UNIVERSITY OF IOANNINA
DEPARTMENT OF MATHEMATICS

Maria Keroglou

Ricci Flow and Sphere Theorems

MASTER THESIS

Ioannina, 2023

The present dissertation was carried out under the postgraduate program of the Department of Mathematics of the University of Ioannina in order to obtain the master degree.

Accepted by the three members of the evaluation committee on November 2, 2023:

Andreas Savas-Halilaj	Associate Professor (Supervisor)
Theodoros Vlachos	Professor
Nikolaos Roidos	Assistant Professor

The acceptance of this dissertation by the Department of Mathematics of the University of Ioannina does not imply the approval of the opinions of the author (section 202 par. 2 Law 5343/1932 and section 50 par. 8 Law 1268/1982).

Statutory Declaration

I lawfully declare with statutorily that the present dissertation was carried out under the international ethical and academical rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis.

Dedicated to my beloved parents Stavros and Lia.

ΕΥΧΑΡΙΣΤΙΕΣ

Αρχικά θα ήθελα να ευχαριστήσω τον επιβλέποντά μου Αναπληρωτή Καθηγητή Ανδρέα Σάββα-Χαλιλάι, για τη συνεργασία, το χρόνο του και την επιλογή αυτού του ιδιαίτερα απαιτητικού αλλά και ενδιαφέροντος θέματος. Επιπλέον, θα ήθελα να ευχαριστήσω τα άλλα δύο μέλη της επιτροπής, τον Καθηγητή Θεόδωρο Βλάχο για όσα μας έμαθε στη διαφορική γεωμετρία και τον Επίκουρο Καθηγητή Νικόλαο Ρόιδο.

Δε θα είχα καταφέρει να αρχίσω και να ολοκληρώσω το μεταπτυχιακό μου χωρίς την πολύτιμη στήριξη της οικογένειάς μου.

Ευχαριστώ επίσης τους συμφοιτητές και φίλους μου στο πανεπιστήμιο για την αλληλοϋποστήριξη και όλες τις ενδιαφέρουσες συζητήσεις που είχαμε.

Τέλος, θα ήθελα να ευχαριστήσω τους πολύ κοντινούς μου ανθρώπους για την κατανόηση τους, την πίστη τους σε μένα καθόλη τη διάρκεια του μεταπτυχιακού και τη συνεισφορά τους στο να γίνομαι καλύτερη.

Abstract

The Ricci flow is a certain weakly parabolic partial differential equation which deforms a given Riemannian metric on a compact manifold in the direction of its Ricci curvature. This particular flow, share similarities to the heat flow, however it is nonlinear and exihibits many phenomena not present in the study of the heat equation. The Ricci flow was introduced by Hamilton in his seminal paper [18] and was used by Hamilton & Perelman in resolution of the Poincare conjecture in dimension 3.

The objective of this master thesis is to present the following result due to Hamilton [18]:

Main Theorem: *Let M^3 be an oriented compact 3-dimensional manifold which admits a smooth Riemannian metric with strictly positive Ricci curvature. Then, M^3 also admits a smooth Riemannian metric of constant positive curvature. In particular, if M^3 is simply connected then it is diffeomorphic to $\mathbb{S}^3$.*

ΠΕΡΙΛΗΨΗ

Η ροή Ricci είναι μια διαδικασία παραμόρφωσης μιας μετρικής Riemann στη διεύθυνση της καμπυλότητας Ricci. Η ροή Ricci είναι ένα ιδιάζων παραβολικού τύπου μη-γραμμικό σύστημα διαφορικών εξισώσεων. Όταν η αρχική συνθήκη είναι ένα συμπαγές πολύπτυγμα Riemann, τότε η ροή Ricci υπάρχει, τουλάχιστον, για ένα μικρό χρονικό διάστημα. Η συγκεκριμένη ροή χρησιμοποιήθηκε από τους R. Hamilton & G. Perelman για την επίλυση της Εικασίας του Poincare.

Στην μεταπτυχιακή διατριβή θα αναλύσουμε τα βασικά στοιχεία περί της ροής Ricci και, μεταξύ άλλων, θα αποδείξουμε το εξής θεώρημα του Hamilton [18]:

Κεντρικό Θεώρημα: *Έστω M^3 προσανατολισμένο, συμπαγές πολύπτυγμα διάστασης 3 το οποίο είναι εφοδιασμένο με λεία μετρική Riemann με αυστηρά θετική καμπυλότητα Ricci. Τότε, το M^3 εφοδιάζεται με μια λεία μετρική Riemann σταθερής καμπυλότητας. Συγκεκριμένα, αν το M^3 είναι απλά συνεκτικό τότε είναι διαφορομορφικό με τη σφαίρα $\mathbb{S}^3$.*

Contents

CHAPTER 1

Preliminaries on Riemannian geometry

The purpose of this chapter, is to set up the notation and to recall fundamental definitions from Riemannian geometry. We follow the exposition in [1], [2], [7], [19], [22] and [26].

