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Global in time smooth solutions for the Euler equation

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The present dissertation thesis was carried out under the postgraduate program of the Department of Mathematics of the University of Ioannina in order to obtain the master degree.

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Statutory Declaration

I lawfully declare here with statutorily that the present dissertation thesis was carried out under the international ethical and academical rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis.

Panagiota Papanikolaou

Αφιερώνεται στους ανθρώπους που ήταν δίπλα μου όλα αυτά τα χρόνια

Ευχαριστίες

Ολοκληρώνοντας τις μεταπτυχιακές μου σπουδές, και ύστερα απο την εκπόνηση της παρούσας μεταπτυχιακής διατριβής, αισθάνομαι την ανάγκη αλλα και την υποχρέωση να ευχαριστήσω τους ανθρωπους που συνέβαλαν στην επίτευξη των στόχων μου και με στήριξαν με κάθε δυνατό τρόπο.
Αρχικά θα ήθελα να ευχαριστήσω τον επιβλέποντα μου, Αναπληρωτή Καθηγητή κ. Ιωάννη Γιαννούλη, για την πολύτιμη βοήθεια του, την καθοδήγηση του, καθώς και για τις ιδιαίτερα επικοδομητικές συζήτησεις πάνω στο θέμα. Θα ήθελα επίσης να τον ευχαριστήσω για την επιλογή του θέματος, αφού κατά την εκπόνηση της διατριβής μου, εφοδιάστηκα με νέες γνώσεις, κάλυψα σε ένα μεγάλο μέρος κενά που μπορεί να υπήρχαν, ενώ ταυτόχρονα εισάχθηκα ομαλά στον «κόσμο» της εξίσωσης Euler. Θα ήθελα επίσης να ευχαριστήσω τον Καθηγητή κ. Ιωάννη Πουρναρά και τον Λέκτορα κ. Κυριάκο Μαυρίδη, για τις παρατηρήσεις τους και την συμβολή τους στην ολοκλήρωση αυτής της εργασίας.
Τέλος θα ήθελα να ευχαριστήσω την οικογένεια μου και τους συγγενείς μου που είναι δίπλα μου σε κάθε μου βήμα, με στηρίζουν και πιστεύουν σε εμένα. Τους φίλους μου, για την άοκνη συμπαράσταση και την υπομονή τους σε αυτό το απαιτητικό ταξίδι.

Περίληψη

Στο πρώτο κεφάλαιο της διατριβής σκοπός μας είναι να δούμε κάποια βασικά στοιχεία της μηχανικής των ρευστών καθώς και την εξαγωγή των εξισώσεων Euler και Navier-Stokes. Θα ασχοληθούμε επίσης με την τοπική ανάλυση του πεδίου ταχύτητας και θα γράψουμε τις εξισώσεις με τη χρήση πινάκων και θα εξαγάγουμε την εξίσωση του στροβιλισμού στις τρείς και στις δύο χωρικές διαστάσεις.

Στο δεύτερο κεφάλαιο θα δούμε δύο σημαντίκες ισοδύναμες διατυπώσεις των εξισώσεων Navier-Stokes και Euler. Την διατύπωση του Leray, διατύπωση, η οποία θα παίξει σημαντικό ρόλο στην απόδειξη της ύπαρξης λύσεων και τη διατύπωση στροβιλισμού-ροής, όπου θα εισάγουμε και τον νόμο Biot-Savart που συνδέει το πεδίο της ταχύτητας με το πεδίο στροβιλισμού του μέσω ενός ολοκληρωτικού τελεστή.

Στο τρίτο κεφάλαιο θα μιλήσουμε για κάποιες βασικές ιδιότητες των λύσεων, αν αυτές υπάρχουν, και θα βρούμε κάποιες βασικές οικογένειες λύσεων.

Στο τέταρτο και πέμπτο κεφάλαιο βρίσκονται τα βασικά αποτελέσματα της διατριβής. Στο τέταρτο κεφάλαιο αναφερόμαστε στην ύπαρξη λείων λύσεων τοπικά στον χρόνο, ενώ στο πέμπτο κεφάλαιο αποδεικνύουμε το κριτήριο των Beale-Kato-Majda και το εφαρμόζουμε για την επέκταση των λύσεων ολικά στον χρόνο στις δύο διαστάσεις.

Σε αυτή τη διατριβή δεν υπάρχει προκαταρκτικό κεφάλαιο ή παράρτημα καθώς οτιδήποτε χρησιμοποιούμε και χρήζει απόδειξης, θα αποδεικνύεται στο εκάστοτε κεφαλαιο.

Σημειώνουμε οτι η διατριβή έχει στηριχθεί κυρίως στο βιβλίο των A. Majda και A. Bertozzi, *Vorticity and Incompressible flow*, Cambridge University Press, 2002. Βλέπε [30]

Abstract

In the first chapter of this thesis our aim is to examine some basic concepts of fluid mechanics,and to derive the Euler and Navier-Stokes equations. We will consider also a local decomposition of the velocity field. Then, we will use matrices to write the equations and we will derive the vorticity equation for three and two spatial dimensions.

In the second chapter we will deal with two important formulations of the Navier-Stokes and the Euler equation, the formulation by Leray, which will play a crucial role in the proof of the existence of smooth solutions, and the vorticity-stream formulation, where we also introduce the Biot-Savart law which links the velocity field to its vorticity through an integral operator.

In the third chapter, we will present some properties of solutions, provided of course any solutions exist, and we will see some exact solutions to the equations.

In the fourth and fifth chapter we present the basic results of this thesis. In the fourth chapter we discuss the existence of smooth solutions locally in time, while in the fifth chapter we prove the well known Beale-Kato-Majda criterion and we apply it in order to extend the solutions globally in time in two dimensions.

The present thesis does not have preliminaries or an appendix, since everything we will use will be proven in each chapter.

We note that this thesis is mainly based on the book of A. Majda and A. Bertozzi, *Vorticity and Incompressible flow*, Cambridge University Press, 2002. See [30]

Contents

CHAPTER 1

Introduction

1.1 Description of fluids, the Eulerian and Lagrangian points of view

We begin with the following definition regarding fluids

Definition 1. *Fluid is a quantity which deforms under the action of a shear stress.*[1]

This deformation or movement of the fluid, which is time dependent, is called flow. If we assume that each particle of the fluid can be decomposed into smaller particles, then we can consider the fluid as a continuum. This assumption allows us to consider the existence of some physical quantities. Given a continuum, as the volume of particles tends to zero the physical quantities tend to become constant. The main idea is to examine the behavior of each particle and derive the results taking the average quantity. We note that we will exclude extreme (chaotic) behaviors.[2]
In order to continue the study of the description of fluids, we will define a mapping named particle trajectory mapping.
Let D be a region containing an incompressible, homogeneous fluid with velocity $u(x, t)$. Regarding the homogeneity, we say that a fluid is homogeneous if it has the same composition throughout its movement. The definition about incompressibility is more complicated. Schematically we see that (figure 1.1) a fluid is incompressible if its volume remains constant throughout its movement. In the next section we will discuss extensively the incompressibility property.

Assume a particle of the fluid with initial position $a = (a_1, a_2, ..., a_N)$. We should mention that the initial position is not necessarily for $t = 0$, but for convenience we will consider this case.
Define $\phi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ so that

$$(a, t) \to x$$

where x will be the position of the particle at time t (x(t)).
Furthermore we assume that ϕ is sufficiently smooth, and given the time and position of a particle, we are able to find the initial position by means of reverse function $\phi^{-1}(x, t) = a$

[1] [2] pg. 2
[2] [16]

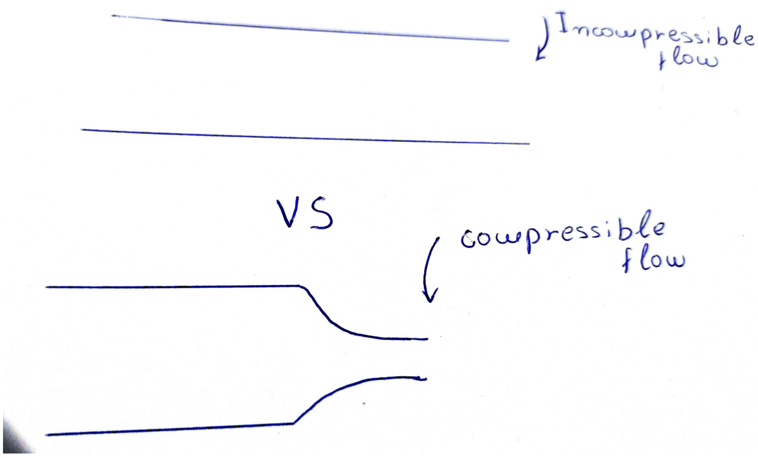

Figure 1.1:

Since the mapping is reversible, its Jacobian isn't equal to zero. So we have

$$J(a,t) = \begin{vmatrix} \frac{\partial \phi_1}{\partial a_1} & \frac{\partial \phi_1}{\partial a_2} & \frac{\partial \phi_1}{\partial a_N} \\ \frac{\partial \phi_2}{\partial a_1} & \frac{\partial \phi_2}{\partial a_2} & \frac{\partial \phi_2}{\partial a_N} \\ \vdots & & \\ \frac{\partial \phi_N}{\partial a_1} & \frac{\partial \phi_N}{\partial a_2} & \frac{\partial \phi_N}{\partial a_N} \end{vmatrix} \neq 0$$

This mapping allows us to examine the movement of a volume of a fluid, meaning that given a quantity of fluid on time $t = 0$, say Ω_o and using the above function, we get[3]

$$\Omega(t) = \phi(\Omega_o, t) = \{x = \phi(a,t) \text{ for } a \in \Omega_o\}.$$

Formally a is called material coordinate or Lagrangian particle marker which indicates what it actually describes i.e. a particular fluid particle. The $x = \phi(a,t)$ is called spatial coordinate and indicates a specific position on $\mathbb{R}^N$.

There are two points of view to describe the fluid's flow. The first one is the Lagrangian, in which we start with a specific particle that we follow throughout its movement and observe its evolution. In fact, we examine its motion path i.e. trajectory with velocity:

$$u(\phi(a,t),t) = \frac{\partial \phi(a,t)}{\partial t}$$

which will be tangential to the trajectory of the particle.

The second one is the Eulerian, in which we study the properties of the fluid in a specific position as a function of time. It is reasonable to wonder if there is a connection between them. The answer is positive and in order to understand this connection, it is enough to observe that the velocity in position x will be equal with the velocity of the particle i.e.

$$u(x,t) = \frac{\partial \phi(a,t)}{\partial t}$$

[3] [33],pg 26

We note here that we will follow the Eulerian description in our study. For the sake of completeness we will discuss some basic properties of the mapping ϕ

Proposition 1.1.1. *So be $\Omega_o \in \mathbb{R}^N$ and let ϕ be a particle trajectory mapping of a smooth velocity field, then*

$$\frac{\partial J}{\partial t} = \text{div}u(\phi(a,t),t)J(a,t), \forall a \in \Omega_o$$

Proof. For $N = 3$, let $a = (a_1, a_2, a_3)$
By the Jacobi formula[4] for the derivative of the determinant of a square matrix we have that

$$\frac{\partial detA(t)}{\partial t} = \text{tr}\left(adj(A(t))\frac{\partial A(t)}{\partial t}\right)$$

So in our case

$$\frac{\partial J}{\partial t} = \text{tr}\left[adj\begin{pmatrix} \frac{\partial \phi_1}{\partial a_1} & \frac{\partial \phi_1}{\partial a_2} & \frac{\partial \phi_1}{\partial a_3} \\ \frac{\partial \phi_2}{\partial a_1} & \frac{\partial \phi_2}{\partial a_2} & \frac{\partial \phi_2}{\partial a_3} \\ \frac{\partial \phi_3}{\partial a_1} & \frac{\partial \phi_3}{\partial a_2} & \frac{\partial \phi_3}{\partial a_3} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \phi_1}{\partial t \partial a_1} & \frac{\partial^2 \phi_1}{\partial t \partial a_2} & \frac{\partial^2 \phi_1}{\partial t \partial a_3} \\ \frac{\partial^2 \phi_2}{\partial t \partial a_1} & \frac{\partial^2 \phi_2}{\partial t \partial a_2} & \frac{\partial^2 \phi_2}{\partial t \partial a_3} \\ \frac{\partial^2 \phi_3}{\partial t \partial a_1} & \frac{\partial^2 \phi_3}{\partial t \partial a_2} & \frac{\partial^2 \phi_3}{\partial t \partial a_3} \end{pmatrix}\right]$$

We will see each matrix individually

$$adj\begin{pmatrix} \frac{\partial \phi_1}{\partial a_1} & \frac{\partial \phi_1}{\partial a_2} & \frac{\partial \phi_1}{\partial a_3} \\ \frac{\partial \phi_2}{\partial a_1} & \frac{\partial \phi_2}{\partial a_2} & \frac{\partial \phi_2}{\partial a_3} \\ \frac{\partial \phi_3}{\partial a_1} & \frac{\partial \phi_3}{\partial a_2} & \frac{\partial \phi_3}{\partial a_3} \end{pmatrix}$$

Name M_{ij} the elements of the adjugate matrix for convenience.
So we have $\begin{pmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}$

$$M_{11} = \frac{\partial \phi_2}{\partial a_2}\frac{\partial \phi_3}{\partial a_3} - \frac{\partial \phi_2}{\partial a_3}\frac{\partial \phi_3}{\partial a_2},$$

$$M_{12} = \frac{\partial \phi_2}{\partial a_3}\frac{\partial \phi_3}{\partial a_1} - \frac{\partial \phi_2}{\partial a_1}\frac{\partial \phi_3}{\partial a_3},$$

$$M_{13} = \frac{\partial \phi_2}{\partial a_1}\frac{\partial \phi_3}{\partial a_2} - \frac{\partial \phi_2}{\partial a_2}\frac{\partial \phi_3}{\partial a_1},$$

$$M_{21} = \frac{\partial \phi_1}{\partial a_3}\frac{\partial \phi_3}{\partial a_2} - \frac{\partial \phi_1}{\partial a_2}\frac{\partial \phi_3}{\partial a_3},$$

[4]proof:
Assume that the determinant of a matrix A is a function $F(A) = F(a_{11}, a_{12}, ..., a_{21}, a_{22}, ..., a_{nn})$ by the chain rule we have that $\frac{d}{dt}f = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial f}{\partial a_{ij}}\frac{\partial a_{ij}}{\partial t}$ we also know that if M_{ij} is the submatrix of a which follows by deleting the ith row and the jth column we have that $A_{ij} = (-1)^{i+j}\det M_{ij}$ and $\det A = \sum_{j=1}^{n} a_{ij}A_{ij}$. Also we see by product rule that $\frac{\partial}{\partial a_{ij}}\det A = \sum_{k=1}^{n}\left(a_{ik}\frac{\partial A_{ik}}{\partial a_{ij}} + A_{ik}\frac{\partial a_{ik}}{\partial a_{ij}}\right) = \sum_{k=1}^{n} A_{ik}\frac{\partial a_{ik}}{\partial a_{ij}}$ furthermore we have that $\frac{\partial a_{ik}}{\partial a_{ij}} = \delta_{kj}$ and thus $\frac{\partial \det A}{\partial t} = \sum_{i=1}^{n}\sum_{j=1}^{n}\det M_{ij}\frac{da_{ij}}{dt} = \text{tr}(adjA\frac{\partial A(t)}{\partial t})$

$$M_{22} = \frac{\partial \phi_1}{\partial a_1}\frac{\partial \phi_3}{\partial a_3} - \frac{\partial \phi_1}{\partial a_3}\frac{\partial \phi_3}{\partial a_1},$$
$$M_{23} = \frac{\partial \phi_1}{\partial a_2}\frac{\partial \phi_3}{\partial a_1} - \frac{\partial \phi_1}{\partial a_1}\frac{\partial \phi_3}{\partial a_2},$$
$$M_{31} = \frac{\partial \phi_1}{\partial a_2}\frac{\partial \phi_2}{\partial a_3} - \frac{\partial \phi_1}{\partial a_3}\frac{\partial \phi_2}{\partial a_2},$$
$$M_{32} = \frac{\partial \phi_1}{\partial a_3}\frac{\partial \phi_2}{\partial a_1} - \frac{\partial \phi_1}{\partial a_1}\frac{\partial \phi_2}{\partial a_3},$$
$$M_{33} = \frac{\partial \phi_1}{\partial a_1}\frac{\partial \phi_2}{\partial a_2} - \frac{\partial \phi_1}{\partial a_2}\frac{\partial \phi_2}{\partial a_1}$$

Now

$$\frac{\partial J}{\partial t} = \begin{pmatrix} \frac{\partial^2 \phi_1}{\partial t \partial a_1} & \frac{\partial^2 \phi_1}{\partial t \partial a_2} & \frac{\partial^2 \phi_1}{\partial t \partial a_3} \\ \frac{\partial^2 \phi_2}{\partial t \partial a_1} & \frac{\partial^2 \phi_2}{\partial t \partial a_2} & \frac{\partial^2 \phi_2}{\partial t \partial a_3} \\ \frac{\partial^2 \phi_3}{\partial t \partial a_1} & \frac{\partial^2 \phi_3}{\partial t \partial a_2} & \frac{\partial^2 \phi_3}{\partial t \partial a_3} \end{pmatrix}$$

So we conclude $trB(t) = b_{11} + b_{22} + b_{33}$, where

$$b_{11} = M_{11}\frac{\partial^2 \phi_1}{\partial t \partial a_1} + M_{21}\frac{\partial^2 \phi_2}{\partial t \partial a_1} + M_{31}\frac{\partial^2 \phi_3}{\partial t \partial a_1},$$
$$b_{22} = M_{12}\frac{\partial^2 \phi_1}{\partial t \partial a_2} + M_{22}\frac{\partial^2 \phi_2}{\partial t \partial a_2} + M_{32}\frac{\partial^2 \phi_3}{\partial t \partial a_2},$$
$$b_{33} = M_{13}\frac{\partial^2 \phi_1}{\partial t \partial a_3} + M_{23}\frac{\partial^2 \phi_2}{\partial t \partial a_3} + M_{33}\frac{\partial^2 \phi_3}{\partial t \partial a_3}$$

I.e.

$$\frac{\partial J}{\partial t} = \sum_{i,j} M_{ij}\frac{\partial^2 \phi_i}{\partial t \partial a_j} \Rightarrow$$
$$\frac{\partial J}{\partial t} = \sum_{i,j} M_{ij}\frac{\partial}{\partial a_j}u_i(a,t) \overset{\text{chain rule}}{\Rightarrow}$$
$$\frac{\partial J}{\partial t} = \sum_{i,j,k} M_{ij}\frac{\partial x_k}{\partial a_j}\frac{\partial u_i}{\partial x_k}$$

Now, we shall prove that $\sum_j M_{ij}\frac{\partial x_k}{\partial a_j} = \delta_{ik}J$, where δ_{ik} is Kronecker delta . Indeed, let's see this for $i = 1$ and the other follows. So $\sum_j M_{1j}\frac{\partial x_k}{\partial a_j}$

- $k = 1$, $\sum_j M_{1j}\frac{\partial x_1}{\partial a_j} =$

$$\frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_2}\frac{\partial x_3}{\partial a_3} - \frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_3}\frac{\partial x_3}{\partial a_2} - \frac{\partial x_1}{\partial a_2}\frac{\partial x_2}{\partial a_1}\frac{\partial x_3}{\partial a_3} + \frac{\partial x_1}{\partial a_2}\frac{\partial x_2}{\partial a_3}\frac{\partial x_3}{\partial a_1} + \frac{\partial x_1}{\partial a_3}\frac{\partial x_2}{\partial a_1}\frac{\partial x_3}{\partial a_2} - \frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_2}\frac{\partial x_3}{\partial a_3} = J$$

- $k = 2$, $\sum_j M_{1j}\frac{\partial x_2}{\partial a_j} =$

$$\frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_2}\frac{\partial x_3}{\partial a_3} - \frac{\partial x_2}{\partial a_1}\frac{\partial x_2}{\partial a_3}\frac{\partial x_3}{\partial a_2} - \frac{\partial x_2}{\partial a_2}\frac{\partial x_2}{\partial a_1}\frac{\partial x_3}{\partial a_3} + \frac{\partial x_2}{\partial a_2}\frac{\partial x_2}{\partial a_3}\frac{\partial x_3}{\partial a_1} + \frac{\partial x_2}{\partial a_3}\frac{\partial x_2}{\partial a_1}\frac{\partial x_3}{\partial a_2} - \frac{\partial x_2}{\partial a_3}\frac{\partial x_2}{\partial a_2}\frac{\partial x_3}{\partial a_1} = 0$$

- $k=3$, $\sum_j M_{1j}\frac{\partial x_3}{\partial a_j} =$

$$\frac{\partial x_3}{\partial a_1}\frac{\partial x_2}{\partial a_2}\frac{\partial x_3}{\partial a_3}-\frac{\partial x_3}{\partial a_1}\frac{\partial x_2}{\partial a_3}\frac{\partial x_3}{\partial a_2}-\frac{\partial x_3}{\partial a_2}\frac{\partial x_2}{\partial a_1}\frac{\partial x_3}{\partial a_3}+\frac{\partial x_3}{\partial a_2}\frac{\partial x_2}{\partial a_3}\frac{\partial x_3}{\partial a_1}+\frac{\partial x_3}{\partial a_3}\frac{\partial x_2}{\partial a_1}\frac{\partial x_3}{\partial a_2}-\frac{\partial x_3}{\partial a_3}\frac{\partial x_2}{\partial a_2}\frac{\partial x_3}{\partial a_1}=0$$

At the end

$$\frac{\partial J}{\partial t}=\sum_{i,j,k}\frac{\partial u_i}{\partial x_k}\delta_{ik}J=\sum_{i,k}\frac{\partial u_i}{\partial x_k}\delta_{ik}J=\sum_{i=k}\frac{\partial u_i}{\partial x_k}J=(\sum_{i}\frac{\partial u_i}{\partial x_k})J=\text{div}uJ$$

And for $N=2$ and let $a=(a_1,a_2)$ for the derivative of a determinant of a square matrix by Jacobi's formula we obtain

$$\frac{\partial detA(t)}{\partial t}=\text{tr}(adj(A(t))\frac{\partial A(t)}{\partial t})$$

So in our case

$$\frac{\partial J}{\partial t}=\text{tr}\left[adj\begin{pmatrix}\frac{\partial\phi_1}{\partial a_1} & \frac{\partial\phi_1}{\partial a_2}\\ \frac{\partial\phi_2}{\partial a_1} & \frac{\partial\phi_2}{\partial a_2}\end{pmatrix}\begin{pmatrix}\frac{\partial^2\phi_1}{\partial t\partial a_1} & \frac{\partial^2\phi_1}{\partial t\partial a_2}\\ \frac{\partial^2\phi_2}{\partial t\partial a_1} & \frac{\partial^2\phi_2}{\partial t\partial a_2}\end{pmatrix}\right]$$

We will see each matrix individually

$$adj\begin{pmatrix}\frac{\partial\phi_1}{\partial a_1} & \frac{\partial\phi_1}{\partial a_2}\\ \frac{\partial\phi_2}{\partial a_1} & \frac{\partial\phi_2}{\partial a_2}\end{pmatrix}$$

Name M_{ij} the elements of the adjugate matrix for convenience.
So we have the matrix $\begin{pmatrix}M_{11} & M_{21}\\ M_{12} & M_{22}\end{pmatrix}$

$$M_{11}=\frac{\partial\phi_2}{\partial a_2},$$

$$M_{12}=-\frac{\partial\phi_2}{\partial a_1},$$

$$M_{21}=-\frac{\partial\phi_1}{\partial a_2},$$

$$M_{22}=\frac{\partial\phi_1}{\partial a_1}$$

Now

$$\frac{\partial J}{\partial t}=\begin{pmatrix}\frac{\partial^2\phi_1}{\partial t\partial a_1} & \frac{\partial^2\phi_1}{\partial t\partial a_2}\\ \frac{\partial^2\phi_2}{\partial t\partial a_1} & \frac{\partial^2\phi_2}{\partial t\partial a_2}\end{pmatrix}$$

So we conclude $trB(t)=b_{11}+b_{22}$ where

$$b_{11}=M_{11}\frac{\partial^2\phi_1}{\partial t\partial a_1}+M_{21}\frac{\partial^2\phi_2}{\partial t\partial a_1},$$

$$b_{22}=M_{12}\frac{\partial^2\phi_1}{\partial t\partial a_2}+M_{22}\frac{\partial^2\phi_2}{\partial t\partial a_2}$$

I.e.

$$\frac{\partial J}{\partial t} = \sum_{i,j} M_{ij} \frac{\partial^2 x_i}{\partial t \partial a_j} \Rightarrow$$

$$\frac{\partial J}{\partial t} = \sum_{i,j} M_{ij} \frac{\partial}{\partial a_j} u_i \overset{\text{chain rule}}{\Rightarrow}$$

$$\frac{\partial J}{\partial t} = \sum_{i,j,k} M_{ij} \frac{\partial x_k}{\partial a_j} \frac{\partial u_i}{\partial x_k}$$

Now, we shall prove that $\sum_j M_{ij} \frac{\partial x_k}{\partial a_j} = \delta_{ik} J$, where δ_{ik} is Kronecker delta. Indeed, lets see this for $i = 1$ so $\sum_j M_{1j} \frac{\partial x_k}{\partial a_j}$

- k=1, $\sum_j M_{1j} \frac{\partial x_1}{\partial a_j} = M_{11} \frac{\partial x_1}{\partial a_1} + M_{12} \frac{\partial x_1}{\partial a_2}$

$$\frac{\partial x_2}{\partial a_2} \frac{\partial x_1}{\partial a_1} - \frac{\partial x_2}{\partial a_1} \frac{\partial x_1}{\partial a_2} = J$$

- k=2, $\sum_j M_{1j} \frac{\partial x_2}{\partial a_j} = M_{11} \frac{\partial x_2}{\partial a_1} + M_{12} \frac{\partial x_2}{\partial a_2}$

$$\frac{\partial x_2}{\partial a_2} \frac{\partial x_2}{\partial a_1} - \frac{\partial x_2}{\partial a_1} \frac{\partial x_2}{\partial a_2} = 0$$

for $i = 2$ we have $\sum_j M_{2j} \frac{\partial x_k}{\partial a_j}$ so

- k=1, $\sum_j M_{2j} \frac{\partial x_1}{\partial a_j} = M_{21} \frac{\partial x_1}{\partial a_1} + M_{22} \frac{\partial x_1}{\partial a_2}$

$$-\frac{\partial x_1}{\partial a_2} \frac{\partial x_1}{\partial a_1} + \frac{\partial x_1}{\partial a_1} \frac{\partial x_1}{\partial a_2} = 0$$

- k=2, $\sum_j M_{2j} \frac{\partial x_2}{\partial a_j} = M_{21} \frac{\partial x_2}{\partial a_1} + M_{22} \frac{\partial x_2}{\partial a_2}$

$$-\frac{\partial x_1}{\partial a_2} \frac{\partial x_2}{\partial a_1} + \frac{\partial x_1}{\partial a_1} \frac{\partial x_2}{\partial a_2} = J$$

At the end

$$\frac{\partial J}{\partial t} = \sum_{i,j,k} \frac{\partial u_i}{\partial x_k} \delta_{ik} J = \sum_{i,k} \frac{\partial u_i}{\partial x_k} \delta_{ik} J = \sum_{i=k} \frac{\partial u_i}{\partial x_k} J = (\sum_i \frac{\partial u_i}{\partial x_k}) J = \mathrm{div} u J$$

□

Proposition 1.1.2. *(Transport formula) Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with smooth boundary. Let ϕ be a particle trajectory mapping of a smooth vector field u. Then for every smooth function $f(x,t)$ we get:*

$$\frac{d}{dt} \int_{\phi(\Omega,t)} f dx = \int_{\phi(\Omega,t)} \frac{\partial f}{\partial t} + \mathrm{div}(fu) dx$$

Proof. We set $x = \phi(a,t)$ We have already seen the Jacobian of this mapping so we change variables and we get

$$\frac{d}{dt}\int_{\phi(\Omega,t)} f dx = \frac{d}{dt}\int_{\Omega} f(\phi(a,t),t)J(a,t)da$$

By Leibniz integral rule we get

$$\frac{d}{dt}\int_{\phi(\Omega,t)} f dx = \int_{\Omega} \frac{\partial}{\partial t}(f(\phi(a,t),t)J(a,t))da$$

By the product rule

$$\frac{d}{dt}\int_{\phi(\Omega,t)} f dx = \int_{\Omega} \left(\frac{\partial}{\partial t} f(\phi(a,t),t)\right) J(a,t) + f(\phi(a,t),t)\frac{\partial J}{\partial t} \Rightarrow$$

$$\frac{d}{dt}\int_{\phi(\Omega,t)} f dx = \int_{\Omega} \nabla f(\phi(a,t),t) \begin{pmatrix} \frac{\partial \phi_1}{\partial t} \\ \frac{\partial \phi_2}{\partial t} \\ \frac{\partial \phi_3}{\partial t} \\ 1 \end{pmatrix} J + f\frac{\partial J}{\partial t} da$$

By the Proposition 1.1.1 we get :

$$\frac{d}{dt}\int_{\phi(\Omega,t)} f dx = \int_{\Omega} \left(\nabla f \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ 1 \end{pmatrix} + f(\nabla \cdot u)\right) J da$$

$$\frac{d}{dt}\int_{\phi(\Omega,t)} f dx = \int_{\Omega} \left(\frac{\partial f}{\partial t} + \nabla \cdot (fu)\right) J da$$

We change variables again and we conclude :

$$\frac{d}{dt}\int_{\phi(\Omega,t)} f dx = \int_{\Omega} \frac{\partial f}{\partial t} + \mathrm{div} f u dx$$

□

1.2 Derivation of the Navier-Stokes and Euler equations

In this section our aim is to derive the equations of motion for an incompressible and homogeneous fluid. We will see in details the 3d case, the 2d case is similar. Assume a flow and let x be a particle of the fluid, where $\vec{n}$ is the outward normal vector and W a neighborhood of the particle.
Furthermore, let us assume that we deal with incompressible flows and homogeneous fluids. We have already see in the previous section the definitions of those concepts. Concerning the incompressibility there are two points of view. Some refer to incompressibility assuming that the density remains constant, while others claim that the material-convective derivative of density remains constant. Below we will check that in our case we can work with both assumptions.

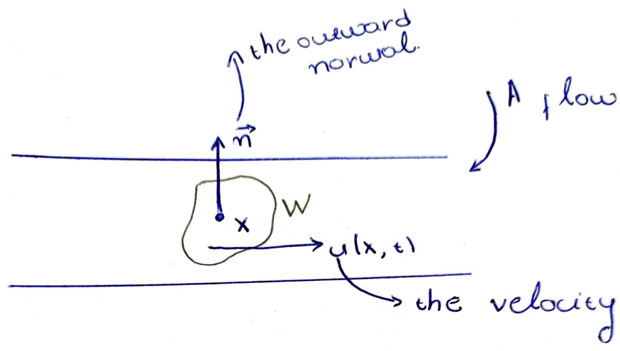

Figure 1.2:

Definition 2. *The convective derivative of a quantity includes spatial and temporal information about motion changes.*
$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{N} u_j \frac{\partial}{\partial x_j}$
It is the derivative taken with respect to a mooing coordinate system,and describes the deform of a fluid under some quantities (such as velocity).

It is known that the movement of objects obeys three basic principles

1. Conservation of mass
2. Conservation of momentum
3. Conservation of energy

We note that we will not deal with the third principle. We begin this investigation with the first principle.
1.Conservation of mass:

$$\frac{d}{dt}\int_W \rho \, dV = -\int_{\vartheta W} \rho u n \, dA$$

By the Leibniz integral rule and Gauss-Green theorem[5]

$$\int_W \frac{\partial \rho}{\partial t} dV = -\int_W \text{div} \rho u \, dV \Rightarrow$$

$$\int_W \frac{\partial \rho}{\partial t} + \text{div} \rho u \, dV = 0$$

But W is a random neighborhood of x, so

$$\frac{\partial \rho}{\partial t} + \text{div} \rho u = 0 \tag{CM}$$

Following the first assumption about incompressibility we have that ρ is constant. So we conclude by (CM) that

$$\text{div} u = 0$$

[5][18] pg 712

. Following the second assumption we have that

$$\frac{D}{Dt}\rho = 0$$

i.e.

$$\frac{\partial}{\partial t}\rho = -\nabla\rho \cdot u$$

. So we conclude by (CM) that $\rho \text{div} u = 0$ and eventually

$$\text{div} u = 0$$

Thus, we notice that the incompressibility together with the conservation of mass gives us the condition that $\text{div} u = 0$
2. Conservation of momentum: Newtons's second law of motion is

$$F = ma \tag{SeL}$$

If we divide with the volume of the fluid, we get

$$\frac{F}{V} = \rho a \tag{SeLV}$$

Where $V = dx_1 dx_2 dx_3$ and a is the acceleration.
Analyzing in each component $\frac{F_{x_i}}{V} = \rho a_{x_i}$ For the velocity field $u = (u_1, u_2, u_3)$ we have

$$a_{x_i} = \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3}$$

So $a = \frac{d}{dt}u$

$$a = \frac{\partial u}{\partial t} + u_1 \frac{\partial u}{\partial x_1} + u_2 \frac{\partial u}{\partial x_2} + u_3 \frac{\partial u}{\partial x_3} \Rightarrow$$

$$a = \frac{\partial u}{\partial t} + \sum_{j}^{N} u_j \frac{\partial u}{\partial x_j} \overset{\text{Def } 2}{\Rightarrow}$$

$$a = \frac{Du}{Dt}$$

There are two kinds of forces exerted on the fluid.

1. The external forces
2. The internal forces

We set F the sum of external and internal forces.
We will not deal with body forces, which can be gravity, magnetic field forces, etc. We will denote these forces as $f_i(\vec{x}, t), i = 1...N$ Assume also an elementary particle of the fluid with a shape of a cube For the internal forces, we denote as $\sigma_{x_i x_j}$ outward pointing stresses named normal stresses and $\tau_{x_i x_j}$ the tangential to the surface stresses named shear stresses (see figure 1. 3). x_i, x_j shows the direction of the stress, meaning that

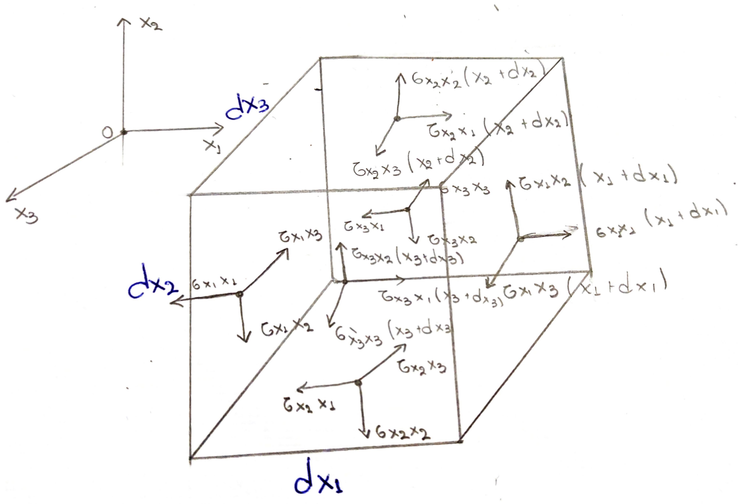

Figure 1.3:

the $stress_{ij}$ is perpendicular to the direction of x_i and parallel to the direction of x_j [6]. In order to find the inner forces, we have to find the resultant force per unit volume. So in the direction of x_1 we have that the force is

$$f_{x_1} = \sigma_{x_1x_1}(x_1 + dx_1)dx_2dx_3 - \sigma_{x_1x_1}(x_1)dx_2dx_3 +$$

$$\tau_{x_2x_1}(x_2 + dx_2)dx_1dx_3 - \tau_{x_2x_1}(x_2)dx_1dx_3 +$$

$$\tau_{x_3x_1}(x_3 + dx_3)dx_1dx_2 - \tau_{x_3x_1}(x_3)dx_1dx_2$$

So per unit volume we have

$$f_{x_1} = \frac{\sigma_{x_1x_1}(x_1 + dx_1) - \sigma_{x_1x_1}(x_1)}{dx_1} +$$

$$\frac{\tau_{x_2x_1}(x_2 + dx_2) - \tau_{x_2x_1}(x_2)}{dx_2} +$$

$$\frac{\tau_{x_3x_1}(x_3 + dx_3) - \tau_{x_3x_1}(x_3)}{dx_3}$$

$$f_{x_1} = \frac{\partial \sigma_{x_1x_1}}{\partial x_1} + \frac{\partial \tau_{x_2x_1}}{\partial x_2} + \frac{\partial \tau_{x_3x_1}}{\partial x_3}$$

In the same way we can find the internal force in the direction of x_2, x_3 $F_{x_1} = f_1 + \frac{\partial \sigma_{x_1x_1}}{\partial x_1} + \frac{\partial \tau_{x_1x_2}}{\partial x_2} + \frac{\partial \tau_{x_1x_3}}{\partial x_3}$. This way we also get the rest of forces in the direction of x_2, x_3

We denote $\bar{T}$ the stresses tensor,which is a 3x3 matrix.

$$\bar{T} = \begin{pmatrix} \sigma_{x_1x_1} & \tau_{x_1x_2} & \tau_{x_1x_3} \\ \tau_{x_2x_1} & \sigma_{x_2x_2} & \tau_{x_2x_3} \\ \tau_{x_3x_1} & \tau_{x_3x_2} & \sigma_{x_3x_3} \end{pmatrix}$$

[6][41]

Consequently the total force is $F = f_j + \sum_i^N \nabla \cdot T_{ij}$, where T_{ij} are the elements of the matrix
Substituting all these into (SeLV) we derive

$$\rho(\frac{\partial u_i}{\partial t} + \sum_j^N u_j \frac{\partial u_i}{\partial x_j}) = f_i + \sum_j^N \frac{\partial t_{ij}}{\partial x_j}$$

We can view t_{ij} component as the total stress, which is the sum of normal and shear stress. Due to the fact that its complicated to solve this equation, since we have to compute each normal or shear stress, we will express t_{ij} as functions of pressure (p), viscosity (μ) and velocity (u).
Normal stresses are the result of the pressure exerted of the fluid so $\sigma_{ij} = -p\delta_{ij}$. Shear stresses tend to cause deformation of the fluid by slippage along a plane. We will use Newtons law for viscosity. In the two dimension case we have

$$\tau_{x_j} = \mu \frac{\partial u_j}{\partial x_i}$$

the generalization of this form in three dimensions will not be presented intricately [7]. It is sufficient to observe that internal friction exists when fluid particles move, so $\tau_{x_i x_j}$ should depends on the space derivatives of velocity. So $\tau_{x_i x_j}$ may assumed as a linear function of the derivatives $\frac{\partial u_i}{\partial x_j}$, and only those, since the shear stress must vanish for constant velocity. Also must vanish when the fluid rotates uniformly, since no viscous forces are exerted in this case. We conclude that $\tau_{x_i x_j} = \mu(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})$. So, for

$$t_{ij} = \sigma_{ij} + \tau_{ij}$$

by replacing the above relations we get :

$$t_{ij} = p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Therefore

$$\rho\left(\frac{\partial u_i}{\partial t} + \sum_j^N u_j \frac{\partial u_i}{\partial x_j}\right) = f_i + \sum_j^3 \frac{\partial}{\partial x_j}\left(p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\right)$$

But the fluid is incompressible so $\text{div} u = 0$ and thus

$$\rho(\frac{\partial u_i}{\partial t} + \sum_j^3 u_j \frac{\partial u_i}{\partial x_j}) = f_i - \frac{\partial p}{\partial x_i} + \mu \sum_j^3 \frac{\partial^2 u_i}{\partial x_j^2}$$

Finally

$$\left(\frac{\partial u_i}{\partial t} + \sum_j^3 u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{1}{\rho} f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta u_i$$

Where ν is the kinematic viscosity . The kinematic viscosity $\nu = \frac{\mu}{\rho}$ expresses the speed of response of the fluid in relation to its tension to remain inert. Kinematic viscosity can be viewed as the reciprocal of Reynold's number

[7] see [12],section 1.2

Definition 3. *Reynold's number is the ratio of inertial forces to viscous forces. It describes if the flow is laminar or turbulent.* $Re = \frac{\rho u L}{\mu}$ *where L is the length of the flow.*

For the 2d case we have that our particle is a square and we have four stresses (plus 4 opposite stresses). We follow the same procedure as above. So assuming that the fluid is ideal, and no body forces exerted, we conclude that incompressible flows of homogeneous fluids in $\mathbb{R}^N$,$N = 2, 3$, are solutions of the system

$$\frac{Du}{Dt} = -\frac{1}{\rho}\nabla p + \nu\Delta u$$
$$\mathrm{div} u = 0$$
$$u(x, 0) = u_0$$

with $u = (u^1, ..., u^N)$ the velocity, and p the scalar pressure For $\nu > 0$ the equation is called Navier-Stokes, for $\nu = 0$ it is known as the Euler equation.
To summarize, if we examine these equations as a mathematical object it is a non linear time dependent system of partial differential equations. The unknown variables are the components of velocity and the pressure. As a physics interest object is we will see each term individually.

- $\frac{Du}{Dt}$ is the total acceleration where $\frac{\partial u}{\partial t}$ is the local acceleration and $\sum_j u_j \frac{\partial u}{\partial x_j}$ is the convective acceleration [8]
- ∇p are the pressure forces
- $\nu\Delta u$ are the viscous forces

We also have the following proposition which gives equivalent concepts for the incompressibility:

Proposition 1.2.1. *The following results are equivalent*

1. *A flow is incompressible*
2. $\mathrm{div} u = 0$
3. $J(a, t) = 1$

Proof. ($1 \implies 2$) it has been proved
($2 \implies 3$) By proposition 1.1.2

$$\frac{\vartheta J}{\vartheta t} = \mathrm{div} u J(a, t)$$
$$\frac{\partial J}{\partial t} = 0J(a, t)$$

[8]Convective acceleration is the acceleration due to movement of the fluid particle to a different position in the flow field

$$\frac{\partial J}{\partial t} = 0$$
$$J = c$$
$$J(a,0) = det(\nabla_a a) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

So $J(a,t) = J(a,0) = 1$
$(3 \implies 1)$

$$vol\phi(\Omega,t) = \int_{\phi(\Omega,t)} dx = \int_{\Omega} J(a,t)da = \int_{\Omega} da = vol\Omega$$

□

1.3 Local behavior of the velocity field

Now we will examine the behavior of the velocity around a specific position (x_0, t_0). Assuming that the velocity field is smooth, we will use Taylor's theorem to expand the velocity field around (x_0, t_0).
Let $h \in \mathbb{R}^3$, then $u(x_0+h, t_0) = u(x_0, t_0) + \nabla u(x_0, t_0)h + O(h^2)$, where O is the Landau big O[9] .
∇u is a 3×3 matrix i.e. $\nabla u = [\frac{\partial u_i}{\partial x_j}]_{ij}$.
So $\nabla u + (\nabla u)^T$ is a symmetric matrix, indeed

$$(\nabla u + (\nabla u)^T)^T = (\nabla u)^T + ((\nabla u)^T)^T = (\nabla u)^T + \nabla u = \nabla u + (\nabla u)^T$$

And $\nabla u - (\nabla u)^T$ is a skew symmetric matrix, indeed

$$(\nabla u - (\nabla u)^T)^T = (\nabla u)^T - ((\nabla u)^T)^T = (\nabla u)^T - \nabla u = -(\nabla u + (\nabla u)^T)$$

We add those matrices and we get

$$(\nabla u + (\nabla u)^T) + (\nabla u - (\nabla u)^T) = 2\nabla u$$

i.e

$$\nabla u = \frac{1}{2}(\nabla u + (\nabla u)^T) + \frac{1}{2}(\nabla u - (\nabla u)^T)$$

We have proved that ∇u is sum of a symmetric and a skew symmetric matrix. We define $D = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and $\Omega = \frac{1}{2}(\nabla u - (\nabla u)^T)$ and we will name D, deformation matrix, and Ω rotation matrix. Below we will see that these names has a physical meaning. [10]
We define $x = x_0 + h$ taking the derivative with respect on t

$$\frac{dx}{dt} = \frac{dh}{dt}$$

[9] [1] pg 154
[10] [3], pg 18

We know that

$$\frac{dx}{dt} = u(x)$$

and by the Taylor expansion above
So $\frac{dh}{dt} \approx u(x_0, t_0) + \nabla u h = u(x_0, t_0) + Dh + \Omega h$ which is a linear equation in terms of h.

- if $\frac{dh}{dt} \approx u(x_0, t)$ then $h(t) = h(0) + u(x_, t_0)t$ which describes infinitesimal translation
- if $\frac{dh}{dt} \approx \Omega h$ We define the vorticity of the velocity field as $\omega = \text{curl} u = \nabla \times u$
  So $\omega = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$
  We will prove that $\Omega h = \frac{1}{2}(\omega \times h)$.

  $$\Omega = \frac{1}{2} \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & 0 & \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} & \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} & 0 \end{pmatrix}$$

  So $\Omega = \frac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$

  $$\Omega h = \frac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

  $$\Omega h = \frac{1}{2} \begin{pmatrix} -\omega_3 h_2 + \omega_2 h_3 \\ \omega_3 h_1 - \omega_1 h_3 \\ -\omega_2 h_1 + \omega_1 h_2 \end{pmatrix}$$

  Now $\omega \times h = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ h_1 & h_2 & h_3 \end{vmatrix} = \begin{pmatrix} -\omega_3 h_2 + \omega_2 h_3 \\ \omega_3 h_1 - \omega_1 h_3 \\ -\omega_2 h_1 + \omega_1 h_2 \end{pmatrix}$ And so is the result. Consequently $\frac{dh}{dt} = \Omega h = \frac{1}{2}(\omega \times h)$ Because of the vector product it is a rotation with angular velocity $\frac{1}{2}|\omega|$
- if $\frac{dh}{dt} = Dh$. Since D is a symmetric matrix there exists eigenvalues λ_i and orthonomal eigenvectors $\bar{e_i}$ so that

  $$D\bar{e_i} = \lambda_i \bar{e_i}$$

  $\bar{e_i}$ is a base so $h(t) = \sum_i h_i \bar{e_i}$
  So $\sum_i \frac{dh_i}{dt} \bar{e_i} = D \sum_i h_i \bar{e_i}$
  Consequently $\frac{dh_i}{dt} = \lambda_i h_i$ i.e. $h_i(t) = h_i(0)e^{\lambda_i t}$
  Which is expansion or contraction along $\bar{e_i}$

To sum up the velocity field is a sum of deformation and vorticity.

$$u(x,t) = u(x_0, t) + D(x,t)(x - x_0) + \frac{1}{2}\omega \times h(x - x_0)$$

After this process we are ready to derive the vorticity equation.

1.4 The vorticity equation

By differentiating the Navier-Stokes equation in three dimensions, we have:

$$\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i \Rightarrow$$

$$\frac{\partial}{\partial x_k}(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j}) = \frac{\partial}{\partial x_k}(-\frac{\partial p}{\partial x_i} + \nu \Delta u_i) \Rightarrow$$

$$\frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_k} + \sum_j \frac{\partial u_j}{\partial x_k}\frac{\partial u_i}{\partial x_j} + \sum_j u_j \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_k} = -\frac{\partial}{\partial x_k}\frac{\partial p}{\partial x_i} + \nu \Delta \frac{\partial u_i}{\partial x_k}$$

We will use the previous results about the behavior of the velocity field, and we will utilize the rotation matrix Ω. We set V the 3×3 matrix with elements $\frac{\partial u_i}{\partial x_k}$ and P the matrix with elements $\frac{\partial}{\partial x_k}\frac{\partial p}{\partial x_i}$
So

$$\frac{\partial V}{\partial t} + \sum_j u_j \frac{\partial V}{\partial x_j} + \sum_j \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_k} = -P + \nu \Delta V \Rightarrow$$

$$\frac{\partial V}{\partial t} + \sum_j u_j \frac{\partial V}{\partial x_j} + V^2 = -P + \nu \Delta V \Rightarrow$$

$$\frac{DV}{Dt} + V^2 = -P + \nu \Delta V \tag{NSM}$$

According to the previous construction we have obtained a symmetric and a skew symmetric matrix $D = \frac{1}{2}(V + V^T) and \Omega = \frac{1}{2}(V - V^T)$, where $V = D + \Omega$

$$V^2 = (D + \Omega)(D + \Omega) = D^2 + D\Omega + \Omega D + \Omega^2 = (D^2 + \Omega^2) + (D\Omega + \Omega D)$$

The matrix $(D^2 + \Omega^2)$ is symmetric. Indeed,

$$(D^2 + \Omega^2)^T = (D^2)^T + (\Omega^2)^T = (DD)^T + (\Omega\Omega)^T = (D^T)^2 + (\Omega^T)^2$$

but D is symmetric and Ω skew symmetric so

$$(D^2 + \Omega^2)^T = D^2 + (-\Omega)^2 = D^2 + \Omega^2$$

The matrix $(D\Omega + \Omega D)$ is skew symmetric. Indeed,

$$(D\Omega + \Omega D)^T = (D\Omega)^T + (\Omega D)^T = \Omega^T D^T + D^T \Omega^T = -\Omega D - D\Omega = -(D\Omega + \Omega D)$$

So for the Navier-Stokes is:

$$\frac{D}{Dt}(D + \Omega) + (\Omega^2 + D^2) + (\Omega D + D\Omega) = -P + \Delta(D + \Omega)$$

We observe that there exists a symmetric part of the equation, which is $\frac{D}{Dt}D + \Omega^2 + D^2 = -P + \nu \Delta D$ and the anti-symmetric part, which is $\frac{D}{Dt}\Omega + (\Omega D + D\Omega) = \nu \Delta \Omega$

From the anti-symmetric part, and since we have proved that the matrix Ω is linked with the vorticity, we will derive the vorticity equation. We choose an $h \in \mathbb{R}^3$, which does not depend on t and consequently on x. We have proved that $\Omega h = \frac{1}{2}\omega \times h$, we will multiply the anti-symmetric part with h.
So, for the first term of equation we obtain

$$\frac{D}{Dt}(\Omega h) = \frac{1}{2}\frac{D}{Dt}(\omega \times h) \Rightarrow$$

$$\frac{D\Omega}{Dt}h = \frac{1}{2}\frac{D\omega}{Dt} \times h$$

We will prove that $(D\Omega + \Omega D)h = \frac{1}{2}(-(D\omega) \times h + \mathrm{tr}(D)(\omega \times h))$ We have already saw that $\Omega = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$
So

$$\Omega D = \frac{1}{2}\begin{pmatrix} d_{31}\omega_2 - d_{21}\omega_3 & d_{32}\omega_2 - d_{22}\omega_3 & d_{33}\omega_2 - d_{23}\omega_3 \\ d_{11}\omega_3 - d_{31}\omega_1 & d_{12}\omega_3 - d_{32}\omega_1 & d_{13}\omega_3 - d_{33}\omega_1 \\ d_{21}\omega_1 - d_{11}\omega_2 & d_{22}\omega_1 - d_{12}\omega_2 & d_{23}\omega_1 - d_{13}\omega_2 \end{pmatrix}$$

Also $(\Omega D)^T = D^T\Omega^T = -D\Omega$
Hence $\Omega D + D\Omega =$

$$\frac{1}{2}\begin{pmatrix} 0 & d_{32}\omega_2 - d_{22}\omega_3 + d_{31}\omega_1 - d_{11}\omega_3 & d_{33}\omega_2 - d_{23}\omega_3 + d_{11}\omega_2 - d_{21}\omega_1 \\ d_{11}\omega_3 - d_{31}\omega_1 - d_{32}\omega_2 + d_{22}\omega_3 & 0 & d_{13}\omega_3 - d_{33}\omega_1 - d_{22}\omega_1 + d_{12}\omega_2 \\ d_{21}\omega_1 - d_{11}\omega_2 - d_{33}\omega_2 + d_{23}\omega_3 & d_{22}\omega_1 - d_{12}\omega_2 - d_{13}\omega_3 + d_{33}\omega_1 & 0 \end{pmatrix}$$

Name q_{ij} the elements of the above matrix for convenience.
We observe that $q_{12} = d_{32}\omega_2 - d_{22}\omega_3 + d_{31}\omega_1 - d_{11}\omega_3 - d_{22}\omega_3 + d_{31}\omega_1 - d_{11}\omega_3 =$

$$d_{33}\omega_3 + d_{31}\omega_1 + d_{32}\omega_2 - \omega_3\mathrm{tr}(D) = [d_{3j}]\omega - \omega_3\mathrm{tr}(D)$$

The other elements follows so

$$(\Omega D + D\Omega)h = \frac{1}{2}(\begin{pmatrix} [d_{3j}]\omega h_2 - [d_{2j}]\omega h_3 \\ [d_{1j}]\omega h_3 - [d_{3j}]\omega h_1 \\ [d_{2j}]\omega h_1 - [d_{1j}]\omega h_2 \end{pmatrix}) - \mathrm{tr}D\begin{pmatrix} -\omega_3 h_2 + \omega_2 h_3 \\ \omega_3 h_1 - \omega_1 h_3 \\ -\omega_2 h_1 + \omega_1 h_2 \end{pmatrix}$$

It means that $(\Omega D + D\Omega)h = \frac{1}{2}(-(D\omega) \times h + \mathrm{tr}(D)(\omega \times h))$.
And now for the last term $\Omega h = \frac{1}{2}\omega \times h$ Therefore

$$\Delta(\Omega h) = \frac{1}{2}\Delta(\omega \times h)$$

since $\Delta h = 0$

$$(\Delta\Omega)h = \frac{1}{2}(\Delta\omega) \times h$$

Finally $\frac{D\Omega}{Dt}h + (\Omega D + D\Omega)h = \nu\Delta\Omega h$

$$\frac{1}{2}\frac{D\omega}{Dt} \times h + \frac{1}{2}\Big(- (D\omega) \times h + \mathrm{tr}(D)(\omega \times h)\Big) = \frac{1}{2}\nu(\Delta\omega) \times h$$

By vector product properties [11]

$$\frac{D\omega}{Dt} \times h - (D\omega) \times h + (\mathrm{tr}(D)\Omega) \times h = \nu\Delta\omega \times h$$

$$\left(\frac{D\omega}{Dt} - (D\omega) + (\mathrm{tr}(D)\Omega)\right) \times h = \nu\Delta\omega \times h$$

$$\frac{D\omega}{Dt} - (D\omega) + (\mathrm{tr}(D)\omega) = \nu\Delta\omega$$

This is the vorticity equation, and since the flow is incompressible i.e. $\mathrm{div}u = 0$ we have that $\frac{\vartheta u_1}{\vartheta x_1} + \frac{\vartheta u_2}{\vartheta x_2} + \frac{\vartheta u_3}{\vartheta x_3} = 0$ i.e $\mathrm{tr}(D) = 0$.
So the vorticity equation reduces to

$$\frac{D\omega}{Dt} = D\omega + \nu\Delta\omega$$

It is obvious that this equation is important because we got ridden of the pressure term. The terms remaining are the velocity and vorticity which are related. For inviscid fluids the vorticity equation reduces to

$$\frac{D\omega}{Dt} = D\omega$$

Remark:In two dimensions the vorticity of the velocity field is not a vector, i.e.
To calculate the curl we assume that in the x_3-direction there is a zero. So we get

$$\omega = \mathrm{curl}u = \nabla \times u = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 \\ u_1 & u_2 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix}$$

It is obvious that the curl in two dimensions points in the direction of x_3, since the first coordinates are 0. We can assume that in this direction there exists an axis of rotation which is perpendicular to a point. From now on we take vorticity as a scalar quantity so $\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$, which measures how much the velocity field rotates around the point. So it is reasonable for someone to wonder what happens with the vorticity equation in that case. We will follow the same procedure again.
For the Navier Stokes equation in two dimensions we get

$$\frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_k} + \sum_j \frac{\partial u_j}{\partial x_k}\frac{\partial u_i}{\partial x_j} + \sum_j u_j \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_k} = -\frac{\partial}{\partial x_k}\frac{\partial p}{\partial x_i} + \nu\Delta\frac{\partial u_i}{\partial x_k}$$

We set V the 2×2 matrix with elements $\frac{\partial u_i}{\partial x_k}$ and P the matrix with elements $\frac{\partial}{\partial x_k}\frac{\partial p}{\partial x_i}$.
(So far besides the dimensions there is no other change).
So

$$\frac{DV}{Dt} + V^2 = -P + \nu\Delta V$$

According to the previous construction we have a symmetric and a skew symmetric

[11] [40], pg 255

matrix $D = \frac{1}{2}(V + V^T)$ and $\Omega = \frac{1}{2}(V - V^T)$ accordingly, where $V = D + \Omega$ with the difference that D and Ω are 2×2 matrices.
So for the Navier-Stokes is:

$$\frac{D}{Dt}(D + \Omega) + (\Omega^2 + D^2) + (\Omega D + D\Omega) = -P + \nu\Delta(D + \Omega)$$

Again we take the skew symmetric part $\frac{D}{Dt}\Omega + (\Omega D + D\Omega) = \nu\Delta\Omega$.
We see that in order to derive the vorticity equation, we make use of this relation $\Omega h = \frac{1}{2}\omega \times h$, where h is a vector in two dimensions, which does not depend on time. Is this equality true in two dimensions?

$$\Omega = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

So $\Omega h = \frac{1}{2}\begin{pmatrix} -\omega h_2 \\ \omega h_1 \end{pmatrix}$

Now $\omega \times h = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ h_1 & h_2 & 0 \end{vmatrix} = \begin{pmatrix} -\omega h_2 \\ \omega h_1 \end{pmatrix}$

So we continue our process, by multiplying the anti-symmetric part with h.
So for the first term of equation

$$\frac{D}{Dt}(\Omega h) = \frac{1}{2}\frac{D}{Dt}(\omega \times h) \Rightarrow$$

$$\frac{D\Omega}{Dt}h = \frac{1}{2}\frac{D\omega}{Dt} \times h$$

And the second term $(D\Omega + \Omega D) = \frac{1}{2}\begin{pmatrix} 0 & -d_{11}\omega - d_{22}\omega \\ d_{11}\omega + d_{22}\omega & 0 \end{pmatrix}$

So $(D\Omega + \Omega D)h = \frac{1}{2}\begin{pmatrix} -h_2\omega \mathrm{tr}(D) \\ h_1\omega \mathrm{tr}(D) \end{pmatrix}$

But the fluid is incompressible so the trace of D is zero so$(D\Omega + \Omega D)h = 0$.
And now the last term $\Omega h = \frac{1}{2}\omega \times h$ therefore

$$\Delta(\Omega h) = \frac{1}{2}\Delta(\omega \times h)$$

since $\Delta h = 0$
So we conclude that

$$\frac{1}{2}\frac{D\omega}{Dt} \times h = \frac{1}{2}\nu(\Delta\omega) \times h$$

I.e $\frac{D\omega}{Dt} = \nu(\Delta\omega)$ and for inviscid fluids $\frac{D\omega}{Dt} = 0$
Which means that in two dimensions for incompressible and inviscid fluids the vorticity of each particle is constant as particle moves.We will discuss this in the next chapters.
In three dimensions we also have that
We know that $\nabla u = D + \Omega$ (remember the construction with the symmetric and skew symmetric matrices), then $\nabla u \cdot \omega = (D + \Omega)\omega$[12] by linear properties of the product

[12] We will also denote $\cdot$ the product between matrices

$\nabla u \cdot \omega = D\omega + \Omega\omega$
It is easy for someone to see that $\Omega\omega = 0$

$$\Omega\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} -\omega_3\omega_2 + \omega_3\omega_2 \\ \omega_1\omega_3 - \omega_1\omega_3 \\ -\omega_2\omega_1 + \omega_1\omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Consequently

$$\frac{D\omega}{Dt} = \nabla u \cdot \omega$$

Next we see the vorticity transport formula which will be very useful in the following chapters.

Proposition 1.4.1. *Assume ϕ a smooth particle trajectory mapping of a smooth velocity field of an incompressible fluid. Then the solution ω of the $\frac{D\omega}{Dt} = \nabla u \cdot \omega$ is $\omega(\phi(a,t),t) = \nabla_a \phi(a,t)\omega_0(a)$*

Proof. We first proof the following lemma.

Lemma 1. *Let $u(x,t)$ a smooth velocity field with $\phi(a,t)$ its particle trajectory mapping so that $\frac{\partial \phi}{\partial t} = u(\phi(a,t),t)$ and $\phi(a,0) = a$. Moreover assume h a smooth vector field then*

$$\frac{Dh}{Dt} = \nabla u \cdot h \iff h(\phi(a,t),t) = \nabla_a \phi(a,t) h_0(a)$$

proof of lemma:

$$\frac{\partial \phi}{\partial t} = u(\phi(a,t),t) \tag{P 2.1.1.1}$$

$$\frac{\partial}{\partial a}\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial a} u(\phi(a,t),t)$$

$$\frac{\partial}{\partial t}\frac{\partial \phi}{\partial a} = \nabla u(\phi(a,t),t)\frac{\partial \phi}{\partial a}$$

So

$$\frac{\partial}{\partial t}\nabla_a \phi(a,t) = \nabla u(\phi(a,t),t)\nabla_a \phi(a,t) \tag{P 2.1.1.2}$$

We multiply with $h_0(a)$ and we get

$$\frac{\partial}{\partial t}\nabla_a \phi(a,t)h_0(a) = \nabla u(\phi(a,t),t)\nabla_a \phi(a,t)h_0(a)$$

We continue with the equation $\frac{Dh}{Dt} = \nabla u \cdot h$. For $u = u(\phi(a,t),t)$ and $h = h(\phi(a,t),t)$ we get

$$\frac{\partial}{\partial t}h(\phi(a,t),t) + \sum u_j \frac{\partial}{\partial x_j}\phi(a,t) = \nabla u(\phi(a,t),t)h(\phi(a,t),t)$$

$$\frac{\partial}{\partial t}h(\phi(a,t),t) = \nabla u(\phi(a,t),t)h(\phi(a,t),t) \tag{P 2.1.1.3}$$

The initial condition for the (P 2.1.1.1) is $\phi(a,0) = a$ so $\nabla_a \phi(a,0) = \nabla_a a$ so the initial condition for (P 2.1.1.2) is $\nabla_a \phi(a,0) h_0(a) = h_0(a)$.
So we have two equations (P 2.1.1.2) and (P 2.1.1.3) differentiated with respect to t and same initial conditions. Therefore the $\nabla_a \phi(a,t) h_0(a)$ and $h(\phi(a,t),t)$ satisfy the same differential equation with same initial condition,because of the uniqueness of the solutions we get $h(\phi(a,t),t) = \nabla_a \phi(a,t) h_0(a)$

Now for the proof of the proposition we substitute $h = \omega$ and the mapping ϕ is the particle trajectory mapping we have defined. This completes the proof. □

In two dimension we have the corresponding proposition:

Proposition 1.4.2. *Let ϕ a smooth particle trajectory mapping of a smooth velocity field.Then the vorticity of an inviscid fluid satisfies:*

$$\omega(\phi(a,t),t) = \omega_0(a)$$

, where $a \in \mathbb{R}^2$

Proof. In two dimensions we have that the particle trajectory mapping is $\phi : a \in \mathbb{R}^2 \to \phi(a,t) \in \mathbb{R}^2$.Its Jacobian determinant is :

$$J_\phi = \begin{vmatrix} \frac{\partial}{\partial a_1}\phi_1 & \frac{\partial}{\partial a_2}\phi_2 \\ \frac{\partial}{\partial a_1}\phi_2 & \frac{\partial}{\partial a_2}\phi_2 \end{vmatrix}$$

Since the fluid is incompressible by proposition 1.2.1 we have that $J_\phi = 1$ so we have that $\frac{\partial}{\partial a_1}\phi_1 \frac{\partial}{\partial a_2}\phi_2 - \frac{\partial}{\partial a_2}\phi_2 \frac{\partial}{\partial a_1}\phi_2 = 1$ we continue by adding a row and a column to the matrix in order to make a 3x3 matrix with the same determinant so we have the matrix

$$\begin{pmatrix} \frac{\partial}{\partial a_1}\phi_1 & \frac{\partial}{\partial a_2}\phi_2 & 0 \\ \frac{\partial}{\partial a_1}\phi_2 & \frac{\partial}{\partial a_2}\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Assuming now that the vorticity is embedded into three dimension (by assuming that the first coordinates are 0), by proposition 1.4.1 we have that

$$\omega(\phi(a,t),t) = \begin{pmatrix} \frac{\partial}{\partial a_1}\phi_1 & \frac{\partial}{\partial a_2}\phi_2 & 0 \\ \frac{\partial}{\partial a_1}\phi_2 & \frac{\partial}{\partial a_2}\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega_0 \end{pmatrix}$$

Thus we conclude that $\omega(\phi(a,t),t) = \omega_0(a)$ □

Remark:We observe that in two dimensions we have better results than in the three dimensions. We see that along particle trajectories the vorticity in two dimensions is conserved. But in three dimensions this is not true, more precisely in three dimensions there exists the matrix $\nabla_a \phi$ which deforms the initial vorticity.
Now we will examine the vortex lines of the fluid.

Definition 4. *Let c be a smooth curve such that* $c = \{y(s) \in \mathbb{R}^N : 0 < s < 1\}$ *will be a vortex line in a fixed time t if its tangent is everywhere parallel to vorticity* ω *i.e.* $\frac{\partial y(s)}{\partial t} = \lambda(s)\omega(y(s), t)),\ \lambda(s) \neq 0.$

We have already seen that thanks to the mapping ϕ we are able to examine the motion of a whole region of fluid in time. Is it possible to examine the behavior of vortex lines in time in a similar way?
Let c be the vortex line for $t = 0$ and $c(t) = \{\phi(y(s), t) \in \mathbb{R}^N : 0 < s < 1\}$ we will prove that the curve $c(t)$ is also a vortex line.
Indeed $\frac{\partial}{\partial t}\phi(y(s), t) = \nabla_a\phi(y(s), t)\frac{\partial}{\partial t}(y(s), t)$ so

$$\frac{\partial}{\partial t}\phi(y(s), t) = \nabla_a\phi(y(s), t)\frac{\partial y(s)}{\partial t}$$

c is a vortex line so

$$\frac{\partial}{\partial t}\phi(y(s), t) = \nabla_a\phi(y(s), t)\lambda(s)\omega(y(s), 0)$$

by proposition 1.4.1

$$\frac{\partial}{\partial t}\phi(y(s), t) = \lambda(s)\omega(\phi(y(s), t))$$

We observe that the tangent of ϕ is parallel to ω. So, since $c(t)$ is a vortex line we understand that for inviscid incompressible fluid vortex lines moves with the fluid.
We will see the next proposition which is similar to the transport formula.

Proposition 1.4.3. *Let c a smooth oriented closed curve and* ϕ *a smooth mapping of a divergence free velocity field u. Then* $\frac{d}{dt}\oint udl = \oint \frac{Du}{Dt}dl$.

Proof. Assume $c(t) = \{\phi(s, t) : 0 \leq s \leq 1\}$ by the definition of closed integral we get

$$\frac{d}{dt}\oint udl = \frac{d}{dt}\int_0^1 u(\phi(s, t), t)\frac{\partial\phi(s, t)}{\partial s}ds$$

By the Leibniz integral rule we get

$$\frac{d}{dt}\int_0^1 u(\phi(s, t), t)\frac{\partial\phi(s, t)}{\partial s} = \int_0^1 \frac{\partial}{\partial t}(u(\phi(s, t), t)\frac{\partial\phi(s, t)}{\partial s})ds$$

$$= \int_0^1 \frac{\partial}{\partial t}u(\phi((s, t), t)\frac{\partial\phi(s, t)}{\partial s} + u(\phi(s, t))\frac{\partial}{\partial t}\frac{\partial\phi(s, t)}{\partial s}ds$$

$$= \int_0^1 \frac{\partial}{\partial t}u(\phi((s, t), t)\frac{\partial\phi(s, t)}{\partial s} + u(\phi(s, t))\frac{\partial}{\partial s}\frac{\partial\phi(s, t)}{\partial t}ds$$

By the definition of the mapping $\frac{\partial\phi(s,t)}{\partial t} = u(\phi(s, t))$

$$= \int_0^1 \frac{\partial}{\partial t}u(\phi((s, t), t)\frac{\partial\phi(s, t)}{\partial s} + u(\phi(s, t))\frac{\partial}{\partial s}u(\phi(s, t))ds$$

$$= \int_0^1 \frac{\partial}{\partial t} u(\phi((s,t),t) \frac{\partial \phi(s,t)}{\partial s} ds + \frac{1}{2} \int_0^1 \frac{\partial}{\partial s} |u(\phi(s,t))|^2 ds$$

So

$$\frac{d}{dt} \oint udl = \int_0^1 \frac{\partial}{\partial t} u(\phi((s,t),t) \frac{\partial \phi(s,t)}{\partial s} ds$$

Now

$$\oint \frac{Du}{Dt} dl = \int_0^1 \frac{Du(\phi(s,t),t)}{Dt} \cdot \frac{\partial}{\partial s} \phi(s,t)$$

$$= \int_0^1 \frac{\partial}{\partial t} u(\phi((s,t),t) \frac{\partial \phi(s,t)}{\partial s} ds$$

To sum up

$$\oint \frac{Du}{Dt} dl = \frac{d}{dt} \oint udl$$

□

Using the above proposition we will prove the next well known theorem

Theorem 1.4.1. *(Kelvin's conservation of circulation)*
Let u be a smooth solution of the Euler equation, then the circulation $\Gamma_{c(t)}$ around a curve $c(t)$ moving with the fluid, i.e

$$\Gamma_{c(t)} = \oint udl$$

is constant in time.

Proof. $\frac{d}{dt} \oint udl = \oint \frac{Du}{Dl} dl$ but u satisfies Euler so

$$\frac{d}{dt} \Gamma_{c(t)} = \oint -\nabla p dl = 0$$

□

We give the following proposition

Proposition 1.4.4. *(Helmholtz's conservation of vorticity flux)*
Let u be a smooth solution of the Euler equation then the vorticity flux along a surface S(t) moving with the fluid i.e.

$$F_{A(t)} = \int_A \omega dS$$

is constant in time.

Proof.

$$\frac{d}{dt} F_{A(t)} =$$

by Stokes theorem

$$= \frac{d}{dt} \oint_{\partial A} uds = 0$$

□

1.5 Conserved quantities

In this section we will see some other quantities that remain constant in time.

First we will see the preserved quantities in $\mathbb{R}^3$
We will assume that u is a smooth solution of the 3d Euler Equation which vanishes sufficiently rapidly as $|x|$ tends to infinity then the following quantities remain constant in time.

- Total flux of velocity [13]

$$V = \int_{\mathbb{R}^3} u dx$$

Proof: For the Euler we have

$$\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i}$$

We integrate both parts over $\mathbb{R}^3$

$$\int_{\mathbb{R}^3} \frac{\partial u_i}{\partial t} + \sum_{j=1} u_j \frac{\partial u_i}{\partial x_j} dx = \int_{\mathbb{R}^3} -\frac{\partial p}{\partial x_i} dx \Rightarrow$$

$$\int_{\mathbb{R}^3} \frac{\partial u_i}{\partial t} + \int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial u_i}{\partial x_j} dx = \int_{\mathbb{R}^3} -\frac{\partial p}{\partial x_i} dx$$

By Leibniz integral rule [14]

$$\frac{d}{dt} \int_{\mathbb{R}^3} u_i dx = -\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial u_i}{\partial x_j} dx - \int_{\mathbb{R}^3} \frac{\partial p}{\partial x_i} dx$$

Lets see each term individually.

$$\int_{\mathbb{R}^3} \frac{\partial p}{\partial x_i} dx \overset{\text{(polar coordinates)}^{15}}{=} \int_0^\infty \int_{\partial B(x_0,r)} \frac{\partial p}{\partial x_i} dS dr$$

Since $\partial B(x_0, r)$ is a closed curve of $\mathbb{R}^2$

$$\int_{\partial B(x_0,r)} \frac{\partial p}{\partial x_i} dS = 0$$

So $\int_{\mathbb{R}^3} \frac{\partial p}{\partial x_i} dx = 0$
We continue with the other term:

$$\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial u_i}{\partial x_j} dx = \sum_{j=1} \int_{\mathbb{R}^3} u_j \frac{\partial u_i}{\partial x_j} dx$$

[13] Total flux is the amount of quantity passing through a surface area per time.
[14] $\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x)) \cdot \frac{d}{dx} b(x) - f(x,a(x)) \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t) dt$

By integration by parts formula we get

$$\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial u_i}{\partial x_j} dx = \sum_{j=1} \int_{\mathbb{R}^3} \frac{\partial u_j}{\partial x_j} u_i dx \Rightarrow$$

$$\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial u_i}{\partial x_j} dx = \int_{\mathbb{R}^3} \sum_{j=1} \frac{\partial u_j}{\partial x_j} u_i dx = \int_{\mathbb{R}^3} u_i \sum_{j=1} \frac{\partial u_j}{\partial x_j} dx = \int_{\mathbb{R}^3} u_i \mathrm{div} u dx$$

We know that we deal with incompressible fluids, $\mathrm{div} u = 0$ so

$$\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial u_i}{\partial x_j} dx = 0$$

So we have that

$$\frac{d}{dt} \int_{\mathbb{R}^3} u_i dx = 0$$

which is true for all i=1,2,3 so

$$\frac{d}{dt} \int_{\mathbb{R}^3} u dx = 0$$

- Total flux of vorticity $W = \int_{\mathbb{R}^3} \omega dx$
  Proof:

$$\frac{d}{dt} W = \frac{d}{dt} \int_{\mathbb{R}^3} \omega dx$$

By Leibniz integral rule

$$\frac{d}{dt} W = \int_{\mathbb{R}^3} \frac{\partial \omega}{\partial t} dx$$

Now we will use the vorticity equation so
$\frac{d}{dt} W = -\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial \omega}{\partial x_j} dx + \int_{\mathbb{R}^3} D\omega dx$ [16]
Again we examine each term individually

$$\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial \omega}{\partial x_j} dx = \sum_{j=1} \int_{\mathbb{R}^3} u_j \frac{\partial \omega}{\partial x_j} dx$$

By integration by parts formula :

$$\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial \omega}{\partial x_j} dx = \sum_{j=1} \int_{\mathbb{R}^3} \frac{\partial u_j}{\partial x_j} \omega dx = \int_{\mathbb{R}^3} \sum_{j=1} \frac{\partial u_j}{\partial x_j} \omega dx = \int_{\mathbb{R}^3} \omega \sum_{j=1} \frac{\partial u_j}{\partial x_j} dx = \int_{\mathbb{R}^3} \omega \mathrm{div} u dx$$

$\mathrm{div} u = 0$ so

$$\int_{\mathbb{R}^3} \sum_{j=1} u_j \frac{\partial \omega}{\partial x_j} dx = 0$$

Lets work with the other term:

$$\int_{\mathbb{R}^3} D\omega dx = \int_{\mathbb{R}^3} \nabla u \cdot \omega = \int_{\mathbb{R}^3} \sum_{j=1} \frac{\partial u}{\partial x_j} \omega_j dx = \sum_{j=1} \int_{\mathbb{R}^3} \frac{\partial u}{\partial x_j} \omega_j dx$$

[16] D is the 3x3 symmetric matrix from our previous construction.

By integration by parts formula

$$\int_{\mathbb{R}^3} D\omega dx = \int_{\mathbb{R}^3} u\frac{\partial \omega_j}{\partial x_j}dx = \int_{\mathbb{R}^3} u\text{div}\omega dx$$

We have that div$\omega = 0$ [17],so

$$\int_{\mathbb{R}^3} D\omega dx = 0$$

It turns out that:

$$\frac{d}{dt}W = 0$$

- The kinetic energy $E = \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 dx$
  Proof:

$$\frac{d}{dt}E = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 dx$$

By Leibniz integral rule

$$\frac{d}{dt}E = \frac{1}{2}\int_{\mathbb{R}^3} \frac{\partial}{\partial t}|u|^2 dx = \int_{\mathbb{R}^3} \frac{1}{2}2u\frac{\partial u}{\partial t}dx$$

We take the Euler equation and multiply with u so

$$\frac{d}{dt}E = -\int_{\mathbb{R}^3} u\sum_{j=1} u_j\frac{\partial u}{\partial x_j}dx - \int_{\mathbb{R}^3} u\nabla p dx$$

We see each term individually

$$\int_{\mathbb{R}^3} u\nabla p = \int_{\mathbb{R}^3} \sum_{j=1} u_j\frac{\partial p}{\partial x_j}dx = \sum_{j=1}\int_{\mathbb{R}^3} u_j\frac{\partial p}{\partial x_j}dx$$

By integration by parts formula

$$\int_{\mathbb{R}^3} u\nabla p dx = \sum_{j=1}\int_{\mathbb{R}^3} \frac{\partial u_j}{\partial x_j}p dx = \int_{\mathbb{R}^3} p\text{div}u = 0x$$

The other term :

$$\int_{\mathbb{R}^3} u\sum_{j=1} u_j\frac{\partial u}{\partial x_j}dx = \int_{\mathbb{R}^3} \sum_{i,j} u_i u_j\frac{\partial u_i}{\partial x_j}dx$$

By integration by parts we get

$$\int_{\mathbb{R}^3} \sum_{i,j} u_i u_j\frac{\partial u_i}{\partial x_j} = \int_{\mathbb{R}^3} \sum_{i,j} u_i u_j\frac{\partial u_i}{\partial x_j} + \int_{\mathbb{R}^3} \sum_{i,j} u_i u_i\frac{\partial u_j}{\partial x_j}$$

I.e

$$2\int_{\mathbb{R}^3} \sum_{i,j} u_i u_j\frac{\partial u_i}{\partial x_j} = 0$$

To conclude

$$\frac{d}{dt}E = 0$$

[17] [40] pg. 255

- The Helicity $H = \int_{\mathbb{R}^3} u \cdot \omega dx$
  Proof:
$$\frac{d}{dt}H = \frac{d}{dt}\int_{\mathbb{R}^3} u \cdot \omega dx$$
  By Leibniz integral formula we have
$$\frac{d}{dt}\int_{\mathbb{R}^3} u \cdot \omega dx = \int_{\mathbb{R}^3} \frac{\partial}{\partial t}(u \cdot \omega) dx = \int_{\mathbb{R}^3} u\frac{\partial \omega}{\partial t} + \omega \frac{\partial u}{\partial t} dx$$
  By the vorticity equation $\frac{\partial \omega}{\partial t} = \nabla u \omega - u \nabla \omega$
  By the Euler equation $\frac{\partial u}{\partial t} = -u\nabla u - \nabla p$ So
$$\frac{dH}{dt} = \int_{\mathbb{R}^3} u(\nabla u - u\nabla\omega) + \omega(-u\nabla u - \nabla p)dx$$
$$= \int_{\mathbb{R}^3} -\nabla p\omega - \omega u \nabla u + u\omega\nabla u - uu\nabla\omega$$
  It is true that $\nabla \cdot (p\omega) = p\text{div}\omega + \omega\nabla p$ but $\text{div}\omega = 0$ because $\text{div}(\text{curl}u) = 0$ so $\nabla(p\omega) = \omega\nabla p$.
  Moreover $u\omega\nabla u = \frac{1}{2}\nabla(|u|^2\omega)$ and $\omega(u\nabla u)+uu\nabla\omega = u(\omega\nabla u+u\nabla\omega)+u(\nabla(u\omega))+ u\omega\text{div}u = \nabla(u(u\omega))$.
  Eventually
$$\frac{dH}{dt} = \int_{\mathbb{R}^3} \nabla \cdot (-p\omega - uu\omega + \frac{1}{2}|u|^2\omega)$$
  By the divergence theorem, this integral is equal to the integral of $(-p\omega - uu\omega + \frac{1}{2}|u|^2\omega)n$ on the boundary of $\mathbb{R}^3$, which is an empty set. So the integral is equal to zero.

