau d_{2\varsigma},
\end{aligned} \tag{A.330}$$

where

$$d_{1\varsigma} = \bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T}, \;\; d_{2\varsigma} = \bar{\boldsymbol{B}}\boldsymbol{Z}'\boldsymbol{\varepsilon}/\sqrt{T}. \tag{A.331}$$

A estimator of $\boldsymbol{\varsigma}$

$$\hat{\varsigma}_A = \left[\sum_{t=1}^{T}(\boldsymbol{z}'_t\varsigma_{GQ})^{-2}\boldsymbol{z}_t\boldsymbol{z}'_t\right]^{-1}\sum_{t=1}^{T}(\boldsymbol{z}'_t\varsigma_{GQ})^{-2}\boldsymbol{z}_t(y_{Ht} - \boldsymbol{x}_{Ht}\tilde{\boldsymbol{\beta}})^2, \tag{A.332}$$

where $\tilde{\boldsymbol{\beta}}$ is the OLS estimator of $\boldsymbol{\beta}$ and $y_{Ht} - \boldsymbol{x}_{Ht}\tilde{\boldsymbol{\beta}} = \tilde{u}_{Ht}$ are the OLS residuals from equation (A.314). By using equations (A.223), (A.324), (A.325) and (A.326) we find that the A estimator of $\boldsymbol{\varsigma}$ is the GLS estimator of $\boldsymbol{\varsigma}$ from the equation (A.325) because

$$\begin{aligned}
\hat{\varsigma}_{GLS} &= (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\tilde{U} \\
&= \left[\sum_{t=1}^{T}(\boldsymbol{z}'_t\varsigma_{GQ})^{-2}\boldsymbol{z}_t\boldsymbol{z}'_t\right]^{-1}\sum_{t=1}^{T}(\boldsymbol{z}'_t\varsigma_{GQ})^{-2}\boldsymbol{z}_t(y_{Ht} - \boldsymbol{x}_{Ht}\tilde{\boldsymbol{\beta}})^2 = \hat{\varsigma}_A,
\end{aligned} \tag{A.333}$$

where

$$\hat{\boldsymbol{\Omega}}_H = \hat{\boldsymbol{\Omega}}_{HGQ} = \text{diag}[(\boldsymbol{z}'_t\hat{\varsigma}_{GQ})^{-1}]. \tag{A.334}$$

From equations (A.223), (1.28), (1.30), (A.325), (A.333) and (A.334) we have

$$\begin{aligned}
\hat{\varsigma}_A &= (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\tilde{U} = (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H(\boldsymbol{Z}\boldsymbol{\varsigma} + \boldsymbol{\upsilon}) \\
&= (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{Z}\boldsymbol{\varsigma} + (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{\upsilon} = \boldsymbol{\varsigma} + (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{\upsilon} \Rightarrow \\
\boldsymbol{\delta}^{A}_{\varsigma} &= \sqrt{T}(\hat{\varsigma}_A - \varsigma) = \sqrt{T}(\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{\upsilon} = \sqrt{T}(\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{\upsilon}/T \\
&= \hat{\bar{\boldsymbol{G}}}_H\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}^2_H\boldsymbol{\upsilon}/\sqrt{T},
\end{aligned} \tag{A.335}$$

where

$$\hat{\bar{\boldsymbol{G}}}_H = (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z}/T)^{-1}, \;\; \hat{\boldsymbol{\Omega}}_H = \hat{\boldsymbol{\Omega}}_{HGQ} = \text{diag}[(z_t'\hat{\varsigma}_{GQ})^{-1}]. \tag{A.336}$$

From Lemmas UR.1, A.34 and equation (A.186), we have

$$\begin{aligned}
\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z}/T &= \boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}/T + 2\tau\sum_{i=1}^{m}(\boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H\boldsymbol{Z}/T)d_{1\varsigma_i}^{GQ} + \omega(\tau^2) \Rightarrow \\
\hat{\bar{\boldsymbol{A}}}_H &= \bar{\boldsymbol{A}}_H + 2\tau\sum_{i=1}^{m}\bar{\boldsymbol{A}}_{H\varsigma_i}d_{1\varsigma_i}^{GQ} + \omega(\tau^2) \Rightarrow \\
\hat{\bar{\boldsymbol{G}}}_H &= (\hat{\bar{\boldsymbol{A}}}_H)^{-1} = [\bar{\boldsymbol{A}}_H + 2\tau\sum_{i=1}^{m}\bar{\boldsymbol{A}}_{H\varsigma_i}d_{1\varsigma_i}^{GQ} + \omega(\tau^2)]^{-1} \\
&= \bar{\boldsymbol{A}}_H^{-1} - 2\tau\sum_{i=1}^{m}\bar{\boldsymbol{A}}_H^{-1}\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{A}}_H^{-1}d_{1\varsigma_i}^{GQ} + \omega(\tau^2) \\
&= \bar{\boldsymbol{G}}_H - 2\tau\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H d_{1\varsigma_i}^{GQ} + \omega(\tau^2),
\end{aligned} \tag{A.337}$$

where

$$\bar{\boldsymbol{A}}_{H\varsigma_i} = \boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{Z}/T. \tag{A.338}$$

Furthermore, by using Lemma A.34 and equation (A.329) we have

$$\begin{aligned}
\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{v}/\sqrt{T} &= \boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2(\bar{\boldsymbol{u}} - \tau\boldsymbol{\varepsilon})/\sqrt{T} \\
&= \boldsymbol{Z}'\boldsymbol{\Omega}_H^2(\bar{\boldsymbol{u}} - \tau\boldsymbol{\varepsilon})/\sqrt{T} + 2\tau\sum_{i=1}^{m}[\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}(\bar{\boldsymbol{u}} - \tau\boldsymbol{\varepsilon})/\sqrt{T}]d_{1\varsigma_i}^{GQ} + \omega(\tau^2) \\
&= \boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T} - \tau\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T} + 2\tau\sum_{i=1}^{m}(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{GQ} + \omega(\tau^2).
\end{aligned} \tag{A.339}$$

By substituting equations (A.337) and (A.339) in equation (A.335) we find that

$$\begin{aligned}
\boldsymbol{\delta}_\varsigma^A &= [\bar{\boldsymbol{G}}_H - 2\tau\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H d_{1\varsigma_i}^{GQ} + \omega(\tau^2)]\cdot \\
&\quad \cdot[\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T} - \tau\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T} + 2\tau\sum_{j=1}^{m}(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_j}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_j}^{GQ} + \omega(\tau^2)] \\
&= \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T}) - \tau\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T}) + 2\tau\sum_{j=1}^{m}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_j}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_j}^{GQ} \\
&\quad -2\tau\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{GQ} + \omega(\tau^2) \\
&= \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T}) - \\
&\quad -\tau[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T}) - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{GQ} + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{GQ}] + \omega(\tau^2) \\
&= \boldsymbol{d}_{1\varsigma}^A - \tau\boldsymbol{d}_{2\varsigma}^A + \omega(\tau^2),
\end{aligned} \tag{A.340}$$

where

$$\begin{aligned} \boldsymbol{d}_{1\varsigma}^{A} &= \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T}) \\ &\text{and} \\ \boldsymbol{d}_{2\varsigma}^{A} &= \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T}) - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{GQ} + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{GQ}. \end{aligned} \quad (A.341)$$

IA estimator of $\boldsymbol{\varsigma}$

$$\hat{\varsigma}_\alpha = \left[\sum_{t=1}^{T}(z_t'\hat{\varsigma}_{\alpha-1})^{-2}z_t z_t'\right]^{-1}\sum_{t=1}^{T}(z_t'\hat{\varsigma}_{\alpha-1})^{-2}z_t(y_{Ht} - \boldsymbol{x}_{Ht}'\hat{\boldsymbol{\beta}}_{\alpha-1})^2, \quad (A.342)$$

where $\hat{\boldsymbol{\beta}}_{\alpha-1}$, $\hat{\varsigma}_{\alpha-1}$ is the feasible GLS estimator of $\boldsymbol{\beta}$ and the corresponding estimator of $\hat{\varsigma}$, according to the previous repetition, and $y_{Ht} - \boldsymbol{x}_{Ht}'\hat{\beta}_{\alpha-1} = \hat{u}_{Ht}$ are the GLS residuals of equation (A.314). Let $\hat{\varsigma}_1 = \hat{\varsigma}_A$. Using equation (1.13) as well as estimator $\hat{\varsigma}_A$ we find $\hat{\boldsymbol{\Omega}}_H$ and using the GLS method we estimate $\hat{\boldsymbol{\beta}}_1 = \hat{\boldsymbol{\beta}}_A$. For $\alpha = 2, 3, \ldots$, we may easily prove that $\hat{\varsigma}_\alpha$ is the GLS estimator of $\hat{\varsigma}$ from the equation

$$\hat{U} = \boldsymbol{Z}\boldsymbol{\varsigma} + \boldsymbol{\upsilon}, \quad (A.343)$$

where

$$\hat{U} = [(\hat{u}_{Ht}^2)_{t=1,\ldots,T}] \quad (A.344)$$

and for υ, equation (A.334) applies. By letting

$$\hat{\boldsymbol{\Omega}}_H = \hat{\boldsymbol{\Omega}}_{H\alpha-1} = \mathrm{diag}[(z_t'\hat{\varsigma}_{\alpha-1})^{-1}], \quad (A.345)$$

we find that

$$\begin{aligned} \boldsymbol{\delta}_{\varsigma}^{GLS} &= (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\hat{U} \\ &= \left[\sum_{t=1}^{T}(z_t'\hat{\varsigma}_{\alpha-1})^{-2}z_t z_t'\right]^{-1}\sum_{t=1}^{T}(z_t'\hat{\varsigma}_{\alpha-1})^{-2}z_t(y_{Ht} - \boldsymbol{x}_{Ht}'\hat{\boldsymbol{\beta}}_{\alpha-1})^2 = \hat{\varsigma}_\alpha. \end{aligned} \quad (A.346)$$

From Lemma A.34 and equations (A.336), (A.344), (A.345), and (A.346) we have

$$\begin{aligned} \hat{\varsigma}_\alpha &= (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\hat{U} = (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2(\boldsymbol{Z}\boldsymbol{\varsigma} + \boldsymbol{\upsilon}) \\ &= (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z}\boldsymbol{\varsigma} + (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{\upsilon} = \boldsymbol{\varsigma} + (\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{\upsilon} \Rightarrow \\ \boldsymbol{\delta}_{\varsigma}^{\alpha} &= \sqrt{T}(\hat{\varsigma}_\alpha - \varsigma) = \sqrt{T}(\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z})^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{\upsilon} \\ &= \sqrt{T}(\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{\upsilon}/T = \hat{\boldsymbol{G}}_H\boldsymbol{Z}'\hat{\boldsymbol{\Omega}}_H^2\boldsymbol{\upsilon}/\sqrt{T}. \end{aligned} \quad (A.347)$$

In the case being studied, $\hat{u}_{Ht}$ are the GLS residuals of equation (A.314) , when matrix $\boldsymbol{\Omega}_H$ is unknown. Working as in the proof of equation (A.320) we get

$$\hat{u}_{Ht} = u_{Ht} - \tau \boldsymbol{x}'_{Ht} \hat{\boldsymbol{G}}_H \boldsymbol{X}'_H \hat{\boldsymbol{\Omega}}_H \boldsymbol{u}_H / \sqrt{T}. \tag{A.348}$$

Taking Lemmas B.2, B.3 of the PHD thesis, (Symeonides, 1991, p.229-234), of into consideration we get that

$$\hat{\boldsymbol{G}}_H = \boldsymbol{G}_H + \omega(\tau), \;\; \hat{\boldsymbol{\Omega}}_H = \boldsymbol{\Omega}_H + \omega(\tau). \tag{A.349}$$

By substituting equation (A.349) in equation (A.348), and taking into account equation (A.320) we find

$$\hat{u}_{Ht} = u_{Ht} - \tau \boldsymbol{x}'_{Ht} \boldsymbol{G}_H \boldsymbol{X}'_H \boldsymbol{\Omega}_H \boldsymbol{u}_H / \sqrt{T} + \omega(\tau) = u_{Ht} - \tau \bar{e}_t + \omega(\tau). \tag{A.350}$$

From equations (1.14), (1.15), (A.326), (A.343) and (A.350) we have that the t-th element of the $T \times 1$ vector $\boldsymbol{\upsilon} = [(\upsilon_t)_{t=1,\dots,T}]$ is

$$\upsilon_t = \hat{u}^2_{Ht} - \boldsymbol{z}'_t \boldsymbol{\varsigma} + \omega(\tau) = u^2_{Ht} - \tau \bar{\varepsilon}_t - \sigma^2_t + \omega(\tau) = (u^2_{Ht} - \sigma^2_t) - \tau \bar{\varepsilon}_t + \omega(\tau) = \bar{u}_t - \tau \bar{\varepsilon}_t, \tag{A.351}$$

where

$$\bar{u}_t = u^2_{Ht} - \sigma^2_t. \tag{A.352}$$

Thus,

$$\boldsymbol{\upsilon} = \bar{\boldsymbol{u}} - \tau \bar{\boldsymbol{\varepsilon}} + \omega(\tau^2), \;\; \bar{\boldsymbol{u}} = [(\bar{u}_t)_{t=1,\dots,T}]. \tag{A.353}$$

Lemmas A.1 and A.34, equations (A.186) and (A.345) and working as in the proof of equation (A.337), we find that

$$\hat{\bar{\boldsymbol{G}}}_H = \bar{\boldsymbol{G}}_H - 2\tau \sum_{i=1}^{m} \bar{\boldsymbol{G}}_H \bar{\boldsymbol{A}}_{H\varsigma_i} \bar{\boldsymbol{G}}_H d^A_{1\varsigma_i} + \omega(\tau^2), \tag{A.354}$$

where matrix was defined in equation (A.338). Furthermore, from Lemma A.34 and equation (A.353) it follows that

$$\begin{aligned}
\boldsymbol{Z}' \hat{\boldsymbol{\Omega}}^2_H \boldsymbol{\upsilon} / \sqrt{T} &= \boldsymbol{Z}' \hat{\boldsymbol{\Omega}}^2_H [\bar{\boldsymbol{u}} - \tau \bar{\boldsymbol{\varepsilon}} + \omega(\tau^2)] / \sqrt{T} + \omega(\tau^2) \\
&= \boldsymbol{Z}' \hat{\boldsymbol{\Omega}}^2_H [\bar{\boldsymbol{u}} - \tau \bar{\boldsymbol{\varepsilon}} + \omega(\tau^2)] / \sqrt{T} + 2\tau \sum_{i=1}^{m} [\boldsymbol{Z}' \boldsymbol{\Omega}_H \boldsymbol{\Omega}_{H\varsigma_i} [\bar{\boldsymbol{u}} - \tau \bar{\boldsymbol{\varepsilon}} + \omega(\tau^2)] / \sqrt{T}] d^A_{1\varsigma_i} + \omega(\tau^2) \\
&= \boldsymbol{Z}' \hat{\boldsymbol{\Omega}}^2_H (\bar{\boldsymbol{u}} - \tau \bar{\boldsymbol{\varepsilon}}) / \sqrt{T} + 2\tau \sum_{i=1}^{m} [\boldsymbol{Z}' \boldsymbol{\Omega}_H \boldsymbol{\Omega}_{H\varsigma_i} (\bar{\boldsymbol{u}} - \tau \bar{\boldsymbol{\varepsilon}}) / \sqrt{T}] d^A_{1\varsigma_i} + \omega(\tau^2) \\
&= \boldsymbol{Z}' \hat{\boldsymbol{\Omega}}^2_H \bar{\boldsymbol{u}} / \sqrt{T} - \tau \boldsymbol{Z}' \hat{\boldsymbol{\Omega}}^2_H \bar{\boldsymbol{\varepsilon}} / \sqrt{T} + 2\tau \sum_{i=1}^{m} (\boldsymbol{Z}' \boldsymbol{\Omega}_H \boldsymbol{\Omega}_{H\varsigma_i} \bar{\boldsymbol{u}} / \sqrt{T}) d^A_{1\varsigma_i} + \omega(\tau^2).
\end{aligned} \tag{A.355}$$

By using equations (A.341) (A.347), (A.354) and (A.355) and working as in proof of equation (A.340), we find that

$$\begin{aligned}\boldsymbol{\delta}_{\varsigma}^{\alpha} &= \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T}) - \\ &-\tau[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T}) - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{A} + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{A}] + \omega(\tau^2) \\ &= \boldsymbol{d}_{1\varsigma}^{A} - \tau\boldsymbol{d}_{2\varsigma}^{\alpha} + \omega(\tau^2), \end{aligned} \tag{A.356}$$

where

$$\begin{aligned}\boldsymbol{d}_{1\varsigma}^{A} &= \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T}) \\ \text{and} & \\ \boldsymbol{d}_{2\varsigma}^{\alpha} &= \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T}) - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{A} + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}^{A}.\end{aligned} \tag{A.357}$$

Proof of Theorem 2. The elements of matrix $\boldsymbol{\Lambda}$ and vector $\boldsymbol{\kappa}$

$$\begin{bmatrix}\lambda_0 & \lambda_{0\rho} & \boldsymbol{\lambda}_{0\varsigma}' \\ \lambda_{0\rho} & \lambda_{\rho\rho} & \boldsymbol{\lambda}_{\rho\varsigma}' \\ \boldsymbol{\lambda}_{0\varsigma} & \boldsymbol{\lambda}_{\rho\varsigma} & \boldsymbol{\Lambda}_{\varsigma\varsigma}\end{bmatrix} = \lim_{T\to\infty} \mathrm{E}\left[\begin{bmatrix}\sigma_0 \\ \rho_1 \\ \boldsymbol{d}_{1\varsigma}\end{bmatrix}(\sigma_0, \rho_1, \boldsymbol{d}_{1\varsigma}')\right] = \lim_{T\to\infty} \mathrm{E}\begin{bmatrix}\sigma_0{}^2 & \sigma_0\rho_1 & \sigma_0\boldsymbol{d}_{1\varsigma}' \\ \sigma_0\rho_1 & \rho_1{}^2 & \rho_1\boldsymbol{d}_{1\varsigma}' \\ \sigma_0\boldsymbol{d}_{1\varsigma} & \rho_1\boldsymbol{d}_{1\varsigma} & \boldsymbol{d}_{1\varsigma}\boldsymbol{d}_{1\varsigma}'\end{bmatrix} \Longrightarrow \tag{A.358}$$

$$\lambda_0 = \lim_{T\to\infty}\mathrm{E}(\sigma_0{}^2),\ \lambda_{0\rho} = \lim_{T\to\infty}\mathrm{E}(\sigma_0\rho_1),\ \boldsymbol{\lambda}_{0\varsigma} = \lim_{T\to\infty}\mathrm{E}(\sigma_0\boldsymbol{d}_{1\varsigma}),\ \lambda_{\rho\rho} = \lim_{T\to\infty}\mathrm{E}(\rho_1\rho_1), \tag{A.359}$$

$$\boldsymbol{\lambda}_{\rho\varsigma} = \lim_{T\to\infty}\mathrm{E}(\rho_1\boldsymbol{d}_{1\varsigma}),\ \boldsymbol{\Lambda}_{\varsigma\varsigma} = \lim_{T\to\infty}\mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}_{1\varsigma}') \tag{A.360}$$

$$\begin{bmatrix}\kappa_0 \\ \kappa_\rho \\ \boldsymbol{\kappa}_\varsigma\end{bmatrix} = \lim_{T\to\infty}\mathrm{E}\begin{bmatrix}\sqrt{T}\sigma_0 + \sigma_1 \\ \sqrt{T}\rho_1 + \rho_2 \\ \sqrt{T}\boldsymbol{d}_{1\varsigma} - \boldsymbol{d}_{2\varsigma}\end{bmatrix} \Longrightarrow \tag{A.361}$$

$$\kappa_0 = \lim_{T\to\infty}\mathrm{E}(\sqrt{T}\sigma_0 + \sigma_1),\ \kappa_\rho = \lim_{T\to\infty}\mathrm{E}(\sqrt{T}\rho_1 + \rho_2),\ \boldsymbol{\kappa}_\varsigma = \lim_{T\to\infty}\mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\varsigma} - \boldsymbol{d}_{2\varsigma}) \tag{A.362}$$

By using equation (A.281) we have

$$\begin{aligned}\mathrm{E}(\sigma_0{}^2) &= \mathrm{E}[(w_0 - a_\rho\rho_1 - \boldsymbol{a}'\boldsymbol{d}_{1\varsigma})^2] \\ &= \mathrm{E}[(w_0 - a_\rho\rho_1)^2 - 2(w_0 - a_\rho\rho_1)\boldsymbol{a}'\boldsymbol{d}_{1\varsigma} + \boldsymbol{a}'\boldsymbol{d}_{1\varsigma}\boldsymbol{d}_{1\varsigma}'\boldsymbol{a}] \\ &= \mathrm{E}[w_0^2 - 2a_\rho w_0\rho_1 + (a_\rho\rho_1)^2 - 2\boldsymbol{a}'w_0\boldsymbol{d}_{1\varsigma} + 2a_\rho\boldsymbol{a}'\rho_1\boldsymbol{d}_{1\varsigma} + \boldsymbol{a}'\boldsymbol{d}_{1\varsigma}\boldsymbol{d}_{1\varsigma}'\boldsymbol{a}] \\ &= \mathrm{E}(w_0^2) - 2a_\rho\,\mathrm{E}(w_0\rho_1) + a_\rho{}^2\,\mathrm{E}(\rho_1{}^2) - 2\boldsymbol{a}'\,\mathrm{E}(w_0\boldsymbol{d}_{1\varsigma}) + 2a_\rho\boldsymbol{a}'\,\mathrm{E}(\rho_1\boldsymbol{d}_{1\varsigma}) \\ &\quad +\boldsymbol{a}'\,\mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}_{1\varsigma}')\boldsymbol{a} \Longrightarrow \end{aligned} \tag{A.363}$$

By using Lemma A.31 and equations (A.359), (A.360) we have

$$\begin{aligned}\lambda_0 &= \lim_{T\to\infty} \mathrm{E}({\sigma_0}^2) = 2 - 2a_\rho \lim_{T\to\infty} \mathrm{E}(w_0\rho_1) + {a_\rho}^2\lambda_{\rho\rho} \\ &\quad -2\boldsymbol{a}' \lim_{T\to\infty} \mathrm{E}(w_0\boldsymbol{d}_{1\varsigma}) - 2a_\rho\boldsymbol{a}'\lambda_{\rho\varsigma} + \boldsymbol{a}'\boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a}. \qquad (A.364)\end{aligned}$$

$$\begin{aligned}\mathrm{E}(\sigma_0\rho_1) &= \mathrm{E}[(w_0 - a_\rho\rho_1 - \boldsymbol{a}'\boldsymbol{d}_{1\varsigma})\rho_1] \\ &= \mathrm{E}[w_0\rho_1 - a_\rho{\rho_1}^2 - \boldsymbol{a}'\rho_1\boldsymbol{d}_{1\varsigma}] \\ &= \mathrm{E}(w_0\rho_1) - a_\rho\,\mathrm{E}({\rho_1}^2) - \boldsymbol{a}'\,\mathrm{E}(\rho_1\boldsymbol{d}_{1\varsigma}) \implies \qquad (A.365)\end{aligned}$$

$$\lambda_{0\rho} = \lim_{T\to\infty} \mathrm{E}(\sigma_0\rho_1) = \lim_{T\to\infty} \mathrm{E}(w_0\rho_1) - a_\rho\lambda_{\rho\rho} - \boldsymbol{a}'\lambda_{\rho\varsigma} \qquad (A.366)$$

$$\begin{aligned}\mathrm{E}(\sigma_0\boldsymbol{d}_{1\varsigma}) &= \mathrm{E}[(w_0 - a_\rho\rho_1 - \boldsymbol{a}'\boldsymbol{d}_{1\varsigma})\boldsymbol{d}_{1\varsigma}] \\ &= \mathrm{E}[\boldsymbol{d}_{1\varsigma}(w_0 - a_\rho\rho_1 - \boldsymbol{a}'\boldsymbol{d}_{1\varsigma})'] \\ &= \mathrm{E}[w_0\boldsymbol{d}_{1\varsigma} - a_\rho\rho_1\boldsymbol{d}_{1\varsigma} - \boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma}\boldsymbol{a}] \\ &= \mathrm{E}(w_0\boldsymbol{d}_{1\varsigma}) - a_\rho\,\mathrm{E}(\rho_1\boldsymbol{d}_{1\varsigma}) - \mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma})\boldsymbol{a} \implies \qquad (A.367)\end{aligned}$$

$$\lambda_{0\varsigma} = \lim_{T\to\infty} \mathrm{E}(\sigma_0\boldsymbol{d}_{1\varsigma}) = \lim_{T\to\infty} \mathrm{E}(w_0\boldsymbol{d}_{1\varsigma}) - a_\rho\lambda_{\rho\varsigma} - \boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a}. \qquad (A.368)$$

From Lemma A.36 and equations (A.300), (A.303), (A.309), (A.312) it follows that for all estimators of ρ examined we can write:

$$\hat{\rho} = \rho + \tau\rho_1 + \omega(\tau^2), \qquad (A.369)$$

where

$$\rho_1 = -\alpha\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR2}\boldsymbol{u}_{AR}/2\sqrt{T}. \qquad (A.370)$$

From Lemma UR.2 and equations (A.187), (A.303) , we have the following results:

$$\mathrm{E}(\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR2}\boldsymbol{u}_{AR}) = \mathrm{tr}\,\boldsymbol{\Omega}_{AR2}{\boldsymbol{\Omega}_{AR}}^{-1} = 2\rho/\alpha. \qquad (A.371)$$

$$\begin{aligned}\mathrm{E}(\rho_1) &= \mathrm{E}(-\alpha\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR2}\boldsymbol{u}_{AR}/2\sqrt{T}) = \frac{-\alpha}{2\sqrt{T}}\,\mathrm{E}(\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR2}\boldsymbol{u}_{AR}) = \frac{-\alpha}{2\sqrt{T}}\,\mathrm{tr}\,\boldsymbol{\Omega}_{AR2}{\boldsymbol{\Omega}_{AR}}^{-1} \\ &= \frac{-\alpha}{2\sqrt{T}}\frac{2\rho}{\alpha} = -\frac{\rho}{\sqrt{T}}. \qquad (A.372)\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}) &= (\mathrm{tr}\,\boldsymbol{\Omega}_{AR}{}^{-1})(tr\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1}) + 2(tr\boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1}) \\
&= (1/\alpha)(\mathrm{tr}\,\boldsymbol{R})(tr\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1}) + 2(tr\boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1}) \\
&= (T/\alpha)(2\rho/\alpha) + 2[-2\rho T/\alpha^2 + O(1)] \\
&= 2\rho T/\alpha^2 - 4\rho T/\alpha^2 + O(1) \\
&= -2\rho T/\alpha^2 + O(1). \qquad \text{(A.373)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{\mathbf{X}_{AR}}\boldsymbol{u}_{AR}) &= \mathrm{tr}\,\bar{\boldsymbol{P}}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} \\
&= \mathrm{tr}(\boldsymbol{I} - \boldsymbol{P}_{\mathbf{X}_{AR}})\boldsymbol{\Omega}_{_{AR}2}(\boldsymbol{I} - \boldsymbol{P}_{\mathbf{X}_{AR}})\boldsymbol{\Omega}_{AR}{}^{-1} \\
&= \mathrm{tr}(\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1} + \boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1}) \\
&= \mathrm{tr}\,\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1} - 2\,\mathrm{tr}\,\boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1} + \mathrm{tr}\,\boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} \\
&= 2\rho/\alpha - 2(n - \mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho + (\mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha - \mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho + O(\tau^2) \\
&= 2\rho/\alpha - 2n/\rho + 2\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}/\rho + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha\rho - \mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}/\rho + O(\tau^2) \\
&= (1/\rho\alpha)(2\rho^2 - 2n\alpha + \alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}) + O(\tau^2) \\
&= (1/\rho\alpha)[2(\rho^2 - n\alpha) + \alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}] + O(\tau^2), \qquad \text{(A.374)}
\end{aligned}
$$

wherefore, since matrices $\boldsymbol{\Omega}_{AR}{}^{-1}$, $\boldsymbol{\Omega}_{_{AR}2}$ and $\boldsymbol{P}_{\mathbf{X}_{AR}}$ are symmetric, we have

$$
\mathrm{tr}\,\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} = \mathrm{tr}\,\boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\mathbf{X}_{AR}} = \mathrm{tr}(\boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\mathbf{X}_{AR}})' = \mathrm{tr}\,\boldsymbol{P}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1}. \qquad \text{(A.375)}
$$

$$
\begin{aligned}
\mathrm{E}(\rho_2) &= -\,\mathrm{E}(\alpha\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{\mathbf{X}_{AR}}\boldsymbol{u}_{AR}/2 - \alpha^2\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}/2T) \\
&= -\frac{\alpha}{2}\,\mathrm{E}(\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\mathbf{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\bar{\boldsymbol{P}}_{\mathbf{X}_{AR}}\boldsymbol{u}_{AR}) + \frac{\alpha^2}{2T}\,\mathrm{E}(\boldsymbol{u}'_{AR}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}) \\
&= \Big[(\alpha^2/2T)(-2\rho T/\alpha^2) \\
&\quad -(\alpha/2\rho\alpha)[2(\rho^2 - n\alpha) + \alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}]\Big] + O(\tau^2) \qquad \text{(A.376)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(\sqrt{T}\rho_1 + \rho_2) &= \sqrt{T}\,\mathrm{E}(\rho_1) + \mathrm{E}(\rho_2) = -\rho + \mathrm{E}(\rho_2) \\
&= -(2\rho/\alpha)(\alpha/2) + (\alpha^2/2T)(-2\rho T/\alpha^2) \\
&\quad -(\alpha/2\rho\alpha)[2(\rho^2 - n\alpha) + \alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}] + O(\tau^2) \\
&= -(1/2\rho)[2(n+3)\rho^2 - 2n + c_1] + O(\tau^2) \qquad \text{(A.377)}
\end{aligned}
$$

where

$$
c_1 = \alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}. \qquad \text{(A.378)}
$$

From Lemma A.36 and equations (A.372),(A.376) we have that

$$\begin{aligned}
\mathrm{E}(\tilde{\rho}_{LS}) &= \mathrm{E}[\rho + \tau(\rho_1 + \tau\rho_2) + \omega(\tau^3)] \\
&= \rho + \tau[\mathrm{E}(\rho_1) + \tau\,\mathrm{E}(\rho_2)] + O(\tau^3) \\
&= \rho + \tau\Big[-(2\rho/\alpha)(\alpha/2\sqrt{T}) + \tau\big[(\alpha^2/2T)(-2\rho T/\alpha^2) \\
&\quad -(\alpha/2\rho\alpha)[2(\rho^2 - n\alpha) + \alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}]\big]\Big] + O(\tau^3) \\
&= \rho - (\tau^2/2\rho)[2\rho^2 + 2\rho^2 + 2[\rho^2 - n(1-\rho^2)] \\
&\quad +\alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}] + O(\tau^3) \\
&= \rho - (\tau^2/2\rho)(6\rho^2 - 2n + 2n\rho^2 \\
&\quad +\alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}) + O(\tau^3) \\
&= \rho - (\tau^2/2\rho)[2(n+3)\rho^2 - 2n + c_1] + O(\tau^3), \qquad \text{(A.379)}
\end{aligned}$$

where $c_1 = \alpha\,\mathrm{tr}\,\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \mathrm{tr}\,\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}$.

From equations (A.304) and (A.379) and the definitions of our model we find

$$\begin{aligned}
\kappa_\rho^{LS} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\delta_\rho{}^{LS}) = \lim_{T\to\infty} \mathrm{E}[T(\tilde{\rho}^{LS} - \rho)] = \lim_{T\to\infty}[\mathrm{E}(\tilde{\rho}^{LS}) - \rho]/\tau^2 \\
&= \lim_{T\to\infty}[\rho - (\tau^2/2\rho)[2(n+3)\rho^2 - 2n + c_1] + O(\tau^3) - \rho]/\tau^2 \\
&= \lim_{T\to\infty}[-(1/2\rho)[2(n+3)\rho^2 - 2n + c_1] + O(T^{-1/2})] \\
&= -[(n+3)\rho + (c_1 - 2n)/2\rho]. \qquad \text{(A.380)}
\end{aligned}$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} \qquad \text{(A.381)}$$

From equation (A.381) we conclude that

$$\begin{aligned}
\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma} &= [\boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}]\boldsymbol{\Omega}_{AR}[\boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}] \\
&= \boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Omega}_{AR}\boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} \\
&\quad +\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} \\
&= \boldsymbol{\Omega}_{AR}{}^{-1} - 2\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} + \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} \\
&= \boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} = \boldsymbol{\Sigma}. \qquad \text{(A.382)}
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\Sigma}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}} &= [\boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}]\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}} \\
&= \boldsymbol{\Omega}_{AR}{}^{-1}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}[\boldsymbol{I} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}] \\
&= \boldsymbol{\Omega}_{AR}{}^{-1}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} + \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} \\
&= \boldsymbol{\Omega}_{AR}{}^{-1}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} + \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR} = \boldsymbol{\Omega}_{AR}{}^{-1}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}. \qquad \text{(A.383)}
\end{aligned}$$

By using Lemma UR.2 and equations (A.187), (A.375), (A.382) and (A.383) we find the following results:

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{u}_{AR}) &= \operatorname{tr}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{\Omega}_{AR}{}^{-1}\\
&= \operatorname{tr}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\\
&= \operatorname{tr}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\\
&= \operatorname{tr}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1}\\
&= \operatorname{tr}(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_{AR}})\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1}\\
&= \operatorname{tr}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1}\\
&= \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{\Omega}_{AR}{}^{-1} - \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1}\\
&= (n-\operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho - (\operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha - \operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho + O(\tau^2)\\
&= (n-\operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} - \operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha + \operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho + O(\tau^2)\\
&= (n-\operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha)/\rho + O(\tau^2). \qquad \text{(A.384)}
\end{aligned}
$$

By using Lemma UR.2 and equations (A.187), (A.375), (A.382) and (A.383) we find the following results:

$$
\begin{aligned}
&\mathrm{E}(\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{u}_{AR}) =\\
=\ & \operatorname{tr}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{\Omega}_{AR}{}^{-1}\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma}\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma}\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}[\boldsymbol{\Omega}_{AR}{}^{-1} - \boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}]\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - \operatorname{tr}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR2}\boldsymbol{X}_{AR}\\
& \cdot(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - \operatorname{tr}\boldsymbol{X}_{AR}\boldsymbol{\Omega}_{AR2}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR})^{-1}\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - \operatorname{tr}\boldsymbol{X}_{AR}(\boldsymbol{\Omega}_{AR1} + \rho\boldsymbol{\Delta})\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - \operatorname{tr}\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR1}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1} - \rho\operatorname{tr}\boldsymbol{X}'_{AR}\boldsymbol{\Delta}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - (1/\rho)\operatorname{tr}\boldsymbol{X}'_{AR}(\boldsymbol{\Omega}_{AR} - \alpha\boldsymbol{I})\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1} + O(1)\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - (1/\rho)[\operatorname{tr}\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1} - \alpha\operatorname{tr}\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR})^{-1}] + O(1)\\
=\ & \operatorname{tr}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{AR}{}^{-1} - (1/\rho)[\operatorname{tr}\boldsymbol{I}_n - \alpha\operatorname{tr}(\boldsymbol{X}'_{AR}\boldsymbol{X}_{AR}/T)(\boldsymbol{X}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{X}_{AR}/T)^{-1}] + O(1)\\
=\ & -(\operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha - \operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho - (n - \alpha\operatorname{tr}\boldsymbol{F}_{AR}\boldsymbol{G}_{AR}) + O(\tau^2)\\
=\ & -(n - \operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha)/\rho + (\alpha\operatorname{tr}\boldsymbol{F}_{AR}\boldsymbol{G}_{AR} - \operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho + O(\tau^2), \qquad \text{(A.385)}
\end{aligned}
$$

wherefore from the equation (A.187) we know that $\boldsymbol{X}'_{AR}\boldsymbol{\Delta X}_{AR} = O(1)$. From equations (A.305), (A.384) and (A.385) we have that

$$\begin{aligned}
\mathrm{E}(\hat{\rho}_{PW}) &= \mathrm{E}[\tilde{\rho}_{LS} - \tau^2\alpha[\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma\Omega}_{AR}\boldsymbol{u}_{AR} + (1/2)\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma\Omega}_{AR}\boldsymbol{u}_{AR}] + \omega(\tau^3)] \\
&= \mathrm{E}(\tilde{\rho}_{LS}) - \tau^2\alpha[\mathrm{E}(\boldsymbol{u}'_{AR}\bar{\boldsymbol{P}}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma\Omega}_{AR}\boldsymbol{u}_{AR}) + (1/2)\,\mathrm{E}(\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR}\boldsymbol{\Sigma P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{P}_{\boldsymbol{X}_{AR}}\boldsymbol{\Sigma\Omega}_{AR}\boldsymbol{u}_{AR})] + O(\tau^3) \\
&= \mathrm{E}(\tilde{\rho}_{LS}) - \tau^2\alpha[(n - \operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha)/\rho \\
&\quad +(1/2)[-(n - \operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha)/\rho + (\alpha\operatorname{tr}\boldsymbol{F}_{AR}\boldsymbol{G}_{AR} - \operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR})/\rho]] + O(\tau^3) \\
&= \mathrm{E}(\tilde{\rho}_{LS}) - (\tau^2\alpha/2\rho)[n - \operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}/\alpha + \alpha\operatorname{tr}\boldsymbol{F}_{AR}\boldsymbol{G}_{AR} - \operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}] + O(\tau^3) \\
&= \mathrm{E}(\tilde{\rho}_{LS}) - (\tau^2/2\rho)[n\alpha - (\operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR} + \alpha\operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}) + \alpha^2\operatorname{tr}\boldsymbol{F}_{AR}\boldsymbol{G}_{AR}] + O(\tau^3) \\
&= \mathrm{E}(\tilde{\rho}_{LS}) - (\tau^2/2\rho)[n\alpha - c_1 - \alpha c_2] + O(\tau^3),
\end{aligned} \tag{A.386}$$

where

$$\begin{aligned}
c_1 &= \operatorname{tr}\boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR} + \alpha\operatorname{tr}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} \\
&\text{and} \\
c_2 &= \alpha\operatorname{tr}\boldsymbol{F}_{AR}\boldsymbol{G}_{AR}.
\end{aligned} \tag{A.387}$$

From equations (A.306), (A.380), and (A.386) and the definitions of the model we have that

$$\begin{aligned}
\kappa_\rho^{GL} &= \kappa_\rho^{PW} = \lim_{T\to\infty}\mathrm{E}(\sqrt{T}\delta_\rho{}^{PW}) = \lim_{T\to\infty}\mathrm{E}[T(\hat{\rho}_{PW} - \rho)] = \lim_{T\to\infty}[\mathrm{E}(\hat{\rho}_{PW}) - \rho]/\tau^2 \\
&= \lim_{T\to\infty}[\mathrm{E}(\tilde{\rho}_{LS}) - \rho - (\tau^2/2\rho)[n\alpha - c_1 + \alpha c_2] + O(\tau^3)]/\tau^2 \\
&= \lim_{T\to\infty}[[\mathrm{E}(\tilde{\rho}_{LS}) - \rho]/\tau^2 - (1/2\rho)[n\alpha - c_1 + \alpha c_2] + O(T^{-1/2})] \\
&= \kappa_\rho^{LS} - \alpha c_2/2\rho + (c_1 - \alpha n)/2\rho.
\end{aligned} \tag{A.388}$$

From the definitions of the Linear Model with Autocorrelated Disturbances we know that $\mathrm{E}(u_{_{AR}t}{}^2) = 1/\alpha$. Thus,

$$\mathrm{E}(u_{_{AR}1}{}^2 + u_{_{AR}T}{}^2) = \mathrm{E}(u_{_{AR}1}{}^2) + \mathrm{E}(u_{_{AR}T}{}^2) = 1/\alpha + 1/\alpha = 2/\alpha. \tag{A.389}$$

Equations (A.386) and (A.389) imply that

$$\begin{aligned}
\mathrm{E}(\hat{\rho}^{ML}) &= \mathrm{E}[\hat{\rho}_{PW} + \tau^2[\rho\alpha(u_{_{AR}1}{}^2 + u_{_{AR}T}{}^2) - \rho] + \omega(\tau^3)] \\
&= \mathrm{E}(\hat{\rho}_{PW}) + \tau^2[\rho\alpha\,\mathrm{E}(u_{_{AR}1}{}^2 + u_{_{AR}T}{}^2) - \rho] + O(\tau^3) = \mathrm{E}(\hat{\rho}_{PW}) + \tau^2(2\rho\alpha/\alpha - \rho) + O(\tau^3) \\
&= \mathrm{E}(\hat{\rho}_{PW}) + \tau^2\rho + O(\tau^3).
\end{aligned} \tag{A.390}$$

From equations (A.308), (A.388) and (A.390)

$$\begin{aligned}
\kappa_\rho^{ML} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\delta_\rho{}^{ML}) = \lim_{T\to\infty} \mathrm{E}[T(\hat{\rho}_{ML}-\rho)] = \lim_{T\to\infty}[\mathrm{E}(\hat{\rho}_{ML})-\rho]/\tau^2 \\
&= \lim_{T\to\infty}[\mathrm{E}(\hat{\rho}_{PW})-\rho+\tau^2\rho+O(\tau^3)]/\tau^2 \\
&= \lim_{T\to\infty}\left[[\mathrm{E}(\hat{\rho}_{PW})-\rho]/\tau^2+\rho+O(T^{-1/2})\right] \\
&= \kappa_\rho^{PW}+\rho = \kappa_\rho^{GL}+\rho. \qquad \text{(A.391)}
\end{aligned}$$

From equations (A.312) and (A.389) we have

$$\begin{aligned}
\mathrm{E}(\hat{\rho}_{DW}) &= \mathrm{E}[\tilde{\rho}_{LS}+\tau^2\alpha(u_{_{AR}1}{}^2+u_{_{AR}T}{}^2)/2+\omega(\tau^3)] \\
&= \mathrm{E}(\tilde{\rho}_{LS})+\tau^2\alpha\,\mathrm{E}(u_{_{AR}1}{}^2+u_{_{AR}T}{}^2)/2+O(\tau^3) \\
&= \mathrm{E}(\tilde{\rho}_{LS})+(\tau^2\alpha/2)(2/\alpha)+O(\tau^3) \\
&= \mathrm{E}(\tilde{\rho}_{LS})+\tau^2+O(\tau^3). \qquad \text{(A.392)}
\end{aligned}$$

By using equations (A.313) and (A.392) we find

$$\begin{aligned}
\kappa_\rho^{DW} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\delta_\rho{}^{DW}) = \lim_{T\to\infty} \mathrm{E}[T(\hat{\rho}_{DW}-\rho)] = \lim_{T\to\infty}[\mathrm{E}(\hat{\rho}_{DW})-\rho]/\tau^2 \\
&= \lim_{T\to\infty}[\mathrm{E}(\tilde{\rho}_{LS})-\rho+\tau^2+O(\tau^3)]/\tau^2 \\
&= \lim_{T\to\infty}\left[[\mathrm{E}(\tilde{\rho}_{LS})-\rho]/\tau^2+1+O(T^{-1/2})\right] \\
&= \kappa_\rho^{LS}+1. \qquad \text{(A.393)}
\end{aligned}$$

From equations (A.235) and (A.239) we have

$$\mathrm{E}(\bar{u}_t) = \mathrm{E}(u_{Ht}{}^2-\sigma_t{}^2) = \sigma_t{}^2-\sigma_t{}^2 = 0 \implies \mathrm{E}(\bar{u}) = \mathrm{E}[(\bar{u}_t)t=1,\ldots,T] = 0. \qquad \text{(A.394)}$$

Also, since $\boldsymbol{d}_{1\varsigma} = d_{1\varsigma}{}^{GQ} = \bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T}$ for $\hat{\boldsymbol{\varsigma}}^{GQ}$ and $\boldsymbol{d}_{1\varsigma} = d_{1\varsigma}{}^{A} = \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})$ for $\hat{\boldsymbol{s}}^A$, $\hat{\boldsymbol{s}}^\alpha$ and $\hat{\boldsymbol{\varsigma}}^{ML}$ subsequently

$$\mathrm{E}(d_{1\varsigma}) = 0. \qquad \text{(A.395)}$$

Thus,

$$\begin{aligned}
\mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\varsigma}-\boldsymbol{d}_{2\varsigma}) &= \sqrt{T}\,\mathrm{E}(d_{1\varsigma})-\mathrm{E}(d_{2\varsigma}) = -\mathrm{E}(d_{2\varsigma}) \implies \\
\boldsymbol{\kappa}_\varsigma &= \lim_{T\to\infty}\mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\varsigma}-\boldsymbol{d}_{2\varsigma}) = -\lim_{T\to\infty}\mathrm{E}(d_{2\varsigma}). \qquad \text{(A.396)}
\end{aligned}$$

By using Lemma A.31, equations (A.281), (A.372),(A.395), and since $\boldsymbol{b} \sim \mathrm{N}(O,\boldsymbol{G})$, $tr\boldsymbol{AG} = \mathrm{tr}\,\boldsymbol{I}_n = n$ we get that

$$\mathrm{E}(\sigma_0) = \mathrm{E}(w_0-a_\rho\rho_1-\boldsymbol{a}'\boldsymbol{d}_{1\varsigma}) = \mathrm{E}(w_0)-a_\rho\,\mathrm{E}(\rho_1)-\boldsymbol{a}'\,\mathrm{E}(\boldsymbol{d}_{1\varsigma}) = a_\rho\frac{\rho}{\sqrt{T}} = -\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T)\frac{\rho}{\sqrt{T}}$$

$$= \quad -\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}/T\frac{\rho}{\sqrt{T}} = -\frac{2\alpha_*}{\rho T}\frac{\rho}{\sqrt{T}} = -\frac{2\alpha_*}{T\sqrt{T}}. \tag{A.397}$$

Therefore,

$$\begin{aligned}
\mathrm{E}(\sqrt{T}\sigma_0+\sigma_1) &= \mathrm{E}(\sqrt{T}\sigma_0)+\mathrm{E}(\sigma_1) = -\sqrt{T}\frac{2\alpha_*}{T\sqrt{T}}+\mathrm{E}(\sigma_1)\\
&= \frac{2\alpha_*}{T}+\mathrm{E}(w_\rho\rho_1+\boldsymbol{w}'\boldsymbol{d}_{1\varsigma}-a_\rho\rho_2+\boldsymbol{a}'\boldsymbol{d}_{2\varsigma}\\
&\quad +\boldsymbol{d}_{1\varsigma}'\bar{\boldsymbol{A}}\boldsymbol{d}_{1\varsigma}+a_{\rho\rho}{\rho_1}^2+\rho_1\boldsymbol{a}_{\rho\varsigma}'\boldsymbol{d}_{1\varsigma}-\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b}+n)\\
&= \frac{2\alpha_*}{T}+\mathrm{E}(w_\rho\rho_1)+\mathrm{E}(\boldsymbol{w}'\boldsymbol{d}_{1\varsigma})-\mathrm{E}(a_\rho\rho_2)+\mathrm{E}(\boldsymbol{a}'\boldsymbol{d}_{2\varsigma})\\
&\quad +\mathrm{E}(\boldsymbol{d}_{1\varsigma}'\bar{\boldsymbol{A}}\boldsymbol{d}_{1\varsigma})+\mathrm{E}(a_{\rho\rho}{\rho_1}^2)+\mathrm{E}(\rho_1\boldsymbol{a}_{\rho\varsigma}'\boldsymbol{d}_{1\varsigma})-\mathrm{E}(\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b})+n\\
&= \frac{2\alpha_*}{T}+\mathrm{E}(w_\rho\rho_1)+\mathrm{E}(\boldsymbol{w}'\boldsymbol{d}_{1\varsigma})-\mathrm{E}(a_\rho\rho_2)+\mathrm{E}(\boldsymbol{a}'\boldsymbol{d}_{2\varsigma})\\
&\quad +\operatorname{tr}\bar{\boldsymbol{A}}\,\mathrm{E}(\boldsymbol{d}_{1\varsigma}'\boldsymbol{d}_{1\varsigma})+\mathrm{E}(a_{\rho\rho}{\rho_1}^2)+\mathrm{E}(\rho_1\boldsymbol{a}_{\rho\varsigma}'\boldsymbol{d}_{1\varsigma})-\operatorname{tr}\boldsymbol{A}\boldsymbol{G}+n\\
&= \frac{2\alpha_*}{T}+\mathrm{E}(w_\rho\rho_1)+\mathrm{E}(\boldsymbol{w}'\boldsymbol{d}_{1\varsigma})-\mathrm{E}(a_\rho\rho_2)+\mathrm{E}(\boldsymbol{a}'\boldsymbol{d}_{2\varsigma})\\
&\quad +\operatorname{tr}\bar{\boldsymbol{A}}\,\mathrm{E}(\boldsymbol{d}_{1\varsigma}'\boldsymbol{d}_{1\varsigma})+\mathrm{E}(a_{\rho\rho}{\rho_1}^2)+\mathrm{E}(\rho_1\boldsymbol{a}_{\rho\varsigma}'\boldsymbol{d}_{1\varsigma})-n+n \Rightarrow\\
\kappa_0 &= \lim_{T\to\infty}\mathrm{E}(\sqrt{T}\sigma_0+\sigma_1) = \lim_{T\to\infty}E(\sqrt{T}\sigma_0)+\mathrm{E}(\sigma_1) = -\lim_{T\to\infty}\sqrt{T}\frac{2\alpha_*}{T\sqrt{T}}+\lim_{T\to\infty}\mathrm{E}(\sigma_1)\\
&= \lim_{T\to\infty}\frac{2\alpha_*}{T}+\lim_{T\to\infty}\mathrm{E}(\sigma_1) = \lim_{T\to\infty}\mathrm{E}(\sigma_1)\\
&= \lim_{T\to\infty}\mathrm{E}(w_\rho\rho_1)+\lim_{T\to\infty}\mathrm{E}(\boldsymbol{w}'\boldsymbol{d}_{1\varsigma})-\lim_{T\to\infty}\mathrm{E}(a_\rho\rho_2)+\lim_{T\to\infty}\mathrm{E}(\boldsymbol{a}'\boldsymbol{d}_{2\varsigma})\\
&\quad +\lim_{T\to\infty}\operatorname{tr}\bar{\boldsymbol{A}}\,\mathrm{E}(\boldsymbol{d}_{1\varsigma}'\boldsymbol{d}_{1\varsigma})+\lim_{T\to\infty}\mathrm{E}(a_{\rho\rho}{\rho_1}^2)+\lim_{T\to\infty}\mathrm{E}(\rho_1\boldsymbol{a}_{\rho\varsigma}'\boldsymbol{d}_{1\varsigma})\\
&= \lim_{T\to\infty}\mathrm{E}(w_\rho\rho_1)+\lim_{T\to\infty}\mathrm{E}(\boldsymbol{w}'\boldsymbol{d}_{1\varsigma})-a_\rho\lim_{T\to\infty}\mathrm{E}(\rho_2)+\boldsymbol{a}'(-\boldsymbol{\kappa}_\varsigma)\\
&\quad +\operatorname{tr}\bar{\boldsymbol{A}}\boldsymbol{\Lambda}_{\varsigma\varsigma}+a_{\rho\rho}\lambda_{\rho\rho}+\boldsymbol{a}_{\rho\varsigma}'\lambda_{\rho\varsigma}.
\end{aligned} \tag{A.398}$$

For the $\hat{\boldsymbol{\varsigma}}_{GQ}$ estimator of $\boldsymbol{\varsigma}$ and the $\tilde{\rho}_{LS}$ estimator of ρ we have that

$$\begin{aligned}
\delta_\varsigma &= {\delta_\varsigma}^{GQ},\ \boldsymbol{d}_{1\varsigma}={\boldsymbol{d}_{1\varsigma}}^{GQ},\ \boldsymbol{d}_{2\varsigma}={\boldsymbol{d}_{2\varsigma}}^{GQ},\\
\delta_\rho &= {\delta_\rho}^{LS},\ \rho_1={\rho_1}^{LS},\ \rho_2={\rho_2}^{LS},\\
{\sigma_0}^{GQ} &= w_0-a_\rho{\rho_1}^{LS}-\boldsymbol{a}'{\boldsymbol{d}_{1\varsigma}}^{GQ}\\
{\sigma_1}^{GQ} &= w_\rho{\rho_1}^{LS}+\boldsymbol{w}'{\boldsymbol{d}_{1\varsigma}}^{GQ}-a_\rho{\rho_2}^{LS}+\boldsymbol{a}'{\boldsymbol{d}_{2\varsigma}}^{GQ}+{\boldsymbol{d}_{1\varsigma}'}^{GQ}\bar{\boldsymbol{A}}{\boldsymbol{d}_{1\varsigma}}^{GQ}+a_{\rho\rho}{\rho_1}^{2LS}+{\rho_1}^{LS}\boldsymbol{a}_{\rho\varsigma}'{\boldsymbol{d}_{1\varsigma}}^{GQ}\\
&\quad -\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b}+n.
\end{aligned} \tag{A.399}$$

For the $\hat{\boldsymbol{\varsigma}}_{A}$ estimator of $\boldsymbol{\varsigma}$ and the $\hat{\rho}_{GL}$ estimator of ρ we have that

$$\begin{aligned}
\delta_\varsigma &= {\delta_\varsigma}^{A},\ \boldsymbol{d}_{1\varsigma}={\boldsymbol{d}_{1\varsigma}}^{A},\ \boldsymbol{d}_{2\varsigma}={\boldsymbol{d}_{2\varsigma}}^{A},\\
\delta_\rho &= {\delta_\rho}^{GL},\ \rho_1={\rho_1}^{GL}={\rho_1}^{LS},\ \rho_2={\rho_2}^{GL},\\
{\sigma_0}^{A} &= w_0-a_\rho{\rho_1}^{GL}-\boldsymbol{a}'{\boldsymbol{d}_{1\varsigma}}^{A}\\
{\sigma_1}^{A} &= w_\rho{\rho_1}^{GL}+\boldsymbol{w}'{\boldsymbol{d}_{1\varsigma}}^{A}-a_\rho{\rho_2}^{GL}+\boldsymbol{a}'{\boldsymbol{d}_{2\varsigma}}^{A}+{\boldsymbol{d}_{1\varsigma}'}^{A}\bar{\boldsymbol{A}}{\boldsymbol{d}_{1\varsigma}}^{A}+a_{\rho\rho}{\rho_1}^{2GL}+{\rho_1}^{GL}\boldsymbol{a}_{\rho\varsigma}'{\boldsymbol{d}_{1\varsigma}}^{A}
\end{aligned}$$

$$-\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b} + n. \tag{A.400}$$

For the $\hat{\varsigma}_{IA}$ and the $\hat{\varsigma}_{ML}$ estimator of $\boldsymbol{\varsigma}$ and $\hat{\rho}_I$ (I=S, GL, IG, ML) estimator of ρ we have that

$$\begin{aligned}
\delta_\varsigma &= \delta_\varsigma{}^\alpha, \ \boldsymbol{d}_{1\varsigma} = \boldsymbol{d}_{1\varsigma}{}^A, \ \boldsymbol{d}_{2\varsigma} = \boldsymbol{d}_{2\varsigma}{}^\alpha, \\
\delta_\rho &= \delta_\rho{}^I, \ \rho_1 = \rho_1{}^I = \rho_1{}^{LS}, \ \rho_2 = \rho_2{}^I, \\
\sigma_0{}^\alpha &= w_0 - a_\rho \rho_1{}^I - \boldsymbol{a}'\boldsymbol{d}_{1\varsigma}{}^A = \sigma_0{}^A \\
\sigma_1{}^\alpha &= w_\rho \rho_1{}^I + \boldsymbol{w}'\boldsymbol{d}_{1\varsigma}{}^A - a_\rho \rho_2{}^I + \boldsymbol{a}'\boldsymbol{d}_{2\varsigma}{}^\alpha + \boldsymbol{d}'_{1\varsigma}{}^A \bar{\boldsymbol{A}} \boldsymbol{d}_{1\varsigma}{}^A + a_{\rho\rho}\rho_1{}^{2I} + \rho_1{}^I \boldsymbol{a}'_{\rho\varsigma}\boldsymbol{d}_{1\varsigma}{}^A \\
&\quad -\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b} + n,
\end{aligned} \tag{A.401}$$

where I is any estimator of ρ.

By using Lemma A.31 we have

$$\begin{aligned}
w_0 &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T - 1) = \sqrt{T}\left(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\psi_t\psi_{t'} - 1\right), \\
w_i &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T + a_i) = \sqrt{T}\left(-\frac{1}{2T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\left[\frac{z_{ti}}{\sigma_t{}^2} + \frac{z_{t'i}}{\sigma_{t'}{}^2}\right]\psi_t\psi_{t'} + a_i\right), \\
w_\rho &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T + a_\rho) = \sqrt{T}\left(\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'_\rho}\psi_t\psi_{t'} + a_\rho\right)
\end{aligned} \tag{A.402}$$

where

$$\psi_t = u_t/\sigma_t. \tag{A.403}$$

Using Lemma A.30 and equations (UR.25), (A.212), and (A.370) we find that

$$\begin{aligned}
\mathrm{E}(w_0\rho_1) &= \mathrm{E}[\sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}/T - 1)(-\alpha\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR}2\boldsymbol{u}_{AR}/2\sqrt{T})] \\
&= -\frac{\alpha}{2}\mathrm{E}[(\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Omega}_{AR}2\boldsymbol{\Sigma}^{-1/2}\boldsymbol{u}/T - \boldsymbol{u}'\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Omega}_{AR}2\boldsymbol{\Sigma}^{-1/2}\boldsymbol{u})] \\
&= -\frac{\alpha}{2}\mathrm{E}[\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}/T - \boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}] \\
&= -\frac{\alpha}{2}\Big(\mathrm{E}[\boldsymbol{u}'\boldsymbol{\Omega}\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}/T] - \mathrm{E}[\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}]\Big) \\
&= -\frac{\alpha}{2}\Big(\operatorname{tr}\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}/T + 2\operatorname{tr}\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}/T - \operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}\Big) \\
&= -\frac{\alpha}{2}\Big(\operatorname{tr}\boldsymbol{I}\operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}/T + 2\operatorname{tr}\boldsymbol{I}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}/T - \operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}\Big) \\
&= -\frac{\alpha}{2}\Big(T\operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}/T + 2\operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}/T - \operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}\Big) \\
&= -\frac{\alpha}{2}2\operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}/T \\
&= -\frac{\alpha}{T}\operatorname{tr}(\boldsymbol{\Omega}_1 + \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2})\boldsymbol{\Omega}^{-1} \\
&= -\frac{\alpha}{T}(\operatorname{tr}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \rho\operatorname{tr}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{R}\alpha\boldsymbol{\Sigma}^{1/2}) \\
&= -\frac{\alpha}{T}(\operatorname{tr}\frac{1}{\rho}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I} - \boldsymbol{R}]\boldsymbol{\Sigma}^{1/2} + \rho\operatorname{tr}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{R}\alpha\boldsymbol{\Sigma}^{1/2})
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\alpha}{T}(\operatorname{tr}\frac{1}{\rho}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}[\boldsymbol{I}-\boldsymbol{R}] + \frac{\rho}{\alpha}\operatorname{tr}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}) \\
&= -\frac{\alpha}{T}(\operatorname{tr}\frac{1}{\rho}(T-T) + \frac{\rho}{\alpha}\operatorname{tr}\boldsymbol{\Delta R}) \\
&= -\frac{\alpha}{T}\frac{\rho}{\alpha}2 \\
&= -\frac{2\rho}{T} = O(T^{-1}).
\end{aligned} \tag{A.404}
$$

Furthermore, by using equations (A.212), (A.370),(UR.25) and Lemma A.30 we have

$$
\begin{aligned}
w_\rho \rho_1 &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T + a_\rho)(-\alpha\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR2}\boldsymbol{u}_{AR}/2\sqrt{T}) \\
&= -\frac{\alpha}{2}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T + a_\rho)(\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{AR}2\boldsymbol{u}_{AR}) = -\frac{\alpha}{2}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_2\boldsymbol{u}_{AR}/T + a_\rho\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_2\boldsymbol{u}_{AR}) \\
&= -\frac{\alpha}{2}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Omega}_{AR}2\boldsymbol{\Sigma}^{-1/2}\boldsymbol{u}/T + a_\rho\boldsymbol{u}'\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Omega}_{AR}2\boldsymbol{\Sigma}^{-1/2}\boldsymbol{u}) \\
&= -\frac{\alpha}{2}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}/T + a_\rho\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}) \implies
\end{aligned} \tag{A.405}
$$

$$
\begin{aligned}
\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1} &= (\boldsymbol{\Omega}_1 - \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2})\boldsymbol{\Omega}^{-1}(\boldsymbol{\Omega}_1 + \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2})\boldsymbol{\Omega}^{-1} \\
&= (\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} - \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Omega}^{-1})(\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Omega}^{-1}) \\
&= (\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} - \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\frac{1}{\alpha}\boldsymbol{\Sigma}^{1/2}\boldsymbol{R}\boldsymbol{\Sigma}^{1/2})(\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \rho\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}\frac{1}{\alpha}\boldsymbol{\Sigma}^{1/2}\boldsymbol{R}\boldsymbol{\Sigma}^{1/2}) \\
&= (\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} - \frac{\rho}{\alpha}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2})(\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \frac{\rho}{\alpha}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2}) \\
&= (\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1})^2 - \frac{\rho}{\alpha}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1} + \frac{\rho}{\alpha}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2} - \left(\frac{\rho}{\alpha}\right)^2(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2})^2 \Rightarrow \\
\operatorname{tr}(\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}) &= \operatorname{tr}(\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1})^2 - \frac{\rho}{\alpha}\operatorname{tr}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1}) + \frac{\rho}{\alpha}\operatorname{tr}(\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2}) - \left(\frac{\rho}{\alpha}\right)^2\operatorname{tr}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2})^2 \\
&= \operatorname{tr}(\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1})^2 - \frac{\rho}{\alpha}\operatorname{tr}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1}) + \frac{\rho}{\alpha}\operatorname{tr}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1}) - \left(\frac{\rho}{\alpha}\right)^2\operatorname{tr}(\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Delta R})^2 \\
&= \operatorname{tr}(\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1})^2 - \left(\frac{\rho}{\alpha}\right)^2\operatorname{tr}(\boldsymbol{X}) \\
&= \operatorname{tr}(\boldsymbol{\Omega}_1\boldsymbol{\Omega}^{-1})^2 + O(1) \\
&= 2T/\alpha + O(1) \implies
\end{aligned} \tag{A.406}
$$

$$
\begin{aligned}
\mathrm{E}(w_\rho\rho_1) &= -\frac{\alpha}{2}\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}/T + a_\rho\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}) \\
&= -\frac{\alpha}{2}\Big(\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}/T) + a_\rho\,\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u})\Big) \\
&= -\frac{\alpha}{2}\Big(\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u}/T) + (-\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T))\,\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_2\boldsymbol{u})\Big) \\
&= -\frac{\alpha}{2}\left[\frac{1}{T}(\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}\operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1} + 2\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}) - \frac{1}{T}\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}\operatorname{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}\right] \\
&= -\frac{\alpha}{2}\left(\frac{2}{T}\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}\right) \\
&= -\frac{\alpha}{T}\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\boldsymbol{\Omega}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\alpha}{T}\left(\frac{2T}{\alpha} + O(1)\right) \\
&= -2 + O(T^{-1}). \qquad \text{(A.407)}
\end{aligned}
$$

By using equation (A.370)

$$\rho_1{}^2 = \frac{\alpha^2}{4T}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR} \implies \qquad \text{(A.408)}$$

$$\mathrm{E}(\rho_1{}^2) = \frac{\alpha^2}{4T}\,\mathrm{E}(\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}) \implies \qquad \text{(A.409)}$$

By using Lemma UR.2 and (A.187) we have

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}\boldsymbol{u}'_{AR}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{u}_{AR}) &= \operatorname{tr}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1}\operatorname{tr}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1} + 2\operatorname{tr}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1} \\
&= (\operatorname{tr}\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1})^2 + 2\operatorname{tr}(\boldsymbol{\Omega}_{_{AR}2}\boldsymbol{\Omega}_{AR}{}^{-1})^2 \\
&= \left(\frac{2\rho}{\alpha}\right)^2 + 2\left(\frac{2T}{\alpha}\right) + O(1) \\
&= \frac{4\rho^2}{\alpha^2} + \frac{4T}{\alpha} + O(1) \implies \qquad \text{(A.410)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(\rho_1{}^2) &= \frac{\alpha^2}{4T}\left[\frac{4\rho^2}{\alpha^2} + \frac{4T}{\alpha} + O(1)\right] \\
&= \frac{\rho^2}{T} + \alpha + O(1) = \alpha + O(T^{-1}) \implies \qquad \text{(A.411)}
\end{aligned}
$$

$$\lambda_{\rho\rho} = \lim_{T\to\infty}\mathrm{E}(\rho_1{}^2) = \alpha. \qquad \text{(A.412)}$$

By using equations (A.329), (A.352) and (A.403)

$$
\begin{aligned}
\bar{u} &= [(\bar{u}_t)_{t=1,\dots,T}] \\
&= [(u_{_H}t^2 - \sigma_t{}^2)_{t=1,\dots,T}] \\
&= [((1-\rho^2)u_t{}^2 - \sigma_t{}^2)_{t=1,\dots,T}] \\
&= [(\sigma_t{}^2[(1-\rho^2)\psi_t{}^2 - 1])_{t=1,\dots,T}]. \qquad \text{(A.413)}
\end{aligned}
$$

By using equations (1.11a), (1.11b) and (1.11c) we have

$$
\begin{aligned}
\psi_t &= \frac{u_t}{\sigma_t} \\
\mathrm{E}(\psi_t) &= \frac{\mathrm{E}(u_t)}{\sigma_t} = 0 \qquad \text{(A.414)} \\
\mathrm{E}(\psi_t{}^2) &= \frac{\mathrm{E}(u_t{}^2)}{\sigma_t{}^2} = \frac{\sigma_t{}^2}{\sigma_t{}^2(1-\rho^2)} = \frac{1}{1-\rho^2} \\
\mathrm{E}(\psi_t\psi_{t'}) &= \frac{\mathrm{E}(u_t u_{t'})}{\sigma_t\sigma_{t'}} = \frac{\sigma_t\sigma_{t'}\rho^{|t-t'|}}{\sigma_t\sigma_{t'}(1-\rho^2)} = \frac{\rho^{|t-t'|}}{1-\rho^2}.
\end{aligned}
$$

By using equations (A.402), (A.413) and (A.414)

$$
\begin{aligned}
w_0\bar{u}_l/\sqrt{T} &= \sqrt{T}\left[\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\psi_t\psi_{t'} - 1\right]\sigma_l^2[(1-\rho^2)\psi_l^2 - 1]/\sqrt{T} \\
&= \frac{1}{T}\sigma_l^2(1-\rho^2)\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\psi_t\psi_{t'}\psi_l^2 - \sigma_l^2\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\psi_t\psi_{t'} \\
&\quad -\sigma_l^2(1-\rho^2)\psi_l^2 + \sigma_l^2 \qquad \text{(A.415)}\\
&\Longrightarrow
\end{aligned}
$$

By using the Isserlis' Theorem (UR.27) which is defined in the Useful Results' chapter and (A.414) we have

$$
\begin{aligned}
\mathrm{E}(\psi_t\psi_{t'}\psi_l^2) &= \mathrm{E}(\psi_t\psi_{t'})\,\mathrm{E}(\psi_l^2) + 2\,\mathrm{E}(\psi_t\psi_l)\,\mathrm{E}(\psi_l\psi_{t'}) \\
&= \frac{\rho^{|t-t'|}}{(1-\rho^2)}\frac{1}{(1-\rho^2)} + 2\frac{\rho^{|t-l|}}{(1-\rho^2)}\frac{\rho^{|l-t'|}}{(1-\rho^2)} \\
&= \frac{\rho^{|t-t'|}}{(1-\rho^2)^2} + 2\frac{\rho^{|t-l|+|l-t'|}}{(1-\rho^2)^2}. \qquad \text{(A.416)}
\end{aligned}
$$

By using equations (A.402) and (A.415) we get

$$
\begin{aligned}
\mathrm{E}(w_0\bar{u}_l/\sqrt{T}) &= \mathrm{E}\Bigg[\frac{1}{T}\sigma_l^2(1-\rho^2)\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\psi_t\psi_{t'}\psi_l^2 - \sigma_l^2\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\psi_t\psi_{t'} \\
&\quad -\sigma_l^2(1-\rho^2)\psi_l^2 + \sigma_l^2\Bigg] \\
&= \frac{1}{T}\sigma_l^2(1-\rho^2)\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\,\mathrm{E}(\psi_t\psi_{t'}\psi_l^2) - \sigma_l^2\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\,\mathrm{E}(\psi_t\psi_{t'}) \\
&\quad -\sigma_l^2(1-\rho^2)\,\mathrm{E}(\psi_l^2) + \sigma_l^2 \\
&= \frac{1}{T}\sigma_l^2(1-\rho^2)\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\left(\frac{\rho^{|t-t'|}}{(1-\rho^2)^2} + 2\frac{\rho^{|t-l|+|l-t'|}}{(1-\rho^2)^2}\right) - \sigma_l^2\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\frac{\rho^{|t-t'|}}{1-\rho^2} \\
&\quad -\sigma_l^2(1-\rho^2)\frac{1}{1-\rho^2} + \sigma_l^2 \\
&= \frac{1}{T}\sigma_l^2\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\left(\frac{\rho^{|t-t'|}}{1-\rho^2} + 2\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\right) - \sigma_l^2\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\frac{\rho^{|t-t'|}}{1-\rho^2} \\
&\quad -\sigma_l^2 + \sigma_l^2 \\
&= \frac{1}{T}\sigma_l^2\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\left(2\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\right) \\
&= \frac{1}{T}\sigma_l^2\sum_{t'=1}^{T}\sum_{t=1}^{T}(\delta_{tt'} + \rho^2\delta_{tt'}(1-\delta_{1t}-\delta_{tT}) - \rho(\delta_{t(t'+1)} + \delta_{(t+1)t'})\left(2\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\right) \\
&= \frac{2}{T}\sigma_l^2\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T}\delta_{tt'}\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2} + \sum_{t'=1}^{T}\sum_{t=1}^{T}\rho^2\delta_{tt'}(1-\delta_{1t}-\delta_{tT})\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2} \\
&\quad -\sum_{t'=1}^{T}\sum_{t=1}^{T}\rho(\delta_{t(t'+1)} + \delta_{(t+1)t'})\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{T}\sigma_l{}^2\Bigg(\sum_{t=1}^{T}\delta_{tt}\frac{\rho^{|t-l|+|l-t|}}{1-\rho^2} + \sum_{t=1}^{T}\rho^2\delta_{tt}(1-\delta_{1t}-\delta_{tT})\frac{\rho^{|t-l|+|l-t|}}{1-\rho^2} \\
&\quad -\sum_{t=2}^{T+1}\rho(\delta_{tt}+\delta_{(t+1)(t-1)})\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1}\rho(\delta_{t(t+1)}+\delta_{(t+1)(t+1)})\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Bigg) \\
&= \frac{2}{T}\sigma_l{}^2\Bigg(\sum_{t=1}^{T}\frac{\rho^{2|t-l|}}{1-\rho^2} + \sum_{t=2}^{T-1}\rho^2\frac{\rho^{2|t-l|}}{1-\rho^2} - \sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Bigg) \\
&= \frac{2}{T}\sigma_l{}^2\Bigg(\sum_{t=1}^{T}\frac{\rho^{2|t-l|}}{1-\rho^2} + \rho^2\frac{\rho^{2|1-l|}}{1-\rho^2} - \rho^2\frac{\rho^{2|1-l|}}{1-\rho^2} + \rho^2\frac{\rho^{2|T-l|}}{1-\rho^2} - \rho^2\frac{\rho^{2|T-l|}}{1-\rho^2} + \sum_{t=2}^{T-1}\rho^2\frac{\rho^{2|t-l|}}{1-\rho^2} \\
&\quad -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Bigg) \\
&= \frac{2}{T}\sigma_l{}^2\Bigg(\sum_{t=1}^{T}\frac{\rho^{2|t-l|}}{1-\rho^2} - \rho^2\frac{\rho^{2(l-1)}}{1-\rho^2} - \rho^2\frac{\rho^{2(T-l)}}{1-\rho^2} + \sum_{t=1}^{T}\rho^2\frac{\rho^{2|t-l|}}{1-\rho^2} - \sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Bigg) \\
&= \frac{2}{T}\sigma_l{}^2\Bigg((1+\rho^2)\sum_{t=1}^{T}\frac{\rho^{2|t-l|}}{1-\rho^2} - \rho^2\frac{\rho^{2(l-1)}}{1-\rho^2} - \rho^2\frac{\rho^{2(T-l)}}{1-\rho^2} - \sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Bigg) \\
&= \frac{2}{T}\sigma_l{}^2\Bigg[\frac{1+\rho^2}{1-\rho^2}\Bigg(\frac{1+\rho^2}{1-\rho^2} - \frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\Bigg) - \frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
&\quad -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Bigg], \qquad \text{(A.417)}
\end{aligned}
$$

which implies that by using equation (A.417) we have

$$\begin{aligned}
\mathrm{E}(w_0\bar{\boldsymbol{u}}/\sqrt{T}) &= \mathrm{E}[(w_0\bar{u}_l/\sqrt{T})_{l=1,\ldots,T}] \\
&= \Bigg[\bigg(\frac{2}{T}{\sigma_l}^2\bigg[\frac{1+\rho^2}{1-\rho^2}\bigg(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\bigg)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
&\quad -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\bigg]\bigg)_{l=1,\ldots,T}\Bigg] \\
&= \Bigg[\bigg(\frac{2}{T}\boldsymbol{z}_l'\varsigma\bigg[\frac{1+\rho^2}{1-\rho^2}\bigg(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\bigg)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
&\quad -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\bigg]\bigg)_{l=1,\ldots,T}\Bigg] \\
&= \boldsymbol{Z}\varsigma\frac{2}{T}\Bigg[\bigg(\frac{1+\rho^2}{1-\rho^2}\bigg(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\bigg)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
&\quad -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\bigg)_{l=1,\ldots,T}\Bigg].
\end{aligned} \tag{A.418}$$

By using equations (A.331), (A.341) and (A.418) we get the following results:

For GQ estimator

$$\begin{aligned}
\mathrm{E}(w_0d_{1\varsigma}) &= \mathrm{E}(w_0{d_{1\varsigma}}^{GQ}) = \mathrm{E}(w_0\bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T}) \\
&= \bar{\boldsymbol{B}}\boldsymbol{Z}'\,\mathrm{E}(w_0\bar{\boldsymbol{u}}/\sqrt{T}) = \bar{\boldsymbol{B}}\boldsymbol{Z}'\,\mathrm{E}[(w_0\bar{u}_l/\sqrt{T})\ l=1,\ldots,T] \\
&= \bar{\boldsymbol{B}}\boldsymbol{Z}'\boldsymbol{Z}\varsigma\frac{2}{T}\Bigg[\bigg(\frac{1+\rho^2}{1-\rho^2}\bigg(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\bigg)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
&\quad -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\bigg)_{l=1,\ldots,T}\Bigg] \\
&= 2\bar{\boldsymbol{B}}\boldsymbol{Z}'\boldsymbol{Z}/T\varsigma\Bigg[\bigg(\frac{1+\rho^2}{1-\rho^2}\bigg(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\bigg)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
&\quad -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\bigg)_{l=1,\ldots,T}\Bigg] \\
&= 2\varsigma\Bigg[\bigg(\frac{1+\rho^2}{1-\rho^2}\bigg(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\bigg)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
&\quad -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\bigg)_{l=1,\ldots,T}\Bigg].
\end{aligned} \tag{A.419}$$

For A estimator

$$\begin{aligned}
\mathrm{E}(w_0d_{1\varsigma}) &= \mathrm{E}(w_0{d_{1\varsigma}}^{A}) = \mathrm{E}(w_0\bar{\boldsymbol{G}}(\boldsymbol{Z}'\boldsymbol{\Omega}^2\bar{\boldsymbol{u}}/\sqrt{T})) \\
&= \bar{\boldsymbol{G}}\boldsymbol{Z}'\boldsymbol{\Omega}^2\,\mathrm{E}(w_0\bar{\boldsymbol{u}}/\sqrt{T}) = \bar{\boldsymbol{G}}\boldsymbol{Z}'\boldsymbol{\Omega}^2\,\mathrm{E}[(w_0\bar{u}_l/\sqrt{T})\ l=1,\ldots,T] \\
&= \bar{\boldsymbol{G}}\boldsymbol{Z}'\boldsymbol{\Omega}^2\boldsymbol{Z}\varsigma\frac{2}{T}\Bigg[\bigg(\frac{1+\rho^2}{1-\rho^2}\bigg(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\bigg)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\ldots,T}\Big] \\
= \;& 2(\mathbf{Z}'\boldsymbol{\Omega}^2\mathbf{Z}/T)^{-1}\mathbf{Z}'\boldsymbol{\Omega}^2\mathbf{Z}/T\varsigma\Big[\Big(\frac{1+\rho^2}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\Big)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
& -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\ldots,T}\Big] \\
= \;& 2\varsigma\Big[\Big(\frac{1+\rho^2}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)})\Big)-\frac{1}{1-\rho^2}(\rho^{2l}+\rho^{2(T-l+1)}) \\
& -\sum_{t=2}^{T+1}\rho\frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2}-\sum_{t=0}^{T-1}\rho\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\ldots,T}\Big].
\end{aligned} \tag{A.420}
$$

By using equations (A.402), (A.413), (A.414) and (A.416) we have

$$
\begin{aligned}
w_i\bar{u}_l/\sqrt{T} &= \sqrt{T}(\boldsymbol{u}'\boldsymbol{\Omega}_{\varsigma_i}\boldsymbol{u}/T+a_i)\bar{u}_l/\sqrt{T} \\
&= \sqrt{T}\Big(-\frac{1}{2T}\sum_{t'=1}^{T}\sum_{t=1}^{T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{t'i}}{\sigma_{t'}{}^2}\Big]\psi_t\psi_{t'}+a_i\Big)\sigma_l{}^2((1-\rho^2)\psi_l{}^2-1)/\sqrt{T} \\
&= (1-\rho^2)\sigma_l{}^2(-1/2T)\sum_{t'=1}^{T}\sum_{t=1}^{T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{t'i}}{\sigma_{t'}{}^2}\Big]\psi_t\psi_{t'}\psi_l{}^2 \\
&\quad +\sigma_l{}^2\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{2T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{t'i}}{\sigma_{t'}{}^2}\Big]\psi_t\psi_{t'}-a_i\sigma_l{}^2+a_i\sigma_l{}^2(1-\rho^2)\psi_l{}^2 \implies
\end{aligned} \tag{A.421}
$$

$$
\begin{aligned}
\mathrm{E}(w_i\bar{u}_l/\sqrt{T}) &= (1-\rho^2)\sigma_l{}^2(-1/2T)\sum_{t'=1}^{T}\sum_{t=1}^{T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{t'i}}{\sigma_{t'}{}^2}\Big]\mathrm{E}(\psi_t\psi_{t'}\psi_l{}^2) \\
&\quad +\sigma_l{}^2\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{2T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{t'i}}{\sigma_{t'}{}^2}\Big]\mathrm{E}(\psi_t\psi_{t'})+a_i\sigma_l{}^2(1-\rho^2)\,\mathrm{E}(\psi_l{}^2)-a_i\sigma_l{}^2 \\
&= (1-\rho^2)\sigma_l{}^2(-1/2T)\sum_{t'=1}^{T}\sum_{t=1}^{T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-t'|}}{(1-\rho^2)^2}+2\frac{\rho^{|t-l|+|l-t'|}}{(1-\rho^2)^2}\Big] \\
&\quad +\sigma_l{}^2\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{2T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{t'i}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-t'|}}{1-\rho^2}\Big]+a_i\sigma_l{}^2-a_i\sigma_l{}^2 \\
&= \sigma_l{}^2(-1/2T)\sum_{t'=1}^{T}\sum_{t=1}^{T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-t'|}}{1-\rho^2}+2\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big] \\
&\quad +\sigma_l{}^2\sum_{t'=1}^{T}\sum_{t=1}^{T}\frac{1}{2T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{t'i}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-t'|}}{1-\rho^2}\Big] \\
&= \sigma_l{}^2(-1/2T)\sum_{t'=1}^{T}\sum_{t=1}^{T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[2\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big] \\
&= -\sigma_l{}^2\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T}r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big] \implies
\end{aligned}
$$

$$\begin{aligned}
\mathrm{E}(\bar{\boldsymbol{u}}\boldsymbol{w}'/\sqrt{T}) &= \mathrm{E}[(w_i\bar{u}_l/\sqrt{T})\ l=1,\ldots,T,\ i=1,\ldots,m] \\
&= \Bigg[\Bigg(-\sigma_l{}^2\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg] \\
&= \Bigg[\Bigg(-z_l'\varsigma\frac{1}{T}\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg] \\
&= -\boldsymbol{Z}\varsigma/T\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg] \implies \qquad (A.422)
\end{aligned}$$

By using equations (A.331), (A.341) and (A.422) we get the following results:

For GQ estimator

$$\begin{aligned}
\mathrm{E}(\boldsymbol{w}'d_{1\varsigma}) &= \mathrm{E}(\boldsymbol{w}'d_{1\varsigma}{}^{GQ}) = \mathrm{E}(\mathrm{tr}\,\boldsymbol{w}'d_{1\varsigma}{}^{GQ}) = \mathrm{E}(\mathrm{tr}\,d_{1\varsigma}{}^{GQ}\boldsymbol{w}') \\
&= \mathrm{tr}\,\mathrm{E}(d_{1\varsigma}{}^{GQ}\boldsymbol{w}') = \mathrm{tr}\,\mathrm{E}[(\bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T})\boldsymbol{w}'] = \mathrm{tr}\,\bar{\boldsymbol{B}}\boldsymbol{Z}'\,\mathrm{E}(\bar{\boldsymbol{u}}\boldsymbol{w}'/\sqrt{T}) \\
&= -\mathrm{tr}\,\bar{\boldsymbol{B}}\boldsymbol{Z}'\boldsymbol{Z}/T\varsigma\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg] \\
&= -\mathrm{tr}(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{Z}/T\varsigma\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg] \\
&= -\mathrm{tr}\,\varsigma\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg]. \qquad (A.423)
\end{aligned}$$

For A estimator

$$\begin{aligned}
\mathrm{E}(\boldsymbol{w}'d_{1\varsigma}) &= \mathrm{E}(\boldsymbol{w}'d_{1\varsigma}{}^{A}) = \mathrm{E}(\mathrm{tr}\,\boldsymbol{w}'d_{1\varsigma}{}^{A}) = \mathrm{E}(\mathrm{tr}\,d_{1\varsigma}{}^{A}\boldsymbol{w}') \\
&= = \mathrm{tr}\,\mathrm{E}(d_{1\varsigma}{}^{A}\boldsymbol{w}') = \mathrm{tr}\,\mathrm{E}[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})\boldsymbol{w}'] = \mathrm{tr}\,\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\,\mathrm{E}(\bar{\boldsymbol{u}}\boldsymbol{w}'/\sqrt{T}) \\
&= -\mathrm{tr}\,\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}/T)\varsigma\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg] \\
&= -\mathrm{tr}(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}/T)^{-1}(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}/T)\varsigma\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg] \\
&= -\mathrm{tr}\,\varsigma\Bigg[\Bigg(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2}+\frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Bigg)_{l=1,\ldots,T,\ i=1,\ldots,m}\Bigg]. \qquad (A.424)
\end{aligned}$$

Calculation of matrix $\boldsymbol{\Lambda}_{\varsigma\varsigma}$

Since $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{G}}_H$ are symmetric matrices, equations (A.186), (A.331), (A.341), (A.351), (A.358) and Lemma A.33 imply the following results:

i. For GQ estimator we have that

$$\boldsymbol{d}_{1\varsigma}{}^{GQ} = \bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{u}/\sqrt{T} \implies \tag{A.425}$$

$$\begin{aligned}
\mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma}) &= \mathrm{E}(\boldsymbol{d}_{1\varsigma}{}^{GQ}\boldsymbol{d}_{1\varsigma}{}^{GQ\prime}) = \mathrm{E}[(\bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T})(\bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T})'] \\
&= \mathrm{E}[\bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}'\boldsymbol{Z}\bar{\boldsymbol{B}}/T] = \bar{\boldsymbol{B}}\boldsymbol{Z}'\,\mathrm{E}(\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}')\boldsymbol{Z}\bar{\boldsymbol{B}}/T \\
&= \bar{\boldsymbol{B}}\boldsymbol{Z}'(2\boldsymbol{\Omega}_H^{-2})\boldsymbol{Z}\bar{\boldsymbol{B}}/T = 2\bar{\boldsymbol{B}}(\boldsymbol{Z}'\boldsymbol{\Omega}_H^{-2}\boldsymbol{Z}/T)\bar{\boldsymbol{B}} \\
&= 2\bar{\boldsymbol{B}}\bar{\boldsymbol{\Gamma}}_H\bar{\boldsymbol{B}} \implies
\end{aligned}$$

$$\boldsymbol{\Lambda}_{\varsigma\varsigma}{}^{GQ} = \lim_{T\to\infty}\mathrm{E}(\boldsymbol{d}_{1\varsigma}{}^{GQ}\boldsymbol{d}_{1\varsigma}{}^{GQ}) = \lim_{T\to\infty} 2\bar{\boldsymbol{B}}\bar{\boldsymbol{\Gamma}}_H\bar{\boldsymbol{B}}. \tag{A.426}$$

Thus, for the GQ estimator of ς, matrix $\boldsymbol{\Lambda}_{\varsigma\varsigma}$ can be estimated as

$$\boldsymbol{\Lambda}_{\varsigma\varsigma} = 2\bar{\boldsymbol{B}}\bar{\boldsymbol{\Gamma}}_H\bar{\boldsymbol{B}}. \tag{A.427}$$

ii. For the A, IA and ML estimators of ς we have that

$$\boldsymbol{d}_{1\varsigma}{}^{A} = \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T} \implies \tag{A.428}$$

$$\begin{aligned}
\mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma}) &= \mathrm{E}(\boldsymbol{d}_{1\varsigma}{}^{A}\boldsymbol{d}_{1\varsigma}{}^{A\prime}) = \mathrm{E}[(\bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})(\bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})'] \\
&= \mathrm{E}[\bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{\boldsymbol{G}}_H/T] = \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2 E(\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}')\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{\boldsymbol{G}}_H/T \\
&= \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2(2\boldsymbol{\Omega}_H^{-2})\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{\boldsymbol{G}}_H/T \\
&= 2\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}/T)\bar{\boldsymbol{G}}_H = 2\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_H\bar{\boldsymbol{G}}_H = 2\bar{\boldsymbol{G}}_H \implies
\end{aligned}$$

$$\boldsymbol{\Lambda}_{\varsigma\varsigma}{}^{A} = \lim_{T\to\infty}\mathrm{E}(\boldsymbol{d}_{1\varsigma}{}^{A}\boldsymbol{d}_{1\varsigma}{}^{A}) = \lim_{T\to\infty} 2\bar{\boldsymbol{G}}_H. \tag{A.429}$$

Thus, for the A, IA and ML estimators of ς, matrix $\boldsymbol{\Lambda}_{\varsigma\varsigma}$ can be estimated as

$$\boldsymbol{\Lambda}_{\varsigma\varsigma}{}^{A} = 2\bar{\boldsymbol{G}}_H. \tag{A.430}$$

We define the $m \times 1$ vectors

$$\boldsymbol{\xi}_H = \sum_{t=1}^{T} v_t z_t/T,\ \boldsymbol{\xi}_{H1} = \sum_{t=1}^{T}\sigma^{-4}v_t z_t/T,\ \boldsymbol{\xi}_{H2} = \sum_{t=1}^{T}\sigma^{-4}\boldsymbol{x}'_{Ht}\boldsymbol{G}_H\boldsymbol{x}_{Ht}z_t/T, \tag{A.431}$$

where $v_t = 2\sigma_t^2 \boldsymbol{x}'_{_{H}t} \boldsymbol{B}_H \boldsymbol{x}_{_{H}t} - \boldsymbol{x}'_{_{H}t} \boldsymbol{B}_H \boldsymbol{\Gamma}_H \boldsymbol{B}_H \boldsymbol{x}_{_{H}t}$.

Calculation of vector $\boldsymbol{\kappa}$

Since matrices $\boldsymbol{\Omega}_H, \boldsymbol{\Omega}_H^{-2}$ and $\boldsymbol{\Omega}_{_{H}\varsigma_i}$ are diagonal, equations (A.186), (A.331), (A.341), (A.351), (A.396), definition (A.431) and Lemma A.33 imply the following results:

i. For GQ estimator we have that

$$\begin{aligned} \mathrm{E}(d_{2\varsigma}) &= \mathrm{E}(d_{2\varsigma}{}^{GQ}) = \mathrm{E}(\bar{\boldsymbol{B}}\boldsymbol{Z}'\boldsymbol{\varepsilon}/\sqrt{T}) = \bar{\boldsymbol{B}}\boldsymbol{Z}'\,\mathrm{E}(\boldsymbol{\varepsilon})/\sqrt{T} = \bar{\boldsymbol{B}}\boldsymbol{Z}'(\boldsymbol{v}/\sqrt{T})/\sqrt{T} \\ &= \bar{\boldsymbol{B}}[(z_t)_{t=1,\dots,T}][(v_t)_{t=1,\dots,T}]/T = \bar{\boldsymbol{B}}\xi_H \implies \\ \boldsymbol{\kappa}_\varsigma &= -\lim_{T\to\infty} \mathrm{E}(\delta_{2\varsigma}) = -\lim_{T\to\infty} \bar{\boldsymbol{B}}\boldsymbol{\xi}_H. \end{aligned} \tag{A.432}$$

Thus, for the GQ estimator of $\boldsymbol{\varsigma}$, κ_ς expressed as

$$\boldsymbol{\kappa}_\varsigma = -\bar{\boldsymbol{B}}\xi_H. \tag{A.433}$$

ii. For A estimator of $\boldsymbol{\varsigma}$ we have that

$$\begin{aligned} \mathrm{E}[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T})] &= \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\,\mathrm{E}(\boldsymbol{\varepsilon})/\sqrt{T} = \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2(\boldsymbol{v}/\sqrt{T})/\sqrt{T} \\ &= \bar{\boldsymbol{G}}_H[(z_t)_{t=1,\dots,T}]\,\mathrm{diag}(\sigma_t{}^{-4})[(v_t)_{t=1,\dots,T}]/T \\ &= \bar{\boldsymbol{G}}_H\sum_{t=1}^{T}\sigma_t{}^{-4}v_t z_t/T = \bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{_{H}1}, \end{aligned} \tag{A.434}$$

$$\begin{aligned} \mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^{GQ}] &= \mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})(\bar{b}_i'\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T})] \\ &= \mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})(\bar{\boldsymbol{u}}'\boldsymbol{Z}\bar{b}_i/\sqrt{T})] = \mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}'\boldsymbol{Z}\bar{b}_i/T)] \\ &= \boldsymbol{Z}'\boldsymbol{\Omega}_H^2\,\mathrm{E}(\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}')\boldsymbol{Z}\bar{b}_i/T = \boldsymbol{Z}'\boldsymbol{\Omega}^2\,\mathrm{E}(\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}')\boldsymbol{Z}\bar{b}_i/T \\ &= \boldsymbol{Z}'\boldsymbol{\Omega}_H^2(2\boldsymbol{\Omega}_H^{-2})\boldsymbol{Z}\bar{b}_i/T = 2(\boldsymbol{Z}'\boldsymbol{Z}/T)\bar{b}_i = 2\bar{\boldsymbol{F}}\bar{b}_i, \end{aligned} \tag{A.435}$$

where $\bar{b}_i$ is i-column of $\bar{\boldsymbol{B}}$ matrix.