1.1 Vector bundles

Let M be a smooth manifold of dimension m. The set of all smooth real valued functions of M will be denoted by $C^\infty(M)$. We would like now to associate to every point $x \in M$ a vector space E_x in such a way that these vector spaces fit together to form another manifold which is then called a vector bundle over M.

Definition 1.1.1. *Let E be a smooth manifold and let $\pi\colon E \to M$ be a smooth surjective map. The triple (E, π, M) is called a* real vector bundle of rank n over M*, if the following three conditions are satisfied:*

(1) *For each $x \in M$, the set $E_x = \pi^{-1}(x)$ possesses a real vector space structure of dimension n.*

(2) *For each $x \in M$, there exists a neighborhood $U \subset M$ around x, and a diffeomorphism $\Phi\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$, with the property $\Phi(y, v) \in E_y$ for all $(y, v) \in U \times \mathbb{R}^n$.*

(3) *The smooth map $\Phi_y : \mathbb{R}^n \to E_y$ given by $\Phi_y(v) = \Phi(y, v)$ is a $\mathbb{R}$-linear isomorphism.*

The manifold E is called the *total manifold*, the map π is said to be the *projection map* and M the *base manifold*. The vector space E_x is called *the fiber* over the point x. When M is fixed and π is known, for simplicity we denote the bundle (E, π, M) only by the letter E.

Definition 1.1.2. *A* smooth section σ of (E, π, M) *is a smooth map $\sigma\colon M \to E$ with the property $\pi \circ \sigma = I$, where I stands for the identity map of M. The space of sections of (E, π, M) is denoted by $\Gamma(E)$.*

If σ, σ_1, $\sigma_2 \in \Gamma(E)$ and $f \in C^\infty(M)$ a smooth function, then we can form new sections $\sigma_1 + \sigma_2$ and $f\sigma$, from point-wise addition and multiplication on each fiber respectively. More precisely, we define

$$(\sigma_1 + \sigma_2)(x) = \sigma_1(x) + \sigma_2(x) \quad \text{and} \quad (f\sigma)(x) = f(x)\,\sigma(x), \quad x \in M.$$

The most simple vector bundle over a manifold M is the trivial vector bundle $M \times \mathbb{R}^k$. Moreover, the tangent bundle TM of a manifold is an example of a vector bundle. The space of sections of TM is usually denoted by $\mathfrak{X}(M)$.

Let (E, π, M) be a vector bundle and U an open subset of the manifold M. A collection $\{\varphi_1, \dots, \varphi_k\}$ of (smooth) sections defined in U is called *frame* if, for every $x \in M$, the vectors $\{\varphi_1(x), \dots, \varphi_k(x)\}$ consists a basis of E_x. If φ is a section, then it can be written in the form

$$\varphi = f_1\varphi_1 + \cdots + f_k\varphi_k.$$

The functions f_j, $j \in \{1, \dots, k\}$, are called the *components* of the section φ with respect to the given frame. It turns out that φ is smooth if and only if its components are smooth; see for example [1, Proposition 2.8].

The class of all vector bundles can be equipped with a category structure. This can be achieved once we introduce the notion of a bundle map.

Definition 1.1.3. *Let (E, π, M) and (F, θ, N) be vector bundles. A pair (f, L) of smooth maps $f : M \to N$ and $L : E \to F$ is called a* bundle map *if:*

(1) *L is fiber preserving, i.e. it holds $f \circ \pi = \theta \circ L$,*

(2) *$L_x = L|_{E_x} : E_x \to F_{f(x)}$ is linear for each $x \in M$.*

If $M \equiv N$, then the map $L : E \to F$ is called a morphism *if (I, L) is a bundle map. A morphism L is called an* isomorphism *if it is invertible.*

One simple method of constructing vector bundles is by restricting bundles on submanifolds or by taking subbundles of other vector bundles. We give the precise definitions below.