Corollary 1.5.1. *In $\mathbb{R}^2$ the above quantities are preserved and the Helicity is equal to zero.*

Now, we will prove that the L^p-norm of the vorticity in $\mathbb{R}^2$ is conserved in time.
Remark: Let $1 \leq |p| \leq \infty$ then the L^p-norm of vorticity in $\mathbb{R}^2$ is $||\omega||_p = (\int_{\mathbb{R}^2} |\omega|^p)^{\frac{1}{p}} dx$
Proof of the Remark:By L^p estimates on the vorticity quation we get:
$$\int_{\mathbb{R}^2} \frac{\partial}{\partial t}\omega \cdot \omega|\omega|^{p-2}dx = -\int_{\mathbb{R}^2}(u\cdot\nabla)\omega\cdot\omega|\omega|^{p-2}dx$$
Thus by Leibniz integral rule and integration by parts we have that
$$\frac{d}{dt}\int_{\mathbb{R}^2}|\omega|^p dx = 0$$
$$\frac{d}{dt}||\omega||_p = \frac{1}{p}(\int_{\mathbb{R}^2}|\omega|^p dx)^{\frac{1}{p}-1}(\frac{d}{dt}\int_{\mathbb{R}^2}|\omega|^p dx) = 0$$

CHAPTER 2

Important reformulations of the Navier-Stokes and Euler equations

2.1 Leray's formulation for incompressible fluids

With the previous matrix construction regarding the Navier Stokes and Euler equation, we derived the vorticity equation. Our aim now is to examine a different approach. Someone may ask, why do we care about those formulations regarding our equations. Firstly, we can get rid of some terms, such as pressure. This is useful since we get rid of an unknown term that is partially differentiated with respect to x. Eventually, we obtain an equation with only one unknown quantity. Secondly, we derive propositions and formulas that will be used in the next chapters. Also, we can study and understand physical properties. We recall equation (NSM)

$$\frac{DV}{Dt} + V^2 = -P + \nu \Delta V$$

where $V = [\frac{\partial u_i}{\partial x_j}]_{ij}$ and P is the Hessian of pressure. Recall also that $\text{div} u = 0$, so we have that $\text{tr}V = 0$. It seems that the only information we have here is about the trace, so for the above matrices equation we take the trace on both parts.

$$\text{tr}(\frac{DV}{Dt} + V^2) = \text{tr}(-P + \nu \Delta V)^1 \Rightarrow$$

$$\text{tr}(\frac{DV}{Dt}) + \text{tr}(V^2) = \text{tr}(-P) + \text{tr}(\nu \Delta V)$$

We will examine each term of this equality.

- $\text{tr}(\frac{DV}{Dt}) = D_{11} + D_{22} + D_{33}$

$$\frac{DV}{Dt} = [\frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_j} + \sum_k u_k \frac{\partial}{\partial x_k}\frac{\partial u_i}{\partial x_j}]_{ij}$$

[1] $\text{tr}(A + B) = \text{tr}A + \text{tr}B$

Therefore $D_{ii} = \frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_i} + \sum_j u_j \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_i}$
So

$$\mathrm{tr}\left(\frac{DV}{Dt}\right) = \frac{\partial}{\partial t}\frac{\partial u_1}{\partial x_1}+\sum_j\left(u_j\frac{\partial}{\partial x_j}\frac{\partial u_1}{\partial x_1}\right)+\frac{\partial}{\partial t}\frac{\partial u_2}{\partial x_2}+\sum_j\left(u_j\frac{\partial}{\partial x_j}\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial}{\partial t}\frac{\partial u_3}{\partial x_3}+\sum_j\left(u_j\frac{\partial}{\partial x_j}\frac{\partial u_3}{\partial x_3}\right) \Rightarrow$$

$$\mathrm{tr}\left(\frac{DV}{Dt}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+\sum_j u_j\frac{\partial}{\partial x_j}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right) \Rightarrow$$

$$\mathrm{tr}\left(\frac{DV}{Dt}\right) = \frac{\partial}{\partial t}\mathrm{div}u + \sum_j u_j \frac{\partial}{\partial x_j}\mathrm{div}u$$

But $\mathrm{div}u = 0$ so

$$\mathrm{tr}(\frac{DV}{Dt}) = 0$$

- $\mathrm{tr}P = \frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2} = \sum_j \frac{\partial^2 p}{\partial x_j^2} = \Delta p$
- $\mathrm{tr}(\Delta V) = \Delta_{11} + \Delta_{22} + \Delta_{33}$

$$\Delta V = [\sum_k \frac{\partial^2}{\partial x_k^2}\frac{\partial u_i}{\partial x_j}]_{ij}$$

So

$$\mathrm{tr}(\Delta V) = \sum_k \frac{\partial^2}{\partial x_k^2}\frac{\partial u_1}{\partial x_1} + \sum_k \frac{\partial^2}{\partial x_k^2}\frac{\partial u_2}{\partial x_2} + \sum_k \frac{\partial^2}{\partial x_k^2}\frac{\partial u_3}{\partial x_3} = \sum_k \frac{\partial^2}{\partial x_k^2}\mathrm{div}u = 0$$

Eventually $-\Delta p = \mathrm{tr}V^2 = \mathrm{tr}(\nabla u)^2 = \sum_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$ [2]
The above equation is a Poisson we know for its solutions the following.[3]
We define $g(x) = \begin{cases} -\frac{1}{2\pi}ln|x| & N=2 \\ \frac{1}{N(N-2)a(N)}\frac{1}{|x|^{N-2}} & N \geq 3 \end{cases}$.
Where a(N) is the volume of the unit sphere in $\mathbb{R}^N$, we also know that for the $\Delta z = f$ Poisson the solution is $z(x) = \int_{\mathbb{R}^N} g(x-y)f(y)dy$ with $|x-y| \neq 0$. In our case $p(x,t) = \int_{\mathbb{R}^N} g(x-y)\mathrm{tr}(\nabla_y u(y,t))^2 dy$.
For $N=2$ we have $\nabla_x p(x,t) = \int_{\mathbb{R}^2} g(x-y)\mathrm{tr}(\nabla_y u(y,t))^2 dy$ where $g(x-y) = -\frac{1}{2\pi}\ln|x-$

[2] $\mathrm{tr}(\nabla u)^2 =$

$$\mathrm{tr}\begin{pmatrix} (\frac{\partial u_1}{\partial x_1})^2 + \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + ... + \frac{\partial u_1}{\partial x_N}\frac{\partial u_N}{\partial x_1} & * & * \\ * & \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + (\frac{\partial u_2}{\partial x_2})^2 + ... + \frac{\partial u_2}{\partial x_N}\frac{\partial u_N}{\partial x_2} & * \\ . & & \\ . & & \\ . & & \\ * & * & \frac{\partial u_N}{\partial x_1}\frac{\partial u_1}{\partial x_N} + \frac{\partial u_N}{\partial x_2}\frac{\partial u_2}{\partial x_N} + ... + (\frac{\partial u_N}{\partial x_N})^2 \end{pmatrix}$$

[3] [18],chapter 2

$y|$.
So by the Leibniz integral rule[4]

$$\nabla_x p(x,t) = \int_{\mathbb{R}^2} \nabla_x((g(x-y)\mathrm{tr}((\nabla_y u(y,t))^2))dy$$

the trace term is constant when we differentiate with respect to x

$$\nabla_x p(x,t) = \int_{\mathbb{R}^2} \mathrm{tr}(\nabla_y u(y,t))^2 \nabla_x g(x-y) dy \Rightarrow$$

$$\nabla_x p(x,t) = \int_{\mathbb{R}^2} \mathrm{tr}(\nabla_y u(y,t))^2 (-\frac{1}{2\pi}\frac{x-y}{|x-y|^2})dy \Rightarrow$$

$$\nabla_x p(x,t) = c_2 \int_{\mathbb{R}^2} \mathrm{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^2} dy$$

For $N \geq 3$ we have $\nabla_x p(x,t) = \int_{\mathbb{R}^N} g(x-y)\mathrm{tr}[(\nabla_y u(y,t))^2]dy$ where $g(x-y) = \frac{1}{N(N-2)a(N)}\frac{1}{|x|^(N-2)}$.
So by Leibniz integral rule

$$\nabla_x p(x,t) = \int_{\mathbb{R}^N} \nabla_x((g(x-y)\mathrm{tr}((\nabla_y u(y,t))^2))dy$$

the trace term is constant when we differentiate with respect to x

$$\nabla_x p(x,t) = \int_{\mathbb{R}^N} \mathrm{tr}(\nabla_y u(y,t))^2 \nabla_x g(x-y) dy \Rightarrow$$

$$\nabla_x p(x,t) = \int_{\mathbb{R}^N} \mathrm{tr}(\nabla_y u(y,t))^2 \frac{1}{\tilde{c_N}} \nabla_x |x|^(2-N) dy \Rightarrow$$

$$\nabla_x p(x,t) = c_N \int_{\mathbb{R}^N} \mathrm{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy$$

I.e. $\forall N$

$$\nabla_x p(x,t) = c_N \int_{\mathbb{R}^N} \mathrm{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy$$

this relation links the pressure with the velocity. So we go back to Navier stokes equation

$$\frac{Du}{Dt} = -\nabla p + \nu \Delta u$$

and we replace the pressure term with the above solution so

$$\frac{Du}{Dt} = -c_N \int_{\mathbb{R}^N} \mathrm{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy + \nu \Delta u$$

Someone can easily observe that we have reach to an equation, which is depended only on time variable t and the unknown term is only the velocity field. We will prove now the following proposition:

[4] There is not a singularity in this integral operator so we proceed with simple calculations

Proposition 2.1.1. *Solving the Navier stokes equation with smooth initial velocity* u_0 *such that* $\text{div}u_0 = 0$ *is equivalent to solving the evolution equation*

$$\frac{Du}{Dt} = -c_N \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy + \nu \Delta u$$

$$u|_{t=0} = u_0$$

Proof. $\Rightarrow$ Assuming that u solves the Navier-Stokes then by the previous construction we are done.
$\Leftarrow$ Assuming now that u solves the evolution equation then for the Navier stokes $\frac{Du}{Dt} = -\nabla p + \nu\Delta u$ we have $-c_N \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy + \nu\Delta u = -\nabla p + \nu \Delta u$ so $\nabla p = c_N \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy$ thus the evolution equation becomes the Navier stokes. So the only remaining equation to be proven is that the vector field is divergence free. Assume now that u is random solution of the above equation we will prove that this solution is divergence free and we can omit the initial condition of incompressibility ($\text{div}u = 0$) assuming all along that $divu_0 = 0$.
Proof:

$$\frac{Du}{Dt} = -c_N \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy + \nu \Delta u \Rightarrow$$

$$\text{div}\Big(\frac{Du}{Dt}\Big) = \text{div}\Big(- c_N \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy + \nu \Delta u\Big) \Rightarrow$$

$$\sum_k \frac{\partial}{\partial x_k}\frac{Du_k}{Dt} = -c \sum_k \frac{\partial}{\partial x_k} \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{x-y}{|x-y|^N} dy + \nu \sum_k \frac{\partial}{\partial x_k} \Delta u_k \Rightarrow$$

$$\sum_k \frac{\partial}{\partial x_k}\Big(\frac{\partial u_k}{\partial t} + \sum_j u_j \frac{\partial u_k}{\partial x_j}\Big) = -c \sum_k \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{\partial}{\partial x_k}\Big(\frac{x-y}{|x-y|^N}\Big) dy + \nu \sum_k \frac{\partial}{\partial x_k} \Delta u_k$$

So we have that

$$\sum_k \frac{\partial}{\partial x_k}\Big(\frac{\partial u_k}{\partial t} + \sum_j u_j \frac{\partial u_k}{\partial x_j}\Big) =$$

$$-\tilde{c} \sum_k \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{\partial}{\partial x_k}\Big(\frac{1}{|x-y|^{N-1}}\Big) dy + \nu \sum_k \frac{\partial}{\partial x_k} \sum_j \frac{\partial^2 u_k}{\partial x_j^2}$$

thus

$$\frac{\partial}{\partial t} \sum_k \frac{\partial u_k}{\partial x_k} + \sum_{j,k} \frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_j} + \sum_{j,k} u_j \frac{\partial}{\partial x_k}\frac{\partial u_k}{\partial x_j} =$$

$$-\tilde{c} \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{\partial}{\partial x_k} \frac{1}{|x-y|^{N-1}} dy + \sum_j \frac{\partial^2}{\partial x_j^2} \sum_k \frac{\partial u_k}{\partial x_k}$$

$$\frac{D}{Dt}(\text{div}u) = -\sum_{j,k} \frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_j} - \tilde{c} \sum_k \int_{\mathbb{R}^N} \text{tr}(\nabla_y u(y,t))^2 \frac{\partial}{\partial x_k}\Big(\frac{1}{|x-y|^{N-1}}\Big) dy + \nu \Delta(\text{div}u)$$

We reach to the relation

$$\frac{D}{Dt}(\text{div}u) = -\sum_{j,k} \frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_j} - \nabla \cdot \nabla p + \nu \Delta(\text{div}u)$$

By footnote 2 on page 30 we see that

$$\frac{D}{Dt}(\text{div}u) = \Delta p - \Delta p + \nu\Delta(\text{div}u) \Rightarrow$$

$$\frac{D}{Dt}(\text{div}u) = \nu\Delta(\text{div}u)$$

So our goal is to prove that if $\text{div}u_0 = 0$ then $\text{div}u = 0$ in any time. We will use energy methods

Lets assume that the above equation has two solutions i.e. $\frac{D}{Dt}(\tilde{u_1}) = \nu\Delta(\tilde{u_1})$ and $\frac{D}{Dt}(\tilde{u_2}) = \nu\Delta(\tilde{u_2})$ where $\tilde{u_1} = \text{div}u_1$ and $\tilde{u_2} = \text{div}u_2$

$$\frac{d}{dt}\int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 dx = \int_{\mathbb{R}^N} \frac{\partial}{\partial t}|\tilde{u_1} - \tilde{u_2}|^2 dx = \int_{\mathbb{R}^N} 2(\tilde{u_1} - \tilde{u_2})\frac{\partial}{\partial t}(\tilde{u_1} - \tilde{u_2})dx$$

Since they are solutions we have

$$\frac{\partial \tilde{u_1}}{\partial t} = -u \cdot \nabla\tilde{u_1} + \nu\Delta\tilde{u_1}$$

and

$$\frac{\partial \tilde{u_2}}{\partial t} = -u \cdot \nabla\tilde{u_2} + \nu\Delta\tilde{u_2}$$

So $\frac{\partial}{\partial t}(\tilde{u_1} - \tilde{u_2}) = -u \cdot \nabla\tilde{u_1} + \nu\Delta\tilde{u_1} + u \cdot \nabla\tilde{u_2} - \nu\Delta\tilde{u_2}$

We continue our calculations by multiplying the above equation with the $\tilde{u_1} - \tilde{u_2}$ on L^2 so we have that

$$\int_{\mathbb{R}^N} \frac{\partial}{\partial t}(\tilde{u_1} - \tilde{u_2}) \cdot (\tilde{u_1} - \tilde{u_2})dx = \int_{\mathbb{R}^N} (\tilde{u_1} - \tilde{u_2}) \cdot (-u \cdot \nabla\tilde{u_1} + \nu\Delta\tilde{u_1} + u \cdot \nabla\tilde{u_2} - \nu\Delta\tilde{u_2})dx$$

By Leibniz integral rule we have that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 dx = \int_{\mathbb{R}^N} 2(\tilde{u_1} - \tilde{u_2}) \cdot (-u \cdot \nabla\tilde{u_1} + \nu\Delta\tilde{u_1} + u \cdot \nabla\tilde{u_2} - \nu\Delta\tilde{u_2})dx$$

$$= 2\int_{\mathbb{R}^N} (\tilde{u_1} - \tilde{u_2})(-u)\nabla(\tilde{u_1} - \tilde{u_2})dx + 2\int_{\mathbb{R}^N} \nu(\tilde{u_1} - \tilde{u_2})\Delta(\tilde{u_1} - \tilde{u_2})dx$$

We will expand on each term individually. Firstly we set $w = |\tilde{u_1} - \tilde{u_2}|^2$

$$\int_{\mathbb{R}^N} (\tilde{u_1} - \tilde{u_2}) \cdot (-u \cdot \nabla(\tilde{u_1} - \tilde{u_2}))dx = -\int_{\mathbb{R}^N} u\nabla|\tilde{u_1} - \tilde{u_2}|^2 dx = -\int_{\mathbb{R}^N} u \cdot \nabla w dx$$

We assume that we examine these solutions on a sufficiently small time interval, in which $\tilde{u_1} - \tilde{u_2}$ has compact support so

$$-\int_{\mathbb{R}^N} u \cdot \nabla w dx = \lim_{r\to\infty}\int_{B(x_0,r)} u \cdot \nabla w dx = \lim_{r\to\infty}\left(\int_{\partial B(x_0,r)} uwn dS - \int_{B(x_0,r)} w\nabla \cdot u dx\right)$$

$$= \lim_{r\to\infty} -\int_{B(x_0,r)} w\nabla \cdot u dx = -\int_{\mathbb{R}^N} w\nabla \cdot u dx$$

The other term

$$\int_{\mathbb{R}^N} \nu(\tilde{u_1} - \tilde{u_2})\Delta(\tilde{u_1} - \tilde{u_2})dx = \nu \lim_{r\to\infty} \int_{B(x_0,r)} (\tilde{u_1} - \tilde{u_2})\Delta(\tilde{u_1} - \tilde{u_2})dx$$

$$= -\nu \lim_{r\to\infty} \int_{B(x_0,r)} \nabla(\tilde{u_1} - \tilde{u_2}) \cdot \nabla(\tilde{u_1} - \tilde{u_2})dx = -\nu \int_{\mathbb{R}^N} |\nabla(\tilde{u_1} - \tilde{u_2})|^2 dx$$

So

$$\frac{d}{dt} \int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 dx = 2 \int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 \nabla \cdot u dx - 2\nu \int_{\mathbb{R}^N} |\nabla(\tilde{u_1} - \tilde{u_2})|^2 dx \Rightarrow$$

$$\frac{d}{dt} \int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 dx \leq 2 \int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 \nabla \cdot u dx$$

Thus

$$\frac{d}{dt} \int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 dx \leq 0$$

We will use the following lemma, in a more easy form. In the next chapters we will see the Gronwall's lemma and we will prove it.

Lemma 2. *(Simplified Gronwall's lemma) Let I=[0,a) be an interval an b a real valued function, and f a differentiable function on I such that* $\frac{d}{dt}f \leq bfinI$ *then* $f \leq f_0 e^{\int_a^t b(s)ds}$

So

$$\int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 dx(t) \leq e^0 \int_{\mathbb{R}^N} |\tilde{u_1(0)} - \tilde{u_2(0)}|^2 dx(0)$$

So assuming that initially $\tilde{u_1(0)} = \tilde{u_2(0)} = 0$ then

$$\int_{\mathbb{R}^N} |\tilde{u_1} - \tilde{u_2}|^2 dx(t) \leq 0$$

i.e. $\tilde{u_1} - \tilde{u_2} = 0 \forall t$ We conclude that the solution of this equation is unique, so if initially our velocity field is divergence free, then div$u$ will be zero at any time. □

Remark :In fact we project the Navier stokes equation on the space of divergence free vector fields in order to eliminate the pressure term. This projection is called the Leray's projection. And this is the Leray's formulation. We will discuss this projection a little more.

Proposition 2.1.2. *Let* $u \in L^2(\mathbb{R}^N)$ *then there exists an orthogonal decomposition such that:*

$$u = w + \nabla q$$

and

$$divw = 0$$

Proof. We define the following sets
$\mathcal{H}(\mathbb{R}^N)$ is the closure with the L^2 norm of the set

$$\mathfrak{C} = \{f \in C_c^\infty(\mathbb{R}^N) : \text{div} f = 0 \; in \; \mathbb{R}^N\}$$

and

$$G(\mathbb{R}^N) = \{f \in \mathbb{R}^N : f = \nabla g \; for \; g \in L^2_{loc}(\mathbb{R}^N)\}$$

So it is sufficient to show that u can decompose into u_1,u_2 where $u_1 \in \mathcal{H}$ and $u_2 \in G$ We know that an element u of a Hilbert space can decompose into two parts[5] i.e. $u+u_1+u_2$ where $u_1 \in D$ and D is a closed subspace of the Hilbert space, and $u_2 \in D^\perp$.
The $L^2(\mathbb{R}^N)$ may be considered as a Hilbert space with inner product :$(f,g) = \int_{\mathbb{R}^N} f \cdot gdx$, so it is sufficient to show that

1. $\mathcal{H}(\mathbb{R}^N)$ is a closed subspace of $L^2(\mathbb{R}^N)$
2. $\mathcal{H}^\perp(\mathbb{R}^N) = G(\mathbb{R}^N)$

The first is immediate from the definition of H, since :
Assume that $f, g \in \mathcal{H}$ and $\lambda \in \mathbb{R}^N$ then $\text{div}(f+g) = \text{div} f + \text{div} g = 0$, furthermore $\text{div}\lambda f = \lambda \text{div} f = 0$ so we are able to define those two relations in $\mathcal{H}$, so it is a subspace and it is closed because is the closure of $\mathfrak{C}$
For the second we have to prove the two inclusions

- $G \subset \mathcal{H}^\perp$
  Let $f \in G$ then f is a square integrable function and $f = \nabla g$ for some $g \in L^2_{loc}(\mathbb{R}^N)$. Assume that $h \in \mathcal{H}$ then

$$\int_{\mathbb{R}^N} f \cdot g = \int_{\mathbb{R}^N} \nabla g \cdot hdx = -\int_{\mathbb{R}^N} g\text{div}h = 0$$

  thus $f \in \mathcal{H}^\perp$

- $\mathcal{H}^\perp \subset G$
  Assume now that $f \in \mathcal{H}^\perp = \{f \in L^2 : \int_{\mathbb{R}^N} f \cdot g = 0 \forall \; g \in \mathcal{H}\}$ we want to find an h such that $f = \nabla h$ and $h \in L^2_{loc}$ so the problem comes down to solving the equation $f = \nabla h$

  Lemma 3. *Let $1 < q < \infty$ and $f \in W^{-1,q}_{loc}$[6] such that $\int_{\mathbb{R}^N} f \cdot vdx = 0$, $\forall v \in \mathfrak{C}(\mathbb{R}^N)$,then there exists a p such that $p \in L^q_{loc}$ and $\nabla p = f$ in the distribution sense*[7].

  So it is sufficient to show that $f \in W^{-1,2}_{loc}(\mathbb{R}^N)$ and also $\int_{\mathbb{R}^N} f \cdot gdx = 0$, $\forall g \in C(\mathbb{R}^N)$

[5][36] pg 94
[6]See the definition of Sobolev spaces on chapter 4
[7][36] pg 73, see section 2.2.2 the discussion about distributions

Assume that $V \subset\subset \mathbb{R}^N$, the space $W^{-1,2}(V)$ is the dual space of $W_0^{1,2}(V)$[8]
For the dual norm we have

$$\|f\|_{W^{-1,2}} = \sup_{\|g\|_{W_0^{1,2}} \leq 1} (f, g)$$

so by Cauchy Schwartz

$$\|f\|_{W^{-1,2}} \leq \sup_{\|g\|_{W_0^{1,2}} \leq 1} \|f\|_{L^2} \|g\|_{L^2} < \infty$$

Now assume a $g \in \mathfrak{C}(\mathbb{R}^N)$ since $\mathcal{H}$ is the closure of $\mathfrak{C}$ with the L^2 norm we have that $g \in \mathcal{H}$ and also $f \in \mathcal{H}^\perp$ so it is immediate that $\int_{\mathbb{R}^N} f \cdot g dx = 0$. So by lemma 3 there exists an $h \in L^2_{loc}$ such that $f = \nabla h$ and this completes the proof for the opposite inclusion.

□

We also have the following proposition for vector fields that are not L^2 but have a vanishing property[9]

Proposition 2.1.3. *Let u be a divergence free vector field in $\mathbb{R}^N$ and q a smooth scalar such that $|u||q| = \mathcal{O}[|x|^{1-N}] as |x| \to \infty$ then $\int_{\mathbb{R}^N} u \nabla q dx = 0$*

I.e. the divergence free vector fields and the gradient of the scalar are orthogonal in L^2.
Remark: The $\mathcal{O}$ is the big O which means that, let f, g two functions such that $f = \mathcal{O}[g]$ then there exists a constant c and a fixed point name x_0 such that $|f(x)| \leq cg(x)$ $\forall x \geq x_0$.

Proof. Let $B(0, r)$ be a sphere in $\mathbb{R}^N$ with surface area $c_N r^{N-1}$ then

$$|u(x)||q(x)| \leq Cr^{2(N-1)} \leq C|x|^{2(N-1)} \leq C|x|^{1-N}$$

Now

$$\int_{|x|\leq r} u \nabla q \, dx = \int_{|x|\leq r} \sum_i u_i \frac{\partial q}{\partial x_i} dx = \sum_i \int_{|x|\leq r} u_i \frac{\partial q}{\partial x_i} dx$$

integration by parts gives

$$= \sum_i \left(- \int_{|x|\leq r} q \frac{\partial u}{\partial x_i} dx + \int_{|x|=r} q u n_i \, dS \right)$$

$$= - \int_{|x|\leq r} q \sum_i \frac{\partial u}{\partial x_i} dx + \int_{|x|=r} q u n \, dS$$

[8][13] pg 291
[9]With this vanishing property we are able to do the decomposition of the velocity field via orthogonality.

since div$u = 0$ we get

$$\int_{|x|\leq r} u\nabla q dx = \int_{|x|=r} qun \, dS$$

As r tends to infinity the last integral tends to zero . □

Note: We define $P : L^2(\mathbb{R}^N) \to H(\mathbb{R}^N)$ such that for a $u \in L^2$ we have $Pu = w$, where P is the Leray or Helmholtz projection of L^2 to $\mathcal{H}$. We define Leray projection as the orthogonal projection from L^2 to the subset of divergence-free vector fields in L^2. We want to write the vector field $b \in L^2$ as $b = P(b) + \nabla p$. Since we want $P(b)$ to be in the space of divergence free vector fields, we take the divergence in the above equality So div$b = \Delta p$ i.e. $\nabla p = -\nabla(-\Delta)^{-1}\text{div}b$.[10] Consequently

$$\mathbb{P}(b) = b + \nabla(-\Delta)^{-1}(\text{div}b)$$

, where Δ^{-1} is the inverse of Laplace operator.
It is easy to see now that we can write the Navier-Stokes equation with the projection term as follows. From the Poisson equation we have

$$-\Delta p = \sum_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = \text{div}\Big(\sum_j u_j \frac{\partial}{\partial x_j} u\Big)$$

So $p = (-\Delta)^{-1}\text{div}(\sum_j u_j \frac{\partial}{\partial x_j} u)$.
Thus by Navier-Stokes $\frac{\partial}{\partial t}u + \sum_j u_j \frac{\partial}{\partial x_j} u = -\nabla p + \nu\Delta u$

$$\frac{\partial}{\partial t}u = -\sum_j u_j \frac{\partial}{\partial x_j} u - \nabla(-\Delta)^{-1}\text{div}\Big(\sum_j u_j \frac{\partial}{\partial x_j} u\Big) + \nu\Delta u$$

$$\frac{\partial}{\partial t}u = \mathbb{P}\Big(-\sum_j u_j \frac{\partial}{\partial x_j} u\Big) + \nu\Delta u$$

2.2 The vorticity-stream formulation for incompressible fluids

In this section we will deal with vorticity equation, which exported in the first chapter. We remember that there are two forms of this equation, one on 2 dimensions and the other on 3 dimensions. In the first case we also observe that we have a scalar equation, since the vorticity on 2d is a scalar quantity, but on 3d is a vector equation.
Furthermore, in the first term of both equations we have the material derivative, which involves the velocity field, which is related to the vorticity through the system

$$\begin{cases} \omega = \nabla \times u \\ \text{div}u = 0 \end{cases}$$

So in this chapter we will try to formulate the vorticity equation into a such way that the only terms are those involving vorticity. So at the end we will have an evolution equation of vorticity.

[10] [10], pg 14

2.2.1 Biot-Savart law

We want to determine the velocity through vorticity i.e. we want to solve $\begin{cases} \omega = \nabla \times u \\ \text{div} u = 0 \end{cases}$.
Since the vorticity is something different in 2 dimensions we will do this two times, one for each dimension. Finally we will see that we will come up to results that have some similarities between them.

2 dimensions

Firstly we will prove that for the divergence free velocity field $u = (u_1, u_2)$ exists a stream function.
Proof : Since div$u = 0$ the vector field $\breve{u} = (-u_2, u_1)$ is conservative, indeed
We define $F = -u_2 dx_1 + u_1 dx_2$
By Green's theorem[11]

$$\oint_C F dl = \iint_D \frac{\partial u_1}{\partial x_1} - (-\frac{\partial u_2}{\partial x_2}) = 0$$

So there exists a gradient field ψ such that $\breve{u} = -\nabla\psi$ so $u_1 = -\frac{\partial \psi}{\partial x_2}$ and $u_2 = \frac{\partial \psi}{\partial x_1}$
Finally $u = \begin{pmatrix} \frac{\partial \psi}{\partial x_2} \\ \frac{\partial \psi}{\partial x_1} \end{pmatrix}$
Next we will calculate the curl of this vector by expanding it to the three dimensions with the third coordinate to be zero.

$$\text{curl} u = \nabla \times u = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} \end{pmatrix}$$

So we define ω be the scalar quantity $\omega = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = \Delta\psi$ This is a Poisson equation, since we examine the 2d case the Newtonian potential is $g(x) = -\frac{1}{2\pi} \ln|x|$ so the solution will be

$$\psi = \int_{\mathbb{R}^2} g(x-y)(-\omega(y))dy$$

$$\psi = \int_{\mathbb{R}^2} \frac{1}{2\pi} \ln|x-y|\omega(y)dy$$

So we will differentiate the above equality with respect to x_2 and x_1 in order to define the vector u

$$\frac{\partial \psi}{\partial x_2} = \frac{1}{2\pi} \frac{\partial}{\partial x_2} \int_{\mathbb{R}^2} \ln|x-y|\omega(y)dy$$

By Leibniz integral rule

$$\frac{\partial \psi}{\partial x_2} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\partial}{\partial x_2} \ln|x-y|\omega(y)dy$$

[11] [1] pg 293

For the term inside the integral we have that

$$\frac{\partial}{\partial x_2}\ln|x-y| = \frac{1}{|x-y|}\frac{\partial}{\partial x_2}|x-y| = \frac{1}{|x-y|}\frac{\partial}{\partial x_2}\sqrt{(x-y)(x-y)}$$

$$= \frac{1}{|x-y|}\frac{1}{2\sqrt{(x-y)(x-y)}}\frac{\partial}{\partial x_2}(x-y)(x-y)$$

$$= \frac{1}{2|x-y|^2}\frac{\partial}{\partial x_2}((x_1-y_1)^2+(x_2-y_2)^2) = \frac{1}{2|x-y|^2}2(x_2-y_2)$$

Finally $\frac{\partial \psi}{\partial x_2} = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_2-y_2}{|x-y|^2}\omega(y)dy$
Similarly $\frac{\partial \psi}{\partial x_1} = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_1-y_1}{|x-y|^2}\omega(y)dy$
Therefore $u_1(x,t) = \int_{\mathbb{R}^2} -\frac{1}{2\pi}\frac{x_2-y_2}{|x-y|^2}\omega(y,t)dy$ and $u_2 = \int_{\mathbb{R}^2}\frac{1}{2\pi}\frac{x_1-y_1}{|x-y|^2}\omega(y,t)dy$
I.e.

$$u(x,t) = \int_{\mathbb{R}^2}\frac{1}{2\pi}\begin{pmatrix} -\frac{x_2-y_2}{|x-y|^2} \\ \frac{x_1-y_1}{|x-y|^2}\end{pmatrix}\omega(y,t)dy \tag{2.1}$$

We define $K_2(x) = \frac{1}{2\pi}\begin{pmatrix} -\frac{x_2}{|x|^2} \\ \frac{x_1}{|x|^2}\end{pmatrix}$
Remark: This is an integral transform, the kernel K_2 is homogeneous of degree -1.
Indeed $K_2(\lambda x) = \frac{1}{2\pi}\begin{pmatrix} -\frac{\lambda x_2}{\lambda|x|^2} \\ \frac{\lambda x_1}{\lambda|x|^2}\end{pmatrix} = \lambda^{-1}K_2(x)$
We will name the formula 2.1 Biot-Savart law.

3 dimensions

Now will we solve the system we saw in the introduction using the following proposition

Proposition 2.2.1. *Let u vanishes sufficiently rapidly as $|x| \to \infty$ then the above system has a smooth solution*

$$u(x,t) = \int_{\mathbb{R}^3}\frac{1}{4\pi}\frac{(x-y)\times\omega(y,t)}{|x-y|^3} \tag{2.2}$$

In order to proof the above we will use the following lemma

Lemma 4 (Helmholtz decomposition on 3 dimensions)**.** [12] *Let F be a vector field vanishes rapidly as $x \to \infty$ then F can decomposed into a curl free and an divergence free component*
I.e. $F = \nabla\Phi + \nabla \times A$

$$\Phi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} F(y)\cdot\nabla_y\frac{1}{|x-y|}dy$$

$$A(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} F(y)\times\nabla_y\frac{1}{|x-y|}dy$$

[12] [29]

Helmholtz decomposition states that any vector field can described as a curl-free term ($\nabla\Phi$) and a divergence free term ($\nabla \times A$)

Proof. Let F a vector field as above we will use the Dirac delta function in three dimensions to expand the integral $\delta_3(x-y) = -\frac{1}{4\pi}\nabla^2\frac{1}{|x-y|}$ [13]
So $F(x) = \int_{\mathbb{R}^3} F(y)\delta_3(x-y)dy$

$$F(x) = \int_{\mathbb{R}^3} -\frac{1}{4\pi}F(y)\nabla^2\frac{1}{|x-y|}$$

Since the gradient is taken with respect to x

$$F(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla^2\frac{F(y)}{|x-y|}dy$$

By Leibniz integral rule

$$F(x) = -\frac{1}{4\pi}\nabla^2\int_{\mathbb{R}^3} \frac{F(y)}{|x-y|}dy$$

We will use the following identity for the Laplace operator $\nabla^2 F = \nabla(\nabla\cdot F) - \nabla\times(\nabla F)$

proof of identity:

$$\nabla(\nabla\cdot F) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_1\partial x_2} + \frac{\partial^2 F_3}{\partial x_1\partial x_3} \\ \frac{\partial^2 F_1}{\partial x_1\partial x_2} + \frac{\partial^2 F_2}{\partial x_2^2} + \frac{\partial^2 F_3}{\partial x_2\partial x_3} \\ \frac{\partial^2 F_1}{\partial x_1\partial x_3} + \frac{\partial^2 F_2}{\partial x_2\partial x_3} + \frac{\partial^2 F_3}{\partial x_3^2} \end{pmatrix}$$

$$\nabla\times F = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$$

$$\nabla\times(\nabla\times F) = \begin{pmatrix} \frac{\partial^2 F_2}{\partial x_2\partial x_1} - \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial x_3^2} + \frac{\partial^2 F_3}{\partial x_3\partial x_1} \\ \frac{\partial^2 F_3}{\partial x_3\partial x_2} - \frac{\partial^2 F_2}{\partial x_3^2} - \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_1\partial x_2} \\ \frac{\partial^2 F_1}{\partial x_1\partial x_3} - \frac{\partial^2 F_3}{\partial x_1^2} - \frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial x_2\partial x_3} \end{pmatrix}$$

So $\nabla(\nabla\cdot F) - \nabla\times(\nabla\times F) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_2^2} + \frac{\partial^2 F_1}{\partial x_3^2} \\ \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial x_3^2} \\ \frac{\partial^2 F_3}{\partial x_1^2} + \frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_3}{\partial x_3^2} \end{pmatrix} = \frac{\partial^2}{\partial x_1^2}F + \frac{\partial^2}{\partial x_2^2}F + \frac{\partial^2}{\partial x_3^2}F = \nabla^2 F$

, so we will create the Φ and A term. Therefore $F(x) = -\frac{1}{4\pi}(\nabla(\nabla\cdot\int_{\mathbb{R}^3}\frac{F(y)}{|x-y|}dy)) + \frac{1}{4\pi}(\nabla\times(\nabla\times\int_{\mathbb{R}^3}\frac{F(y)}{|x-y|}dy))$

- First term:

$$\nabla\cdot\int_{\mathbb{R}^3}\frac{F(y)}{|x-y|}dy = \int_{\mathbb{R}^3}\nabla\cdot\frac{F(y)}{|x-y|}dy$$

[13]For the derivation of this form check [26], pg.35

We will use the property $\mathrm{div}(fF) = f\mathrm{div}F + F \cdot \nabla f$

$$\nabla \cdot \int_{\mathbb{R}^3} \frac{F(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \frac{1}{|x-y|} \mathrm{div}F(y) + F(y) \cdot \nabla \frac{1}{|x-y|} dy$$

The first term is zero since the $\mathrm{div}F(y) = 0$ because the divergence is taken with respect to x.
So $\nabla \cdot \int_{\mathbb{R}^3} \frac{F(y)}{|x-y|} dy = \int_{\mathbb{R}^3} F(y) \cdot \nabla \frac{1}{|x-y|} dy = -\int_{\mathbb{R}^3} F(y) \cdot \nabla_y \frac{1}{|x-y|} dy$ [14]

- Second term:

$$\nabla \times \int_{\mathbb{R}^3} \frac{F(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \nabla \times \frac{F(y)}{|x-y|} dy$$

We will use the property $\mathrm{curl}(fF) = f\mathrm{curl}F + \nabla f \times F$

$$\nabla \times \int_{\mathbb{R}^3} \frac{F(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \frac{1}{|x-y|} (\nabla \times F(y) + \nabla \frac{1}{|x-y|} \times F(y) dy$$

The first term is zero since $\nabla \times F(y) = 0$ because the curl is taken with respect to x.

$$\nabla \times \int_{\mathbb{R}^3} \frac{F(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \nabla \frac{1}{|x-y|} \times F(y) dy = -\int_{\mathbb{R}^3} \nabla_y \frac{1}{|x-y|} \times F(y) dy$$

After all $F(x) = \frac{-1}{4\pi} \nabla (-\int_{\mathbb{R}^3} F(y) \cdot \nabla_y \frac{1}{|x-y|} dy) + \frac{1}{4\pi} \nabla \times (-\int_{\mathbb{R}^3} \nabla_y \frac{1}{|x-y|} \times F(y) dy)$

$$F(x) = \nabla (\frac{1}{4\pi} \int_{\mathbb{R}^3} F(y) \cdot \nabla_y \frac{1}{|x-y|} dy) + \nabla \times (\frac{1}{4\pi} \int_{\mathbb{R}^3} F(y) \times \nabla_y \frac{1}{|x-y|} dy)$$

□

Lemma 5 (Helmholtz corollary). *Let F be a vector field of which we know its divergence and its curl functions, namely $P = \nabla \cdot F$ and $Q = \nabla \times F$, then*

$$F(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{P(y)(x-y)}{|x-y|^3} dy - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{Q(y) \times (x-y)}{|x-y|^3} dy$$

Proof. By Helmholtz decomposition $F(x) = \nabla (\frac{1}{4\pi} \int_{\mathbb{R}^3} F(y) \cdot \nabla_y \frac{1}{|x-y|} dy) + \nabla \times (\frac{1}{4\pi} \int_{\mathbb{R}^3} F(y) \times \nabla_y \frac{1}{|x-y|} dy)$
We have $F(y) \cdot \nabla_y \frac{1}{|x-y|} = \nabla_y (\frac{1}{|x-y|} F) - \frac{1}{|x-y|} \nabla_y \cdot F(y)$ and $F(y) \times \nabla_y \frac{1}{|x-y|} = -\nabla_y \times \frac{F(y)}{|x-y|} + \frac{1}{|x-y|} \nabla_y \times F(y)$
Consequently $F(x) = \nabla (\frac{1}{4\pi} (\int_{\mathbb{R}^3} \nabla_y (\frac{F(y)}{|x-y|} dy) - \int_{\mathbb{R}^3} \frac{1}{|x-y|} \nabla_y \cdot F(y) dy)) + \nabla \times (\frac{1}{4\pi} (-\int_{\mathbb{R}^3} \nabla_y \times (\frac{F(y)}{|x-y|} dy) + \int_{\mathbb{R}^3} \frac{1}{|x-y|} \nabla_y \times F(y) dy)$
By divergence and Stokes theorems :

$$F(x) = -\frac{1}{4\pi} \nabla \int_{\mathbb{R}^3} \frac{\nabla_y \cdot F(y)}{|x-y|} dy + \frac{1}{4\pi} \nabla \times \int_{\mathbb{R}^3} \frac{\nabla_y \times F(y)}{|x-y|} dy$$

[14] $\nabla \frac{1}{|x-y|} = -\frac{x-y}{|x-y|^3}$ and $\nabla_y \frac{1}{|x-y|} = \frac{x-y}{|x-y|^3}$

$$F(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla \frac{\nabla_y \cdot F(y)}{|x-y|} dy + \frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla \times \frac{\nabla_y \times F(y)}{|x-y|} dy \Rightarrow$$

$$F(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\nabla_y \cdot F(y))(x-y)}{|x-y|^3} dy + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times(\nabla_y \times F(y))}{|x-y|^3} dy$$

After all $F(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{P(y)(x-y)}{|x-y|^3} dy + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times Q(y)}{|x-y|^3} dy$ □

Proof of Proposition 2.2.1. Using lemma 5 for the velocity field u where $P = \mathrm{div} u = 0$ and $Q = \mathrm{curl} u = \omega$ we get $u(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\omega(y,t)}{|x-y|^3} dy$ □

We use again the terminology we use in the 2d flows and name the formula 2.2 Biot-Savart law. We also define $K_3(x)h = \frac{1}{4\pi}\frac{x\times h}{|x|^3}$ and $h \in \mathbb{R}^3$ which is a homogeneous kernel of degree -2.
Indeed $K_3(\lambda x)h = \frac{1}{4\pi}\frac{\lambda x\times h}{|\lambda x|^3} = \lambda^{-2}K_3(x)$.
Closing this subsection we observe that for both cases we have a non-local operator given by convolution. The kernels we deal with are homogeneous of degree 1-N with a singularity at the origin. This integral operator is defined on $\mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$.
In the next chapter, we will use this solution to formulate the vorticity equation. We emphasize the fact that the Biot-Savart law has not occurred by the vorticity equation and up to now has no relation to it. So in the next subsection, we will see some differences in the process of the formulation which do not arise from the Biot Savart law but from the form of the vorticity equation.

2.2.2 Vorticity-Stream Formulation

Let us recall the form of vorticity equation $\begin{cases} \frac{\partial}{\partial t}\omega + \sum_j u_j \frac{\partial}{\partial x_j}\omega = \nu\Delta\omega & (2d) \\ \frac{\partial}{\partial t}\omega + \sum_j u_j \frac{\partial}{\partial x_j}\omega = \nabla u\omega + \nu\Delta\omega & (3d) \end{cases}$
It is clear that in two dimensions we will just substitute the velocity by the Biot Savart law on the vorticity equation and we are done. But in three dimensions there are two terms containing velocity, the problem is for the term involving the gradient of u. The truth is that we cannot differentiate the velocity field since the derivative of the kernel by doing heuristic calculations will be of degree -N. So it's singularity is of type $\frac{1}{|x|^N}$ which is not integrable on $\mathbb{R}^N$[15]. So since the classical concept of differentiation fails we will develop the following theory for the distribution derivative.

Distribution derivative

The main idea for using the distributions is to generalize concepts which do not exist in the classic sense(solutions, derivatives etc). To do this we replace the function f by

[15]

$$\int_{B(0,r)} \frac{1}{|x|^a} dx = \int_0^r \int_{\partial B(0,t)} \frac{1}{|x|^a} dS(x) dt = \int_0^r \frac{1}{t^a}\int_{\partial B(0,t)} 1 dS(x) dt = \int_0^r \frac{1}{t^a} t^{n-1} n a(n) dt = c r^{n-a}$$

which is bounded for $a < n$

an integral using test functions.
So a distribution is a functional: $f[\phi] = \int_{\mathbb{R}^N} f(x)\phi(x)dx$ where $\phi \in C_c^\infty(\mathbb{R}^N)$. We will say that $f \in \acute{D}(\mathbb{R}^N)$, where $\acute{D}$ is the space of distributions, which is the space of linear functionals on the vector space of test functions (D).
So we will define the derivative of the distribution as the linear functional :

$$D^{|a|}f[\phi] = \int_{\mathbb{R}^N} \phi D^{|a|} f dx$$

integration by parts gives that,

$$D^{|a|}f[\phi] = \int_{\mathbb{R}^N} \phi D^{|a|} f dx = -\int_{\mathbb{R}^N} (D^{|a|}\phi) f dx$$

To develop the theory we need we will take the following case :

$$\partial_{x_j} f[\phi] = \int_{\mathbb{R}^N} \frac{\partial f}{\partial x_j}\phi dx = -\int_{\mathbb{R}^N} f\frac{\partial \phi}{\partial x_j}$$

We remember that our problem was to differentiate a function homogeneous of degree 1-N which is $C^\infty(\mathbb{R}^N/\{0\})$, that is $f(\lambda x) = \lambda^{1-N} f(x) \forall \lambda > 0$ so we will prove the following proposition

Proposition 2.2.2. *Let f be a function as above, then its distribution derivative is given by the following formula*

$$\int_{\mathbb{R}^N} \frac{\partial f}{\partial x_j}\phi dx = -\int_{\mathbb{R}^N} f\frac{\partial \phi}{\partial x_j} = \text{P.V.} \int_{\mathbb{R}^N} \frac{\partial f}{\partial x_j}\phi dx - c_j \int_{\mathbb{R}^N} \delta\phi dx, \forall \phi \in C_c^\infty(\mathbb{R}^N)$$

Notes:

- The first term is Cauchy's principle value (we will explain this in the proof)
- $c_j = \int_{|x|=1} f(x)x_j dS$
- $\int_{\mathbb{R}^N} \delta\phi dx$ is the Dirac distribution [16]. We have that $\int_{\mathbb{R}^N} \delta\phi dx = 0$

Proof. First of all in order to define the distribution derivative we will prove that homogeneous functions of degree 1-N smooth away 0, say f, are locally integrable.

Lemma 6. $f \in L^1_{loc}(\mathbb{R}^N)$

proof of lemma : I want to prove that $\forall K \subset\subset \mathbb{R}^N$ then $\int_K |f(x)|dx < \infty$
$\forall x$ holds $x = |x|\frac{x}{|x|}$, we define $r = |x|$ and $y = \frac{x}{|x|}$, $y \in S^{N-1}$ (observe that $|\frac{x}{|x|}| = \frac{|x|}{|x|} = 1$)
We set $x = ry$, where $r \in \mathbb{R}^+$ and $y \in \Lambda = \{y \in S^{N-1} : ry \in K\}$ [a]
So

$$\int_K |f(x)|dx = \int_0^t \int_\Lambda |f(ry)|r^{N-1} dS(y)dr$$

[16][24] pg 7 and 29

f is homogeneous of degree 1-N

$$= \int_0^t \int_\Lambda |f(y)| \frac{1}{r^{N-1}} r^{N-1} dS(y) dr = \int_0^t \int_\Lambda |f(y)| dS(y) dr \ \leq \int_0^t c r^{N-1}$$

$1 - N < 1$ so $\int_0^t c r^{N-1} < \infty$

[a] Λ is the projection onto the sphere of a slice of K

So now we can define the derivative in the distribution sense as following :

$$\partial_{x_j} f[\phi] = \int_{\mathbb{R}^N} \frac{\partial f(x)}{\partial x_j} \phi(x) dx = - \int_{\mathbb{R}^N} f(x) \frac{\partial \phi(x)}{\partial x_j} dx, \forall \phi \in C_c^\infty(\mathbb{R}^N)$$

We will deal with the second integral:
We define $g_N = \chi_{[|x| \geq \frac{1}{N}]} f(x) \frac{\partial}{\partial_{x_j}} \phi$, where $\chi_{[|x| \geq \frac{1}{N}]}$ is the characteristic function of the set $\{|x| \geq \frac{1}{N}\}$ i.e.

$$\chi = \chi_{[|x| \geq \frac{1}{N}]} = \begin{cases} 1 & x \in \mathbb{R}^N \backslash \bar{B}(0, \frac{1}{N}) \\ 0 & x \in \bar{B}(0, \frac{1}{N}) \end{cases}$$

We will use the dominated convergence theorem (DCT) which states: Let g_N be a sequence of measurable functions(1), so that $g_N \to g$ as $N \to \infty$ a.e. x(2). If there exist a function f integrable with the property $|g_N(x)| \leq f(x)$(3), then $\int_{\mathbb{R}^N} g_N(x) dx \to \int_{\mathbb{R}^N} g(x) dx$[17].
(1)First of all our g_N is measurable since f(x) is continuous a.e. x, ϕ is $C_c^\infty(\mathbb{R}^N)$ and χ is the characteristic function.
Now we will prove (2) i.e.

$$\forall x \in \mathbb{R}^N \backslash \{0\}, \text{and}, \forall \epsilon > 0, \exists N_0 \in \mathbb{N} : \forall N \geq N_0, \text{holds}, |\chi f(x) \frac{\partial}{\partial_{x_j}} \phi(x) - f(x) \frac{\partial}{\partial_{x_j}} \phi(x)| < \epsilon$$

Let $\epsilon > 0$ since $\frac{1}{N} \to 0$ as $N \to \infty$ we have that

$$\forall \zeta, \exists N_1 : \forall N \geq N_1, \frac{1}{N} < \zeta$$

We choose $\zeta = |x|$ thus $\exists N_1$ which depends on |x|, such that $\frac{1}{N} < |x|$
Thus from a N_1 and then $x \in \mathbb{R}^N \backslash \bar{B}(0, \frac{1}{N})$.
We set $N_0 = N_1$, and we get $\forall \epsilon > 0, \exists N_0 = N_1 : \forall N \geq N_1, \text{holds}, |\chi f(x) \frac{\partial}{\partial_{x_j}} \phi(x) - f(x) \frac{\partial}{\partial_{x_j}} \phi(x)| = |f(x) \frac{\partial}{\partial_{x_j}} \phi(x) - f(x) \frac{\partial}{\partial_{x_j}} \phi(x)| = 0 < \epsilon$.
We will prove the (3).

$$|g_n(x)| = |\chi f(x) \frac{\partial}{\partial_{x_j}} \phi(x)| \leq |f(x) \frac{\partial}{\partial_{x_j}} \phi(x)|$$

The last function is integrable since ϕ is smooth and has compact support, then it's derivative has compact support, so outside of this compact set is zero, and by the lemma

[17] [38] pg 67

6 f is integrable on compact subsets of $\mathbb{R}^N$.
Since all the conditions of the DCT are satisfied, we have

$$\int_{\mathbb{R}^N} \chi f(x) \frac{\partial}{\partial_{x_j}} \phi(x) dx \to \int_{\mathbb{R}^N} f(x) \frac{\partial}{\partial_{x_j}} \phi(x) dx, \text{as} N \to \infty$$

$$\int_{|x| \geq \frac{1}{N}} f(x) \frac{\partial}{\partial_{x_j}} \phi(x) dx \ to \int_{\mathbb{R}^N} f(x) \frac{\partial}{\partial_{x_j}} \phi(x) dx$$

After all, for $\epsilon = \frac{1}{N}, \text{for}, N \to \infty, \text{we have}, \epsilon \searrow 0$ we conclude to:

$$\int_{\mathbb{R}^N} f(x) \frac{\partial}{\partial_{x_j}} \phi(x) dx = \lim_{\epsilon \searrow 0} \int_{|x| \geq \epsilon} f(x) \frac{\partial}{\partial_{x_j}} \phi(x) dx$$

By integration by parts

$$\lim_{\epsilon \searrow 0} \int_{|x| \geq \epsilon} f(x) \frac{\partial}{\partial_{x_j}} \phi(x) dx = \lim_{\epsilon \searrow 0} \left(- \int_{|x| \geq \epsilon} \frac{\partial f(x)}{\partial x_j} \phi(x) dx + \int_{|x| = \epsilon} f(x) \phi(x) n_j dS \right)$$

- For the first term we define Cauchy's principal value In the Riemannian integration the range of integration is finite, so the improper integrals used for infinite range or unbounded functions. Cauchy principal value serves for define a value for the improper integrals[18]. We assume that f is integrable outside of a small ball of radius ϵ. We define the principal value of this function as: $\text{P.V.} \int_{\mathbb{R}^N} f dx = \lim_{\epsilon \searrow 0} \int_{|x| \geq \epsilon} f dx$
  So in our case

$$\lim_{\epsilon \searrow 0} \int_{|x| \geq \epsilon} \frac{\partial f(x)}{\partial x_j} \phi(x) dx = \text{P.V.} \int_{\mathbb{R}^N} \frac{\partial f(x)}{\partial x_j} \phi(x) dx$$

- For the second term

$$\int_{|x| = \epsilon} f(x) \phi(x) n_j dS = \int_{|x| = \epsilon} f \phi \frac{x_j}{|x|} dS$$

  We will do a change of variables $x = \epsilon y$ so

$$\int_{|y| = 1} \frac{1}{\epsilon^{N-1}} f(y) \phi(\epsilon y) y_j \epsilon^{N-1} dS(y)$$

  So, $\lim_{\epsilon \searrow 0} \int_{|x| = \epsilon} f(x) \phi(x) n_j dS = \lim_{\epsilon \searrow 0} \int_{|x| = 1} f(x) \phi(\epsilon x) x_j dS(x) = \int_{|x| = 1} f(x) \phi(0) x_j dS(x)$

Eventually we have proof that

$$\lim_{\epsilon \searrow 0} \int_{|x| \geq \epsilon} f(x) \frac{\partial}{\partial x_j} \phi(x) dx = -\text{P.V.} \int_{\mathbb{R}^N} \frac{\partial f(x)}{\partial x_j} \phi(x) dx + \phi(0) \int_{|x| = 1} f(x) x_j dS(x)$$

□

[18] [27]

Remark: The function $\frac{\partial}{\partial x_j} f$ is homogeneous of degree -N. This follows from the proposition below.

Proposition 2.2.3. *If the distribution $f[\phi]$ is homogeneous of degree a then f is homogeneous of degree a.* [19]

Proof. **Definition**: A distribution is $f[\phi]$ is homogeneous of degree a if $f[\phi_\lambda] = \lambda^{-a-N} f[\phi]$
So $\int_{\mathbb{R}^N} f(x)\phi(\lambda x)dx = \lambda^{-a-N}\int_{\mathbb{R}^N} f(x)\phi(x)dx$
By changing variables we get:

$$\int_{\mathbb{R}^N} \lambda^{-N} f(\lambda^{-1}x)\phi(x)dx = \lambda^{-a-N}\int_{\mathbb{R}^N} f(x)\phi(x)dx$$

$$\int_{\mathbb{R}^N} (f(\lambda^{-1}x) - \lambda^{-a} f(x))\phi(x)dx = 0$$

This is true for every test function, so

$$f(\lambda^{-1}x) - \lambda^{-a} f(x) = 0$$

i.e $f(\lambda x) = \lambda^a f(x)$ □

So for the function:

$$\partial_{x_j} f[\phi_\lambda] = -\int_{\mathbb{R}^N} f(x)\frac{\partial}{\partial x_j}\phi(\lambda x)dx = -\int_{\mathbb{R}^N} f(\frac{y}{\lambda})\lambda\frac{\partial}{\partial y_j}\phi(y)\frac{1}{\lambda^N}dy$$

$$= -\int_{\mathbb{R}^N} \lambda^{N-1} f(y)\lambda\frac{1}{\lambda^N}\frac{\partial}{\partial y_j}\phi(y)dy = -\int_{\mathbb{R}^N} f(x)\frac{\partial}{\partial x_j}\phi(x) = \partial_{x_j} f[\phi(x)]$$

So the distribution is homogeneous of degree zero i.e. $-a - N = 0$, $a = -N$
We close this sub-subsection with two propositions, that will by useful in the following sections.

Proposition 2.2.4. *Let $f \in C^\infty(\mathbb{R}^N \backslash \{0\})$ homogeneous of degree 1-N, then the function $\frac{\partial}{\partial x_j} f$ has mean-value zero on the unit sphere.*

Note:The mean-value of a function is $\frac{1}{s_{N-1}(r)}\int_{\partial B(x,r)} f dS$, where s^{N-1} is the surface area of N-1 sphere. So we need to prove that

$$\frac{1}{s_{N-1}}\int_{|x|=1}\frac{\partial}{\partial x_j} f dS = 0$$

i.e.

$$\int_{|x|=1}\frac{\partial}{\partial x_j} f dS = 0$$

[19] [31]

Proof. We want to see what happens for the mean value on the unit sphere so we will use a cut-off function to mollify our function and get rid of the singularity. Inside of a sphere, we will have something that is integrable everywhere and outside of the sphere we will have the zero function. We set

$$p(r) = \begin{cases} sinr & |r| \leq \pi \\ 0 & |r| > 2\pi \end{cases}$$

And we define $g(x) = p(|x|)\frac{\partial}{\partial x_j}f(x)$ with domain of the function $D_g = \{|x| \leq \pi\} \cup \{|x| > 2\pi\}$
We will examine the integrability of g(x)

- outside of the sphere of radius 2π, $g(x) = 0$
- inside of the sphere of radius π the only problem is on the singularity $x = 0$, since p has compact support and the function $\frac{\partial}{\partial x_j}f(x)$ is smooth away zero, and homogeneous of degree -N. It's singularity is of type $\frac{1}{|x|^N}$ but we know that $\frac{sin|x|}{|x|^N}$ is integrable in all $\mathbb{R}^N$.

So

$$\int_{|x|\leq\pi\cup|x|>2\pi} p(|x|)\frac{\partial}{\partial x_j}f(x)dx = \int_{|x|\leq\pi} p(|x|)\frac{\partial}{\partial x_j}f(x)dx + \int_{|x|>2\pi} p(|x|)\frac{\partial}{\partial x_j}f(x)dx$$

$$= \int_{|x|\leq\pi} p(|x|)\frac{\partial}{\partial x_j}f(x)dx = -\int_{|x|\leq\pi} \frac{\partial p(|x|)}{\partial x_j}f(x) + \int_{|x|=\pi} p(|x|)f(x)\frac{x_j}{|x|}dx$$

I.e.

$$\int_{|x|\leq\pi} p(|x|)\frac{\partial}{\partial x_j}f(x)dx + \int_{|x|\leq\pi} f(x)\frac{\partial p(|x|)}{\partial x_j}\frac{x_j}{|x|}dx = 0$$

- $\int_{|x|\leq\pi} p(|x|)\frac{\partial}{\partial x_j}f(x)dx = \int_0^\pi \int_{|x|=r} p(r)\frac{\partial}{\partial x_j}f(rx)r^{N-1}dS(x)dr$

  $$= \int_0^\pi \int_{|x|=r} \frac{p(r)}{r}\frac{\partial}{\partial x_j}f(x)dS(x)dr$$

  By Fubini's theorem: $\int_{|x|=r} \frac{\partial}{\partial x_j}f(x)\int_0^\pi \frac{p(r)}{r}drdS(x)$
  $\int_0^\pi \frac{p(r)}{r}dr = \int_0^\pi \frac{sinr}{r} = c \neq 0$
  So $\int_{|x|\leq\pi} p(|x|)\frac{\partial}{\partial x_j}f(x)dx = c\int_{|x|=r} \frac{\partial}{\partial x_j}f(x)dS(x) = c\int_{|x|=1} \frac{\partial}{\partial x_j}f(x)dS(x)$

- $\int_{|x|=\pi} \acute{p}(|x|)\frac{x_j}{|x|}f(x)dx = \int_0^\pi \int_{|x|=r} \acute{p}(r)x_j r^{N-1}f(rx)dS(x)dr$

  $$= \int_0^\pi \int_{|x|=r} \acute{p}x_j f(x)dS(x)dr$$

  By Fubini's theorem: $\int_{|x|=r} x_j f(x)\int_0^\pi \acute{p}(r)drdS(x)$
  We calculate $\int_0^\pi \acute{p}(r)dr = \sin\pi - \sin 0 = 0$
  So $\int_{|x|=\pi} \acute{p}(|x|)\frac{x_j}{|x|}f(x)dx = 0$

To conclude $c\int_{|x|=1}\frac{\partial}{\partial x_j}f(x)dS(x) = 0$ □

The kernels we have are homogeneous of degree -N smooth away zero, like the above functions. We will prove the following proposition.

Proposition 2.2.5. *Let $P(x) = \nabla K(x)$ in the distribution sense - we will say that P is the gradient kernel. If P is homogeneous of degree -N, smooth away zero, and has the mean value property, that we have discuss above. Then P(x) defines a singular integral operator through the convolution.*

$$P[f](x) = \lim_{\epsilon\searrow 0}\int_{|x-y|\geq\epsilon} P(x-y)f(y)dy$$

Note: A singular integral operator is of form $T[f](x) = \int K(x-y)f(y)dy$, with singularity on the diagonal of type $\frac{1}{|x-y|^N}$,(SIO). A singular integral operator of convolution type is a singular integral operator exists on $\mathbb{R}^N$ through convolution by distributions.

Proof. We will prove that the mean value property is enough to ensure the existence of the above limit. [20]
We define $P_\epsilon[f](x) = \int_{|y|\geq\epsilon} P(y)f(x-y)dy$, where $f \in C_c^\infty$
Since $\epsilon \searrow 0$ we assume that $\epsilon = \frac{1}{N}$ as $N \to \infty$, so $\frac{1}{N} \leq 1$ we get.

$$\int_{|y|\geq\epsilon} P(y)f(x-y)dy = \int_{|y|\geq 1} P(y)f(x-y)dy + \int_{\epsilon\leq|y|\leq 1} P(y)f(x-y)dy$$

- $\lim_{\epsilon\searrow 0} = \int_{|y|\geq 1} P(y)f(x-y)dy$ exists since on this range both functions are integrable, the matter is on range containing zero.
- $\lim_{\epsilon\searrow 0} int_{|y|\geq 1}P(y)f(x-y)dy$ Since f has compact support in this compact support we give a value k

  $$= k\lim_{\epsilon\searrow 0}\int_\epsilon^1\int_{\partial B(0,1)} r^{N-1}P(ry)dS(y)dr = k\lim_{\epsilon\searrow 0}\int_\epsilon^1\int_{\partial B(0,1)} r^{N-1}\frac{1}{r^N}P(y)dS(y)dr$$

  Since $\frac{1}{r}$ is not integrable on a small ball containing zero, we need $\int_{|y|=1} P(y)dS(y)$ to be zero.

□

Calculating the gradient of the velocity field in the distribution sense

In this sub-subsection, we will apply the previous theory to handle the problem of differentiating the velocity u which is determined by the vorticity through Biot-Savart law.

Proposition 2.2.6. *Let $u = \int_{\mathbb{R}^N} K_N(x-y)\omega(y,t)dy$ then*

[20] [38] pg38-45

- *In 2d the $\nabla u(x)$, in the distribution sense, is defined as*

$$\nabla u(x) = \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2} \frac{k(x-y)}{|x-y|^2}\omega(y)dy + \frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Where $k(x)$ is the 2×2 matrix kernel: $\frac{1}{|x|^2}\begin{pmatrix} 2x_1x_2 & x_2^2 - x_1^2 \\ x_2^2 - x_1^2 & -2x_1x_2 \end{pmatrix}$

- *In 3d the ∇u, in the distribution sense, is defined as :*

$$[\nabla u(x)]h = -\text{P.V.}\int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{\omega(y)\times h}{|x-y|^3} + \frac{3}{4\pi}\frac{\{[(x-y)\times\omega(y)]\otimes(x-y)h\}}{|x-y|^5}dy + \frac{1}{3}\omega(x)\times h$$

We see that the proposition has two cases, thus we will move on with the proof individually for each case.

Proof. For this proof we will use the proposition 2.2.2

- For the 2 dimensions : $u(x) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y)dy$ We can write the convolution as an inner product $u(x) =< \check{K_2}(y), \omega(y) >$, where $\check{K_2}(y)$ is $K_2(-y)$ translated by x. We want to calculate the j-th derivative of u in the distribution sense.

$$\partial_{x_j} u(x) = \partial_{x_j} < \check{K_2}(y), \omega(y) > = < \partial_{x_j}\check{K_2}(y), \omega(y) > = \partial_{x_j}\check{K}_2[\omega](x)$$

So we get

$$\partial_{x_j} u(x) = \text{P.V.}\int_{\mathbb{R}^2} \frac{\partial}{\partial x_j}K_2(x-y)\omega(y)dy - \omega(x)\int_{|v|=1} K_2(v)z_j dS(v)$$

We will continue with a trivial way i.e. calculating each term of ∇u

  - $\partial_{x_1} u_1 = \text{P.V.}\int_{\mathbb{R}^2} \frac{\partial}{\partial x_1}\left(-\frac{1}{2\pi}\frac{x_2-y_2}{|x-y|^2}\right)\omega(y)dy - \omega(x)\int_{|v|=1} -\frac{1}{2\pi}\frac{v_2}{|v|^2}v_1 dS(v)$
    The first term:

$$-\frac{\partial}{\partial x_1}\frac{x_2-y_2}{|x-y|^2} = \frac{x_2-y_2}{|x-y|^4}\frac{\partial}{\partial x_1}|x-y|^2 = 2\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^4}$$

    So we have $\text{P.V.}\int_{\mathbb{R}^2}\frac{1}{2\pi}\left(2\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^4}\right)\omega(y)dy$
    The second term:

$$\int_{|v|=1} -\frac{1}{2\pi}\frac{v_2}{|v|^2}v_1 dS(v) = -\frac{1}{2\pi}\int_0^{2\pi}\cos\theta\sin\theta d\theta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin 2\theta}{2}d\theta = 0$$

    I.e. $\partial_{x_1} u_1 = \text{P.V.}\int_{\mathbb{R}^2}\frac{1}{2\pi}\left(2\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^4}\right)\omega(y)dy$

- $\partial_{x_1} u_2(x) = \text{P.V.} \int_{\mathbb{R}^2} \frac{\partial}{\partial x_1} \left(\frac{1}{2\pi} \frac{x_1-y_1}{|x-y|^2} \right) \omega(y)dy - \omega(x) \int_{|v|=1} \frac{1}{2\pi} \frac{v_1}{|v|^2} v_1 dS(v)$
  The first term :
  $$\frac{\partial}{\partial x_1} \frac{x_1 - y_1}{|x-y|^2} = \frac{|x-y|^2 - 2(x_1-y_1)^2}{|x-y|^4} = \frac{(x_2-y_2)^2 - (x_1-y_1)^2}{|x-y|^4}$$
  So we have $\text{P.V.} \int_{\mathbb{R}^2} \frac{1}{2\pi} \left(\frac{(x_2-y_2)^2-(x_1-y_1)^2}{|x-y|^4} \right) \omega(y)dy$
  The second term:
  $$\int_{|v|=1} \frac{1}{2\pi} \frac{v_1}{|v|^2} v_1 dS(v) = \frac{1}{2\pi} \int_0^{2\pi} \cos^2\theta d\theta = \frac{1}{2\pi}\pi$$
  I.e. $\partial_{x_1} u_2(x) = \text{P.V.} \int_{\mathbb{R}^2} \frac{1}{2\pi} \left(\frac{(x_2-y_2)^2-(x_1-y_1)^2}{|x-y|^4} \right) \omega(y)dy - \frac{1}{2}\omega(x)$
- $\partial_{x_2} u_1 = \text{P.V.} \int_{\mathbb{R}^2} \frac{\partial}{\partial x_2} \left(-\frac{1}{2\pi} \frac{x_2-y_2}{|x-y|^2} \right) \omega(y)dy - \omega(x) \int_{|v|=1} -\frac{1}{2\pi} \frac{v_2}{|v|^2} v_2 dS(v)$
  The first term:
  $$-\frac{\partial}{\partial x_2} \frac{x_2 - y_2}{|x-y|^2} = -\frac{|x-y|^2 - 2(x_2-y_2)^2}{|x-y|^4} = \frac{(x_2-y_2)^2 - (x_1-y_1)^2}{|x-y|^4}$$
  So we have $\text{P.V.} \int_{\mathbb{R}^2} \frac{1}{2\pi} \left(\frac{(x_2-y_2)^2-(x_1-y_1)^2}{|x-y|^4} \right) \omega(y)dy$
  The second term :
  $$\int_{|v|=1} -\frac{1}{2\pi} \frac{v_2}{|v|^2} v_2 dS(v) = \int_0^{2\pi} -\frac{1}{2\pi} \sin^2\theta d\theta = -\frac{1}{2\pi}\pi$$
  I.e. $\partial_{x_2} u_1 = \text{P.V.} \int_{\mathbb{R}^2} \frac{1}{2\pi} \left(\frac{(x_2-y_2)^2-(x_1-y_1)^2}{|x-y|^4} \right) \omega(y)dy + \frac{1}{2}\omega(x)$
- $\partial_{x_2} u_2(x) = \text{P.V.} \int_{\mathbb{R}^2} \frac{\partial}{\partial x_2} \left(\frac{1}{2\pi} \frac{x_1-y_1}{|x-y|^2} \right) \omega(y)dy - \omega(x) \int_{|v|=1} \frac{1}{2\pi} \frac{v_1}{|v|^2} v_2 dS(v)$
  The first term:
  $$\frac{\partial}{\partial x_2} \frac{x_1 - y_1}{|x-y|^2} = -2\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^4}$$
  So we have $\text{P.V.} \int_{\mathbb{R}^2} \frac{1}{2\pi} \left(-2\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^4} \right) \omega(y)dy$
  The second term:
  $$\int_{|v|=1} -\frac{1}{2\pi} = -\frac{1}{2\pi} \int_0^{2\pi} \cos\theta \sin\theta d\theta = 0$$
  I.e. $\partial_{x_2} u_2(x) = \text{P.V.} \int_{\mathbb{R}^2} \frac{1}{2\pi} \left(-2\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^4} \right) \omega(y)dy$

Finally

$$\nabla u(x) = \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}^2} \frac{1}{|x-y|^4} \begin{pmatrix} 2(x_1-y_1)(x_2-y_2) & (x_2-y_2)^2 - (x_1-y_1)^2 \\ (x_2-y_2)^2 - (x_1-y_1)^2 & -2(x_1-y_1)(x_2-y_2) \end{pmatrix} \omega(y)dy$$
$$+ \frac{\omega(x)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- For the 3 dimensions:The kernel we have define is a 3×3 matrix which acts on a vector in $\mathbb{R}^3$. So in order to write the convolution as an inner product we will multiply with a vector h.

$$u(x)h = \int_{\mathbb{R}^3} K_3(x-y)h\omega(y)dy =< \check{K_3}(y)h, \omega(y) >$$

Thus the j-th derivative of u in the distribution sense is:

$$(\partial_{x_j} u(x))h =< \partial_{x_j} \check{K_3}(y)h, \omega(y) >= \partial_{x_j} \check{K}_3[\omega](x)$$

By proposition 2 we get:

$$\partial_{x_j} uh = \text{P.V.} \int_{\mathbb{R}^3} \frac{\partial}{\partial x_j}(K_3(x-y)h)\omega(y)dy - \omega(x) \int_{|v|=1} K_3(v)hv_j dS(v)$$

So

$$[\nabla u]h = \text{P.V.} \int_{\mathbb{R}^3} \nabla_x \left(\frac{1}{4\pi} \frac{(x-y) \times h}{|x-y|^3} \right) \omega(y)dy$$

$$-\frac{1}{4\pi} \int_{|v|=1} \left[\left(\frac{v \times h}{|v|^3} \right) v_1, \left(\frac{v \times h}{|v|^3} \right) v_2, \left(\frac{v \times h}{|v|^3} \right) v_3 \right] \omega(x)ds(v)$$

- The first term:

$$\int_{\mathbb{R}^3} \nabla_x \left(\frac{1}{4\pi} \frac{(x-y) \times h}{|x-y|^3} \right) \omega(y)dy = \int_{\mathbb{R}^3} \nabla_x \left(\frac{1}{4\pi} \frac{[(x-y) \times h]\omega(y)}{|x-y|^3} \right) i_d dy$$

$$= \frac{1}{4\pi} \nabla_x \left(\frac{(x-y) \times \omega(y)}{|x-y|^3} \right) hdy$$

By the well known property of tabla operator ($\nabla \frac{A}{\phi} = \frac{\phi \nabla A - \nabla \phi \otimes A}{\phi^2}$) we get:

$$\nabla_x \left(\frac{(x-y) \times \omega(y)}{|x-y|^3} \right) = \left(\frac{|x-y|^3 \nabla_x [(x-y) \times \omega(y)] - [(x-y) \times \omega(y)] \otimes \nabla_x |x-y|^3}{|x-y|^6} \right)$$

$$= \frac{\nabla_x [(x-y) \times \omega(y)]}{|x-y|^3} - \frac{[(x-y) \times \omega(y)] \otimes \nabla_x |x-y|^3}{|x-y|^6}$$

We will do the calculations separately : 1. $\frac{\nabla_x [(x-y) \times \omega(y)]}{|x-y|^3}$

$$\nabla_x [(x-y) \times \omega(y)] = \nabla_x \begin{pmatrix} (x_2-y_2)\omega_3 - (x_3-y_3)\omega_2 \\ (x_3-y_3)\omega_1 - (x_1-y_1)\omega_3 \\ (x_1-y_1)\omega_2 - (x_2-y_2)\omega_1 \end{pmatrix} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

We name the last matrix W so we have: $\frac{\nabla_x [(x-y) \times \omega(y)]}{|x-y|^3} = \frac{1}{|x-y|^3} W$

2. $\frac{[(x-y) \times \omega(y)] \otimes \nabla_x |x-y|^3}{|x-y|^6}$

$$= \frac{[(x-y) \times \omega(y)] \otimes 3|x-y|(x-y)}{|x-y|^6} = 3 \frac{[(x-y) \times \omega(y)] \otimes (x-y)}{|x-y|^5}$$

So we have

$$\int_{\mathbb{R}^3} \frac{1}{4\pi}\left[\frac{1}{|x-y|^3}W - 3\frac{[(x-y)\times\omega(y)]\otimes(x-y)}{|x-y|^5}\right]hdy$$

$$= \int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{1}{|x-y|^3}Wh - \frac{3}{4\pi}\frac{[(x-y)\times\omega(y)]\otimes(x-y)}{|x-y|^5}hdy$$

$$\left(Wh = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} \omega_3 h_2 - \omega_2 h_3 \\ -\omega_3 h_1 + \omega_1 h_3 \\ \omega_2 h_1 - \omega_1 h_2 \end{pmatrix} = -\omega \times h \right)$$

Finally

$$\text{P.V.}\int_{\mathbb{R}^3} \nabla_x\left(\frac{1}{4\pi}\frac{(x-y)\times h}{|x-y|^3}\right)\omega(y)dy =$$

$$-\text{P.V.}\int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{\omega(y)\times h}{|x-y|^3} + \frac{3}{4\pi}\frac{[(x-y)\times\omega(y)]\otimes(x-y)}{|x-y|^5}hdy$$

– The second term :
We will continue with this part, $[(v\times\omega)v_1, (v\times\omega)v_2, (v\times\omega)v_2]h$

$$= \begin{pmatrix} v_1v_2\omega_3 - v_1v_3\omega_2 & v_2v_2\omega_3 - v_2v_3\omega_2 & v_2v_3\omega_3 - v_3v_3\omega_2 \\ v_1v_3\omega_1 - v_1v_1\omega_3 & v_2v_3\omega_1 - v_1v_2\omega_3 & v_3v_3\omega_1 - v_1v_3\omega_3 \\ v_1v_1\omega_2 - v_2v_1\omega_1 & v_1v_1\omega_2 - v_2v_2\omega_1 & v_1v_3\omega_2 - v_2v_3\omega_1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$$= \begin{pmatrix} v_1v_2\omega_3h_1 - v_1v_3\omega_2h_1 + v_2v_2\omega_3h_2 - v_2v_3\omega_2h_2 + v_2v_3\omega_3h_3 - v_3v_3\omega_2h_3 \\ v_1v_3\omega_1h_1 - v_1v_1\omega_3h_1 + v_2v_3\omega_1h_2 - v_1v_2\omega_3h_2 + v_3v_3\omega_1h_3 - v_1v_3\omega_3h_3 \\ v_1v_1\omega_2h_1 - v_2v_1\omega_1h_1 + v_1v_2\omega_2h_2 - v_2v_2\omega_1h_2 + v_1v_3\omega_2h_3 - v_2v_3\omega_1h_3 \end{pmatrix}$$

So for

$$\frac{1}{4\pi}\int_{|v|=1}\frac{1}{|v|^3}\begin{pmatrix} v_1v_2\omega_3h_1 - v_1v_3\omega_2h_1 + v_2v_2\omega_3h_2 - v_2v_3\omega_2h_2 + v_2v_3\omega_3h_3 - v_3v_3\omega_2h_3 \\ v_1v_3\omega_1h_1 - v_1v_1\omega_3h_1 + v_2v_3\omega_1h_2 - v_1v_2\omega_3h_2 + v_3v_3\omega_1h_3 - v_1v_3\omega_3h_3 \\ v_1v_1\omega_2h_1 - v_2v_1\omega_1h_1 + v_1v_2\omega_2h_2 - v_2v_2\omega_1h_2 + v_1v_3\omega_2h_3 - v_2v_3\omega_1h_3 \end{pmatrix}dS(v)$$

we calculate:

$$\int_{|v|=1}\frac{1}{|v|^3}v_1v_1dS(v) = 2\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sin^2\theta\cos^2\phi\sin\theta d\phi d\theta = 2\frac{2\pi}{3} = \frac{4\pi}{3},$$

$$\int_{|v|=1}\frac{1}{|v|^3}v_1v_2dS(v) = 2\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sin^3\theta\sin\phi\cos\phi d\phi d\theta = 0,$$

$$\int_{|v|=1}\frac{1}{|v|^3}v_1v_3dS(v) = 2\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sin^2\theta\cos\theta\cos\phi d\phi d\theta = 0,$$

$$\int_{|v|=1}\frac{1}{|v|^3}v_2v_3dS(v) = 2\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sin^2\theta\cos\theta\sin\phi d\phi d\theta = 0,$$

$$\int_{|v|=1}\frac{1}{|v|^3}v_2v_2dS(v) = 2\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sin^3\theta\sin^2\phi d\phi d\theta = 2\frac{2\pi}{3} = \frac{4\pi}{3},$$

$$\int_{|v|=1} \frac{1}{|v|^3} v_3 v_3 dS(v) = 2\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin\theta\cos^2\theta d\phi d\theta = 2\frac{2\pi}{3} = \frac{4\pi}{3}$$

Thus:

$$\frac{1}{4\pi}\int_{|v|=1} \frac{1}{|v|^3} \begin{pmatrix} v_1v_2\omega_3h_1 - v_1v_3\omega_2h_1 + v_2v_2\omega_3h_2 - v_2v_3\omega_2h_2 + v_2v_3\omega_3h_3 - v_3v_3\omega_2h_3 \\ v_1v_3\omega_1h_1 - v_1v_1\omega_3h_1 + v_2v_3\omega_1h_2 - v_1v_2\omega_3h_2 + v_3v_3\omega_1h_3 - v_1v_3\omega_3h_3 \\ v_1v_1\omega_2h_1 - v_2v_1\omega_1h_1 + v_1v_2\omega_2h_2 - v_2v_2\omega_1h_2 + v_1v_3\omega_2h_3 - v_2v_3\omega_1h_3 \end{pmatrix} dS(v)$$

$$= \frac{1}{4\pi}\begin{pmatrix} \frac{4\pi}{3}\omega_3h_2 - \frac{4\pi}{3}\omega_2h_3 \\ -\frac{4\pi}{3}\omega_3h_1 + \frac{4\pi}{3}\omega_1h_3 \\ \frac{4\pi}{3}\omega_2h_1 - \frac{4\pi}{3}\omega_1h_2 \end{pmatrix} = \frac{1}{3}h \times \omega$$

After those trivial calculations we conclude to :

$$[\nabla u]h = -\text{P.V.}\int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{\omega(y)\times h}{|x-y|^3} + \frac{3}{4\pi}\frac{[(x-y)\times\omega(y)]\otimes(x-y)}{|x-y|^5} h dy + \frac{1}{3}\omega \times h$$

□

We remind that in the first chapter, we have made a construction with a symmetric and an anti-symmetric matrix. For the symmetric one $D = \frac{1}{2}(\nabla u + (\nabla u)^T)$ we will close this sub-subsection proving the following proposition.