By working as in equation (A.435) we get

$$\begin{aligned} \mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{_{H}\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^{GQ}] &= \mathrm{E}(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{_{H}\varsigma_i}\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}'\boldsymbol{Z}\bar{b}_i/T) = \boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{_{H}\varsigma_i}\,\mathrm{E}(\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}')\boldsymbol{Z}\bar{b}_i/T \\ &= \boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{_{H}\varsigma_i}(2\boldsymbol{\Omega}_H^{-2})\boldsymbol{Z}\bar{b}_i/T = 2(\boldsymbol{Z}'\boldsymbol{\Omega}_{_{H}\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T)\bar{b}_i. \end{aligned} \tag{A.436}$$

By combining equations (A.341), (A.396), (A.434), (A.435) and (A.436) we find that

$$\begin{aligned} \mathrm{E}(d_{2\varsigma}) &= \mathrm{E}(d_{2\varsigma}{}^{A}) \\ &= \mathrm{E}[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T} + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{_{H}\varsigma_i}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^{GQ} - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{_{H}\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^{GQ}] \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{\varepsilon}/\sqrt{T}] + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H\,\mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^{GQ}] - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\,\mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^{GQ}] \\
&= \bar{\boldsymbol{G}}_H\xi_{H}1 + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H2\bar{\boldsymbol{F}}\bar{b}_i - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H2(\boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T)\bar{b}_i \implies \\
\boldsymbol{\kappa}_\varsigma &= -\lim_{T\to\infty}\mathrm{E}(\delta_{2\varsigma}) \\
&= -\lim_{T\to\infty}[\bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H}1 + 4\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{F}}\bar{b}_i - 4\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T)\bar{b}_i] \\
&= -\lim_{T\to\infty}\Big[\bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H}1 + 4\sum_{i=1}^{m}[\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{F}}\bar{b}_i - \bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T)\bar{b}_i]\Big] \\
&= -\lim_{T\to\infty}\Big[\bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H}1 + 4\bar{\boldsymbol{G}}_H\sum_{i=1}^{m}[\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H e_i - (\boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T)\bar{b}_i]\Big] \\
&= -\lim_{T\to\infty}\Big[\bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H}1 + 4\bar{\boldsymbol{G}}_H\sum_{i=1}^{m}[\bar{\boldsymbol{A}}_{H\varsigma_i}g_{H}i - (\boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T)\bar{b}_i]\Big].
\end{aligned}
\tag{A.437}
$$

Thus, for the A estimator of $\boldsymbol{\varsigma}$, $\boldsymbol{\kappa}$ can be estimated as

$$
\boldsymbol{\kappa}_\varsigma = -\bar{\boldsymbol{G}}_H\xi_{H}1 - 4\bar{\boldsymbol{G}}_H\sum_{i=1}^{m}[\bar{\boldsymbol{A}}_{H\varsigma_i}g_{H}i - (\boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T)\bar{b}_i \tag{A.438}
$$

where $\bar{\boldsymbol{A}}_{H\varsigma_i} = \boldsymbol{Z}'\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{\Omega}_H{}^{-1}\boldsymbol{Z}/T, \bar{\boldsymbol{g}}_i$ is the i-th column of matrix $\bar{\boldsymbol{G}}_H$ and $\bar{b}_i$ is the i-th column of matrix $\bar{\boldsymbol{B}}_H$. Moreover,

$$
\bar{\boldsymbol{F}}\bar{\boldsymbol{B}} = \bar{\boldsymbol{F}}(\bar{b}_1,\ldots,\bar{b}_m) = (\bar{\boldsymbol{F}}\bar{b}_1,\ldots,\bar{\boldsymbol{F}}\bar{b}_m) = \boldsymbol{I}_m \Rightarrow \bar{\boldsymbol{F}}\bar{b}_i = e_i, \tag{A.439}
$$

where e_i is the i-th column of matrix $\boldsymbol{I}_m$.

iii. For the IA and ML estimators of $\boldsymbol{\varsigma}$ we have that

$$
\begin{aligned}
\mathrm{E}[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{\varepsilon}}/\sqrt{T})] &= \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\,\mathrm{E}(\bar{\boldsymbol{\varepsilon}})/\sqrt{T} = \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2(\boldsymbol{X}_H\boldsymbol{G}_H\boldsymbol{x}_{Ht}/\sqrt{T})/\sqrt{T} \\
&= \bar{\boldsymbol{G}}_H[(z_t)_{t=1,\ldots,T}]\,\mathrm{diag}(\sigma_t{}^{-4})[(\boldsymbol{x}'_{Ht}\boldsymbol{G}_H\boldsymbol{x}_{Ht})_{t=1,\ldots,T}]/T \\
&= \bar{\boldsymbol{G}}_H\sum_{t=1}^{T}\sigma_t{}^{-4}\boldsymbol{x}'_{Ht}\boldsymbol{G}_H\boldsymbol{x}_{Ht}z_t/T = \bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H}2,
\end{aligned}
\tag{A.440}
$$

$$
\begin{aligned}
\mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^{A}] &= \mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})(\bar{g}_i'\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})] \\
&= \mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})(\bar{\boldsymbol{u}}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{g}_i/\sqrt{T})] = \mathrm{E}(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{g}_i/T) \\
&= \boldsymbol{Z}'\boldsymbol{\Omega}_H^2\,\mathrm{E}(\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}')\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{g}_i/T = \boldsymbol{Z}'\boldsymbol{\Omega}_H^2 2\boldsymbol{\Omega}_H^{-2}\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{g}_i/T = 2\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{g}_i/T \\
&= 2\bar{A}_H\bar{g}_{H}i,
\end{aligned}
\tag{A.441}
$$

where $\bar{g}_i$ is i-column of $\bar{\boldsymbol{G}}_H$ matrix.

By working as in equation (A.441) we get

$$\begin{aligned} \mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^A] &= \mathrm{E}(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}'\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{g}_i/T) \\ &= \boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}2\boldsymbol{\Omega}_H{}^{-2}\boldsymbol{\Omega}_H^2\boldsymbol{Z}\bar{g}_i/T = 2\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\boldsymbol{Z}\bar{g}_i/T \\ &= 2\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{g}_{Hi}. \end{aligned} \tag{A.442}$$

From equations (A.357), (A.440), (A.441) and (A.442)

$$\begin{aligned} \mathrm{E}(d_{2\varsigma}) &= \mathrm{E}(d_{2\varsigma}{}^{\alpha}) \\ &= \mathrm{E}[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{\varepsilon}}/\sqrt{T}) + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^A - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^A] \\ &= \mathrm{E}[\bar{\boldsymbol{G}}_H(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{\varepsilon}}/\sqrt{T})] + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H\,\mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^A] \\ &\quad -2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\,\mathrm{E}[(\boldsymbol{Z}'\boldsymbol{\Omega}_H\boldsymbol{\Omega}_{H\varsigma_i}\bar{\boldsymbol{u}}/\sqrt{T})d_{1\varsigma_i}{}^A] \\ &= \bar{\boldsymbol{G}}_H\xi_{H2} + 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H2\bar{\boldsymbol{A}}_H\bar{g}_i - 2\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H2\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{g}_i \\ &= \bar{\boldsymbol{G}}_H\xi_{H2} + 4\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_H\bar{g}_i - 4\sum_{i=1}^{m}\bar{\boldsymbol{G}}_H\bar{\boldsymbol{A}}_{H\varsigma_i}\bar{g}_i = \bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H2} \implies \\ \boldsymbol{\kappa}_{\varsigma} &= -\lim_{T\to\infty}\mathrm{E}(\delta_{2\varsigma}) = -\lim_{T\to\infty}\bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H2}. \end{aligned} \tag{A.443}$$

Thus, for *IA* and *ML* estimators of ς we have that

$$\boldsymbol{\kappa}_{\varsigma} = -\bar{\boldsymbol{G}}_H\boldsymbol{\xi}_{H2}. \tag{A.444}$$

Some useful results

$$\begin{aligned} \rho_1 &= -\alpha\boldsymbol{u}_{AR}'\boldsymbol{\Omega}_{AR2}\boldsymbol{u}_{AR}/2\sqrt{T} \\ &= \frac{-\alpha}{2\sqrt{T}}\boldsymbol{u}'\boldsymbol{\Sigma}^{-1/2}[2\rho\boldsymbol{I}-\boldsymbol{D}]\boldsymbol{\Sigma}^{-1/2}\boldsymbol{u} \\ &= \frac{-\alpha}{2\sqrt{T}}[2\rho\sum_{t=1}^{T}\frac{u_t{}^2}{\sigma_t{}^2} - 2\sum_{t=1}^{T-1}\frac{u_t}{\sigma_t}\frac{u_{t+1}}{\sigma_{t+1}}] \\ &= \frac{-\alpha}{2\sqrt{T}}\Big[2\rho\sum_{t=1}^{T}\psi_t{}^2 - 2\sum_{t=1}^{T-1}\psi_t\psi_{t+1}] \\ &= \frac{-\alpha}{\sqrt{T}}[\rho\sum_{t=1}^{T}\psi_t{}^2 - \sum_{t=1}^{T-1}\psi_t\psi_{t+1}\Big]. \end{aligned} \tag{A.445}$$

By using (A.413) and (A.445) we get

$$\begin{aligned}
\rho_1\bar{u}_l/\sqrt{T} &= \frac{-\alpha}{\sqrt{T}}\Big[\rho\sum_{t=1}^{T}\psi_t{}^2-\sum_{t=1}^{T-1}\psi_t\psi_{t+1}]\sigma_l{}^2[(1-\rho^2)\psi_l^2-1]/\sqrt{T} \\
&= \frac{-\alpha}{T}[\rho\sigma_l{}^2(1-\rho^2)\sum_{t=1}^{T}\psi_t{}^2\psi_l{}^2-\rho\sigma_l{}^2\sum_{t=1}^{T}\psi_t{}^2-\sigma_l{}^2(1-\rho^2)\sum_{t=1}^{T-1}\psi_t\psi_{t+1}\psi_l{}^2 \\
&\quad +\sigma_l{}^2\sum_{t=1}^{T-1}\psi_t\psi_{t+1}\Big], \qquad (A.446)
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathrm{E}(\rho_1\bar{u}_l/\sqrt{T}) &= \frac{-\alpha}{T}\Big[\rho\sigma_l{}^2(1-\rho^2)\sum_{t=1}^{T}\mathrm{E}(\psi_t{}^2\psi_l{}^2)-\rho\sigma_l{}^2\sum_{t=1}^{T}\mathrm{E}(\psi_t{}^2)-\sigma_l{}^2(1-\rho^2)\sum_{t=1}^{T-1}\mathrm{E}(\psi_t\psi_{t+1}\psi_l{}^2) \\
&\quad +\sigma_l{}^2\sum_{t=1}^{T-1}\mathrm{E}(\psi_t\psi_{t+1})\Big] \\
&= \frac{-\alpha}{T}\Big[\rho\sigma_l{}^2(1-\rho^2)\sum_{t=1}^{T}(\frac{1}{1-\rho^2}+2\frac{\rho^{2|t-l|}}{(1-\rho^2)^2})-\rho\sigma_l{}^2\sum_{t=1}^{T}\frac{1}{1-\rho^2} \\
&\quad -\sigma_l{}^2(1-\rho^2)\sum_{t=1}^{T-1}(\frac{\rho}{(1-\rho^2)^2}+2\frac{\rho^{|t-l|+|l-t-1|}}{(1-\rho^2)^2})+\sigma_l{}^2\sum_{t=1}^{T-1}\frac{\rho}{(1-\rho^2)^2}\Big] \\
&= \frac{-\alpha}{T}\Big[\rho\sigma_l{}^2\sum_{t=1}^{T}\Big(\frac{1}{1-\rho^2}+2\frac{\rho^{2|t-l|}}{1-\rho^2}\Big)-\rho\sigma_l{}^2\sum_{t=1}^{T}\frac{1}{1-\rho^2} \\
&\quad -\sigma_l{}^2\sum_{t=1}^{T-1}\Big(\frac{\rho}{1-\rho^2}+2\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)+\sigma_l{}^2\sum_{t=1}^{T-1}\frac{\rho}{1-\rho^2}\Big] \\
&= \frac{-\alpha\sigma_l{}^2}{T}\Big[\frac{\rho T}{1-\rho^2}+2\rho\sum_{t=1}^{T}\frac{\rho^{2|t-l|}}{1-\rho^2}-\frac{\rho T}{1-\rho^2} \\
&\quad -\frac{(T-1)\rho}{1-\rho^2}-2\sum_{t=1}^{T-1}\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}+\frac{(T-1)\rho}{1-\rho^2}\Big] \\
&= \frac{-\alpha\sigma_l{}^2}{T}\Big[2\rho\sum_{t=1}^{T}\frac{\rho^{2|t-l|}}{1-\rho^2}-2\sum_{t=1}^{T-1}\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big] \\
&= \frac{-\alpha\sigma_l{}^2}{T}\Big[\frac{2\rho}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})\Big)-2\sum_{t=1}^{T-1}\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big] \\
&= \frac{-2\alpha\sigma_l{}^2}{T}\Big[\frac{\rho}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2}-\frac{1}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})\Big)-\sum_{t=1}^{T-1}\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big]. \qquad (A.447)
\end{aligned}$$

By using equation (A.447) we get

$$\begin{aligned}
\mathrm{E}(\rho_1\bar{\boldsymbol{u}}/\sqrt{T}) &= \mathrm{E}[(\rho_1\bar{u}_l/\sqrt{T})_{l=1,\dots,T}] \\
&= \Big[\Big(\frac{-2\alpha\sigma_l{}^2}{T}\Big[\frac{(1+\rho^2)\rho}{(1-\rho^2)^2}-\frac{\rho}{(1-\rho^2)^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big]\Big)_{l=1,\dots,T}\Big] \\
&= \boldsymbol{Z}\boldsymbol{\varsigma}/T\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2}-\frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big]. \qquad (A.448)
\end{aligned}$$

By using equations(A.425), (A.428) and (A.448) we have

$$\begin{aligned}
\mathrm{E}(\rho_1 d_{1\varsigma}) &= \mathrm{E}(\rho_1 {d_{1\varsigma}}^{GQ}) = \mathrm{E}(\rho_1 \bar{\boldsymbol{B}}\boldsymbol{Z}'\bar{\boldsymbol{u}}/\sqrt{T}) \\
&= \bar{\boldsymbol{B}}\boldsymbol{Z}'\,\mathrm{E}(\rho_1\bar{\boldsymbol{u}}/\sqrt{T}) = \bar{\boldsymbol{B}}\boldsymbol{Z}'\,\mathrm{E}[(\rho_1\bar{u}_l/\sqrt{T})_{l=1,\dots,T}] \\
&= \bar{\boldsymbol{B}}\boldsymbol{Z}'\boldsymbol{Z}/T\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2}-\frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \\
&= \varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2}-\frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big]. \qquad (A.449)
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(\rho_1 d_{1\varsigma}) &= \mathrm{E}(\rho_1 {d_{1\varsigma}}^{A}) = \mathrm{E}(\rho_1 \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\bar{u}/\sqrt{T}) \\
&= \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\,\mathrm{E}(\rho_1\bar{\boldsymbol{u}}/\sqrt{T}) = \bar{\boldsymbol{G}}_H\boldsymbol{Z}'\boldsymbol{\Omega}_H^2\,\mathrm{E}[(\rho_1\bar{u}_l/\sqrt{T})_{l=1,\dots,T}] \\
&= \bar{\boldsymbol{G}}\boldsymbol{Z}'\boldsymbol{\Omega}^2\boldsymbol{Z}\varsigma\frac{2}{T}\varsigma\Big[\Big(-\Big[\frac{(1+\rho^2)\rho}{1-\rho^2}-\frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \\
&= \varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2}-\frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \Longrightarrow \qquad (A.450)
\end{aligned}$$

$$\lambda_{\rho\varsigma} = \lim_{T\to\infty}\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2}-\frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big]. \qquad (A.451)$$

From Lemmas A.31 and UR.2 and equation (A.197) we have

$$\begin{aligned}
a_\rho &= -\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_\rho\boldsymbol{u}/T) = -\operatorname{tr}\boldsymbol{\Omega}_\rho\boldsymbol{\Omega}^{-1} = O(T^{-1}) = O(\tau^2) \Longrightarrow \\
a_\rho^2 &= O(\tau^4). \qquad (A.452)
\end{aligned}$$

From Lemmas A.31 and UR.2 and equation (A.203) we have

$$\begin{aligned}
a_{\rho\rho} &= \frac{1}{2}\mathrm{E}(\boldsymbol{u}'\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{u}/T) = \operatorname{tr}\boldsymbol{\Omega}_{\rho\rho}\boldsymbol{\Omega}^{-1} \\
&= \frac{1}{2}\Big[\frac{2}{\alpha}-\frac{4}{\alpha T}\Big] = \frac{1}{\alpha}-\frac{2}{\alpha T}. \qquad (A.453)
\end{aligned}$$

For the parameters (1.33) the following results hold:

By using Lemma A.31 and equations (A.366), (A.404), (A.412) (A.451) and (A.452) we have

$$\begin{aligned}
\lambda_{0\rho} &= \lim_{T\to\infty}\mathrm{E}(w_0\rho_1) - a_\rho\lambda_{\rho\rho} - \boldsymbol{a}'\lambda_{\rho\varsigma} \\
&= \lim_{T\to\infty}O(T^{-1}) - \alpha O(\tau^2) - \boldsymbol{a}'\lim_{T\to\infty}\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2}-\frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \\
&= -\boldsymbol{a}'\lim_{T\to\infty}\varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2}-\frac{\rho}{1-\rho^2}(\rho^{2l}-\rho^{2T-2l+2})-\sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] + O(\tau^2). \qquad (A.454)
\end{aligned}$$

By using Lemma A.31 and equations (A.368), (A.419), (A.420), (A.451) and (A.452) we have

$$\begin{aligned}
\lambda_{0\varsigma} &= \lim_{T\to\infty} \mathrm{E}(w_0 \boldsymbol{d}_{1\varsigma}) - a_\rho \lambda_{\rho\varsigma} - \boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a} \\
&= \lim_{T\to\infty} 2\varsigma\Big[\Big(\frac{1+\rho^2}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2} - \frac{1}{1-\rho^2}(\rho^{2l} + \rho^{2(T-l+1)})\Big) - \frac{1}{1-\rho^2}(\rho^{2l} + \rho^{2(T-l+1)}) \\
&\quad - \sum_{t=2}^{T+1} \rho \frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1} \rho \frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\dots,T}\Big] \\
&\quad - a_\rho \lim_{T\to\infty} \varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l} - \rho^{2T-2l+2}) - \sum_{t=1}^{T-1} \rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \\
&\quad - \boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a} \\
&= \lim_{T\to\infty} 2\varsigma\Big[\Big(\frac{1+\rho^2}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2} - \frac{1}{1-\rho^2}(\rho^{2l} + \rho^{2(T-l+1)})\Big) - \frac{1}{1-\rho^2}(\rho^{2l} + \rho^{2(T-l+1)}) \\
&\quad - \sum_{t=2}^{T+1} \rho \frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1} \rho \frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\dots,T}\Big] \\
&\quad - O(\tau^2) \lim_{T\to\infty} \varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l} - \rho^{2T-2l+2}) - \sum_{t=1}^{T-1} \rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \\
&\quad - \boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a}. \qquad \text{(A.455)}
\end{aligned}$$

By using Lemma A.31 and equations (A.404), (A.412), (A.419), (A.420), (A.451) and (A.452) we have

$$\begin{aligned}
\lambda_0 &= \lim_{T\to\infty} \mathrm{E}(\sigma_0{}^2) = 2 - 2a_\rho \lim_{T\to\infty} \mathrm{E}(w_0\rho_1) + a_\rho{}^2 \lambda_{\rho\rho} \\
&\quad - 2\boldsymbol{a}' \lim_{T\to\infty} \mathrm{E}(w_0 \boldsymbol{d}_{1\varsigma}) - 2a_\rho \boldsymbol{a}' \lambda_{\rho\varsigma} + \boldsymbol{a}'\boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a} \\
&= 2 - 2a_\rho \lim_{T\to\infty} O(T^{-1}) + \alpha O(\tau^4) \\
&\quad - 2\boldsymbol{a}' \lim_{T\to\infty} 2\varsigma\Big[\Big(\frac{1+\rho^2}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2} - \frac{1}{1-\rho^2}(\rho^{2l} + \rho^{2(T-l+1)})\Big) \\
&\quad - \frac{1}{1-\rho^2}(\rho^{2l} + \rho^{2(T-l+1)}) - \sum_{t=2}^{T+1} \rho \frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1} \rho \frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\dots,T}\Big] \\
&\quad - 2a_\rho \boldsymbol{a}' \lim_{T\to\infty} \varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l} - \rho^{2T-2l+2}) - \sum_{t=1}^{T-1} \rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] + \boldsymbol{a}'\boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a} \\
&= 2 - 4\boldsymbol{a}' \lim_{T\to\infty} \varsigma\Big[\Big(\frac{1+\rho^2}{1-\rho^2}\Big(\frac{1+\rho^2}{1-\rho^2} - \frac{1}{1-\rho^2}(\rho^{2l} + \rho^{2(T-l+1)})\Big) \\
&\quad - \frac{1}{1-\rho^2}(\rho^{2l} + \rho^{2(T-l+1)}) - \sum_{t=2}^{T+1} \rho \frac{\rho^{|t-l|+|l-t+1|}}{1-\rho^2} - \sum_{t=0}^{T-1} \rho \frac{\rho^{|t-l|+|l-t-1|}}{1-\rho^2}\Big)_{l=1,\dots,T}\Big] \\
&\quad - 2O(\tau^2)\boldsymbol{a}' \lim_{T\to\infty} \varsigma\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l} - \rho^{2T-2l+2}) - \sum_{t=1}^{T-1} \rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \\
&\quad + \boldsymbol{a}'\boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a} + O(\tau^4). \qquad \text{(A.456)}
\end{aligned}$$

By using Lemma A.31 and equations (A.376), (A.407), (A.412), (A.423), (A.424) and (A.451) we have

$$
\begin{aligned}
\kappa_0 &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\sigma_0 + \sigma_1) \\
&= \lim_{T\to\infty} \mathrm{E}(w_\rho \rho_1) + \lim_{T\to\infty} \mathrm{E}(\boldsymbol{w}'\boldsymbol{d}_{1\varsigma}) - \lim_{T\to\infty} a_\rho \,\mathrm{E}(\rho_2) + \boldsymbol{a}'(-\kappa_\varsigma) \\
&\quad + \operatorname{tr} \bar{\boldsymbol{A}}\boldsymbol{\Lambda}_{\varsigma\varsigma} + a_{\rho\rho}\lambda_{\rho\rho} + \boldsymbol{a}'_{\rho\varsigma}\lambda_{\rho\varsigma} \\
&= -2 + \lim_{T\to\infty} O(T^{-1}) + \lim_{T\to\infty}\Big(-\operatorname{tr} \boldsymbol{\varsigma}\Big[\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2} + \frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Big)_{l=1,\dots,T,\ i=1,\dots,m}\Big]\Big) \\
&\quad -a_\rho \lim_{T\to\infty}\Big[-(\alpha/2\rho\alpha)[2(\rho^2 - n\alpha) + \alpha \operatorname{tr} \boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \operatorname{tr} \boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}]] + O(\tau^2)\Big] \\
&\quad +\alpha\Big[\frac{1}{\alpha} - \frac{2}{\alpha T}\Big] - \boldsymbol{a}'\kappa_\varsigma + \operatorname{tr} \bar{\boldsymbol{A}}\boldsymbol{\Lambda}_{\varsigma\varsigma} \\
&\quad +\boldsymbol{a}'_{\rho\varsigma}\lim_{T\to\infty}\boldsymbol{\varsigma}\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l} - \rho^{2T-2l+2}) - \sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \\
&= -2 + \lim_{T\to\infty} O(T^{-1}) + \lim_{T\to\infty}\Big(-\operatorname{tr} \boldsymbol{\varsigma}\Big[\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2} + \frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Big)_{l=1,\dots,T,\ i=1,\dots,m}\Big]\Big) \\
&\quad -O(\tau^2) \lim_{T\to\infty}\Big[-(\alpha/2\rho\alpha)[2(\rho^2 - n\alpha) + \alpha \operatorname{tr} \boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \operatorname{tr} \boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}]] + O(\tau^2)\Big] \\
&\quad +\alpha\Big[\frac{1}{\alpha} - \frac{2}{\alpha T}\Big] - \boldsymbol{a}'\kappa_\varsigma + \operatorname{tr} \bar{\boldsymbol{A}}\boldsymbol{\Lambda}_{\varsigma\varsigma} \\
&\quad +\boldsymbol{a}'_{\rho\varsigma}\lim_{T\to\infty}\boldsymbol{\varsigma}\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l} - \rho^{2T-2l+2}) - \sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big] \\
&= -1 + \lim_{T\to\infty}\Big(-\operatorname{tr} \boldsymbol{\varsigma}\Big[\Big(\sum_{t'=1}^{T}\sum_{t=1}^{T} r_{*tt'}\Big[\frac{z_{ti}}{\sigma_t{}^2} + \frac{z_{'ti}}{\sigma_{t'}{}^2}\Big]\Big[\frac{\rho^{|t-l|+|l-t'|}}{1-\rho^2}\Big]\Big)_{l=1,\dots,T,\ i=1,\dots,m}\Big]\Big) - \boldsymbol{a}'\kappa_\varsigma + \operatorname{tr} \bar{\boldsymbol{A}}\boldsymbol{\Lambda}_{\varsigma\varsigma} \\
&\quad -O(\tau^2) \lim_{T\to\infty}\Big[-(\alpha/2\rho\alpha)[2(\rho^2 - n\alpha) + \alpha \operatorname{tr} \boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR} + \operatorname{tr} \boldsymbol{A}_{AR}\boldsymbol{B}_{AR}\boldsymbol{\Gamma}_{AR}\boldsymbol{B}_{AR}]] + O(\tau^2)\Big] \\
&\quad +\boldsymbol{a}'_{\rho\varsigma}\lim_{T\to\infty}\boldsymbol{\varsigma}\Big[\Big(-2\Big[\frac{(1+\rho^2)\rho}{1-\rho^2} - \frac{\rho}{1-\rho^2}(\rho^{2l} - \rho^{2T-2l+2}) - \sum_{t=1}^{T-1}\rho^{|t-l|+|l-t-1|}\Big]\Big)_{l=1,\dots,T}\Big].
\end{aligned} \tag{A.457}
$$

□

Subtracting autocorrelation or heteroskedasticity respectively, the cross elements λ_0, are simplified as follows:

i. If there is no autocorrelation, $\rho = 0$. Then,

$$\lambda_{\rho\varsigma} = 0. \tag{A.458}$$

$$\lambda_{0\rho} = O(\tau^2). \tag{A.459}$$

$$\lambda_{0\varsigma} = -\boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a}. \tag{A.460}$$

$$\lambda_0 = 2 - 4\boldsymbol{a}'\boldsymbol{\varsigma} + \boldsymbol{a}'\boldsymbol{\Lambda}_{\varsigma\varsigma}\boldsymbol{a} + O(\tau^4). \tag{A.461}$$

$$\kappa_0 = -1 - \boldsymbol{a}'\kappa_\varsigma + \operatorname{tr} \bar{\boldsymbol{A}}\boldsymbol{\Lambda}_{\varsigma\varsigma} + O(\tau^4) \tag{A.462}$$

ii. If there is no heteroskedasticity, $\varsigma = 0$. Then,

$$\lambda_{\rho\varsigma} = 0. \tag{A.463}$$

$$\lambda_{0\rho} = O(\tau^2). \tag{A.464}$$

$$\lambda_{0\varsigma} = 0. \tag{A.465}$$

$$\lambda_0 = 2 + O(\tau^4). \tag{A.466}$$

$$\kappa_0 = -1 + O(\tau^4). \tag{A.467}$$

Appendix B

Matrix $\boldsymbol{\Omega}$

Equations (3.28b) and (3.28c) imply that $\boldsymbol{\Omega}^{-1} = \boldsymbol{P}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)\boldsymbol{P}'$ where $\boldsymbol{\Sigma} = [(\sigma_{ij})_{i,j=1,\dots,M}]$ and $\boldsymbol{P} = [(\delta_{ij}\boldsymbol{P}_i)_{i,j=1,\dots,M}]$ is a block diagonal matrix. Let $\boldsymbol{P}^{-1}$ and $\boldsymbol{P}'^{-1}$ be the inverse of $\boldsymbol{P}$ and $\boldsymbol{P}'$ respectively and let $\boldsymbol{\Sigma}^{-1} = [(\sigma^{ij})_{i,j=1,\dots,M}]$ be the inverse of $\boldsymbol{\Sigma}$.

Then by using (3.29) we find that

$$\begin{aligned}
\boldsymbol{\Omega}^{-1} = \boldsymbol{P}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)\boldsymbol{P}' &= \begin{bmatrix} \boldsymbol{P}_1 & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M \end{bmatrix} \begin{bmatrix} \sigma_{11}\boldsymbol{I}_T & \dots & \sigma_{1M}\boldsymbol{I}_T \\ \vdots & & \vdots \\ \sigma_{M1}\boldsymbol{I}_T & \dots & \sigma_{MM}\boldsymbol{I}_T \end{bmatrix} \begin{bmatrix} \boldsymbol{P}'_1 & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}'_M \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{11}\boldsymbol{P}_1\boldsymbol{P}'_1 & \dots & \sigma_{1M}\boldsymbol{P}_1\boldsymbol{P}'_M \\ \vdots & & \vdots \\ \sigma_{M1}\boldsymbol{P}_M\boldsymbol{P}'_1 & \dots & \sigma_{MM}\boldsymbol{P}_M\boldsymbol{P}'_M \end{bmatrix} = \begin{bmatrix} \sigma_{11}\boldsymbol{R}_{11} & \dots & \sigma_{1M}\boldsymbol{R}_{1M} \\ \vdots & & \vdots \\ \sigma_{M1}\boldsymbol{R}_{M1} & \dots & \sigma_{MM}\boldsymbol{R}_{MM} \end{bmatrix} \\
&= [(\sigma_{ij}\boldsymbol{P}_i\boldsymbol{P}'_j)_{i,j=1,\dots,M}] = [(\sigma_{ij}\boldsymbol{R}_{ij})_{i,j=1,\dots,M}].
\end{aligned} \tag{B.1}$$

Equation (B.1) implies that

$$\begin{aligned}
\boldsymbol{\Omega} = \boldsymbol{P}'^{-1}(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{P}^{-1} &= \begin{bmatrix} \boldsymbol{P}_1'^{-1} & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M'^{-1} \end{bmatrix} \begin{bmatrix} \sigma^{11}\boldsymbol{I}_T & \dots & \sigma^{1M}\boldsymbol{I}_T \\ \vdots & & \vdots \\ \sigma^{M1}\boldsymbol{I}_T & \dots & \sigma^{MM}\boldsymbol{I}_T \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_1^{-1} & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \sigma^{11}\boldsymbol{P}_1'^{-1}\boldsymbol{P}_1^{-1} & \dots & \sigma^{1M}\boldsymbol{P}_1'^{-1}\boldsymbol{P}_M^{-1} \\ \vdots & & \vdots \\ \sigma^{M1}\boldsymbol{P}_M'^{-1}\boldsymbol{P}_1^{-1} & \dots & \sigma^{MM}\boldsymbol{P}_M'^{-1}\boldsymbol{P}_M^{-1} \end{bmatrix} = \begin{bmatrix} \sigma^{11}\boldsymbol{R}^{11} & \dots & \sigma^{1M}\boldsymbol{R}^{1M} \\ \vdots & & \vdots \\ \sigma^{M1}\boldsymbol{R}^{M1} & \dots & \sigma^{MM}\boldsymbol{R}^{MM} \end{bmatrix} \\
&= [(\sigma^{ij}\boldsymbol{P}_i'^{-1}\boldsymbol{P}_j^{-1})_{i,j=1,\dots,M}] = [(\sigma^{ij}\boldsymbol{R}^{ij})_{i,j=1,\dots,M}],
\end{aligned} \tag{B.2}$$

where

$$\boldsymbol{R}^{ij} = \boldsymbol{P}_i'^{-1}\boldsymbol{P}_j^{-1} \;\; (i, j = 1, \dots, M). \tag{B.3}$$

Matrices $\boldsymbol{R}_{ij}$, $\boldsymbol{R}_{ii}$, $\boldsymbol{R}^{ij}$ $\boldsymbol{R}^{ii}$ and their Derivatives with respect to the elements ρ_i, ρ_j

Equation (3.21) imply that

$$\boldsymbol{R}^{ij} = \boldsymbol{P}_i'^{-1}\boldsymbol{P}_j^{-1} = \begin{bmatrix} (1-\rho_i^2)^{1/2} & -\rho_i & 0 & \dots & 0 \\ 0 & 1 & -\rho_i & \dots & 0 \\ \vdots & & & & -\rho_i \\ 0 & & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} (1-\rho_j^2)^{1/2} & 0 & 0 & \dots & 0 \\ -\rho_j & 1 & 0 & \dots & 0 \\ 0 & -\rho_j & 1 & & 0 \\ 0 & \dots & 0 & -\rho_j & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1-\rho_i^2)^{1/2}(1-\rho_j^2)^{1/2} + \rho_i\rho_j & -\rho_i & 0 & \dots & 0 \\ -\rho_j & 1+\rho_i\rho_j & \ddots & & \\ 0 & & \ddots & & \\ \vdots & & & 1+\rho_i\rho_j & -\rho_i \\ 0 & \dots & 0 & -\rho_j & 1 \end{bmatrix}. \tag{B.4}$$

Obviously,

$$\boldsymbol{R}^{ii} = \boldsymbol{P}_i'^{-1}\boldsymbol{P}_i^{-1} = \begin{bmatrix} 1 & -\rho_i & 0 & \dots & 0 \\ -\rho_i & 1+\rho_i^2 & \ddots & & \\ 0 & & \ddots & & \\ \vdots & & & 1+\rho_i^2 & -\rho_i \\ 0 & \dots & 0 & -\rho_i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\rho_i^2 & 0 & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1+\rho_i^2 \end{bmatrix} - \begin{bmatrix} 0 & \rho_i & & \dots & 0 \\ \rho_i & \ddots & & & \\ & & & & \\ & & & \ddots & \rho_i \\ & & & \rho_i & 0 \end{bmatrix} - \begin{bmatrix} \rho_i^2 & & & 0 \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ 0 & & & \rho_i \end{bmatrix}$$

$$= (1+\rho_i^2)\boldsymbol{I}_T - \rho_i\boldsymbol{D} - \rho_i^2\boldsymbol{\Delta}, \tag{B.5}$$

where $\boldsymbol{I}_T$ is the identity matrix, $\boldsymbol{D}$ is a $T \times T$ matrix with elements 1 if $|t-t'| = 1$ and zeros elsewhere, and $\boldsymbol{\Delta}$ is a $T \times T$ matrix with elements 1 in (1,1)-st and (T,T)-th positions and zeros elsewhere.

It can be easily seen that Φ^{ii} is the inverse of Φ_{ii} ($\forall i$) since

$$\boldsymbol{R}^{ii}\boldsymbol{R}_{ii} = \boldsymbol{R}_{ii}\boldsymbol{R}^{ii} = \boldsymbol{I}. \tag{B.6}$$

Moreover,

$$\boldsymbol{R}_{\rho_i}{}^{ii} = \frac{\partial \boldsymbol{R}^{ii}}{\partial \rho_i} = 2\rho_i\boldsymbol{I}_T - \boldsymbol{D} - 2\rho_i\boldsymbol{\Delta}, \tag{B.7}$$

$$\boldsymbol{R}_{\rho_i\rho_i}{}^{ii} = \frac{\partial^2 \boldsymbol{R}^{ii}}{\partial \rho_i{}^2} = 2\boldsymbol{I}_T - 2\boldsymbol{\Delta} = 2(\boldsymbol{I}_T - \boldsymbol{\Delta}), \tag{B.8}$$

$$\boldsymbol{R}_{\rho_j}{}^{ii} = \frac{\partial \boldsymbol{R}^{ii}}{\partial \rho_j} = 0,\ \boldsymbol{R}_{\rho_j\rho_j}{}^{ii} = \frac{\partial^2 \boldsymbol{R}^{ii}}{\partial \rho_j{}^2} = 0,\ \boldsymbol{R}_{\rho_i\rho_j}{}^{ii} = \frac{\partial^2 \boldsymbol{R}^{ii}}{\partial \rho_j \partial \rho_i} = 0,\ (\forall i \neq j). \tag{B.9}$$

Define the $T \times T$ matrix $\boldsymbol{D}_j$ with (t,t')-th element equals 1 if $t - t' = 1$ and zeros elsewhere, and the $T \times T$ matrix $\boldsymbol{D}_i$ with (t,t')-th element equals 1 if $t - t' = -1$. Also, define $T \times T$ matrix $\boldsymbol{\Delta}_{11}$ with 1 in (1,1)-st position and zeros elsewhere and define $T \times T$ matrix $\boldsymbol{\Delta}_{TT}$ with 1 in (T,T)-st position and zeros elsewhere.

Then (B.4) implies that

$$\boldsymbol{R}^{ij} = (1 + \rho_i\rho_j)\boldsymbol{I}_T - \rho_i\boldsymbol{D}_i - \rho_j\boldsymbol{D}_j - \rho_i\rho_j\boldsymbol{\Delta}_{TT} + [(1-\rho_i{}^2)^{1/2}(1-\rho_j{}^2)^{1/2} - 1]\boldsymbol{\Delta}_{11}. \tag{B.10}$$

Note that $\boldsymbol{R}^{ij}$ is not the inverse of $\boldsymbol{R}_{ij}$, since

$$\boldsymbol{R}_{ij}{}^{-1} = \begin{bmatrix} 1 & -\rho_i & 0 & \dots & 0 \\ -\rho_j & 1+\rho_i\rho_j & \ddots & & \\ 0 & & \ddots & & \\ \vdots & & & 1+\rho_i\rho_j & -\rho_i \\ 0 & \dots & 0 & -\rho_j & 1 \end{bmatrix} = (1 + \rho_i\rho_j)\boldsymbol{I}_T - \rho_i\boldsymbol{D}_i - \rho_j\boldsymbol{D}_j - \rho_i\rho_j\boldsymbol{\Delta}. \tag{B.11}$$

Moreover, since

$$\begin{aligned} \frac{\partial}{\rho_i}(1-\rho_i{}^2)^{1/2}(1-\rho_j{}^2)^{1/2} &= -\frac{1}{2}(1-\rho_i{}^2)^{-1/2}(1-\rho_j{}^2)^{1/2}2\rho_i \\ &= -\rho_i(1-\rho_i{}^2)^{-1/2}(1-\rho_j{}^2)^{1/2} = \xi'_{(i)j}, \end{aligned} \tag{B.12}$$

$$\begin{aligned} \frac{\partial^2}{\rho_i{}^2}(1-\rho_i{}^2)^{1/2}(1-\rho_j{}^2)^{1/2} &= -(1-\rho_i{}^2)^{-1/2}(1-\rho_j{}^2)^{1/2} - \rho_i{}^2(1-\rho_i{}^2)^{-3/2}(1-\rho_j{}^2)^{1/2} \\ &= -(1-\rho_i{}^2)^{-3/2}(1-\rho_j{}^2)^{1/2}[1-\rho_i{}^2+\rho_i{}^2] \\ &= -(1-\rho_i{}^2)^{-3/2}(1-\rho_j{}^2)^{1/2} = \xi''_{(i)j}, \end{aligned} \tag{B.13}$$

$$\begin{aligned} \frac{\partial^2}{\rho_i\rho_j}(1-\rho_i{}^2)^{1/2}(1-\rho_j{}^2)^{1/2} &= -\rho_i\frac{1}{2}(-2\rho_j)(1-\rho_i{}^2)^{-1/2}(1-\rho_j{}^2)^{-1/2} \\ &= \rho_i\rho_j(1-\rho_i{}^2)^{-1/2}(1-\rho_j{}^2)^{-1/2} = \xi''_{(i)(j)}, \end{aligned} \tag{B.14}$$

and

$$\frac{\partial a_{ij}}{\partial \rho_\mu} = 0,\ \frac{\partial^2 a_{ij}}{\partial \rho_\mu{}^2} = 0,\ \frac{\partial^2 a_{ij}}{\partial \rho_\mu \partial \rho_i} = 0,\ \frac{\partial^2 a_{ij}}{\partial \rho_\mu \partial \rho_j} = 0,\ (\forall \mu \neq i \quad \mu \neq j), \tag{B.15}$$

where $a_{ij} = (1 - \rho_i^2)^{1/2}(1 - \rho_j^2)^{1/2}$.

We find

$$\boldsymbol{R}^{ij}{}_{\rho_i} = \frac{\partial \boldsymbol{R}^{ij}}{\partial \rho_i} = \rho_j \boldsymbol{I}_T - \boldsymbol{D}_i - \rho_j \boldsymbol{\Delta}_{TT} + \xi'_{(i)j}\boldsymbol{\Delta}_{11}, \tag{B.16}$$

$$\boldsymbol{R}^{ij}{}_{\rho_j} = \frac{\partial \boldsymbol{R}^{ij}}{\partial \rho_j} = \rho_i \boldsymbol{I}_T - \boldsymbol{D}_j - \rho_i \boldsymbol{\Delta}_{TT} + \xi'_{(j)i}\boldsymbol{\Delta}_{11}, \tag{B.17}$$

$$\boldsymbol{R}^{ij}{}_{\rho_i\rho_i} = \xi''_{(i)j}\boldsymbol{\Delta}_{11}, \tag{B.18}$$

$$\boldsymbol{R}^{ij}{}_{\rho_j\rho_j} = \xi''_{(j)i}\boldsymbol{\Delta}_{11}, \tag{B.19}$$

$$\boldsymbol{R}^{ij}{}_{\rho_i\rho_j} = \boldsymbol{I}_T - \boldsymbol{\Delta}_{TT} + \xi''_{(i)(j)}\boldsymbol{\Delta}_{11}, \tag{B.20}$$

$$\boldsymbol{R}^{ij}{}_{\rho_\mu} = 0,\ \boldsymbol{R}^{ij}{}_{\rho_\mu\rho_\mu} = 0,\ \boldsymbol{R}^{ij}{}_{\rho_\mu\rho_i} = 0,\ \boldsymbol{R}^{ij}{}_{\rho_\mu\rho_j} = 0,\ (\forall \mu \neq i \wedge \mu \neq j). \tag{B.21}$$

By using equation (B.6) we find that

$$\begin{aligned} \boldsymbol{I} = \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega} &= [(\sigma_{i\kappa}\boldsymbol{R}_{i\kappa})_{i,\kappa=1,\dots,M}][(\sigma^{\kappa j}\boldsymbol{R}^{\kappa j})_{\kappa,j=1,\dots,M}] \\ &= \left[\Big(\sum_{\kappa=1}^{M} \sigma_{i\kappa}\sigma^{\kappa j}\boldsymbol{R}_{i\kappa}\boldsymbol{R}^{\kappa j}\Big)_{i,j=1,\dots,M}\right], \end{aligned} \tag{B.22}$$

which implies that

$$\sum_{\kappa=1}^{M} \sigma_{i\kappa}\sigma^{\kappa i}\boldsymbol{R}_{i\kappa}\boldsymbol{R}^{\kappa i} = \boldsymbol{I}, \tag{B.23}$$

and

$$\sum_{\kappa=1}^{M} \sigma_{i\kappa}\sigma^{\kappa j}\boldsymbol{R}_{i\kappa}\boldsymbol{R}^{\kappa j} = 0,\ (\forall i \neq j). \tag{B.24}$$

Similarly, since $\boldsymbol{I} = \boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}$ we find that

$$\sum_{\kappa=1}^{M} \sigma^{i\kappa}\sigma_{\kappa i}\boldsymbol{R}^{i\kappa}\boldsymbol{R}_{\kappa i} = \boldsymbol{I}, \tag{B.25}$$

and

$$\sum_{\kappa=1}^{M} \sigma^{i\kappa}\sigma_{\kappa j}\boldsymbol{R}^{i\kappa}\boldsymbol{R}_{\kappa j} = 0,\ (\forall i \neq j). \tag{B.26}$$

Along the same lines, since $\boldsymbol{I} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}$ we find that

$$\sum_{\kappa=1}^{M} \sigma_{i\kappa}\sigma^{\kappa i} = \sum_{\kappa=1}^{M} \sigma^{i\kappa}\sigma_{\kappa i} = 1, \tag{B.27}$$

and

$$\sum_{\kappa=1}^{M} \sigma_{i\kappa}\sigma^{\kappa j} = \sum_{\kappa=1}^{M} \sigma^{i\kappa}\sigma_{\kappa j} = 0,\ (\forall i \neq j), \tag{B.28}$$

Equation (B.23) implies that

$$\begin{aligned} \boldsymbol{I} &= \sum_{\kappa=1}^{M} \sigma_{i\kappa}\sigma^{\kappa i}\boldsymbol{R}_{i\kappa}\boldsymbol{R}^{\kappa i} = \sigma_{ii}\sigma^{ii}\boldsymbol{R}_{ii}\boldsymbol{R}^{ii} + \sum_{\kappa\neq i} \sigma_{i\kappa}\sigma^{\kappa i}\boldsymbol{R}_{i\kappa}\boldsymbol{R}^{\kappa i} \\ &= \sigma_{ii}\sigma^{ii}\boldsymbol{I} + \sum_{\kappa\neq i} \sigma_{i\kappa}\sigma^{\kappa i}\boldsymbol{R}_{i\kappa}\boldsymbol{R}^{\kappa i} \end{aligned} \quad \text{(B.29)}$$

$$\Rightarrow (1-\sigma_{ii}\sigma^{ii})\boldsymbol{I} = \sum_{\kappa\neq i} \sigma_{i\kappa}\sigma^{\kappa i}\boldsymbol{R}_{i\kappa}\boldsymbol{R}^{\kappa i} \quad \text{(B.30)}$$

$$\Rightarrow \sigma_{ii}\sigma^{ii}\boldsymbol{I} = \boldsymbol{I} - \sum_{\kappa\neq i} \sigma_{i\kappa}\sigma^{\kappa i}\boldsymbol{R}_{i\kappa}\boldsymbol{R}^{\kappa i}. \quad \text{(B.31)}$$

Similarly, equation (B.25) implies that

$$\begin{aligned} \boldsymbol{I} &= \sum_{\kappa=1}^{M} \sigma^{i\kappa}\sigma_{\kappa i}\boldsymbol{R}^{i\kappa}\boldsymbol{R}_{\kappa i} = \sigma^{ii}\sigma_{ii}\boldsymbol{R}^{ii}\boldsymbol{R}_{ii} + \sum_{\kappa\neq i} \sigma^{i\kappa}\sigma_{\kappa i}\boldsymbol{R}^{i\kappa}\boldsymbol{R}_{\kappa i} \\ &= \sigma^{ii}\sigma_{ii}\boldsymbol{I} + \sum_{\kappa\neq i} \sigma^{i\kappa}\sigma_{\kappa i}\boldsymbol{R}^{i\kappa}\boldsymbol{R}_{\kappa i} \end{aligned} \quad \text{(B.32)}$$

$$\Rightarrow (1-\sigma^{ii}\sigma_{ii})\boldsymbol{I} = \sum_{\kappa\neq i} \sigma^{i\kappa}\sigma_{\kappa i}\boldsymbol{R}^{i\kappa}\boldsymbol{R}_{\kappa i} \quad \text{(B.33)}$$

$$\Rightarrow \sigma^{ii}\sigma_{ii}\boldsymbol{I} = \boldsymbol{I} - \sum_{\kappa\neq i} \sigma^{i\kappa}\sigma_{\kappa i}\boldsymbol{R}^{i\kappa}\boldsymbol{R}_{\kappa i}. \quad \text{(B.34)}$$

Derivatives of $\boldsymbol{\Omega}$ with respect to the element ρ_μ

Since, $\boldsymbol{\Omega} = [(\sigma^{ij}\boldsymbol{R}^{ij})_{i,j=1,\ldots,M}]$ we find that

$$\begin{aligned} \boldsymbol{\Omega}_{\rho_\mu} = \frac{\partial\boldsymbol{\Omega}}{\partial\rho_\mu} &= [(\sigma^{ij}\boldsymbol{R}_{\rho_\mu}{}^{ij})_{i,j=1,\ldots,M}] = [see(\text{B.7}), (\text{B.16}), (\text{B.17})] \\ &= [(\delta_{\mu i}\sigma^{ij}\boldsymbol{R}_{\rho_\mu}{}^{ij} + \delta_{j\mu}\sigma^{ij}\boldsymbol{R}_{\rho_\mu}{}^{ij} + \delta_{\mu i}\delta_{j\mu}\sigma^{ij}\boldsymbol{R}_{\rho_\mu}{}^{ij})_{i,j}] \\ &= [(\delta_{\mu i}\sigma^{\mu j}\boldsymbol{R}_{\rho_\mu}{}^{\mu j} + \delta_{j\mu}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu} + \delta_{\mu i}\delta_{j\mu}\sigma^{\mu\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu})_{i,j}], \end{aligned} \quad \text{(B.35)}$$

$$\begin{aligned} \boldsymbol{\Omega}_{\rho_\mu\rho_\mu} = \frac{\partial^2\boldsymbol{\Omega}}{\partial{\rho_\mu}^2} &= [see(\text{B.8}), (\text{B.18}), (\text{B.19})] \\ &= [(\delta_{\mu i}\sigma^{\mu j}\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu j} + \delta_{j\mu}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{i\mu} + \delta_{\mu i}\delta_{j\mu}\sigma^{\mu\mu}\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu})_{i,j}], \end{aligned} \quad \text{(B.36)}$$

$$\begin{aligned} \boldsymbol{\Omega}_{\rho_\mu\rho_{\mu'}} = \frac{\partial^2\boldsymbol{\Omega}}{\partial\rho_\mu\rho_{\mu'}} &= [see(\text{B.9}), (\text{B.20}), (\text{B.21})] \\ &= [(\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu j}\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{\mu j} + \delta_{\mu' i}\delta_{j\mu}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{i\mu} + \delta_{\mu i}\delta_{j\mu}\delta_{\mu' i}\sigma^{\mu\mu}\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{\mu\mu})_{i,j}] \\ &= [(\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\mu'}\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{\mu\mu'} + \delta_{\mu' i}\delta_{j\mu}\sigma^{\mu'\mu}\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{\mu'\mu} + 0)_{i,j}]. \end{aligned} \quad \text{(B.37)}$$

Derivatives of $\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T$ and $\boldsymbol{\Omega}$ with respect to the element σ^{ii}

Since,

$$\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T = [(\sigma^{ij}\boldsymbol{I}_T)_{i,j=1,\dots,M}], \tag{B.38}$$

and

$$\varsigma = \text{vec}(\boldsymbol{\Sigma}^{-1}) = [(\varsigma_{ij})_{i,j=1,\dots,M^2}], \tag{B.39}$$

we find

$$\begin{aligned}\frac{\partial}{\partial\varsigma_{(\mu\mu')}}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T) &= \frac{\partial}{\partial\sigma^{\mu\mu'}}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T) = \left[\left(\frac{\partial\sigma^{ij}\boldsymbol{I}_T}{\partial\sigma^{\mu\mu'}}\right)_{i,j=1,\dots,M}\right]\\ &= [(\delta_{\mu i}\delta_{j\mu'}\boldsymbol{I}_T)_{i,j=1,\dots,M}] = [(\delta_{\mu i}\delta_{j\mu'})_{i,j=1,\dots,M}]\otimes \boldsymbol{I}_T\\ &= \boldsymbol{\Delta}_{(\mu\mu')}\otimes \boldsymbol{I}_T,\end{aligned} \tag{B.40}$$

where $\boldsymbol{\Delta}_{(\mu\mu')}$ is a $(M\times M)$ matrix with 1 in the $(\mu\mu')$-th position and zeros elsewhere.

$$\begin{aligned}\frac{\partial^2}{\partial\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T) &= \frac{\partial}{\partial\varsigma_{(\nu\nu')}}\left[\frac{\partial}{\partial\varsigma_{(\mu\mu')}}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T)\right]\\ &= [(\partial\delta_{\mu i}\delta_{j\mu'}\boldsymbol{I}_T/\partial\varsigma_{(\nu\nu')})_{i,j=1,\dots,M}] = 0.\end{aligned} \tag{B.41}$$

Since $\boldsymbol{\Omega} = \boldsymbol{P}'^{-1}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}$, (B.40) implies that

$$\begin{aligned}\boldsymbol{\Omega}_{\varsigma_{(\mu\mu')}} &= \frac{\partial\boldsymbol{\Omega}}{\partial\varsigma_{(\mu\mu')}} = \frac{\partial\boldsymbol{\Omega}}{\partial\sigma^{\mu\mu'}} = \boldsymbol{P}'^{-1}\left[\frac{\partial}{\partial\sigma^{\mu\mu'}}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T)\right]\boldsymbol{P}^{-1} = \boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\mu\mu'}\otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}\\ &= \boldsymbol{\Delta}_{\mu\mu'}\otimes \boldsymbol{P}_i'^{-1}\boldsymbol{P}_j{}^{-1} = [(\delta_{\mu i}\delta_{j\mu'}\boldsymbol{P}_i'^{-1}\boldsymbol{P}_j{}^{-1})_{i,j=1,\dots,M}]\\ &= [(\delta_{\mu i}\delta_{j\mu'}\boldsymbol{R}^{ij})_{i,j=1,\dots,M}] = [(\delta_{\mu i}\delta_{j\mu'}\boldsymbol{R}^{\mu\mu'})_{i,j=1,\dots,M}].\end{aligned} \tag{B.42}$$

Similarly, (B.41) implies that

$$\begin{aligned}\boldsymbol{\Omega}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \frac{\partial^2\boldsymbol{\Omega}}{\partial\varsigma_{(\mu\mu')}\partial\varsigma_{(\nu\nu')}} = \frac{\partial^2\boldsymbol{\Omega}}{\partial\sigma^{\mu\mu'}\partial\sigma^{\nu\nu'}} = \frac{\partial}{\partial\sigma^{\nu\nu'}}\left(\frac{\partial\boldsymbol{\Omega}}{\partial\sigma^{\mu\mu'}}\right)\\ &= [(\partial\delta_{\mu i}\delta_{j\mu'}\boldsymbol{R}^{ij}/\partial\sigma^{\nu\nu'})_{i,j=1,\dots,M}] = 0.\end{aligned} \tag{B.43}$$

The Second-order cross derivatives and useful matrices

Equations (B.7),(B.16),(B.17) and (B.42) imply that

$$\begin{aligned}\boldsymbol{\Omega}_{\rho_\mu\varsigma_{(\nu\nu')}} &= \boldsymbol{\Omega}_{\varsigma_{(\nu\nu')}\rho_\mu} = \frac{\partial}{\partial\rho_\mu}\left(\frac{\partial\boldsymbol{\Omega}}{\partial\sigma^{\nu\nu'}}\right) = \frac{\partial}{\partial\rho_\mu}(\boldsymbol{\Delta}_{\nu\nu'}\otimes \boldsymbol{P}_i'^{-1}\boldsymbol{P}_j{}^{-1}) = [(\partial\delta_{\nu i}\delta_{j\nu'}\boldsymbol{R}^{ij}/\partial\rho_\mu)_{i,j}]\\ &= [(\delta_{\nu i}\delta_{j\nu'}(\delta_{\mu i}\boldsymbol{R}_{\rho_\mu}{}^{\mu j} + \delta_{j\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu} + \delta_{\mu i}\delta_{j\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}))_{ij}]\\ &= [(\delta_{\nu i}\delta_{j\nu'}\delta_{\mu i}\boldsymbol{R}_{\rho_\mu}{}^{\mu j} + \delta_{\nu i}\delta_{j\nu'}\delta_{j\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu} + \delta_{\nu i}\delta_{j\nu'}\delta_{\mu i}\delta_{j\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu})_{ij}]\\ &= [(\delta_{\nu i}\delta_{j\nu'}\delta_{\mu\nu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\nu'} + \delta_{\nu i}\delta_{j\nu'}\delta_{\nu'\mu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\mu} + \delta_{\nu i}\delta_{j\nu'}\delta_{\mu\nu}\delta_{\nu'\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu})_{ij}].\end{aligned} \tag{B.44}$$

Obviously, $\boldsymbol{\Omega}_{\rho_\mu \varsigma_{(\nu\nu')}} = 0$ $(\forall \nu \neq \mu$ and $\forall \nu' \neq \mu')$ and $\boldsymbol{\Omega}_{\rho_\mu \varsigma_{(\nu\nu')}} = \boldsymbol{\Omega}_{\varsigma_{(\nu\nu')}\rho_\mu}$.

$$\begin{aligned}
\boldsymbol{\Omega}^*{}_{\rho_{\mu'}\rho_\mu} &= \boldsymbol{\Omega}_{\rho_{\mu'}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{\rho_\mu} = \boldsymbol{\Omega}^*{}_{\rho_{\mu'}\rho_\mu} = \boldsymbol{\Omega}'_{\rho_\mu}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}'_{\rho_{\mu'}} = \boldsymbol{\Omega}^{*\prime}{}_{\rho_\mu\rho_{\mu'}} \\
&= [(\sigma^{i\kappa}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa})_{i,\kappa=1,\ldots,M}][(\sigma_{\kappa l}\boldsymbol{R}_{\kappa l})_{\kappa,l=1,\ldots,M}][(\sigma^{lj}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj})_{l,j=1,\ldots,M}] \\
&= \left[\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\right)_{i,j=1,\ldots,M}\right].
\end{aligned} \tag{B.45}$$

But, equations (B.7), (B.16) and (B.17) imply that

$$\boldsymbol{R}_{\rho_\mu}{}^{i\kappa} = \delta_{\mu i}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa} + \delta_{\kappa\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu} + \delta_{\mu i}\delta_{\kappa\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}, \tag{B.46}$$

and

$$\boldsymbol{R}_{\rho_{\mu'}}{}^{lj} = \delta_{\mu' l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} + \delta_{j\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} + \delta_{\mu' l}\delta_{j\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}. \tag{B.47}$$

Therefore,

$$\begin{aligned}
\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj} &= [\delta_{\mu i}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa} + \delta_{\kappa\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu} + \delta_{\mu i}\delta_{\kappa\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}]\boldsymbol{R}_{\kappa l}[\delta_{\mu' l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} + \delta_{j\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} + \delta_{\mu' l}\delta_{j\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}] \\
&= \delta_{\mu i}\delta_{\mu' l}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} + \delta_{\mu i}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} + \delta_{\mu i}\delta_{\mu' l}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
&\quad + \delta_{\kappa\mu}\delta_{\mu' l}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} + \delta_{\kappa\mu}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} + \delta_{\kappa\mu}\delta_{\mu' l}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
&\quad + \delta_{\mu i}\delta_{\kappa\mu}\delta_{\mu' l}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} + \delta_{\mu i}\delta_{\kappa\mu}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} \\
&\quad + \delta_{\mu i}\delta_{\kappa\mu}\delta_{\mu' l}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}.
\end{aligned} \tag{B.48}$$

Moreover,

$$\begin{aligned}
(1)\quad &\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\mu i}\delta_{\mu' l}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
= \quad &\sum_{\kappa=1}^{M}\delta_{\mu i}\sigma^{\mu\kappa}\sigma_{\kappa\mu'}\sigma^{\mu' j}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
= \quad &\left[\sum_{\kappa=1}^{M}\sigma^{\mu\kappa}\sigma_{\kappa\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa\mu'}\right]\delta_{\mu i}\sigma^{\mu' j}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j},
\end{aligned} \tag{B.49}$$

$$\begin{aligned}
(2)\quad &\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\mu i}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} \\
= \quad &\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\kappa}\sigma_{\kappa l}\sigma^{l\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'},
\end{aligned} \tag{B.50}$$

$$
\begin{aligned}
(3)\quad & \sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\mu i}\delta_{\mu' l}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
= \quad & \sum_{\kappa=1}^{M}\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\kappa}\sigma_{\kappa\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
= \quad & \left[\sum_{\kappa=1}^{M}\sigma^{\mu\kappa}\sigma_{\kappa\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa\mu'}\right]\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'},
\end{aligned}
\tag{B.51}
$$

$$
\begin{aligned}
(4)\quad & \sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\kappa\mu}\delta_{\mu' l}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
= \quad & \sigma^{i\mu}\sigma_{\mu\mu'}\sigma^{\mu' j}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j},
\end{aligned}
\tag{B.52}
$$

$$
\begin{aligned}
(5)\quad & \sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\kappa\mu}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} \\
= \quad & \sum_{l=1}^{M}\delta_{j\mu'}\sigma^{i\mu}\sigma_{\mu l}\sigma^{lj}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\mu l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} \\
= \quad & \left[\sum_{l=1}^{M}\sigma_{\mu l}\sigma^{l\mu'}\boldsymbol{R}_{\mu l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'}\right]\delta_{j\mu'}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu},
\end{aligned}
\tag{B.53}
$$

$$
\begin{aligned}
(6)\quad & \sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\kappa\mu}\delta_{\mu' l}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
= \quad & \delta_{j\mu'}\sigma^{i\mu}\sigma_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'},
\end{aligned}
\tag{B.54}
$$

$$
\begin{aligned}
(7)\quad & \sum_{\kappa=1}^{T}\sum_{l=1}^{T}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\mu i}\delta_{\kappa\mu}\delta_{\mu' l}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
= \quad & \delta_{\mu i}\sigma^{\mu\mu}\sigma_{\mu\mu'}\sigma^{\mu' j}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j},
\end{aligned}
\tag{B.55}
$$

$$
\begin{aligned}
(8)\quad & \sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\mu i}\delta_{\kappa\mu}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} \\
= \quad & \sum_{l=1}^{M}\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\mu}\sigma_{\mu l}\sigma^{l\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} \\
= \quad & \left[\sum_{l=1}^{M}\sigma_{\mu l}\sigma^{l\mu'}\boldsymbol{R}_{\mu l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'}\right]\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu},
\end{aligned}
\tag{B.56}
$$

$$\begin{aligned}(9)\quad & \sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\delta_{\mu i}\delta_{\kappa\mu}\delta_{\mu' l}\delta_{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\ = \quad & \delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\mu}\sigma_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}.\end{aligned} \tag{B.57}$$

Equations (B.45), (B.48) and (B.49) through (B.57) imply that

$$\begin{aligned}
& \sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj} \\
= \; & \left[\sum_{\kappa=1}^{M}\sigma^{\mu\kappa}\sigma_{\kappa\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa\mu'}\right]\delta_{\mu i}\sigma^{\mu' j}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
& +\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\kappa}\sigma_{\kappa l}\sigma^{l\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'} \\
& +\left[\sum_{\kappa=1}^{M}\sigma^{\mu\kappa}\sigma_{\kappa\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa\mu'}\right]\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
& +\sigma^{i\mu}\sigma_{\mu\mu'}\sigma^{\mu' j}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
& +\left[\sum_{l=1}^{M}\sigma_{\mu l}\sigma^{l\mu'}\boldsymbol{R}_{\mu l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'}\right]\delta_{j\mu'}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu} \\
& +\delta_{j\mu'}\sigma^{i\mu}\sigma_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
& +\delta_{\mu i}\sigma^{\mu\mu}\sigma_{\mu\mu'}\sigma^{\mu' j}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
& +\left[\sum_{l=1}^{M}\sigma_{\mu l}\sigma^{l\mu'}\boldsymbol{R}_{\mu l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu}\right]\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu} \\
& +\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\mu}\sigma_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
= \; & \sigma^{i\mu}\sigma_{\mu\mu'}\sigma^{\mu' j}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
& +\delta_{\mu i}\left[\left[\sum_{\kappa=1}^{M}\sigma^{\mu\kappa}\sigma_{\kappa\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa\mu'}\right]+\sigma^{\mu\mu}\sigma_{\mu\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu'}\right]\sigma_{\mu' j}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu' j} \\
& +\delta_{j\mu'}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\left[\left[\sum_{l=1}^{M}\sigma_{\mu l}\sigma^{l\mu'}\boldsymbol{R}_{\mu l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'}\right]+\sigma_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu\mu'}\right] \\
& +\delta_{\mu i}\delta_{j\mu'}\left[\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{\mu\kappa}\sigma_{\kappa l}\sigma^{l\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'}\right] \\
& +\delta_{\mu i}\delta_{j\mu'}\left[\sum_{\kappa=1}^{M}\sigma^{\mu\kappa}\sigma_{\kappa\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\kappa}\boldsymbol{R}_{\kappa\mu'}\right]\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} \\
& +\delta_{\mu i}\delta_{j\mu'}\left[\sum_{l=1}^{M}\sigma_{\mu l}\sigma^{l\mu'}\boldsymbol{R}_{\mu l}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'}\right]\sigma^{\mu\mu}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu} \\
& +\delta_{\mu i}\delta_{j\mu'}\sigma^{\mu\mu}\sigma_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'} = w_{ij}.
\end{aligned} \tag{B.58}$$

Therefore equations (B.45) and (B.58) imply that

$$\boldsymbol{\Omega}^{*}{}_{\rho_\mu\rho_{\mu'}} = [(w_{ij})_{i,j=1,\dots,M}]. \tag{B.59}$$

$$\begin{aligned}
\boldsymbol{\Omega}^*{}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \boldsymbol{\Omega}_{\varsigma_{(\mu\mu')}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{\varsigma_{(\nu\nu')}} = \text{[see (B.40) and (B.42)]} \\
&= \boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\mu\mu'} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}\boldsymbol{P}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{P}'\boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\nu\nu'} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\mu\mu'} \otimes \boldsymbol{I}_T)(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)(\boldsymbol{\Delta}_{\nu\nu'} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\mu\mu'}\boldsymbol{\Sigma}\boldsymbol{\Delta}_{\nu\nu'} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}'^{-1}(\sigma_{\mu'\nu}\boldsymbol{\Delta}_{\mu\nu'} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}, \qquad \text{(B.60)}
\end{aligned}$$

because

$$\begin{aligned}
\boldsymbol{\Delta}_{\mu\mu'}\boldsymbol{\Sigma}\boldsymbol{\Delta}_{\nu\nu'} &= [(\delta_{\mu i}\delta_{\kappa\mu'})_{i,\kappa=1,\ldots,M}][(\sigma_{\kappa l})_{\kappa,l=1,\ldots,M}][(\delta_{\nu l}\delta_{j\nu'})_{l,j=1,\ldots,M}] \\
&= \left[\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\delta_{\mu i}\delta_{\kappa\mu'}\sigma_{\kappa l}\delta_{\nu l}\delta_{j\nu'}\right)_{i,j=1,\ldots,M}\right] \\
&= [(\delta_{\mu i}\sigma_{\mu'\nu}\delta_{j\nu'})_{i,j=1,\ldots,M}] \\
&= \sigma_{\mu'\nu}[(\delta_{\mu i}\delta_{j\nu'})_{i,j=1,\ldots,M}] \\
&= \sigma_{\mu'\nu}\boldsymbol{\Delta}_{\mu\nu'}. \qquad \text{(B.61)}
\end{aligned}$$

Equations (B.60) and (B.61) imply that

$$\begin{aligned}
\boldsymbol{\Omega}^*{}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \sigma_{\mu'\nu}\boldsymbol{\Delta}_{\mu\nu'} \otimes \boldsymbol{P}'_i{}^{-1}\boldsymbol{P}_j^{-1} \\
&= [(\delta_{\mu i}\sigma_{\mu'\nu}\delta_{j\nu'}\boldsymbol{P}'_\mu{}^{-1}\boldsymbol{P}_{\nu'}{}^{-1})_{i,j=1,\ldots,M}] \\
&= [(\delta_{\mu i}\delta_{j\nu'}\sigma_{\mu'\nu}\boldsymbol{R}^{\mu\nu'})_{i,j=1,\ldots,M}], \qquad \text{(B.62)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Omega}^*{}_{\rho_\mu\varsigma_{(\nu\nu')}} &= \boldsymbol{\Omega}_{\rho_\mu}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{\varsigma_{(\nu\nu')}} \\
&= [(\sigma^{i\kappa}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa})_{i,\kappa=1,\ldots,M}][(\sigma_{\kappa l}\boldsymbol{R}_{\kappa l})_{\kappa,l=1,\ldots,M}][(\delta_{\nu i}\delta_{j\nu'}\boldsymbol{R}^{lj})_{l,j=1,\ldots,M}] \\
&= \left[\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}^{lj}\delta_{\nu l}\delta_{j\nu'}\right)_{i,j=1,\ldots,M}\right] \\
&= \left[\left(\sum_{\kappa=1}^{M}\delta_{j\nu'}\sigma^{i\kappa}\sigma_{\kappa\nu}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu j}\right)_{i,j=1,\ldots,M}\right] \\
&= \left[\left(\sum_{\kappa=1}^{M}\delta_{j\nu'}\sigma^{i\kappa}\sigma_{\kappa\nu}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\right)_{i,j=1,\ldots,M}\right] \\
&= \left[\left(\left(\sum_{\kappa=1}^{M}\sigma^{i\kappa}\sigma_{\kappa\nu}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\right)\delta_{j\nu'}\boldsymbol{R}^{\nu\nu'}\right)_{i,j=1,\ldots,M}\right]. \qquad \text{(B.63)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\boldsymbol{\Omega^*}_{\varsigma_{(\nu\nu')}\rho_\mu} &= \boldsymbol{\Omega}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{\rho_\mu} \\
&= [(\delta_{\nu i}\delta_{\kappa\nu'}\boldsymbol{R}^{i\kappa})_{i,\kappa=1,\ldots,M}][(\sigma_{\kappa l}\boldsymbol{R}_{\kappa l})_{\kappa,l=1,\ldots,M}][(\sigma^{lj}\boldsymbol{R}_{\rho_\mu}{}^{lj})_{l,j=1,\ldots,M}] \\
&= \left[\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\delta_{\nu i}\delta_{\kappa\nu'}\sigma_{\kappa l}\sigma^{lj}\boldsymbol{R}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\right)_{i,j=1,\ldots,M}\right] \\
&= \left[\left(\sum_{l=1}^{M}\delta_{\nu i}\sigma_{\nu' l}\sigma^{lj}\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\right)_{i,j=1,\ldots,M}\right] \\
&= \left[\left(\left(\sum_{l=1}^{M}\sigma_{\nu' l}\sigma^{lj}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\right)\delta_{\nu i}\boldsymbol{R}^{\nu\nu'}\right)_{i,j=1,\ldots,M}\right].
\end{aligned} \tag{B.64}$$

Define the $(n \times n)$ matrix

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T = [(\boldsymbol{X}_i')_{i=1,\ldots,M}][(\sigma^{ij}\boldsymbol{R}^{ij})_{i,j=1,\ldots,M}][(\boldsymbol{X}_j)_{j=1,\ldots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}^{ij}\boldsymbol{X}_j/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{B}_{ij},
\end{aligned} \tag{B.65}$$

where

$$\boldsymbol{B}_{ij} = \boldsymbol{X}_i'\boldsymbol{R}^{ij}\boldsymbol{X}_j/T. \tag{B.66}$$

Therefore,

$$\begin{aligned}
\boldsymbol{A}_{\rho_\mu} &= \frac{\partial \boldsymbol{A}}{\partial \rho_\mu} = \partial(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\rho_\mu = \boldsymbol{X}'(\partial\boldsymbol{\Omega}/\partial\rho_\mu)\boldsymbol{X}/T \\
&= \boldsymbol{X}'\boldsymbol{\Omega}_{\rho_\mu}\boldsymbol{X}/T = [(\boldsymbol{X}_i')_{i=1,\ldots,M}][(\sigma^{ij}\boldsymbol{R}_{\rho_\mu}{}^{ij})_{i,j=1,\ldots,M}][(\boldsymbol{X}_j)_{j=1,\ldots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T,
\end{aligned} \tag{B.67}$$

$$\begin{aligned}
\boldsymbol{A}_{\rho_\mu\rho_{\mu'}} &= \frac{\partial^2 \boldsymbol{A}}{\partial\rho_\mu\partial\rho_{\mu'}} = \partial^2(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\rho_\mu\partial\rho_{\mu'} = \boldsymbol{X}'(\partial^2\boldsymbol{\Omega}/\partial\rho_\mu\partial\rho_{\mu'})\boldsymbol{X}/T \\
&= \boldsymbol{X}'\boldsymbol{\Omega}_{\rho_\mu\rho_{\mu'}}\boldsymbol{X}/T = [(\boldsymbol{X}_i')_{i=1,\ldots,M}][(\sigma^{ij}\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij})_{i,j=1,\ldots,M}][(\boldsymbol{X}_j)_{j=1,\ldots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j/T,
\end{aligned} \tag{B.68}$$

$$\begin{aligned}
\boldsymbol{A^*}_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{X}'\boldsymbol{\Omega^*}_{\rho_\mu\rho_{\mu'}}\boldsymbol{X}/T \\
&= \left[(\boldsymbol{X}_i')_{i=1,\ldots,M}\right]\left[\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\right)_{i,j=1,\ldots,M}\right]\left[(\boldsymbol{X}_j)_{j=1,\ldots,M}\right]/T
\end{aligned}$$

$$= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/T = [\text{see(B.59)}]$$

$$= \sum_{i=1}^{M}\sum_{j=1}^{M}\boldsymbol{X}_i' w_{ij}\boldsymbol{X}_j/T, \qquad \text{(B.69)}$$

$$\begin{aligned}\boldsymbol{A}_{\varsigma_{(\mu\mu')}} &= \frac{\partial \boldsymbol{A}}{\partial\varsigma_{(\mu\mu')}} = \partial(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\sigma^{\mu\mu'} = \boldsymbol{X}'(\partial\boldsymbol{\Omega}/\partial\sigma^{\mu\mu'})\boldsymbol{X}/T\\
&= \boldsymbol{X}'\boldsymbol{\Omega}_{\varsigma_{(\mu\mu')}}\boldsymbol{X}/T = [\text{see(B.42)}]\\
&= [(\boldsymbol{X}_i')_{i=1,\dots,M}][(\delta_{\mu i}\delta_{j\mu'}\boldsymbol{R}^{\mu\mu'})_{i,j=1,\dots,M}][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T\\
&= \boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}/T = [\text{see(B.66)}]\\
&= \boldsymbol{B}_{\mu\mu'}, \qquad \text{(B.70)}\end{aligned}$$

$$\begin{aligned}\boldsymbol{A}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \frac{\partial^2\boldsymbol{A}}{\partial\varsigma_{(\mu\mu')}\partial\varsigma_{(\nu\nu')}} = \partial^2(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\sigma^{\mu\mu'}\partial\sigma^{\nu\nu'}\\
&= \boldsymbol{X}'(\partial^2\boldsymbol{\Omega}/\partial\sigma^{\mu\mu'}\partial\sigma^{\nu\nu'})\boldsymbol{X}/T = \boldsymbol{X}'\boldsymbol{\Omega}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{X}/T = [\text{see(B.43)}] = 0, \qquad \text{(B.71)}\end{aligned}$$

$$\begin{aligned}\boldsymbol{A}^*{}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \boldsymbol{X}'\boldsymbol{\Omega}^*{}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{X}/T = [\text{see(B.62)}]\\
&= [(\boldsymbol{X}_i')_{i=1,\dots,M}][(\delta_{\mu i}\delta_{j\nu'}\sigma_{\mu'\nu}\boldsymbol{R}^{\mu\nu'})_{i,j=1,\dots,M}][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T\\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{\mu i}\delta_{j\nu'}\sigma_{\mu'\nu}\boldsymbol{X}_i'\boldsymbol{R}^{\mu\nu'}\boldsymbol{X}_j/T\\
&= \sigma_{\mu'\nu}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\nu'}\boldsymbol{X}_{\nu'}/T = [\text{see(B.66)}]\\
&= \sigma_{\mu'\nu}\boldsymbol{B}_{\mu\nu'}\\
&= \sigma_{\mu'\nu}\boldsymbol{A}_{\varsigma_{(\mu\nu')}}\ , \qquad \text{(B.72)}\end{aligned}$$

$$\begin{aligned}\boldsymbol{A}_{\rho_\mu\varsigma_{(\nu\nu')}} &= \frac{\partial^2\boldsymbol{A}}{\partial\rho_\mu\partial\varsigma_{(\nu\nu')}} = \partial^2(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\rho_\mu\partial\varsigma_{(\nu\nu')}\\
&= \boldsymbol{X}'(\partial^2\boldsymbol{\Omega}/\partial\rho_\mu\partial\varsigma_{(\nu\nu')})\boldsymbol{X}/T\\
&= [(\boldsymbol{X}_i')_{i=1,\dots,M}][(\delta_{\nu i}\delta_{j\nu'}\boldsymbol{R}_{\rho_\mu}{}^{ij})_{i,j=1,\dots,M}][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T\\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{\nu i}\delta_{j\nu'}\sigma_{\mu'\nu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T = \boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}/T, \qquad \text{(B.73)}\end{aligned}$$

$$\boldsymbol{A}^*{}_{\rho_\mu\varsigma_{(\nu\nu')}} = \boldsymbol{X}'\boldsymbol{\Omega}^*{}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{X}/T = \boldsymbol{X}'\boldsymbol{\Omega}_{\rho_\mu}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{\varsigma_{(\nu\nu')}}\boldsymbol{X}/T = [\text{see(B.63)}]$$

$$
\begin{aligned}
&= [(\boldsymbol{X}_i')_{i=1,\dots,M}]\left[\left(\sum_{\kappa=1}^{M} \delta_{j\nu'}\sigma^{i\kappa}\sigma_{\kappa\nu}\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\right)_{i,j=1,\dots,M}\right][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M} \delta_{j\nu'}\sigma^{i\kappa}\sigma_{\kappa\nu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_j/T \\
&= \sum_{i=1}^{M}\sum_{\kappa=1}^{M} \sigma^{i\kappa}\sigma_{\kappa\nu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}/T. \qquad \text{(B.74)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\boldsymbol{A}^*{}_{\varsigma_{(\nu\nu')}\rho_\mu} &= \boldsymbol{X}'\boldsymbol{\Omega}^*{}_{\varsigma_{(\nu\nu')}\rho_\mu}\boldsymbol{X}/T = \boldsymbol{X}'\boldsymbol{\Omega}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{\rho_\mu}\boldsymbol{X}/T = [\text{see(B.64)}] \\
&= [(\boldsymbol{X}_i')_{i=1,\dots,M}]\left[\left(\sum_{l=1}^{M} \delta_{\nu i}\sigma_{\nu' l}\sigma^{lj}\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\right)_{i,j=1,\dots,M}\right][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{l=1}^{M} \delta_{\nu i}\sigma_{\nu' l}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j/T \\
&= \sum_{j=1}^{M}\sum_{l=1}^{M} \sigma_{\nu' l}\sigma^{lj}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j/T. \qquad \text{(B.75)}
\end{aligned}
$$

Define the $n \times n$ matrices

$$
\boldsymbol{G} = \boldsymbol{A}^{-1} \text{ and } \boldsymbol{\Xi} = \boldsymbol{G}\boldsymbol{Q}\boldsymbol{G}, \qquad \text{(B.76)}
$$

where

$$
\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T \text{ and } \boldsymbol{Q} = \boldsymbol{H}'(\boldsymbol{H}\boldsymbol{G}\boldsymbol{H}')^{-1}\boldsymbol{H}. \qquad \text{(B.77)}
$$

By using equations (B.76) and (B.77) we find the following results:

1.

$$
\begin{aligned}
\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi} &= [\text{see(B.67)}] \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T \Rightarrow \\
\operatorname{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}) &= \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\operatorname{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T). \qquad \text{(B.78)}
\end{aligned}
$$

2.

$$
\begin{aligned}
\boldsymbol{A}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi} &= [\text{see(B.68)}] \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T \Rightarrow \\
\operatorname{tr}(\boldsymbol{A}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}) &= \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\operatorname{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T). \qquad \text{(B.79)}
\end{aligned}
$$

3.

$$\begin{aligned}
\boldsymbol{A}^{*}{}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi} &= [\text{see(B.69)}] \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi}/T \Rightarrow \\
\operatorname{tr}(\boldsymbol{A}^{*}{}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}) &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\operatorname{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi}/T).
\end{aligned} \quad \text{(B.80)}$$

4.

$$\begin{aligned}
\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{\Xi} &= [\text{see(B.70)}] \\
&= \boldsymbol{B}_{\mu\mu'}\boldsymbol{\Xi} \Rightarrow \\
\operatorname{tr}(\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{\Xi}) &= \operatorname{tr}(\boldsymbol{B}_{\mu\mu'}\boldsymbol{\Xi}) = \operatorname{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T).
\end{aligned} \quad \text{(B.81)}$$

5. Since

$$\begin{aligned}
\boldsymbol{A}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{\Xi} &= 0 = [\text{see(B.71)}] \Rightarrow \\
\operatorname{tr}(\boldsymbol{A}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) &= 0.
\end{aligned} \quad \text{(B.82)}$$

6. Since

$$\begin{aligned}
\boldsymbol{A}^{*}{}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{\Xi} &= [\text{see(B.72)}] \\
&= \sigma_{\mu'\nu}\boldsymbol{A}_{\varsigma_{(\mu\nu')}}\boldsymbol{\Xi} \Rightarrow \\
\operatorname{tr}(\boldsymbol{A}^{*}{}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) &= \sigma_{\mu'\nu}\operatorname{tr}(\boldsymbol{A}_{\varsigma_{(\mu\nu')}}\boldsymbol{\Xi}) = [\text{see(B.81)}] \\
&= \sigma_{\mu'\nu}\operatorname{tr}(\boldsymbol{B}_{\mu\nu'}\boldsymbol{\Xi}) = \sigma_{\mu'\nu}\operatorname{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T).
\end{aligned} \quad \text{(B.83)}$$

7.

$$\begin{aligned}
\boldsymbol{A}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{\Xi} &= [\text{see(B.73)}] \\
&= \boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T \Rightarrow \\
\operatorname{tr}(\boldsymbol{A}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) &= \operatorname{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T).
\end{aligned} \quad \text{(B.84)}$$

8.

$$\begin{aligned}
\boldsymbol{A}^{*}{}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{\Xi} &= [\text{see(B.74)}] \\
&= \sum_{i=1}^{M}\sum_{\kappa=1}^{M}\sigma^{i\kappa}\sigma_{\kappa\nu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T \Rightarrow
\end{aligned}$$

$$\mathrm{tr}(\boldsymbol{A^*}_{\rho_\mu \varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) = \sum_{i=1}^{M}\sum_{\kappa=1}^{M} \sigma^{i\kappa}\sigma_{\kappa\nu}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T). \tag{B.85}$$

9.

$$\begin{aligned}\boldsymbol{A^*}_{\varsigma_{(\nu\nu')}\rho_\mu}\boldsymbol{\Xi} &= [\text{see(B.75)}]\\ &= \sum_{j=1}^{M}\sum_{l=1}^{M} \sigma_{\nu' l}\sigma^{lj}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi}/T \Rightarrow\\ \mathrm{tr}(\boldsymbol{A^*}_{\varsigma_{(\nu\nu')}\rho_\mu}\boldsymbol{\Xi}) &= \sum_{j=1}^{M}\sum_{l=1}^{M} \sigma_{\nu' l}\sigma^{lj}\,\mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi}/T). \qquad \text{(B.86)}\end{aligned}$$

10.

$$\begin{aligned}\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}} &= [\text{see(B.67)}]\\ &= \left(\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T\right)\boldsymbol{G}\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{\kappa l}\boldsymbol{X}_\kappa'\boldsymbol{R}_{\rho_{\mu'}}{}^{\kappa l}\boldsymbol{X}_l/T\right)\\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{ij}\sigma^{\kappa l}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{X}_\kappa'\boldsymbol{R}_{\rho_{\mu'}}{}^{\kappa l}\boldsymbol{X}_l/T^2 \Rightarrow\\ \boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}}\boldsymbol{\Xi} &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{ij}\sigma^{\kappa l}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{X}_\kappa'\boldsymbol{R}_{\rho_{\mu'}}{}^{\kappa l}\boldsymbol{X}_l\boldsymbol{\Xi}/T^2 \Rightarrow\\ \mathrm{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}}\boldsymbol{\Xi}) &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{ij}\sigma^{\kappa l}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{X}_\kappa'\boldsymbol{R}_{\rho_{\mu'}}{}^{\kappa l}\boldsymbol{X}_l\boldsymbol{\Xi}/T^2). \qquad \text{(B.87)}\end{aligned}$$

11. Similarly, by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\mathrm{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_{\mu'}}\boldsymbol{\Xi}) = \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{ij}\sigma^{\kappa l}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}\boldsymbol{X}_\kappa'\boldsymbol{R}_{\rho_{\mu'}}{}^{\kappa l}\boldsymbol{X}_l\boldsymbol{\Xi}/T^2). \tag{B.88}$$

12.

$$\begin{aligned}\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}} &= [\text{see(B.70)}]\\ &= \boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'} \Rightarrow\\ \boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi} &= \boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}\boldsymbol{\Xi} \Rightarrow\\ \mathrm{tr}(\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) &= \mathrm{tr}(\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}\boldsymbol{\Xi}) = [\text{see(B.66)}]\\ &= \mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T^2). \qquad \text{(B.89)}\end{aligned}$$

13. Similarly, by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\begin{aligned}\operatorname{tr}(\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) &= \operatorname{tr}(\boldsymbol{B}_{\mu\mu'}\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu'}\boldsymbol{\Xi}) \\ &= \operatorname{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T^2).\end{aligned} \quad \text{(B.90)}$$

14.

$$\begin{aligned}\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}} &= \text{[see (B.67) and (B.70)]} \\ &= \left(\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T\right)\boldsymbol{G}\boldsymbol{B}_{\nu\nu'} \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}/T \Rightarrow \text{[see (B.66)]} \\ \boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi} &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T^2 \Rightarrow \\ \operatorname{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\operatorname{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T^2).\end{aligned} \quad \text{(B.91)}$$

15. Similarly, by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\operatorname{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) = \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\operatorname{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}/T^2). \quad \text{(B.92)}$$

16.

$$\begin{aligned}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{G}\boldsymbol{A}_{\rho_\mu} &= \text{[see (B.67) and (B.70)]} \\ &= \boldsymbol{B}_{\nu\nu'}\boldsymbol{G}\left(\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T\right) \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{B}_{\nu\nu'}\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T \Rightarrow \text{[see (B.66)]} \\ \boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi} &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T^2 \Rightarrow \\ \operatorname{tr}(\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}) &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\operatorname{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T^2).\end{aligned} \quad \text{(B.93)}$$

17. Similarly, by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\operatorname{tr}(\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}) = \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\operatorname{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T^2). \quad \text{(B.94)}$$

Proof. [Proof of Theorem 3]

i a. From equations (B.68), (B.69) and (B.87) we have that

$$\begin{aligned}
\boldsymbol{C}_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{A}^*{}_{\rho_\mu\rho_{\mu'}} - 2\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}} + \boldsymbol{A}_{\rho_\mu\rho_{\mu'}}/2 \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/T \\
&\quad -2\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{ij}\sigma^{\kappa l}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{X}_\kappa'\boldsymbol{R}_{\rho_{\mu'}}{}^{\kappa l}\boldsymbol{X}_l/T^2 \\
&\quad +\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j/2T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/T \\
&\quad -2\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{jl}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_j'\boldsymbol{R}_{\rho_{\mu'}}{}^{jl}\boldsymbol{X}_l/T^2 \\
&\quad +\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j/2T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\sigma_{\kappa l}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa l}\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/T \\
&\quad -2\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_l'\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/T^2 \\
&\quad +\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j/2T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa l}\boldsymbol{R}_{\kappa l} - 2\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/T \\
&\quad +\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j/2T. \qquad \text{(B.95)}
\end{aligned}$$

ii a. From equation (B.87) by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\begin{aligned}
\boldsymbol{D}_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_{\mu'}}/2 \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{ij}\sigma^{\kappa l}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}\boldsymbol{X}_\kappa'\boldsymbol{R}_{\rho_{\mu'}}{}^{\kappa l}\boldsymbol{X}_l/2T^2 \\
&= [\text{by interchanging } j \leftrightarrow k \text{ and } j \leftrightarrow l] \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{X}_\kappa\boldsymbol{\Xi}\boldsymbol{X}_l'\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j/2T^2. \qquad \text{(B.96)}
\end{aligned}$$

iii a.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{G} &= [\text{see (B.67)}] = \boldsymbol{G}\left(\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T\right)\boldsymbol{G} \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}/T. \end{aligned} \tag{B.97}$$

iv a.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G} &= [\text{see (B.95)}] \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa l}\boldsymbol{R}_{\kappa l} - 2\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j\boldsymbol{G}/T \\ &\quad + \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}/2T. \end{aligned} \tag{B.98}$$

i b.

$$\begin{aligned} \boldsymbol{C}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \boldsymbol{A}^*{}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} - 2\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}} + \boldsymbol{A}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}/2 \\ &= \sigma_{\mu'\nu}\boldsymbol{A}_{\varsigma_{(\mu\nu')}} - 2\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}} \\ &= \sigma_{\mu'\nu}\boldsymbol{B}_{\mu\nu'} - 2\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}. \end{aligned} \tag{B.99}$$

ii b.From equation (B.89) by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\begin{aligned} \boldsymbol{D}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}/2 \\ &= \boldsymbol{B}_{\mu\mu'}\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu'}/2. \end{aligned} \tag{B.100}$$

iii b.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G} &= [\text{see (B.70)}] \\ &= \boldsymbol{G}\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}. \end{aligned} \tag{B.101}$$

iv b.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{C}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{G} &= [\text{see (B.99)}] \\ &= \sigma_{\mu'\nu}\boldsymbol{G}\boldsymbol{B}_{\mu\nu'}\boldsymbol{G} - 2\boldsymbol{G}\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}\boldsymbol{G}. \end{aligned} \tag{B.102}$$

i c. From equations (B.73), (B.74) and (B.91)

$$
\begin{aligned}
\boldsymbol{C}_{\rho_\mu \varsigma_{(\nu\nu')}} &= \boldsymbol{A}^*{}_{\rho_\mu \varsigma_{(\nu\nu')}} - 2\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}} + \boldsymbol{A}_{\rho_\mu \varsigma_{(\nu\nu')}}/2 \\
&= \sum_{i=1}^{M}\sum_{\kappa=1}^{M} \sigma^{i\kappa}\sigma_{\kappa\nu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}/T \\
&\quad -2\sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T\boldsymbol{G}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}/T) \\
&\quad +\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}/2T \\
&= [\text{by interchanging } j \leftrightarrow k] \\
&= \sum_{i=1}^{M}\sum_{\kappa=1}^{M} \sigma^{i\kappa}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa\nu}\boldsymbol{R}_{\kappa\nu} - 2\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_\nu'/T]\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}/T \\
&\quad +\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}/2T.
\end{aligned}
\tag{B.103}
$$

ii c. From equation (B.91), by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$
\begin{aligned}
\boldsymbol{D}_{\rho_\mu \varsigma_{(\nu\nu')}} &= \boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}/2 \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}/2T^2.
\end{aligned}
\tag{B.104}
$$

iii c.

$$
\begin{aligned}
\boldsymbol{G}\boldsymbol{C}_{\rho_\mu \varsigma_{(\nu\nu')}}\boldsymbol{G} &= [\text{see (B.103)}] \\
&= \sum_{i=1}^{M}\sum_{\kappa=1}^{M} \sigma^{i\kappa}\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa\nu}\boldsymbol{R}_{\kappa\nu} - 2\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_\nu'/T]\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}/T \\
&\quad +\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}/2T.
\end{aligned}
\tag{B.105}
$$

i d. From equations (B.73), (B.75) and (B.93)

$$
\begin{aligned}
\boldsymbol{C}_{\varsigma_{(\nu\nu')}\rho_\mu} &= \boldsymbol{A}^*{}_{\varsigma_{(\nu\nu')}\rho_\mu} - 2\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{G}\boldsymbol{A}_{\rho_\mu} + \boldsymbol{A}_{\varsigma_{(\nu\nu')}\rho_\mu}/2 \\
&= \sum_{j=1}^{M}\sum_{l=1}^{M} \sigma_{\nu' l}\sigma^{lj}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j/T \\
&\quad -2(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}/T)\boldsymbol{G}\sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/T \\
&\quad +\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}/2T \\
&= [\text{by interchanging } i \leftrightarrow l] \\
&= \sum_{l=1}^{M}\sum_{j=1}^{M} \sigma^{lj}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}[\sigma_{\nu' l}\boldsymbol{R}_{\nu' l} - 2\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j/T \\
&\quad +\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}/2T.
\end{aligned}
\tag{B.106}
$$

ii d. From (B.93) by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\begin{aligned} \boldsymbol{D}_{\varsigma_{(\nu\nu')}\rho_\mu} &= \boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_\mu}/2 \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j/2T^2 \end{aligned} \tag{B.107}$$

iii d.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{C}_{\varsigma_{(\nu\nu')}\rho_\mu}\boldsymbol{G} &= [\text{see (B.106)}] \\ &= \sum_{l=1}^{M}\sum_{j=1}^{M}\sigma^{lj}\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}[\sigma_{\nu' l}\boldsymbol{R}_{\nu' l} - 2\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{G}/T \\ &\quad + \boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}/2T. \end{aligned} \tag{B.108}$$

1. a. The μ-th element of the $((M+M^2)\times 1)$ vector $\boldsymbol{l}$ is

$$\begin{aligned} l_{\rho_\mu} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} = [\text{see (B.97)}] \\ &= \frac{\boldsymbol{e}'}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}\left(\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}/T\right)\frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}} \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{h}/T, \end{aligned} \tag{B.109}$$

where

$$\boldsymbol{h} = \frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}. \tag{B.110}$$

2. a. Similarly, the (μ, μ')-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ is

$$\begin{aligned} l_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\ &= \frac{\boldsymbol{e}'}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G}\frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}} = \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G}\boldsymbol{h} = [\text{see (B.98)}] \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\sigma^{i\kappa}\sigma^{lj}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa l}\boldsymbol{R}_{\kappa l} - 2\boldsymbol{X}_\kappa\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_{\mu'}}{}^{lj}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{h}/T \\ &\quad + \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu\rho_{\mu'}}{}^{ij}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{h}/2T. \end{aligned} \tag{B.111}$$

3. a. The μ-th element of the $((M+M^2)\times 1)$ vector $\boldsymbol{c}$ is

$$\begin{aligned} c_{\rho_\mu} &= tr(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}) = [\text{see (B.78)}] \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\,\mathrm{tr}(\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi}/T). \end{aligned} \tag{B.112}$$