Definition 1.1.4. *Let (E, π, M) be a vector bundle over the manifold M. If $\Sigma \subset M$ is a submanifold of M, then the triple $(\pi^{-1}(\Sigma), \pi, \Sigma)$ is called the* restricted bundle. *Often we denote the restricted bundle only by the symbol $E|_\Sigma$.*

Definition 1.1.5. *Let (E, π, M) be a vector bundle over the manifold M. A vector bundle (V, π_1, M) is a called* sub-bundle *of (E, π, M) if the following three conditions are satisfied:*

(1) *V is a submanifold of the total space E,*

(2) *$V_x = V \cap E_x$, for each every $x \in M$,*

(3) *$\pi_1 = \pi|_V$.*

1.2 Connections and metrics

The investigation of geometric properties of vector bundles requires the notion of the directional derivative. Here we give the basic facts about metrics and associated to them connections.

Definition 1.2.1. *A* (*linear*) connection *on a vector bundle E over the manifold M is a map $\nabla^E \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, written $\nabla^E(X,\varphi) = \nabla^E_X\varphi$, satisfying the properties:*

(1) *For every $X, Y \in \mathfrak{X}(M)$ and $\varphi \in \Gamma(E)$, it holds*

$$\nabla^E_{X+Y}\varphi = \nabla^E_X\varphi + \nabla^E_Y\varphi.$$

(2) *For every $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ and $\varphi \in \Gamma(E)$, it holds*

$$\nabla^E_{fX}\varphi = f\nabla^E_X\varphi.$$

(3) *For every $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ and $\varphi_1, \varphi_2 \in \Gamma(E)$, it holds*

$$\nabla^E_X(\varphi_1 + \varphi_2) = \nabla^E_X\varphi_1 + \nabla^E_X\varphi_2.$$

(4) *For every $X \in \mathfrak{X}(M)$, $\varphi \in \Gamma(E)$ and $f \in C^\infty(M)$, it holds*

$$\nabla^E_X(f\varphi) = (Xf)\,\varphi + f\nabla^E_X\varphi.$$

The usual directional derivative in the Euclidean space is a connection. With respect to this connection, any constant vector field on the Euclidean space is parallel. Hence, we give the following general definition. Another important fact is the following:

Proposition 1.2.2. *Let E be a vector bundle over M equipped with a connection ∇^E. If $\varphi \in \Gamma(E)$ and $X, Y \in \mathfrak{X}(M)$ such that $X_x = Y_x$ at a point $x \in M$, then*

$$\nabla^E_X\varphi|_x = \nabla^E_Y\varphi|_x.$$

For that reason, often we write

$$\nabla^E_X\varphi|_x \equiv \nabla^E_{X_x}\varphi.$$

Definition 1.2.3. *Let (E, π, M) be a vector bundle equipped with a connection ∇^E. A section $\varphi \in \Gamma(E)$ is said to be* parallel with respect to ∇^E *if $\nabla^E_X\varphi = 0$, for each vector field $X \in \mathfrak{X}(M)$.*

We can define higher derivatives of sections of a vector bundle over a manifold M whose tangent bundle TM is equipped with a connection.

Definition 1.2.4. *Let (E, π, M) be a vector bundle over a manifold M and assume that E equipped with a connection ∇^E and TM with a connection ∇^M. For each $X, Y \in \mathfrak{X}(M)$, the map $\nabla^2_{X,Y} : \Gamma(E) \to \Gamma(E)$, given by*

$$\nabla^2_{X,Y}\varphi = \nabla^E_X \nabla^E_Y \varphi - \nabla^E_{\nabla^M_X Y}\varphi,$$

is called the second covariant derivative of φ*, with respect to the directions X and Y. By coupling the connections ∇^M and ∇^E, one may define, the k^{th}* derivative ∇^k of $\varphi \in \Gamma(E)$.

To each connection, we associate an operator which measures the non commutativity of the second covariant derivative.

Definition 1.2.5. *Let (E, π, M) be a vector bundle over M. Assume that the bundles E and TM are equipped with connections ∇^E and ∇^M, respectively. The linear operator $R^E : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, given by*

$$R^E(X, Y, \varphi) = \nabla^2_{X,Y}\varphi - \nabla^2_{Y,X}\varphi,$$

for each $X, Y \in \mathfrak{X}(M)$ and $\varphi \in \Gamma(E)$, is called the curvature tensor associated with ∇^E.

Now let us turn our attention to vector bundles equipped with a Riemannian metric structure.