Proposition 2.2.7. *Let $u \in \mathbb{R}^3$, defined by the Biot-Savart law, the gradient of this field defined by the above proposition and $\omega = \text{curl} u$ then for the deformation matrix D we have:*

$$D(x) = \text{P.V.}\int_{\mathbb{R}^3} P(x-y)\omega(y)dy$$

where

$$P(x)h = \frac{-3}{8\pi}\frac{[(x\times h)\otimes x] + [x \otimes (x \times h)]}{|x|^5}$$

Proof. We take the matrix identity from the above proof:

$$\nabla u(x) = \text{P.V.}\int_{\mathbb{R}^3} \frac{1}{4\pi}\nabla_x\left(\frac{(x-y)\times\omega(y)}{|x-y|^3}\right) dy$$

So

$$\frac{1}{4\pi}\nabla_x\left(\frac{(x-y)\times\omega(y)}{|x-y|^3}\right) = \frac{1}{4\pi}\frac{1}{|x-y|^3}W - \frac{3}{4\pi}\frac{1}{|x-y|^5}R$$

Where

$$W = \begin{pmatrix} 0 & \omega_3(y) & -\omega_2(y) \\ -\omega_3(y) & 0 & \omega_1(y) \\ \omega_2(y) & -\omega_1(y) & 0 \end{pmatrix}$$

and

$$R = [(x-y)\times\omega(y)](x-y)^T$$

$$\nabla u(x) = \text{P.V.}\int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{1}{|x-y|^3}W - \frac{3}{4\pi}\frac{1}{|x-y|^5}Rdy$$

and

$$[\nabla u(x)]^T = \text{P.V.} \int_{\mathbb{R}^3} \frac{1}{4\pi} \frac{1}{|x-y|^3} W^T - \frac{3}{4\pi} \frac{1}{|x-y|^5} R^T dy$$

So

$$\nabla u + (\nabla u)^T = \text{P.V.} \int_{\mathbb{R}^3} \frac{1}{4\pi} \frac{1}{|x-y|^3} W - \frac{3}{4\pi} \frac{1}{|x-y|^5} R + \frac{1}{4\pi} \frac{1}{|x-y|^3} W^T - \frac{3}{4\pi} \frac{1}{|x-y|^5} R^T dy$$

$$= \text{P.V.} \int_{\mathbb{R}^3} \frac{1}{4\pi} \frac{1}{|x-y|^3} (W + W^T) - \frac{3}{4\pi} \frac{1}{|x-y|^5} (R + R^T) dy$$

$$= \text{P.V.} \int_{\mathbb{R}^3} -\frac{3}{4\pi} \frac{1}{|x-y|^5} [[(x-y) \times \omega(y)](x-y)^T + (x-y)[(x-y) \times \omega(y)]^T] dy$$

$$= \text{P.V.} \int_{\mathbb{R}^3} -\frac{3}{4\pi} \frac{1}{|x-y|^5} [[(x-y) \times \omega(y)] \otimes (x-y) + (x-y) \otimes [(x-y) \times \omega(y)]] dy$$

Finally

$$D = \text{P.V.} \int_{\mathbb{R}^3} -\frac{3}{8\pi} \frac{1}{|x-y|^5} [[(x-y) \times \omega(y)] \otimes (x-y) + (x-y) \otimes [(x-y) \times \omega(y)]] dy$$

□

The formulation

In this last sub-subsection, we will use the previous tools and we will end up with the result.
For the following proofs we will use this lemma:

Lemma 7. *Let u be a smooth velocity field in $\mathbb{R}^N$ which is divergence free and vanishes rapidly as $|x| \nearrow \infty$. Assume a vector field solving the above equation:*

$$\frac{D}{Dt} b = \nabla u \cdot {}^{21} b + \nu \Delta b \tag{1}$$

which is a convection-diffusion equation[22]*. Then the* div$b$ *solves the scalar equation*

$$\frac{D}{Dt} \text{div} b = \nu \Delta \text{div} b$$

proof of lemma:
We take the divergence of (1) and we get :

$$\text{div} \frac{Db}{Dt} = \text{div}(\nabla u \cdot b + \nu \Delta b)$$

We will see each term individually

[21] matrices product
[22] [39], Chapter 3

- $\text{div}\frac{Db}{Dt}$

$$\nabla \cdot \begin{pmatrix} \frac{\partial}{\partial t}b_1 + \sum_j u_j \frac{\partial}{\partial x_j} b_1 \\ \vdots \\ \frac{\partial}{\partial t}b_N + \sum_j u_j \frac{\partial}{\partial x_j} b_N \end{pmatrix} = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial t}b_1 + \sum_j u_j \frac{\partial}{\partial x_j} b_1\right) + \ldots$$

$$+ \frac{\partial}{\partial x_N}\left(\frac{\partial}{\partial t}b_N + \sum_j u_j \frac{\partial}{\partial x_j} b_N\right)$$

$$= \frac{\partial}{\partial t}\left[\frac{\partial b_1}{\partial x_1} + \ldots + \frac{\partial b_N}{\partial x_N}\right] + \sum_j \left[\frac{\partial}{\partial x_1}\left(u_j \frac{\partial b_1}{\partial x_j}\right) + \ldots\right.$$

$$\left. + \frac{\partial}{\partial x_N}\left(u_j \frac{\partial b_N}{\partial x_j}\right)\right]$$

$$= \frac{\partial}{\partial t}\text{div} b + \frac{\partial u_j}{\partial x_1}\frac{\partial b_1}{\partial x_j} + u_j \frac{\partial^2 b_1}{\partial x_1 \partial x_j} + \ldots + \frac{\partial u_j}{\partial x_N} + u_j \frac{\partial^2 b_N}{\partial x_N \partial x_j}$$

$$= \frac{\partial}{\partial t}\text{div} b + \sum_j \left(\frac{\partial u_j}{\partial x_1}\frac{\partial b_1}{\partial x_j} + \ldots + \frac{\partial u_j}{\partial x_N}\frac{\partial b_N}{\partial x_j}\right)$$

$$+ \sum_j u_j \left(\frac{\partial^2 b_1}{\partial x_1 \partial x_j} + \ldots + \frac{\partial^2 b_N}{\partial x_N \partial x_j}\right)$$

So

$$\text{div}\frac{Db}{Dt} = \frac{\partial}{\partial t}\text{div} b + \sum_j u_j \frac{\partial}{\partial x_j}\text{div} b + \sum_{i,j} \frac{\partial u_j}{\partial x_1}\frac{\partial b_1}{\partial x_j}$$

- $\text{div}(\nabla u \cdot b)$

$$\nabla u \cdot b = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \ldots & \frac{\partial u_1}{\partial x_N} \\ \vdots & & \\ \frac{\partial u_N}{\partial x_1} & \ldots & \frac{\partial u_N}{\partial x_N} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} b_1 \frac{\partial u_1}{\partial x_1} + \ldots + b_N \frac{\partial u_1}{\partial x_N} \\ \vdots \\ b_1 \frac{\partial u_N}{\partial x_1} + \ldots + b_N \frac{\partial u_N}{\partial x_N} \end{pmatrix}$$

So

$$\text{div}(\nabla u b) = \frac{\partial}{\partial x_1}\left[b_1 \frac{\partial u_1}{\partial x_1} + \ldots + b_N \frac{\partial u_1}{\partial x_N}\right] + \ldots$$

$$+ \frac{\partial}{\partial x_N}\left[b_1 \frac{\partial u_N}{\partial x_1} + \ldots + b_N \frac{\partial u_N}{\partial x_N}\right]$$

$$= \frac{\partial}{\partial x_1}\left(\sum_j b_j \frac{\partial u_1}{\partial x_j}\right) + \ldots + \frac{\partial}{\partial x_N}\left(\sum_j b_j \frac{\partial u_N}{\partial x_j}\right)$$

$$= \left(\sum_j \frac{\partial b_j}{\partial x_1} \frac{\partial u_1}{\partial x_j} + \ldots + \sum_j \frac{\partial b_j}{\partial x_N} \frac{\partial u_N}{\partial x_j} \right)$$

$$+ \left(\sum_j b_j \frac{\partial^2 u_1}{\partial x_1 \partial x_j} + \ldots + \sum_j b_j \frac{\partial^2 u_N}{\partial x_N \partial x_j} \right)$$

$$= \sum_{i,j} \frac{\partial b_j}{\partial x_i} \frac{\partial u_1}{\partial x_j} + \sum_j b_j \frac{\partial}{\partial x_j} \left[\frac{\partial u_1}{\partial x_1} + \ldots + \frac{\partial u_N}{\partial x_N} \right]$$

Since u is divergence free

$$\operatorname{div}(\nabla u \cdot b) = \sum_{i,j} \frac{\partial b_j}{\partial x_i} \frac{\partial u_1}{\partial x_j}$$

- $\operatorname{div}(\nu \Delta b)$

$$\nu \Delta b = v\operatorname{div} \left(\sum_j \frac{\partial^2 b}{\partial x_j^2} \right) = \nu \sum_i \sum_j \frac{\partial}{\partial x_i} \frac{\partial^2 b_i}{\partial x_j^2}$$

$$= \nu \sum_j \frac{\partial^2}{\partial x_j^2} \operatorname{div} b = \nu \Delta(\operatorname{div} b)$$

Finally

$$\frac{\partial}{\partial t} \operatorname{div} b + (u \cdot \nabla) \operatorname{div} b + \sum_{i,j} \frac{\partial u_j}{\partial x_i} \frac{\partial b_i}{\partial x_j} = \sum_{i,j} \frac{\partial u_j}{\partial x_i} \frac{\partial b_i}{\partial x_j} + \nu \Delta \operatorname{div} b$$

$$\frac{D}{Dt} \operatorname{div} b = \nu \Delta \operatorname{div} b$$

Our result is summarized in the following two propositions.

Proposition 2.2.8. [23] *Let u be a smooth velocity field vanishing rapidly as $|x|$ tends to infinity then the 2d Navier-Stokes equations*

$$\begin{cases} \frac{\partial}{\partial t} u + \sum_j u_j \frac{\partial}{\partial x_j} u = -\nabla p + \nu \Delta u & u \in \mathbb{R}^2 \\ \operatorname{div} u = 0 & (x,t) \in \mathbb{R}^2 \times [0, \infty] \\ u|_{t=0} = u_0 \end{cases} \tag{N.S}$$

are equivalent to the vorticity-stream formulation, which is a scalar evolution equation of ω

$$\begin{cases} \frac{\partial}{\partial t} \omega + \sum_j \check{K}_2[\omega] \frac{\partial}{\partial x_j} \omega = \nu \Delta \omega \\ \omega|_{t=0} = \omega_0 = \operatorname{curl} u_0 \end{cases} \tag{V.S.F}$$

Where $u = \check{K}_2[\omega] = \int_{\mathbb{R}^2} K_2(x-y) \omega(y) dy$ with $K_2(x) = \frac{1}{2\pi} \left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)^T$

[23] [42]

Remark:In (V.S.F) we have no information about the pressure, which is an unknown quantity in the (N.S). So if we find a solution of the (V.S.F) then by the Biot-Savart law we will determine the velocity field. In the first section of this chapter, we have seen Leray's formulation, we will determine the pressure through velocity by the formula $\Delta p = -\sum_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$.

Proof. ($\Rightarrow$) Let u be a solution of (N.S) we will prove that $\omega = \text{curl}u$ solves the (V.S.F). Since div$u = 0$ and curl$u = \omega$, by the Biot-Savart law, we obtain the velocity field through vorticity. Furthermore, the initial condition is satisfied, so we have:

$$\frac{\partial}{\partial t}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) + u_1\frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) + u_2\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \nu\sum_j \frac{\partial^2}{\partial x_j^2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

We will do the calculations

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{\partial u_2}{\partial x_1} - \frac{\partial}{\partial t}\frac{\partial u_1}{\partial x_2} + u_1\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_1} - u_1\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_2} \\ &+u_2\frac{\partial}{\partial x_2}\frac{\partial u_2}{\partial x_1} - u_2\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} - \nu\sum_j\frac{\partial^2}{\partial x_j^2}\frac{\partial u_2}{\partial x_1} + \nu\sum_j\frac{\partial^2}{\partial x_j^2}\frac{\partial u_1}{\partial x_2} = 0 \end{aligned}$$

So

$$\begin{aligned} &\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial t} - \frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial t} + u_1\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_1} - u_1\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_2} \\ &+u_2\frac{\partial}{\partial x_2}\frac{\partial u_2}{\partial x_1} - u_2\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} - \nu\sum_j\frac{\partial^2}{\partial x_j^2}\frac{\partial u_2}{\partial x_1} + \nu\sum_j\frac{\partial^2}{\partial x_j^2}\frac{\partial u_1}{\partial x_2} = 0 \end{aligned}$$

We substitute the two first terms by the Navier-Stokes

$$\begin{aligned} &\frac{\partial}{\partial x_1}\left(-u_1\frac{\partial u_2}{\partial x_1} - u_2\frac{\partial u_2}{\partial x_2} + \nu\frac{\partial^2 u_2}{\partial x_1^2} + \nu\frac{\partial^2 u_2}{\partial x_2^2} - \frac{\partial p}{\partial x_2}\right) \\ &-\frac{\partial}{\partial x_2}\left(-u_1\frac{\partial u_1}{\partial x_1} - u_2\frac{\partial u_1}{\partial x_2} + \nu\frac{\partial^2 u_1}{\partial x_1^2} + \nu\frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial p}{\partial x_1}\right) \end{aligned}$$

$$+u_1\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_1} - u_1\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_2} + u_2\frac{\partial}{\partial x_2}\frac{\partial u_2}{\partial x_1} - u_2\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} - \nu\sum_j\frac{\partial^2}{\partial x_j^2}\frac{\partial u_2}{\partial x_1} + \nu\sum_j\frac{\partial^2}{\partial x_j^2}\frac{\partial u_1}{\partial x_2} = 0$$

So

$$\begin{aligned} &-\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} - u_1\frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} - u_2\frac{\partial^2 u_2}{\partial x_1\partial x_2} + \nu\frac{\partial^2}{\partial x_1^2}\frac{\partial u_2}{\partial x_1} + \nu\frac{\partial^2}{\partial x_2^2}\frac{\partial u_2}{\partial x_1} - \frac{\partial^2 p}{\partial x_1\partial x_2} \\ &+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_1} + u_1\frac{\partial^2 u_1}{\partial x_1\partial x_2} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2} + u_2\frac{\partial^2 u_1}{\partial x_2^2} - \nu\frac{\partial^2}{\partial x_1^2}\frac{\partial u_1}{\partial x_2} - \nu\frac{\partial^2}{\partial x_2^2}\frac{\partial u_1}{\partial x_2} + \frac{\partial^2 p}{\partial x_2\partial x_1} \end{aligned}$$

$$+u_1\frac{\partial^2 u_2}{\partial x_1^2} - u_1\frac{\partial^2 u_1}{\partial x_1\partial x_2} + u_2\frac{\partial^2 u_2}{\partial x_1\partial x_2} - u_2\frac{\partial^2 u_2}{\partial x_2^2} - v\frac{\partial^2}{\partial x_1^2}\frac{\partial u_2}{\partial x_1} - \nu\frac{\partial^2}{\partial x_2^2}\frac{\partial u_2}{\partial x_1} + \nu\frac{\partial^2}{\partial x_1^2}\frac{\partial u_1}{\partial x_2} + \nu\frac{\partial^2}{\partial x_2^2}\frac{\partial u_1}{\partial x_2} = 0$$

Thus

$$-\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2} = 0 \Rightarrow$$

$$-\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)=0\Rightarrow$$

$$-\frac{\partial u_2}{\partial x_1}\text{div}u+\frac{\partial u_1}{\partial x_2}\text{div}u=0$$

I.e. $0=0$ which is always true, so $\omega=\text{curl}u$ solves the vorticity equation
($\Leftarrow$) Let's do the opposite. Assume that there is an ω which solves the vorticity equation

$$\frac{D\omega}{Dt}=\nu\Delta\omega,$$

We will prove that the velocity u given by the Biot-Savart law is a solution for the (N.S). Since the velocity is a SIO the derivatives are in the distribution sense.
Firstly we will prove that this velocity field u satisfies the incompressibility condition.

$$\text{div}u=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}$$

$$=\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\frac{2(x_1-y_1)(x_2-y_2)}{|x-y|^4}\omega(y)dy-\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\frac{2(x_1-y_1)(x_2-y_2)}{|x-y|^4}\omega(y)dy=0$$

To continue with the proof we will use the lemma 7. ω solves the vorticity equation, we also have that $\nabla u\cdot$[24]$\omega=0$ if we assume that $u=(u_1,u_2,0)^T$ and $\omega=(0,0,\omega)$.
Therefore ω solves the $\frac{D\omega}{Dt}=\nabla u\omega+\nu\Delta\omega$

So from the lemma we proved we have $\begin{cases}\frac{D}{Dt}\text{div}\omega=\nu\Delta\text{div}\omega\\ \text{div}\omega|_{t=0}=0\end{cases}$ which is a scalar parabolic equation, by the uniqueness of its solutions we get $\text{div}\omega=0$. Thus ω is of form $\omega=\nabla\times function$ and since $\omega_0=\text{curl}u_0$ we set $\omega=\nabla\times u$
Now that we have prove that $\omega=\nabla\times u$ we will prove that the vorticity equation gives the first equation of Navier-Stokes.

$$\frac{D\omega}{Dt}=\nu\Delta\omega$$

$$\frac{D}{Dt}\text{curl}u=\nu\Delta\text{curl}u$$

So

$$\text{curl}\left(\frac{Du}{Dt}-\nu\Delta u\right)=0$$

Thus $\frac{Du}{Dt}-\nu\Delta u$ is of form $\frac{Du}{Dt}-\nu\Delta u=\nabla function$ so we get this function to be -p $\frac{Du}{Dt}-\nu\Delta u=\nabla(-p)$ which is the first equation of Navier-Stokes □

Now we will see the 3d case

Proposition 2.2.9. *Let u be a smooth velocity field vanishing rapidly as $|x|$ tends to infinity then the 3d Navier-Stokes equations*

$$\begin{cases}\frac{\partial}{\partial t}u+\sum_j u_j\frac{\partial}{\partial x_j}u=-\nabla p+\nu\Delta u & u\in\mathbb{R}^3\\ \text{div}u=0 & (x,t)\in\mathbb{R}^3\times[0,\infty]\\ u|_{t=0}=u_0\end{cases}\tag{N.S}$$

[24]matrices product

are equivalent to the vorticity-stream formulation, which is an evolution equation of ω

$$\begin{cases} \frac{\partial}{\partial t}\omega + \sum_j \check{K}_3[\omega]\frac{\partial}{\partial x_j}\omega = \nabla \check{K}_3[\omega]\omega + \nu\Delta\omega \\ \omega|_{t=0} = \omega_0 = \text{curl}u_0 \end{cases} \quad \text{(V.S.F)}$$

Where $u = \check{K}_3[\omega] = \int_{\mathbb{R}^2} K_3(x-y)\omega(y)dy$ *with* $K_3(x)h = \frac{1}{4\pi}\frac{x\times h}{|x|^3}$

The proof of this proposition has the same logic as the proof for the two dimensions.

Proof. ($\Rightarrow$) Assuming that u is a solution for the (N.S), we will prove that $\omega = \text{curl}u$ is a solution for the (V.S.F). Since $\text{div}u = 0$ and $\omega = \text{curl}u$ the Biot-Savart law defines the velocity field u. Furthermore, the initial condition is satisfied, so we have

$$\frac{\partial}{\partial t}\text{curl}u + \sum_j u_j \frac{\partial}{\partial x_j}\text{curl}u = \nabla u \text{curl}u + \nu\Delta \text{curl}u$$

Let's start the calculations

$$\text{curl}u = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix}$$

Additionally

$$\nabla u \text{curl}u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \frac{\partial u_1}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\ \frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \frac{\partial u_2}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \frac{\partial u_2}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\ \frac{\partial u_3}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \frac{\partial u_3}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \frac{\partial u_3}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \end{pmatrix}$$

Since it is just calculations, we will check the first component of this vector equation and the other follows.

$$\frac{\partial}{\partial t}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + u_1\frac{\partial}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + u_2\frac{\partial}{\partial x_2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + u_3\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$

$$= \frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \frac{\partial u_1}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) + \nu\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$

$$+\nu\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \nu\frac{\partial^2}{\partial x_3^2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$

We substitute the first terms by the Navier-Stokes equation

$$\frac{\partial}{\partial x_2}\left(-u_1\frac{\partial u_3}{\partial x_1} - u_2\frac{\partial u_3}{\partial x_2} - u_3\frac{\partial u_3}{\partial x_3} + \nu\frac{\partial^2 u_3}{\partial x_1^2} + \nu\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial^2 u_3}{\partial x_3^2} - \frac{\partial p}{\partial x_3}\right)$$

$$-\frac{\partial}{\partial x_3}\left(-u_1\frac{\partial u_2}{\partial x_1} - u_2\frac{\partial u_2}{\partial x_2} - u_3\frac{\partial u_2}{\partial x_3} + \nu\frac{\partial^2 u_2}{\partial x_1^2} + \nu\frac{\partial^2 u_2}{\partial x_2^2} + \nu\frac{\partial^2 u_2}{\partial x_3^2} - \frac{\partial p}{\partial x_2}\right)$$

$$+u_1\frac{\partial}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + u_2\frac{\partial}{\partial x_2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + u_3\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$

$$-\frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)-\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)-\frac{\partial u_1}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)$$
$$-\nu\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)-\nu\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)-\nu\frac{\partial^2}{\partial x_3^2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)=0$$

So

$$-\frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1}-u_1\frac{\partial^2 u_3}{\partial x_1\partial x_2}-\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_2}-u_2\frac{\partial^2 u_3}{\partial x_2^2}-\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}-u_3\frac{\partial^2 u_3}{\partial x_2\partial x_3}$$
$$+\nu\frac{\partial^3 u_3}{\partial x_1^2\partial x_2}+\nu\frac{\partial^3 u_3}{\partial x_2^3}+\nu\frac{\partial^3 u_3}{\partial x_2\partial x_3^2}-\frac{\partial^2 p}{\partial x_2\partial x_3}+\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1}+u_1\frac{\partial^2 u_2}{\partial x_1\partial x_3}$$
$$+\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_2}+u_2\frac{\partial^2 u_2}{\partial x_2\partial x_3}+\frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_3}+u_3\frac{\partial^2 u_2}{\partial x_3^2}-\nu\frac{\partial^3 u_2}{\partial x_1^2\partial x_3}-\nu\frac{\partial^3 u_2}{\partial x_2^2\partial x_3}-\nu\frac{\partial^3 u_2}{\partial x_3^3}$$
$$+\frac{\partial^2 p}{\partial x_2\partial x_3}+u_1\frac{\partial^2 u_3}{\partial x_1\partial x_2}-u_1\frac{\partial^2 u_2}{\partial x_1\partial x_3}+u_2\frac{\partial^2 u_3}{\partial x_2^2}-u_2\frac{\partial^2 u_2}{\partial x_2\partial x_3}+u_3\frac{\partial^2 u_3}{\partial x_2\partial x_3}$$
$$-u_3\frac{\partial^2 u_2}{\partial x_3^2}-\frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_2}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_3}-\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1}$$
$$+\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_2}-\nu\frac{\partial^3 u_3}{\partial x_1^2\partial x_2}+\nu\frac{\partial^3 u_2}{\partial x_1^2\partial x_3}-\nu\frac{\partial^3 u_3}{\partial x_2^3}+\nu\frac{\partial^3 u_3}{\partial x_1^2\partial x_2}-\nu\frac{\partial^3 u_3}{\partial x_2\partial x_3^2}+\nu\frac{\partial^3 u_2}{\partial x_3^3}=0$$

Hence

$$-\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_2}-\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}+\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_3}-\frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_2}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_3}=0\Rightarrow$$
$$\frac{\partial u_3}{\partial x_2}\mathrm{div}u+\frac{\partial u_2}{\partial x_3}\mathrm{div}u=0$$

I.e. $0=0$ which always holds, so the ω_1 satisfies the first component of the vorticity equation. Doing the other calculations we have the same results also for ω_2 and ω_3 so we conclude that $\omega=\mathrm{curl}u$ solves the vorticity equation.
($\Leftarrow$) Let's do the opposite. Assuming that there is an ω which solves the vorticity equation $\frac{\partial}{\partial t}\omega+\sum_j \check{K}_3[\omega]\frac{\partial}{\partial x_j}\omega=\nabla\check{K}_3[\omega]\omega+\nu\Delta\omega$ we will prove that the u, which we determine by the Biot-Savart law is a solution for the (N.S).
Remark: u is a SIO all the derivatives are in the distribution sense.
Firstly, we will prove that u satisfies the incompressibility condition.

$$\mathrm{div}u=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=\mathrm{tr}[\nabla u]^{25}=\mathrm{P.V.}\int_{\mathbb{R}^3}\frac{1}{4\pi}\frac{1}{|x-y|^3}\mathrm{tr}W-\frac{3}{4\pi}\frac{1}{|x-y|5}\mathrm{tr}R dy$$

We can easily see that $\mathrm{tr}W=0$ let us calculate the trace of matrix R

$$\mathrm{tr}R=\mathrm{tr}\begin{pmatrix} (x_1-y_1)(x_2-y_2)\omega_3(y) & (x_2-y_2)^2\omega_3(y) & (x_2-y_2)(x_3-y_3)\omega_3(y) \\ -(x_1-y_1)(x_3-y_3)\omega_2(y) & -(x_2-y_2)(x_3-y_3)\omega_2(y) & -(x_3-y_3)^2\omega_2(y) \\ \\ (x_1-y_1)(x_3-y_3)\omega_1(y) & (x_3-y_3)(x_2-y_2)\omega_1(y) & (x_3-y_3)^2\omega_1(y) \\ -(x_1-y_1)^2\omega_3(y) & -(x_1-y_1)(x_2-y_2)\omega_3(y) & -(x_1-y_1)(x_3-y_3)\omega_3(y) \\ \\ (x_1-y_1)^2\omega_2(y) & (x_1-y_1)(x_2-y_2)\omega_2(y) & (x_1-y_1)(x_3-y_3)\omega_2(y) \\ -(x_1-y_1)(x_2-y_2)\omega_1(y) & -(x_2-y_2)^2\omega_1(y) & -(x_2-y_2)(x_3-y_3)\omega_1(y) \end{pmatrix}$$

[25] see proposition 3.2.7

$$= (x_1 - y_1)(x_2 - y_2)\omega_3(y) - (x_1 - y_1)(x_3 - y_3)\omega_2(y) + (x_3 - y_3)(x_2 - y_2)\omega_1(y)$$
$$-(x_1 - y_1)(x_2 - y_2)\omega_3(y) + (x_1 - y_1)(x_3 - y_3)\omega_2(y) - (x_2 - y_2)(x_3 - y_3)\omega_1(y) = 0$$

Thus $\mathrm{div}u = 0$

Now we can use the lemma since ω solves the vorticity the $\mathrm{div}\omega$ solves the scalar parabolic equation

$$\frac{D}{Dt}\mathrm{div}\omega = \nu\Delta\mathrm{div}\omega$$

with initial condition $\mathrm{div}\omega|_{t=0} = 0$ so by the uniqueness of solutions $\mathrm{div}\omega = 0$, so ω is of form $\nabla \times function$ by taking into account the initial condition of (V.S.F), we have $\omega = \mathrm{curl}u$

We use the vorticity equation and substitute the ω with $\mathrm{curl}u$ and we aim to reach the Navier-Stokes.

$$\frac{D}{Dt}\mathrm{curl}u = \nabla u\mathrm{curl}u + \nu\Delta\mathrm{curl}u$$

We will write down the curl of this vector field, but we will not do substitutions since its derivative in the distribution sense is a little bit complicated.

$$\mathrm{curl}u = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix}$$

We will see each term individually :

- $\frac{\partial}{\partial t}\mathrm{curl}u = \mathrm{curl}\frac{\partial u}{\partial t}$
- $\sum_j u_j \frac{\partial}{\partial x_j}\mathrm{curl}u = \sum_j u_j \mathrm{curl}\frac{\partial u}{\partial x_j}$
  By the identity $\nabla \times (fF) = f(\nabla \times F) + (\nabla f) \times F$ we get

$$\sum_j u_j \frac{\partial}{\partial x_j}\mathrm{curl}u = \sum_j \left[\nabla \times \left(u_j \frac{\partial u}{\partial x_j}\right) - \nabla u_j \times \frac{\partial u}{\partial x_j}\right]$$
$$= \sum_j \mathrm{curl}\left(u_j \frac{\partial u}{\partial x_j}\right) - \sum_j \nabla u_j \times \frac{\partial u}{\partial x_j}$$

  For the second term :

$$\nabla u_1 \times \frac{\partial u}{\partial x_1} + \nabla u_2 \times \frac{\partial u}{\partial x_2} + \nabla u_3 \times \frac{\partial u}{\partial x_3}$$

$$= \begin{pmatrix} \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3} - \frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\frac{\partial u_1}{\partial x_3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) - \frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - \frac{\partial u_1}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\ -\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - \frac{\partial u_2}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - \frac{\partial u_2}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\ -\frac{\partial u_3}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - \frac{\partial u_3}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - \frac{\partial u_3}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \end{pmatrix} = -\nabla u \cdot {}^{26}\mathrm{curl}u$$

[26] matrices product

Finally

$$\sum_j u_j \frac{\partial}{\partial x_j} \mathrm{curl} u = \sum_j \mathrm{curl}\left(u_j \frac{\partial u}{\partial x_j}\right) - (-\nabla u \cdot \mathrm{curl} u)$$

- $\nu \Delta \mathrm{curl} u = \nu \sum_j \frac{\partial^2}{\partial x_j^2} \mathrm{curl} u = \mathrm{curl}(\nu \Delta u)$

Therefore we get that

$$\frac{D}{Dt} \mathrm{curl} u = \nabla u \cdot \mathrm{curl} u + \nu \Delta \mathrm{curl} u$$

is

$$\mathrm{curl}\left(\frac{\partial u}{\partial t} + \sum_j u_j \frac{\partial u}{\partial x_j} - \nu \Delta u\right) = \nabla u \cdot \mathrm{curl} u - \nabla u \mathrm{curl} u = 0$$

So $\frac{Du}{Dt} - \nu \Delta u$ is of form $\nabla function$. Assume that this function is -p then

$$\frac{Du}{Dt} = -\nabla p + \nu \Delta u$$

which is the Navier-Stokes □

We note that the formulation for the Euler arises by taking $\nu = 0$.
Firstly we have the corresponding lemma

Lemma 8. *Let u be a smooth velocity field in $\mathbb{R}^N$ which is divergence free and vanishes rapidly as $|x| \nearrow \infty$. Assume a vector field solving the above equation:*

$$\frac{D}{Dt} b = \nabla u \cdot {}^{27} b \tag{1}$$

which is a convection-diffusion equation[28]*. Then the* div$b$ *solves the scalar equation*

$$\frac{D}{Dt} \mathrm{div} b = 0$$

proof of lemma:
We take the divergence of (1) and we get :

$$\mathrm{div} \frac{Db}{Dt} = \mathrm{div}(\nabla u \cdot b)$$

We will see each term individually

- $\mathrm{div} \frac{Db}{Dt}$

$$\nabla \cdot \begin{pmatrix} \frac{\partial}{\partial t} b_1 + \sum_j u_j \frac{\partial}{\partial x_j} b_1 \\ \vdots \\ \frac{\partial}{\partial t} b_N + \sum_j u_j \frac{\partial}{\partial x_j} b_N \end{pmatrix} = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial t} b_1 + \sum_j u_j \frac{\partial}{\partial x_j} b_1\right) + \dots$$

[27] matrices product
[28] [39], Chapter 3

$$+\frac{\partial}{\partial x_N}\left(\frac{\partial}{\partial t}b_N+\sum_j u_j\frac{\partial}{\partial x_j}b_N\right)$$

$$=\frac{\partial}{\partial t}\left[\frac{\partial b_1}{\partial x_1}+\ldots+\frac{\partial b_N}{\partial x_N}\right]+\sum_j\left[\frac{\partial}{\partial x_1}\left(u_j\frac{\partial b_1}{\partial x_j}\right)+\ldots+\frac{\partial}{\partial x_N}\left(u_j\frac{\partial b_N}{\partial x_j}\right)\right]$$

$$=\frac{\partial}{\partial t}\text{div}b+\frac{\partial u_j}{\partial x_1}\frac{\partial b_1}{\partial x_j}+u_j\frac{\partial^2 b_1}{\partial x_1\partial x_j}+\ldots+\frac{\partial u_j}{\partial x_N}+u_j\frac{\partial^2 b_N}{\partial x_N\partial x_j}$$

$$=\frac{\partial}{\partial t}\text{div}b+\sum_j\left(\frac{\partial u_j}{\partial x_1}\frac{\partial b_1}{\partial x_j}+\ldots+\frac{\partial u_j}{\partial x_N}\frac{\partial b_N}{\partial x_j}\right)$$

$$+\sum_j u_j\left(\frac{\partial^2 b_1}{\partial x_1\partial x_j}+\ldots+\frac{\partial^2 b_N}{\partial x_N\partial x_j}\right)$$

So

$$\text{div}\frac{Db}{Dt}=\frac{\partial}{\partial t}\text{div}b+\sum_j u_j\frac{\partial}{\partial x_j}\text{div}b+\sum_{i,j}\frac{\partial u_j}{\partial x_1}\frac{\partial b_1}{\partial x_j}$$

- $\text{div}(\nabla u\cdot b)$

$$\nabla ub=\begin{pmatrix}\frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_N}\\ \vdots & & \\ \frac{\partial u_N}{\partial x_1} & \cdots & \frac{\partial u_N}{\partial x_N}\end{pmatrix}\begin{pmatrix}b_1\\ \vdots\\ b_N\end{pmatrix}=\begin{pmatrix}b_1\frac{\partial u_1}{\partial x_1}+\ldots+b_N\frac{\partial u_1}{\partial x_N}\\ \vdots\\ b_1\frac{\partial u_N}{\partial x_1}+\ldots+b_N\frac{\partial u_N}{\partial x_N}\end{pmatrix}$$

So

$$\text{div}(\nabla ub)=\frac{\partial}{\partial x_1}\left[b_1\frac{\partial u_1}{\partial x_1}+\ldots+b_N\frac{\partial u_1}{\partial x_N}\right]+\ldots$$

$$+\frac{\partial}{\partial x_N}\left[b_1\frac{\partial u_N}{\partial x_1}+\ldots+b_N\frac{\partial u_N}{\partial x_N}\right]$$

$$=\frac{\partial}{\partial x_1}\left(\sum_j b_j\frac{\partial u_1}{\partial x_j}\right)+\ldots+\frac{\partial}{\partial x_N}\left(\sum_j b_j\frac{\partial u_N}{\partial x_j}\right)$$

$$=\left(\sum_j\frac{\partial b_j}{\partial x_1}\frac{\partial u_1}{\partial x_j}+\ldots+\sum_j\frac{\partial b_j}{\partial x_N}\frac{\partial u_N}{\partial x_j}\right)$$

$$+\left(\sum_j b_j\frac{\partial^2 u_1}{\partial x_1\partial x_j}+\ldots+\sum_j b_j\frac{\partial^2 u_N}{\partial x_N\partial x_j}\right)$$

$$=\sum_{i,j}\frac{\partial b_j}{\partial x_i}\frac{\partial u_1}{\partial x_j}+\sum_j b_j\frac{\partial}{\partial x_j}\left[\frac{\partial u_1}{\partial x_1}+\ldots+\frac{\partial u_N}{\partial x_N}\right]$$

Since u is divergence free

$$\text{div}(\nabla u \cdot b) = \sum_{i,j} \frac{\partial b_j}{\partial x_i} \frac{\partial u_1}{\partial x_j}$$

Finally

$$\frac{\partial}{\partial t}\text{div}b + (u \cdot \nabla)\text{div}b + \sum_{i,j} \frac{\partial u_j}{\partial x_i} \frac{\partial b_i}{\partial x_j} = \sum_{i,j} \frac{\partial u_j}{\partial x_i} \frac{\partial b_i}{\partial x_j}$$

$$\frac{D}{Dt}\text{div}b = 0$$

We continue with the two dimensions, we have the following proposition

Proposition 2.2.10. [29] *Let u be a smooth velocity field vanishing rapidly as $|x|$ tends to infinity then the 2d Euler equation*

$$\begin{cases} \frac{\partial}{\partial t}u + \sum_j u_j \frac{\partial}{\partial x_j} u = -\nabla p & u \in \mathbb{R}^2 \\ \text{div}u = 0 & (x,t) \in \mathbb{R}^2 \times [0,\infty] \\ u|_{t=0} = u_0 \end{cases} \tag{E}$$

are equivalent to the vorticity-stream formulation, which is a scalar evolution equation of ω

$$\begin{cases} \frac{\partial}{\partial t}\omega + \sum_j \check{K}_2[\omega] \frac{\partial}{\partial x_j} \omega = 0 \\ \omega|_{t=0} = \omega_0 = \text{curl}u_0 \end{cases} \tag{V.E}$$

Where $u = \check{K}_2[\omega] = \int_{\mathbb{R}^2} K_2(x-y)\omega(y)dy$ with $K_2(x) = \frac{1}{2\pi}\left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)^T$

Remark:In (V.E) we have no information about the pressure, which is an unknown quantity in the (N.S). So if we solve the (V.S.F) then by the Biot-Savart law we will determine the velocity field. In the first section of this chapter, we have seen the Leray formulation, we will determine the pressure by the formula $\Delta p = -\sum_{i,j} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$. The proof has the same logic as the proof for the Navier-Stokes.

Proof. ($\Rightarrow$) Let u be a solution of (E) we will prove that $\omega = \text{curl}u$ solves the (V.E). Since $\text{div}u = 0$ and $\text{curl}u = \omega$, the Biot-Savart law determines the velocity, furthermore, the initial condition is satisfied, so we have:

$$\frac{\partial}{\partial t}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) + u_1 \frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) + u_2 \frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = 0$$

We will do the calculations

$$\frac{\partial}{\partial t}\frac{\partial u_2}{\partial x_1} - \frac{\partial}{\partial t}\frac{\partial u_1}{\partial x_2} + u_1 \frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_1} - u_1 \frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_2}$$

[29] [42]

$$+u_2\frac{\partial}{\partial x_2}\frac{\partial u_2}{\partial x_1}-u_2\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2}=0$$

So

$$\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial t}-\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial t}+u_1\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_1}-u_1\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_2}$$

$$+u_2\frac{\partial}{\partial x_2}\frac{\partial u_2}{\partial x_1}-u_2\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2}=0$$

We substitute the two first terms by the (E)

$$\frac{\partial}{\partial x_1}\left(-u_1\frac{\partial u_2}{\partial x_1}-u_2\frac{\partial u_2}{\partial x_2}-\frac{\partial p}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(-u_1\frac{\partial u_1}{\partial x_1}-u_2\frac{\partial u_1}{\partial x_2}-\frac{\partial p}{\partial x_1}\right)$$

$$+u_1\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_1}-u_1\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_2}+u_2\frac{\partial}{\partial x_2}\frac{\partial u_2}{\partial x_1}-u_2\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2}=0$$

So

$$-\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1}-u_1\frac{\partial^2 u_2}{\partial x_1^2}-\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}-u_2\frac{\partial^2 u_2}{\partial x_1\partial x_2}-\frac{\partial^2 p}{\partial x_1\partial x_2}$$

$$+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_1}+u_1\frac{\partial^2 u_1}{\partial x_1\partial x_2}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2}+u_2\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 p}{\partial x_2\partial x_1}$$

$$+u_1\frac{\partial^2 u_2}{\partial x_1^2}-u_1\frac{\partial^2 u_1}{\partial x_1\partial x_2}+u_2\frac{\partial^2 u_2}{\partial x_1\partial x_2}-u_2\frac{\partial^2 u_2}{\partial x_2^2}=0$$

Thus

$$-\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1}-\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2}=0$$

$$-\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)=0$$

$$-\frac{\partial u_2}{\partial x_1}\text{div}u+\frac{\partial u_1}{\partial x_2}\text{div}u=0$$

I.e. $0=0$,so $\omega=\text{curl}u$ solves the vorticity equation
($\Leftarrow$) Let's do the opposite. Assume that there is an ω, which solves the vorticity equation

$$\frac{D\omega}{Dt}=0$$

we will prove that the velocity u which is given by the Biot-Savart law is a solution for the (N.S). From now on since the velocity is a SIO the derivatives are taken in the distribution sense.
Firstly we will prove that u satisfies the incompressibility condition.

$$\text{div}u=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}$$

$$=\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\frac{2(x_1-y_1)(x_2-y_2)}{|x-y|^4}\omega(y)dy-\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}^2}\frac{2(x_1-y_1)(x_2-y_2)}{|x-y|^4}\omega(y)dy=0$$

From the lemma 8 we proved we have $\begin{cases} \frac{D}{Dt}\text{div}\omega = 0 \\ \text{div}\omega|_{t=0} = 0 \end{cases}$ which is a scalar parabolic equation so by the uniqueness of its solutions div$\omega = 0$. Thus ω is of form $\omega = \nabla \times function$ and since $\omega_0 = \text{curl}u_0$ we set $\omega = \nabla \times u$
Now that we have prove that $\omega = \nabla \times u$ we will prove that the vorticity equation gives the first equation of Euler.

$$\frac{D\omega}{Dt} = 0$$

$$\frac{D}{Dt}\text{curl}u = 0$$

So

$$\text{curl}\left(\frac{Du}{Dt}\right) = 0$$

Thus $\frac{Du}{Dt}$ is of form $\frac{Du}{Dt} = \nabla function$ so we get this function to be -p $\frac{Du}{Dt} = \nabla(-p)$ which actually is the first equation of Euler. □

In the 3 dimensions we have

Proposition 2.2.11. *Let u be a smooth velocity field vanishing rapidly as $|x|$ tends to infinity then the 3d Euler equation*

$$\begin{cases} \frac{\partial}{\partial t}u + \sum_j u_j \frac{\partial}{\partial x_j}u = -\nabla p & u \in \mathbb{R}^3 \\ \text{div}u = 0 & (x,t) \in \mathbb{R}^3 \times [0,\infty] \\ u|_{t=0} = u_0 \end{cases} \tag{E}$$

are equivalent to the vorticity-stream formulation, which is an evolution equation of ω

$$\begin{cases} \frac{\partial}{\partial t}\omega + \sum_j \check{K_3}[\omega] \frac{\partial}{\partial x_j}\omega = \nabla \check{K_3}[\omega]\omega \\ \omega|_{t=0} = \omega_0 = \text{curl}u_0 \end{cases} \tag{V.E}$$

Where $u = \check{K_3}[\omega] = \int_{\mathbb{R}^2} K_3(x-y)\omega(y)dy$ with $K_3(x)h = \frac{1}{4\pi}\frac{x \times h}{|x|^3}$

The proof of this proposition has the same logic as the proof for the two dimensions.

Proof. ($\Rightarrow$) Let u be a solution for the (E), we will prove that $\omega = \text{curl}u$ is a solution for the (V.E). Since div$u = 0$ and $\omega = \text{curl}u$, the velocity field, u, is determined by the Biot-Savart law. Furthermore, the initial condition is satisfied, so now we have:

$$\frac{\partial}{\partial t}\text{curl}u + \sum_j u_j \frac{\partial}{\partial x_j}\text{curl}u = \nabla u \text{curl}u$$

Let's start the calculations

$$\text{curl}u = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix}$$

Additionally

$$\nabla u \text{curl} u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)+\frac{\partial u_1}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) \\ \frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+\frac{\partial u_2}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)+\frac{\partial u_2}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) \\ \frac{\partial u_3}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+\frac{\partial u_3}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)+\frac{\partial u_3}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) \end{pmatrix}$$

We will check the first component of this vector equation and the other follows.

$$\frac{\partial}{\partial t}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+u_1\frac{\partial}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+u_2\frac{\partial}{\partial x_2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+u_3\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)$$
$$=\frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)+\frac{\partial u_1}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)$$

We substitute the first terms by the (E) equation

$$\frac{\partial}{\partial x_2}\left(-u_1\frac{\partial u_3}{\partial x_1}-u_2\frac{\partial u_3}{\partial x_2}-u_3\frac{\partial u_3}{\partial x_3}-\frac{\partial p}{\partial x_3}\right)$$
$$-\frac{\partial}{\partial x_3}\left(-u_1\frac{\partial u_2}{\partial x_1}-u_2\frac{\partial u_2}{\partial x_2}-u_3\frac{\partial u_2}{\partial x_3}-\frac{\partial p}{\partial x_2}\right)$$
$$+u_1\frac{\partial}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+u_2\frac{\partial}{\partial x_2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+u_3\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)$$
$$-\frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)-\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)-\frac{\partial u_1}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)=0$$

So

$$-\frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1}-u_1\frac{\partial^2 u_3}{\partial x_1\partial x_2}-\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_2}-u_2\frac{\partial^2 u_3}{\partial x_2^2}-\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}-u_3\frac{\partial^2 u_3}{\partial x_2\partial x_3}$$
$$-\frac{\partial^2 p}{\partial x_2\partial x_3}+\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1}+u_1\frac{\partial^2 u_2}{\partial x_1\partial x_3}+\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_2}+u_2\frac{\partial^2 u_2}{\partial x_2\partial x_3}+\frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_3}+u_3\frac{\partial^2 u_2}{\partial x_3^2}$$
$$+\frac{\partial^2 p}{\partial x_2\partial x_3}+u_1\frac{\partial^2 u_3}{\partial x_1\partial x_2}-u_1\frac{\partial^2 u_2}{\partial x_1\partial x_3}+u_2\frac{\partial^2 u_3}{\partial x_2^2}-u_2\frac{\partial^2 u_2}{\partial x_2\partial x_3}+u_3\frac{\partial^2 u_3}{\partial x_2\partial x_3}$$
$$-u_3\frac{\partial^2 u_2}{\partial x_3^2}-\frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_2}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_3}-\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_2}=0$$

Hence

$$-\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_2}-\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}+\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_3}-\frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_2}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_3}=0\Rightarrow$$
$$\frac{\partial u_3}{\partial x_2}\text{div}u+\frac{\partial u_2}{\partial x_3}\text{div}u=0$$

I.e. $0=0$, so the ω_1 satisfies the first component of the vorticity equation. Doing the other calculations, we have the same results also for ω_2 and ω_3 for the second and third components, so we conclude that $\omega=\text{curl}u$ solves the vorticity equation.
($\Leftarrow$) Let's do the opposite. Assume that there is an ω which solves the vorticity equation

$\frac{\partial}{\partial t}\omega + \sum_j \check{K}_3[\omega]\frac{\partial}{\partial x_j}\omega = \nabla\check{K}_3[\omega]\omega$ we will prove that,u determined by the Biot-Savart law is a solution for the (N.S).

Remark: u is a SIO, so the derivatives are in the distribution sense.

Firstly, we will prove that u satisfies the incompressibility condition.

$$\text{div}u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \text{tr}[\nabla u]^{30} = \text{P.V.} \int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{1}{|x-y|^3}\text{tr}W - \frac{3}{4\pi}\frac{1}{|x-y|5}\text{tr}Rdy$$

We can easily see that $\text{tr}W = 0$, we will calculate the trace of matrix R

$$\text{tr}R = \text{tr}\begin{pmatrix} (x_1-y_1)(x_2-y_2)\omega_3(y) & (x_2-y_2)^2\omega_3(y) & (x_2-y_2)(x_3-y_3)\omega_3(y) \\ -(x_1-y_1)(x_3-y_3)\omega_2(y) & -(x_2-y_2)(x_3-y_3)\omega_2(y) & -(x_3-y_3)^2\omega_2(y) \\ \\ (x_1-y_1)(x_3-y_3)\omega_1(y) & (x_3-y_3)(x_2-y_2)\omega_1(y) & (x_3-y_3)^2\omega_1(y) \\ -(x_1-y_1)^2\omega_3(y) & -(x_1-y_1)(x_2-y_2)\omega_3(y) & -(x_1-y_1)(x_3-y_3)\omega_3(y) \\ \\ (x_1-y_1)^2\omega_2(y) & (x_1-y_1)(x_2-y_2)\omega_2(y) & (x_1-y_1)(x_3-y_3)\omega_2(y) \\ -(x_1-y_1)(x_2-y_2)\omega_1(y) & -(x_2-y_2)^2\omega_1(y) & -(x_2-y_2)(x_3-y_3)\omega_1(y) \end{pmatrix}$$

$$= (x_1-y_1)(x_2-y_2)\omega_3(y) - (x_1-y_1)(x_3-y_3)\omega_2(y) + (x_3-y_3)(x_2-y_2)\omega_1(y)$$

$$-(x_1-y_1)(x_2-y_2)\omega_3(y) + (x_1-y_1)(x_3-y_3)\omega_2(y) - (x_2-y_2)(x_3-y_3)\omega_1(y) = 0$$

Thus $\text{div}u = 0$

Now we can use the lemma 8 since ω solves the vorticity the $\text{div}\omega$ solves the scalar parabolic equation

$$\frac{D}{Dt}\text{div}\omega = 0$$

with initial condition $\text{div}\omega|_{t=0} = 0$ so by the uniqueness of solutions $\text{div}\omega = 0$, so ω is of form $\nabla \times function$ by taking into account the initial condition of (V.E), we have $\omega = \text{curl}u$

We take again the vorticity equation and we substitute the ω with $\text{curl}u$ and our aim is to reach to the Euler equation.

$$\frac{D}{Dt}\text{curl}u = \nabla u \cdot \text{curl}u$$

We will write down the curl of this vector field, but we will not do substitutions since its derivative in the distribution sense, has a complicated form.

$$\text{curl}u = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix}$$

We will see each term individually :

- $\frac{\partial}{\partial t}\text{curl}u = \text{curl}\frac{\partial u}{\partial t}$

[30] see proposition 3.2.7

- $\sum_j u_j \frac{\partial}{\partial x_j}\mathrm{curl}u = \sum_j u_j \mathrm{curl}\frac{\partial u}{\partial x_j}$
  By the identity $\nabla \times (fF) = f(\nabla \times F) + (\nabla f) \times F$ we get

$$\sum_j u_j \frac{\partial}{\partial x_j}\mathrm{curl}u = \sum_j \left[\nabla \times \left(u_j \frac{\partial u}{\partial x_j}\right) - \nabla u_j \times \frac{\partial u}{\partial x_j}\right]$$

$$= \sum_j \mathrm{curl}\left(u_j \frac{\partial u}{\partial x_j}\right) - \sum_j \nabla u_j \times \frac{\partial u}{\partial x_j}$$

For the second term :

$$\nabla u_1 \times \frac{\partial u}{\partial x_1} + \nabla u_2 \times \frac{\partial u}{\partial x_2} + \nabla u_3 \times \frac{\partial u}{\partial x_3}$$

$$= \begin{pmatrix} \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3} - \frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\frac{\partial u_1}{\partial x_3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) - \frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - \frac{\partial u_1}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\ -\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - \frac{\partial u_2}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - \frac{\partial u_2}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\ -\frac{\partial u_3}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - \frac{\partial u_3}{\partial x_2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - \frac{\partial u_3}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \end{pmatrix} = -\nabla u \cdot {}^{31}\mathrm{curl}u$$

Finally

$$\sum_j u_j \frac{\partial}{\partial x_j}\mathrm{curl}u = \sum_j \mathrm{curl}\left(u_j \frac{\partial u}{\partial x_j}\right) - (-\nabla u \cdot \mathrm{curl}u)$$

Therefore we get that

$$\frac{D}{Dt}\mathrm{curl}u = \nabla u \cdot \mathrm{curl}u$$

is

$$\mathrm{curl}\left(\frac{\partial u}{\partial t} + \sum_j u_j \frac{\partial u}{\partial x_j}\right) = \nabla u \cdot \mathrm{curl}u - \nabla u \cdot \mathrm{curl}u = 0$$

So $\frac{Du}{Dt}$ is of form $\nabla function$. Assume that this function is -p then

$$\frac{Du}{Dt} = -\nabla p$$

which is the Euler equation. □

Finishing this chapter, we observe that those two formulations have some similarities. In Leray's formulation, we link the velocity with the pressure while, in the vorticity-stream formulation, we link the velocity with the vorticity, and in both cases, we have reached an integral operator. These formulations will play a crucial role in the further analysis, specifically in Chapter 4.

[31] matrices product

CHAPTER 3

A PRIORI ESTIMATES VIA ENERGY METHODS

In the first chapter, we discussed some physical properties of the fluids, and we have seen that for inviscid fluids the kinetic energy is a conserved quantity. For viscous fluids, we have the following result:

Proposition 3.0.1. *Assume that u is a smooth solution of the Navier Stokes, vanishing rapidly as $|x|$ tends to infinity. Then*

$$\frac{d}{dt}E = -\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

Proof. We know that $E = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx$
So

$$\frac{d}{dt}E = \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx\right)$$

By Leibniz integral rule we obtain

$$\frac{d}{dt}E = \frac{1}{2}\int_{\mathbb{R}^N} \frac{\partial}{\partial t}|u|^2 dx = \frac{1}{2}\int_{\mathbb{R}^N} 2u \cdot \frac{\partial u}{\partial t} dx$$

By Navier stokes we get

$$\int_{\mathbb{R}^N} 2u \cdot \left(\sum_j u_j \frac{\partial u}{\partial x_j} - \nabla p + \nu \Delta u\right) dx$$

I.e.

$$2\int_{\mathbb{R}^N} -u \cdot \sum_j u_j \frac{\partial u}{\partial x_j} - u \cdot \nabla p - \nu(u \cdot \Delta u) dx$$

We will see each term individually

- $\int_{\mathbb{R}^N} u \cdot \nabla p = \int_{\mathbb{R}^N} \sum_j u_j \frac{\partial p}{\partial x_j}\, dx = \sum_j \int_{\mathbb{R}^N} u_j \frac{\partial p}{\partial x_j}\, dx = \sum_j \int_{\mathbb{R}^N} \frac{\partial u_j}{\partial x_j} p\, dx = \int_{\mathbb{R}^N} p \operatorname{div} u\, dx = 0$
- $\int_{\mathbb{R}^N} u \cdot \sum_j u_j \frac{\partial u}{\partial x_j}\, dx = \int_{\mathbb{R}^N} \sum_i u_i \sum_j \frac{\partial u_i}{\partial x_j}\, dx = \sum_{i,j} \int_{\mathbb{R}^N} u_i u_j \frac{\partial u_i}{\partial x_j}\, dx$
  Consequently
  $$\int_{\mathbb{R}^N} u_i u_j \frac{\partial u_i}{\partial x_j}\, dx$$

by integration by parts gives

$$\int_{\mathbb{R}^N} \frac{\partial}{\partial x_j}(u_i u_j)u_i \, dx = \int_{\mathbb{R}^N} \left(u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_j} \right) u_i \, dx$$

Finally $\int_{\mathbb{R}^N} u \sum_j u_j \frac{\partial u}{\partial x_j} dx = 0$

- $\int_{\mathbb{R}^N} u \cdot \Delta u dx = \int_{\mathbb{R}^N} \nabla u \cdot \nabla u \, dx = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$

So

$$\frac{d}{dt}E = -\nu \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

□

Remark :

1. If $\nu = 0$, then we have th same rsult as in chapter 1.
2. Since viscosity is a positive quantity, the sign of the derivative of the kinetic energy on time is negative, so the function of kinetic energy is decreasing through time.
3. By the simplified Gronwall lemma in Chapter 1 we have that

$$\int_{\mathbb{R}^N} |u|^2 dx \leq u_0 e^0 = u_0$$

The fact that we have proved a bound for the kinetic energy, gives us the idea to continue with energy methods(L^2 estimates).

3.1 Basic energy method for the solutions of the Navier-Stokes and Euler equations

Proposition 3.1.1. *Assume that v and w are two smooth solutions of Navier Stokes equation with external forces F_v and F_w inL^2 and the same viscosity $\nu \geq 0$. Furthermore we suppose that these solutions exists on a common time interval $[0, T]$, and for fixed time decay fast enough so that $v, w \in L^2(\mathbb{R}^N)$*

Proof. Let $\tilde{u} = v - w, \tilde{p} = p_v - p_w, \tilde{F} = F_v - F_w$ Since v, w are solutions to the Navier Stokes we get

$$\begin{cases} \frac{D}{Dt}v - \frac{D}{Dt}w = -\nabla p_v + \nabla p_w + \nu \Delta v - \nu \Delta w \\ \text{div} v - \text{div} w = 0 \\ v|_{t=0} = w|_{t=0} = v_0 - w_0 \end{cases}$$

We will use the first equation

$$\frac{\partial}{\partial t}v + (v \cdot \nabla)v - \frac{\partial}{\partial t}w - (w \cdot \nabla)w = -\nabla(p_v - p_w) + \nu \Delta(v - w) + (F_v - F_w) \Rightarrow$$

$$\frac{\partial}{\partial t}\tilde{u} + (v\cdot\nabla)v - (w\cdot\nabla)w = -\nabla\tilde{p} + \nu\Delta\tilde{u} + \tilde{F}$$

We want all terms of this equation, to be related with the $\tilde{u}$ so

$$(v\cdot\nabla)v - (w\cdot\nabla)w = (v\cdot\nabla)v - (v\cdot\nabla)w + (v\cdot\nabla)w - (w\cdot\nabla)w$$

$$= (v\cdot\nabla)(v-w) + [(v-w)\cdot\nabla]w = (v\cdot\nabla)\tilde{u} + (\tilde{u}\cdot\nabla)w$$

Eventually

$$\begin{cases} \frac{\partial}{\partial t}\tilde{u}(v\cdot\nabla)\tilde{u} + (\tilde{u}\cdot\nabla)w = -\nabla\tilde{p} + \nu\Delta\tilde{u} + \tilde{F} \\ \mathrm{div}\tilde{u} = 0 \\ \tilde{u}|_{t=0} = \tilde{u}_0 \end{cases}$$

We continue by doing a usual step when we do energy methods, i.e. we multiply the first equation with the $\tilde{u}$ on L^2, so we get

$$\int_{\mathbb{R}^N}\left(\frac{\partial}{\partial t}\tilde{u} + (v\cdot\nabla)\tilde{u} + (\tilde{u}\cdot\nabla)w\right)\cdot\tilde{u}\,dx = \int_{\mathbb{R}^N}(-\nabla\tilde{p} + \nu\Delta\tilde{u} + \tilde{F})\cdot\tilde{u}\,dx \Rightarrow$$

$$\int_{\mathbb{R}^N}\frac{\partial}{\partial t}\tilde{u}\cdot\tilde{u}\,dx + \int_{\mathbb{R}^N}[(v\cdot\nabla)\tilde{u}]\cdot\tilde{u}\,dx + \int_{\mathbb{R}^N}[(\tilde{u}\cdot\nabla)w]\cdot\tilde{u}\,dx = -\int_{\mathbb{R}^N}\nabla\tilde{p}\cdot\tilde{u}\,dx + \nu\int_{\mathbb{R}^N}\Delta\tilde{u}\cdot\tilde{u}\,dx + \int_{\mathbb{R}^N}\tilde{F}\cdot\tilde{u}\,dx$$

We see that

- $\int_{\mathbb{R}^N}\Delta\tilde{u}\cdot\tilde{u}\,dx = -\int_{\mathbb{R}^N}\nabla\tilde{u}\cdot\nabla\tilde{u}\,dx$
- $\int_{\mathbb{R}^N}\nabla\tilde{p}\cdot\tilde{u}\,dx = \sum_j\int_{\mathbb{R}^N}\frac{\partial}{\partial x_j}\tilde{p}\,\tilde{u}_j\,dx$
  Integration by parts gives $-\sum_j\int_{\mathbb{R}^N}\tilde{p}\frac{\partial}{\partial x_j}\tilde{u}_j\,dx = -\int_{\mathbb{R}^N}\tilde{p}\,\mathrm{div}\tilde{u}\,dx = 0$
- $\int_{\mathbb{R}^N}[(v\cdot\nabla)\tilde{u}]\cdot\tilde{u}\,dx = \sum_{i,j}\int_{\mathbb{R}^N}v_j\frac{\partial}{\partial x_j}\tilde{u}_i\tilde{u}_i dx$
  Lets examine the $\int_{\mathbb{R}^N}v_j\frac{\partial}{\partial x_j}\tilde{u}_i\tilde{u}_i\,dx$
  Integration by parts gives $\int_{\mathbb{R}^N}\frac{\partial}{\partial x_j}(\tilde{u}_i v_j)\tilde{u}_i\,dx = -\int_{\mathbb{R}^N}\tilde{u}_i^2\frac{\partial}{\partial x_j}v_j\,dx - \int_{\mathbb{R}^N}\tilde{u}_i v_j\frac{\partial}{\partial x_j}\tilde{u}_i\,dx$
  So
  $$2\sum_{i,j}\int_{\mathbb{R}^N}\tilde{u}_i v_j\frac{\partial}{\partial x_j}\tilde{u}_i\,dx = -\sum_{i,j}\int_{\mathbb{R}^N}\tilde{u}_i^2\frac{\partial}{\partial x_j}v_j\,dx = -\sum_i\int_{\mathbb{R}^N}\tilde{u}_i^2\mathrm{div}v\,dx = 0$$

So we have

$$\int_{\mathbb{R}^N}\frac{\partial}{\partial t}\tilde{u}\cdot\tilde{u}\,dx + \nu\int_{\mathbb{R}^N}\nabla\tilde{u}\cdot\nabla\tilde{u}\,dx = -\int_{\mathbb{R}^N}[(\tilde{u}\cdot\nabla)w]\cdot\tilde{u}\,dx + \int_{\mathbb{R}^N}\tilde{F}\cdot\tilde{u}\,dx$$

I.e.

$$\left(\int_{\mathbb{R}^N}|\tilde{u}|^2\,dx\right)^{\frac{1}{2}}\frac{d}{dt}\left(\int_{\mathbb{R}^N}|\tilde{u}|^2\,dx\right)^{\frac{1}{2}} + \nu\int_{\mathbb{R}^N}\nabla\tilde{u}\nabla\cdot\tilde{u}\,dx = -\int_{\mathbb{R}^N}[(\tilde{u}\cdot\nabla)w]\cdot\tilde{u}\,dx + \int_{\mathbb{R}^N}\tilde{F}\cdot\tilde{u}\,dx \Rightarrow$$

$$\|\tilde{u}\|_{L^2}\frac{d}{dt}\|\tilde{u}\|_{L^2} + \nu\int_{\mathbb{R}^N}|\nabla\tilde{u}|^2dx \leq \int_{\mathbb{R}^N}[(\tilde{u}\cdot\nabla)w]\cdot\tilde{u}\,dx + \int_{\mathbb{R}^N}\tilde{F}\cdot\tilde{u}\,dx \Rightarrow$$

$$\|\tilde{u}\|_{L^2}\frac{d}{dt}\|\tilde{u}\|_{L^2}+\nu\left[\left(\int_{\mathbb{R}^N}|\nabla\tilde{u}|^2dx\right)^{\frac{1}{2}}\right]^2\leq\int_{\mathbb{R}^N}[(\tilde{u}\cdot\nabla)w]\cdot\tilde{u}\,dx+\int_{\mathbb{R}^N}\tilde{F}\cdot\tilde{u}\,dx$$

For the integrals on the right side of this inequality we will use Holder inequality [1]
It follows that

$$\int_{\mathbb{R}^N}\tilde{F}\cdot\tilde{u}\,dx\leq\Big|\int_{\mathbb{R}^N}\tilde{F}\cdot\tilde{u}\,dx\Big|\leq\int_{\mathbb{R}^N}|\tilde{F}\cdot\tilde{u}|\,dx$$

We know that $\tilde{u}$ and $\tilde{F}\in L^2$ so $\int_{\mathbb{R}^N}\tilde{F}\tilde{u}dx\leq\|\tilde{F}\|_{L^2}\|\tilde{u}\|_{L^2}$
Furthermore

$$\int_{\mathbb{R}^N}[(\tilde{u}\cdot\nabla)w]\cdot\tilde{u}\,dx=\int_{\mathbb{R}^N}(\nabla w\;\tilde{u})\cdot\tilde{u}dx$$

Since w decay fast enough such that $w\in L^2$ we have that $\nabla w\in L^\infty$ [2], additionally $\tilde{u}\in L^2$ so $\int_{\mathbb{R}^N}[(\tilde{u}\cdot\nabla)w]\cdot\tilde{u}\,dx\leq\|\nabla w\|_{L^\infty}\|\tilde{u}\|_{L^2}\|\tilde{u}\|_{L^2}=\|\nabla w\|_{L^\infty}\|\tilde{u}\|^2_{L^2}$
We conclude to this relation

$$\|\tilde{u}\|_{L^2}\frac{d}{dt}\|\tilde{u}\|_{L^2}\leq\|\nabla w\|_{L^\infty}\|\tilde{u}\|^2_{L^2}+\|\tilde{F}\|_{L^2}\|\tilde{u}\|_{L^2}$$

Assume that $\|\tilde{u}\|_{L^2}\neq 0$ we have that :

$$\frac{d}{dt}\|\tilde{u}\|_{L^2}\leq\|\nabla w\|_{L^\infty}\|\tilde{u}\|_{L^2}+\|\tilde{F}\|_{L^2}$$

Now we will use the following lemma

Lemma 9 (Gronwall s lemma). [3] *Let I be an interval on the real line and u,q,c non negative continuous functions on I with c differentiable if*

$$q(t)\leq c(t)+\int_a^t u(s)q(s)\,ds,\forall t\in I \tag{H}$$

then

$$q(t)\leq c(t)+\int_a^t c(s)u(s)e^{\int_s^t u(r)\,dr}\,ds,t\in I \tag{R}$$

proof of lemma
We define $p(t)=e^{-\int_a^t u(s)\,ds}\int_a^t u(s)q(s)\,ds$ and $p(a)=0$ Then

$$\frac{d}{dt}p=u(t)e^{-\int_a^t u(s)\,ds}\left(q(t)-\int_a^t u(s)q(s)\,ds\right)$$

By the H we have

$$\frac{d}{dt}p\leq u(t)e^{-\int_a^t u(s)\,ds}\left(c(t)+\int_a^t u(s)q(s)\,ds-\int_a^t u(s)q(s)\,ds\right)$$

[1] Assume two measurable functions f and g and let $p,q\in[1,\infty]$ with $\frac{1}{p}+\frac{1}{q}=1$ then $\|fg\|_{L^1}\leq\|f\|_{L^p}\|g\|_{L^q}$

[2] here ∇w is a matrix, but this is not a problem since, let's think the ∇w as a vector i.e. $\nabla w=\left(\frac{\partial}{\partial x_1}w,...,\frac{\partial}{\partial x_N}w\right)$, then the L infinity norm is the supremum of the magnitudes of $\frac{\partial}{\partial x_i}w$, which is, of course, the definition we have about the L infinity norm of a matrix i.e The L infinity norm of a square matrix is the supremum of the absolute row sums

[3] [23]

The integral of non negative function is an increasing function so we get

$$p(t) - p(a) \leq \int_a^t c(s)u(s)e^{-\int_a^s u(r)\,dr}\,ds$$

So since $p(a) = 0$

$$p(t) \leq \int_a^t c(s)u(s)e^{-\int_a^s u(r)\,dr}\,ds$$

Substituting p(t) with its equal we have

$$e^{-\int_a^t u(s)\,ds}\int_a^t u(s)q(s)\,ds \leq \int_a^t c(s)u(s)e^{-\int_a^s u(r)\,dr}\,ds$$

$$\int_a^t u(s)q(s)\,ds \leq e^{\int_a^t u(s)\,ds}\int_a^t c(s)u(s)e^{-\int_a^s u(r)\,dr}\,ds$$

Therefore

$$\int_a^t u(s)q(s)\,ds \leq \int_a^t c(s)u(s)e^{\int_t^s u(r)\,dr}\,ds$$

Again by H

$$q(t) - c(t) \leq \int_a^t c(s)u(s)e^{\int_t^s u(r)\,dr}\,ds$$

How will we apply this lemma to our relation?
For notation convenience we set

$$\|\tilde{u}\|_{L^2} = q(t)$$

$$\|\nabla w\|_{L^\infty} = p$$

$$\|\tilde{F}\|_{L^2} = z(t)$$

Thus

$$\frac{d}{dt}q(t) \leq pq(t) + z(t)$$

We integrate over (0,t)

$$\int_0^t \frac{d}{ds}q(s)\,ds \leq \int_0^t pq(s)\,ds + \int_0^t z(s)\,ds$$

$$q(t) \leq q(0) + \int_0^t pq(s)\,ds + \int_0^t z(s)\,ds$$

We set $c(t) = q(0) + \int_0^t z(s)\,ds$ So we reached to the relation

$$q(t) \leq c(t) + \int_0^t pq(s)\,ds$$

By the lemma

$$q(t) \leq c(t) + \int_0^t pc(s)e^{\int_s^t p\,dr}\,ds \Rightarrow$$

[4]

$$q(t) \leq c(t) - \int_0^t c(s) \frac{d}{ds} e^{\int_s^t p\, dr}\, ds$$

By integration by parts to the right side we have

$$q(t) \leq c(t) - \left[c(s) e^{\int_s^t p\, dr} \right]_{s=t, s=0} + \int_0^t \left(\frac{d}{ds} c(s) \right) e^{\int_s^t p\, dr}\, ds$$

$$q(t) \leq c(t) - c(t) e^{\int_s^t p\, dr} + c(0) e^{\int_0^t p\, dr} + \int_0^t \left(\frac{d}{ds} c(s) \right) e^{\int_s^t p\, dr}\, ds \Rightarrow$$

[5]

$$q(t) \leq c(0) e^{\int_0^t p\, dr} + \int_0^t z(s) e^{\int_s^t p\, dr}\, ds \Rightarrow$$

$$q(t) \leq q(0) e^{\int_0^t p\, dr} + \int_0^t z(s) e^{\int_0^t p\, dr - \int_0^s p\, dr}\, ds$$

Consequently

$$q(t) \leq q(0) e^{\int_0^t p\, dr} + e^{\int_0^t p\, dr} \int_0^t z(s) e^{-ps} ds \Rightarrow$$

[6]

$$q(t) \leq q(0) e^{\int_0^t p\, dr} + e^{\int_0^t p\, dr} \int_0^t z(s)\, ds$$

We have reached to

$$q(t) \leq e^{\int_0^t p\, du} \left(q(0) + \int_0^t z(u)\, du \right)$$

Taking the supremum over this relation

$$\sup_{0 \leq t \leq T} q(t) \leq \sup_{0 \leq t \leq T} \left[e^{\int_0^t p\, du} \left(q(0) + \int_0^t z(u)\, du \right) \right]$$

Let us check the right part, because our functions are non negative we use the fact that $\sup ab \leq \sup a \sup b$

$$\sup_{0 \leq t \leq T} \left[e^{\int_0^t p\, du} \left(q(0) + \int_0^t z(u)\, du \right) \right] \leq \sup_{0 \leq t \leq T} e^{\int_0^t p\, du} \sup_{0 \leq t \leq T} \left(q(0) + \int_0^t z(u)\, du \right) \Rightarrow$$

[7]

$$\sup_{0 \leq t \leq T} e^{\int_0^t p\, du} \sup_{0 \leq t \leq T} \left(q(0) + \int_0^t z(u)\, du \right) \leq \left(q(0) + \int_0^T z(u)\, du \right) e^{\int_0^T p\, du}$$

---

[4]we see that

$$\frac{d}{ds} e^{\int_s^t p\, dr} = e^{\int_s^t p\, dr} \frac{d}{ds} \int_s^t p\, dr = e^{\int_s^t p\, dr} \left(p \frac{d}{ds} t - p \frac{d}{ds} s + \int_s^t \frac{\partial}{\partial s} p\, dr \right) = -p e^{\int_s^t p\, dr}$$

[5]We see that $\frac{d}{ds} c(s) = \frac{d}{ds} \left(\int_0^s z(u)\, du + q(0) \right) = \frac{d}{ds} \int_0^s z(u)\, du = z(s)$

[6]since p, s are positive $\frac{1}{e^{p} s} < 1$

[7]The integral of non negative functions is an increasing function

Finally

$$\sup_{0\le t\le T}\|\tilde{u}\|_{L^2} \le \left(\|\tilde{u}|_{t=0}\|_{L^2} + \int_0^T \|\tilde{F}\|_{L^2}\,dt\right) e^{\int_0^T \|\nabla w\|_{L^\infty}\,dt} \tag{Ene}$$

□

Remark

1. In this proof, we observe that the viscosity ν is absent, so this energy estimate for the solutions of the Navier-Stokes equation, holds for the solutions of the Euler.

2. If there are two solutions for the same problem, we have the same initial conditions and external forces, so the above estimate imposes the uniqueness of the solutions. Let u_1, u_2 be the solutions then $\tilde{F} = 0$ and $\tilde{u}|_{t=0} = 0$ so we get that $\sup_{0\le t\le T}\|\tilde{u}\|_{L^2} \le 0$ which gives that $u_1 = u_2$ a.e.x

3. For viscous fluids from this estimate, we can seek a gradient control for the solutions.
   We take the relation

$$\|\tilde{u}\|_{L^2}\frac{d}{dt}\|\tilde{u}\|_{L^2} + \nu\|\nabla\tilde{u}\|_{L^2}^2 \le \|\nabla w\|_{L^\infty}\|\tilde{u}\|_{L^2}^2 + \|\tilde{F}\|_{L^2}\|\tilde{u}\|_{L^2}$$

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}\|_{L^2}^2 + \nu\|\nabla\tilde{u}\|_{L^2}^2 \le \|\nabla w\|_{L^\infty}\|\tilde{u}\|_{L^2}^2 + \|\tilde{F}\|_{L^2}\|\tilde{u}\|_{L^2}$$

   We integrate this relation over time

$$\int_0^T \frac{1}{2}\frac{d}{dt}\|\tilde{u}\|_{L^2}^2\,dt + \nu\int_0^T \|\nabla\tilde{u}\|_{L^2}^2\,dt \le \int_0^T \|\nabla w\|_{L^\infty}\|\tilde{u}\|_{L^2}^2\,dt + \int_0^T \|\tilde{F}\|_{L^2}\|\tilde{u}\|_{L^2}\,dt$$

$$\frac{1}{2}\left(\|\tilde{u}|_{t=T}\|_{L^2}^2 - \|\tilde{u}|_{t=0}\|_{L^2}^2\right) + \nu\int_0^T \|\nabla\tilde{u}\|_{L^2}^2\,dt \le$$

$$\left(\sup_{0\le t\le T}\|\tilde{u}\|_{L^2}^2\right)\int_0^T \|\nabla w\|_{L^\infty}\,dt + \left(\sup_{0\le t\le T}\|\tilde{u}\|_{L^2}\right)\int_0^T \|\tilde{F}\|_{L^2}\,dt$$

   So

$$\nu\int_0^T \|\nabla\tilde{u}\|_{L^2}^2\,dt \le \|\tilde{u}|_{t=0}\|_{L^2}^2 + \left(\sup_{0\le t\le T}\|\tilde{u}\|_{L^2}\right)^2\int_0^T \|\nabla w\|_{L^\infty}\,dt$$

$$+ \sup_{0\le t\le T}\|\tilde{u}\|_{L^2}\int_0^T \|\tilde{F}\|_{L^2}\,dt$$

   So by the energy estimate above

$$\|\tilde{u}|_{t=0}\|_{L^2}^2 + \left(\sup_{0\le t\le T}\|\tilde{u}\|_{L^2}\right)^2\int_0^T \|\nabla w\|_{L^\infty}\,dt + \sup_{0\le t\le T}\|\tilde{u}\|_{L^2}\int_0^T \|\tilde{F}\|_{L^2}\,dt$$

$$\le \|\tilde{u}|_{t=0}\|_{L^2}^2 + \int_0^T \|\nabla w\|_{L^\infty}\,dt\; e^{2\int_0^T \|\nabla w\|_{L^\infty}\,dt}\left(\|\tilde{u}|_{t=0}\|_{L^2} + \int_0^T \|\tilde{F}\|_{L^2}\,dt\right)^2$$

$$+\int_0^T \|\tilde{F}\|_{L^2}\, dt\, e^{\int_0^T \|\nabla w\|_{L^\infty}\, dt} \left(\|\tilde{u}|_{t=0}\|_{L^2} + \int_0^T \|\tilde{F}\|_{L^2}\, dt \right)$$

We do the calculations

$$\nu \int_0^T \|\nabla \tilde{u}\|_{L^2}^2\, dt \leq \|\tilde{u}|_{t=0}\|_{L^2}^2 + \int_0^T \|\nabla w\|_{L^\infty}\, dt\, e^{2\int_0^T \|\nabla w\|_{L^\infty}\, dt}\, \|\tilde{u}|_{t=0}\|_{L^2}^2$$

$$+2\|\tilde{u}|_{t=0}\|_{L^2}^2 \int_0^T \|\nabla w\|_{L^\infty}\, dt\, e^{2\int_0^T \|\nabla w\|_{L^\infty}\, dt} \int_0^T \|\tilde{F}\|_{L^2}\, dt$$

$$+\left(\int_0^T \|\tilde{F}\|_{L^2}\, dt \right)^2 \int_0^T \|\nabla w\|_{L^\infty}\, dt e^{2\int_0^T \|\nabla w\|_{L^\infty}\, dt}$$

$$+\int_0^T \|\tilde{F}\|_{L^2}\, dt\, \|\tilde{u}|_{t=0}\|_{L^2} e^{\int_0^T \|\nabla w\|_{L^\infty}\, dt} + \left(\int_0^T \|\tilde{F}\|_{L^2}\, dt \right)^2 e^{\int_0^T \|\nabla w\|_{L^\infty}\, dt}$$

So

$$\nu \int_0^T \|\nabla \tilde{u}\|_{L^2}^2\, dt \leq \|\tilde{u}|_{t=0}\|_{L^2}^2 \left(1 + \int_0^T \|\nabla w\|_{L^\infty}\, dt\, e^{\int_0^T \|\nabla w\|_{L^\infty}\, dt} \right)$$

$$+2\|\tilde{u}|_{t=0}\|_{L^2} \int_0^T \|\tilde{F}\|_{L^2}\, dt \left(\int_0^T \|\nabla w\|_{L^\infty}\, dt\, e^{2\int_0^T \|\nabla w\|_{L^\infty}\, dt} + \frac{1}{2} e^{\int_0^T \|\nabla w\|_{L^\infty}\, dt} \right)$$

$$+\left(\int_0^T \|\tilde{F}\|_{L^2}\, dt \right)^2 \left(\int_0^T \|\nabla w\|_{L^\infty}\, dt\, e^{2\int_0^T \|\nabla w\|_{L^\infty}\, dt} + e^{\int_0^T \|\nabla w\|_{L^\infty}\, dt} \right) \Rightarrow$$

$$\nu \int_0^T \|\nabla \tilde{u}\|_{L^2}^2\, dt \leq \left(e^{\int_0^T \|\nabla w\|_{L^\infty}} + \int_0^T \|\nabla w\|_{L^\infty}\, dt\, e^{\int_0^T \|\nabla w\|_{L^\infty}} \right)$$

$$\left[\|\tilde{u}|_{t=0}\|_{L^2}^2 + 2\|\tilde{u}|_{t=0}\|_{L^2} \int_0^T \|\tilde{F}\|_{L^2}\, dt + \left(\int_0^T \|\tilde{F}\|_{L^2}\, dt \right)^2 \right]$$

Finally

$$\int_0^T \|\nabla \tilde{u}\|_{L^2}^2\, dt \leq c(w,T) \left(\|\tilde{u}|_{t=0}\|_{L^2} + \int_0^T \|\tilde{F}\|_{L^2}\, dt \right)^2$$

In the next section we will use the energy estimate (Ene) to approximate the solutions of the Euler by the solutions of the Navier Stokes.

3.2 Approximation of the inviscid flow by viscous flows for $\nu << 1$

It is clear that we talk about the properties of the solutions. In this section, we will deal with the relation between the solutions of the Euler and the Navier Stokes. We will see some examples, and then we will conclude with a more general result. Firstly, we will derive some exact solutions for the 3-dimensional case based on the matrix construction in the first chapter.