4. a. The (μ, μ')-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{C}$ is

$$
\begin{aligned}
c_{\rho_\mu \rho_{\mu'}} &= \mathrm{tr}(\boldsymbol{C}_{\rho_\mu \rho_{\mu'}} \boldsymbol{\Xi}) = [\text{see (B.95)}] \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M} \sigma^{i\kappa}\sigma_{\kappa l}\sigma^{lj}\, \mathrm{tr}(\boldsymbol{X}_i' \boldsymbol{R}_{\rho_\mu}{}^{i\kappa} \boldsymbol{R}_{\kappa l} \boldsymbol{R}_{\rho_{\mu'}}{}^{lj} \boldsymbol{X}_j \boldsymbol{\Xi})/T \\
&\quad -2\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M} \sigma^{i\kappa}\sigma^{lj}\, \mathrm{tr}(\boldsymbol{X}_i' \boldsymbol{R}_{\rho_\mu}{}^{i\kappa} \boldsymbol{X}_\kappa \boldsymbol{G} \boldsymbol{X}_l' \boldsymbol{R}_{\rho_{\mu'}}{}^{lj} \boldsymbol{X}_j \boldsymbol{\Xi})/T^2 \\
&\quad +\sum_{i=1}^{M}\sum_{j=1}^{M} \sigma^{ij}\, \mathrm{tr}(\boldsymbol{X}_i' \boldsymbol{R}_{\rho_\mu \rho_{\mu'}}{}^{ij} \boldsymbol{X}_j \boldsymbol{\Xi})/2T. \qquad \text{(B.113)}
\end{aligned}
$$

5. a. The (μ, μ')-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{D}$ is

$$
\begin{aligned}
d_{\rho_\mu \rho_{\mu'}} &= \mathrm{tr}(\boldsymbol{D}_{\rho_\mu \rho_{\mu'}} \boldsymbol{\Xi}) = [\text{see (B.96)}] \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{\kappa=1}^{M}\sum_{l=1}^{M} \sigma^{i\kappa}\sigma^{lj}\, \mathrm{tr}(\boldsymbol{X}_i' \boldsymbol{R}_{\rho_\mu}{}^{i\kappa} \boldsymbol{X}_\kappa \boldsymbol{\Xi} \boldsymbol{X}_l' \boldsymbol{R}_{\rho_{\mu'}}{}^{lj} \boldsymbol{X}_j \boldsymbol{\Xi})/2T^2. \qquad \text{(B.114)}
\end{aligned}
$$

1. b. The $(\mu\mu')$-th element of the $((M+M^2)\times 1)$ vector $\boldsymbol{l}$ is

$$
\begin{aligned}
l_{\varsigma(\mu\mu')} &= [\text{see (B.101)}] = \boldsymbol{e}'\boldsymbol{G}\boldsymbol{A}_{\varsigma(\mu\mu')}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\
&= \frac{\boldsymbol{e}'}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}\boldsymbol{G}\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}} \\
&= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{h} = [\text{see (B.66), (B.70)}] \\
&= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{h}/T. \qquad \text{(B.115)}
\end{aligned}
$$

2. b. Similarly, the $((\mu\mu'),(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ is

$$
\begin{aligned}
l_{\varsigma(\mu\mu')\varsigma(\nu\nu')} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{\varsigma(\mu\mu')\varsigma(\nu\nu')}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\
&= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\varsigma(\mu\mu')\varsigma(\nu\nu')}\boldsymbol{G}\boldsymbol{h} = [\text{see (B.102)}] \\
&= \boldsymbol{h}'(\sigma_{\mu'\nu}\boldsymbol{G}\boldsymbol{B}_{\mu\nu'}\boldsymbol{G} - 2\boldsymbol{G}\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}\boldsymbol{G})\boldsymbol{h} \\
&= \sigma_{\mu'\nu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{B}_{\mu\nu'}\boldsymbol{G}\boldsymbol{h} - 2\boldsymbol{h}'\boldsymbol{G}\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}\boldsymbol{G}\boldsymbol{h} \\
&= \sigma_{\mu'\nu}l_{\varsigma(\mu\nu')} - 2\boldsymbol{h}'\boldsymbol{G}\boldsymbol{B}_{\mu\mu'}\boldsymbol{G}\boldsymbol{B}_{\nu\nu'}\boldsymbol{G}\boldsymbol{h} \\
&= [\text{see (B.70) and (B.106)}] \\
&= \sigma_{\mu'\nu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/T - 2\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/T^2. \qquad \text{(B.116)}
\end{aligned}
$$

3. b.The $(\mu\mu')$-th element of the $((M+M^2)\times 1)$ vector $\boldsymbol{c}$ is

$$
\begin{aligned}
c_{\varsigma(\mu\mu')} &= \mathrm{tr}(\boldsymbol{A}_{\varsigma(\mu\mu')}\boldsymbol{\Xi}) = [\text{see (B.81)}] \\
&= \mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi})/T. \qquad \text{(B.117)}
\end{aligned}
$$

4. b.The $((\mu\mu'),(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned}
c_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \mathrm{tr}(\boldsymbol{C}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) = [\text{see (B.99)}] \\
&= \sigma_{\mu'\nu}\,\mathrm{tr}(\boldsymbol{A}_{\varsigma_{(\mu\nu')}}\boldsymbol{\Xi}) - 2\,\mathrm{tr}(\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{G}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) \\
&= [\text{see(B.81) and (B.89)}] \\
&= \sigma_{\mu'\nu}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/T - 2(\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{X}'_\nu \boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/T^2. \qquad \text{(B.118)}
\end{aligned}$$

5. b. The $((\mu\mu'),(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{D}$ is

$$\begin{aligned}
d_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}} &= \mathrm{tr}(\boldsymbol{D}_{\varsigma_{(\mu\mu')}\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) = [\text{see (B.100)}] = \mathrm{tr}(\boldsymbol{A}_{\varsigma_{(\mu\mu')}}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi})/2 \\
&= \mathrm{tr}(\boldsymbol{B}_{\mu\mu'}\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu'}\boldsymbol{\Xi})/2 = [\text{see (B.66)}] \\
&= \mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}\boldsymbol{X}'_\nu \boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/2T^2. \qquad \text{(B.119)}
\end{aligned}$$

1. c. Similarly, the $(\mu,(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ is

$$\begin{aligned}
l_{\rho_\mu\varsigma_{(\nu\nu')}} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\
&= \frac{\boldsymbol{e}'}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{G}\frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}} \\
&= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{G}\boldsymbol{h} = [\text{see (B.103)}] \\
&= \sum_{i=1}^{M}\sum_{\kappa=1}^{M}\sigma^{i\kappa}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_i \boldsymbol{R}_{\rho_\mu}{}^{i\kappa}[\sigma_{\kappa\nu}\boldsymbol{R}_{\kappa\nu} - 2\boldsymbol{X}_\kappa \boldsymbol{G}\boldsymbol{X}'_\nu/T]\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/T \\
&\quad + \boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\nu \boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/2T. \qquad \text{(B.120)}
\end{aligned}$$

2. c.The $(\mu,(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned}
c_{\rho_\mu\varsigma_{(\nu\nu')}} &= [\text{see (B.103)}] = \mathrm{tr}(\boldsymbol{C}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) \\
&= \sum_{i=1}^{M}\sum_{\kappa=1}^{M}\sigma^{i\kappa}\sigma_{\kappa\nu}\,\mathrm{tr}(\boldsymbol{X}'_i \boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{R}_{\kappa\nu}\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/T \\
&\quad -2\sum_{i=1}^{M}\sum_{\kappa=1}^{M}\sigma^{i\kappa}\,\mathrm{tr}(\boldsymbol{X}'_i \boldsymbol{R}_{\rho_\mu}{}^{i\kappa}\boldsymbol{X}_\kappa \boldsymbol{G}\boldsymbol{X}'_\nu \boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/T^2 \\
&\quad + \mathrm{tr}(\boldsymbol{X}'_\nu \boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/2T. \qquad \text{(B.121)}
\end{aligned}$$

3. c. The $(\mu,(\nu\nu'))$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{D}$ is

$$\begin{aligned}
d_{\rho_\mu\varsigma_{(\nu\nu')}} &= \mathrm{tr}(\boldsymbol{D}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) = [\text{see (B.104)}] = \mathrm{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\varsigma_{(\nu\nu')}}/2) \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\,\mathrm{tr}(\boldsymbol{X}'_i \boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j \boldsymbol{\Xi}\boldsymbol{X}'_\nu \boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/2T^2. \qquad \text{(B.122)}
\end{aligned}$$

1. d. The $((\nu\nu'),\mu)$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{L}$ is

$$\begin{aligned}
l_{\varsigma_{(\nu\nu')}\rho_\mu} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{\varsigma_{(\nu\nu')}\rho_\mu}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\
&= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\varsigma_{(\nu\nu')}\rho_\mu}\boldsymbol{G}\boldsymbol{h} = \text{[see (B.108)]} \\
&= \sum_{l=1}^{M}\sum_{j=1}^{M}\sigma^{lj}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}[\sigma_{\nu' l}\boldsymbol{R}_{\nu' l} - 2\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_l'/T]\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{G}\boldsymbol{h}/T \\
&\quad +\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{h}/2T. \qquad \text{(B.123)}
\end{aligned}$$

2. d.The $((\nu\nu'),\mu)$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned}
c_{\varsigma_{(\nu\nu')}\rho_\mu} &= \text{tr}(\boldsymbol{C}_{\rho_\mu\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}) = \text{[see (B.106)]} \\
&= \sum_{l=1}^{M}\sum_{j=1}^{M}\sigma^{lj}\sigma_{\nu' l}\,\text{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{R}_{\nu' l}\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi})/T \\
&\quad -2\sum_{l=1}^{M}\sum_{j=1}^{M}\sigma^{lj}\,\text{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{G}\boldsymbol{X}_l'\boldsymbol{R}_{\rho_\mu}{}^{lj}\boldsymbol{X}_j\boldsymbol{\Xi})/T^2 \\
&\quad +\text{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi})/2T. \qquad \text{(B.124)}
\end{aligned}$$

3. d.The $((\nu\nu'),\mu)$-th element of the $((M+M^2)\times(M+M^2))$ matrix $\boldsymbol{D}$ is

$$\begin{aligned}
d_{\varsigma_{(\nu\nu')}\rho_\mu} &= \text{tr}(\boldsymbol{D}_{\varsigma_{(\nu\nu')}\rho_\mu}\boldsymbol{\Xi}) = \text{[see (B.107)]} = \text{tr}(\boldsymbol{A}_{\varsigma_{(\nu\nu')}}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_\mu}/2) \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma^{ij}\,\text{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu'}\boldsymbol{X}_{\nu'}\boldsymbol{\Xi}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{ij}\boldsymbol{X}_j\boldsymbol{\Xi})/2T^2. \qquad \text{(B.125)}
\end{aligned}$$

□

Lemma B.1. For all estimators $\hat{\boldsymbol{B}}_I$, (I=UL, RL, GL, IG, ML) of B the following results hold:

$$\hat{\boldsymbol{B}}_I = \boldsymbol{B} + \tau\boldsymbol{B}_1{}^I + \omega(\tau^2), \qquad \text{(B.126)}$$

where

$$\boldsymbol{B}_1{}^{UL} = (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T}, \qquad \text{(B.127)}$$

$$\text{vec}(\boldsymbol{B}_1{}^{RL}) = \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*/T)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon}/\sqrt{T}, \qquad \text{(B.128)}$$

$$\begin{aligned}
\text{vec}(\boldsymbol{B}_1{}^{GL}) &= \text{vec}(\boldsymbol{B}_1{}^{IG}) = \text{vec}(\boldsymbol{B}_1{}^{ML}) \\
&= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T}. \qquad \text{(B.129)}
\end{aligned}$$

Proof of Lemma B.1. i.

$$\begin{aligned}\hat{\boldsymbol{B}}_{UL} &= (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{Y}_* = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{Z}\boldsymbol{B}+\boldsymbol{E}) \\ &= \boldsymbol{B} + (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E} = \boldsymbol{B} + \tau(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} \\ &= \boldsymbol{B} + \tau\boldsymbol{B_1}^{UL}. \end{aligned} \tag{B.130}$$

ii. Since

$$\text{vec}(\boldsymbol{B}) = \begin{bmatrix}\boldsymbol{b}_1\\ \vdots \\ \boldsymbol{b}_M\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Psi}_1\boldsymbol{\beta}\\ \vdots \\ \boldsymbol{\Psi}_M\boldsymbol{\beta}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Psi}_1\\ \vdots \\ \boldsymbol{\Psi}_M\end{bmatrix}\boldsymbol{\beta} = \boldsymbol{\Psi}\boldsymbol{\beta}, \tag{B.131}$$

by vectorizing (3.38) we take

$$\begin{aligned}\boldsymbol{y}_* &= \text{vec}(\boldsymbol{Y}_*) = \text{vec}(\boldsymbol{Z}\boldsymbol{B}+\boldsymbol{E}) = \text{vec}(\boldsymbol{Z}\boldsymbol{B}) + \text{vec}(\boldsymbol{E}) \\ &= (\boldsymbol{I}\otimes\boldsymbol{Z})\,\text{vec}(\boldsymbol{B}) + \boldsymbol{\varepsilon} = (\boldsymbol{I}\otimes\boldsymbol{Z})\boldsymbol{\Psi}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \boldsymbol{X}_*\boldsymbol{\beta} + \boldsymbol{\varepsilon}. \end{aligned} \tag{B.132}$$

Thus,

$$\begin{aligned}\text{vec}(\hat{\boldsymbol{B}}_{RL}) &= \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{y}_* \\ &= \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'(\boldsymbol{X}_*\boldsymbol{\beta}+\boldsymbol{\varepsilon}) = \boldsymbol{\Psi}\boldsymbol{\beta} + \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon} \\ &= \boldsymbol{\Psi}\boldsymbol{\beta} + \tau\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*/T)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon}/\sqrt{T} = \text{vec}(\boldsymbol{B}) + \tau\,\text{vec}(\boldsymbol{B_1}^{RL}) \Rightarrow \qquad \text{(B.133)} \\ \Rightarrow \hat{\boldsymbol{B}}_{RL} &= \boldsymbol{B} + \tau\boldsymbol{B_1}^{RL}. \qquad \text{(B.134)} \end{aligned}$$

iii. For any consistent estimator $\hat{\boldsymbol{\Sigma}}^{-1}$ of $\boldsymbol{\Sigma}^{-1}$ it holds that

$$\hat{\boldsymbol{\Sigma}}^{-1} = \boldsymbol{\Sigma}^{-1} + \omega(\tau), \tag{B.135}$$

which implies that

$$(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T) = (\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T) + \omega(\tau). \tag{B.136}$$

Therefore,

$$\begin{aligned}\text{vec}(\hat{\boldsymbol{B}}_{GL}) &= \boldsymbol{\Psi}(\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{y}_* \\ &= \boldsymbol{\Psi}(\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T)(\boldsymbol{X}_*\boldsymbol{\beta}+\boldsymbol{\varepsilon}) \\ &= \boldsymbol{\Psi}\boldsymbol{\beta} + \tau\boldsymbol{\Psi}[\boldsymbol{X}_*'((\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)+\omega(\tau))\boldsymbol{X}_*/T]^{-1}\boldsymbol{X}_*'((\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)+\omega(\tau))\boldsymbol{\varepsilon}/\sqrt{T} \\ &= \text{vec}(\boldsymbol{B}) + \tau\boldsymbol{\Psi}[(\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T) + \tau\omega(\tau^2)]^{-1}[(\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T}) + \omega(\tau^2)] \\ &= \text{vec}(\boldsymbol{B}) + \tau\boldsymbol{\Psi}[(\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T)^{-1} + \tau\omega(\tau^2)][(\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T}) + \omega(\tau^2)] \\ &= \text{vec}(\boldsymbol{B}) + \tau\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T]^{-1}\boldsymbol{X}_*(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T} + \omega(\tau^2) \\ &= \text{vec}(\boldsymbol{B}) + \tau\,\text{vec}(\boldsymbol{B_1}^{GL}) + \omega(\tau^2) \Rightarrow \end{aligned} \tag{B.137}$$

$$\hat{\boldsymbol{B}}_{GL} = \boldsymbol{B} + \tau \boldsymbol{B}_1{}^{GL} + \omega(\tau^2). \tag{B.138}$$

Since $\hat{\boldsymbol{B}}_{IG}$ and $\hat{\boldsymbol{B}}_{ML}$ are the outcome of iterative use of the GL-estimation process, equation (B.138) implies that

$$\hat{\boldsymbol{B}}_{IG} = \boldsymbol{B} + \tau \boldsymbol{B}_1{}^{IG} + \omega(\tau^2) \tag{B.139}$$

and

$$\hat{\boldsymbol{B}}_{ML} = \boldsymbol{B} + \tau \boldsymbol{B}_1{}^{ML} + \omega(\tau^2), \tag{B.140}$$

where

$$\text{vec}(\boldsymbol{B}_1{}^{IG}) = \text{vec}(\boldsymbol{B}_1{}^{ML}) = \text{vec}(\boldsymbol{B}_1{}^{GL}). \tag{B.141}$$

So, equations (B.130), (B.133), (B.137), (B.139), (B.140) and (B.141) complete the proof.

□

Lemma B.2. For any conformable matrix $\boldsymbol{\Gamma}$ lemma B.1 implies that

$$\lim_{T\to\infty} T\,\text{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] = \lim_{T\to\infty} \text{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]. \tag{B.142}$$

Proof of Lemma B.2.

$$\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL} = (B + \tau \boldsymbol{B}_1{}^I + \omega(\tau^2)) - (\boldsymbol{B} + \tau \boldsymbol{B}_1{}^{UL}) = \tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2) \Rightarrow \tag{B.143}$$

$$\begin{aligned}
(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL}) &= [\tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2)]'\boldsymbol{\Gamma}[\tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2)] \\
&= \tau^2(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^3) \Rightarrow \\
T\,\text{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] &= \text{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})] + O(\tau) \Rightarrow \\
\lim_{T\to\infty} T\,\text{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] &= \lim_{T\to\infty} \text{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]. \qquad \text{(B.144)}
\end{aligned}$$

□

Lemma B.3. Since the rows $\boldsymbol{\varepsilon}_t'$ $(t = 1, \ldots, T)$ of $\boldsymbol{E}$ are independent $\mathcal{N}_M(\boldsymbol{0}, \boldsymbol{\Sigma})$ vectors, the matrix $\boldsymbol{E}'\boldsymbol{E}$ has a Wishart distribution with weight matrix $\boldsymbol{\Sigma}$ and T degrees of freedom i.e,

$$\boldsymbol{E}'\boldsymbol{E} \sim \mathcal{W}(\boldsymbol{\Sigma}, T), \;\; \mathrm{E}(\boldsymbol{E}'\boldsymbol{E}) = T\boldsymbol{\Sigma}. \tag{B.145}$$

Then,

$$\mathrm{E}(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}) = T(M + T + 1)\boldsymbol{\Sigma}. \tag{B.146}$$

Proof of Lemma B.3.

$$\boldsymbol{E}'\boldsymbol{E} = (\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_T)\begin{bmatrix}\boldsymbol{\varepsilon}_1' \\ \vdots \\ \boldsymbol{\varepsilon}_T'\end{bmatrix} = \sum_{t=1}^{T} \boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' \tag{B.147}$$

$$\begin{aligned}\Rightarrow \boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E} \quad &= \quad \sum_{t=1}^{T} \boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\sum_{t'=1}^{T}\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}' \\ = \sum_{t=1}^{T}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' \quad &+ \quad \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}'\end{aligned} \tag{B.148}$$

where $\boldsymbol{\varepsilon}_t'$ and $\boldsymbol{\varepsilon}_{t'}'$ are independent $\mathcal{N}_M(\boldsymbol{0}, \boldsymbol{\Sigma})$ vectors for $t \neq t'$.

Let $\boldsymbol{g}$ be any arbitrary $(M \times 1)$ non-stochastic vector. Then,

$$\begin{aligned}\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{g} \quad &= \quad \mathrm{tr}(\boldsymbol{g}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{g}) \\ &= \quad \mathrm{tr}(\boldsymbol{\varepsilon}_t'\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t) = \boldsymbol{\varepsilon}_t'\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t \Rightarrow \\ \mathrm{E}(\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{g}) \quad &= \quad \mathrm{E}(\boldsymbol{\varepsilon}_t'\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t) \\ &= \quad [\text{ see Magnus and Neudecker, 1979, p.389}] \\ &= \quad \mathrm{tr}\,(\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\Sigma})\,\mathrm{tr}\,(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}) + 2\,\mathrm{tr}\,(\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}) \\ &= \quad \mathrm{tr}(\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g})\,\mathrm{tr}\,(\boldsymbol{I}_M) + 2\,\mathrm{tr}\,(\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g}) \\ &= \quad M\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} + 2\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} \\ &= \quad (M + 2)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g}.\end{aligned} \tag{B.149}$$

Since $\boldsymbol{\varepsilon}_t'$ and $\boldsymbol{\varepsilon}_{t'}'$ are independent vectors for $t \neq t'$, equations (B.145) and (B.146) imply that

$$\begin{aligned}
\mathrm{E}[\boldsymbol{g}'(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E})\boldsymbol{g}] &= \mathrm{E}\left[\boldsymbol{g}'\left(\sum_{t=1}^{T}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' + \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}'\right)\boldsymbol{g}\right] \\
&= \sum_{t=1}^{T}\mathrm{E}[\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{g}] + \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\mathrm{E}[\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}')\boldsymbol{g}] \\
&= \sum_{t=1}^{T}\mathrm{E}[\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{g}] + \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\boldsymbol{g}'\,\mathrm{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{\Sigma}^{-1}\,\mathrm{E}(\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}')\boldsymbol{g} \\
&= \sum_{t=1}^{T}(M+2)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} + \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{g} \\
&= T(M+2)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} + T(T-1)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} \\
&= T(M+T+1)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g}. \qquad \text{(B.150)}
\end{aligned}$$

Since $\boldsymbol{g}$ is any arbitrary non-stochastic vector, equation (B.147) implies that

$$\begin{aligned}
\mathrm{E}[\boldsymbol{g}'(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E})\boldsymbol{g}] &= \boldsymbol{g}'\,\mathrm{E}[\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}]\boldsymbol{g} = T(M+T+1)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} \\
\Rightarrow \mathrm{E}[\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}] &= T(M+T+1)\boldsymbol{\Sigma}. \qquad \text{(B.151)}
\end{aligned}$$

□

Lemma B.4. Let $\hat{E}_I$ be the residuals of the regression equation

$$\boldsymbol{Y}_* = \boldsymbol{Z}\boldsymbol{B} + \boldsymbol{E}, \quad (B.152)$$

when the $\hat{B}_I$ (I=UL, RL, GL, IG, ML) estimator is used. Lemma B.1 implies that

$$\begin{aligned} \hat{\boldsymbol{E}}_I &= \boldsymbol{Y}_* - \boldsymbol{Z}\hat{\boldsymbol{B}}_I = \boldsymbol{Z}\boldsymbol{B} + \boldsymbol{E} - \boldsymbol{Z}(\boldsymbol{B} + \tau\boldsymbol{B}_1{}^I + \omega(\tau^2)) \\ &= \boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I + \omega(\tau^2). \end{aligned} \quad (B.153)$$

For the $\hat{\boldsymbol{\Sigma}}_I$ (I=UL, RL, GL, IG, ML) estimator of $\boldsymbol{\Sigma}$ it holds that

$$\begin{aligned} \hat{\boldsymbol{\Sigma}}_I &= \hat{\boldsymbol{E}}_I'\hat{\boldsymbol{E}}_I/T = [\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I + \omega(\tau^2)]'[\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I + \omega(\tau^2)]/T \\ &= [\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I]'[\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I]/T + \omega(\tau^4) \\ &= [\boldsymbol{E}' - \tau\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'][\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I]/T + \omega(\tau^4) \\ &= \boldsymbol{E}'\boldsymbol{E}/T - \tau\boldsymbol{E}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/T - \tau\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{E}/T + \tau^2\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/T + \omega(\tau^4) \\ &= \boldsymbol{E}'\boldsymbol{E}/T - \tau^2\boldsymbol{E}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/\sqrt{T} - \tau^2\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} + \tau^2\boldsymbol{B}_1{}^{I\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)\boldsymbol{B}_1{}^I + \omega(\tau^4) \\ &= \boldsymbol{E}'\boldsymbol{E}/T + \tau^2[\boldsymbol{B}_1{}^{I\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)\boldsymbol{B}_1{}^I - \boldsymbol{E}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/\sqrt{T} - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T}] + \omega(\tau^4). \end{aligned} \quad (B.154)$$

By using equation (B.127) we find that

$$\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} = \boldsymbol{B}_1{}^{I\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} = \boldsymbol{B}_1{}^{I\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)\boldsymbol{B}_1{}^{UL}. \quad (B.155)$$

Similarly,

$$\boldsymbol{E}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/\sqrt{T} = \boldsymbol{B}_1{}^{UL\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)\boldsymbol{B}_1{}^I. \quad (B.156)$$

Since $\boldsymbol{\Gamma} = \boldsymbol{Z}'\boldsymbol{Z}/T$, equations (B.154), (B.155) and (B.156) imply that

$$\begin{aligned} \hat{\boldsymbol{\Sigma}}_I &= \boldsymbol{E}'\boldsymbol{E}/T + \tau^2[\boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL}] + \omega(\tau^4) \\ &= \boldsymbol{\Sigma} - \tau\sqrt{T}\boldsymbol{\Sigma} + \tau\sqrt{T}\boldsymbol{E}'\boldsymbol{E}/T + \tau^2[\boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL}] + \omega(\tau^4). \end{aligned} \quad (B.157)$$

The following result holds:

$$\begin{aligned} & \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL\prime} = \boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL} \\ = \; & \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL} + \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL} - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL} \\ = \; & (\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) - [(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T}]'(\boldsymbol{Z}'\boldsymbol{Z}/T)[(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T}] \\ = \; & (\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}(\boldsymbol{Z}'\boldsymbol{Z}/T)(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/T \\ = \; & (\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E} \\ = \; & (\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}, \end{aligned} \quad (B.158)$$

where $\boldsymbol{P}_Z = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'$. Thus, equations (B.157) and (B.158) imply that

$$\begin{aligned}\hat{\boldsymbol{\Sigma}}_I &= \boldsymbol{\Sigma} + \tau[\sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma})] + \tau^2[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}] + \omega(\tau^4)\\ &= \boldsymbol{\Sigma} + \tau\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}_2^I + \omega(\tau^3)\\ &= \boldsymbol{\Sigma} + \tau(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I) + \omega(\tau^3), \end{aligned} \tag{B.159}$$

where

$$\boldsymbol{\Sigma}_1 = \sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma}) \tag{B.160}$$

and

$$\boldsymbol{\Sigma}_2^I = (\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}. \tag{B.161}$$

Equation (B.159) implies that

$$\begin{aligned}\hat{\boldsymbol{\Sigma}}_I^{-1} &= [\boldsymbol{\Sigma} + \tau(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I) + \omega(\tau^3)]^{-1}\\ &= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1} + \tau^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1} + \omega(\tau^3)\\ &= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} - \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1} + \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} + \omega(\tau^3)\\ &= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} + \tau^2[\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1}] + \omega(\tau^3)\\ &= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} + \tau^2[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}] + \omega(\tau^3)\\ &= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{S}_1 + \tau^2\boldsymbol{S}_2{}^I + \omega(\tau^3), \end{aligned} \tag{B.162}$$

where

$$\boldsymbol{S}_1 = \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}, \tag{B.163}$$

$$\boldsymbol{S}_2{}^I = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}. \tag{B.164}$$

Moreover, the following results hold:

i.

$$\begin{aligned}\mathrm{E}(\boldsymbol{\Sigma}_1) &= \mathrm{E}[\sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma})] = \sqrt{T}[\mathrm{E}(\boldsymbol{E}'\boldsymbol{E})/T - \boldsymbol{\Sigma}] = [\text{see (B.145)}]\\ &= \sqrt{T}[T\boldsymbol{\Sigma}/T - \boldsymbol{\Sigma}] = 0. \end{aligned} \tag{B.165}$$

ii. Since $\boldsymbol{E}'\boldsymbol{E} \sim \mathcal{W}(\boldsymbol{\Sigma}, T)$ and since $\boldsymbol{P}_Z = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'$ is idempotent with

$$\operatorname{rank}(\boldsymbol{P}_Z) = \operatorname{tr}(\boldsymbol{P}_Z) = \operatorname{tr}[\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'] = \operatorname{tr}[(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{Z}] = \operatorname{tr}\boldsymbol{I}_K = K, \tag{B.166}$$

it follows that

$$\boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E} \sim \mathcal{W}(\boldsymbol{\Sigma}, K). \tag{B.167}$$

Furthermore,

$$\mathrm{E}(\boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}) = \mathrm{tr}\,(\boldsymbol{P}_Z)\boldsymbol{\Sigma} = K\boldsymbol{\Sigma} \text{ [see Magnus and Neudecker, 1979]}. \tag{B.168}$$

iii.

$$\mathrm{E}(\boldsymbol{S}_1) = \mathrm{E}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}) = \boldsymbol{\Sigma}^{-1}\,\mathrm{E}(\boldsymbol{\Sigma}_1)\boldsymbol{\Sigma}^{-1} = 0 \text{ [see (B.165)]}. \tag{B.169}$$

iv.

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1) &= \mathrm{E}[\,\sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}\,\sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma})] \\
&= \mathrm{E}[T(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}/T^2 + \boldsymbol{\Sigma} - \boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{E}'\boldsymbol{E}/T)] \\
&= \mathrm{E}(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}/T + T\boldsymbol{\Sigma} - 2\boldsymbol{E}'\boldsymbol{E}) \\
&= \mathrm{E}(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E})/T - 2\,\mathrm{E}(\boldsymbol{E}'\boldsymbol{E}) + T\boldsymbol{\Sigma} \\
&= T(M + T + 1)\boldsymbol{\Sigma}/T - 2T\boldsymbol{\Sigma} + T\boldsymbol{\Sigma} \\
&= M\boldsymbol{\Sigma} + T\boldsymbol{\Sigma} + \boldsymbol{\Sigma} - 2T\boldsymbol{\Sigma} + T\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(M + 1).
\end{aligned} \tag{B.170}$$

v.

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\Sigma}_2^I) &= \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}] \\
&= \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})] - \mathrm{E}[\boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}] \\
&= \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})] - K\boldsymbol{\Sigma}
\end{aligned} \tag{B.171}$$

$$\begin{aligned}
\Rightarrow \mathrm{E}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1}) &= \boldsymbol{\Sigma}^{-1}\,\mathrm{E}(\boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1} \\
&= \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1} - K\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} \\
&= \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1} - K\boldsymbol{\Sigma}^{-1}.
\end{aligned} \tag{B.172}$$

vi. Thus equationS (B.164), (B.170) and (B.172) imply that

$$\begin{aligned}
\mathrm{E}(\boldsymbol{S}_2{}^I) &= \mathrm{E}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}] \\
&= \mathrm{E}[\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1}] \\
&= \boldsymbol{\Sigma}^{-1}\,\mathrm{E}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1)\boldsymbol{\Sigma}^{-1} - \mathrm{E}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1}) \\
&= (M + 1)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} + K\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1} \\
&= (M + K + 1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1}.
\end{aligned} \tag{B.173}$$

Lemma B.5. We estimate the model

$$\boldsymbol{y}_* = \boldsymbol{X}_*\boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{B.174}$$

by using the I estimation process, and we estimate $(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)$ by using the estimator

$$(\hat{\boldsymbol{\Sigma}}_I^{-1} \otimes \boldsymbol{I}_T). \tag{B.175}$$

Then by using (B.175) we estimate (B.174) via the GL-estimation method. Let $\hat{\boldsymbol{\Sigma}}_I$ the estimation of $\boldsymbol{\Sigma}$ by using the GL residuals, $\hat{\boldsymbol{\varepsilon}}_{GL} = \text{vec}(\hat{\boldsymbol{E}}_{GL})$ say, from equation (B.174) i.e.,

$$\hat{\boldsymbol{\Sigma}}_I = \hat{\boldsymbol{E}}'_{GL}\hat{\boldsymbol{E}}_{GL}/T. \tag{B.176}$$

Let $\hat{\boldsymbol{\beta}}_{GL}$ be the GL estimator of $\boldsymbol{\beta}$ in (B.174). For the $\hat{\sigma}_I^2$ (I=UL, RL, GL, IG, ML) estimator of σ^2 holds that

$$\begin{aligned}
\hat{\sigma}_I^2 &= (\boldsymbol{y}_* - \boldsymbol{X}_*\hat{\boldsymbol{\beta}}_{GL})'(\hat{\boldsymbol{\Sigma}}_I^{-1} \otimes \boldsymbol{I}_T)(\boldsymbol{y}_* - \boldsymbol{X}_*\hat{\boldsymbol{\beta}}_{GL})/(TM-n)\\
&= \hat{\boldsymbol{\varepsilon}}'_{GL}(\hat{\boldsymbol{\Sigma}}_I^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{\varepsilon}}_{GL}/(TM-n)\\
&= [\text{vec}(\hat{\boldsymbol{E}}_{GL})]'(\hat{\boldsymbol{\Sigma}}_I^{-1} \otimes \boldsymbol{I}_T)[\text{vec}(\hat{\boldsymbol{E}}_{GL})]/(TM-n)\\
&= \text{tr}\,[\hat{\boldsymbol{E}}_{GL}(\hat{\boldsymbol{\Sigma}}_I^{-1})'\hat{\boldsymbol{E}}'_{GL}]/(TM-n) = \text{tr}\,\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{E}}'_{GL}\hat{\boldsymbol{E}}_{GL}/(TM-n)\\
&= \text{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}T\hat{\boldsymbol{\Sigma}}_J)/(TM-n) = \text{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J)/((TM-n)/T)\\
&= \text{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J)/(M-n/T) = \text{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J)/(M-\tau^2 n).
\end{aligned} \tag{B.177}$$

By using equations (B.159), (B.160) and (B.161) we take

$$\hat{\boldsymbol{\Sigma}}_J = \boldsymbol{\Sigma} + \tau\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}_2^J + \omega(\tau^3), \tag{B.178}$$

where

$$\boldsymbol{\Sigma}_1 = \sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma}) \tag{B.179}$$

and

$$\boldsymbol{\Sigma}_2^J = (\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}. \tag{B.180}$$

Then, equations (B.162),(B.164), (B.177) and (B.178) imply that

$$\begin{aligned}
\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J &= [\boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{S}_1 + \tau^2\boldsymbol{S}_2{}^I + \omega(\tau^3)][\boldsymbol{\Sigma} + \tau\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}_2^J + \omega(\tau^3)]\\
&= \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} + \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^J - \tau\boldsymbol{S}_1\boldsymbol{\Sigma} - \tau^2\boldsymbol{S}_1\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{S}_2{}^I\boldsymbol{\Sigma} + \omega(\tau^3)\\
&= \boldsymbol{I}_M + \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^J - \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} - \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} + \omega(\tau^3)\\
&= \boldsymbol{I}_M + \tau^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I) + \omega(\tau^3) \Rightarrow
\end{aligned} \tag{B.181}$$

$$\begin{aligned}\mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J) &= \mathrm{tr}\,\boldsymbol{I}_M + \tau^2\,\mathrm{tr}\,[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)] + \omega(\tau^3)\\ &= M + \tau^2\,\mathrm{tr}\,[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)] + \omega(\tau^3) \Rightarrow \end{aligned} \tag{B.182}$$

$$\hat{\sigma}_I^2 = \mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J)/(M-\tau^2 n) = [M + \tau^2\,\mathrm{tr}\,[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)]]/(M-\tau^2 n) + \omega(\tau^3). \tag{B.183}$$

Moreover,

$$\begin{aligned}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I) &= \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)\\ &= \boldsymbol{\Sigma}^{-1}[(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I) - (\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^J)]\\ &= \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^J)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\\ &= \boldsymbol{S}_2^I\boldsymbol{\Sigma} - \boldsymbol{S}_2^J\boldsymbol{\Sigma} = (\boldsymbol{S}_2^I - \boldsymbol{S}_2^J)\boldsymbol{\Sigma} \Rightarrow \end{aligned} \tag{B.184}$$

$$\mathrm{tr}\,[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)] = \mathrm{tr}\,(\boldsymbol{S_2}^I - \boldsymbol{S_2}^J)\boldsymbol{\Sigma} \tag{B.185}$$

Thus, equations (B.183) and (B.185) imply that

$$\hat{\sigma}_I^2 = [M + \tau^2\,\mathrm{tr}[(\boldsymbol{S_2}^I - \boldsymbol{S_2}^J)\boldsymbol{\Sigma}]]/(M-\tau^2 n) + \omega(\tau^3). \tag{B.186}$$

Lemma B.6. Define the $M \times M$ matrices

$$\boldsymbol{M}_I = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{S_2}^I) \tag{B.187}$$

and

$$\boldsymbol{\Delta}_I = \lim_{T\to\infty} T\,\mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})]\ (\mathrm{I{=}UL,\ RL,\ GL,\ IG,\ ML}) \tag{B.188}$$

and the $(M^2 \times M^2)$ matrix N with elements

$$\nu_{(ij)(kl)} = \sigma_{i\kappa}\sigma_{jl} + \sigma_{il}\sigma_{j\kappa}\ (i, j, \kappa, l = 1, \ldots, M). \tag{B.189}$$

The following results hold:

i.

$$\begin{aligned}\boldsymbol{M}_I &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{S_2}^I) = (\mathrm{B.173})\\ &= (M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}[\lim_{T\to\infty} \mathrm{E}[(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})]]\boldsymbol{\Sigma}^{-1}\\ &= [\text{see Lemma (B.2)}]\\ &= (M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}[\lim_{T\to\infty} T\,\mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})]]\boldsymbol{\Sigma}^{-1}\\ &= [\text{see (B.188)}] = (M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1} \Rightarrow \end{aligned} \tag{B.190}$$

$$\begin{aligned}(\boldsymbol{M}_I - \boldsymbol{M}_{GL})\boldsymbol{\Sigma} &= [(M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1} - (M+K+1)\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_{GL}\boldsymbol{\Sigma}^{-1}]\boldsymbol{\Sigma} \\ &= (\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_{GL}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1})\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL} - \boldsymbol{\Delta}_I)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} \\ &= \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL} - \boldsymbol{\Delta}_I). \end{aligned} \tag{B.191}$$

ii.

$$\begin{aligned}\mathrm{E}[(\boldsymbol{S_2}^I - \boldsymbol{S_2}^J)\boldsymbol{\Sigma}] &= [\mathrm{E}(\boldsymbol{S_2}^I) - \mathrm{E}(\boldsymbol{S_2}^J)]\boldsymbol{\Sigma} = [\text{see} \quad (\text{B.173})] \\ &= [(M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})]\boldsymbol{\Sigma}^{-1} \\ &\quad -(M+K+1)\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B_1}^J - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^J - \boldsymbol{B_1}^{UL})]\boldsymbol{\Sigma}^{-1}]\boldsymbol{\Sigma} \\ &= -\boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})]\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} \\ &\quad +\boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B_1}^J - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^J - \boldsymbol{B_1}^{UL})]\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} \Rightarrow \end{aligned} \tag{B.192}$$

$$\begin{aligned}\lim_{T\to\infty}\mathrm{E}[(\boldsymbol{S_2}^I - \boldsymbol{S_2}^J)\boldsymbol{\Sigma}] &= -\boldsymbol{\Sigma}^{-1}\lim_{T\to\infty}\mathrm{E}[(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})] \\ &\quad +\boldsymbol{\Sigma}^{-1}\lim_{T\to\infty}\mathrm{E}[(\boldsymbol{B_1}^J - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^J - \boldsymbol{B_1}^{UL})] = [\text{see Lemma B.2}] \\ &= -\boldsymbol{\Sigma}^{-1}\lim_{T\to\infty}\mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] \\ &\quad +\boldsymbol{\Sigma}^{-1}\lim_{T\to\infty}\mathrm{E}[(\hat{\boldsymbol{B}}_J - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_J - \hat{\boldsymbol{B}}_{UL})] = [\text{see} \quad (\text{B.188})] \\ &= -\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I + \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_J = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_J - \boldsymbol{\Delta}_I) = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL} - \boldsymbol{\Delta}_I), \end{aligned} \tag{B.193}$$

because the I estimation method is the GL method.

iii. Moreover,

$$\begin{aligned}\boldsymbol{S}_1 &= \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} \Rightarrow \\ \mathrm{vec}(\boldsymbol{S}_1) &= [(\boldsymbol{\Sigma}^{-1})' \otimes \boldsymbol{\Sigma}^{-1}]\,\mathrm{vec}(\boldsymbol{\Sigma}_1) = (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\,\mathrm{vec}(\boldsymbol{\Sigma}_1) \Rightarrow \\ (\mathrm{vec}(\boldsymbol{S}_1))(\mathrm{vec}(\boldsymbol{S}_1))' &= [(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\,\mathrm{vec}(\boldsymbol{\Sigma}_1)][(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\,\mathrm{vec}(\boldsymbol{\Sigma}_1)]' \\ &= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})(\mathrm{vec}(\boldsymbol{\Sigma}_1))(\mathrm{vec}(\boldsymbol{\Sigma}_1))'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}). \end{aligned} \tag{B.194}$$

Since $\boldsymbol{E}'\boldsymbol{E} \sim \mathcal{W}(\boldsymbol{\Sigma}, T)$ and $\mathrm{E}(\boldsymbol{E}'\boldsymbol{E}) = T\boldsymbol{\Sigma}$, equation (B.160) implies that the matrix

$$\boldsymbol{W} = \sqrt{T}\boldsymbol{\Sigma}_1 = T(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma}) = \boldsymbol{E}'\boldsymbol{E} - T\boldsymbol{\Sigma} \tag{B.195}$$

is a Wishart matrix in deviations from its expected value. Let w_{ij} be the (i, j)-th element of $\boldsymbol{W}$.

Then, since σ_{ij} is the (i, j)-th element of $\boldsymbol{\Sigma}$, by using definition (B.189) and following Zellner, 1971 p.389, (B.58), we find that

$$\mathrm{cov}(w_{ij}w_{kl}) = \mathrm{E}(w_{ij}w_{kl}) = T(\sigma_{i\kappa}\sigma_{jl} + \sigma_{il}\sigma_{j\kappa}) = T\nu_{(ij)(kl)}. \tag{B.196}$$

Then, (B.194), (B.195) and (B.196) imply that

$$\begin{aligned}
\mathrm{E}[(\mathrm{vec}(\boldsymbol{S}_1))(\mathrm{vec}(\boldsymbol{S}_1))'] &= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\,\mathrm{E}[(\mathrm{vec}(\boldsymbol{\Sigma}_1))(\mathrm{vec}(\boldsymbol{\Sigma}_1))'](\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}) \\
&= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})(1/T)\,\mathrm{E}[(\mathrm{vec}(\sqrt{T}\boldsymbol{\Sigma}_1))(\mathrm{vec}(\sqrt{T}\boldsymbol{\Sigma}_1))'](\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}) \\
&= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})(1/T)\,\mathrm{E}[(\mathrm{vec}(\boldsymbol{W}))(\mathrm{vec}(\boldsymbol{W}))'](\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}) \\
&= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})(1/T)\,\mathrm{E}[(w_{ij}w_{kl})_{(ij),(kl)=1,\ldots,M^2}](\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}) \\
&= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})(1/T)[(\mathrm{E}(w_{ij}w_{kl}))_{(ij),(kl)}](\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}) \\
&= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})(1/T)[(T\nu_{(ij)(kl)})_{(ij),(kl)}](\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}) \\
&= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})([(\nu_{(ij)(kl)})_{(ij),(kl)}](\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}) \\
&= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\boldsymbol{N}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}) \Rightarrow
\end{aligned} \tag{B.197}$$

$$\lim_{T\to\infty} \mathrm{E}[(\mathrm{vec}(\boldsymbol{S}_1))(\mathrm{vec}(\boldsymbol{S}_1))'] = (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\boldsymbol{N}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}). \tag{B.198}$$

Lemma B.7. Calculation of $\boldsymbol{\Delta}_I$ (I=UL,RL,GL, IG, ML)

Since, $\boldsymbol{y}_* = \mathrm{vec}(\boldsymbol{Y}_*)$, $\boldsymbol{X}_* = (\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}$, $\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{E})$ and $\mathrm{vec}(\boldsymbol{B}) = \boldsymbol{\Psi\beta}$ where $\boldsymbol{y}_*$, $\boldsymbol{\varepsilon}$ are $(TM \times 1)$ vectors and $(\boldsymbol{I}_M \otimes \boldsymbol{Z})$, $\boldsymbol{\Psi}$ and $\boldsymbol{X}_*$ are $TM \times Mk$, $Mk \times n$ and $TM \times n$ matrices, respectively, the following results hold:

(i)

$$\begin{aligned}
\boldsymbol{B}_1{}^{UL} &= (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} = T(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} \\
&= \sqrt{T}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E} \Rightarrow
\end{aligned} \tag{B.199}$$

$$\begin{aligned}
\mathrm{vec}(\boldsymbol{B}_1{}^{UL}) &= \mathrm{vec}[\sqrt{T}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E}] \\
&= \sqrt{T}\,\mathrm{vec}[(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E}] \\
&= \sqrt{T}[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']\,\mathrm{vec}(\boldsymbol{E}) \\
&= \sqrt{T}[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']\boldsymbol{\varepsilon}.
\end{aligned} \tag{B.200}$$

(ii)

$$\mathrm{vec}(\boldsymbol{B}_1{}^{RL}) = \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*/T)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon}/\sqrt{T} = \sqrt{T}\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon}. \tag{B.201}$$

(iii) Similarly,

$$\begin{aligned}\mathrm{vec}(\boldsymbol{B}_1{}^{GL}) &= \mathrm{vec}(\boldsymbol{B}_1{}^{IG}) = \mathrm{vec}(\boldsymbol{B}_1{}^{ML}) = \\ &= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T} \\ &= \sqrt{T}\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}.\end{aligned} \tag{B.202}$$

Moreover,

$$\begin{aligned}\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL} &= \tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2) = [\text{see (B.143)}] \Rightarrow \\ \sqrt{T}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL}) &= \sqrt{T}[\tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2)] \\ &= (\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau) \Rightarrow\end{aligned} \tag{B.203}$$

$$\begin{aligned}\mathrm{vec}[\sqrt{T}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] &= \sqrt{T}\,\mathrm{vec}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL}) \\ &= \mathrm{vec}(\boldsymbol{B}_1^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau).\end{aligned} \tag{B.204}$$

Define the matrix $\boldsymbol{\Phi}_I$ such that

$$\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}). \tag{B.205}$$

Then equations (B.204) and (B.205) imply that

$$\sqrt{T}\,\mathrm{vec}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL}) = \sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} + \omega(\tau). \tag{B.206}$$

By using equations (B.199), (B.200), (B.201), and (B.205), we find the following results:

I For $I = UL$

$$\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} = \sqrt{T}\boldsymbol{\Phi}_{UL}\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{B}_1{}^{UL} - \boldsymbol{B}_1{}^{UL}) = 0 \Rightarrow \boldsymbol{\Phi}_{UL} = 0. \tag{B.207}$$

II For $I = RL$

$$\begin{aligned}\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} &= \sqrt{T}\boldsymbol{\Phi}_{RL}\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{B}_1{}^{RL} - \boldsymbol{B}_1{}^{UL}) = \sqrt{T}\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon} - \sqrt{T}[\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}']\boldsymbol{\varepsilon} \\ &= \sqrt{T}[\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*' - [\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}']]\boldsymbol{\varepsilon} \Rightarrow \\ \boldsymbol{\Phi}_{RL} &= \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*' - [\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'].\end{aligned} \tag{B.208}$$

III Similarly, for $I = GL, IG, ML$

$$\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} = \sqrt{T}\boldsymbol{\Phi}_{GL}\boldsymbol{\varepsilon} = \sqrt{T}\boldsymbol{\Phi}_{IG}\boldsymbol{\varepsilon} = \sqrt{T}\boldsymbol{\Phi}_{ML}\boldsymbol{\varepsilon}$$

$$
\begin{aligned}
&= \operatorname{vec}(\boldsymbol{B_1}^{GL} - \boldsymbol{B_1}^{UL}) = \operatorname{vec}(\boldsymbol{B_1}^{IG} - \boldsymbol{B_1}^{UL}) = \operatorname{vec}(\boldsymbol{B_1}^{ML} - \boldsymbol{B_1}^{UL}) \\
&= \sqrt{T}[\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T) - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}']]\boldsymbol{\varepsilon} \Rightarrow \\
\boldsymbol{\Phi}_{GL} &= \boldsymbol{\Phi}_{IG} = \boldsymbol{\Phi}_{ML} = \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T) - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']. \quad \text{(B.209)}
\end{aligned}
$$

Let $\boldsymbol{l}$ be any arbitrary $M \times 1$ vector and let $\boldsymbol{L} = \boldsymbol{l}\boldsymbol{l}'$ be any $(M \times M)$ symmetric matrix i.e.,

$$
\boldsymbol{l} = [(l_i)_{i=1,\dots,M}] \quad \text{(B.210)}
$$

and

$$
\begin{aligned}
\boldsymbol{L} &= [(l_{ij})_{i,j=1,\dots,M}] = \boldsymbol{l}\boldsymbol{l}' = \begin{bmatrix} l_1 \\ \vdots \\ l_M \end{bmatrix} (l_1, \dots, l_M) = \begin{bmatrix} l_1 l_1 & \dots & l_1 l_M \\ \vdots & & \vdots \\ l_M l_1 & \dots & l_M l_M \end{bmatrix} \\
&= [(l_i l_j)_{i,j=1,\dots,M}] \Rightarrow \\
l_{ij} &= l_i l_j \;\; (i,j = 1,\dots,M). \quad \text{(B.211)}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\boldsymbol{l}'(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})\boldsymbol{l} &= \operatorname{tr}[\boldsymbol{l}'(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})\boldsymbol{l}] \\
&= \operatorname{tr}[(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})\boldsymbol{l}\boldsymbol{l}'] \\
&= \operatorname{tr}[(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})\boldsymbol{L}] \\
&= [\operatorname{vec}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})]' \operatorname{vec}[\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})\boldsymbol{L}] \\
&= [\operatorname{vec}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})]'(\boldsymbol{L}' \otimes \boldsymbol{\Gamma})[\operatorname{vec}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})] \\
&= [\operatorname{vec}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})]'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})[\operatorname{vec}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})]. \quad \text{(B.212)}
\end{aligned}
$$

By using equations (B.205), and (B.212) and since $\mathrm{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \boldsymbol{\Sigma} \otimes \boldsymbol{I}_T$, we find that

$$
\begin{aligned}
\boldsymbol{l}'(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})\boldsymbol{l} &= (\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon})'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})(\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon}) = \\
T\boldsymbol{\varepsilon}'\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} &= T\operatorname{tr}(\boldsymbol{\varepsilon}'\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I\boldsymbol{\varepsilon}) \\
&= T\operatorname{tr}(\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') \Rightarrow \\
\mathrm{E}[\boldsymbol{l}'(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})\boldsymbol{l}] &= T\operatorname{tr}(\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I\,\mathrm{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')) \\
&= T\operatorname{tr}(\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)). \quad \text{(B.213)}
\end{aligned}
$$

Then, Lemma B.2 and equations (B.188) and (B.213) imply that

$$
\begin{aligned}
\boldsymbol{l}'\boldsymbol{\Delta}_I\boldsymbol{l} &= \boldsymbol{l}' \lim_{T\to\infty} \mathrm{E}[(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})]\boldsymbol{l} \\
&= \lim_{T\to\infty} \mathrm{E}[\boldsymbol{l}'(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^{I} - \boldsymbol{B_1}^{UL})\boldsymbol{l}] \\
&= \lim_{T\to\infty} [T\operatorname{tr}(\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)]. \quad \text{(B.214)}
\end{aligned}
$$

The following results hold:

(a) Equations (B.207) and (B.214) imply that

$$\boldsymbol{l}'\boldsymbol{\Delta}_{UL}\boldsymbol{l} = \lim_{T\to\infty}[T\,\mathrm{tr}(\boldsymbol{\Phi}'_{UL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{UL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T))] = 0 \Rightarrow \boldsymbol{\Delta}_{UL} = 0. \tag{B.215}$$

(b) Since $\boldsymbol{X}'_* = [\boldsymbol{X}'_{1*},\ldots,\boldsymbol{X}'_{M*}]$ we take

$$\begin{aligned}
\boldsymbol{X}'_*\boldsymbol{X}_* &= [\boldsymbol{X}'_{1*},\ldots,\boldsymbol{X}'_{M*}]\begin{bmatrix}\boldsymbol{X}_{1*}\\ \vdots \\ \boldsymbol{X}_{M*}\end{bmatrix} = \sum_{\mu=1}^{M}\boldsymbol{X}'_{\mu*}\boldsymbol{X}_{\mu*} \Rightarrow \\
(\boldsymbol{X}'_*\boldsymbol{X}_*)^{-1} &= \left(\sum_{\mu=1}^{M}\boldsymbol{X}'_{\mu*}\boldsymbol{X}_{\mu*}\right)^{-1} \Rightarrow \\
\boldsymbol{\Psi}(\boldsymbol{X}'_*\boldsymbol{X}_*)^{-1}\boldsymbol{X}'_* &= \begin{bmatrix}\boldsymbol{\Psi}_1\\ \vdots \\ \boldsymbol{\Psi}_M\end{bmatrix}\left(\sum_{\mu=1}^{M}\boldsymbol{X}'_{\mu*}\boldsymbol{X}_{\mu*}\right)^{-1}[\boldsymbol{X}'_{1*},\ldots,\boldsymbol{X}'_{M*}] \\
&= \left[\left(\boldsymbol{\Psi}_i\left(\sum_{\mu=1}^{M}\boldsymbol{X}'_{\mu*}\boldsymbol{X}_{\mu*}\right)^{-1}\boldsymbol{X}'_{j*}\right)_{i,j}\right].
\end{aligned} \tag{B.216}$$

Moreover,

$$\begin{aligned}
[\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'] &= \begin{bmatrix}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' & & 0\\ & \ddots & \\ 0 & & (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\end{bmatrix} = \mathrm{diag}[((\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}')_i] \\
&= [(\delta_{ij}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}')_{i,j}].
\end{aligned} \tag{B.217}$$

Therefore,

$$\begin{aligned}
\boldsymbol{\Phi}_{RL} &= \boldsymbol{\Psi}(\boldsymbol{X}'_*\boldsymbol{X}_*)^{-1}\boldsymbol{X}'_* - [\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'] \\
&= \left[\left(\boldsymbol{\Psi}_i\left(\sum_{\mu=1}^{M}\boldsymbol{X}'_{\mu*}\boldsymbol{X}_{\mu*}\right)^{-1}\boldsymbol{X}'_{j*} - \delta_{ij}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\right)_{i,j}\right] \\
&= [(\boldsymbol{\Phi}_{ij}{}^{RL})_{i,j}],
\end{aligned} \tag{B.218}$$

where

$$\boldsymbol{\Phi}_{ij}{}^{RL} = \boldsymbol{\Psi}_i\left(\sum_{\mu=1}^{M}\boldsymbol{X}'_{\mu*}\boldsymbol{X}_{\mu*}\right)^{-1}\boldsymbol{X}'_{j*} - \delta_{ij}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\,. \tag{B.219}$$

Thus,

$$\boldsymbol{\Phi}'_{RL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma}) = \left[(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime})_{i,\kappa}\right]\left[[(l_{\kappa q})_{\kappa q}]\otimes\boldsymbol{\Gamma}\right] = \left[(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime})_{i,\kappa}\right]\left[(l_{\kappa q}\boldsymbol{\Gamma})_{\kappa q}\right] = \left[\left(\sum_{\kappa=1}^{M} l_{\kappa q}\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\right)_{i,q}\right] \tag{B.220}$$

and

$$\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T) = [(\boldsymbol{\Phi}_{q\mu}{}^{RL})_{q,\mu}][[(\sigma_{\mu j})_{\mu,j}]\otimes\boldsymbol{I}_T] = [(\boldsymbol{\Phi}_{q\mu}{}^{RL})_{q,\mu}][(\sigma_{\mu j}\boldsymbol{I}_T)_{\mu,j}] = \left[\left(\sum_{\mu=1}^{M}\sigma_{\mu j}\boldsymbol{\Phi}_{q\mu}{}^{RL}\right)_{q,j}\right]. \tag{B.221}$$

Then, equations (B.220) and (B.221) imply that

$$\begin{aligned}\boldsymbol{\Phi}'_{RL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T) &= \left[\left(\sum_{\kappa=1}^{M} l_{\kappa q}\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\right)_{i,q}\right]\left[\left(\sum_{\mu=1}^{M}\sigma_{\mu j}\boldsymbol{\Phi}_{q\mu}{}^{RL}\right)_{q,j}\right]\\ &= \left[\left(\sum_{q=1}^{M}\sum_{\kappa=1}^{M}\sum_{\mu=1}^{M} l_{\kappa q}\sigma_{\mu j}\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL}\right)_{i,j}\right]\Rightarrow\end{aligned} \tag{B.222}$$

$$\begin{aligned}\Rightarrow \operatorname{tr}[\boldsymbol{\Phi}'_{RL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)] &= \operatorname{tr}\left[\left(\sum_{q=1}^{M}\sum_{\kappa=1}^{M}\sum_{\mu=1}^{M} l_{\kappa q}\sigma_{\mu j}\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL}\right)_{i,j}\right]\\ &= \sum_{i=1}^{M}\sum_{q=1}^{M}\sum_{\kappa=1}^{M}\sum_{\mu=1}^{M} l_{\kappa q}\sigma_{\mu i}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL}).\end{aligned} \tag{B.223}$$

Since $\boldsymbol{X}_{i*} = \boldsymbol{Z}\boldsymbol{\Psi}_i$ and $\boldsymbol{\Gamma} = (\boldsymbol{Z}'\boldsymbol{Z}/T)$, equation (B.219) implies that

$$\begin{aligned}\boldsymbol{\Phi}_{i\kappa}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}^{RL} &= [\boldsymbol{\Psi}_i\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\kappa*} - \delta_{i\kappa}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']'(\boldsymbol{Z}'\boldsymbol{Z}/T)\cdot[\boldsymbol{\Psi}_q\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\mu*} - \delta_{q\mu}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']\\
&= [\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{\Psi}'_i - \delta_{i\kappa}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}](\boldsymbol{Z}'\boldsymbol{Z})[\boldsymbol{\Psi}_q\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\mu*} - \delta_{q\mu}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']/T\\
&= [\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{\Psi}'_i\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\Psi}_q\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\mu*} - \delta_{q\mu}\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{\Psi}'_i(\boldsymbol{Z}'\boldsymbol{Z})(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\\
&\quad -\delta_{i\kappa}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}(\boldsymbol{Z}'\boldsymbol{Z})\boldsymbol{\Psi}_q\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\mu*} + \delta_{i\kappa}\delta_{q\mu}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}(\boldsymbol{Z}'\boldsymbol{Z})(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']/T\\
&= [\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}(\boldsymbol{Z}\boldsymbol{\Psi}_i)'(\boldsymbol{Z}\boldsymbol{\Psi}_q)\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\mu*} - \delta_{q\mu}\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}(\boldsymbol{Z}\boldsymbol{\Psi}_i)'\\
&\quad -\delta_{i\kappa}(\boldsymbol{Z}\boldsymbol{\Psi}_q)\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\mu*} + \delta_{i\kappa}\delta_{q\mu}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']/T\\
&= [\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}(\boldsymbol{X}_{i*})'(\boldsymbol{X}_{q*})\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\mu*} - \delta_{q\mu}\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}(\boldsymbol{X}_{i*})'\end{aligned}$$

$$
\begin{aligned}
&-\delta_{i\kappa}(\boldsymbol{X}_{q*})\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}_{\mu*}' + \delta_{i\kappa}\delta_{q\mu}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']/T \Rightarrow \\
\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}) = {} & \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}_{i*}'\boldsymbol{X}_{q*}/T)\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}_{\mu*}'\boldsymbol{X}_{\kappa*}/T)\right]/T \\
&-\delta_{q\mu}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}_{i*}'\boldsymbol{X}_{\kappa*}/T)\right]/T \\
&-\delta_{i\kappa}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}_{\mu*}'\boldsymbol{X}_{q*}/T)\right]/T \\
&+\delta_{i\kappa}\delta_{q\mu}\operatorname{tr}(\boldsymbol{P}_Z)/T. \qquad \text{(B.224)}
\end{aligned}
$$

Since $\boldsymbol{Z}$ is $T \times k$, equation (B.166) implies that

$$\operatorname{tr}(\boldsymbol{P}_Z) = k. \qquad \text{(B.225)}$$

Since $\boldsymbol{X}_{i*} = \boldsymbol{P}_i{}^{-1}\boldsymbol{X}_i$, $\boldsymbol{X}_{j*} = \boldsymbol{P}_j{}^{-1}\boldsymbol{X}_j$, and since $\boldsymbol{P}_i{}^{-1\prime}\boldsymbol{P}_j{}^{-1} = \boldsymbol{R}^{ij}$, we find that for any $i, j = 1, \ldots, M$

$$\boldsymbol{X}_{i*}'\boldsymbol{X}_{j*}/T = \boldsymbol{X}_i'\boldsymbol{P}_i{}^{-1\prime}\boldsymbol{P}_j{}^{-1}\boldsymbol{X}_j/T = \boldsymbol{X}_i'\boldsymbol{R}^{ij}\boldsymbol{X}_j/T = \boldsymbol{B}_{ij} \ \text{[see (B.66)]}. \qquad \text{(B.226)}$$

Therefore,

$$
\begin{aligned}
\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T &= \sum_{p=1}^{M}\boldsymbol{B}_{pp} \Rightarrow \\
\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1} &= \left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}. \qquad \text{(B.227)}
\end{aligned}
$$

So,

$$\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}_{i*}'\boldsymbol{X}_{\kappa*}/T)\right] = \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{i\kappa}\right] \qquad \text{(B.228)}$$

and similarly

$$\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}_{\mu*}'\boldsymbol{X}_{q*}/T)\right] = \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{\mu q}\right]. \qquad \text{(B.229)}$$

Furthermore,

$$
\begin{aligned}
&\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}_{i*}'\boldsymbol{X}_{q*}/T)\left(\sum_{p=1}^{M}\boldsymbol{X}_{p*}'\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}_{\mu*}'\boldsymbol{X}_{\kappa*}/T)\right] \\
= {} & \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{iq}\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{\mu\kappa}\right]. \qquad \text{(B.230)}
\end{aligned}
$$

Thus, equations (B.224), (B.225), (B.228), (B.229), and (B.230) imply that

$$\begin{aligned}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}^{RL}) &= \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{iq}\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{\mu\kappa}\right]/T-\delta_{q\mu}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{i\kappa}\right]/T\\ &\quad -\delta_{i\kappa}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{\mu q}\right]/T+\delta_{i\kappa}\delta_{q\mu}K/T. \qquad \text{(B.231)}\end{aligned}$$

Since $l_{kq} = l_k l_q$ (see (B.211)), equations (B.210) and (B.223) imply that

$$\begin{aligned}\operatorname{tr}[\boldsymbol{\Phi}_{RL}'(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)] &= \sum_{i=1}^{M}\sum_{q=1}^{M}\sum_{\kappa=1}^{M}\sum_{\mu=1}^{M} l_{\kappa q}\sigma_{\mu i}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL})\\ &= \sum_{i=1}^{M}\sum_{q=1}^{M}\sum_{\kappa=1}^{M}\sum_{\mu=1}^{M} l_{\kappa}\sigma_{\mu i}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL})l_q\\ &= \boldsymbol{l}'\left[\left(\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL})\right)_{k,q}\right]\boldsymbol{l}\Rightarrow\\ \boldsymbol{l}'\boldsymbol{\Delta}_{RL}\boldsymbol{l} &= \lim_{T\to\infty}[T(\boldsymbol{\Phi}_{RL}'(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T))]\\ &= \lim_{T\to\infty}\boldsymbol{l}'\left[\left(\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}T\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL})\right)_{k,q}\right]\boldsymbol{l}\\ &= \boldsymbol{l}'\lim_{T\to\infty}\left[\left(\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}T\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL})\right)_{k,q}\right]\boldsymbol{l}\Rightarrow\\ \boldsymbol{\Delta}_{RL} &= \lim_{T\to\infty}\left[\left(\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}T\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL})\right)_{k,q}\right]. \qquad \text{(B.232)}\end{aligned}$$

By using (B.231) we find

$$\begin{aligned}\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}T\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{q\mu}{}^{RL}) &= \sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{iq}\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{\mu\kappa}\right]\\ &\quad -\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}\delta_{q\mu}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{i\kappa}\right]-\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}\delta_{i\kappa}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{\mu q}\right]\\ &\quad +\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}\delta_{i\kappa}\delta_{q\mu}K\\ &= \sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{iq}\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{\mu\kappa}\right]-\sum_{i=1}^{M}\sigma_{qi}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{i\kappa}\right]\\ &\quad -\sum_{\mu=1}^{M}\sigma_{\mu\kappa}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\boldsymbol{B}_{\mu q}\right]+\sigma_{q\kappa}K. \qquad \text{(B.233)}\end{aligned}$$

So, equations (B.232) and (B.233) imply that

$$\begin{aligned}\boldsymbol{\Delta}_{RL} &= \Bigg[\Big(\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{iq}\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{\mu\kappa}\Big] \\ &\quad -\sum_{i=1}^{M}\sigma_{qi}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{i\kappa}\Big]-\sum_{\mu=1}^{M}\sigma_{\mu\kappa}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{\mu q}\Big]+\sigma_{q\kappa}K\Big)_{k,q}\Bigg]. \end{aligned} \tag{B.234}$$

(c) Since $\boldsymbol{X}_* = (\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}$ and $\boldsymbol{X}_{\mu *} = \boldsymbol{Z}\boldsymbol{\Psi}_\mu$ $(\mu = 1, \ldots, M)$ (see (3.42)), we find that

$$\begin{aligned}\boldsymbol{\Psi}'(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Psi} &= \boldsymbol{\Psi}'(\boldsymbol{L}\otimes(\boldsymbol{Z}'\boldsymbol{Z}/T))\boldsymbol{\Psi} = \boldsymbol{\Psi}'(\boldsymbol{L}\otimes(\boldsymbol{Z}'\boldsymbol{Z}))\boldsymbol{\Psi}/T \\ &= \boldsymbol{\Psi}'[\boldsymbol{I}_M\otimes\boldsymbol{Z}'][\boldsymbol{L}\otimes\boldsymbol{I}_T][\boldsymbol{I}_M\otimes\boldsymbol{Z}]\boldsymbol{\Psi}/T \\ &= [(\boldsymbol{I}_M\otimes\boldsymbol{Z})\boldsymbol{\Psi}]'[\boldsymbol{L}\otimes\boldsymbol{I}_T][(\boldsymbol{I}_M\otimes\boldsymbol{Z})\boldsymbol{\Psi}]/T \\ &= \boldsymbol{X}_*'[\boldsymbol{L}\otimes\boldsymbol{I}_T]\boldsymbol{X}_*/T = [(\boldsymbol{X}_{i*}')_i][(l_{ij}\boldsymbol{I}_T)_{i,j}][(\boldsymbol{X}_{j*})_j]/T \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M} l_{ij}(\boldsymbol{X}_{i*}'\boldsymbol{X}_{j*}/T) = \sum_{i=1}^{M}\sum_{j=1}^{M} l_{ij}(\boldsymbol{X}_i'\boldsymbol{P}_i^{-1\prime}\boldsymbol{P}_j^{-1}\boldsymbol{X}_j/T) \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M} l_{ij}(\boldsymbol{X}_i'\boldsymbol{R}^{ij}\boldsymbol{X}_j/T) = \sum_{i=1}^{M}\sum_{j=1}^{M} l_{ij}\boldsymbol{B}_{ij}. \end{aligned} \tag{B.235}$$

The following result holds:

$$\begin{aligned}\boldsymbol{\Phi}_{GL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)\boldsymbol{\Phi}_{GL}' &= [\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T) - [\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']](\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)\cdot \\ &\quad [\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T) - [\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']]' \\ &= [\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T) - [\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']](\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)\cdot \\ &\quad [(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' - [\boldsymbol{I}_M\otimes\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}]] \\ &= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*\cdot \\ &\quad [\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\ &\quad -\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)[\boldsymbol{I}_M\otimes\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\ &\quad -[\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'](\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\ &\quad +[\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'](\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)[\boldsymbol{I}_M\otimes\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\ &= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\ &\quad -\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}[(\boldsymbol{I}_M\otimes\boldsymbol{Z})\boldsymbol{\Psi}]'[\boldsymbol{I}_M\otimes\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\ &\quad -[\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'][(\boldsymbol{I}_M\otimes\boldsymbol{Z})\boldsymbol{\Psi}][\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\ &\quad +[\boldsymbol{\Sigma}\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}(\boldsymbol{Z}'\boldsymbol{Z})(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\ &= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\ &\quad -\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}'(\boldsymbol{I}_M\otimes\boldsymbol{Z}')[\boldsymbol{I}_M\otimes\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\ &\quad -[\boldsymbol{I}_M\otimes(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'](\boldsymbol{I}_M\otimes\boldsymbol{Z})\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \end{aligned}$$

$$
\begin{aligned}
& +[\Sigma \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\
= \;& \boldsymbol{\Psi}[\boldsymbol{X}_*'(\Sigma^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\
& -\boldsymbol{\Psi}[\boldsymbol{X}_*'(\Sigma^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}'(\boldsymbol{I}_M \otimes \boldsymbol{I}_K) \\
& -(\boldsymbol{I}_M \otimes \boldsymbol{I}_K)\boldsymbol{\Psi}[\boldsymbol{X}_*'(\Sigma^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' + [\Sigma \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}]. \qquad \text{(B.236)}
\end{aligned}
$$

Since $\boldsymbol{X}_* = \boldsymbol{P}^{-1}\boldsymbol{X}$, and $\boldsymbol{\Omega}^{-1} = \boldsymbol{P}(\Sigma \otimes \boldsymbol{I}_T)\boldsymbol{P}'$, we find that

$$
\begin{aligned}
\boldsymbol{X}_*'(\Sigma^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_* &= \boldsymbol{X}'\boldsymbol{P}^{-1\prime}(\Sigma^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}\boldsymbol{X} \\
&= \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X} \Rightarrow \qquad \text{(B.237)}
\end{aligned}
$$

$$
[\boldsymbol{X}_*'(\Sigma^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X})^{-1}. \qquad \text{(B.238)}
$$

Also, since $\boldsymbol{\Gamma} = (\boldsymbol{Z}'\boldsymbol{Z}/T)$, $\boldsymbol{A} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)$, and $\boldsymbol{G} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X})^{-1} = \boldsymbol{A}^{-1}$, by using equations (B.236), (B.237) and (B.238) we find that

$$
\begin{aligned}
T\boldsymbol{\Phi}_{GL}(\Sigma \otimes \boldsymbol{I}_T)\boldsymbol{\Phi}_{GL}' &= \boldsymbol{\Psi}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{\Psi}' - \boldsymbol{\Psi}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{\Psi}'(\boldsymbol{I}_M \otimes \boldsymbol{I}_K) \\
&\quad -(\boldsymbol{I}_M \otimes \boldsymbol{I}_K)\boldsymbol{\Psi}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{\Psi}' + [\Sigma \otimes (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}] \\
&= \boldsymbol{\Psi}\boldsymbol{G}\boldsymbol{\Psi}' - \boldsymbol{\Psi}\boldsymbol{G}\boldsymbol{\Psi}' - \boldsymbol{\Psi}\boldsymbol{G}\boldsymbol{\Psi}' + (\Sigma \otimes \boldsymbol{G}^{-1}) = (\Sigma \otimes \boldsymbol{G}^{-1}) - \boldsymbol{\Psi}\boldsymbol{G}\boldsymbol{\Psi}' . \text{(B.239)}
\end{aligned}
$$

Moreover, since $\boldsymbol{\Omega} = [(\sigma^{ij}\boldsymbol{R}^{ij})_{i,j=1,\dots,M}]$ we take

$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T = [(\boldsymbol{X}_i')_i][(\sigma^{ij}\boldsymbol{R}^{ij})_{i,j}][(\boldsymbol{X}_j)_j]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma_{ij}(\boldsymbol{X}_i'\boldsymbol{R}^{ij}\boldsymbol{X}_j/T) = \sum_{i=1}^{M}\sum_{j=1}^{M}\sigma_{ij}\boldsymbol{B}_{ij} \Rightarrow \\
\boldsymbol{G} &= (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1} = \boldsymbol{A}^{-1} = (\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma_{ij}\boldsymbol{B}_{ij})^{-1}. \qquad \text{(B.240)}
\end{aligned}
$$

Thus, by using equations (B.235), (B.236), (B.239) and (B.240) we find that

$$
\begin{aligned}
T\,\mathrm{tr}\,\boldsymbol{\Phi}_{GL}'(\Sigma \otimes \boldsymbol{I}_T)\boldsymbol{\Phi}_{GL}'(\boldsymbol{L} \otimes \boldsymbol{\Gamma}) &= \mathrm{tr}\,(\Sigma\boldsymbol{L} \otimes \boldsymbol{I}_K) - \mathrm{tr}[\boldsymbol{G}\boldsymbol{\Psi}'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Psi}] \\
&= \mathrm{tr}\,(\Sigma\boldsymbol{L})\,\mathrm{tr}\,(\boldsymbol{I}_K) - \mathrm{tr}\left[\left(\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma_{ij}\boldsymbol{B}_{ij}\right)^{-1}\left(\sum_{i=1}^{M}\sum_{j=1}^{M}l_{ij}\boldsymbol{B}_{ij}\right)\right] \\
&= K\,\mathrm{tr}\,(\Sigma\boldsymbol{l}'\boldsymbol{l}) - \mathrm{tr}\left[\left(\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma_{ij}\boldsymbol{B}_{ij}\right)^{-1}\left(\sum_{i=1}^{M}\sum_{j=1}^{M}l_{ij}\boldsymbol{B}_{ij}\right)\right] \\
&= K\,\mathrm{tr}\,(\boldsymbol{l}'\Sigma\boldsymbol{l}) - \mathrm{tr}\left[\sum_{i=1}^{M}\sum_{j=1}^{M}l_{ij}\boldsymbol{G}\boldsymbol{B}_{ij}\right] \\
&= \boldsymbol{l}'(K\Sigma)\boldsymbol{l} - \sum_{i=1}^{M}\sum_{j=1}^{M}l_i\,\mathrm{tr}\,(\boldsymbol{G}\boldsymbol{B}_{ij})l_j \\
&= \boldsymbol{l}'(K\Sigma)\boldsymbol{l} - \boldsymbol{l}'[(\mathrm{tr}\,(\boldsymbol{G}\boldsymbol{B}_{ij}))_{i,j}]\boldsymbol{l}
\end{aligned}
$$

$$= \quad \boldsymbol{l}'[K\boldsymbol{\Sigma} - [(\mathrm{tr}\,(\boldsymbol{G}\boldsymbol{B}_{ij}))_{i,j}]]\boldsymbol{l} \Rightarrow \tag{B.241}$$

For any arbitrary vector $\boldsymbol{l}$

$$\begin{aligned} \boldsymbol{l}'\boldsymbol{\Delta}_{GL}\boldsymbol{l} &= \boldsymbol{l}'\boldsymbol{\Delta}_{IG}\boldsymbol{l} = \boldsymbol{l}'\boldsymbol{\Delta}_{ML}\boldsymbol{l} \\ &= \lim_{T\to\infty}[T\,\mathrm{tr}\,\boldsymbol{\Phi}'_{GL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{GL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)] \\ &= \lim_{T\to\infty}[\boldsymbol{l}'[K\boldsymbol{\Sigma} - [(\mathrm{tr}\,(\boldsymbol{G}\boldsymbol{B}_{ij}))_{i,j}]]\boldsymbol{l}] \\ &= \boldsymbol{l}'[K\boldsymbol{\Sigma} - [(\mathrm{tr}\,(\boldsymbol{G}\boldsymbol{B}_{ij}))_{i,j}]]\boldsymbol{l} \Rightarrow \\ & \boldsymbol{\Delta}_{GL} = \boldsymbol{\Delta}_{IG} = \boldsymbol{\Delta}_{ML} = K\boldsymbol{\Sigma} - \left[\left(\mathrm{tr}\left[\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma_{ij}\boldsymbol{B}_{ij}\right]^{-1}\boldsymbol{B}_{ij}\right)_{i,j}\right]. \end{aligned} \tag{B.242}$$

Lemma B.8. The LS estimator $\tilde{\rho}_\mu$ of ρ_μ admits the stochastic expansion

$$\tilde{\rho}_\mu = \rho_\mu + \tau{\rho_\mu}^{(1)} + \tau^2{\rho_\mu}^{(2)} + \omega(\tau^3), \tag{B.243}$$

where

$${\rho_\mu}^{(1)} = -(\rho_\mu{\boldsymbol{D}_\mu}^{(1)} - {N_\mu}^{(1)}) \tag{B.244}$$

and

$${\rho_\mu}^{(2)} = {N_\mu}^{(2)} - {N_\mu}^{(1)}{\boldsymbol{D}_\mu}^{(1)} + \rho_\mu[({\boldsymbol{D}_\mu}^{(1)})^2 + {\boldsymbol{D}_\mu}^{(2)}]. \tag{B.245}$$

Proof of Lemma B.8. Since

$$\tilde{\rho}_\mu = \sum_{t=2}^{T}\tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu}/\sum_{t=1}^{T}\tilde{u}_{t\mu}^2 = \sum_{t=1}^{T-1}\tilde{u}_{t\mu}\tilde{u}_{(t+1)\mu}/\sum_{t=1}^{T}\tilde{u}_{t\mu}^2 = N_\mu/D_\mu, \tag{B.246}$$

where

$$N_\mu = \frac{1}{2}\tilde{\boldsymbol{u}}'_\mu\boldsymbol{D}\tilde{\boldsymbol{u}}_\mu/T{\sigma_{u_\mu}}^2 \tag{B.247}$$

and

$$D_\mu = \tilde{\boldsymbol{u}}'_\mu\tilde{\boldsymbol{u}}_\mu/T{\sigma_{u_\mu}}^2, \tag{B.248}$$

where

$$u_{t\mu} \sim \mathcal{N}(0, \sigma_{\mu\mu}/(1-\rho_\mu^2)) \Rightarrow {\sigma_{u_\mu}}^2 = \sigma_{\mu\mu}/(1-\rho_\mu^2) \tag{B.249}$$

and $\boldsymbol{D}$ is a matrix with (t,t')-th element equal to 1 if $|t-t'| = 1$ and zero elsewhere.

Let $\tilde{\boldsymbol{\beta}}$ be the LS estimator of $\boldsymbol{\beta}$ in the (μ)-th equation

$$\boldsymbol{y}_\mu = \boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{u}_\mu. \tag{B.250}$$

Then,

$$\begin{aligned}\tilde{\boldsymbol{u}}_\mu &= \boldsymbol{y}_\mu - \boldsymbol{X}_\mu\tilde{\boldsymbol{\beta}} = \boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{u}_\mu - \boldsymbol{X}_\mu\tilde{\boldsymbol{\beta}} \\ &= \boldsymbol{u}_\mu - \tau\sqrt{T}\boldsymbol{X}_\mu(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) \\ &= \boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu, \end{aligned} \tag{B.251}$$

where

$$\begin{aligned}\boldsymbol{\theta}_\mu &= \sqrt{T}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \sqrt{T}[(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{y}_\mu - \boldsymbol{\beta}] \\ &= \sqrt{T}[(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu(\boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{u}_\mu) - \boldsymbol{\beta}] \\ &= \sqrt{T}(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{X}_\mu\boldsymbol{\beta} + \sqrt{T}(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu - \sqrt{T}\boldsymbol{\beta} \\ &= \sqrt{T}(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu = (\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu/\sqrt{T} \Rightarrow \end{aligned} \tag{B.252}$$

$$\boldsymbol{X}'_\mu\boldsymbol{u}_\mu/\sqrt{T} = (\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)\boldsymbol{\theta}_\mu. \tag{B.253}$$

But, equation (B.251) implies that

$$\begin{aligned}\tilde{\boldsymbol{u}}'_\mu\boldsymbol{D}\tilde{\boldsymbol{u}}_\mu &= (\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu)'\boldsymbol{D}(\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu) \\ &= (\boldsymbol{u}'_\mu - \tau\boldsymbol{\theta}'_\mu\boldsymbol{X}'_\mu)\boldsymbol{D}(\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu) \\ &= \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu - 2\boldsymbol{\theta}'_\mu(\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/\sqrt{T}) + \boldsymbol{\theta}'_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)\boldsymbol{\theta}_\mu. \end{aligned} \tag{B.254}$$

Then by using equations (B.247), (B.252) and (B.254) we find that

$$\begin{aligned}N_\mu &= \tilde{\boldsymbol{u}}'_\mu\boldsymbol{D}\tilde{\boldsymbol{u}}_\mu/2T{\sigma_{u_\mu}}^2 \\ &= \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - 2[\boldsymbol{u}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)^{-1}/\sqrt{T}][\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/\sqrt{T}]/2T{\sigma_{u_\mu}}^2 \\ &\quad +[\boldsymbol{u}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)^{-1}/\sqrt{T}](\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{X}_\mu/T)[(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu/\sqrt{T}]/2T{\sigma_{u_\mu}}^2 \\ &= \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - \tau^2\boldsymbol{u}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/{\sigma_{u_\mu}}^2 \\ &\quad +\tau^2\boldsymbol{u}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu/2{\sigma_{u_\mu}}^2 \\ &= \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - \tau^2\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu/{\sigma_{u_\mu}}^2 \\ &\quad +\tau^2\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/2{\sigma_{u_\mu}}^2 \\ &= \rho_\mu - \rho_\mu + \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 + \tau^2(\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/2 - \boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu)/{\sigma_{u_\mu}}^2 \\ &= \rho_\mu + \tau[\sqrt{T}(\boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - \rho_\mu)] + \tau^2(\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/2 - \boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu)/{\sigma_{u_\mu}}^2 \\ &= \rho_\mu + \tau {N_\mu}^{(1)} + \tau^2 {N_\mu}^{(2)}, \end{aligned} \tag{B.255}$$

where

$${N_\mu}^{(1)} = \sqrt{T}(\boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - \rho_\mu) = \sqrt{T}\left(\sum_{t=1}^{T-1} u_{t\mu}u_{(t+1)\mu}/2T{\sigma_{u_\mu}}^2 - \rho_\mu\right) \tag{B.256}$$

and

$$N_\mu{}^{(2)} = (\boldsymbol{u}'_\mu \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{D} \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{u}_\mu / 2 - \boldsymbol{u}'_\mu \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{D} \boldsymbol{u}_\mu) / \sigma_{u_\mu}{}^2. \tag{B.257}$$

Similarly, equations (B.251), (B.252) and (B.253) imply that

$$\begin{aligned}
\tilde{\boldsymbol{u}}'_\mu \tilde{\boldsymbol{u}}_\mu &= (\boldsymbol{u}_\mu - \tau \boldsymbol{X}_\mu \boldsymbol{\theta}_\mu)'(\boldsymbol{u}_\mu - \tau \boldsymbol{X}_\mu \boldsymbol{\theta}_\mu) = (\boldsymbol{u}'_\mu - \tau \boldsymbol{\theta}'_\mu \boldsymbol{X}'_\mu)(\boldsymbol{u}_\mu - \tau \boldsymbol{X}_\mu \boldsymbol{\theta}_\mu) \\
&= \boldsymbol{u}'_\mu \boldsymbol{u}_\mu - 2\boldsymbol{\theta}'_\mu (\boldsymbol{X}'_\mu \boldsymbol{u}_\mu / \sqrt{T}) + \boldsymbol{\theta}'_\mu (\boldsymbol{X}'_\mu \boldsymbol{X}_\mu / T) \boldsymbol{\theta}_\mu \\
&= \boldsymbol{u}'_\mu \boldsymbol{u}_\mu - 2\boldsymbol{\theta}'_\mu (\boldsymbol{X}'_\mu \boldsymbol{X}_\mu / T) \boldsymbol{\theta}_\mu + \boldsymbol{\theta}'_\mu (\boldsymbol{X}'_\mu \boldsymbol{X}_\mu / T) \boldsymbol{\theta}_\mu \\
&= \boldsymbol{u}'_\mu \boldsymbol{u}_\mu - \boldsymbol{\theta}'_\mu (\boldsymbol{X}'_\mu \boldsymbol{X}_\mu / T) \boldsymbol{\theta}_\mu \\
&= \boldsymbol{u}'_\mu \boldsymbol{u}_\mu - (\boldsymbol{u}'_\mu \boldsymbol{X}_\mu / \sqrt{T})(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu / T)^{-1} (\boldsymbol{X}'_\mu \boldsymbol{X}_\mu / T)(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu / T)^{-1} (\boldsymbol{X}'_\mu \boldsymbol{u}_\mu / \sqrt{T}) \\
&= \boldsymbol{u}'_\mu \boldsymbol{u}_\mu - \boldsymbol{u}'_\mu \boldsymbol{X}_\mu (\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1} \boldsymbol{X}'_\mu \boldsymbol{u}_\mu = \boldsymbol{u}'_\mu \boldsymbol{u}_\mu - \boldsymbol{u}'_\mu \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{u}_\mu.
\end{aligned} \tag{B.258}$$

Thus, equations (B.248) and (B.258) imply that

$$\begin{aligned}
\boldsymbol{D}_\mu &= \tilde{\boldsymbol{u}}'_\mu \tilde{\boldsymbol{u}}_\mu / T\sigma_{u_\mu}{}^2 = \boldsymbol{u}'_\mu \boldsymbol{u}_\mu / T\sigma_{u_\mu}{}^2 - \boldsymbol{u}'_\mu \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{u}_\mu / T\sigma_{u_\mu}{}^2 \\
&= 1 - 1 + \boldsymbol{u}'_\mu \boldsymbol{u}_\mu / T\sigma_{u_\mu}{}^2 - \boldsymbol{u}'_\mu \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{u}_\mu / T\sigma_{u_\mu}{}^2 \\
&= 1 + \tau[\sqrt{T}(\boldsymbol{u}'_\mu \boldsymbol{u}_\mu / T\sigma_{u_\mu}{}^2 - 1)] - \tau^2 \boldsymbol{u}'_\mu \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{u}_\mu / \sigma_{u_\mu}{}^2 \\
&= 1 + \tau \boldsymbol{D}_\mu{}^{(1)} - \tau^2 \boldsymbol{D}_\mu{}^{(2)},
\end{aligned} \tag{B.259}$$

where

$$\boldsymbol{D}_\mu{}^{(1)} = \sqrt{T}(\boldsymbol{u}'_\mu \boldsymbol{u}_\mu / T\sigma_{u_\mu}{}^2 - 1) \tag{B.260}$$

and

$$\boldsymbol{D}_\mu{}^{(2)} = \boldsymbol{u}'_\mu \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{u}_\mu / \sigma_{u_\mu}{}^2. \tag{B.261}$$

Thus, by using equation (B.259) we find that

$$\begin{aligned}
\boldsymbol{D}_\mu &= 1 + \tau(\boldsymbol{D}_\mu{}^{(1)} - \tau \boldsymbol{D}_\mu{}^{(2)}) \Rightarrow \\
\boldsymbol{D}_\mu{}^{-1} &= [1 + \tau(\boldsymbol{D}_\mu{}^{(1)} - \tau \boldsymbol{D}_\mu{}^{(2)})]^{-1} = 1 - \tau(\boldsymbol{D}_\mu{}^{(1)} - \tau \boldsymbol{D}_\mu{}^{(2)}) + \tau^2 (\boldsymbol{D}_\mu{}^{(1)} - \tau \boldsymbol{D}_\mu{}^{(2)})^2 + \omega(\tau^3) \\
&= 1 - \tau \boldsymbol{D}_\mu{}^{(1)} + \tau^2 [(\boldsymbol{D}_\mu{}^{(1)})^2 + \boldsymbol{D}_\mu{}^{(2)}] + \omega(\tau^3).
\end{aligned} \tag{B.262}$$

By using equations (B.246), (B.255) and (B.262) we find that

$$\begin{aligned}
\tilde{\rho}_\mu &= \boldsymbol{N}_\mu \boldsymbol{D}_\mu{}^{-1} = (\rho_\mu + \tau N_\mu{}^{(1)} + \tau^2 N_\mu{}^{(2)})[1 - \tau \boldsymbol{D}_\mu{}^{(1)} + \tau^2 [(\boldsymbol{D}_\mu{}^{(1)})^2 + \boldsymbol{D}_\mu{}^{(2)}] + \omega(\tau^3)] \\
&= \rho_\mu - \tau \rho_\mu \boldsymbol{D}_\mu{}^{(1)} + \tau^2 \rho_\mu [(\boldsymbol{D}_\mu{}^{(1)})^2 + \boldsymbol{D}_\mu{}^{(2)}] + \tau N_\mu{}^{(1)} - \tau^2 N_\mu{}^{(1)} \boldsymbol{D}_\mu{}^{(1)} + \tau^2 N_\mu{}^{(2)} + \omega(\tau^3) \\
&= \rho_\mu - \tau(\rho_\mu \boldsymbol{D}_\mu{}^{(1)} - N_\mu{}^{(1)}) + \tau^2 [N_\mu{}^{(2)} - N_\mu{}^{(1)} \boldsymbol{D}_\mu{}^{(1)} + \rho_\mu [(\boldsymbol{D}_\mu{}^{(1)})^2 + \boldsymbol{D}_\mu{}^{(2)}]] + \omega(\tau^3) \\
&= \rho_\mu + \tau(\rho_\mu{}^{(1)} + \tau \rho_\mu{}^{(2)}) + \omega(\tau^3),
\end{aligned} \tag{B.263}$$

where

$$\rho_\mu{}^{(1)} = -(\rho_\mu \boldsymbol{D}_\mu{}^{(1)} - N_\mu{}^{(1)}) \tag{B.264}$$

and

$$\rho_\mu{}^{(2)} = N_\mu{}^{(2)} - N_\mu{}^{(1)}\boldsymbol{D}_\mu{}^{(1)} + \rho_\mu[(\boldsymbol{D}_\mu{}^{(1)})^2 + \boldsymbol{D}_\mu{}^{(2)}]. \tag{B.265}$$

Since

$$\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_\mu \boldsymbol{P}'_\mu = \frac{1}{1-\rho_\mu{}^2}\begin{bmatrix} 1 & \rho_\mu & \dots & \rho_\mu{}^{T-1} \\ \rho_\mu & & & \\ \vdots & & & \\ \rho_\mu{}^{T-1} & & \dots & 1 \end{bmatrix}, \tag{B.266}$$

it is straightforward that

$$\boldsymbol{R}_{\mu\mu}{}^{-1} = \boldsymbol{P}'_\mu{}^{-1}\boldsymbol{P}_\mu{}^{-1} = \boldsymbol{R}^{\mu\mu} = (1+\rho_\mu{}^2)\boldsymbol{I}_T - \rho_\mu \boldsymbol{D} - \rho_\mu{}^2 \boldsymbol{\Delta} \text{ [see (B.5)]}. \tag{B.267}$$

Then,

$$\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu} = \partial \boldsymbol{R}^{\mu\mu}/\partial\rho_\mu = 2\rho_\mu \boldsymbol{I}_T - \boldsymbol{D} - 2\rho_\mu \boldsymbol{\Delta} \text{ [see (B.7)]} \tag{B.268}$$

and

$$\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu} = \partial^2 \boldsymbol{R}^{\mu\mu}/\partial\rho_\mu{}^2 = 2\boldsymbol{I}_T - 2\boldsymbol{\Delta} = 2(\boldsymbol{I}_T - \boldsymbol{\Delta}) \text{ [see (B.8)]}. \tag{B.269}$$

Define the $(T \times T)$ matrices

$$\boldsymbol{R}_i{}^{\mu\mu} = \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu} + i\rho_\mu \boldsymbol{\Delta}, \;\; \boldsymbol{R}_{ii}{}^{\mu\mu} = \boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu} + i\boldsymbol{\Delta} \;\; (i = 1, 2). \tag{B.270}$$

Then,

$$\begin{aligned} \boldsymbol{R}_2{}^{\mu\mu} &= \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu} + 2\rho_\mu \boldsymbol{\Delta} = 2\rho_\mu \boldsymbol{I}_T - \boldsymbol{D} - 2\rho_\mu \boldsymbol{\Delta} + 2\rho_\mu \boldsymbol{\Delta} \\ &= 2\rho_\mu \boldsymbol{I}_T - \boldsymbol{D}. \end{aligned} \tag{B.271}$$

The quantities $\rho_\mu{}^{(1)}$ and $\rho_\mu{}^{(2)}$ can be written as functions of $\boldsymbol{R}_2{}^{\mu\mu}$ as follows:

i.

$$\begin{aligned} \rho_\mu{}^{(1)} &= -(\rho_\mu \boldsymbol{D}_\mu{}^{(1)} - \boldsymbol{N}_\mu{}^{(1)}) = \text{[see (B.256) and(B.260)]} \\ &= -[\rho_\mu \sqrt{T}(\boldsymbol{u}'_\mu \boldsymbol{u}_\mu / T\sigma_{\boldsymbol{u}_\mu}{}^2 - 1) - \sqrt{T}(\boldsymbol{u}'_\mu \boldsymbol{D}\boldsymbol{u}_\mu / 2T\sigma_{\boldsymbol{u}_\mu}{}^2 - \rho_\mu)] \\ &= -\sqrt{T}(2\rho_\mu \boldsymbol{u}'_\mu \boldsymbol{u}_\mu - \boldsymbol{u}'_\mu \boldsymbol{D}\boldsymbol{u}_\mu)/2T\sigma_{\boldsymbol{u}_\mu}{}^2 \\ &= -\boldsymbol{u}'_\mu(2\rho_\mu \boldsymbol{I}_T - \boldsymbol{D})\boldsymbol{u}_\mu / 2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2 = \text{[see (B.271)]} \\ &= -\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu / 2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2. \end{aligned} \tag{B.272}$$

ii.

$$\begin{aligned}
\rho_\mu{}^{(2)} &= \boldsymbol{N}_\mu{}^{(2)} - \boldsymbol{N}_\mu{}^{(1)}\boldsymbol{D}_\mu{}^{(1)} + \rho_\mu[(\boldsymbol{D}_\mu{}^{(1)})^2 + \boldsymbol{D}_\mu{}^{(2)}] \\
&= \boldsymbol{N}_\mu{}^{(2)} - \boldsymbol{N}_\mu{}^{(1)}\boldsymbol{D}_\mu{}^{(1)} + \rho_\mu(\boldsymbol{D}_\mu{}^{(1)})^2 + \rho_\mu\boldsymbol{D}_\mu{}^{(2)} \\
&= \boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)} + \boldsymbol{D}_\mu{}^{(1)}(\rho_\mu\boldsymbol{D}_\mu{}^{(1)} - \boldsymbol{N}_\mu{}^{(1)}) = [\text{see (B.272)}] \\
&= \boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)} - \boldsymbol{D}_\mu{}^{(1)}[-(\rho_\mu\boldsymbol{D}_\mu{}^{(1)} - \boldsymbol{N}_\mu{}^{(1)})] = [\text{see (B.272)}] \\
&= \boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)} - \boldsymbol{D}_\mu{}^{(1)}\rho_\mu{}^{(1)}. \qquad \text{(B.273)}
\end{aligned}$$

By using equations (B.256), (B.257), (B.260), (B.261), and (B.271) we find that

$$\begin{aligned}
& 2\sigma_{\boldsymbol{u}_\mu}{}^2(\boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)}) = \\
=\; & 2\sigma_{\boldsymbol{u}_\mu}{}^2[(\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/2 - \boldsymbol{u}_{\mu'}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu)/\sigma_{\boldsymbol{u}_\mu}{}^2 + \rho_\mu\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/\sigma_{\boldsymbol{u}_\mu}{}^2] \\
=\; & \boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu - 2\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu + 2\rho_\mu\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu \\
=\; & \boldsymbol{u}'_\mu(\boldsymbol{I}_T - \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu})\boldsymbol{D}(\boldsymbol{I}_T - \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu})\boldsymbol{u}_\mu - 2\boldsymbol{u}'_\mu(\boldsymbol{I}_T - \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu})\boldsymbol{D}\boldsymbol{u}_\mu + 2\rho_\mu\boldsymbol{u}'_\mu(\boldsymbol{I}_T - \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu})\boldsymbol{u}_\mu \\
=\; & \boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{D}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu - 2\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu - 2\boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu \\
& +2\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu + 2\rho_\mu\boldsymbol{u}'_\mu\boldsymbol{u}_\mu - 2\rho_\mu\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu \\
=\; & \boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{D}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu - 2\rho_\mu\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + 2\rho_\mu\boldsymbol{u}'_\mu\boldsymbol{u}_\mu - \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu \\
& (\text{since } \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \text{ is idempotent}) \\
=\; & \boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}(\boldsymbol{D} - 2\rho_\mu\boldsymbol{I}_T)\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu(2\rho_\mu\boldsymbol{I}_T - \boldsymbol{D})\boldsymbol{u}_\mu \\
=\; & -\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu. \qquad \text{(B.274)}
\end{aligned}$$

Similarly, equations (B.249), (B.260), (B.261), (B.273), and (B.274) imply that

$$\begin{aligned}
& 2\sigma_{\boldsymbol{u}_\mu}{}^2\rho_\mu{}^{(2)} = 2\sigma_{\boldsymbol{u}_\mu}{}^2[(\boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)}) - \boldsymbol{D}_\mu{}^{(1)}\rho_\mu{}^{(1)}] \\
=\; & 2\sigma_{\boldsymbol{u}_\mu}{}^2(\boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)}) - 2\sigma_{\boldsymbol{u}_\mu}{}^2\boldsymbol{D}_\mu{}^{(1)}\rho_\mu{}^{(1)} \\
=\; & -\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu + 2\sigma_{\boldsymbol{u}_\mu}{}^2\sqrt{T}(\boldsymbol{u}'_\mu\boldsymbol{u}_\mu/T\sigma_{\boldsymbol{u}_\mu}{}^2 - 1)(\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2) \\
=\; & -\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{u}_\mu\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/T\sigma_{\boldsymbol{u}_\mu}{}^2 - \boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu \Rightarrow \\
\rho_\mu{}^{(2)} =\; & -\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/2\sigma_{\boldsymbol{u}_\mu}{}^2 + \boldsymbol{u}'_\mu\boldsymbol{u}_\mu\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2T\sigma_{\boldsymbol{u}_\mu}{}^4. \qquad \text{(B.275)}
\end{aligned}$$

□

Lemma B.9. The following results hold:

i) By using (B.243) the sampling error of the Least Squares estimator of ρ_μ is

$$\begin{aligned}
{\delta_{\rho_\mu}}^{LS} &= \frac{\tilde{\rho}_\mu - \rho_\mu}{\tau} = \sqrt{T}(\tilde{\rho}_\mu - \rho_\mu) = [\text{see (B.243)}] \\
&= \sqrt{T}[\rho_\mu + \tau({\rho_\mu}^{(1)} + \tau{\rho_\mu}^{(2)}) + \omega(\tau^3) - \rho_\mu] \\
&= {\rho_\mu}^{(1)} + \tau{\rho_\mu}^{(2)} + \omega(\tau^2) \\
&= {d_{(1)\mu}}^{LS} + \tau{d_{(2)\mu}}^{LS} + \omega(\tau^2), \qquad \text{(B.276)}
\end{aligned}$$

where

$${d_{(1)\mu}}^{LS} = {\rho_\mu}^{(1)} = -\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu / 2\sqrt{T}{\sigma_{\boldsymbol{u}_\mu}}^2 \qquad \text{(B.277)}$$

and

$${d_{(2)\mu}}^{LS} = {\rho_\mu}^{(2)} = -\boldsymbol{u}'_\mu \bar{\boldsymbol{P}}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/2{\sigma_{\boldsymbol{u}_\mu}}^2 + \boldsymbol{u}'_\mu\boldsymbol{u}_\mu\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2T{\sigma_{\boldsymbol{u}_\mu}}^4. \qquad \text{(B.278)}$$

ii) The iterative Prais-Winsten estimator of ρ_μ is (see Magee, 1985, p. 279-281)

$$\begin{aligned}
\hat{\rho}_\mu^{PW} &= \tilde{\rho}^{LS} - \tau^2\frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2] \\
&\quad +\omega(\tau^3), \qquad \text{(B.279)}
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{V} &= \boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu = [\boldsymbol{I} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{R}_{\mu\mu}]\boldsymbol{R}^{\mu\mu} \\
&= \boldsymbol{W}_{\mu\mu}\boldsymbol{R}^{\mu\mu} \qquad \text{(B.280)}
\end{aligned}$$

and

$$\boldsymbol{W}_{\mu\mu} = \boldsymbol{I} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{R}_{\mu\mu}. \qquad \text{(B.281)}$$

The iterative Prais-Winsten estimator of ρ_μ is equal to its GL estimator, i.e., $\hat{\rho}^{PW} = \hat{\rho}^{GL}$. Thus, by using equations (B.279), (B.280), and (B.281), the sampling error of iterative Prais-Winsten estimator of ρ_μ is

$$\begin{aligned}
{\delta_{\rho_\mu}}^{GL} = {\delta_{\rho_\mu}}^{PW} &= \sqrt{T}(\hat{\rho}_\mu^{PW} - \rho_\mu) \\
&= [(\tilde{\rho}_\mu^{LS} - \rho_\mu) - \tau^2\frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}'_\mu\bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2] \\
&\quad +\omega(\tau^3)]/\tau \\
&= {\delta_{\rho_\mu}}^{LS} - \tau\frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}'_\mu\bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2] + \omega(\tau^2) \\
&= {d_{(1)\mu}}^{LS} + \tau[{d_{(2)\mu}}^{LS} - \frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}'_\mu\bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]] + \omega(\tau^2)
\end{aligned}$$

$$= d_{(1)\mu}{}^{LS} + \tau d_{(2)\mu}{}^{GL} + \omega(\tau^2), \quad \text{(B.282)}$$

where

$$d_{(2)\mu}{}^{GL} = -\boldsymbol{u}'_\mu \bar{P}_{\mathbf{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \bar{P}_{\mathbf{X}_\mu} \boldsymbol{u}_\mu / 2\sigma_{u_\mu}{}^2 + \boldsymbol{u}'_\mu \boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu / 2T\sigma_{u_\mu}{}^4$$
$$-\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}'_\mu \bar{P}_{\mathbf{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\mathbf{X}_\mu} \boldsymbol{V} \boldsymbol{R}^{\mu\mu} \boldsymbol{u}_\mu + \boldsymbol{u}'_\mu \boldsymbol{R}^{\mu\mu} \boldsymbol{V} \boldsymbol{P}_{\mathbf{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\mathbf{X}_\mu} \boldsymbol{V} \boldsymbol{R}^{\mu\mu} \boldsymbol{u}_\mu / 2]. \quad \text{(B.283)}$$

iii) The ML estimator of ρ_μ is

$$\hat{\rho}_\mu^{ML} = \hat{\rho}_\mu^{PW} + \tau^2[\rho_\mu \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}(u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu] + \omega(\tau^3). \quad \text{(B.284)}$$

(see Beach and MacKinnon, 1978 p. 52-54, Magee, 1985 p. 281-284).