Proposition 3.2.1. *Let $\tilde{D}(t)$ be 3×3 real symmetric matrix with trace zero.*
If we determine the vorticity by the ode

$$\frac{d}{dt}\omega(t) = \tilde{D}(t)\omega(t)$$

with initial vorticity $\omega|_{t=0} = \omega_0$
and the skew symmetric matrix Ω by

$$\Omega\, h = \frac{1}{2}\omega \times h$$

then

$$u(x,t) = \frac{1}{2}\omega(t) \times x + \tilde{D}(t) \times x$$

and

$$p(x,t) = -\frac{1}{2}[\frac{d}{dt}\tilde{D}(t) + \tilde{D}(t)^2 + \Omega(t)^2]x \cdot x$$

are solutions for the 3 dimensional Navier Stokes.

Proof. Recall that for D and P determined in the same as in matrix construction on chapter 1 we have that

$$\frac{D}{Dt}D + D^2 + \Omega^2 = -P + \nu\Delta D$$

$$\frac{D}{Dt}\Omega + \Omega D + D\Omega = \nu\Delta\Omega$$

We have to understand that this proposition gives us a specific class of solutions, so we have no problem assuming that there exists a matrix D that depends only on time. We will denote this matrix as $\tilde{D}$, furthermore, we assume the vorticity of the velocity field u, which corresponds to the above matrix $\tilde{D}$, is $\omega(t)$
We define $u(x,t) = \frac{1}{2}\omega(t) \times x + \tilde{D}x$ the vorticity of this vector field is

$$\omega(x,t) = \nabla\times u(x,t) = \frac{1}{2}\left(\omega(t)(\nabla \cdot x) - x(\nabla \cdot \omega(t)) + (x \cdot \nabla)\omega(t) - (\omega(t) \cdot \nabla)x\right) + \nabla\times(\tilde{D}(t)\cdot x) \Rightarrow$$

$$\omega(x,t) = \frac{1}{2}(3\omega(t) - \omega(t)) + \nabla \times (\tilde{D}(t) \cdot x)$$

Let's examine $\nabla \times (\tilde{D}(t) \cdot x)$

$$\nabla \times (\tilde{D}(t) \cdot x) = \begin{pmatrix} \frac{\partial}{\partial x_2}(x_1\, d_{31}(t) + x_2\, d_{32}(t) + x_3\, d_{33}(t)) \\ -\frac{\partial}{\partial x_3}(x_1\, d_{21}(t) + x_2\, d_{22}(t) + x_3\, d_{23}(t)) \\ \\ \frac{\partial}{\partial x_3}(x_1\, d_{11}(t) + x_2\, d_{12}(t) + x_3\, d_{13}(t)) \\ -\frac{\partial}{\partial x_1}(x_1\, d_{21}(t) + x_2\, d_{22}(t) + x_3\, d_{23}(t)) \\ \\ \frac{\partial}{\partial x_1}(x_1\, d_{21}(t) + x_2\, d_{22}(t) + x_3\, d_{23}(t)) \\ -\frac{\partial}{\partial x_2}(x_1\, d_{11}(t) + x_2\, d_{12}(t) + x_3\, d_{13}(t)) \end{pmatrix}$$

So

$$\nabla \times (\tilde{D}(t) \cdot x) = \begin{pmatrix} d_{32}(t) - d_{23}(t) \\ d_{13}(t) - d_{31}(t) \\ d_{21}(t) - d_{12}(t) \end{pmatrix} = 0^{8}$$

Consequently $\omega(x,t) = \omega(t)$
So by the vorticity equation we conclude that $\frac{d}{dt}\omega(t) = \tilde{D}(t)\omega(t)$
By the symmetric part and since $\tilde{D}$ has only time dependence we have that $\frac{d}{dt}\tilde{D}(t) + \tilde{D}(t)^2 + \Omega(t)^2 = -P(t)$ we also know that P is the Hessian matrix of pressure so $p(x,t) = \frac{1}{2}(P(t)x) \cdot x$[9]. Eventually $p(x,t) = -\frac{1}{2}[\frac{d}{dt}\tilde{D}(t) + \tilde{D}(t)^2 + \Omega(t)^2]x \cdot x$ □

Now we will use this proposition to see some examples

Example 1 Let $D = \begin{pmatrix} -\gamma_1 & 0 & 0 \\ 0 & -\gamma_2 & 0 \\ 0 & 0 & \gamma_1 + \gamma_2 \end{pmatrix}$ where $\gamma_{ij} > 0$
Then we define $\omega(t)$ by the system of odes $\frac{d}{dt}\omega = D(t)\omega$ with initial value $\omega|_{t=0} = \omega_0 = 0$

$$\begin{cases} \frac{d}{dt}\omega_1 = -\gamma_1\omega_1 \\ \frac{d}{dt}\omega_2 = -\gamma_2\omega_2 \\ \frac{d}{dt}\omega_3 = (\gamma_1 + \gamma_2)\omega_3 \end{cases}$$

Therefore

$$\begin{cases} \omega_1 = c_1 e^{-\gamma_1 \, t} \\ \omega_2 = c_2 e^{-\gamma_2 \, t} \\ \omega_3 = c_3 e^{(\gamma_1+\gamma_2) \, t} \end{cases}$$

Substituting with t=0 we see that $c_i = 0$ so $\omega_i = 0$
By the above proposition $u(x,t) = \frac{1}{2}\omega(t) \times x + D(t) \times x = \begin{pmatrix} -\gamma_1 \, x_1 \\ -\gamma_2 \, x_2 \\ (\gamma_1 + \gamma_2) \, x_3 \end{pmatrix}$
Furthermore $\Omega \cdot h = \frac{1}{2}\omega \times h \; \forall h$ so Ω is the null matrix.
So we can compute the pressure $p(x,t) = -\frac{1}{2}[\frac{d}{dt}\tilde{D}(t) + \tilde{D}(t)^2 + \Omega(t)^2]x \cdot x = -\frac{1}{2}(\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2 + (\gamma_1 + \gamma_2)^2 x_3^2)$
Example 2 In this example we will see the case where $\gamma_1 = \gamma$ and $\gamma_2 = -\gamma$ so the matrix $D = \begin{pmatrix} -\gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So by the previous example we have that $u(x,t) = \begin{pmatrix} -\gamma x_1 \\ \gamma x_2 \\ 0 \end{pmatrix}$ and $p(x,t) = -\frac{1}{2}\gamma^2(x_1^2 + x_2^2)$
We will expand this solution by assuming that in the coordinate we have a function which only depends on the first coordinate of x i.e x_1 an the time t.
So we seek a velocity

$$u(x,t) = \begin{pmatrix} -\gamma x_1 \\ \gamma x_2 \\ u_3(x_1, t) \end{pmatrix}$$

[8] $\tilde{D}$ is a symmetric matrix
[9] $H(p(x,t)) = J(\nabla p(x,t))^T$ so $\nabla p(x,t) = P(t)x$ and $J(P(t)x)^T = P(t)$

For the pressure and the first two components of velocity, we know from example 2 that they satisfy the Navier Stokes equation, so we will check the term u_3

$$\frac{\partial}{\partial t}u_3(x_1,t)+u_1\frac{\partial}{\partial x_1}u_3(x_1,t)+u_2\frac{\partial}{\partial x_2}u_3(x_1,t)+u_3\frac{\partial}{\partial x_3}u_3(x_1,t)=\frac{\partial}{\partial x_3}p+\nu\Delta u_3(x_1,t)$$

$$\frac{\partial}{\partial t}u_3(x_1,t)-\gamma x_1\frac{\partial}{\partial x_1}u_3(x_1,t)=\nu\frac{\partial^2}{\partial x_1^2}u_3(x_1,t)$$

Assuming now that $\gamma=0$ then we have a pair of solutions $u(x,t)=(0,0,u_3(x_1,t))$ and $p(x,t)=0$.
Now we will see two cases

- $\nu=0$ [10] then
$$\frac{\partial}{\partial t}u_3(x_1,t)=0$$
so $u_3(x_1,t)=u_3(x_1,0)$
- $\nu>0$ then
$$\frac{\partial}{\partial t}u_3(x_1,t)=\nu\frac{\partial^2}{\partial x_1^2}u_3(x_1,t)$$
The above equation is the heat equation in one dimension so
$$u_3(x_1,t)=\frac{1}{\sqrt{4\pi vt}}\int_{\mathbb{R}}e^{\frac{-(x_1-y_1)^2}{4vt}}u_3(x_1,0)dy_1$$

So far, we have two specific solutions for the Euler and the Navier-Stokes. Now, by the following proposition, we will do an energy estimate for the difference of these solutions.

Proposition 3.2.2. *Let the initial value for the velocity $u_0(x)$ be a decreasing function such that $|u_0(x)|+|\nabla u_0(x)|\leq M$. Assume that u_E is a solution as above for the Euler and u_N is a solution as above for the Navier Stokes, then*

$$|u_E(x,t)-u_N(x,t)|\leq cM(vt)^{\frac{1}{2}}$$

Proof.

$$|u^E(x_1,t)-u^N(x_1,t)|=|(0,0,u_3(x_1,0))-(0,0,u_3^N(x_1,t))|$$

So we need to estimate $|u_3(x_1,0)-u_3^N(x_1,t)|$
Thus

$$|u_3^N(x_1,t)-u_3(x_1,0)|=\left|\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi vt}}e^{\frac{-(x_1-y_1)^2}{4vt}}u_3(x_1,0)\,dy_1-u_3(x_1,0)\right|$$

$$=\left|\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi vt}}e^{-\frac{y_1^2}{4vt}}u_3(x_1+y_1,0)\,dy_1-u_3(x_1,0)\right|$$

[10] It is possible for someone to question whether or not we are capable of achieving this. To clarify, proposition 4.2.1 provides us with two solutions for the Navier Stokes. It is beneficial for our situation that we have taken D and ω to be solely time-dependent, as the term with viscosity disappears in both cases.

We set $\zeta = \frac{y_1}{(vt)^{\frac{1}{2}}}$ so $d\zeta = \frac{1}{(vt)^{\frac{1}{2}}} dy_1$
We get

$$= \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi vt}} e^{-\frac{\zeta^2}{4}} u_3(x_1 + (vt)^{\frac{1}{2}}\zeta)(vt)^{\frac{1}{2}} d\zeta - u_3(x_1, 0) \right|$$

$$= \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi}} \left(e^{-\frac{\zeta^2}{4}} u_3(x_1 + (vt)^{\frac{1}{2}}\zeta, 0) - u_3(x_1, 0) \right) d\zeta \right|$$

$$\leq c \int_{\mathbb{R}} e^{\frac{-\zeta^2}{4}} |u_3(x_1 + (vt)^{\frac{1}{2}}\zeta, 0) - u_3(x_1, 0)| d\zeta$$

$$\leq c \int_{\mathbb{R}} e^{-\frac{\zeta^2}{4}} (|u_3(x_1 + (vt)^{\frac{1}{2}}\zeta, 0)| - |u_3(x_1, 0)|) d\zeta$$

We set for convenience $g(x) = |u_3(x, 0)|$ and $b = x_1 + (vt)^{\frac{1}{2}}\zeta$ and $a = x_1$.
So $|u_3^N(x_1, t) - u_3(x_1, 0)| \leq c \int_{\mathbb{R}} e^{-\frac{\zeta^2}{4}} \frac{g(b)-g(a)}{b-a} (vt)^{\frac{1}{2}} \zeta d\zeta$.
By the mean value theorem there exists $d \in (a, b)$ so that $g'(d) = \frac{g(b)-g(a)}{b-a}$.
So we conclude that $|u_3^N(x_1, t) - u_3(x_1, 0)| \leq c(vt)^{\frac{1}{2}} g'(d)$.
We substitute

$$|u_3^N(x_1, t) - u_3(x_1, 0)| \leq c(vt)^{\frac{1}{2}} \left| \frac{\partial}{\partial x_1} u_3(x_1, 0) \right|.$$

By the hypothesis $|\nabla u_3(x_1, 0)| \leq M$.
In conclusion $|u_3^E(x_1, t) - u_3^N(x_1, t)| \leq c(vt)^{\frac{1}{2}} M$. □

Remark:By the above estimate, if $\nu \searrow 0$, then the solution we construct for the inviscid fluids approximates the solution for the viscous fluids.
By this example, we have the intuition that the solutions of the Euler may approximate the solutions of the Navier Stokes. The answer for smooth solutions of Euler is affirmative.

Proposition 3.2.3. *Assume that we have the Navier Stokes and the corresponding Euler equation. We denote that u^ν and u^0 the solutions of each equation respectively. Furthermore we assume that for viscosity $0 \leq \nu \leq \nu_0$ exists on a common time interval and vanishes rapidly as $|x| \to \infty$ then*

$$\sup_{0 \leq t \leq T} \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)} \leq c(u^0, T)\nu T$$

Proof. We start from the Navier Stokes, and in the energy estimate we have proved in the first section, we substitute $u_1 = u^0$ the solution of Navier Stokes with external force $F_1^0 = F_1 = -\nu \Delta u^0$ and $u_2 = u^\nu$ and external force $F_2^\nu = F_2 = 0$

$$\sup_{0 \leq t \leq T} \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)} \leq \left[\|(u^\nu - u^0)|_{t=0}\|_{L^2(\mathbb{R}^N)} + \int_0^T \| - \nu \Delta u^0\|_{L^2(\mathbb{R}^N)} dt \right] e^{\int_0^T \|\nabla u^0\|_{L^\infty(\mathbb{R}^N)}} \Rightarrow$$

$$\sup_{0 \leq t \leq T} \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)} \leq \nu T \sup_{0 \leq t \leq T} (\|\Delta u^0\|_{L^2(\mathbb{R}^N)}) e^{\int_0^T \|\nabla u^0\|_{L^\infty} dt}$$

$$\sup_{0\le t\le T} \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)} \le \nu T \left(\sup_{0\le t\le T} \|\Delta u^0\|_{L^2(\mathbb{R}^N)} e^{T\|\nabla^0\|_{L^\infty}} \right) \Rightarrow$$

$$\sup_{0\le t\le T} \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)} \le \nu T c(u^0, T)$$

□

We can also find a gradient control by Remark 3 in the previous section. Indeed

$$\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2(\mathbb{R}^N)} dt$$

By holder inequality we get

$$\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2(\mathbb{R}^N)} dt \le \left(\int_0^T 1dt\right)^{\frac{1}{2}} \left(\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2}^2\right)^{\frac{1}{2}} \Rightarrow$$

$$\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2(\mathbb{R}^N)} dt \le T^{\frac{1}{2}} \left(\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2}^2\right)^{\frac{1}{2}}$$

$$\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2(\mathbb{R}^N)} dt \le T^{\frac{1}{2}} \frac{1}{\nu^{\frac{1}{2}}} c(u^0, T) \left(\|(u^\nu - u^0)|_{t=0}\|_{L^2}^2 + \int_0^T \|\nu \Delta u^0\|_{L^2}^2 dt\right)^{\frac{1}{2}} \Rightarrow$$

$$\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2(\mathbb{R}^N)} dt \le T^{\frac{1}{2}}\ \nu^{\frac{1}{2}} (T \sup_{0\le t\le T} \|\Delta u^0\|_{L^2})^2 \Rightarrow$$

$$\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2(\mathbb{R}^N)} dt \le C\ \nu^{\frac{1}{2}} T^{\frac{3}{2}}$$

NOTES:

1. We aim to find a convergence between those two solutions, but the main question is under which norm. By the above proposition, we have proved that for $0 \le \nu \le \nu_0$ the $\sup_{0\le t\le T} \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)}$ is bounded. So we have prove that $u^0 \to u^\nu$ with $L^\infty\{[0,T]; L^2(\mathbb{R}^N)\}$ norm.

2. Similarly $\nabla u^0 \to \nabla u^\nu$ with $L^1\{[0,T]; L^2(\mathbb{R}^N)\}$ norm.

3. We recall the notation of big-O. Let two functions f, g then $f(x) = O(G(x))$ with $x \to a \iff |f(x)| \le M g(x)\ \forall x\ 0 \le x - a \le d$
So we have $\sup_{0\le t\le T} \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)} = O(\nu)$ for $\nu \searrow 0$ and we will say that the convergence is of order 1 concerning ν. We observe that for the specific solutions we have construct $\sup_{0\le t\le T} \|u^E - u^N\|_{L^2(\mathbb{R}^N)} = O(\nu^{\frac{1}{2}})$ for $\nu \searrow 0$ so the convergence is of order $\frac{1}{2}$ with respect to ν. By the estimate we have done for the gradients, we see that $\int_0^T \|\nabla(u^\nu - u^0)\|_{L^2(\mathbb{R}^N)} dt \le O(\nu^{\frac{1}{2}})$

4. We know that $\int_0^T \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)} \leq \sup_{0\leq t\leq T} \|u^\nu - u^0\|_{L^2(\mathbb{R}^N)}$ so concerning $L^1\{[0,T], L^2(\mathbb{R}^N)\}$ the convergence of solutions is of order 1, and the gradients of them is of order $\frac{1}{2}$. So we have a better convergence in the second case since the terms are "closer", which is not a strange result if someone considers how the L^2 norm behaves on gradients.

With these remarks, we close this section and continue with the last section of this chapter, which mainly deals with the energy and the estimate we made in the first section.

3.3 The energy in two dimensions

For the basic energy estimate we have done, in the first section, we assumed that the velocity field vanishes rapidly as $|x| \nearrow \infty$ so that $u \in L^2(\mathbb{R}^N)$.
This is a strong assumption since we asking for the kinetic energy to be finite. From a physical point of view and in three dimensions, this is true, but in two dimensions we can find a very common counter-example.
This is the example of the velocity fields with vorticity of compact support.

Lemma 10. *Let $x, y \in \mathbb{R}^N$ if $|x| \geq 2R$ and $|y| \leq R$ with $R > 0$ then*

$$|x-y|^{-N} = |x|^{-N}\left(1 - 2\frac{x\cdot y}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^{-\frac{N}{2}}$$

And if $|x| \to \infty$ then

$$|x-y|^{-N} = |x|^{-N} + O(|x|^{-N-1})$$

Proof of lemma:We will prove this lemma by induction.
For $N = 1$ we will prove that $\frac{1}{|x-y|} = \frac{1}{|x|}\frac{1}{\sqrt{\left(1+2\frac{x\cdot y}{|x|^2}+\frac{|y|^2}{|x|^2}\right)}}$

- If $x \leq -2R$ and $-R \leq y \leq 0$ we get $x - y \leq -R$
  If $y = 0$ we are done since we get $\frac{1}{|x|} = \frac{1}{|x|}$
  If $y \neq 0$ then

$$\frac{1}{y-x} = -\frac{1}{x}\frac{1}{\sqrt{1-2\frac{xy}{x^2}+\frac{y^2}{x^2}}}$$

$$\frac{1}{y-x} = -\frac{1}{x}\frac{|x|}{|x-y|}$$

$$\frac{1}{y-x} = \frac{1}{y-x}$$

  So in that case the equality holds.

- If $x \leq -2R$ and $0 < y \leq R$ we get $x - y \leq -2R$ then

$$\frac{1}{y-x} = -\frac{1}{x}\frac{|x|}{|x-y|}$$

$$\frac{1}{y-x} = \frac{1}{y-x}$$

  So in that case the equality holds.

- If $x \geq 2R$ and $-R \leq y < 0$ we get $y - x \leq -2R$ so $-(y-x) = x - y \geq 2R$ then

$$\frac{1}{x-y} = \frac{1}{x}\frac{|x|}{|x-y|}$$

$$\frac{1}{x-y} = \frac{1}{x-y}$$

  So in that case the equality holds.

- If $x \geq 2R$ and $0 < y \leq R$ we get $y - x \leq -R$ so $x - y \geq R$ then

$$\frac{1}{x-y} = \frac{1}{x}\frac{|x|}{|x-y|}$$

$$\frac{1}{x-y} = \frac{1}{x-y}$$

  So in that case the equality holds.

For the inductive step we assume that the equality holds for $N = k$ so we have that

$$|x-y|^{-k} = |x|^{-k}\left(1 - 2\frac{x \cdot y}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^{\frac{-k}{2}}$$

We will examine the case when $N = k + 1$, we want to prove that

$$|x-y|^{-k-1} = |x|^{-k-1}\left(1 - 2\frac{x \cdot y}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^{\frac{-k-1}{2}}$$

By the inductive step and the case n=1 we know that

$$|x-y|^{-k}\,|x-y|^{-1} = \left(|x|^{-k}\left(1 - 2\frac{x \cdot y}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^{\frac{-k}{2}}\right)$$

$$\cdot\left(|x|^{-1}\left(1 - 2\frac{x \cdot y}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^{\frac{-1}{2}}\right)$$

$$= |x|^{-k-1}\,\left(1 - 2\frac{x \cdot y}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^{\frac{-k-1}{2}}$$

Now for the second part of this lemma we will use this result
We set $|x| = a$, $|y| = b$ and $x \cdot y = ab\cos\theta$ then

$$\frac{1}{|x-y|^N} = \frac{1}{a^N}\left(1 - 2\frac{b}{a}\cos\theta + \left(\frac{b}{a}\right)^2\right)^{-\frac{N}{2}}$$

We set $\frac{b}{a} = p$ so

$$\frac{1}{|x-y|^N} = \frac{1}{a^N}(1 - 2p\cos\theta + p^2)^{-\frac{N}{2}}$$

We now set $z = p^2 - 2p\cos\theta$
By Taylor expansion for $(1+z)^{\frac{-N}{2}}$ for $z \in \mathbb{R}$ around $z_0 = 0$ we have that

$$(1+z)^{\frac{-N}{2}} = 1 + ((1+z)^{\frac{-N}{2}})'|_{z_0}\, z + ((1+z)^{\frac{-N}{2}})''|_{z_0}\, \frac{z^2}{2!} + ...$$

$$= 1 - \frac{N}{2}z + \frac{N(N+2)}{8}z^2 + ...$$

If $z \to 0$ we have $(1+z)^{\frac{-N}{2}} = 1 + O(z)$
Consequently for $|y| \leq R$ and $|x| \to \infty$ we have that $z \to 0$ so by substituting

$$|x-y|^{-N} = |x|^{-N}(1 + |x|^{-1}) = |x|^{-N} + O(|x|^{-N-1})$$

Someone may wonder why we need this lemma. The answer is easy if we recall the form of the velocity fields given by Biot Savart law, which kernels have singularity along the diagonal.
Assume now that the vorticity ω is smooth and has compact support i.e. $\text{supp}\omega \subset \{|y| \leq R\}$
We will start with the three dimensions:

- $u_3(x,t) = \int_{\mathbb{R}^3} K_3(x-y)\omega(y,t)dy = \int_{\mathbb{R}^3} \frac{(x-y)\times\omega(y,t)}{|x-y|^3}dy.$
  We know that this kernel is homogeneous of degree -2 so $K_3 \sim \frac{1}{|x-y|^2}$.
  Thus $u_3(x,t) \sim \int_{\mathbb{R}^3} \frac{1\times\omega(y,t)}{|x-y|^2}dy.$
  !!!Now we will use the lemma we have $|y| \leq R$ and we want to see what happens when $|x| \nearrow \infty$ so :

$$\int_{\mathbb{R}^3} \frac{1\times\omega(y,t)}{|x-y|^2}dy = \int_{\mathbb{R}^3} [|x|^{-2} + O(|x|^{-3})](1\times\omega(y,t))dy$$

$$= (|x|^{-2} + O(|x|^{-3})) \int_{\mathbb{R}^3} \begin{pmatrix} \omega_3(y,t) - \omega_2(y,t) \\ \omega_1(y,t) - \omega_3(y,t) \\ \omega_2(y,t) - \omega_1(y,t) \end{pmatrix} dy$$

$$= (|x|^{-2} + O(|x|^{-3})) \int_{|y|\leq R} \begin{pmatrix} \omega_3(y,t) - \omega_2(y,t) \\ \omega_1(y,t) - \omega_3(y,t) \\ \omega_2(y,t) - \omega_1(y,t) \end{pmatrix} dy$$

$$\leq c(|x|^{-2} + O(|x|^{-3}))$$

We conclude that $u_3(x,t) \sim O(|x|^{-2})$
So for the kinetic energy for $|x| \geq 2R$ we get

$$\int_{|x|\geq 2R} |u_3(x,t)|^2 dx \sim \int_R^\infty r^{-4} r^2 dr = \frac{1}{R}$$

- $u_2(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy$.
  We know that this kernel is homogeneous of degree -1 so $K_2 \sim \frac{1}{|x-y|}$.
  Thus $u_2(x,t) \sim \int_{\mathbb{R}^2} \frac{1}{|x-y|}\omega(y,t)dy$.
  By the lemma 10 we have that

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|}\omega(y,t)dy = [|x|^{-1} + O(|x|^{-2})] \int_{\mathbb{R}^2} \omega(y,t)dy$$

$$= [|x|^{-1} + O(|x|^{-2})] \int_{|y|\leq R} \omega(y,t)dy \leq c[|x|^{-1} + O(|x|^{-2})].$$

We conclude that $u_2(x,t) \sim O(|x|^{-1})$.
So for the kinetic energy for $|x| \geq 2R$ we get

$$\int_{|x|\geq 2R} |u_2(x,t)|^2 dx \sim \int_R^\infty r^{-2} r dr = \infty$$

Remark:By the previous procedure we have that $u_2(x,t) \sim \frac{1}{|x|}\int_{\mathbb{R}^2} \omega(y,t)dy + O(|x|^{-2})$ so if $\int_{\mathbb{R}^2} \omega(y,t)dy = 0$ we have that $u_2(x,t) \sim O(|x|^{-2})$ thus

$$\int_{|x|\geq 2R} |u_2(x,t)|^2 dx \sim \int_R^\infty r^{-4} r dr = \int_R^\infty r^{-3} dr = \frac{1}{2R^2}$$

So we have following proposition :

Proposition 3.3.1. *Let $u(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ with vorticity of compact support then*

$$\int_{\mathbb{R}^2} u^2 dx \leq \infty \iff \int_{\mathbb{R}^2} \omega(y,t)dy = 0$$

Proof. ($\Leftarrow$) We have this from the remark. ($\Rightarrow$) $\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy\right) dx = c$
So $\int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy = 0$

$$K_2(x)\int_{\mathbb{R}^2} \omega(y,t)dy = 0$$

Therefore $\int_{\mathbb{R}^2} \omega(y,t)dy = 0$ □

Remark:This proposition holds for all the velocity fields on $\mathbb{R}^2$.

For velocity fields with vorticity of compact support and mean value zero, we solve the problem of finite kinetic energy. We would like a more global result, but the only result we have is that the kinetic energy in 2 dimensions is locally finite.
So the energy estimate we have made fails in most cases of 2d, thus we will derive a new estimate

Energy estimate for 2 dimensions

We will do the following decomposition, in order to use energy methods.

Definition 5. *Let u be an incompressible smooth velocity field on* $\mathbb{R}^2$*. Then u has radial energy decomposition if there exist a smooth radial vorticity* $\bar{\omega}(|x|)$ *such that*

$$u(x,t) = v(x,t) + b(x,t)$$

- $v \in L^2(\mathbb{R}^2)$ *and* $\mathrm{div} v = 0$
- *b(x,t)is defined via* $\bar{\omega}$ *by*

$$b(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \frac{1}{|x|^2} \int_0^{|x|} s\bar{\omega}(s,t)ds$$

Remark

1. This decomposition is not unique, if we choose another radial vorticity, then we will have another decomposition.
2. Recall that the total flux of vorticity for the solutions of the Euler equation is constant in time.Thus we have that $\int_0^{|x|} s\bar{\omega}(s,t)ds = \int_0^{|x|} s\bar{\omega}(s,0)ds$ so we can find b(x,t) from the initial value of radial vorticity.

   Lemma 11. [11] *Any smooth incompressible velocity field with vorticity in* L^1 *has radial energy decomposition*

We proceed now with the seeking of a class of solutions for Euler and Navier Stokes, which have radial vorticity.
We search for steady flows. The flow of a fluid is steady if its velocity and all values depending on its substance are independent of time[12]. So now we will see the relation between stream functions and steady flows.
We begin with the vorticity equation in two dimensions.
Recall that:

$$\frac{\partial}{\partial t}\omega + u_1 \frac{\partial}{\partial x_1}\omega + u_2 \frac{\partial}{\partial x_2} = 0 \tag{V}$$

We know that since the field is incompressible there exist a stream function ψsuch that $u_1 = -\frac{\partial}{\partial t}\psi$ and $u_2 = \frac{\partial}{\partial t}\psi$.
We substitute those two in (V) and we get

$$\frac{\partial}{\partial t}\omega - \frac{\partial}{\partial x_2}\psi \frac{\partial}{\partial x_1}\omega + \frac{\partial}{\partial x_1}\psi \frac{\partial}{\partial x_2}\omega = 0$$

Since the steady flow

$$-\frac{\partial}{\partial x_2}\psi \frac{\partial}{\partial x_1}\omega + \frac{\partial}{\partial x_1}\psi \frac{\partial}{\partial x_2}\omega = 0 \Leftrightarrow$$

[11] See [30] lemma 3.2
[12] [8] pg 72

$$\begin{vmatrix} \frac{\partial}{\partial x_1}\psi & \frac{\partial}{\partial x_2}\psi \\ \frac{\partial}{\partial x_1}\omega & \frac{\partial}{\partial x_2}\omega \end{vmatrix} = 0$$

This matrix is the Jacobian of the field (ψ, ω), by this we take that $\nabla\psi$ and $\nabla\omega$ are parallel[13]. This means that if we have a level curve of ψ in a specific point, then $\nabla\psi$ is c and parallel to this curve, thus $\nabla\omega$ is parallel to this curve, so ω along this level curve is constant. Doing this procedure for every level curve of ψ, we conclude that there is a function F so that $\omega = F(\psi)$.
We also know that $\omega = \Delta\psi$. Consequently, a stream function defines a steady solution to the 2 dimensional Euler equation $\iff$ $\Delta\psi = F(\psi)$
So now we will search for our radial vorticity, we know that $\Delta\psi = \bar{\omega}(|x|)$ we know that the Laplace operator is invariant under rotation indeed

In N dimensions for rotation we know there exist an orthogonal matrix B such that $x' = BX$ and $BB^T = B^T B = I$ we have that

$$x'_k = \sum_i b_{ki} x_i$$

for the relation between derivatives after this translation we compute

$$\frac{\partial}{\partial x_i} = \sum_k \frac{\partial}{\partial x_i} x'_k \frac{\partial}{\partial x'_k} = \sum_k b_{ki} \frac{\partial}{\partial x'_k}$$

and for the second derivatives we have that

$$\frac{\partial^2}{\partial x_i^2} = \sum_k b_{ki} \frac{\partial}{\partial x'_k} \sum_l b_{li} \frac{\partial}{\partial x'_l} = \sum_{k,l} b_{ki} b_{li} \frac{\partial^2}{\partial x'_k \partial x'_l}$$

So for the Laplace operator we see that

$$\Delta_x = \sum_i \frac{\partial^2}{\partial x_i^2} = \sum_i \sum_{k,l} b_{ki} b_{li} \frac{\partial^2}{\partial x'_k \partial x'_l} = \sum_{k,l} \left(\sum_i b_{ki} b_{li} \right) \frac{\partial^2}{\partial x'_k \partial x'_l}$$

$$= \sum_{k,l} \delta_{kl} \frac{\partial^2}{\partial x'_k \partial x'_l}$$

where δ_{kl} thus $\Delta_x = \sum_k \frac{\partial^2}{\partial {x'_k}^2} = \Delta_{x'}$

Therefore the solutions of the Laplace equation will be invariant under rotation, this means that the stream function will be radial .

Lemma 12. *Assume that $\omega_0(|x|)$ then this radial vorticity defines a steady solution to the 2D Euler equation.*

[13] we expand them in 3 dimensions with the 3d coordinate be a zero so $\nabla\psi \times \nabla\omega = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1}\psi & \frac{\partial}{\partial x_2}\psi & 0 \\ \frac{\partial}{\partial x_1}\omega & \frac{\partial}{\partial x_2}\omega & 0 \end{vmatrix} = 0$

Proof of lemma:It is sufficient to show that the determinant of the Jacobian of the $(\psi_0, \Delta\psi_0)$ is zero. This is true since the gradient of radial functions on a point pointing away or towards the origin[a]

[a]Let f be a radial function then there exist a function g so that $f = g(x^2 + y^2)$ so for a random (x_0, y_0) we see that $\nabla f(x_0, y_0) = (2x_0 g', 2y_0 g')$

So now we will find the stream function, we set $|x| = r$ $\begin{cases} x_1 = r\cos\phi \\ x_2 = r\sin\phi \end{cases}$ and we will find the Laplace operator in polar coordinates.
The Jacobian of this change to polar coordinates is:

$$J = \begin{pmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{pmatrix}$$

and the Jacobian of the inverse is:

$$J^{-} = \begin{pmatrix} \cos\phi & -\frac{\sin\phi}{r} \\ \sin\phi & \frac{\cos\phi}{r} \end{pmatrix}.$$

So for the derivatives we have $\begin{cases} \frac{\partial}{\partial x_1} = \frac{\partial r}{\partial x_1}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial x_1}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial x_2} = \frac{\partial r}{\partial x_2}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial x_2}\frac{\partial}{\partial\phi} \end{cases}$ and for the Laplacian we get:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \cos^2\phi\frac{\partial^2}{\partial r^2} - \cos\phi\frac{\partial}{\partial r}\frac{\sin\phi}{r}\frac{\partial}{\partial\phi} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\cos\phi\frac{\partial}{\partial r} + \frac{\partial\phi}{r}\frac{\partial}{\partial\phi}\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}$$

$$+\sin^2\phi\frac{\partial^2}{\partial r^2} + \sin\phi\frac{\partial}{\partial r}\frac{\cos\phi}{r}\frac{\partial}{\partial\phi} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\frac{\cos\phi}{r}\frac{\partial}{\partial\phi} \Rightarrow$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi}.$$

So now we have that

$$\frac{\partial^2}{\partial r^2}\psi(r) + \frac{1}{r}\frac{\partial}{\partial r}\psi(r) = \omega(r)$$

So $\frac{\partial}{\partial r}\psi(r) = \frac{1}{r}\int_0^r s\bar{\omega}(s)ds$.

We will determine the velocity $\begin{cases} u_1 = -\frac{\partial\psi}{\partial x_2} = -\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_2} = -\frac{x_2}{r}\frac{\partial\psi}{\partial r} \\ u_2 = \frac{\partial\psi}{\partial x_1} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_1} = \frac{x_1}{r}\frac{\partial\psi}{\partial r} \end{cases}$

Thus $u(x) = \begin{pmatrix} \frac{-x_2}{r} \\ \frac{x_1}{r} \end{pmatrix} \frac{1}{r}\int_0^r s\bar{\omega}(s)ds$ replacing $r = |x|$ we have

$$u(x) = \begin{pmatrix} \frac{-x_2}{|x|^2} \\ \frac{x_1}{|x|^2} \end{pmatrix} \int_0^{|x|} s\bar{\omega}(s)ds$$

Remark:The solution is steady thus $u(x,t) = u(x)$
The only thing to examine now is the viscous fluids. We start again with the vorticity equation

$$\frac{\partial}{\partial t}\omega(x,t) + \sum_j u_j\frac{\partial}{\partial x_j}\omega(X,t) = \nu\Delta\omega(x,t)$$

Again the velocity field is incompressible so we have

$$\frac{\partial}{\partial t}\omega(x,t) + \begin{vmatrix} \frac{\partial \psi}{\partial x_1} & \frac{\partial \psi}{\partial x_2} \\ \frac{\partial \omega}{\partial x_1} & \frac{\partial \omega}{\partial x_2} \end{vmatrix} = \nu \Delta \omega(x,t)$$

Since the solution ψ is also radial the determinant is zero. Thus we have $\frac{\partial \omega}{\partial t} = \nu \Delta \omega$
This is a heat equation we assume that the initial vorticity is $\omega|_{t=0} = \omega_0(|x|)$, therefore $\omega(x,t) = \frac{1}{4\pi v t}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4vt}}\omega(|y|)dy$, we see that the vorticity depends on x by $|x|$ so its a radial vorticity. We use again the Laplace operator in polar coordinates so we have that

$$u(x) = \begin{pmatrix} \frac{-x_2}{|x|^2} \\ \frac{x_1}{|x|^2} \end{pmatrix} \int_0^{|x|} s\omega_0(s)ds$$

So far we have find a solution for Euler and Navier-Stokes which is defined by a radial vorticity, so this solution will play the role we want for the decomposition.

Proposition 3.3.2. *Every smooth solution u(x,t) of the Euler or the Navier-Stokes with initial vorticity* $\omega_0 \in L^1$ *has radial energy decomposition*

Proof. We choose any radial vorticity with initial value such that

$$\int_{\mathbb{R}^2} \bar{\omega}_0(x)dx = \int_{\mathbb{R}^2} \omega_0(x)dx$$

In the previous work, we have defined an exact solution with radial vorticity. So "b" is the radial eddy above and is the same as in the definition, so in order to complete the proof, we seek a v such that

$$v(x,t) = u(x,t) - b(x,t)$$

, if this v is div free and has finite kinetic energy we are done.
Firstly we know that u and b solve the Euler or the Navier Stokes so $\text{div}u = \text{div}b = 0$
So

$$\text{div}v(x,t) = \text{div}u(x,t) - \text{div}b(x,t) \Rightarrow$$

$$\text{div}v(x,t) = 0$$

So know we want to see if u has finite kinetic energy .

$$\omega_v(x,t) = \omega(x,t) - \bar{\omega}(x,t) \Rightarrow$$

$$\int_{\mathbb{R}^2} \omega_v(x,t)dx = \int_{\mathbb{R}^2} \omega(x,t)dx - \int_{\mathbb{R}^2} \bar{\omega}(x,t)dx \Rightarrow$$

$$\int_{\mathbb{R}^2} \omega_v(x,t)dx = \int_{\mathbb{R}^2} \omega_0(x)dx - \int_{\mathbb{R}^2} \bar{\omega}_0(x)dx = 0$$

And by the Remark below the proposition 3.3.1 we have the desired result □

Remark:The radial vorticity on the above decomposition and consequently the velocity field b are known. So we will deal only with the field $v(x,t)$ Since u and b are solutions of N-S we have that

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + F$$

$$\underbrace{\frac{\partial b}{\partial t}}_{=0} + (b \cdot \nabla)b = -\nabla p_b + \nu \Delta b$$

We subtract those relations and we get :

$$\frac{\partial u}{\partial t} - \frac{\partial b}{\partial t} + (u \cdot \nabla)u - (b \cdot \nabla)b = -\nabla p + \nabla p_b + \nu \Delta u - \nu \Delta b + F$$

$$\frac{\partial v}{\partial t} + (u \cdot \nabla)u - (b \cdot \nabla)b - (b \cdot \nabla)u + (b \cdot \nabla)u - (b \cdot \nabla)b = -\nabla p + \nabla p_b + \nu \Delta v + F$$

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)u + (b \cdot \nabla)(u - b) = -\nabla p + \nabla p_b + \nu \Delta v + F$$

So we reach to the relation[14]:

$$\frac{\partial v}{\partial t} + (b \cdot \nabla)v + (v \cdot \nabla)b + (v \cdot \nabla)v = -\nabla p + \nabla p_b + \nu \Delta v + F \tag{1}$$

Thus we have the following energy estimate

Proposition 3.3.3. *Assume that u_1, u_2 are to smooth solutions for Navier-Stokes with radial energy decomposition $u_i = v_i + b_i$ external forces F_i and pressures p_i then we have the following energy estimate*

$$\sup_{0 \le t \le T} \|v_1 - v_2\|_{L^2} \le e^{\int_0^T \|\nabla v_2\|_{L^\infty} + \|\nabla b_1\|_{L^\infty}} \|(v_1 - v_2)|t = 0\|_{L^2}$$

$$+ e^{\int_0^T \|\nabla v_2\|_{L^\infty} + \|\nabla b_1\|_{L^\infty}} \int_0^T \|F_1 - F_2\|_{L^2} + \|b_1 - b_2\|_{L^\infty} \|\nabla v_2\|_{L^2} + \|\nabla b_1 - \nabla b_2\|_{L^\infty} \|v_2\|_{L^2} dt$$

Proof. We define $\tilde{v} = v_1 - v_2$, $\tilde{b} = b_1 - b_2$,$\tilde{F} = F_1 - F_2$, $\tilde{p} = p_1 - p_2$ and $\tilde{p}_b = p_{b_1} - p_{b_2}$ We will find a relation between them. By the relation (1) above we have that:

$$\frac{\partial v_1}{\partial t} + (b_1 \cdot \nabla)v_1 + (v_1 \cdot \nabla)b_1 + (v_1 \cdot \nabla)v_1 = -\nabla p_1 - \nabla p_{b_1} + \nu \Delta v_1 + F_1$$

and

$$\frac{\partial v_2}{\partial t} + (b_2 \cdot \nabla)v_2 + (v_2 \cdot \nabla)b_2 + (v_2 \cdot \nabla)v_2 = -\nabla p_2 - \nabla p_{b_2} + \nu \Delta v_2 + F_2$$

Thus

$$\frac{\partial v_1}{\partial t} - \frac{\partial v_2}{\partial t} + (b_1 \cdot \nabla)v_1 + (v_1 \cdot \nabla)b_1 + (v_1 \cdot \nabla)v_1 - (b_2 \cdot \nabla)v_2 - (v_2 \cdot \nabla)b_2 - (v_2 \cdot \nabla)v_2$$

[14] b(x,t) is a solution to Navier Stokes with external force zero

$$= -\nabla p_1 - \nabla p_{b_1} + \nu\Delta v_1 + F_1 + \nabla p_2 + \nabla p_{b_2} - \nu\Delta v_2 - F_2$$

Our aim is to reach a relation with most orders be the tilded ones.

$$\frac{\partial \tilde{v}}{\partial t} + (v_1 \cdot \nabla)v_1 - (v_1 \cdot \nabla)v_2 + (v_1 \cdot \nabla)v_2 - (v_2 \cdot \nabla)v_2 + (b_1 \cdot \nabla)v_1 - (b_1 \cdot \nabla)v_2 + (b_1 \cdot \nabla)v_2$$

$$-(b_2 \cdot \nabla)v_2 + (v_1 \cdot \nabla)b_1 - (v_2 \cdot \nabla)b_1 + (v_2 \cdot \nabla)b_1 - (v_2 \cdot \nabla)b_2 = -\nabla\tilde{p} - \nabla\tilde{p}_b + \nu\Delta\tilde{v} + \tilde{F}$$

Thus

$$\frac{\partial \tilde{v}}{\partial t} + (v_1 \cdot \nabla)(v_1 - v_2) + [(v_1 - v_2) \cdot \nabla]v_2 + (b_1 \cdot \nabla)(v_1 - v_2) + [(b_1 - b_2) \cdot \nabla]v_2$$

$$+(v_1 - v_2) \cdot \nabla b_1 + v_2 \cdot \nabla(b_1 - b_2) = -\nabla\tilde{p} - \nabla\tilde{p}_b + \nu\Delta\tilde{v} + \tilde{F}$$

We conclude that

$$\frac{\partial \tilde{v}}{\partial t} + (v_1 \cdot \nabla)\tilde{v} + (\tilde{v} \cdot \nabla)v_2 + (b_1 \cdot \nabla)\tilde{v} + (\tilde{b} \cdot \nabla)v_2 + (\tilde{v} \cdot \nabla)b_1 + (v_2 \cdot \nabla)\tilde{b} = -\nabla\tilde{p} - \nabla\tilde{p}_b + \nu\Delta\tilde{v} + \tilde{F} \tag{2}$$

The procedure we will follow is exactly the same as in the previous estimate

$$\int_{\mathbb{R}^2} \left(\frac{\partial \tilde{v}}{\partial t} + (v_1 \cdot \nabla)\tilde{v} + (\tilde{v} \cdot \nabla)v_2 + (b_1 \cdot \nabla)\tilde{v} + (\tilde{b} \cdot \nabla)v_2 + (\tilde{v} \cdot \nabla)b_1 + (v_2 \cdot \nabla)\tilde{b} \right) \tilde{v}dx$$

$$= \int_{\mathbb{R}^2} \left(-\nabla\tilde{p} - \nabla\tilde{p}_b + \nu\Delta\tilde{v} + \tilde{F} \right) \tilde{v}dx$$

We compute

- $$\int_{\mathbb{R}^2} \nabla\tilde{p} \cdot \tilde{v}dx = -\int_{\mathbb{R}^2} \tilde{p}\mathrm{div}\tilde{v} = 0$$

- $$\int_{\mathbb{R}^2} \Delta\tilde{v} \cdot \tilde{v}dx = -\int_{\mathbb{R}^2} \nabla\tilde{v} \cdot \nabla\tilde{v}$$

- $$\int_{\mathbb{R}^2} \tilde{v}\frac{\partial \tilde{v}}{\partial t}dx = \left(\int_{\mathbb{R}^2} |\tilde{v}|^2 dx \right)^{\frac{1}{2}} \frac{d}{dt} \left(\int_{\mathbb{R}^2} |\tilde{v}|^2 dx \right)$$

- $$\int_{\mathbb{R}^2} [(v_1 \cdot \nabla)\tilde{v}]\tilde{v}dx = \int_{\mathbb{R}^2} \sum_{i,j} v_{1_j}\tilde{v}_i \frac{\partial}{\partial x_j}\tilde{v}_i dx = \sum_{i,j} \int_{\mathbb{R}^2} v_{1_j}\tilde{v}_i \frac{\partial}{\partial x_j}\tilde{v}_i dx$$

  Thus by integration by parts

  $$\int_{\mathbb{R}^2} v_{1_j}\tilde{v}_i \frac{\partial}{\partial x_j}\tilde{v}_i dx = -\int_{\mathbb{R}^2} \tilde{v}_i^2 \frac{\partial v_{1_j}}{\partial x_j} dx - \int_{\mathbb{R}^2} \tilde{v}_1 v_{1_j} \frac{\partial v_i}{\partial x_j}$$

  So

  $$\int_{\mathbb{R}^2} [(v_1 \cdot \nabla)\tilde{v}]\tilde{v}dx = 0$$

- $$\int_{\mathbb{R}^2}[(b_1\cdot\nabla)\tilde{v}]\tilde{v}=0$$

  as above.

So we have

$$\|\tilde{v}\|_{L^2}\frac{d}{dt}\|\tilde{v}\|_{L^2}+\nu\|\nabla\tilde{v}\|_{L^2}^2\leq\int_{\mathbb{R}^2}(\nabla v_2\tilde{v})\cdot\tilde{v}dx+\int_{\mathbb{R}^2}(\nabla v_2\tilde{b})\cdot\tilde{v}dx$$

$$+\int_{\mathbb{R}^2}(\nabla b_1\tilde{v})\cdot\tilde{v}dx+\int_{\mathbb{R}^2}(\nabla\tilde{b}v_2)\cdot\tilde{v}+\int_{\mathbb{R}^2}\tilde{F}\cdot\tilde{v}dx$$

By Holder inequality on the right part we have that

$$\|\tilde{v}\|_{L^2}\frac{d}{dt}\|\tilde{v}\|_{L^2}\leq\|\tilde{v}\|_{L^2}\ \|\tilde{v}\|_{L^2}\ \|\nabla v_2\|_{L^\infty}+\|\tilde{v}\|_{L^2}\ \|\nabla v_2\|_{L^2}\ \|\tilde{b}\|_{L^\infty}$$

$$+\|\tilde{v}\|_{L^2}\ \|\tilde{v}\|_{L^2}\ \|\nabla b_1\|_{L^\infty}+\|\tilde{v}\|_{L^2}\ \|v_2\|_{L^2}\ \|\nabla b\|_{L^\infty}+\|\tilde{F}\|_{L^2}\ \|\tilde{v}\|_{L^2}$$

We will apply Gronwall s lemma to the relation:

$$\frac{d}{dt}\|\tilde{v}\|_{L^2}\leq\|\tilde{v}\|_{L^2}\ \|\nabla v_2\|_{L^\infty}+\|\nabla v_2\|_{L^2}\ \|\tilde{b}\|_{L^\infty}+\|\tilde{v}\|_{L^2}\ \|\nabla b_1\|_{L^\infty}+\|v_2\|_{L^2}\ \|\nabla\tilde{b}\|_{L^\infty}+\|\tilde{F}\|_{L^2}$$

$$=(\|\nabla v_2\|_{L^\infty}+\|\nabla b_1\|_{L^\infty})\,\|\tilde{v}\|_{L^2}+\Big(\|\nabla v_2\|_{L^2}\ \|\tilde{b}\|_{L^\infty}+\|v_2\|_{L^2}\ \|\nabla\tilde{b}\|_{L^\infty}+\|\tilde{F}\|_{L^2}\Big)$$

We set $q=\|\tilde{v}\|_{L^2(\mathbb{R}^N)}$, $p=\|\nabla v_2\|_{L^\infty}+\|\nabla b_1\|_{L^\infty}$ and $z=\|\nabla v_2\|_{L^2}\ \|\tilde{b}\|_{L^\infty}+\|v_2\|_{L^2}\ \|\nabla\tilde{b}\|_{L^\infty}+\|\tilde{F}\|_{L^2}$ So we have that

$$\sup_{0\leq t\leq T}q(t)\leq\left[q(0)+\int_0^T zdt\right]e^{\int_0^T pdt}$$

Substituting everything we conclude that

$$\sup_{0\leq t\leq T}\|v_1-v_2\|_{L^2}\leq e^{\int_0^T\|\nabla v_2\|_{L^\infty}+\|\nabla b_1\|_{L^\infty}}\ \|(v_1-v_2)|t=0\|_{L^2}$$

$$+e^{\int_0^T\|\nabla v_2\|_{L^\infty}+\|\nabla b_1\|_{L^\infty}}\int_0^T\|F_1-F_2\|_{L^2}+\|b_1-b_2\|_{L^\infty}\|\nabla v_2\|_{L^2}+\|\nabla b_1-\nabla b_2\|_{L^\infty}\|v_2\|_{L^2}dt$$

□

Remark:

1. As before this estimate does not depend on viscosity so this is also an energy estimate for the solutions of Euler.

2. By this energy estimate, we obtain from the result of uniqueness in the 2 dimensions case. Indeed if we assume two solutions to the Navier Stokes equation, with sam initial data and external forces, we have that $\tilde{F} = 0$,$\tilde{v}_0 = 0$, and if we choose the same radial vorticity, then $\tilde{b} = 0$, so we have that $\sup_{0\leq t\leq T} \|v_1 - v_2\|_{L^2} \leq 0$ i.e. $v_1 = v_2$ and $u_1 = v_1 + b = v_2 + b + u_2$.
   Furthermore, we have an estimate for the supremum of the difference of solutions of the Navier-Stokes and the Euler. We suppose that the solution of Euler u^0 is a solution to Navier Stokes with external force $-\nu\Delta u_0$ and the solution of the Navier Stokes u^ν with external force zero then we get that $\sup_{0\leq t\leq T} \|u - u^0\|_{L^2} \leq c(T)\nu$.

3. We can also find a gradient control for solutions of the Navier Stokes, indeed we use the relation:

$$\frac{1}{2}\frac{d}{dt}\|\tilde{v}\|_{L^2} + \nu\|\nabla\tilde{v}\|^2_{L^2} \leq \|\tilde{v}\|^2_{L^2} \left(\|\nabla v_2\|_{L^\infty} + \|\nabla b_1\|_{L^\infty}\right)$$

$$+\|\tilde{v}\|_{L^2} \left(\|\tilde{b}\|_{L^\infty}\|\nabla v_2\|_{L^2} + \|v_2\|_{L^2}\|\nabla\tilde{b}\|_{L^\infty} + \|\tilde{F}\|_{L^2}\right)$$

   We set $p = \|\nabla v_2\|_{L^\infty} + \|\nabla b_1\|_{L^\infty}$ and $q = \|\tilde{b}\|_{L^\infty}\|\nabla v_2\|_{L^2} + \|v_2\|_{L^2}\|\nabla\tilde{b}\|_{L^\infty} + \|\tilde{F}\|_{L^2}$
   We integrate over time and we do Holder on the right side so

$$\nu\int_0^T \|\nabla\tilde{v}\|^2_{L^2}dt \leq \|\tilde{v}|_{t=0}\|^2_{L^2} + \left(\sup_{0\leq t\leq T}\|\tilde{v}\|_{L^2}\right)^2 + \int_0^T \|p\|_{L^\infty}dt + \sup_{0\leq t\leq T}\|\tilde{v}\|_{L^2}\int_0^T qdt$$

   We do the calculations and we have that

$$\nu\int_0^T \|\nabla\tilde{v}\|^2_{L^2}dt \leq c(p,T)\left[\nu\int_0^T \|\tilde{v}|_{t=0}\|^2_{L^2} + \|\tilde{b}\|_{L^\infty}\|\nabla v_2\|_{L^2} + \|v_2\|_{L^2}\|\nabla\tilde{b}\|_{L^\infty}dt\right]$$

Summarizing in this section, we have seen some basic properties of the solutions of our equations, and, we have derived some important classes of solutions. But do we have a general result for the existence of solutions? We will answer this question in the next chapters.

CHAPTER 4

Existence of local in time smooth solutions

In the previous chapter, we used energy estimates to obtain some interesting results about the properties of "hypothetical" solutions to the Euler and Navier Stokes equations. However, we are yet to find a general result for the existence of these solutions. We have only been able to find exact solutions through examples. In this chapter, we will discuss the existence of the solution locally in time.
Before we begin the search for these solutions, we will provide a brief introduction to some functional spaces and basic tools that we will need for the following proofs.

4.1 Preliminaries

Sobolev spaces

Definition 6. *We define the m-th order, L_p Sobolev space in $\mathbb{R}^N$ with $m, p \in \mathbb{Z}_0^+$ to be the space of functions which are p-integrable and their distribution derivatives up to order m are p-integrable. We denote $W^{m,p}(\mathbb{R}^N)$.*

For $p = 2$ we denote the Sobolev space of square-integrable functions together with all their distribution derivatives as $H^m(\mathbb{R}^N)$.
We aim to generalize those spaces for $m \in \mathbb{R}^N$. We will do this generalization via Fourier transform[1]. This idea is not arbitrary and this is because for L^2 functions by Plancherels theorem[2], and the fact that the space $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$ we can extend the Fourier transform as a function for $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$.
Indeed let $g \in L^2(\mathbb{R}^N)$ since $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$ we can find a sequence $\{g_n\}_{n=1}^{\infty} \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ such that $\|g - g_n\|_{L^2(\mathbb{R}^N)} \to 0$.
We define $g_n(x) = g(x)\chi_{[-n,n]}(x)$. We have that $g_n \in L^1(\mathbb{R}^N)$ since

$$\int_{\mathbb{R}^N} |g_n(x)| dx = \int_{\mathbb{R}^N} |g(x)\chi_{[-n,n]}(x)$$

[1]Let $f \in L^1(\mathbb{R}^N)$ we define the Fourier transform of this function to be $\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-i\xi x} f(x)dx$
For a more detailed discussion about Fourier transform see [22]

[2]Plancherels theorem: Let $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ then $\hat{f} \in L^2(\mathbb{R}^N)$ and $\|\hat{f}\|_{L^2(\mathbb{R}^N)} = \|f\|_{L^2(\mathbb{R}^N)}$

By Holders inequality, and the fact that g is square integrable we obtain:

$$\int_{\mathbb{R}^N} |g_n(x)|dx \leq c\|g\|_{L^2(\mathbb{R}^N)} < \infty$$

Furthermore g_n is a Cauchy sequence in $L^2(\mathbb{R}^N)$ because it converges in $L^2(\mathbb{R}^N)$, so

$$\forall \epsilon > 0 \; \exists n_0 \in \mathbb{N} \text{ such that } \forall n, m > \mathbb{N} \text{ we have that } \|g_n - g_m\|_{L^2(\mathbb{R}^N)} < \epsilon$$

By Plancherels theorem, since $g_n - g_m \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ we have that $\|\widehat{g_n - g_m}\|_{L^2(\mathbb{R}^N)} = \|\hat{g_n} - \hat{g_m}\|_{L^2(\mathbb{R}^N)} = \|g_n - g_m\|_{L^2(\mathbb{R}^N)}$ [3]. So $\hat{g_n}$ is a Cauchy sequence. The space $L^2(\mathbb{R}^N)$ is a Banach space so since $\hat{g_n}$ is Cauchy in this space, there exist a $F \in L^2(\mathbb{R}^N)$ such that $\hat{g_n} \to F$. So we define the Fourier transform of g in $L^2(\mathbb{R}^N)$ to be $\hat{g} = F$.
Remark:It is not possible to generalize L^p Sobolev spaces with Fourier transform due to the lack of a good relation between norms, as provided by Plancherel's theorem
To achieve our goal, we must take into consideration that the space of distribution derivatives, known as D', is too vast to offer a clear definition for the Fourier transform. Hence, we bring in the space of tempered distributions, referred to as S'[4]. This space is the dual of the space of rapidly decreasing functions, known as S. i.e.

$$S(\mathbb{R}^N) = \left\{ f \in C^\infty(\mathbb{R}^N) : \sup_{x \in \mathbb{R}^N} |x^a D^b f(x)| < \infty \; \forall \text{ multindicies } a, b \geq 0 \right\}$$

The space S' includes all the functionals $f[\phi] = \int_{\mathbb{R}^N} f\phi dx$ with $\phi \in S$
Note:This space is named space of tempered distribution because of the polynomial (tempered) growth of distributions.
Another great advantage of Fourier transform is that, loosely speaking Fourier transform exchanges differentiation with multiplications i.e. we have the following proposition.

Proposition 4.1.1. *Let $D^a f \in L^1$ or L^2 $\forall a \geq 0$ then $\widehat{D^a} f(\xi) = (i\xi)^a \hat{f}(\xi)$*

Proof. By the definition of Fourier transform we have that:

$$\widehat{D^a} f(\xi) = \int_{\mathbb{R}^N} e^{-i\xi x} D^a f(x) dx$$

By integration by parts we get

$$\int_{\mathbb{R}^N} e^{-i\xi x} D^a f(x) = (-1)^{|a|} \int_{\mathbb{R}^N} D^a(e^{-i\xi x}) \; f(x)dx = (i\xi)^a \hat{f}(\xi)$$

So we conclude that :

$$\widehat{D^a} f(\xi) = (i\xi)^a \hat{f}(\xi)$$

□

[3]Let $f, g \in L^1(\mathbb{R}^N)$ then $\widehat{f-g}(\xi) = \int_{\mathbb{R}^N} e^{-i\xi x}(f(x) - g(x))dx = \int_{\mathbb{R}^N} e^{-i\xi x} f(x)dx - \int_{\mathbb{R}^N} e^{-i\xi x} g(x)dx = \hat{f}(\xi) - \hat{g}(\xi)$

[4]as described in [20], Chapters 7,9

Note: Using the Fourier transform we have "turned" derivatives to products, hence the assumption of smoothness can be replaced with the assumption of rapidly decaying. We define the Fourier transform for $f \in S$ to be $\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-i\xi x} f(x)dx$.

Proposition 4.1.2. *The Fourier transform is a tempered distribution.*

Proof. Let $g, \phi \in S$

$$\int_{\mathbb{R}^N} \hat{g}(x)\phi(x)dx = \int_{\mathbb{R}^N} \phi(x) \int_{\mathbb{R}^N} e^{-iyx} g(y)dydx$$

By Fubini's theorem we have that

$$\int_{\mathbb{R}^N} \hat{g}(x)\phi(x)dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-iyx}\phi(x)dxg(y)dy = \int_{\mathbb{R}^N} \hat{\phi}(x)g(x)dx$$

□

We define for $f \in S'$ the Fourier transform $\hat{f} \in S'$ to be the distribution defined by

$$\hat{f}[\phi] = f[\hat{\phi}]\ \forall \phi \in S$$

All the above results together with the following theorem show us which path to follow for the desire generalization.

Theorem 4.1.1. [5] $f \in H^m(\mathbb{R}^N) \iff (1+|\xi|^2)^{\frac{m}{2}}\hat{f}(\xi) \in L^2(\mathbb{R}^N)$

Proof. By Plancherels's theorem we have that

$$\|D^a f\|_{L^2(\mathbb{R}^N)} = \|\widehat{D^a f}\|_{L^2(\mathbb{R}^N)} = \|(i\xi)^a \hat{f}(\xi)\|_{L^2(\mathbb{R}^N)}$$

Consequently taking the sum above this relation we get that

$$\sum_{|a|\le m} \|D^a f\|^2_{L^2(\mathbb{R}^N)} = \sum_{|a|\le m} \int_{\mathbb{R}^N} |\xi^a|^2 |\hat{f}(\xi)|^2 d\xi \tag{T 4.1.1}$$

We will use this lemma

Lemma 13. *There exist positive constants, name c_1, c_2 such that*

$$c_1(1+|\xi|^2)^m \le \sum_{|a|\le m} |\xi^a|^2 \le c_2(1+|\xi|^2)^m$$

proof of lemma:
We know that

$$\begin{cases} |\xi^a| \le 1 & \text{for } |\xi| \le 1 \\ |\xi^a| \le |\xi|^{|a|} \le |\xi|^m & \text{for } |\xi| \ge 1 \text{ and } |a| \le m \end{cases}$$

So we have that $\sum_{|a|\le m} |\xi^a|^2 \le c_2 \max\{1, |\xi|^{2m}\} \le c_2(1+|\xi|^2)^m$ and we are done with the right inequality.
We continue in order to prove the left inequality, we have that $|\xi|^{2m} \le c\sum_{j=1}^{n} |\xi_j^m|^2$

[5] [21], pg 301

so

$$(1+|\xi|^2)^m \leq 2^m \max\{1, |\xi|^{2m}\} \leq 2^m c(1+\sum_{j=1}^{n} |\xi_j^m|^2) \leq \tilde{c} \sum_{|a|\leq m} |\xi^a|^2$$

Setting $c_1 = \frac{1}{\tilde{c}}$ we have proved the left inequality.

($\Rightarrow$) Assume that $f \in H^m$ then $\|f\|_{H^m} < \infty$ where $\|f\|_{H^m} = \left(\sum_{|a|\leq m} \|D^a f\|_{L^2}^2\right)^{\frac{1}{2}}$
We have that

$$\int_{\mathbb{R}^N} |(1+|\xi|^2)^{\frac{m}{2}} \hat{f}|^2 d\xi = \int_{\mathbb{R}^N} (1+|\xi|^2)^m |\hat{f}|^2 d\xi$$

Using the above lemma we get that

$$\int_{\mathbb{R}^N} |(1+|\xi|^2)^{\frac{m}{2}} \hat{f}|^2 d\xi \leq \frac{1}{c_1} \int_{\mathbb{R}^N} \sum_{|a|\leq m} |\xi^a|^2 |\hat{f}|^2 d\xi$$

By the relation (T 4.1.1)

$$\int_{\mathbb{R}^N} |(1+|\xi|^2)^{\frac{m}{2}} \hat{f}|^2 d\xi \leq \frac{1}{c_1} \sum_{|a|\leq m} \|D^a f\|_{L^2(\mathbb{R}^N)}^2 = \|f\|_{H^m} < \infty$$

So $\left(\int_{\mathbb{R}^N} |(1+|\xi|^2)^{\frac{m}{2}} \hat{f}|^2 d\xi\right)^{\frac{1}{2}} < \infty$
($\Leftarrow$) Assume now that $(1+|\xi|^2)^{\frac{m}{2}} \hat{f}(\xi) \in L^2$ then $\int_{\mathbb{R}^N} (1+|\xi|^2)^m |\hat{f}|^2 d\xi < \infty$
By the above lemma we have that $\frac{1}{c_2} \int_{\mathbb{R}^N} \sum_{|a|\leq m} |\xi^a|^2 \hat{f}(\xi) d\xi < \infty$
So by relation (T 1.1.1) we get that

$$\frac{1}{c_2} \sum_{|a|\leq m} \|D^a f\|_{L^2}^2 < \infty$$

I.e.

$$\left(\sum_{|a|\leq m} \|D^a f\|_{L^2}^2\right)^{\frac{1}{2}} = \|f\|_{H^m} < \infty$$

□

Finally we define H^s for $s \in \mathbb{R}^N$ as

$$H^s = \left\{ f \in S'(\mathbb{R}^N) : \|f\|_{H^s} = \left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s d\xi\right)^{\frac{1}{2}} < \infty \right\}$$

We now continue with some properties in Sobolev spaces.

Theorem 4.1.2 (Sobolev embedding)**.** *The space H^{s+k} with $s > \frac{n}{2}$ and $k \in \mathbb{Z}_0^+$ is continuously embedded to C^k i.e.$\exists c > 0$ such that*

$$\|f\|_{C^k} \leq c\|f\|_{H^{s+k}}$$

Proof. We remind that the norm in space C^k is $\|u\|_{C^k} = \sup_{|a|\le k}\sup_{x\in\mathbb{R}^N}|D^a f(x)|$
For $|a| \le k$ we have that $\int_{\mathbb{R}^N}|\widehat{D^a f}|d\xi = \int_{\mathbb{R}^N}|(i\xi)^a\hat{f}(\xi)|d\xi$
So

$$\int_{\mathbb{R}^N}|\widehat{D^a f}|d\xi \le \int_{\mathbb{R}^N}(1+|\xi|^2)^{\frac{k}{2}}|\hat{f}(\xi)|d\xi = \int_{\mathbb{R}^N}(1+|\xi|^2)^{\frac{k}{2}+\frac{s}{2}}|\hat{f}(\xi)|(1+|\xi|^2)^{\frac{-s}{2}}d\xi$$

By Cauchy -Schwartz inequality we get

$$\int_{\mathbb{R}^N}(1+|\xi|^2)^{\frac{k}{2}+\frac{s}{2}}|\hat{f}(\xi)|(1+|\xi|^2)^{\frac{-s}{2}}d\xi$$

$$\le \left(\int_{\mathbb{R}^N}(1+|\xi|^2)^{k+s}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}(1+|\xi|^2)^{-s}d\xi\right)^{\frac{1}{2}}$$

$$= \|f\|_{H^{s+k}}\left(\int_{\mathbb{R}^N}(1+|\xi|^2)^{-s}d\xi\right)^{\frac{1}{2}}$$

For the integral, by using polar coordinates we have that

$$\int_{\mathbb{R}^N}(1+|\xi|^2)^{-s}d\xi = c\int_0^\infty (1+r^2)^{-s}r^{N-1}dr$$

$$= \left[\int_0^1 (1+r^2)^{-s}r^{N-1}dr + \int_1^\infty (1+r^2)^{-s}r^{N-1}dr\right]$$

$$\le {}^6 c\left(\int_0^1 r^{N-1}dr + \int_1^\infty \frac{1}{r^{2s}}r^{N-1}dr\right)$$

$$\le c\left(\left[\frac{r^N}{N}\right]_0^1 + \int_1^\infty r^{N-2s-1}dr\right)$$

$$\le c\left(1 + \lim_{t\to\infty}\left[\frac{r^{N-2s}}{N-2s}\right]_1^t\right)$$

For $s > \frac{N}{2}$ the limit quantity is finite, thus we assume that the integral takes a value, say C.
Consequently

$$\|\widehat{D^a f}\|_{L^1} = \int_{\mathbb{R}^N}|\widehat{D^a f}|d\xi \le C\|f\|_{H^{s+k}}$$

By Fourier inversion theorem[7] we get that $\sup_{x\in\mathbb{R}^N}|D^a f| \le \|\widehat{D^a f}\|_{L^1}$.
So $\sup_{|a|\le k}\sup_{x\in\mathbb{R}^N}|D^a f(x)| \le C\|f\|_{H^{s+k}}$ ☐

[6] It is true that $r^2 \le r^2+1$ and $(r^2)^s \le (r^2+1)^s$

[7] Let f be a continuous and integrable function with Fourier transform $\hat{f}(\xi)$ then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}d\xi$
Proof:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(y)e^{-i\xi y}dy e^{i\xi x}d\xi =$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)\int_{-\infty}^{\infty}e^{-i\xi y}e^{i\xi x}d\xi dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)\widehat{e^{i\xi x}}dy =$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)2\pi\delta(y-x)dy = f(x)$$

We also have the following proposition which generalize the Leibniz rule for the product of Sobolev functions.

Proposition 4.1.3 (Leibniz rule)**.** [8] *Let U be an open subset of $\mathbb{R}^N$ and p, p' that satisfy the relation $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in W^{k,p}(U)$ and $g \in W^{k,p'}(U)$ then $f \cdot g \in W^{k,1}(U)$ and*

$$D^a(f \cdot g) = \sum_{0 \le m \le a} D^m f \cdot D^{a-m} g \tag{P 4.1.3}$$

Proof. The space $W^{k,p}(U) \cap C^\infty(U)$ is dense in $W^{k,p}(U)$so there exists a sequence $\{f_n\}_{n=1}^\infty \in W^{k,p}(U) \cap C^\infty(U)$ such that $f_n \to f$ in $W^{k,p}(U)$.
Let $g \in W^{k,p'}(U)$ and ϕ a test function. Then

- For $|a| = 1$, D^a is a first order derivative so

$$\int_U (f_n g) D^a \phi dx = \int_U g \left(f_n \frac{\partial}{\partial x_i} \phi \right) dx$$

$$= - \int_U g \phi \frac{\partial f_n}{\partial x_i} dx + \int_U g \frac{\partial}{\partial x_i} (f_n \phi) dx$$

So

$$\frac{\partial}{\partial} (f_n \cdot g) = g \frac{\partial}{\partial x_i} f_n + f_n \frac{\partial}{\partial x_i} g$$

For $n \to \infty$ we get $\frac{\partial}{\partial x_i}(f \cdot g) = g \frac{\partial}{\partial x_i} f + f \frac{\partial}{\partial x_i} g$

- Assume that the relation (P 4.1.3) is true for all multindicies up to order k.

- We set $|a| = |b + c| = |b| + |c| = l + 1$ with $l \le k$ then

$$\int_U (f_n g) D^a \phi dx = (-1)^{|b|} \int_U D^b (f_n g) D^c \phi dx$$

$$= (-1)^{|b|} (-1)^{|c|} \int_U D^c (D^b f_n g) \phi dx$$

$$= (-1)^{|a|} \int_U D^c \left(\sum_{m \le b} \binom{b}{c} D^m f_n D^{b-m} g \right) \phi dx$$

$$= (-1)^{|a|} \int_U \sum_{m \le b} \binom{b}{c} \left[D^m f_n D^{c-m} g + D^{c+m} f_n D^{b-m} g \right] \phi dx$$

Consequently we have that

$$\int_U (f_n g) D^a \phi dx$$

$$= (-1)^{|a|} \int_U \left[\sum_{m \le b} \binom{b}{c} D^m f_n D^{a-m} g + \sum_{m \le b} \binom{b}{m-c} D^m f_n D^{a-m} g + \binom{b}{b} D^m f_n D^a g \right] \phi dx$$

[8] [6], pg 124

$$= (-1)^{|a|} \int_U \sum_{m\leq a} \binom{a}{m} D^m f_n D^{a-m} g dx$$

We conclude that $D^a(f_n \cdot g) = \sum_{m\leq a} \binom{a}{m} D^m f_n \cdot D^{a-m} g$ for $n \to \infty$ then $D^a(f \cdot g) = \sum_{m\leq a} \binom{a}{m} D^m f \cdot D^{a-m} g$

□

Remark:We know that there exists an extension operator E from $W^{k,p}(U)$ to $W^{k,p}(\mathbb{R}^N)$[9] so that for $f \in W^{k,p}(U)$ we have

$$\tilde{f} = Ef = f \ a.e. \ U$$

and

$$D^a \tilde{f} = D^a f$$

So we are able to extend the above result in $\mathbb{R}^N$. Now we will prove the following estimates for the H^m norm of the product of Sobolev functions.

Proposition 4.1.4. *$\forall m \in \mathbb{Z}_0^+$,$\exists c$ such that $\forall u, v \in L^\infty(\mathbb{R}^N) \cap H^m(\mathbb{R}^N)$ then*
(i)

$$\|u \cdot v\|_{H^m} \leq c\{\|u\|_{L^\infty}\|v\|_{H^m} + \|u\|_{H^m}\|v\|_{L^\infty}\} \qquad \text{(P 4.1.4 i)}$$

(ii)

$$\sum_{0\leq m\leq a} \|D^a(u \cdot v) - uD^a v\|_{L^2} \leq c\{\|\nabla u\|_{L^\infty}\|v\|_{H^{m-1}} + \|u\|_{H^m}\|v\|_{L^\infty}\} \qquad \text{(P 4.1.4 ii)}$$

Proof. (i) Assume $u, v \in H^m(\mathbb{R}^N)$ we have by the above proposition that

$$D^a(u \cdot v) = \sum_{b\leq a} c_a \, D^b u \cdot D^{a-b} v$$

for all multindices a,b
So for the L^2 norm of the derivative of the product we get that

$$\|D^a(u \cdot v)\|_{L^2} = \|\sum_{b\leq a} c_a \, D^b u \cdot D^{a-b} v\|_{L^2}$$

We choose $|a| = s \in [0, m]$ and s is an integer, then we have that $s - |b| = |a - b|$ and we apply the Holder inequality with conjugates $\frac{|b|}{2s}$ and $\frac{|a-b|}{2s}$ So we have that

$$D^a(u \cdot v) = \sum_{b\leq s} c_s \, \|D^b u\|_{L^{\frac{2s}{|b|}}} \, \|D^{a-b} v\|_{L^{\frac{2s}{|a-b|}}}$$

[9] [4] section 5.17

We will estimate each derivative by the Gagliardo-Nirenberg inequality[10]:

In our case $u \in L^\infty(\mathbb{R}^N) \cap H^m(\mathbb{R}^N)$ so for $p = \frac{2s}{|b|}$,$q = \infty$,$k = s$,$r = 2$,$j = |b|$ and $\theta = \frac{|b|}{s}$ we have that

$$\|D^b u\|_{L^{\frac{2s}{|b|}}} \leq c_b \|D^s u\|_{L^2}^{\frac{|b|}{s}} \|u\|_{L^\infty}^{1-\frac{|b|}{s}}$$

Similarly for $v \in L^\infty(\mathbb{R}^N) \cap H^m(\mathbb{R}^N)$ so for $p = \frac{2s}{|a-b|}$,$q = \infty$,$k = s$,$r = 2$,$j = |a-b|$ and $\theta = \frac{|a-b|}{s}$ we have that

$$\|D^{a-b} v\|_{L^{\frac{2s}{|a-b|}}} \leq c_{a,b} \|D^s v\|_{L^2}^{\frac{|a-b|}{s}} \|u\|_{L^\infty}^{1-\frac{|a-b|}{s}}$$

We conclude that

$$D^a(u \cdot v) \leq c_s \sum_{b \leq s} \left(c_b \|D^s u\|_{L^2}^{\frac{|b|}{s}} \|u\|_{L^\infty}^{1-\frac{|b|}{s}} c_{a,b} \|D^s v\|_{L^2}^{\frac{|a-b|}{s}} \|u\|_{L^\infty}^{1-\frac{|a-b|}{s}} \right)$$

It is also true that $1 - \frac{|b|}{m} = \frac{|a-b|}{m}$ we get that

$$D^a(u \cdot v) \leq c_s \sum_{b \leq s} \tilde{c}_{a,b} \left[\left(\|v\|_{L^\infty} \|D^s u\|_{L^2}\right)^{\frac{|b|}{s}} \left(\|u\|_{L^\infty} \|D^s v\|_{L^2}\right)^{\frac{|a-b|}{s}} \right]$$

Thus

$$D^a(u \cdot v) \leq C \left(\|v\|_{L^\infty} \|D^a u\|_{L^2} + \|u\|_{L^\infty} \|D^a v\|_{L^2}\right) \tag{1}$$

This equation is true for all a=s we will take the sum of this relation over all $s \in [0, m]$

- $|a| = 0$ then we get

$$\|u \cdot v\|_{L^2} \leq C \left(\|v\|_{L^\infty} \|u\|_{L^2} + \|u\|_{L^\infty} \|v\|_{L^2}\right)$$

- $|a| \leq 1$ then we get

$$\sum_{a \leq 1} \|D^a(u \cdot v)\|_{L^2}^2 = \|u \cdot v\|_{L^2}^2 + \|D(u \cdot v)\|_{L^2}^2$$

  Therefore by the first case and relation (1)

$$\sum_{a \leq 1} \|D^a(u \cdot v)\|_{L^2}^2 \leq C \left[\left(\|v\|_{L^\infty} \|u\|_{L^2} + \|u\|_{L^\infty} \|v\|_{L^2}\right)^2 + \left(\|v\|_{L^\infty} \|Du\|_{L^2} + \|u\|_{L^\infty} \|Dv\|_{L^2}\right)^2 \right]$$

[10] [19]
Assume that $f \in L^q \cap W^{k,r}$ with $1 \leq q \leq \infty$ and a $j \in \mathbb{N}$ such that $j < k$.
If

$$\frac{1}{p} = \frac{j}{N} + \theta \left(\frac{1}{r} - \frac{k}{N} \right) + \frac{1-\theta}{q}$$

Then there exists a positive constant c such that :

$$\|D^j f\|_{L^p} \leq c_p \|D^k f\|_{L^r}^\theta \|u\|_{L^q}^{1-\theta}$$

So

$$\sum_{a\leq 1}\|D^a(u\cdot v)\|_{L^2}^2 \leq C\left(\|v\|_{L^\infty}\|u\|_{L^2}+\|u\|_{L^\infty}\|v\|_{L^2}+\|v\|_{L^\infty}\|Du\|_{L^2}+\|u\|_{L^\infty}\|Dv\|_{L^2}\right)^2$$

Since the function of square root is an increasing function we have that

$$\left(\sum_{a\leq 1}\|D^a(u\cdot v)\|_{L^2}^2\right)^{\frac{1}{2}} \leq C\left(\|v\|_{L^\infty}\|u\|_{L^2}+\|u\|_{L^\infty}\|v\|_{L^2}+\|v\|_{L^\infty}\|Du\|_{L^2}+\|u\|_{L^\infty}\|Dv\|_{L^2}\right)$$

Consequently

$$\|u\cdot v\|_{H^1} \leq C\left[\|v\|_{L^\infty}\left(\|u\|_{L^2}+\|Du\|_{L^2}\right)+\|u\|_{L^\infty}\left(\|v\|_{L^2}+\|Dv\|_{L^2}\right)\right]$$

Follows that

$$\|u\cdot v\|_{H^1} \leq C\left(\|v\|_{L^\infty}\|u\|_{H^1}+\|u\|_{L^\infty}\|v\|_{H^1}\right)$$

- $|a| \leq 2$ then we get

$$\sum_{a\leq 2}\|D^a(u\cdot v)\|_{L^2}^2 = \|u\cdot v\|_{L^2}^2+\|D(u\cdot v)\|_{L^2}^2+\|D^2(u\cdot v)\|_{L^2}^2$$

By the two previous cases and relation 1 we get

$$\sum_{a\leq 2}\|D^a(u\cdot v)\|_{L^2}^2 \leq C\left(\|v\|_{L^\infty}\|u\|_{L^2}+\|u\|_{L^\infty}\|v\|_{L^2}\right)^2+C\left(\|v\|_{L^\infty}\|Du\|_{L^2}+\|u\|_{L^\infty}\|Dv\|_{L^2}\right)^2$$

$$+C\left(\|v\|_{L^\infty}\|D^2u\|_{L^2}+\|u\|_{L^\infty}\|D^2v\|_{L^2}\right)^2$$

Therefore

$$\left(\sum_{a\leq 2}\|D^a(u\cdot v)\|_{L^2}^2\right)^{\frac{1}{2}} \leq C\Big(\|v\|_{L^\infty}\|u\|_{L^2}+\|u\|_{L^\infty}\|v\|_{L^2}$$

$$+\|v\|_{L^\infty}\|Du\|_{L^2}+\|u\|_{L^\infty}\|Dv\|_{L^2}+\|v\|_{L^\infty}\|D^2u\|_{L^2}+\|u\|_{L^\infty}\|D^2v\|_{L^2}\Big)$$

Consequently

$$\|u\cdot v\|_{H^2} \leq C\left(\|v\|_{L^\infty}\|u\|_{H^2}+\|u\|_{L^\infty}\|v\|_{H^2}\right)$$

Repeating this procedure m times we conclude that

$$\|u\cdot v\|_{H^m} \leq C\left(\|v\|_{L^\infty}\|u\|_{H^m}+\|u\|_{L^\infty}\|v\|_{H^m}\right)$$

Which completes the proof of (i) we continue with the proof of (ii):
For $|a| = s$ we have that

$$\|D^a(u\cdot v)-uD^av\|_{L^2} \leq \sum_{b+c=s-1} c\|D^{b+1}u\cdot D^cv\|_{L^2}$$

We set $Du = \nabla u = h$ and we get that

$$\|D^a(u \cdot v) - uD^a v\|_{L^2} \leq \sum_{b+c=s-1} c\|D^b u \cdot D^c v\|_{L^2}$$

Recall the steps of the proof of (i)

$$\sum_{b+c=s-1} \|D^b u \cdot D^c v\|_{L^2} \leq C_s \left(\|v\|_{L^\infty}\|D^{s-1}h\|_{L^2} + \|h\|_{L^\infty}\|D^{s-1}v\|_{L^2}\right)$$

We take the sum above all $s \leq m$ so

$$\sum_{a \leq m} \|D^a(u \cdot v) - uD^a v\|_{L^2} \leq c_m \left(\|v\|_{L^\infty}\|h\|_{H^{m-1}} + \|h\|_{L^\infty}\|v\|_{H^{m-1}}\right)$$

Substituting h with ∇u we conclude that

$$\sum_{a \leq m} \|D^a(u \cdot v) - uD^a v\|_{L^2} \leq c_m \left(\|v\|_{L^\infty}\|u\|_{H^m} + \|\nabla u\|_{L^\infty}\|v\|_{H^{m-1}}\right)$$

□

Proposition 4.1.5. $\forall s > \frac{N}{2}$ *where s is a real number, then H^s is a Banach algebra.*

Proof. To prove that H^s is a Banach algebra it is sufficient to show that for $u, v \in H^s$ it is true that

$$\|u \cdot v\|_{H^s} \leq c\|u\|_{H^s}\|v\|_{H^s}$$

By the definition of the H^s norm via Fourier transform we have that

$$\|u \cdot v\|_{H^s} = \left(\int_{\mathbb{R}^N} |\widehat{u \cdot v}(\xi)|^2 \left(1 + |\xi|^2\right)^s d\xi\right)^{\frac{1}{2}}$$

For the Fourier transform of the product[11], it is true that:

$$\widehat{u \cdot v}(\xi) = \int_{\mathbb{R}^N} e^{-i\xi x} \left(u(x) \cdot v(x)\right) dx$$

$$= \int_{\mathbb{R}^N} \frac{1}{2\pi} \int_{\mathbb{R}^N} \hat{u}(p) e^{ipx} dp v(x) e^{-i\xi x} dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^N} \hat{u}(p) \int_{\mathbb{R}^N} v(x) e^{-i(\xi - p)x} dx dp$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^N} \hat{u}(p)\hat{v}(\xi - p) dp = \frac{1}{2\pi}\hat{u}(\xi) * \hat{v}(\xi)$$

So

$$|\widehat{u \cdot v}(\xi)|^2 \left(1 + |\xi|^2\right)^s = \frac{1}{4\pi^2}\left(1 + |\xi|^2\right)^s \left|\int_{\mathbb{R}^N} \hat{u}(\xi - p)\hat{v}(p) dp\right|^2$$

[11] [7] Chapter 11

$$\leq \left(1+|\xi|^2\right)^s \left(\int_{\mathbb{R}^N} |\hat{u}(\xi-p)\hat{v}(p)|\,dp\right)^2$$

Furthermore, it is true that :

$$\left(1+|\xi|^2\right)^s \leq \left(1+2|\xi-p|^2+2|p|^2\right)^s$$

$$e\left(4+2|\xi-p|^2+2|p|^2\right)^s = 2^s\left[\left(1+|\xi-p|^2\right)+\left(1+|\xi|^2\right)\right]^s$$

$$\leq 2^s 2^{s-1}\left(\left(1+|\xi-p|^2\right)^s+\left(1+|\xi|^2\right)^s\right)$$

Consequently, we have that

$$|\widehat{u\cdot v}(\xi)|^2\left(1+|\xi|^2\right)^s \leq c\left(1+|\xi-p|^2\right)^s\left(\int_{\mathbb{R}^N}|\hat{u}(\xi-p)\hat{v}(p)|\,dp\right)^2$$

$$+c\left(1+|\xi|^2\right)^s\left(\int_{\mathbb{R}^N}|\hat{u}(\xi-p)\hat{v}(p)|\,dp\right)^2$$

By Holder inequality and if we set $g=\left(1+|\xi|^2\right)^{\frac{s}{2}}$, we get that:

$$|\widehat{u\cdot v}(\xi)|^2\left(1+|\xi|^2\right)^s \leq c\left(|g\hat{u}|*|\hat{v}|\right)^2+c\left(|\hat{u}|*|g\hat{v}|\right)^2$$

Thus $\|u\cdot v\|_{H^s}^2 \leq \|g\hat{u}\|_{L^2}^2\|\hat{v}\|_{L^1}^2+\|g\hat{v}\|_{L^2}^2\|\hat{u}\|_{L^1}^2$

Lemma 14. *Assume that $f\in H^s$ with $s>\frac{N}{2}$ then*

$$\|\hat{f}\|_{L^1}\leq C\|f\|_{H^s}$$

proof of lemma:

$$\|\hat{f}\|_{L^1}=\int_{\mathbb{R}^N}|\hat{f}(\xi)|d\xi=\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^{\frac{s}{2}}|\hat{f}(\xi)|\left(1+|\xi|^2\right)^{\frac{-s}{2}}d\xi$$

$$\leq \|\left(1+|\xi|^2\right)^{\frac{s}{2}}|\hat{f}(\xi)|\|_{L^2}\|\left(1+|\xi|^2\right)^{\frac{-s}{2}}\|_{L^2}$$

We will check the term

$$\|\left(1+|\xi|^2\right)^{\frac{-s}{2}}\|_{L^2}\leq\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^{-s}d\xi$$

$$=\int_0^\infty cr^{N-1}\frac{1}{(1+r^2)^s}dr$$

$$\leq c\int_0^1 r^{N-1}dr+c\int_1^\infty\frac{1}{r^{2s}}r^{N-1}dr$$

$$\leq c\left(1+\lim_{t\to\infty}\frac{t^{N-2s}}{N-2s}+\frac{1}{N-2s}\right)=K<\infty$$

It follows that

$$\|u \cdot v\|_{H^s}^2 \leq C\|u\|_{H^s}^2 \|v\|_{H^s}^2$$

I.e.

$$\|u \cdot v\|_{H^s} \leq c\|u\|_{H^s} \|v\|_{H^s}$$

□

From now on we will mostly search for solutions in Sobolev space H^m, where energy methods can be applied. Our approach to finding solutions for the Euler and Navier-Stokes equations locally in time is to first regularize the equations through convolution with mollifiers. This will give us a regularized solution for the problem, which we can then use to approximate and find a solution for the initial problem. To understand the properties of mollifiers better, let's list some of their key features.