Thus, by using equation (B.284), the sampling error of ML estimator of ρ_μ is

$$\delta_{\rho_\mu}{}^{ML} = \sqrt{T}(\hat{\rho}_\mu^{ML} - \rho_\mu) = [(\hat{\rho}_\mu^{PW} - \rho_\mu) + \tau^2[\rho_\mu(1-\rho_\mu{}^2)(u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu] + \omega(\tau^3)]/\tau$$
$$= \delta_{\rho_\mu}{}^{PW} + \tau[\rho_\mu \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}(u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu] + \omega(\tau^2)$$
$$= d_{(1)\mu}{}^{LS} + \tau[d_{(2)\mu}{}^{GL} + \rho_\mu \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}(u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu] + \omega(\tau^2)$$
$$= d_{(1)\mu}{}^{LS} + \tau d_{(2)\mu}{}^{ML} + \omega(\tau^2), \quad \text{(B.285)}$$

where

$$d_{(2)\mu}{}^{ML} = d_{(2)\mu}{}^{GL} + \rho_\mu \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}(u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu. \quad \text{(B.286)}$$

iv) The Durbin-Watson estimator of ρ_μ is

$$\hat{\rho}_\mu^{DW} = 1 - D_{W_\mu}/2, \quad \text{(B.287)}$$

where D_{W_μ} is the Durbin-Watson statistic, i.e.,

$$D_{W_\mu} = \frac{\sum_{t=2}^T (\tilde{u}_{t\mu} - \tilde{u}_{(t-1)\mu})^2}{\sum_{t=1}^T \tilde{u}_{t\mu}^2} = \frac{\sum_{t=2}^T \tilde{u}_{t\mu}^2 + \sum_{t=2}^T \tilde{u}_{(t-1)\mu}^2 - 2\sum_{t=2}^T \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu}}{\sum_{t=1}^T \tilde{u}_{t\mu}^2}$$
$$= \frac{\sum_{t=1}^T \tilde{u}_{t\mu}^2 - \tilde{u}_{1\mu}^2 + \sum_{t=1}^T \tilde{u}_{t\mu}^2 - \tilde{u}_{T\mu}^2 - 2\sum_{t=2}^T \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu}}{\sum_{t=1}^T \tilde{u}_{t\mu}^2}$$
$$= \frac{2\sum_{t=1}^T \tilde{u}_{t\mu}^2 - (2\sum_{t=2}^T \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu} + \tilde{u}_{1\mu}^2 + \tilde{u}_{T\mu}^2)}{\sum_{t=1}^T \tilde{u}_{t\mu}^2}$$
$$= 2 - \frac{2\sum_{t=2}^T \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu} + \tilde{u}_{1\mu}^2 + \tilde{u}_{T\mu}^2}{\sum_{t=1}^T \tilde{u}_{t\mu}^2}. \quad \text{(B.288)}$$

Equations (B.287) and (B.288) imply that

$$\begin{aligned}
\hat{\rho}_{\mu}^{DW} &= 1-\left[1-\frac{\sum_{t=2}^{T}\tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu}+(\tilde{u}_{1\mu}^{2}+\tilde{u}_{T\mu}^{2})/2}{\sum_{t=1}^{T}\tilde{u}_{t\mu}^{2}}\right] \\
&= \frac{\sum_{t=2}^{T}\tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu}}{\sum_{t=1}^{T}\tilde{u}_{t\mu}^{2}}+\frac{(\tilde{u}_{1\mu}^{2}+\tilde{u}_{T\mu}^{2})/2}{\sum_{t=1}^{T}\tilde{u}_{t\mu}^{2}} \\
&= \tilde{\rho}_{\mu}^{LS}+\frac{(\tilde{u}_{1\mu}^{2}+\tilde{u}_{T\mu}^{2})/2T\sigma_{u_{\mu}}^{2}}{\sum_{t=1}^{T}(\tilde{u}_{t\mu}^{2}/T)(1/\sigma_{u_{\mu}}^{2})} \\
&= \tilde{\rho}_{\mu}^{LS}+\frac{1}{2T}\frac{(\tilde{u}_{1\mu}^{2}+\tilde{u}_{T\mu}^{2})}{\tilde{\sigma}_{u_{\mu}}^{2}} \\
&= \tilde{\rho}_{\mu}^{LS}+\tau^{2}(1-\rho_{\mu}{}^{2})(u_{1\mu}{}^{2}+u_{T\mu}{}^{2})/2\sigma_{\mu\mu}+\omega(\tau^{3}), \qquad \text{(B.289)}
\end{aligned}$$

because $\tilde{u}_{t\mu}$ is a consistent estimator of $u_{t\mu}$ and so $\sum_{t=1}^{T}\tilde{u}_{t\mu}^{2}/T$ is a consistent estimator of $\sigma_{u_{t\mu}}{}^{2}$ with an error of order $\omega(\tau^3)$. Therefore, equation (B.289) implies that the sampling error of DW estimator of ρ_μ is

$$\begin{aligned}
\delta_{\rho_{\mu}}{}^{DW} &= \sqrt{T}(\hat{\rho}_{\mu}^{DW}-\rho_{\mu}) = [(\tilde{\rho}_{\mu}^{LS}-\rho_{\mu})+\tau^{2}\frac{(1-\rho_{\mu}{}^{2})}{\sigma_{\mu\mu}}(u_{1\mu}{}^{2}+u_{T\mu}{}^{2})/2+\omega(\tau^{3})]/\tau = (\textit{see}\ \text{(B.276)}) \\
&= \delta_{\rho_{\mu}}{}^{LS}+\tau\frac{(1-\rho_{\mu}{}^{2})}{\sigma_{\mu\mu}}(u_{1\mu}{}^{2}+u_{T\mu}{}^{2})/2+\omega(\tau^{2}) \\
&= d_{(1)\mu}{}^{LS}+\tau[d_{(2)\mu}{}^{LS}+\frac{(1-\rho_{\mu}{}^{2})}{\sigma_{\mu\mu}}(u_{1\mu}{}^{2}+u_{T\mu}{}^{2})/2]+\omega(\tau^{2}) \\
&= d_{(1)\mu}{}^{LS}+\tau d_{(2)\mu}{}^{DW}+\omega(\tau^{2}), \qquad \text{(B.290)}
\end{aligned}$$

where

$$d_{(2)\mu}{}^{DW} = d_{(2)\mu}{}^{LS}+\frac{(1-\rho_{\mu}{}^{2})}{\sigma_{\mu\mu}}(u_{1\mu}{}^{2}+u_{T\mu}{}^{2})/2. \qquad \text{(B.291)}$$

Lemma B.10. The following results hold:

i) Equations (B.267), (B.268), and (B.270) imply that

$$\begin{aligned}
\boldsymbol{R}_{1}{}^{\mu\mu} &= \boldsymbol{R}_{\rho_{\mu}}{}^{\mu\mu}+\rho_{\mu}\boldsymbol{\Delta} = 2\rho_{\mu}\boldsymbol{I}_{T}-\boldsymbol{D}-2\rho_{\mu}\boldsymbol{\Delta}+\rho_{\mu}\boldsymbol{\Delta} = 2\rho_{\mu}\boldsymbol{I}_{T}-\boldsymbol{D}-\rho_{\mu}\boldsymbol{\Delta} \\
&= \frac{1}{\rho_{\mu}}[2\rho_{\mu}{}^{2}\boldsymbol{I}_{T}-\rho_{\mu}\boldsymbol{D}-\rho_{\mu}{}^{2}\boldsymbol{\Delta}] = \frac{1}{\rho_{\mu}}[\boldsymbol{I}_{T}+\rho_{\mu}{}^{2}\boldsymbol{I}_{T}-\rho_{\mu}\boldsymbol{D}-\rho_{\mu}{}^{2}\boldsymbol{\Delta}-\boldsymbol{I}_{T}+\rho_{\mu}{}^{2}\boldsymbol{I}_{T}] \\
&= \frac{1}{\rho_{\mu}}[(1+\rho_{\mu}{}^{2})\boldsymbol{I}_{T}-\rho_{\mu}\boldsymbol{D}-\rho_{\mu}{}^{2}\boldsymbol{\Delta}-(1-\rho_{\mu}{}^{2})\boldsymbol{I}_{T}] \\
&= \frac{1}{\rho_{\mu}}[\boldsymbol{R}^{\mu\mu}-(1-\rho_{\mu}{}^{2})\boldsymbol{I}_{T}], \qquad \text{(B.292)}
\end{aligned}$$

which implies that

$$\begin{aligned}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} &= \frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{I}_T]\boldsymbol{R}_{\mu\mu} = \frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu}\boldsymbol{R}_{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] \\ &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]. \end{aligned} \tag{B.293}$$

Then, equations (B.270) and (B.271) imply that

$$\begin{aligned}\boldsymbol{R}_2{}^{\mu\mu} &= \boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu \boldsymbol{\Delta} \Rightarrow \\ \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} &= (\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta})\boldsymbol{R}_{\mu\mu} = \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} \\ &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}. \end{aligned} \tag{B.294}$$

Furthermore,

$$\begin{aligned}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 &= [\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}][\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}] \\ &= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu{}^2\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}. \end{aligned} \tag{B.295}$$

ii)

$$\begin{aligned}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu} &= \bar{\boldsymbol{P}}_{X_\mu}[\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}]\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu} \\ &= \bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_1{}^{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{\Delta}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu}. \end{aligned} \tag{B.296}$$

Similarly,

$$\begin{aligned}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} &= \bar{\boldsymbol{P}}_{X_\mu}[\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}]\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} \\ &= \bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} + \rho_\mu\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{\Delta}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} \end{aligned} \tag{B.297}$$

and

$$\begin{aligned}\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} &= \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{X_\mu}[\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}]\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} \\ &= \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} + \rho_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{X_\mu}\boldsymbol{\Delta}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}. \end{aligned} \tag{B.298}$$

iii) Then, by using (B.266)

$$\operatorname{tr}\boldsymbol{R}_{\mu\mu} = \frac{1}{1-\rho_\mu{}^2}\sum_{t=1}^{T} 1 = \frac{T}{1-\rho_\mu{}^2}, \tag{B.299}$$

we find that

$$\operatorname{tr}[(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] = (1-\rho_\mu{}^2)\operatorname{tr}\boldsymbol{R}_{\mu\mu} = T. \tag{B.300}$$

By using equations (B.293) and (B.300) we find that

$$\operatorname{tr}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) = \frac{1}{\rho_\mu}[\operatorname{tr}\boldsymbol{I}_T - (1-{\rho_\mu}^2)\operatorname{tr}\boldsymbol{R}_{\mu\mu}] = \frac{1}{\rho_\mu}[T-T] = 0. \tag{B.301}$$

Let δ_{ij} be the (i,j)-th element of $\boldsymbol{\Delta}$. Then, $\delta_{ij} = 1$ for $i = j = 1$ and $i = j = T$ and $\delta_{ij} = 0$ elsewhere. Moreover, the (i,j)-th element of $\boldsymbol{R}_{\mu\mu}$ is $\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-j|}$. Then, the (i,j)-th element of $\boldsymbol{\Delta R}_{\mu\mu}$ is

$$\delta_{ij}{}^* = \sum_{\kappa=1}^{T}\delta_{i\kappa}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|\kappa-j|} = \delta_{ii}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-j|}, \tag{B.302}$$

because $\delta_{i\kappa} = 0$ for $\kappa \neq i$. Therefore, equation (B.302) implies that

$$\begin{aligned}\operatorname{tr}\boldsymbol{\Delta R}_{\mu\mu} &= \sum_{i=1}^{T}\delta_{ii}{}^* = \sum_{i=1}^{T}\delta_{ii}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-i|} = \frac{1}{1-{\rho_\mu}^2}\sum_{i=1}^{T}\delta_{ii} \\ &= \frac{1}{1-{\rho_\mu}^2}(\delta_{11}+\delta_{TT}) = \frac{2}{1-{\rho_\mu}^2}\,.\end{aligned} \tag{B.303}$$

The (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}$ is

$$\begin{aligned}\tilde{\delta}_{ij} &= \sum_{\kappa=1}^{T}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-\kappa|}\delta_{\kappa j}{}^* = \sum_{\kappa=1}^{T}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-\kappa|}\delta_{\kappa\kappa}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|\kappa-j|} \\ &= \frac{1}{(1-{\rho_\mu}^2)^2}{\rho_\mu}^{|i-1|+|1-j|}\delta_{11} + \frac{1}{(1-{\rho_\mu}^2)^2}{\rho_\mu}^{|i-T|+|T-j|}\delta_{TT} \\ &= \frac{1}{(1-{\rho_\mu}^2)^2}({\rho_\mu}^{i+j-2} + {\rho_\mu}^{2T-i-j}),\end{aligned} \tag{B.304}$$

which implies that

$$\begin{aligned}\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}) &= \sum_{i=1}^{T}\tilde{\delta}_{ii} = \frac{1}{(1-{\rho_\mu}^2)^2}\left[\sum_{i=1}^{T}{\rho_\mu}^{2(i-1)} + \sum_{i=1}^{T}{\rho_\mu}^{2(T-i)}\right] \\ &= \frac{1}{(1-{\rho_\mu}^2)^2}\left[\sum_{i=1}^{T}{\rho_\mu}^{2(i-1)} + \sum_{j=1}^{T}{\rho_\mu}^{2(j-1)}\right] = [\text{defining the index } j = T-i+1] \\ &= \frac{1}{(1-{\rho_\mu}^2)^2}2\sum_{i=1}^{T}{\rho_\mu}^{2(i-1)} \\ &= \frac{2}{(1-{\rho_\mu}^2)^2}\sum_{i=1}^{T}{\rho_\mu}^{2(i-1)} = [\text{defining the index } j = i-1] \\ &= \frac{2}{(1-{\rho_\mu}^2)^2}\sum_{j=0}^{T-1}{\rho_\mu}^{2j} = [\text{defining } r = {\rho_\mu}^2] \\ &= \frac{2}{(1-{\rho_\mu}^2)^2}\sum_{j=0}^{T-1}r^j = \frac{2}{(1-{\rho_\mu}^2)^2}\frac{1-r^T}{1-r} \\ &= \frac{2}{(1-{\rho_\mu}^2)^2}\frac{1-{\rho_\mu}^{2T}}{(1-{\rho_\mu}^2)} = \frac{2(1-{\rho_\mu}^{2T})}{(1-{\rho_\mu}^2)^3}.\end{aligned} \tag{B.305}$$

Along the same lines as in equation (B.302) we find that the (i, j)-th element of the $(\boldsymbol{\Delta R}_{\mu\mu})^2$ is

$$\begin{aligned}
\delta_{ij}^{\circ} &= \sum_{\kappa=1}^{T} \delta_{i\kappa}{}^{*}\delta_{\kappa j}{}^{*} = \sum_{\kappa=1}^{T} \delta_{ii} \frac{1}{1-\rho_\mu{}^2} \rho_\mu{}^{|i-\kappa|} \delta_{\kappa\kappa} \frac{1}{1-\rho_\mu{}^2} \rho_\mu{}^{|\kappa-j|} \\
&= \delta_{ii} \frac{1}{(1-\rho_\mu{}^2)^2} \sum_{\kappa=1}^{T} \delta_{\kappa\kappa} \rho_\mu{}^{|i-\kappa|+|\kappa-j|} = \delta_{ii} \frac{1}{(1-\rho_\mu{}^2)^2} (\rho_\mu{}^{|i-1|+|1-j|} \delta_{11} + \rho_\mu{}^{|i-T|+|T-j|} \delta_{TT}) \\
&= \delta_{ii} \frac{1}{(1-\rho_\mu{}^2)^2} (\rho_\mu{}^{i+j-2} + \rho_\mu{}^{2T-i-j}), \qquad \text{(B.306)}
\end{aligned}$$

which implies that

$$\begin{aligned}
\operatorname{tr}[(\boldsymbol{\Delta R}_{\mu\mu})^2] &= \sum_{i=1}^{T} \delta_{ii}^{\circ} = \sum_{i=1}^{T} \delta_{ii} \frac{1}{(1-\rho_\mu{}^2)^2} (\rho_\mu{}^{2(i-1)} + \rho_\mu{}^{2(T-i)}) \\
&= \delta_{11} \frac{1}{(1-\rho_\mu{}^2)^2} (\rho_\mu{}^{2(1-1)} + \rho_\mu{}^{2(T-1)}) + \delta_{TT} \frac{1}{(1-\rho_\mu{}^2)^2} (\rho_\mu{}^{2(T-1)} + \rho_\mu{}^{2(T-T)}) \\
&= \frac{2}{(1-\rho_\mu{}^2)^2} (1 + \rho_\mu{}^{2(T-1)}). \qquad \text{(B.307)}
\end{aligned}$$

By using equation (B.306) we find that the (i, j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2$ is

$$\begin{aligned}
\delta_{\hat{ij}} &= \sum_{\kappa=1}^{T} \frac{1}{1-\rho_\mu{}^2} \rho_\mu{}^{|i-\kappa|} \delta_{\kappa j}^{\circ} = \sum_{\kappa=1}^{T} \frac{1}{1-\rho_\mu{}^2} \rho_\mu{}^{|i-\kappa|} \delta_{\kappa\kappa} \frac{1}{(1-\rho_\mu{}^2)^2} (\rho_\mu{}^{\kappa+j-2} + \rho_\mu{}^{2T-\kappa-j}) \\
&= \delta_{11} \frac{1}{(1-\rho_\mu{}^2)^3} \rho_\mu{}^{|i-1|} (\rho_\mu{}^{j-1} + \rho_\mu{}^{2T-j-1}) + \delta_{TT} \frac{1}{(1-\rho_\mu{}^2)^3} \rho_\mu{}^{|i-T|} (\rho_\mu{}^{T+j-2} + \rho_\mu{}^{T-j}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3} (\rho_\mu{}^{i+j-2} + \rho_\mu{}^{2T+i-j-2} + \rho_\mu{}^{2T-i+j-2} + \rho_\mu{}^{2T-i-j}), \qquad \text{(B.308)}
\end{aligned}$$

which implies that

$$\begin{aligned}
\operatorname{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2] &= \sum_{i=1}^{T} \delta_{\hat{ii}} = \frac{1}{(1-\rho_\mu{}^2)^3} \sum_{i=1}^{T} (\rho_\mu{}^{2i-2} + 2\rho_\mu{}^{2T-2} + \rho_\mu{}^{2T-2i}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3} [2T\rho_\mu{}^{2T-2} + \sum_{i=1}^{T} \rho_\mu{}^{2(i-1)} + \sum_{i=1}^{T} \rho_\mu{}^{2(T-i)}] \\
&= [\text{defining the indeces } j = i-1 \text{ and } \kappa = T-i] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3} [2T\rho_\mu{}^{2(T-1)} + \sum_{j=0}^{T-1} \rho_\mu{}^{2j} + \sum_{\kappa=0}^{T-1} \rho_\mu{}^{2\kappa}] \\
&= \frac{2}{(1-\rho_\mu{}^2)^3} [T\rho_\mu{}^{2(T-1)} + \sum_{j=0}^{T-1} \rho_\mu{}^{2j}] = [\text{defining } r = \rho_\mu{}^2] \\
&= \frac{2}{(1-\rho_\mu{}^2)^3} [T\rho_\mu{}^{2(T-1)} + \sum_{j=0}^{T-1} r^j] = \frac{2}{(1-\rho_\mu{}^2)^3} \left[T\rho_\mu{}^{2(T-1)} + \frac{1-r^T}{1-r}\right] \\
&= \frac{2}{(1-\rho_\mu{}^2)^3} \left[T\rho_\mu{}^{2(T-1)} + \frac{1-\rho_\mu{}^{2T}}{1-\rho_\mu{}^2}\right]. \qquad \text{(B.309)}
\end{aligned}$$

By using equations (B.302) and (B.306) we find that the (i, j)-th element of the matrix $(\boldsymbol{\Delta R}_{\mu\mu})^3 = \boldsymbol{\Delta R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2$ is

$$\begin{aligned}
\delta_{ij}{}^{+} &= \sum_{\kappa=1}^{T} \delta_{i\kappa}{}^{*}\delta_{\kappa j}{}^{\circ} = \sum_{\kappa=1}^{T} \delta_{ii}\frac{1}{1-\rho_\mu{}^2}\rho_\mu{}^{|i-\kappa|}\delta_{\kappa\kappa}\frac{1}{(1-\rho_\mu{}^2)^2}(\rho_\mu{}^{\kappa+j-2}+\rho_\mu{}^{2T-\kappa-j}) \\
&= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)^3}[\delta_{11}\rho_\mu{}^{|i-1|}(\rho_\mu{}^{j-1}+\rho_\mu{}^{2T-1-j})+\delta_{TT}\rho_\mu{}^{|i-T|}(\rho_\mu{}^{T+j-2}+\rho_\mu{}^{T-j})] \\
&= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)^3}[\rho_\mu{}^{i+j-2}+\rho_\mu{}^{2T+i-j-2}+\rho_\mu{}^{2T-i+j-2}+\rho_\mu{}^{2T-i-j}], \qquad \text{(B.310)}
\end{aligned}$$

which implies that

$$\begin{aligned}
\text{tr}\,[(\boldsymbol{\Delta R}_{\mu\mu})^3] &= \sum_{i=1}^{T}\delta_{ii}{}^{+} = \sum_{i=1}^{T}\delta_{ii}\frac{1}{(1-\rho_\mu{}^2)^3}(\rho_\mu{}^{2(i-1)}+2\rho_\mu{}^{2(T-1)}+\rho_\mu{}^{2(T-i)}) \\
&= \delta_{11}\frac{1}{(1-\rho_\mu{}^2)^3}(\rho_\mu{}^{2(1-1)}+3\rho_\mu{}^{2(T-1)})+\delta_{TT}\frac{1}{(1-\rho_\mu{}^2)^3}(3\rho_\mu{}^{2(T-1)}+\rho_\mu{}^{2(T-T)}) \\
&= \frac{2}{(1-\rho_\mu{}^2)^3}(1+3\rho_\mu{}^{2(T-1)}). \qquad \text{(B.311)}
\end{aligned}$$

Let w_{ij} be the (i, j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}{}^3$. Then, the (i, j)-th element of the matrix $\boldsymbol{\Delta R}_{\mu\mu}{}^3$ is

$$\delta_{ij}{}^{\ddagger} = \sum_{\kappa=1}^{T}\delta_{i\kappa}w_{\kappa j} = \delta_{ii}w_{ij}, \qquad \text{(B.312)}$$

because $\delta_{i\kappa} = 0\ \forall\ \kappa \neq i$. Therefore,

$$\text{tr}[\boldsymbol{\Delta R}_{\mu\mu}{}^3] = \sum_{i=1}^{T}\delta_{ii}{}^{\ddagger} = \sum_{i=1}^{T}\delta_{ii}w_{ii} = \delta_{11}w_{11}+\delta_{TT}w_{TT} = w_{11}+w_{TT}. \qquad \text{(B.313)}$$

Let w_{ll} be the l-diagonal element of matrix $\boldsymbol{R}_{\mu\mu}{}^3$, i.e.,

$$\begin{aligned}
w_{ll} &= \sum_{m=1}^{T}\sum_{\kappa=1}^{T}\frac{1}{(1-\rho_\mu{}^2)^3}\rho_\mu{}^{|l-\kappa|+|\kappa-m|+|m-l|} \\
&= \sum_{i=1-l}^{T-l}\sum_{j=1-l}^{T-l}\frac{1}{(1-\rho_\mu{}^2)^3}\rho_\mu{}^{|i|+|j|+|j-i|}, \qquad \text{(B.314)}
\end{aligned}$$

where $i = m - l$ and $j = \kappa - l$ with $i, j = 1 - l, \ldots, T - l$, and $j - i = \kappa - l - m + l = \kappa - m$.

Figure 1

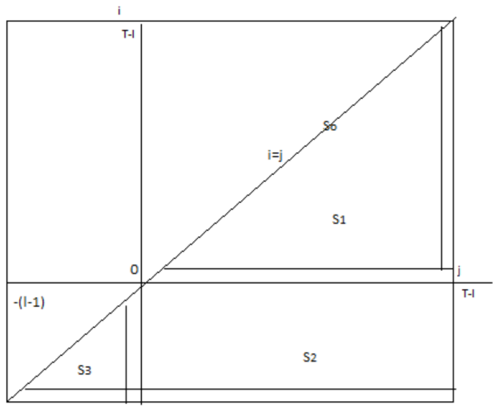

Figure 1 implies that

$$w_{ll} = 2(S_1 + S_2 + S_3) - S_0, \tag{B.315}$$

where

(i)

$$\begin{aligned} S_0 &= \sum_{i=1-l}^{T-l} \frac{1}{(1-\rho_\mu{}^2)^3}\rho_\mu{}^{2|i|} = \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{i=1-l}^{T-l} r^{|i|} = [\text{by defining } r = \rho_\mu{}^2] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1+r}{1-r} - \frac{1}{1-r}(r^l + r^{T-l+1})\right]. \end{aligned} \tag{B.316}$$

(ii)

$$\begin{aligned} S_1 &= \sum_{i=0}^{T-l}\sum_{j=i}^{T-l} \frac{1}{(1-\rho_\mu{}^2)^3}\rho_\mu{}^{i+j+j-i} = \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{i=0}^{T-l}\left[\sum_{j=0}^{T-l}\rho_\mu{}^{2j} - \sum_{j=0}^{i-1}\rho_\mu{}^{2j}\right] = [\text{by defining } r = \rho_\mu{}^2] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{i=0}^{T-l}\left[\sum_{j=0}^{T-l} r^j - \sum_{j=0}^{i-1} r^j\right] = \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{i=0}^{T-l}\left[\frac{1-r^{T-l+1}}{1-r} - \frac{1-r^{i-1+1}}{1-r}\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{i=0}^{T-l}\left[\frac{r^i - r^{T-l+1}}{1-r}\right] = \frac{1}{(1-\rho_\mu{}^2)^3}\frac{\sum_{i=0}^{T-l} r^i - \sum_{i=0}^{T-l} r^{T-l+1}}{1-r} \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1-r^{T-l+1}}{(1-r)^2} - \frac{(T-l+1)r^{T-l+1}}{1-r}\right]. \end{aligned} \tag{B.317}$$

(iii)

$$\begin{aligned}
S_2 &= \sum_{i=1-l}^{-1}\sum_{j=i}^{T-l}\frac{1}{(1-\rho_\mu{}^2)^3}\rho_\mu{}^{-i+j+j-i} = \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{i=1-l}^{-1}\sum_{j=1}^{T-l}\rho_\mu{}^{-2i}\rho_\mu{}^{2j}\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{i=1-l}^{-1}\rho_\mu{}^{-2i}\sum_{j=1}^{T-l}\rho_\mu{}^{2j} = \text{[by setting } k=-i \text{ with } k=1,\ldots,l-1]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{k=1}^{l-1}\rho_\mu{}^{2k}\sum_{j=1}^{T-l}\rho_\mu{}^{2j} = \text{[by defining } r=\rho_\mu{}^2]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{k=1}^{l-1}r^k\sum_{j=1}^{T-l}r^j\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\cdot\frac{r(1-r^{T-l})}{1-r}\cdot\frac{r(1-r^{l-1})}{1-r}\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{r^2}{(1-r)^2}[1+r^{T-1}-r^{T-l}-r^{l-1}]. \qquad \text{(B.318)}
\end{aligned}$$

(iv)

$$\begin{aligned}
S_3 &= \sum_{i=1-l}^{0}\sum_{j=i}^{0}\frac{1}{(1-\rho_\mu{}^2)^3}[\rho_\mu{}^{-i+j+j-i}-1]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\sum_{i=1-l}^{0}\rho_\mu{}^{-2i}(i+1)-1\right] = \text{[by setting } k=-i \text{ with } k=0,\ldots,l-1]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\sum_{k=0}^{l-1}(1-k)\rho_\mu{}^{2k}-1\right] = \text{[by defining } r=\rho_\mu{}^2]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\sum_{k=0}^{l-1}r^k-\sum_{k=0}^{l-1}kr^k-1\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1-r^{l-1+1}}{1-r}-\frac{r[1-(l-1+1)r^{l-1}]+(l-1)r^{l-1+1}]}{(1-r)^2}-1\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{(1-r)(1-r^l)-r+lr^l-(l-1)r^{l+1}-(1-r)^2}{(1-r)^2}\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1-r-r^l+r^{l+1}-r+lr^l-lr^{l+1}+r^{l+1}+-1+2r-r^2}{(1-r)^2}\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{-r^2+(l-1)r^l-(l-2)r^{l+1}}{(1-r)^2}. \qquad \text{(B.319)}
\end{aligned}$$

By combining equations (B.317), (B.318), and (B.319) we find that

$$\begin{aligned}
S_* &= S_1+S_2+S_3\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1-r^{T-l+1}}{(1-r)^2}-\frac{(T-l+1)r^{T-l+1}}{1-r}+\frac{r^2}{(1-r)^2}[1+r^{T-1}-r^{T-l}-r^{l-1}]\right]\\
&\quad+\frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{-r^2+(l-1)r^l-(l-2)r^{l+1}}{(1-r)^2}\right] \Longrightarrow
\end{aligned}$$

$$
\begin{aligned}
S_* &= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1-r^{T-l+1}-(T-l+1)(1-r)r^{T-l+1}+r^2+r^{T-1+2}-r^{T-l+2}-r^{l-1+2}-r^2+(l-1)r^l-(l-2)r^{l+1}\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1-r^{T-l+1}-(T-l+1)r^{T-l+1}+(T-l+1)r^{T-l+2}+r^{T+1}-r^{T-l+2}-r^{l+1}+(l-1)r^l-(l-2)r^{l+1}\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1+r^{T+1}+(l-1)r^l-(T-l+2)r^{T-l+1}+(T-l)r^{T-l+2}-(l-1)r^{l+1}\right]. \qquad \text{(B.320)}
\end{aligned}
$$

Equation (B.320) implies that

$$
\begin{aligned}
\sum_{l=1}^{T} S_* &= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{l=1}^{T}(1-r)^{-2}\left[1+r^{T+1}+(l-1)r^l-(T-l+2)r^{T-l+1}+(T-l)r^{T-l+2}-(l-1)r^{l+1}\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[T(1+r^{T+1})+s_1+s_2+s_3+s_4\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{T(1+r^{T+1})}{(1-r)^2}+\frac{s_1+s_2+s_3+s_4}{(1-r)^2}\right], \qquad \text{(B.321)}
\end{aligned}
$$

where the quantities s_1, s_2, s_3, s_4 are computed as follows:

(I)

$$
\begin{aligned}
s_1 &= \sum_{l=1}^{T}(l-1)r^l = \sum_{l=1}^{T} lr^l - \sum_{l=1}^{T} r^l = \frac{r[1-(T+1)r^T+Tr^{T+1}]}{(1-r)^2} - \frac{r(1-r^T)}{1-r}\\
&= \frac{r-(T+1)r^{T+1}+Tr^{T+2}-r(1-r)(1-r^T)}{(1-r)^2}\\
&= \frac{r-(T+1)r^{T+1}+Tr^{T+2}-r(1-r-r^T+r^{T+1})}{(1-r)^2}\\
&= \frac{r-(T+1)r^{T+1}+Tr^{T+2}-r+r^2+r^{T+1}-r^{T+2}}{(1-r)^2}\\
&= \frac{r^2-(T)r^{T+1}+(T-1)r^{T+2}}{(1-r)^2}. \qquad \text{(B.322)}
\end{aligned}
$$

(II) By setting $i = T - l$ with $i = 0, \ldots, T-1$ we find that

$$
\begin{aligned}
s_2 &= \sum_{l=1}^{T} -(T-l+2)r^{T-l+1} = -\sum_{i=0}^{T-1}(i+2)r^{i+1} = -r\sum_{i=0}^{T-1}(i+2)r^i\\
&= -r\left[\sum_{i=0}^{T-1} ir^i + 2\sum_{i=0}^{T-1} r^i\right] = -r\left[\frac{r[1-Tr^{T-1}+(T-1)r^T]}{(1-r)^2}+\frac{2(1-r^T)}{1-r}\right]\\
&= -r\frac{r-(T)r^T+(T-1)r^{T+1}+2(1-r)(1-r^T)}{(1-r)^2}\\
&= \frac{-r^2+Tr^{T+1}-(T-1)r^{T+2}+2r(1-r-r^T+r^{T+1})}{(1-r)^2}\\
&= \frac{-r^2+Tr^{T+1}-(T-1)r^{T+2}-2r+2r^2+2r^{T+1}-2r^{T+2}}{(1-r)^2}\\
&= \frac{-2r+r^2+(T+2)r^{T+1}-(T+1)r^{T+2}}{(1-r)^2}. \qquad \text{(B.323)}
\end{aligned}
$$

(III) Similarly, by using the index $i = T - l$ with $i = 0, \ldots, T-1$ we find that

$$\begin{aligned} s_3 &= \sum_{l=1}^{T} (T-l) r^{T-l+2} = \sum_{i=0}^{T-1} i r^{i+2} = r^2 \sum_{i=0}^{T-1} i r^i \\ &= r^2 \frac{r[1 - T r^{T-1} + (T-1) r^T]}{(1-r)^2} \\ &= \frac{r^3 - T r^{T+2} + (T-1) r^{T+3}}{(1-r)^2}. \end{aligned} \tag{B.324}$$

(IV) By setting $k = l - 1$ with $k = 0, \ldots, T-1$ we find that

$$\begin{aligned} s_4 &= \sum_{l=1}^{T} -(l-1) r^{l+1} = -\sum_{l=1}^{T} (l-1) r^{(l-1)+2} = -\sum_{k=0}^{T-1} k r^{k+2} = -r^2 \sum_{k=0}^{T-1} k r^k \\ &= -r^2 \frac{r[1 - T r^{T-1} + (T-1) r^T]}{(1-r)^2} \\ &= \frac{-r^3 + T r^{T+2} - (T-1) r^{T+3}}{(1-r)^2}. \end{aligned} \tag{B.325}$$

Since equations (B.324) and (B.325) imply that

$$s_4 = -s_3, \tag{B.326}$$

by using equations (B.322) and (B.323) we find that

$$\begin{aligned} s_1 + s_2 + s_3 + s_4 &= s_1 + s_2 + s_3 - s_3 \\ &= (1-r)^{-2} [r^2 - T r^{T+1} + (T-1) r^{T+2} - 2r + r^2 + (T+2) r^{T+1} - (T+1) r^{T+2}] \\ &= (1-r)^{-2} [2r^2 - 2r + 2r^{T+1} - 2r^{T+2}] \\ &= 2(1-r)^{-2} [r^2 - r + r^{T+1} - r^{T+2}]. \end{aligned} \tag{B.327}$$

By setting $i = T - l + 1$ with $i = 1, \ldots, T$ and by using equation (B.316) we find that

$$\begin{aligned} \sum_{l=1}^{T} S_0 &= \sum_{l=1}^{T} \frac{1}{(1-\rho_\mu{}^2)^3} \left[\frac{1+r}{1-r} - \frac{1}{1-r} (r^l + r^{T-l+1}) \right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \left[\frac{T(1+r)}{1-r} - \frac{1}{1-r} \left(\sum_{l=1}^{T} r^l + \sum_{l=1}^{T} r^{T-l+1} \right) \right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \left[\frac{T(1+r)}{1-r} - \frac{1}{1-r} \left(\sum_{l=1}^{T} r^l + \sum_{i=1}^{T} r^i \right) \right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \left[\frac{T(1+r)}{1-r} - \frac{2}{1-r} \sum_{l=1}^{T} r^l \right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \left[\frac{T(1+r)}{1-r} - \frac{2}{1-r} \frac{r(1-r^T)}{1-r} \right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \left[\frac{T(1+r)}{1-r} - \frac{2r(1-r^T)}{(1-r)^2} \right]. \end{aligned} \tag{B.328}$$

By using equations (B.315), (B.316), and (B.320) we find that the l-diagonal element of the matrix $\boldsymbol{R}_{\mu\mu}{}^3$ is

$$
\begin{aligned}
w_{ll} &= 2(S_1 + S_2 + S_3) - S_0 = 2S_* - S_0 \\
&= \frac{1}{(1-\rho_\mu{}^2)^3} 2(1-r)^{-2}\left[1 + r^{T+1} + (l-1)r^l - (T-l+2)r^{T-l+1} + (T-l)r^{T-l+2} - (l-1)r^{l+1}\right] \\
&\quad - \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1+r}{1-r} - \frac{1}{1-r}(r^l + r^{T-l+1})\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[2 + 2r^{T+1} + 2(l-1)r^l - 2(T-l+2)r^{T-l+1} + 2(T-l)r^{T-l+2} - 2(l-1)r^{l+1}\right] \\
&\quad - \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}[(1+r)(1-r) + (1-r)(r^l + r^{T-l+1})] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[2 + 2r^{T+1} + 2(l-1)r^l - 2(T-l+2)r^{T-l+1} + 2(T-l)r^{T-l+2} - 2(l-1)r^{l+1}\right] \\
&\quad - \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1 - r^2 - r^l - r^{T-l+1} + r^{l+1} + r^{T-l+2}\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1 + 2r^{T+1} + (2l-1)r^l - (2l-1)r^{l+1} + r^2\right] \\
&\quad - \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[(2(T-l)+3)r^{T-l+1} - (2(T-l)-1)r^{T-l+2}\right].
\end{aligned}
\tag{B.329}
$$

By omitting terms that tend to zero as $T \to \infty$ and since $r = \rho_\mu{}^2$ with $|r| < 1$, we find that

$$
\begin{aligned}
w_{11} &= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1 + r - r^2 + r^2\right] + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1+r}{(1-r)^2} + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^2} + o(T^{-1}) \\
&= \frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^5} + o(T^{-1}).
\end{aligned}
\tag{B.330}
$$

Similarly,

$$
\begin{aligned}
w_{TT} &= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1 + r^2 - 3r - r^2\right] + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1-3r}{(1-r)^2} + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1-3\rho_\mu{}^2}{(1-\rho_\mu{}^2)^2} + o(T^{-1}) \\
&= \frac{1-3\rho_\mu{}^2}{(1-\rho_\mu{}^2)^5} + o(T^{-1}).
\end{aligned}
\tag{B.331}
$$

Thus, equations (B.313), (B.330), and (B.331) imply that

$$\begin{aligned}
\mathrm{tr}[\boldsymbol{\Delta R}_{\mu\mu}{}^3] &= w_{11} + w_{TT} = \frac{1+\rho_\mu{}^2+1-3\rho_\mu{}^2}{(1-\rho_\mu{}^2)^5} + o(T^{-1}) \\
&= \frac{2-2\rho_\mu{}^2}{(1-\rho_\mu{}^2)^5} + o(T^{-1}) \\
&= \frac{2(1-\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^5} + o(T^{-1}) \\
&= \frac{2}{(1-\rho_\mu{}^2)^4} + o(T^{-1}). \qquad \text{(B.332)}
\end{aligned}$$

Moreover, by using equations (B.315), (B.320), (B.321), (B.327), and (B.328) we find that the trace of the matrix $\boldsymbol{R}_{\mu\mu}{}^3/T$ is

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^3)/T &= \frac{1}{T}\sum_{l=1}^{T} w_{ll} = \frac{1}{T}\sum_{l=1}^{T}[2(S_1+S_2+S_3)-S_0] = \frac{1}{T}[\sum_{l=1}^{T} 2S_* - \sum_{l=1}^{T} S_0] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{2}{T}\left[\frac{T(1+r^{T+1})}{(1-r)^2} + \frac{2(1-r)^{-2}(r^2-r+r^{T+1}-r^{T+2})}{(1-r)^2}\right] \\
&\quad -\frac{1}{(1-\rho_\mu{}^2)^3}\frac{1}{T}\left[\frac{T(1+r)}{1-r} - \frac{2r(1-r^T)}{(1-r)^2}\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{2(1+r^{T+1})}{(1-r)^2} + \frac{4(r^2-r+r^{T+1}-r^{T+2})}{(1-r)^4} - \frac{1+r}{1-r} + \frac{2r(1-r^T)}{T(1-r)^2}\right]. \qquad \text{(B.333)}
\end{aligned}$$

By omitting terms that tend to zero as $T \to \infty$ and since $r = \rho_\mu{}^2$ with $|r| < 1$, we find that

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^3)/T &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{2}{(1-r)^2} - \frac{1+r}{1-r} + o(T^{-1})\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{2-(1+r)(1-r)}{(1-r)^2} + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{2-1+r^2}{(1-r)^2} + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1+\rho_\mu{}^4}{(1-\rho_\mu{}^2)^2} + o(T^{-1}) = \frac{1+\rho_\mu{}^4}{(1-\rho_\mu{}^2)^5} + o(T^{-1}). \qquad \text{(B.334)}
\end{aligned}$$

Finally, note that in all traces examined in this Lemma, there appear terms of the form $T^n r^T$ where n is a positive integer. Since $r = \rho_\mu{}^2$ with $0 \le r < 1$,

$$\lim_{T\to\infty} T^n r^T = \lim_{T\to\infty} \frac{T^n}{r^{-T}} = \frac{\infty}{\infty}. \qquad \text{(B.335)}$$

By applying L'Hospital rule we find that

$$\begin{aligned}
\lim_{T\to\infty} T^n r^T &= \lim_{T\to\infty}\frac{T^n}{r^{-T}} = \lim_{T\to\infty}\frac{\partial T^n/\partial T}{\partial r^{-T}/\partial T} = \frac{n}{-lnr}\lim_{T\to\infty}\frac{T^{n-1}}{r^{-T}} = \ldots \\
&= \frac{n!}{(-lnr)^n}\lim_{T\to\infty}\frac{1}{r^{-T}} = \frac{n!}{(-lnr)^n}\lim_{T\to\infty} r^T = 0. \qquad \text{(B.336)}
\end{aligned}$$

Therefore, since all terms of the form $T^n r^T$ tend to zero as $T \to \infty$, all the traces computed in this Lemma are bounded as $T \to \infty$.

Furthermore, the first regularity condition implies that the matrices

$$\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu} \boldsymbol{X}_\mu / T \text{ and } \boldsymbol{X}'_\mu \boldsymbol{X}_\mu / T \tag{B.337}$$

converge to non-singular matrices as $T \to \infty$.

Let x_{ij} and δ_{ij} be the (i, j)-th element of the matrices $\boldsymbol{X}_\mu$ and $\boldsymbol{\Delta}$ respectively. Then equation (B.337) implies that the element x_{ij} $(i = 1, \dots, T;\ j = 1, \dots, n)$ are bounded.
The following results hold:

(a) The (i, j)-th element of the matrix $\boldsymbol{X}'_\mu \boldsymbol{\Delta} \boldsymbol{X}_\mu$ is

$$\begin{aligned} \eta_{ij} &= \sum_{t=1}^{T} \sum_{s=1}^{T} x_{it} \delta_{ts} x_{sj} = \sum_{t=1}^{T} x_{it} \delta_{tt} x_{tj} = x_{i1} \delta_{11} x_{1j} + x_{iT} \delta_{TT} x_{Tj} \\ &= x_{i1} x_{1j} + x_{iT} x_{Tj}, \end{aligned} \tag{B.338}$$

which is bounded and consequently the matrix

$$\boldsymbol{X}'_\mu \boldsymbol{\Delta} \boldsymbol{X}_\mu / T = O(T^{-1}). \tag{B.339}$$

(b) By defining the indeces $k = s - 1$ $(k = 1, \dots, T - 1)$ and $l = T - s$ $(l = 1, \dots, T - 1)$, the (i, j)-th element of the matrix $\boldsymbol{X}'_\mu \boldsymbol{\Delta} \boldsymbol{R}^{\mu\mu} \boldsymbol{X}_\mu$ is (see (B.302))

$$\begin{aligned} \eta_{ij}{}^{*} &= \sum_{t=1}^{T} \sum_{s=1}^{T} x_{it} \delta_{ts}{}^{*} x_{sj} = \sum_{s=1}^{T} x_{it} \delta_{tt} \frac{1}{1 - \rho_\mu{}^2} \rho_\mu{}^{|t-s|} x_{sj} \\ &= \sum_{s=1}^{T} \left[x_{i1} \delta_{11} \frac{1}{1 - \rho_\mu{}^2} \rho_\mu{}^{|1-s|} x_{sj} + x_{iT} \delta_{TT} \frac{1}{1 - \rho_\mu{}^2} \rho_\mu{}^{|T-s|} x_{sj} \right] \\ &= \frac{1}{1 - \rho_\mu{}^2} \left[x_{i1} \left(\sum_{s=1}^{T} x_{sj} \rho_\mu{}^{s-1} \right) + x_{iT} \left(\sum_{s=1}^{T} x_{sj} \rho_\mu{}^{T-s} \right) \right] = \\ &= \frac{1}{1 - \rho_\mu{}^2} \left[x_{i1} \left(\sum_{k=0}^{T-1} x_{(k+1)j} \rho_\mu{}^{k} \right) + x_{iT} \left(\sum_{l=0}^{T-1} x_{(l+1)j} \rho_\mu{}^{l} \right) \right] \\ &= \frac{1}{1 - \rho_\mu{}^2} (x_{i1} + x_{iT}) \left(\sum_{l=1}^{T-1} x_{(l+1)j} \rho_\mu{}^{l} \right). \end{aligned} \tag{B.340}$$

Since $\boldsymbol{X}'_\mu$ is bounded, i.e., $\forall\, l$ $(l = 1, \dots, T - 1)$ it holds that

$$\begin{aligned} &|x_{(l+1)j}| \leq q < \infty \Rightarrow \\ &\Rightarrow \left| \sum_{l=0}^{T-1} x_{(l+1)j} \rho_\mu{}^{l} \right| \leq \sum_{l=0}^{T-1} \left| x_{(l+1)j} \right| \left| \rho_\mu{}^{l} \right| \leq q \sum_{l=0}^{T-1} \left| \rho_\mu{}^{l} \right| = q \frac{1 - |\rho_\mu|^T}{1 - |\rho_\mu|}, \end{aligned} \tag{B.341}$$

which implies that $\eta_{ij}{}^*$ is bounded for every $(i, j = 1, \ldots, n)$ and so the matrix

$$\boldsymbol{X}'_\mu \boldsymbol{\Delta} \boldsymbol{R}^{\mu\mu} \boldsymbol{X}_\mu / T = O(T^{-1}). \tag{B.342}$$

Along the same lines we can prove that

$$\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu} \boldsymbol{\Delta} \boldsymbol{X}_\mu / T = O(T^{-1}). \tag{B.343}$$

(c) The (i, j)-th element of the matrix $\boldsymbol{\Delta} \boldsymbol{R}^{\mu\mu} \boldsymbol{\Delta}$ is (see (B.302))

$$\eta_{\tilde{i}\tilde{j}} = \sum_{k=1}^{T} \delta_{ik}{}^* \delta_{kj} = \delta_{ij}{}^* \delta_{jj}, \tag{B.344}$$

which implies that the (i, j)-th element of the matrix $\boldsymbol{X}'_\mu \boldsymbol{\Delta} \boldsymbol{R}^{\mu\mu} \boldsymbol{\Delta} \boldsymbol{X}_\mu$ is

$$\begin{aligned}
\eta_{ij}{}^\circ &= \sum_{t=1}^{T} \sum_{s=1}^{T} x_{it} \eta_{\tilde{t}\tilde{s}} x_{sj} = \sum_{t=1}^{T} \sum_{s=1}^{T} x_{it} \delta_{ts}{}^* \delta_{ss} x_{sj} = [\text{see (B.302)}] \\
&= \sum_{t=1}^{T} \sum_{s=1}^{T} x_{it} \delta_{tt} \frac{1}{1-\rho_\mu{}^2} \rho_\mu{}^{|t-s|} \delta_{ss} x_{sj} \\
&= \frac{1}{1-\rho_\mu{}^2} \sum_{t=1}^{T} x_{it} \delta_{tt} \sum_{s=1}^{T} \delta_{ss} \rho_\mu{}^{|t-s|} x_{sj} \\
&= \frac{1}{1-\rho_\mu{}^2} \sum_{t=1}^{T} x_{it} \delta_{tt} (\delta_{11} \rho_\mu{}^{|t-1|} x_{1j} + \delta_{TT} \rho_\mu{}^{|t-T|} x_{Tj}) \\
&= \frac{1}{1-\rho_\mu{}^2} \left[x_{i1} \delta_{11} (\rho_\mu{}^{1-1} x_{1j} + \rho_\mu{}^{T-1} x_{Tj}) + x_{iT} \delta_{TT} (\rho_\mu{}^{T-1} x_{1j} + \rho_\mu{}^{T-T} x_{Tj}) \right] \\
&= \frac{1}{1-\rho_\mu{}^2} \left[x_{i1} (x_{1j} + \rho_\mu{}^{T-1} x_{1j}) + x_{iT} (\rho_\mu{}^{T-1} x_{1j} + x_{Tj}) \right].
\end{aligned} \tag{B.345}$$

Thus, equation (B.345) implies that $\eta_{ij}{}^\circ$ is bounded so that

$$\boldsymbol{X}'_\mu \boldsymbol{\Delta} \boldsymbol{R}^{\mu\mu} \boldsymbol{\Delta} \boldsymbol{X}_\mu / T = O(T^{-1}). \tag{B.346}$$

(d) The (i, j)-th element of the matrix $\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu} \boldsymbol{X}_\mu$ is

$$\eta_{ij}{}^+ = \sum_{t=1}^{T} \sum_{s=1}^{T} x_{it} \frac{1}{1-\rho_\mu{}^2} \rho_\mu{}^{|t-s|} x_{sj} = \frac{1}{1-\rho_\mu{}^2} \sum_{t=1}^{T} \sum_{s=1}^{T} x_{it} \rho_\mu{}^{|t-s|} x_{sj} \tag{B.347}$$

and it is bounded given that x_{it} and x_{sj} are bounded for every $i, j = 1, \ldots, n$ and every $t, s = 1, \ldots, T$. Therefore,

$$\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu} \boldsymbol{X}_\mu / T = O(T^{-1}). \tag{B.348}$$

By using equations (B.294), (B.301), and (B.303) we find that

$$\begin{aligned}\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \operatorname{tr}[(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu \boldsymbol{\Delta})\boldsymbol{R}_{\mu\mu}] = \operatorname{tr}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu \operatorname{tr}(\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}) \\ &= 0 + \frac{2\rho_\mu}{1-\rho_\mu{}^2} = \frac{2\rho_\mu}{1-\rho_\mu{}^2}. \end{aligned} \tag{B.349}$$

Similarly, by using equation (B.293) we find the following results:

(a)

$$\begin{aligned}\rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} &= \rho_\mu \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} \\ &= \boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} \Rightarrow \end{aligned} \tag{B.350}$$

$$\begin{aligned}\operatorname{tr}(\rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}) &= \operatorname{tr}(\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}) = \text{(see (B.303) and (B.305))} \\ &= \frac{2}{1-\rho_\mu{}^2} - (1-\rho_\mu{}^2)\frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^3} \\ &= \frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^2}. \end{aligned} \tag{B.351}$$

(b)

$$\begin{aligned}\operatorname{tr}(\rho_\mu \boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \operatorname{tr}(\rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}) \\ &= \frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^2}. \end{aligned} \tag{B.352}$$

(c) By using equation (B.307) we find that

$$\begin{aligned}\operatorname{tr}(\rho_\mu{}^2 \boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}) &= \operatorname{tr}[\rho_\mu{}^2(\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu})^2] = \rho_\mu{}^2 \operatorname{tr}[(\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu})^2] \\ &= \frac{2\rho_\mu{}^2}{(1-\rho_\mu{}^2)^2}(1+\rho_\mu{}^{2(T-1)}). \end{aligned} \tag{B.353}$$

(d) Moreover, by using equation (B.293) we find that

$$\begin{aligned}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]\frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] \\ &= \frac{1}{\rho_\mu{}^2}[\boldsymbol{I}_T - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2]. \end{aligned} \tag{B.354}$$

Defining $j = k - i$ with $j = 1 - i, \ldots, T - i$ and setting $j = T - i + 1$ with $j = 1, \ldots, T$, let v_{ll} be the l-diagonal element of matrix $\boldsymbol{R}_{\mu\mu}{}^2$, i.e.,

$$\begin{aligned}
S(i) &= \sum_{k=1}^{T} \frac{1}{(1-\rho_\mu{}^2)^2} \rho_\mu{}^{|i-k|+|k-i|} = \\
&= \sum_{j=1-i}^{T-i} \frac{1}{(1-\rho_\mu{}^2)^2} \rho_\mu{}^{2|j|} = (\text{defining } r = \rho_\mu{}^2) \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \sum_{j=1-i}^{T-i} r^{|j|} = \frac{1}{(1-\rho_\mu{}^2)^2} \left[\sum_{j=1-i}^{-1} r^{|j|} + \sum_{j=0}^{T-i} r^j \right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \left[\sum_{j+i=1}^{i-1} r^{|j+i|} + \sum_{j=0}^{T-i} r^j \right] = \frac{1}{(1-\rho_\mu{}^2)^2} \left[\sum_{k=1}^{i-1} r^k + \sum_{j=0}^{T-i} r^j \right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \left[\frac{r(1-r^{(i-1)})}{1-r} + \frac{1-r^{T-i+1}}{1-r} \right] = \frac{1}{(1-\rho_\mu{}^2)^2} \frac{r - r^i + 1 - r^{T-i+1}}{1-r} \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \left[\frac{1+r}{1-r} - \frac{r^i + r^{T-i+1}}{1-r} \right] = \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \left[\frac{1+r}{1-r} - \frac{r^i + r^j}{1-r} \right] = \frac{1}{(1-\rho_\mu{}^2)^2} \left[\frac{1+r}{1-r} - \frac{2r^i}{1-r} \right]. \qquad \text{(B.355)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_{\mu\mu}{}^2)/T &= \sum_{i=1}^{T} S(i)/T = \sum_{i=1}^{T} \left[\frac{1}{(1-\rho_\mu{}^2)^2} \left[\frac{1+r}{1-r} - \frac{2r^i}{1-r} \right] \right] /T \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \left[\frac{T(1+r)}{1-r} - \frac{2}{1-r} \sum_{i=1}^{T} r^i \right] /T \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \left[\frac{1+r}{1-r} - \frac{2}{T(1-r)} \frac{r(1-r^T)}{(1-r)} \right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \left[\frac{1+r}{1-r} - \frac{2r(1-r^T)}{T(1-r)^2} \right] \qquad \text{(B.356)}
\end{aligned}$$

and omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_{\mu\mu}{}^2)/T &= \frac{1}{(1-\rho_\mu{}^2)^2} \frac{1+r}{1-r} + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^2} \frac{1+\rho_\mu{}^2}{1-\rho_\mu{}^2} + o(T^{-1}) \\
&= \frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^3} + o(T^{-1}). \qquad \text{(B.357)}
\end{aligned}$$

By combining equations (B.300), (B.354), and (B.357) we find that

$$\begin{aligned}
\mathrm{tr}[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2] &= \frac{1}{\rho_\mu{}^2}[\mathrm{tr}\,(\boldsymbol{I}_T) - 2\,\mathrm{tr}\,[(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] + (1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^2)] \\
&= \frac{1}{\rho_\mu{}^2}\left[T - 2T + (1-\rho_\mu{}^2)^2 T\frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^3} + o(1)\right] \\
&= \frac{1}{\rho_\mu{}^2}\left[-T + T\frac{1+\rho_\mu{}^2}{1-\rho_\mu{}^2} + o(1)\right] = \frac{T}{\rho_\mu{}^2}\left[\frac{-1+\rho_\mu{}^2+1+\rho_\mu{}^2}{1-\rho_\mu{}^2}\right] + o(1) \\
&= \frac{2T\rho_\mu{}^2}{\rho_\mu{}^2(1-\rho_\mu{}^2)} + o(1) = \frac{2T}{1-\rho_\mu{}^2} + o(1). \qquad \text{(B.358)}
\end{aligned}$$

By combining equations (B.295), (B.351), (B.352), (B.353), and (B.358) we find that

$$\begin{aligned}
\mathrm{tr}[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \mathrm{tr}\,[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T + \mathrm{tr}\,(\rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T \\
&\quad + \mathrm{tr}\,(\rho_\mu\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T + \mathrm{tr}\,(\rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T \\
&= \frac{2}{1-\rho_\mu{}^2} + o(T^{-1}) + \frac{2}{T}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^2}\right] \\
&\quad + \frac{2\rho_\mu{}^2}{T(1-\rho_\mu{}^2)^2}(1+\rho_\mu{}^{2(T-1)}) \qquad \text{(B.359)}
\end{aligned}$$

and omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned}
&\mathrm{tr}[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T = \frac{2}{1-\rho_\mu{}^2} + o(T^{-1}) \Rightarrow \\
&\mathrm{tr}[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2] = \frac{2T}{1-\rho_\mu{}^2} + o(1). \qquad \text{(B.360)}
\end{aligned}$$

(e) By using equations (B.294) and (B.295) we take

$$\begin{aligned}
(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3 &= (\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \left[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\right]\cdot \\
&\quad \left[\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}\right] \\
&= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3 + \rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\
&\quad +\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\
&\quad +\rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\
&\quad +\rho_\mu{}^3\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} \\
&= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3 + \rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\
&\quad +\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 \\
&\quad +\rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu{}^2(\boldsymbol{\Delta R}_{\mu\mu})^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\
&\quad +\rho_\mu{}^3(\boldsymbol{\Delta R}_{\mu\mu})^3, \qquad \text{(B.361)}
\end{aligned}$$

which implies that

$$\begin{aligned}\operatorname{tr}[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3] &= \operatorname{tr}[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3] + 3\operatorname{tr}[\rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}] \\ &\quad +3\operatorname{tr}[\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2] + \operatorname{tr}[\rho_\mu{}^3(\boldsymbol{\Delta R}_{\mu\mu})^3]. \end{aligned} \tag{B.362}$$

Since, equation (B.354) implies that

$$\begin{aligned}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu} &= \frac{1}{\rho_\mu{}^2}[\boldsymbol{I}_T - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2]\boldsymbol{\Delta R}_{\mu\mu} \\ &= \frac{1}{\rho_\mu{}^2}[\boldsymbol{\Delta R}_{\mu\mu} - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2\boldsymbol{\Delta R}_{\mu\mu}], \end{aligned} \tag{B.363}$$

it follows that

$$\begin{aligned}\operatorname{tr}[\rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}] &= \frac{\rho_\mu}{\rho_\mu{}^2}[\operatorname{tr}\boldsymbol{\Delta R}_{\mu\mu} - 2(1-\rho_\mu{}^2)\operatorname{tr}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\operatorname{tr}\boldsymbol{R}_{\mu\mu}{}^2\boldsymbol{\Delta R}_{\mu\mu}] \\ &= [\text{see (B.303), (B.305), and (B.332)}] \\ &= \frac{1}{\rho_\mu}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^2)(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^3} + \frac{2(1-\rho_\mu{}^2)^2}{(1-\rho_\mu{}^2)^4} + O(T^{-1})\right] \\ &= \frac{1}{\rho_\mu}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^2} + \frac{2}{(1-\rho_\mu{}^2)^2} + O(T^{-1})\right] \\ &= \frac{1}{\rho_\mu}\left[\frac{2-2\rho_\mu{}^2-2+2\rho_\mu{}^{2T}+2}{(1-\rho_\mu{}^2)^2}\right] + O(T^{-1}) \\ &= \frac{2(1-\rho_\mu{}^2+\rho_\mu{}^{2T})}{\rho_\mu(1-\rho_\mu{}^2)^2} + O(T^{-1}) \Rightarrow \end{aligned} \tag{B.364}$$

$$\operatorname{tr}[\rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}]/T = \frac{2(1-\rho_\mu{}^2+\rho_\mu{}^{2T})}{T\rho_\mu(1-\rho_\mu{}^2)^2} + O(1). \tag{B.365}$$

Moreover, since equation (B.293) implies that

$$\begin{aligned}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2 &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}](\boldsymbol{\Delta R}_{\mu\mu})^2 \\ &= \frac{1}{\rho_\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2 - \frac{(1-\rho_\mu{}^2)}{\rho_\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2, \end{aligned} \tag{B.366}$$

it follows that

$$\begin{aligned}\operatorname{tr}[\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2] &= \rho_\mu\operatorname{tr}[(\boldsymbol{\Delta R}_{\mu\mu})^2] - \rho_\mu(1-\rho_\mu{}^2)\operatorname{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2] \\ &= [\text{see (B.307) and (B.309)}] \\ &= \frac{2\rho_\mu}{(1-\rho_\mu{}^2)^2}(1+\rho_\mu{}^{2(T-1)}) - \frac{2\rho_\mu(1-\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^3}\left[T\rho_\mu{}^{2(T-1)} + \frac{1-\rho_\mu{}^{2T}}{1-\rho_\mu{}^2}\right] \\ &= \frac{2\rho_\mu}{(1-\rho_\mu{}^2)^2}\left[1+\rho_\mu{}^{2(T-1)} - T\rho_\mu{}^{2(T-1)} - \frac{(1-\rho_\mu{}^{2T})}{1-\rho_\mu{}^2}\right] \Rightarrow \end{aligned} \tag{B.367}$$

$$\mathrm{tr}\,[\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2]/T = \frac{2\rho_\mu}{(1-\rho_\mu{}^2)^2}\left[\frac{1}{T}\left[1+\rho_\mu{}^{2(T-1)}-\frac{(1-\rho_\mu{}^{2T})}{1-\rho_\mu{}^2}\right]-\rho_\mu{}^{2(T-1)}\right]. \quad \text{(B.368)}$$

By using equations (B.293) and (B.354) we find that

$$\begin{aligned}
(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3 &= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 \\
&= \frac{1}{\rho_\mu}[\boldsymbol{I}_T-(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]\frac{1}{\rho_\mu{}^2}[\boldsymbol{I}_T-2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}+(1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2] \\
&= \frac{1}{\rho_\mu{}^3}[\boldsymbol{I}_T-2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}+(1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2-(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} \\
&\quad +2(1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2-(1-\rho_\mu{}^2)^3\boldsymbol{R}_{\mu\mu}{}^3] \\
&= \frac{1}{\rho_\mu{}^3}[\boldsymbol{I}_T-3(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}+3(1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2-(1-\rho_\mu{}^2)^3\boldsymbol{R}_{\mu\mu}{}^3] \qquad \text{(B.369)}
\end{aligned}$$

and by using equations (B.299), (B.334), and (B.357) we find that

$$\begin{aligned}
\mathrm{tr}\,[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3] &= \frac{1}{\rho_\mu{}^3}\,\mathrm{tr}[\boldsymbol{I}_T-3(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}+3(1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2-(1-\rho_\mu{}^2)^3\boldsymbol{R}_{\mu\mu}{}^3] \\
&= \frac{1}{\rho_\mu{}^3}[\mathrm{tr}\,\boldsymbol{I}_T-3(1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu})+3(1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^2)-(1-\rho_\mu{}^2)^3\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^3)] \\
&= \frac{1}{\rho_\mu{}^3}[T-3(1-\rho_\mu{}^2)\frac{T}{1-\rho_\mu{}^2}+3(1-\rho_\mu{}^2)^2\frac{(1+\rho_\mu{}^2)T}{(1-\rho_\mu{}^2)^3}-(1-\rho_\mu{}^2)^3\frac{(1+\rho_\mu{}^4)T}{(1-\rho_\mu{}^2)^5}+o(1)] \\
&= \frac{1}{\rho_\mu{}^3}\left[T-3T+3T\frac{(1+\rho_\mu{}^2)}{(1-\rho_\mu{}^2)}-T\frac{(1+\rho_\mu{}^4)}{(1-\rho_\mu{}^2)^2}\right]+o(1) \\
&= \frac{T}{\rho_\mu{}^3}\left[\frac{-2(1-\rho_\mu{}^2)^2+3(1-\rho_\mu{}^4)-1-\rho_\mu{}^4}{(1-\rho_\mu{}^2)^2}\right]+o(1) \\
&= \frac{T}{\rho_\mu{}^3}\left[\frac{-2+4\rho_\mu{}^2-2\rho_\mu{}^4+3-3\rho_\mu{}^4-1-\rho_\mu{}^4}{(1-\rho_\mu{}^2)^2}\right]+o(1) \\
&= \frac{T}{\rho_\mu{}^3}\left[\frac{4\rho_\mu{}^2-6\rho_\mu{}^4}{(1-\rho_\mu{}^2)^2}\right]+o(1)=\frac{2T}{\rho_\mu}\frac{(2-3\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2}+o(1). \qquad \text{(B.370)}
\end{aligned}$$

By combining equations (B.311), (B.362), (B.365), (B.368), and (B.370) we find that

$$\begin{aligned}
tr[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3]/T &= \mathrm{tr}\,[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3]/T+3\,\mathrm{tr}[\rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}]/T \\
&\quad +3\,\mathrm{tr}[\rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2]/T+\mathrm{tr}\,[\rho_\mu{}^3(\boldsymbol{\Delta R}_{\mu\mu})^3]/T \\
&= \frac{2}{\rho_\mu}\frac{(2-3\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2}+o(T^{-1}) \\
&\quad +3\left[\frac{2(\rho_\mu{}^{2T}-\rho_\mu{}^2+1)}{T\rho_\mu(1-\rho_\mu{}^2)}\right]+o(1) \\
&\quad +3\frac{2\rho_\mu}{(1-\rho_\mu{}^2)^2}\left[\frac{1}{T}\left[1+\rho_\mu{}^{2(T-1)}-\frac{(1-\rho_\mu{}^{2T})}{1-\rho_\mu{}^2}\right]-\rho_\mu{}^{2(T-1)}\right] \\
&\quad +\frac{2\rho_\mu{}^3}{T(1-\rho_\mu{}^2)^3}(1+3\rho_\mu{}^{2(T-1)}) \qquad \text{(B.371)}
\end{aligned}$$

and omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned} tr[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3]/T &= \frac{2}{\rho_\mu}\frac{(2-3\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2} + o(T^{-1}) \Rightarrow \\ tr[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3] &= \frac{2T}{\rho_\mu}\frac{(2-3\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2} + o(1). \end{aligned} \quad \text{(B.372)}$$

(f) Equation (B.292) implies that

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}\frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{I}_T] = \frac{1}{\rho_\mu}[\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{P}_{\boldsymbol{X}_\mu}] \quad \text{(B.373)}$$

and since $\boldsymbol{P}_{\boldsymbol{X}_\mu}$ is orthogonal projector into the spaces spanned by the columns of the matrix $\boldsymbol{X}_\mu$, we have that

$$\begin{aligned} \boldsymbol{P}_{\boldsymbol{X}_\mu} &= \boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu' \Rightarrow \\ \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}) &= \mathrm{tr}\,[(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{X}_\mu] = \mathrm{tr}\,\boldsymbol{I}_n = n, \end{aligned} \quad \text{(B.374)}$$

from which we find that

$$\begin{aligned} \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}) &= \frac{1}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}^{\mu\mu}) - (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu})] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}) - (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu})] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,[(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T)(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}] - (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu})] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,[(\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1})] - (1-\rho_\mu{}^2)n], \end{aligned} \quad \text{(B.375)}$$

where

$$\boldsymbol{B}_{\mu\mu} = \boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T \text{ and } \boldsymbol{F}_{\mu\mu} = \boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T. \quad \text{(B.376)}$$

Then, equation (B.294) implies that

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}) = \boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}, \quad \text{(B.377)}$$

which implies that

$$\begin{aligned} \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}) &= \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}) + \rho_\mu\,\mathrm{tr}\,[\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{\Delta}] \\ &= \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}) + \rho_\mu\,\mathrm{tr}\,[(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{X}_\mu/T)] = [\textit{see}\ \text{(B.339)}] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,[(\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1})] - (1-\rho_\mu{}^2)n] + O(T^{-1}). \end{aligned} \quad \text{(B.378)}$$

Moreover, equation (B.293) implies that

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}\frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] = \frac{1}{\rho_\mu}[\boldsymbol{P}_{\boldsymbol{X}_\mu} - (1-\rho_\mu{}^2)\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}] \Rightarrow \tag{B.379}$$

$$\begin{aligned}
\operatorname{tr}[\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}] &= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu})] \\
&= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}[(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)]] \\
&= \frac{1}{\rho_\mu}[n - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})],
\end{aligned} \tag{B.380}$$

where

$$\boldsymbol{\Theta}_{\mu\mu} = \boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T. \tag{B.381}$$

Thus,

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} = [\text{see (B.294)}] = \boldsymbol{P}_{\boldsymbol{X}_\mu}[\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}]\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}, \tag{B.382}$$

which implies that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\operatorname{tr}[\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}] \\
&= \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\operatorname{tr}[(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)] \\
&= [\text{see (B.342)}] = \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + O(T^{-1}) \\
&= \frac{1}{\rho_\mu}[n - (1-\rho_\mu{}^2)\operatorname{tr}[(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]] + O(T^{-1}).
\end{aligned} \tag{B.383}$$

Furthermore, equation (B.292) implies that since $\boldsymbol{P}_{\boldsymbol{X}_\mu}$ is idempotent, we find

$$\begin{aligned}
\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} &= \boldsymbol{P}_{\boldsymbol{X}_\mu}\frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{I}_T]\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} \\
&= \frac{1}{\rho_\mu}[\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}],
\end{aligned} \tag{B.384}$$

which implies that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) &= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu})] \\
&= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu})] \\
&\quad -\frac{1}{\rho_\mu}[(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu})] \\
&= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T)(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)] \\
&\quad -\frac{1}{\rho_\mu}[(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)] \implies
\end{aligned}$$

$$\mathrm{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) = \frac{1}{\rho_\mu}[\mathrm{tr}\,\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu} - (1-\rho_\mu{}^2)\,\mathrm{tr}\,\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}]. \tag{B.385}$$

Moreover, by using equation (B.294) we find that

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta})\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} \Rightarrow \tag{B.386}$$

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) &= \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\,\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\,\mathrm{tr}\,[\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}] \\
&= \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\,\mathrm{tr}\,[(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{X}_\mu/T)(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)] \\
&= [\text{see (B.339)}] = \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) + O(T^{-1}) \\
&= \frac{1}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) - (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1}).
\end{aligned} \tag{B.387}$$

(g) By using equation (B.293) we find that

$$\begin{aligned}
\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} &= \boldsymbol{R}_{\mu\mu}\frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] = \frac{1}{\rho_\mu}[\boldsymbol{R}_{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}{}^2] \Rightarrow \\
\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \frac{1}{\rho_\mu}[\mathrm{tr}(\boldsymbol{R}_{\mu\mu}) - (1-\rho_\mu{}^2)\,\mathrm{tr}(\boldsymbol{R}_{\mu\mu}{}^2)] = [\text{see (B.299) and (B.356)}] \\
&= \frac{1}{\rho_\mu}\left[\frac{T}{1-\rho_\mu{}^2} - (1-\rho_\mu{}^2)T\frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^3}\right] + o(1) \\
&= \frac{T}{\rho_\mu}\left[\frac{(1-\rho_\mu{}^2) - (1+\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2}\right] + o(1) \\
&= \frac{T}{\rho_\mu}\left[\frac{-2\rho_\mu{}^2}{(1-\rho_\mu{}^2)^2}\right] + o(1) \\
&= \left[\frac{-2T\rho_\mu}{(1-\rho_\mu{}^2)^2}\right] + o(1).
\end{aligned} \tag{B.388}$$

Then, equation (B.294) implies that

$$\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} = \boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta})\boldsymbol{R}_{\mu\mu} = \boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} \Rightarrow \tag{B.389}$$

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T &= \mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T + \rho_\mu\,\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu})/T \\
&= [\text{see (B.305)}] = \frac{-2\rho_\mu}{(1-\rho_\mu{}^2)^2} + \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^3T} + o(T^{-1})
\end{aligned} \tag{B.390}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T = \frac{-2\rho_\mu}{(1-\rho_\mu{}^2)^2} + o(T^{-1}) \Rightarrow$$

$$\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) = \frac{-2T\rho_\mu}{(1-\rho_\mu{}^2)^2} + o(1) \tag{B.391}$$

By using equation (B.354) we find that

$$\begin{aligned}\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 &= \boldsymbol{R}_{\mu\mu}\frac{1}{\rho_\mu{}^2}[\boldsymbol{I}_T - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2] \\ &= \frac{1}{\rho_\mu{}^2}[\boldsymbol{R}_{\mu\mu} - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}{}^2 + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^3] \Rightarrow\end{aligned} \tag{B.392}$$

$$\begin{aligned}\mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2] &= \frac{1}{\rho_\mu{}^2}[\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}) - 2(1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^2) + (1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^3)] \\ &= [\text{see (B.299), (B.334) and (B.357)}] \\ &= \frac{1}{\rho_\mu{}^2}\left[\frac{T}{1-\rho_\mu{}^2} - 2(1-\rho_\mu{}^2)\frac{(1+\rho_\mu{}^2)T}{(1-\rho_\mu{}^2)^3} + (1-\rho_\mu{}^2)^2\frac{(1+\rho_\mu{}^4)T}{(1-\rho_\mu{}^2)^5} + o(1)\right] \\ &= \frac{1}{\rho_\mu{}^2(1-\rho_\mu{}^2)}\left[T - 2\frac{(1+\rho_\mu{}^2)T}{(1-\rho_\mu{}^2)} + \frac{(1+\rho_\mu{}^4)T}{(1-\rho_\mu{}^2)^2} + o(1)\right] \\ &= \frac{1}{\rho_\mu{}^2(1-\rho_\mu{}^2)^3}[T(1-\rho_\mu{}^2)^2 - 2T + 2\rho_\mu{}^4T + T + \rho_\mu{}^4T] + o(1) \\ &= \frac{1}{\rho_\mu{}^2(1-\rho_\mu{}^2)^3}[T - 2\rho_\mu{}^2T + \rho_\mu{}^4T - 2T + 2\rho_\mu{}^4T + T + \rho_\mu{}^4T] + o(1) \\ &= \frac{T}{\rho_\mu{}^2(1-\rho_\mu{}^2)^3}[4\rho_\mu{}^4 - 2\rho_\mu{}^2] + o(1) \\ &= \frac{2T\rho_\mu{}^2(2\rho_\mu{}^2-1)}{\rho_\mu{}^2(1-\rho_\mu{}^2)^3} + o(1) \\ &= \frac{2T(2\rho_\mu{}^2-1)}{(1-\rho_\mu{}^2)^3} + o(1).\end{aligned} \tag{B.393}$$

Then, equation (B.295) implies that

$$\begin{aligned}\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 &= \boldsymbol{R}_{\mu\mu}[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}] \\ &= \boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\ &\quad +\rho_\mu{}^2\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2 \Rightarrow\end{aligned} \tag{B.394}$$

$$\begin{aligned}\mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T + \rho_\mu\,\mathrm{tr}[\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}]/T \\ &\quad +\rho_\mu\,\mathrm{tr}[\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}]/T + \rho_\mu{}^2\,\mathrm{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2]/T.\end{aligned} \tag{B.395}$$

But, equation (B.293) implies the following results:

$$\begin{aligned} \boldsymbol{R}_{\mu\mu}(\boldsymbol{R_1}^{\mu\mu}\boldsymbol{R}_{\mu\mu})\boldsymbol{\Delta R}_{\mu\mu} &= \boldsymbol{R}_{\mu\mu}\frac{1}{\rho_\mu}[\boldsymbol{I}_T-(1-{\rho_\mu}^2)\boldsymbol{R}_{\mu\mu}]\boldsymbol{\Delta R}_{\mu\mu} \\ &= \frac{1}{\rho_\mu}[\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}-(1-{\rho_\mu}^2)(\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}] \Rightarrow \end{aligned} \tag{B.396}$$

$$\begin{aligned} \rho_\mu\,\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R_1}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T &= \rho_\mu\frac{1}{\rho_\mu}[\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})-(1-{\rho_\mu}^2)\,\mathrm{tr}({\boldsymbol{R}_{\mu\mu}}^2\boldsymbol{\Delta R}_{\mu\mu})]/T \\ &= \mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T-(1-{\rho_\mu}^2)\,\mathrm{tr}({\boldsymbol{R}_{\mu\mu}}^2\boldsymbol{\Delta R}_{\mu\mu})/T \\ &= \mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T-(1-{\rho_\mu}^2)\,\mathrm{tr}({\boldsymbol{\Delta R}_{\mu\mu}}^3)/T. \end{aligned} \tag{B.397}$$

Moreover,

$$\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}(\boldsymbol{R_1}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) = \boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\frac{1}{\rho_\mu}[\boldsymbol{I}_T-(1-{\rho_\mu}^2)\boldsymbol{R}_{\mu\mu}] = \frac{1}{\rho_\mu}[\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}-(1-{\rho_\mu}^2)\boldsymbol{R}_{\mu\mu}{\boldsymbol{\Delta R}_{\mu\mu}}^2] \Rightarrow \tag{B.398}$$

$$\begin{aligned} \rho_\mu\,\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R_1}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T &= \rho_\mu\frac{1}{\rho_\mu}[\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})-(1-\rho_\mu^2)\,\mathrm{tr}(\boldsymbol{R}_{\mu\mu}{\boldsymbol{\Delta R}_{\mu\mu}}^2)]/T \\ &= \mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T-(1-{\rho_\mu}^2)\,\mathrm{tr}({\boldsymbol{\Delta R}_{\mu\mu}}^3)/T. \end{aligned} \tag{B.399}$$

Thus, equations (B.393), (B.397), and (B.399) imply that

$$\begin{aligned} \mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R_1}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T+2\,\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T \\ &\quad -2(1-{\rho_\mu}^2)\,\mathrm{tr}({\boldsymbol{\Delta R}_{\mu\mu}}^3)/T+{\rho_\mu}^2\,\mathrm{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2]/T \\ &= [\text{see (B.305), (B.309), (B.332) and (B.393)}] \\ &= \frac{2(2{\rho_\mu}^2-1)}{(1-{\rho_\mu}^2)^3}+o(T^{-1})+2\frac{2(1-{\rho_\mu}^{2T})}{T(1-{\rho_\mu}^2)^3} \\ &\quad -2(1-{\rho_\mu}^2)\frac{2}{T(1-{\rho_\mu}^2)^4}+o(T^{-2}) \\ &\quad +{\rho_\mu}^2\frac{2}{(1-{\rho_\mu}^2)^3T}\left[T{\rho_\mu}^{2(T-1)}+\frac{1-{\rho_\mu}^{2T}}{1-{\rho_\mu}^2}\right] \end{aligned} \tag{B.400}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned} \mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \frac{2(2{\rho_\mu}^2-1)}{(1-{\rho_\mu}^2)^3}+o(T^{-1}) \Rightarrow \\ \mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2] &= \frac{2(2{\rho_\mu}^2-1)T}{(1-{\rho_\mu}^2)^3}+o(1). \end{aligned} \tag{B.401}$$

(h)

$$\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu} = (\boldsymbol{I}-\boldsymbol{P}_{X_\mu})\boldsymbol{R}_2{}^{\mu\mu}(\boldsymbol{I}-\boldsymbol{P}_{X_\mu})\boldsymbol{R}_{\mu\mu}$$
$$= \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} - \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu} - \boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}, \qquad \text{(B.402)}$$

where $\bar{\boldsymbol{P}}_{X_\mu} = \boldsymbol{I}-\boldsymbol{P}_{X_\mu}$. Since $\boldsymbol{R}_2{}^{\mu\mu}, \boldsymbol{R}_{\mu\mu}, \boldsymbol{P}_{X_\mu}$ are symmetric matrices the following results holds:

$$\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) = \operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}) = \operatorname{tr}[(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu})'] = \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}), \qquad \text{(B.403)}$$

which implies that

$$\operatorname{tr}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu} = \operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) - 2\operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})$$
$$+ \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) = [\text{see (B.349), (B.380) and (B.387)}]$$
$$= \frac{2\rho_\mu}{1-\rho_\mu{}^2} - \frac{2}{\rho_\mu}[n-(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1})$$
$$+\frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1})$$
$$= \frac{2\rho_\mu}{1-\rho_\mu{}^2} - \frac{2n}{\rho_\mu} + \frac{(1-\rho_\mu{}^2)}{\rho_\mu}\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \frac{1}{\rho_\mu}\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + O(T^{-1})$$
$$= \frac{1}{\rho_\mu}\left[\frac{2(\rho_\mu{}^2 - n(1-\rho_\mu{}^2))}{1-\rho_\mu{}^2} + (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\right]$$
$$+O(T^{-1}). \qquad \text{(B.404)}$$

By using equation (B.280) the following results hold:

(1)

$$\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{V} = [\boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu']\boldsymbol{R}^{\mu\mu}[\boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu']$$
$$= \boldsymbol{R}_{\mu\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{R}_{\mu\mu} - \boldsymbol{R}_{\mu\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu'$$
$$-\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{R}^{\mu\mu} + \boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu'$$
$$= \boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu' = \boldsymbol{V}. \qquad \text{(B.405)}$$

(2)

$$\boldsymbol{V}\bar{\boldsymbol{P}}_{X_\mu} = [\boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu'][\boldsymbol{I}-\boldsymbol{P}_{X_\mu}]$$
$$= \boldsymbol{R}_{\mu\mu} - \boldsymbol{R}_{\mu\mu}\boldsymbol{P}_{X_\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu'$$
$$+\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X_\mu'\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}X_\mu'$$
$$= \boldsymbol{R}_{\mu\mu}[\boldsymbol{I}-\boldsymbol{P}_{X_\mu}] = \boldsymbol{R}_{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}. \qquad \text{(B.406)}$$

Thus,

$$\begin{aligned}
\text{tr}\,(\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}) &= \text{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\bar{\boldsymbol{P}}_{X_\mu}) = [\text{see (B.406)}] \\
&= \text{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}) = \text{tr}\,(\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \text{tr}\,[(\boldsymbol{I}-\boldsymbol{P}_{X_\mu})\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}] \\
&= \text{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \text{tr}\,(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \text{because the matrices } \boldsymbol{P}_{X_\mu},\ \boldsymbol{R}_{\mu\mu} \text{ and } \boldsymbol{R}_2{}^{\mu\mu} \text{ are symmetric are equal to} \\
&= \text{tr}\,(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}) - \text{tr}\,(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \text{tr}\,[(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu})'] - \text{tr}\,(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \text{tr}\,(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) - \text{tr}\,(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) = [\text{see (B.383) and (B.387)}] \\
&= \frac{1}{\rho_\mu}[n-(1-\rho_\mu{}^2)\,\text{tr}\,[(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]] \\
&\quad -\frac{1}{\rho_\mu}[\text{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) - (1-\rho_\mu{}^2)\,\text{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1}) \\
&= \frac{1}{\rho_\mu}[n - \text{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1}). \qquad \text{(B.407)}
\end{aligned}$$

Moreover, equations (B.280), (B.292), (B.294) (B.339), (B.387), and (B.406) imply that

$$
\begin{aligned}
& \operatorname{tr}(\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) = \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{V}) \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}) \\
= \; & \operatorname{tr}[\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}[\boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu]] \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu) \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \operatorname{tr}(\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}X'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu) \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \operatorname{tr}[(\boldsymbol{X}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{X}_\mu)(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}] \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \operatorname{tr}[(\boldsymbol{X}'_\mu(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta})\boldsymbol{X}_\mu)(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}] \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \operatorname{tr}[(\boldsymbol{X}'_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{X}_\mu)(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}] \\
& -\rho_\mu \operatorname{tr}[(\boldsymbol{X}'_\mu\boldsymbol{\Delta}\boldsymbol{X}_\mu)(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}] \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) \\
& - \operatorname{tr}[\boldsymbol{X}'_\mu\frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{I}_T]\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}] - O(1) \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \frac{1}{\rho_\mu}\operatorname{tr}[(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}] \\
& + \frac{(1-\rho_\mu{}^2)}{\rho_\mu}\operatorname{tr}\boldsymbol{X}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}] + O(1) \\
= \; & \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{I}_n) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T)^{-1}] + O(1) \\
= \; & \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] \\
& - \frac{1}{\rho_\mu}[n - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1})] + O(T^{-1}) \\
= \; & \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) - n] + \frac{(1-\rho_\mu{}^2)}{\rho_\mu}[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}) - \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] \\
& + O(T^{-1}). \qquad \text{(B.408)}
\end{aligned}
$$

Lemma B.11. By using Magnus and Neudecker, 1979 we can prove the following results:

(i) By using equation (B.349) we have

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu) &= \sigma_{\mu\mu}\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \frac{2\rho_\mu\sigma_{\mu\mu}}{1-\rho_\mu{}^2}. \qquad \text{(B.409)}
\end{aligned}
$$

(ii) By using equations (B.349) and (B.360), and omitting terms that tend to zero as $T \to \infty$, we find that

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}_\mu' \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu \boldsymbol{u}_\mu' \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu)/T &= [\text{see (UR.2)}] \\
&= \sigma_{\mu\mu}{}^2[\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{\Omega}_{\mu\mu})\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{\Omega}_{\mu\mu}) + 2\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{\Omega}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{\Omega}_{\mu\mu})]/T \\
&= \sigma_{\mu\mu}{}^2[[\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})]^2 + 2[\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]]/T \\
&= \left(\frac{2\rho_\mu\sigma_{\mu\mu}}{1-\rho_\mu{}^2}\right)^2/T + 2\left[\frac{2T\sigma_{\mu\mu}{}^2}{1-\rho_\mu{}^2} + O(1)\right]/T \\
&= \frac{4\sigma_{\mu\mu}{}^2}{1-\rho_\mu{}^2} + O(T^{-1}) \Rightarrow
\end{aligned}
\tag{B.410}
$$

$$
\mathrm{E}(\boldsymbol{u}_\mu' \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu \boldsymbol{u}_\mu' \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu) = \frac{4T\sigma_{\mu\mu}{}^2}{1-\rho_\mu{}^2} + O(1), \tag{B.411}
$$

because equation (B.349) implies that

$$
\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T = \frac{2\rho_\mu}{1-\rho_\mu{}^2} = O(T^{-1}) \Rightarrow \operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) = O(1). \tag{B.412}
$$

(iii) Equations (B.299),(B.349), and(B.391) imply that

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}_\mu' \boldsymbol{u}_\mu \boldsymbol{u}_\mu' \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu) &= \sigma_{\mu\mu}{}^2 \operatorname{tr}(\boldsymbol{I}\boldsymbol{R}_{\mu\mu})\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + 2\sigma_{\mu\mu}{}^2\operatorname{tr}(\boldsymbol{I}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \sigma_{\mu\mu}{}^2\left[\frac{T}{1-\rho_\mu{}^2}\cdot\frac{2\rho_\mu}{1-\rho_\mu{}^2} + 2\left(\frac{-2T\rho_\mu}{(1-\rho_\mu{}^2)^2} + O(1)\right)\right] \\
&= \sigma_{\mu\mu}{}^2\left[\frac{2T\rho_\mu}{(1-\rho_\mu{}^2)^2} - \frac{4T\rho_\mu}{(1-\rho_\mu{}^2)^2}\right] + O(1) \\
&= -\frac{2T\rho_\mu\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2} + O(1).
\end{aligned}
\tag{B.413}
$$

(iv) Equation (B.404) implies that

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}_\mu' \bar{\boldsymbol{P}}_{X_\mu} \boldsymbol{R}_2{}^{\mu\mu} \bar{\boldsymbol{P}}_{X_\mu} \boldsymbol{u}_\mu) &= \sigma_{\mu\mu}\operatorname{tr}(\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \sigma_{\mu\mu}\frac{1}{\rho_\mu}\left[\frac{2(\rho_\mu{}^2 - n(1-\rho_\mu{}^2))}{1-\rho_\mu{}^2} + (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\right] \\
&\quad + O(T^{-1}).
\end{aligned}
\tag{B.414}
$$

(v) Equation (B.407) implies that

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}_\mu' \bar{\boldsymbol{P}}_{X_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{X_\mu} \boldsymbol{V}\boldsymbol{R}^{\mu\mu} \boldsymbol{u}_\mu) &= \sigma_{\mu\mu}\operatorname{tr}(\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \sigma_{\mu\mu}\operatorname{tr}(\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}) \\
&= \frac{\sigma_{\mu\mu}}{\rho_\mu}[n - \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1}).
\end{aligned}
\tag{B.415}
$$

(vi)

$$\begin{aligned}
\mathrm{E}(\boldsymbol{u}'_{\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{X_{\mu}}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_{\mu}}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_{\mu}) &= \sigma_{\mu\mu}\,\mathrm{tr}\,(\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{X_{\mu}}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_{\mu}}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \sigma_{\mu\mu}\,\mathrm{tr}\,(\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{X_{\mu}}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_{\mu}}\boldsymbol{V}) = [\text{see (B.408)}] \\
&= \frac{\sigma_{\mu\mu}}{\rho_{\mu}}[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) - n] \\
&\quad + \frac{\sigma_{\mu\mu}(1-\rho_{\mu}{}^2)}{\rho_{\mu}}[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}) - \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] \\
&\quad + O(T^{-1}).
\end{aligned} \tag{B.416}$$

(vii)

$$\mathrm{E}(\boldsymbol{u}'_{\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_{\mu}\boldsymbol{u}'_{\mu'}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_{\mu'}) = \sigma_{\mu\mu}\sigma_{\mu'\mu'}[\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})\,\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) + 2\,\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'})]. \tag{B.417}$$

Since (B.294) implies that

$$\begin{aligned}
(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})(\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) &= [\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_{\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}][\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'} + \rho_{\mu'}\boldsymbol{\Delta}\boldsymbol{R}_{\mu'\mu'}] \\
&= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) + \rho_{\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'} + \rho_{\mu'}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu'\mu'} \\
&\quad + \rho_{\mu}\rho_{\mu'}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu'\mu'},
\end{aligned} \tag{B.418}$$

it follows that

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) &= \mathrm{tr}\,(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) + \rho_{\mu}\,\mathrm{tr}(\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) \\
&\quad + \rho_{\mu'}\,\mathrm{tr}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu'\mu'}) + \rho_{\mu}\rho_{\mu'}\,\mathrm{tr}(\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu'\mu'}).
\end{aligned} \tag{B.419}$$

Moreover, (B.293) implies that

$$\begin{aligned}
(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})(\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) &= \frac{1}{\rho_{\mu}}[\boldsymbol{I}_T - (1-\rho_{\mu}{}^2)\boldsymbol{R}_{\mu\mu}]\frac{1}{\rho_{\mu'}}[\boldsymbol{I}_T - (1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'}] \\
&= \frac{1}{\rho_{\mu}\rho_{\mu'}}[\boldsymbol{I}_T - (1-\rho_{\mu}{}^2)\boldsymbol{R}_{\mu\mu} - (1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'} \\
&\quad + (1-\rho_{\mu}{}^2)(1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'}].
\end{aligned} \tag{B.420}$$

vii.a Since the (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}$ is $\frac{1}{(1-\rho_{\mu}{}^2)}\rho_{\mu}{}^{|i-j|}$, the (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'}$ is

$$e_{ij} = \sum_{k=1}^{T}\frac{1}{(1-\rho_{\mu}{}^2)(1-\rho_{\mu'}{}^2)}\rho_{\mu}{}^{|i-k|}\rho_{\mu'}{}^{|k-j|}. \tag{B.421}$$

Therefore, the i-diagonal element of the matrix $\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'}$ is

$$e_{ii} = \frac{1}{(1-\rho_{\mu}{}^2)(1-\rho_{\mu'}{}^2)}\sum_{k=1}^{T}\rho_{\mu}{}^{|i-k|}\rho_{\mu'}{}^{|k-i|}\,. \tag{B.422}$$

Define the index $j = k - i$ ($j = 1-i, \dots, T-i$) and set $r = \rho_\mu \rho_{\mu'}$. Then,

$$\begin{aligned}
\sum_{k=1}^{T} \rho_\mu{}^{|i-k|} \rho_{\mu'}{}^{|k-i|} &= \sum_{j=1-i}^{T-i} \rho_\mu{}^{|j|} \rho_{\mu'}{}^{|j|} = \sum_{j=1-i}^{T-i} (\rho_\mu \rho_{\mu'})^{|j|} \\
&= \sum_{j=1-i}^{T-i} r^{|j|} = \sum_{j=1-i}^{-1} r^{|j|} + \sum_{j=0}^{T-i} r^j = \sum_{j+i=1}^{i-1} r^{|j+i|} + \sum_{j=0}^{T-i} r^j \\
&= \sum_{k=1}^{i-1} r^k + \sum_{j=0}^{T-i} r^j = \frac{r(1-r^{(i-1)})}{1-r} + \frac{1-r^{T-i+1}}{1-r} = \frac{r - r^i + 1 - r^{T-i+1}}{1-r} \\
&= \frac{(1+r) - (r^i + r^{T-i+1})}{1-r} = [\text{setting } j = T-i+1 \ (j = 1, \dots, T)] \\
&= \frac{(1+r) - (r^i + r^j)}{1-r} = \frac{1 + r - 2r^i}{1-r}.
\end{aligned} \tag{B.423}$$

Thus, equations (B.422) and (B.423) imply that

$$e_{ii} = \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)} \frac{1+r-2r^i}{1-r} \Rightarrow \tag{B.424}$$

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'})/T &= \sum_{i=1}^{T} e_{ii}/T = \frac{1}{T(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)} \sum_{i=1}^{T} \left[\frac{1+r}{1-r} - \frac{2r^i}{1-r}\right] \\
&= \frac{1}{T(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)} \left[\frac{T(1+r)}{1-r} - \frac{2}{1-r}\sum_{i=1}^{T} r^i\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)} \left[\frac{1+r}{1-r} - \frac{2r}{T(1-r)}\frac{r(1-r^T)}{1-r}\right],
\end{aligned} \tag{B.425}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'})/T = \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)} \frac{1+r}{1-r} + o(T^{-1}) \Rightarrow \tag{B.426}$$

$$\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'}) = \frac{T(1+\rho_\mu\rho_{\mu'})}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)(1-\rho_\mu\rho_{\mu'})} + o(1). \tag{B.427}$$

By combining equations (B.299), (B.420), and (B.427) we find that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) &= \frac{1}{\rho_\mu\rho_{\mu'}}[\operatorname{tr}(\boldsymbol{I}_T) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu\mu})] \\
&\quad - \frac{1}{\rho_\mu\rho_{\mu'}}[(1-\rho_{\mu'}{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu'\mu'}) - (1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'})] \\
&= \frac{1}{\rho_\mu\rho_{\mu'}}\left[T - (1-\rho_\mu{}^2)\frac{T}{(1-\rho_\mu{}^2)} - (1-\rho_{\mu'}{}^2)\frac{T}{(1-\rho_{\mu'}{}^2)}\right] \\
&\quad + \frac{1}{\rho_\mu\rho_{\mu'}}\left[\frac{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)T(1+\rho_\mu\rho_{\mu'})}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)(1-\rho_\mu\rho_{\mu'})} + o(1)\right] \\
&= \frac{1}{\rho_\mu\rho_{\mu'}}\left[-T + \frac{T(1+\rho_\mu\rho_{\mu'})}{(1-\rho_\mu\rho_{\mu'})}\right] + o(1) \implies
\end{aligned}$$

$$\begin{aligned}\mathrm{tr}\,(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) &= \frac{1}{\rho_\mu\rho_{\mu'}}\left[\frac{-T+T\rho_\mu\rho_{\mu'}+T+T\rho_\mu\rho_{\mu'}}{(1-\rho_\mu\rho_{\mu'})}\right]+o(1)\\ &= \frac{2T}{1-\rho_\mu\rho_{\mu'}}+o(1). \end{aligned} \tag{B.428}$$

vii.b Let δ_{ij} be the (i,j)-th element of the matrix $\boldsymbol{\Delta}$. Then, $\delta_{11}=\delta_{TT}=1$ and $\delta_{ij}=0\ \forall\, i,j\neq 1$ and $i,j\neq T$. Moreover, let $\frac{1}{1-\rho_\mu{}^2}\rho_\mu{}^{|i-j|}$ be the (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}$. Then, the (i,j)-th element of the matrix $\boldsymbol{\Delta R}_{\mu\mu}$ is (see (B.302))

$$\delta_{ij}{}^* = \delta_{ii}\frac{1}{1-\rho_\mu{}^2}\rho_\mu{}^{|i-j|} \tag{B.429}$$

Since equation (B.293) implies that

$$\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'} = \frac{1}{\rho_{\mu'}}[\boldsymbol{I}_T-(1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'}], \tag{B.430}$$

we find that

$$\begin{aligned}\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu} &= \frac{1}{\rho_{\mu'}}[\boldsymbol{I}_T-(1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'}]\boldsymbol{\Delta R}_{\mu\mu}\\ &= \frac{1}{\rho_{\mu'}}[\boldsymbol{\Delta R}_{\mu\mu}-(1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}]. \end{aligned} \tag{B.431}$$

The (i,j)-th element of the matrix $\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}$ is

$$\begin{aligned}\delta_{ij}{}^{**} &= \sum_{k=1}^{T}\frac{1}{(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|i-k|}\delta_{kj}{}^* = \sum_{k=1}^{T}\frac{1}{(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|i-k|}\delta_{kk}\frac{1}{(1-\rho_\mu{}^2)}\rho_\mu{}^{|k-j|}\\ &= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|i-1|}\rho_\mu{}^{|1-j|}+\frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|i-T|}\rho_\mu{}^{|T-j|}\\ &= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[\rho_{\mu'}{}^{|i-1|}\rho_\mu{}^{|1-j|}+\rho_{\mu'}{}^{|i-T|}\rho_\mu{}^{|T-j|}] \end{aligned} \tag{B.432}$$

and the i-diagonal element of the matrix $\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}$ is

$$\begin{aligned}\delta_{ii}{}^{**} &= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[\rho_{\mu'}{}^{|i-1|}\rho_\mu{}^{|1-i|}+\rho_{\mu'}{}^{|i-T|}\rho_\mu{}^{|T-i|}]\\ &= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[(\rho_\mu\rho_{\mu'})^{|i-1|}+(\rho_\mu\rho_{\mu'})^{|T-i|}]. \end{aligned} \tag{B.433}$$

Therefore, setting $j = T - i + 1\,(j = 1, \ldots, T)$ and $r = \rho_\mu \rho_{\mu'}$), and setting $l = i - 1\,(l = 0, \ldots, T-1)$, equation (B.433) implies that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}) &= \sum_{i=1}^{T} \delta_{ii}^{**} \\
&= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\left[\sum_{i=1}^{T}(\rho_\mu\rho_{\mu'})^{i-1} + \sum_{i=1}^{T}(\rho_\mu\rho_{\mu'})^{T-i}\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\left[\sum_{i=1}^{T} r^{i-1} + \sum_{j=1}^{T} r^{j-1}\right] = \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\sum_{i=1}^{T} r^{i-1} \\
&= \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\sum_{l=0}^{T-1} r^{l} = \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\frac{1-r^T}{1-r} \\
&= \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\frac{1-(\rho_\mu\rho_{\mu'})^T}{1-\rho_\mu\rho_{\mu'}} = \frac{2[1-(\rho_\mu\rho_{\mu'})^T]}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)(1-\rho_\mu\rho_{\mu'})}. \qquad \text{(B.434)}
\end{aligned}$$

Therefore, equations (B.303), (B.431), and (B.434) imply that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}) &= \frac{1}{\rho_{\mu'}}[\operatorname{tr}(\boldsymbol{\Delta R}_{\mu\mu}) - (1-\rho_{\mu'}{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu})] \\
&= \frac{1}{\rho_{\mu'}}\left[\frac{2}{1-\rho_\mu{}^2} - (1-\rho_{\mu'}{}^2)\frac{2[1-(\rho_\mu\rho_{\mu'})^T]}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)(1-\rho_\mu\rho_{\mu'})}\right] \\
&= \frac{1}{\rho_{\mu'}}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2[1-(\rho_\mu\rho_{\mu'})^T]}{(1-\rho_\mu{}^2)(1-\rho_\mu\rho_{\mu'})}\right]. \qquad \text{(B.435)}
\end{aligned}$$

vii.c By using equation (B.302) we find that the (i,j)-th element of the matrix $\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu'\mu'}$ is

$$\begin{aligned}
\delta_{ij}{}^{\circ\circ} &= \sum_{k=1}^{T}\delta_{ik}{}^{*}\delta_{kj}{}^{*} = \sum_{\kappa=1}^{T}\delta_{ii}\frac{1}{(1-\rho_\mu{}^2)}\rho_\mu{}^{|i-k|}\delta_{kk}\frac{1}{(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|k-j|} \\
&= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[\rho_\mu{}^{|i-1|}\rho_{\mu'}{}^{|1-j|} + \rho_\mu{}^{|i-T|}\rho_{\mu'}{}^{|T-j|}], \qquad \text{(B.436)}
\end{aligned}$$

which implies that the i-diagonal element of the matrix $\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu'\mu'}$ is

$$\begin{aligned}
\delta_{ii}{}^{\circ\circ} &= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[\rho_\mu{}^{|i-1|}\rho_{\mu'}{}^{|1-i|} + \rho_\mu{}^{|i-T|}\rho_{\mu'}{}^{|T-i|}] \\
&= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[(\rho_\mu\rho_{\mu'})^{|i-1|} + (\rho_\mu\rho_{\mu'})^{|i-T|}]. \qquad \text{(B.437)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\operatorname{tr}(\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu'\mu'}) = \sum_{i=1}^{T}\delta_{ii}{}^{\circ\circ} = \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\sum_{i=1}^{T}\delta_{ii}[(\rho_\mu\rho_{\mu'})^{|i-1|} + (\rho_\mu\rho_{\mu'})^{|i-T|}] \\
&= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\sum_{i=1}^{T}[\delta_{11}[(\rho_\mu\rho_{\mu'})^{|1-1|} + (\rho_\mu\rho_{\mu'})^{|T-1|}] + \delta_{TT}[(\rho_\mu\rho_{\mu'})^{|T-1|} + (\rho_\mu\rho_{\mu'})^{|T-T|}]] \\
&= \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[1 + (\rho_\mu\rho_{\mu'})^{T-1}]. \qquad \text{(B.438)}
\end{aligned}$$

Thus, equations (B.419), (B.428), (B.435), and (B.438) imply that

$$\begin{aligned} \mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'})/T &= \frac{2T/T}{(1-\rho_\mu\rho_{\mu'})} + o(T^{-1}) \\ &+\rho_\mu \frac{1}{\rho_{\mu'}}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2[1-(\rho_\mu\rho_{\mu'})^T]}{(1-\rho_\mu{}^2)(1-\rho_\mu\rho_{\mu'})}\right]/T \\ &+\rho_{\mu'} \frac{1}{\rho_\mu}\left[\frac{2}{1-\rho_{\mu'}{}^2} - \frac{2[1-(\rho_{\mu'}\rho_\mu)^T]}{(1-\rho_{\mu'}{}^2)(1-\rho_{\mu'}\rho_\mu)}\right]/T \\ &+\frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[1+(\rho_\mu\rho_{\mu'})^{T-1}]/T, \end{aligned} \tag{B.439}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned} \mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'})/T &= \frac{2}{1-\rho_\mu\rho_{\mu'}} + o(T^{-1}) \Rightarrow \\ \mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) &= \frac{2T}{1-\rho_\mu\rho_{\mu'}} + o(1). \end{aligned} \tag{B.440}$$

Therefore, equations (B.417), (B.349), and (B.440) imply that

$$\begin{aligned} \mathrm{E}(\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu\boldsymbol{u}_{\mu'}'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_{\mu'})/T &= \frac{2\rho_\mu\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}\frac{2\rho_{\mu'}\sigma_{\mu'\mu'}}{(1-\rho_{\mu'}{}^2)}/T \\ &+2\left[\frac{2\sigma_{\mu\mu}\sigma_{\mu'\mu'}}{1-\rho_\mu\rho_{\mu'}} + o(T^{-1})\right] \end{aligned} \tag{B.441}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\mathrm{E}(\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu\boldsymbol{u}_{\mu'}'\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{u}_{\mu'}) = \frac{4T\sigma_{\mu\mu}\sigma_{\mu'\mu'}}{1-\rho_\mu\rho_{\mu'}} + o(1). \tag{B.442}$$

Lemma B.12. The following results hold:

Let ε_{ti} be the (t,i)-th element of the matrix $\boldsymbol{E}$. Then, the (i,j)-th element of the matrix $\boldsymbol{E}'\boldsymbol{E}/T$ is

$$e_{ij} = \sum_{t=1}^{T} \varepsilon_{it}\varepsilon_{tj}/T, \tag{B.443}$$

Since σ_{ij} is the (i,j)-th element of the matrix $\boldsymbol{\Sigma}$, by using equations (B.160) and (B.443) we find that the (i,j)-th element of the matrix $\boldsymbol{\Sigma}_1$ is

$$\sigma_{ij}{}^{(1)} = \sqrt{T}\left(\sum_{t=1}^{T} \varepsilon_{it}\varepsilon_{tj}/T - \sigma_{ij}\right) = \sqrt{T}(e_{ij} - \sigma_{ij}). \tag{B.444}$$

Moreover, since σ^{ij} is the (i,j)-th element of the matrix $\boldsymbol{\Sigma}^{-1}$, by using equation (B.163) we find that the (i,j)-th element of the matrix $\boldsymbol{S}_1$ is

$$\begin{aligned} s_{ij}{}^{(1)} &= \sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma_{kl}{}^{(1)}\sigma^{lj} = \sqrt{T}\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}(e_{kl}-\sigma_{kl})\sigma^{lj} \\ &= \sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}e_{kl}\sigma^{lj} - \sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma_{kl}\sigma^{lj}\right]. \end{aligned} \tag{B.445}$$

Since $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}^{-1}$, the (i,j)-th elements of the matrices $\boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}$ are identical, i.e.,

$$\sigma^{ij} = \sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma_{kl}\sigma^{lj}. \tag{B.446}$$

Thus, equations (B.445) and (B.446) imply that

$$s_{ij}{}^{(1)} = \sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}e_{kl}\sigma^{lj} - \sigma^{ij}\right]. \tag{B.447}$$

Since equation (B.443) implies that

$$e_{ij} = \boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_j/T \tag{B.448}$$

where $\boldsymbol{e}_i$ is the i-th column of the matrix $\boldsymbol{E}$ we find that

$$s_{ij}{}^{(1)} = \sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l/T)\sigma^{lj} - \sigma^{ij}\right]. \tag{B.449}$$

Therefore the (i,j)-th element of $(1\times M^2)$ vector $[\text{vec}(\boldsymbol{S}_1)]'$ is

$$s_{(ij)}^{(1)} = s_{ij}{}^{(1)} = \sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l/T)\sigma^{lj} - \sigma^{ij}\right]. \tag{B.450}$$

Equation (3.22) implies that

$$\boldsymbol{u}_\mu' = \boldsymbol{\varepsilon}_\mu'\boldsymbol{P}_\mu' \text{ and } \boldsymbol{u}_\mu = \boldsymbol{P}_\mu\boldsymbol{\varepsilon}_\mu \Rightarrow \tag{B.451}$$

$$\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu = \boldsymbol{\varepsilon}_\mu'\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu\boldsymbol{\varepsilon}_\mu. \tag{B.452}$$

By using Lemma UR.2 and since (3.17b) implies that (see Magnus and Neudecker, 1979)

$$\text{E}(\boldsymbol{\varepsilon}_\mu'\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu\boldsymbol{\varepsilon}_\mu) = \sigma_{\mu\mu}\,\text{tr}\,(\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu\boldsymbol{I}_T) = \sigma_{\mu\mu}\,\text{tr}\,(\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu), \tag{B.453}$$

we find that

$$\begin{aligned}\mathrm{E}(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_\mu'\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu\boldsymbol{\varepsilon}_\mu) &= \mathrm{tr}\,(\sigma_{kl}\boldsymbol{I}_T)\,\mathrm{tr}\,(\sigma_{\mu\mu}\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu) + 2\,\mathrm{tr}\,(\sigma_{kl}\sigma_{\mu\mu}\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu)\\ &= \sigma_{kl}\sigma_{\mu\mu}T\,\mathrm{tr}\,(\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu) + 2\sigma_{kl}\sigma_{\mu\mu}\,\mathrm{tr}\,(\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu)\\ &= \sigma_{kl}\sigma_{\mu\mu}(T+2)\,\mathrm{tr}\,(\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu), \end{aligned} \tag{B.454}$$

and since (B.1) implies that

$$\boldsymbol{P}_\mu\boldsymbol{P}_\mu' = \boldsymbol{R}_{\mu\mu}, \tag{B.455}$$

by combining equations (B.349) (B.454), and (B.455) we find that

$$\begin{aligned}\mathrm{E}(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu) &= \mathrm{E}(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_\mu'\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu\boldsymbol{\varepsilon}_\mu) = \sigma_{kl}\sigma_{\mu\mu}(T+2)\,\mathrm{tr}\,(\boldsymbol{P}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu)\\ &= \sigma_{kl}\sigma_{\mu\mu}(T+2)\,\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_\mu\boldsymbol{P}_\mu') = \sigma_{kl}\sigma_{\mu\mu}(T+2)\,\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})\\ &= \sigma_{kl}\sigma_{\mu\mu}(T+2)\frac{2\rho_\mu}{1-{\rho_\mu}^2}. \end{aligned} \tag{B.456}$$

Therefore,

$$\begin{aligned}\mathrm{E}[(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l/T)\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu] &= \mathrm{E}(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)/T\\ &= \sigma_{kl}\sigma_{\mu\mu}\frac{2\rho_\mu}{1-{\rho_\mu}^2}(T+2)/T\\ &= \sigma_{kl}\sigma_{\mu\mu}\frac{2\rho_\mu}{1-{\rho_\mu}^2} + \sigma_{kl}\sigma_{\mu\mu}\frac{4\rho_\mu}{1-{\rho_\mu}^2}/T \end{aligned} \tag{B.457}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\mathrm{E}[(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l/T)\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu] = \sigma_{kl}\sigma_{\mu\mu}\frac{2\rho_\mu}{1-{\rho_\mu}^2} + O(T^{-1}). \tag{B.458}$$

Equation (B.450) implies that

$$s_{(ij)}^{(1)}\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu = \sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma^{lj}[(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l/T)\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu] - \sigma^{ij}\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu\right]. \tag{B.459}$$

Equations (B.409), (B.446), and (B.459) imply that

$$\begin{aligned}\mathrm{E}(s_{(ij)}^{(1)}\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu) &= \sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma^{lj}\,\mathrm{E}[(\boldsymbol{\varepsilon}_k'\boldsymbol{\varepsilon}_l/T)\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu] - \sigma^{ij}\,\mathrm{E}(\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)\right]\\ &= \sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma^{lj}\sigma_{kl}\sigma_{\mu\mu}\frac{2\rho_\mu}{1-{\rho_\mu}^2} - \sigma^{ij}\sigma_{\mu\mu}\frac{2\rho_\mu}{1-{\rho_\mu}^2} + O(T^{-1})\right]\\ &= \sqrt{T}\sigma_{\mu\mu}\frac{2\rho_\mu}{1-{\rho_\mu}^2}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma_{kl}\sigma^{lj} - \sigma^{ij}\right] + O(T^{-1/2})\\ &= O(T^{-1/2}) \Rightarrow \end{aligned} \tag{B.460}$$

$$\lim_{T\to\infty} \mathrm{E}(s_{(ij)}^{(1)}\boldsymbol{u}'_{\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_{\mu}) = 0. \tag{B.461}$$

Proof of Theorem 4. Define the $((1 + M + M^2) \times 1)$ vector

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_0 \\ \boldsymbol{\delta}_\rho \\ \boldsymbol{\delta}_\varsigma \end{bmatrix} \tag{B.462}$$

where for $\sigma = 1$

$$\delta_0 = \frac{\hat{\sigma}^2 - \sigma^2}{\tau\sigma^2} = \frac{\hat{\sigma}^2 - 1}{\tau} \tag{B.463}$$

is a scalar,

$$\boldsymbol{\delta}_\rho = [(\delta_{\rho_\mu})_{\mu=1,\ldots,M}] \tag{B.464}$$

is a $M \times 1$ vector with element

$$\delta_{\rho_\mu} = \frac{\hat{\rho}_\mu - \rho_\mu}{\tau} \tag{B.465}$$

and

$$\boldsymbol{\delta}_\varsigma = [(\delta_{\varsigma_{\mu\mu'}})_{\mu\mu'=1,\ldots,M^2}] \tag{B.466}$$

is a $(M^2 \times 1)$ vector with elements

$$\delta_{\varsigma_{\mu\mu'}} = \frac{\hat{\varsigma}_{\mu\mu'} - \varsigma_{\mu\mu'}}{\tau}. \tag{B.467}$$

Moreover, $\boldsymbol{\delta}$ admits a stochastic expansion of the form

$$\boldsymbol{\delta} = \boldsymbol{d}_1 + \tau\boldsymbol{d}_2 + \omega(\tau^2) \tag{B.468}$$

which implies that $\delta_0, \boldsymbol{\delta}_\rho$ and $\boldsymbol{\delta}_\varsigma$ admit the following stochastic expansions:

$$\delta_0 = \sigma_0 + \tau\sigma_1 + \omega(\tau^2) \tag{B.469}$$

$$\boldsymbol{\delta}_\rho = \boldsymbol{d}_{1\rho} + \tau\boldsymbol{d}_{2\rho} + \omega(\tau^2) \tag{B.470}$$

$$\boldsymbol{\delta}_\varsigma = \boldsymbol{d}_{1\varsigma} + \tau\boldsymbol{d}_{2\varsigma} + \omega(\tau^2), \tag{B.471}$$

where σ_0 and σ_1 are scalars, $\boldsymbol{d}_{1\rho}$ and $\boldsymbol{d}_{2\rho}$ are $(M \times 1)$ vectors and $\boldsymbol{d}_{1\varsigma}$ and $\boldsymbol{d}_{2\varsigma}$ are $(M^2 \times 1)$ vectors.

Define the scalars λ_0 and κ_0 the $(M \times 1)$ vectors $\boldsymbol{\lambda}_\rho$ and $\boldsymbol{\kappa}_\rho$, the $(M^2 \times 1)$ vectors $\boldsymbol{\lambda}_\varsigma$ and $\boldsymbol{\kappa}_\varsigma$, the $(M \times M)$ matrix $\boldsymbol{\Lambda}_\rho$, the $(M^2 \times M^2)$ matrix $\boldsymbol{\Lambda}_\varsigma$, the $(M^2 \times M)$ matrix $\boldsymbol{\Lambda}_{\varsigma\rho}$ and the $(M \times M^2)$ matrix $\boldsymbol{\Lambda}_{\rho\varsigma}$ by the following relations:

$$\begin{bmatrix} \lambda_0 & \boldsymbol{\lambda}'_\rho & \boldsymbol{\lambda}'_\varsigma \\ \boldsymbol{\lambda}_\rho & \boldsymbol{\Lambda}_\rho & \boldsymbol{\Lambda}_{\rho\varsigma} \\ \boldsymbol{\lambda}_\varsigma & \boldsymbol{\Lambda}_{\varsigma\rho} & \boldsymbol{\Lambda}_\varsigma \end{bmatrix} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_1\boldsymbol{d}'_1); \begin{bmatrix} \kappa_0 \\ \boldsymbol{\kappa}_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix} = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_1 + \boldsymbol{d}_2) \tag{B.472}$$

By combining equations (B.468), (B.469), (B.470), (B.471), and (B.472) we find that

$$\begin{bmatrix} \lambda_0 & \boldsymbol{\lambda}'_\rho & \boldsymbol{\lambda}'_\varsigma \\ \boldsymbol{\lambda}_\rho & \boldsymbol{\Lambda}_\rho & \boldsymbol{\Lambda}_{\rho\varsigma} \\ \boldsymbol{\lambda}_\varsigma & \boldsymbol{\Lambda}_{\varsigma\rho} & \boldsymbol{\Lambda}_\varsigma \end{bmatrix} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_1\boldsymbol{d}'_1) = \lim_{T\to\infty} \mathrm{E}\left[\begin{bmatrix} \sigma_0 \\ \boldsymbol{d}_{1\rho} \\ \boldsymbol{d}_{1\varsigma} \end{bmatrix} [\sigma_0\ \boldsymbol{d}'_{1\rho}\ \boldsymbol{d}'_{1\varsigma}]\right]$$

$$= \lim_{T\to\infty} \mathrm{E}\begin{bmatrix} {\sigma_0}^2 & \sigma_0\boldsymbol{d}'_{1\rho} & \sigma_0\boldsymbol{d}'_{1\varsigma} \\ \sigma_0\boldsymbol{d}_{1\rho} & \boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\rho} & \boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\varsigma} \\ \sigma_0\boldsymbol{d}_{1\varsigma} & \boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\rho} & \boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma} \end{bmatrix} \tag{B.473}$$

which implies that

$$\lambda_0 = \lim_{T\to\infty} \mathrm{E}({\sigma_0}^2), \tag{B.474}$$

$$\boldsymbol{\lambda}_\rho = \lim_{T\to\infty} \mathrm{E}(\sigma_0\boldsymbol{d}_{1\rho}), \tag{B.475}$$

$$\boldsymbol{\lambda}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\sigma_0\boldsymbol{d}_{1\varsigma}), \tag{B.476}$$

$$\boldsymbol{\Lambda}_\rho = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\rho}), \tag{B.477}$$

$$\boldsymbol{\Lambda}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma}), \tag{B.478}$$

$$\boldsymbol{\Lambda}_{\varsigma\rho} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\rho}), \tag{B.479}$$

$$\boldsymbol{\Lambda}_{\rho\varsigma} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\varsigma}). \tag{B.480}$$

Obviously $\boldsymbol{\Lambda}_{\varsigma\rho} = \boldsymbol{\Lambda}'_{\rho\varsigma}$.

Similarly,

$$\begin{bmatrix} \kappa_0 \\ \boldsymbol{\kappa}_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix} = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_1 + \boldsymbol{d}_2) = \lim_{T\to\infty} \mathrm{E}\begin{bmatrix} \sqrt{T}\sigma_0 + \sigma_1 \\ \sqrt{T}\boldsymbol{d}_{1\rho} + \boldsymbol{d}_{2\rho} \\ \sqrt{T}\boldsymbol{d}_{1\varsigma} + \boldsymbol{d}_{2\varsigma} \end{bmatrix} \tag{B.481}$$

which implies that

$$\kappa_0 = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\sigma_0 + \sigma_1), \tag{B.482}$$

$$\boldsymbol{\kappa}_\rho = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\rho} + \boldsymbol{d}_{2\rho}), \tag{B.483}$$

$$\boldsymbol{\kappa}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\varsigma} + \boldsymbol{d}_{2\varsigma}). \tag{B.484}$$

The estimator $\hat{\boldsymbol{\varsigma}}_I$ (I=UL, RL, GL, IG, ML) of $\boldsymbol{\varsigma}$ is

$$\begin{aligned} \hat{\boldsymbol{\varsigma}}_I &= \mathrm{vec}[[(\boldsymbol{Y}_* - \boldsymbol{Z}\hat{\boldsymbol{B}}_I)'(\boldsymbol{Y}_* - \boldsymbol{Z}\hat{\boldsymbol{B}}_I)/T]^{-1}] = \mathrm{vec}[(\hat{\boldsymbol{E}}'_I\hat{\boldsymbol{E}}_I/T)^{-1}] \\ &= \mathrm{vec}(\hat{\boldsymbol{\Sigma}}_I^{-1}) = [\text{see (B.162)}] \\ &= \mathrm{vec}[\boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{S}_1 + \tau^2{\boldsymbol{S}_2}^I + \omega(\tau^3)] \\ &= \mathrm{vec}(\boldsymbol{\Sigma}^{-1}) - \tau\,\mathrm{vec}(\boldsymbol{S}_1) + \tau^2\,\mathrm{vec}({\boldsymbol{S}_2}^I) + \omega(\tau^3) \\ &= \boldsymbol{\varsigma} - \tau\,\mathrm{vec}(\boldsymbol{S}_1) + \tau^2\,\mathrm{vec}({\boldsymbol{S}_2}^I) + \omega(\tau^3) \Rightarrow \end{aligned} \tag{B.485}$$

$$\begin{aligned}\boldsymbol{\delta}_{\varsigma} &= \frac{\hat{\varsigma}_I - \varsigma}{\tau} = -\operatorname{vec}(\boldsymbol{S}_1) + \tau \operatorname{vec}({\boldsymbol{S}_2}^I) + \omega(\tau^2) \\ &= \boldsymbol{d}_{1\varsigma} + \tau \boldsymbol{d}_{2\varsigma} + \omega(\tau^2), \end{aligned} \tag{B.486}$$

where

$$\boldsymbol{d}_{1\varsigma} = -\operatorname{vec}(\boldsymbol{S}_1) \text{ and } \boldsymbol{d}_{2\varsigma} = \operatorname{vec}({\boldsymbol{S}_2}^I). \tag{B.487}$$

By using equations (B.186) and (B.463) we find that

$$\begin{aligned}{\delta_0}^I &= (\hat{\sigma}_I^2 - 1)/\tau = \hat{\sigma}_I^2/\tau - 1/\tau \\ &= [M + \tau^2 \operatorname{tr}[({\boldsymbol{S}_2}^I - {\boldsymbol{S}_2}^J)\boldsymbol{\Sigma}]]/(M - \tau^2 n)\tau - \frac{1}{\tau} + \omega(\tau^2) \\ &= [M/\tau + \tau \operatorname{tr}[({\boldsymbol{S}_2}^I - {\boldsymbol{S}_2}^J)\boldsymbol{\Sigma}]]/(M - \tau^2 n) - \frac{1}{\tau} + \omega(\tau^2). \end{aligned} \tag{B.488}$$

By using Lemma UR.1 we find that

$$\begin{aligned}1/(M - \tau^2 n) &= (M - \tau^2 n)^{-1} = [M(1 - \tau^2 n/M)]^{-1} = M^{-1}(1 - \tau^2 n/M)^{-1} \\ &= M^{-1}[1 + \tau^2 n/M + \omega(\tau^4)] = (1 + \tau^2 n/M)/M + \omega(\tau^4). \end{aligned} \tag{B.489}$$

Thus, equations (B.476) and (B.477) imply that

$$\begin{aligned}{\delta_0}^I &= [M/\tau + \tau \operatorname{tr}[({\boldsymbol{S}_2}^I - {\boldsymbol{S}_2}^J)\boldsymbol{\Sigma}]][(1 + \tau^2 n/M)/M + \omega(\tau^4)] - 1/\tau + \omega(\tau^2) \\ &= [1/\tau + \tau \operatorname{tr}[({\boldsymbol{S}_2}^I - {\boldsymbol{S}_2}^J)\boldsymbol{\Sigma}/M]](1 + \tau^2 n/M) - 1/\tau + \omega(\tau^2) \\ &= 1/\tau + \tau n/M + \tau \operatorname{tr}[({\boldsymbol{S}_2}^I - {\boldsymbol{S}_2}^J)\boldsymbol{\Sigma}]/M - 1/\tau + \omega(\tau^2) \\ &= \tau[\operatorname{tr}[({\boldsymbol{S}_2}^I - {\boldsymbol{S}_2}^J)\boldsymbol{\Sigma}] + n]/M + \omega(\tau^2). \end{aligned} \tag{B.490}$$

By combining equations (B.469) and (B.490) we find that

$$\sigma_0 = 0,\ \sigma_1 = [\operatorname{tr}[({\boldsymbol{S}_2}^I - {\boldsymbol{S}_2}^J)\boldsymbol{\Sigma}] + n]/M. \tag{B.491}$$

By using equations (B.474), (B.475), (B.476) and since $\sigma_0 = 0$ (see (B.491)) we find that

$$\lambda_0 = \lim_{T\to\infty} \mathrm{E}({\sigma_0}^2) = 0, \tag{B.492}$$

$$\boldsymbol{\lambda}_{\rho} = \lim_{T\to\infty} \mathrm{E}(\sigma_0 \boldsymbol{d}_{1\rho}) = 0, \tag{B.493}$$

$$\boldsymbol{\lambda}_{\varsigma} = \lim_{T\to\infty} \mathrm{E}(\sigma_0 \boldsymbol{d}_{1\varsigma}) = 0. \tag{B.494}$$

Moreover, equations (B.198), (B.478), and (B.487) imply that

$$\begin{aligned}\boldsymbol{\Lambda}_{\varsigma} &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma}) = \lim_{T\to\infty} \mathrm{E}[(\operatorname{vec}(\boldsymbol{S}_1))(\operatorname{vec}(\boldsymbol{S}_1))'] \\ &= (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1})\boldsymbol{N}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}). \end{aligned} \tag{B.495}$$

By using equations (B.469) and (B.491) we find that

$$\begin{aligned}\kappa_0 &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\sigma_0+\sigma_1) = \lim_{T\to\infty}\mathrm{E}(\sigma_1) = \lim_{T\to\infty}\mathrm{E}[\mathrm{tr}[(\boldsymbol{S_2}^I-\boldsymbol{S_2}^J)\boldsymbol{\Sigma}]+n]/M\\ &= \mathrm{tr}[\lim_{T\to\infty}\mathrm{E}[(\boldsymbol{S_2}^I-\boldsymbol{S_2}^J)\boldsymbol{\Sigma}]]/M+n/M = [\text{see (B.193)}]\\ &= \mathrm{tr}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL}-\boldsymbol{\Delta}_I)]/M+n/M \text{ (I=UL, RL, GL, IG, ML)},\end{aligned} \tag{B.496}$$

where

$$\begin{aligned}\boldsymbol{\Delta}_{UL} &= 0\,[\text{see (B.215)}],\\ \boldsymbol{\Delta}_{RL} &= \Bigg[\Big(\sum_{i=1}^{M}\sum_{\mu=1}^{M}\sigma_{\mu i}\,\mathrm{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{iq}\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{\mu\kappa}\Big]\\ &\quad -\sum_{i=1}^{M}\sigma_{qi}\,\mathrm{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{i\kappa}\Big]-\sum_{\mu=1}^{M}\sigma_{\mu\kappa}\,\mathrm{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{\mu q}\Big]+\sigma_{q\kappa}K\Big)_{k,q}\Bigg]\,[\text{see (B.234)}],\\ \boldsymbol{\Delta}_{GL} &= \boldsymbol{\Delta}_{IG}=\boldsymbol{\Delta}_{ML}=K\boldsymbol{\Sigma}-\Bigg[\Big(\mathrm{tr}\Big[\sum_{i=1}^{M}\sum_{j=1}^{M}\sigma_{ij}\boldsymbol{B}_{ij}\Big]^{-1}\boldsymbol{B}_{ij}\Big)_{i,j}\Bigg]\,[\text{see (B.242)}].\end{aligned} \tag{B.497}$$

Furthermore, equations (B.484) and (B.487) imply that

$$\begin{aligned}\boldsymbol{\kappa}_\varsigma &= \lim_{T\to\infty}\mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\varsigma}+\boldsymbol{d}_{2\varsigma}) = \lim_{T\to\infty}\mathrm{E}(\sqrt{T}(-\mathrm{vec}(\boldsymbol{S}_1))+(\mathrm{vec}(\boldsymbol{S_2}^I)))\\ &= \lim_{T\to\infty}\mathrm{E}[\mathrm{vec}[\sqrt{T}(-\boldsymbol{S}_1)+\boldsymbol{S_2}^I]]\\ &= \mathrm{vec}[\lim_{T\to\infty}\mathrm{E}[\sqrt{T}(-\boldsymbol{S}_1)+\boldsymbol{S_2}^I]]\\ &= \mathrm{vec}[\lim_{T\to\infty}[-\sqrt{T}\,\mathrm{E}(\boldsymbol{S}_1)+\mathrm{E}(\boldsymbol{S_2}^I)]] = [\text{see (B.169)}]\\ &= \mathrm{vec}[\lim_{T\to\infty}\mathrm{E}(\boldsymbol{S_2}^I)] = [\text{see (B.190)}]\\ &= \mathrm{vec}[(M+K+1)\boldsymbol{\Sigma}^{-1}-\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1}],\end{aligned} \tag{B.498}$$

where $\boldsymbol{\Delta}_{UL}$, $\boldsymbol{\Delta}_{RL}$ and $\boldsymbol{\Delta}_{GL}=\boldsymbol{\Delta}_{IG}=\boldsymbol{\Delta}_{ML}$ have been defined in (B.497).

For the I estimator of ρ_μ (I=LS, GL, PW, ML,DW) equations (B.276), (B.277), (B.282), (B.285), and (B.290) imply that

$$\begin{aligned}d_{(1)\mu}{}^{LS} &= d_{(1)\mu}{}^{GL}=d_{(1)\mu}{}^{ML}=d_{(1)\mu}{}^{DW}=-\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2\\ &= -\tau\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2=d_{(1)\mu}.\end{aligned} \tag{B.499}$$

Therefore,

$$d_{(1)\mu}d'_{(1)\mu}=\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/4T\sigma_{\boldsymbol{u}_\mu}{}^4. \tag{B.500}$$

Moreover, for $\mu\neq\mu'$ we find that

$$d_{(1)\mu}d'_{(1)\mu'}=\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu\boldsymbol{u}'_{\mu'}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{u}_{\mu'}/4T\sigma_{\boldsymbol{u}_\mu}{}^2\sigma_{\boldsymbol{u}_{\mu'}}{}^2. \tag{B.501}$$

Equations (B.477) and (B.500) imply that, since $\sigma_{\boldsymbol{u}_\mu}{}^2 = \frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}$ the μ-diagonal element of the matrix $\boldsymbol{\Lambda_\rho}$ is

$$\begin{aligned}
\lim_{T\to\infty} \mathrm{E}(d_{(1)\mu}d'_{(1)\mu}) &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)/4T\sigma_{\boldsymbol{u}_\mu}{}^4 \\
&= [\text{see (B.411)}] = \lim_{T\to\infty}\left[\frac{4T\sigma_{\mu\mu}{}^2}{1-\rho_\mu{}^2} + O(1)\right]/4T\sigma_{\boldsymbol{u}_\mu}{}^4 \\
&= \lim_{T\to\infty}\left[\frac{4T\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)4T\sigma_{\boldsymbol{u}_\mu}{}^4} + O(T^{-1})\right] \\
&= \frac{4T\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)4T\frac{\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2}} = (1-\rho_\mu{}^2). \qquad \text{(B.502)}
\end{aligned}$$

Similarly, equations (B.442) and (B.501) imply that, since $\sigma_{\boldsymbol{u}_\mu}{}^2 = \frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}$, $\sigma_{\boldsymbol{u}_{\mu'}}{}^2 = \frac{\sigma_{\mu'\mu'}}{(1-\rho_{\mu'}{}^2)}$, for $\mu \neq \mu'$ the $\mu\mu'$-th off-diagonal element of the matrix $\boldsymbol{\Lambda_\rho}$ is

$$\begin{aligned}
\lim_{T\to\infty} \mathrm{E}(d_{(1)\mu}d'_{(1)\mu'}) &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu \boldsymbol{u}'_{\mu'} \boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{u}_{\mu'})/4T\sigma_{\boldsymbol{u}_\mu}{}^2\sigma_{\boldsymbol{u}_{\mu'}}{}^2 \\
&= \lim_{T\to\infty} \frac{4T\sigma_{\mu\mu}\sigma_{\mu'\mu'}}{(1-\rho_\mu\rho_{\mu'})4T\sigma_{\boldsymbol{u}_\mu}{}^2\sigma_{\boldsymbol{u}_{\mu'}}{}^2} + O(T^{-1}) \\
&= \frac{\sigma_{\mu\mu}\sigma_{\mu'\mu'}}{(1-\rho_\mu\rho_{\mu'})\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}\frac{\sigma_{\mu'\mu'}}{(1-\rho_{\mu'}{}^2)}} = \frac{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}{(1-\rho_\mu\rho_{\mu'})}. \qquad \text{(B.503)}
\end{aligned}$$

Moreover, for the J estimator of ρ_μ (I=LS, GL, PW, ML,DW) it holds that since

$$d_{(1)\mu} = d_{(1)\mu}{}^{LS} = d_{(1)\mu}{}^{GL} = d_{(1)\mu}{}^{ML} = d_{(1)\mu}{}^{DW} = -\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2, \qquad \text{(B.504)}$$

the following results holds:

$$\begin{aligned}
\mathrm{E}(\sqrt{T}d_{(1)\mu}) &= -\mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sigma_{\boldsymbol{u}_\mu}{}^2) = \frac{-2\rho_\mu\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}/\frac{2\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)} = -\rho_\mu \Rightarrow \\
\lim_{T\to\infty} \mathrm{E}(\sqrt{T}d_{(1)\mu}) &= -\rho_\mu. \qquad \text{(B.505)}
\end{aligned}$$

By using equations (B.278), (B.413), and (B.414) we find that

$$\begin{aligned}
\mathrm{E}(d_{(2)\mu}{}^{LS}) &= -\mathrm{E}(\boldsymbol{u}'_\mu \bar{\boldsymbol{P}}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu)/2\sigma_{\boldsymbol{u}_\mu}{}^2 + \mathrm{E}(\boldsymbol{u}'_\mu\boldsymbol{u}_\mu\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)/2T\sigma_{\boldsymbol{u}_\mu}{}^4 \\
&= -\frac{\sigma_{\mu\mu}}{2\rho_\mu}\left[\frac{2(\rho_\mu{}^2 - n(1-\rho_\mu{}^2))}{1-\rho_\mu{}^2}/\sigma_{\boldsymbol{u}_\mu}{}^2 + (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})/\sigma_{\boldsymbol{u}_\mu}{}^2 + \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})/\sigma_{\boldsymbol{u}_\mu}{}^2\right] \\
&\quad +O(T^{-1}) - \left[\frac{2T\rho_\mu\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2} + O(1)\right]/2T\sigma_{\boldsymbol{u}_\mu}{}^4 \\
&= -\frac{1}{2\rho_\mu}\left[\frac{2\rho_\mu{}^2\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)\sigma_{\boldsymbol{u}_\mu}{}^2} - \frac{2n\sigma_{\mu\mu}}{\sigma_{\boldsymbol{u}_\mu}{}^2} + (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\frac{\sigma_{\mu\mu}}{\sigma_{\boldsymbol{u}_\mu}{}^2}\right] \\
&\quad -\frac{1}{2\rho_\mu}\left[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\frac{\sigma_{\mu\mu}}{\sigma_{\boldsymbol{u}_\mu}{}^2} + \frac{2\rho_\mu^2\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2\sigma_{\boldsymbol{u}_\mu}{}^4}\right] + O(T^{-1}) \\
&= -\frac{1}{2\rho_\mu}\left[\frac{2\rho_\mu{}^2\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}} - \frac{2n\sigma_{\mu\mu}}{\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}} + (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\frac{\sigma_{\mu\mu}}{\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}}\right]
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2\rho_\mu}\left[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\frac{\sigma_{\mu\mu}}{\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}} + \frac{2\rho_\mu^2\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2\frac{\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2}}\right] + O(T^{-1}) \\
= & -\frac{1}{2\rho_\mu}[4\rho_\mu{}^2 - 2n(1-\rho_\mu{}^2) + (1-\rho_\mu{}^2)^2\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})(1-\rho_\mu{}^2)] \\
& +O(T^{-1}).
\end{aligned} \tag{B.506}
$$

By combining equations (B.483), (B.505), and (B.506) we find that

$$
\begin{aligned}
\kappa_{\rho_\mu} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}d_{1\mu} + d_{2\mu}{}^{LS}) \\
&= \lim_{T\to\infty}\left[-\frac{1}{2\rho_\mu}[2\rho_\mu{}^2 + 4\rho_\mu{}^2 - 2n(1-\rho_\mu{}^2) + (1-\rho_\mu{}^2)^2\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})(1-\rho_\mu{}^2)] + O(T^{-1})\right] \\
&= -\frac{1}{2\rho_\mu}[6\rho_\mu{}^2 - 2n - 2n\rho_\mu{}^2 + (1-\rho_\mu{}^2)^2\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})(1-\rho_\mu{}^2)] \\
&= -\frac{1}{2\rho_\mu}[2\rho_\mu{}^2(3+n) - 2n + (1-\rho_\mu{}^2)((1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}))] \\
&= -[\rho_\mu(3+n) + (2n-c_1)/2\rho_\mu],
\end{aligned} \tag{B.507}
$$

where

$$
c_1 = (1-\rho_\mu{}^2)((1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})). \tag{B.508}
$$

By using equations (B.278) and (B.283) we find that

$$
d_{(2)\mu}{}^{GL} = d_{(2)\mu}{}^{LS} - \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}_\mu'\bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]. \tag{B.509}
$$

Therefore, equations (B.415) and (B.416) imply that

$$
\begin{aligned}
& \mathrm{E}\left(-\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}_\mu'\bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]\right) \\
= & -\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\frac{\sigma_{\mu\mu}}{\rho_\mu}[n - \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] \\
& +\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\frac{\sigma_{\mu\mu}}{\rho_\mu}[n - \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2 \\
& -\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\frac{\sigma_{\mu\mu}(1-\rho_\mu{}^2)}{\rho_\mu}[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1}) - \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2 + O(T^{-1}) \\
= & -\frac{(1-\rho_\mu{}^2)}{\rho_\mu}[n - \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2 - \frac{(1-\rho_\mu{}^2)^2}{\rho_\mu}[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1}) - \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2 + O(T^{-1}) \Rightarrow \\
& \lim_{T\to\infty}\mathrm{E}\left(-\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}_\mu'\bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]\right) \\
= & -\frac{(1-\rho_\mu{}^2)}{\rho_\mu}[n - \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2 \\
= & (1-\rho_\mu{}^2)[(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2\rho_\mu \\
& -(1-\rho_\mu{}^2)^2\operatorname{tr}(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1})/2\rho_\mu - (1-\rho_\mu{}^2)^2 n/2\rho_\mu \\
= & [c_1 - (1-\rho_\mu{}^2)n]/2\rho_\mu - (1-\rho_\mu{}^2)c_2/2\rho_\mu,
\end{aligned} \tag{B.510}
$$

where

$$c_2 = (1 - {\rho_\mu}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1}). \tag{B.511}$$

Thus, from equations (B.505), (B.506), (B.507), (B.509), and (B.510) we find that

$$\begin{aligned} {\kappa_{\rho_\mu}}^{GL} &= {\kappa_{\rho_\mu}}^{PW} = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}d_{1\mu} + {d_{2\mu}}^{GL}) = \lim_{T\to\infty} \mathrm{E}[(\sqrt{T}d_{1\mu} + {d_{2\mu}}^{LS}) \\ &\quad - \frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}_\mu'\bar{P}_{\mathrm{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathrm{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathrm{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathrm{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]] \\ &= {\kappa_{\rho_\mu}}^{LS} - (1-{\rho_\mu}^2)c_2/2\rho_\mu + [c_1 - (1-{\rho_\mu}^2)n]/2\rho_\mu. \end{aligned} \tag{B.512}$$

By using equations (B.283) and (B.286) we find that

$$d_{(2)\mu}{}^{ML} = d_{(2)\mu}{}^{GL} + \rho_\mu\frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}({u_{1\mu}}^2 + {u_{T\mu}}^2) - \rho_\mu. \tag{B.513}$$

Since

$$\mathrm{E}({u_{1\mu}}^2 + {u_{T\mu}}^2) = \mathrm{E}({u_{1\mu}}^2) + \mathrm{E}({u_{T\mu}}^2) = {\sigma_{\boldsymbol{u}_\mu}}^2 + {\sigma_{\boldsymbol{u}_\mu}}^2 = 2{\sigma_{\boldsymbol{u}_\mu}}^2 = \frac{2\sigma_{\mu\mu}}{(1-{\rho_\mu}^2)}, \tag{B.514}$$

we find that

$$\begin{aligned} \mathrm{E}(d_{(2)\mu}{}^{ML}) &= \mathrm{E}(d_{(2)\mu}{}^{GL}) + \rho_\mu\frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}\frac{2\sigma_{\mu\mu}}{(1-{\rho_\mu}^2)} - \rho_\mu \\ &= \mathrm{E}(d_{(2)\mu}{}^{GL}) + 2\rho_\mu - \rho_\mu = \mathrm{E}(d_{(2)\mu}{}^{GL}) + \rho_\mu. \end{aligned} \tag{B.515}$$

Thus, by combining equations (B.505), (B.512), (B.513) and (B.515) we find that

$$\begin{aligned} {\kappa_{\rho_\mu}}^{ML} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}d_{1\mu} + {d_{2\mu}}^{ML}) = \lim_{T\to\infty} \mathrm{E}[(\sqrt{T}d_{1\mu} + {d_{2\mu}}^{GL}) + \rho_\mu] \\ &= {\kappa_{\rho_\mu}}^{GL} + \rho_\mu = {\kappa_{\rho_\mu}}^{PW} + \rho_\mu. \end{aligned} \tag{B.516}$$

By using equations (B.278) (B.291) we find that

$$\begin{aligned} \mathrm{E}(d_{(2)\mu}{}^{DW}) &= \mathrm{E}(d_{(2)\mu}{}^{LS}) + \frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}(\mathrm{E}({u_{1\mu}}^2) + \mathrm{E}({u_{T\mu}}^2))/2 = [\text{see (B.514)}] \\ &= \mathrm{E}(d_{(2)\mu}{}^{LS}) + \frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}\frac{2\sigma_{\mu\mu}/2}{(1-{\rho_\mu}^2)} \\ &= \mathrm{E}(d_{(2)\mu}{}^{LS}) + 1. \end{aligned} \tag{B.517}$$

Thus, by combining equations (B.505), (B.507), and (B.517) we find that

$$\begin{aligned} {\kappa_{\rho_\mu}}^{DW} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}d_{1\mu} + {d_{2\mu}}^{DW}) = \lim_{T\to\infty} \mathrm{E}[(\sqrt{T}d_{1\mu} + {d_{2\mu}}^{LS}) + \frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}({u_{1\mu}}^2 + {u_{T\mu}}^2)/2] \\ &= {\kappa_{\rho_\mu}}^{LS} + 1. \end{aligned} \tag{B.518}$$

By using equations (B.470), (B.471), and (B.472) we define the $M \times M^2$ matrix $\boldsymbol{\Lambda}_{\rho\varsigma}$ as follows:

$$\boldsymbol{\Lambda}_{\rho\varsigma} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\varsigma}), \tag{B.519}$$

where the μ-th element $M \times 1$ vector $\boldsymbol{d}_{1\rho}$ is

$$d_{(1)\mu} = -\frac{\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu}{2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2} = -\frac{\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu}{2\sqrt{T}\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}} = -\frac{1-\rho_\mu{}^2}{2\sqrt{T}\sigma_{\mu\mu}}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu) \tag{B.520}$$

and the $1 \times M^2$ vector $\boldsymbol{d}'_{1\varsigma}$ is defined as

$$\boldsymbol{d}'_{1\varsigma} = [-\operatorname{vec}(\boldsymbol{S}_1)]'. \tag{B.521}$$

From equation (B.450) we have that the (ij)-th element of $\boldsymbol{d}'_{1\varsigma}$ is

$$-s^{(1)}_{(ij)} = -\sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma^{lj}(\boldsymbol{\varepsilon}'_k\boldsymbol{\varepsilon}_l/T) - \sigma^{ij}\right] \tag{B.522}$$

with $(ij) = 1, \ldots, M^2$.

By combining equations (B.520) and (B.521) we find that the $(\mu, (ij))$-th element of the $(M \times M^2)$ matrix $\boldsymbol{\Lambda}_{\rho\varsigma}$ is

$$\begin{aligned} d_{(1)\mu}(-s^{(1)}_{(ij)}) &= \left[-\frac{1-\rho_\mu{}^2}{2\sqrt{T}\sigma_{\mu\mu}}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)\right]\left[-\sqrt{T}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma^{lj}(\boldsymbol{\varepsilon}'_k\boldsymbol{\varepsilon}_l/T) - \sigma^{ij}\right]\right] \\ &= \frac{(1-\rho_\mu{}^2)}{2\sigma_{\mu\mu}}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma^{lj}[(\boldsymbol{\varepsilon}'_k\boldsymbol{\varepsilon}_l/T)(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)] - \sigma^{ij}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)\right], \end{aligned} \tag{B.523}$$

which implies that by using equation (B.460) we find that

$$\begin{aligned} \mathrm{E}(d_{(1)\mu}(-s^{(1)}_{(ij)})) &= \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\left[\sum_{k=1}^{M}\sum_{l=1}^{M}\sigma^{ik}\sigma^{lj}\,\mathrm{E}[(\boldsymbol{\varepsilon}'_k\boldsymbol{\varepsilon}_l/T)(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)] - \sigma^{ij}\,\mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)\right] \\ &= O(T^{-1/2}) \Rightarrow \\ &\boldsymbol{\Lambda}_{\rho\varsigma} = \lim_{T\to\infty} \mathrm{E}(d_{(1)\mu}(-s^{(1)}_{(ij)})) = 0. \end{aligned} \tag{B.524}$$

Finally, we find that

$$\boldsymbol{\Lambda}_{\varsigma\rho} = \boldsymbol{\Lambda}'_{\rho\varsigma} = 0. \tag{B.525}$$

□

Appendix C

Matrix $\boldsymbol{\Omega}$

Equations (5.26b) and (5.26c) imply that $\boldsymbol{\Omega}^{-1} = \boldsymbol{P}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{P}'$ where $\boldsymbol{\Sigma} = [(\delta_{ij}\sigma_{ii})_{i,j=1,\dots,M}]$ and $\boldsymbol{P} = [(\delta_{ij}\boldsymbol{P}_i)_{i,j=1,\dots,M}]$ is a block diagonal matrix. Let $\boldsymbol{P}^{-1}$ and $\boldsymbol{P}'^{-1}$ be the inverse of $\boldsymbol{P}$ and $\boldsymbol{P}'$ respectively and let $\boldsymbol{\Sigma}^{-1} = [(\delta_{ij}\sigma^{ii})_{i,j=1,\dots,M}]$ be the inverse of $\boldsymbol{\Sigma}$.

Then by using equations (5.17b) and (5.22) we find that

$$\begin{aligned}
\boldsymbol{\Omega}^{-1} = \boldsymbol{P}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{P}' &= \begin{bmatrix} \boldsymbol{P}_1 & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M \end{bmatrix} \begin{bmatrix} \sigma_{11}\boldsymbol{I}_T & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \sigma_{MM}\boldsymbol{I}_T \end{bmatrix} \begin{bmatrix} \boldsymbol{P}'_1 & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}'_M \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{11}\boldsymbol{P}_1\boldsymbol{P}'_1 & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \sigma_{MM}\boldsymbol{P}_M\boldsymbol{P}'_M \end{bmatrix} = \begin{bmatrix} \sigma_{11}\boldsymbol{R}_{11} & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \sigma_{MM}\boldsymbol{R}_{MM} \end{bmatrix} \\
&= [(\delta_{ij}\sigma_{ij}\boldsymbol{R}_{ij})_{i,j=1,\dots,M}] = [(\delta_{ij}\sigma_{ii}\boldsymbol{R}_{ii})_{i,j=1,\dots,M}].
\end{aligned} \tag{C.1}$$

Equation (C.1) implies that

$$\begin{aligned}
\boldsymbol{\Omega} = \boldsymbol{P}'^{-1}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1} &= \begin{bmatrix} \boldsymbol{P}'^{-1}_1 & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}'^{-1}_M \end{bmatrix} \begin{bmatrix} \sigma^{11}\boldsymbol{I}_T & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \sigma^{MM}\boldsymbol{I}_T \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_1^{-1} & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \boldsymbol{P}_M^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \sigma^{11}\boldsymbol{P}'^{-1}_1\boldsymbol{P}_1^{-1} & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \sigma^{MM}\boldsymbol{P}'^{-1}_M\boldsymbol{P}_M^{-1} \end{bmatrix} = \begin{bmatrix} \sigma^{11}\boldsymbol{R}^{11} & \dots & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & \dots & \sigma^{MM}\boldsymbol{R}^{MM} \end{bmatrix} \\
&= [(\delta_{ij}\sigma^{ij}\boldsymbol{R}^{ij})_{i,j=1,\dots,M}] = [(\delta_{ij}\sigma^{ii}\boldsymbol{R}^{ii})_{i,j=1,\dots,M}],
\end{aligned} \tag{C.2}$$

where

$$\boldsymbol{R}^{ii} = \boldsymbol{P}'^{-1}_i\boldsymbol{P}_i^{-1} \;\; (i = 1, \dots, M). \tag{C.3}$$

The Matrices $\boldsymbol{R}_{ii}$, $\boldsymbol{R}^{ii}$ and their Derivatives with respect to the elements ρ_i, ρ_j

Equation (5.19) imply that

$$\boldsymbol{R}^{ii} = \boldsymbol{P}_i'^{-1}\boldsymbol{P}_i^{-1} = \begin{bmatrix} (1-\rho_i^2)^{1/2} & -\rho_i & 0 & \dots & 0 \\ 0 & 1 & -\rho_i & \dots & 0 \\ \vdots & & & & -\rho_i \\ 0 & & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} (1-\rho_i^2)^{1/2} & 0 & 0 & \dots & 0 \\ -\rho_i & 1 & 0 & \dots & 0 \\ 0 & -\rho_i & 1 & & 0 \\ 0 & \dots & 0 & -\rho_i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1-\rho_i^2)^{1/2}(1-\rho_i^2)^{1/2} + \rho_i\rho_i & -\rho_i & 0 & \dots & 0 \\ -\rho_i & 1+\rho_i\rho_i & \ddots & & \\ 0 & & \ddots & & \\ \vdots & & & 1+\rho_i\rho_i & -\rho_i \\ 0 & \dots & 0 & -\rho_i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\rho_i & 0 & \dots & 0 \\ -\rho_i & 1+\rho_i^2 & \ddots & & \\ 0 & & \ddots & & \\ \vdots & & & 1+\rho_i^2 & -\rho_i \\ 0 & \dots & 0 & -\rho_i & 1 \end{bmatrix}. \tag{C.4}$$

Obviously,

$$\begin{aligned} \boldsymbol{R}^{ii} &= \begin{bmatrix} 1+\rho_i^2 & 0 & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1+\rho_i^2 \end{bmatrix} - \begin{bmatrix} 0 & \rho_i & \dots & 0 \\ \rho_i & \ddots & & \\ & & \ddots & \rho_i \\ & & \rho_i & 0 \end{bmatrix} - \begin{bmatrix} \rho_i^2 & & & 0 \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ 0 & & & \rho_i \end{bmatrix} \\ &= (1+\rho_i^2)\boldsymbol{I}_T - \rho_i\boldsymbol{D} - \rho_i^2\boldsymbol{\Delta}, \end{aligned} \tag{C.5}$$

where $\boldsymbol{I}_T$ is the $T \times T$ identity matrix, $\boldsymbol{D}$ is a $T \times T$ matrix with elements 1 if $|t - t'| = 1$ and zeros elsewhere, and $\boldsymbol{\Delta}$ is a $T \times T$ matrix with elements 1 in (1,1)-st and (T,T)-th positions and zeros elsewhere.

It can be easily seen that

$$\boldsymbol{R}^{ii}\boldsymbol{R}_{ii} = \boldsymbol{R}_{ii}\boldsymbol{R}^{ii} = \boldsymbol{I}. \tag{C.6}$$

Moreover,

$$\boldsymbol{R}_{\rho_i}{}^{ii} = \frac{\partial \boldsymbol{R}^{ii}}{\partial \rho_i} = 2\rho_i\boldsymbol{I}_T - \boldsymbol{D} - 2\rho_i\boldsymbol{\Delta}, \tag{C.7}$$

$$\boldsymbol{R}_{\rho_i\rho_i}{}^{ii} = \frac{\partial^2 \boldsymbol{R}^{ii}}{\partial \rho_i{}^2} = 2\boldsymbol{I}_T - 2\boldsymbol{\Delta} = 2(\boldsymbol{I}_T - \boldsymbol{\Delta}), \tag{C.8}$$

$$\boldsymbol{R}_{\rho_j}{}^{ii} = \frac{\partial \boldsymbol{R}^{ii}}{\partial \rho_j} = 0,\ \boldsymbol{R}_{\rho_j\rho_j}{}^{ii} = \frac{\partial^2 \boldsymbol{R}^{ii}}{\partial \rho_j{}^2} = 0,\ \boldsymbol{R}_{\rho_i\rho_j}{}^{ii} = \frac{\partial^2 \boldsymbol{R}^{ii}}{\partial \rho_j \partial \rho_i} = 0,\ (\forall i \neq j). \tag{C.9}$$

The Derivatives of $\boldsymbol{\Omega}$ with respect to the element ρ_μ

Since, $\boldsymbol{\Omega} = [(\delta_{ij}\sigma^{ii}\boldsymbol{R}^{ii})_{i,j=1,\ldots,M}]$ we find that

$$\begin{aligned}
\boldsymbol{\Omega}_{\rho_\mu} = \frac{\partial \boldsymbol{\Omega}}{\partial \rho_\mu} &= [(\partial \delta_{ij}\sigma^{ii}\boldsymbol{R}^{ii}/\partial \rho_\mu)] \\
&= [(\delta_{ij}\sigma^{ii}\boldsymbol{R}_{\rho_\mu}{}^{ii})_{i,j=1,\ldots,M}] \\
&= [(\delta_{ij}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu})_{i,j}] = [see(\text{C.7})] \\
&= [(\delta_{\mu i}\sigma^{i\mu}(2\rho_i\boldsymbol{I}_T - \boldsymbol{D} - 2\rho_i\boldsymbol{\Delta}))_{i,\mu=1,\ldots,M}],
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
\boldsymbol{\Omega}_{\rho_\mu\rho_\mu} = \frac{\partial^2 \boldsymbol{\Omega}}{\partial \rho_\mu{}^2} &= [(\partial \delta_{ij}\sigma^{ii}\boldsymbol{R}_{\rho_\mu}{}^{ii}/\partial \rho_\mu)] \\
&= [(\delta_{ij}\sigma^{ii}\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{ii})_{i,j=1,\ldots,M}] \\
&= [(\delta_{ij}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{i\mu})_{i,j}] = [see(\text{C.8})] \\
&= [(\delta_{\mu i}\sigma^{i\mu}(2(\boldsymbol{I}_T - \boldsymbol{\Delta})))_{i,\mu=1,\ldots,M}].
\end{aligned} \tag{C.11}$$

The Derivatives of $\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T$ and $\boldsymbol{\Omega}$ with respect to the element σ^{ii}

Since,

$$\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T = [(\delta_{ij}\sigma^{ii}\boldsymbol{I}_T)_{i,j=1,\ldots,M}], \tag{C.12}$$

and

$$\varsigma = [(\sigma^{ii})_{i=1,\ldots,M}], \tag{C.13}$$

we find

$$\begin{aligned}
\frac{\partial}{\partial \sigma^{\mu\mu}}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T) &= \left[\left(\frac{\partial \delta_{ij}\sigma^{ii}\boldsymbol{I}_T}{\partial \sigma^{\mu\mu}}\right)_{i,j=1,\ldots,M}\right] \\
&= [(\delta_{ij}\delta_{\mu i}\boldsymbol{I}_T)_{i,j=1,\ldots,M}] = [(\delta_{\mu i}\delta_{\mu j})_{i,j=1,\ldots,M}] \otimes \boldsymbol{I}_T \\
&= \boldsymbol{\Delta}_{(\mu\mu)} \otimes \boldsymbol{I}_T,
\end{aligned} \tag{C.14}$$

where $\boldsymbol{\Delta}_{(\mu\mu)}$ is a $(M \times M)$ matrix with 1 in the $(\mu\mu)$-th position and zeros elsewhere.

$$\begin{aligned}\frac{\partial^2}{\partial\sigma^{\mu\mu}\sigma^{\nu\nu}}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T) &= \frac{\partial}{\partial\sigma^{\nu\nu}}\left[\frac{\partial}{\partial\sigma^{\mu\mu}}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T)\right]\\ &= [(\partial\delta_{\mu j}\delta_{\mu i}\boldsymbol{I}_T/\partial\sigma^{\nu\nu})_{i,j=1,\dots,M}] = 0. \end{aligned} \quad (C.15)$$

Since $\boldsymbol{\Omega} = \boldsymbol{P}'^{-1}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}$, equation (C.14) implies that

$$\begin{aligned}\boldsymbol{\Omega}_{(\mu\mu)} &= \frac{\partial\boldsymbol{\Omega}}{\partial\sigma^{\mu\mu}} = \boldsymbol{P}'^{-1}\left[\frac{\partial}{\partial\sigma^{\mu\mu}}(\boldsymbol{\Sigma}^{-1}\otimes \boldsymbol{I}_T)\right]\boldsymbol{P}^{-1} = \boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\mu\mu}\otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}\\ &= \boldsymbol{\Delta}_{\mu\mu}\otimes \boldsymbol{P}_i'^{-1}\boldsymbol{P}_j^{-1} = [(\delta_{\mu i}\delta_{j\mu}\boldsymbol{P}_i'^{-1}\boldsymbol{P}_j^{-1})_{i,j=1,\dots,M}]\\ &= [(\delta_{\mu i}\delta_{j\mu}\boldsymbol{R}^{ij})_{i,j=1,\dots,M}] = [(\delta_{\mu i}\delta_{j\mu}\boldsymbol{R}^{\mu\mu})_{i,j=1,\dots,M}]. \end{aligned} \quad (C.16)$$

Similarly, equation (C.15) implies that

$$\begin{aligned}\boldsymbol{\Omega}_{(\mu\mu)(\nu\nu)} &= \frac{\partial^2\boldsymbol{\Omega}}{\partial\sigma^{\mu\mu}\partial\sigma^{\nu\nu}} = \frac{\partial}{\partial\sigma^{\nu\nu}}\left(\frac{\partial\boldsymbol{\Omega}}{\partial\sigma^{\mu\mu}}\right)\\ &= [(\partial\delta_{\mu i}\delta_{j\mu}\boldsymbol{R}^{\mu\mu}/\partial\sigma^{\nu\nu})_{i,j=1,\dots,M}] = 0. \end{aligned} \quad (C.17)$$

The Second-order cross derivatives and useful matrices

Equations (C.10) and (C.16) imply that

$$\begin{aligned}\boldsymbol{\Omega}_{(\nu\nu)\rho_\mu} &= \boldsymbol{\Omega}_{\rho_\mu(\nu\nu)} = \frac{\partial}{\partial\sigma^{\nu\nu}}\left(\frac{\partial\boldsymbol{\Omega}}{\partial\rho_\mu}\right) = \frac{\partial\boldsymbol{\Omega}_{\nu\nu}}{\partial\rho_\mu} = \frac{\partial}{\partial\rho_\mu}(\boldsymbol{\Delta}_{\nu\nu}\otimes \boldsymbol{P}_i'^{-1}\boldsymbol{P}_j^{-1})\\ &= [(\partial\delta_{\nu i}\delta_{j\nu}\boldsymbol{R}^{\nu\nu}/\partial\rho_\mu)_{i,j}] = [(\delta_{\nu i}\delta_{j\nu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu})_{i,j=1,\dots,M}]\\ &= [(\delta_{\mu\nu}\delta_{\nu i}\delta_{j\nu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu})_{i,j=1,\dots,M}]. \end{aligned} \quad (C.18)$$

Equations (C.1) and (C.10) imply that

$$\begin{aligned}\boldsymbol{\Omega}^*{}_{\rho_{\mu'}\rho_\mu} &= \boldsymbol{\Omega}_{\rho_{\mu'}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{\rho_\mu} = \boldsymbol{\Omega}^*{}_{\rho_{\mu'}\rho_\mu} = \boldsymbol{\Omega}'_{\rho_\mu}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}'_{\rho_{\mu'}} = \boldsymbol{\Omega}^{*\prime}{}_{\rho_\mu\rho_{\mu'}}\\ &= [(\delta_{i\kappa}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu})_{i,\kappa=1,\dots,M}][(\delta_{\kappa l}\sigma_{\kappa\kappa}\boldsymbol{R}_{\kappa\kappa})_{\kappa,l=1,\dots,M}][(\delta_{lj}\delta_{l\mu'}\sigma^{l\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'})_{l,j=1,\dots,M}]\\ &= \left[\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\delta_{i\kappa}\delta_{\mu i}\delta_{\kappa l}\delta_{lj}\delta_{l\mu'}\sigma^{i\mu}\sigma_{\kappa\kappa}\sigma^{l\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\kappa\kappa}\boldsymbol{R}_{\rho_{\mu'}}{}^{l\mu'}\right)_{i,j=1,\dots,M}\right]\\ &= \left[\left(\delta_{ij}\delta_{\mu i}\delta_{\mu' j}\sigma^{i\mu}\sigma_{ii}\sigma^{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{ii}\boldsymbol{R}_{\rho_{\mu'}}{}^{j\mu'}\right)_{i,j=1,\dots,M}\right]. \end{aligned} \quad (C.19)$$

$$\begin{aligned}\boldsymbol{\Delta}_{\mu\mu}\boldsymbol{\Sigma}\boldsymbol{\Delta}_{\nu\nu} &= [(\delta_{\mu i}\delta_{\mu\kappa})_{i,\kappa=1,\dots,M}][(\delta_{\kappa l}\sigma_{\kappa\kappa})_{\kappa,l=1,\dots,M}][(\delta_{\nu l}\delta_{\nu j})_{l,j=1,\dots,M}]\\ &= \left[\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\delta_{\mu i}\delta_{\mu\kappa}\delta_{\kappa l}\delta_{\nu l}\delta_{\nu j}\sigma_{\kappa\kappa}\right)_{i,j}\right]\\ &= [(\delta_{\mu i}\delta_{\mu\nu}\delta_{\nu j}\sigma_{\mu\mu})_{i,j}] = \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{\Delta}_{\mu\nu} \end{aligned} \quad (C.20)$$

Equations (C.1), (C.16) and (C.20) imply that

$$\begin{aligned}
\boldsymbol{\Omega}^*{}_{(\mu\mu)(\nu\nu)} &= \boldsymbol{\Omega}_{(\mu\mu)}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(\nu\nu)} \\
&= \boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\mu\mu}\otimes\boldsymbol{I}_T)\boldsymbol{P}^{-1}\boldsymbol{P}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)\boldsymbol{P}'\boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\nu\nu}\otimes\boldsymbol{I}_T)\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\mu\mu}\otimes\boldsymbol{I}_T)(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)(\boldsymbol{\Delta}_{\nu\nu}\otimes\boldsymbol{I}_T)\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}'^{-1}(\boldsymbol{\Delta}_{\mu\mu}\boldsymbol{\Sigma}\boldsymbol{\Delta}_{\nu\nu}\otimes\boldsymbol{I}_T)\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}'^{-1}(\delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{\Delta}_{\mu\nu}\otimes\boldsymbol{I}_T)\boldsymbol{P}^{-1}. \qquad \text{(C.21)}
\end{aligned}$$

Equations (C.1), (C.10) and (C.16) imply that

$$\begin{aligned}
\boldsymbol{\Omega}^*{}_{(\nu\nu)\rho_\mu} &= \boldsymbol{\Omega}^*{}_{\rho_\mu(\nu\nu)} = \boldsymbol{\Omega}_{\rho_\mu}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(\nu\nu)} \\
&= [(\delta_{i\kappa}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu})_{i,\kappa=1,\dots,M}][(\delta_{\kappa l}\sigma_{\kappa\kappa}\boldsymbol{R}_{\kappa\kappa})_{\kappa,l=1,\dots,M}][(\delta_{\nu l}\delta_{j\nu}\boldsymbol{R}^{lj})_{l,j=1,\dots,M}] \\
&= \left[\left(\sum_{\kappa=1}^{M}\sum_{l=1}^{M}\delta_{i\kappa}\delta_{\mu i}\delta_{\kappa l}\delta_{\nu l}\delta_{j\nu}\sigma^{i\mu}\sigma_{\kappa\kappa}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{\kappa\kappa}\boldsymbol{R}^{lj}\right)_{ij}\right] \\
&= \left[\left(\delta_{\mu i}\delta_{i\nu}\delta_{j\nu}\sigma^{i\mu}\sigma_{ii}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{ii}\boldsymbol{R}^{\nu\nu}\right)_{ij}\right] \\
&= \left[\left(\delta_{\mu i}\delta_{i\nu}\delta_{j\nu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\right)_{ij=1,\dots,M}\right]. \qquad \text{(C.22)}
\end{aligned}$$

Define the $n\times n$ matrix

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T = [(\boldsymbol{X}_i')_{i=1,\dots,M}][(\delta_{ij}\sigma^{ij}\boldsymbol{R}^{ij})_{i,j=1,\dots,M}][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{ij}\sigma^{ij}\boldsymbol{X}_i'\boldsymbol{R}^{ij}\boldsymbol{X}_j/T \\
&= \sum_{i=1}^{M}\sigma^{ii}\boldsymbol{X}_i'\boldsymbol{R}^{ii}\boldsymbol{X}_i/T \\
&= \sum_{i=1}^{M}\sigma^{ii}\boldsymbol{B}_{ii}, \qquad \text{(C.23)}
\end{aligned}$$

where

$$\boldsymbol{B}_{ii} = \boldsymbol{X}_i'\boldsymbol{R}^{ii}\boldsymbol{X}_i/T. \qquad \text{(C.24)}$$

Therefore, by using equations (C.10), (C.11) and (C.19) we have that

$$\begin{aligned}
\boldsymbol{A}_{\rho_\mu} &= \frac{\partial\boldsymbol{A}}{\partial\rho_\mu} = \partial(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\rho_\mu = \boldsymbol{X}'(\partial\boldsymbol{\Omega}/\partial\rho_\mu)\boldsymbol{X}/T \\
&= \boldsymbol{X}'\boldsymbol{\Omega}_{\rho_\mu}\boldsymbol{X}/T = [(\boldsymbol{X}_i')_{i=1,\dots,M}][(\delta_{ij}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu})_{i,j=1,\dots,M}][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{ij}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{X}_j/T = \sum_{i=1}^{M}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{X}_i/T \\
&= \sigma^{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T, \qquad \text{(C.25)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}_{\rho_\mu\rho_\mu} &= \frac{\partial^2 \boldsymbol{A}}{\partial\rho_\mu\partial\rho_\mu} = \partial^2(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\rho_\mu\partial\rho_\mu = \boldsymbol{X}'(\partial^2\boldsymbol{\Omega}/\partial\rho_\mu\partial\rho_\mu)\boldsymbol{X}/T \\
&= \boldsymbol{X}'\boldsymbol{\Omega}_{\rho_\mu\rho_\mu}\boldsymbol{X}/T = [(\boldsymbol{X}'_i)_{i=1,\ldots,M}][(\delta_{ij}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{i\mu})_{i,j=1,\ldots,M}][(\boldsymbol{X}_j)_{j=1,\ldots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{ij}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{X}'_i\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{i\mu}\boldsymbol{X}_j/T \\
&= \sum_{i=1}^{M}\delta_{\mu i}\sigma^{i\mu}\boldsymbol{X}'_i\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{i\mu}\boldsymbol{X}_i/T \\
&= \sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T, \qquad \text{(C.26)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}^*{}_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{X}'\boldsymbol{\Omega}^*{}_{\rho_\mu\rho_{\mu'}}\boldsymbol{X}/T \\
&= \left[(\boldsymbol{X}'_i)_{i=1,\ldots,M}\right]\left[\left(\delta_{ij}\delta_{\mu i}\delta_{\mu' j}\sigma^{i\mu}\sigma_{ii}\sigma^{j\mu'}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{ii}\boldsymbol{R}_{\rho_{\mu'}}{}^{j\mu'}\right)_{i,j=1,\ldots,M}\right]\left[(\boldsymbol{X}_j)_{j=1,\ldots,M}\right]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta\delta_{\mu i}\delta_{\mu' j}\sigma^{i\mu}\sigma_{ii}\sigma^{j\mu'}\boldsymbol{X}'_i\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{R}_{ii}\boldsymbol{R}_{\rho_{\mu'}}{}^{j\mu'}\boldsymbol{X}_j/T \\
&= \delta_{\mu\mu'}\sigma^{\mu\mu}\sigma_{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/T \\
&= \delta_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/T. \qquad \text{(C.27)}
\end{aligned}$$

Also, by using equations (C.16), (C.17) and (C.21) we have

$$\begin{aligned}
\boldsymbol{A}_{(\mu\mu)} &= \frac{\partial \boldsymbol{A}}{\partial\sigma^{\mu\mu}} = \partial(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\sigma^{\mu\mu} = \boldsymbol{X}'(\partial\boldsymbol{\Omega}/\partial\sigma^{\mu\mu})\boldsymbol{X}/T \\
&= \boldsymbol{X}'\boldsymbol{\Omega}_{\mu\mu}\boldsymbol{X}/T \\
&= [(\boldsymbol{X}'_i)_{i=1,\ldots,M}][(\delta_{\mu i}\delta_{j\mu}\boldsymbol{R}^{\mu\mu})_{i,j=1,\ldots,M}][(\boldsymbol{X}_j)_{j=1,\ldots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{\mu i}\delta_{j\mu}\boldsymbol{X}'_i\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_j = \boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T \\
&= = [\text{see(C.24)}] = \boldsymbol{B}_{\mu\mu}, \qquad \text{(C.28)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}_{(\mu\mu)(\nu\nu)} &= \frac{\partial^2 \boldsymbol{A}}{\partial\sigma^{\mu\mu}\partial\sigma^{\nu\nu}} = \partial^2(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\sigma^{\mu\mu}\partial\sigma^{\nu\nu} \\
&= \boldsymbol{X}'(\partial^2\boldsymbol{\Omega}/\partial\sigma^{\mu\mu}\partial\sigma^{\nu\nu})\boldsymbol{X}/T = \boldsymbol{X}'\boldsymbol{\Omega}_{(\mu\mu)(\nu\nu)}\boldsymbol{X}/T = [\text{see(C.17)}] = 0, \qquad \text{(C.29)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}^*{}_{(\mu\mu)(\nu\nu)} &= \boldsymbol{X}'\boldsymbol{\Omega}^*{}_{(\mu\mu)(\nu\nu)}\boldsymbol{X}/T = [\text{see(C.20)}] \\
&= [(\boldsymbol{X}_i')_{i=1,\dots,M}][(\delta_{\mu i}\delta_{j\nu'}\sigma_{\mu'\nu}\boldsymbol{R}^{\mu\nu'})_{i,j=1,\dots,M}][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{\mu i}\delta_{\mu\nu}\delta_{j\nu}\sigma_{\mu\mu}\boldsymbol{X}_i'\boldsymbol{R}^{ij}\boldsymbol{X}_j/T \\
&= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\nu}\boldsymbol{X}_\nu/T = [\text{see(C.20)}] \\
&= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T \\
&= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{B}_{\mu\mu} \\
&= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{A}_{(\mu\mu)}. \qquad \text{(C.30)}
\end{aligned}$$

Moreover by using equations (C.18)

$$\begin{aligned}
\boldsymbol{A}_{\rho_\mu(\nu\nu)} &= \boldsymbol{A}_{(\nu\nu)\rho_\mu} = \frac{\partial^2\boldsymbol{A}}{\partial\rho_\mu\partial\sigma^{(\nu\nu)}} = \partial^2(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)/\partial\rho_\mu\partial\sigma^{(\nu\nu)} \\
&= \boldsymbol{X}'(\partial^2\boldsymbol{\Omega}/\partial\rho_\mu\partial\sigma^{(\nu\nu)})\boldsymbol{X}/T \\
&= [(\boldsymbol{X}_i')_{i=1,\dots,M}][(\delta_{\mu\nu}\delta_{\nu i}\delta_{j\nu}\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu})_{i,j=1,\dots,M}][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{\mu\nu}\delta_{\nu i}\delta_{j\nu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_j/T = \delta_{\mu\nu}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/T, \qquad \text{(C.31)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}^*{}_{(\nu\nu)\rho_\mu} &= \boldsymbol{A}^*{}_{\rho_\mu(\nu\nu)} = \boldsymbol{X}'\boldsymbol{\Omega}^*{}_{\rho_\mu(\nu\nu)}\boldsymbol{X}/T = \boldsymbol{X}'\boldsymbol{\Omega}_{\rho_\mu}\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{(\nu\nu)}\boldsymbol{X}/T = [\text{see(C.22)}] \\
&= [(\boldsymbol{X}_i')_{i=1,\dots,M}]\left[\left(\delta_{\mu i}\delta_{i\nu}\delta_{j\nu}\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\right)_{ij=1,\dots,M}\right][(\boldsymbol{X}_j)_{j=1,\dots,M}]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{\mu i}\delta_{i\nu}\delta_{j\nu}\boldsymbol{X}_i'\boldsymbol{R}_{\rho_\mu}{}^{i\mu}\boldsymbol{X}_j/T \\
&= \delta_{\mu\nu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T. \qquad \text{(C.32)}
\end{aligned}$$

Define the $n\times n$ matrices

$$\boldsymbol{G} = \boldsymbol{A}^{-1} \text{ and } \boldsymbol{\Xi} = \boldsymbol{G}\boldsymbol{Q}\boldsymbol{G}, \qquad \text{(C.33)}$$

where

$$\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T \text{ and } \boldsymbol{Q} = \boldsymbol{H}'(\boldsymbol{H}\boldsymbol{G}\boldsymbol{H}')^{-1}\boldsymbol{H}. \qquad \text{(C.34)}$$

By using (C.33) and (C.34) we find the following results:

1.

$$\begin{aligned}
\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi} &= [\text{see(C.25)}] \\
&= \sigma^{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T \Rightarrow
\end{aligned}$$

$$\mathrm{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}) = \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T). \tag{C.35}$$

2.

$$\begin{aligned}\boldsymbol{A}_{\rho_\mu\rho_\mu}\boldsymbol{\Xi} &= [\text{see(C.26)}]\\ &= \sigma^{\mu\mu}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T \Rightarrow\\ \mathrm{tr}(\boldsymbol{A}_{\rho_\mu\rho_\mu}\boldsymbol{\Xi}) &= \delta_{\mu\mu}\sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T). \end{aligned}\tag{C.36}$$

3.

$$\begin{aligned}\boldsymbol{A}^*{}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi} &= [\text{see(C.27)}]\\ &= \delta_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T \Rightarrow\\ \mathrm{tr}(\boldsymbol{A}^*{}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}) &= \delta_{\mu\mu'}\sigma^{\mu'\mu'}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T). \end{aligned}\tag{C.37}$$

4.

$$\begin{aligned}\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi} &= [\text{see(C.28)}]\\ &= \boldsymbol{B}_{\mu\mu}\boldsymbol{\Xi} \Rightarrow\\ \mathrm{tr}(\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi}) &= \mathrm{tr}(\boldsymbol{B}_{\mu\mu}\boldsymbol{\Xi}) = \mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T). \end{aligned}\tag{C.38}$$

5. Since

$$\begin{aligned}\boldsymbol{A}_{(\mu\mu)(\nu\nu)}\boldsymbol{\Xi} &= 0 = [\text{see(C.29)}] \Rightarrow\\ \mathrm{tr}(\boldsymbol{A}_{(\mu\mu)(\nu\nu)}\boldsymbol{\Xi}) &= 0. \end{aligned}\tag{C.39}$$

6. Since

$$\begin{aligned}\boldsymbol{A}^*{}_{(\mu\mu)(\nu\nu)}\boldsymbol{\Xi} &= [\text{see(C.30)}]\\ &= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi} \Rightarrow\\ \mathrm{tr}(\boldsymbol{A}^*{}_{(\mu\mu)(\nu\nu)}\boldsymbol{\Xi}) &= \delta_{\mu\nu}\sigma_{\mu\mu}\,\mathrm{tr}(\boldsymbol{A}_{\mu\mu}\boldsymbol{\Xi}) = [\text{see(B.81)}]\\ &= \delta_{\mu\nu}\sigma_{\mu\mu}\,\mathrm{tr}(\boldsymbol{B}_{\mu\mu}\boldsymbol{\Xi}) = \delta_{\mu\nu}\sigma_{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T). \end{aligned}\tag{C.40}$$

7.

$$\begin{aligned}\boldsymbol{A}_{\rho_\mu(\nu\nu)}\boldsymbol{\Xi} &= [\text{see(C.31)}]\\ &= \boldsymbol{X}'_\nu \boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/T \Rightarrow\\ \mathrm{tr}(\boldsymbol{A}_{\rho_\mu(\nu\nu)}\boldsymbol{\Xi}) &= \mathrm{tr}(\boldsymbol{X}'_\nu \boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/T). \end{aligned}\tag{C.41}$$

8.

$$\begin{aligned} \boldsymbol{A}^*{}_{\rho_\mu(\nu\nu)}\boldsymbol{\Xi} &= [\text{see(C.32)}] \\ &= \delta_{\mu\nu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T \Rightarrow \\ \text{tr}(\boldsymbol{A}^*{}_{\rho_\mu(\nu\nu)}\boldsymbol{\Xi}) &= \delta_{\mu\nu}\,\text{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T). \end{aligned} \tag{C.42}$$

9.