Mollifiers

Definition 7. *Assume that* $\rho \in C_c^\infty$ *is a radial function with* $\rho \geq 0$ *and* $\int_{\mathbb{R}^N} \rho(|x|)dx = 1$. *We call* ρ *standard mollifies.*
We also define for $\epsilon > 0$ *the function* $\rho_\epsilon(x) = \rho\left(\frac{x}{\epsilon}\right)$

Definition 8. *Assume that* $u \in L^p(\mathbb{R}^N)$ *with* $1 \leq p \leq \infty$ *we define the mollification of this function as* $J_\epsilon u = \rho_\epsilon * u$ *i.e.*

$$J_\epsilon u = \epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy$$

Proposition 4.1.6. [12] *Assume* $J_\epsilon u$ *a mollification as above then* $J_\epsilon u \in C^\infty$

Proof. We will prove this argument bi induction
Assume that $x \in \mathbb{R}^N$ and h sufficiently small we define $x + he_i \in \mathbb{R}^N$ and we get

$$\frac{J_\epsilon u(x+he_i) - J\epsilon u(x)}{h} = \epsilon^{-N} \int_{\mathbb{R}^N} \frac{1}{h}\left[\rho\left(\frac{x+he_i-y}{\epsilon}\right) - \rho\left(\frac{x-y}{\epsilon}\right)\right] u(y)dy$$

Since $\rho \in C_c^\infty$, let K be its compact support then it is true that

$$\frac{1}{h}\left[\rho\left(\frac{x+he_i-y}{\epsilon}\right) - \rho\left(\frac{x-y}{\epsilon}\right)\right] \xrightarrow[h\to 0]{uniformly\ on\ K} \frac{1}{\epsilon}\frac{\partial}{\partial x_i}\rho\left(\frac{x-y}{\epsilon}\right)$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial x_i} J_\epsilon u(x) &= \lim_{h\to 0} \epsilon^{-N} \int_{\mathbb{R}^N} \frac{1}{h}\left[\rho\left(\frac{x+he_i-y}{\epsilon}\right) - \rho\left(\frac{x-y}{\epsilon}\right)\right] u(y)dy \\ &= \epsilon^{-N} \int_K \frac{1}{h}\left[\rho\left(\frac{x+he_i-y}{\epsilon}\right) - \rho\left(\frac{x-y}{\epsilon}\right)\right] u(y)dy \\ &= \epsilon^{-N} \int_K \frac{1}{\epsilon}\frac{\partial}{\partial x_i}\rho\left(\frac{x-y}{\epsilon}\right) u(y)dy\end{aligned}$$

[12] [18] pg 714

$$= \frac{1}{\epsilon^{N+1}} \int_{\mathbb{R}^N} \frac{\partial}{\partial x_i} \rho \left(\frac{x-y}{\epsilon} \right) u(y) dy$$

For the second derivative we have that

$$\frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_i} J_\epsilon u(x) \right) = \lim_{h \to 0} \frac{\frac{\partial}{\partial x_i} J_\epsilon u(x+he_j) - \frac{\partial}{\partial x_i} J_\epsilon u(x)}{h}$$

$$= \frac{1}{\epsilon^{N+1}} \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}^N} \left[\frac{\partial}{\partial x_i} \rho \left(\frac{x+he_j-y}{\epsilon} \right) - \frac{\partial}{\partial x_i} \rho \left(\frac{x-y}{\epsilon} \right) \right] u(y) dy$$

$$= \frac{1}{\epsilon^{N+1}} \lim_{h \to 0} \frac{1}{h} \int_{K} \left[\frac{\partial}{\partial x_i} \rho \left(\frac{x+he_j-y}{\epsilon} \right) - \frac{\partial}{\partial x_i} \rho \left(\frac{x-y}{\epsilon} \right) \right] u(y) dy$$

Since $\rho \in C_c^\infty$, let K be its compact support then it is true that

$$\frac{1}{h} \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_i} \rho \left(\frac{x+he_j-y}{\epsilon} \right) - \frac{\partial}{\partial x_i} \rho \left(\frac{x-y}{\epsilon} \right) \right] \xrightarrow[h \to 0]{uniformly\ on\ K} \frac{1}{\epsilon} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \rho \left(\frac{x-y}{\epsilon} \right)$$

Thus

$$\frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_i} J_\epsilon u(x) \right) = \frac{1}{\epsilon^{N+2}} \int_K \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \rho \left(\frac{x-y}{\epsilon} \right) u(y) dy$$

$$= \frac{1}{\epsilon^{N+2}} \int_{\mathbb{R}^N} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \rho \left(\frac{x-y}{\epsilon} \right) u(y) dy$$

Induction hypothesis: Assume that for $|a| = k$ it is true that

$$D^a J_\epsilon u(x) = \frac{1}{\epsilon^{N+k}} \int_{\mathbb{R}^N} D^a \rho \left(\frac{x-y}{\epsilon} \right) u(y) dy$$

Then for $|b| = |a| + 1 = k + 1$ we have that

$$D^b J_\epsilon u(x) = D(D^a J_\epsilon u(x)) = \frac{\partial}{\partial x_i} \left(\frac{1}{\epsilon^{N+k}} \int_{\mathbb{R}^N} D^a \rho \left(\frac{x-y}{\epsilon} \right) u(y) dy \right)$$

By Leibniz integral rule

$$D^b J_\epsilon u(x) = \frac{1}{\epsilon^{N+k+1}} \int_{\mathbb{R}^N} D^{a+1} \rho \left(\frac{x-y}{\epsilon} \right) u(y) dy = \frac{1}{\epsilon^{N+k+1}} \int_{\mathbb{R}^N} D^b \rho \left(\frac{x-y}{\epsilon} \right) u(y) dy$$

□

Proposition 4.1.7. *Let $u \in C(\mathbb{R}^N)$, then $J_\epsilon u(x) \to u(x)$ uniformly on any compact subset of $\mathbb{R}^N$ and $\|J_\epsilon u(x)\|_{L^\infty} \leq \|u\|_{L^\infty}$*

Proof. Assume that $x \in \mathbb{R}^N$, then

$$J_\epsilon u(x) - u(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho \left(\frac{x-y}{\epsilon} \right) u(y) dy - u(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho \left(\frac{y}{\epsilon} \right) u(x-y) dy - u(x)$$

Since $\int_{\mathbb{R}^N} \rho dx = 1$ with a simple chance of variables$x = \frac{y}{\epsilon}$ we have that $\int_{\mathbb{R}^N} \epsilon^{-N} \rho \left(\frac{y}{\epsilon} \right) dy = 1$, so

$$J_\epsilon u(x) - u(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho \left(\frac{y}{\epsilon} \right) u(x-y) dy - \epsilon^{-N} \int_{\mathbb{R}^N} \rho \left(\frac{y}{\epsilon} \right) dy u(x)$$

$$= \epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{y}{\epsilon}\right)[u(x-y)-u(x)]\,dy$$

We assume that the support of the mollifier $\rho$is a subset of a ball of radius r then

$$J_\epsilon u(x) - u(x) = \epsilon^{-N} \int_{B(0,r)} \rho\left(\frac{y}{\epsilon}\right)[u(x-y)-u(x)]\,dy$$

So we have that $\forall \eta > 0$,$\exists \epsilon_0 = \frac{r}{\eta}$ such that $|J_\epsilon u(x) - u(x)| \leq \eta \ \forall x \in K \subset\subset \mathbb{R}^N$
Now for the inequality we have that

$$|J_\epsilon u(x)| = |\int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy|$$

So

$$\sup_{x\in\mathbb{R}^N} |J_\epsilon u(x)| \leq \sup_{x\in\mathbb{R}^N} |u(x) \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) dy|$$

i.e.

$$\|J_\epsilon u(x)\|_{L^\infty} \leq \|u\|_{L^\infty}$$

□

Proposition 4.1.8. *Mollifiers commute with distribution derivatives.*

Proof. We have that $D^a J_\epsilon u(x) = D^a \left[\epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy\right]$

$$= \epsilon^{-N} \int_{\mathbb{R}^N} D_x^a \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy$$

[13]

$$= \epsilon^{-N}(-1)^a \int_{\mathbb{R}^N} D_y^a \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy$$

By integration by parts we have that

$$D^a J_\epsilon u(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) D_y^a u(y)dy = J_\epsilon(D^a u)(x)$$

□

Proposition 4.1.9. *Let $u \in L^p(\mathbb{R}^N)$ and $v \in L^q(\mathbb{R}^N)$ with $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then*

$$\int_{\mathbb{R}^N} J\epsilon u(x) \cdot v(x)dx = \int_{\mathbb{R}^N} u \cdot J_\epsilon v(x)dx$$

Proof.

$$\int_{\mathbb{R}^N} J\epsilon u(x) \cdot v(x)dx = \int_{\mathbb{R}^N} \epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy \cdot v(x)dx$$

$$= \epsilon^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) v(x)u(y)dydx$$

[13]Since ρ is a a radial function it is true that $D_x^a \rho\left(\frac{x-y}{\epsilon}\right) = (-1)^a D_y^a \rho\left(\frac{x-y}{\epsilon}\right)$

$$= \epsilon^{-N} u(y) \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) v(x)dxdy$$
$$= \int_{\mathbb{R}^N} u(x) \cdot J_\epsilon v(x)dx$$

□

Proposition 4.1.10. *Let $f \in L^p(\mathbb{R}^N)$ and $1 \leq p < \infty$ then*

$$\lim_{\epsilon \searrow 0} \|J_\epsilon f(x) - f(x)\|_{L^p} = 0$$

Proof. It is true that

$$|J_\epsilon u(x) - u(x)| = \left| \int_{\mathbb{R}^N} \epsilon^{-N} \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy - u(x) \right|$$
$$= \left| \epsilon^{-N} \rho\left(\frac{y}{\epsilon}\right) [u(x-y) - u(x)]\, dy \right|$$
$$\leq \int_{\mathbb{R}^N} \left| \epsilon^{-N} \rho\left(\frac{y}{\epsilon}\right) [u(x-y) - u(x)] \right| dy$$
$$\leq \int_{\mathbb{R}^N} \left| \epsilon^{-N} \rho\left(\frac{y}{\epsilon}\right) \right| |[u(x-y) - u(x)]|\, dy$$

Consequently,

$$\|J_\epsilon u - u\|_{L^p} \leq \left\{ \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} |u(x-y) - u(x)| \left| \epsilon^{-N} \rho\left(\frac{y}{\epsilon}\right) \right| dx \right]^p dy \right\}^{\frac{1}{p}}$$

By Minkowski's integral inequality: [14] We have that

$$\|J_\epsilon u - u\|_{L^p} \leq \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} |u(x-y) - u(x)|^p \left| \epsilon^{-N} \rho\left(\frac{y}{\epsilon}\right) \right|^p dx \right] \frac{1}{p} dy$$
$$\leq \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} |u(x-y) - u(x)|^p dx \right] \frac{1}{p} \left| \epsilon^{-N} \rho\left(\frac{y}{\epsilon}\right) \right| dy$$
$$\leq \int_{\mathbb{R}^N} \|u_{-y}(x) - u(x)\|_{L^p} \left| \epsilon^{-N} \rho\left(\frac{y}{\epsilon}\right) \right| dy$$

Now, we will examine the $\|u_{-y}(x) - u(x)\|_{L^p}$. For the sake of simpler notation, we define $g(y) = \|u_{-y}(x) - u(x)\|_{L^p}$. This function is continuous and bounded

- Continuous:

  Lemma 15. *The space $C_c^\infty(\mathbb{R}^N)$ is dense on $L^p(\mathbb{R}^N)$*

---

[14] [21] section 6.3
Let $f \in L^p$ with $1 \leq p < \infty$ then

$$\left[\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |f(x,y)|dy \right)^p dx \right] \frac{1}{p} \leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |f(x,y)|^p dx \right)^{\frac{1}{p}} dy$$

proof of lemma
We are on the measurable space $(\mathbb{R}^N, m)$, where m is the Lebesgue measure. Assume that S is the class of all measurable simple functions s(x) such that $m(\{s(x) \neq 0\}) < \infty$, S is dense on L^p[a]. By Lusin's theorem[b], we have that for $s_n \in S$ and $\eta > 0$ there exist a $g_n \in C_c(\mathbb{R}^N)$ such that $g_n(x) = s_n(x)$ except of a set of measure η and $|g_n| \leq \sup_{x\in\mathbb{R}^N} |g_n| \leq \|s_n\|_{L^\infty}$. Hence $\|g_n - s_n\|_{L^p} \leq \frac{\tilde{\eta}}{4}$. Since S is dense on L^p we have that $\|s_n - s\|_{L^p} \leq \frac{\tilde{\eta}}{4}$with $s \in L^p$, thus $\|g_n - s\|_{L^p} \leq \|g_n - s_n\|_{L^p} + \|s_n - s\|_{L^p} \leq \frac{\tilde{\eta}}{2}$, so we have that C_c is dense on L^p
It is true that $g_n^\epsilon(x) = g_n * \rho_\epsilon$ has compact support , also g_n has compact support then $\|g_n^\epsilon - g_n\|_{L^p}^p = \int_{\mathbb{R}^N} |g_n^\epsilon - g_n|^p dx \leq m(B) \sup_{x\in B} |g_n^\epsilon - g_n|^p$. By proposition(1.0.7.) we have that $\sup_{x\in B} |g_n^\epsilon - g_n|^p \leq \frac{\tilde{\eta}}{2}$, therefore $\sup_{x\in B} |g_n^\epsilon - g_n|^p \leq \frac{\tilde{\eta}}{2}$. Consequently $\|g - g_n^\epsilon\|_{L^p} \leq \|g - g_n\|_{L^p} + \|g_n^\epsilon - g_n\|_{L^p} \leq \eta$.
By proposition (1.1.6.) we know that $g_n^\epsilon \in C_c^\infty$ so C_c^∞ is dense in L^p

[a] [35] pg 67
[b] Let F be a measurable function on $\mathbb{R}^N$ such that $F(x) = 0$ for $x \notin A$ with $m(A) < \infty$ and $\eta > 0$ then there exists a function $g \in C_c(\mathbb{R}^N)$ such that $f(x) = g(x)$ except of a set of measure smaller than η and $\sup_{x\in\mathbb{R}^N} |g(x)| \leq \sup_{x\in\mathbb{R}^N} |f(x)|$

thus there exists an $h_n \in C_c^\infty$ such that $\|h_n - u\|_{L^p} \leq \frac{\delta}{3}$, furthermore due to the fact that h_n is continuous we have that $|h_n(x_1) - h_n(x_2) \leq \frac{\delta}{3}$, and since$h_n$ has compact support we reach to $\|h_n(x_1) - h_n(x_2)\|\|_{L^p} \leq \left(\frac{\delta}{3}\right)^p$.
So

$$\|g(x_1) - g(x_2)\|_{L^p} = \|u(x - x_1) - u(x - x_2)\|_{L^p} \leq$$
$$\|u(x - x_1) - h_n(x - x_1)\|_{L^p} + \|h_n(x - x_1) - h_n(x - x_2)\|_{L^p}$$
$$+\|h_n(x - x_2) - u(x - x_2)\|_{L^p} \leq \delta$$

, which proves the continuity.

- It is bounded $\|g\|_{L^p}^p \leq \|u(x - y) - u(x)\|_{L^p}^p \leq 2^p \|u\|_{L^p}^p = M$

Now we re able to use dominated convergence theorem since g is bounded and integrable and ρ is integrable so we reach to the fact that

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}^N} g(y) \rho\left(\frac{y}{\epsilon}\right) = 0$$

Thus $\lim_{\epsilon\searrow 0} \|J_\epsilon u(x) - u(x)\|_{L^p} = 0$ □

Proposition 4.1.11. *Let $u \in H^s(\mathbb{R}^N)$ then*

$$\|J_\epsilon u - u\|_{H^{s-1}} \leq c\epsilon \|u\|_{H^s}$$

Proof.

Lemma 16. *Assume that $f \in L^2$ then*

$$\widehat{J_\epsilon f}(\xi) = \hat{\rho}(\epsilon\xi) \hat{f}(\xi)$$

proof of lemma:

$$\widehat{J_\epsilon f}(\xi) = \int_{\mathbb{R}^N} e^{-i\xi x} \int_{\mathbb{R}^N} \epsilon^{-N} \rho\left(\frac{x-y}{\epsilon}\right) f(y)dy\, dx$$

$$= \int_{\mathbb{R}^N} f(y) \int_{\mathbb{R}^N} \epsilon^{-N} e^{-i\xi x} \rho\left(\frac{x-y}{\epsilon}\right) dx\, dy$$

We define $x = \epsilon w + y$ thus

$$\widehat{J_\epsilon f}(\xi) = \int_{\mathbb{R}^N} f(y) \int_{\mathbb{R}^N} \epsilon^{-N} e^{i\xi(\epsilon w + y)} \rho(w) \epsilon^N dwdy$$

$$= \int_{\mathbb{R}^N} f(y) \int_{\mathbb{R}^N} e^{-i\epsilon\xi w} e^{-i\xi y} \rho(w) dw\, dy$$

By Fubini's theorem we get

$$\widehat{J_\epsilon f}(\xi) = \int_{\mathbb{R}^N} e^{-i\xi y} f(y)dy \int_{\mathbb{R}^N} e{-}i\xi w \rho(w)dw$$

$$= \hat{\rho}(\epsilon\xi)\hat{f}(\xi)$$

$$\|J_\epsilon f - f\|_{H^{s-1}} = \left(\int_{\mathbb{R}^N} (1+|\xi|^2)^{s-1} |\widehat{J_\epsilon f - f}|^2 d\xi\right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathbb{R}^N} (1+|\xi|^2)^{s-1} |\hat{f}(\xi)(\hat{\rho}(\epsilon\xi)-1)|^2 d\xi\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\mathbb{R}^N} \left(\frac{(\rho(\epsilon\xi)-1)^2}{1+|\xi|^2}\right) (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$\leq \left(\left\|\frac{(\hat{\rho}(\epsilon\xi)-1)^2}{1+|\xi|^2}\right\|_\infty\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$= \left(\left\|\frac{(\hat{\rho}(\epsilon\xi)-1)^2}{1+|\xi|^2}\right\|_\infty\right)^{\frac{1}{2}} \|f\|_{H^s}$$

We observe that :

$$\frac{(\hat{\rho}(\epsilon\xi)-1)^2}{1+|\xi|^2} \leq \frac{(\hat{\rho}(\epsilon\xi)-1)^2}{|\xi|^2} \leq \frac{x|\epsilon\xi|^2}{|\xi|^2} \leq c\epsilon^2$$

Thus we conclude that

$$\|J_\epsilon f - f\|_{H^{s-1}} \leq c\epsilon \|f\|_{H^s}$$

□

Proposition 4.1.12. *Let $u \in H^s(\mathbb{R}^N)$ then*

$$\lim_{\epsilon \searrow 0} \|J_\epsilon f - f\|_{H^s} = 0$$

Proof.

$$\|J_\epsilon u(x) - u(x)\|_{H^s} = \|(1+|\xi|^2)^{\frac{s}{2}}(J_\epsilon \widehat{u(x) - u}(x))\|_{L^2}$$

$$\leq \|C \sum_{a \leq \frac{s}{2}} |\xi^a|^2 (J_\epsilon \widehat{u(x) - u}(x))\|_{L^2} \leq \check{c}\|J_\epsilon \widehat{u(x) - u}(x))\|_{L^2}$$

By Plancherel's we have:

$$\|J_\epsilon u(x) - u(x)\|_{H^s} \leq \check{c}\|J_\epsilon u(x) - u(x)\|_{L^2}$$

By proposition (1.1.10.) we get that

$$\lim_{\epsilon \searrow 0} \|J_\epsilon f - f\|_{H^s} = 0$$

□

Proposition 4.1.13. *Let $u \in H^m(\mathbb{R}^N)$ and $k \in \mathbb{Z}_0^+$ then*

$$\|J_\epsilon u\|_{H^{m+k}} \leq \frac{c_k}{\epsilon^k}\|u\|_{H^m} \tag{P 4.1.13 i}$$

and

$$\|J_\epsilon D^k u\|_{L^\infty} \leq \frac{c_k}{\epsilon^{\frac{N}{2}+k}}\|u\|_{L^2} \tag{P 4.1.13 ii}$$

Proof.

$$\|J_\epsilon u\|_{H^{m+k}} = \left(\sum_{a \leq m+k} \|D^a J_\epsilon u\|_{L^2}^2\right)^{\frac{1}{2}}$$

We assume that $|b| \leq m$ and $|c| \leq k$ and we set $|a| = |b| + |c|$ So we have

$$D^a J_\epsilon u = D^{b+c} J_\epsilon u = D^b D^c J_\epsilon u$$

By Leibniz integral rule

$$= D^b D^c \int_{\mathbb{R}^N} \epsilon^{-N} \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy$$

$$= D^b \epsilon^{-N} \int_{\mathbb{R}^N} D_x^c \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy$$

$$= \epsilon^{-c-N} D^b \int_{\mathbb{R}^N} \rho_c\left(\frac{x-y}{\epsilon}\right) u(y)dy$$

$$= \epsilon^{-c-N} \int_{\mathbb{R}^N} D_x^b \rho_c\left(\frac{x-y}{\epsilon}\right) u(y)dy$$

By integration by parts

$$D^a J_\epsilon u = \epsilon^{-c-N} \int_{\mathbb{R}^N} \rho_c\left(\frac{x-y}{\epsilon}\right) D_y^b u(y)dy$$

$$|D^a J_\epsilon|^2 = \left|\epsilon^{-c-N}\int_{\mathbb{R}^N}\rho_c\left(\frac{x-y}{\epsilon}\right)D^b_y u(y)dy\right|^2$$

$$\leq \epsilon^{-2c}\int_{\mathbb{R}^N}\epsilon^{-2N}\left|\rho_c\left(\frac{x-y}{\epsilon}\right)\right|^2\left|D^b_y u(y)\right|^2 dy$$

$$\leq \epsilon^{-2c}\int_{\mathbb{R}^N}\epsilon^{-N}\left|\rho_c\left(\frac{x-y}{\epsilon}\right)\right| dy\int_{\mathbb{R}^N}\epsilon^{-N}\left|\rho_c\left(\frac{x-y}{\epsilon}\right)\right|\left|D^b_y u(y)\right|^2 dy$$

Therefore

$$D^a J_\epsilon u \leq \frac{C_c}{\epsilon^{2c}}\epsilon^{-N}\int_{\mathbb{R}^N}\left|\rho_c\left(\frac{x-y}{\epsilon}\right)\right|\left|D^b_y u(y)\right|^2 dy$$

So by those trivial calculations we have

$$\|J_\epsilon u\|_{H^{m+k}} \leq \sum_{\substack{|b|\leq m\\|c|\leq k}}\int_{\mathbb{R}^N}\frac{C_c}{\epsilon^{2c}}\epsilon^{-N}\int_{\mathbb{R}^N}\left|\rho_c\left(\frac{x-y}{\epsilon}\right)\right|\left|D^b_y u(y)\right|^2 dy\, dx$$

By Fubini's theorem

$$\|J_\epsilon u\|_{H^{m+k}} \leq \sum_{\substack{|b|\leq m\\|c|\leq k}}\frac{C_c}{\epsilon^{2c}}\int_{\mathbb{R}^N}|D^b_y u(y)|^2\int_{\mathbb{R}^N}\epsilon^{-N}\rho_c\left(\frac{x-y}{\epsilon}\right)dx\, dy$$

$$\leq \frac{c_k}{\epsilon^k}\sum_{|b|\leq m}\int_{\mathbb{R}^N}|D^b_y u(y)|^2 dy$$

$$\leq \frac{c_k}{\epsilon^k}\|u\|_{H^m}$$

For the second part of the proof we have that

$$|J_\epsilon D^k u| = |D^k J_\epsilon u|$$

$$= \left|\epsilon^{-N}D^k\int_{\mathbb{R}^N}\rho\left(\frac{x-y}{\epsilon}\right)u(y)dy\right|$$

By Leibniz integral rule

$$= \epsilon^{-N}\left|\int_{\mathbb{R}^N}D^k_x\rho\left(\frac{x-y}{\epsilon}\right)u(y)dy\right|$$

$$\leq \epsilon^{-N}\epsilon^{-k}\int_{\mathbb{R}^N}\left|\rho_k\left(\frac{x-y}{\epsilon}\right)u(y)\right|dy$$

$$\leq \epsilon^{-N-k}\int_{\mathbb{R}^N}\left|\rho_k\left(\frac{x-y}{\epsilon}\right)\right||u(y)|\, dy$$

By Holder inequality

$$\leq \left(\int_{\mathbb{R}^N}\left|\rho_k\left(\frac{x-y}{\epsilon}\right)\right|^2 dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|u(y)|^2 dy\right)^{\frac{1}{2}}$$

$$\leq \epsilon^{-\frac{N}{2}}\epsilon^{-k}\left[\epsilon^{-\frac{N}{2}}\int_{\mathbb{R}^N}\left|\rho_k\left(\frac{x-y}{\epsilon}\right)\right|^2 dy\right]^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|u(y)|^2\right)^{\frac{1}{2}}$$

Consequently

$$|J_\epsilon D^k u| \leq \frac{c_k}{\epsilon^{\frac{N}{2}}+k}\left(\int_{\mathbb{R}^N}|u(y)|^2 dy\right)^{\frac{1}{2}}$$

So we conclude

$$\|J_\epsilon D^k u\|_{L^\infty} \leq \frac{c_k}{\epsilon^{\frac{N}{2}+k}}\|u\|_{L^2}$$

□

In this section, we will briefly discuss Leray's projection in Sobolev spaces. As we learned in chapter 3, $P : L^2(\mathbb{R}^N) \to \mathcal{H}(\mathbb{R}^N)$ is defined for a $u \in L^2$. It is worth noting that for an $m \in Z_0^+$ and $u \in H^m$, we can extend this decomposition. This means that for $u \in H^m$, there exists a unique orthogonal decomposition $u = w + \nabla q$. We also define the projection operator $P : H^m \to V^m$ and $w = Pu$, where V^m is the space of divergence-free functions.

Proposition 4.1.14. *Assume that $m \in \mathbb{Z}_0^+$, $u \in H^m$ with the above decomposition, and P the Leray's projection, then*

1. Pu , $\nabla q \in H^m$
2. $\int_{\mathbb{R}^N} Pu\ \nabla q dx = 0$
3. $\|Pu\|_{H^m}^2 + \|\nabla q\|_{H^m}^2 = \|u\|_{H^m}^2$
4. $P(D^a u) = D^a(Pu), \forall |a| \leq m$
5. $P(J_\epsilon u) = J_\epsilon(Pu)$, $\epsilon > 0$
6. $(Pu, v)_{H^m} = (u, Pv)_{H^m}$
7. $\|P\| = 1$

Proof. We remind that the space H^m is a Hilbert space with inner product $(u, v)_{H^m} = \sum_{|a|\leq m}\int_{\mathbb{R}^N}(D^a u)\cdot(D^a v)dx$ We will prove each property individually, the property (2) is obvious since $Pu = w$ and $w, \nabla u$ are perpendicular.
1.The space $C_C^\infty(\mathbb{R}^N)$ is dense in H^0 so there exist a sequence u_n such that $u_n \to u$ in H^0 We have that $u_n = w_n + \nabla q_n$ i.e. $\Delta q_n = \text{div}u_n$, this is a Poisson equation so we have that:

$$\nabla q_n = \int_{|y|\leq R}\frac{x-y}{|x-y|^N}\text{div}u_n(y)dy$$

We have proved in lemma 10 that for $|x| \to \infty$ and $|y| \leq R$ then

$$|x-y|^{-N} = |x|^{-N} + O(|x|^{-N-1})$$

thus

$$\nabla q_n(x) = c_N\frac{x}{|x|^N}\int_{|y|\leq R}\text{div}u_n(y)dy + O(|x|^{-N})$$

I.e. $\nabla q_n(x) \approx O(|x|^{-N})$
So we have that for $|x| \leq 2R$

$$\int_{\mathbb{R}^N} |\nabla q_n|^2 dx \leq c \int_{2R}^{\infty} r^{-2N} r^{N-1} dr < \infty$$

It follows that $\nabla q_n(x) \in L^2$ for $q_n = \chi_{[|x| \geq \frac{1}{n}]} q(x)$ by dominated convergence theorem we conclude that $q \in L^2 = H^0$ also we set $w = u - \nabla q$ and consequently $w \in L^2 = H^0$
By the property 4 we have this result for all $m \in \mathbb{Z}_0^+$

4.We start with the relation $u = Pu + \nabla q$ we differentiate this relation and we get that:

$$D^a u = D^a(w + \nabla q)$$

$$D^a u = D^a w + D^a(\nabla q)$$

It is also true that

$$\operatorname{div}(D^a w) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} D^a w$$

$$= \sum_{i=1}^{N} D^a \frac{\partial}{\partial x_i} w$$

$$= D^a \sum_{i=1}^{N} \frac{\partial}{\partial x_i} w = 0$$

So we define the $P(D^a u) = D^a w$ and we get that

$$D^a(Pu) = D^a w = P(D^a u)$$

5. We adopt the same procedure as above

$$J_\epsilon u = J_\epsilon(w + \nabla q)$$

$$J_\epsilon u = J_\epsilon w + J_\epsilon(\nabla q)$$

It is also true that

$$\operatorname{div}(J_\epsilon w) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} J_\epsilon w$$

By proposition 4.1.8 we get that

$$= \sum_{i=1}^{N} J_\epsilon \frac{\partial}{\partial x_i} w$$

$$= J_\epsilon \left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i} w \right) = 0$$

So we define $P(J_\epsilon u) = J_\epsilon w$ and we get that

$$J_\epsilon(Pu) = J_\epsilon w = P(J_\epsilon u)$$

3.

$$\begin{aligned}\|u\|_{H^m}^2 &= \sum_{|a|\leq m} \|u\|_{L^2}^2 \\ &= \sum_{|a|\leq m} (Pu + \nabla q, Pu + \nabla q)_{L^2} \\ &= \sum_{|a|\leq m} [(Pu, Pu)_L^2 + 2(Pu, \nabla q)_{L^2} + (\nabla q, \nabla q)]\end{aligned}$$

By property 2. we get that

$$= \sum_{|a|\leq m} \left(\|Pu\|_{L^2}^2 + \|\nabla q\|_{L^2}^2\right)$$

Thus

$$\|u\|_{H^m}^2 = \|Pu\|_{H^m}^2 + \|\nabla q\|_{H^m}^2$$

6. Assume $u, v \in H^m$ then this property Holds since P is orthogonal projection.

7. $\|Pu\|^2 = \|Pu\|\|Pu\|$ by Pythagoras theorem we know that $\|Pu\| \leq \|u\|$ so we have that

$$\frac{\|Pu\|}{\|u\|} \leq 1$$

Recall that

$$\begin{aligned}\|P\| &= \sup_{\substack{u\in H^m \\ \|u\|\neq 0}} \frac{\|Pu\|}{\|u\|} \\ &\leq \sup_{\substack{u\in H^m \\ \|u\|\neq 0}} 1 = 1\end{aligned}$$

Furthermore since P is orthogonal projection it is true that $P = P^2$ so

$$\|Pu\| = \|P^2 u\| = \|P(Pu)\| \leq \|P\|\|Pu\|$$

Consequently $\|P\| = 1$ □

4.2 Existence of smooth solutions for the regularized equations

In this section, we will deal with Banach space valued functions[15]. Our maps are $f : [a, b] \to B$ with B a Banach space. Those spaces are referred to as Bochner spaces,

[15] [25]

and we have some more general concepts about differentiation (Frechet derivative) and integrability (Bochner integrable). These concepts do not differ significantly from what we already know about derivatives and integrals. For instance, we say that a function $f : [a,b] \to B$ is Bochner integrable if $\int_a^b \|f\|_B < \infty$. We will give more details about these spaces as and when we need them.

The regularized equations

The strategy we will follow is known. We start with non-smooth functions and through convolution with smooth kernels, we end up smooth functions. When dealing with the Euler and Navier-Stokes equations, we aim to mollify them and obtain approximations. In the previous chapter, our theory relied on energy estimates, and we plan to continue using energy methods to prove results.
We consider the following mollification of the Navier Stokes equation,

$$\frac{\partial}{\partial t}u_\epsilon + J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon) = -\nabla p_\epsilon + \nu J_\epsilon\left(\Delta J_\epsilon u_\epsilon\right)^{16} \tag{NS_ϵ}$$

Using the Leray's formulation[17] we project this equation to the closed[18] space of H^s divergence free functions,i.e. $V^s = \{u \in H^s : \text{div} u = 0\}$
So we have that

$$\frac{\partial}{\partial t}u_\epsilon + P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\} = \nu J_\epsilon(\Delta J_\epsilon u_\epsilon) \tag{$L-NS_\epsilon$}$$

We set

$$F_\epsilon(u_\epsilon) = \nu J_\epsilon(\Delta J_\epsilon u_\epsilon) - P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}$$

So we reach to

$$\frac{\partial}{\partial t}u_\epsilon = F_\epsilon(u_\epsilon)$$

By assuming that the initial value of the mollified function of velocity is equal to the initial value of the function of velocity we reach to an initial value problem

$$\begin{cases} \frac{\partial}{\partial t}u_\epsilon = F_\epsilon(u_\epsilon) \\ u_\epsilon(*,0) = u_\epsilon(0) = u_0 \end{cases} \tag{IVP}$$

Note that the above equation is an ODE, since we consider F_ϵ to be known. The goal is to find the flow of u_ϵ given that F_ϵ. The problem at hand is to solve the initial value problem on V^s. Additionally, this is an autonomous ODE, because F_ϵ is a function depending only on space. Although, the term $F_\epsilon(u_\epsilon)$ depends on time, since u_ϵ is time-dependent. The well-known Picard-Lindelof theorem will be employed for solving this problem. This theorem has the benefit of having strong results despite not having very strict conditions.

[16] We denote u_ϵ the velocity field, because the regularized solution will depend on the choice of ϵ, meaning that if we choose another ϵ, we will take another u_ϵ

[17] see chapter 2

[18] [36] pg 87

Theorem 4.2.1 (Picard-Lindelof)**.** *Assume that B is a Banach space and F is a locally Lipschitz mapping $F : B \to B$ then for $u_0 \in B$ there exists a time T such that the initial value problem*

$$\begin{cases} \frac{d}{dt}u(t) = F(u(t)) \\ u|_{t=0} = u_0 \end{cases}$$

has unique local solution on $C^1([0,T), B)$[19]
***Remark**:This theorem gives us a classical solution.*

Proof. [20] So we want to solve

$$\begin{cases} \frac{d}{dt}u(t) = F(u(t)), t \in [0,T) \\ u|_{t=0} = u_0 \end{cases}$$

We will find a u which satisfy the following integral equation

$$u(t) = u_0 + \int_0^t F(u(s))ds$$

where the integral here is a Bochner integral
Since F is locally Lipschitz we assume that this is true in an open set O up to time T
We assume for $k > 0$ the set

$$\mathbf{X} = \{u \in C([0,T), B) : \sup_{0 \le t \le T} e^{-kt} \|u\|_B < \infty\}$$

The space $\mathbf{X}$ is a Banach space with norm $\| \cdot \|_X = \sup_{0 \le t < T} e^{-kt} \| \cdot \|_B$
Indeed: Assume that $\{u_n(t)\}$ is a Cauchy sequence in $\mathbf{X}$.
Step 1: We will prove that $u(t_0) = \lim_{n\to\infty} u_n(t_0)$ is well defined $\forall t_0 \in [0,T)$
Let t_0 since u_n is a Cauchy sequence we have that

$$\forall \delta > 0 \; \exists n_0 \in \mathbb{N} \; \forall n, m \ge n_0 \text{ we have } \|u_n(t) - u_m(t)\|_X < \delta$$

Thus

$$e^{-kt_0} \|u_n(t_0) - u_m t_0\|_B < \delta$$

I.e. $\|u_n(t_0) - u_m(t_0)\|_B < \delta'$ thus $u_n(t_0)$ is a Cauchy sequence, and B is a Banach space so this sequence converges so $u(t_0)$ is well defined.
Step 2: We will prove that $u_n(t) \to u(t)$ in $\mathbf{X}$
We have

$$\|u_n(t) - u(t)\|_X \le \sup_{0 \le t \le T} e^{-kt} \|u_n(t) - u(t)\|_B$$

Since B is a Banach space we have that $\|u_n - u_m\|_X \le \sup_{0 \le t \le T} e^{-kt}\delta$ and $\sup_{0 \le t \le T} e^{-kt} \le 1$ thus

$$\forall \delta > 0 \; \exists n_1 \; \forall n \ge n_1 \; : \; \|u_n - u_m\| < \delta$$

[19] the space $C^m([a,b], B) = \{f : [a,b] \to B \text{ is m times differentiable}\}$the derivative is the Frechet derivative.
[20] [13] pg 184

Step 3: We define the following mapping and we will show that it is a contraction
Let $u \in \mathbf{X}$ we define Φ such that $\Phi u(t) = u_0 + \int_0^t F(u(s))ds$ with $t \in [0, T)$ So we have that

$$\|\Phi u - \Phi v\|_X = \sup_{0 \le t < T} e^{-kt} \| \int_0^t Fu - Fv\|_B ds$$

F is locally Lipschitz so

$$\|\Phi u - \Phi v\_X| \le \sup_{0 \le t < T} e^{-kt} \int_0^t L\|u - v\|_B ds$$

Thus

$$\|\Phi u - \Phi v\|_X \le \frac{L}{k} \|u - v\|_X$$

So in the open set O we fix $k > L$ so this mapping is a contraction so we can use Banach's fixed point theorem[21]

Banach Fixed point theorem:Assume that G is a mapping from a Banach space B to itself which is a contraction then G has a fixed point i.e.$Gu = u$
proof: Let u_0 be a known initial value we set $u_1 = G(u_0)$ then we set $u_2 = G(u_1)$ we continue this process and we create a recursive sequence u_n such that $u_{n+1} = G(u_n)$
This sequence is a Cauchy sequence, indeed:

$$\|u_{m+1} - u_m\|_B = \|G(u_m) - G(u_{m-1})\|_B$$

but G is a contraction so

$$\|u_{m+1} - u_m\|_B \le K\|u_m - u_{m-1}\|_B$$

we continue the same way i.e.

$$\|u_m - u_{m-1}\|_B \le K^2\|u_{m-1} - u_{m-2}\|_B$$

and we conclude to the following inequality

$$\|u_{m+1} - u_m\|_B \le K^m\|u_1 - u_0\|_B$$

Without loss of generality we assume that $n \ge m$ so by triangle inequality we have

$$\|u_m - u_n\|_B \le \|u_m - u_{m+1}\|_B + \|u_{m+1} - u_{m+2}\|_B + ... + \|u_{n-1} - u_n\|_B$$
$$\le \|u_1 - u_0\|_B \left(K^m + K^{m+1} + ... + K^{n-1}\right)$$
$$= K^m \frac{1 - K^{n-m}}{1 - K} \|u_1 - u_0\|_B$$

So we have that

$$\|u_m - u_n\|_B \le \frac{K^m}{1 - K} \|u_1 - u_0\|_B$$

[21] [5] pg 5

Consequently choosing m sufficiently large u_n is a Cauchy sequence. Furthermore B is a Banach space so this sequence converges, i.e there exists a $u \in B$ such that $u_n \to u$ This u is a fixed point, indeed:

$$\|u - G(u)\|_B \leq \|u - u_n\|_B + \|u_n - G(u)\|_B$$

$$= \|u - u_n\|_B + \|G(u_{n-1}) - G(u)\|_B$$

$$\leq \|u - u_n\|_B + K\|u_{n-1} - u\|_B$$

So we have that for n large enough $\|u - G(u)\|_B = 0$ which completes the proof.

And we get that there exist a u such that $\Phi u = u$ so we have solve the integral equation. This solution is unique since if we assume two different solutions of the integral equation, name u, v we have that

$$\|u - v\|_B \leq \int_0^t L\|u - v\|_B ds$$

So $\|u - v\|_B = 0$ i.e $u = v$ □

So firstly we will talk about local in time solution for the regularized problem.

Lemma 17. *The space V^m , $m \in \mathbb{Z}_0^+$, with the norm of H^m space, is a Banach space*

Proof. It is a closed subset of a Banach space, and we know that a closed subspace of a Banach space is Banach space. □

4.2.1 3 dimensions

We will seperate the analysis into 2D and 3D, based on energy estimates. For 2D, we will use radial finite energy decomposition. Our goal here is to prove a theorem on **three dimensions** that affirms the local existence of solutions for the regularized Navier-Stokes equation.

Theorem 4.2.2. *Let $u_0 \in V^m$,$m \in Z_0^+$ the initial value of the velocity field u_ϵ,then*

1. *$\forall \epsilon > 0$ there exist a unique solution $u_\epsilon \in C^1([0,T), V^m)$ of the (IVP).*
2. *On any time interval $[0,T]$ on which the solution exists in $C^1([0,T), V^m)$ we have that :*

$$\sup_{0 \leq t \leq T} \|u_\epsilon\|_{L^2} \leq \|u_0\|_{L^2}$$

Remarks:
1. The time T depends on ϵ and the initial data.
2.The energy estimate that is given in the second part of the theorem depends neither on ϵ nor on viscosity. So the regularization we choose satisfies our will to achieve an energy estimate independent of ϵ. Furthermore, if we compare this energy estimate with the energy estimate in the previous chapter we see that for $u_1 = u_\epsilon$, $u_2 = 0$, $F_1, F_2 = 0$, $u_0^1 = u_0$ and $u_0^2 = 0$, those two estimates are identical, so our energy estimate here is optimal.

Proof. Note: In the upcoming proofs, we will utilize numerous constants that arise from the estimates in the preceding propositions. To simplify notation, we will refer to all of these constants as "c", with the understanding that each one depends solely on the values of "m" and "k".

1. For the first part of the theorem we will use the Picard Lindelof theorem, so we only need data for the F_ϵ. We will prove that $F_\epsilon : V^m \to V^m$ and also it is locally Lipschitz. So recall that

$$F_\epsilon(u_\epsilon) = \nu J_\epsilon(\Delta J_\epsilon u_\epsilon) - P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}$$

Assume that $u_\epsilon \in V^m$ we will prove that $F_\epsilon(u_\epsilon) \in V^m$ i.e. $F_\epsilon(u_\epsilon) \in H^m$ and $\operatorname{div} F_\epsilon(u_\epsilon) = 0$
So

$$\|F_\epsilon(u_\epsilon)\|_{H^m} = \|\nu J_\epsilon(\Delta J_\epsilon u_\epsilon) - P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}\|_{H^m}$$

By triangle inequality we have that

$$\leq \|\nu J_\epsilon(\Delta J_\epsilon u_\epsilon)\|_{H^m} + \|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}\|_{H^m}$$

We will estimate each term individually :

$$\|\nu J_\epsilon^2(\Delta u_\epsilon)\|_{H^m} \overset{proposition 1.1.8}{=} \|\nu\Delta(J_\epsilon^2 u_\epsilon)\|_{H^m} = \|\nu J_\epsilon^2 u\|_{H^{m+2}}$$

By proposition 4.1.13

$$\leq \frac{c\nu}{\epsilon^2}\|J_\epsilon u_\epsilon\|_{H^m}$$

$$\leq \frac{c\nu}{\epsilon^2}\|u_\epsilon\|_{H^m}$$

Since $u_\epsilon \in H^m$ we have that

$$\|\nu J_\epsilon^2(\Delta u_\epsilon)\|_{H^m} < \infty$$

And the other term

$$\|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}\|_{H^m}$$

$$\leq \|P\|_{H^m}\|J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\|_{H^m}$$

By proposition 4.1.13

$$\leq c\|\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon\|_{H^m}$$

By proposition 4.1.4

$$\leq c\left\{\|\nabla J_\epsilon u_\epsilon\|_{H^m}\|J_\epsilon u_\epsilon\|_{L^\infty} + \|\nabla J_\epsilon u_\epsilon\|_{L^\infty}\|J_\epsilon u_\epsilon\|_{H^m}\right\}$$

By proposition 4.1.13

$$\leq c\left\{\|J_\epsilon u_\epsilon\|_{H^{m+1}} c\|u_\epsilon\|_{L^2} + \frac{c}{\epsilon^{\frac{5}{2}}}\|u_\epsilon\|_{L^2} c\|u_\epsilon\|_{H^m}\right\}$$

$$\leq c(\epsilon)\|u_\epsilon\|_{H^m}^2$$

Since $u_\epsilon \in H^m$ we have that

$$\|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}\|_{H^m} < \infty$$

Thus $\|F_\epsilon(u_\epsilon)\|_{H^m} < \infty$
Furthermore

$$\mathrm{div} F_\epsilon(u_\epsilon) = \mathrm{div}\left(vJ_\epsilon^2\Delta u_\epsilon\right) - \mathrm{div} P\left(J_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right]\right) = 0$$

So we have prove that $F_\epsilon(u_\epsilon) \in V^m$ i.e. $F_\epsilon : V^m \to V^m$
Now we will prove that this mapping is locally Lipschitz

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon)\|_{H^m}$$

$$= \|vJ_\epsilon(\Delta J_\epsilon u_\epsilon) - P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\} - \nu J_\epsilon(\Delta J_\epsilon v_\epsilon) + P\left\{J_\epsilon\left[(J_\epsilon v_\epsilon)\cdot\nabla\right](J_\epsilon v_\epsilon)\right\}\|_{H^m}$$

By triangle inequality and by add and subtract the term $P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon v_\epsilon)\right\}$

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon)\|_{H^m} \le \|vJ_\epsilon^2\Delta(u_\epsilon - v_\epsilon)\|_{H^m}$$

$$+\|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon(u_\epsilon - v_\epsilon))\right\}\|_{H^m}$$

$$+\|P\left\{J_\epsilon\left[(J_\epsilon(u_\epsilon - v_\epsilon))\cdot\nabla\right](J_\epsilon v_\epsilon)\right\}\|_{H^m}$$

We will see each term individually

- $\|vJ_\epsilon^2\Delta(u_\epsilon - v_\epsilon)\|_{H^m} = \nu\|J_\epsilon^2(u_\epsilon - v_\epsilon)\|_{H^{m+2}}$ By proposition 4.1.13

$$\le \frac{c}{\epsilon^2}\|J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^m}$$

$$le\frac{c}{\epsilon^2}\|u_\epsilon - v_\epsilon\|_{H^m}$$

- $\|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon(u_\epsilon - v_\epsilon))\right\}\|_{H^m}$

$$\le c\|\nabla J_\epsilon(u_\epsilon - v_\epsilon)\cdot J_\epsilon u_\epsilon\|_{H^m}$$

  By proposition 4.1.4

$$\le c\left\{\|\nabla J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^m}\|J_\epsilon u_\epsilon\|_{L^\infty} + \|J_\epsilon u_\epsilon\|_{H^m}\|\nabla J_\epsilon(u_\epsilon - v_\epsilon)\|_{L^\infty}\right\}$$

$$\le c\left\{\|J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^{m+1}}\|J_\epsilon u_\epsilon\|_{L^\infty} + \|J_\epsilon u_\epsilon\|_{H^m}\|J_\epsilon\nabla(u_\epsilon - v_\epsilon)\|_{L^\infty}\right\}$$

  By proposition 4.1.13

$$\le c\left\{\frac{c}{\epsilon}\|u_\epsilon - v_\epsilon\|_{H^m}\frac{c}{\epsilon^{\frac{3}{2}}}\|u_\epsilon\|_{L^2} + c\|u_\epsilon\|_{H^m}\frac{c}{\epsilon^{\frac{5}{2}}}\|u_\epsilon - v_\epsilon\|_{L^2}\right\}$$

$$\le \frac{c}{\epsilon^{\frac{5}{2}}}\|u_\epsilon\|_{H^m}\|u_\epsilon - v_\epsilon\|_{H^m}$$

- $\|P\left\{J_\epsilon\left[(J_\epsilon(u_\epsilon - v_\epsilon))\cdot\nabla\right](J_\epsilon v_\epsilon)\right\}\|_{H^m}$

$$\leq c\|(\nabla J_\epsilon v_\epsilon)\cdot J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^m}$$

By proposition 4.1.4

$$\leq c\left\{\|\nabla J_\epsilon v_\epsilon\|_{H^m}\|J_\epsilon(u_\epsilon - v_\epsilon)\|_{L^\infty} + \|\nabla J_\epsilon v_\epsilon\|_{L^\infty}\|J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^m}\right\}$$

By proposition 4.1.13

$$\leq c\left\{\frac{c}{\epsilon}\|v_\epsilon\|_{H^m}\frac{c}{\epsilon^{\frac{3}{2}}}\|u_\epsilon - v_\epsilon\|_{L^2} + \frac{c}{\epsilon^{\frac{5}{2}}}\|v_\epsilon\|_{L^2}c\|u_\epsilon - v_\epsilon\|_{H^m}\right\}$$

$$\leq \frac{c}{\epsilon^{\frac{5}{2}}}\|v_\epsilon\|_{H^m}\|u_\epsilon - v_\epsilon\|_{H^m}$$

Consequently

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon)\|_{H^m} \leq \frac{c}{\epsilon^2}\|u_\epsilon - v_\epsilon\|_{H^m} + \frac{c}{\epsilon^{\frac{5}{2}}}\left(\|u_\epsilon\|_{H^m} + \|v_\epsilon\|_{H^m}\right)\|u_\epsilon - v_\epsilon\|_{H^m}$$

Assume that $O^M = \{u \in V^m : \|u\|_{H^m} \leq M\}$ then we conclude that

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon)\|_{H^m} \leq c_M\|u_\epsilon - v_\epsilon\|_{H^m}$$

Thus F_ϵ is locally Lipschitz.
By the Picard-Lindelof theorem there exist a unique solution $u_\epsilon \in C^1\left([0,T), V^m\right)$

2. For the second part of the theorem we will follow the well known steps for the energy methods.
We begin with the equation of (IVP) and we multiply with the solution u_ϵ in L^2:

$$\int_{\mathbb{R}^3}\left(\frac{d}{dt}u_\epsilon - F_\epsilon(u_\epsilon)\right)\cdot u_\epsilon dx = 0$$

$$\int_{\mathbb{R}^3}\frac{1}{2}\frac{d}{dt}|u_\epsilon|^2 dx = \int_{\mathbb{R}^3} u_\epsilon \cdot F_\epsilon(u_\epsilon)dx$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|_{L^2}^2 = \int_{\mathbb{R}^3} u_\epsilon \cdot F_\epsilon(u_\epsilon)dx$$

Now we will process with the integral on the right part :

$$\int_{\mathbb{R}^3} u_\epsilon \cdot F_\epsilon(u_\epsilon)dx = \int_{\mathbb{R}^3} u_\epsilon \cdot \left[vJ_\epsilon^2(\Delta u_\epsilon) - PJ_\epsilon\left(J_\epsilon u_\epsilon \cdot \nabla J_\epsilon u_\epsilon\right)\right]dx$$

$$\int_{\mathbb{R}^3} vu_\epsilon \cdot J_\epsilon^2(\Delta u_\epsilon)dx - \int_{\mathbb{R}^3} u_\epsilon \cdot J_\epsilon P\left[(J_\epsilon u_\epsilon \cdot \nabla)J_\epsilon u_\epsilon\right]dx$$

By proposition 4.1.9 we have that the mollifiers are symmetric so

$$= \nu\int_{\mathbb{R}^3} J_\epsilon u_\epsilon \cdot \Delta(J_\epsilon u_\epsilon)dx - \int_{\mathbb{R}^3} J_\epsilon u_\epsilon \cdot P\left[(J_\epsilon u_\epsilon \cdot \nabla)J_\epsilon u_\epsilon\right]dx$$

For the first integral we have that

$$\nu \int_{\mathbb{R}^3} J_\epsilon u_\epsilon \cdot \Delta(J_\epsilon u_\epsilon)dx = -\nu \int_{\mathbb{R}^3} \nabla(J_\epsilon u_\epsilon) \cdot \nabla(J_\epsilon u_\epsilon)dx$$

$$= -\nu \int_{\mathbb{R}^3} |\nabla J_\epsilon u_\epsilon|^2 dx = \nu \|\nabla J_\epsilon u_\epsilon\|_{L^2}^2$$

For the second integral we have that

$$-\int_{\mathbb{R}^3} J_\epsilon u_\epsilon \cdot P\left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon\right] dx \leq -\int_{\mathbb{R}^3} J_\epsilon u_\epsilon \cdot \left[J_\epsilon u_\epsilon \cdot \nabla\right] J_\epsilon u_\epsilon = 0$$

Consequently

$$\frac{d}{dt}\|u_\epsilon\|_{L^2}^2 \leq 0$$

So by simplified Gronwall we have that

$$\|u_\epsilon\|_{L^2}^2 \leq \|u_0\|_{L^2}^2$$

Eventually taking the supremum we conclude that:

$$\sup_{0 \leq t \leq T} \|u_\epsilon\|_{L^2} \leq \|u_0\|_{L^2}$$

□

We have currently achieved a smooth solution for the regularized problem within a certain time frame. However, it is now necessary to determine if we can attain a global solution for the same problem. Fortunately, the answer is affirmative.
The idea of continuous extension of solutions is the following assume that we have a solution on the interval $[0, T)$ in B if $\lim_{t \to T^-} u(t)$ exists then we set $\tilde{u_0} = u(T) \in B$. This quantity becomes the new initial value, and we find a solution for the updated initial value problem for the interval $[T, T + c)$. The only concern is whether this new solution effectively solves the initial problem. We define $\tilde{u}$ the new solution, where

$$\tilde{u}(t =) \begin{cases} u(t) & \text{for } t \in [0, T) \\ u(T) & \text{for } t \in [T, T + c) \end{cases}$$

We have that $u, \tilde{u}$ satisfy an ivp,so

$$u(t) = \tilde{u}(t) = u_0 + \int_0^t F(\tilde{u}(s))ds \text{ for } \ 0 \leq t < T \tag{1}$$

and

$$\tilde{u}(t) = \tilde{u_0} + \int_T^t F(\tilde{u}(s))ds \text{ for } T \leq t < T + c \tag{2}$$

By relation (1) we have that

$$\lim_{t \to T^-} \tilde{u}(t) = \lim_{t \to T^-} u_0 + \int_0^t F(\tilde{u}(s))ds$$

$$= u_0 + \lim_{t \to T^-} \int_0^t F(\tilde{u}(s))ds$$
$$= u_0 + \int_0^T F(\tilde{u}(s))ds$$

By relation (2) we have that

$$\tilde{u}(t) = u(T) + \int_T^t F(\tilde{u}(s))ds$$
$$= u_0 + \int_0^T F(\tilde{u}(s))ds + \int_T^t F(\tilde{u}(s))ds$$

So $\tilde{u}(t)$ solves the first ivp in the interval $[0, T+c)$. We continue this process until we find a maximum time T'. This process may stop for two reasons the first one is that for $T' < \infty$ the limit may not exist then we say that the solution is not continuous, the second one is that the limit may exist but the solution does not belong on the Banach space, in both cases we will say that the solution blows up.
So we summarize in the following theorem

Theorem 4.2.3. *Assume that B is a Banach space and $F : B \to B$ a locally Lipschitz mapping then the unique solution u of the ivp* $\begin{cases} \frac{d}{dt}u = F(u) \\ u|_{t=0} = u_0 \in B \end{cases}$ *either exist globally in time or for $T < \infty$ the solution blows up.*

Proof. Assume that T^* is the maximum time of existence of the solution u(t). Assume also that $T^* < \infty$ and the solution does not blow up We have that $\lim_{t \to T^{*-}} = u(T^*) \in B$. We set $t = t - T^{*-}$ and we have initial value $u(T^*)$ for the ivp $\begin{cases} \frac{d}{dt}u = F(u) \\ u|_{t=T^*} = u(T^*) \end{cases}$
By Picard Lindelof we find a solution which satisfy the first ivp for time $t - T^* + c$ which contradicts the fact that T^* is the maximum time so $T^* = \infty$ □

Note: This extension is true on N-dimensions So in our case we have that u_ϵ satisfies the following inequality

$$\|F_\epsilon\|_{u_\epsilon}\|_{H^m} \leq c_M \|u_\epsilon\|_{H^m}$$

So since it is a solution of the (IVP) we reach to

$$\frac{d}{dt}\|u_\epsilon\|_{H^m} \leq c_M \|u_\epsilon\|_{H^m}$$

By Gronwall's lemma in differential form

If $\frac{d}{dt}u(t) \leq b(t)u(u)$ (H) for $t \in I$ then

$$u(t) \leq u(a)e^{\int_a^t b(s)ds} \ (R)$$

proof:
Let $v(t) = e^{\int_a^t b(s)ds}$ it is true that v satisfies the differential equation $\frac{d}{dt}v(t) = b(t)v(t)$ (1) with initial value $v(a) = 1$ and $v(t) > 0$ We define the quotient $\frac{u(t)}{v(t)}$ and

we have that

$$\frac{d}{dt}\frac{u(t)}{v(t)} = \frac{\frac{d}{dt}u(t)v(t) - u(t)\frac{d}{dt}v(t)}{v^2(t)}$$

By relation (1) it follows that

$$\frac{d}{dt}\frac{u(t)}{v(t)} = \frac{\frac{d}{dt}u(t) - b(t)u(t)}{v(t)} \leq 0$$

by the (H).
So the function of the quotient is decreasing thus for $a \leq t$ we have that

$$\frac{u(t)}{v(t)} \leq \frac{u(a)}{v(a)}$$

Thus we reach to the desired result i.e. $u(t) \leq u(a)v(t) = u(a)e^{\int_a^t b(s)ds}$

we conclude that $\|u_\epsilon\|_{H^m} \leq e^{c_M T}$ which is an apriori bound, thus for $T < \infty$and we will not have a blow up so by the above theorem we have a global solution.
Conclusions: The regularized problem for the Navier Stokes equation has a smooth global in time solution on space V^m in 3 dimensions.
Remark:We have exactly the same result for the Euler equation, indeed
We consider the following mollification of the Euler equation

$$\frac{\partial}{\partial t}u_\epsilon + J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon) = -\nabla p_\epsilon \tag{E_ϵ}$$

Using the Leray's formulation[22] we project this equation to the closed[23] space of H^s divergence free functions i.e. $V^s = \{u \in H^s : \text{div}u = 0\}$
So we have that

$$\frac{\partial}{\partial t}u_\epsilon + P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\} = 0 \tag{$L-E_\epsilon$}$$

We set

$$F_\epsilon(u_\epsilon) = -P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}$$

So we reach to

$$\frac{\partial}{\partial t}u_\epsilon = F_\epsilon(u_\epsilon)$$

By assuming that the initial value of the mollified function of velocity is equal to the initial value of the function of velocity we reach to an initial value problem

$$\begin{cases}\frac{\partial}{\partial t}u_\epsilon = F_\epsilon(u_\epsilon) \\ u_\epsilon(*,0) = u_\epsilon(0) = u_0\end{cases} \tag{$IVP*$}$$

So we have the following theorem which gives a local in time solution for the regularized Euler:

Theorem 4.2.4. *Let $u_0 \in V^m$,$m \in Z_0^+$ the initial value of the velocity field u_ϵ, then*

[22]see chapter 2
[23][36] pg 87

1. *$\forall \epsilon > 0$ there exist a unique solution $u_\epsilon \in C^1([0,T), V^m)$ of the (IVP∗).*
2. *On any time interval $[0,T]$ on which the solution exists in $C^1([0,T), V^m)$ we have that :*

$$\sup_{0\le t\le T} \|u_\epsilon\|_{L^2} \le \|u_0\|_{L^2}$$

Proof. We have the same statements as the proof of the theorem above, with some little adaptations occurring due to the lack of the viscosity term.

1. For the first part of the theorem we will use the Picard Lindelof theorem, so we only need data for the F_ϵ. We will prove that $F_\epsilon : V^m \to V^m$ and also it is locally Lipschitz. So recall that

$$F_\epsilon(u_\epsilon) = -P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}$$

Assume that $u_\epsilon \in V^m$ we will prove that $F_\epsilon(u_\epsilon) \in V^m$ i.e. $F_\epsilon(u_\epsilon) \in H^m$ and $\text{div}F_\epsilon(u_\epsilon) = 0$
So

$$\|F_\epsilon(u_\epsilon)\|_{H^m} = \| - P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}\|_{H^m}$$

i.e. we have that

$$\le \|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}\|_{H^m}$$

We can easily see that

$$\|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}\|_{H^m}$$

$$\le \|P\|_{H^m}\|J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\|_{H^m}$$

By proposition 4.1.13

$$\le c\|\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon\|_{H^m}$$

By proposition 4.1.4

$$\le c\left\{\|\nabla J_\epsilon u_\epsilon\|_{H^m}\|J_\epsilon u_\epsilon\|_{L^\infty} + \|\nabla J_\epsilon u_\epsilon\|_{L^\infty}\|J_\epsilon u_\epsilon\|_{H^m}\right\}$$

By proposition 4.1.13

$$\le c\left\{\|J_\epsilon u_\epsilon\|_{H^{m+1}}c\|u_\epsilon\|_{L^2} + \frac{c}{\epsilon^{\frac{5}{2}}}\|u_\epsilon\|_{L^2}c\|u_\epsilon\|_{H^m}\right\}$$

$$\le c(\epsilon)\|u_\epsilon\|^2_{H^m}$$

Since $u_\epsilon \in H^m$ we have that

$$\|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\}\|_{H^m} < \infty$$

Thus $\|F_\epsilon(u_\epsilon)\|_{H^m} < \infty$
Furthermore

$$\text{div}F_\epsilon(u_\epsilon) = -\text{div}P\left(J_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right]\right) = 0$$

So we have prove that $F_\epsilon(u_\epsilon) \in V^m$ i.e. $F_\epsilon : V^m \to V^m$
Now we will prove that this mapping is locally Lipschitz

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon)\|_{H^m}$$
$$= \| - P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon u_\epsilon)\right\} + P\left\{J_\epsilon\left[(J_\epsilon v_\epsilon)\cdot\nabla\right](J_\epsilon v_\epsilon)\right\}\|_{H^m}$$

By triangle inequality and by add and subtract the term $P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon v_\epsilon)\right\}$

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon)\|_{H^m} \leq \|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon(u_\epsilon - v_\epsilon))\right\}\|_{H^m}$$
$$+\|P\left\{J_\epsilon\left[(J_\epsilon(u_\epsilon - v_\epsilon))\cdot\nabla\right](J_\epsilon v_\epsilon)\right\}\|_{H^m}$$

We will see each term individually

- $\|P\left\{J_\epsilon\left[(J_\epsilon u_\epsilon)\cdot\nabla\right](J_\epsilon(u_\epsilon - v_\epsilon))\right\}\|_{H^m}$

$$\leq c\|\nabla J_\epsilon(u_\epsilon - v_\epsilon)\cdot J_\epsilon u_\epsilon\|_{H^m}$$

  By proposition 4.1.4

$$\leq c\left\{\|\nabla J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^m}\|J_\epsilon u_\epsilon\|_{L^\infty} + \|J_\epsilon u_\epsilon\|_{H^m}\|\nabla J_\epsilon(u_\epsilon - v_\epsilon)\|_{L^\infty}\right\}$$
$$\leq c\left\{\|J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^{m+1}}\|J_\epsilon u_\epsilon\|_{L^\infty} + \|J_\epsilon u_\epsilon\|_{H^m}\|J_\epsilon\nabla(u_\epsilon - v_\epsilon)\|_{L^\infty}\right\}$$

  By proposition 4.1.13

$$\leq c\left\{\frac{c}{\epsilon}\|u_\epsilon - v_\epsilon\|_{H^m}\frac{c}{\epsilon^{\frac{3}{2}}}\|u_\epsilon\|_{L^2} + c\|u_\epsilon\|_{H^m}\frac{c}{\epsilon^{\frac{5}{2}}}\|u_\epsilon - v_\epsilon\|_{L^2}\right\}$$
$$\leq \frac{c}{\epsilon^{\frac{5}{2}}}\|u_\epsilon\|_{H^m}\|u_\epsilon - v_\epsilon\|_{H^m}$$

- $\|P\left\{J_\epsilon\left[(J_\epsilon(u_\epsilon - v_\epsilon))\cdot\nabla\right](J_\epsilon v_\epsilon)\right\}\|_{H^m}$

$$\leq c\|(\nabla J_\epsilon v_\epsilon)\cdot J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^m}$$

  By proposition 4.1.4

$$\leq c\left\{\|\nabla J_\epsilon v_\epsilon\|_{H^m}\|J_\epsilon(u_\epsilon - v_\epsilon)\|_{L^\infty} + \|\nabla J_\epsilon v_\epsilon\|_{L^\infty}\|J_\epsilon(u_\epsilon - v_\epsilon)\|_{H^m}\right\}$$

  By proposition 4.1.13

$$\leq c\left\{\frac{c}{\epsilon}\|v_\epsilon\|_{H^m}\frac{c}{\epsilon^{\frac{3}{2}}}\|u_\epsilon - v_\epsilon\|_{L^2} + \frac{c}{\epsilon^{\frac{5}{2}}}\|v_\epsilon\|_{L^2}c\|u_\epsilon - v_\epsilon\|_{H^m}\right\}$$
$$\leq \frac{c}{\epsilon^{\frac{5}{2}}}\|v_\epsilon\|_{H^m}\|u_\epsilon - v_\epsilon\|_{H^m}$$

Consequently

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon)\|_{H^m} \leq \frac{c}{\epsilon^{\frac{5}{2}}}\left(\|u_\epsilon\|_{H^m} + \|v_\epsilon\|_{H^m}\right)\|u_\epsilon - v_\epsilon\|_{H^m}$$

Assume that $O^M = \{u \in V^m : \|u\|_{H^m} \leq M\}$ then we conclude that

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon)\|_{H^m} \leq \tilde{c}_M\|u_\epsilon - v_\epsilon\|_{H^m}$$

Thus F_ϵ is locally Lipschitz.
By Picard-Lindelof theorem there exist a unique solution $u_\epsilon \in C^1\left([0,T), V^m\right)$

2. For the second part of the theorem we will follow the well known steps for the energy methods.
   We begin with the equation of (IVP) and we multiply with the solution u_ϵ in L^2:
$$\int_{\mathbb{R}^3} \left(\frac{d}{dt} u_\epsilon - F_\epsilon(u_\epsilon) \right) \cdot u_\epsilon dx = 0$$
$$\int_{\mathbb{R}^3} \frac{1}{2} \frac{d}{dt} |u_\epsilon|^2 dx = \int_{\mathbb{R}^3} u_\epsilon \cdot F_\epsilon(u_\epsilon) dx$$
   Thus
$$\frac{1}{2} \frac{d}{dt} \|u_\epsilon\|_{L^2}^2 = \int_{\mathbb{R}^3} u_\epsilon \cdot F_\epsilon(u_\epsilon) dx$$
   Now we will process with the integral on the right part :
$$\int_{\mathbb{R}^3} u_\epsilon \cdot F_\epsilon(u_\epsilon) dx = -\int_{\mathbb{R}^3} u_\epsilon \cdot P J_\epsilon \left(J_\epsilon u_\epsilon \cdot \nabla J_\epsilon u_\epsilon \right) dx$$
   By proposition 4.1.9 we have that the mollifiers are symmetric so
$$= -\int_{\mathbb{R}^3} J_\epsilon u_\epsilon \cdot P \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] dx$$
$$\leq -\int_{\mathbb{R}^3} J_\epsilon u_\epsilon \cdot \left[J_\epsilon u_\epsilon \cdot \nabla \right] J_\epsilon u_\epsilon = 0$$
   Consequently
$$\frac{d}{dt} \|u_\epsilon\|_{L^2}^2 \leq 0$$
   So by simplified Gronwall we have that
$$\|u_\epsilon\|_{L^2}^2 \leq \|u_0\|_{L^2}^2$$
   Eventually taking the supremum we conclude that:
$$\sup_{0 \leq t \leq T} \|u_\epsilon\|_{L^2} \leq \|u_0\|_{L^2}$$

□

To conclude we can extend this solution since we have that u_ϵ satisfies the following inequality
$$\|F_\epsilon\|_{u_\epsilon}\|_{H^m} \leq \tilde{c}_M \|u_\epsilon\|_{H^m}$$
So since it is a solution of the (IVP) we reach to
$$\frac{d}{dt} \|u_\epsilon\|_{H^m} \leq \tilde{c}_M \|u_\epsilon\|_{H^m}$$
By Gronwall's lemma in differential form we conclude that $\|u_\epsilon\|_{H^m} \leq e^{\tilde{c}_M T}$ which is an apriori bound, thus for $T < \infty$ we will not have a blow up so by the theorem 4.2.3 we have a global solution.
Conclusions: The regularized problem for the Euler equation has a smooth global in time solution on space V^m in 3 dimensions.

4.2.2 2 dimensions

In this chapter, we will continue discussing the 2-dimensional case using the finite radial energy decomposition we introduced in the previous chapter. According to lemma 11, we know that every incompressible velocity field u in 2 dimensions, with $\omega \in L^1$, has a finite radial energy decomposition, meaning that u can be expressed as the sum of y and b, where $y \in L^2$ is the term we are interested in, and b is a known quantity. We know that the y satisfy the following equation:

$$\frac{\partial}{\partial t} y + (y \cdot \nabla)y + (b \cdot \nabla)y + (y \cdot \nabla)b = -\nabla p + \nu \Delta y \tag{NSy}$$

We consider the following mollification for the (NSy) assuming that $u_\epsilon = y_\epsilon + b$[24]

$$\frac{\partial}{\partial t} y_\epsilon + J_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon + (b \cdot \nabla) J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla) b\right] = -\nabla p_\epsilon + \nu J_\epsilon^2 (\Delta y_\epsilon) \tag{NSy_ϵ}$$

Using the Leray's formulation we project this equation to the closed space of H^s divergence free functions i.e. $V^s = \{u \in H^s : \text{div} u = 0\}$ and we have that

$$\frac{\partial}{\partial t} y_\epsilon + P\left\{J_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon + (b \cdot \nabla) J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla) b\right]\right\} = \nu J_\epsilon^2 (\Delta y_\epsilon) \tag{$L - NSy_\epsilon$}$$

We set

$$F_\epsilon(y_\epsilon) = -P\left\{J_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon + (b \cdot \nabla) J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla) b\right]\right\} + \nu J_\epsilon^2 (\Delta y_\epsilon)$$

By assuming that the initial value of the mollified y is equal to the initial value of y we reach to an initial value problem

$$\begin{cases} \frac{\partial}{\partial t} y_\epsilon = F_\epsilon(y_\epsilon) \\ y_\epsilon(*, 0) = y_\epsilon(0) = y_0 \end{cases} \tag{IVP}$$

So now we have reach to the point where we seek for solutions for the above (IVP), so we have the following theorem

Theorem 4.2.5. *Let $y_0 \in V^m$,$m \in Z_0^+$ the initial value of the field y_ϵ, then*

1. *$\forall \epsilon > 0$ there exist a unique solution $y_\epsilon \in C^1([0,T), V^m)$ of the (IVP).*
2. *On any time interval $[0,T]$ on which the solution exists in $C^1([0,T), V^m)$ we have that :*

$$\sup_{0 \leq t \leq T} \|y_\epsilon\|_{L^2} \leq \|y_0\|_{L^2} e^{\int_0^T c \|\nabla b\|_{L^\infty} dt}$$

Proof. 1. For the first part of the theorem: using the Picard Lindelof theorem, we only require data for F_ϵ, so will show that $F_\epsilon : V^m \to V^m$ and it is locally Lipschitz.

Assume a $y_\epsilon \in V^m$ we have that

$$\|F_\epsilon(y_\epsilon)\|_{H^m} \leq \|\nu J_\epsilon^2 (\Delta y_\epsilon)\|_{H^m} + \|P\left\{J_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon + (b \cdot \nabla) J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla) b\right]\right\}\|_{H^m}$$

We will estimate each term individually

[24] In the 2d case we solve this equation for y_ϵ, observe that it isn't the same equation as in three dimensions. Also observe that we don't mollify the term b, since this is a known quantity.

- $$\|\nu J_\epsilon^2(\Delta y_\epsilon)\|_{H^m} = \nu\|J_\epsilon^2 u_\epsilon\|_{H^{m+2}}$$

  By the proposition 4.1.13

  $$\le \frac{c\nu}{\epsilon^2}\|J_\epsilon u_\epsilon\|_{H^m}$$
  $$\le \frac{c\nu}{\epsilon^2}\|u_\epsilon\|_{H^m}$$

  Since $y_\epsilon \in H^m$ we have that

  $$\|\nu J_\epsilon^2(\Delta y_\epsilon)\|_{H^m} < \infty$$

- $$\|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon\cdot\nabla)b\right]\right\}\|_{H^m}$$
  $$\le \|J_\epsilon\left[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon\cdot\nabla)b\right]\|_{H^m}$$

  By triangle inequality and proposition 1.1.13 we have that

  $$\le c\left[\|(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon\|_{H^m} + \|(b\cdot\nabla)J_\epsilon y_\epsilon\|_{H^m} + \|(J_\epsilon y_\epsilon\cdot\nabla)b\|_{H^m}\right]$$

  By proposition 4.1.13 we have that

  $$\le c\left[\|y_\epsilon\|_{H^m}^2 + \|y_\epsilon\|_{H^m}\left(\|b\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right)\right]$$

  Thus we have that

  $$\|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon\cdot\nabla)b\right]\right\}\|_{H^m} < \infty$$

So $\|F_\epsilon(y_\epsilon)\|_{H^m} < \infty$, furthermore $\text{div}F_\epsilon(y_\epsilon) = 0$ so we have that $F_\epsilon : V^m \to V^m$
Now we will prove that this function is locally Lipschitz
Assume u_ϵ, v_ϵ with radial energy decomposition $u_\epsilon = y_\epsilon + b$ and $v_\epsilon = \bar{y}_\epsilon + b$ so we have that

$$\|F_\epsilon(y_\epsilon) - F_\epsilon(\bar{y}_\epsilon)\|_{H^m} = \|\nu J_\epsilon^2(\Delta y_\epsilon) - P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon\cdot\nabla)b\right]\right\}$$
$$-\nu J_\epsilon^2(\Delta\bar{y}_\epsilon) + P\left\{J_\epsilon\left[(J_\epsilon\bar{y}_\epsilon\cdot\nabla)J_\epsilon\bar{y}_\epsilon + (b\cdot\nabla)J_\epsilon\bar{y}_\epsilon + (J_\epsilon\bar{y}_\epsilon\cdot\nabla)b\right]\right\}\|_{H^m}$$

By triangle inequality and by add and subtract the term: $P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon)\cdot\nabla\right](J_\epsilon\bar{y}_\epsilon)\right\}$ we have that

$$\le \nu\|J_\epsilon^2\Delta(y_\epsilon - \bar{y}_\epsilon)\|_{H^m} + \|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon)\cdot\nabla\right](J_\epsilon(y_\epsilon - \bar{y}_\epsilon))\right\}\|_{H^m}$$
$$+\|P\left\{J_\epsilon\left[(J_\epsilon(y_\epsilon - \bar{y}_\epsilon))\cdot\nabla\right](J_\epsilon\bar{y}_\epsilon)\right\}\|_{H^m} + \|P\left\{J_\epsilon\left[(b\cdot\nabla)(J_\epsilon(y_\epsilon - \bar{y}_\epsilon))\right]\right\}\|_{H^m}$$
$$+\|P\left\{J_\epsilon\left[(J_\epsilon(y_\epsilon - \bar{y}_\epsilon))\cdot\nabla\right]b\right\}\|_{H^m}$$

We will see each of these 5 terms individually

- $$\nu\|J_\epsilon^2\Delta(y_\epsilon - \bar{y}_\epsilon)\|_{H^m} = \nu\|J_\epsilon^2(y_\epsilon - \bar{y}_\epsilon)\|_{H^{m+2}}$$

  By proposition 4.1.13

  $$\le \frac{c}{\epsilon^2}\|J_\epsilon(y_\epsilon - \bar{y}_\epsilon)\|_{H^m}$$
  $$\le \|y_\epsilon - \bar{y}_\epsilon\|_{H^m}$$

- $$\|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon)\cdot\nabla\right]\left(J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\right)\right\}\|_{H^m}$$

  By proposition 4.1.13 and 4.1.15

  $$\leq c\|\nabla J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\cdot J_\epsilon y_\epsilon\|_{H^m}$$

  By proposition 4.1.4

  $$\leq c\left(\|\nabla J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{L^\infty}\|J_\epsilon y_\epsilon\|_{H^m}+\|\nabla J_\epsilon(y_\epsilon-\bar{y}_\epsilon\|_{H^m})\|J_\epsilon y-\epsilon\|_{L^\infty}\right)$$

  By proposition 4.1.13

  $$\leq c\left(\frac{c}{\epsilon}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}\frac{c}{\epsilon^2}\|y_\epsilon\|_{L^2}+c\|y_\epsilon\|_{H^m}\frac{c}{\epsilon^3}\|y_\epsilon-\bar{y}_\epsilon\|_{L^2}\right)$$

  $$\leq\frac{c^3}{\epsilon^3}\|y_\epsilon\|_{H^m}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}$$

- $$\|P\left\{J_\epsilon\left[(J_\epsilon(y_\epsilon-\bar{y}_\epsilon))\cdot\nabla\right]\left(J_\epsilon\bar{y}_\epsilon\right)\right\}\|_{H^m}$$

  By proposition 4.1.13 and 4.1.15

  $$\leq c\|\nabla J_\epsilon\bar{y}_\epsilon\cdot J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{H^m}$$

  By proposition 4.1.4

  $$\leq c\left(\|\nabla J_\epsilon\bar{y}_\epsilon\|_{L^\infty}\|J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{H^m}+\|J_\epsilon\bar{y}_\epsilon\|_{H^m}\|J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{L^\infty}\right)$$

  By proposition 4.1.13

  $$\leq c\left(\frac{c}{\epsilon^3}\|\bar{y}_\epsilon\|_{L^2}c\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}+\frac{c}{\epsilon}\|\bar{y}_\epsilon\|_{H^m}\frac{c}{\epsilon^2}\|y_\epsilon-\bar{y}_\epsilon\|_{L^2}\right)$$

  $$\leq\frac{c^3}{\epsilon^3}\|\bar{y}_\epsilon\|_{H^m}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}$$

- $$\|P\left\{J_\epsilon\left[(b\cdot\nabla)(J_\epsilon(y_\epsilon-\bar{y}_\epsilon))\right]\right\}\|_{H^m}$$

  By proposition 4.1.13 and 4.1.15

  $$\leq c\|\nabla J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\cdot b\|_{H^m}$$

  Thus

  $$\leq c\|\nabla J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{H^m}\|b\|_{L^\infty}$$

  By proposition 4.1.13

  $$\leq\frac{c}{\epsilon}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}\|b\|_{L^\infty}$$

  $$\leq\frac{c}{\epsilon}\|b\|_{L^\infty}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}$$

- $$\|P\left\{J_\epsilon\left[(J_\epsilon(y_\epsilon-\bar{y_\epsilon}))\cdot\nabla\right]b\right\}\|_{H^m}$$

  By proposition 4.1.13 and 4.1.15

  $$\leq c\|\nabla b\cdot J_\epsilon(y_\epsilon-\bar{y_\epsilon})\|_{H^m}$$

  Thus

  $$\leq c^2\|\nabla b\|_{L^\infty}\|y_\epsilon-\bar{y_\epsilon}\|_{H^m}$$

So in the set $O^M=\{y\in V^m:\|y\|_{H^m}\leq M\}$

$$\|F_\epsilon(y_\epsilon)-F_\epsilon(\bar{y_\epsilon})\|_{H^m}\leq L_{M,b}\|y_\epsilon-\bar{y_\epsilon}\|_{H^m}$$

Thus F_ϵ is local Lipschitz so by Picard Lindelof theorem there exist a unique solution of the (IVP), say $y_\epsilon\in C^1([0,T),V^m)$

2. For the second part of the theorem we have that

$$\int_{\mathbb{R}^2}y_\epsilon\cdot\frac{d}{dt}y_\epsilon dx=\int_{\mathbb{R}^2}y_\epsilon\cdot F_\epsilon(y_\epsilon)dx$$

$$\frac{1}{2}\frac{d}{dt}\|y_\epsilon\|_{L^2}^2=\int_{\mathbb{R}^2}y_\epsilon\cdot F_\epsilon(y_\epsilon)dx$$

For the integral on the right side we have that

$$\int_{\mathbb{R}^2}y_\epsilon\cdot F_\epsilon(y_\epsilon)dx=\int_{\mathbb{R}^2}y_\epsilon\cdot\left\{-P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon+(b\cdot\nabla)J_\epsilon y_\epsilon+(J_\epsilon y_\epsilon\cdot\nabla)b\right]\right\}+\nu J_\epsilon^2(\Delta y_\epsilon)\right\}$$

$$=\nu\int_{\mathbb{R}^2}J_\epsilon y_\epsilon\cdot\Delta J_\epsilon y_\epsilon dx+\frac{1}{2}\int_{\mathbb{R}^2}\left[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon\right]J_\epsilon y_\epsilon dx$$

$$+\frac{1}{2}\int_{\mathbb{R}^2}\left[(b\cdot\nabla)J_\epsilon y_\epsilon\right]J_\epsilon y_\epsilon dx+\frac{1}{2}\int_{\mathbb{R}^2}\left[(J_\epsilon y_\epsilon\cdot\nabla)b\right]J_\epsilon y_\epsilon dx$$

So

$$\leq-\nu\int_{\mathbb{R}^2}\nabla J_\epsilon y_\epsilon\cdot\nabla J_\epsilon y_\epsilon dx+\frac{1}{2}\int_{\mathbb{R}^2}\left[(b\cdot\nabla)J_\epsilon y_\epsilon\right]J_\epsilon y_\epsilon dx+\frac{1}{2}\int_{\mathbb{R}^2}\left[(J_\epsilon y_\epsilon\cdot\nabla)b\right]J_\epsilon y_\epsilon dx$$

[25]

$$\leq\frac{1}{4}\|\nabla b\|_{L^\infty}\|y_\epsilon\|_{L^2}^2$$

Thus we conclude to the relation

$$\frac{d}{dt}\|y_\epsilon\|_{L^2}\leq C\|\nabla b\|_{L^\infty}\|y_\epsilon\|_{L^2}$$

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[25] Since $-\nu\int_{\mathbb{R}^2}\nabla J_\epsilon y_\epsilon\cdot\nabla J_\epsilon y_\epsilon dx=-\nu\|\nabla J_\epsilon y_\epsilon\|_{L^2}^2<0$
Also

$$\frac{1}{2}\int_{\mathbb{R}^2}\left[(b\cdot\nabla)J_\epsilon y_\epsilon\right]J_\epsilon y_\epsilon dx=-\frac{1}{2}\int_{\mathbb{R}^2}J_\epsilon y_\epsilon\cdot\nabla(b\cdot J_\epsilon y_\epsilon)dx=-\frac{1}{2}\int_{\mathbb{R}^2}\left[(b\cdot\nabla)J_\epsilon y_\epsilon\right]J_\epsilon y_\epsilon dx-\frac{1}{2}\int_{\mathbb{R}^2}\left[(J_\epsilon y_\epsilon\cdot\nabla)b\right]J_\epsilon y_\epsilon dx\Rightarrow$$

$$\int_{\mathbb{R}^2}\left[(b\cdot\nabla)J_\epsilon y_\epsilon\right]J_\epsilon y_\epsilon dx=-\frac{1}{2}\int_{\mathbb{R}^2}\left[(J_\epsilon y_\epsilon\cdot\nabla)b\right]J_\epsilon y_\epsilon dx$$

By Gronwall's lemma in the differential form we have that

$$\|y_\epsilon\|_{L^2} \leq \|y_\epsilon|_{t=0}\|_{L^2} e^{\int_0^t C\|\nabla b\|_{L^\infty}}$$

Taking the supremum over this relation, together with the fact that the integral of a positive quantity as a function is an increasing one we conclude that

$$\sup_{0\leq t\leq T} \|y_\epsilon\|_{L^2} \leq \|y_0\|_{L^2} e^{\int_0^T c\|\nabla b\|_{L^\infty} dt}$$

□

Remark: In the previous chapter, the basic energy estimate in two dimensions is the same as the energy estimate mentioned above. Therefore, we have once again achieved an optimal energy estimate. It is worth noting that our bound in two dimensions has a time dependence, but in three dimensions, this is not the case.
So far we have find a local solution for the regularized problem we cam also extend this solution on 2 dimension since by the fact that F_ϵ is Lipschitz we have the following inequality

$$\|F_\epsilon(y_\epsilon)\|_{H^m} \leq L_{M,b}\|y_\epsilon\|_{H^m}$$

Also y_ϵ satisfies the IVP so we have

$$\frac{d}{dt}\|y_\epsilon\|_H^m \leq L_{M,b}\|y_\epsilon\|_{H^m}$$

So by Gronwall lemma for $T < \infty$ we find an apriori bound $e^{L_{M,b}T}$ so the case of blow up is not true and thus $T = \infty$
Conclusions:The regularized problem for the Navier Stokes equation has a smooth global in time solution on V^m in 2 dimensions. Why?

We have find a solution for the $L - NSy_\epsilon$ say y_ϵ we know that$y_\epsilon = u_\epsilon - b$ So we substitute to the $L - NSy_\epsilon$ the $u_\epsilon - b$ and we have that

$$\frac{\partial}{\partial t}u_\epsilon - \frac{\partial}{\partial t}b + PJ_\epsilon\Bigg[(J_\epsilon u_\epsilon \cdot \nabla)J_\epsilon u_\epsilon$$

$$-(J_\epsilon u_\epsilon \cdot \nabla)b - (b \cdot \nabla)J_\epsilon u_\epsilon + (b \cdot \nabla)b$$

$$+(b \cdot \nabla)J_\epsilon u_\epsilon - (b \cdot \nabla)b + (J_\epsilon u_\epsilon \cdot \nabla)b - (b \cdot \nabla)b\Bigg] = \nu J_\epsilon^2 \Delta u_\epsilon - \nu\Delta b$$

We know that b is an exact solution of the Navier Stokes equation so we conclude that

$$\frac{\partial}{\partial t}u_\epsilon + PJ_\epsilon\Big[(J_\epsilon u_\epsilon \cdot \nabla)J_\epsilon u_\epsilon\Big] = \nu J_\epsilon^2 \Delta u_\epsilon$$

This is the regularized Navier Stokes in two dimensions so we see that u_ϵ satisfies this equation, so its a solution.

Remark: We have exactly the same results for the Euler equation indeed:

$$\frac{\partial}{\partial t}y + (y \cdot \nabla)y + (b \cdot \nabla)y + (y \cdot \nabla)b = -\nabla p \tag{Ey}$$

We consider the following mollification for the (Eu) assuming that $u_\epsilon = y_\epsilon + b$

$$\frac{\partial}{\partial t}y_\epsilon + J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b \cdot \nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right] = -\nabla p_\epsilon \tag{Ey_ϵ}$$

Using the Leray's formulation we project this equation to the closed space of H^s divergence free functions i.e. $V^s = \{u \in H^s : \text{div}u = 0\}$ and we have that

$$\frac{\partial}{\partial t}y_\epsilon + P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b \cdot \nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]\right\} = 0 \tag{$L - Ey_\epsilon$}$$

We set

$$F_\epsilon(y_\epsilon) = -P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b \cdot \nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]\right\}$$

By assuming that the initial value of the mollified y is equal to the initial value of y we reach to an initial value problem

$$\begin{cases} \frac{\partial}{\partial t}y_\epsilon = F_\epsilon(y_\epsilon) \\ y_\epsilon(*, 0) = y_\epsilon(0) = y_0 \end{cases} \tag{IVP}$$

So now we have reach to the point where we seek for solutions for the above (IVP), so we have the following theorem Let $y_0 \in V^m$,$m \in Z_0^+$ the initial value of the field y_ϵ, then

1. $\forall \epsilon > 0$ there exist a unique solution $y_\epsilon \in C^1\left([0, T), V^m\right)$ of the (IVP).
2. On any time interval $[0, T]$ on which the solution exists in $C^1\left([0, T), V^m\right)$ we have that :
$$\sup_{0 \le t \le T} \|y_\epsilon\|_{L^2} \le \|y_0\|_{L^2} e^{\int_0^T c\|\nabla b\|_{L^\infty} dt}$$

Proof. 1. For the first part of the theorem we will prove that $F_\epsilon : V^m \to V^m$ and also it is locally Lipschitz
Assume a $y_\epsilon \in V^m$ we have that

$$\|F_\epsilon(y_\epsilon)\|_{H^m} \le \|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b \cdot \nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]\right\}\|_{H^m}$$

We can easily see that

2.
$$\begin{aligned} &\|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b \cdot \nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]\right\}\|_{H^m} \\ &\le \|J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b \cdot \nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right] \end{aligned}$$

By triangle inequality and proposition 4.1.13 we have that

$$\le c\left[\|(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\|_{H^m} + \|(b \cdot \nabla)J_\epsilon y_\epsilon\|_{H^m} + \|(J_\epsilon y_\epsilon \cdot \nabla)b\|_{H^m}\right]$$

By proposition 4.1.4

$$\leq c\|\nabla J_\epsilon y_\epsilon\|_{L^\infty}\|J_\epsilon y_\epsilon\|_{H^m} + c\|\nabla J_\epsilon y_\epsilon\|_{H^m}\|J_\epsilon y_\epsilon\|_{L^\infty}$$

$$+c\|\nabla J_\epsilon y_\epsilon\|_{H^m}\|b\|_{L^\infty} + c\|\nabla b\|_{L^\infty}\|J_\epsilon y_\epsilon\|_{H^m}$$

By proposition 4.1.13 we have that

$$\leq c\left[\|y_\epsilon\|_{H^m}^2 + \|y_\epsilon\|_{H^m}\left(\|b\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right)\right]$$

Thus we have that

$$\|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon\cdot\nabla)b\right]\right\}\|_{H^m} < \infty$$

So $\|F_\epsilon(y_\epsilon)\|_{H^m} < \infty$, furthermore $\text{div}F_\epsilon(y_\epsilon) = 0$ so we have that $F_\epsilon : V^m \to V^m$
Now we will prove that this function is locally Lipschitz
Assume u_ϵ, v_ϵ with radial energy decomposition $u_\epsilon = y_\epsilon + b$ and $v_\epsilon = \bar{y_\epsilon} + b$ so we have that

$$\|F_\epsilon(y_\epsilon) - F_\epsilon(\bar{y_\epsilon})\|_{H^m} = \| - P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon\cdot\nabla)b\right]\right\}$$

$$+P\left\{J_\epsilon\left[(J_\epsilon\bar{y_\epsilon}\cdot\nabla)J_\epsilon\bar{y_\epsilon} + (b\cdot\nabla)J_\epsilon\bar{y_\epsilon} + (J_\epsilon\bar{y_\epsilon}\cdot\nabla)b\right]\right\}\|_{H^m}$$

By triangle inequality and by add and subtract the term: $P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon)\cdot\nabla\right](J_\epsilon\bar{y_\epsilon})\right\}$ we have that

$$\leq \|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon)\cdot\nabla\right](J_\epsilon(y_\epsilon - \bar{y_\epsilon}))\right\}\|_{H^m} + \|P\left\{J_\epsilon\left[(J_\epsilon(y_\epsilon - \bar{y_\epsilon}))\cdot\nabla\right](J_\epsilon\bar{y_\epsilon})\right\}\|_{H^m}$$

$$+\|P\left\{J_\epsilon\left[(b\cdot\nabla)(J_\epsilon(y_\epsilon - \bar{y_\epsilon}))\right]\right\}\|_{H^m} + \|P\left\{J_\epsilon\left[(J_\epsilon(y_\epsilon - \bar{y_\epsilon}))\cdot\nabla\right]b\right\}\|_{H^m}$$

We will see each of these terms individually

- $$\|P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon)\cdot\nabla\right](J_\epsilon(y_\epsilon - \bar{y_\epsilon}))\right\}\|_{H^m}$$

  By proposition 4.1.13 and 4.1.15

  $$\leq c\|\nabla J_\epsilon(y_\epsilon - \bar{y_\epsilon})\cdot J_\epsilon y_\epsilon\|_{H^m}$$

  By proposition 4.1.4

  $$\leq c\left(\|\nabla J_\epsilon(y_\epsilon - \bar{y_\epsilon})\|_{L^\infty}\|J_\epsilon y_\epsilon\|_{H^m} + \|\nabla J_\epsilon(y_\epsilon - \bar{y_\epsilon}\|_{H^m})\|J_\epsilon y - \epsilon\|_{L^\infty}\right)$$

  By proposition 4.1.13

  $$\leq c\left(\frac{c}{\epsilon}\|y_\epsilon - \bar{y_\epsilon}\|_{H^m}\frac{c}{\epsilon^2}\|y_\epsilon\|_{L^2} + c\|y_\epsilon\|_{H^m}\frac{c}{\epsilon^3}\|y_\epsilon - \bar{y_\epsilon}\|_{L^2}\right)$$

  $$\leq \frac{c^3}{\epsilon^3}\|y_\epsilon\|_{H^m}\|y_\epsilon - \bar{y_\epsilon}\|_{H^m}$$

- $$\|P\left\{J_\epsilon\left[(J_\epsilon(y_\epsilon-\bar{y}_\epsilon))\cdot\nabla\right](J_\epsilon\bar{y}_\epsilon)\right\}\|_{H^m}$$

  By proposition 4.1.13 and 4.1.15

  $$\le c\|\nabla J_\epsilon\bar{y}_\epsilon\cdot J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{H^m}$$

  By proposition 4.1.4

  $$\le c\left(\|\nabla J_\epsilon\bar{y}_\epsilon\|_{L^\infty}\|J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{H^m}+\|J_\epsilon\bar{y}_\epsilon\|_{H^m}\|J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{L^\infty}\right)$$

  By proposition 4.1.13

  $$\le c\left(\frac{c}{\epsilon^3}\|\bar{y}_\epsilon\|_{L^2}c\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}+\frac{c}{\epsilon}\|\bar{y}_\epsilon\|_{H^m}\frac{c}{\epsilon^2}\|y_\epsilon-\bar{y}_\epsilon\|_{L^2}\right)$$

  $$\le\frac{c^3}{\epsilon^3}\|\bar{y}_\epsilon\|_{H^m}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}$$

- $$\|P\left\{J_\epsilon\left[(b\cdot\nabla)(J_\epsilon(y_\epsilon-\bar{y}_\epsilon))\right]\right\}\|_{H^m}$$

  By proposition 4.1.13 and 4.1.15

  $$\le c\|\nabla J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\cdot b\|_{H^m}$$

  $$\le c\|\nabla J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|_{H^m}\|b\|_{L^\infty}$$

  $$\le c\left(\frac{c}{\epsilon}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}\|b\|_{L^\infty}\right)$$

  $$\le\frac{c^2}{\epsilon}\|b\|_{L^\infty}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}$$

- $$\|P\left\{J_\epsilon\left[(J_\epsilon(y_\epsilon-\bar{y}_\epsilon))\cdot\nabla\right]b\right\}\|_{H^m}$$

  By proposition 4.1.13 and 4.1.15

  $$\le c\|\nabla b\cdot J_\epsilon(y_\epsilon-\bar{y}_\epsilon)\|$$

  By proposition 4.1.13

  $$\le c^2\|\nabla b\|_{L^\infty}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}$$

So in the set $O^M=\{y\in V^m:\|y\|_{H^m}\le M\}$

$$\|F_\epsilon(y_\epsilon)-F_\epsilon(\bar{y}_\epsilon)\|_{H^m}\le L^*_{M,b}\|y_\epsilon-\bar{y}_\epsilon\|_{H^m}$$

Thus F_ϵ is locally Lipschitz so by Picard Lindelof theorem there exist a unique solution of the (IVP), say $y_\epsilon\in C^1([0,T),V^m)$

3. For the second part of the theorem we have that

$$\int_{\mathbb{R}^2} y_\epsilon \cdot \frac{d}{dt} y_\epsilon dx = \int_{\mathbb{R}^2} y_\epsilon \cdot F_\epsilon(y_\epsilon) dx$$

$$\frac{1}{2}\frac{d}{dt}\|y_\epsilon\|_{L^2}^2 = \int_{\mathbb{R}^2} y_\epsilon \cdot F_\epsilon(y_\epsilon) dx$$

For the integral on the right side we have that

$$\int_{\mathbb{R}^2} y_\epsilon \cdot F_\epsilon(y_\epsilon) dx = -\int_{\mathbb{R}^2} y_\epsilon \cdot P\left\{J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon + (b\cdot\nabla) J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla) b\right]\right\}$$

$$\frac{1}{2}\int_{\mathbb{R}^2} \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right] J_\epsilon y_\epsilon dx$$

$$\frac{1}{2}\int_{\mathbb{R}^2} \left[(b\cdot\nabla) J_\epsilon y_\epsilon\right] J_\epsilon y_\epsilon dx + \frac{1}{2}\int_{\mathbb{R}^2} \left[(J_\epsilon y_\epsilon \cdot) b\right] J_\epsilon y_\epsilon dx$$

So by footnote 26

$$\leq \frac{1}{4}\int_{\mathbb{R}^2} \left[(J_\epsilon y_\epsilon \cdot) b\right] J_\epsilon y_\epsilon dx$$

$$\leq C\|\nabla b\|_{L^\infty}\|y_\epsilon\|_{L^2}^2$$

Thus we conclude to the relation

$$\frac{d}{dt}\|y_\epsilon\|_{L^2} \leq C\|\nabla b\|_{L^\infty}\|y_\epsilon\|_{L^2}$$

By Gronwall's lemma in the differential form we have that

$$\|y_\epsilon\|_{L^2} \leq \|y_\epsilon|_{t=0}\|_{L^2} e^{\int_0^t C\|\nabla b\|_{L^\infty}}$$

Taking the supremum over this relation, together with the fact that the integral of a positive quantity as a function is an increasing one we conclude that

$$\sup_{0\leq t\leq T} \|y_\epsilon\|_{L^2} \leq \|y_0\|_{L^2} e^{\int_0^T c\|\nabla b\|_{L^\infty} dt}$$

□

So far we have find a local solution for the regularized problem we cam also extend this solution on 2 dimension since by the fact that F_ϵ is Lipschitz we have the following inequality

$$\|F_\epsilon(y_\epsilon)\|_{H^m} \leq L_{M,b}^*\|y_\epsilon\|_{H^m}$$

Also y_ϵ satisfies the IVP so we have

$$\frac{d}{dt}\|u_\epsilon\|_H^m \leq L_{M,b}^*\|y_\epsilon\|_{H^m}$$

So by Gronwall lemma for $T < \infty$ we find an apriori bound $e^{L_{M,b}^* T}$ so the case of blow up is not true and thus $T = \infty$

Conclusions:The regularized problem for the Euler equation has a smooth global in time solution on V^m in 2 dimensions.

We have find a solution for the $L - Ey_\epsilon$ say y_ϵ we know that$y_\epsilon = u_\epsilon - b$ So we substitute to the $L - Ey_\epsilon$ the $u_\epsilon - b$ and we have that

$$\frac{\partial}{\partial t}u_\epsilon - \frac{\partial}{\partial t}b + PJ_\epsilon\Bigg[(J_\epsilon u_\epsilon \cdot \nabla)J_\epsilon u_\epsilon$$

$$-(J_\epsilon u_\epsilon \cdot \nabla)b - (b \cdot \nabla)J_\epsilon u_\epsilon + (b \cdot \nabla)b$$

$$+(b \cdot \nabla)J_\epsilon u_\epsilon - (b \cdot \nabla)b + (J_\epsilon u_\epsilon \cdot \nabla)b - (b \cdot \nabla)b\Bigg] = 0$$

We know that b is an exact solution of the Euler equation so we conclude that

$$\frac{\partial}{\partial t}u_\epsilon + PJ_\epsilon\Big[(J_\epsilon u_\epsilon \cdot \nabla)J_\epsilon u_\epsilon\Big] = 0$$

This is the regularized Euler in two dimensions so we see that u_ϵ satisfies this equation, so its a solution.

4.3 Existence of smooth solutions as the limit of the regularized solutions

The primary concern in this section is whether and how we can solve the Navier-Stokes and Euler equations through the regularized ones. We need to determine the circumstances under which this is possible and the time interval required. We will examine two cases: one for three dimensions and one for two dimensions, as we did before.

4.3.1 3 dimensions

The following theorem summarizes the main result which answers all the previous questions, for its proof we will need several steps which we will discuss below

Theorem 4.3.1. *Assume that* $u_0 \in V^m$ *with* $m \geq \left[\frac{3}{2}\right] + 2 = 3$[26] *then*

1. *There exists a time* T *with upper bound which depends on the initial value i.e.*

$$T \leq \frac{1}{c_m \|u_0\|_{H^m}}$$

such that $\forall \nu \geq 0$ *there exists a unique solution* $u_v \in C\left([0,T], C^2(\mathbb{R}^3)\right) \cap C^1\left([0,T], C(\mathbb{R}^3)\right)$ *for the Euler and Navier-Stokes equations in the Leray's form.*

2. *The solutions* u_v *and* u_ϵ *satisfies the following estimates*

$$\sup_{0 \leq t \leq T} \|u_\epsilon\|_{H^m} \leq \frac{\|u_0\|_{H^m}}{1 - c_m T \|u_0\|_{H^m}}$$

[26]In the case of the Euler equation we can take $m \geq \left[\frac{N}{2}\right] + 1 = 2$

$$\sup_{0\le t\le T}\|u_v\|_{H^m}\le\frac{\|u_0\|_{H^m}}{1-c_mT\|u_0\|_{H^m}}$$

3. The solutionsu$_\epsilon$, u_v is uniformly bounded on the spaces

- $L^2\left([0,T],V^m(\mathbb{R}^3)\right)$
- $L^\infty\left([0,T],V^m(\mathbb{R}^3)\right)$
- $Lip\left([0,T],V^{m-2}(\mathbb{R}^3)\right)$
- $C_W\left([0,T],V^m(\mathbb{R}^3)\right)$

For the proof we will do the following steps, each of these steps needs several results to get proved:
Step 1: We will show the energy estimate $\sup_{0\le t\le T}\|u_\epsilon\|_{H^m}\le\frac{\|u_0\|_{H^m}}{1-c_mT\|u_0\|_{H^m}}$
Step 2:We will prove that the family of the regularizes solutions forms a Cauchy sequence on $C\left([0,T],L^2(\mathbb{R}^3)\right)$
Step 3: We will prove strong convergence in all the intermediate norms of the high norm of the space $C\left([0,T],H^m(\mathbb{R}^3)\right)$
Step 4:Via weak convergences we will prove the third part of the theorem

We also note that in the previous chapters we have an estimate which its constant depends on ϵ in the denominator and M this is not so useful here since we will send ϵ to zero in order to find the limit solution. So we will prove the following lemma which give us an estimate with no bad dependence on ϵ

Lemma 18. *Assume that $u_0\in V^m$ then for the solution u_ϵ of the (IVP) it is true that*

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|^2_{H^m}+\nu\|J_\epsilon\nabla u_\epsilon\|^2_{H^m}\le c_m\|\nabla J_\epsilon u_\epsilon\|_{L^\infty}\|u_\epsilon\|^2_{H^m}$$

Proof. Since u_ϵ is a solution of the (IVP) we have

$$\frac{d}{dt}u_\epsilon=\nu J_\epsilon^2\Delta u_\epsilon-PJ_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right]$$

We observe that the estimate we want to achieve is about Sobolev norms so we differentiate for any multiindex a the above relation and we have that

$$D^a\left(\frac{d}{dt}u_\epsilon\right)=\nu D^a\left(vJ_\epsilon^2\Delta u_\epsilon\right)-D^a\left(PJ_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right]\right)$$

We will do energy methods and we will multiply this relation with the $D^a u_\epsilon$ in L^2
Then

$$\int_{\mathbb{R}^3}D^a\left(\frac{d}{dt}u_\epsilon\right)\cdot D^au_\epsilon dx=\nu\int_{\mathbb{R}^3}D^a\left(J_\epsilon^2\Delta u_\epsilon\right)\cdot D^au_\epsilon dx-\int_{\mathbb{R}^3}D^a\left\{PJ_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right]\right\}\cdot D^au_\epsilon dx$$

Thus by proposition 4.1.8 and 4.1.15(6)

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|D^au_\epsilon|^2dx=\nu\int_{\mathbb{R}^3}\Delta(D^aJ_\epsilon u_\epsilon)\cdot D^aJ_\epsilon u_\epsilon-\int_{\mathbb{R}^3}D^a\left\{PJ_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right]\right\}\cdot D^au_\epsilon dx$$

So

$$\frac{1}{2}\frac{d}{dt}\|D^a u_\epsilon\|_{L^2}^2 = -\nu \int_{\mathbb{R}^3} D^a J_\epsilon u_\epsilon \cdot D^a J_\epsilon u_\epsilon dx - \int_{\mathbb{R}^3} D^a \left\{ PJ_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] \right\} \cdot D^a u_\epsilon dx$$

Consequently

$$\frac{1}{2}\frac{d}{dt}\|D^a u_\epsilon\|_{L^2}^2 + \nu \|D^a J_\epsilon \nabla u_\epsilon\|_{L^2}^2 = -\int_{\mathbb{R}^3} D^a \left\{ PJ_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] \right\} \cdot D^a u_\epsilon dx$$

Now we will deal with the integral on the right side of the above relation, we see that by ass and subtract the term $\int_{\mathbb{R}^3} PJ_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) D^a J_\epsilon u_\epsilon \right] D^a u_\epsilon dx$

$$-\int_{\mathbb{R}^3} D^a \left\{ PJ_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] \right\} \cdot D^a u_\epsilon dx$$

$$= -\int_{\mathbb{R}^3} PJ_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) D^a J_\epsilon u_\epsilon \right] D^a u_\epsilon dx$$

$$-\int_{\mathbb{R}^3} \left[D^a \left\{ PJ_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] \right\} - PJ_\epsilon \left[(J_\epsilon u_\epsilon \nabla \cdot) D^a J_\epsilon u_\epsilon \right] \right] \cdot D^a u_\epsilon$$

We have that $\int_{\mathbb{R}^3} PJ_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) D^a J_\epsilon u_\epsilon \right] D^a u_\epsilon dx = \frac{1}{2} \int_{\mathbb{R}^3} (J_\epsilon u_\epsilon) \cdot \left(\nabla \left(J_\epsilon D^a J_\epsilon u_\epsilon \right)^2 \right) dx$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} \operatorname{div} J_\epsilon u_\epsilon |J_\epsilon D^a u_\epsilon|^2 dx = 0$$

Also by Holders inequality, proposition 4.1.8 and 4.1.15 it follows that

$$-\int_{\mathbb{R}^3} D^a \left\{ PJ_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] \right\} \cdot D^a u_\epsilon dx$$

$$\leq \|D^a u_\epsilon\|_{L^2} \|PJ_\epsilon \left\{ D^a \left(\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon \right) - D^a (\nabla J_\epsilon u_\epsilon) \cdot J_\epsilon u_\epsilon \right\} \|_{L^2}$$

By proposition 4.1.13

$$\leq c_m \|D^a u_\epsilon\|_{L^2} \|D^a \left(\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon \right) - D^a (\nabla J_\epsilon u_\epsilon) \cdot J_\epsilon u_\epsilon\|_{L^2}$$

We reach to the relation

$$\frac{1}{2}\frac{d}{dt}\|D^a u_\epsilon\|_{L^2}^2 + \nu \|D^a J_\epsilon \nabla u_\epsilon\|_{L^2}^2 \leq c_m \|u_\epsilon\|_{H^a} \|D^a \left(\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon \right) - D^a (\nabla J_\epsilon u_\epsilon) \cdot J_\epsilon u_\epsilon\|_{L^2}$$

Since $a \leq m$ we get

$$\frac{1}{2}\frac{d}{dt}\|D^a u_\epsilon\|_{L^2}^2 + \nu \|D^a J_\epsilon \nabla u_\epsilon\|_{L^2}^2 \leq c_m \|u_\epsilon\|_{H^m} \|D^a \left(\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon \right) - D^a (\nabla J_\epsilon u_\epsilon) \cdot J_\epsilon u_\epsilon\|_{L^2}$$

We take the sum over this relation,

$$\sum_{|a| \leq m} \frac{1}{2}\frac{d}{dt}\|D^a u_\epsilon\|_{L^2}^2 + \sum_{|a| \leq m} \nu \|D^a J_\epsilon \nabla u_\epsilon\|_{L^2}^2$$

$$\leq c_m \|u_\epsilon\|_{H^m} \sum_{|a| \leq m} \|D^a \left(\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon \right) - D^a (\nabla J_\epsilon u_\epsilon) \cdot J_\epsilon u_\epsilon\|_{L^2}$$

We use the proposition 4.1.4(2) and we have that

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|^2_{H^m} + \nu\|J_\epsilon\nabla u_\epsilon\|^2_{H^m} \leq c_m\|u_\epsilon\|_{H^m}2c_m\|\nabla J_\epsilon u_\epsilon\|_{L^\infty}\|J_\epsilon u_\epsilon\|_{H^m}$$

So we conclude that

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|^2_{H^m} + \nu\|J_\epsilon\nabla u_\epsilon\|^2_{H^m} \leq C_m\|\nabla J_\epsilon u_\epsilon\|_{L^\infty}\|u_\epsilon\|^2_{H^m}$$

□

Proof. Theorem 4.3.1
Step 1:By the above lemma we have that

$$\frac{d}{dt}\|u_\epsilon\|_{H^m} \leq C_m\|\nabla J_\epsilon u_\epsilon\|_{L^\infty}\|u_\epsilon\|_{H^m} \tag{PrS1}$$

For the norm $\|J_\epsilon u_\epsilon\|_{L^\infty}$ we will use the Sobolev embedding theorem 4.1.2 we are on the case where $s \geq \frac{N}{2} + 2$ so our k=2 so by the theorem $\sup_{|a|\leq 2}\|D^a u\|_{L^\infty} \leq c\|u\|_{H^s}$, thus we have the following estimate

$$\|\nabla J_\epsilon u_\epsilon\|_{L^\infty} \leq \sup_{|a|\leq 2}\|D^a J_\epsilon u_\epsilon|_{L^\infty} \leq \|J_\epsilon u_\epsilon\|_{H^m}$$

Now we can use proposition 4.1.13 and we have that $\|J_\epsilon u_\epsilon\|_{H^m} \leq \frac{c}{\epsilon^0}\|u_\epsilon\|_{H^m}$
Eventually put all this together with the relation (PrS1) we have that

$$\frac{d}{dt}\|u_\epsilon\|_{H^m} \leq C\|u_\epsilon\|^2_{H^m}$$

Note:The constant C depends only by m
So now we will find the solution which satisfies this inequality

Assume that b(t) solves the differential equation

$$\frac{d}{dt}b(t) = cb^2(t)$$

with initial value $b(0) = \|u_0\|_{H^m}$ Firstly we want to find a relation between our function and b(t)
Then by combine those two relations we have that

$$\frac{d}{dt}\|u_\epsilon\|_{H^m} - \frac{d}{dt}b(t) \leq c\|u_\epsilon\|^2_{H^m} - cb^2(t)$$

I.e.

$$\frac{d}{dt}\left(\|u_\epsilon\|_{H^m} - b(t)\right) \leq c\left(\|u_\epsilon\|^2_{H^m} - b^2(t)\right)$$

We set $y(t) = \|u_\epsilon\|_{H^m} - b(t)$ thus we have that

$$\frac{d}{dt}y(t) \leq c\left(\|u_\epsilon\|_{H^m} + b(t)\right)y(t)$$

We also set $q(t) = (\|u_\epsilon\|_{H^m} + b(t))$
So we have the inequality

$$\frac{d}{dt}y(y) \leq cq(t)y(t)$$

By Gronwall's inequality in differential form we have that

$$y(t) \leq y(0)e^{\int_0^t cq(s)ds}$$

where $y(0) = \|u_0\|_{H^m} - \|u_0\|_{H^m} = 0$
Consequently $\|u_\epsilon(t)\|_{H^m} \leq b(t)$.
Now we will find this b by solving the initial value problem, we are sure that we have a solution since we have a first order ivp.

$$\begin{cases} \frac{d}{dt}b(t) = cb^2(t) \\ b(0) = \|u_0\|_{H^m} \end{cases}$$

For convenience we interpret the differentiation with respect to t with $'$
So $b'(t) = cb^2(t)$ so for $b(t) \neq 0$ we have that

$$\int_0^t \frac{b'(s)}{b^2(s)} = \int_0^t cds$$

$$\frac{-1}{b(t)} + \frac{1}{b(0)} = ct$$

$$\frac{1}{b(t)} = \frac{1 - b(0)ct}{b(0)}$$

Eventually $b(t) = \frac{b(0)}{1-b(0)ct} = \frac{\|u_0\|_{H^m}}{1-ct\|u_0\|_{H^m}}$

So we have the result i.e.

$$\|u_\epsilon\|_{H^m} \leq \frac{\|u_0\|_{H^m}}{1 - c_m t\|u_0\|_{H^m}}$$

Taking the supremum over this relation we have that

$$\sup_{0\leq t\leq T} \|u_\epsilon\|_{H^m} \leq \sup_{0\leq t\leq T} \frac{\|u_0\|_{H^m}}{1 - c_m t\|u_0\|_{H^m}}$$

For the supremum of the right quantity: we have that as t takes its supremum the denominator gets smaller so the fraction gets bigger, so the supremum of this quotient is achieved when takes it supremum so

$$\sup_{0\leq t\leq T} \|u_\epsilon\|_{H^m} \leq \frac{\|u_0\|_{H^m}}{1 - c_m T\|u_0\|_{H^m}}$$

It is obvious that this bound is uniform since so far we have no dependence on ϵ, also we want to secure that the right part has a supremum .
So for $T \leq \frac{1}{\bar{c}_m\|u_0\|_{H^m}}$ u_ϵ is uniformly bounded on $C([0,T], H^m)$

Step 2:The family u_ϵ forms a Cauchy sequence[27] on $C\left([0,T],L^2\right)$
In order to prove this we will try to estimate the L^2 norm of the difference $u_\epsilon - u_{\epsilon'}$ via energy methods. We start with the equation of (IVP) and assume that $u_\epsilon, u_{\epsilon'}$satisfy this equation we have that

$$\frac{d}{dt}u_\epsilon - \frac{d}{dt}u_{\epsilon'} = F_\epsilon(u_\epsilon) - F_{\epsilon'}(u_{\epsilon'})$$

So we have the following equation

$$\frac{d}{dt}\left(u_\epsilon - u_{\epsilon'}\right) = \nu(J_\epsilon^2\Delta u_\epsilon - J_{\epsilon'}^2\Delta u_{\epsilon'})$$
$$-\left\{PJ_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right] + PJ_{\epsilon'}\left[(J_{\epsilon'}u_{\epsilon'}\cdot\nabla)J_{\epsilon'}u_{\epsilon'}\right]\right\}$$

We multiply this equation with the difference $u_\epsilon - u_{\epsilon'}$ in the L^2 and we have that

$$\int_{\mathbb{R}^3}\frac{d}{dt}(u_\epsilon - u_{\epsilon'})\cdot(u_\epsilon - u_{\epsilon'})dx$$
$$= \nu\int_{\mathbb{R}^3}(J_\epsilon^2\Delta u_\epsilon - J_{\epsilon'}^2\Delta u_{\epsilon'})\cdot(u_\epsilon - u_{\epsilon'})dx$$
$$-\int_{\mathbb{R}^3}\left\{PJ_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right] + PJ_{\epsilon'}\left[(J_{\epsilon'}u_{\epsilon'}\cdot\nabla)J_{\epsilon'}u_{\epsilon'}\right]\right\}\cdot(u_\epsilon - u_{\epsilon'})dx$$

So we have that

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon - u_{\epsilon'}\|_{L^2}^2 = \nu\int_{\mathbb{R}^3}(J_\epsilon^2\Delta u_\epsilon - J_{\epsilon'}^2\Delta u_{\epsilon'})\cdot(u_\epsilon - u_{\epsilon'})dx$$
$$+\int_{\mathbb{R}^3}\left\{PJ_\epsilon\left[(J_\epsilon u_\epsilon\cdot\nabla)J_\epsilon u_\epsilon\right] - PJ_{\epsilon'}\left[(J_{\epsilon'}u_{\epsilon'}\cdot\nabla)J_{\epsilon'}u_{\epsilon'}\right]\right\}\cdot(u_\epsilon - u_{\epsilon'})dx$$

For the terms on the right side of the above relation we will do the estimates separately:[28]

1.
$$\nu\int_{\mathbb{R}^3}(J_\epsilon^2\Delta u_\epsilon - J_{\epsilon'}^2\Delta u_{\epsilon'})\cdot(u_\epsilon - u_{\epsilon'})dx = \nu\int_{\mathbb{R}^3}J_\epsilon^2\Delta u_\epsilon\cdot(u_\epsilon - u_{\epsilon'}) - J_{\epsilon'}^2\Delta u_{\epsilon'}\cdot(u_\epsilon - u_{\epsilon'})dx$$

   We want to maintain the differences so we will not deal with them. We proceed by add and subtract the term $J_{\epsilon'}^2\Delta u_\epsilon\cdot(u_\epsilon - u_{\epsilon'})$ so we have that

$$\nu\int_{\mathbb{R}^3}(J_\epsilon^2\Delta u_\epsilon - J_{\epsilon'}^2\Delta u_{\epsilon'})\cdot(u_\epsilon - u_{\epsilon'})dx$$

[27]How this family forms a sequence since ϵ is a positive real?We set $\epsilon = \frac{1}{n}$ and we construct a sequence such that :

$$u_1 = u_1$$
$$u_2 = u_{\frac{1}{2}}$$
...
$$u_n = u_{\frac{1}{n}} = u_\epsilon$$

[28]We will use the tools we will need, observe here that we want our estimates to have a "good" dependance from ϵ or ϵ'

$$= \int_{\mathbb{R}^3} J_\epsilon^2 \Delta u_\epsilon \cdot (u_\epsilon - u_{\epsilon'}) - J_{\epsilon'}^2 \Delta u_\epsilon \cdot (u_\epsilon - u_{\epsilon'}) + J_{\epsilon'}^2 \Delta u_\epsilon \cdot (u_\epsilon - u_{\epsilon'}) - J_{\epsilon'}^2 \Delta u_{\epsilon'} \cdot (u_\epsilon - u_{\epsilon'}) dx$$

$$= \nu \left[\int_{\mathbb{R}^3} \left(J_\epsilon^2 - J_{\epsilon'}^2 \right) \Delta u_\epsilon \cdot (u_\epsilon - u_{\epsilon'}) dx + \int_{\mathbb{R}^3} J_{\epsilon'}^2 \Delta (u_\epsilon - u\epsilon') \cdot (u_\epsilon - u_{\epsilon'}) dx \right]$$

For the second integral[29] and proposition 4.1.9 we have that

$$\int_{\mathbb{R}^3} J_{\epsilon'}^2 \Delta (u_\epsilon - u\epsilon') \cdot (u_\epsilon - u_{\epsilon'}) dx = - \int_{\mathbb{R}^3} J_{\epsilon'} \nabla (u_\epsilon - u_{\epsilon'}) \cdot J_{\epsilon'} \nabla (u_\epsilon - u_{\epsilon'})$$

$$= - \int_{\mathbb{R}^3} |J_{\epsilon'} \nabla (u_\epsilon - u_{\epsilon'})|^2 dx = - \| J_{\epsilon'} \nabla (u_\epsilon - u_{\epsilon'}) \|_{L^2}^2 \leq 0$$

Thus

$$\nu \int_{\mathbb{R}^3} (J_\epsilon^2 \Delta u_\epsilon - J_{\epsilon'}^2 \Delta u_{\epsilon'}) \cdot (u_\epsilon - u_{\epsilon'}) dx \leq \nu \int_{\mathbb{R}^3} \left(J_\epsilon^2 - J_{\epsilon'}^2 \right) \Delta u_\epsilon \cdot (u_\epsilon - u_{\epsilon'}) dx$$

By Holders inequality we have that

$$\nu \int_{\mathbb{R}^3} (J_\epsilon^2 \Delta u_\epsilon - J_{\epsilon'}^2 \Delta u_{\epsilon'}) \cdot (u_\epsilon - u_{\epsilon'}) dx \leq \nu \| \left(J_\epsilon^2 - J_{\epsilon'}^2 \right) \Delta u_\epsilon \|_{L^2} \| u_\epsilon - u_{\epsilon'} \|_{L^2}$$

By triangle inequality we have

$$\leq \nu \left(\| J_\epsilon^2 \Delta u_\epsilon \|_{L^2} + \| J_{\epsilon'}^2 \Delta u_\epsilon \|_{L^2} \right) \| u_\epsilon - u_{\epsilon'} \|_{L^2}$$

$$= \nu \left(\| J_\epsilon^2 u_\epsilon \|_{H^2} + \| J_{\epsilon'}^2 u_\epsilon \|_{H^2} \right) \| u_\epsilon - u_{\epsilon'} \|_{L^2}$$

By proposition 4.1.13

$$\leq \nu c \left(\epsilon \| u_\epsilon \|_{H^3} + \epsilon' \| u_\epsilon \|_{H^3} \right) \| u_\epsilon - u_{\epsilon'} \|_{L^2}$$

$$\leq \nu c \max(\epsilon, \epsilon') \| u_\epsilon \|_{H^3} \| u_\epsilon - u_{\epsilon'} \|_{L^2}$$

So for this term we conclude that

$$\nu \int_{\mathbb{R}^3} (J_\epsilon^2 \Delta u_\epsilon - J_{\epsilon'}^2 \Delta u_{\epsilon'}) \cdot (u_\epsilon - u_{\epsilon'}) dx \leq \nu c \max(\epsilon, \epsilon') \| u_\epsilon \|_{H^m} \| u_\epsilon - u_{\epsilon'} \|_{L^2}$$

2.

$$\int_{\mathbb{R}^3} \{ P J_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] - P J_{\epsilon'} \left[(J_{\epsilon'} u_{\epsilon'} \cdot \nabla) J_{\epsilon'} u_{\epsilon'} \right] \} \cdot (u_\epsilon - u_{\epsilon'}) dx$$

$$\leq \int_{\mathbb{R}^3} \{ J_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] - J_{\epsilon'} \left[(J_{\epsilon'} u_{\epsilon'} \cdot \nabla) J_{\epsilon'} u_{\epsilon'} \right] \} \cdot (u_\epsilon - u_{\epsilon'}) dx$$

We will add and subtract the term $J_{\epsilon'} \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right]$

$$\leq \int_{\mathbb{R}^3} J_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] - J_{\epsilon'} \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right]$$

$$+ J_{\epsilon'} \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon \right] - J_{\epsilon'} \left[(J_{\epsilon'} u_{\epsilon'} \cdot \nabla) J_{\epsilon'} u_{\epsilon'} \right]$$

[29] [18] pg712 theorem 3

$$\cdot(u_\epsilon - u_{\epsilon'})dx$$

So we have two integrals:

$$\int_{\mathbb{R}^3} (J_\epsilon - J_{\epsilon'}) \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon\right] \cdot (u_\epsilon - u_{\epsilon'})dx$$

and

$$\int_{\mathbb{R}^3} J_{\epsilon'} \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon - (J_{\epsilon'} u_{\epsilon'} \cdot \nabla) J_{\epsilon'} u_{\epsilon'}\right] \cdot (u_\epsilon - u_{\epsilon'})dx$$

In the second integral we add and subtract the terms $(J_{\epsilon'} u_\epsilon \cdot \nabla)\, J_\epsilon u_\epsilon$, $(J_{\epsilon'} u_{\epsilon'} \cdot \nabla)\, J_\epsilon u_\epsilon$, $(J_\epsilon u_{\epsilon'} \cdot \nabla)\, J_{\epsilon'} u_\epsilon$

So eventually we will estimate the following 5 integrals:

(a) $= \int_{\mathbb{R}^3} (J_\epsilon - J_{\epsilon'}) \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon\right] \cdot (u_\epsilon - u_{\epsilon'})dx$

(b) $= \int_{\mathbb{R}^3} J_{\epsilon'} \left\{\left[(J_\epsilon - J_{\epsilon'})\, u_\epsilon \cdot \nabla\right] J_\epsilon u_\epsilon\right\} \cdot (u_\epsilon - u_{\epsilon'})dx$

(c) $= \int_{\mathbb{R}^3} J_{\epsilon'} \left\{\left[J_{\epsilon'}(u_\epsilon - u_{\epsilon'}) \cdot \nabla\right] J_\epsilon u_\epsilon\right\} \cdot (u_\epsilon - u_{\epsilon'})dx$

(d) $= \int_{\mathbb{R}^3} J_\epsilon \left\{J_{\epsilon'} \left[(u_\epsilon \cdot \nabla)(J_\epsilon - J_{\epsilon'}) u_\epsilon\right]\right\} \cdot (u_\epsilon - u_{\epsilon'})dx$

(e) $= \int_{\mathbb{R}^3} J_{\epsilon'} \left\{J_{\epsilon'} \left[(u_{\epsilon'} \cdot \nabla) J_{\epsilon'}(u_\epsilon - u_{\epsilon'})\right]\right\} \cdot (u_\epsilon - u_{\epsilon'})dx$

So let's start the estimates

$$(a) \leq \|\, (J_\epsilon - J_{\epsilon'}) \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon\right] \|_{L^2} \|u_\epsilon - u_{\epsilon'}\|_{L^2}$$

For the first norm we add and subtract the term $(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon$ by triangle inequality we have that

$$\|\, (J_\epsilon - J_{\epsilon'}) \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon\right] \|_{L^2} \leq \|J_\epsilon \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon\right] - (J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon\|_{H^0} + \|J_{\epsilon'} \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon - (J_\epsilon u_\epsilon \cdot$$

By proposition 4.1.11 we have that

$$\leq c\epsilon \|\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon\|_{H^1} + c\epsilon' \|\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon\|_{H^1}$$

$$\leq 2c \max(\epsilon, \epsilon') \|\nabla J_\epsilon u_\epsilon \cdot J_\epsilon u_\epsilon\|_{H^1}$$

By proposition 4.1.4 we have that

$$\leq c \max(\epsilon, \epsilon') \left\{\|\nabla J_\epsilon u_\epsilon\|_{L^\infty} \|J_\epsilon u_\epsilon\|_{H^1} + \|J_\epsilon u_\epsilon\|_{L^\infty} \|\nabla J_\epsilon u_\epsilon\|_{H^1}\right\}$$

By proposition 4.1.7and the fact that $m \geq 3$ we have that

$$\leq c \max(\epsilon, \epsilon') \left(\|\nabla u_\epsilon\|_{L^\infty} \|u_\epsilon\|_{H^m} + \|u_\epsilon\|_{L^\infty} \|u_\epsilon\|_{H^m}\right)$$

By the Sobolev embedding theorem we have that $\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty} \leq \sup_{|a| \leq 2} \|D^a u\|_{L^\infty} \leq c_m \|u\|_{H_m}$ Thus as a result

$$\|\, (J_\epsilon - J_{\epsilon'}) \left[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon\right] \|_{L^2} \leq c \max(\epsilon, \epsilon') \|u_\epsilon\|^2_{H^m}$$

So $(a) \leq c_a \max(\epsilon, \epsilon') \|u_\epsilon\|^2_{H^m} \|u_\epsilon - u_{\epsilon'}\|_{L^2}$

$$(b) \leq \|J_{\epsilon'} \left\{\left[(J_\epsilon - J_{\epsilon'})\, u_\epsilon \cdot \nabla\right] J_\epsilon u_\epsilon\right\} \|_{L^2} \|u_\epsilon - u_{\epsilon'}\|_{L^2}$$

For the first norm by proposition 4.1.13 we have that

$$\|J_{\epsilon'} \left\{\left[(J_\epsilon - J_{\epsilon'})\, u_\epsilon \cdot \nabla\right] J_\epsilon u_\epsilon\right\} \|_{L^2} \leq \frac{c}{\epsilon^0} \|\, \left[(J_\epsilon - J_{\epsilon'})\, u_\epsilon \cdot \nabla\right] J_\epsilon u_\epsilon\|_{H^0}$$

$$\leq c\,\|\left[(J_\epsilon - J_{\epsilon'})\,u_\epsilon \cdot \nabla\right] J_\epsilon u_\epsilon\|_{H^4}$$

Since $H^4(\mathbb{R}^3)$ is a Banach algebra we have:

$$c \leq \|\nabla J_\epsilon u_\epsilon\|_{H^4}\|(J_\epsilon - J_{\epsilon'})u_\epsilon\|_{H^4}$$

So by proposition and 4.1.13 we have that

$$\leq c\|u_\epsilon\|_{H^m}\bigg(\|J_\epsilon u_\epsilon - u_\epsilon\|_{H^3} + \|J_{\epsilon'} u_\epsilon - u_\epsilon\|_{H^3}\bigg)$$

By proposition 4.1.11

$$\leq c\|u_\epsilon\|_{H^m}\left(\epsilon\|u_\epsilon\|_{H^m} + \epsilon'\|u_\epsilon\|_{H^m}\right)$$

$$\|J_{\epsilon'}\left\{\left[(J_\epsilon - J_{\epsilon'})\,u_\epsilon \cdot \nabla\right] J_\epsilon u_\epsilon\right\}\|_{L^2} \leq c\max(\epsilon,\epsilon')\|u_\epsilon\|^2_{H^m}$$

Consequently $(b) \leq c_b \max(\epsilon,\epsilon')\|u_\epsilon\|^2_{H^m}\|u_\epsilon - u_{\epsilon'}\|_{L^2}$

$$(c) \leq \|J_{\epsilon'}\left\{\left[J_{\epsilon'}(u_\epsilon - u_{\epsilon'}) \cdot \nabla\right] J_\epsilon u_\epsilon\right\}\|_{L^2}\|u_\epsilon - u_{\epsilon'}\|_{L^2}$$

For the first norm by the proposition 4.1.13 we have that

$$\|J_{\epsilon'}\left\{\left[J_{\epsilon'}(u_\epsilon - u_{\epsilon'}) \cdot \nabla\right] J_\epsilon u_\epsilon\right\}\|_{L^2} \leq c\|\nabla J_\epsilon u_\epsilon \cdot J_{\epsilon'}(u_\epsilon - u_{\epsilon'})_{H^0}$$

By the proposition 4.1.4

$$\leq c\left\{\|\nabla J_\epsilon u_\epsilon\|_{L^\infty}\|J_{\epsilon'}(u_\epsilon - u_{\epsilon'})\|_{H^0} + \|\nabla J_\epsilon u_\epsilon\|_{H^0}\|J_{\epsilon'}(u_\epsilon - u_{\epsilon'})\|_{L^\infty}\right\}$$

By the proposition 4.1.13 and the Sobolev embedding we have that

$$\|J_{\epsilon'}\left\{\left[J_{\epsilon'}(u_\epsilon - u_{\epsilon'}) \cdot \nabla\right] J_\epsilon u_\epsilon\right\}\|_{L^2} \leq c\|u_\epsilon\|_{H^m}\|u_\epsilon - u_{\epsilon'}\|_{L^2}$$

It follows that $(c) \leq c_c\|u_\epsilon\|_{H^m}\|u_\epsilon - u_{\epsilon'}\|^2_{L^2}$ For (d) we have an estimate as in (b) i.e.

$$(d) \leq \|J_\epsilon\left\{J_{\epsilon'}\left[(u_\epsilon \cdot \nabla)(J_\epsilon - J_{\epsilon'})u_\epsilon\right]\right\}\|_{L^2}\|u_\epsilon - u_{\epsilon'}\|_{L^2}$$

For the first norm by the proposition 4.1.13 we have that

$$\|J_\epsilon\left\{J_{\epsilon'}\left[(u_\epsilon \cdot \nabla)(J_\epsilon - J_{\epsilon'})u_\epsilon\right]\right\}\|_{L^2} \leq c\|\nabla(J_\epsilon - J_{\epsilon'})u_\epsilon \cdot u_\epsilon\|_{H^0}$$

$$\leq c\|\nabla(J_\epsilon - J_{\epsilon'})u_\epsilon \cdot u_\epsilon\|_{H^4}$$

The space $H^4(\mathbb{R}^3)$ is a Banach algebra we have:

$$\leq c\|u_\epsilon\|_{H^m}\|(J_\epsilon - J_{\epsilon'})u_\epsilon\|_{H^5}$$

$$\leq c\|u_\epsilon\|_{H^m}\left(\|J_\epsilon u_\epsilon - u_\epsilon\|_{H^m} + \|J_{\epsilon'} u_\epsilon - u_\epsilon\|_{H^m}\right)$$

By proposition 4.1.11 we have

$$\leq c\|u_\epsilon\|_{H^m}\left(\epsilon\|u_\epsilon\|_{H^m} + \epsilon'\|u_\epsilon\|_{H^m}\right)$$

So

$$\|J_\epsilon\left\{J_{\epsilon'}\left[(u_\epsilon \cdot \nabla)(J_\epsilon - J_{\epsilon'})u_\epsilon\right]\right\}\|_{L^2} \leq c\max(\epsilon,\epsilon')\|u_\epsilon\|^2_{H^m}$$

As a consequence $(d) \leq c_d \max(\epsilon,\epsilon')\|u_\epsilon\|^2_{H^m}\|u_\epsilon - u_{\epsilon'}\|_{L^2}$ Now for the last integral (e) we have that

$$(e) = \frac{1}{2}\int_{\mathbb{R}^3}(J_{\epsilon'}u_{\epsilon'} \cdot \nabla)|J_{\epsilon'}(u_\epsilon - u_{\epsilon'})|^2 dx = -\frac{1}{2}\int_{\mathbb{R}^3}\operatorname{div}(J_{\epsilon'}u_{\epsilon'})|J_{\epsilon'}(u_\epsilon - u_{\epsilon'})|^2 dx = 0$$

So we combine all this relations and we have that :

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon - u_{\epsilon'}\|_{L^2}^2 \leq \nu c \max(\epsilon,\epsilon')\|u_\epsilon\|_{H^m}\|u_\epsilon - u_{\epsilon'}\|_{L^2} + c_a \max(\epsilon,\epsilon')\|u_\epsilon\|_{H^m}^2\|u_\epsilon - u_{\epsilon'}\|_{L^2}$$

$$c_b \max(\epsilon,\epsilon')\|u_\epsilon\|_{H^m}^2\|u_\epsilon - u_{\epsilon'}\|_{L^2} + c_c\|u_\epsilon\|_{H^m}\|u_\epsilon - u_{\epsilon'}\|_{L^2}^2 + c_d \max(\epsilon,\epsilon')\|u_\epsilon\|_{H^m}^2\|u_\epsilon - u_{\epsilon'}\|_{L^2}$$

By **Step 1** we have a uniform bound for $\|u_\epsilon\|_{H^m}$ we set this bound M, recall that this bound depends on time and initial value.If $M \geq 1$ then we have $M \leq M^2$, if $M < 1$ then $M^2 < M$. Assume also that C is the maximum constant of all. And set C_M the constant which depends on C and M or M^2 then we have the relation :

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon - u_{\epsilon'}\|_{L^2}^2 \leq C_M\|u_\epsilon - u_{\epsilon'}\|_{L^2}\left[\max(\epsilon,\epsilon') + \|u_\epsilon - u_{\epsilon'}\|_{L^2}\right]$$

So

$$\frac{d}{dt}\|u_\epsilon - u_{\epsilon'}\|_{L^2} \leq C_M\left[\max(\epsilon,\epsilon') + \|u_\epsilon - u_{\epsilon'}\|_{L^2}\right]$$

We take the integral with respect to t

$$\|u_\epsilon - u_{\epsilon'}\|_{L^2} \leq \int_0^t C_M\left[\max(\epsilon,\epsilon') + \|u_\epsilon - u_{\epsilon'}\|_{L^2}\right] ds$$

By simple calculations

$$\|u_\epsilon - u_{\epsilon'}\|_{L^2} \leq C_M \max(\epsilon,\epsilon')t + \int_0^t C_M\|u_\epsilon - u_{\epsilon'}\|_{L^2} ds$$

We set $a(t) = C_M \max(\epsilon,\epsilon')t$ so by Gronwall's lemma we have

$$\|u_\epsilon - u_{\epsilon'}\|_{L^2} \leq a(t)\int_0^t C_M a(s)e^{\int_s^t C_M dr} ds$$

$$\leq C_M \max(\epsilon,\epsilon')\left(2t + \frac{1}{C_M} - \frac{1}{C_M}e^{C_M t}\right)$$

It is true that $e^x << \leq x+1$ thus $e^{C_M t} \leq C_M t + 1$

$$\|u_\epsilon - u_{\epsilon'}\|_{L^2} \leq C_M \max(\epsilon,\epsilon')(2t-1) \leq 2tC_M \max(\epsilon,\epsilon')$$

Taking the supremum over this relation we have

$$\sup_{0\leq t\leq T}\|u_\epsilon - u_{\epsilon'}\|_{L^2} \leq C_{M,T}\max(\epsilon,\epsilon')$$

With out loss generality we assume that $\epsilon' < \epsilon$ So eventually $\|u_\epsilon - u_{\epsilon'}\|_{L^2} \leq C_{M,T}\epsilon$ thus sequence is Cauchy on $C([0,T],L^2(\mathbb{R}^3))$ As an intermediate step we prove that the space $C([0,T],L^2)$ is Banach.

Assume that a_n is a Cauchy sequence on $C([0,T],L^2)$ then we have that $\forall\eta > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n,m \geq n_0$: $\sup_{0\leq t\leq T}\|a_n - a_m\|_{L^2} \leq \eta$ we will show that this sequence converges. The space L^2 is Banach indeed assume that f_n is a Cauchy sequence on L^2 i.e. $\forall\delta > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n,m \geq n_0$: $\sup_{0\leq t\leq T}\|f_n - f_m\|_{L^2} \leq \delta$, so we may assume a subsequence f_{n_k} such that $\|f_{n_{k+1}} - f_{n_k}\| \leq \frac{1}{2^k}$ for $k \geq 1$ We also assume the series $f(x) = f_{n_1} + \sum_1^\infty \left(f_{n_{k+1}} - f_{n_k}\right)$ and $g(x) = |f_{n_1}| + \sum_1^\infty |f_{n_{k+1}} - f_{n_k}|$. It is obvious that $|f(x)| \leq g(x)$ For the sequence

of partial sum of g we have that $S_N^g = |f_{n_1}| + \sum_1^N |f_{n_{k+1}} - f_{n_k}|$
Thus $\|S_N^g\|_{L^2} = \left(\int_{\mathbb{R}^3} \left||f_{n_1}| + \sum_1^N |f_{n_{k+1}} - f_{n_k}|\right|^2 dx\right)^{\frac{1}{2}}$ So $\|S_N^g\|_{L^2} \leq \|f_{n_1}\|_{L^2} + \sum_1^N \|f_{n_{k+1}} - f_{n_k}\|_{L^2}$ Consequently for $n \to \infty$ we have that $\|g\|_{L^2} < \infty$,so $\|f\|_{L^2} < \infty$ thus $f \in L^2$. We also have that $f_{n_k} \to f$ in L^2 thus $\|f_n - f\|_{L^2} \leq \|f_n - f_{nk}\|_{L^2} + \|f_{n_k} - f\|_{L^2}\delta'$ So we have that $\forall \eta > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0$: $\|a_n - a\|_{L^2} \leq \eta$ taking the supremum we have that $\forall \eta > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0$: $\sup_{0 \leq t \leq T} \|a_n - a\|_{L^2} \leq \eta$

So we have that u_ϵ is a Cauchy sequence in a Banach space thus this sequence converges i.e $u_\epsilon \to u_v$
Step 3:The main tool here is the interpolation spaces[30]. An interpolation space is a space layed between two Banach spaces. Loosely speaking the idea is the following, assume two Banach spaces X,Y which are continuously embedded to a Hausdorff topological space Z. We define the spaces $T_1 = X \cap Y$ and $T_2 = X + Y$ with norms $\|.\|_{T_1} \max(\|.\|_X, \|, \|_Y)$ and $\|.\| = \inf\{\|.\|_X + \|.\|_Y\}$. Via the interpolation method we seek for all sets S such that $X \cap Y \subset S \subset X + Y$. A well known result is that of the interpolation in L^p spaces[31]. Where we have the result:
Let $1 \leq p < q < r \leq \infty$ such that for $\theta \in (0,1)$: $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$ then $\|u\|_{L^q} \leq \|u\|_{L^p}^{\theta} \|u\|_{L^r}^{1-\theta}$. This gives the interpolation $L^p \cap L^r \subset L^q$. We observe that the interpolation theory gives relation between norms so our interest is to have a result us above in L^2 Sobolev spaces.

Proposition 4.3.1. *Let $s \in \mathbb{R}^+$ then $\forall u \in H^s$ and $0 < r < s$ we have that*

$$\|u\|_{H^r} \leq \|u\|_{L^2}^{1-\frac{r}{s}} \|u\|_{H^s}^{\frac{r}{s}}$$

proof of proposition 4.3.1 We will prove the following:Assume an r that is a convex combination of x and y i.e. $r = (1-\lambda)x + \lambda y$ for $\lambda \in (0,1)$ then $\|u\|_{H^r} \leq \|u(x)\|_{H^x}^{1-\lambda} \|u\|_{H^y}^{\lambda}$
For the H^r norm with $r > 0$ we have that

$$\|u\|_{H^r} = \|(1+|\xi|^2)^{\frac{r}{2}} \hat{u}(\xi)\|_{L^2} = \|(1+|\xi|^2)^{\frac{(1-\lambda)x+\lambda y}{2}} \hat{u}(\xi)\|_{L^2}$$

$$(1+|\xi|^2)^{\frac{(1-\lambda)x}{2}} (1+|\xi|^2)^{\frac{\lambda y}{2}} \hat{u}(\xi)\|_{L^2} = (1+|\xi|^2)^{\frac{(1-\lambda)x}{2}} \hat{u}^{1-\lambda} (1+|\xi|^2)^{\frac{\lambda y}{2}} \hat{u}^{\lambda}\|_{L^2}$$

Assume p,q,we will define them later such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, by Holder we have that

$$\|u\|_{H^r} \leq \|(1+|\xi|^2)^{\frac{x}{2}} \hat{u}\|_{L^p}^{1-\lambda} \|(1+|\xi|^2)^{\frac{y}{2}} \hat{u}\|_{L^q}^{\lambda}$$

For $p = \frac{2}{1-\lambda}$ and $q = \frac{2}{\lambda}$ we have that

$$\|u\|_{H^r} \leq \left[\left(\int_{\mathbb{R}^3} \left|(1+|\xi|^2)^{\frac{x}{2}} \hat{u}\right|^{1-\lambda}\right)^{\frac{2}{1-\lambda}} d\xi\right]^{\frac{1-\lambda}{2}} \left[\left(\int_{\mathbb{R}^3} \left|(1+|\xi|^2)^{\frac{y}{2}} \hat{u}\right|^{\lambda}\right)^{\frac{2}{\lambda}} d\xi\right]^{\frac{\lambda}{2}}$$

[30] [11]
[31] [4] Chapter 2

$$\|u\|_{H^r} \leq \left(\|(1+|\xi|^2)^{\frac{x}{2}}\hat{u}\|_{L^2}\right)^{1-\lambda}\left(\|(1+|\xi|^2)^{\frac{y}{2}}\hat{u}\|_{L^2}\right)^{\lambda}$$

$$\|u\|_{H^r} \leq \|u\|_{H^x}^{1-\lambda}\|u\|_{H^y}^{\lambda}$$

So in our case for $x = 0$, $\lambda = \frac{r}{s}$, and $y = s$ we have that $r = (1-\lambda)x + \lambda y$ thus by the previous result

$$\|u\|_{H^r} \leq \|u\|_{H^0}^{1-\frac{r}{s}}\|u\|_{H^s}^{\frac{r}{s}}$$

We need this relation since we have proved convergence in $C([0,T],L^2)$ and also we have a uniform bound in $C([0,T],H^m)$,and we are interested to find convergence for the intermediate norms of H^mnorm. So we estimate the $H^{m'}$ norm of the difference $u_\epsilon - u_v$ where $m' \leq m$. By the above proposition we can easily see that

$$\|u_\epsilon - u_v\|_{H^{m'}} \leq \|u_\epsilon - u_v\|_{L^2}^{1-\frac{m'}{m}}\|u_\epsilon - u_v\|_{H^m}^{\frac{m'}{m}}$$

Taking the supremum over this relation we have that

$$\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{H^{m'}} \leq \sup_{0\leq t\leq T}\left\{\|u_\epsilon - u_v\|_{L^2}^{1-\frac{m'}{m}}\|u_\epsilon - u_v\|_{H^m}^{\frac{m'}{m}}\right\}$$

Thus $\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{H^{m'}} \leq \sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{L^2}^{1-\frac{m'}{m}}\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{H^m}^{\frac{m'}{m}}$
By the **Step 2** we have that $\|u_\epsilon - u_{\epsilon'}\|_{L^2} \leq C_{M,T}\epsilon$, we also now that $\sup ab \leq \sup a \sup b$ it follows that $\sup a^k \leq (\sup a)^k$.
Consequently $\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{L^2}^{1-\frac{m'}{m}} \leq \left(\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{L^2}\right)^{1-\frac{m'}{m}}$.
We also know that the function x^a is increasing when $a > 0$, so we have that $\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{L^2}^{1-\frac{m'}{m}} \leq (c\epsilon)^{1-\frac{m'}{m}}$.
By the **Step 1** we have that $\sup_{0\leq t\leq T}\|u_\epsilon\|_{H^m} \leq \frac{\|u_0\|_{H^m}}{1-c_mT\|u_0\|_{H^m}}$ and it is also true that $\|u_v\|_{H^m} \leq \limsup_{\epsilon\to 0}\|u_\epsilon\|_{H^m} \leq \limsup_{\epsilon\to 0}\frac{\|u_0\|_{H^m}}{1-c_mT\|u_0\|_{H^m}} \leq \frac{\|u_0\|_{H^m}}{1-c_mT\|u_0\|_{H^m}}$
Thus we have also an uniform bound for the H^m norm of u_v i.e.

$$\sup_{0\leq t\leq T}\|u_v\|_{H^m} \leq \frac{\|u_0\|_{H^m}}{1-c_mT\|u_0\|_{H^m}}$$

It is also true by triangle inequality that $\sup(a+b) \leq \sup a + \sup b$ so $\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{H^m}^{\frac{m'}{m}} \leq \left(\sup_{0\leq t\leq T}\|u_\epsilon\|_H^m + \sup_{0\leq t\leq T}\|u_v\|_{H^m}\right) = M^{\frac{m'}{m}}$ Eventually we combine the above relations and we have that

$$\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{H^{m'}} \leq C(u_0,T,m,m')\epsilon^{1-\frac{m'}{m}}$$

So we have proved that we have the converge of u_ϵ to u_v in all spaces $C([0,T],V^{m'})$ with $m' < m$.
We choose $m' > \frac{3}{2}+2$, so by the Sobolev embedding theorem we have that $\|u_\epsilon - u_v\|_{C^2} \leq \|u_\epsilon - u_v\|_{H^{m'}}$, taking the supremum over this relation we have that $\sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{C^2} \leq \sup_{0\leq t\leq T}\|u_\epsilon - u_v\|_{H^{m'}}$.
It follows that $u_\epsilon \to u_v$ in $C([0,T],C^2)$
To complete the step 3 we have to show that we also have a convergence in $C^1([0,T],C)$.
We will need the following lemma

Lemma 19. *Assume we have $v_n \to v$ in $C([0,T],X)$ and $v_n' = f_n \to f$ in $C([0,T],Y)$, where $X \subset Y$ Hilbert spaces. We want to show that then $v' = f \in C([0,T],Y)$.*[32]

proof of lemma: Adapting classical results on uniform convergence to Hilbert-space-valued functions.
First of all, from $v_n \to v$ in $C([0,T],X)$ and $X \subset Y$ we obtain $v_n \to v$ in $C([0,T],Y)$.

Theorem 4.3.2. [a] *Let $\epsilon > 0$. Since $v_n' \to f$ in $C([0,T],Y)$, there exists an $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have*

$$\|v_n'(s) - v_m'(s)\|_Y < \epsilon \quad \forall\ s \in [0,T]$$

Then, by the Mean Value Theorem for Hilbert-space-valued functions, [b] *applied to each function $v_n - v_m \in C([0,T],Y)$ on any interval between $s,t \in [0,T]$, we obtain*

$$\|v_n(s) - v_m(s) - (v_n(t) - v_m(t))\|_Y \leq \epsilon|t-s| \quad \forall\ t,s \in [0,T], \quad \forall\ n,m \geq n_0$$

Let now $t \in [0,T]$ be fixed and define $\phi_n(s) := \frac{v_n(s)-v_n(t)}{s-t} \in Y$ and $\phi(s) := \frac{v(s)-v(t)}{s-t} \in Y$, with $s \in [0,T] \setminus \{t\} =: E$, such that, since $v_n \in C^1([0,T],Y)$,

$$\lim_{s\to t} \phi_n(s) = v_n'(t) \quad \text{in} \quad Y$$

Moreover,by the above theorem we get

$$\|\phi_n(s) - \phi_m(s)\|_Y \leq \epsilon, \quad \forall\ s \in E, \quad \forall\ n,m \geq n_0$$

such that (ϕ_n) converges in Y uniformly on $E = [0,T]\setminus\{t\}$, and since (v_n) converges in Y uniformly on $[0,T] \supset E$ to v, we obtain that $\phi_n \to \phi$ in Y, uniformly on E.
Now from $\phi_n \to \phi$ in Y, uniformly on $E = [0,T] \setminus \{t\}$, and $v_n'(t) \to f(t) \in Y$, we obtain that for any $\epsilon > 0$ there exists an $n \in \mathbb{N}$ and a corresponding $\delta > 0$ such that $\|v_n'(t) - f(t)\|_Y < \frac{\epsilon}{3}$, $\|\phi_n(s) - \phi(s)\|_Y < \frac{\epsilon}{3}$ $\forall\ s \in E$, $\|\phi_n(s) - v_n'(t)\|_Y < \frac{\epsilon}{3}$ $\forall\ s \in E \cap \{0 < |s-t| < \delta\}$, and thus, $\|\phi(s) - f(t)\|_Y < \epsilon$ for all $s \in E \cap \{0 < |s-t| < \delta\}$. This yields $\lim_{s\to t} \phi(s) = f(t)$.
So $v'(t) = \lim_{s\to t} \phi(s) = \lim_{n\to\infty} v_n'(t) = f(t)$ in Y

[a] This is the Theorem 7.17 pg 152 on [42], adapted to the case of functions with values on a Hilbert space

[b] This is a trivial adaptation of Theorem 5.19, pg 113,[34] for functions with values in $\mathbb{R}^n$, which we present here for the sake of completeness.
Assume $v : [a,b] \to Y$ is continuous and $v : (a,b) \to Y$ is differentiable for a Hilbert space Y. Then, there exists $t \in (a,b)$ such that $\|v(b)-v(a)\|_Y \leq \|v'(t)\|_Y(b-a)$. Indeed, set $z := v(b)-v(a)$. Then, $\phi(t) := (z, v(t))_Y$ is continuous on $[a,b]$ and differentiable on (a,b) with $\phi'(t) = (z, v'(t))_Y$.

[32] Since $C([0,T],X) \subset C([0,T],Y)$ for $X \subset Y$, this implies then $v \in C([0,T],X) \cap C^1([0,T],Y)$.