$$\begin{aligned} \boldsymbol{A}^*{}_{(\nu\nu)\rho_\mu}\boldsymbol{\Xi} &= [\text{see(C.32)}] \\ &= \delta_{\mu\nu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T \Rightarrow \\ \text{tr}(\boldsymbol{A}^*{}_{\rho_\mu(\nu\nu)}\boldsymbol{\Xi}) &= \delta_{\mu\nu}\,\text{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T). \end{aligned} \tag{C.43}$$

10.

$$\begin{aligned} \boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}} &= [\text{see(C.25)}] \\ &= \left(\sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{i\mu\mu}\boldsymbol{X}_\mu/T\right)\boldsymbol{G}\left(\sigma^{\mu'\mu'}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/T\right) \\ &= \sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/T^2 \Rightarrow \\ \boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}}\boldsymbol{\Xi} &= \sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T^2 \\ \text{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}}\boldsymbol{\Xi}) &= \sigma^{\mu\mu}\sigma^{\mu'\mu'}\,\text{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T^2). \end{aligned} \tag{C.44}$$

11. Similarly, by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\text{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_{\mu'}}\boldsymbol{\Xi}) = \sigma^{\mu\mu}\sigma^{\mu'\mu'}\,\text{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T^2). \tag{C.45}$$

12.

$$\begin{aligned} \boldsymbol{A}_{(\mu\mu)}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)} &= [\text{see(C.28)}] \\ &= \boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu} \Rightarrow \\ \boldsymbol{A}_{(\mu\mu)}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi} &= \boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu}\boldsymbol{\Xi} \Rightarrow \\ \text{tr}(\boldsymbol{A}_{(\mu\mu)}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}) &= \text{tr}(\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu}\boldsymbol{\Xi}) = [\text{see(C.24)}] \\ &= \text{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/T^2). \end{aligned} \tag{C.46}$$

13. Similarly, by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\text{tr}(\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}) = \text{tr}(\boldsymbol{B}_{\mu\mu}\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu}\boldsymbol{\Xi}) = \text{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/T^2). \tag{C.47}$$

14.

$$\begin{aligned}
\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)} &= \text{[see (C.25) and (C.28)]} \\
&= \left(\sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T\right)\boldsymbol{G}\boldsymbol{B}_{\nu\nu} \\
&= \sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{B}_{\nu\nu}/T \Rightarrow \text{[see (C.24)]} \\
\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi} &= \sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/T^2 \Rightarrow \\
\mathrm{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}) &= \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/T^2). \qquad \text{(C.48)}
\end{aligned}$$

15. Similarly, by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\mathrm{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}) = \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/T^2). \qquad \text{(C.49)}$$

16.

$$\begin{aligned}
\boldsymbol{A}_{(\nu\nu)}\boldsymbol{G}\boldsymbol{A}_{\rho_\mu} &= \text{[see (C.25) and (C.28)]} \\
&= \boldsymbol{B}_{\nu\nu}\boldsymbol{G}\left(\sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T\right) \\
&= \sigma^{\mu\mu}\boldsymbol{B}_{\nu\nu}\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T \Rightarrow \text{[see (C.24)]} \\
\boldsymbol{A}_{(\nu\nu)}\boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi} &= \sigma^{\mu\mu}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T^2 \Rightarrow \\
\mathrm{tr}(\boldsymbol{A}_{(\nu\nu)}\boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}) &= \mathrm{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}) \\
&= \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T^2). \qquad \text{(C.50)}
\end{aligned}$$

17. Similarly, by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\mathrm{tr}(\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}) = \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T^2). \qquad \text{(C.51)}$$

Proof. [Proof of Theorem 5]

i a. From (C.26), (C.27) and (C.44) we have that

$$\begin{aligned}
\boldsymbol{C}_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{A}^*{}_{\rho_\mu\rho_{\mu'}} - 2\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{\rho_{\mu'}} + \boldsymbol{A}_{\rho_\mu\rho_{\mu'}}/2 \\
&= \delta_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/T - 2\sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/T^2 \\
&\quad + \delta_{\mu\mu'}\sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/2T. \qquad \text{(C.52)}
\end{aligned}$$

ii a. From (C.44), by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\begin{aligned}
\boldsymbol{D}_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_{\mu'}}/2 \\
&= \sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}'_{\mu'}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}/2T^2. \qquad \text{(C.53)}
\end{aligned}$$

iii a.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{G} &= [\text{see (C.25)}] = \boldsymbol{G}\left(\sigma^{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T\right)\boldsymbol{G} \\ &= \sigma^{\mu\mu}\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}/T. \end{aligned} \tag{C.54}$$

iv a.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G} &= [\text{see (C.52)}] = \delta_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}/T \\ & -2\sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}_{\mu'}'\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}/T^2 + \delta_{\mu\mu'}\sigma^{\mu\mu}\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}/2T. \end{aligned} \tag{C.55}$$

i b. From (C.29), (C.30) and (C.46) we have that

$$\begin{aligned} \boldsymbol{C}_{(\mu\mu)(\nu\nu)} &= \boldsymbol{A}^*{}_{(\mu\mu)(\nu\nu)} - 2\boldsymbol{A}_{(\mu\mu)}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)} + \boldsymbol{A}_{(\mu\mu)(\nu\nu)}/2 \\ &= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{A}_{(\mu\mu)} - 2\boldsymbol{A}_{(\mu\mu)}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)} \\ &= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{B}_{\mu\mu} - 2\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu}. \end{aligned} \tag{C.56}$$

ii b.From (C.46) by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\begin{aligned} \boldsymbol{D}_{(\mu\mu)(\nu\nu)} &= \boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi}\boldsymbol{A}_{(\nu\nu)}/2 \\ &= \boldsymbol{B}_{\mu\mu}\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu}/2. \end{aligned} \tag{C.57}$$

iii b.

$$\boldsymbol{G}\boldsymbol{A}_{(\mu\mu)}\boldsymbol{G} = [\text{see (C.28)}] = \boldsymbol{G}\boldsymbol{B}_{\mu\mu}\boldsymbol{G}. \tag{C.58}$$

iv b.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{C}_{(\mu\mu)(\nu\nu)}\boldsymbol{G} &= [\text{see (C.56)}] \\ &= \boldsymbol{G}[\delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{B}_{\mu\mu} - 2\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu}]\boldsymbol{G}. \end{aligned} \tag{C.59}$$

i c. From (C.31), (C.32) and (C.48)

$$\begin{aligned} \boldsymbol{C}_{\rho_\mu(\nu\nu)} &= \boldsymbol{A}^*{}_{\rho_\mu(\nu\nu)} - 2\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)} + \boldsymbol{A}_{\rho_\mu(\nu\nu)}/2 \\ &= \delta_{\mu\nu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T \\ & \quad -2\sigma^{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{B}_{\nu\nu}/T \\ & \quad +\delta_{\mu\nu}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/2T \\ &= \delta_{\mu\nu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T - 2\sigma^{\mu\mu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu/T^2 \\ & \quad +\delta_{\mu\nu}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/2T. \end{aligned} \tag{C.60}$$

ii c.From (C.48), by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\begin{aligned} \boldsymbol{D}_{\rho_\mu(\nu\nu)} &= \boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{(\nu\nu)}/2 \\ &= \sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu}/2T \\ &= \sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu/2T^2. \end{aligned} \tag{C.61}$$

iii c.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{C}_{\rho_\mu(\nu\nu)}\boldsymbol{G} &= [\text{see (C.60)}] \\ &= \boldsymbol{G}\left[\delta_{\mu\nu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T - 2\sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu/T^2 + \delta_{\mu\nu}\boldsymbol{X}'_\nu\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/2T\right]\boldsymbol{G}. \end{aligned} \tag{C.62}$$

i d.From (C.31), (C.32) and (C.50)

$$\begin{aligned} \boldsymbol{C}_{(\nu\nu)\rho_\mu} &= \boldsymbol{A}^*{}_{(\nu\nu)\rho_\mu} - 2\boldsymbol{A}_{(\nu\nu)}\boldsymbol{G}\boldsymbol{A}_{\rho_\mu} + \boldsymbol{A}_{(\nu\nu)\rho_\mu}/2 \\ &= \delta_{\mu\nu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T \\ &\quad -2\sigma^{\mu\mu}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T^2 \\ &\quad +\delta_{\mu\nu}\boldsymbol{X}'_\nu\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/2T. \end{aligned} \tag{C.63}$$

ii d.From (C.50), by substituting $\boldsymbol{\Xi}$ for $\boldsymbol{G}$ we find that

$$\begin{aligned} \boldsymbol{D}_{(\nu\nu)\rho_\mu} &= \boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_\mu}/2 \\ &= \sigma^{\mu\mu}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/2T^2. \end{aligned} \tag{C.64}$$

iii d.

$$\begin{aligned} \boldsymbol{G}\boldsymbol{C}_{(\nu\nu)\rho_\mu}\boldsymbol{G} &= [\text{see (C.63)}] \\ &= \boldsymbol{G}\left[\delta_{\mu\nu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T - 2\sigma^{\mu\mu}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T^2 + \delta_{\mu\nu}\boldsymbol{X}'_\nu\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/2T\right]\boldsymbol{G}. \end{aligned} \tag{C.65}$$

1. a. The μ-th element of the $((M+M)\times 1)$ vector $\boldsymbol{l}$ is

$$\begin{aligned} l_{\rho_\mu} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{A}_{\rho_\mu}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} = [\text{see (C.54)}] \\ &= \frac{\boldsymbol{e}'}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}\left(\sigma^{\mu\mu}\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}/T\right)\frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}} \\ &= \sigma^{\mu\mu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{h}/T, \end{aligned} \tag{C.66}$$

where

$$\boldsymbol{h} = \frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}. \tag{C.67}$$

2. a. Similarly, the $(\mu\mu')$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{L}$ is

$$\begin{aligned} l_{\rho_\mu\rho_{\mu'}} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\ &= \frac{\boldsymbol{e}'}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G}\frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}} = \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{G}\boldsymbol{h} = [\text{see (C.52)}] \\ &= \delta_{\mu\mu'}\sigma^{\mu'\mu'}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{h}/T - 2\sigma^{\mu\mu}\sigma^{\mu'\mu'}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}_{\mu'}'\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{G}\boldsymbol{h}/T^2 \\ &\quad +\delta_{\mu\mu'}\sigma^{\mu\mu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{h}/2T. \end{aligned} \tag{C.68}$$

3. a. The μ-th element of the $((M+M)\times 1)$ vector $\boldsymbol{c}$ is

$$\begin{aligned} c_{\rho_\mu} &= tr(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}) = [\text{see (C.35)}] \\ &= \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T). \end{aligned} \tag{C.69}$$

4. a. The $(\mu\mu')$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned} c_{\rho_\mu\rho_{\mu'}} &= \mathrm{tr}(\boldsymbol{C}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}) = [\text{see (C.52)}] \\ &= \delta_{\mu\mu'}\sigma^{\mu'\mu'}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T) - 2\sigma^{\mu\mu}\sigma^{\mu'\mu'}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}_{\mu'}'\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/T^2) \\ &\quad +\delta_{\mu\mu'}\sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/2T). \end{aligned} \tag{C.70}$$

5. a. The $(\mu\mu')$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{D}$ is

$$\begin{aligned} d_{\rho_\mu\rho_{\mu'}} &= \mathrm{tr}(\boldsymbol{D}_{\rho_\mu\rho_{\mu'}}\boldsymbol{\Xi}) = [\text{see (C.53)}] \\ &= \sigma^{\mu\mu}\sigma^{\mu'\mu'}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}_{\mu'}'\boldsymbol{R}_{\rho_{\mu'}}{}^{\mu'\mu'}\boldsymbol{X}_{\mu'}\boldsymbol{\Xi}/2T^2). \end{aligned} \tag{C.71}$$

1. b. The $(\mu\mu)$-th element of the $((M+M)\times 1)$ vector $\boldsymbol{l}$ is

$$\begin{aligned} l_{(\mu\mu)} &= [\text{see (C.28)}] = \boldsymbol{e}'\boldsymbol{G}\boldsymbol{A}_{(\mu\mu)}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\ &= \frac{\boldsymbol{e}'}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}\boldsymbol{G}\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}} \\ &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{h} = [\text{see (C.24), (C.28)}] \\ &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{h}/T. \end{aligned} \tag{C.72}$$

2. b. Similarly, the $((\mu\mu),(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{L}$ is

$$\begin{aligned} l_{(\mu\mu)(\nu\nu)} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{(\mu\mu)(\nu\nu)}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\ &= \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{(\mu\mu)(\nu\nu)}\boldsymbol{G}\boldsymbol{h} = [\text{see (C.59)}] \\ &= \boldsymbol{h}'\boldsymbol{G}[\delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{B}_{\mu\mu} - 2\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu}]\boldsymbol{G}\boldsymbol{h} \\ &= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{h} - 2\boldsymbol{h}'\boldsymbol{G}\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu}\boldsymbol{G}\boldsymbol{h} \\ &= \delta_{\mu\nu}\sigma_{\mu\mu}l_{(\mu\mu)} - 2\boldsymbol{h}'\boldsymbol{G}\boldsymbol{B}_{\mu\mu}\boldsymbol{G}\boldsymbol{B}_{\nu\nu}\boldsymbol{G}\boldsymbol{h} \implies \end{aligned}$$

$$
\begin{aligned}
l_{(\mu\mu)(\nu\nu)} &= \text{[see (C.24) and (C.72)]} \\
&= \delta_{\mu\nu}\sigma_{\mu\mu}\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{h}/T - 2\boldsymbol{h}'\boldsymbol{G}\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{G}\boldsymbol{h}/T^2. \qquad (C.73)
\end{aligned}
$$

3. b.The $(\mu\mu)$-th element of the $((M+M)\times 1)$ vector $\boldsymbol{c}$ is

$$
\begin{aligned}
c_{(\mu\mu)} &= \mathrm{tr}(\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi}) = \text{[see (C.38)]} \\
&= \mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T). \qquad (C.74)
\end{aligned}
$$

4. b.The $((\mu\mu),(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{C}$ is

$$
\begin{aligned}
c_{(\mu\mu)(\nu\nu)} &= \mathrm{tr}(\boldsymbol{C}_{(\mu\mu)(\nu\nu)}\boldsymbol{\Xi}) = \text{[see (C.56)]} \\
&= \delta_{\mu\nu}\sigma_{\mu\mu}\,\mathrm{tr}(\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi}) - 2\,\mathrm{tr}(\boldsymbol{A}_{\varsigma(\mu\mu)}\boldsymbol{G}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}) \\
&= \text{[see(C.24)]} \\
&= \delta_{\mu\nu}\sigma_{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi})/T - 2(\mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/T^2). \qquad (C.75)
\end{aligned}
$$

5. b. The $((\mu\mu),(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{D}$ is

$$
\begin{aligned}
d_{(\mu\mu)(\nu\nu)} &= \mathrm{tr}(\boldsymbol{D}_{(\mu\mu)(\nu\nu)}\boldsymbol{\Xi}) = \text{[see (C.57)]} = \mathrm{tr}(\boldsymbol{A}_{(\mu\mu)}\boldsymbol{\Xi}\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi})/2 \\
&= \mathrm{tr}(\boldsymbol{B}_{\mu\mu}\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu}\boldsymbol{\Xi})/2 = \text{[see (C.24)]} \\
&= \mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/2T^2). \qquad (C.76)
\end{aligned}
$$

1. c. Similarly the $(\mu,(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{L}$ is

$$
\begin{aligned}
l_{\rho_\mu(\nu\nu)} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu(\nu\nu)}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} \\
&= \frac{\boldsymbol{e}'}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}}\boldsymbol{G}\boldsymbol{C}_{\rho_\mu(\nu\nu)}\boldsymbol{G}\frac{\boldsymbol{e}}{(\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e})^{1/2}} = \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{\rho_\mu(\nu\nu)}\boldsymbol{G}\boldsymbol{h} = \text{[see (C.62)]} \\
&= \boldsymbol{h}'\boldsymbol{G}\left[\delta_{\mu\nu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T - 2\sigma^{\mu\mu}\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu/T^2 + \delta_{\mu\nu}\boldsymbol{X}'_\nu\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/T\right]\boldsymbol{G}\boldsymbol{h}. \qquad (C.77)
\end{aligned}
$$

2. c.The $(\mu,(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{C}$ is

$$
\begin{aligned}
c_{\rho_\mu(\nu\nu)} &= \text{[see (C.60)]} = \mathrm{tr}(\boldsymbol{C}_{\rho_\mu(\nu\nu)}\boldsymbol{\Xi}) \\
&= \delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T) \\
&\quad -2\sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{G}\boldsymbol{B}_{\nu\nu}\boldsymbol{\Xi}/T) \\
&\quad +\delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}'_\nu\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/2T) \\
&= \text{[see (C.24)]} = \delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T) \\
&\quad -2\sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}'_\mu\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}'_\nu\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/2T^2) \\
&\quad +\delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}'_\nu\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/2T). \qquad (C.78)
\end{aligned}
$$

3. c. The $(\mu,(\nu\nu))$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{D}$ is

$$\begin{aligned} d_{\rho_\mu(\nu\nu)} &= \mathrm{tr}(\boldsymbol{D}_{\rho_\mu(\nu\nu)}\boldsymbol{\Xi}) = [\text{see (C.61)}] = \mathrm{tr}(\boldsymbol{A}_{\rho_\mu}\boldsymbol{\Xi}\boldsymbol{A}_{(\nu\nu)}/2) \\ &= \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{B}_{\nu\nu}\boldsymbol{\Xi}/2T) \\ &= \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/2T^2). \end{aligned} \tag{C.79}$$

1. d. The $((\nu\nu),\mu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{L}$ is

$$\begin{aligned} l_{(\nu\nu)\rho_\mu} &= \boldsymbol{e}'\boldsymbol{G}\boldsymbol{C}_{(\nu\nu)\rho_\mu}\boldsymbol{G}\boldsymbol{e}/\boldsymbol{e}'\boldsymbol{G}\boldsymbol{e} = \boldsymbol{h}'\boldsymbol{G}\boldsymbol{C}_{(\nu\nu)\rho_\mu}\boldsymbol{G}\boldsymbol{h} = [\text{see (C.65)}] \\ &= \boldsymbol{h}'\boldsymbol{G}\left[\delta_{\mu\nu}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T - 2\sigma^{\mu\mu}\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu/T^2 + \delta_{\mu\nu}\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu/2T\right]\boldsymbol{G}\boldsymbol{h}. \end{aligned} \tag{C.80}$$

2. d.The $((\nu\nu),\mu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{C}$ is

$$\begin{aligned} c_{(\nu\nu)\rho_\mu} &= \mathrm{tr}(\boldsymbol{C}_{(\nu\nu)\rho_\mu}\boldsymbol{\Xi}) = [\text{see (C.63)}] \\ &= \delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T) \\ &\quad -2\sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{G}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/T^2) \\ &\quad +\delta_{\mu\nu}\,\mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}_{\rho_\mu}{}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}/2T). \end{aligned} \tag{C.81}$$

3. d.The $((\nu\nu),\mu)$-th element of the $((M+M)\times(M+M))$ matrix $\boldsymbol{D}$ is

$$\begin{aligned} d_{(\nu\nu)\rho_\mu} &= \mathrm{tr}(\boldsymbol{D}_{(\nu\nu)\rho_\mu}\boldsymbol{\Xi}) = [\text{see (C.64)}] = \mathrm{tr}(\boldsymbol{A}_{(\nu\nu)}\boldsymbol{\Xi}\boldsymbol{A}_{\rho_\mu}/2) \\ &= \sigma^{\mu\mu}\,\mathrm{tr}(\boldsymbol{X}_\nu'\boldsymbol{R}^{\nu\nu}\boldsymbol{X}_\nu\boldsymbol{\Xi}\boldsymbol{X}_\mu'\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu}\boldsymbol{X}_\mu\boldsymbol{\Xi}/2T^2). \end{aligned} \tag{C.82}$$

□

Lemma C.1. For all estimators $\hat{\boldsymbol{B}}_I$, (I=UL, RL, GL, IG, ML) of B the following results hold:

$$\hat{\boldsymbol{B}}_I = \boldsymbol{B} + \tau\boldsymbol{B}_1{}^I + \omega(\tau^2), \tag{C.83}$$

where

$$\boldsymbol{B}_1{}^{UL} = (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T}, \tag{C.84}$$

$$\mathrm{vec}(\boldsymbol{B}_1{}^{RL}) = \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*/T)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon}/\sqrt{T}, \tag{C.85}$$

$$\begin{aligned} \mathrm{vec}(\boldsymbol{B}_1{}^{GL}) &= \mathrm{vec}(\boldsymbol{B}_1{}^{IG}) = \mathrm{vec}(\boldsymbol{B}_1{}^{ML}) \\ &= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T}. \end{aligned} \tag{C.86}$$

Proof of Lemma C.1. i.

$$\begin{aligned}\hat{\boldsymbol{B}}_{UL} &= (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{Y}_* = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{Z}\boldsymbol{B}+\boldsymbol{E}) \\ &= \boldsymbol{B} + (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E} = \boldsymbol{B} + \tau(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} \\ &= \boldsymbol{B} + \tau \boldsymbol{B_1}^{UL}. \end{aligned} \tag{C.87}$$

ii. Since

$$\operatorname{vec}(\boldsymbol{B}) = \begin{bmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_M \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}_1\boldsymbol{\beta} \\ \vdots \\ \boldsymbol{\Psi}_M\boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}_1 \\ \vdots \\ \boldsymbol{\Psi}_M \end{bmatrix}\boldsymbol{\beta} = \boldsymbol{\Psi}\boldsymbol{\beta}, \tag{C.88}$$

by vectorizing (5.34) we take

$$\begin{aligned}\boldsymbol{y}_* &= \operatorname{vec}(\boldsymbol{Y}_*) = \operatorname{vec}(\boldsymbol{Z}\boldsymbol{B}+\boldsymbol{E}) = \operatorname{vec}(\boldsymbol{Z}\boldsymbol{B}) + \operatorname{vec}(\boldsymbol{E}) \\ &= (\boldsymbol{I}\otimes\boldsymbol{Z})\operatorname{vec}(\boldsymbol{B}) + \boldsymbol{\varepsilon} = (\boldsymbol{I}\otimes\boldsymbol{Z})\boldsymbol{\Psi}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \boldsymbol{X}_*\boldsymbol{\beta} + \boldsymbol{\varepsilon}. \end{aligned} \tag{C.89}$$

Thus,

$$\begin{aligned}\operatorname{vec}(\hat{\boldsymbol{B}}_{RL}) &= \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{y}_* \\ &= \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'(\boldsymbol{X}_*\boldsymbol{\beta}+\boldsymbol{\varepsilon}) = \boldsymbol{\Psi}\boldsymbol{\beta} + \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon} \\ &= \boldsymbol{\Psi}\boldsymbol{\beta} + \tau\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*/T)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon}/\sqrt{T} = \operatorname{vec}(\boldsymbol{B}) + \tau\operatorname{vec}(\boldsymbol{B_1}^{RL}) \Rightarrow \qquad \text{(C.90)} \\ \Rightarrow \hat{\boldsymbol{B}}_{RL} &= \boldsymbol{B} + \tau\boldsymbol{B_1}^{RL}. \qquad \text{(C.91)}\end{aligned}$$

iii. For any consistent estimator $\hat{\boldsymbol{\Sigma}}^{-1}$ of $\boldsymbol{\Sigma}^{-1}$ it holds that

$$\hat{\boldsymbol{\Sigma}}^{-1} = \boldsymbol{\Sigma}^{-1} + \omega(\tau), \tag{C.92}$$

which implies that

$$(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T) = (\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T) + \omega(\tau). \tag{C.93}$$

Therefore,

$$\begin{aligned}\operatorname{vec}(\hat{\boldsymbol{B}}_{GL}) &= \boldsymbol{\Psi}(\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{y}_* \\ &= \boldsymbol{\Psi}(\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\boldsymbol{I}_T)(\boldsymbol{X}_*\boldsymbol{\beta}+\boldsymbol{\varepsilon}) \\ &= \boldsymbol{\Psi}\boldsymbol{\beta} + \tau\boldsymbol{\Psi}[\boldsymbol{X}_*'((\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)+\omega(\tau))\boldsymbol{X}_*/T]^{-1}\boldsymbol{X}_*'((\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)+\omega(\tau))\boldsymbol{\varepsilon}/\sqrt{T} \\ &= \operatorname{vec}(\boldsymbol{B}) + \tau\boldsymbol{\Psi}[(\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T) + \tau\omega(\tau^2)]^{-1}[(\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T}) + \omega(\tau^2)] \\ &= \operatorname{vec}(\boldsymbol{B}) + \tau\boldsymbol{\Psi}[(\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T)^{-1} + \tau\omega(\tau^2)][(\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T}) + \omega(\tau^2)] \\ &= \operatorname{vec}(\boldsymbol{B}) + \tau\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{X}_*/T]^{-1}\boldsymbol{X}_*(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T} + \omega(\tau^2) \\ &= \operatorname{vec}(\boldsymbol{B}) + \tau\operatorname{vec}(\boldsymbol{B_1}^{GL}) + \omega(\tau^2) \Rightarrow \end{aligned} \tag{C.94}$$

$$\hat{\boldsymbol{B}}_{GL} \quad = \quad \boldsymbol{B} + \tau \boldsymbol{B}_1{}^{GL} + \omega(\tau^2). \tag{C.95}$$

Since $\hat{\boldsymbol{B}}_{IG}$ and $\hat{\boldsymbol{B}}_{ML}$ are the outcome of iterative use of the GL-estimation process, equation (C.94) implies that

$$\hat{\boldsymbol{B}}_{IG} = \boldsymbol{B} + \tau \boldsymbol{B}_1{}^{IG} + \omega(\tau^2) \tag{C.96}$$

and

$$\hat{\boldsymbol{B}}_{ML} = \boldsymbol{B} + \tau \boldsymbol{B}_1{}^{ML} + \omega(\tau^2), \tag{C.97}$$

where

$$\mathrm{vec}(\boldsymbol{B}_1{}^{IG}) = \mathrm{vec}(\boldsymbol{B}_1{}^{ML}) = \mathrm{vec}(\boldsymbol{B}_1{}^{GL}). \tag{C.98}$$

So, equations ((C.87), (C.90), (C.94), (C.96), (C.97) and (C.98)) complete the proof.

□

Lemma C.2. For any conformable matrix $\boldsymbol{\Gamma}$ lemma C.1 implies that

$$\lim_{T\to\infty} T\,\mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] = \lim_{T\to\infty} \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]. \tag{C.99}$$

Proof of Lemma C.2.

$$\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL} = (B + \tau \boldsymbol{B}_1{}^I + \omega(\tau^2)) - (\boldsymbol{B} + \tau \boldsymbol{B}_1{}^{UL}) = \tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2) \Rightarrow \tag{C.100}$$

$$\begin{aligned}
(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL}) &= [\tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2)]'\boldsymbol{\Gamma}[\tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2)] \\
&= \tau^2(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^3) \Rightarrow \\
T\,\mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] &= \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})] + O(\tau) \Rightarrow \\
\lim_{T\to\infty} T\,\mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] &= \lim_{T\to\infty} \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})].
\end{aligned} \tag{C.101}$$

□

Lemma C.3. Since the rows $\boldsymbol{\varepsilon}_t'$ $(t = 1, \ldots, T)$ of $\boldsymbol{E}$ are independent $\mathcal{N}_M(\boldsymbol{0}, \boldsymbol{\Sigma})$ vectors, the matrix $\boldsymbol{E}'\boldsymbol{E}$ has a Wishart distribution with weight matrix $\boldsymbol{\Sigma}$ and T degrees of freedom i.e,

$$\boldsymbol{E}'\boldsymbol{E} \sim \mathcal{W}(\boldsymbol{\Sigma}, T), \;\; \mathrm{E}(\boldsymbol{E}'\boldsymbol{E}) = T\boldsymbol{\Sigma}. \tag{C.102}$$

Then,

$$\mathrm{E}(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}) = T(M + T + 1)\boldsymbol{\Sigma}. \tag{C.103}$$

Proof of Lemma C.3.

$$\boldsymbol{E}'\boldsymbol{E} = (\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_T)\begin{bmatrix}\boldsymbol{\varepsilon}_1' \\ \vdots \\ \boldsymbol{\varepsilon}_T'\end{bmatrix} = \sum_{t=1}^{T} \boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' \tag{C.104}$$

$$\begin{aligned} \Rightarrow \boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E} \quad &= \quad \sum_{t=1}^{T} \boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\sum_{t'=1}^{T}\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}' \\ = \sum_{t=1}^{T}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' \quad &+ \quad \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}' \end{aligned} \tag{C.105}$$

where $\boldsymbol{\varepsilon}_t'$ and $\boldsymbol{\varepsilon}_{t'}'$ are independent $\mathcal{N}_M(\boldsymbol{0}, \boldsymbol{\Sigma})$ vectors for $t \neq t'$.

Let $\boldsymbol{g}$ be any arbitrary $(M \times 1)$ non-stochastic vector. Then,

$$\begin{aligned} \boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{g} &= \operatorname{tr}(\boldsymbol{g}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{g}) \\ &= \operatorname{tr}(\boldsymbol{\varepsilon}_t'\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t) = \boldsymbol{\varepsilon}_t'\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t \Rightarrow \\ \mathrm{E}(\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{g}) &= \mathrm{E}(\boldsymbol{\varepsilon}_t'\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t) \\ &= \text{[see Magnus and Neudecker, 1979 p.389]} \\ &= \operatorname{tr}(\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\Sigma})\operatorname{tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}) + 2\operatorname{tr}(\boldsymbol{g}\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}) \\ &= \operatorname{tr}(\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g})\operatorname{tr}(\boldsymbol{I}_M) + 2\operatorname{tr}(\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g}) \\ &= M\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} + 2\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} \\ &= (M+2)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g}. \end{aligned} \tag{C.106}$$

Since $\boldsymbol{\varepsilon}_t'$ and $\boldsymbol{\varepsilon}_{t'}'$ are independent vectors for $t \neq t'$, equations (C.102) and (C.103) imply that

$$\begin{aligned} \mathrm{E}[\boldsymbol{g}'(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E})\boldsymbol{g}] &= \mathrm{E}\left[\boldsymbol{g}'\left(\sum_{t=1}^{T}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' + \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}'\right)\boldsymbol{g}\right] \\ &= \sum_{t=1}^{T}\mathrm{E}[\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{g}] + \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\mathrm{E}[\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}')\boldsymbol{g}] \\ &= \sum_{t=1}^{T}\mathrm{E}[\boldsymbol{g}'(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{g}] + \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\boldsymbol{g}'\,\mathrm{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')\boldsymbol{\Sigma}^{-1}\,\mathrm{E}(\boldsymbol{\varepsilon}_{t'}\boldsymbol{\varepsilon}_{t'}')\boldsymbol{g} \\ &= \sum_{t=1}^{T}(M+2)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} + \sum_{\substack{t=1 \\ t\neq t'}}^{T}\sum_{t'=1}^{T}\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{g} \\ &= T(M+2)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} + T(T-1)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} \\ &= T(M+T+1)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g}. \end{aligned} \tag{C.107}$$

Since $\boldsymbol{g}$ is any arbitrary non-stochastic vector, equation (C.104) implies that

$$\begin{aligned} \mathrm{E}[\boldsymbol{g}'(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E})\boldsymbol{g}] &= \boldsymbol{g}'\,\mathrm{E}[\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}]\boldsymbol{g} = T(M+T+1)\boldsymbol{g}'\boldsymbol{\Sigma}\boldsymbol{g} \\ \Rightarrow \mathrm{E}[\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}] &= T(M+T+1)\boldsymbol{\Sigma}. \end{aligned} \tag{C.108}$$

□

Lemma C.4. Let $\hat{E}_I$ be the residuals of the regression equation

$$\boldsymbol{Y}_* = \boldsymbol{Z}\boldsymbol{B} + \boldsymbol{E}, \tag{C.109}$$

when the $\hat{B}_I$ (I=UL, RL, GL, IG, ML) estimator is used. Lemma C.1 implies that

$$\begin{aligned} \hat{\boldsymbol{E}}_I &= \boldsymbol{Y}_* - \boldsymbol{Z}\hat{\boldsymbol{B}}_I = \boldsymbol{Z}\boldsymbol{B} + \boldsymbol{E} - \boldsymbol{Z}(\boldsymbol{B} + \tau\boldsymbol{B}_1{}^I + \omega(\tau^2)) \\ &= \boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I + \omega(\tau^2). \end{aligned} \tag{C.110}$$

For the $\hat{\boldsymbol{\Sigma}}_I$ (I=UL, RL, GL, IG, ML) estimator of $\boldsymbol{\Sigma}$ it holds that

$$\begin{aligned} \hat{\boldsymbol{\Sigma}}_I &= \hat{\boldsymbol{E}}_I'\hat{\boldsymbol{E}}_I/T = [\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I + \omega(\tau^2)]'[\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I + \omega(\tau^2)]/T \\ &= [\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I]'[\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I]/T + \omega(\tau^4) \\ &= [\boldsymbol{E}' - \tau\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'][\boldsymbol{E} - \tau\boldsymbol{Z}\boldsymbol{B}_1{}^I]/T + \omega(\tau^4) \\ &= \boldsymbol{E}'\boldsymbol{E}/T - \tau\boldsymbol{E}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/T - \tau\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{E}/T + \tau^2\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/T + \omega(\tau^4) \\ &= \boldsymbol{E}'\boldsymbol{E}/T - \tau^2\boldsymbol{E}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/\sqrt{T} - \tau^2\boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} + \tau^2\boldsymbol{B}_1{}^{I\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)\boldsymbol{B}_1{}^I + \omega(\tau^4) \\ &= \boldsymbol{E}'\boldsymbol{E}/T + \tau^2[\boldsymbol{B}_1{}^{I\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)\boldsymbol{B}_1{}^I - \boldsymbol{E}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/\sqrt{T} - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T}] + \omega(\tau^4). \end{aligned} \tag{C.111}$$

By using equation (C.84) we find that

$$\begin{aligned} \boldsymbol{B}_1{}^{I\prime}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} &= \boldsymbol{B}_1{}^{I\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} \\ &= \boldsymbol{B}_1{}^{I\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)\boldsymbol{B}_1{}^{UL}. \end{aligned} \tag{C.112}$$

Similarly,

$$\boldsymbol{E}'\boldsymbol{Z}\boldsymbol{B}_1{}^I/\sqrt{T} = \boldsymbol{B}_1{}^{UL\prime}(\boldsymbol{Z}'\boldsymbol{Z}/T)\boldsymbol{B}_1{}^I. \tag{C.113}$$

Since $\boldsymbol{\Gamma} = \boldsymbol{Z}'\boldsymbol{Z}/T$, equations (C.111), (C.112) and (C.113) imply that

$$\begin{aligned} \hat{\boldsymbol{\Sigma}}_I &= \boldsymbol{E}'\boldsymbol{E}/T + \tau^2[\boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL}] + \omega(\tau^4) \\ &= \boldsymbol{\Sigma} - \tau\sqrt{T}\boldsymbol{\Sigma} + \tau\sqrt{T}\boldsymbol{E}'\boldsymbol{E}/T + \tau^2[\boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL}] + \omega(\tau^4). \end{aligned} \tag{C.114}$$

The following result holds:

$$\begin{aligned}
& \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL} \\
= \;& \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{I\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL} + \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL} - \boldsymbol{B}_1{}^{UL\prime}\boldsymbol{\Gamma}\boldsymbol{B}_1{}^{UL} \\
= \;& (\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL}) - [(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T}]'(\boldsymbol{Z}'\boldsymbol{Z}/T)[(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T}] \\
= \;& (\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}(\boldsymbol{Z}'\boldsymbol{Z}/T)(\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/T \\
= \;& (\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E} \\
= \;& (\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E},
\end{aligned} \tag{C.115}$$

where $\boldsymbol{P}_Z = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'$. Thus, equations (C.114) and (C.115) imply that

$$\begin{aligned}
\hat{\boldsymbol{\Sigma}}_I &= \boldsymbol{\Sigma} + \tau[\sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma})] + \tau^2[(\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}] + \omega(\tau^4) \\
&= \boldsymbol{\Sigma} + \tau\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}_2^I + \omega(\tau^3) \\
&= \boldsymbol{\Sigma} + \tau(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I) + \omega(\tau^3),
\end{aligned} \tag{C.116}$$

where

$$\boldsymbol{\Sigma}_1 = \sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma}) \tag{C.117}$$

and

$$\boldsymbol{\Sigma}_2^I = (\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^{I} - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}. \tag{C.118}$$

Equation (C.116) implies that

$$\begin{aligned}
\hat{\boldsymbol{\Sigma}}_I^{-1} &= [\boldsymbol{\Sigma} + \tau(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I) + \omega(\tau^3)]^{-1} \\
&= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1} + \tau^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1 + \tau\boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1} + \omega(\tau^3) \\
&= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} - \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1} + \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} + \omega(\tau^3) \\
&= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} + \tau^2[\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1}] + \omega(\tau^3) \\
&= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} + \tau^2[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}] + \omega(\tau^3) \\
&= \boldsymbol{\Sigma}^{-1} - \tau\boldsymbol{S}_1 + \tau^2\boldsymbol{S}_2{}^I + \omega(\tau^3),
\end{aligned} \tag{C.119}$$

where

$$\boldsymbol{S}_1 = \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}, \tag{C.120}$$

$$\boldsymbol{S}_2{}^I = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}. \tag{C.121}$$

Moreover, the following results hold:

i.

$$\begin{aligned} \mathrm{E}(\boldsymbol{\Sigma}_1) &= \mathrm{E}[\sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma})] = \sqrt{T}[\mathrm{E}(\boldsymbol{E}'\boldsymbol{E})/T - \boldsymbol{\Sigma}] = [\text{see (C.102)}] \\ &= \sqrt{T}[T\boldsymbol{\Sigma}/T - \boldsymbol{\Sigma}] = 0. \end{aligned} \tag{C.122}$$

ii. Since $\boldsymbol{E}'\boldsymbol{E} \sim \mathcal{W}(\boldsymbol{\Sigma}, T)$ and since $\boldsymbol{P}_Z = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'$ is idempotent with

$$\text{rank}(\boldsymbol{P}_Z) = \text{tr}\,(\boldsymbol{P}_Z) = \text{tr}\,[\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'] = \text{tr}\,[(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{Z}] = \text{tr}\,\boldsymbol{I}_K = K, \tag{C.123}$$

it follows that

$$\boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E} \sim \mathcal{W}(\boldsymbol{\Sigma}, K). \tag{C.124}$$

Furthermore,

$$\mathrm{E}(\boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}) = \text{tr}\,(\boldsymbol{P}_Z)\boldsymbol{\Sigma} = K\boldsymbol{\Sigma}\ [\text{see Magnus and Neudecker, 1979}]. \tag{C.125}$$

iii.

$$\mathrm{E}(\boldsymbol{S}_1) = \mathrm{E}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}) = \boldsymbol{\Sigma}^{-1}\,\mathrm{E}(\boldsymbol{\Sigma}_1)\boldsymbol{\Sigma}^{-1} = 0\ [\text{see (C.122)}]. \tag{C.126}$$

iv.

$$\begin{aligned} \mathrm{E}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1) &= \mathrm{E}[\sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}\sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma})] \\ &= \mathrm{E}[T(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}/T^2 + \boldsymbol{\Sigma} - \boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{E}'\boldsymbol{E}/T)] \\ &= \mathrm{E}(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E}/T + T\boldsymbol{\Sigma} - 2\boldsymbol{E}'\boldsymbol{E}) \\ &= \mathrm{E}(\boldsymbol{E}'\boldsymbol{E}\boldsymbol{\Sigma}^{-1}\boldsymbol{E}'\boldsymbol{E})/T - 2\,\mathrm{E}(\boldsymbol{E}'\boldsymbol{E}) + T\boldsymbol{\Sigma} \\ &= T(M + T + 1)\boldsymbol{\Sigma}/T - 2T\boldsymbol{\Sigma} + T\boldsymbol{\Sigma} \\ &= M\boldsymbol{\Sigma} + T\boldsymbol{\Sigma} + \boldsymbol{\Sigma} - 2T\boldsymbol{\Sigma} + T\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(M + 1). \end{aligned} \tag{C.127}$$

v.

$$\begin{aligned} \mathrm{E}(\boldsymbol{\Sigma}_2^I) &= \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}] \\ &= \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})] - \mathrm{E}[\boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}] \\ &= \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})] - K\boldsymbol{\Sigma} \end{aligned} \tag{C.128}$$

$$\begin{aligned} \Rightarrow \mathrm{E}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1}) &= \boldsymbol{\Sigma}^{-1}\,\mathrm{E}(\boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1} \\ &= \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1} - K\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} \\ &= \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1} - K\boldsymbol{\Sigma}^{-1}. \end{aligned} \tag{C.129}$$

vi. Thus equationS (C.121), (C.127) and (C.129) imply that

$$\begin{aligned}
\mathrm{E}(\boldsymbol{S}_2{}^I) &= \mathrm{E}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}] \\
&= \mathrm{E}[\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1}] \\
&= \boldsymbol{\Sigma}^{-1}\,\mathrm{E}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1)\boldsymbol{\Sigma}^{-1} - \mathrm{E}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^I\boldsymbol{\Sigma}^{-1}) \\
&= (M+1)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} + K\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1} \\
&= (M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1}. \qquad \text{(C.130)}
\end{aligned}$$

Lemma C.5. We estimate the model

$$\boldsymbol{y}_* = \boldsymbol{X}_*\boldsymbol{\beta} + \boldsymbol{\varepsilon} \qquad \text{(C.131)}$$

by using the I estimation process, and we estimate $(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)$ by using the estimator

$$(\hat{\boldsymbol{\Sigma}}_I{}^{-1} \otimes \boldsymbol{I}_T). \qquad \text{(C.132)}$$

Then by using (C.132) we estimate (C.131) via the GL-estimation method. Let $\hat{\boldsymbol{\Sigma}}_I$ the estimation of $\boldsymbol{\Sigma}$ by using the GL residuals, $\hat{\boldsymbol{\varepsilon}}_{GL} = \mathrm{vec}(\hat{\boldsymbol{E}}_{GL})$ say, from equation (C.131) i.e.,

$$\hat{\boldsymbol{\Sigma}}_I = \hat{\boldsymbol{E}}'_{GL}\hat{\boldsymbol{E}}_{GL}/T. \qquad \text{(C.133)}$$

Let $\hat{\boldsymbol{\beta}}_{GL}$ be the GL estimator of $\boldsymbol{\beta}$ in (C.131). For the $\hat{\sigma}_I^2$ (I=UL, RL, GL, IG, ML) estimator of σ^2 holds that

$$\begin{aligned}
\hat{\sigma}_I^2 &= (\boldsymbol{y}_* - \boldsymbol{X}_*\hat{\boldsymbol{\beta}}_{GL})'(\hat{\boldsymbol{\Sigma}}_I{}^{-1} \otimes \boldsymbol{I}_T)(\boldsymbol{y}_* - \boldsymbol{X}_*\hat{\boldsymbol{\beta}}_{GL})/(TM-n) \\
&= \hat{\boldsymbol{\varepsilon}}'_{GL}(\hat{\boldsymbol{\Sigma}}_I{}^{-1} \otimes \boldsymbol{I}_T)\hat{\boldsymbol{\varepsilon}}_{GL}/(TM-n) \\
&= [\mathrm{vec}(\hat{\boldsymbol{E}}_{GL})]'(\hat{\boldsymbol{\Sigma}}_I{}^{-1} \otimes \boldsymbol{I}_T)[\mathrm{vec}(\hat{\boldsymbol{E}}_{GL})]/(TM-n) \\
&= \mathrm{tr}\,[\hat{\boldsymbol{E}}_{GL}(\hat{\boldsymbol{\Sigma}}_I{}^{-1})'\hat{\boldsymbol{E}}'_{GL}]/(TM-n) = \mathrm{tr}\,\hat{\boldsymbol{\Sigma}}_I{}^{-1}\hat{\boldsymbol{E}}'_{GL}\hat{\boldsymbol{E}}_{GL}/(TM-n) \\
&= \mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}T\hat{\boldsymbol{\Sigma}}_J)/(TM-n) = \mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J)/((TM-n)/T) \\
&= \mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J)/(M-n/T) = \mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J)/(M-\tau^2 n). \qquad \text{(C.134)}
\end{aligned}$$

By using equations (C.116), (C.117) and (C.118) we take

$$\hat{\boldsymbol{\Sigma}}_J = \boldsymbol{\Sigma} + \tau\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}_2^J + \omega(\tau^3), \qquad \text{(C.135)}$$

where

$$\boldsymbol{\Sigma}_1 = \sqrt{T}(\boldsymbol{E}'\boldsymbol{E}/T - \boldsymbol{\Sigma}) \qquad \text{(C.136)}$$

and

$$\boldsymbol{\Sigma}_2^J = (\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL}) - \boldsymbol{E}'\boldsymbol{P}_Z\boldsymbol{E}. \qquad \text{(C.137)}$$

Then, equations (C.119),(C.121), (C.134) and (C.135) imply that

$$\begin{aligned}
\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J &= [\boldsymbol{\Sigma}^{-1} - \tau \boldsymbol{S}_1 + \tau^2 \boldsymbol{S}_2{}^I + \omega(\tau^3)][\boldsymbol{\Sigma} + \tau\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}_2^J + \omega(\tau^3)] \\
&= \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} + \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^J - \tau\boldsymbol{S}_1\boldsymbol{\Sigma} - \tau^2\boldsymbol{S}_1\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{S}_2{}^I\boldsymbol{\Sigma} + \omega(\tau^3) \\
&= \boldsymbol{I}_M + \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_2^J - \tau\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} - \tau^2\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 + \tau^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} + \omega(\tau^3) \\
&= \boldsymbol{I}_M + \tau^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I) + \omega(\tau^3) \Rightarrow \qquad \text{(C.138)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J) &= \mathrm{tr}\,\boldsymbol{I}_M + \tau^2\,\mathrm{tr}\,[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)] + \omega(\tau^3) \\
&= M + \tau^2\,\mathrm{tr}\,[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)] + \omega(\tau^3) \Rightarrow \qquad \text{(C.139)}
\end{aligned}$$

$$\hat{\sigma}_I^2 = \mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}_I^{-1}\hat{\boldsymbol{\Sigma}}_J)/(M - \tau^2 n) = [M + \tau^2\,\mathrm{tr}\,[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)]]/(M - \tau^2 n) + \omega(\tau^3). \qquad \text{(C.140)}$$

Moreover,

$$\begin{aligned}
\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I) &= \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I) \\
&= \boldsymbol{\Sigma}^{-1}[(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I) - (\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^J)] \\
&= \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^I)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2^J)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} \\
&= \boldsymbol{S}_2^I\boldsymbol{\Sigma} - \boldsymbol{S}_2^J\boldsymbol{\Sigma} = (\boldsymbol{S}_2^I - \boldsymbol{S}_2^J)\boldsymbol{\Sigma} \Rightarrow \qquad \text{(C.141)}
\end{aligned}$$

$$\mathrm{tr}\,[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}_2^J - \boldsymbol{\Sigma}_2^I)] = \mathrm{tr}\,(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma} \qquad \text{(C.142)}$$

Thus, equations (C.140) and (C.142) imply that

$$\hat{\sigma}_I^2 = [M + \tau^2\,\mathrm{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}]]/(M - \tau^2 n) + \omega(\tau^3). \qquad \text{(C.143)}$$

Lemma C.6. Define the $M \times M$ matrices

$$\boldsymbol{M}_I = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{S}_2{}^I) \qquad \text{(C.144)}$$

and

$$\boldsymbol{\Delta}_I = \lim_{T\to\infty} T\,\mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] \text{ (I=UL, RL, GL, IG, ML)} \qquad \text{(C.145)}$$

The following results hold:

i.

$$
\begin{aligned}
\boldsymbol{M}_I &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{S_2}^I) = \text{(C.130)}\\
&= (M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}[\lim_{T\to\infty} \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]]\boldsymbol{\Sigma}^{-1}\\
&= [\text{see Lemma (C.2)}]\\
&= (M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}[\lim_{T\to\infty} T\,\mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})]]\boldsymbol{\Sigma}^{-1}\\
&= [\text{see (C.145)}] = (M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1} \Rightarrow
\end{aligned}
\tag{C.146}
$$

$$
\begin{aligned}
(\boldsymbol{M}_I - \boldsymbol{M}_{GL})\boldsymbol{\Sigma} &= [(M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1} - (M+K+1)\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_{GL}\boldsymbol{\Sigma}^{-1}]\boldsymbol{\Sigma}\\
&= (\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_{GL}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1})\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL} - \boldsymbol{\Delta}_I)\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\\
&= \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL} - \boldsymbol{\Delta}_I).
\end{aligned}
\tag{C.147}
$$

ii.

$$
\begin{aligned}
\mathrm{E}[(\boldsymbol{S_2}^I - \boldsymbol{S_2}^J)\boldsymbol{\Sigma}] &= [\mathrm{E}(\boldsymbol{S_2}^I) - \mathrm{E}(\boldsymbol{S_2}^J)]\boldsymbol{\Sigma} = [\text{see (C.130)}]\\
&= [(M+K+1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1}\\
&\quad -(M+K+1)\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1}]\boldsymbol{\Sigma}\\
&= -\boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\\
&\quad +\boldsymbol{\Sigma}^{-1}\,\mathrm{E}[(\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL})]\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} \Rightarrow
\end{aligned}
\tag{C.148}
$$

$$
\begin{aligned}
\lim_{T\to\infty} \mathrm{E}[(\boldsymbol{S_2}^I - \boldsymbol{S_2}^J)\boldsymbol{\Sigma}] &= -\boldsymbol{\Sigma}^{-1} \lim_{T\to\infty} \mathrm{E}[(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL})]\\
&\quad +\boldsymbol{\Sigma}^{-1} \lim_{T\to\infty} \mathrm{E}[(\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B}_1{}^J - \boldsymbol{B}_1{}^{UL})] = [\text{see Lemma C.2}]\\
&= -\boldsymbol{\Sigma}^{-1} \lim_{T\to\infty} \mathrm{E}[(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})]\\
&\quad +\boldsymbol{\Sigma}^{-1} \lim_{T\to\infty} \mathrm{E}[(\hat{\boldsymbol{B}}_J - \hat{\boldsymbol{B}}_{UL})'\boldsymbol{\Gamma}(\hat{\boldsymbol{B}}_J - \hat{\boldsymbol{B}}_{UL})] = [\text{see (C.145)}]\\
&= -\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I + \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_J = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_J - \boldsymbol{\Delta}_I) = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL} - \boldsymbol{\Delta}_I),
\end{aligned}
\tag{C.149}
$$

because the I estimation method is the GL method.

iii. Moreover,

$$
\begin{aligned}
S_1 &= \Sigma^{-1}\Sigma_1\Sigma^{-1} = \sqrt{T}(\Sigma^{-1}E'E\Sigma^{-1}/T - \Sigma^{-1}\Sigma\Sigma^{-1}) \\
&= \sqrt{T}(\Sigma^{-1}E'E\Sigma^{-1}/T - \Sigma^{-1}) \\
&= \sqrt{T}\left[[(\delta_{ik}\sigma_{ii}{}^{-1})_{i,k=1,\dots,M}][(\delta_{kl}\varepsilon_k'\varepsilon_l/T)_{k,l=1,\dots,M}][(\delta_{lj}\sigma_{ll}{}^{-1})_{l,j=1,\dots,M}] - [(\delta_{ij}\sigma_{ii}{}^{-1})_{i,j=1,\dots,M}]\right] \\
&= \sqrt{T}\left[\Big[\Big(\sum_{k=1}^{M}\sum_{l=1}^{M}\delta_{ik}\delta_{kl}\delta_{lj}\sigma_{ii}{}^{-1}\sigma_{ll}{}^{-1}\varepsilon_k'\varepsilon_l/T\Big)_{ij}\Big] - [(\delta_{ij}\sigma_{ii}{}^{-1})_{i,j}]\right] \\
&= \sqrt{T}\Big[(\delta_{ij}\sigma_{ii}{}^{-1}\sigma_{jj}{}^{-1}\varepsilon_i'\varepsilon_j/T - \delta_{ij}\sigma_{ii}{}^{-1})_{i,j}\Big]. \qquad \text{(C.150)}
\end{aligned}
$$

Moreover, we define the ii-th elements of matrix S_1

$$s_{(ii)}^{(1)} = \sqrt{T}[\sigma_{ii}{}^{-1}(\sigma_{ii}{}^{-1}\varepsilon_i'\varepsilon_i/T - 1)], \qquad \text{(C.151)}$$

and

$$s_1 = [(s_{(ii)}^{(1)})_{i=1,\dots,M}]' \qquad \text{(C.152)}$$

Since $E'E \sim \mathcal{W}(\Sigma, T)$, $\varepsilon_t' \sim \mathcal{N}_M(0, \Sigma)$, $\varepsilon_i \sim \mathcal{N}_M(0, \sigma_{ii}I_T)$ and $\mathrm{E}(E'E) = T\Sigma$ we have that

$$
\begin{aligned}
\mathrm{E}(\varepsilon_i\varepsilon_i') &= \sigma_{ii}I_T, \qquad &\text{(C.153)} \\
\mathrm{E}(\varepsilon_t\varepsilon_t') &= \Sigma, \qquad &\text{(C.154)} \\
\mathrm{E}(\varepsilon_i'\varepsilon_i) &= T\sigma_{ii}, \qquad &\text{(C.155)}
\end{aligned}
$$

and equation (C.117) implies that the matrix

$$W = \sqrt{T}\Sigma_1 = T(E'E/T - \Sigma) = E'E - T\Sigma \qquad \text{(C.156)}$$

is a Wishart diagonal matrix in deviations from its expected value. Let w_{ii} be the (i, i)-th element of W. Then, since σ_{ii} is the (i, i)-th element of Σ, following Zellner, 1971 p.389, (B.58), we find that

$$
\begin{aligned}
\sigma_{ii}{}^{(1)} &= \sqrt{T}(\varepsilon_i'\varepsilon_i/T - \sigma_{ii}) \Rightarrow \mathrm{E}(\sigma_{ii}{}^{(1)}) = 0 \qquad &\text{(C.157)} \\
w_{ii} &= \sqrt{T}\sigma_{ii}{}^{(1)} = T(\varepsilon_i'\varepsilon_i/T - \sigma_{ii}) \Rightarrow E(w_{ii}) = 0 \qquad &\text{(C.158)}
\end{aligned}
$$

and by using Theorem UR.1 we have

$$
\begin{aligned}
\mathrm{cov}(w_{ii}w_{jj}) &= \mathrm{E}(w_{ii}w_{jj}) = T(\sigma_{ij}\sigma_{ij} + \sigma_{ij}\sigma_{ij}) = 0 \qquad &\text{(C.159)} \\
\mathrm{cov}(w_{ii}w_{ii}) &= \mathrm{E}(w_{ii}w_{ii}) = T(\sigma_{ii}\sigma_{ii} + \sigma_{ii}\sigma_{ii}) = 2T\sigma_{ii}{}^2 \qquad &\text{(C.160)}
\end{aligned}
$$

$$\begin{aligned}
\mathrm{E}[\boldsymbol{s}_1\boldsymbol{s}_1'] &= \mathrm{E}\left[\begin{bmatrix} s_{11}{}^{(1)} \\ s_{22}{}^{(1)} \\ \vdots \\ s_{MM}{}^{(1)} \end{bmatrix} \cdot [s_{11}{}^{(1)}\ s_{22}{}^{(1)} \ldots s_{MM}{}^{(1)}]\right] \\
&= \mathrm{E}\begin{bmatrix} s_{11}{}^{(1)2} & s_{11}{}^{(1)}s_{22}{}^{(1)} & \ldots & s_{11}{}^{(1)}s_{MM}{}^{(1)} \\ s_{22}{}^{(1)}s_{11}{}^{(1)} & s_{22}{}^{(1)2} & s_{22}{}^{(1)}s_{33}{}^{(1)} & \ldots \\ \vdots & & \ddots & \\ s_{MM}{}^{(1)}s_{11}{}^{(1)} & \ldots & & s_{MM}{}^{(1)2} \end{bmatrix}. \qquad \text{(C.161)}
\end{aligned}$$

By using Lemma (UR.2) and equation (C.151) and since $\boldsymbol{\varepsilon}_i \sim \mathcal{N}(0, \sigma_{ii}\boldsymbol{I}_T)$ we have

$$\mathrm{E}(\boldsymbol{\varepsilon}_i'\boldsymbol{I}_T\boldsymbol{\varepsilon}_i) = \mathrm{tr}(\sigma_{ii}\boldsymbol{I}_T) = T\sigma_{ii} \qquad \text{(C.162)}$$

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\varepsilon}_i'\boldsymbol{I}_T\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\boldsymbol{I}_T\boldsymbol{\varepsilon}_i) &= \mathrm{tr}(\sigma_{ii}\boldsymbol{I}_T)\,\mathrm{tr}(\sigma_{ii}\boldsymbol{I}_T) + 2\,\mathrm{tr}(\sigma_{ii}\boldsymbol{I}_T\sigma_{ii}\boldsymbol{I}_T) = (T\sigma_{ii})(T\sigma_{ii}) + 2(T\sigma_{ii}{}^2) \\
&= T^2\sigma_{ii}{}^2 + 2T\sigma_{ii}{}^2 = (T^2 + 2T)\sigma_{ii}{}^2 \Rightarrow \qquad \text{(C.163)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[(s_{ii}{}^{(1)})^2] &= \mathrm{E}[\sqrt{T}[\sigma_{ii}{}^{-1}(\sigma_{ii}{}^{-1}\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i/T - 1)]]^2 \\
&= T\,\mathrm{E}[\sigma_{ii}{}^{-2}(\sigma_{ii}{}^{-2}\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i/T^2 - 2\sigma_{ii}{}^{-1}\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i/T + 1)] \\
&= T\sigma_{ii}{}^{-4}\,\mathrm{E}(\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i)^2/T^2 - 2T\sigma_{ii}{}^{-3}\,\mathrm{E}(\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i)/T + T\sigma_{ii}{}^{-2} \\
&= T\sigma_{ii}{}^{-4} \cdot \frac{(T^2 + 2T)\sigma_{ii}{}^2}{T^2} - 2T\sigma_{ii}{}^{-3}\frac{T\sigma_{ii}}{T} + T\sigma_{ii}{}^{-2} \\
&= 2\sigma_{ii}{}^{-2}. \qquad \text{(C.164)}
\end{aligned}$$

$$E[(s_{ii}{}^{(1)}s_{jj}{}^{(1)})] = 0. \qquad \text{(C.165)}$$

By using equations (C.164) and (C.165), equation (C.161) can be written as

$$\mathrm{E}[\boldsymbol{s}_1\boldsymbol{s}_1'] = \begin{bmatrix} 2\sigma_{11}{}^{-2} & 0 & \ldots & 0 \\ 0 & 2\sigma_{22}{}^{-2} & & 0 \\ & & \ddots & \\ 0 & \ldots & 0 & 2\sigma_{MM}{}^{-2} \end{bmatrix} \Rightarrow \qquad \text{(C.166)}$$

$$\lim_{T\to\infty} \mathrm{E}[\boldsymbol{s}_1\boldsymbol{s}_1'] = \lim_{T\to\infty} \begin{bmatrix} 2\sigma_{11}{}^{-2} & 0 & \dots & 0 \\ 0 & 2\sigma_{22}{}^{-2} & & 0 \\ & & \ddots & \\ 0 & \dots & 0 & 2\sigma_{MM}{}^{-2} \end{bmatrix}. \tag{C.167}$$

Lemma C.7. Calculation of $\boldsymbol{\Delta}_I$ (I=UL,RL,GL, IG, ML)

Since, $\boldsymbol{y}_* = \mathrm{vec}(\boldsymbol{Y}_*)$, $\boldsymbol{X}_* = (\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}$, $\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{E})$ and $\mathrm{vec}(\boldsymbol{B}) = \boldsymbol{\Psi\beta}$ where $\boldsymbol{y}_*$, $\boldsymbol{\varepsilon}$ are $(TM \times 1)$ vectors and $(\boldsymbol{I}_M \otimes \boldsymbol{Z})$, $\boldsymbol{\Psi}$ and $\boldsymbol{X}_*$ are $TM \times Mk$, $Mk \times n$ and $TM \times n$ matrices, respectively, the following results hold:

(i)

$$\begin{aligned} \boldsymbol{B}_1{}^{UL} &= (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} = T(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E}/\sqrt{T} \\ &= \sqrt{T}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E} \Rightarrow \end{aligned} \tag{C.168}$$

$$\begin{aligned} \mathrm{vec}(\boldsymbol{B}_1{}^{UL}) &= \mathrm{vec}[\sqrt{T}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E}] \\ &= \sqrt{T}\,\mathrm{vec}[(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{E}] \\ &= \sqrt{T}[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']\,\mathrm{vec}(\boldsymbol{E}) \\ &= \sqrt{T}[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']\boldsymbol{\varepsilon}. \end{aligned} \tag{C.169}$$

(ii)

$$\mathrm{vec}(\boldsymbol{B}_1{}^{RL}) = \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*/T)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon}/\sqrt{T} = \sqrt{T}\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon}. \tag{C.170}$$

(iii) Similarly,

$$\begin{aligned} \mathrm{vec}(\boldsymbol{B}_1{}^{GL}) &= \mathrm{vec}(\boldsymbol{B}_1{}^{IG}) = \mathrm{vec}(\boldsymbol{B}_1{}^{ML}) \\ &= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*/T]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{\varepsilon}/\sqrt{T} \\ &= \sqrt{T}\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{\varepsilon}. \end{aligned} \tag{C.171}$$

Moreover,

$$\begin{aligned} \hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL} &= \tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2) = [\text{see (C.100)}] \\ &\Rightarrow \sqrt{T}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL}) = \sqrt{T}[\tau(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau^2)] \\ &= (\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}) + \omega(\tau) \Rightarrow \end{aligned} \tag{C.172}$$

$$\mathrm{vec}[\sqrt{T}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL})] = \sqrt{T}\,\mathrm{vec}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL}) = \mathrm{vec}(\boldsymbol{B}_I - \boldsymbol{B}_{UL}) + \omega(\tau). \tag{C.173}$$

Define the matrix $\boldsymbol{\Phi}_I$ such that

$$\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} = \text{vec}(\boldsymbol{B}_1{}^I - \boldsymbol{B}_1{}^{UL}). \quad \text{(C.174)}$$

Then equations (C.173) and (C.174) imply that

$$\sqrt{T}\,\text{vec}(\hat{\boldsymbol{B}}_I - \hat{\boldsymbol{B}}_{UL}) = \sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} + \omega(\tau). \quad \text{(C.175)}$$

By using equations (C.168), (C.169), (C.170), and (C.174), we find the following results:

I For $I = UL$

$$\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} = \sqrt{T}\boldsymbol{\Phi}_{UL}\boldsymbol{\varepsilon} = \text{vec}(\boldsymbol{B}_1{}^{UL} - \boldsymbol{B}_1{}^{UL}) = 0 \Rightarrow \boldsymbol{\Phi}_{UL} = 0. \quad \text{(C.176)}$$

II For $I = RL$

$$\begin{aligned}
\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} &= \sqrt{T}\boldsymbol{\Phi}_{RL}\boldsymbol{\varepsilon} = \text{vec}(\boldsymbol{B}_1{}^{RL} - \boldsymbol{B}_1{}^{UL}) = \sqrt{T}\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{\varepsilon} - \sqrt{T}[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}']\boldsymbol{\varepsilon} \\
&= \sqrt{T}[\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*' - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}']]\boldsymbol{\varepsilon} \Rightarrow \\
\boldsymbol{\Phi}_{RL} &= \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*' - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}']. \quad \text{(C.177)}
\end{aligned}$$

III Similarly, for $I = GL, IG, ML$

$$\begin{aligned}
\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} &= \sqrt{T}\boldsymbol{\Phi}_{GL}\boldsymbol{\varepsilon} = \sqrt{T}\boldsymbol{\Phi}_{IG}\boldsymbol{\varepsilon} = \sqrt{T}\boldsymbol{\Phi}_{ML}\boldsymbol{\varepsilon} \\
&= \text{vec}(\boldsymbol{B}_1{}^{GL} - \boldsymbol{B}_1{}^{UL}) = \text{vec}(\boldsymbol{B}_1{}^{IG} - \boldsymbol{B}_1{}^{UL}) = \text{vec}(\boldsymbol{B}_1{}^{ML} - \boldsymbol{B}_1{}^{UL}) \\
&= \sqrt{T}[\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T) - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}\boldsymbol{Z}']]\boldsymbol{\varepsilon} \Rightarrow \\
\boldsymbol{\Phi}_{GL} &= \boldsymbol{\Phi}_{IG} = \boldsymbol{\Phi}_{ML} = \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T) - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']. \quad \text{(C.178)}
\end{aligned}$$

Let $\boldsymbol{l}$ be any arbitrary $M \times 1$ vector and let $\boldsymbol{L} = \boldsymbol{l}\boldsymbol{l}'$ be any symmetric matrix i.e.,

$$\boldsymbol{l} = [(l_i)_{i=1,\ldots,M}] \quad \text{(C.179)}$$

and

$$\begin{aligned}
\boldsymbol{L} &= [(l_{ij})_{i,j=1,\ldots,M}] = \boldsymbol{l}\boldsymbol{l}' = \begin{bmatrix} l_1 \\ \vdots \\ l_M \end{bmatrix}(l_1, \ldots, l_M) = \begin{bmatrix} l_1 l_1 & \ldots & l_1 l_M \\ \vdots & & \vdots \\ l_M l_1 & \ldots & l_M l_M \end{bmatrix} \\
&= [(l_i l_j)_{i,j=1,\ldots,M}] \Rightarrow \\
l_{ij} &= l_i l_j \;\; (i, j = 1, \ldots, M). \quad \text{(C.180)}
\end{aligned}$$

Then,

$$
\begin{aligned}
\boldsymbol{l}'(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})\boldsymbol{l} &= \operatorname{tr}[\boldsymbol{l}'(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})\boldsymbol{l}] \\
&= \operatorname{tr}[(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})\boldsymbol{l}\boldsymbol{l}'] \\
&= \operatorname{tr}[(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})\boldsymbol{L}] \\
&= [\operatorname{vec}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})]' \operatorname{vec}[\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})\boldsymbol{L}] \\
&= [\operatorname{vec}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})]'(\boldsymbol{L}' \otimes \boldsymbol{\Gamma})[\operatorname{vec}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})] \\
&= [\operatorname{vec}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})]'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})[\operatorname{vec}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})]. \qquad \text{(C.181)}
\end{aligned}
$$

By using equations (C.174), and (C.181) and since $\mathrm{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \boldsymbol{\Sigma} \otimes \boldsymbol{I}_T$, we find that

$$
\begin{aligned}
\boldsymbol{l}'(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})\boldsymbol{l} &= (\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon})(\boldsymbol{L} \otimes \boldsymbol{\Gamma})(\sqrt{T}\boldsymbol{\Phi}_I\boldsymbol{\varepsilon}) = \\
T\boldsymbol{\varepsilon}'\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I\boldsymbol{\varepsilon} &= T\operatorname{tr}(\boldsymbol{\varepsilon}'\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I\boldsymbol{\varepsilon}) \\
&= T\operatorname{tr}(\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') \Rightarrow \\
\mathrm{E}[\boldsymbol{l}'(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})\boldsymbol{l}] &= T\operatorname{tr}(\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I\,\mathrm{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')) \\
&= T\operatorname{tr}(\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)). \qquad \text{(C.182)}
\end{aligned}
$$

Then, Lemma C.2 and equations (C.145) and (C.182) imply that

$$
\begin{aligned}
\boldsymbol{l}'\boldsymbol{\Delta}_I\boldsymbol{l} &= \boldsymbol{l}' \lim_{T\to\infty} \mathrm{E}[(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})]\boldsymbol{l} \\
&= \lim_{T\to\infty} \mathrm{E}[\boldsymbol{l}'(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})'\boldsymbol{\Gamma}(\boldsymbol{B_1}^I - \boldsymbol{B_1}^{UL})\boldsymbol{l}] \\
&= \lim_{T\to\infty} [T\operatorname{tr}(\boldsymbol{\Phi}_I'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_I(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)]. \qquad \text{(C.183)}
\end{aligned}
$$

The following results hold:

(a) Equations (C.176) and (C.183) imply that

$$
\boldsymbol{l}'\boldsymbol{\Delta}_{UL}\boldsymbol{l} = \lim_{T\to\infty}[T(\boldsymbol{\Phi}_{UL}'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Phi}_{UL}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T))] = 0 \Rightarrow \boldsymbol{\Delta}_{UL} = 0. \qquad \text{(C.184)}
$$

(b) Since $\boldsymbol{X}_*' = [\boldsymbol{X}_{1*}', \ldots, \boldsymbol{X}_{M*}']$ we take

$$
\begin{aligned}
\boldsymbol{X}_*'\boldsymbol{X}_* &= [\boldsymbol{X}_{1*}', \ldots, \boldsymbol{X}_{M*}'] \begin{bmatrix} \boldsymbol{X}_{1*} \\ \vdots \\ \boldsymbol{X}_{M*} \end{bmatrix} = \sum_{\mu=1}^{M} \boldsymbol{X}_{\mu*}'\boldsymbol{X}_{\mu*} \Rightarrow \\
(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1} &= \left(\sum_{\mu=1}^{M} \boldsymbol{X}_{\mu*}'\boldsymbol{X}_{\mu*} \right)^{-1} \Rightarrow
\end{aligned}
$$

$$\begin{aligned}
\boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*' &= \begin{bmatrix} \boldsymbol{\Psi}_1 \\ \vdots \\ \boldsymbol{\Psi}_M \end{bmatrix} \left(\sum_{\mu=1}^{M} \boldsymbol{X}_{\mu*}'\boldsymbol{X}_{\mu*} \right)^{-1} [\boldsymbol{X}_{1*}', \ldots, \boldsymbol{X}_{M*}'] \\
&= \left[\left(\boldsymbol{\Psi}_i \left(\sum_{\mu=1}^{M} \boldsymbol{X}_{\mu*}'\boldsymbol{X}_{\mu*} \right)^{-1} \boldsymbol{X}_{j*}' \right)_{i,j} \right]. \qquad \text{(C.185)}
\end{aligned}$$

Moreover,

$$\begin{aligned}
[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'] &= \begin{bmatrix} (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' & & 0 \\ & \ddots & \\ 0 & & (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' \end{bmatrix} = \mathrm{diag}[((\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}')_i] \\
&= [(\delta_{ij}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}')_{i,j}]. \qquad \text{(C.186)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\boldsymbol{\Phi}_{RL} &= \boldsymbol{\Psi}(\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*' - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'] \\
&= \left[\left(\boldsymbol{\Psi}_i \left(\sum_{\mu=1}^{M} \boldsymbol{X}_{\mu*}'\boldsymbol{X}_{\mu*} \right)^{-1} \boldsymbol{X}_{j*}' - \delta_{ij}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' \right)_{i,j} \right] \\
&= [(\boldsymbol{\Phi}_{ij}{}^{RL})_{i,j}], \qquad \text{(C.187)}
\end{aligned}$$

where

$$\boldsymbol{\Phi}_{ij}{}^{RL} = \boldsymbol{\Psi}_i \left(\sum_{\mu=1}^{M} \boldsymbol{X}_{\mu*}'\boldsymbol{X}_{\mu*} \right)^{-1} \boldsymbol{X}_{j*}' - \delta_{ij}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' \,. \qquad \text{(C.188)}$$

Thus,

$$\begin{aligned}
\boldsymbol{\Phi}_{RL}'(\boldsymbol{L} \otimes \boldsymbol{\Gamma}) &= \left[(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime})_{i,\kappa} \right] \left[[(l_{\kappa q})_{\kappa q}] \otimes \boldsymbol{\Gamma} \right] = \left[(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime})_{i,\kappa} \right] \left[(l_{\kappa q}\boldsymbol{\Gamma})_{\kappa q} \right] \\
&= \left[\left(\sum_{\kappa=1}^{M} l_{\kappa q} \boldsymbol{\Phi}_{i\kappa}{}^{RL\prime} \boldsymbol{\Gamma} \right)_{i,q} \right] \qquad \text{(C.189)}
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T) &= [(\boldsymbol{\Phi}_{q\mu}{}^{RL})_{q,\mu}][[(\delta_{\mu j}\sigma_{\mu\mu})_{\mu,j}] \otimes \boldsymbol{I}_T] = [(\boldsymbol{\Phi}_{q\mu}{}^{RL})_{q,\mu}][(\delta_{\mu j}\sigma_{\mu\mu}\boldsymbol{I}_T)_{\mu,j}] \\
&= \left[\left(\sum_{\mu=1}^{M} \delta_{\mu j}\sigma_{\mu\mu} \boldsymbol{\Phi}_{q\mu}{}^{RL} \right)_{q,j} \right] = [(\sigma_{jj}\boldsymbol{\Phi}_{qj}{}^{RL})_{q,j}]. \qquad \text{(C.190)}
\end{aligned}$$

Then, equations (C.189) and (C.190) imply that

$$\begin{aligned}\boldsymbol{\Phi}'_{RL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T) &= \left[\left(\sum_{\kappa=1}^{M} l_{\kappa q}\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\right)_{i,q}\right]\left[\left(\sigma_{jj}\boldsymbol{\Phi}_{qj}{}^{RL}\right)_{q,j}\right] \\ &= \left[\left(\sum_{q=1}^{M}\sum_{\kappa=1}^{M} l_{\kappa q}\sigma_{jj}\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qj}{}^{RL}\right)_{i,j}\right] \Rightarrow \qquad \text{(C.191)}\end{aligned}$$

$$\begin{aligned}\Rightarrow \operatorname{tr}[\boldsymbol{\Phi}'_{RL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)] &= \operatorname{tr}\left[\left(\sum_{q=1}^{M}\sum_{\kappa=1}^{M} l_{\kappa q}\sigma_{jj}\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qj}{}^{RL}\right)_{i,j}\right] \\ &= \sum_{i=1}^{M}\sum_{q=1}^{M}\sum_{\kappa=1}^{M} l_{\kappa q}\sigma_{ii}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL}). \qquad \text{(C.192)}\end{aligned}$$

Since $\boldsymbol{X}_{i*} = \boldsymbol{Z}\boldsymbol{\Psi}_i$ and $\boldsymbol{\Gamma} = (\boldsymbol{Z}'\boldsymbol{Z}/T)$, equation (C.188) implies that

$$\begin{aligned}\boldsymbol{\Phi}_{i\kappa}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}^{RL} &= [\boldsymbol{\Psi}_i\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{\kappa*} - \delta_{i\kappa}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']'(\boldsymbol{Z}'\boldsymbol{Z}/T)\cdot[\boldsymbol{\Psi}_q\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{i*} - \delta_{qi}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'] \\ &= [\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{\Psi}'_i - \delta_{i\kappa}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}](\boldsymbol{Z}'\boldsymbol{Z})[\boldsymbol{\Psi}_q\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{i*} - \delta_{qi}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']/T \\ &= [\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{\Psi}'_i\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\Psi}_q\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{i*} - \delta_{qi}\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{\Psi}'_i(\boldsymbol{Z}'\boldsymbol{Z})(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' \\ &\quad -\delta_{i\kappa}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}(\boldsymbol{Z}'\boldsymbol{Z})\boldsymbol{\Psi}_q\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{i*} + \delta_{i\kappa}\delta_{qi}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}(\boldsymbol{Z}'\boldsymbol{Z})(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']/T \\ &= [\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}(\boldsymbol{Z}\boldsymbol{\Psi}_i)'(\boldsymbol{Z}\boldsymbol{\Psi}_q)\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{i*} - \delta_{qi}\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}(\boldsymbol{Z}\boldsymbol{\Psi}_i)' \\ &\quad -\delta_{i\kappa}(\boldsymbol{Z}\boldsymbol{\Psi}_q)\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{i*} + \delta_{i\kappa}\delta_{qi}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']/T \\ &= [\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}(\boldsymbol{X}_{i*})'(\boldsymbol{X}_{q*})\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{i*} - \delta_{qi}\boldsymbol{X}_{\kappa*}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}(\boldsymbol{X}_{i*})' \\ &\quad -\delta_{i\kappa}(\boldsymbol{X}_{q*})\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}\right)^{-1}\boldsymbol{X}'_{i*} + \delta_{i\kappa}\delta_{qi}\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']/T \Rightarrow \\ \operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL}) &= \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}'_{i*}\boldsymbol{X}_{q*}/T)\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}'_{i*}\boldsymbol{X}_{\kappa*})\right]/T \\ &\quad -\delta_{qi}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}'_{i*}\boldsymbol{X}_{\kappa*}/T)\right]/T\end{aligned}$$

$$-\operatorname{tr}\left[\delta_{i\kappa}\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}'_{i*}\boldsymbol{X}_{q*}/T)\right]/T$$
$$+\delta_{i\kappa}\delta_{qi}\operatorname{tr}(\boldsymbol{P}_Z)/T. \quad \text{(C.193)}$$

Since $\boldsymbol{Z}$ is $T \times k$, equation (C.123) implies that

$$\operatorname{tr}(\boldsymbol{P}_Z) = k. \quad \text{(C.194)}$$

Since $\boldsymbol{X}_{i*} = \boldsymbol{P}_i^{-1}\boldsymbol{X}_i$, $\boldsymbol{X}_{j*} = \boldsymbol{P}_j^{-1}\boldsymbol{X}_j$, and since $\boldsymbol{P}_i^{-1\prime}\boldsymbol{P}_j^{-1} = \delta_{ij}\boldsymbol{R}^{ij}$, we find that for any $i, j = 1, \ldots, M$

$$\boldsymbol{X}'_{i*}\boldsymbol{X}_{j*}/T = \boldsymbol{X}'_i\boldsymbol{P}_i^{-1\prime}\boldsymbol{P}_j^{-1}\boldsymbol{X}_j/T = \delta_{ij}\boldsymbol{X}'_i\boldsymbol{R}^{ij}\boldsymbol{X}_j/T = \delta_{ij}\boldsymbol{B}_{ij} = \boldsymbol{B}_{ii} \ \text{[see (C.24)]}. \quad \text{(C.195)}$$

Therefore,

$$\begin{aligned}\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T &= \sum_{p=1}^{M}\boldsymbol{B}_{pp} \Rightarrow \\ \left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1} &= \left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}. \end{aligned} \quad \text{(C.196)}$$

So,

$$\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}'_{i*}\boldsymbol{X}_{\kappa*}/T)\right] = \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\delta_{i\kappa}\boldsymbol{B}_{i\kappa}\right] \quad \text{(C.197)}$$

and similarly

$$\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}'_{i*}\boldsymbol{X}_{q*}/T)\right] = \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\delta_{iq}\boldsymbol{B}_{iq}\right]. \quad \text{(C.198)}$$

Furthermore,

$$\begin{aligned}&\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}'_{i*}\boldsymbol{X}_{q*}/T)\left(\sum_{p=1}^{M}\boldsymbol{X}'_{p*}\boldsymbol{X}_{p*}/T\right)^{-1}(\boldsymbol{X}'_{i*}\boldsymbol{X}_{\kappa*}/T)\right] \\ =\ &\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\delta_{iq}\boldsymbol{B}_{iq}\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\delta_{i\kappa}\boldsymbol{B}_{i\kappa}\right].\end{aligned} \quad \text{(C.199)}$$

Thus, equations (C.193), (C.194), (C.197), (C.198), and (C.199) imply that

$$\begin{aligned}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}^{RL}) &= \operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\delta_{iq}\boldsymbol{B}_{iq}\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\delta_{i\kappa}\boldsymbol{B}_{i\kappa}\right]/T - \delta_{qi}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\delta_{i\kappa}\boldsymbol{B}_{i\kappa}\right]/T \\ &\quad -\delta_{i\kappa}\operatorname{tr}\left[\left(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\right)^{-1}\delta_{iq}\boldsymbol{B}_{iq}\right]/T + \delta_{i\kappa}\delta_{qi}K/T.\end{aligned} \quad \text{(C.200)}$$

Since $l_{kq} = l_k l_q$ (see (C.180)), equations(C.179) and (C.192) imply that

$$
\begin{aligned}
\operatorname{tr}[\boldsymbol{\Phi}'_{RL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)] &= \sum_{i=1}^{M}\sum_{q=1}^{M}\sum_{\kappa=1}^{M} l_{\kappa q}\sigma_{ii}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL})\\
&= \sum_{i=1}^{M}\sum_{q=1}^{M}\sum_{\kappa=1}^{M} l_{\kappa}\sigma_{ii}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL})l_q\\
&= \boldsymbol{l}'\left[\left(\sum_{i=1}^{M}\sigma_{ii}\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL})\right)_{k,q}\right]\boldsymbol{l}\Rightarrow\\
\boldsymbol{l}'\boldsymbol{\Delta}_{RL}\boldsymbol{l} &= \lim_{T\to\infty}[T(\boldsymbol{\Phi}'_{RL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{RL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T))]\\
&= \lim_{T\to\infty}\boldsymbol{l}'\left[\left(\sum_{i=1}^{M}\sigma_{ii}T\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL})\right)_{k,q}\right]\boldsymbol{l}\\
&= \boldsymbol{l}'\lim_{T\to\infty}\left[\left(\sum_{i=1}^{M}\sigma_{ii}T\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL})\right)_{k,q}\right]\boldsymbol{l}\Rightarrow\\
\boldsymbol{\Delta}_{RL} &= \lim_{T\to\infty}\left[\left(\sum_{i=1}^{M}\sigma_{ii}T\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL})\right)_{k,q}\right].
\end{aligned}
\tag{C.201}
$$

By using (C.200) we find

$$
\begin{aligned}
\sum_{i=1}^{M}\sigma_{ii}T\operatorname{tr}(\boldsymbol{\Phi}_{i\kappa}{}^{RL\prime}\boldsymbol{\Gamma}\boldsymbol{\Phi}_{qi}{}^{RL}) &= \sum_{i=1}^{M}\sigma_{ii}\Bigg[\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\delta_{iq}\boldsymbol{B}_{iq}\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\delta_{i\kappa}\boldsymbol{B}_{i\kappa}\Big]-\delta_{qi}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\delta_{i\kappa}\boldsymbol{B}_{i\kappa}\Big]\\
&\quad-\delta_{i\kappa}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\delta_{iq}\boldsymbol{B}_{iq}\Big]+\delta_{i\kappa}\delta_{qi}K\Bigg]\\
&= \delta_{q\kappa}\sigma_{qq}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{qq}\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{q\kappa}\Big]-\delta_{q\kappa}\sigma_{qq}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{q\kappa}\Big]\\
&\quad-\delta_{q\kappa}\sigma_{\kappa\kappa}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{\kappa q}\Big]+\delta_{q\kappa}\sigma_{qq}K.
\end{aligned}
\tag{C.202}
$$

So, equations (C.201) and (C.202) imply that

$$
\begin{aligned}
\boldsymbol{\Delta}_{RL} &= \Bigg[\Bigg(\delta_{q\kappa}\sigma_{qq}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{qq}\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{q\kappa}\Big]-\delta_{q\kappa}\sigma_{qq}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{q\kappa}\Big]\\
&\quad-\delta_{q\kappa}\sigma_{\kappa\kappa}\operatorname{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{\kappa q}\Big]+\delta_{q\kappa}\sigma_{qq}K\Bigg)_{k,q}\Bigg].
\end{aligned}
\tag{C.203}
$$

(c) Since $\boldsymbol{X}_* = (\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}$ and $\boldsymbol{X}_{\mu *} = \boldsymbol{Z}\boldsymbol{\Psi}_\mu$ $(\mu = 1, \ldots, M)$, we find that

$$\begin{aligned}
\boldsymbol{\Psi}'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Psi} &= \boldsymbol{\Psi}'(\boldsymbol{L} \otimes (\boldsymbol{Z}'\boldsymbol{Z}/T))\boldsymbol{\Psi} = \boldsymbol{\Psi}'(\boldsymbol{L} \otimes (\boldsymbol{Z}'\boldsymbol{Z}))\boldsymbol{\Psi}/T \\
&= \boldsymbol{\Psi}'[\boldsymbol{I}_M \otimes \boldsymbol{Z}'][\boldsymbol{L} \otimes \boldsymbol{I}_T][\boldsymbol{I}_M \otimes \boldsymbol{Z}]\boldsymbol{\Psi}/T \\
&= [(\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}]'[\boldsymbol{L} \otimes \boldsymbol{I}_T][(\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}]/T \\
&= \boldsymbol{X}_*'[\boldsymbol{L} \otimes \boldsymbol{I}_T]\boldsymbol{X}_*/T = [(\boldsymbol{X}_{i*}')_i][(l_{ij}\boldsymbol{I}_T)_{i,j}][(\boldsymbol{X}_{j*})_j]/T \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} l_{ij}(\boldsymbol{X}_{i*}'\boldsymbol{X}_{j*}/T) = \sum_{i=1}^{M}\sum_{j=1}^{M} l_{ij}(\boldsymbol{X}_i'\boldsymbol{P}_i^{-1\prime}\boldsymbol{P}_j^{-1}\boldsymbol{X}_j/T) \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} \delta_{ij}l_{ij}(\boldsymbol{X}_i'\boldsymbol{R}^{ij}\boldsymbol{X}_j/T) = \sum_{i=1}^{M}\sum_{j=1}^{M} \delta_{ij}l_{ij}\boldsymbol{B}_{ij} = \sum_{i=1}^{M} l_{ii}\boldsymbol{B}_{ii}.
\end{aligned} \tag{C.204}$$

The following result holds:

$$\begin{aligned}
\boldsymbol{\Phi}_{GL}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{\Phi}_{GL}' &= [\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T) - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']](\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T) \cdot \\
&\quad [\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T) - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']]' \\
&= [\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T) - [\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}']](\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T) \cdot \\
&\quad [(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' - [\boldsymbol{I}_M \otimes \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}]] \\
&= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_* \cdot \\
&\quad [\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\
&\quad -\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)[\boldsymbol{I}_M \otimes \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\
&\quad -[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'](\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\
&\quad +[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'](\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)[\boldsymbol{I}_M \otimes \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\
&= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\
&\quad -\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}[(\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}]'[\boldsymbol{I}_M \otimes \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\
&\quad -[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'][(\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}][\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\
&\quad +[\boldsymbol{\Sigma} \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}(\boldsymbol{Z}'\boldsymbol{Z})(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\
&= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\
&\quad -\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}'(\boldsymbol{I}_M \otimes \boldsymbol{Z}')[\boldsymbol{I}_M \otimes \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\
&\quad -[\boldsymbol{I}_M \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'](\boldsymbol{I}_M \otimes \boldsymbol{Z})\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\
&\quad +[\boldsymbol{\Sigma} \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}] \\
&= \boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' \\
&\quad -\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}'(\boldsymbol{I}_M \otimes \boldsymbol{I}_K) \\
&\quad -(\boldsymbol{I}_M \otimes \boldsymbol{I}_K)\boldsymbol{\Psi}[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1}\boldsymbol{\Psi}' + [\boldsymbol{\Sigma} \otimes (\boldsymbol{Z}'\boldsymbol{Z})^{-1}].
\end{aligned} \tag{C.205}$$

Since $\boldsymbol{X}_* = \boldsymbol{P}^{-1}\boldsymbol{X}$, and $\boldsymbol{\Omega}^{-1} = \boldsymbol{P}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{P}'$, we find that

$$\begin{aligned} \boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_* &= \boldsymbol{X}'\boldsymbol{P}^{-1\prime}(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{P}^{-1}\boldsymbol{X} \\ &= \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X} \Rightarrow \end{aligned} \tag{C.206}$$

$$[\boldsymbol{X}_*'(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_T)\boldsymbol{X}_*]^{-1} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X})^{-1}. \tag{C.207}$$

Also, since $\boldsymbol{\Gamma} = (\boldsymbol{Z}'\boldsymbol{Z}/T)$, $\boldsymbol{A} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)$, and $\boldsymbol{G} = (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X})^{-1} = \boldsymbol{A}^{-1}$, by using equations (C.205), (C.206) and (C.207) we find that

$$\begin{aligned} T\boldsymbol{\Phi}_{GL}(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{\Phi}_{GL}' &= \boldsymbol{\Psi}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{\Psi}' - \boldsymbol{\Psi}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{\Psi}'(\boldsymbol{I}_M \otimes \boldsymbol{I}_K) \\ &\quad -(\boldsymbol{I}_M \otimes \boldsymbol{I}_K)\boldsymbol{\Psi}(\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1}\boldsymbol{\Psi}' + [\boldsymbol{\Sigma} \otimes (\boldsymbol{Z}'\boldsymbol{Z}/T)^{-1}] \\ &= \boldsymbol{\Psi}\boldsymbol{G}\boldsymbol{\Psi}' - \boldsymbol{\Psi}\boldsymbol{G}\boldsymbol{\Psi}' - \boldsymbol{\Psi}\boldsymbol{G}\boldsymbol{\Psi}' + (\boldsymbol{\Sigma} \otimes \boldsymbol{G}^{-1}) = (\boldsymbol{\Sigma} \otimes \boldsymbol{G}^{-1}) - \boldsymbol{\Psi}\boldsymbol{G}\boldsymbol{\Psi}' . \end{aligned} \tag{C.208}$$

Moreover, since $\boldsymbol{\Omega} = [(\delta_{ij}\sigma^{ii}\boldsymbol{R}^{ii})_{i,j=1,\dots,M}]$ we take

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T = [(\boldsymbol{X}_i')_i][(\delta_{ij}\sigma^{ii}\boldsymbol{R}^{ii})_{i,j}][(\boldsymbol{X}_j)_j]/T \\ &= \sum_{i=1}^{M}\sigma_{ii}(\boldsymbol{X}_i'\boldsymbol{R}^{ii}\boldsymbol{X}_i/T) = \sum_{i=1}^{M}\sigma_{ii}\boldsymbol{B}_{ii} \Rightarrow \\ \boldsymbol{G} &= (\boldsymbol{X}'\boldsymbol{\Omega}\boldsymbol{X}/T)^{-1} = \boldsymbol{A}^{-1} = (\sum_{i=1}^{M}\sigma_{ii}\boldsymbol{B}_{ii})^{-1}. \end{aligned} \tag{C.209}$$

Thus,

$$\begin{aligned} T\,\mathrm{tr}\,\boldsymbol{\Phi}_{GL}'(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_T)\boldsymbol{\Phi}_{GL}'(\boldsymbol{L} \otimes \boldsymbol{\Gamma}) &= \mathrm{tr}\,(\boldsymbol{\Sigma}\boldsymbol{L} \otimes \boldsymbol{I}_K) - \mathrm{tr}[\boldsymbol{G}\boldsymbol{\Psi}'(\boldsymbol{L} \otimes \boldsymbol{\Gamma})\boldsymbol{\Psi}] \\ &= \mathrm{tr}\,(\boldsymbol{\Sigma}\boldsymbol{L})\,\mathrm{tr}\,(\boldsymbol{I}_K) - \mathrm{tr}\left[\left(\sum_{i=1}^{M}\sigma_{ii}\boldsymbol{B}_{ii}\right)^{-1}\left(\sum_{i=1}^{M}l_{ii}\boldsymbol{B}_{ii}\right)\right] \\ &= K\,\mathrm{tr}\,(\boldsymbol{\Sigma}\boldsymbol{l}'\boldsymbol{l}) - \mathrm{tr}\left[\left(\sum_{i=1}^{M}\sigma_{ii}\boldsymbol{B}_{ii}\right)^{-1}\left(\sum_{i=1}^{M}l_{ii}\boldsymbol{B}_{ii}\right)\right] \\ &= K\,\mathrm{tr}\,(\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l}) - \mathrm{tr}\left[\sum_{i=1}^{M}l_{ii}\boldsymbol{G}\boldsymbol{B}_{ii}\right] \\ &= \boldsymbol{l}'(K\boldsymbol{\Sigma})\boldsymbol{l} - \sum_{i=1}^{M}l_i\,\mathrm{tr}\,(\boldsymbol{G}\boldsymbol{B}_{ii})l_i \\ &= \boldsymbol{l}'(K\boldsymbol{\Sigma})\boldsymbol{l} - \boldsymbol{l}'[(\mathrm{tr}\,(\boldsymbol{G}\boldsymbol{B}_{ii}))_{i,i}]\boldsymbol{l} \\ &= \boldsymbol{l}'[K\boldsymbol{\Sigma} - [(\mathrm{tr}\,(\boldsymbol{G}\boldsymbol{B}_{ii}))_{i,i}]]\boldsymbol{l} \Rightarrow \end{aligned} \tag{C.210}$$

For any arbitrary vector $\boldsymbol{l}$

$$\begin{aligned}
\boldsymbol{l}'\boldsymbol{\Delta}_{GL}\boldsymbol{l} &= \boldsymbol{l}'\boldsymbol{\Delta}_{IG}\boldsymbol{l} = \boldsymbol{l}'\boldsymbol{\Delta}_{ML}\boldsymbol{l} \\
&= \lim_{T\to\infty}[T \operatorname{tr} \boldsymbol{\Phi}'_{GL}(\boldsymbol{L}\otimes\boldsymbol{\Gamma})\boldsymbol{\Phi}_{GL}(\boldsymbol{\Sigma}\otimes\boldsymbol{I}_T)] \\
&= \lim_{T\to\infty}[\boldsymbol{l}'[K\boldsymbol{\Sigma} - [(\operatorname{tr}(\boldsymbol{G}\boldsymbol{B}_{ii}))_{i,i}]]\boldsymbol{l}] \\
&= \boldsymbol{l}'[K\boldsymbol{\Sigma} - [(\operatorname{tr}(\boldsymbol{G}\boldsymbol{B}_{ii}))_{i,i}]]\boldsymbol{l} \Rightarrow \\
& \boldsymbol{\Delta}_{GL} = \boldsymbol{\Delta}_{IG} = \boldsymbol{\Delta}_{ML} = K\boldsymbol{\Sigma} - \left[\left(\operatorname{tr}\Big[\sum_{i=1}^{M}\sigma_{ii}\boldsymbol{B}_{ii}\Big]^{-1}\boldsymbol{B}_{ii}\right)_{i,i}\right].
\end{aligned} \tag{C.211}$$

Lemma C.8. The LS estimator $\tilde{\rho}_\mu$ of ρ_μ admits the stochastic expansion

$$\tilde{\rho}_\mu = \rho_\mu + \tau{\rho_\mu}^{(1)} + \tau^2{\rho_\mu}^{(2)} + \omega(\tau^3), \tag{C.212}$$

where

$${\rho_\mu}^{(1)} = -(\rho_\mu{\boldsymbol{D}_\mu}^{(1)} - {N_\mu}^{(1)}) \tag{C.213}$$

and

$${\rho_\mu}^{(2)} = {N_\mu}^{(2)} - {N_\mu}^{(1)}{\boldsymbol{D}_\mu}^{(1)} + \rho_\mu[({\boldsymbol{D}_\mu}^{(1)})^2 + {\boldsymbol{D}_\mu}^{(2)}]. \tag{C.214}$$

Proof of Lemma C.8. Since

$$\tilde{\rho}_\mu = \sum_{t=2}^{T}\tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu} \Big/ \sum_{t=1}^{T}\tilde{u}^2_{t\mu} = \sum_{t=1}^{T-1}\tilde{u}_{t\mu}\tilde{u}_{(t+1)\mu} \Big/ \sum_{t=1}^{T}\tilde{u}^2_{t\mu} = N_\mu/D_\mu, \tag{C.215}$$

where

$$N_\mu = \frac{1}{2}\tilde{\boldsymbol{u}}'_\mu\boldsymbol{D}\tilde{\boldsymbol{u}}_\mu/T{\sigma_{u_\mu}}^2 \tag{C.216}$$

and

$$D_\mu = \tilde{\boldsymbol{u}}'_\mu\tilde{\boldsymbol{u}}_\mu/T{\sigma_{u_\mu}}^2, \tag{C.217}$$

where

$$u_{t\mu} \sim \mathcal{N}(0, \sigma_{\mu\mu}/(1-\rho^2_\mu)) \Rightarrow {\sigma_{u_\mu}}^2 = \sigma_{\mu\mu}/(1-\rho^2_\mu) \tag{C.218}$$

and $\boldsymbol{D}$ is a matrix with (t, t')-th element equal to 1 if $|t - t'| = 1$ and zero elsewhere.