Applying on ϕ the Mean Value Theorem for real-valued functions[c] we obtain the existence of a $t \in (a,b)$ such that
$$\phi(b) - \phi(a) = \phi'(t)(b-a) = (z, v'(t))_Y (b-a)$$
Using $\phi(b) - \phi(a) = (z, v(b))_Y - (z, v(a))_Y = (z, v(b) - v(a))_Y = \|z\|_Y^2$, the Cauchy inequality, and dividing by $\|z\|_Y > 0$ (for $z = 0$ nothing has to be shown), we obtain $\|v(b) - v(a)\|_Y = \|z\|_Y \leq \|v'(t)\|_Y (b-a)$

We will apply this lemma in our case.
For this we recall the flow $\frac{d}{dt} u_\epsilon = F_\epsilon(u_\epsilon)$.
We know that a $\lim_{\epsilon \to 0} F_\epsilon u_\epsilon = \Delta u_v - P[(u_v \cdot \nabla) u_v]$ in $V^{m'-2}$ since (we denote $u_v = u$)

$$\|\nu J_\epsilon^2 \Delta u_\epsilon - P J_\epsilon[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon] - \nu \Delta u + P[(u \cdot \nabla) u]\|_{H^{m'-2}}$$

By the triangle inequality we have that

$$\leq \|\nu J_\epsilon^2 \Delta u_\epsilon - \nu \Delta u\|_{H^{m'-2}} + \|P J_\epsilon[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon] - P[(u \cdot \nabla) u]\|_{H^{m'-2}}$$

$$\leq \nu \|J_\epsilon u_\epsilon - u\|_{H^{m'}} + \|J_\epsilon[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon] - [(u \cdot \nabla) u]\|_{H^{m'-2}}$$

We add and subtract some terms in order to reach to some terms we can estimate by the previous results

$$= \nu \|J_\epsilon u_\epsilon - J_\epsilon u + J_\epsilon u - u\|_{H^{m'}} + \|J_\epsilon[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon] - J_\epsilon[(u \cdot \nabla) u] + J_\epsilon[(u \cdot \nabla) u] - [(u \cdot \nabla) u]\|_{H^{m'-2}}$$

Again by triangle inequality we have

$$\leq \nu \|J_\epsilon u_\epsilon - J_\epsilon u\|_{H^{m'}} + \nu \|J_\epsilon u - u\|_{H^{m'}}$$

$$+ \underbrace{\|J_\epsilon[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon] - [(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon]\|_{H^{m'-2}}}_{(1)} + \underbrace{\|(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon - (u \cdot u)\|_{H^{m'-2}}}_{(2)}$$

By the proposition 4.1.11 , 4.1.12 , 4.1.13 and step 2 we have that the first two terms converge in $H^{m'}$ so we continue with (1) and (2).

$$(1) \leq \|J_\epsilon[(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon] - [(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon]\|_{H^{m'}}$$

By proposition 4.1.12, and steps 1,2,3 we have that $(1) \to 0$
For (2) we have

$$(2) \leq \underbrace{\|(J_\epsilon u_\epsilon \cdot \nabla) J_\epsilon u_\epsilon - (J_\epsilon u_\epsilon \cdot \nabla) u_\epsilon\|_{H^{m'-2}}}_{(2a)} + \underbrace{\|(J_\epsilon u_\epsilon \cdot \nabla) u_\epsilon - (u \cdot \nabla) u\|_{H^{m'-2}}}_{(2b)}$$

So $(2a)$: $\|(J_\epsilon u_\epsilon \cdot \nabla)(J_\epsilon u_\epsilon - u_\epsilon)\|_{H^{m'-2}} \leq \|\nabla (J_\epsilon u_\epsilon - u_\epsilon)\|_{H^{m'-2}} \|J_\epsilon u_\epsilon\|_{H^{m'-2}} \leq \epsilon M \|J_\epsilon u_\epsilon - u_\epsilon\|_{H^{m'}}$
By proposition 4.1.12 and steps 1,2,3 we have that $(2a) \to 0$ We continue with $(2b)$ by adding and subtracting some terms we have

$$(2b) \leq \|(J_\epsilon u_\epsilon \cdot \nabla) u_\epsilon - (J_\epsilon u_\epsilon \cdot \nabla) u\|_{H^{m'-2}} + \|(J_\epsilon u_\epsilon \cdot \nabla) u - (u_\epsilon \cdot \nabla) u\|_{H^{m'-2}} + \|(u_\epsilon \cdot \nabla) u - (u \cdot \nabla) u\|_{H^{m'-2}}$$

$$\leq \|u_\epsilon - u\|_{H^{m'-1}} \|u_\epsilon\|_{H^{m'}} + \|u\|_{H^{m'-1}} \|J_\epsilon u_\epsilon - u_\epsilon\|_{H^{m'-2}} + \|u\|_{H^{m'-1}} \|u_\epsilon - u\|_{H^{m'-2}}$$

By proposition 4.1.12 and steps 1,2,3 we have that $(2b) \to 0$
So we conclude that $\frac{d}{dt}u_\epsilon \to F(u)$ in $C([0,T], V^{m'-2})$. Invoking now $v^\epsilon \to v$ in $C([0,T], H^{m'})$, we obtain from Lemma19 above, that $\partial_t u = F = \nu\Delta u - P(u \cdot \nabla u) \in C([0,T], H^{m'-2})$
This means that the (strong) time-derivative $\partial_t u$ exists and is continuous in the respective spaces, such that $u \in C([0,T], H^{m'}) \cap C^1([0,T], H^{m'-2})$ satisfies the Navier-Stokes and Euler equations. For m' like before we have by the Sobolev embedding theorem that $u_\epsilon, u_u \in C([0,T], C^2) \cap C^1([0,T], C)$[33]
Step 4:We start by giving the definitions for the spaces in the third part of the theorem.

- $L^\infty([0,T], V^m) = \{f : [0,T] \to V^m, ess\sup_{0\le t\le T} \|u(x)\|_{H^m} \le \infty\}$
- $Lip([0,T], V^{m-2}) = \{f : [0,T] \to V^{m-2}, \text{f is Lipschitz}\}$
- $C_w([0,T], V^m)$ Assume that w is a weak topology on V^m [34]

$$C_w([0,T], V^m) =$$

$$\{f : [0,T] \to (V^m, w),\ f\ continuous, i.e. for \phi \in (V^m)^*\ the\ function t \to \phi(f(t))\text{is continuous}\}$$

We also define the weak convergence.

Definition 9. *Assume that B is a Banach space and u_n a sequence in B we will say that u_n (strongly) converge to u in B if $\|u_n - u\|_B \to 0$. If H is a Hilbert space we will say that u_n (weakly) converge to u in H if $\forall \phi in H$ we have that $(u_n, \phi)_H \to (u, \phi)_H$ and we will write $u_n \rightharpoonup u$*

We will also need the following theorem

Theorem 4.3.3. [35] *Let H be a Hilbert space and u_n a bounded sequence on H, then there exists a subsequence u_{k_n} of u_n which weakly converge on H.*

proof of theorem 4.3.2,Assume a u_n is a bounded sequence of H. We define H_o is the closure of the set $S = span(u_1, u_2, ..., u_n)$, recall that the span of u_n contains all the linear combinations of the elements of u_n. The space H_o is separable indeed : The space S is separable since $S = \sum_{n=1}^{k} \lambda_n u_n$, $\lambda_n \in \mathbb{R}$, thus every non empty subset of S contains at least one element of u_n. The space H_o is the closure of S. Since S is separable there exists a D dense and countable subset of S. Let U open subset of H_o and a $x \in U$ then $x \in H_o$ thus $\exists x_n \in S$ such that $x_n \to x$ i.e.

[33] The u_v is a classical solution

[34] Assume that H is a Hilbert space and H^*its dual,for $\phi \in H^*$ we define $\rho(x) = |(x, \phi)_X|$ this is a semi norm since

- $\rho(x) \ge 0$, by the definition
- $\rho(x + y) = |(x + y, \phi)_X| = |(x, \phi)_X + |(y, \phi)_X| \le |(x, \phi)_X| + |(y, \phi)_X| = \rho(x) + \rho(y)$
- for $s > 0$, $\rho(sx) = |(sx, \phi)_X| = |s(x, \phi)_X| = |s||(x, \phi)_X| = s\rho(x)$

The family of semi norms forms a topology,this is the weak topology on X.

[35] [32] pg 313

$U \cap S \neq \emptyset$. The $U \cap S$ is an open subset of S and D dense on S thus

$$U \cap D = U \cap (S \cap D) = (U \cap S) \cap D \neq \emptyset$$

So H_o has countable dense subset,which gives that H_o is separable.
We set H_o^* the dual of H^o which contains all the functionals $[g_n](f) = (u_n, f)$ where $u_n \in H$ and $f \in H_o$, we have that g_n is bounded since

$$|g_n| = |(u_n, f)|$$

By Cauchy Schwartz inequality we have

$$|g_n| \leq \|u_n\|\|f\|$$

Lemma 20. *This is the Helley's theorem*[a]
Assume that X is a separable normed space and T_n a sequence on its dual X^* which is bounded (i.e. there exists a $M \geq 0$ such that $|T_n(f)| \leq M\|f\|, \forall f \in X$) then $\exists T_{n_k}$ subsequence of T_n and a $T \in X^*$ such that $T_{n_k}(f) \to T(f)$, $\forall f \in X$

proof of lemma:
Assume that X is separable, then there exist a countable dense subset $D = f_{i}{}_{i=1}^{\infty}$ Our aim is to create a sequence on reals in order to use the classic result of Bolzano Weierstrass,i.e. that every bounded sequence has a convergence subsequence. The $T_n f$ is on the dual so it is a functional.

- We have that the sequence $T_n(f_1)$ is bounded, recall by the hypothesis of the lemma $T_n(f_1) \leq M\|f_1\|$, and since we refer to functionals it is true that $T_n(f_1) \subset \mathbb{R}$. Thus by the Bolzano Weierstrass we have that there exists a subsequence that converges. Assume that $k(1, n)$ is an increasing sequence, then we define $T_{k(1,n)}(f_1)$ be s subsequence of $T_n(f_1)$ and a_1 to be it limit for $n \to \infty$
- We proceed with $T_n(f_2)$, this is a bounded sequence, assume that $k(2, n)$ is an increasing sequence, then we define $T_{k(2,n)}(f_2)$ be s subsequence of $T_n(f_2)$ and a_2 to be it limit for $n \to \infty$
- We continue until take all f_i and this way we have create an increasing sequence $k(j, n)_{j=1}^{\infty}$ and a sequence of reals a_j such that $T_{k(j,n)}(f_j) \to a_j$ as $n \to \infty$ We set $n_s = k(., s)$ then $\forall j$, n_s it is a subsequence of $k(j, s)$. We also know that if a sequence converge all its subsequences converge in the same limit thus $T_{n_s}(f_j) \to a_j$.

Now we also have that T_{n_s} is a Cauchy sequence since $|T_{n_s} - T_{n'_s}| \leq 2M\|f\|$ thus the sequence converge say to a $T(f)$ (end of proof of lemma).
So by lemma 19 we have that there exists a subsequence $g_{n_k}(f) \to g_o$, where $g_o \in H_o$

Lemma 21. *This is the Rierz-Frechet representation theorem*
Assume that H is a Hilbert space. We define the operator $T : H \to H^*$ such that $[Th]u = (h, u)_H$, $\forall h \in H$, then T is a linear isometrics of H to H^*

proof of lemma
Let $h \in H$,first we will prove the linearity oh Th:
Let $\forall k, l \in \mathbb{R}$ and $u, v \in H$ then

$$[Th](ku + lv) = (h, ku + lv)_H$$

$$(h, ku)_H + (h, lv)_H = k(h, u)_H + l(h, v)_H$$

$$[Th]u + [Th]v$$

Furthermore Th is bounded since by Cauchy-Schwartz we have that $[Th]u = (h, u)_H \leq \|h\|\|u\|$
Also $[Th]h = (h, h)_H = \|h\|$,so Th is an isometry.
T is linear since $[T(kh_1 + lh_2)](u) = (kh_1 + lh_2, u)_H = k[Th_1]u + l[Th_2]u$. And we have that $[T0]u = (0, u)_H = 0$, $\forall u \in H$.
Assume now a functional $u_o \neq 0 \ \in H^*$,u_o is linear so it is continuous, so for 0 which is close we have that the inverse image of u_o is also closed , $u_o^{-1}(0) = \ker(u_o)$. Since $u_o \neq 0$,$\ker(u_o)$ is a non trivial subspace of H thus $H = \ker(u_o) \bigoplus \ker(u_o)^{\perp}$. So we have that $\exists h' \in H$ such that $(\ker(u_o), h')_H = 0$. We define $h_o = u_o(h')h'$ and for $h \in H$ we have that $h - \frac{u_o(h)}{u_o(h')}h' \in \ker(u_o)$ thus $\left(h - \frac{u_o(h)}{u_o(h')}h', h'\right)_H = 0$,so we have that $u_o(h) = (h, u_o(h')h')_H = (h, h_o) = [Th_o]h$. So we have that T is a linear isometry $T : H \to H^*$. (end of proof)
Thus by lemma 20 we have that there exists a $p_o \in H_o$ such that $g_o = [Tp_o]$. Consequently $\lim_{n\to\infty}(g_{n_k}, f)_H = (g_o, f)_H$,$\forall f \in H_o$. So we have reached to the result that $g_{n_k} \rightharpoonup g_o$ in H_o. It remains to show that $g_{n_k} \rightharpoonup g_o$ in H.
SinceH_o is a closed subspace of H we have that $H = H_o \bigoplus H_o^{\perp}$. Assume that Π is the orthogonal projection of H toH_o then for $f \in H$it is true that $(g_{n_k}, (I_d - \Pi)f)_H = 0$ since $(I_d - \Pi)H = \Pi(H)^{\perp} = H_o^{\perp}$, furthermore for$f_o \in H_o$ $(f_o, (I_d - \Pi)f)_H = 0$. Also we see that $(g_{n_k}, f)_H = (g_{n_k}, \Pi f)_H$ and also $(g_o, f)_H = (g_o, \Pi f)_H$. Thus it is obvious that for$f \in H$ and $\Pi f \in H_o$

$$\lim_{n\to\infty}(g_{n_k}, f)_H = \lim_{n\to\infty}(g_{n_k}, \Pi f)_H = (g_o, \Pi f)_H = (g_o, f)_H$$

which completes the proof that $g_{n_k} \rightharpoonup g_o$ in H.

[a] [32] pg 171

We also have the following proposition

Proposition 4.3.2. [36] *Let H be a normed space and $u_n \rightharpoonup u$in H, then*

$$\|u\| \leq \lim_{n\to\infty} \inf \|u_n\|$$

[36] [18] pg 723

proof of proposition: By Cauchy-Schwartz we have that $(u, u_n) \leq \|u\|\|u_n\|$,so

$$\lim_{n\to\infty} \inf(u, u_n) \leq \|u\| \lim_{n\to\infty} \inf \|u_n\|$$

. Then since $u_n \rightharpoonup u$ we have that $\lim_{n\to\infty} \inf(u, u_n) = \lim_{n\to\infty}(u, u_n)$.
Furthermore

$$\|u\|^2 = (u, u) = \lim_{n\to\infty}(u, u_n) \leq \|u\| \lim_{n\to\infty} \inf \|u_n\|$$

Which verifies that $\|u\| \leq \lim_{n\to\infty} \inf \|u_n\|$

We also have that in an finite dimension normed space X, weakly convergence gives strong convergence,indeed

Since X has a finite dimension, say k, there exists an orthomonal basis $e_1, ...e_k$ and we can write its elements as a linear combination of the elements of the basis,i.e.

$$u_n = \sum_{i=1}^{k} a_n^i e_i$$

and

$$u = \sum_{i=1}^{k} a^i e_i$$

We have by the weak convergence that for all functionals f on the dual space of X $(u_n, f) \to (u, f)$let the functional $g(\sum_{i=1}^{k} a^i e^i) = a^i$ then we have that $\|u_n - u\| = \|\sum_1^k (a_n^i - a^i)e_i\| \to 0$

We close with some word for the space L^∞ . The space L^∞ is the dual of L^1 By Helly's theorem we have that for a sequence $u_n \in L^\infty$ there exist subsequence u_{n_k} that weakly $*$ converges. So now we have all the basic arguments which we will need we proceed with the proof of step 4.

- The space L^2 is Hilbert, in addition the sequence u_ϵ is bounded in $L^2([0, T], V^m)$ indeed $\left(\int_0^T \|u_\epsilon\|_{H^m}^2 dt\right)^{\frac{1}{2}} \leq \int_0^T \|u_\epsilon\|_{H^m}$. By the **Step 1** we have an upper bound for the $\|u_\epsilon\|_{H^m}$ thus

  $$\left(\int_0^T \|u_\epsilon\|_{H^m}^2 dt\right)^{\frac{1}{2}} \leq M^2 T = C < \infty$$

  Thus by theorem 4.3.2 we have that there exist a subsequence such that converges to a u. This u is the u_v we have found in the step 2 since for $m' < m$we have that $V^m \subset V^{m'}$ and we know that the limit of a subsequence is unique. So $u_{\epsilon_k} \rightharpoonup u$. Also by proposition 5.3.2 we have that

  $$\|u_v\|_{L^2([0,T],V^m)} \leq \lim\inf \|u_{\epsilon_k}\|_{L^2([0,T],V^m)} \leq C$$

- The sequence u_ϵ is bounded in $L^\infty([0,T],V^m)$ since

$$ess \sup_{0\le t\le T} \|u_\epsilon\|_{H^m} \le ess \sup_{0\le t\le T} M = M < \infty$$

We know that since $L^\infty([0,T],V^m)$ is a reflexive, seperable, Banach space[37], there exists a weakly$*$ convergence subsequence to u_v we also have that

$$\|u_v\|_{L^\infty([0,T],V^m)} \le \limsup \|u_{\epsilon_k}\|_{L^\infty([0,T],V^m)} \le M < \infty$$

- By the step 2 we have that $\|\frac{d}{dt}u_\epsilon\|_{H^m} \le c\nu\|u_\epsilon\|_{H^m}+c\|u_\epsilon\|^2_{H^m}$ thus $\|\frac{d}{dt}u_\epsilon\|_{H^m} \le C_M$

Lemma 22. *Assume that B is a Banach space and $f:[a,b]\to X$ then if the derivative of f is bounded, f is Lipschitz.*

Remark:The derivative in this spaces is the Frechet which is the general case of the usual derivative we use on $\mathbb{R}^N$ i.e. assume that $f;X\to Y$ then f is Frechet differentiable if

$$\lim_{h\to 0}\frac{\|f(x+h)-f(x)-[Df](h)\|_Y}{\|h\|_X}=0$$

proof of lemma: Assume that the derivative is bounded by a constant, say M. By the fundamental theorem of calculus we have that

$$\|f(b)-f(a)\|_B \le \|\int_a^b Df(x)dx\|_B \le M\|b-a\|_B$$

So in our case it follows that $u_\epsilon \in Lip\left([0,T],V^{m-2}\right)$. As far as concern u_v we have the following arguments: By the previous work we have that $\frac{\partial}{\partial t}u_\epsilon \to \frac{\partial}{\partial t}u$ in $C([0,T],V^{m-2})$ thus it is true that $\|\frac{\partial}{\partial t}u_v\|_{H^{m-2}}$ is bounded so by the lemma 22 we have that $u_v \in Lip\left([0,T],V^{m-2}\right)$

- We have already prove that $u_\epsilon \to u_v$ in $C([0,T],V^{m'})$, thus $\|u_\epsilon - u_v\|_{H^{m'}} \le \sup_{0\le t\le T}\|u_\epsilon - u_v\|_{H^{m'}} \le \eta$ so it occurs that $(u_\epsilon - u_v,\phi)\to 0$ for any $\phi \in V^{-m'}(1)$[38]. The space $V^{-m'}$ is dense in V^{-m} so $\forall y\in V^{-m}$, there exists a sequence ϕ_n in $V^{-m'}$ such that $\phi_n \to y$. So by (1) we have that $(u_\epsilon - u_v,\phi_n)\to 0$. We want to prove that $(u_\epsilon - u_v,\phi)\to(u_\epsilon - u_v,y)$ this is true since $|(u_\epsilon - u_v,\phi_n - y)| \le \|u_\epsilon - u_v\|_{H_{-m}}\|\phi_n - y\|_{H^{-m}} \le \eta'$. So we have that for $y\in V^{-m}$ that $(u_\epsilon - u_v,y)\to 0$, we know that u_ϵ, u_v are continuous so they are and weakly continuous and this completes the proof of step 4.

As far as concerned the uniqueness of the solution it occurs by the previous chapter where we have proved that if the solution exists, it is unique. □

[37][10] pg 189

[38]The dual of H^m is the H^{-m}

So far by the proposition above we see that we have our solution is on $C([0,T],C^2)\cap C^1([0,T],C)$ if we want to speak in terms of Sobolev spaces on $C([0,T],V^{m'})\cap C^1([0,T],V^{m'-2})$ for $m'<m$. Now we will prove the following theorem, which gives us continuity in the high H^m norm.

Theorem 4.3.4. *Assume that u_v is a solution as described above, then $u_v \in C([0,T),V^m)\cap C^1([0,T),V^{m-2})$*

Proof. As in **Step 3** of the previous proof we will firstly prove that $u_v = u \in C([0,T),V^m)$ and then follows that $u \in C^1([0,T),V^{m-2})$. In **Step 4**, we have proved the weak continuity -with respect to time, of our solution in V^m, we want to prove that $\lim_{\delta\to 0} u(t+\delta)\to u(t)$ i.e. $\|u(t+\delta)-u(t)\|_{H^m}\le \eta$ for $\delta\le\beta$ we know that $\|u(t+\delta)-u(t)\|_{H^m}\le (u(t+\delta)-u(t),u(t+\delta)-u(t))_{H^m}\le \left|\|u(t+\delta)\|^2_{H^m}-\|u(t)\|^2_{H^m}\right|$ so it is enough to show that $\|u\|_{H^m}$ is a continuous function.
We start with the right continuity on which is the same for $\nu=0$ and $\nu>0$.
For t=0 by the previous theorem we have that

$$\|u\|_{H^m}\le \frac{\|u_0\|_{H^m}}{1-c_m t\|u_o\|_{H^m}}$$

so

$$\lim_{t\to 0^+}\sup\|u\|_{H^m}\le \lim_{t\to 0^+}\sup\frac{\|u_0\|_{H^m}}{1-c_m t\|u_0\|_{H^m}}\le\|u_0\|_{H^m}$$

. By step 4 we have that $u_t \rightharpoonup u_0$ thus by proposition 5.3.2 we have that

$$\|u_0\|_{H^m}\le \lim_{t\to 0^+}\inf\|u\|_{H^m}$$

It is also true that $\lim_{t\to 0^+}\inf\le \lim_{t\to 0^+}\sup$ so combining those three relations we have that

$$\lim_{t\to 0^+}\|u\|^m_H=\lim_{t\to 0^+}\sup\|u\|_{H^m}=\lim_{t\to 0^+}\inf\|u\|_{H^m}=\|u_0\|_{H^m}$$

So we have that $\|u\|_{H^m}$ is strongly right continuous on 0.
For the left continuity we have to see each case individually

- $\nu=0$ The Euler equation is time reversible, indeed:
  Recall the Euler equation in 3 dimensions

$$\frac{\partial}{\partial t}u+(u\cdot\nabla)u=-\nabla p$$

  We set $u(x,t)=-v(x,t)$ and $p(x,t)=-\bar p(x,-t)$ thus we have that

$$\frac{\partial}{\partial t}u(x,t)=\frac{\partial}{\partial t}(-v(x,-t))=-\frac{\partial}{\partial t}v(x,-t)=\frac{\partial v}{\partial t}(x,-t)$$

  The other derivatives in the equation does not change with this substitution since they are derivatives with respect to x i.e. we have that

$$\frac{\partial v}{\partial t}(x,-t)+(-v\cdot\nabla)-v(x,-t)=-\nabla\bar p(x,-t)$$

So now we set $u_v(-t) = \bar{u}(t)$ and we have the same arguments:
By the previous theorem we have that

$$\|\bar{u}\|_{H^m} \leq \frac{\|\bar{u}_0\|_{H^m}}{1 - c_m t \|\bar{u}_0\|_{H^m}}$$

so

$$\lim_{t\to 0^-} \sup \|\bar{u}\|_{H^m} \leq \lim_{t\to 0^-} \sup \frac{\|\bar{u}_0\|_{H^m}}{1 - c_m t \|\bar{u}_0\|_{H^m}} \leq \|\bar{u}_0\|_{H^m}$$

. By step 4 we have that $\bar{u}_t \rightharpoonup \bar{u}_0$ thus by proposition 5.3.2 we have that

$$\|\bar{u}_0\|_{H^m} \leq \lim_{t\to 0^-} \inf \|\bar{u}\|_{H^m}$$

It is also true that $\lim_{t\to 0^-} \inf \leq \lim_{t\to 0^-} \sup$ so combining those three relations we have that

$$\lim_{t\to 0^-} \|\bar{u}\|_H^m = \lim_{t\to 0^-} \sup \|\bar{u}\|_{H^m} = \lim_{t\to 0^-} \inf \|\bar{u}\|_{H^m} = \|\bar{u}_0\|_{H^m}$$

So we have that $\|\bar{u}\|_{H^m}$ is strongly left continuous on 0. Thus the function

$$u = \begin{cases} u & [0, T) \\ \bar{u} & (-T, 0) \end{cases}$$

we have that $\|u\|_{H^m}$ is strongly continuous on 0. Now we will prove that $\|u\|_{H^m}$ is continuous for every$t \in (0, T)$, let T_0 be a random time and $u(x, T_0) = u_{T_0}$ the solution on this time. We know then that $\|u_{T_0}\|_{H^m} \leq M$ so $u_{T_0} \in V^m$ and thus we can use u_{T_0} as an initial value for the new IVP

$$\begin{cases} \frac{d}{dt} u_\epsilon = F_\epsilon(u_\epsilon) & t \in [T_0, T) \\ u_\epsilon(T_0) \end{cases}$$

Assume that $\bar{u}_\epsilon$ is the solution of the above ivp, which we are sure that exist since we have prove that we can find a global solution on the first IVP. By lemma 18 we have that

$$\frac{d}{dt}\frac{1}{2}\|\bar{u}_\epsilon\|^2_{H^m} \leq c_m \|J_\epsilon \nabla \bar{u}_\epsilon\|_{L^\infty} \|\bar{u}_\epsilon\|^2_{H^m}$$

. Following the same process as in **Step 1** we have that

$$\|\bar{u}_\epsilon\|_{H^m} \leq \frac{\|u_{T_0}\|_{H^m}}{1 - c_m t \|u_{T_0}\|_{H^m}}$$

So by **Step 2** we have a solution $\bar{u}$. So we have again the same arguments as above and we have the continuity of $\|u\|_{H^m}$ in all the interval $[0, T)$

- The Navier Stokes equation is not time reversible, so we will follow another strategy.In order to prove the right continuity on every $t \in [0, T)$ we follow the same arguments as in te case of $t = 0$.As far as concerned the left continuity, we know that u_ϵ is bounded on $L^2([0, T], V^m)$ and thus in $L^2([0, T], V^{m+1})$, this is a Hilbert space so by lemma 19 we have that there exists a subsequence that converge on

V^{m+1}. Assume now a $T_0 \in (0,T]$ we will prove the left continuity . We choose $\tilde{T}$ such that $0 < \tilde{T} < T_0$ and $u(\tilde{T}) \in V^{m+1}$ with$\tilde{T} = T_0 - \delta$. With initial value $u(\tilde{T})$ and $m = m+1$ in the theorem 4.3.1 we have that for $m' < m+1$ and $T' \geq \tilde{T}$ there exists a solution $u \in C([\tilde{T}, T'], V^{m'})$. For $m' = m$ and $\delta = 0$ we have the left continuity on T_0 and since T_0 is arbitrary we have the left continuity in all the interval $[0,T)$

So we conclude that $\|u\|_{H^m}$ is continuous thus $u_v \in C([0,T), V^m)$ and $\frac{d}{dt}u \in C([0,T), V^{m-2})$ □

The proof of the following theorem is based on the discussion about the continuity of solutions in the previous chapter. Here the following proposition summarizes the result for the existence of maximum interval for the existence of the solution u_v.

Proposition 4.3.3. *Let $u_0 \in V^m$ with $m \geq 3$ and $\nu \geq 0$, then there exists a maximum interval $[0,T^*]$ that the solution u_v described in theorem 5.3.1, exists. T^*maybe the infinity ,otherwise for $T^* < \infty$ we will have $\lim_{t\to T^*} \|u\|_{H^m} = \infty$*

Proof. Assume that $T^* \leq \infty$ is the maximum time and $\lim_{t\to T^*} \|u\|_{H^m} = \infty$, then we have already seen that we can extend the interval of existence, say $[0, T^* + \delta)$ which contradicts the initial hypothesis. □

4.3.2 2 dimensions

We will use again the radial energy decomposition, the following theorem will give a solution y which is the limit of y_ϵ.

Theorem 4.3.5. *Let $u_0 \in V^m$ with $m \geq \left[\frac{2}{2}\right] + 2 = 3$ then*

1. *There exists a time T with upper bound which depends on the initial value i.e.*

$$T \leq \frac{\ln\left(\frac{\|\nabla b\|_{L^\infty} + \|y_0\|_{H^m}}{\|y_0\|_{H^m}}\right)}{c_m \|\nabla b\|_{L^\infty}}$$

such that $\forall \nu \geq 0$ there exists a $y_v \in C\left([0,T], C^2(\mathbb{R}^2)\right) \cap C^1\left([0,T], C(\mathbb{R}^2)\right)$ which defines a unique solution for the Euler and Navier-Stokes equations in the Leray's form.

2. *The solutions y_v and y_ϵ satisfies the following estimates*

$$\sup_{0\leq t\leq T} \|y_\epsilon\|_{H^m} \leq \frac{\|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}{1 + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}$$

$$\sup_{0\leq t\leq T} \|y_v\|_{H^m} \leq \frac{\|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}{1 + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}$$

3. *The solutionsy_ϵ, y_v are uniformly bounded on the spaces*

- $L^2\left([0,T], V^m(\mathbb{R}^2)\right)$
- $L^\infty\left([0,T], V^m(\mathbb{R}^2)\right)$
- $Lip\left([0,T], V^{m-2}(\mathbb{R}^2)\right)$
- $C_W\left([0,T], V^m(\mathbb{R}^2)\right)$

For the proof of the theorem we will do exactly the same things as in the 3 dimensions case we will prove four steps and one lemma which give us an estimate with no bad dependence on ϵ.

Lemma 23. *Assume that $y_0 \in V^m$ then for the solution y_ϵ of the (IVP) it is true that*

$$\frac{d}{dt}\|y_\epsilon\|_{H^m} \leq c_m\left[\|\nabla J_\epsilon y_\epsilon\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right]\|u_\epsilon\|_{H^m}$$

Proof. Since y_ϵ is the solution of (IVP) in the 2 dimensions we have that

$$\frac{d}{dt}y_\epsilon = F_\epsilon(y_\epsilon)$$

We differentiate this relation over D^a for any multiindex and we get

$$D^a\left(\frac{d}{dt}y_\epsilon\right) = D^a\left(F_\epsilon(y_\epsilon)\right)$$

We continue by multiplying this relation with $D^a y_\epsilon$ in $L^2(\mathbb{R}^2)$ and we have that

$$\int_{\mathbb{R}^2} D^a\left(\frac{d}{dt}y_\epsilon\right) \cdot D^a y_\epsilon dx = \int_{\mathbb{R}^2} D^a\left(F_\epsilon(y_\epsilon)\right) \cdot D^a y_\epsilon dx$$

We substitute F_ϵ and we reach to the relation

$$\int_{\mathbb{R}^2} D^a\left(\frac{d}{dt}y_\epsilon\right) \cdot D^a y_\epsilon dx = \nu \int_{\mathbb{R}^2} D^a\left(J_\epsilon^2 \Delta y_\epsilon\right) \cdot D^a y_\epsilon dx$$

$$-\int_{\mathbb{R}^2} D^a\left\{PJ_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]\right\} \cdot D^a y_\epsilon dx$$

By simple calculations and the Leibniz integral rule we have that

$$\frac{1}{2}\frac{d}{dt}\|D^a y_\epsilon\|_{L^2}^2 = -\nu\|D^{a+1}J_\epsilon y_\epsilon\|_{L^2}^2$$

$$-\int_{\mathbb{R}^2} D^a\left\{PJ_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]\right\} \cdot D^a y_\epsilon dx$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|D^a y_\epsilon\|_{L^2}^2 \leq \int_{\mathbb{R}^2} D^a\left\{PJ_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]\right\} \cdot D^a y_\epsilon dx \tag{L20}$$

Now we will deal with the integral on the right hand of the above relation, we will separate it in two integrals which will estimate individually

$$I_1 = \int_{\mathbb{R}^2} D^a \left\{PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right]\right\} \cdot D^a y_\epsilon dx$$

$$I_2 = \int_{\mathbb{R}^2} D^a \left\{PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)b + (b \cdot \nabla)J_\epsilon y_\epsilon\right]\right\} \cdot D^a y_\epsilon dx$$

For the I_1 we sum and subtract the term $PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right]$, thus

$$I_1 = \int_{\mathbb{R}^2} D^a \left\{PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right]\right\} - PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right] + PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right] \cdot D^a y_\epsilon dx$$

We also have that $\int_{\mathbb{R}^2} PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right] \cdot D^a y_\epsilon dx \leq \int_{\mathbb{R}^2} \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right] \cdot D^a y_\epsilon dx \leq 0$

So

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}^2} \left[D^a \left\{PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right]\right\} - PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right]\right] \cdot D^a y_\epsilon dx \\ &\leq \|D^a y_\epsilon\|_{L^2} \|D^a \left\{PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right]\right\} - PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right] \|_{L^2} \end{aligned}$$

For the I_2 we have that

$$I_2 \leq c \int_{\mathbb{R}^2} D^a [(J_\epsilon y_\epsilon \cdot \nabla)b J_\epsilon y_\epsilon] \cdot D^a y_\epsilon \, dx, \quad c > 1$$

Since

By the properties of the projection P and the mollifier J_ϵ we write the integral in the following form:

$$I_2 = \int_{\mathbb{R}^2} D^a \{[(J_\epsilon y_\epsilon \cdot \nabla)b] + [(b \cdot \nabla)J_\epsilon y_\epsilon]\} D^a J_\epsilon y_\epsilon dx$$

We set

$$I_2^1 = \int_{\mathbb{R}^2} D^a [(b \cdot \nabla)J_\epsilon y_\epsilon] D^a J_\epsilon y_\epsilon$$

and

$$I_2^2 = \int_{\mathbb{R}^2} D^a [(J_\epsilon y_\epsilon \cdot \nabla)b] D^a J_\epsilon y_\epsilon$$

Our goal is to prove that $I_2 \leq c I_2^2$

- When $a = 0$ we get

$$I_2^1 = \int_{\mathbb{R}^2} [(b \cdot \nabla)J_\epsilon y_\epsilon] J_\epsilon y_\epsilon$$

and

$$I_2^2 = \int_{\mathbb{R}^2} [(J_\epsilon y_\epsilon \cdot \nabla)b] J_\epsilon y_\epsilon$$

By footnote 26 page 135 we recall that

$$I_2 = \frac{1}{2} I_2^2$$

- When $a = 1$ we get

$$I_2^1 = \int_{\mathbb{R}^2} \nabla[(b \cdot \nabla) J_\epsilon y_\epsilon] \nabla J_\epsilon y_\epsilon$$

$$I_2^1 = \underbrace{\int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \; b \; \nabla J_\epsilon y_\epsilon \; dx}_{[I_2^1]_1} + \underbrace{\int_{\mathbb{R}^2} \nabla J_\epsilon y_\epsilon \; \nabla b \; \nabla J_\epsilon y_\epsilon}_{[I_2^1]_2}$$

and

$$I_2^2 = \int_{\mathbb{R}^2} \nabla[(J_\epsilon y_\epsilon \cdot \nabla) b] \nabla J_\epsilon y_\epsilon$$

$$I_2^2 = \underbrace{\int_{\mathbb{R}^2} D^2 b \; J_\epsilon y_\epsilon \; \nabla J_\epsilon y_\epsilon \; dx}_{[I_2^2]_1} + \underbrace{\int_{\mathbb{R}^2} \nabla J_\epsilon y_\epsilon \; \nabla b \; \nabla J_\epsilon y_\epsilon}_{[I_2^2]_2}$$

We easily observe that $[I_2^1]_2 = [I_2^2]_2$.
For the integral $[I_2^1]_1$, using integration by parts we have:

$$\int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \; b \; \nabla J_\epsilon y_\epsilon \; dx = -\int_{\mathbb{R}^2} \nabla J_\epsilon y_\epsilon \; \nabla b \nabla J_\epsilon y_\epsilon \; dx - \int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \; b \; \nabla J_\epsilon y_\epsilon \; dx$$

Thus $2[I_2^1]_1 = 2\int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \; b \; \nabla J_\epsilon y_\epsilon \; dx = -\int_{\mathbb{R}^2} \nabla J_\epsilon y_\epsilon \; \nabla b \nabla J_\epsilon y_\epsilon \; dx = -[I_2^2]_2$
So we conclude that $I_2 = -\frac{1}{2}[I_2^2]_2 + [I_2^2]_2 + [I_2^2]_1 + [I_2^2]_2 = [I_2^2]_1 + \frac{3}{2}[I_2^2]_2 \leq c I_2^2$

- When $a = 2$

$$I_2^1 = \int_{\mathbb{R}^2} D^2[(b \cdot \nabla) J_\epsilon y_\epsilon] D^2 J_\epsilon y_\epsilon$$

$$I_2^1 = \underbrace{\int_{\mathbb{R}^2} D^3 J_\epsilon y_\epsilon \; b \; D^2 J_\epsilon y_\epsilon \; dx}_{[I_2^1]_1} + \underbrace{2\int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \; \nabla b \; D^2 J_\epsilon y_\epsilon dx}_{[I_2^1]_2}$$

$$+ \underbrace{\int_{\mathbb{R}^2} \nabla J_\epsilon y_\epsilon \; D^2 b \; D^2 J_\epsilon y_\epsilon \; dx}_{[I_2^1]_3}$$

and

$$I_2^2 = \int_{\mathbb{R}^2} D^2[(J_\epsilon y_\epsilon \cdot \nabla) b] D^2 J_\epsilon y_\epsilon$$

$$I_2^2 = \underbrace{\int_{\mathbb{R}^2} D^3 b \; J_\epsilon y_\epsilon \; D^2 J_\epsilon y_\epsilon \; dx}_{[I_2^2]_1} + \underbrace{2\int_{\mathbb{R}^2} \nabla J_\epsilon y_\epsilon \; D^2 b \; D^2 J_\epsilon y_\epsilon \; dx}_{[I_2^2]_2}$$

$$+\underbrace{\int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \, \nabla b \, D^2 J_\epsilon y_\epsilon dx}_{[I_2^2]_3}$$

So $[I_2^1]_2 = [I_2^2]_3$, $[I_2^1]_3 = [I_2^2]_2$
For the integral $[I_2^1]_1$, using integration by parts we have:

$$\int_{\mathbb{R}^2} D^3 J_\epsilon y_\epsilon \, b \, D^2 J_\epsilon y_\epsilon dx = -\int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \, \nabla b \nabla D^2 J_\epsilon y_\epsilon dx - \int_{\mathbb{R}^2} D^3 J_\epsilon y_\epsilon \, b \, D^2 J_\epsilon y_\epsilon dx$$

Thus

$$2[I_2^1]_1 = 2\int_{\mathbb{R}^2} D^3 J_\epsilon y_\epsilon \, b \, D^2 J_\epsilon y_\epsilon \, dx = -\int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \, \nabla b \, D^2 J_\epsilon y_\epsilon dx = -[I_2^2]_3$$

So we conclude that

$$I_2 = -\frac{1}{2}[I_2^2]_3 + 2[I_2^2]_3 + [I_2^2]_2 + [I_2^2]_1 + 2[I_2^2]_2 + [I_2^2]_3 \leq cI_2^2$$

- When $a = 3$

$$I_2^1 = \int_{\mathbb{R}^2} D^3[(b \cdot \nabla) J_\epsilon y_\epsilon] D^3 J_\epsilon y_\epsilon$$

$$I_2^1 = \underbrace{\int_{\mathbb{R}^2} D^4 J_\epsilon y_\epsilon \, b \, D^3 J_\epsilon y_\epsilon dx}_{[I_2^1]_1} + 3\underbrace{\int_{\mathbb{R}^2} D^3 J_\epsilon y_\epsilon \, \nabla b \, D^3 J_\epsilon y_\epsilon \, dx}_{[I_2^1]_2}$$

$$+3\underbrace{\int_{\mathbb{R}^2} D^2 J_\epsilon y_\epsilon \, D^2 b \, D^3 J_\epsilon y_\epsilon}_{[I_2^1]_3} + \underbrace{\int_{\mathbb{R}^2} \nabla J_\epsilon y_\epsilon \, D^3 b \, D^3 J_\epsilon y_\epsilon}_{[I_2^1]_4}$$

and

$$I_2^2 = \int_{\mathbb{R}^2} D^3[(J_\epsilon y_\epsilon \cdot \nabla) b] D^3 J_\epsilon y_\epsilon$$

$$I_2^2 = \underbrace{\int_{\mathbb{R}^2} D^4 b \, J_\epsilon y_\epsilon \, D^3 J_\epsilon y_\epsilon}_{[I_2^2]_1} + 3\underbrace{\int_{\mathbb{R}^2} D^3 b \, \nabla J_\epsilon y_\epsilon \, D^3 J_\epsilon y_\epsilon}_{[I_2^2]_2}$$

$$+3\underbrace{\int_{\mathbb{R}^2} D^2 b \, D^2 J_\epsilon y_\epsilon \, D^3 J_\epsilon y_\epsilon \, dx}_{[I_2^2]_3} + \underbrace{\int_{\mathbb{R}^2} \nabla b \, D^3 J_\epsilon y_\epsilon \, D^3 J_\epsilon y_\epsilon \, dx}_{[I_2^2]_4}$$

It is obvious that: $[I_2^1]_4 = [I_2^2]_2$, $[I_2^1]_3 = [I_2^2]_3$, $[I_2^1]_2 = [I_2^2]_4$.
For the integral $[I_2^1]_1$, using integration by parts we have:

$$\int_{\mathbb{R}^2} D^4 J_\epsilon y_\epsilon \, b \, D^3 J_\epsilon y_\epsilon = -\int_{\mathbb{R}^2} D^3 J_\epsilon y_\epsilon \, \nabla b \, D^3 J_\epsilon y_\epsilon dx - \int_{\mathbb{R}^2} D^3 J_\epsilon y_\epsilon \, b \, D^4 J_\epsilon y_\epsilon \, dx$$

So

$$2[I_2^1]_1 = 2\int_{\mathbb{R}^2} D^4 J_\epsilon y_\epsilon \; b \; D^3 J_\epsilon y_\epsilon = -\int_{\mathbb{R}^2} \nabla b \; D^3 J_\epsilon y_\epsilon \; D^3 J_\epsilon y_\epsilon \; dx = [I_2^2]_4$$

Thus

$$I_2 = -\frac{1}{2}[I_2^2]_4 + 3[I_2^2]_4 + [I_2^2]_3 + [I_2^2]_2 + [I_2^2]_1 + 3[I_2^2]_2 + 3[I_2^2]_3 + [I_2^2]_4 \leq cI_2^2$$

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- When $a = k$ for the integral I_2^1 will occur $k+1$ integrals and for I_2^2 will also occur $k+1$ integrals. The $2, ..., k+1$ integrals of I_2^1 are equal to some of the integrals of I_2^2. For the integral $[I_2^1]_1$, using integration by parts we have that $[I_2^1]_1 = [I_2^2]_{k+1}$. Thus we conclude that $I_2 \leq cI_2^2$

So by Holders inequality we have $I_2 \leq c\|D^a J_\epsilon y_\epsilon\|_{L^2}\|D^a[(J_\epsilon y_\epsilon \cdot \nabla)b]\|_{L^2}$
By combining all the above relations with (L20) we get :

$$\frac{1}{2}\frac{d}{dt}\|D^a y_\epsilon\|_{L^2}^2 \leq \|D^a y_\epsilon\|_{L^2}$$

$$\left[\|D^a \left\{PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right]\right\} - PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right]\|_{L^2} + \|D^a \left[(J_\epsilon y_\epsilon \cdot \nabla)b\right]\|_{L^2}\right]$$

Since $a \leq m$ we have

$$\frac{1}{2}\frac{d}{dt}\|D^a y_\epsilon\|_{L^2}^2 \leq \|y_\epsilon\|_{H^m}$$

$$\left[\|D^a \left\{PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right]\right\} - PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right]\|_{L^2} + \|(J_\epsilon y_\epsilon \cdot \nabla)b\|_{H^m}\right]$$

We sum over this relation and we get

$$\frac{1}{2}\frac{d}{dt}\sum_{|a|\leq m}\|D^a y_\epsilon\|_{L^2}^2 \leq \|y_\epsilon\|_{H^m}$$

$$\sum_{|a|\leq m}\left[\|D^a \left\{PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right]\right\} - PJ_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right]\|_{L^2} + \|(J_\epsilon y_\epsilon \cdot \nabla)b\|_{H^m}\right]$$

By the proposition 4.1.13 we have

$$\frac{1}{2}\frac{d}{dt}\|y_\epsilon\|_{H^m}^2 \leq \|y_\epsilon\|_{H^m}$$

$$c\sum_{|a|\leq m}\left[\|D^a \left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right] - \left[(J_\epsilon y_\epsilon \cdot \nabla)D^a J_\epsilon y_\epsilon\right]\|_{L^2} + \|(J_\epsilon y_\epsilon \cdot \nabla)b\|_{H^m}\right]$$

And by proposition 4.1.4

$$\frac{1}{2}\frac{d}{dt}\|y_\epsilon\|_{H^m}^2 \leq c\|y_\epsilon\|_{H^m}$$

$$\left[\|\nabla J_\epsilon y_\epsilon\|_{L^\infty}\|J_\epsilon y_\epsilon\|_{H^m} + \|J_\epsilon y_\epsilon\|_{H^m}\|\nabla J_\epsilon y_\epsilon\|_{L^\infty} + \|\nabla b\|_{L^\infty}\|J_\epsilon y_\epsilon\|_{H^m}\right]$$

By the proposition 4.1.13

$$\frac{1}{2}\frac{d}{dt}\|y_\epsilon\|^2_{H^m} \leq c\|y_\epsilon\|^2_{H^m}\left[\|\nabla J_\epsilon\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right]$$

So eventually we have that

$$\frac{d}{dt}\|y_\epsilon\|_{H^m} \leq c\|y_\epsilon\|_{H^m}\left[\|\nabla J_\epsilon\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right]$$

□

We recall the four steps we will need in order to prove the theorem 4.3.4
Step 1: We will show the energy estimate

$$\sup_{0\leq t\leq T}\|y_\epsilon\|_{H^m} \leq \frac{\|y_0\|_{H^m}e^{c_m\|\nabla b\|_{L^\infty}T}}{1+\|y_0\|_{H^m} - \|y_0\|_{H^m}e^{c_m\|\nabla b\|_{L^\infty}T}}$$

Step 2:We will prove that the family of the regularized solutions forms a Cauchy sequence on $C\left([0,T],L^2(\mathbb{R}^2)\right)$
Step 3: We will prove strong convergence in all the intermediate norms of the high norm of the space $C\left([0,T],H^m(\mathbb{R}^2)\right)$
Step 4:Via weak convergences we will prove the third part of the theorem

Proof. Of theorem 4.3.4
Step 1:By lemma 22 we have that

$$\frac{d}{dt}\|y_\epsilon\|_{H^m} \leq c\|y_\epsilon\|_{H^m}\left[\|\nabla J_\epsilon\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right]$$

We set $c_b = c_m\|\nabla b\|_{L^\infty}$ and by the Sobolev embedding theorem we have that

$$\frac{d}{dt}\|y_\epsilon\|_{H^m} \leq c_m\|y_\epsilon\|^2_{H^m} + c_b\|y_\epsilon\|_{H^m}$$

So now we will find the solution which satisfies this inequality

Assume that q(t) solves the differential equation

$$\frac{d}{dt}q(t) = c_m q^2(t) + c_b q(t)$$

with initial value $q(0) = \|y_0\|_{H^m}$. Firstly we will find a relation between our function $\|y_\epsilon\|_{H^m}$ and q(t) We have that

$$\frac{d}{dt}(\|y_\epsilon\|_{H^m} - q(t)) = c_m(\|y_\epsilon\|^2_{H^m} - q^2(t)) + c_b(\|y_\epsilon\|_{H^m} - q(t))$$

We set $z(t) = \|y_\epsilon\|_{H^m} - q(t)$
It follows that

$$\frac{d}{dt}z(t) \leq c_m(\|y_\epsilon\|_{H^m} + q(t))z(t) + c_b z(t)$$

We also set $p(t) = c_m(\|y_\epsilon\|_{H^m} + q(t)) + c_b$, and we reach to the following inequality

$$\frac{d}{dt}z(t) \leq p(t)z(t)$$

By Gronwall in differential form we have that

$$z(t) \leq z(0)e^{\int_0^t p(s)ds}$$

We have that $z(0) = \|y_0\|_{H^m} - q(0) = 0$ thus $z(t) \leq 0$ i.e. $\|y_\epsilon\|_{H^m} \leq q(t)$ so now we will define q(t) so we will solve the initial value problem:

$$\begin{cases} \frac{d}{dt}q(t) = c_m q^2(t) + c_b q(t) \\ q(0) = \|y_0\|_{H^m} \end{cases}$$

This is a Bernoulli differential equation[a] so we have that if we set $k(t) = \frac{1}{q(t)}$ we will reach to a linear first order ordinary differential equation,i.e.

$$\begin{cases} \frac{d}{dt}k(t) + c_b k(t) = -c_m \\ k(0) = \frac{1}{q(0)} = \frac{1}{\|y_0\|_{H^m}} \end{cases}$$

With solution $k(t) = e^{-\int_0^t c_b ds}\big[c + \int_0^t -c_m e^{\int_t^s c_b du} ds\big]$, we can easily calculate that $\int_0^t c_b ds = c_b t$ and $\int_s^t c_b du = c_b t - c_b s$ so we have that

$$k(t) = e^{-c_b t}\big[c + \int_0^t -c_m e^{c_b t} e^{-c_b s} ds\big]$$

So

$$k(t) = ce^{-c_b t} - \frac{c_m}{c_b} + \frac{c_m}{c_b e^{c_b t}}$$

Thus we have that $q(t) = \frac{c_b e^{c_b t}}{cc_b - c_m e^{c_b t} + c_m}$ for $t = 0$ we have that the $c = \frac{1}{\|y_0\|_{H^m}}$ Eventually $q(t) = \frac{c_b e^{c_b t}\|y_0\|_{H^m}}{c_b - c_m\|y_0\|_{H^m} e^{c_b t} + c_m\|y_0\|_{H^m}}$

[a] [2] pg 32

So we have that

$$\|y_\epsilon\|_{H^m} \leq \frac{\|y_0\|_{H^m} e^{c_m\|\nabla b\|_{L^\infty} t}}{c_m\|\nabla b\|_{L^\infty} + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m\|\nabla b\|_{L^\infty} t}}$$

Taking the supremum over this relation we have for the right quantity is an invreasing function of t, so the supremum of this fraction is achieved when t=T, so we conclude that

$$\sup_{0\leq t\leq T} \|y_\epsilon\|_{H^m} \leq \frac{\|y_0\|_{H^m} e^{c_m\|\nabla b\|_{L^\infty} T}}{c_m\|\nabla b\|_{L^\infty} + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m\|\nabla b\|_{L^\infty} T}}$$

To ensure that the right quantity has a supremum we need

$$T \leq \frac{\ln(c_m\|\nabla b\|_{L^\infty} + \|y_0\|_{H^m}) - \ln(\|y_0\|_{H^m})}{c_m\|\nabla b\|_{L^\infty}}$$

Step 2: In order to prove this step we will estimate the $\sup_{0\leq t\leq T}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$ via energy methods. Assume that $y_\epsilon, y_{\epsilon'}$ satisfy the (IVP) on the 2 dimensions, then we have that $\frac{d}{dt}y_\epsilon = F_\epsilon(y_\epsilon)$ and $\frac{d}{dt}y_{\epsilon'} = F_{\epsilon'}(y_{\epsilon'})$, we subtract those two relations and we get

$$\frac{d}{dt}(y_\epsilon - y_{\epsilon'}) = F_\epsilon(y_\epsilon) - F_{\epsilon'}(y_{\epsilon'})$$

We multiply this relation with $y_\epsilon - y_{\epsilon'}$ in L^2 and we get

$$\int_{\mathbb{R}^2} \frac{d}{dt}(y_\epsilon - y_{\epsilon'}) \cdot (y_\epsilon - y_{\epsilon'})dx = \int_{\mathbb{R}^2} [F_\epsilon(y_\epsilon) - F_{\epsilon'}(y_{\epsilon'})] \cdot (y_\epsilon - y_{\epsilon'})dx$$

So

$$\frac{1}{2}\frac{d}{dt}\|y_\epsilon - y_{\epsilon'}\|_{L^2}^2 = \int_{\mathbb{R}^2} [F_{\epsilon'}(y_{\epsilon'}) - F_{\epsilon'}(y_{\epsilon'})] \cdot (y_\epsilon - y_{\epsilon'})dx$$

Now we will deal with the integral on the right side of this equality. Firstly we have that

$$F_{\epsilon'}(y_{\epsilon'}) - F_{\epsilon'}(y_{\epsilon'}) = \nu J_\epsilon^2 \Delta y_\epsilon - PJ_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]$$
$$-\nu J_{\epsilon'}^2 \Delta y_{\epsilon'} + PJ_{\epsilon'}\left[(J_{\epsilon'} y_{\epsilon'} \cdot \nabla)J_{\epsilon'} y_{\epsilon'} + (b\cdot\nabla)J_{\epsilon'} y_{\epsilon'} + (J_{\epsilon'} y_{\epsilon'} \cdot \nabla)b\right]$$

So we have to estimate the integrals

$$\int_{\mathbb{R}^2} \nu(J_\epsilon^2\Delta y_\epsilon - J_{\epsilon'}^2\Delta y_{\epsilon'}) \cdot (y_\epsilon - y_{\epsilon'})dx$$

and

$$\int_{\mathbb{R}^2} P\Bigg[J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon \cdot \nabla)b\right]$$
$$-J_{\epsilon'}\left[(J_{\epsilon'} y_{\epsilon'} \cdot \nabla)J_{\epsilon'} y_{\epsilon'} + (b\cdot\nabla)J_{\epsilon'} y_{\epsilon'} + (J_{\epsilon'} y_{\epsilon'} \cdot \nabla)b\right]\Bigg] \cdot (y_\epsilon - y_{\epsilon'})dx$$

For the first integral we sum and subtract the term $J_{\epsilon'}^2\Delta y_\epsilon$ so

$$\int_{\mathbb{R}^2} \nu(J_\epsilon^2\Delta y_\epsilon - J_{\epsilon'}^2\Delta y_{\epsilon'}) \cdot (y_\epsilon - y_{\epsilon'})dx = \nu\int_{\mathbb{R}^2}\left[J_\epsilon^2\Delta y_\epsilon - J_{\epsilon'}^2\Delta y_\epsilon + J_{\epsilon'}^2\Delta y_\epsilon - J_{\epsilon'}^2\Delta y_{\epsilon'}\right] \cdot (y_\epsilon - y_{\epsilon'})dx$$

$$= \nu\Big[\int_{\mathbb{R}^2}(J_\epsilon^2 - J_\epsilon^2)\Delta y_\epsilon(y_\epsilon - y_{\epsilon'})dx$$
$$-\int_{\mathbb{R}^2} J_{\epsilon'}^2\left(\nabla(y_\epsilon - y_{\epsilon'})\cdot\nabla(y_\epsilon - y_{\epsilon'})\right)dx\Big]$$
$$\leq \nu\int_{\mathbb{R}^2}(J_\epsilon^2 - J_{\epsilon'}^2)\Delta y_\epsilon \cdot (y_\epsilon - y_{\epsilon'})dx$$
$$\leq \nu\|(J_\epsilon^2 - J_{\epsilon'}^2)\Delta y_\epsilon\|_{L^2}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

By triangle inequality we have that

$$\leq \nu\left(\|J_\epsilon^2\Delta y_\epsilon\|_{L^2} + \|J_{\epsilon'}^2\Delta y_\epsilon\|_{L^2}\right)\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

$$\le \nu \left(\|J_\epsilon^2 y_\epsilon\|_{H^2} + \|J_{\epsilon'} y_\epsilon\|_{H^2} \right) \|y_\epsilon - y_{\epsilon'}\|_{L^2}$$
$$\le \nu c \left(\epsilon \|y_\epsilon\|_{H^3} + \epsilon' \|y_\epsilon\|_{H^3} \right) \|y_\epsilon - y_{\epsilon'}\|_{L^2}$$
$$\le \nu c \max(\epsilon, \epsilon') \|y_\epsilon\|_{H^m} \|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

For the second integral we have the following three integrals:

$$I_1 = \int_{\mathbb{R}^2} P\big[J_\epsilon[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon] - J_{\epsilon'}[(J_\epsilon y_{\epsilon'} \cdot \nabla) J_{\epsilon'} y_{\epsilon'}]\big] \cdot (y_\epsilon - y_{\epsilon'}) dx$$

$$I_2 = \int_{\mathbb{R}^2} P\big[J_\epsilon[(b \cdot \nabla) J_\epsilon y_\epsilon] - J_{\epsilon'}[(b \cdot \nabla) J_{\epsilon'} y_{\epsilon'}]\big] \cdot (y_\epsilon - y_{\epsilon'}) dx$$
$$I_3 = P\big[J_\epsilon[(J_\epsilon y_\epsilon \cdot \nabla) b] - J_{\epsilon'}[(J_\epsilon y_{\epsilon'} \cdot \nabla) b]\big] \cdot (y_\epsilon - y_{\epsilon'}) dx$$

We will see each integral individually, for the first integral we sum and subtract the term $J_{\epsilon'}[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon]$, so

$$\int_{\mathbb{R}^2} \{P J_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right] - P J_{\epsilon'} \left[(J_{\epsilon'} y_{\epsilon'} \cdot \nabla) J_{\epsilon'} y_{\epsilon'}\right]\} \cdot (y_\epsilon - y_{\epsilon'}) dx$$

$$\le \int_{\mathbb{R}^2} \{J_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right] - J_{\epsilon'} \left[(J_{\epsilon'} y_{\epsilon'} \cdot \nabla) J_{\epsilon'} y_{\epsilon'}\right]\} \cdot (y_\epsilon - y_{\epsilon'}) dx$$

We will add and subtract the term $J_{\epsilon'} \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right]$

$$\le \int_{\mathbb{R}^2} J_\epsilon \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right] - J_{\epsilon'} \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right]$$
$$+ J_{\epsilon'} \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right] - J_{\epsilon'} \left[(J_{\epsilon'} y_{\epsilon'} \cdot \nabla) J_{\epsilon'} y_{\epsilon'}\right]$$
$$\cdot (y_\epsilon - y_{\epsilon'}) dx$$

So we have two integrals:

$$\int_{\mathbb{R}^2} (J_\epsilon - J_{\epsilon'}) \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right] \cdot (y_\epsilon - y_{\epsilon'}) dx$$

and

$$\int_{\mathbb{R}^2} J_{\epsilon'} \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon - (J_{\epsilon'} y_{\epsilon'} \cdot \nabla) J_{\epsilon'} y_{\epsilon'}\right] \cdot (y_\epsilon - y_{\epsilon'}) dx$$

In the second integral we add and subtract the terms $(J_{\epsilon'} y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon$, $(J_{\epsilon'} y_{\epsilon'} \cdot \nabla) J_\epsilon y_\epsilon$, $(J_\epsilon y_{\epsilon'} \cdot \nabla) J_{\epsilon'} y_\epsilon$

So eventually we will estimate the following 5 integrals:

(a) $= \int_{\mathbb{R}^2} (J_\epsilon - J_{\epsilon'}) \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right] \cdot (y_\epsilon - y_{\epsilon'}) dx$

(b) $= \int_{\mathbb{R}^2} J_{\epsilon'} \left\{\left[(J_\epsilon - J_{\epsilon'}) y_\epsilon \cdot \nabla\right] J_\epsilon y_\epsilon\right\} \cdot (y_\epsilon - y_{\epsilon'}) dx$

(c) $= \int_{\mathbb{R}^2} J_{\epsilon'} \left\{\left[J_{\epsilon'}(y_\epsilon - y_{\epsilon'}) \cdot \nabla\right] J_\epsilon y_\epsilon\right\} \cdot (y_\epsilon - y_{\epsilon'}) dx$

(d) $= \int_{\mathbb{R}^2} J_\epsilon \left\{J_{\epsilon'} \left[(y_\epsilon \cdot \nabla)(J_\epsilon - J_{\epsilon'}) y_\epsilon\right]\right\} \cdot (y_\epsilon - y_{\epsilon'}) dx$

(e) $= \int_{\mathbb{R}^2} J_{\epsilon'} \left\{J_{\epsilon'} \left[(y_{\epsilon'} \cdot \nabla) J_{\epsilon'}(y_\epsilon - y_{\epsilon'})\right]\right\} \cdot (y_\epsilon - y_{\epsilon'}) dx$

So let's start the estimates

$$(a) \le \| (J_\epsilon - J_{\epsilon'}) \left[(J_\epsilon y_\epsilon \cdot \nabla) J_\epsilon y_\epsilon\right] \|_{L^2} \|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

For the first norm we add and subtract the term $(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon$ by triangle inequality we have that

$\| \left(J_\epsilon - J_{\epsilon'}\right)\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right] \|_{L^2} \leq \|J_\epsilon\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right] - (J_\epsilon u_\epsilon \cdot \nabla)J_\epsilon u_\epsilon\|_{H^0} + \|J_{\epsilon'}\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right] - (J_\epsilon u_\epsilon \cdot \nabla)J_\epsilon u_\epsilon\|$

By proposition 4.1.11 we have that

$$\leq c\epsilon\|\nabla J_\epsilon y_\epsilon \cdot J_\epsilon y_\epsilon\|_{H^1} + c\epsilon'\|\nabla J_\epsilon y_\epsilon \cdot J_\epsilon y_\epsilon\|_{H^1}$$

$$\leq 2c\max(\epsilon, \epsilon')\|\nabla J_\epsilon y_\epsilon \cdot J_\epsilon y_\epsilon\|_{H^1}$$

By proposition 4.1.4 we have that

$$\leq c\max(\epsilon, \epsilon')\left\{\|\nabla J_\epsilon y_\epsilon\|_{L^\infty}\|J_\epsilon y_\epsilon\|_{H^1} + \|J_\epsilon y_\epsilon\|_{L^\infty}\|\nabla J_\epsilon y_\epsilon\|_{H^1}\right\}$$

By proposition 4.1.7and the fact that $m \geq 3$ we have that

$$\leq c\max(\epsilon, \epsilon')\left(\|\nabla y_\epsilon\|_{L^\infty}\|y_\epsilon\|_{H^m} + \|y_\epsilon\|_{L^\infty}\|y_\epsilon\|_{H^m}\right)$$

By the Sobolev embedding theorem we have that $\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty} \leq \sup_{|a|\leq 2}\|D^a u\|_{L^\infty} \leq c_m\|u\|_{H_m}$ Thus as a result

$$\| \left(J_\epsilon - J_{\epsilon'}\right)\left[(J_\epsilon y_\epsilon \cdot \nabla)J_\epsilon y_\epsilon\right] \|_{L^2} \leq c\max(\epsilon, \epsilon')\|y_\epsilon\|_{H^m}^2$$

So $(a) \leq c_a \max(\epsilon, \epsilon')\|y_\epsilon\|_{H^m}^2\|y_\epsilon - y_{\epsilon'}\|_{L^2}$

$$(b) \leq \|J_{\epsilon'}\left\{\left[\left(J_\epsilon - J_{\epsilon'}\right)y_\epsilon \cdot \nabla\right]J_\epsilon y_\epsilon\right\}\|_{L^2}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

For the first norm by proposition 4.1.13 we have that

$$\|J_{\epsilon'}\left\{\left[\left(J_\epsilon - J_{\epsilon'}\right)y_\epsilon \cdot \nabla\right]J_\epsilon y_\epsilon\right\}\|_{L^2} \leq \frac{c}{\epsilon^0}\|\left[\left(J_\epsilon - J_{\epsilon'}\right)y_\epsilon \cdot \nabla\right]J_\epsilon y_\epsilon\|_{H^0}$$

$$\leq c\left[\left(J_\epsilon - J_{\epsilon'}\right)y_\epsilon \cdot \nabla\right]J_\epsilon y_\epsilon\|_{H^4}$$

Since $H^4(\mathbb{R}^2)$ is a Banach algebra we have:

$$c \leq \|\nabla J_\epsilon y_\epsilon\|_{H^4}\|(J_\epsilon - J_{\epsilon'})y_\epsilon\|_{H^4}$$

So by proposition and 4.1.13 we have that

$$\leq c\|y_\epsilon\|_{H^m}\left(\|J_\epsilon y_\epsilon - y_\epsilon\|_{H^3} + \|J_{\epsilon'}y_\epsilon - y_\epsilon\|_{H^3}\right)$$

By proposition 4.1.11

$$\leq c\|y_\epsilon\|_{H^m}\left(\epsilon\|y_\epsilon\|_{H^m} + \epsilon'\|y_\epsilon\|_{H^m}\right)$$

$$\|J_{\epsilon'}\left\{\left[\left(J_\epsilon - J_{\epsilon'}\right)y_\epsilon \cdot \nabla\right]J_\epsilon y_\epsilon\right\}\|_{L^2} \leq c\max(\epsilon, \epsilon')\|y_\epsilon\|_{H^m}^2$$

Consequently $(b) \leq c_b \max(\epsilon, \epsilon')\|y_\epsilon\|_{H^m}^2\|y_\epsilon - y_{\epsilon'}\|_{L^2}$

$$(c) \leq \|J_{\epsilon'}\left\{\left[J_{\epsilon'}(y_\epsilon - y_{\epsilon'}) \cdot \nabla\right]J_\epsilon y_\epsilon\right\}\|_{L^2}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

For the first norm by the proposition 4.1.13 we have that

$$\|J_{\epsilon'}\left\{\left[J_{\epsilon'}(y_\epsilon - y_{\epsilon'}) \cdot \nabla\right]J_\epsilon y_\epsilon\right\}\|_{L^2} \leq c\|\nabla J_\epsilon y_\epsilon \cdot J_{\epsilon'}(y_\epsilon - y_{\epsilon'})_{H^0}$$

By the proposition 4.1.4

$$\leq c\left\{\|\nabla J_\epsilon y_\epsilon\|_{L^\infty}\|J_{\epsilon'}(y_\epsilon - y_{\epsilon'})\|_{H^0} + \|\nabla J_\epsilon y_\epsilon\|_{H^0}\|J_{\epsilon'}(y_\epsilon - y_{\epsilon'})\|_{L^\infty}\right\}$$

By the proposition 4.1.13 and the Sobolev embedding we have that

$$\|J_{\epsilon'}\left\{\left[J_{\epsilon'}(y_\epsilon - y_{\epsilon'})\cdot\nabla\right]J_\epsilon y_\epsilon\right\}\|_{L^2} \leq c\|y_\epsilon\|_{H^m}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

It follows that $(c) \leq c_c\|y_\epsilon\|_{H^m}\|y_\epsilon - y_{\epsilon'}\|^2_{L^2}$ For (d) we have an estimate as in (b) i.e.

$$(d) \leq \|J_\epsilon\left\{J_{\epsilon'}\left[(y_\epsilon\cdot\nabla)(J_\epsilon - J_{\epsilon'})y_\epsilon\right]\right\}\|_{L^2}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

For the first norm by the proposition 4.1.13 we have that

$$\|J_\epsilon\left\{J_{\epsilon'}\left[(y_\epsilon\cdot\nabla)(J_\epsilon - J_{\epsilon'})y_\epsilon\right]\right\}\|_{L^2} \leq c\|\nabla(J_\epsilon - J_{\epsilon'})y_\epsilon\cdot y_\epsilon\|_{H^0}$$

$$\leq c\|\nabla(J_\epsilon - J_{\epsilon'})y_\epsilon\cdot y_\epsilon\|_{H^4}$$

The space $H^4(\mathbb{R}^2)$ is a Banach algebra we have:

$$\leq c\|y_\epsilon\|_{H^m}\|(J_\epsilon - J_{\epsilon'})y_\epsilon\|_{H^5}$$

$$\leq c\|y_\epsilon\|_{H^m}\left(\|J_\epsilon y_\epsilon - y_\epsilon\|_{H^m} + \|J_{\epsilon'}y_\epsilon - y_\epsilon\|_{H^m}\right)$$

By proposition 4.1.11 we have

$$\leq c\|y_\epsilon\|_{H^m}\left(\epsilon\|y_\epsilon\|_{H^m} + \epsilon'\|y_\epsilon\|_{H^m}\right)$$

So

$$\|J_\epsilon\left\{J_{\epsilon'}\left[(y_\epsilon\cdot\nabla)(J_\epsilon - J_{\epsilon'})y_\epsilon\right]\right\}\|_{L^2} \leq c\max(\epsilon,\epsilon')\|y_\epsilon\|^2_{H^m}$$

As a consequence $(d) \leq c_d\max(\epsilon,\epsilon')\|y_\epsilon\|^2_{H^m}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$ Now for the last integral (e) we have that

$$(e) = \frac{1}{2}\int_{\mathbb{R}^3}(J_{\epsilon'}y_{\epsilon'}\cdot\nabla)|J_{\epsilon'}(y_\epsilon - u_{\epsilon'})|^2dx = -\frac{1}{2}\int_{\mathbb{R}^3}\operatorname{div}(J_{\epsilon'}y_{\epsilon'})|J_{\epsilon'}(y_\epsilon - u_{\epsilon'})|^2dx = 0$$

For the I_2 we sum and subtract the term $J_{\epsilon'}[(b\cdot\nabla)J_\epsilon y_\epsilon]$, so

$$I_2 = \int_{\mathbb{R}^2}P\big[J_\epsilon[(b\cdot\nabla)J_\epsilon y_\epsilon] - J_{\epsilon'}[(b\cdot\nabla)J_\epsilon y_\epsilon] + J_{\epsilon'}[(b\cdot\nabla)J_\epsilon y_\epsilon] - J_{\epsilon'}[(b\cdot\nabla)J_{\epsilon'}y_{\epsilon'}]\big]$$

$$\cdot(y_\epsilon - y_{\epsilon'})dx$$

$$\leq \int_{\mathbb{R}^2}(J_\epsilon - J_{\epsilon'})[(b\cdot\nabla)J_\epsilon y_\epsilon]\cdot(y_\epsilon - y_{\epsilon'})dx$$

$$+\int_{\mathbb{R}^2}J_{\epsilon'}\big[(b\cdot\nabla)J_\epsilon y_\epsilon - (b\cdot\nabla)J_{\epsilon'}y_{\epsilon'}\big]\cdot(y_\epsilon - y_{\epsilon'})dx$$

We will see each integral individually

$$I_2^1 = \int_{\mathbb{R}^2}(J_\epsilon - J_{\epsilon'})[(b\cdot\nabla)J_\epsilon y_\epsilon]\cdot(y_\epsilon - y_{\epsilon'})dx$$

and

$$I_2^2 = \int_{\mathbb{R}^2} J_{\epsilon'}\big[(b\cdot\nabla)J_\epsilon y_\epsilon - (b\cdot\nabla)J_{\epsilon'}y_{\epsilon'}\big]\cdot(y_\epsilon - y_{\epsilon'})dx$$

So for the first one we have that

$$I_2^1 \leq \|(J_\epsilon - J_{\epsilon'})[(b\cdot\nabla)J_\epsilon y_\epsilon]\|_{L^2}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

We add and subtract the term $(b\cdot\nabla)J_\epsilon y_\epsilon$ and by the triange inequality we have that

$$\leq \big[\|J_\epsilon[(b\cdot\nabla)J_\epsilon y_\epsilon]\|_{L^2} + \|J_{\epsilon'}[(b\cdot\nabla)J_\epsilon y_\epsilon]\|_{L^2}\big]\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

By the proposition 1.1.11 we have that

$$\leq c\big[\epsilon\|\nabla J_\epsilon y_\epsilon\cdot b\|_{H^1} + \epsilon'\|\nabla J_\epsilon y_\epsilon\cdot b\|_{H^1}\big]\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

$$\leq 2c\max(\epsilon,\epsilon')\|\nabla J_\epsilon y_\epsilon\cdot b\|_{H^1}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

$$\leq 2c\max(\epsilon,\epsilon')\|b\|_{L^\infty}\|J_\epsilon y_\epsilon\|_{H^2}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

We conclude that

$$I_2^1 \leq c_{2,1}\max(\epsilon,\epsilon')\|b\|_{L^\infty}\|y_\epsilon\|_{H^m}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

For the I_2^2 we have that

$$I_2^2 \leq \|J_{\epsilon'}\big[(b\cdot\nabla)(J_\epsilon - J_{\epsilon'})y_\epsilon\big]\|_{L^2}\big]\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

By the proposition 4.1.13

$$\leq c\|\nabla(J_\epsilon - J_{\epsilon'})y_\epsilon\cdot b\|_{H^0}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

$$\leq c\|\nabla(J_\epsilon - J_{\epsilon'})y_\epsilon\|_{H^0}\|b\|_{L^\infty}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

$$\leq c\|(J_\epsilon - J_{\epsilon'})y_\epsilon\|_{H^1}\|b\|_{L^\infty}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

We add and subtract the term y_ϵ, and by the triangle inequality we have

$$\leq c\big[\|J_\epsilon y_\epsilon - y_\epsilon\|_{H^1}\|b\|_{L^\infty} + \|J_{\epsilon'}y_\epsilon - y_\epsilon\|_{H^1}\|b\|_{L^\infty}\big]\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

By the proposition 4.1.11 we have

$$\leq c\big[\epsilon\|b\|_{L^\infty}\epsilon\|y_\epsilon\|_{H^2} + \epsilon' + \|b\|_{L^\infty}\epsilon'\|y_\epsilon\|_{H^2}\big]\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

Thus

$$I_2^2 \leq c_{2,2}\max(\epsilon,\epsilon')\|y_\epsilon\|_{H^m}\|b\|_{L^\infty}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

So we conclude that

$$I_2 \leq C\max(\epsilon,\epsilon')\|y_\epsilon\|_{H^m}\|b\|_{L^\infty}\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

Combining all this together we have that

$$\frac{d}{dt}\|y_\epsilon - y_{\epsilon'}\|_{L^2} \leq C\big[\max(\epsilon,\epsilon')(\|y_\epsilon\|_{H^m} + \|y_{\epsilon'}\|_{H^m} + \|b\|_{L^\infty} + \|\nabla b\|_{L^\infty}) + \|y_\epsilon - y_{\epsilon'}\|_{L^2}\big]$$

By **Step 1** we have found a uniform bound for the H^m norm of the regularized solutions, say M

$$\frac{d}{dt}\|y_\epsilon - y_{\epsilon'}\|_{L^2} \leq CM\big[\max(\epsilon,\epsilon')M + \|\nabla b\|_{L^\infty} + \|b\|_{L^\infty}\big] + CM\|y_\epsilon - y_{\epsilon'}\|_{L^2}$$

We integrate with respect to t and we have that

$$\|y_\epsilon - y_{\epsilon'}\|_{L^2} \leq C_M \int_0^t \big[\max(\epsilon,\epsilon') + \|\nabla b\|_{L^\infty}\big]dx + \int_{\mathbb{R}^2} C_M\|y_\epsilon - y_{\epsilon'}\|_{L^2}dx$$

We set $a(t) = C_M\int_0^t \big[\max(\epsilon,\epsilon') + \|b\|_{L^\infty} + \|\nabla b\|_{L^\infty}\big]dx$ so by Gronwall we have that

$$\|y_\epsilon - y_{\epsilon'}\|_{L^2} \leq a(t) + \int_0^t C_M a(s)e^{\int_s^t C_M\, dr}ds$$

$$\|y_\epsilon - y_{\epsilon'}\|_{L^2} \leq \max(\epsilon,\epsilon')f(t)$$

Where $f(t) = t+\int_0^t \|b\|_{L^\infty}+\|\nabla b\|_{L^\infty}dx + C_M\int_0^t e^{\int_s^t C_M\, dr}ds + C_M\int_0^t \|b\|_{L^\infty}+\|\nabla b\|_{L^\infty}e^{\int_s^t C_M\, dr}ds$
Taking the supremum over this relation we have that $\sup_{0\leq t\leq T} f(t) = L(T)$ thus

$$\sup_{0\leq t\leq T}\|y_\epsilon - y_{\epsilon'}\|_{L^2} \leq L(T)\max(\epsilon,\epsilon') \leq \tilde{\epsilon}$$

with out loss of convergence $\epsilon' < \epsilon$, thus so the family y_ϵ forms a Cauchy sequence on $C\left([0,T], L^2(\mathbb{R}^2)\right)$ which is a Banach space and it follows that y_ϵ converges to y_v **Step 3** By the proposition 4.3.1 we have that for m'such that $0 \leq m' \leq m$ we have that

$$\|y_\epsilon - y_v\|_{H^{m'}} \leq \|y_\epsilon - y_v\|_{L^2}^{1-\frac{m'}{m}}\|y_\epsilon - y_v\|_{H^m}^{\frac{m'}{m}}$$

Taking the supremum over this relation we have that

$$\sup_{0\leq t\leq T}\|y_\epsilon - y_v\|_{H^{m'}} \leq \sup_{0\leq t\leq T}\left\{\|y_\epsilon - y_v\|_{L^2}^{1-\frac{m'}{m}}\|y_\epsilon - y_v\|_{H^m}^{\frac{m'}{m}}\right\}$$

Thus $\sup_{0\leq t\leq T}\|y_\epsilon - y_v\|_{H^{m'}} \leq \sup_{0\leq t\leq T}\|y_\epsilon - y_v\|_{L^2}^{1-\frac{m'}{m}} \sup_{0\leq t\leq T}\|y_\epsilon - y_v\|_{H^m}^{\frac{m'}{m}}$
By the **Step 2** we have that $\|y_\epsilon - y_{\epsilon'}\|_{L^2} \leq C_{M,T}\epsilon$, we also now that $\sup ab \leq \sup a \sup b$ it follows that $\sup a^k \leq (\sup a)^k$.
Consequently $\sup_{0\leq t\leq T}\|y_\epsilon - y_v\|_{L^2}^{1-\frac{m'}{m}} \leq \left(\sup_{0\leq t\leq T}\|y_\epsilon - y_v\|_{L^2}\right)^{1-\frac{m'}{m}}$.
We also know that the function x^a is increasing when $a > 0$, so we have that $\sup_{0\leq t\leq T}\|y_\epsilon - y_v\|_{L^2}^{1-\frac{m'}{m}} \leq (c\epsilon)^{1-\frac{m'}{m}}$.
By the **Step 1** we have that $\sup_{0\leq t\leq T}\|y_\epsilon\|_{H^m} \leq \frac{\|y_0\|_{H^m}e^{c_m\|\nabla b\|_{L^\infty}T}}{c_m\|\nabla b\|_{L^\infty}+\|y_0\|_{H^m}-\|y_0\|_{H^m}e^{c_m\|\nabla b\|_{L^\infty}T}}$
and it is also true that $\|y_v\|_{H^m} \leq \limsup_{\epsilon\to 0}\|y_\epsilon\|_{H^m} \leq \frac{\|y_0\|_{H^m}e^{c_m\|\nabla b\|_{L^\infty}T}}{c_m\|\nabla b\|_{L^\infty}+\|y_0\|_{H^m}-\|y_0\|_{H^m}e^{c_m\|\nabla b\|_{L^\infty}T}}$
Thus we have also an uniform bound for the H^m norm of y_v i.e.

$$\sup_{0\leq t\leq T}\|y_v\|_{H^m} \leq \frac{\|y_0\|_{H^m}e^{c_m\|\nabla b\|_{L^\infty}T}}{c_m\|\nabla b\|_{L^\infty} + \|y_0\|_{H^m} - \|y_0\|_{H^m}e^{c_m\|\nabla b\|_{L^\infty}T}}$$

It is also true by triangle inequality that $\sup(a+b) \leq \sup a + \sup b$ so $\sup_{0\leq t\leq T} \|y_\epsilon - y_v\|_{H^m}^{\frac{m'}{m}} \leq \left(\sup_{0\leq t\leq T} \|y_\epsilon\|_H^m + \sup_{0\leq t\leq T} \|y_v\|_{H^m}\right) = M^{\frac{m'}{m}}$ Eventually we combine the above relations we have that

$$\sup_{0\leq t\leq T} \|y_\epsilon - y_v\|_{H^{m'}} \leq C(y_0, T, m, m')\epsilon^{1-\frac{m'}{m}}$$

So we have proved that we have the converge of y_ϵ to y_v in all spaces $C([0,T], V^{m'})$ with $m' < m$.
We choose $m' > \frac{3}{2}+2$, so by the Sobolev embedding theorem we have that $\|y_\epsilon - y_v\|_{C^2} \leq \|y_\epsilon - y_v\|_{H^{m'}}$, taking the supremum over this relation we have that $\sup_{0\leq t\leq T} \|y_\epsilon - y_v\|_{C^2} \leq \sup_{0\leq t\leq T} \|y_\epsilon - y_v\|_{H^{m'}}$.
It follows that $y_\epsilon \to y_v$ in $C([0,T], C^2)$
To complete the step 3 we have to show that we also have a convergence in $C^1([0,T], C)$, we recall the flow $\frac{d}{dt}y_\epsilon = F_\epsilon(y_\epsilon)$.
We know that a $\lim_{\epsilon\to 0} F_\epsilon y_\epsilon = \Delta y_v - P[(y_v\cdot\nabla)y_v + (b\cdot\nabla)y_v + (y_v\cdot\nabla)b]$ in $V^{m'-2}$ since (we denote $y_v = y$)

$$\|\nu J_\epsilon^2 \Delta y_\epsilon - PJ_\epsilon[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon\cdot\nabla)b] - \nu\Delta y + P[(y\cdot\nabla)y + (b\cdot\nabla)y + (y\cdot\nabla)b]\|_{H^{m'-2}}$$

By the triangle inequality we have that

$$\leq \|\nu J_\epsilon^2 \Delta y_\epsilon - \nu\Delta y\|_{H^{m'-2}} + \|PJ_\epsilon[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon + (b\cdot\nabla)J_\epsilon y_\epsilon + (J_\epsilon y_\epsilon\cdot\nabla)b] - P[(y\cdot\nabla)y + (b\cdot\nabla)y + (y\cdot\nabla)b]\|_{H^{m'-2}}$$

Firstly we will see the terms, which do not include the b. We add and subtract some terms in order to reach to some terms we can estimate by the previous results

$$= \nu\|J_\epsilon y_\epsilon - J_\epsilon y + J_\epsilon y - y\|_{H^{m'}} + \|J_\epsilon[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon] - J_\epsilon[(y\cdot\nabla)y] + J_\epsilon[(y\cdot\nabla)y] - [(y\cdot\nabla)y]\|_{H^{m'-2}}$$

Again by triangle inequality we have

$$\leq \nu\|J_\epsilon y_\epsilon - J_\epsilon y\|_{H^{m'}} + \nu\|J_\epsilon y - y\|_{H^{m'}}$$

$$+ \underbrace{\|J_\epsilon[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon] - [(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon]\|_{H^{m'-2}}}_{(1)} + \underbrace{\|(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon - (y\cdot y)\|_{H^{m'-2}}}_{(2)}$$

By the proposition 4.1.11, 4.1.12 ,4.1.13 and step 2 we have that the first two terms converge in $H^{m'}$ so we continue with (1) and (2).

$$(1) \leq \|J_\epsilon[(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon] - [(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon]\|_{H^{m'}}$$

By proposition 4.1.12, and steps 1,2,3 we have that $(1) \to 0$
For (2) we have

$$(2) \leq \underbrace{\|(J_\epsilon y_\epsilon\cdot\nabla)J_\epsilon y_\epsilon - (J_\epsilon y_\epsilon\cdot\nabla)y_\epsilon\|_{H^{m'-2}}}_{(2a)} + \underbrace{\|(J_\epsilon y_\epsilon\cdot\nabla)y_\epsilon - (y\cdot\nabla)y\|_{H^{m'-2}}}_{(2b)}$$

So $(2a) : \|(J_\epsilon y_\epsilon\cdot\nabla)(J_\epsilon y_\epsilon - y_\epsilon)\|_{H^{m'-2}} \leq \|\nabla(J_\epsilon y_\epsilon - y_\epsilon)\|_{H^{m'-2}}\|J_\epsilon y_\epsilon\|_{H^{m'-2}} \leq \epsilon M\|J_\epsilon y_\epsilon - y_\epsilon\|_{H^{m'}}$

By proposition 4.1.12 and steps 1,2,3 we have that $(2a) \to 0$ We continue with $(2b)$ by adding and subtracting some terms we have

$$(2b) \leq \|(J_\epsilon y_\epsilon \cdot \nabla) y_\epsilon - (J_\epsilon y_\epsilon \cdot \nabla) y\|_{H^{m'-2}} + \|(J_\epsilon y_\epsilon \cdot \nabla) y - (y_\epsilon \cdot \nabla) y\|_{H^{m'-2}} + \|(y_\epsilon \cdot \nabla) y - (y \cdot \nabla) y\|_{H^{m'-2}}$$

$$\leq \|y_\epsilon - y\|_{H^{m'-1}} \|y_\epsilon\|_{H^{m'}} + \|y\|_{H^{m'-1}} \|J_\epsilon y_\epsilon - y_\epsilon\|_{H^{m'-2}} + \|y\|_{H^{m'-1}} \|y_\epsilon - y\|_{H^{m'-2}}$$

By proposition 4.1.12 and steps 1,2,3 we have that $(2b) \to 0$
We continue with the terms that include the term b

$$\|J_\epsilon[(b \cdot \nabla) J_\epsilon y_\epsilon] - (b \cdot \nabla) y\|_{H^{m'-2}} \leq$$

$$\underbrace{\|J_\epsilon[(b \cdot \nabla) J_\epsilon y_\epsilon] - (b \cdot \nabla) J_\epsilon y_\epsilon\|_{H^{m'-2}}}_{(1)} + \underbrace{\|(b \cdot \nabla) J_\epsilon y_\epsilon - (b \cdot \nabla) y\|_{H^{m'-2}}}_{(2)}$$

By proposition 4.1.12, we have that $(1) \to 0$
For (2) we have that

$$(2) \leq \|(b \cdot \nabla) J_\epsilon y_\epsilon - (b \cdot \nabla) y_\epsilon\|_{H^{m'-2}} + \|(b \cdot \nabla) y_\epsilon - (b \cdot \nabla) y\|_{H^{m'-2}}$$

$$\leq \|b\|_{L^\infty} \left(\|\nabla (J_\epsilon y_\epsilon - y_\epsilon)\|_{H^{m'-2}} + \|\nabla (y_\epsilon - y)\|_{H^{m'-2}} \right)$$

$$\leq \|b\|_{L^\infty} \left(\|J_\epsilon y_\epsilon - y_\epsilon\|_{H^{m'-1}} + \|y_\epsilon - y\|_{H^{m'-1}} \right)$$

By proposition 4.1.12, and steps 1,2,3 we have that $(2) \to 0$
The other term is

$$\|J_\epsilon[(J_\epsilon y_\epsilon \cdot \nabla) b] - (y \cdot \nabla) b\|_{H^{m'-2}} \leq \underbrace{\|J_\epsilon[(J_\epsilon y_\epsilon \cdot \nabla) b] - (J_\epsilon y_\epsilon \cdot \nabla) b\|_{H^{m'-2}}}_{(1)} + \underbrace{\|(J_\epsilon y_\epsilon \cdot \nabla) b - (y \cdot \nabla) b\|_{H^{m'-2}}}_{(2)}$$

$$\leq \|\nabla b\|_{L^\infty} \left(\|J_\epsilon y_\epsilon - y_\epsilon\|_{H^{m'-2}} + \|y_\epsilon - y\|_{H^{m'-2}} \right)$$

By proposition 4.1.12, we have that $(1) \to 0$

$$(2) \leq \|(J_\epsilon y_\epsilon \cdot \nabla) b - (y_\epsilon \cdot \nabla) b\|_{H^{m'-2}} + \|(y_\epsilon \cdot \nabla) b - (y \cdot \nabla) b\|_{H^{m'-2}}$$

By proposition 4.1.12, and steps 1,2,3 we have that $(2) \to 0$
So we conclude that $\frac{d}{dt} y_\epsilon \to F(y)$ in $C([0,T], V^{m'-2})$. Invoking now $y_\epsilon \to y$ in $C([0,T], H^{m'})$, we obtain $\partial_t y = F = \nu \Delta y - P(y \cdot \nabla y) \in C([0,T], H^{m'-2})$
This means that the (strong) time-derivative $\partial_t y$ exists and is continuous in the respective spaces, such that $v \in C([0,T], H^{m'}) \cap C^1([0,T], H^{m'-2})$ satisfies the Navier-Stokes and Euler equations. For m' like before we have by the Sobolev embedding theorem that $y_\epsilon, y_v \in C([0,T], C^2) \cap C^1([0,T], C)$[39]
Step 4:We will use weak convergence in order to prove this step

[39]The y_v is a classical solution

- The space L^2 is Hilbert, in addition the sequence y_ϵ is bounded in $L^2([0,T],V^m)$ indeed $\left(\int_0^T \|y_\epsilon\|_{H^m}^2 dt\right)^{\frac{1}{2}} \leq \int_0^T \|y_\epsilon\|_{H^m}$. By the **Step 1** we have an upper bound for the $\|y_\epsilon\|_{H^m}$ thus

$$\left(\int_0^T \|y_\epsilon\|_{H^m}^2 dt\right)^{\frac{1}{2}} \leq M^2T = C < \infty$$

  Thus by theorem 4.3.2 we have that there exist a subsequence such that converges to a y. This y is the y_v we have found in the step 2 since for $m' < m$we have that $V^m \subset V^{m'}$ and we know that the limit of a subsequence is unique. So $y_{\epsilon_k} \rightharpoonup y$. Also by proposition 4.3.2 we have that

$$\|y_v\|_{L^2([0,T],V^m)} \leq \liminf \|y_{\epsilon_k}\|_{L^2([0,T],V^m)} \leq C$$

- The sequence y_ϵ is bounded in $L^\infty([0,T],V^m)$ since $ess\sup_{0\leq t\leq T} \|y_\epsilon\|_{H^m} \leq ess\sup_{0\leq t\leq T} M = M < \infty$. We know that there exists a weakly$*$ convergence subsequence to y_v we also have that

$$\|y_v\|_{L^\infty([0,T],V^m)} \leq \limsup \|y_{\epsilon_k}\|_{L^\infty([0,T],V^m)} \leq M < \infty$$

- By the step 2 we have that $\|\frac{d}{dt}y_\epsilon\|_{H^m} \leq cv\|y_\epsilon\|_{H^m} + c\|y_\epsilon\|_{H^m}^2$ thus $\|\frac{d}{dt}\|_{H^m} \leq C_M$ So it follows that $y_\epsilon \in Lip\left([0,T],V^{m-2}\right)$ we also have that $\|\frac{d}{dt}y_v\|_{H^m} \leq C_M$ thus $y_v \in Lip\left([0,T],V^{m-2}\right)$

- We have already prove that $y_\epsilon \to y_v$ in $C([0,T],V^{m'})$,thus $\|y_\epsilon - y_v\|_{H^{m'}} \leq \sup_{0\leq t\leq T} \|y_\epsilon - y_v\|_{H^{m'}} \leq \eta$ so it occurs that $(y_\epsilon - y_v, \phi) \to 0$ for any $\phi \in V^{-m'}(1)$[40]. The space $V^{-m'}$ is dense in V^{-m} so $\forall g \in V^{-m}$, there exists a sequence ϕ_n in $V^{-m'}$ such that $\phi_n \to g$. So by (1) we have that $(y_\epsilon - y_v, \phi_n) \to 0$. We want to prove that $(y_\epsilon - y_v, \phi) \to (y_\epsilon - y_v, g)$ this is true since $|(y_\epsilon - y_v, \phi_n - g)| \leq \|y_\epsilon - y_v\|_{H_{-m}}\|\phi_n - g\|_{H^{-m}} \leq \eta'$. So we have that for $g \in V^{-m}$ that $(y_\epsilon - y_v, g) \to 0$, we know that y_ϵ, y_v are continuous so they are and weakly continuous and this completes the proof of step 4.