Let $\tilde{\boldsymbol{\beta}}$ be the LS estimator of $\boldsymbol{\beta}$ in the (μ)-th equation

$$\boldsymbol{y}_\mu = \boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{u}_\mu. \tag{C.219}$$

Then,

$$\begin{aligned}
\tilde{\boldsymbol{u}}_\mu &= \boldsymbol{y}_\mu - \boldsymbol{X}_\mu\tilde{\boldsymbol{\beta}} = \boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{u}_\mu - \boldsymbol{X}_\mu\tilde{\boldsymbol{\beta}} \\
&= \boldsymbol{u}_\mu - \tau\sqrt{T}\boldsymbol{X}_\mu(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu,
\end{aligned} \tag{C.220}$$

where

$$\begin{aligned}\boldsymbol{\theta}_\mu &= \sqrt{T}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \sqrt{T}[(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu \boldsymbol{y}_\mu - \boldsymbol{\beta}] \\ &= \sqrt{T}[(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu(\boldsymbol{X}_\mu\boldsymbol{\beta} + \boldsymbol{u}_\mu) - \boldsymbol{\beta}] \\ &= \sqrt{T}(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{X}_\mu\boldsymbol{\beta} + \sqrt{T}(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu - \sqrt{T}\boldsymbol{\beta} \\ &= \sqrt{T}(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu = (\boldsymbol{X}'_\mu \boldsymbol{X}_\mu/T)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu/\sqrt{T} \Rightarrow \end{aligned} \tag{C.221}$$

$$\boldsymbol{X}'_\mu\boldsymbol{u}_\mu/\sqrt{T} = (\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)\boldsymbol{\theta}_\mu. \tag{C.222}$$

But, equation (C.220) implies that

$$\begin{aligned}\tilde{\boldsymbol{u}}'_\mu\boldsymbol{D}\tilde{\boldsymbol{u}}_\mu &= (\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu)'\boldsymbol{D}(\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu) \\ &= (\boldsymbol{u}'_\mu - \tau\boldsymbol{\theta}'_\mu\boldsymbol{X}'_\mu)\boldsymbol{D}(\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu) \\ &= \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu - 2\boldsymbol{\theta}'_\mu(\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/\sqrt{T}) + \boldsymbol{\theta}'_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)\boldsymbol{\theta}_\mu. \end{aligned} \tag{C.223}$$

Then by using equations (C.216), (C.221) and (C.223) we find that

$$\begin{aligned} N_\mu &= \tilde{\boldsymbol{u}}'_\mu\boldsymbol{D}\tilde{\boldsymbol{u}}_\mu/2T{\sigma_{u_\mu}}^2 \\ &= \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - 2[\boldsymbol{u}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)^{-1}/\sqrt{T}][\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/\sqrt{T}]/2T{\sigma_{u_\mu}}^2 \\ &\quad +[\boldsymbol{u}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)^{-1}/\sqrt{T}](\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{X}_\mu/T)[(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu/T)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu/\sqrt{T}]/2T{\sigma_{u_\mu}}^2 \\ &= \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - \tau^2\boldsymbol{u}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/{\sigma_{u_\mu}}^2 \\ &\quad +\tau^2\boldsymbol{u}'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{D}\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{u}_\mu/2{\sigma_{u_\mu}}^2 \\ &= \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - \tau^2\boldsymbol{u}'_\mu\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu/{\sigma_{u_\mu}}^2 \\ &\quad +\tau^2\boldsymbol{u}'_\mu\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/2{\sigma_{u_\mu}}^2 \\ &= \rho_\mu - \rho_\mu + \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 + \tau^2(\boldsymbol{u}'_\mu\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/2 - \boldsymbol{u}'_\mu\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu)/{\sigma_{u_\mu}}^2 \\ &= \rho_\mu + \tau[\sqrt{T}(\boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - \rho_\mu)] + \tau^2(\boldsymbol{u}'_\mu\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/2 - \boldsymbol{u}'_\mu\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu)/{\sigma_{u_\mu}}^2 \\ &= \rho_\mu + \tau{N_\mu}^{(1)} + \tau^2{N_\mu}^{(2)}, \end{aligned} \tag{C.224}$$

where

$${N_\mu}^{(1)} = \sqrt{T}(\boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T{\sigma_{u_\mu}}^2 - \rho_\mu) = \sqrt{T}\left(\sum_{t=1}^{T-1} u_{t\mu}u_{(t+1)\mu}/2T{\sigma_{u_\mu}}^2 - \rho_\mu\right) \tag{C.225}$$

and

$${N_\mu}^{(2)} = (\boldsymbol{u}'_\mu\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/2 - \boldsymbol{u}'_\mu\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu)/{\sigma_{u_\mu}}^2. \tag{C.226}$$

Similarly, equations (C.220), (C.221) and (C.222) imply that

$$\begin{aligned}
\tilde{\boldsymbol{u}}_\mu'\tilde{\boldsymbol{u}}_\mu &= (\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu)'(\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu) = (\boldsymbol{u}_\mu' - \tau\boldsymbol{\theta}_\mu'\boldsymbol{X}_\mu')(\boldsymbol{u}_\mu - \tau\boldsymbol{X}_\mu\boldsymbol{\theta}_\mu)\\
&= \boldsymbol{u}_\mu'\boldsymbol{u}_\mu - 2\boldsymbol{\theta}_\mu'(\boldsymbol{X}_\mu'\boldsymbol{u}_\mu/\sqrt{T}) + \boldsymbol{\theta}_\mu'(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)\boldsymbol{\theta}_\mu\\
&= \boldsymbol{u}_\mu'\boldsymbol{u}_\mu - 2\boldsymbol{\theta}_\mu'(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)\boldsymbol{\theta}_\mu + \boldsymbol{\theta}_\mu'(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)\boldsymbol{\theta}_\mu\\
&= \boldsymbol{u}_\mu'\boldsymbol{u}_\mu - \boldsymbol{\theta}_\mu'(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)\boldsymbol{\theta}_\mu\\
&= \boldsymbol{u}_\mu'\boldsymbol{u}_\mu - (\boldsymbol{u}_\mu'\boldsymbol{X}_\mu/\sqrt{T})(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{u}_\mu/\sqrt{T})\\
&= \boldsymbol{u}_\mu'\boldsymbol{u}_\mu - \boldsymbol{u}_\mu'\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{u}_\mu = \boldsymbol{u}_\mu'\boldsymbol{u}_\mu - \boldsymbol{u}_\mu'\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu. \qquad \text{(C.227)}
\end{aligned}$$

Thus, equations (C.217) and (C.227) imply that

$$\begin{aligned}
\boldsymbol{D}_\mu &= \tilde{\boldsymbol{u}}_\mu'\tilde{\boldsymbol{u}}_\mu/T{\sigma_{u_\mu}}^2 = \boldsymbol{u}_\mu'\boldsymbol{u}_\mu/T{\sigma_{u_\mu}}^2 - \boldsymbol{u}_\mu'\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/T{\sigma_{u_\mu}}^2\\
&= 1 - 1 + \boldsymbol{u}_\mu'\boldsymbol{u}_\mu/T{\sigma_{u_\mu}}^2 - \boldsymbol{u}_\mu'\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/T{\sigma_{u_\mu}}^2\\
&= 1 + \tau[\sqrt{T}(\boldsymbol{u}_\mu'\boldsymbol{u}_\mu/T{\sigma_{u_\mu}}^2 - 1)] - \tau^2\boldsymbol{u}_\mu'\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/{\sigma_{u_\mu}}^2\\
&= 1 + \tau{\boldsymbol{D}_\mu}^{(1)} - \tau^2{\boldsymbol{D}_\mu}^{(2)}, \qquad \text{(C.228)}
\end{aligned}$$

where

$${\boldsymbol{D}_\mu}^{(1)} = \sqrt{T}(\boldsymbol{u}_\mu'\boldsymbol{u}_\mu/T{\sigma_{u_\mu}}^2 - 1) \qquad \text{(C.229)}$$

and

$${\boldsymbol{D}_\mu}^{(2)} = \boldsymbol{u}_\mu'\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{u}_\mu/{\sigma_{u_\mu}}^2. \qquad \text{(C.230)}$$

Thus, by using equation (C.228) we find that

$$\begin{aligned}
\boldsymbol{D}_\mu &= 1 + \tau({\boldsymbol{D}_\mu}^{(1)} - \tau{\boldsymbol{D}_\mu}^{(2)}) \Rightarrow\\
{\boldsymbol{D}_\mu}^{-1} &= [1 + \tau({\boldsymbol{D}_\mu}^{(1)} - \tau{\boldsymbol{D}_\mu}^{(2)})]^{-1} = 1 - \tau({\boldsymbol{D}_\mu}^{(1)} - \tau{\boldsymbol{D}_\mu}^{(2)}) + \tau^2({\boldsymbol{D}_\mu}^{(1)} - \tau{\boldsymbol{D}_\mu}^{(2)})^2 + \omega(\tau^3)\\
&= 1 - \tau{\boldsymbol{D}_\mu}^{(1)} + \tau^2[({\boldsymbol{D}_\mu}^{(1)})^2 + {\boldsymbol{D}_\mu}^{(2)}] + \omega(\tau^3). \qquad \text{(C.231)}
\end{aligned}$$

By using equations (C.215), (C.224) and (C.231) we find that

$$\begin{aligned}
\tilde{\rho}_\mu &= \boldsymbol{N}_\mu{\boldsymbol{D}_\mu}^{-1} = (\rho_\mu + \tau{N_\mu}^{(1)} + \tau^2{N_\mu}^{(2)})[1 - \tau{\boldsymbol{D}_\mu}^{(1)} + \tau^2[({\boldsymbol{D}_\mu}^{(1)})^2 + {\boldsymbol{D}_\mu}^{(2)}] + \omega(\tau^3)]\\
&= \rho_\mu - \tau\rho_\mu{\boldsymbol{D}_\mu}^{(1)} + \tau^2\rho_\mu[({\boldsymbol{D}_\mu}^{(1)})^2 + {\boldsymbol{D}_\mu}^{(2)}] + \tau{N_\mu}^{(1)} - \tau^2{N_\mu}^{(1)}{\boldsymbol{D}_\mu}^{(1)} + \tau^2{N_\mu}^{(2)} + \omega(\tau^3)\\
&= \rho_\mu - \tau(\rho_\mu{\boldsymbol{D}_\mu}^{(1)} - {N_\mu}^{(1)}) + \tau^2[{N_\mu}^{(2)} - {N_\mu}^{(1)}{\boldsymbol{D}_\mu}^{(1)} + \rho_\mu[({\boldsymbol{D}_\mu}^{(1)})^2 + {\boldsymbol{D}_\mu}^{(2)}]] + \omega(\tau^3)\\
&= \rho_\mu + \tau({\rho_\mu}^{(1)} + \tau{\rho_\mu}^{(2)}) + \omega(\tau^3), \qquad \text{(C.232)}
\end{aligned}$$

where

$${\rho_\mu}^{(1)} = -(\rho_\mu{\boldsymbol{D}_\mu}^{(1)} - {N_\mu}^{(1)}) \qquad \text{(C.233)}$$

and

$$\rho_\mu{}^{(2)} = N_\mu{}^{(2)} - N_\mu{}^{(1)}\boldsymbol{D}_\mu{}^{(1)} + \rho_\mu[(\boldsymbol{D}_\mu{}^{(1)})^2 + \boldsymbol{D}_\mu{}^{(2)}]. \tag{C.234}$$

Since

$$\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_\mu \boldsymbol{P}'_\mu = \frac{1}{1-\rho_\mu{}^2}\begin{bmatrix} 1 & \rho_\mu & \dots & \rho_\mu{}^{T-1} \\ \rho_\mu & & & \\ \vdots & & & \\ \rho_\mu{}^{T-1} & & \dots & 1, \end{bmatrix}, \tag{C.235}$$

it is straightforward that

$$\boldsymbol{R}_{\mu\mu}{}^{-1} = \boldsymbol{P}'_\mu{}^{-1}\boldsymbol{P}_\mu{}^{-1} = \boldsymbol{R}^{\mu\mu} = (1+\rho_\mu{}^2)\boldsymbol{I}_T - \rho_\mu \boldsymbol{D} - \rho_\mu{}^2\boldsymbol{\Delta} \text{ [see (C.5)]}. \tag{C.236}$$

Then,

$$\boldsymbol{R}_{\rho_\mu}{}^{\mu\mu} = \partial \boldsymbol{R}^{\mu\mu}/\partial\rho_\mu = 2\rho_\mu \boldsymbol{I}_T - \boldsymbol{D} - 2\rho_\mu\boldsymbol{\Delta} \text{ [see (C.7)]} \tag{C.237}$$

and

$$\boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu} = \partial^2 \boldsymbol{R}^{\mu\mu}/\partial\rho_\mu{}^2 = 2\boldsymbol{I}_T - 2\boldsymbol{\Delta} = 2(\boldsymbol{I}_T - \boldsymbol{\Delta}) \text{ [see (C.8)]}. \tag{C.238}$$

Define the $(T \times T)$ matrices

$$\boldsymbol{R}_i{}^{\mu\mu} = \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu} + i\rho_\mu\boldsymbol{\Delta}, \;\; \boldsymbol{R}_{ii}{}^{\mu\mu} = \boldsymbol{R}_{\rho_\mu\rho_\mu}{}^{\mu\mu} + i\boldsymbol{\Delta} \;\; (i=1,2). \tag{C.239}$$

Then,

$$\begin{aligned} \boldsymbol{R}_2{}^{\mu\mu} &= \boldsymbol{R}_{\rho_\mu}{}^{\mu\mu} + 2\rho_\mu\boldsymbol{\Delta} = 2\rho_\mu\boldsymbol{I}_T - \boldsymbol{D} - 2\rho_\mu\boldsymbol{\Delta} + 2\rho_\mu\boldsymbol{\Delta} \\ &= 2\rho_\mu\boldsymbol{I}_T - \boldsymbol{D}. \end{aligned} \tag{C.240}$$

The quantities $\rho_\mu{}^{(1)}$ and $\rho_\mu{}^{(2)}$ can be written as functions of $\boldsymbol{R}_2{}^{\mu\mu}$ as follows:

i.

$$\begin{aligned} \rho_\mu{}^{(1)} &= -(\rho_\mu\boldsymbol{D}_\mu{}^{(1)} - \boldsymbol{N}_\mu{}^{(1)}) = \text{[see (C.225) and(C.229)]} \\ &= -[\rho_\mu\sqrt{T}(\boldsymbol{u}'_\mu\boldsymbol{u}_\mu/T\sigma_{\boldsymbol{u}_\mu}{}^2 - 1) - \sqrt{T}(\boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu/2T\sigma_{\boldsymbol{u}_\mu}{}^2 - \rho_\mu)] \\ &= -\sqrt{T}(2\rho_\mu\boldsymbol{u}'_\mu\boldsymbol{u}_\mu - \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu)/2T\sigma_{\boldsymbol{u}_\mu}{}^2 \\ &= -\boldsymbol{u}'_\mu(2\rho_\mu\boldsymbol{I}_T - \boldsymbol{D})\boldsymbol{u}_\mu/2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2 = \text{[see (C.240)]} \\ &= -\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2. \end{aligned} \tag{C.241}$$

ii.

$$
\begin{aligned}
\rho_\mu{}^{(2)} &= \boldsymbol{N}_\mu{}^{(2)} - \boldsymbol{N}_\mu{}^{(1)}\boldsymbol{D}_\mu{}^{(1)} + \rho_\mu[(\boldsymbol{D}_\mu{}^{(1)})^2 + \boldsymbol{D}_\mu{}^{(2)}] \\
&= \boldsymbol{N}_\mu{}^{(2)} - \boldsymbol{N}_\mu{}^{(1)}\boldsymbol{D}_\mu{}^{(1)} + \rho_\mu(\boldsymbol{D}_\mu{}^{(1)})^2 + \rho_\mu\boldsymbol{D}_\mu{}^{(2)} \\
&= \boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)} + \boldsymbol{D}_\mu{}^{(1)}(\rho_\mu\boldsymbol{D}_\mu{}^{(1)} - \boldsymbol{N}_\mu{}^{(1)}) = [\text{see (C.241)}] \\
&= \boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)} - \boldsymbol{D}_\mu{}^{(1)}[-(\rho_\mu\boldsymbol{D}_\mu{}^{(1)} - \boldsymbol{N}_\mu{}^{(1)})] = [\text{see (C.241)}] \\
&= \boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)} - \boldsymbol{D}_\mu{}^{(1)}\rho_\mu{}^{(1)}. \qquad \text{(C.242)}
\end{aligned}
$$

By using equations(C.225), (C.226), (C.229), (C.230), and (C.240) we find that

$$
\begin{aligned}
& 2\sigma_{\boldsymbol{u}_\mu}{}^2(\boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)}) \\
= \;& 2\sigma_{\boldsymbol{u}_\mu}{}^2[(\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/2 - \boldsymbol{u}_{\mu'}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu)/\sigma_{\boldsymbol{u}_\mu}{}^2 + \rho_\mu\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/\sigma_{\boldsymbol{u}_\mu}{}^2] \\
= \;& \boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu - 2\boldsymbol{u}_{\mu'}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu + 2\rho_\mu\boldsymbol{u}'_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu \\
= \;& \boldsymbol{u}'_\mu(\boldsymbol{I}_T - \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu})\boldsymbol{D}(\boldsymbol{I}_T - \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu})\boldsymbol{u}_\mu - 2\boldsymbol{u}'_\mu(\boldsymbol{I}_T - \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu})\boldsymbol{D}\boldsymbol{u}_\mu + 2\rho_\mu\boldsymbol{u}'_\mu(\boldsymbol{I}_T - \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu})\boldsymbol{u}_\mu \\
= \;& \boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{D}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu - 2\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu - 2\boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu \\
& +2\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{D}\boldsymbol{u}_\mu + 2\rho_\mu\boldsymbol{u}'_\mu\boldsymbol{u}_\mu - 2\rho_\mu\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu \\
= \;& \boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{D}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu - 2\rho_\mu\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + 2\rho_\mu\boldsymbol{u}'_\mu\boldsymbol{u}_\mu - \boldsymbol{u}'_\mu\boldsymbol{D}\boldsymbol{u}_\mu \\
& (\text{since } \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \text{ is idempotent}) \\
= \;& \boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}(\boldsymbol{D} - 2\rho_\mu\boldsymbol{I}_T)\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu(2\rho_\mu\boldsymbol{I}_T - \boldsymbol{D})\boldsymbol{u}_\mu \\
= \;& -\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu. \qquad \text{(C.243)}
\end{aligned}
$$

Similarly, equations (C.218), (C.229), (C.230), (C.242), and (C.243) imply that

$$
\begin{aligned}
& 2\sigma_{\boldsymbol{u}_\mu}{}^2\rho_\mu{}^{(2)} = 2\sigma_{\boldsymbol{u}_\mu}{}^2[(\boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)}) - \boldsymbol{D}_\mu{}^{(1)}\rho_\mu{}^{(1)}] \\
= \;& 2\sigma_{\boldsymbol{u}_\mu}{}^2(\boldsymbol{N}_\mu{}^{(2)} + \rho_\mu\boldsymbol{D}_\mu{}^{(2)}) - 2\sigma_{\boldsymbol{u}_\mu}{}^2\boldsymbol{D}_\mu{}^{(1)}\rho_\mu{}^{(1)} \\
= \;& -\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu + 2\sigma_{\boldsymbol{u}_\mu}{}^2\sqrt{T}(\boldsymbol{u}'_\mu\boldsymbol{u}_\mu/T\sigma_{\boldsymbol{u}_\mu}{}^2 - 1)(\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2) \\
= \;& -\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{u}_\mu\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/T\sigma_{\boldsymbol{u}_\mu}{}^2 - \boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu \Rightarrow \\
\rho_\mu{}^{(2)} = \;& -\boldsymbol{u}'_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{u}_\mu/2\sigma_{\boldsymbol{u}_\mu}{}^2 + \boldsymbol{u}'_\mu\boldsymbol{u}_\mu\boldsymbol{u}'_\mu\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2T\sigma_{\boldsymbol{u}_\mu}{}^4. \qquad \text{(C.244)}
\end{aligned}
$$

□

Lemma C.9. The following results hold:

i) By using (C.212) the sampling error of the Least Squares estimator of ρ_μ is

$$\begin{aligned}
\delta_{\rho_\mu}{}^{LS} &= \frac{\tilde{\rho}_\mu - \rho_\mu}{\tau} = \sqrt{T}(\tilde{\rho}_\mu - \rho_\mu) = [\text{see (C.212)}] \\
&= \sqrt{T}[\rho_\mu + \tau(\rho_\mu{}^{(1)} + \tau\rho_\mu{}^{(2)}) + \omega(\tau^3) - \rho_\mu] \\
&= \rho_\mu{}^{(1)} + \tau\rho_\mu{}^{(2)} + \omega(\tau^2) \\
&= d_{(1)\mu}{}^{LS} + \tau d_{(2)\mu}{}^{LS} + \omega(\tau^2), \qquad \text{(C.245)}
\end{aligned}$$

where

$$d_{(1)\mu}{}^{LS} = \rho_\mu{}^{(1)} = -\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu / 2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2 \qquad \text{(C.246)}$$

and

$$d_{(2)\mu}{}^{LS} = \rho_\mu{}^{(2)} = -\boldsymbol{u}'_\mu \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \boldsymbol{u}_\mu / 2\sigma_{\boldsymbol{u}_\mu}{}^2 + \boldsymbol{u}'_\mu \boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu / 2T\sigma_{\boldsymbol{u}_\mu}{}^4. \qquad \text{(C.247)}$$

ii) The iterative Prais-Winsten estimator of ρ_μ is (see Magee, 1985)

$$\begin{aligned}
\hat{\rho}_\mu^{PW} &= \tilde{\rho}^{LS} - \tau^2 \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} [\boldsymbol{u}'_\mu \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2] \\
&\quad + \omega(\tau^3), \qquad \text{(C.248)}
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{V} &= \boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu = [\boldsymbol{I} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{R}_{\mu\mu}]\boldsymbol{R}^{\mu\mu} \\
&= \boldsymbol{W}_{\mu\mu}\boldsymbol{R}^{\mu\mu} \qquad \text{(C.249)}
\end{aligned}$$

and

$$\boldsymbol{W}_{\mu\mu} = \boldsymbol{I} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu\boldsymbol{R}_{\mu\mu}. \qquad \text{(C.250)}$$

The iterative Prais-Winsten estimator of ρ_μ is equal to its GL estimator, i.e., $\hat{\rho}^{PW} = \hat{\rho}^{GL}$. Thus, by using equations (C.248), (C.249), and (C.250), the sampling error of iterative Prais-Winsten estimator of ρ_μ is

$$\begin{aligned}
\delta_{\rho_\mu}{}^{GL} = \delta_{\rho_\mu}{}^{PW} &= \sqrt{T}(\hat{\rho}_\mu^{PW} - \rho_\mu) \\
&= [(\tilde{\rho}_\mu^{LS} - \rho_\mu) - \tau^2 \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} [\boldsymbol{u}'_\mu \bar{P}_{\boldsymbol{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2] \\
&\quad + \omega(\tau^3)]/\tau \\
&= \delta_{\rho_\mu}{}^{LS} - \tau \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} [\boldsymbol{u}'_\mu \bar{P}_{\boldsymbol{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2] + \omega(\tau^2) \\
&= d_{(1)\mu}{}^{LS} + \tau[d_{(2)\mu}{}^{LS} - \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} [\boldsymbol{u}'_\mu \bar{P}_{\boldsymbol{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\boldsymbol{X}_\mu} \boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]] + \omega(\tau^2)
\end{aligned}$$

$$= \quad d_{(1)\mu}{}^{LS} + \tau d_{(2)\mu}{}^{GL} + \omega(\tau^2), \tag{C.251}$$

where

$$\begin{aligned} d_{(2)\mu}{}^{GL} &= -\boldsymbol{u}'_\mu \bar{P}_{\mathbf{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \bar{P}_{\mathbf{X}_\mu} \boldsymbol{u}_\mu / 2\sigma_{u_\mu}{}^2 + \boldsymbol{u}'_\mu \boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu / 2T\sigma_{u_\mu}{}^4 \\ &- \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} [\boldsymbol{u}'_\mu \bar{P}_{\mathbf{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\mathbf{X}_\mu} \boldsymbol{V} \boldsymbol{R}^{\mu\mu} \boldsymbol{u}_\mu + \boldsymbol{u}'_\mu \boldsymbol{R}^{\mu\mu} \boldsymbol{V} \boldsymbol{P}_{\mathbf{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\mathbf{X}_\mu} \boldsymbol{V} \boldsymbol{R}^{\mu\mu} \boldsymbol{u}_\mu / 2]. \end{aligned} \tag{C.252}$$

iii) The ML estimator of ρ_μ is

$$\hat{\rho}_\mu^{ML} = \hat{\rho}_\mu^{PW} + \tau^2[\rho_\mu \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} (u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu] + \omega(\tau^3). \tag{C.253}$$

(see Beach and MacKinnon, 1978, Magee, 1985).

Thus, by using equation (C.253), the sampling error of ML estimator of ρ_μ is

$$\begin{aligned} \delta_{\rho_\mu}{}^{ML} &= \sqrt{T}(\hat{\rho}_\mu^{ML} - \rho_\mu) = [(\hat{\rho}_\mu^{PW} - \rho_\mu) + \tau^2[\rho_\mu(1-\rho_\mu{}^2)(u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu] + \omega(\tau^3)]/\tau \\ &= \delta_{\rho_\mu}{}^{PW} + \tau[\rho_\mu \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} (u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu] + \omega(\tau^2) \\ &= d_{(1)\mu}{}^{LS} + \tau[d_{(2)\mu}{}^{GL} + \rho_\mu \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} (u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu] + \omega(\tau^2) \\ &= d_{(1)\mu}{}^{LS} + \tau d_{(2)\mu}{}^{ML} + \omega(\tau^2), \end{aligned} \tag{C.254}$$

where

$$d_{(2)\mu}{}^{ML} = d_{(2)\mu}{}^{GL} + \rho_\mu \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}} (u_{1\mu}{}^2 + u_{T\mu}{}^2) - \rho_\mu. \tag{C.255}$$

iv) The Durbin-Watson estimator of ρ_μ is

$$\hat{\rho}_\mu^{DW} = 1 - D_{W_\mu}/2, \tag{C.256}$$

where D_{W_μ} is the Durbin-Watson statistic, i.e.,

$$\begin{aligned} D_{W_\mu} &= \frac{\sum_{t=2}^T (\tilde{u}_{t\mu} - \tilde{u}_{(t-1)\mu})^2}{\sum_{t=1}^T \tilde{u}_{t\mu}^2} = \frac{\sum_{t=2}^T \tilde{u}_{t\mu}^2 + \sum_{t=2}^T \tilde{u}_{(t-1)\mu}^2 - 2\sum_{t=2}^T \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu}}{\sum_{t=1}^T \tilde{u}_{t\mu}^2} \\ &= \frac{\sum_{t=1}^T \tilde{u}_{t\mu}^2 - \tilde{u}_{1\mu}^2 + \sum_{t=1}^T \tilde{u}_{t\mu}^2 - \tilde{u}_{T\mu}^2 - 2\sum_{t=2}^T \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu}}{\sum_{t=1}^T \tilde{u}_{t\mu}^2} \\ &= \frac{2\sum_{t=1}^T \tilde{u}_{t\mu}^2 - (2\sum_{t=2}^T \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu} + \tilde{u}_{1\mu}^2 + \tilde{u}_{T\mu}^2)}{\sum_{t=1}^T \tilde{u}_{t\mu}^2} \\ &= 2 - \frac{2\sum_{t=2}^T \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu} + \tilde{u}_{1\mu}^2 + \tilde{u}_{T\mu}^2}{\sum_{t=1}^T \tilde{u}_{t\mu}^2}. \end{aligned} \tag{C.257}$$

Equations (C.256) and (C.257) imply that

$$\begin{aligned}
\hat{\rho}_\mu^{DW} &= 1 - \left[1 - \frac{\sum_{t=2}^{T} \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu} + (\tilde{u}_{1\mu}^2 + \tilde{u}_{T\mu}^2)/2}{\sum_{t=1}^{T} \tilde{u}_{t\mu}^2}\right] \\
&= \frac{\sum_{t=2}^{T} \tilde{u}_{t\mu}\tilde{u}_{(t-1)\mu}}{\sum_{t=1}^{T} \tilde{u}_{t\mu}^2} + \frac{(\tilde{u}_{1\mu}^2 + \tilde{u}_{T\mu}^2)/2}{\sum_{t=1}^{T} \tilde{u}_{t\mu}^2} \\
&= \tilde{\rho}_\mu^{LS} + \frac{(\tilde{u}_{1\mu}^2 + \tilde{u}_{T\mu}^2)/2T\sigma_{u_\mu}^2}{\sum_{t=1}^{T}(\tilde{u}_{t\mu}^2/T)(1/\sigma_{u_\mu}^2)} \\
&= \tilde{\rho}_\mu^{LS} + \frac{1}{2T}\frac{(\tilde{u}_{1\mu}^2 + \tilde{u}_{T\mu}^2)}{\tilde{\sigma}_{u_\mu}^2} \\
&= \tilde{\rho}_\mu^{LS} + \tau^2(1-{\rho_\mu}^2)({u_{1\mu}}^2 + {u_{T\mu}}^2)/2\sigma_{\mu\mu} + \omega(\tau^3), \qquad \text{(C.258)}
\end{aligned}$$

because $\tilde{u}_{t\mu}$ is a consistent estimator of $u_{t\mu}$ and so $\sum_{t=1}^{T} \tilde{u}_{t\mu}^2/T$ is a consistent estimator of ${\sigma_{u_{t\mu}}}^2$ with an error of order $\omega(\tau^3)$. Therefore, (C.258) implies that the sampling error of DW estimator of ρ_μ is

$$\begin{aligned}
{\delta_{\rho_\mu}}^{DW} &= \sqrt{T}(\hat{\rho}_\mu^{DW} - \rho_\mu) = [(\tilde{\rho}_\mu^{LS} - \rho_\mu) + \tau^2\frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}({u_{1\mu}}^2 + {u_{T\mu}}^2)/2 + \omega(\tau^3)]/\tau = (\textit{see}\ \text{(C.245)}) \\
&= {\delta_{\rho_\mu}}^{LS} + \tau\frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}({u_{1\mu}}^2 + {u_{T\mu}}^2)/2 + \omega(\tau^2) \\
&= {d_{(1)\mu}}^{LS} + \tau[{d_{(2)\mu}}^{LS} + \frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}({u_{1\mu}}^2 + {u_{T\mu}}^2)/2] + \omega(\tau^2) \\
&= {d_{(1)\mu}}^{LS} + \tau {d_{(2)\mu}}^{DW} + \omega(\tau^2), \qquad \text{(C.259)}
\end{aligned}$$

where

$$ {d_{(2)\mu}}^{DW} = {d_{(2)\mu}}^{LS} + \frac{(1-{\rho_\mu}^2)}{\sigma_{\mu\mu}}({u_{1\mu}}^2 + {u_{T\mu}}^2)/2. \qquad \text{(C.260)}$$

Lemma C.10. The following results hold:

i) Equations (C.236), (C.237), and (C.239) imply that

$$\begin{aligned}
{\boldsymbol{R}_1}^{\mu\mu} &= {\boldsymbol{R}_{\rho_\mu}}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta} = 2\rho_\mu\boldsymbol{I}_T - \boldsymbol{D} - 2\rho_\mu\boldsymbol{\Delta} + \rho_\mu\boldsymbol{\Delta} = 2\rho_\mu\boldsymbol{I}_T - \boldsymbol{D} - \rho_\mu\boldsymbol{\Delta} \\
&= \frac{1}{\rho_\mu}[2{\rho_\mu}^2\boldsymbol{I}_T - \rho_\mu\boldsymbol{D} - {\rho_\mu}^2\boldsymbol{\Delta}] = \frac{1}{\rho_\mu}[\boldsymbol{I}_T + {\rho_\mu}^2\boldsymbol{I}_T - \rho_\mu\boldsymbol{D} - {\rho_\mu}^2\boldsymbol{\Delta} - \boldsymbol{I}_T + {\rho_\mu}^2\boldsymbol{I}_T] \\
&= \frac{1}{\rho_\mu}[(1+{\rho_\mu}^2)\boldsymbol{I}_T - \rho_\mu\boldsymbol{D} - {\rho_\mu}^2\boldsymbol{\Delta} - (1-{\rho_\mu}^2)\boldsymbol{I}_T] \\
&= \frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu} - (1-{\rho_\mu}^2)\boldsymbol{I}_T], \qquad \text{(C.261)}
\end{aligned}$$

which implies that

$$\begin{aligned} \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} &= \frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{I}_T]\boldsymbol{R}_{\mu\mu} = \frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu}\boldsymbol{R}_{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] \\ &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]. \end{aligned} \tag{C.262}$$

Then, equations (C.239) and (C.240) imply that

$$\begin{aligned} \boldsymbol{R}_2{}^{\mu\mu} &= \boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta} \Rightarrow \\ \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} &= (\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta})\boldsymbol{R}_{\mu\mu} = \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} \\ &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}. \end{aligned} \tag{C.263}$$

Furthermore,

$$\begin{aligned} (\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 &= [\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}][\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}] \\ &= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu{}^2\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}. \end{aligned} \tag{C.264}$$

ii)

$$\begin{aligned} \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} &= \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}[\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}]\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} \\ &= \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}. \end{aligned} \tag{C.265}$$

Similarly,

$$\begin{aligned} \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} &= \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}[\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}]\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} \\ &= \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} + \rho_\mu\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} \end{aligned} \tag{C.266}$$

and

$$\begin{aligned} \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} &= \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}_\mu}[\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}]\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} \\ &= \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu} + \rho_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}. \end{aligned} \tag{C.267}$$

iii) Then, by using (C.235)

$$\operatorname{tr}\boldsymbol{R}_{\mu\mu} = \frac{1}{1-\rho_\mu{}^2}\sum_{t=1}^{T}1 = \frac{T}{1-\rho_\mu{}^2}, \tag{C.268}$$

we find that

$$\operatorname{tr}[(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] = (1-\rho_\mu{}^2)\operatorname{tr}\boldsymbol{R}_{\mu\mu} = T. \tag{C.269}$$

By using equations (C.262) and (C.269) we find that

$$\operatorname{tr}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) = \frac{1}{\rho_\mu}[\operatorname{tr}\boldsymbol{I}_T - (1-{\rho_\mu}^2)\operatorname{tr}\boldsymbol{R}_{\mu\mu}] = \frac{1}{\rho_\mu}[T-T] = 0. \tag{C.270}$$

Let δ_{ij} be the (i,j)-th element of $\boldsymbol{\Delta}$. Then, $\delta_{ij} = 1$ for $i = j = 1$ and $i = j = T$ and $\delta_{ij} = 0$ elsewhere. Moreover, the (i,j)-th element of $\boldsymbol{R}_{\mu\mu}$ is $\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-j|}$. Then, the (i,j)-th element of $\boldsymbol{\Delta R}_{\mu\mu}$ is

$$\delta_{ij}{}^* = \sum_{\kappa=1}^{T}\delta_{i\kappa}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|\kappa-j|} = \delta_{ii}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-j|}, \tag{C.271}$$

because $\delta_{i\kappa} = 0$ for $\kappa \neq i$. Therefore, equation (C.271) implies that

$$\begin{aligned}\operatorname{tr}\boldsymbol{\Delta R}_{\mu\mu} &= \sum_{i=1}^{T}\delta_{ii}{}^* = \sum_{i=1}^{T}\delta_{ii}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-i|} = \frac{1}{1-{\rho_\mu}^2}\sum_{i=1}^{T}\delta_{ii} \\ &= \frac{1}{1-{\rho_\mu}^2}(\delta_{11}+\delta_{TT}) = \frac{2}{1-{\rho_\mu}^2}\,.\end{aligned} \tag{C.272}$$

The (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}$ is

$$\begin{aligned}\tilde{\delta}_{ij} &= \sum_{\kappa=1}^{T}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-\kappa|}\delta_{\kappa j}{}^* = \sum_{\kappa=1}^{T}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|i-\kappa|}\delta_{\kappa\kappa}\frac{1}{1-{\rho_\mu}^2}{\rho_\mu}^{|\kappa-j|} \\ &= \frac{1}{(1-{\rho_\mu}^2)^2}{\rho_\mu}^{|i-1|+|1-j|}\delta_{11} + \frac{1}{(1-{\rho_\mu}^2)^2}{\rho_\mu}^{|i-T|+|T-j|}\delta_{TT} \\ &= \frac{1}{(1-{\rho_\mu}^2)^2}({\rho_\mu}^{i+j-2} + {\rho_\mu}^{2T-i-j}),\end{aligned} \tag{C.273}$$

which implies that

$$\begin{aligned}\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}) &= \sum_{i=1}^{T}\tilde{\delta}_{ii} = \frac{1}{(1-{\rho_\mu}^2)^2}\left[\sum_{i=1}^{T}{\rho_\mu}^{2(i-1)} + \sum_{i=1}^{T}{\rho_\mu}^{2(T-i)}\right] \\ &= \frac{1}{(1-{\rho_\mu}^2)^2}\left[\sum_{i=1}^{T}{\rho_\mu}^{2(i-1)} + \sum_{j=1}^{T}{\rho_\mu}^{2(j-1)}\right] = [\text{defining the index } j = T-i+1] \\ &= \frac{1}{(1-{\rho_\mu}^2)^2}2\sum_{i=1}^{T}{\rho_\mu}^{2(i-1)} \\ &= \frac{2}{(1-{\rho_\mu}^2)^2}\sum_{i=1}^{T}{\rho_\mu}^{2(i-1)} = [\text{defining the index } j = i-1] \\ &= \frac{2}{(1-{\rho_\mu}^2)^2}\sum_{j=0}^{T-1}{\rho_\mu}^{2j} = [\text{defining } r = {\rho_\mu}^2] \\ &= \frac{2}{(1-{\rho_\mu}^2)^2}\sum_{j=0}^{T-1}r^j = \frac{2}{(1-{\rho_\mu}^2)^2}\frac{1-r^T}{1-r} \\ &= \frac{2}{(1-{\rho_\mu}^2)^2}\frac{1-{\rho_\mu}^{2T}}{(1-{\rho_\mu}^2)} = \frac{2(1-{\rho_\mu}^{2T})}{(1-{\rho_\mu}^2)^3}.\end{aligned} \tag{C.274}$$

Along the same lines as in equation (C.271) we find that the (i, j)-th element of the $(\boldsymbol{\Delta R}_{\mu\mu})^2$ is

$$\begin{aligned}\delta_{ij}^{\circ} &= \sum_{\kappa=1}^{T}\delta_{i\kappa}{}^{*}\delta_{\kappa j}{}^{*} = \sum_{\kappa=1}^{T}\delta_{ii}\frac{1}{1-\rho_{\mu}{}^{2}}\rho_{\mu}{}^{|i-\kappa|}\delta_{\kappa\kappa}\frac{1}{1-\rho_{\mu}{}^{2}}\rho_{\mu}{}^{|\kappa-j|}\\ &= \delta_{ii}\frac{1}{(1-\rho_{\mu}{}^{2})^{2}}\sum_{\kappa=1}^{T}\delta_{\kappa\kappa}\rho_{\mu}{}^{|i-\kappa|+|\kappa-j|} = \delta_{ii}\frac{1}{(1-\rho_{\mu}{}^{2})^{2}}(\rho_{\mu}{}^{|i-1|+|1-j|}\delta_{11}+\rho_{\mu}{}^{|i-T|+|T-j|}\delta_{TT})\\ &= \delta_{ii}\frac{1}{(1-\rho_{\mu}{}^{2})^{2}}(\rho_{\mu}{}^{i+j-2}+\rho_{\mu}{}^{2T-i-j}),\end{aligned} \tag{C.275}$$

which implies that

$$\begin{aligned}\mathrm{tr}\,[(\boldsymbol{\Delta R}_{\mu\mu})^{2}] &= \sum_{i=1}^{T}\delta_{ii}^{\circ} = \sum_{i=1}^{T}\delta_{ii}\frac{1}{(1-\rho_{\mu}{}^{2})^{2}}(\rho_{\mu}{}^{2(i-1)}+\rho_{\mu}{}^{2(T-i)})\\ &= \delta_{11}\frac{1}{(1-\rho_{\mu}{}^{2})^{2}}(\rho_{\mu}{}^{2(1-1)}+\rho_{\mu}{}^{2(T-1)}) + \delta_{TT}\frac{1}{(1-\rho_{\mu}{}^{2})^{2}}(\rho_{\mu}{}^{2(T-1)}+\rho_{\mu}{}^{2(T-T)})\\ &= \frac{2}{(1-\rho_{\mu}{}^{2})^{2}}(1+\rho_{\mu}{}^{2(T-1)}).\end{aligned} \tag{C.276}$$

By using equation (C.275) we find that the (i, j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2$ is

$$\begin{aligned}\delta_{\hat{i}\hat{j}} &= \sum_{\kappa=1}^{T}\frac{1}{1-\rho_{\mu}{}^{2}}\rho_{\mu}{}^{|i-\kappa|}\delta_{\kappa j}^{\circ} = \sum_{\kappa=1}^{T}\frac{1}{1-\rho_{\mu}{}^{2}}\rho_{\mu}{}^{|i-\kappa|}\delta_{\kappa\kappa}\frac{1}{(1-\rho_{\mu}{}^{2})^{2}}(\rho_{\mu}{}^{\kappa+j-2}+\rho_{\mu}{}^{2T-\kappa-j})\\ &= \delta_{11}\frac{1}{(1-\rho_{\mu}{}^{2})^{3}}\rho_{\mu}{}^{|i-1|}(\rho_{\mu}{}^{j-1}+\rho_{\mu}{}^{2T-j-1}) + \delta_{TT}\frac{1}{(1-\rho_{\mu}{}^{2})^{3}}\rho_{\mu}{}^{|i-T|}(\rho_{\mu}{}^{T+j-2}+\rho_{\mu}{}^{T-j})\\ &= \frac{1}{(1-\rho_{\mu}{}^{2})^{3}}(\rho_{\mu}{}^{i+j-2}+\rho_{\mu}{}^{2T+i-j-2}+\rho_{\mu}{}^{2T-i+j-2}+\rho_{\mu}{}^{2T-i-j}),\end{aligned} \tag{C.277}$$

which implies that

$$\begin{aligned}\mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^{2}] &= \sum_{i=1}^{T}\delta_{\hat{i}\hat{i}} = \frac{1}{(1-\rho_{\mu}{}^{2})^{3}}\sum_{i=1}^{T}(\rho_{\mu}{}^{2i-2}+2\rho_{\mu}{}^{2T-2}+\rho_{\mu}{}^{2T-2i})\\ &= \frac{1}{(1-\rho_{\mu}{}^{2})^{3}}[2T\rho_{\mu}{}^{2T-2}+\sum_{i=1}^{T}\rho_{\mu}{}^{2(i-1)}+\sum_{i=1}^{T}\rho_{\mu}{}^{2(T-i)}]\\ &= [\text{defining the indexes } j=i-1 \text{ and } \kappa=T-i]\\ &= \frac{1}{(1-\rho_{\mu}{}^{2})^{3}}[2T\rho_{\mu}{}^{2(T-1)}+\sum_{j=0}^{T-1}\rho_{\mu}{}^{2j}+\sum_{\kappa=0}^{T-1}\rho_{\mu}{}^{2\kappa}]\\ &= \frac{2}{(1-\rho_{\mu}{}^{2})^{3}}[T\rho_{\mu}{}^{2(T-1)}+\sum_{j=0}^{T-1}\rho_{\mu}{}^{2j}] = [\text{defining } r=\rho_{\mu}{}^{2}]\\ &= \frac{2}{(1-\rho_{\mu}{}^{2})^{3}}[T\rho_{\mu}{}^{2(T-1)}+\sum_{j=0}^{T-1}r^{j}] = \frac{2}{(1-\rho_{\mu}{}^{2})^{3}}\left[T\rho_{\mu}{}^{2(T-1)}+\frac{1-r^{T}}{1-r}\right]\\ &= \frac{2}{(1-\rho_{\mu}{}^{2})^{3}}\left[T\rho_{\mu}{}^{2(T-1)}+\frac{1-\rho_{\mu}{}^{2T}}{1-\rho_{\mu}{}^{2}}\right].\end{aligned} \tag{C.278}$$

By using equations (C.271) and (C.275) we find that the (i,j)-th element of the matrix $(\boldsymbol{\Delta R}_{\mu\mu})^3 = \boldsymbol{\Delta R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2$ is

$$\begin{aligned}
\delta_{ij}{}^{+} &= \sum_{\kappa=1}^{T} \delta_{i\kappa}{}^{*}\delta_{\kappa j}{}^{\circ} = \sum_{\kappa=1}^{T} \delta_{ii}\frac{1}{1-\rho_\mu{}^2}\rho_\mu{}^{|i-\kappa|}\delta_{\kappa\kappa}\frac{1}{(1-\rho_\mu{}^2)^2}(\rho_\mu{}^{\kappa+j-2}+\rho_\mu{}^{2T-\kappa-j}) \\
&= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)^3}[\delta_{11}\rho_\mu{}^{|i-1|}(\rho_\mu{}^{j-1}+\rho_\mu{}^{2T-1-j})+\delta_{TT}\rho_\mu{}^{|i-T|}(\rho_\mu{}^{T+j-2}+\rho_\mu{}^{T-j})] \\
&= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)^3}[\rho_\mu{}^{i+j-2}+\rho_\mu{}^{2T+i-j-2}+\rho_\mu{}^{2T-i+j-2}+\rho_\mu{}^{2T-i-j}], \qquad \text{(C.279)}
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathrm{tr}\,[(\boldsymbol{\Delta R}_{\mu\mu})^3] &= \sum_{i=1}^{T}\delta_{ii}{}^{+} = \sum_{i=1}^{T}\delta_{ii}\frac{1}{(1-\rho_\mu{}^2)^3}(\rho_\mu{}^{2(i-1)}+2\rho_\mu{}^{2(T-1)}+\rho_\mu{}^{2(T-i)}) \\
&= \delta_{11}\frac{1}{(1-\rho_\mu{}^2)^3}(\rho_\mu{}^{2(1-1)}+3\rho_\mu{}^{2(T-1)})+\delta_{TT}\frac{1}{(1-\rho_\mu{}^2)^3}(3\rho_\mu{}^{2(T-1)}+\rho_\mu{}^{2(T-T)}) \\
&= \frac{2}{(1-\rho_\mu{}^2)^3}(1+3\rho_\mu{}^{2(T-1)}). \qquad \text{(C.280)}
\end{aligned}$$

Let w_{ij} be the (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}{}^3$. Then, the (i,j)-th element of the matrix $\boldsymbol{\Delta R}_{\mu\mu}{}^3$ is

$$\delta_{ij}{}^{\ddagger} = \sum_{\kappa=1}^{T}\delta_{i\kappa}w_{\kappa j} = \delta_{ii}w_{ij}, \qquad \text{(C.281)}$$

because $\delta_{i\kappa} = 0 \ \forall\ \kappa \neq i$. Therefore,

$$\mathrm{tr}[\boldsymbol{\Delta R}_{\mu\mu}{}^3] = \sum_{i=1}^{T}\delta_{ii}{}^{\ddagger} = \sum_{i=1}^{T}\delta_{ii}w_{ii} = \delta_{11}w_{11}+\delta_{TT}w_{TT} = w_{11}+w_{TT}. \qquad \text{(C.282)}$$

Let w_{ll} be the l-diagonal element of matrix $\boldsymbol{R}_{\mu\mu}{}^3$, i.e.,

$$\begin{aligned}
w_{ll} &= \sum_{m=1}^{T}\sum_{\kappa=1}^{T}\frac{1}{(1-\rho_\mu{}^2)^3}\rho_\mu{}^{|l-\kappa|+|\kappa-m|+|m-l|} \\
&= \sum_{i=1-l}^{T-l}\sum_{j=1-l}^{T-l}\frac{1}{(1-\rho_\mu{}^2)^3}\rho_\mu{}^{|i|+|j|+|j-i|}, \qquad \text{(C.283)}
\end{aligned}$$

where $i = m - l$ and $j = \kappa - l$ with $i, j = 1 - l, \ldots, T - l$, and $j - i = \kappa - l - m + l = \kappa - m$.

Figure 1

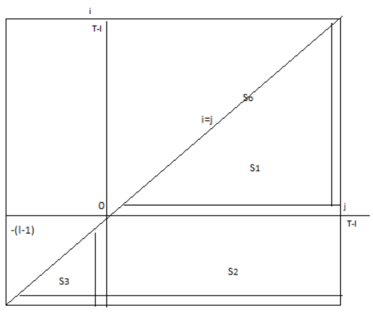

Figure 1 implies that

$$w_{ll} = 2(S_1 + S_2 + S_3) - S_0, \tag{C.284}$$

where

(i)

$$\begin{aligned} S_0 &= \sum_{i=1-l}^{T-l} \frac{1}{(1-\rho_\mu{}^2)^3} \rho_\mu{}^{2|i|} = \frac{1}{(1-\rho_\mu{}^2)^3} \sum_{i=1-l}^{T-l} r^{|i|} = [\text{by defining } r = \rho_\mu{}^2] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \left[\frac{1+r}{1-r} - \frac{1}{1-r}(r^l + r^{T-l+1}) \right]. \end{aligned} \tag{C.285}$$

(ii)

$$\begin{aligned} S_1 &= \sum_{i=0}^{T-l} \sum_{j=i}^{T-l} \frac{1}{(1-\rho_\mu{}^2)^3} \rho_\mu{}^{i+j+j-i} = \frac{1}{(1-\rho_\mu{}^2)^3} \sum_{i=0}^{T-l} \left[\sum_{j=0}^{T-l} \rho_\mu{}^{2j} - \sum_{j=0}^{i-1} \rho_\mu{}^{2j} \right] = [\text{by defining } r = \rho_\mu{}^2] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \sum_{i=0}^{T-l} \left[\sum_{j=0}^{T-l} r^j - \sum_{j=0}^{i-1} r^j \right] = \frac{1}{(1-\rho_\mu{}^2)^3} \sum_{i=0}^{T-l} \left[\frac{1-r^{T-l+1}}{1-r} - \frac{1-r^{i-1+1}}{1-r} \right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \sum_{i=0}^{T-l} \left[\frac{r^i - r^{T-l+1}}{1-r} \right] = \frac{1}{(1-\rho_\mu{}^2)^3} \frac{\sum_{i=0}^{T-l} r^i - \sum_{i=0}^{T-l} r^{T-l+1}}{1-r} \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \left[\frac{1-r^{T-l+1}}{(1-r)^2} - \frac{(T-l+1)r^{T-l+1}}{1-r} \right]. \end{aligned} \tag{C.286}$$

(iii)

$$\begin{aligned} S_2 &= \sum_{i=1-l}^{-1} \sum_{j=i}^{T-l} \frac{1}{(1-\rho_\mu{}^2)^3} \rho_\mu{}^{-i+j+j-i} = \frac{1}{(1-\rho_\mu{}^2)^3} \sum_{i=1-l}^{-1} \sum_{j=1}^{T-l} \rho_\mu{}^{-2i} \rho_\mu{}^{2j} \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} \sum_{i=1-l}^{-1} \rho_\mu{}^{-2i} \sum_{j=1}^{T-l} \rho_\mu{}^{2j} = [\text{by setting } k = -i \text{ with } k = 1, \ldots, l-1] \end{aligned}$$

$$
\begin{aligned}
S_2 &= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{k=1}^{l-1}\rho_\mu{}^{2k}\sum_{j=1}^{T-l}\rho_\mu{}^{2j} = \text{[by defining } r=\rho_\mu{}^2]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{k=1}^{l-1} r^k\sum_{j=1}^{T-l} r^j\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\cdot\frac{r(1-r^{T-l})}{1-r}\cdot\frac{r(1-r^{l-1})}{1-r}\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{r^2}{(1-r)^2}[1+r^{T-1}-r^{T-l}-r^{l-1}]. \qquad \text{(C.287)}
\end{aligned}
$$

(iv)

$$
\begin{aligned}
S_3 &= \sum_{i=1-l}^{0}\sum_{j=i}^{0}\frac{1}{(1-\rho_\mu{}^2)^3}[\rho_\mu{}^{-i+j-j-i}-1]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\sum_{i=1-l}^{0}\rho_\mu{}^{-2i}(i+1)-1\right] = \text{[by setting } k=-i \text{ with } k=0,\ldots,l-1]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\sum_{k=0}^{l-1}(1-k)\rho_\mu{}^{2k}-1\right] = \text{[by defining } r=\rho_\mu{}^2]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\sum_{k=0}^{l-1} r^k-\sum_{k=0}^{l-1} kr^k-1\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1-r^{l-1+1}}{1-r}-\frac{r[1-(l-1+1)r^{l-1}]+(l-1)r^{l-1+1}]}{(1-r)^2}-1\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{(1-r)(1-r^l)-r+lr^l-(l-1)r^{l+1}-(1-r)^2}{(1-r)^2}\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1-r-r^l+r^{l+1}-r+lr^l-lr^{l+1}+r^{l+1}+-1+2r-r^2}{(1-r)^2}\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{-r^2+(l-1)r^l-(l-2)r^{l+1}}{(1-r)^2}. \qquad \text{(C.288)}
\end{aligned}
$$

By combining equations (C.286), (C.287), and (C.288) we find that

$$
\begin{aligned}
S_* &= S_1+S_2+S_3\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1-r^{T-l+1}}{(1-r)^2}-\frac{(T-l+1)r^{T-l+1}}{1-r}+\frac{r^2}{(1-r)^2}[1+r^{T-1}-r^{T-l}-r^{l-1}]\right]\\
&\quad+\frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{-r^2+(l-1)r^l-(l-2)r^{l+1}}{(1-r)^2}\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1-r^{T-l+1}-(T-l+1)(1-r)r^{T-l+1}+r^2+r^{T-1+2}-r^{T-l+2}-r^{l-1+2}-r^2+(l-1)r^l-(l-2)r^{l+1}\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1-r^{T-l+1}-(T-l+1)r^{T-l+1}+(T-l+1)r^{T-l+2}+r^{T+1}-r^{T-l+2}-r^{l+1}+(l-1)r^l-(l-2)r^{l+1}\right]\\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1+r^{T+1}+(l-1)r^l-(T-l+2)r^{T-l+1}+(T-l)r^{T-l+2}-(l-1)r^{l+1}\right]. \qquad \text{(C.289)}
\end{aligned}
$$

Equation (C.289) implies that

$$\begin{aligned}\sum_{l=1}^{T} S_* &= \frac{1}{(1-\rho_\mu{}^2)^3}\sum_{l=1}^{T}(1-r)^{-2}\left[1+r^{T+1}+(l-1)r^l-(T-l+2)r^{T-l+1}+(T-l)r^{T-l+2}-(l-1)r^{l+1}\right]\\ &= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[T(1+r^{T+1})+s_1+s_2+s_3+s_4\right]\\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{T(1+r^{T+1})}{(1-r)^2}+\frac{s_1+s_2+s_3+s_4}{(1-r)^2}\right],\end{aligned} \tag{C.290}$$

where the quantities s_1, s_2, s_3, s_4 are computed as follows:

(I)

$$\begin{aligned}s_1 &= \sum_{l=1}^{T}(l-1)r^l=\sum_{l=1}^{T} lr^l-\sum_{l=1}^{T} r^l=\frac{r[1-(T+1)r^T+Tr^{T+1}]}{(1-r)^2}-\frac{r(1-r^T)}{1-r}\\ &= \frac{r-(T+1)r^{T+1}+Tr^{T+2}-r(1-r)(1-r^T)}{(1-r)^2}\\ &= \frac{r-(T+1)r^{T+1}+Tr^{T+2}-r(1-r-r^T+r^{T+1})}{(1-r)^2}\\ &= \frac{r-(T+1)r^{T+1}+Tr^{T+2}-r+r^2+r^{T+1}-r^{T+2}}{(1-r)^2}\\ &= \frac{r^2-(T)r^{T+1}+(T-1)r^{T+2}}{(1-r)^2}.\end{aligned} \tag{C.291}$$

(II) By setting $i=T-l$ with $i=0,\dots,T-1$ we find that

$$\begin{aligned}s_2 &= \sum_{l=1}^{T}-(T-l+2)r^{T-l+1}=-\sum_{i=0}^{T-1}(i+2)r^{i+1}=-r\sum_{i=0}^{T-1}(i+2)r^i\\ &= -r\left[\sum_{i=0}^{T-1} ir^i+2\sum_{i=0}^{T-1} r^i\right]=-r\left[\frac{r[1-Tr^{T-1}+(T-1)r^T]}{(1-r)^2}+\frac{2(1-r^T)}{1-r}\right]\\ &= -r\frac{r-(T)r^T+(T-1)r^{T+1}+2(1-r)(1-r^T)}{(1-r)^2}\\ &= \frac{-r^2+Tr^{T+1}-(T-1)r^{T+2}+2r(1-r-r^T+r^{T+1})}{(1-r)^2}\\ &= \frac{-r^2+Tr^{T+1}-(T-1)r^{T+2}-2r+2r^2+2r^{T+1}-2r^{T+2}}{(1-r)^2}\\ &= \frac{-2r+r^2+(T+2)r^{T+1}-(T+1)r^{T+2}}{(1-r)^2}.\end{aligned} \tag{C.292}$$

(III) Similarly, by using the index $i=T-l$ with $i=0,\dots,T-1$ we find that

$$\begin{aligned}s_3 &= \sum_{l=1}^{T}(T-l)r^{T-l+2}=\sum_{i=0}^{T-1} ir^{i+2}=r^2\sum_{i=0}^{T-1} ir^i\\ &= r^2\frac{r[1-Tr^{T-1}+(T-1)r^T]}{(1-r)^2}\\ &= \frac{r^3-Tr^{T+2}+(T-1)r^{T+3}}{(1-r)^2}.\end{aligned} \tag{C.293}$$

(IV) By setting $k = l - 1$ with $k = 0, \dots, T-1$ we find that

$$\begin{aligned} s_4 &= \sum_{l=1}^{T} -(l-1)r^{l+1} = -\sum_{l=1}^{T}(l-1)r^{(l-1)+2} = -\sum_{k=0}^{T-1} kr^{k+2} = -r^2 \sum_{k=0}^{T-1} kr^k \\ &= -r^2 \frac{r[1 - Tr^{T-1} + (T-1)r^T]}{(1-r)^2} \\ &= \frac{-r^3 + Tr^{T+2} - (T-1)r^{T+3}}{(1-r)^2}. \end{aligned} \quad (C.294)$$

Since equations (C.293) and (C.294) imply that

$$s_4 = -s_3, \quad (C.295)$$

by using equations (C.291) and (C.292) we find that

$$\begin{aligned} s_1 + s_2 + s_3 + s_4 &= s_1 + s_2 + s_3 - s_3 \\ &= (1-r)^{-2}[r^2 - Tr^{T+1} + (T-1)r^{T+2} - 2r + r^2 + (T+2)r^{T+1} - (T+1)r^{T+2}] \\ &= (1-r)^{-2}[2r^2 - 2r + 2r^{T+1} - 2r^{T+2}] \\ &= 2(1-r)^{-2}[r^2 - r + r^{T+1} - r^{T+2}]. \end{aligned} \quad (C.296)$$

By setting $i = T - l + 1$ with $i = 1, \dots, T$ and by using equation (C.285) we find that

$$\begin{aligned} \sum_{l=1}^{T} S_0 &= \sum_{l=1}^{T} \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1+r}{1-r} - \frac{1}{1-r}(r^l + r^{T-l+1})\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{T(1+r)}{1-r} - \frac{1}{1-r}\left(\sum_{l=1}^{T} r^l + \sum_{l=1}^{T} r^{T-l+1}\right)\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{T(1+r)}{1-r} - \frac{1}{1-r}\left(\sum_{l=1}^{T} r^l + \sum_{i=1}^{T} r^i\right)\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{T(1+r)}{1-r} - \frac{2}{1-r}\sum_{l=1}^{T} r^l\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{T(1+r)}{1-r} - \frac{2}{1-r}\frac{r(1-r^T)}{1-r}\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{T(1+r)}{1-r} - \frac{2r(1-r^T)}{(1-r)^2}\right]. \end{aligned} \quad (C.297)$$

By using equations (C.284), (C.285), and (C.289) we find that the l-diagonal element of the matrix $\boldsymbol{R}_{\mu\mu}{}^3$ is

$$\begin{aligned} w_{ll} &= 2(S_1 + S_2 + S_3) - S_0 = 2S_* - S_0 \\ &= \frac{1}{(1-\rho_\mu{}^2)^3} 2(1-r)^{-2}\left[1 + r^{T+1} + (l-1)r^l - (T-l+2)r^{T-l+1} + (T-l)r^{T-l+2} - (l-1)r^{l+1}\right] \\ &\quad - \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{1+r}{1-r} - \frac{1}{1-r}(r^l + r^{T-l+1})\right] \implies \end{aligned}$$

$$\begin{aligned}
w_{ll} &= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[2+2r^{T+1}+2(l-1)r^l-2(T-l+2)r^{T-l+1}+2(T-l)r^{T-l+2}-2(l-1)r^{l+1}\right] \\
&\quad -\frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}[(1+r)(1-r)+(1-r)(r^l+r^{T-l+1})] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[2+2r^{T+1}+2(l-1)r^l-2(T-l+2)r^{T-l+1}+2(T-l)r^{T-l+2}-2(l-1)r^{l+1}\right] \\
&\quad -\frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1-r^2-r^l-r^{T-l+1}+r^{l+1}+r^{T-l+2}\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1+2r^{T+1}+(2l-1)r^l-(2l-1)r^{l+1}+r^2\right] \\
&\quad -\frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[(2(T-l)+3)r^{T-l+1}-(2(T-l)-1)r^{T-l+2}\right]. \qquad \text{(C.298)}
\end{aligned}$$

By omitting terms that tend to zero as $T \to \infty$ and since $r = \rho_\mu{}^2$ with $|r| < 1$, we find that

$$\begin{aligned}
w_{11} &= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1+r-r^2+r^2\right]+o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1+r}{(1-r)^2}+o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^2}+o(T^{-1}) \\
&= \frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^5}+o(T^{-1}). \qquad \text{(C.299)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
w_{TT} &= \frac{1}{(1-\rho_\mu{}^2)^3}(1-r)^{-2}\left[1+r^2-3r-r^2\right]+o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1-3r}{(1-r)^2}+o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1-3\rho_\mu{}^2}{(1-\rho_\mu{}^2)^2}+o(T^{-1}) \\
&= \frac{1-3\rho_\mu{}^2}{(1-\rho_\mu{}^2)^5}+o(T^{-1}). \qquad \text{(C.300)}
\end{aligned}$$

Thus, equations (C.282), (C.299), and (C.300) imply that

$$\begin{aligned}
\mathrm{tr}[\boldsymbol{\Delta R}_{\mu\mu}{}^3] &= w_{11}+w_{TT} = \frac{1+\rho_\mu{}^2+1-3\rho_\mu{}^2}{(1-\rho_\mu{}^2)^5}+o(T^{-1}) \\
&= \frac{2-2\rho_\mu{}^2}{(1-\rho_\mu{}^2)^5}+o(T^{-1}) \\
&= \frac{2(1-\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^5}+o(T^{-1}) \\
&= \frac{2}{(1-\rho_\mu{}^2)^4}+o(T^{-1}). \qquad \text{(C.301)}
\end{aligned}$$

Moreover, by using equations (C.284), (C.289), (C.290), (C.296), and (C.297) we find that the trace of the matrix $\boldsymbol{R}_{\mu\mu}{}^3/T$ is

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^3)/T &= \frac{1}{T}\sum_{l=1}^{T} w_{ll} = \frac{1}{T}\sum_{l=1}^{T}[2(S_1+S_2+S_3)-S_0] = \frac{1}{T}[\sum_{l=1}^{T}2S_* - \sum_{l=1}^{T}S_0] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{2}{T}\left[\frac{T(1+r^{T+1})}{(1-r)^2} + \frac{2(1-r)^{-2}(r^2-r+r^{T+1}-r^{T+2})}{(1-r)^2}\right] \\
&\quad -\frac{1}{(1-\rho_\mu{}^2)^3}\frac{1}{T}\left[\frac{T(1+r)}{1-r} - \frac{2r(1-r^T)}{(1-r)^2}\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{2(1+r^{T+1})}{(1-r)^2} + \frac{4(r^2-r+r^{T+1}-r^{T+2})}{(1-r)^4} - \frac{1+r}{1-r} + \frac{2r(1-r^T)}{T(1-r)^2}\right]. \qquad \text{(C.302)}
\end{aligned}$$

By omitting terms that tend to zero as $T \to \infty$ and since $r = \rho_\mu{}^2$ with $|r| < 1$, we find that

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^3)/T &= \frac{1}{(1-\rho_\mu{}^2)^3}\left[\frac{2}{(1-r)^2} - \frac{1+r}{1-r} + o(T^{-1})\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{2-(1+r)(1-r)}{(1-r)^2} + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{2-1+r^2}{(1-r)^2} + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^3}\frac{1+\rho_\mu{}^4}{(1-\rho_\mu{}^2)^2} + o(T^{-1}) = \frac{1+\rho_\mu{}^4}{(1-\rho_\mu{}^2)^5} + o(T^{-1}). \qquad \text{(C.303)}
\end{aligned}$$

Finally, note that in all traces examined in this Lemma, there appear terms of the form $T^n r^T$ where n is a positive integer. Since $r = \rho_\mu{}^2$ with $0 \le r < 1$,

$$\lim_{T\to\infty} T^n r^T = \lim_{T\to\infty}\frac{T^n}{r^{-T}} = \frac{\infty}{\infty}. \qquad \text{(C.304)}$$

By applying L'Hospital rule we find that

$$\begin{aligned}
\lim_{T\to\infty} T^n r^T &= \lim_{T\to\infty}\frac{T^n}{r^{-T}} = \lim_{T\to\infty}\frac{\partial T^n/\partial T}{\partial r^{-T}/\partial T} = \frac{n}{-lnr}\lim_{T\to\infty}\frac{T^{n-1}}{r^{-T}} = \dots \\
&= \frac{n!}{(-lnr)^n}\lim_{T\to\infty}\frac{1}{r^{-T}} = \frac{n!}{(-lnr)^n}\lim_{T\to\infty} r^T = 0. \qquad \text{(C.305)}
\end{aligned}$$

Therefore, since all terms of the form $T^n r^T$ tend to zero as $T \to \infty$, all the traces computed in this Lemma are bounded as $T \to \infty$.

Furthermore, the first regularity condition implies that the matrices

$$\boldsymbol{X}_\mu' \boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T \text{ and } \boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T \qquad \text{(C.306)}$$

converge to non-singular matrices as $T \to \infty$.

Let x_{ij} and δ_{ij} be the (i,j)-th element of the matrices $\boldsymbol{X}_\mu$ and $\boldsymbol{\Delta}$ respectively. Then equation (C.306) implies that the element x_{ij} $(i = 1,\dots,T;\ j = 1,\dots,n)$ are bounded.

The following results hold:

(a) The (i,j)-th element of the matrix $\boldsymbol{X}_\mu' \boldsymbol{\Delta} \boldsymbol{X}_\mu$ is

$$\begin{aligned} \eta_{ij} &= \sum_{t=1}^{T}\sum_{s=1}^{T} x_{it}\delta_{ts}x_{sj} = \sum_{t=1}^{T} x_{it}\delta_{tt}x_{tj} = x_{i1}\delta_{11}x_{1j} + x_{iT}\delta_{TT}x_{Tj} \\ &= x_{i1}x_{1j} + x_{iT}x_{Tj}, \end{aligned} \tag{C.307}$$

which is bounded and consequently the matrix

$$\boldsymbol{X}_\mu' \boldsymbol{\Delta} \boldsymbol{X}_\mu / T = O(T^{-1}). \tag{C.308}$$

(b) By defining the indexes $k = s-1$ $(k = 1,\ldots,T-1)$ and $l = T-s$ $(l = 1,\ldots,T-1)$, the (i,j)-th element of the matrix $\boldsymbol{X}_\mu' \boldsymbol{\Delta} \boldsymbol{R}^{\mu\mu} \boldsymbol{X}_\mu$ is (see (C.271))

$$\begin{aligned} \eta_{ij}^{*} &= \sum_{t=1}^{T}\sum_{s=1}^{T} x_{it}\delta_{ts}^{*}x_{sj} = \sum_{s=1}^{T} x_{it}\delta_{tt}\frac{1}{1-\rho_\mu^{2}}\rho_\mu^{|t-s|}x_{sj} \\ &= \sum_{s=1}^{T}\left[x_{i1}\delta_{11}\frac{1}{1-\rho_\mu^{2}}\rho_\mu^{|1-s|}x_{sj} + x_{iT}\delta_{TT}\frac{1}{1-\rho_\mu^{2}}\rho_\mu^{|T-s|}x_{sj}\right] \\ &= \frac{1}{1-\rho_\mu^{2}}\left[x_{i1}\left(\sum_{s=1}^{T} x_{sj}\rho_\mu^{s-1}\right) + x_{iT}\left(\sum_{s=1}^{T} x_{sj}\rho_\mu^{T-s}\right)\right] = \\ &= \frac{1}{1-\rho_\mu^{2}}\left[x_{i1}\left(\sum_{k=0}^{T-1} x_{(k+1)j}\rho_\mu^{k}\right) + x_{iT}\left(\sum_{l=0}^{T-1} x_{(l+1)j}\rho_\mu^{l}\right)\right] \\ &= \frac{1}{1-\rho_\mu^{2}}(x_{i1}+x_{iT})\left(\sum_{l=1}^{T-1} x_{(l+1)j}\rho_\mu^{l}\right). \end{aligned} \tag{C.309}$$

Since $\boldsymbol{X}_\mu'$ is bounded, i.e., $\forall\, l$ $(l = 1,\ldots,T-1)$ it holds that

$$\begin{aligned} &|x_{(l+1)j}| \leq q < \infty \Rightarrow \\ &\Rightarrow \left|\sum_{l=0}^{T-1} x_{(l+1)j}\rho_\mu^{l}\right| \leq \sum_{l=0}^{T-1}\left|x_{(l+1)j}\right|\left|\rho_\mu^{l}\right| \leq q\sum_{l=0}^{T-1}\left|\rho_\mu^{l}\right| = q\frac{1-|\rho_\mu|^{T}}{1-|\rho_\mu|}, \end{aligned} \tag{C.310}$$

which implies that η_{ij}^{*} is bounded for every $(i,j = 1,\ldots,n)$ and so the matrix

$$\boldsymbol{X}_\mu' \boldsymbol{\Delta} \boldsymbol{R}^{\mu\mu} \boldsymbol{X}_\mu / T = O(T^{-1}). \tag{C.311}$$

Along the same lines we can prove that

$$\boldsymbol{X}_\mu' \boldsymbol{R}^{\mu\mu} \boldsymbol{\Delta} \boldsymbol{X}_\mu / T = O(T^{-1}). \tag{C.312}$$

(c) The (i,j)-th element of the matrix $\boldsymbol{\Delta} \boldsymbol{R}^{\mu\mu} \boldsymbol{\Delta}$ is (see (C.271))

$$\eta_{\tilde{i}\tilde{j}} = \sum_{k=1}^{T} \delta_{ik}^{*}\delta_{kj} = \delta_{ij}^{*}\delta_{jj}, \tag{C.313}$$

which implies that the (i,j)-th element of the matrix $\boldsymbol{X}'_\mu \boldsymbol{\Delta R}^{\mu\mu}\boldsymbol{\Delta X}_\mu$ is

$$\begin{aligned}
\eta_{ij}{}^\circ &= \sum_{t=1}^{T}\sum_{s=1}^{T} x_{it}\eta_{\tilde{t}\tilde{s}}x_{sj} = \sum_{t=1}^{T}\sum_{s=1}^{T} x_{it}\delta_{ts}{}^{*}\delta_{ss}x_{sj} = [\text{see (C.271)}] \\
&= \sum_{t=1}^{T}\sum_{s=1}^{T} x_{it}\delta_{tt}\frac{1}{1-\rho_\mu{}^2}\rho_\mu{}^{|t-s|}\delta_{ss}x_{sj} \\
&= \frac{1}{1-\rho_\mu{}^2}\sum_{t=1}^{T} x_{it}\delta_{tt}\sum_{s=1}^{T}\delta_{ss}\rho_\mu{}^{|t-s|}x_{sj} \\
&= \frac{1}{1-\rho_\mu{}^2}\sum_{t=1}^{T} x_{it}\delta_{tt}(\delta_{11}\rho_\mu{}^{|t-1|}x_{1j} + \delta_{TT}\rho_\mu{}^{|t-T|}x_{Tj}) \\
&= \frac{1}{1-\rho_\mu{}^2}\left[x_{i1}\delta_{11}(\rho_\mu{}^{1-1}x_{1j} + \rho_\mu{}^{T-1}x_{Tj}) + x_{iT}\delta_{TT}(\rho_\mu{}^{T-1}x_{1j} + \rho_\mu{}^{T-T}x_{Tj})\right] \\
&= \frac{1}{1-\rho_\mu{}^2}\left[x_{i1}(x_{1j} + \rho_\mu{}^{T-1}x_{1j}) + x_{iT}(\rho_\mu{}^{T-1}x_{1j} + x_{Tj})\right]. \qquad \text{(C.314)}
\end{aligned}$$

Thus, equation (C.314) implies that $\eta_{ij}{}^\circ$ is bounded so that

$$\boldsymbol{X}'_\mu\boldsymbol{\Delta R}^{\mu\mu}\boldsymbol{\Delta X}_\mu/T = O(T^{-1}). \qquad \text{(C.315)}$$

(d) The (i,j)-th element of the matrix $\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu$ is

$$\eta_{ij}{}^{+} = \sum_{t=1}^{T}\sum_{s=1}^{T} x_{it}\frac{1}{1-\rho_\mu{}^2}\rho_\mu{}^{|t-s|}x_{sj} = \frac{1}{1-\rho_\mu{}^2}\sum_{t=1}^{T}\sum_{s=1}^{T} x_{it}\rho_\mu{}^{|t-s|}x_{sj} \qquad \text{(C.316)}$$

and it is bounded given that x_{it} and x_{sj} are bounded for every $i,j = 1,\dots,n$ and every $t,s = 1,\dots,T$. Therefore,

$$\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T = O(T^{-1}). \qquad \text{(C.317)}$$

By using equations (C.263), (C.270), and (C.272) we find that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \operatorname{tr}[(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta})\boldsymbol{R}_{\mu\mu}] = \operatorname{tr}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\operatorname{tr}(\boldsymbol{\Delta R}_{\mu\mu}) \\
&= 0 + \frac{2\rho_\mu}{1-\rho_\mu{}^2} = \frac{2\rho_\mu}{1-\rho_\mu{}^2}. \qquad \text{(C.318)}
\end{aligned}$$

Similarly, by using equation (C.262) we find the following results:

(a)

$$\begin{aligned}
\rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} &= \rho_\mu\frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]\boldsymbol{\Delta R}_{\mu\mu} \\
&= \boldsymbol{\Delta R}_{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} \Rightarrow \qquad \text{(C.319)}
\end{aligned}$$

$$\operatorname{tr}(\rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}) = \operatorname{tr}(\boldsymbol{\Delta R}_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}) = (\text{see (C.272) and (C.274)})$$

$$\begin{aligned}\mathrm{tr}\,(\rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}) &= \frac{2}{1-\rho_\mu{}^2} - (1-\rho_\mu{}^2)\frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^3} \\ &= \frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^2}. \end{aligned} \tag{C.320}$$

(b)

$$\begin{aligned}\mathrm{tr}\,(\rho_\mu \boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \mathrm{tr}\,(\rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}) \\ &= \frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^2}. \end{aligned} \tag{C.321}$$

(c) By using equation (C.276) we find that

$$\begin{aligned}\mathrm{tr}\,(\rho_\mu{}^2 \boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}) &= \mathrm{tr}\,[\rho_\mu{}^2(\boldsymbol{\Delta R}_{\mu\mu})^2] = \rho_\mu{}^2\,\mathrm{tr}\,[(\boldsymbol{\Delta R}_{\mu\mu})^2] \\ &= \frac{2\rho_\mu{}^2}{(1-\rho_\mu{}^2)^2}(1+\rho_\mu{}^{2(T-1)}). \end{aligned} \tag{C.322}$$

(d) Moreover, by using equation (C.262) we find that

$$\begin{aligned}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]\frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] \\ &= \frac{1}{\rho_\mu{}^2}[\boldsymbol{I}_T - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2]. \end{aligned} \tag{C.323}$$

Defining $j = k - i$ with $j = 1-i, \dots, T-i$ and setting $j = T-i+1$ with $j = 1, \dots, T$, let v_{ll} be the l-diagonal element of matrix $\boldsymbol{R}_{\mu\mu}{}^2$, i.e.,

$$\begin{aligned} S(i) &= \sum_{k=1}^{T} \frac{1}{(1-\rho_\mu{}^2)^2}\rho_\mu{}^{|i-k|+|k-i|} = \\ &= \sum_{j=1-i}^{T-i} \frac{1}{(1-\rho_\mu{}^2)^2}\rho_\mu{}^{2|j|} = (\text{defining } r = \rho_\mu{}^2) \\ &= \frac{1}{(1-\rho_\mu{}^2)^2}\sum_{j=1-i}^{T-i} r^{|j|} = \frac{1}{(1-\rho_\mu{}^2)^2}\left[\sum_{j=1-i}^{-1} r^{|j|} + \sum_{j=0}^{T-i} r^j\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^2}\left[\sum_{j+i=1}^{i-1} r^{|j+i|} + \sum_{j=0}^{T-i} r^j\right] = \frac{1}{(1-\rho_\mu{}^2)^2}\left[\sum_{k=1}^{i-1} r^k + \sum_{j=0}^{T-i} r^j\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)^2}\left[\frac{r(1-r^{(i-1)})}{1-r} + \frac{1-r^{T-i+1}}{1-r}\right] = \frac{1}{(1-\rho_\mu{}^2)^2}\frac{r - r^i + 1 - r^{T-i+1}}{1-r} \\ &= \frac{1}{(1-\rho_\mu{}^2)^2}\left[\frac{1+r}{1-r} - \frac{r^i + r^{T-i+1}}{1-r}\right] = \\ &= \frac{1}{(1-\rho_\mu{}^2)^2}\left[\frac{1+r}{1-r} - \frac{r^i + r^j}{1-r}\right] = \frac{1}{(1-\rho_\mu{}^2)^2}\left[\frac{1+r}{1-r} - \frac{2r^i}{1-r}\right]. \end{aligned} \tag{C.324}$$

Therefore,

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^2)/T &= \sum_{i=1}^{T} S(i)/T = \sum_{i=1}^{T}\left[\frac{1}{(1-\rho_\mu{}^2)^2}\left[\frac{1+r}{1-r} - \frac{2r^i}{1-r}\right]\right]/T \\
&= \frac{1}{(1-\rho_\mu{}^2)^2}\left[\frac{T(1+r)}{1-r} - \frac{2}{1-r}\sum_{i=1}^{T} r^i\right]/T \\
&= \frac{1}{(1-\rho_\mu{}^2)^2}\left[\frac{1+r}{1-r} - \frac{2}{T(1-r)}\frac{r(1-r^T)}{(1-r)}\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)^2}\left[\frac{1+r}{1-r} - \frac{2r(1-r^T)}{T(1-r)^2}\right] \qquad \text{(C.325)}
\end{aligned}$$

and omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^2)/T &= \frac{1}{(1-\rho_\mu{}^2)^2}\frac{1+r}{1-r} + o(T^{-1}) \\
&= \frac{1}{(1-\rho_\mu{}^2)^2}\frac{1+\rho_\mu{}^2}{1-\rho_\mu{}^2} + o(T^{-1}) \\
&= \frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^3} + o(T^{-1}). \qquad \text{(C.326)}
\end{aligned}$$

By combining equations (C.269), (C.323), and (C.326) we find that

$$\begin{aligned}
\mathrm{tr}[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2] &= \frac{1}{\rho_\mu{}^2}[\mathrm{tr}\,(\boldsymbol{I}_T) - 2\,\mathrm{tr}\,[(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] + (1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^2)] \\
&= \frac{1}{\rho_\mu{}^2}\left[T - 2T + (1-\rho_\mu{}^2)^2 T\frac{1+\rho_\mu{}^2}{(1-\rho_\mu{}^2)^3} + o(1)\right] \\
&= \frac{1}{\rho_\mu{}^2}\left[-T + T\frac{1+\rho_\mu{}^2}{1-\rho_\mu{}^2} + o(1)\right] = \frac{T}{\rho_\mu{}^2}\left[\frac{-1+\rho_\mu{}^2+1+\rho_\mu{}^2}{1-\rho_\mu{}^2}\right] + o(1) \\
&= \frac{2T\rho_\mu{}^2}{\rho_\mu{}^2(1-\rho_\mu{}^2)} + o(1) = \frac{2T}{1-\rho_\mu{}^2} + o(1). \qquad \text{(C.327)}
\end{aligned}$$

By combining equations (C.264), (C.320), (C.321), (C.322), and (C.327) we find that

$$\begin{aligned}
\mathrm{tr}[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \mathrm{tr}\,[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T + \mathrm{tr}\,(\rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T \\
&\quad + \mathrm{tr}\,(\rho_\mu \boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T + \mathrm{tr}\,(\rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T \\
&= \frac{2}{1-\rho_\mu{}^2} + o(T^{-1}) + \frac{2}{T}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^2}\right] \\
&\quad + \frac{2\rho_\mu{}^2}{T(1-\rho_\mu{}^2)^2}(1+\rho_\mu{}^{2(T-1)}) \qquad \text{(C.328)}
\end{aligned}$$

and omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned} \mathrm{tr}[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \frac{2}{1-\rho_\mu{}^2} + o(T^{-1}) \Rightarrow \\ \mathrm{tr}[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2] &= \frac{2T}{1-\rho_\mu{}^2} + o(1). \end{aligned} \tag{C.329}$$

(e) By using equations (C.263) and (C.264) we take

$$\begin{aligned} (\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3 &= (\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) \\ &= \left[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 + \rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu \boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\right] \cdot \\ &\quad \left[\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}\right] \\ &= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3 + \rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\ &\quad +\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\ &\quad +\rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\ &\quad +\rho_\mu{}^3\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} \\ &= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3 + \rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu \boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\ &\quad +\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 \\ &\quad +\rho_\mu{}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu{}^2(\boldsymbol{\Delta R}_{\mu\mu})^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\ &\quad +\rho_\mu{}^3(\boldsymbol{\Delta R}_{\mu\mu})^3, \end{aligned} \tag{C.330}$$

which implies that

$$\begin{aligned} \mathrm{tr}\,[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3] &= \mathrm{tr}\,[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3] + 3\,\mathrm{tr}\,[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}] \\ &\quad +3\,\mathrm{tr}\,[\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2] + \mathrm{tr}\,[\rho_\mu{}^3(\boldsymbol{\Delta R}_{\mu\mu})^3]. \end{aligned} \tag{C.331}$$

Since, equation (C.323) implies that

$$\begin{aligned} (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu} &= \frac{1}{\rho_\mu{}^2}[\boldsymbol{I}_T - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2]\boldsymbol{\Delta R}_{\mu\mu} \\ &= \frac{1}{\rho_\mu{}^2}[\boldsymbol{\Delta R}_{\mu\mu} - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2\boldsymbol{\Delta R}_{\mu\mu}], \end{aligned} \tag{C.332}$$

it follows that

$$\begin{aligned}
\operatorname{tr}[\rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}] &= \frac{\rho_\mu}{\rho_\mu{}^2}[\operatorname{tr}\boldsymbol{\Delta R}_{\mu\mu} - 2(1-\rho_\mu{}^2)\operatorname{tr}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\operatorname{tr}\boldsymbol{R}_{\mu\mu}{}^2\boldsymbol{\Delta R}_{\mu\mu}] \\
&= [\text{see (C.272), (C.274), and (C.301)}] \\
&= \frac{1}{\rho_\mu}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^2)(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^3} + \frac{2(1-\rho_\mu{}^2)^2}{(1-\rho_\mu{}^2)^4} + O(T^{-1})\right] \\
&= \frac{1}{\rho_\mu}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2(1-\rho_\mu{}^{2T})}{(1-\rho_\mu{}^2)^2} + \frac{2}{(1-\rho_\mu{}^2)^2} + O(T^{-1})\right] \\
&= \frac{1}{\rho_\mu}\left[\frac{2-2\rho_\mu{}^2-2+2\rho_\mu{}^{2T}+2}{(1-\rho_\mu{}^2)^2}\right] + O(T^{-1}) \\
&= \frac{2(1-\rho_\mu{}^2+\rho_\mu{}^{2T})}{\rho_\mu(1-\rho_\mu{}^2)^2} + O(T^{-1}) \Rightarrow \qquad \text{(C.333)}
\end{aligned}$$

$$\operatorname{tr}[\rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}]/T = \frac{2(1-\rho_\mu{}^2+\rho_\mu{}^{2T})}{T\rho_\mu(1-\rho_\mu{}^2)^2} + O(1). \qquad \text{(C.334)}$$

Moreover, since equation (C.262) implies that

$$\begin{aligned}
\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2 &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}](\boldsymbol{\Delta R}_{\mu\mu})^2 \\
&= \frac{1}{\rho_\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2 - \frac{(1-\rho_\mu{}^2)}{\rho_\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2, \qquad \text{(C.335)}
\end{aligned}$$

it follows that

$$\begin{aligned}
\operatorname{tr}[\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2] &= \rho_\mu\operatorname{tr}[(\boldsymbol{\Delta R}_{\mu\mu})^2] - \rho_\mu(1-\rho_\mu{}^2)\operatorname{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2] \\
&= [\text{see (C.276) and (C.278)}] \\
&= \frac{2\rho_\mu}{(1-\rho_\mu{}^2)^2}(1+\rho_\mu{}^{2(T-1)}) - \frac{2\rho_\mu(1-\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^3}\left[T\rho_\mu{}^{2(T-1)} + \frac{1-\rho_\mu{}^{2T}}{1-\rho_\mu{}^2}\right] \\
&= \frac{2\rho_\mu}{(1-\rho_\mu{}^2)^2}\left[1+\rho_\mu{}^{2(T-1)} - T\rho_\mu{}^{2(T-1)} - \frac{(1-\rho_\mu{}^{2T})}{1-\rho_\mu{}^2}\right] \Rightarrow \qquad \text{(C.336)}
\end{aligned}$$

$$\operatorname{tr}[\rho_\mu{}^2\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2]/T = \frac{2\rho_\mu}{(1-\rho_\mu{}^2)^2}\left[\frac{1}{T}\left[1+\rho_\mu{}^{2(T-1)} - \frac{(1-\rho_\mu{}^{2T})}{1-\rho_\mu{}^2}\right] - \rho_\mu{}^{2(T-1)}\right]. \qquad \text{(C.337)}$$

By using equations (C.262) and (C.323) we find that

$$\begin{aligned}
(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3 &= (\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 \\
&= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]\frac{1}{\rho_\mu{}^2}[\boldsymbol{I}_T - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2] \\
&= \frac{1}{\rho_\mu{}^3}[\boldsymbol{I}_T - 2(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + (1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2 - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} \\
&\quad +2(1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2 - (1-\rho_\mu{}^2)^3\boldsymbol{R}_{\mu\mu}{}^3] \\
&= \frac{1}{\rho_\mu{}^3}[\boldsymbol{I}_T - 3(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + 3(1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2 - (1-\rho_\mu{}^2)^3\boldsymbol{R}_{\mu\mu}{}^3] \qquad \text{(C.338)}
\end{aligned}$$

and by using equations (C.268), (C.303), and (C.326) we find that

$$\begin{aligned}
\mathrm{tr}\,[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3] &= \frac{1}{\rho_\mu{}^3}\,\mathrm{tr}[\boldsymbol{I}_T - 3(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} + 3(1-\rho_\mu{}^2)^2\boldsymbol{R}_{\mu\mu}{}^2 - (1-\rho_\mu{}^2)^3\boldsymbol{R}_{\mu\mu}{}^3] \\
&= \frac{1}{\rho_\mu{}^3}[\mathrm{tr}\,\boldsymbol{I}_T - 3(1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}) + 3(1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^2) - (1-\rho_\mu{}^2)^3\,\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}{}^3)] \\
&= \frac{1}{\rho_\mu{}^3}[T - 3(1-\rho_\mu{}^2)\frac{T}{1-\rho_\mu{}^2} + 3(1-\rho_\mu{}^2)^2\frac{(1+\rho_\mu{}^2)T}{(1-\rho_\mu{}^2)^3} - (1-\rho_\mu{}^2)^3\frac{(1+\rho_\mu{}^4)T}{(1-\rho_\mu{}^2)^5} + o(1)] \\
&= \frac{1}{\rho_\mu{}^3}\left[T - 3T + 3T\frac{(1+\rho_\mu{}^2)}{(1-\rho_\mu{}^2)} - T\frac{(1+\rho_\mu{}^4)}{(1-\rho_\mu{}^2)^2}\right] + o(1) \\
&= \frac{T}{\rho_\mu{}^3}\left[\frac{-2(1-\rho_\mu{}^2)^2 + 3(1-\rho_\mu{}^4) - 1 - \rho_\mu{}^4}{(1-\rho_\mu{}^2)^2}\right] + o(1) \\
&= \frac{T}{\rho_\mu{}^3}\left[\frac{-2 + 4\rho_\mu{}^2 - 2\rho_\mu{}^4 + 3 - 3\rho_\mu{}^4 - 1 - \rho_\mu{}^4}{(1-\rho_\mu{}^2)^2}\right] + o(1) \\
&= \frac{T}{\rho_\mu{}^3}\left[\frac{4\rho_\mu{}^2 - 6\rho_\mu{}^4}{(1-\rho_\mu{}^2)^2}\right] + o(1) = \frac{2T}{\rho_\mu}\frac{(2-3\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2} + o(1).
\end{aligned} \tag{C.339}$$

By combining equations (C.278), (C.331), (C.334), (C.337), and (C.339) we find that

$$\begin{aligned}
tr[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3]/T &= \mathrm{tr}\,[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3]/T + 3\,\mathrm{tr}[\rho_\mu(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}]/T \\
&\quad +3\,\mathrm{tr}[\rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2]/T + \mathrm{tr}\,[\rho_\mu{}^3(\boldsymbol{\Delta R}_{\mu\mu})^3]/T \\
&= \frac{2}{\rho_\mu}\frac{(2-3\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2} + o(T^{-1}) \\
&\quad +3\left[\frac{2(\rho_\mu{}^{2T} - \rho_\mu{}^2 + 1)}{T\rho_\mu(1-\rho_\mu{}^2)}\right] + o(1) \\
&\quad +3\frac{2\rho_\mu}{(1-\rho_\mu{}^2)^2}\left[\frac{1}{T}\left[1 + \rho_\mu{}^{2(T-1)} - \frac{(1-\rho_\mu{}^{2T})}{1-\rho_\mu{}^2}\right] - \rho_\mu{}^{2(T-1)}\right] \\
&\quad +\frac{2\rho_\mu{}^3}{T(1-\rho_\mu{}^2)^3}(1 + 3\rho_\mu{}^{2(T-1)})
\end{aligned} \tag{C.340}$$

and omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned}
tr[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3]/T &= \frac{2}{\rho_\mu}\frac{(2-3\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2} + o(T^{-1}) \Rightarrow \\
tr[(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^3] &= \frac{2T}{\rho_\mu}\frac{(2-3\rho_\mu{}^2)}{(1-\rho_\mu{}^2)^2} + o(1).
\end{aligned} \tag{C.341}$$

(f) Equation (C.261) implies that

$$\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu} = \boldsymbol{P}_{\mathbf{X}_\mu}\frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{I}_T] = \frac{1}{\rho_\mu}[\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{P}_{\mathbf{X}_\mu}] \tag{C.342}$$

and since $\boldsymbol{P}_{\mathbf{X}_\mu}$ is orthogonal projector into the spaces spanned by the columns of the matrix $\boldsymbol{X}_\mu$, we have that

$$\begin{aligned} \boldsymbol{P}_{\boldsymbol{X}_\mu} &= \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu \Rightarrow \\ \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}) &= \mathrm{tr}\,[(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu \boldsymbol{X}_\mu] = \mathrm{tr}\,\boldsymbol{I}_n = n, \end{aligned} \tag{C.343}$$

from which we find that

$$\begin{aligned} \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}) &= \frac{1}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}^{\mu\mu}) - (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu})] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu}) - (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu})] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,[(\boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T)(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu/T)^{-1}] - (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu})] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,[(\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1})] - (1-\rho_\mu{}^2)n], \end{aligned} \tag{C.344}$$

where

$$\boldsymbol{B}_{\mu\mu} = \boldsymbol{X}'_\mu \boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T \text{ and } \boldsymbol{F}_{\mu\mu} = \boldsymbol{X}'_\mu \boldsymbol{X}_\mu/T. \tag{C.345}$$

Then, equation (C.263) implies that

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu \boldsymbol{\Delta}) = \boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu \boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}, \tag{C.346}$$

which implies that

$$\begin{aligned} \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}) &= \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}) + \rho_\mu\,\mathrm{tr}\,[\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu \boldsymbol{\Delta}] \\ &= \mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}) + \rho_\mu\,\mathrm{tr}\,[(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}'_\mu \boldsymbol{\Delta}\boldsymbol{X}_\mu/T)] = [\textit{see}\ (\text{C.308})] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,[(\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1})] - (1-\rho_\mu{}^2)n] + O(T^{-1}). \end{aligned} \tag{C.347}$$

Moreover, equation (C.262) implies that

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}\frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] = \frac{1}{\rho_\mu}[\boldsymbol{P}_{\boldsymbol{X}_\mu} - (1-\rho_\mu{}^2)\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}] \Rightarrow \tag{C.348}$$

$$\begin{aligned} \mathrm{tr}\,[\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}] &= \frac{1}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}) - (1-\rho_\mu{}^2)\,\mathrm{tr}(\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu)^{-1}\boldsymbol{X}'_\mu \boldsymbol{R}_{\mu\mu})] \\ &= \frac{1}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{P}_{\boldsymbol{X}_\mu}) - (1-\rho_\mu{}^2)\,\mathrm{tr}\,[(\boldsymbol{X}'_\mu \boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}'_\mu \boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)]] \\ &= \frac{1}{\rho_\mu}[n - (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})], \end{aligned} \tag{C.349}$$

where

$$\boldsymbol{\Theta}_{\mu\mu} = \boldsymbol{X}'_\mu \boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T. \tag{C.350}$$

Thus,

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} = [\text{see (C.263)}] = \boldsymbol{P}_{\boldsymbol{X}_\mu}[\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta}]\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}, \tag{C.351}$$

which implies that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu \operatorname{tr}[\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}] \\
&= \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu \operatorname{tr}[(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)] \\
&= [\text{see (C.311)}] = \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + O(T^{-1}) \\
&= \frac{1}{\rho_\mu}[n - (1-\rho_\mu{}^2)\operatorname{tr}[(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]] + O(T^{-1}).
\end{aligned} \tag{C.352}$$

Furthermore, equation (C.261) implies that since $\boldsymbol{P}_{\boldsymbol{X}_\mu}$ is idempotent, we find

$$\begin{aligned}
\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} &= \boldsymbol{P}_{\boldsymbol{X}_\mu}\frac{1}{\rho_\mu}[\boldsymbol{R}^{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{I}_T]\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} \\
&= \frac{1}{\rho_\mu}[\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} - (1-\rho_\mu{}^2)\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}],
\end{aligned} \tag{C.353}$$

which implies that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) &= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu})] \\
&= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu})] \\
&\quad -\frac{1}{\rho_\mu}[(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu})] \\
&= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu/T)(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)] \\
&\quad -\frac{1}{\rho_\mu}[(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)] \\
&= \frac{1}{\rho_\mu}[\operatorname{tr}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu} - (1-\rho_\mu{}^2)\operatorname{tr}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}].
\end{aligned} \tag{C.354}$$

Moreover, by using equation (C.263) we find that

$$\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta})\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} = \boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu} \Rightarrow \tag{C.355}$$

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) &= \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{\Delta}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) \\
&= \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\operatorname{tr}[\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{X}_\mu(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu)^{-1}\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}] \\
&= \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) + \rho_\mu\operatorname{tr}[(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{\Delta}\boldsymbol{X}_\mu/T)(\boldsymbol{X}_\mu'\boldsymbol{X}_\mu/T)^{-1}(\boldsymbol{X}_\mu'\boldsymbol{R}_{\mu\mu}\boldsymbol{X}_\mu/T)] \\
&= [\text{see (C.308)}] = \operatorname{tr}(\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{P}_{\boldsymbol{X}_\mu}\boldsymbol{R}_{\mu\mu}) + O(T^{-1}) \\
&= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1}).
\end{aligned} \tag{C.356}$$

(g) By using equation (C.262) we find that

$$\begin{aligned}
\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} &= \boldsymbol{R}_{\mu\mu}\frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-{\rho_\mu}^2)\boldsymbol{R}_{\mu\mu}] = \frac{1}{\rho_\mu}[\boldsymbol{R}_{\mu\mu} - (1-{\rho_\mu}^2){\boldsymbol{R}_{\mu\mu}}^2] \Rightarrow \\
\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{R}_{\mu\mu}) - (1-{\rho_\mu}^2)\operatorname{tr}({\boldsymbol{R}_{\mu\mu}}^2)] = [\text{see (C.268) and (C.325)}] \\
&= \frac{1}{\rho_\mu}\left[\frac{T}{1-{\rho_\mu}^2} - (1-{\rho_\mu}^2)T\frac{1+{\rho_\mu}^2}{(1-{\rho_\mu}^2)^3}\right] + o(1) \\
&= \frac{T}{\rho_\mu}\left[\frac{(1-{\rho_\mu}^2)-(1+{\rho_\mu}^2)}{(1-{\rho_\mu}^2)^2}\right] + o(1) \\
&= \frac{T}{\rho_\mu}\left[\frac{-2{\rho_\mu}^2}{(1-{\rho_\mu}^2)^2}\right] + o(1) \\
&= \left[\frac{-2T\rho_\mu}{(1-{\rho_\mu}^2)^2}\right] + o(1).
\end{aligned} \tag{C.357}$$

Then, equation (C.263) implies that

$$\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} = \boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu} + \rho_\mu\boldsymbol{\Delta})\boldsymbol{R}_{\mu\mu} = \boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + \rho_\mu\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu} \Rightarrow \tag{C.358}$$

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T &= \operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T + \rho_\mu\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta}\boldsymbol{R}_{\mu\mu})/T \\
&= [\text{see (C.274)}] = \frac{-2\rho_\mu}{(1-{\rho_\mu}^2)^2} + \frac{2(1-{\rho_\mu}^{2T})}{(1-{\rho_\mu}^2)^3 T} + o(T^{-1})
\end{aligned} \tag{C.359}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T &= \frac{-2\rho_\mu}{(1-{\rho_\mu}^2)^2} + o(T^{-1}) \Rightarrow \\
\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) &= \frac{-2T\rho_\mu}{(1-{\rho_\mu}^2)^2} + o(1)
\end{aligned} \tag{C.360}$$

By using equation (C.323) we find that

$$\begin{aligned}
\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 &= \boldsymbol{R}_{\mu\mu}\frac{1}{{\rho_\mu}^2}[\boldsymbol{I}_T - 2(1-{\rho_\mu}^2)\boldsymbol{R}_{\mu\mu} + (1-{\rho_\mu}^2)^2{\boldsymbol{R}_{\mu\mu}}^2] \\
&= \frac{1}{{\rho_\mu}^2}[\boldsymbol{R}_{\mu\mu} - 2(1-{\rho_\mu}^2){\boldsymbol{R}_{\mu\mu}}^2 + (1-{\rho_\mu}^2)^2{\boldsymbol{R}_{\mu\mu}}^3] \Rightarrow
\end{aligned} \tag{C.361}$$

$$\begin{aligned}
\mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2] &= \frac{1}{{\rho_\mu}^2}[\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}) - 2(1-{\rho_\mu}^2)\,\mathrm{tr}\,({\boldsymbol{R}_{\mu\mu}}^2) + (1-{\rho_\mu}^2)^2\,\mathrm{tr}\,({\boldsymbol{R}_{\mu\mu}}^3)] \\
&= [\text{see (C.268), (C.303) and (C.326)}] \\
&= \frac{1}{{\rho_\mu}^2}\left[\frac{T}{1-{\rho_\mu}^2} - 2(1-{\rho_\mu}^2)\frac{(1+{\rho_\mu}^2)T}{(1-{\rho_\mu}^2)^3} + (1-{\rho_\mu}^2)^2\frac{(1+{\rho_\mu}^4)T}{(1-{\rho_\mu}^2)^5} + o(1)\right] \\
&= \frac{1}{{\rho_\mu}^2(1-{\rho_\mu}^2)}\left[T - 2\frac{(1+{\rho_\mu}^2)T}{(1-{\rho_\mu}^2)} + \frac{(1+{\rho_\mu}^4)T}{(1-{\rho_\mu}^2)^2} + o(1)\right] \\
&= \frac{1}{{\rho_\mu}^2(1-{\rho_\mu}^2)^3}[T(1-{\rho_\mu}^2)^2 - 2T + 2{\rho_\mu}^4T + T + {\rho_\mu}^4T] + o(1) \\
&= \frac{1}{{\rho_\mu}^2(1-{\rho_\mu}^2)^3}[T - 2{\rho_\mu}^2T + {\rho_\mu}^4T - 2T + 2{\rho_\mu}^4T + T + {\rho_\mu}^4T] + o(1) \\
&= \frac{T}{{\rho_\mu}^2(1-{\rho_\mu}^2)^3}[4{\rho_\mu}^4 - 2{\rho_\mu}^2] + o(1) \\
&= \frac{2T{\rho_\mu}^2(2{\rho_\mu}^2-1)}{{\rho_\mu}^2(1-{\rho_\mu}^2)^3} + o(1) \\
&= \frac{2T(2{\rho_\mu}^2-1)}{(1-{\rho_\mu}^2)^3} + o(1). \qquad \text{(C.362)}
\end{aligned}$$

Then, equation (C.264) implies that

$$\begin{aligned}
\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 &= \boldsymbol{R}_{\mu\mu}[(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} + {\rho_\mu}^2\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}] \\
&= \boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2 + \rho_\mu\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} + \rho_\mu\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu} \\
&\quad + {\rho_\mu}^2\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2 \Rightarrow \qquad \text{(C.363)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \mathrm{tr}\,[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T + \rho_\mu\,\mathrm{tr}[\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}]/T \\
&\quad + \rho_\mu\,\mathrm{tr}[\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}]/T + {\rho_\mu}^2\,\mathrm{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2]/T. \qquad \text{(C.364)}
\end{aligned}$$

But, equation (C.262) implies the following results:

$$\begin{aligned}
\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})\boldsymbol{\Delta R}_{\mu\mu} &= \boldsymbol{R}_{\mu\mu}\frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-{\rho_\mu}^2)\boldsymbol{R}_{\mu\mu}]\boldsymbol{\Delta R}_{\mu\mu} \\
&= \frac{1}{\rho_\mu}[\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu} - (1-{\rho_\mu}^2)(\boldsymbol{R}_{\mu\mu})^2\boldsymbol{\Delta R}_{\mu\mu}] \Rightarrow \qquad \text{(C.365)}
\end{aligned}$$

$$\begin{aligned}
\rho_\mu\,\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T &= \rho_\mu\frac{1}{\rho_\mu}[\mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}) - (1-{\rho_\mu}^2)\,\mathrm{tr}({\boldsymbol{R}_{\mu\mu}}^2\boldsymbol{\Delta R}_{\mu\mu})]/T \\
&= \mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T - (1-{\rho_\mu}^2)\,\mathrm{tr}({\boldsymbol{R}_{\mu\mu}}^2\boldsymbol{\Delta R}_{\mu\mu})/T \\
&= \mathrm{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T - (1-{\rho_\mu}^2)\,\mathrm{tr}({\boldsymbol{\Delta R}_{\mu\mu}}^3)/T. \qquad \text{(C.366)}
\end{aligned}$$

Moreover,

$$\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) = \boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\frac{1}{\rho_\mu}[\boldsymbol{I}_T-(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}] = \frac{1}{\rho_\mu}[\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}-(1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}{}^2] \Rightarrow \quad \text{(C.367)}$$

$$\begin{aligned}\rho_\mu \operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T &= \rho_\mu\frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})-(1-\rho_\mu^2)\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu}{}^2)]/T\\ &= \operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T-(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{\Delta R}_{\mu\mu}{}^3)/T. \qquad \text{(C.368)}\end{aligned}$$

Thus, equations (C.362), (C.366), and (C.368) imply that

$$\begin{aligned}\operatorname{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \operatorname{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T+2\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu\mu})/T\\ &\quad -2(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{\Delta R}_{\mu\mu}{}^3)/T+\rho_\mu{}^2\operatorname{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{\Delta R}_{\mu\mu})^2]/T\\ &= [\text{see (C.274), (C.320), (C.343) and (C.362)}]\\ &= \frac{2(2\rho_\mu{}^2-1)}{(1-\rho_\mu{}^2)^3}+o(T^{-1})+2\frac{2(1-\rho_\mu{}^{2T})}{T(1-\rho_\mu{}^2)^3}\\ &\quad -2(1-\rho_\mu{}^2)\frac{2}{T(1-\rho_\mu{}^2)^4}+o(T^{-2})\\ &\quad +\rho_\mu{}^2\frac{2}{(1-\rho_\mu{}^2)^3T}\left[T\rho_\mu{}^{2(T-1)}+\frac{1-\rho_\mu{}^{2T}}{1-\rho_\mu{}^2}\right] \qquad \text{(C.369)}\end{aligned}$$

and by omitting terms that tend to zero as $T\to\infty$ we find that

$$\begin{aligned}\operatorname{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]/T &= \frac{2(2\rho_\mu{}^2-1)}{(1-\rho_\mu{}^2)^3}+o(T^{-1}) \Rightarrow\\ \operatorname{tr}[\boldsymbol{R}_{\mu\mu}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2] &= \frac{2(2\rho_\mu{}^2-1)T}{(1-\rho_\mu{}^2)^3}+o(1). \qquad \text{(C.370)}\end{aligned}$$

(h)

$$\begin{aligned}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu} &= (\boldsymbol{I}-\boldsymbol{P}_{X_\mu})\boldsymbol{R}_2{}^{\mu\mu}(\boldsymbol{I}-\boldsymbol{P}_{X_\mu})\boldsymbol{R}_{\mu\mu}\\ &= \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}-\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}-\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}+\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}, \qquad \text{(C.371)}\end{aligned}$$

where $\bar{\boldsymbol{P}}_{X_\mu}=\boldsymbol{I}-\boldsymbol{P}_{X_\mu}$. Since $\boldsymbol{R}_2{}^{\mu\mu},\boldsymbol{R}_{\mu\mu},\boldsymbol{P}_{X_\mu}$ are symmetric matrices the following results holds:

$$\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu})=\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu})=\operatorname{tr}[(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu})']=\operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}), \qquad \text{(C.372)}$$

which implies that

$$\begin{aligned}
\operatorname{tr}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_{\mu\mu} &= \operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) - 2\operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu})\\
&\quad + \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) = [\text{see (C.318), (C.352) and (C.356)}]\\
&= \frac{2\rho_\mu}{1-\rho_\mu{}^2} - \frac{2}{\rho_\mu}[n-(1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1})\\
&\quad + \frac{1}{\rho_\mu}[\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1})\\
&= \frac{2\rho_\mu}{1-\rho_\mu{}^2} - \frac{2n}{\rho_\mu} + \frac{(1-\rho_\mu{}^2)}{\rho_\mu}\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \frac{1}{\rho_\mu}\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + O(T^{-1})\\
&= \frac{1}{\rho_\mu}\left[\frac{2(\rho_\mu{}^2 - n(1-\rho_\mu{}^2))}{1-\rho_\mu{}^2} + (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\right]\\
&\quad + O(T^{-1}).
\end{aligned}\tag{C.373}$$

By using equation (C.249) the following results hold:

(1)

$$\begin{aligned}
\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{V} &= [\boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu]\boldsymbol{R}^{\mu\mu}[\boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu]\\
&= \boldsymbol{R}_{\mu\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{R}_{\mu\mu} - \boldsymbol{R}_{\mu\mu}\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu\\
&\quad - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu\boldsymbol{R}_{\mu\mu}\boldsymbol{R}^{\mu\mu} + \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu\\
&= \boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu = \boldsymbol{V}.
\end{aligned}\tag{C.374}$$

(2)

$$\begin{aligned}
\boldsymbol{V}\bar{\boldsymbol{P}}_{X_\mu} &= [\boldsymbol{R}_{\mu\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu][\boldsymbol{I} - \boldsymbol{P}_{X_\mu}]\\
&= \boldsymbol{R}_{\mu\mu} - \boldsymbol{R}_{\mu\mu}\boldsymbol{P}_{X_\mu} - \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu\\
&\quad + \boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{X}_\mu)^{-1}X'_\mu\boldsymbol{X}_\mu(\boldsymbol{X}'_\mu\boldsymbol{X}_\mu)^{-1}X'_\mu\\
&= \boldsymbol{R}_{\mu\mu}[\boldsymbol{I} - \boldsymbol{P}_{X_\mu}] = \boldsymbol{R}_{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}.
\end{aligned}\tag{C.375}$$

Thus,

$$\begin{aligned}
\operatorname{tr}(\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}) &= \operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{V}\bar{\boldsymbol{P}}_{X_\mu}) = [\text{see (C.375)}]\\
&= \operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}\bar{\boldsymbol{P}}_{X_\mu}) = \operatorname{tr}(\bar{\boldsymbol{P}}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu})\\
&= \operatorname{tr}[(\boldsymbol{I} - \boldsymbol{P}_{X_\mu})\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}]\\
&= \operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu}) - \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu})\\
&= \text{because the matrices } \boldsymbol{P}_{X_\mu},\ \boldsymbol{R}_{\mu\mu} \text{ and } \boldsymbol{R}_2{}^{\mu\mu} \text{ are symmetric are equal to}\\
&= \operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}) - \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu})\\
&= \operatorname{tr}[(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu})'] - \operatorname{tr}(\boldsymbol{P}_{X_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{X_\mu}\boldsymbol{R}_{\mu\mu})
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr}(\bar{P}_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} V) &= \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} R_{\mu\mu}) - \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) = \text{[see (C.352) and (C.356)]} \\
&= \frac{1}{\rho_\mu}[n - (1-\rho_\mu{}^2)\operatorname{tr}[(F_{\mu\mu}{}^{-1}\Theta_{\mu\mu})]] \\
&\quad - \frac{1}{\rho_\mu}[\operatorname{tr}(F_{\mu\mu}{}^{-1} B_{\mu\mu} F_{\mu\mu}{}^{-1}\Theta_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(F_{\mu\mu}{}^{-1}\Theta_{\mu\mu})] + O(T^{-1}) \\
&= \frac{1}{\rho_\mu}[n - \operatorname{tr}(F_{\mu\mu}{}^{-1} B_{\mu\mu} F_{\mu\mu}{}^{-1}\Theta_{\mu\mu})] + O(T^{-1}).
\end{aligned} \tag{C.376}$$

Moreover, equations (C.249), (C.261), (C.263) (C.308), (C.356), and (C.375) imply that

$$\begin{aligned}
& \operatorname{tr}(R^{\mu\mu} V P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} V R^{\mu\mu} R_{\mu\mu}) = \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} V R^{\mu\mu} V) \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} V) \\
= \; & \operatorname{tr}[P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu}[R_{\mu\mu} - X_\mu (X'_\mu R^{\mu\mu} X_\mu)^{-1} X'_\mu]] \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) - \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} X_\mu (X'_\mu R^{\mu\mu} X_\mu)^{-1} X'_\mu) \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) - \operatorname{tr}(X_\mu (X'_\mu X_\mu)^{-1} X'_\mu R_2{}^{\mu\mu} X_\mu (X'_\mu R^{\mu\mu} X_\mu)^{-1} X'_\mu) \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) - \operatorname{tr}[(X'_\mu R_2{}^{\mu\mu} X_\mu)(X'_\mu R^{\mu\mu} X_\mu)^{-1}] \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) - \operatorname{tr}[(X'_\mu (R_1{}^{\mu\mu} + \rho_\mu \Delta) X_\mu)(X'_\mu R^{\mu\mu} X_\mu)^{-1}] \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) - \operatorname{tr}[(X'_\mu R_1{}^{\mu\mu} X_\mu)(X'_\mu R^{\mu\mu} X_\mu)^{-1}] \\
& -\rho_\mu \operatorname{tr}[(X'_\mu \Delta X_\mu)(X'_\mu R^{\mu\mu} X_\mu)^{-1}] \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) \\
& - \operatorname{tr}[X'_\mu \frac{1}{\rho_\mu}[R^{\mu\mu} - (1-\rho_\mu{}^2) I_T] X_\mu (X'_\mu R^{\mu\mu} X_\mu)^{-1}] - O(1) \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) - \frac{1}{\rho_\mu}\operatorname{tr}[(X'_\mu R^{\mu\mu} X_\mu)(X'_\mu R^{\mu\mu} X_\mu)^{-1}] \\
& + \frac{(1-\rho_\mu{}^2)}{\rho_\mu}\operatorname{tr} X'_\mu X_\mu (X'_\mu R^{\mu\mu} X_\mu)^{-1}] + O(1) \\
= \; & \operatorname{tr}(P_{X_\mu} R_2{}^{\mu\mu} P_{X_\mu} R_{\mu\mu}) - \frac{1}{\rho_\mu}[\operatorname{tr}(I_n) - (1-\rho_\mu{}^2)\operatorname{tr}(X'_\mu X_\mu/T)(X'_\mu R^{\mu\mu} X_\mu/T)^{-1}] + O(1) \\
= \; & \frac{1}{\rho_\mu}[\operatorname{tr}(F_{\mu\mu}{}^{-1} B_{\mu\mu} F_{\mu\mu}{}^{-1}\Theta_{\mu\mu}) - (1-\rho_\mu{}^2)\operatorname{tr}(F_{\mu\mu}{}^{-1}\Theta_{\mu\mu})] \\
& - \frac{1}{\rho_\mu}[n - (1-\rho_\mu{}^2)\operatorname{tr}(F_{\mu\mu} B_{\mu\mu}^{-1})] + O(T^{-1}) \\
= \; & \frac{1}{\rho_\mu}[\operatorname{tr}(F_{\mu\mu}{}^{-1} B_{\mu\mu} F_{\mu\mu}{}^{-1}\Theta_{\mu\mu}) - n] + \frac{(1-\rho_\mu{}^2)}{\rho_\mu}[\operatorname{tr}(F_{\mu\mu}{}^{-1} B_{\mu\mu}) - \operatorname{tr}(F_{\mu\mu}{}^{-1}\Theta_{\mu\mu})] \\
& + O(T^{-1}).
\end{aligned} \tag{C.377}$$

Lemma C.11. By using Magnus and Neudecker, 1979 we can prove the following results:

(i) By using equation(C.318) we have

$$\begin{aligned} \mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R_2}^{\mu\mu} \boldsymbol{u}_\mu) &= \sigma_{\mu\mu} \operatorname{tr}(\boldsymbol{R_2}^{\mu\mu} \boldsymbol{R}_{\mu\mu}) \\ &= \frac{2\rho_\mu \sigma_{\mu\mu}}{1-{\rho_\mu}^2}. \end{aligned} \tag{C.378}$$

(ii) By using equations (C.318) and (C.329), and omitting terms that tend to zero as $T \to \infty$, we find that

$$\begin{aligned} \mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R_2}^{\mu\mu} \boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R_2}^{\mu\mu} \boldsymbol{u}_\mu)/T &= [\text{see (UR.2)}] \\ &= {\sigma_{\mu\mu}}^2[\operatorname{tr}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{\Omega}_{\mu\mu}) \operatorname{tr}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{\Omega}_{\mu\mu}) + 2\operatorname{tr}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{\Omega}_{\mu\mu}\boldsymbol{R_2}^{\mu\mu}\boldsymbol{\Omega}_{\mu\mu})]/T \\ &= {\sigma_{\mu\mu}}^2[[\operatorname{tr}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu})]^2 + 2[\operatorname{tr}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu})^2]]/T \\ &= \left(\frac{2\rho_\mu\sigma_{\mu\mu}}{1-{\rho_\mu}^2}\right)^2/T + 2\left[\frac{2T{\sigma_{\mu\mu}}^2}{1-{\rho_\mu}^2} + O(1)\right]/T \\ &= \frac{4{\sigma_{\mu\mu}}^2}{1-{\rho_\mu}^2} + O(T^{-1}) \Rightarrow \end{aligned} \tag{C.379}$$

$$\mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R_2}^{\mu\mu} \boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R_2}^{\mu\mu} \boldsymbol{u}_\mu) = \frac{4T{\sigma_{\mu\mu}}^2}{1-{\rho_\mu}^2} + O(1), \tag{C.380}$$

because equation (C.318) implies that

$$\operatorname{tr}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu})/T = \frac{2\rho_\mu}{1-{\rho_\mu}^2} = O(T^{-1}) \Rightarrow \operatorname{tr}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) = O(1). \tag{C.381}$$

(iii) Equations (C.268),(C.318), and(C.360) imply that

$$\begin{aligned} \mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R_2}^{\mu\mu} \boldsymbol{u}_\mu) &= {\sigma_{\mu\mu}}^2 \operatorname{tr}(\boldsymbol{I}\boldsymbol{R}_{\mu\mu}) \operatorname{tr}(\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) + 2{\sigma_{\mu\mu}}^2 \operatorname{tr}(\boldsymbol{I}\boldsymbol{R}_{\mu\mu}\boldsymbol{R_2}^{\mu\mu}\boldsymbol{R}_{\mu\mu}) \\ &= {\sigma_{\mu\mu}}^2\left[\frac{T}{1-{\rho_\mu}^2}\cdot\frac{2\rho_\mu}{1-{\rho_\mu}^2} + 2\left(\frac{-2T\rho_\mu}{(1-{\rho_\mu}^2)^2} + O(1)\right)\right] \\ &= {\sigma_{\mu\mu}}^2\left[\frac{2T\rho_\mu}{(1-{\rho_\mu}^2)^2} - \frac{4T\rho_\mu}{(1-{\rho_\mu}^2)^2}\right] + O(1) \\ &= -\frac{2T\rho_\mu{\sigma_{\mu\mu}}^2}{(1-{\rho_\mu}^2)^2} + O(1). \end{aligned} \tag{C.382}$$

(iv) Equation (C.373) implies that

$$\begin{aligned} \mathrm{E}(\boldsymbol{u}'_\mu \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \boldsymbol{R_2}^{\mu\mu} \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \boldsymbol{u}_\mu) &= \sigma_{\mu\mu} \operatorname{tr}(\bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \boldsymbol{R_2}^{\mu\mu} \bar{\boldsymbol{P}}_{\boldsymbol{X}_\mu} \boldsymbol{R}_{\mu\mu}) \\ &= \sigma_{\mu\mu}\frac{1}{\rho_\mu}\left[\frac{2({\rho_\mu}^2 - n(1-{\rho_\mu}^2))}{1-{\rho_\mu}^2} + (1-{\rho_\mu}^2)\operatorname{tr}({\boldsymbol{F}_{\mu\mu}}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \operatorname{tr}({\boldsymbol{F}_{\mu\mu}}^{-1}\boldsymbol{B}_{\mu\mu}{\boldsymbol{F}_{\mu\mu}}^{-1}\boldsymbol{\Theta}_{\mu\mu})\right] \\ &\quad + O(T^{-1}). \end{aligned} \tag{C.383}$$

(v) Equation (C.376) implies that

$$\begin{aligned}\mathrm{E}(\boldsymbol{u}'_\mu \bar{\boldsymbol{P}}_{X_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{X_\mu} \boldsymbol{V} \boldsymbol{R}^{\mu\mu} \boldsymbol{u}_\mu) &= \sigma_{\mu\mu}\,\mathrm{tr}\,(\bar{\boldsymbol{P}}_{X_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{X_\mu} \boldsymbol{V} \boldsymbol{R}^{\mu\mu} \boldsymbol{R}_{\mu\mu}) \\ &= \sigma_{\mu\mu}\,\mathrm{tr}\,(\bar{\boldsymbol{P}}_{X_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{X_\mu} \boldsymbol{V}) \\ &= \frac{\sigma_{\mu\mu}}{\rho_\mu}[n - \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1} \boldsymbol{B}_{\mu\mu} \boldsymbol{F}_{\mu\mu}{}^{-1} \boldsymbol{\Theta}_{\mu\mu})] + O(T^{-1}). \qquad \text{(C.384)}\end{aligned}$$

(vi)

$$\begin{aligned}\mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}^{\mu\mu} \boldsymbol{V} \boldsymbol{P}_{X_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{X_\mu} \boldsymbol{V} \boldsymbol{R}^{\mu\mu} \boldsymbol{u}_\mu) &= \sigma_{\mu\mu}\,\mathrm{tr}\,(\boldsymbol{R}^{\mu\mu} \boldsymbol{V} \boldsymbol{P}_{X_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{X_\mu} \boldsymbol{V} \boldsymbol{R}^{\mu\mu} \boldsymbol{R}_{\mu\mu}) \\ &= \sigma_{\mu\mu}\,\mathrm{tr}\,(\boldsymbol{R}^{\mu\mu} \boldsymbol{V} \boldsymbol{P}_{X_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{X_\mu} \boldsymbol{V}) = [\text{see (C.377)}] \\ &= \frac{\sigma_{\mu\mu}}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1} \boldsymbol{B}_{\mu\mu} \boldsymbol{F}_{\mu\mu}{}^{-1} \boldsymbol{\Theta}_{\mu\mu}) - n] \\ &\quad + \frac{\sigma_{\mu\mu}(1-\rho_\mu{}^2)}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1} \boldsymbol{B}_{\mu\mu}) - \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1} \boldsymbol{\Theta}_{\mu\mu})] \\ &\quad + O(T^{-1}). \qquad \text{(C.385)}\end{aligned}$$

(vii)

$$\mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu \boldsymbol{u}'_{\mu'} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_{\mu'}) = \sigma_{\mu\mu}\sigma_{\mu'\mu'}[\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{R}_{\mu\mu})\,\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'}) + 2\,\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{R}_{\mu\mu} \boldsymbol{R}_2{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'})]. \qquad \text{(C.386)}$$

Since (C.263) implies that

$$\begin{aligned}(\boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{R}_{\mu\mu})(\boldsymbol{R}_2{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'}) &= [\boldsymbol{R}_1{}^{\mu\mu} \boldsymbol{R}_{\mu\mu} + \rho_\mu \boldsymbol{\Delta R}_{\mu\mu}][\boldsymbol{R}_1{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'} + \rho_{\mu'} \boldsymbol{\Delta R}_{\mu'\mu'}] \\ &= (\boldsymbol{R}_1{}^{\mu\mu} \boldsymbol{R}_{\mu\mu} \boldsymbol{R}_1{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'}) + \rho_\mu \boldsymbol{\Delta R}_{\mu\mu} \boldsymbol{R}_1{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'} + \rho_{\mu'} \boldsymbol{R}_1{}^{\mu\mu} \boldsymbol{R}_{\mu\mu} \boldsymbol{\Delta R}_{\mu'\mu'} \\ &\quad + \rho_\mu \rho_{\mu'} \boldsymbol{\Delta R}_{\mu\mu} \boldsymbol{\Delta R}_{\mu'\mu'}, \qquad \text{(C.387)}\end{aligned}$$

it follows that

$$\begin{aligned}\mathrm{tr}\,(\boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{R}_{\mu\mu} \boldsymbol{R}_2{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'}) &= \mathrm{tr}\,(\boldsymbol{R}_1{}^{\mu\mu} \boldsymbol{R}_{\mu\mu} \boldsymbol{R}_1{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'}) + \rho_\mu\,\mathrm{tr}(\boldsymbol{\Delta R}_{\mu\mu} \boldsymbol{R}_1{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'}) \\ &\quad + \rho_{\mu'}\,\mathrm{tr}(\boldsymbol{R}_1{}^{\mu\mu} \boldsymbol{R}_{\mu\mu} \boldsymbol{\Delta R}_{\mu'\mu'}) + \rho_\mu \rho_{\mu'}\,\mathrm{tr}(\boldsymbol{\Delta R}_{\mu\mu} \boldsymbol{\Delta R}_{\mu'\mu'}). \qquad \text{(C.388)}\end{aligned}$$

Moreover, (C.262) implies that

$$\begin{aligned}(\boldsymbol{R}_1{}^{\mu\mu} \boldsymbol{R}_{\mu\mu})(\boldsymbol{R}_1{}^{\mu'\mu'} \boldsymbol{R}_{\mu'\mu'}) &= \frac{1}{\rho_\mu}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu}]\frac{1}{\rho_{\mu'}}[\boldsymbol{I}_T - (1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'}] \\ &= \frac{1}{\rho_\mu \rho_{\mu'}}[\boldsymbol{I}_T - (1-\rho_\mu{}^2)\boldsymbol{R}_{\mu\mu} - (1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'} \\ &\quad + (1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu\mu} \boldsymbol{R}_{\mu'\mu'}]. \qquad \text{(C.389)}\end{aligned}$$

vii.a Since the (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}$ is $\frac{1}{(1-\rho_\mu{}^2)}\rho_\mu{}^{|i-j|}$, the (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'}$ is

$$e_{ij} = \sum_{k=1}^{T} \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\rho_\mu{}^{|i-k|}\rho_{\mu'}{}^{|k-j|}. \tag{C.390}$$

Therefore, the i-diagonal element of the matrix $\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'}$ is

$$e_{ii} = \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\sum_{k=1}^{T}\rho_\mu{}^{|i-k|}\rho_{\mu'}{}^{|k-i|}]\,. \tag{C.391}$$

Define the index $j = k - i\,(j = 1-i, \ldots, T-i)$ and set $r = \rho_\mu\rho_{\mu'}$. Then,

$$\begin{aligned}
\sum_{k=1}^{T}\rho_\mu{}^{|i-k|}\rho_{\mu'}{}^{|k-i|} &= \sum_{j=1-i}^{T-i}\rho_\mu{}^{|j|}\rho_{\mu'}{}^{|j|} = \sum_{j=1-i}^{T-i}(\rho_\mu\rho_{\mu'})^{|j|} \\
&= \sum_{j=1-i}^{T-i} r^{|j|} = \sum_{j=1-i}^{-1} r^{|j|} + \sum_{j=0}^{T-i} r^j = \sum_{j+i=1}^{i-1} r^{|j+i|} + \sum_{j=0}^{T-i} r^j \\
&= \sum_{k=1}^{i-1} r^k + \sum_{j=0}^{T-i} r^j = \frac{r(1-r^{(i-1)})}{1-r} + \frac{1-r^{T-i+1}}{1-r} = \frac{r - r^i + 1 - r^{T-i+1}}{1-r} \\
&= \frac{(1+r)-(r^i + r^{T-i+1})}{1-r} = [\text{setting } j = T-i+1\ (j=1,\ldots,T)] \\
&= \frac{(1+r)-(r^i+r^j)}{1-r} = \frac{1+r-2r^i}{1-r}.
\end{aligned} \tag{C.392}$$

Thus, equations (C.391) and (C.392) imply that

$$e_{ii} = \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\frac{1+r-2r^i}{1-r} \Rightarrow \tag{C.393}$$

$$\begin{aligned}
\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'})/T &= \sum_{i=1}^{T} e_{ii}/T = \frac{1}{T(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\sum_{i=1}^{T}\left[\frac{1+r}{1-r} - \frac{2r^i}{1-r}\right] \\
&= \frac{1}{T(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\left[\frac{T(1+r)}{1-r} - \frac{2}{1-r}\sum_{i=1}^{T} r^i\right] \\
&= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\left[\frac{1+r}{1-r} - \frac{2r}{T(1-r)}\frac{r(1-r^T)}{1-r}\right],
\end{aligned} \tag{C.394}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'})/T = \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\frac{1+r}{1-r} + o(T^{-1}) \Rightarrow \tag{C.395}$$

$$\mathrm{tr}\,(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'}) = \frac{T(1+\rho_\mu\rho_{\mu'})}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)(1-\rho_\mu\rho_{\mu'})} + o(1). \tag{C.396}$$

By combining equations (C.268), (C.389), and (C.396) we find that

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_1{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) &= \frac{1}{\rho_\mu\rho_{\mu'}}[\operatorname{tr}(\boldsymbol{I}_T) - (1-\rho_\mu{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu\mu})] \\
&\quad -\frac{1}{\rho_\mu\rho_{\mu'}}[(1-\rho_{\mu'}{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu'\mu'}) - (1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)\operatorname{tr}(\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_{\mu'\mu'})] \\
&= \frac{1}{\rho_\mu\rho_{\mu'}}\left[T - (1-\rho_\mu{}^2)\frac{T}{(1-\rho_\mu{}^2)} - (1-\rho_{\mu'}{}^2)\frac{T}{(1-\rho_{\mu'}{}^2)}\right] \\
&\quad +\frac{1}{\rho_\mu\rho_{\mu'}}\left[\frac{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)T(1+\rho_\mu\rho_{\mu'})}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)(1-\rho_\mu\rho_{\mu'})} + o(1)\right] \\
&= \frac{1}{\rho_\mu\rho_{\mu'}}\left[-T + \frac{T(1+\rho_\mu\rho_{\mu'})}{(1-\rho_\mu\rho_{\mu'})}\right] + o(1) \\
&= \frac{1}{\rho_\mu\rho_{\mu'}}\left[\frac{-T + T\rho_\mu\rho_{\mu'} + T + T\rho_\mu\rho_{\mu'}}{(1-\rho_\mu\rho_{\mu'})}\right] + o(1) \\
&= \frac{2T}{1-\rho_\mu\rho_{\mu'}} + o(1). \qquad \text{(C.397)}
\end{aligned}$$

vii.b Let δ_{ij} be the (i,j)-th element of the matrix $\boldsymbol{\Delta}$. Then, $\delta_{11} = \delta_{TT} = 1$ and $\delta_{ij} = 0 \; \forall \; i, j \neq 1$ and $i, j \neq T$. Moreover, let $\frac{1}{1-\rho_\mu{}^2}\rho_\mu{}^{|i-j|}$ be the (i,j)-th element of the matrix $\boldsymbol{R}_{\mu\mu}$. Then, the (i,j)-th element of the matrix $\boldsymbol{\Delta R}_{\mu\mu}$ is (see (C.271))

$$\delta_{ij}{}^* = \delta_{ii}\frac{1}{1-\rho_\mu{}^2}\rho_\mu{}^{|i-j|} \qquad \text{(C.398)}$$

Since equation (C.262) implies that

$$\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'} = \frac{1}{\rho_{\mu'}}[\boldsymbol{I}_T - (1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'}], \qquad \text{(C.399)}$$

we find that

$$\begin{aligned}
\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu} &= \frac{1}{\rho_{\mu'}}[\boldsymbol{I}_T - (1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'}]\boldsymbol{\Delta R}_{\mu\mu} \\
&= \frac{1}{\rho_{\mu'}}[\boldsymbol{\Delta R}_{\mu\mu} - (1-\rho_{\mu'}{}^2)\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}]. \qquad \text{(C.400)}
\end{aligned}$$

The (i,j)-th element of the matrix $\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}$ is

$$\begin{aligned}
\delta_{ij}{}^{**} &= \sum_{k=1}^{T}\frac{1}{(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|i-k|}\delta_{kj}{}^* = \sum_{k=1}^{T}\frac{1}{(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|i-k|}\delta_{kk}\frac{1}{(1-\rho_\mu{}^2)}\rho_\mu{}^{|k-j|} \\
&= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|i-1|}\rho_\mu{}^{|1-j|} + \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|i-T|}\rho_\mu{}^{|T-j|} \\
&= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[\rho_{\mu'}{}^{|i-1|}\rho_\mu{}^{|1-j|} + \rho_{\mu'}{}^{|i-T|}\rho_\mu{}^{|T-j|}] \qquad \text{(C.401)}
\end{aligned}$$

and the i-diagonal element of the matrix $\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}$ is

$$\begin{aligned}\delta_{ii}^{**} &= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[\rho_{\mu'}{}^{|i-1|}\rho_\mu{}^{|1-i|} + \rho_{\mu'}{}^{|i-T|}\rho_\mu{}^{|T-i|}] \\ &= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[(\rho_\mu\rho_{\mu'})^{|i-1|} + (\rho_\mu\rho_{\mu'})^{|T-i|}]. \qquad \text{(C.402)}\end{aligned}$$

Therefore, setting $j = T-i+1\,(j = 1,\dots,T)$ and $(r = \rho_\mu\rho_{\mu'})$, and setting $l = i-1\,(l = 0,\dots,T-1)$, equation (C.402) implies that

$$\begin{aligned}\mathrm{tr}\,(\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}) &= \sum_{i=1}^{T}\delta_{ii}^{**} \\ &= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\left[\sum_{i=1}^{T}(\rho_\mu\rho_{\mu'})^{i-1} + \sum_{i=1}^{T}(\rho_\mu\rho_{\mu'})^{T-i}\right] \\ &= \frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\left[\sum_{i=1}^{T}r^{i-1} + \sum_{j=1}^{T}r^{j-1}\right] = \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\sum_{i=1}^{T}r^{i-1} \\ &= \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\sum_{l=0}^{T-1}r^{l} = \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\frac{1-r^T}{1-r} \\ &= \frac{2}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}\frac{1-(\rho_\mu\rho_{\mu'})^T}{1-\rho_\mu\rho_{\mu'}} = \frac{2[1-(\rho_\mu\rho_{\mu'})^T]}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)(1-\rho_\mu\rho_{\mu'})}. \qquad \text{(C.403)}\end{aligned}$$

Therefore, equations (C.272), (C.400), and (C.403) imply that

$$\begin{aligned}\mathrm{tr}\,(\boldsymbol{R}_1{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu}) &= \frac{1}{\rho_{\mu'}}[\mathrm{tr}(\boldsymbol{\Delta R}_{\mu\mu}) - (1-\rho_{\mu'}{}^2)\,\mathrm{tr}(\boldsymbol{R}_{\mu'\mu'}\boldsymbol{\Delta R}_{\mu\mu})] \\ &= \frac{1}{\rho_{\mu'}}\left[\frac{2}{1-\rho_\mu{}^2} - (1-\rho_{\mu'}{}^2)\frac{2[1-(\rho_\mu\rho_{\mu'})^T]}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)(1-\rho_\mu\rho_{\mu'})}\right] \\ &= \frac{1}{\rho_{\mu'}}\left[\frac{2}{1-\rho_\mu{}^2} - \frac{2[1-(\rho_\mu\rho_{\mu'})^T]}{(1-\rho_\mu{}^2)(1-\rho_\mu\rho_{\mu'})}\right]. \qquad \text{(C.404)}\end{aligned}$$

vii.c By using equation (C.271) we find that the (i,j)-th element of the matrix $\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu'\mu'}$ is

$$\begin{aligned}\delta_{ij}^{\circ\circ} &= \sum_{k=1}^{T}\delta_{ik}{}^*\delta_{kj}{}^* = \sum_{\kappa=1}^{T}\delta_{ii}\frac{1}{(1-\rho_\mu{}^2)}\rho_\mu{}^{|i-k|}\delta_{kk}\frac{1}{(1-\rho_{\mu'}{}^2)}\rho_{\mu'}{}^{|k-j|} \\ &= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[\rho_\mu{}^{|i-1|}\rho_{\mu'}{}^{|1-j|} + \rho_\mu{}^{|i-T|}\rho_{\mu'}{}^{|T-j|}], \qquad \text{(C.405)}\end{aligned}$$

which implies that the i-diagonal element of the matrix $\boldsymbol{\Delta R}_{\mu\mu}\boldsymbol{\Delta R}_{\mu'\mu'}$ is

$$\begin{aligned}\delta_{ii}^{\circ\circ} &= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[\rho_\mu{}^{|i-1|}\rho_{\mu'}{}^{|1-i|} + \rho_\mu{}^{|i-T|}\rho_{\mu'}{}^{|T-i|}] \\ &= \delta_{ii}\frac{1}{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}[(\rho_\mu\rho_{\mu'})^{|i-1|} + (\rho_\mu\rho_{\mu'})^{|i-T|}]. \qquad \text{(C.406)}\end{aligned}$$

Therefore,

$$
\begin{aligned}
\operatorname{tr}(\Delta\boldsymbol{R}_{\mu\mu}\Delta\boldsymbol{R}_{\mu'\mu'}) &= \sum_{i=1}^{T}\delta_{ii}{}^{\circ\circ} = \frac{1}{(1-\rho_{\mu}{}^2)(1-\rho_{\mu'}{}^2)}\sum_{i=1}^{T}\delta_{ii}[(\rho_\mu\rho_{\mu'})^{|i-1|}+(\rho_\mu\rho_{\mu'})^{|i-T|}] \\
= & \frac{1}{(1-\rho_{\mu}{}^2)(1-\rho_{\mu'}{}^2)}\sum_{i=1}^{T}[\delta_{11}[(\rho_\mu\rho_{\mu'})^{|1-1|}+(\rho_\mu\rho_{\mu'})^{|T-1|}]+\delta_{TT}[(\rho_\mu\rho_{\mu'})^{|T-1|}+(\rho_\mu\rho_{\mu'})^{|T-T|}]] \\
= & \frac{2}{(1-\rho_{\mu}{}^2)(1-\rho_{\mu'}{}^2)}[1+(\rho_\mu\rho_{\mu'})^{T-1}]. \qquad (C.407)
\end{aligned}
$$

Thus, equations (C.388), (C.397), (C.404), and (C.407) imply that

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'})/T \quad = \quad & \frac{2T/T}{(1-\rho_\mu\rho_{\mu'})}+o(T^{-1}) \\
& +\rho_\mu\frac{1}{\rho_{\mu'}}\left[\frac{2}{1-\rho_\mu{}^2}-\frac{2[1-(\rho_\mu\rho_{\mu'})^T]}{(1-\rho_\mu{}^2)(1-\rho_\mu\rho_{\mu'})}\right]/T \\
& +\rho_{\mu'}\frac{1}{\rho_{\mu}}\left[\frac{2}{1-\rho_{\mu'}{}^2}-\frac{2[1-(\rho_{\mu'}\rho_{\mu})^T]}{(1-\rho_{\mu'}{}^2)(1-\rho_{\mu'}\rho_{\mu})}\right]/T \\
& +\frac{2}{(1-\rho_{\mu}{}^2)(1-\rho_{\mu'}{}^2)}[1+(\rho_\mu\rho_{\mu'})^{T-1}]/T, \qquad (C.408)
\end{aligned}
$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'})/T &= \frac{2}{1-\rho_\mu\rho_{\mu'}}+o(T^{-1}) \Rightarrow \\
\operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{R}_{\mu\mu}\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{R}_{\mu'\mu'}) &= \frac{2T}{1-\rho_\mu\rho_{\mu'}}+o(1). \qquad (C.409)
\end{aligned}
$$

Therefore, equations (C.386), (C.318), and (C.409) imply that

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu\boldsymbol{u}_{\mu'}'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_{\mu'})/T \quad = \quad & \frac{2\rho_\mu\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}\frac{2\rho_{\mu'}\sigma_{\mu'\mu'}}{(1-\rho_{\mu'}{}^2)}/T \\
& +2\left[\frac{2\sigma_{\mu\mu}\sigma_{\mu'\mu'}}{1-\rho_\mu\rho_{\mu'}}+o(T^{-1})\right] \qquad (C.410)
\end{aligned}
$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$
\mathrm{E}(\boldsymbol{u}_\mu'\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu\boldsymbol{u}_{\mu'}'\boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{u}_{\mu'}) = \frac{4T\sigma_{\mu\mu}\sigma_{\mu'\mu'}}{1-\rho_\mu\rho_{\mu'}}+o(1). \qquad (C.411)
$$

Lemma C.12. The following results hold:

Let ε_{ti} be the (t,i)-th element of the matrix $\boldsymbol{E}$. Then, the (i,i)-th element of the matrix $\boldsymbol{E}'\boldsymbol{E}/T$ is

$$e_{ii} = \sum_{t=1}^{T} \varepsilon_{it}\varepsilon_{ti}/T. \tag{C.412}$$

Since σ_{ii} is the (i,i)-th element of the matrix $\boldsymbol{\Sigma}$, by using equations (C.117) and (C.412) we find that the (i,i)-th element of the matrix $\boldsymbol{\Sigma}_1$ is

$$\sigma_{ii}{}^{(1)} = \sqrt{T}\left(\sum_{t=1}^{T} \varepsilon_{it}\varepsilon_{ti}/T - \sigma_{ii}\right) = \sqrt{T}(e_{ii} - \sigma_{ii}). \tag{C.413}$$

Moreover, since σ^{ii} is the (i,i)-th element of the matrix $\boldsymbol{\Sigma}^{-1}$, by using equation (??) we find that the (i,i)-th element of the matrix $\boldsymbol{S}_1$ is

$$\begin{aligned} s_{ii}{}^{(1)} &= [(\delta_{i\kappa}\sigma^{i\kappa})_{i,\kappa=1,\ldots,M}][(\delta_{\kappa l}\sigma_{\kappa l}{}^{(1)})_{\kappa,l=1,\ldots,M}][(\delta_{lj}\sigma^{lj})_{l,j=1,\ldots,M}] \\ &= \sum_{k=1}^{M}\sum_{l=1}^{M} \delta_{i\kappa}\delta_{\kappa l}\delta_{lj}\sigma^{ik}\sigma_{kl}{}^{(1)}\sigma^{lj} = \delta_{ij}\sigma^{ii}\sigma_{ij}{}^{(1)}\sigma^{jj} \\ &= \sqrt{T}[\sigma^{ii}e_{ii}\sigma^{ii} - \sigma^{ii}\sigma_{ii}\sigma^{ii}] = \sqrt{T}[\sigma^{ii}e_{ii}\sigma^{ii} - \sigma^{ii}]. \end{aligned} \tag{C.414}$$

Since $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}^{-1}$, the (i,i)-th elements of the matrices $\boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}$ are identical, i.e.,

$$\begin{aligned} \sigma^{ii} &= [(\delta_{i\kappa}\sigma^{i\kappa})_{i,\kappa=1,\ldots,M}][(\delta_{\kappa l}\sigma_{\kappa l})_{\kappa,l=1,\ldots,M}][(\delta_{lj}\sigma^{lj})_{l,j=1,\ldots,M}] \\ &= \sum_{k=1}^{M}\sum_{l=1}^{M} \delta_{i\kappa}\delta_{\kappa l}\delta_{lj}\sigma^{ik}\sigma_{kl}\sigma^{lj} = \delta_{ij}\sigma^{ii}\sigma_{ij}\sigma^{jj} \\ &= \sigma^{ii}\sigma_{ii}\sigma^{ii} = \sigma^{ii}. \end{aligned} \tag{C.415}$$

Thus, equations (C.414) and (C.415) imply that

$$s_{ii}{}^{(1)} = \sqrt{T}[\sigma^{ii}e_{ii}\sigma^{ii} - \sigma^{ii}]. \tag{C.416}$$

Since equation (C.412) implies that

$$e_{ii} = \boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i/T \tag{C.417}$$

where $\boldsymbol{e}_i$ is the i-th column of the matrix $\boldsymbol{E}$ we find that

$$s_{ii}{}^{(1)} = \sqrt{T}[\sigma^{ii}(\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i/T)\sigma^{ii} - \sigma^{ii}]. \tag{C.418}$$

Therefore the (i,i)-th element of $(1 \times M)$ vector s_1 is

$$s_{(ii)}^{(1)} = s_{ii}{}^{(1)} = \sqrt{T}[\sigma^{ii}(\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i/T)\sigma^{ii} - \sigma^{ii}]. \tag{C.419}$$

Equation (3.22) implies that

$$\boldsymbol{u}'_\mu = \boldsymbol{\varepsilon}'_\mu \boldsymbol{P}'_\mu \text{ and } \boldsymbol{u}_\mu = \boldsymbol{P}_\mu \boldsymbol{\varepsilon}_\mu \Rightarrow \tag{C.420}$$

$$\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu = \boldsymbol{\varepsilon}'_\mu \boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu \boldsymbol{\varepsilon}_\mu. \tag{C.421}$$

By using Lemma UR.2 and since (5.15b) implies that (see Magnus and Neudecker, 1979, p.389)

$$\mathrm{E}(\boldsymbol{\varepsilon}'_\mu \boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu \boldsymbol{\varepsilon}_\mu) = \sigma_{\mu\mu} \operatorname{tr}(\boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu \boldsymbol{I}_T) = \sigma_{\mu\mu} \operatorname{tr}(\boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu), \tag{C.422}$$

we find that

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\varepsilon}'_i \boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}'_\mu \boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu \boldsymbol{\varepsilon}_\mu) &= \text{(see Magnus and Neudecker, 1979 p.389)} \\
&= \operatorname{tr}(\sigma_{ii} \boldsymbol{I}_T) \operatorname{tr}(\sigma_{\mu\mu} \boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu) + 2 \operatorname{tr}(\sigma_{ii}\sigma_{\mu\mu} \boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu) \\
&= \sigma_{ii}\sigma_{\mu\mu} T \operatorname{tr}(\boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu) + 2\sigma_{ii}\sigma_{\mu\mu} \operatorname{tr}(\boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu) \\
&= \sigma_{ii}\sigma_{\mu\mu}(T+2) \operatorname{tr}(\boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu),
\end{aligned} \tag{C.423}$$

and since (C.1) implies that

$$\boldsymbol{P}_\mu \boldsymbol{P}'_\mu = \boldsymbol{R}_{\mu\mu}, \tag{C.424}$$

by combining equations (C.318) (C.423), and (C.424) we find that

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\varepsilon}'_i \boldsymbol{\varepsilon}_i \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu) &= \mathrm{E}(\boldsymbol{\varepsilon}'_i \boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}'_\mu \boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu \boldsymbol{\varepsilon}_\mu) = \sigma_{ii}\sigma_{\mu\mu}(T+2) \operatorname{tr}(\boldsymbol{P}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu) \\
&= \sigma_{ii}\sigma_{\mu\mu}(T+2) \operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_\mu \boldsymbol{P}'_\mu) = \sigma_{ii}\sigma_{\mu\mu}(T+2) \operatorname{tr}(\boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{R}_{\mu\mu}) \\
&= \sigma_{ii}\sigma_{\mu\mu}(T+2) \frac{2\rho_\mu}{1-\rho_\mu{}^2}.
\end{aligned} \tag{C.425}$$

Therefore,

$$\begin{aligned}
\mathrm{E}[(\boldsymbol{\varepsilon}'_i \boldsymbol{\varepsilon}_i / T) \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu] &= \mathrm{E}(\boldsymbol{\varepsilon}'_i \boldsymbol{\varepsilon}_i \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu)/T \\
&= \sigma_{ii}\sigma_{\mu\mu} \frac{2\rho_\mu}{1-\rho_\mu{}^2} (T+2)/T \\
&= \sigma_{ii}\sigma_{\mu\mu} \frac{2\rho_\mu}{1-\rho_\mu{}^2} + \sigma_{ii}\sigma_{\mu\mu} \frac{4\rho_\mu}{1-\rho_\mu{}^2} / T
\end{aligned} \tag{C.426}$$

and by omitting terms that tend to zero as $T \to \infty$ we find that

$$\mathrm{E}[(\boldsymbol{\varepsilon}'_i \boldsymbol{\varepsilon}_i / T) \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu] = \sigma_{ii}\sigma_{\mu\mu} \frac{2\rho_\mu}{1-\rho_\mu{}^2} + O(T^{-1}). \tag{C.427}$$

Equation (C.419) implies that

$$s^{(1)}_{(ii)} \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu = \sqrt{T}[(\sigma^{ii})^2 (\boldsymbol{\varepsilon}'_i \boldsymbol{\varepsilon}_i / T) \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu - \sigma^{ii} \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu]. \tag{C.428}$$

Equations (C.378), (C.415), and (C.428) imply that

$$\begin{aligned}
\mathrm{E}(s^{(1)}_{(ii)}\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu) &= \sqrt{T}[(\sigma^{ii})^2\,\mathrm{E}[(\boldsymbol{\varepsilon}'_i\boldsymbol{\varepsilon}_i/T)\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu] - \sigma^{ii}\,\mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)] \\
&= \sqrt{T}[(\sigma^{ii})^2\sigma_{ii}\sigma_{\mu\mu}\frac{2\rho_\mu}{1-\rho_\mu{}^2} - \sigma^{ii}\sigma_{\mu\mu}\frac{2\rho_\mu}{1-\rho_\mu{}^2} + O(T^{-1})] \\
&= O(T^{-1/2}) \Rightarrow
\end{aligned} \tag{C.429}$$

$$\lim_{T\to\infty}\mathrm{E}(s^{(1)}_{(ii)}\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu) = 0. \tag{C.430}$$

Proof of Theorem 6. Define the $((1+M+M)\times 1)$ vector

$$\boldsymbol{\delta} = \begin{bmatrix}\delta_0\\ \boldsymbol{\delta}_\rho\\ \boldsymbol{\delta}_\varsigma\end{bmatrix} \tag{C.431}$$

where for $\sigma = 1$

$$\delta_0 = \frac{\hat{\sigma}^2-\sigma^2}{\tau\sigma^2} = \frac{\hat{\sigma}^2-1}{\tau} \tag{C.432}$$

is a scalar,

$$\boldsymbol{\delta}_\rho = [(\delta_{\rho_\mu})_{\mu=1,\dots,M}] \tag{C.433}$$

is a $M\times 1$ vector with element

$$\delta_{\rho_\mu} = \frac{\hat{\rho}_\mu-\rho_\mu}{\tau} \tag{C.434}$$

and

$$\boldsymbol{\delta}_\varsigma = [(\delta_{\sigma^{\mu\mu}})_{\mu\mu=1,\dots,M}] \tag{C.435}$$

is a $(M\times 1)$ vector with elements

$$\delta_{\sigma^{\mu\mu}} = \frac{\hat{\sigma}^{\mu\mu}-\sigma^{\mu\mu}}{\tau}. \tag{C.436}$$

Moreover, $\boldsymbol{\delta}$ admits the following stochastic expansions:

$$\boldsymbol{\delta} = \boldsymbol{d}_1 + \tau\boldsymbol{d}_2 + \omega(\tau^2) \tag{C.437}$$

which implies that $\delta_0, \boldsymbol{\delta}_\rho$ and $\boldsymbol{\delta}_\varsigma$ admits a stochastic expansion of the form

$$\delta_0 = \sigma_0 + \tau\sigma_1 + \omega(\tau^2) \tag{C.438}$$

$$\boldsymbol{\delta}_\rho = \boldsymbol{d}_{1\rho} + \tau\boldsymbol{d}_{2\rho} + \omega(\tau^2) \tag{C.439}$$

$$\boldsymbol{\delta}_\varsigma = \boldsymbol{d}_{1\varsigma} + \tau\boldsymbol{d}_{2\varsigma} + \omega(\tau^2), \tag{C.440}$$

where σ_0 and σ_1 are scalars, $\boldsymbol{d}_{1\rho}$ and $\boldsymbol{d}_{2\rho}$ are $(M\times 1)$ vectors and $\boldsymbol{d}_{1\varsigma}$ and $\boldsymbol{d}_{2\varsigma}$ are $(M\times 1)$ vectors.

Define the scalars λ_0 and κ_0 the $(M\times 1)$ vectors $\boldsymbol{\lambda}_\rho$ and $\boldsymbol{\kappa}_\rho$, the $(M\times 1)$ vectors $\boldsymbol{\lambda}_\varsigma$ and $\boldsymbol{\kappa}_\varsigma$, the $(M\times M)$ matrix $\boldsymbol{\Lambda}_\rho$, the $(M\times M)$ matrix $\boldsymbol{\Lambda}_\varsigma$, the $(M\times M)$ matrix $\boldsymbol{\Lambda}_{\varsigma\rho}$ and the $(M\times M)$ matrix $\boldsymbol{\Lambda}_{\rho\varsigma}$ by the following relations:

$$\begin{bmatrix} \lambda_0 & \boldsymbol{\lambda}'_\rho & \boldsymbol{\lambda}'_\varsigma \\ \boldsymbol{\lambda}_\rho & \boldsymbol{\Lambda}_\rho & \boldsymbol{\Lambda}_{\rho\varsigma} \\ \boldsymbol{\lambda}_\varsigma & \boldsymbol{\Lambda}_{\varsigma\rho} & \boldsymbol{\Lambda}_\varsigma \end{bmatrix} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_1\boldsymbol{d}'_1); \begin{bmatrix} \kappa_0 \\ \boldsymbol{\kappa}_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix} = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_1 + \boldsymbol{d}_2) \tag{C.441}$$

By combining equations (C.437), (C.438), (C.439), (C.440), and (C.441) we find that

$$\begin{bmatrix} \lambda_0 & \boldsymbol{\lambda}'_\rho & \boldsymbol{\lambda}'_\varsigma \\ \boldsymbol{\lambda}_\rho & \boldsymbol{\Lambda}_\rho & \boldsymbol{\Lambda}_{\rho\varsigma} \\ \boldsymbol{\lambda}_\varsigma & \boldsymbol{\Lambda}_{\varsigma\rho} & \boldsymbol{\Lambda}_\varsigma \end{bmatrix} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_1\boldsymbol{d}'_1) = \lim_{T\to\infty} \mathrm{E}\left[\begin{bmatrix} \sigma_0 \\ \boldsymbol{d}_{1\rho} \\ \boldsymbol{d}_{1\varsigma} \end{bmatrix} [\sigma_0\ \boldsymbol{d}'_{1\rho}\ \boldsymbol{d}'_{1\varsigma}]\right]$$

$$= \lim_{T\to\infty} \mathrm{E}\begin{bmatrix} {\sigma_0}^2 & \sigma_0\boldsymbol{d}'_{1\rho} & \sigma_0\boldsymbol{d}'_{1\varsigma} \\ \sigma_0\boldsymbol{d}_{1\rho} & \boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\rho} & \boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\varsigma} \\ \sigma_0\boldsymbol{d}_{1\varsigma} & \boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\rho} & \boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma} \end{bmatrix} \tag{C.442}$$

which implies that

$$\lambda_0 = \lim_{T\to\infty} \mathrm{E}({\sigma_0}^2), \tag{C.443}$$

$$\boldsymbol{\lambda}_\rho = \lim_{T\to\infty} \mathrm{E}(\sigma_0\boldsymbol{d}_{1\rho}), \tag{C.444}$$

$$\boldsymbol{\lambda}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\sigma_0\boldsymbol{d}_{1\varsigma}), \tag{C.445}$$

$$\boldsymbol{\Lambda}_\rho = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\rho}), \tag{C.446}$$

$$\boldsymbol{\Lambda}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma}), \tag{C.447}$$

$$\boldsymbol{\Lambda}_{\varsigma\rho} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\rho}), \tag{C.448}$$

$$\boldsymbol{\Lambda}_{\rho\varsigma} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\varsigma}). \tag{C.449}$$

Obviously $\boldsymbol{\Lambda}_{\varsigma\rho} = \boldsymbol{\Lambda}'_{\rho\varsigma}$.

Similarly,

$$\begin{bmatrix} \kappa_0 \\ \boldsymbol{\kappa}_\rho \\ \boldsymbol{\kappa}_\varsigma \end{bmatrix} = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_1 + \boldsymbol{d}_2) = \lim_{T\to\infty} \mathrm{E}\begin{bmatrix} \sqrt{T}\sigma_0 + \sigma_1 \\ \sqrt{T}\boldsymbol{d}_{1\rho} + \boldsymbol{d}_{2\rho} \\ \sqrt{T}\boldsymbol{d}_{1\varsigma} + \boldsymbol{d}_{2\varsigma} \end{bmatrix} \tag{C.450}$$

which implies that

$$\kappa_0 = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\sigma_0 + \sigma_1), \tag{C.451}$$

$$\boldsymbol{\kappa}_\rho = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\rho} + \boldsymbol{d}_{2\rho}), \tag{C.452}$$

$$\boldsymbol{\kappa}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\varsigma} + \boldsymbol{d}_{2\varsigma}). \tag{C.453}$$

By using equations (C.119), the estimator $\hat{\boldsymbol{\varsigma}}_I$ (I=UL, RL, GL, IG, ML) of $\boldsymbol{\varsigma}$ is

$$\begin{aligned}\hat{\boldsymbol{\Sigma}}_I^{-1} &= (\hat{\boldsymbol{E}}_I'\hat{\boldsymbol{E}}_I/T)^{-1} = \boldsymbol{\Sigma}^{-1} - \tau \boldsymbol{S}_1 + \tau^2 \boldsymbol{S}_2{}^I + \omega(\tau^3) \Rightarrow \\ \hat{\boldsymbol{\varsigma}}_I &= [(\hat{\sigma}_I^{ii})_{ii=1,\dots,M}] \\ &= [(\sigma^{ii} - \tau \boldsymbol{s}_{ii}{}^1 + \tau^2 \boldsymbol{s}_{2ii}{}^I)_{ii=1,\dots,M}] \\ &= \boldsymbol{\varsigma} - \tau \boldsymbol{s}_1 + \tau^2 \boldsymbol{s}_2{}^I + \omega(\tau^3) \Rightarrow \end{aligned} \tag{C.454}$$

$$\begin{aligned}\boldsymbol{\delta}_\varsigma &= \frac{\hat{\boldsymbol{\varsigma}}_I - \boldsymbol{\varsigma}}{\tau} = -\boldsymbol{s}_1 + \tau \boldsymbol{s}_2{}^I + \omega(\tau^2) \\ &= \boldsymbol{d}_{1\varsigma} + \tau \boldsymbol{d}_{2\varsigma} + \omega(\tau^2), \end{aligned} \tag{C.455}$$

where

$$\boldsymbol{d}_{1\varsigma} = -\boldsymbol{s}_1 \text{ and } \boldsymbol{d}_{2\varsigma} = \boldsymbol{s}_2{}^I. \tag{C.456}$$

By using equations (C.143) and (C.432) we find that

$$\begin{aligned}\delta_0{}^I &= (\hat{\sigma}_I^2 - 1)/\tau = \hat{\sigma}_I^2/\tau - 1/\tau \\ &= [M + \tau^2 \operatorname{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}]]/(M - \tau^2 n)\tau - \frac{1}{\tau} + \omega(\tau^2) \\ &= [M/\tau + \tau \operatorname{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}]]/(M - \tau^2 n) - \frac{1}{\tau} + \omega(\tau^2). \end{aligned} \tag{C.457}$$

By using Lemma UR.1 we find that

$$\begin{aligned}1/(M - \tau^2 n) &= (M - \tau^2 n)^{-1} = [M(1 - \tau^2 n/M)]^{-1} = M^{-1}(1 - \tau^2 n/M)^{-1} \\ &= M^{-1}[1 + \tau^2 n/M + \omega(\tau^4)] = (1 + \tau^2 n/M)/M + \omega(\tau^4). \end{aligned} \tag{C.458}$$

Thus, equations (C.445) and (C.446) imply that

$$\begin{aligned}\delta_0{}^I &= [M/\tau + \tau \operatorname{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}]][(1 + \tau^2 n/M)/M + \omega(\tau^4)] - 1/\tau + \omega(\tau^2) \\ &= [1/\tau + \tau \operatorname{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}/M]](1 + \tau^2 n/M) - 1/\tau + \omega(\tau^2) \\ &= 1/\tau + \tau n/M + \tau \operatorname{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}]/M - 1/\tau + \omega(\tau^2) \\ &= \tau[\operatorname{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}] + n]/M + \omega(\tau^2). \end{aligned} \tag{C.459}$$

By combining equations (C.438) and (C.459) we find that

$$\sigma_0 = 0,\ \sigma_1 = [\operatorname{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}] + n]/M. \tag{C.460}$$

By using equations (C.443), (C.444), (C.445) and since $\sigma_0 = 0$ (see (C.460)) we find that

$$\lambda_0 = \lim_{T\to\infty} \mathrm{E}(\sigma_0{}^2) = 0, \tag{C.461}$$

$$\boldsymbol{\Lambda}_\rho = \lim_{T\to\infty} \mathrm{E}(\sigma_0 \boldsymbol{d}_{1\rho}) = 0, \tag{C.462}$$

$$\boldsymbol{\Lambda}_\varsigma = \lim_{T\to\infty} \mathrm{E}(\sigma_0 \boldsymbol{d}_{1\varsigma}) = 0. \tag{C.463}$$

Moreover, equations (C.167), (C.447), and (C.456) imply that

$$\begin{aligned}
\boldsymbol{\Lambda}_\varsigma &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\varsigma}\boldsymbol{d}'_{1\varsigma}) = \lim_{T\to\infty} \mathrm{E}[\boldsymbol{s}_1\boldsymbol{s}'_1] \\
&= \lim_{T\to\infty} \begin{bmatrix} 2\sigma_{11}{}^{-2} & 0 & \dots & 0 \\ 0 & 2\sigma_{22}{}^{-2} & & 0 \\ & & \ddots & \\ 0 & \dots & 0 & 2\sigma_{MM}{}^{-2} \end{bmatrix} \\
&= \boldsymbol{\Sigma}^{-2}.
\end{aligned} \tag{C.464}$$

By using equations (C.438) and (C.460) we find that

$$\begin{aligned}
\kappa_0 &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\sigma_0 + \sigma_1) = \lim_{T\to\infty} \mathrm{E}(\sigma_1) = \lim_{T\to\infty} \mathrm{E}[\mathrm{tr}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}] + n]/M \\
&= \mathrm{tr}[\lim_{T\to\infty} \mathrm{E}[(\boldsymbol{S}_2{}^I - \boldsymbol{S}_2{}^J)\boldsymbol{\Sigma}]]/M + n/M = [\text{see (C.149)}] \\
&= \mathrm{tr}[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Delta}_{GL} - \boldsymbol{\Delta}_I)]/M + n/M \ (\text{I=UL, RL, GL, IG, ML}),
\end{aligned} \tag{C.465}$$

where

$$\begin{aligned}
\boldsymbol{\Delta}_{UL} &= 0 \ [\text{see (C.184)}], \\
\boldsymbol{\Delta}_{RL} &= \Bigg[\Bigg(\delta_{q\kappa}\sigma_{qq}\,\mathrm{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{qq}\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{q\kappa}\Big] - \delta_{q\kappa}\sigma_{qq}\,\mathrm{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{q\kappa}\Big] \\
&\quad -\delta_{q\kappa}\sigma_{\kappa\kappa}\,\mathrm{tr}\Big[\Big(\sum_{p=1}^{M}\boldsymbol{B}_{pp}\Big)^{-1}\boldsymbol{B}_{\kappa q}\Big] + \delta_{q\kappa}\sigma_{qq}K\Bigg)_{k,q}\Bigg] \ [\text{see (C.203)}], \\
\boldsymbol{\Delta}_{GL} &= \boldsymbol{\Delta}_{IG} = \boldsymbol{\Delta}_{ML} = K\boldsymbol{\Sigma} - \Bigg[\Bigg(\mathrm{tr}\Big[\sum_{i=1}^{M}\sigma_{ii}\boldsymbol{B}_{ii}\Big]^{-1}\boldsymbol{B}_{ii}\Bigg)_{i,i}\Bigg] \ [\text{see (C.211)}].
\end{aligned} \tag{C.466}$$

Furthermore, equations (C.453) and (C.456) imply that

$$\begin{aligned}
\lim_{T\to\infty} \mathrm{E}[\sqrt{T}(-\boldsymbol{S}_1) + \boldsymbol{S}_2{}^I] &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{S}_2{}^I) = [\text{see (C.146)}] \\
&= (M + K + 1)\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_I\boldsymbol{\Sigma}^{-1}
\end{aligned} \tag{C.467}$$

$$\begin{aligned}
\boldsymbol{\kappa}_\varsigma &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}\boldsymbol{d}_{1\varsigma} + \boldsymbol{d}_{2\varsigma}) = \lim_{T\to\infty} \mathrm{E}(\sqrt{T}(-\boldsymbol{s}_1) + \boldsymbol{s}_2{}^I) \\
&= [\text{see (C.467)}] = [((M + K + 1)\sigma^{ii} - \sigma^{ii}\mathrm{d}_{ii}{}^I\sigma^{ii})_{i=1,\dots,M}],
\end{aligned} \tag{C.468}$$

where d_{ii}^{I} is the (ii)-th element of matrix $\boldsymbol{\Delta}_I$. $\boldsymbol{\Delta}_{UL}$, $\boldsymbol{\Delta}_{RL}$ and $\boldsymbol{\Delta}_{GL} = \boldsymbol{\Delta}_{IG} = \boldsymbol{\Delta}_{ML}$ have been defined in (C.466).

For the I estimator of ρ_μ (I=LS, GL, PW, ML,DW) equations (C.245), (C.246), (C.251), (C.254), and (C.259) imply that

$$\begin{aligned} d_{(1)\mu}{}^{LS} &= d_{(1)\mu}{}^{GL} = d_{(1)\mu}{}^{ML} = d_{(1)\mu}{}^{DW} = -\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2 \\ &= -\tau\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2 = d_{(1)\mu}. \end{aligned} \tag{C.469}$$

Therefore,

$$d_{(1)\mu}d'_{(1)\mu} = \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/4T\sigma_{\boldsymbol{u}_\mu}{}^4. \tag{C.470}$$

Moreover, for $\mu \neq \mu'$ we find that

$$d_{(1)\mu}d'_{(1)\mu'} = \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu \boldsymbol{u}'_{\mu'} \boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{u}_\mu/4T\sigma_{\boldsymbol{u}_\mu}{}^2\sigma_{\boldsymbol{u}_{\mu'}}{}^2. \tag{C.471}$$

Equations (C.446) and (C.470) imply that the μ-diagonal element of the matrix $\boldsymbol{\Lambda}_{\boldsymbol{\rho}}$ is

$$\begin{aligned} \lim_{T\to\infty} \mathrm{E}(d_{(1)\mu}d'_{(1)\mu}) &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu \boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu)/4T\sigma_{\boldsymbol{u}_\mu}{}^4 \\ &= [\text{see (C.380)}] = \lim_{T\to\infty}\left[\frac{4T\sigma_{\mu\mu}{}^2}{1-\rho_\mu{}^2} + O(1)\right]/4T\sigma_{\boldsymbol{u}_\mu}{}^4 \\ &= \lim_{T\to\infty}\left[\frac{4T\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)4T\sigma_{\boldsymbol{u}_\mu}{}^4} + O(T^{-1})\right] = \left(\text{since } \sigma_{\boldsymbol{u}_\mu}{}^2 = \frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}\right) \\ &= \frac{4T\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)4T\frac{\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2}} = (1-\rho_\mu{}^2). \end{aligned} \tag{C.472}$$

Similarly, equations (C.411) and (C.471) imply that since $\sigma_{\boldsymbol{u}_\mu}{}^2 = \frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}$, $\sigma_{\boldsymbol{u}_{\mu'}}{}^2 = \frac{\sigma_{\mu'\mu'}}{(1-\rho_{\mu'}{}^2)}$, for $\mu \neq \mu'$ the $\mu\mu'$-th off-diagonal element of the matrix $\boldsymbol{\Lambda}_{\boldsymbol{\rho}}$ is

$$\begin{aligned} \lim_{T\to\infty} \mathrm{E}(d_{(1)\mu}d'_{(1)\mu'}) &= \lim_{T\to\infty} \mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu \boldsymbol{u}'_{\mu'} \boldsymbol{R}_2{}^{\mu'\mu'}\boldsymbol{u}_{\mu'})/4T\sigma_{\boldsymbol{u}_\mu}{}^2\sigma_{\boldsymbol{u}_{\mu'}}{}^2 \\ &= \lim_{T\to\infty}\frac{4T\sigma_{\mu\mu}\sigma_{\mu'\mu'}}{(1-\rho_\mu\rho_{\mu'})4T\sigma_{\boldsymbol{u}_\mu}{}^2\sigma_{\boldsymbol{u}_{\mu'}}{}^2} + O(T^{-1}) \\ &= \frac{\sigma_{\mu\mu}\sigma_{\mu'\mu'}}{(1-\rho_\mu\rho_{\mu'})\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}\frac{\sigma_{\mu'\mu'}}{(1-\rho_{\mu'}{}^2)}} = \frac{(1-\rho_\mu{}^2)(1-\rho_{\mu'}{}^2)}{(1-\rho_\mu\rho_{\mu'})}. \end{aligned} \tag{C.473}$$

Moreover, for the J estimator of ρ_μ (I=LS, GL, PW, ML,DW) it holds that since

$$d_{(1)\mu} = d_{(1)\mu}{}^{LS} = d_{(1)\mu}{}^{GL} = d_{(1)\mu}{}^{ML} = d_{(1)\mu}{}^{DW} = -\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2, \tag{C.474}$$

the following results holds:

$$\mathrm{E}(\sqrt{T}d_{(1)\mu}) = -\mathrm{E}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu/2\sigma_{\boldsymbol{u}_\mu}{}^2) = \frac{-2\rho_\mu\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}/\frac{2\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)} = -\rho_\mu \Rightarrow$$

$$\lim_{T\to\infty} \mathrm{E}(\sqrt{T} d_{(1)\mu}) \quad = \quad -\rho_\mu. \tag{C.475}$$

By using equations (C.247), (C.382), and (C.383) we find that

$$\begin{aligned}
\mathrm{E}(d_{(2)\mu}{}^{LS}) &= -\mathrm{E}(\boldsymbol{u}_\mu' \bar{\boldsymbol{P}}_{\mathbf{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \bar{\boldsymbol{P}}_{\mathbf{X}_\mu} \boldsymbol{u}_\mu)/2\sigma_{\boldsymbol{u}_\mu}{}^2 + \mathrm{E}(\boldsymbol{u}_\mu' \boldsymbol{u}_\mu \boldsymbol{u}_\mu' \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{u}_\mu)/2T\sigma_{\boldsymbol{u}_\mu}{}^4 \\
&= -\frac{\sigma_{\mu\mu}}{2\rho_\mu}\left[\frac{2(\rho_\mu{}^2 - n(1-\rho_\mu{}^2))}{1-\rho_\mu{}^2}/\sigma_{\boldsymbol{u}_\mu}{}^2 + (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})/\sigma_{\boldsymbol{u}_\mu}{}^2 + \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})/\sigma_{\boldsymbol{u}_\mu}{}^2\right] \\
&\quad + O(T^{-1}) - \left[\frac{2T\rho_\mu\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2} + O(1)\right]/2T\sigma_{\boldsymbol{u}_\mu}{}^4 \\
&= -\frac{1}{2\rho_\mu}\left[\frac{2\rho_\mu{}^2\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)\sigma_{\boldsymbol{u}_\mu}{}^2} - \frac{2n\sigma_{\mu\mu}}{\sigma_{\boldsymbol{u}_\mu}{}^2} + (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\frac{\sigma_{\mu\mu}}{\sigma_{\boldsymbol{u}_\mu}{}^2}\right] \\
&\quad - \frac{1}{2\rho_\mu}\left[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\frac{\sigma_{\mu\mu}}{\sigma_{\boldsymbol{u}_\mu}{}^2} + \frac{2\rho_\mu^2\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2\sigma_{\boldsymbol{u}_\mu}{}^4}\right] + O(T^{-1}) \\
&= -\frac{1}{2\rho_\mu}\left[\frac{2\rho_\mu{}^2\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}} - \frac{2n\sigma_{\mu\mu}}{\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}} + (1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\frac{\sigma_{\mu\mu}}{\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}}\right] \\
&\quad - \frac{1}{2\rho_\mu}\left[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})\frac{\sigma_{\mu\mu}}{\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}} + \frac{2\rho_\mu^2\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2\frac{\sigma_{\mu\mu}{}^2}{(1-\rho_\mu{}^2)^2}}\right] + O(T^{-1}) \\
&= -\frac{1}{2\rho_\mu}[4\rho_\mu{}^2 - 2n(1-\rho_\mu{}^2) + (1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})(1-\rho_\mu{}^2)] \\
&\quad + O(T^{-1}).
\end{aligned} \tag{C.476}$$

By combining equations (C.452), (C.475), and (C.476) we find that

$$\begin{aligned}
\kappa_{\rho_\mu} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T} d_{1\mu} + d_{2\mu}{}^{LS}) \\
&= \lim_{T\to\infty}\left[-\frac{1}{2\rho_\mu}[4\rho_\mu{}^2 - 2n(1-\rho_\mu{}^2) + (1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})(1-\rho_\mu{}^2)] + O(T^{-1})\right] \\
&= -\frac{1}{2\rho_\mu}[6\rho_\mu{}^2 - 2n - 2n\rho_\mu{}^2 + (1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})(1-\rho_\mu{}^2)] \\
&= -\frac{1}{2\rho_\mu}[2\rho_\mu{}^2(3+n) - 2n + (1-\rho_\mu{}^2)((1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}))] \\
&= -[\rho_\mu(3+n) + (2n - c_1)/2\rho_\mu],
\end{aligned} \tag{C.477}$$

where

$$c_1 = (1-\rho_\mu{}^2)((1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu}) + \mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})). \tag{C.478}$$

By using equations (C.247) and (C.252) we find that

$$d_{(2)\mu}{}^{GL} = d_{(2)\mu}{}^{LS} - \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}_\mu' \bar{P}_{\mathbf{X}_\mu} \boldsymbol{R}_2{}^{\mu\mu} \boldsymbol{P}_{\mathbf{X}_\mu} \boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}_\mu' \boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]. \tag{C.479}$$

Therefore, equations (C.384) and (C.385) imply that

$$\begin{aligned}
&\mathrm{E}\left(-\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}'_\mu \bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]\right)\\
= & -\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\frac{\sigma_{\mu\mu}}{\rho_\mu}[n-\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]\\
& +\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\frac{\sigma_{\mu\mu}}{\rho_\mu}[n-\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2\\
& -\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\frac{\sigma_{\mu\mu}(1-\rho_\mu{}^2)}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1})-\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2+O(T^{-1})\\
= & -\frac{(1-\rho_\mu{}^2)}{\rho_\mu}[n-\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2-\frac{(1-\rho_\mu{}^2)^2}{\rho_\mu}[\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1})-\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2+O(T^{-1})\Rightarrow\\
&\lim_{T\to\infty}\mathrm{E}\left(-\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}'_\mu \bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]\right)\\
= & -\frac{(1-\rho_\mu{}^2)}{\rho_\mu}[n-\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})+(1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1})-(1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2\\
= & (1-\rho_\mu{}^2)[(1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})+\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{B}_{\mu\mu}\boldsymbol{F}_{\mu\mu}{}^{-1}\boldsymbol{\Theta}_{\mu\mu})]/2\rho_\mu\\
& -(1-\rho_\mu{}^2)^2\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1})/2\rho_\mu-(1-\rho_\mu{}^2)^2 n/2\rho_\mu\\
= & [c_1-(1-\rho_\mu{}^2)n]/2\rho_\mu-(1-\rho_\mu{}^2)c_2/2\rho_\mu, \qquad \text{(C.480)}
\end{aligned}$$

where

$$c_2=(1-\rho_\mu{}^2)\,\mathrm{tr}\,(\boldsymbol{F}_{\mu\mu}\boldsymbol{B}_{\mu\mu}^{-1}). \qquad \text{(C.481)}$$

Thus, from equations (C.475), (C.476), (C.477), (C.479), and (C.480) we find that

$$\begin{aligned}
\kappa_{\rho_\mu}{}^{GL} &= \kappa_{\rho_\mu}{}^{PW}=\lim_{T\to\infty}\mathrm{E}(\sqrt{T}d_{1\mu}+d_{2\mu}{}^{GL})=\lim_{T\to\infty}\mathrm{E}(\sqrt{T}d_{1\mu}+d_{2\mu}{}^{LS})\\
&\quad -\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}[\boldsymbol{u}'_\mu \bar{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu + \boldsymbol{u}'_\mu\boldsymbol{R}^{\mu\mu}\boldsymbol{V}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{P}_{\mathbf{X}_\mu}\boldsymbol{V}\boldsymbol{R}^{\mu\mu}\boldsymbol{u}_\mu/2]\\
&= \kappa_{\rho_\mu}{}^{LS}-(1-\rho_\mu{}^2)c_2/2\rho_\mu+[c_1-(1-\rho_\mu{}^2)n]/2\rho_\mu. \qquad \text{(C.482)}
\end{aligned}$$

By using equations (C.252) and (C.255) we find that

$$d_{(2)\mu}{}^{ML}=d_{(2)\mu}{}^{GL}+\rho_\mu\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}(u_{1\mu}{}^2+u_{T\mu}{}^2)-\rho_\mu. \qquad \text{(C.483)}$$

Since

$$\mathrm{E}(u_{1\mu}{}^2+u_{T\mu}{}^2)=\mathrm{E}(u_{1\mu}{}^2)+\mathrm{E}(u_{T\mu}{}^2)=\sigma_{\boldsymbol{u}_\mu}{}^2+\sigma_{\boldsymbol{u}_\mu}{}^2=2\sigma_{\boldsymbol{u}_\mu}{}^2=\frac{2\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}, \qquad \text{(C.484)}$$

we find that

$$\begin{aligned}
\mathrm{E}(d_{(2)\mu}{}^{ML}) &= \mathrm{E}(d_{(2)\mu}{}^{GL})+\rho_\mu\frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\frac{2\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}-\rho_\mu\\
&= \mathrm{E}(d_{(2)\mu}{}^{GL})+2\rho_\mu-\rho_\mu=\mathrm{E}(d_{(2)\mu}{}^{GL})+\rho_\mu. \qquad \text{(C.485)}
\end{aligned}$$

Thus, by combining equations (C.475), (C.482), (C.483) and (C.485) we find that

$$\begin{aligned}
\kappa_{\rho_\mu}{}^{ML} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}d_{1\mu} + d_{2\mu}{}^{ML}) = \lim_{T\to\infty} \mathrm{E}[(\sqrt{T}d_{1\mu} + d_{2\mu}{}^{GL}) + \rho_\mu] \\
&= \kappa_{\rho_\mu}{}^{GL} + \rho_\mu = \kappa_{\rho_\mu}{}^{PW} + \rho_\mu. \qquad (C.486)
\end{aligned}$$

By using equations (C.247) (C.260) we find that

$$\begin{aligned}
\mathrm{E}(d_{(2)\mu}{}^{DW}) &= \mathrm{E}(d_{(2)\mu}{}^{LS}) + \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}(\mathrm{E}(u_{1\mu}{}^2) + \mathrm{E}(u_{T\mu}{}^2))/2 = [\text{see (C.484)}] \\
&= \mathrm{E}(d_{(2)\mu}{}^{LS}) + \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}\frac{2\sigma_{\mu\mu}/2}{(1-\rho_\mu{}^2)} \\
&= \mathrm{E}(d_{(2)\mu}{}^{LS}) + 1. \qquad (C.487)
\end{aligned}$$

Thus, by combining equations (C.475), (C.477), and (C.487) we find that

$$\begin{aligned}
\kappa_{\rho_\mu}{}^{DW} &= \lim_{T\to\infty} \mathrm{E}(\sqrt{T}d_{1\mu} + d_{2\mu}{}^{DW}) = \lim_{T\to\infty} \mathrm{E}[(\sqrt{T}d_{1\mu} + d_{2\mu}{}^{LS}) + \frac{(1-\rho_\mu{}^2)}{\sigma_{\mu\mu}}(u_{1\mu}{}^2 + u_{T\mu}{}^2)/2] \\
&= \kappa_{\rho_\mu}{}^{LS} + 1. \qquad (C.488)
\end{aligned}$$

By using equations (C.440), (C.442), and (C.449) we define the $M \times M$ matrix $\boldsymbol{\Lambda}_{\rho\varsigma}$ as follows:

$$\boldsymbol{\Lambda}_{\rho\varsigma} = \lim_{T\to\infty} \mathrm{E}(\boldsymbol{d}_{1\rho}\boldsymbol{d}'_{1\varsigma}), \qquad (C.489)$$

where the μ-th element $M \times 1$ vector $\boldsymbol{d}_{1\rho}$ is

$$d_{(1)\mu} = -\frac{\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu}{2\sqrt{T}\sigma_{\boldsymbol{u}_\mu}{}^2} = -\frac{\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu}{2\sqrt{T}\frac{\sigma_{\mu\mu}}{(1-\rho_\mu{}^2)}} = -\frac{1-\rho_\mu{}^2}{2\sqrt{T}\sigma_{\mu\mu}}(\boldsymbol{u}'_\mu \boldsymbol{R}_2{}^{\mu\mu}\boldsymbol{u}_\mu) \qquad (C.490)$$

and the $1 \times M$ vector $\boldsymbol{d}'_{1\varsigma}$ is defined as

$$\boldsymbol{d}'_{1\varsigma} = [-\boldsymbol{s}_1]'. \qquad (C.491)$$

From equation (C.419) we have that the (ii)-th element of $\boldsymbol{d}'_{1\varsigma}$ is

$$-s^{(1)}_{(ii)} = -\sqrt{T}[\sigma^{ii}(\boldsymbol{\varepsilon}'_i\boldsymbol{\varepsilon}_i/T)\sigma^{ii} - \sigma^{ii}] \qquad (C.492)$$

with $(ii) = 1, \ldots, M$.

By combining equations (C.490) and (C.492) we find that the $(\mu, (ii))$-th element of the $(M \times M)$ matrix $\boldsymbol{\Lambda}_{\rho\varsigma}$ is

$$\begin{aligned} d_{(1)\mu}(-s^{(1)}_{(ii)}) &= \left[-\frac{1-\rho_{\mu}{}^{2}}{2\sqrt{T}\sigma_{\mu\mu}}(\boldsymbol{u}'_{\mu}\boldsymbol{R}_{2}{}^{\mu\mu}\boldsymbol{u}_{\mu})\right]\left[-\sqrt{T}[\sigma^{ii}(\boldsymbol{\varepsilon}'_{i}\boldsymbol{\varepsilon}_{i}/T)\sigma^{ii}-\sigma^{ii}]\right] \\ &= \frac{(1-\rho_{\mu}{}^{2})}{2\sigma_{\mu\mu}}\left[\sigma^{ii}(\boldsymbol{\varepsilon}'_{i}\boldsymbol{\varepsilon}_{i}/T)(\boldsymbol{u}'_{\mu}\boldsymbol{R}_{2}{}^{\mu\mu}\boldsymbol{u}_{\mu})-\sigma^{ii}(\boldsymbol{u}'_{\mu}\boldsymbol{R}_{2}{}^{\mu\mu}\boldsymbol{u}_{\mu})\right], \end{aligned} \tag{C.493}$$

which implies that by using equation (C.429) we find that

$$\begin{aligned} \mathrm{E}(d_{(1)\mu}(-s^{(1)}_{(ii)})) &= \frac{(1-\rho_{\mu}{}^{2})}{\sigma_{\mu\mu}}\left[\sigma^{ii}\,\mathrm{E}[(\boldsymbol{\varepsilon}'_{i}\boldsymbol{\varepsilon}_{i}/T)(\boldsymbol{u}'_{\mu}\boldsymbol{R}_{2}{}^{\mu\mu}\boldsymbol{u}_{\mu})]-\sigma^{ii}\,\mathrm{E}(\boldsymbol{u}'_{\mu}\boldsymbol{R}_{2}{}^{\mu\mu}\boldsymbol{u}_{\mu})\right] \\ &= O(T^{-1/2}) \Rightarrow \\ &\boldsymbol{\Lambda}_{\rho\varsigma} = \lim_{T\to\infty}\mathrm{E}(d_{(1)\mu}(-s^{(1)}_{(ii)})) = 0. \end{aligned} \tag{C.494}$$

Finally, we find that

$$\boldsymbol{\Lambda}_{\varsigma\rho} = \boldsymbol{\Lambda}'_{\rho\varsigma} = 0. \tag{C.495}$$

□

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