As far as concerned the uniqueness of the solution it occurs by the previous chapter remark 2 where we have proved that if the solution exist it is unique. □

So far by the proposition above we see that we have our y is on $C([0,T],C^2) \cap C^1([0,T],C)$ if we want to speak in terms of Sobolev spaces on $C([0,T],V^{m'}) \cap C^1([0,T],V^{m'-2})$ for $m' < m$. Now we will prove the following theorem, which gives us continuity in the high H^m norm.

Theorem 4.3.6. *Assume that y_v is a solution as described above, then $y_v \in C([0,T),V^m) \cap C^1([0,T),V^{m-2})$*

[40]The dual of H^m is the H^{-m}

Proof. As in **Step 3** of the previous proof we will firstly prove that $y_v = y \in C([0,T), V^m)$ and then follows that $y \in C^1([0,T), V^{m-2})$. In **Step 4**, we have proved the weak continuity -with respect to time, of our solution in V^m, we want to prove that $\lim_{\delta\to 0} u(t+\delta) \to y(t)$ i.e. $\|y(t+\delta) - y(t)\|_{H^m} \leq \eta$ for $\delta \leq \beta$ we know that $\|y(t+\delta) - y(t)\|_{H^m} \leq (y(t+\delta) - y(t), y(t+\delta) - y(t))_{H^m} \leq \left| \|y(t+\delta)\|^2_{H^m} - \|y(t)\|^2_{H^m} \right|$ so it is enough to show that $\|y\|_{H^m}$ is a continuous function.
We start with the right continuity on which is the same for $v = 0$ and $v > 0$.
For t=0 by the previous theorem we have that

$$\|y\|_{H^m} \leq \frac{\|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}{c_m \|\nabla b\|_{L^\infty} + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}$$

so

$$\lim_{t\to 0^+} \sup \|y\|_{H^m} \leq \lim_{t\to 0^+} \sup \frac{\|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}{c_m \|\nabla b\|_{L^\infty} + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}} \leq \|y_0\|_{H^m}$$

. By step 4 we have that $y_t \rightharpoonup y_0$ thus by proposition 4.3.2 we have that

$$\|y_0\|_{H^m} \leq \lim_{t\to 0^+} \inf \|y\|_{H^m}$$

It is also true that $\lim_{t\to 0^+} \inf \leq \lim_{t\to 0^+} \sup$ so combining those three relations we have that

$$\lim_{t\to 0^+} \|y\|^m_H = \lim_{t\to 0^+} \sup \|y\|_{H^m} = \lim_{t\to 0^+} \inf \|y\|_{H^m} = \|y_0\|_{H^m}$$

So we have that $\|y\|_{H^m}$ is strongly right continuous on 0.
For the left continuity we have to see each case individually

- $v = 0$ The Euler equation is time reversible, indeed:
  Recall the Euler equation in 2 dimensions

$$\frac{\partial}{\partial t} y + (y \cdot \nabla) y + (b \cdot \nabla) y + (y \cdot \nabla) b = -\nabla p$$

  We set $y(x,t) = -v(x,t)$ and $p(x,t) = -\bar{p}(x,-t)$ and $b = -\bar{b}$ thus we have that

$$\frac{\partial}{\partial t} y(x,t) = \frac{\partial}{\partial t}(-v(x,-t)) = -\frac{\partial}{\partial t} v(x,-t) = \frac{\partial v}{\partial t}(x,-t)$$

  The other derivatives in the equation does not change with this substitution since they are derivatives with respect to x i.e. we have that

$$\frac{\partial v}{\partial t}(x,-t) + (-v \cdot \nabla) - v(x,-t) + (\bar{b} \cdot \nabla) v + (v \cdot \nabla \bar{b}) b = -\nabla \bar{p}(x,-t)$$

  So now we set $y_v(-t) = \bar{y}(t)$ and we have the same arguments:
  By the previous theorem we have that

$$\|\bar{y}\|_{H^m} \leq \frac{\|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}{1 + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}$$

so

$$\lim_{t\to 0^-} \sup \|\bar{y}\|_{H^m} \le \lim_{t\to 0^-} \sup \frac{\|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}{1 + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}} \le \|\bar{y}_0\|_{H^m}$$

. By step 4 we have that $\bar{y}_t \rightharpoonup \bar{y}_0$ thus by proposition 4.3.2 we have that

$$\|\bar{y}_0\|_{H^m} \le \lim_{t\to 0^-} \inf \|\bar{y}\|_{H^m}$$

It is also true that $\lim_{t\to 0^-} \inf \le \lim_{t\to 0^-} \sup$ so combining those three relations we have that

$$\lim_{t\to 0^-} \|\bar{y}\|_H^m = \lim_{t\to 0^-} \sup \|\bar{y}\|_{H^m} = \lim_{t\to 0^-} \inf \|\bar{y}\|_{H^m} = \|\bar{u}_0\|_{H^m}$$

So we have that $\|\bar{y}\|_{H^m}$ is strongly left continuous on 0. Thus the function

$$u = \begin{cases} y & [0,T) \\ \bar{y} & (-T,0) \end{cases}$$

we have that $\|y\|_{H^m}$ is strongly continuous on 0. Now we will prove that $\|y\|_{H^m}$ is continuous for every$t \in (0,T)$, let T_0 be a random time and $y(x,T_0) = y_{T_0}$ the solution on this time. We know then that $\|y_{T_0}\|_{H^m} \le M$ so $y_{T_0} \in V^m$ and thus we can use y_{T_0} as an initial value for the new IVP

$$\begin{cases} \frac{d}{dt} y_\epsilon = F_\epsilon(y_\epsilon) & t \in [T_0, T) \\ y_\epsilon(T_0) \end{cases}$$

Assume that $\bar{y}_\epsilon$ is the solution of the above ivp, which we are sure that exist since we have prove that we can find a global solution on the first IVP. By lemma 18 we have that

$$\frac{d}{dt}\frac{1}{2}\|\bar{y}_\epsilon\|_{H^m}^2 \le c_m \|J_\epsilon \nabla \bar{y}_\epsilon\|_{L^\infty} \|\bar{y}_\epsilon\|_{H^m}^2$$

Following the same process as in **Step 1** we have that

$$\|\bar{y}_\epsilon\|_{H^m} \le \frac{\|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}{1 + \|y_0\|_{H^m} - \|y_0\|_{H^m} e^{c_m \|\nabla b\|_{L^\infty} T}}$$

So by **Step 2** we have a solution $\bar{y}$. So we have again the same arguments as above and we have the continuity of $\|y\|_{H^m}$ in all the interval $[0,T)$

- The Navier Stokes equation is not time reversible, so we will follow another strategy. We know that y_ϵ is bounded on $L^2([0,T], V^m)$ and thus in $L^2([0,T], V^{m+1})$, this is a Hilbert space so by lemma 19 we have that there exists a subsequence that converge on V^{m+1}. Assume now a $T_0 \in (0,T]$ we will prove the left continuity . We choose $\tilde{T}$ such that $0 < \tilde{T} < T_0$ and $u(\tilde{T}) \in V^{m+1}$ with$\tilde{T} = T_0 - \delta$. With initial value $u(\tilde{T})$ and $m = m+1$ in the theorem 4.3.1 we have that for $m' < m+1$ and $T' \ge \tilde{T}$ there exists a solution $y \in C([\tilde{T}, T'], V^{m'})$. For $m' = m$ and $\delta = 0$ we have the left continuity on T_0 and since T_0 is arbitrary we have the left continuity in all the interval $[0,T)$

So we conclude that $\|y\|_{H^m}$ is continuous thus $y_v \in C([0,T), V^m)$ and $\frac{d}{dt}y \in C([0,T), V^{m-2})$ □

Note : As before (the regularized case) we have prove that there exist a y which solves the equation

$$\frac{\partial}{\partial t} + P\big[(y \cdot \nabla)y + (b \cdot \nabla)y + (y \cdot \nabla)b\big] = \nu \Delta y$$

and we have b an exact solution so we conclude that $u = y + b$ solves the Leray's form of the Navies Stokes equation The proof of the previous theorem is based on the discussion about the continuity of solutions in the previous chapter. Here the following proposition summarizes the result for the existence of maximum interval for the existence of the solution y_v.

Proposition 4.3.4. *Let $y_0 \in V^m$ with $m \geq 3$ and $v \geq 0$, then there exists a maximum interval $[0, T^*]$ that the solution y_v described in theorem 4.3.1, exists. T^*maybe the infinity, otherwise for $T^* < \infty$ we will have $\lim_{t \to T^*} \|y\|_{H^m} = \infty$*

Proof. Assume that $T^* \leq \infty$ is the maximum time and $\lim_{t \to T^*} \|y\|_{H^m} = \infty$, then we have already seen that we can extend the interval of existence, say $[0, T^* + \delta)$ which contradicts the initial hypothesis. □

CHAPTER 5

Existence of global in time smooth solutions

In the previous chapter we found a solution locally in time. The last proposition was about the existence of this solution globally in time. More specifically we saw that if the quantity $\lim_{t\to T^*} \|u(t)\|_{H^m} \leq \infty$ for $T^* < \infty$ then the maximum interval of existence is the $[0, \infty)$. In this Chapter we will see a criterion, which gives a sufficient condition for the above relation to hold. This criterion is the well known Beale-Kato-Majda blow up criterion which links the accumulation of vorticity with the global existence of solutions. Firstly we will prove this criterion and later we will apply this theorem.

5.1 Beale-Kato-Majda criterion

Theorem 5.1.1. [1] *Assume that $u_0 \in V^m$ with $m \geq \left[\frac{N}{2}\right] + 2$ and u is a solution of the Euler or the Navier-Stokes as described in the previous chapter and ω is its vorticity defined as $\omega = \mathrm{curl} u$. Assume that there exists a time T^* such that the above solution can not be extended continuously in time. We also assume that T^* is the first time that this happens, then $\int_0^{T^*} \|\omega(t)\|_{L^\infty} dt = \infty$*

Recall that via Biot-Savart law we have export the velocity by vorticity with a non local operator. We see that on two dimensions and three dimension we have different homogeneous kernels of degree (1-N). So for the proof we will have two cases the 3d one and the 2d

Proof of the criterion

By the proposition 4.3.4 we have that since $T^* < \infty$ is the first time that we can not extend our solutions it is true that $\lim_{t\to T^*} u(t) = \infty$. Assume that $\int_0^{T^*} \|\omega(t)\|_{L^\infty} dt = M < \infty$ we will prove that in this case$\|u(t)\|_{H^m} \leq C$ for $t \leq T^*$ and then $\lim_{t\to T^*} u(t) < \infty$ which is a contradiction.

[1] [9]

3 dimensions

We will prove the following steps
Step 1:We will find an energy estimate for the H^m norm of the velocity in terms of $\|\nabla u\|_{L^\infty}$
Step 2:We will find an L^2 estimate for the vorticity in terms of $\|\omega\|_{L^\infty}$.
Step 3:We will bound the $\|\nabla u\|_{L^\infty}$ in terms of vorticity
Step 4:We will combine all the above steps and we will show that if $\int_0^{T^*} \|\omega(t)\|_{L^\infty} dt$ stays bounded then $\|u\|_{H^m}$ stays bounded.

Step 1. We will prove that $\frac{d}{dt}\|u\|_{H^m} \leq c\|\nabla u\|_{L^\infty}\|u\|_{H^m}$
It is true that u satisfies the the Navier Stokes in the Leray's formulation so we have that

$$\frac{\partial}{\partial t}u = \nu\Delta u - P[(u\cdot\nabla)u]$$

We differentiate this relation over

$$D^a\left(\frac{\partial}{\partial t}u\right) = \nu D^a(\Delta u) - D^a(P[(u\cdot\nabla)u])$$

We multiply with $D^a u$ in L^2 and we get

$$\int_{\mathbb{R}^3} D^a\left(\frac{\partial}{\partial t}u\right)\cdot D^a u dx = \int_{\mathbb{R}^3} D^a(\Delta u)\cdot D^a u dx - \int_{\mathbb{R}^3} D^a(P[(u\cdot\nabla)u])\cdot D^a u dx$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|D^a u\|_{L^2}^2 = -\nu\|\nabla D^a u\|_{L^2}^2 - \int_{\mathbb{R}^3} D^a(P[(u\cdot\nabla)u])\cdot D^a u dx$$

$$\frac{1}{2}\frac{d}{dt}\|D^a u\|_{L^2}^2 \leq \int_{\mathbb{R}^3} D^a(P[(u\cdot\nabla)u])\cdot D^a u dx$$

To the integral on the right side we add and subtract the term $P[(u\cdot\nabla)D^a u]$ and we have that since $\int_{\mathbb{R}^3} P[(u\cdot\nabla)D^a u] \leq 0$ that

$$\frac{1}{2}\frac{d}{dt}\|D^a u\|_{L^2}^2 \leq \int_{\mathbb{R}^3} \left[D^a(P[(u\cdot\nabla)u]) - P[(u\cdot\nabla)D^a u]\right]\cdot D^a u dx$$

So we have that

$$\frac{1}{2}\frac{d}{dt}\|D^a u\|_{L^2}^2 \leq \|u\|_{H^m}\|D^a(P[(u\cdot\nabla)u]) - P[(u\cdot\nabla)D^a u]\|_{L^2}$$

Taking the sum over this relation

$$\frac{1}{2}\frac{d}{dt}\sum_{|a|\leq m}\|D^a u\|_{L^2}^2 \leq \|u\|_{H^m}\sum_{|a|\leq m}\|D^a(P[(u\cdot\nabla)u]) - P[(u\cdot\nabla)D^a u]\|_{L^2}$$

By proposition 4.1.13 we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^m}^2 \leq c_m\|\nabla u\|_{L^\infty}\|\nabla u\|_{H^{m-1}} + \|u\|_{H^m}\|\nabla u\|_{L^\infty}$$

So it follows that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^m}^2 \leq c_m \|\nabla u\|_{L^\infty} \|u\|_{H^m}^2$$

Eventually

$$\frac{d}{dt}\|u\|_{H^m} \leq c_m \|\nabla u\|_{L^\infty} \|u\|_{H^m}$$

Now by Gronwall's lemma in differential form we have that

$$\|u\|_{H^m} \leq \|u_0\|_{H^m} e^{\int_0^t c\|\nabla u\|_{L^\infty} ds}$$

□

Step 2. Let $\omega = \text{curl} u$ where u is the above velocity field, we know for the vorticity equation in 3 dimensions that

$$\frac{\partial}{\partial t}\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega$$

We multiply with ω in L^2 and we have that

$$\int_{\mathbb{R}^3} \frac{\partial}{\partial t}\omega \cdot \omega dx + \int_{\mathbb{R}^3} (u \cdot \nabla)\omega \cdot \omega dx = \int_{\mathbb{R}^3} (\omega \cdot \nabla) u \cdot \omega dx + \nu \int_{\mathbb{R}^3} \Delta\omega \cdot \omega dx$$

We have that $\int_{\mathbb{R}^3}(u \cdot \nabla)\omega \cdot \omega dx = 0$ and $\nu \int_{\mathbb{R}^3} \Delta\omega \cdot \omega dx = -\nu \int_{\mathbb{R}^3} \|\nabla\omega\|^2 dx \leq 0$ so

$$\int_{\mathbb{R}^3} \frac{\partial}{\partial t}\omega \cdot \omega dx \leq \int_{\mathbb{R}^3} (\omega \cdot \nabla) u \cdot \omega dx$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 \leq \|\nabla u\|_{L^2} \|\omega\|_{L^\infty} \|\omega\|_{L^2}$$

Lemma 24. *It is true that in three dimensions that* $\|u\|_{L^2} \leq c\|\omega\|_{L^2}$*, where* $\omega = \text{curl} u$

By Plancherels theorem we recall that if $f \in L^2$ then $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$
Also it is true that $\widehat{\nabla u} = \int_{\mathbb{R}^3} e^{-i\xi x} \cdot \nabla u dx$
We see that for j-th element, with integration by parts that:

$$\int_{\mathbb{R}^3} e^{-i\xi x} \cdot \frac{\partial}{\partial x_j} u dx = -\int_{\mathbb{R}^3} \frac{\partial}{\partial x_j} e^{-i\xi x} \cdot u dx = i\xi_j \hat{u}$$

Since we have that $\frac{\partial}{\partial x_j} e^{-i\xi x} = (-i\xi)\frac{\partial}{\partial x_j} x e^{-i\xi x} = -i\xi_j e^{-i\xi x}$
Thus we have that $\widehat{\nabla u}(\xi) = i\xi \hat{u}(\xi)$
It is also true that

$$\omega = \begin{pmatrix} \frac{\partial}{\partial x_2} u_3 - \frac{\partial}{\partial x_3} u_2 \\ \frac{\partial}{\partial x_3} u_1 - \frac{\partial}{\partial x_1} u_3 \\ \frac{\partial}{\partial x_1} u_2 - \frac{\partial}{\partial x_2} u_1 \end{pmatrix}$$

Also for the Fourier transform of vorticity we have that

$$\hat{\omega}(\xi) = \int_{\mathbb{R}^3} e^{-i\xi x} \omega(x) dx$$

We will check each :

- $\omega_1 = \frac{\partial}{\partial x_2} u_3 - \frac{\partial}{\partial x_3} u_2$ So

$$\int_{\mathbb{R}^3} e^{-i\xi x} \left(\frac{\partial}{\partial x_2} u_3 - \frac{\partial}{\partial x_3} u_2 \right) dx$$
$$= \int_{\mathbb{R}^3} e^{-i\xi x} \frac{\partial}{\partial x_2} u_3 dx - \int_{\mathbb{R}^3} e^{-i\xi x} \frac{\partial}{\partial x_3} u_2 dx$$
$$i\xi_2 \widehat{u}_3 - i\xi_3 \widehat{u}_2$$

- $\omega_2 = \frac{\partial}{\partial x_3} u_1 - \frac{\partial}{\partial x_1} u_3$ So

$$\int_{\mathbb{R}^3} e^{-i\xi x} \left(\frac{\partial}{\partial x_3} u_1 - \frac{\partial}{\partial x_1} u_3 \right) dx$$
$$= \int_{\mathbb{R}^3} e^{-i\xi x} \frac{\partial}{\partial x_3} u_1 dx - \int_{\mathbb{R}^3} e^{-i\xi x} \frac{\partial}{\partial x_1} u_3 dx$$
$$i\xi_3 \widehat{u}_1 - i\xi_1 \widehat{u}_3$$

- $\omega_3 = \frac{\partial}{\partial x_1} u_2 - \frac{\partial}{\partial x_2} u_1$ So

$$\int_{\mathbb{R}^3} e^{-i\xi x} \left(\frac{\partial}{\partial x_1} u_2 - \frac{\partial}{\partial x_2} u_1 \right) dx$$
$$= \int_{\mathbb{R}^3} e^{-i\xi x} \frac{\partial}{\partial x_1} u_2 dx - \int_{\mathbb{R}^3} e^{-i\xi x} \frac{\partial}{\partial x_2} u_1 dx$$
$$i\xi_1 \widehat{u}_2 - i\xi_2 \widehat{u}_1$$

Thus $\hat{\omega}(\xi) = \begin{pmatrix} i\xi_2\widehat{u}_3 - i\xi_3\widehat{u}_2 \\ i\xi_3\widehat{u}_1 - i\xi_1\widehat{u}_3 \\ i\xi_1\widehat{u}_2 - i\xi_2\widehat{u}_1 \end{pmatrix} = \begin{pmatrix} 0 & -i\xi_3 & i\xi_2 \\ i\xi_3 & 0 & -i\xi_1 \\ -i\xi_2 & i\xi_1 & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} = S\hat{u}(\xi)$
We have that $\det S = \xi_1^2 + \xi_2^2 + \xi_3^2 \neq 0$ so there exists the inverse matrix say S^{-1} we will find the inverse
We know that $S^{-1} \frac{1}{\det S} adj(S)$ where $adj(S)$ is the adjugate matrix of S.

$$adj(S) = \begin{pmatrix} \xi_1^2 & -\xi_1\xi_2 & \xi_1\xi_3 \\ -\xi_1\xi_2 & \xi^2 & -\xi_2\xi_3 \\ \xi_1\xi_3 & -\xi_2\xi_3 & \xi_3^2 \end{pmatrix}$$

So the inverse is a 3×3 symmetric matrix and we know that the norm of symmetric matrix is the maximum of the magnitude of its eigenvalues. Assume that $C = \max |\lambda_1|, |\lambda_2|, |\lambda_3|$ We conclude that since $\hat{u}(\xi) = S^{-1}\hat{\omega}(\xi)$

$$\|\hat{u}(\xi)\|_{L^2} = \|S^{-1}\hat{\omega}(\xi)\|_{L^2}$$

$$\leq \|S^{-1}\|_{L^\infty}\|\hat{\omega}(\xi)\|_{L^2}$$
$$\leq \|S^{-1}\|\|\hat{\omega}(\xi)\|_{L^2}$$
$$\leq c\|\hat{\omega}(\xi)\|_{L^2}$$

By the lemma 23 we have that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 \leq c\|\omega\|_{L^\infty}\|\omega\|_{L^2}^2$$

Thus

$$\frac{d}{dt}\|\omega\|_{L^2} \leq c\|\omega\|_{L^\infty}\|\omega\|_{L^2}$$

And by the Gronwall's inequality in the differential form we have that

$$\|\omega\|_{L^2} \leq \|\omega_0\|_{L^2} e^{\int_0^t c\|\omega\|_{L^2} ds}$$

□

Step 3. Assume that $\omega \in L^2 \cap L^\infty \cap C^{0,\gamma}$ then in three dimensions via Biot-Savart law we have that

$$u(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} K_3(x-y)\omega(y)dy = \frac{1}{4\pi}\int_{\mathbb{R}^3} K_3(y)\omega(x-y)dy$$

On chapter 3 we will show that the kernel $P_3 = \nabla K_3$ defines a SIO through convolution, by proposition 3.17 we have that

$$\nabla u(x,t) = c\omega(x) + P_3\omega(x) \tag{Rs3}$$

where $P_3\omega(x) = \text{P.V.} \int_{\mathbb{R}^3} \nabla K_3(y)\omega(x-y)dy$ and P_3 is homogeneous of degree -3 with mean value zero[2].

Assume the cut-off function $\rho(|x|) = \begin{cases} 1 & |x| \leq R_0 \\ 0 & |x| \geq 2R_0 \end{cases}$, we will define R_0 later in the proof

We have that $P_3\omega(x) = \text{P.V.} \int_{\mathbb{R}^3} \nabla K_3(y)\omega(x-y)dy$

$$= \text{P.V.} \int_{\mathbb{R}^3} \nabla K_3(y)\rho(|y|)\omega(x-y) + \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy$$

$$= \text{P.V.} \int_{\mathbb{R}^3} \nabla K_3(y)\rho(|y|)\omega(x-y)dy + \text{P.V.} \int_{\mathbb{R}^3} \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy$$

So we will see each integral individually:

For $\epsilon \leq R_0$ we have that

$$I_1 = \text{P.V.} \int_{\mathbb{R}^3} \nabla K_3(y)\rho(|y|)\omega(x-y)dy$$

$$= \text{P.V.} \left(\int_{|y|\leq\epsilon} \nabla K_3(y)\rho(|y|)\omega(x-y)dy + \int_{|y|\geq\epsilon} \nabla K_3(y)\rho(|y|)\omega(x-y)dy \right)$$

[2]See chapter 2 for more information

The singularity of the kernel is on 0 so the second integral is well defined thus

$$= \text{P.V.} \int_{|y| \leq \epsilon} \nabla K_3(y)\rho(|y|)\omega(x-y)dy + \int_{|y| \geq \epsilon} \nabla K_3(y)\rho(|y|)\omega(x-y)dy$$

Again we will see each term individually

- $I_1^1 = \text{P.V.} \int_{|y| \leq \epsilon} \nabla K_3(y)\rho(|y|)\omega(x-y)dy$
  Since P_3 has mean value zero it is true that P.V. $\int_{\mathbb{R}^3} P_3(y)\rho\omega(x)dy = \omega(x)\int_{\mathbb{R}^3} P_3(y) = 0$ so we have that

$$I_1^1 = \text{P.V.} \int_{|y| \leq \epsilon} \nabla K_3(y)\rho(|y|)\omega(x-y) - \nabla K_3(y)\rho(|y|)\omega(x)dy$$

$$= \text{P.V.} \int_{|y| \leq \epsilon} \nabla K_3(y)\rho(|y|)(\omega(x-y) - \omega(x))$$

  Thus

$$|I_1^1| \leq \int_{|y| \leq \epsilon} |\nabla K_3(y)\rho(|y|)||\omega(x-y) - \omega(x)|dy$$

  We multiply and divide with the $|y|^\gamma$, for $0 < \gamma < 1$ so

$$\leq \int_{|y| \leq \epsilon} |\nabla K_3(y)\rho(|y|)| \frac{|\omega(x-y) - \omega(x)|}{|y|^\gamma} |y|^\gamma dy$$

$$\leq \int_{|y| \leq \epsilon} |y|^\gamma |\nabla K_3(y)\rho(|y|)| \|\omega\|_{C^{0,\gamma}} dy$$

$$\leq \|\omega\|_{C^{0,\gamma}} \int_{|y| \leq \epsilon} |y|^\gamma |\nabla K_3(y)|dy$$

$$\leq c\|\omega\|_{C^{0,\gamma}} \int_{|y| \leq \epsilon} |y|^\gamma \frac{1}{|y|^3} dy$$

$$\leq c\|\omega\|_{C^{0,\gamma}} \int_0^\epsilon r^\gamma \frac{1}{r^3} r^2 dr$$

  So

$$|I_1^1| \leq c_{1,1}\|\omega\|_{C^{0,\gamma}} \epsilon^\gamma$$

- $I_1^2 = \int_{|y| \geq \epsilon} \nabla K_3(y)\rho(|y|)\omega(x-y)dy$

$$= \int_{2\epsilon \leq |y| \leq R_0} \nabla K_3(y)\rho(|y|)\omega(x-y)dy + \int_{|y| > R_0} \nabla K_3(y)\rho(|y|)\omega(x-y)dy$$

  We will see each integral individually:

$$I_1^{2a} = \int_{2\epsilon \leq |y| \leq R_0} \nabla K_3(y)\rho(|y|)\omega(x-y)dy$$

  So we have that

$$|I_1^{2a}| \leq \int_{2\epsilon \leq |y| \leq R_0} |\nabla K_3(y)\rho(|y|)||\omega(x-y)|dy$$

$$\leq c\|\omega\|_{L^\infty} \int_{2\epsilon \leq |y| \leq R_0} |\nabla K_3(y) \rho(|y|)||\omega(x-y)|dy$$

$$\leq c\|\omega\|_{L^\infty} \int_{2\epsilon \leq |y| \leq R_0} \frac{1}{|y|^3} dy$$

$$\leq c\|\omega\|_{L^\infty} \int_{2\epsilon}^{R_0} \frac{1}{r^3} r^2 dr$$

So

$$|I_1^{2a}| \leq c_{1,2a}\|\omega\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right)$$

And for the other one we have that

$$I_1^{2b} = \int_{|y|>R_0} \nabla K_3(y)\rho(|y|)\omega(x-y)dy$$

So we have that

$$|I_1^{2b}| \leq \int_{|y|>R_0} |\nabla K_3(y)\rho(|y|)||\omega(x-y)|dy$$

$$\leq \|\omega\|_{L^2} \left(\int_{|y|>R_0} |\nabla K_3(y)\rho(|y|)|^2 dy\right)^{\frac{1}{2}}$$

$$\leq c\|\omega\|_{L^2} \left(\int_{|y|>2R_0} \left|\frac{1}{|y|^3}\right|^2 dy\right)^{\frac{1}{2}}$$

$$\leq c\|\omega\|_{L^2} \left(\int_{2R_0}^{\infty} \frac{1}{r^6} r^2 dr\right)^{\frac{1}{2}}$$

$$|I_1^{2b}| \leq c_{1,2b}\|\omega\|_{L^2} R_0^{-\frac{3}{2}}$$

So for the first integral we have that

$$|I_1| \leq c_{1,1}\|\omega\|_{C^{0,\gamma}}\epsilon^\gamma + c_{1,2a}\|\omega\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + c_{1,2b}\|\omega\|_{L^2} R_0^{-\frac{3}{2}}$$

For the second integral we have that for $\epsilon \leq R_0$

$$I_2 = \text{P.V.} \int_{\mathbb{R}^3} \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy$$

$$= \text{P.V.} \left(\int_{|y|\leq\epsilon} \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy + \int_{|y|\geq\epsilon} \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy\right)$$

$$= \text{P.V.} \int_{|y|\leq\epsilon} \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy$$

$$+ \int_{2\epsilon\leq|y|\leq R_0} \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy + \int_{|y|>R_0} \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy$$

By the definition of the cut off function we have that the first2 integrals are zero so we proceed with the last one and we have that

$$|\int_{|y|>R_0} \nabla K_3(y)(1-\rho(|y|))\omega(x-y)dy| \leq \int_{|y|>R_0} |\nabla K_3(y)(1-\rho(|y|))||\omega(x-y)|dy$$

$$\leq \|\omega\|_{L^2} c \left(\int_{2R_0}^{\infty} \frac{1}{r^6} r^2 dr\right)$$

So $|I_2| \leq c_2 \|\omega\|_{L^2} R_0^{-\frac{3}{2}}$ Thus we have that

$$|P_3\omega(x)| \leq c_{1,1}\|\omega\|_{C^{0,\gamma}}\epsilon^{\gamma} + c_{1,2a}\|\omega\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + c_{1,2b}\|\omega\|_{L^2} R_0^{-\frac{3}{2}} + c_2\|\omega\|_{L^2} R_0^{-\frac{3}{2}}$$

$$\leq c'\big(\|\omega\|_{C^{0,\gamma}}\epsilon^{\gamma} + \|\omega\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega\|_{L^2} R_0^{-\frac{3}{2}}\big)$$

We set $R_0 = 1$ then

$$|P_3\omega(x)| \leq c'\big(\|\omega\|_{C^{0,\gamma}}\epsilon^{\gamma} + \|\omega\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega\|_{L^2}\big)$$

Remark we will not deal with the R_0 inside of the ln, because we want to estimate it together with 2ϵ.
So by (Rs3)we have that

$$\|\nabla u\|_{L^\infty} \leq c\|\omega\|_{L^\infty} + c'\big(\|\omega\|_{C^{0,\gamma}}\epsilon^{\gamma} + \|\omega\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega\|_{L^2}\big)$$

By the Sobolev embedding we have that $\|\omega\|_{C^{0,\gamma}} \leq c\|\omega\|_{H^2}$ so we have that

$$\|\nabla u\|_{L^\infty} \leq C\left(\|\omega\|_{L^\infty} + \|\omega\|_{H^2}\epsilon^{\gamma} + \|\omega\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega\|_{L^2}\right)$$

Lemma 25. *Assume that u is a velocity field in 3 dimensions and ω its vorticity. Then for $s > 0$ then*

$$\|\omega\|_{H^{s-1}} \leq c\|u\|_{H^s}$$

proof of lemma:
We have seen on the proof of lemma 23 that $\hat{\omega}(\xi) = S\hat{u}(\xi)$ we know that for the matrix S that $|S| = \xi_3^2 + \xi_2^2 + \xi_1^2 + \xi_3^2 + \xi_2^2 + \xi_1^2 = 2|\xi|$ Thus we have that

$$\|\omega\|_{H^{s-1}} = \left(\int_{\mathbb{R}^3} |\hat{\omega}(\xi)|^2(1+|\xi|^2)^{s-1}d\xi\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\mathbb{R}^3} 4|\xi|^2|\hat{u}(\xi)|^2(1+|\xi|^2)^{s-1}d\xi\right)^{\frac{1}{2}}$$

$$c \leq \left(\int_{\mathbb{R}^3} (1+|\xi|^2)|\hat{u}(\xi)|^2(1+|\xi|^2)^{s-1}d\xi\right)$$

So we conclude that
$$\|\omega\|_{s-1} \leq c\|u\|_{H^{s-1}}$$

So by the lemma 24 we have that

$$\|\nabla u\|_{L^\infty} \leq C\left(\|\omega\|_{L^\infty} + \|u\|_{H^3}\epsilon^\gamma + \|\omega\|_{L^\infty}\ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega\|_{L^2}\right)$$

Now if $\|u\|_{H^3} \leq 1$,we assume that $\frac{R_0}{2\epsilon} = \|u\|_{H^3}$ and $\epsilon = \frac{1}{2}$ so we have that

$$\|\nabla u\|_{L^\infty} \leq C\left(\|\omega\|_{L^\infty} + 1 + \|\omega\|_{L^\infty}\ln(\|u\|_{H^3}) + \|\omega\|_{L^2}\right)$$

It follows that

$$\|\nabla u\|_{L^\infty} \leq C\left(1 + \|\omega\|_{L^\infty}(1 + \ln\|u\|_{H^3}) + \|\omega\|_{L^2}\right)$$

If $\|u\|_{H^3} > 1$ we assume that $2\epsilon = \frac{1}{\|u\|_{H^3}}$ and $\frac{R_0}{2\epsilon} = \|u\|_{H^3}$ then

$$\|\nabla u\|_{L^\infty} \leq C\left(1 + \|\omega\|_{L^\infty}(1 + \ln\|u\|_{H^3}) + \|\omega\|_{L^2}\right)$$

We conclude that

$$\|\nabla u\|_{L^\infty} \leq C\left(1 + \|\omega\|_{L^\infty}(1 + \ln\|u\|_{H^3}) + \|\omega\|_{L^2}\right) \tag{RSnab}$$

□

Step 4. In this step we will combine all the previous results. By the hypothesis we have that $\int_0^{T^*}\|\omega(t)\|_{L^\infty}dt = M$, we also set $m(t) = \|\omega\|_{L^\infty}$. Also by step 2 we have that $\|\omega\|_{L^2} \leq \|\omega(0)\|_{L^2}e^{\int_0^t\|\omega\|_{L^\infty}ds}$, for $t < T^*$ we have that $\|\omega\|_{L^2} \leq \|\omega_0\|_{L^2}e^{cM} \leq k$
So in the (RSnab) we have that

$$\|\nabla u\|_{L^\infty} \leq C\left(1 + m(t)(1 + \ln\|u\|_{H^3}) + k\right)$$
$$\|\nabla u\|_{L^\infty} \leq C\left(\delta + m(t)(1 + \ln\|u\|_{H^3})\right)$$

where $\delta = 1 + k$ and $k \geq 0$. Thus we have that

$$\|\nabla u\|_{L^\infty} \leq C\left(\delta + m(t)(\delta + \ln\|u\|_{H^3})\right)$$

We also know that if we assume the function $\ln^+(x) = \begin{cases} ln(x) & x \geq 1 \\ 0 & x < 1 \end{cases}$ then its is true that $\ln(x) \leq \ln^+(x)$ so we have that

$$\|\nabla u\|_{L^\infty} \leq C\left(\delta + m(t)(\delta + \ln^+\|u\|_{H^3})\right)$$

Furthermore we know that $ln^+(x) \geq 0$, $\forall x$, and it is an increasing function thus we have that
$$\|\nabla u\|_{L^\infty} \leq C\left((\delta + \ln^+\|u\|_{H^m}) + m(t)(\delta + \ln^+\|u\|_{H^m})\right)$$
So we have that:
$$\|\nabla u\|_{L^\infty} \leq C(1 + m(t))(\delta + \ln^+\|u\|_{H^m}) \tag{RSnab1}$$

By the Step 1 we have that

$$\|u\|_{H^m} \leq \|u_0\|_{H^m} e^{c\int_0^t \|\nabla u\|_{L^\infty} ds}$$

So

$$\begin{aligned}\ln^+(\|u\|_{H^m}) &\leq \ln^+(\|u_0\|_{H^m} e^{c\int_0^t \|\nabla u\|_{L^\infty} ds}) \\ &= \ln^+ \|u_0\|_{H^m} + c\int_0^t \|\nabla u\|_{L^\infty} ds\end{aligned}$$

Thus by (RSnab1) we have that

$$\|\nabla u\|_{L^\infty} \leq C(1+m(t))(\delta + \ln^+ \|u_0\|_{H^m} + c\int_0^t \|\nabla u\|_{L^\infty} ds)$$

We set $q = \delta + \ln^+ \|u_0\|_{H^m}$ so we have that

$$\|\nabla u\|_{L^\infty} \leq C(1+m(t))(q + c\int_0^t \|\nabla u\|_{L^\infty} ds)$$

We set $a(t) = C(1+m(t))$ so

$$\|\nabla u\|_{L^\infty} \leq a(t)(q + c\int_0^t \|\nabla u\|_{L^\infty} ds)$$

$$\|\nabla u\|_{L^\infty} \leq qa(t) + ca(t)\int_0^t \|\nabla u\|_{L^\infty} ds$$

By Gronwall inequality[3] we have that

$$\|\nabla u\|_{L^\infty} \leq qa(t) + a(t)\int_0^t cqa(s) e^{\int_t^s ca(r)dr} ds$$

$$\begin{aligned}&\leq qa(t) + qa(t)\int_0^t -\frac{d}{ds}\left(e^{\int_t^s ca(r)dr}\right) ds \\ &\leq qa(t) - qa(t)\left[e^{\int_t^s ca(r)dr}\right]_{s=0}^{s=t} \\ &\leq qa(t) - qa(t)\left(e^{\int_t^t ca(r)dr} - e^{\int_0^t ca(r)dr}\right) \\ &\leq qa(t) - qa(t) + qa(t)e^{\int_0^t ca(r)dr} \\ &\leq qa(t)e^{\int_0^t ca(r)dr}\end{aligned}$$

Thus

$$\|\nabla u\|_{L^\infty} \leq qa(t)e^{\int_0^t ca(r)dr}$$

By step 1 again we have that

$$\|u\|_{H^m} \leq \|u_0\|_{H^m} e^{c\int_0^t qa(s)e^{\int_0^s ca(r)dr} ds}$$

[3] [17] theorem 11

$$\|u\|_{H^m} \le \|u_0\|_{H^m} e^{c\int_0^t qC(1+m(s))e^{\int_0^s cC(1+m(r))dr} ds}$$

$$\|u\|_{H^m} \le \|u_0\|_{H^m} e^{c\int_0^t C'(1+m(s))e^{\int_0^s \tilde{C}(1+m(r))dr} ds}$$

Now we have for $\tilde{C}\int_0^s (1+m(r))dr$ and $s \le T^*$ that

$$\tilde{C}\int_0^s (1+m(r))dr \le \tilde{C}\int_0^{T^*} (1+m(r))dr = \tilde{C}(T^* + \int_0^{T^*} m(r)dr) = \tilde{C}(T^*+M)$$

Thus $e^{\tilde{C}(T^*+M)} = Q \in (0,+\infty)$, furthermore for $c\int_0^t C'(1+m(s))e^{\int_0^s \tilde{C}(1+m(r))dr} ds$ for $s < T$ we have by the above that

$$c\int_0^t C'(1+m(s))e^{\int_0^s \tilde{C}(1+m(r))dr} ds \le c\int_0^t C'(1+m(s))Q \le \bar{C}\int_0^t 1+m(s)ds$$

for $t \le T^*$ we have that $\int_0^t 1+m(s)ds \le \int_0^{T^*} 1+m(s)ds = T^*+M$ thus

$$c\int_0^t C'(1+m(s))e^{\int_0^s \tilde{C}(1+m(r))dr} ds \le \bar{C}(T^*+M) = Q' \in (0,+\infty)$$

Now for the $e^{c\int_0^t C'(1+m(s))e^{\int_0^s \tilde{C}(1+m(r))dr} ds}$ and $s \le t \le T^*$ we have that

$$e^{c\int_0^t C'(1+m(s))e^{\int_0^s \tilde{C}(1+m(r))dr} ds} \le e^{\int_0^{T^*} Q'} = e^{Q'T^*} + 1$$

So we conclude that

$$\|u\|_{H^m} \le \|u_0\|_{H^m}(e^{Q'T^*}+1) \le R$$

which is a apriori bound so the $\lim_{t\to T^*} \|u\|_{H^m} < \infty$ which is a contradiction to the initial hypothesis, which occurs from the hypothesis that $\int_0^{T^*} \|\omega\|_{L^\infty} < \infty$ thus we conclude that $\int_0^{T^*} \|\omega\|_{L^\infty} = \infty$ □

2 dimensions

[4] For the 2 dimensions we will use again the radial energy decomposition for the velocity field u. Recall that $u = y + b$ where b is defined by the initial vorticity via Biot-Savart law and also $\omega_b \in C^\infty \cap L^2$. So we have seen in the previous chapter that we can find a y and define the solution u.

Again we will prove 4 steps

Step 1:We will find an energy estimate for the H^m norm of y in terms of $\|\nabla y\|_{L^\infty}$
Step 2:We will bound the $\|\nabla y\|_{L^\infty}$ in terms of vorticity its vorticity
Step 3:We will find an L^2 estimate for the vorticity
Step 4:We will combine all the above steps and we will show that if $\int_0^{T^*} \|\omega_y(t)\|_{L^\infty} dt$ stays bounded then $\|y_t\|_{H^m}$ stays bounded.

[4] [28]

Step 1. We have that y satisfy the equation

$$\frac{\partial}{\partial t}y = \nu\Delta y - P[(y\cdot\nabla)y + (b\cdot\nabla)y + (y\cdot\nabla)b]$$

We differentiate this relation and we have that

$$D^a\left(\frac{\partial}{\partial t}y\right) = D^a\left(\nu\Delta y - P[(y\cdot\nabla)y + (b\cdot\nabla)y + (y\cdot\nabla)b]\right)$$

I.e.

$$D^a\left(\frac{\partial}{\partial t}y\right) = D^a(\nu\Delta y) - D^a(P[(y\cdot\nabla)y + (b\cdot\nabla)y + (y\cdot\nabla)b])$$

We multiply with $D^a u$ in L^2, thus

$$\int_{\mathbb{R}^2} D^a\left(\frac{\partial}{\partial t}y\right)\cdot D^a u dx = \nu\int_{\mathbb{R}^2} D^a(\Delta y)\cdot D^a u dx - \int_{\mathbb{R}^2} D^a(P[(y\cdot\nabla)y+(b\cdot\nabla)y+(y\cdot\nabla)b])\cdot D^a u dx$$

So

$$\frac{1}{2}\frac{d}{dt}\|D^a u\|_{L^2}^2 \leq -\nu\|\nabla D^a u\|_{L^2}^2 - \int_{\mathbb{R}^2} D^a(P[(y\cdot\nabla)y + (b\cdot\nabla)y + (y\cdot\nabla)b])\cdot D^a u dx$$

We will deal with the integral on the right side, we will separate it in three integrals which will estimate individually

$$I_1 = \int_{\mathbb{R}^2} D^a P\left((y_\epsilon\cdot\nabla)y\right)\cdot D^a y dx$$

$$I_2 = c\int_{\mathbb{R}^2} D^a P\left((y\cdot\nabla)b\right)\cdot D^a y dx^5$$

For the I_1 we sum and subtract the term $P\left[(y\cdot\nabla)D^a\cdot y\right]$, thus

$$I_1 = \int_{\mathbb{R}^2} D^a\left\{P\left[(y\cdot\nabla)y\right]\right\} - P\left[(y\cdot\nabla)D^a y\right] + P\left[(y\cdot\nabla)D^a y\right]\cdot D^a y dx$$

We also have that $\int_{\mathbb{R}^2} P\left[(y\cdot\nabla)D^a y\right]\cdot D^a y dx \leq 0$
So

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}^2}\left[D^a\left\{P\left[(y\cdot\nabla)y\right]\right\} - P\left[(y\cdot\nabla)D^a y\right]\right]\cdot D^a y dx \\ &\leq \|D^a y\|_{L^2}\|D^a\left\{P\left[(y\cdot\nabla)y\right]\right\} - P\left[(y\cdot\nabla)D^a y\right]\|_{L^2} \end{aligned}$$

For the I_2 we have that

$$I_2 \leq \|D^a y\|_{L^2}\|D^a\left\{P\left[(y_\epsilon\cdot\nabla)b\right]\right\}\|_{L^2}$$

By combining all the above relations with (L11) we get :

$$\frac{1}{2}\frac{d}{dt}\|D^a y\|_{L^2}^2 \leq \|D^a y\|_{L^2}$$

[5] See pg 164-167

$$\left[\|D^a \{P[(y\cdot\nabla)y]\} - P[(y\cdot\nabla)D^a y]\|_{L^2} + \|D^a\{P[(y\cdot\nabla)b]\}\|_{L^2}\right]$$

We sum over this relation and we get

$$\frac{1}{2}\frac{d}{dt}\sum_{|a|\le m}\|D^a y\|^2_{L^2} \le \sum_{|a|\le m}\|D^a y\|_{L^2}$$

$$\sum_{|a|\le m}\left[\|D^a \{P[(y\cdot\nabla)y]\} - P[(y\cdot\nabla)D^a y]\|_{L^2} + \|D^a\{P[(y\cdot\nabla)b]\}\|_{L^2}\right]$$

By the proposition 4.1.4

$$\frac{1}{2}\frac{d}{dt}\|y\|^2_{H^m} \le c\|y\|_{H^m}$$

$$\left[\|\nabla y\|_{L^\infty}\|y\|_{H^m} + \|y\|_{H^m}\|\nabla y\|_{L^\infty} + \|\nabla b\|_{L^\infty}\|y\|_{H^m}\right]$$

So eventually we have that

$$\frac{d}{dt}\|y\|_{H^m} \le c\|y\|_{H^m}\left[\|\nabla y\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right]$$

By Gronwall's lemma in differential form we have that

$$\|y\|_{H^m} \le \|u_0\|_{H^m} e^{\int_0^t c(\|\nabla y\|_{L^\infty}+\|\nabla b\|_{L^\infty})}$$

□

Step 2. We know that$\text{div}y = 0$ so there exist a stream function z such that $\Delta z = \omega_y$ by the study in chapter 3 we know that we can find a singular kernel such that

$$y = \int_{\mathbb{R}^2} K_2(p)\omega_y(x-p)dp$$

we also have that the kernel $P_2 = \nabla K_2$ defines a SIO through convolution, and that

$$\nabla y(x) = c\omega_y(x) + P_2\omega_y(x) \tag{Rs2}$$

Assume the cut-off function $\rho(|x|) = \begin{cases} 1 & |x| \le R_0 \\ 0 & |x| \ge 2R_0 \end{cases}$, we will define R_0 later in the proof

We have that $P_2\omega_y(x) = \text{P.V.}\int_{\mathbb{R}^2}\nabla K_2(p)\omega(x-p)dp$

$$= \text{P.V.}\int_{\mathbb{R}^2}\nabla K_2(p)\rho(|p|)\omega(x-p) + \nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp$$

$$= \text{P.V.}\int_{\mathbb{R}^2}\nabla K_2(p)\rho(|p|)\omega(x-p)dp + \text{P.V.}\int_{\mathbb{R}^2}\nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp$$

So we will see each integral individually: Remark:We will hand the following integrals and we will use several tools, since we dont't want to do a rough estimate which will

give us infinity
For $\epsilon \leq R_0$ we have that

$$I_1 = \text{P.V.} \int_{\mathbb{R}^2} \nabla K_2(p)\rho(|p|)\omega_y(x-p)dp$$

$$= \text{P.V.} \left(\int_{|p|\leq\epsilon} \nabla K_2(p)\rho(|p|)\omega_y(x-p)dp + \int_{|p|\geq\epsilon} \nabla K_2(p)\rho(|p|)\omega_y(x-p)dp \right)$$

The singularity of the kernel is on 0 so the second integral is well defined thus

$$= \text{P.V.} \int_{|p|\leq\epsilon} \nabla K_2(p)\rho(|p|)\omega_(x-p)dy + \int_{|p|\geq\epsilon} \nabla K_2(p)\rho(|p|)\omega_(x-p)dp$$

Again we will see each term individually

- $I_1^1 = \text{P.V.} \int_{|p|\leq\epsilon} \nabla K_2(p)\rho(|p|)\omega(x-p)dp$
  Since P_2 has mean value zero it is true that P.V. $\int_{\mathbb{R}^2} P_2(p)\rho\omega_y(x)dp = \omega_y(x)\int_{\mathbb{R}^2} P_2(p)dp = 0$ so we have that

  $$I_1^1 = \text{P.V.} \int_{|p|\leq\epsilon} \nabla K_2(p)\rho(|p|)\omega_y(x-p) - \nabla K_2(p)\rho(|p|)\omega_y(x)dy$$

  $$= \text{P.V.} \int_{|p|\leq\epsilon} \nabla K_2(p)\rho(|p|)(\omega_y(x-p) - \omega_y(x))$$

  Thus

  $$|I_1^1| \leq \int_{|p|\leq\epsilon} |\nabla K_2(p)\rho(|p|)||\omega_y(x-p) - \omega_y(x)|dp$$

  We multiply and divide with the $|p|^\gamma$, for $0 < \gamma < 1$ so

  $$\leq \int_{|p|\leq\epsilon} |\nabla K_2(p)\rho(|p|)| \frac{|\omega_y(x-p) - \omega_y(x)|}{|p|^\gamma} |p|^\gamma dp$$

  $$\leq \int_{|p|\leq\epsilon} |p|^\gamma |\nabla K_2(p)\rho(|p|)| \|\omega_y\|_{C^{0,\gamma}} dp$$

  $$\leq \|\omega_y\|_{C^{0,\gamma}} \int_{|p|\leq\epsilon} |p|^\gamma |\nabla K_2(p)| dp$$

  $$\leq c\|\omega_y\|_{C^{0,\gamma}} \int_{|p|\leq\epsilon} |p|^\gamma \frac{1}{|p|^2} dy$$

  $$\leq c\|\omega_y\|_{C^{0,\gamma}} \int_0^\epsilon r^\gamma \frac{1}{r^2} r dr$$

  So

  $$|I_1^1| \leq c_{1,1}\|\omega\|_{C^{0,\gamma}} \epsilon^\gamma$$

- $I_1^2 = \int_{|p|\geq\epsilon} \nabla K_2(p)\rho(|p|)\omega_y(x-p)dp$

$$= \int_{2\epsilon\leq|p|\leq R_0} \nabla K_2(p)\rho(|y|)\omega_y(x-p)dp + \int_{|p|>R_0} \nabla K_3(p)\rho(|p|)\omega_y(x-p)dp$$

We will see each integral individually:

$$I_1^{2a} = \int_{2\epsilon\leq|p|\leq R_0} \nabla K_2(p)\rho(|p|)\omega_y(x-p)dp$$

So we have that

$$|I_1^{2a}| \leq \int_{2\epsilon\leq|p|\leq R_0} |\nabla K_2(p)\rho(|p|)||\omega_y(x-p)|dp$$

$$\leq c\|\omega_y\|_{L^\infty} \int_{2\epsilon\leq|p|\leq R_0} |\nabla K_2(p)\rho(|p|)||\omega_y(x-p)|dp$$

$$\leq c\|\omega_y\|_{L^\infty} \int_{2\epsilon\leq|p|\leq R_0} \frac{1}{|p|^2}dy$$

$$\leq c\|\omega_y\|_{L^\infty} \int_{2\epsilon}^{R_0} \frac{1}{r^2}rdr$$

So

$$|I_1^{2a}| \leq c_{1,2a}\|\omega_y\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right)$$

And for the other one we have that

$$I_1^{2b} = \int_{|p|>R_0} \nabla K_2(p)\rho(|p|)\omega_(x-p)dp$$

So we have that

$$|I_1^{2b}| \leq \int_{|p|>R_0} |\nabla K_2(p)\rho(|p|)||\omega_y(x-p)|dp$$

$$\leq \|\omega_y\|_{L^2} \left(\int_{|p|>R_0} |\nabla K_2(p)\rho(|p|)|^2 dy\right)^{\frac{1}{2}}$$

$$\leq c\|\omega_y\|_{L^2} \left(\int_{|p|>2R_0} \left|\frac{1}{|p|^2}\right|^2 dy\right)^{\frac{1}{2}}$$

$$\leq c\|\omega_y\|_{L^2} \left(\int_{2R_0}^{\infty} \frac{1}{r^4}rdr\right)^{\frac{1}{2}}$$

$$|I_1^{2b}| \leq c_{1,2b}\|\omega_y\|_{L^2} R_0^{-\frac{3}{2}}$$

So for the first integral we have that

$$|I_1| \leq c_{1,1}\|\omega_y\|_{C^{0,\gamma}}\epsilon^\gamma + c_{1,2a}\|\omega_y\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + c_{1,2b}\|\omega_y\|_{L^2} R_0^{-\frac{3}{2}}$$

For the second integral we have that for $\epsilon \leq R_0$

$$I_2 = \text{P.V.} \int_{\mathbb{R}^2} \nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp$$

$$= \text{P.V.} \left(\int_{|p|\leq\epsilon} \nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp + \int_{|p|\geq\epsilon} \nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp \right)$$

$$= \text{P.V.} \int_{|p|\leq\epsilon} \nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp$$

$$+ \int_{2\epsilon\leq|p|\leq R_0} \nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp + \int_{|p|>R_0} \nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp$$

By the definition of the cut off function we have that the first 2 integrals are zero so we proceed with the last one and we have that

$$|\int_{|p|>R_0} \nabla K_2(p)(1-\rho(|p|))\omega_y(x-p)dp| \leq \int_{|p|>R_0} |\nabla K_3(p)(1-\rho(|p|))||\omega_y(x-p)|dp$$

$$\leq \|\omega_y\|_{L^2} c \left(\int_{2R_0}^\infty \frac{1}{r^4} r dr \right)$$

So $|I_2| \leq c_2\|\omega_y\|_{L^2} R_0^{-\frac{3}{2}}$ Thus we have that

$$|P_2\omega_y(x)| \leq c_{1,1}\|\omega_y\|_{C^{0,\gamma}}\epsilon^\gamma + c_{1,2a}\|\omega_y\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + c_{1,2b}\|\omega_y\|_{L^2} R_0^{-\frac{3}{2}} + c_2\|\omega_y\|_{L^2} R_0^{-\frac{3}{2}}$$

$$\leq c'\big(\|\omega_y\|_{C^{0,\gamma}}\epsilon^\gamma + \|\omega_y\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega_y\|_{L^2} R_0^{-\frac{3}{2}}\big)$$

We set $R_0 = 1$ then

$$|P_3\omega_y(x)| \leq c'\big(\|\omega_y\|_{C^{0,\gamma}}\epsilon^\gamma + \|\omega_y\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega_y\|_{L^2}\big)$$

Remark we don't touch the R_0 inside of the ln, because we want to estimate it together with 2ϵ.
So by (Rs2)we have that

$$\|\nabla y\|_{L^\infty} \leq c\|\omega_y\|_{L^\infty} + c'\big(\|\omega_y\|_{C^{0,\gamma}}\epsilon^\gamma + \|\omega_y\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega_y\|_{L^2}\big)$$

By the Sobolev embedding we have that $\|\omega_y\|_{C^{0,\gamma}} \leq c\|\omega_y\|_{H^2}$ so we have that

$$\|\nabla y\|_{L^\infty} \leq C\left(\|\omega_y\|_{L^\infty} + \|\omega_y\|_{H^2}\epsilon^\gamma + \|\omega_y\|_{L^\infty} \ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega_y\|_{L^2}\right)$$

Lemma 26. *Assume that u is a velocity field in 2 dimension and ωits vorticity . Then $s > 0$ we have that*

$$\|\omega\|_{H^{s-1}} \leq c\|u\|_{H^s}$$

proof of lemma: In the two dimensions the vorticity of a velocity field is a scalar quantity given as $\omega = \frac{\partial}{\partial x_1}u_2 - \frac{\partial}{\partial x_2}u_1$
Also

$$\int_{\mathbb{R}^2} e^{-i\xi x} \cdot \frac{\partial}{\partial x_j} u dx = -\int_{\mathbb{R}^2} \frac{\partial}{\partial x_j} e^{-i\xi x} \cdot u dx = i\xi_j \hat{u}$$

So we have that $\hat{\omega}(\xi) = \int_{\mathbb{R}^2} e^{-i\xi x}\left(\frac{\partial}{\partial x_1}u_2 - \frac{\partial}{\partial x_2}u_1\right)$

$$= i\xi_1\hat{u}_2 - i\xi_2\hat{u}_1 = \begin{pmatrix} 0 & i\xi_1 \\ -i\xi_2 & 0 \end{pmatrix}\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$$

For the matrix S we have that $|S| = \xi_1^2 + \xi_2^2 = |\xi|$ Thus we have that

$$\|\omega\|_{H^{s-1}} = \left(\int_{\mathbb{R}^3} |\hat{\omega}(\xi)|^2(1+|\xi|^2)^{s-1}d\xi\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\mathbb{R}^3} |\xi|^2|\hat{u}(\xi)|^2(1+|\xi|^2)^{s-1}d\xi\right)^{\frac{1}{2}}$$

$$c \leq \left(\int_{\mathbb{R}^3} (1+|\xi|^2)|\hat{u}(\xi)|^2(1+|\xi|^2)^{s-1}d\xi\right)$$

So we conclude that

$$\|\omega\|_{s-1} \leq c\|u\|_{H^{s-1}}$$

So by the lemma 25 we have that

$$\|\nabla y\|_{L^\infty} \leq C\left(\|\omega_y\|_{L^\infty} + \|y\|_{H^3}\epsilon^\gamma + \|\omega_y\|_{L^\infty}\ln\left(\frac{R_0}{2\epsilon}\right) + \|\omega_y\|_{L^2}\right)$$

Now if $\|y\|_{H^3} \leq 1$, we assume that $\frac{R_0}{2\epsilon} = \|y\|_{H^3}$ and $\epsilon = \frac{1}{2}$ so we have that

$$\|\nabla y\|_{L^\infty} \leq C\left(\|\omega_y\|_{L^\infty} + 1 + \|\omega\|_{L^\infty}\ln(\|y\|_{H^3}) + \|\omega_y\|_{L^2}\right)$$

It follows that

$$\|\nabla y\|_{L^\infty} \leq C\left(1 + \|\omega_y\|_{L^\infty}(1 + \ln\|y\|_{H^3}) + \|\omega_y\|_{L^2}\right)$$

If $\|y\|_{H^3} > 1$ we assume that $2\epsilon = \frac{1}{\|y\|_{H^3}}$ and $\frac{R_0}{2\epsilon} = \|u\|_{H^3}$ then

$$\|\nabla y\|_{L^\infty} \leq C\left(1 + \|\omega_y\|_{L^\infty}(1 + \ln\|y\|_{H^3}) + \|\omega_y\|_{L^2}\right)$$

We conclude that

$$\|\nabla y\|_{L^\infty} \leq C\left(1 + \|\omega_y\|_{L^\infty}(1 + \ln\|y\|_{H^3}) + \|\omega_y\|_{L^2}\right) \tag{RSnab}$$

□

Step 3. Now as far as concerned the L^2 norm of vorticity we have by the radial energy decomposition that $u = y + b$ so $\omega = \omega_y + \omega_b$ where b is a known radial vorticity. We know that ω_y satisfies the equation $\frac{\partial}{\partial t}\omega_y + (y \cdot \nabla)\omega_y + (y \cdot \nabla)\omega_b + (b \cdot \nabla)\omega_y = \nu \Delta \omega_y$,so by multiplying this equation with ω_y in L^2 we have that

$$\int_{\mathbb{R}^2} \frac{\partial}{\partial t}\omega_y \cdot \omega_y dx + \int_{\mathbb{R}^2} (y \cdot \nabla)\omega_y \cdot \omega_y dx + \int_{\mathbb{R}^2} (y \cdot \nabla)\omega_b \cdot \omega_y dx + \int_{\mathbb{R}^2} (b \cdot \nabla)\omega_y \cdot \omega_y dx = -v \|\nabla \omega_y\|_{L^2}^2$$

We know that $\frac{1}{2}\int_{\mathbb{R}^2} (y \cdot \nabla)\omega_y \cdot \omega_y dx \leq 0$, $\int_{\mathbb{R}^2} (b \cdot \nabla)\omega_y \cdot \omega_y dx \leq 0$ and also $\int_{\mathbb{R}^2} (y \cdot \nabla)\omega_b \cdot \omega_y dx \leq \|\omega_y\|_{L^2} \|y\|_{L^2} \|\nabla \omega_b\|_{L^\infty}$ It follows that

$$\frac{1}{2}\frac{d}{dt}\|\omega_y\|_{L^2}^2 \leq \|\omega_y\|_{L^2} \|y\|_{L^2} \|\nabla \omega_b\|_{L^\infty}$$

Lemma 27. *It is true that in two dimensions that* $\|\nabla u\|_{L^2} \leq c\|\omega\|_{L^2}$, *where* $\omega = \mathrm{curl} u$

proof of lemma

By lemma 25 we have that $\hat{\omega}(\xi) = \begin{pmatrix} 0 & i\xi_1 \\ -i\xi_2 & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$ For the determinant of the matrix we know that for $\xi \neq 0$ that $\det S \neq 0$ so the inverse matrix is $S^{-1} = \frac{1}{\xi_1 \xi_2} \begin{pmatrix} 0 & -i\xi_1 \\ i\xi_2 & 0 \end{pmatrix}$ we have that $|S^{-1}| = |\xi| = c$ We conclude that since $\hat{u}(\xi) = S{-}1\hat{\omega}(\xi)$

$$\begin{aligned} \|\hat{u}(\xi)\|_{L^2} &= \|S^{-1}\hat{\omega}(\xi)\|_{L^2} \\ &\leq \|S^{-1}\| \|\hat{\omega}(\xi)\|_{L^2} \\ &\leq c\|\hat{\omega}(\xi)\|_{L^2} \end{aligned}$$

So by lemma 26 we have that

$$\frac{1}{2}\frac{d}{dt}\|\omega_y\|_{L^2}^2 \leq c\|\omega_y\|_{L^2}^2 \|\nabla \omega_b\|_{L^\infty}$$

$$\frac{d}{dt}\|\omega_y\|_{L^2} \leq c\|\omega_b\|_{L^\infty} \|\omega_y\|_{L^2}$$

So by Gronwall

$$\|\omega_y\|_{L^2} \leq \|\omega_y(0)\|_{L^2} e^{\int_0^t \|\nabla \omega_b\|_{L^\infty}} = f(t)$$

where $f(t)$ for finite t is finite □

Step 4. in this step we will combine all the previous results. By the hypothesis we have that $\int_0^{T^*} \|\omega(t)\|_{L^\infty} dt = M$, we also set $m(t) = \|\omega_y\|_{L^\infty}$. Also by step 3 we have that $\|\omega\|_{L^2} \leq f(t)$ so we go to the relation (RSnab) and we substitute, so we have

$$\|\nabla y\|_{L^\infty} \leq C\left(1 + m(t)(1 + \ln\|y\|_{H^3}) + f(t)\right)$$

$$\|\nabla y\|_{L^\infty} \leq C\left(\delta(t) + m(t)(1 + \ln\|u\|_{H^3})\right)$$

where $\delta(t) = f(t) + C$ where $\delta(t)$ for finite t is finiteand $C \geq 0$ We also know that if we assume the function $\ln^+(x) = \begin{cases} ln(x) & x \geq 1 \\ 0 & x < 1 \end{cases}$ then its is true that $\ln(x) \leq \ln^+(x)$ so we have that

$$\|\nabla y\|_{L^\infty} \leq C\left(\delta(t) + m(t)(\delta(t) + \ln^+\|y\|_{H^3})\right)$$

Furthermore we know that $ln^+(x) \geq 0$, $\forall x$, and it is an increasing function thus we have that

$$\|\nabla y\|_{L^\infty} \leq C\left((\delta(t) + \ln^+ \|y\|_{H^m}) + m(t)(\delta(t) + \ln^+ \|y\|_{H^m})\right)$$

So we have that:

$$\|\nabla y\|_{L^\infty} \leq C(1 + m(t))(\delta(t) + \ln^+ \|y\|_{H^m}) \tag{RSnab1}$$

By step 1 we have that

$$\|y\|_{H^m} \leq \|y_0\|_{H^m} e^{c\int_0^t \|\nabla y\|_{L^\infty} + \|\nabla b\|_{L^\infty} ds}$$

So

$$\ln^+(\|y\|_{H^m}) \leq \ln^+(\|y_0\|_{H^m} e^{c\int_0^t \|\nabla y\|_{L^\infty} + \|\nabla b\|_{L^\infty} ds})$$

$$= \ln^+ \|y_0\|_{H^m} + c\int_0^t \|\nabla y\|_{L^\infty} + \|\nabla b\|_{L^\infty} ds$$

Thus by (RSnab1) we have that

$$\|\nabla y\|_{L^\infty} \leq C(1 + m(t))(\delta(t) + \ln^+ \|y_0\|_{H^m} + c\int_0^t \|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} ds)$$

We set $q(t) = \delta(t) + \ln^+ \|u_0\|_{H^m}$ so we have that

$$\|\nabla y\|_{L^\infty} \leq C(1 + m(t))(q(t) + c\int_0^t \|\nabla y\|_{L^\infty} + \|\nabla b\|_{L^\infty} ds)$$

We set $a(t) = C(1 + m(t))$ so

$$\|\nabla y\|_{L^\infty} \leq a(t)(q(t) + c\int_0^t \|\nabla y\|_{L^\infty} + \|\nabla b\|_{L^\infty} ds)$$

We set $g(t) = q(t)a(t) + ca(t)\int_0^t \|\nabla b\|_{L^\infty} du$ and we have that

$$\|\nabla y\|_{L^\infty} \leq g(t) + ca(t)\int_0^t \|\nabla y\|_{L^\infty} ds$$

By Gronwall's lemma we have that

$$\|\nabla y\|_{L^\infty} \leq g(t) + ca(t)\int_0^t g(s) e^{\int_s^t ca(t)dr} ds$$

We expand this term again

$$\|\nabla y\|_{L^\infty} \leq q(t)a(t) + ca(t)\int_0^t \|\nabla b\|_{L^\infty} du$$

$$+ca(t)\int_0^t \left(q(s)a(s) + ca(s)\int_0^s \|\nabla b\|_{L^\infty} du\right) e^{\int_s^t ca(t)dr} ds$$

We set $\int_0^{t^*} \|\nabla b\|_{L^\infty} du = U(t)$ where $U(t)$ is finite when t is finite thus

$$\|\nabla y\|_{L^\infty} \leq q(t)a(t) + ca(t)U(t) + ca(t)\int_0^{T^*} (qa(s) + ca(s)U(t))\, e^{\int_s^{T^*} ca(t)dr} ds$$

It follows that

$$\|\nabla y\|_{L^\infty} \leq qa(t) + ca(t)U + ca(t)(qM + UcMe^{cM})$$

$$\|\nabla y\|_{L^\infty} \leq \mathbf{C(t)}a(t)$$

where $\mathbf{C(t)} = q(t) + cU(t) + cq(t)M + U(t)c^2Me^{cM}$
Again by step 1 we have that

$$\|y\|_{H^m} \leq \|y_0\|_{H^m} e^{c\int_0^t \|\nabla u\|_{L^\infty} ds}$$

$$\|y\|_{H^m} \leq \|y_0\|_{H^m} e^{c\int_0^t \mathbf{C(s)}a(s)ds}$$

For $T^* < \infty$ we have that the above quantity is bounded this is an apriori bound so the $\lim_{t\to T^*} \|y\|_{H^m} < \infty$ which is a contradiction to the initial hypothesis, which occurs from the hypothesis that $\int_0^{T^*} \|\omega\|_{L^\infty} < \infty$ thus we conclude that $\int_0^{T^*} \|\omega_y\|_{L^\infty} = \infty$ □

5.2 Global solutions in two dimensions

In two dimension the Beale Kato Majda criterion gives global solutions since:

Theorem 5.2.1. *Assume that u_0 is a 2d initial velocity field, with radial energy decomposition $u_0 = v_0 + b$ with $v_0 \in H^m$, $m > 3$ and $curlb = \omega_0(|x|) \in C^\infty \cap L^2$ then there exists a unique smooth global solution $u(x,t) = v(x,t) + b(x,t)$ to the 2d Euler or Navier Stokes equation with $v(x,t) \in C([0,\infty), H^m)$ and b an exact eddy solution.*

Proof. By the previous chapters discussion about local existance we know that there exists a solution y, locally in time for the equation, $\partial_t y + (y\cdot\nabla)y + (y\cdot\nabla)b + (b\cdot\nabla)y = 0$
So our aim is to extend this solution global in time, so by the Beale-Kato-Majda criterion, we have to show that the norm $\|\omega_y\|_{L^1([0,T],L^\infty)}$ remains bounded for every finite time T.
By proposition 1.4.2 we know that $\omega(X(a,t),t) = \omega_0(a)$, so $\|\omega_y(s)\|_{L^\infty} = \|\omega_y(0)\|_{L^\infty}$ and thus

$$\int_0^T \|\omega_y\|_{L^\infty} ds = \int_0^T \|\omega_y(0)\|_{L^\infty} dt = CT$$

So now we have a global solution y to the above equation and a steady solution b to the Euler equation. We set $u = y + b$ we have that u is a global in time solution to the Euler equation, on the space $V^m + b$ □

In three dimensions we don't know if we can bound the quantity $\int_0^{T^*} \|\omega(x,t)\|_{L^\infty}$ and this is an open problem.
Someone may wonder why we have greater results in two dimensions than three, since

the velocity in 2 dimension has not such good properties. To be honest we choose to find solutions via energy methods so the problem in two dimension was that velocity fields did not have finite kinetic energy, but at the end of the day using the radial energy decomposition we have exact the same results for two and three dimensions i.e. the local existence of smooth solutions. The tool we use here to globally extend them, is the vorticity, in two dimensions vorticity is scalar and vorticity equation is just a scalar transport equation. While in three dimension the vorticity is a vector field and the vorticity equation has a more complicated form and contains also the term $(\omega \cdot \nabla)u$. This term gives us the information that the vector of vorticity is deformed by the matrix∇u. So in two dimensions we can easily find a bound, but in three dimensions the case is more complicated.It has been proved by Constantin-Fefferman-Majda[6] the CFM blow up criterion where the vorticity direction vector is inolved.

[6] [10]

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