Definition 1.2.6. *A* Riemannian metric on a vector bundle E over M *is a smooth map $g_E : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$, such that its restriction to the fibers is a positive definite inner product.*

Often we denote Riemannian metrics by the symbol $\langle \cdot\,, \cdot \rangle$. It is known that every vector bundle admits a Riemannian metric. The proof uses the partition of unity to glue local Riemannian metrics on each fiber; see for example [2].

Definition 1.2.7. *Let E be a vector bundle of rank k over M equipped with a connection ∇^E and a Riemannian metric g_E.*

(1) *We say that ∇^E is* compatible with the Riemannian metric g_E *if it satisfies*

$$X g_E(\varphi_1, \varphi_2) = g_E\big(\nabla^E_X \varphi_1, \varphi_2\big) + g_E\big(\varphi_1, \nabla^E_X \varphi_2\big),$$

for each $X \in \mathfrak{X}(M)$ and $\varphi_1, \varphi_2 \in \Gamma(E)$. A vector bundle E endowed with both of these structures is called Riemannian vector bundle *endowed with a compatible connection.*

(2) *We say that a set of sections $\{\varphi_1, \dots, \varphi_k\}$ defined in an open neighborhood of M, consists a* local orthonormal frame, with respect to g_E *if and only if*

$$g_E(\varphi_i, \varphi_j) = \delta_{ij},$$

for each $i, j \in \{1, \dots, k\}$.

Using the Gram-Schmidt process we can always find local orthonormal frames of sections in a Riemannian vector bundle.

1.3 The induced bundle

There is a natural way to differentiate sections along curves on M. More precisely, suppose that $\gamma : [0,1] \to M$ is a smooth (not-necessarily regular) curve and let E be a rank k vector bundle over M. Moreover, suppose that the bundle E is equipped with a Riemannian metric g_{E} and a compatible connection ∇^E. Let φ be a section defined only along the image of γ. If $\{\varphi_1, \dots, \varphi_k\}$ is a local orthonormal frame around $x_0 \in M$, then φ can be decomposed as

$$(\varphi \circ \gamma)(t) = \sum_{j=1}^{k} f_j(t)(\varphi_j \circ \gamma)(t), \quad t \in (0,1), \tag{1.1}$$

where the functions $f_j : [0,1] \to \mathbb{R}$, $j \in \{1, \dots, k\}$, are called the *components* of φ with respect to the given frame. We say that φ is *smooth along* γ if its components given in (1.1) are smooth. The question now is how to define the derivative of φ in the direction of γ'. Since we require the directional derivative to satisfy the properties of Definition 1.2.1, we see that the only possible way to define it is via the formula:

$$\nabla^{\gamma}_{\gamma'(t)}\varphi = \sum_{j=1}^{k} f_j'(t)(\varphi_j \circ \gamma)(t) + \sum_{j=1}^{k} f_j(t)\nabla^{E}_{\gamma'(t)}\varphi_j, \quad t \in (0,1). \tag{1.2}$$

It can be very easily checked that the above definition does not depend on the choice of the local frame. It is not hard to see that, if φ_1 and φ_2 are smooth sections along γ, then

$$(g_{\mathrm{E}}(\varphi_1, \varphi_2))' = g_{\mathrm{E}}(\nabla^{\gamma}_{\gamma'}\varphi_1, \varphi_2) + g_{\mathrm{E}}(\varphi_1, \nabla^{\gamma}_{\gamma'}\varphi_2). \tag{1.3}$$

Definition 1.3.1. *A section φ along a smooth curve is called* parallel *if*

$$\nabla^{\gamma}_{\gamma'}\varphi \equiv 0. \tag{1.4}$$

Note that the ODE (1.4) is of first order and linear. From the standard theory of ODEs, we can easily prove that the initial value problem

$$\begin{cases} \nabla^{\gamma}_{\gamma'(t)}\varphi &= 0, \quad t \in (0,1), \\ \varphi_{\gamma(0)} &= \varphi_0, \end{cases}$$

has a unique solution, which can be extended up to $\gamma(1)$. The obtained section is called the *parallel transport* of φ_0 along the curve γ. With the use of the formula (1.3), we can show that the parallel transport preserves the lengths and and the angles of sections. Consequently, given two points x_0 and y_0 on the manifold M and a smooth curve γ joining them, then the parallel transport gives rise to a linear isometry $P_\gamma : E_{x_0} \to E_{y_0}$.

We would like now to extend the above formulation for sections defined along the image of smooth maps. Suppose that Σ and M are smooth manifolds, (E,π,M) is a vector bundle of rank k over M and $f:\Sigma\to M$ is a smooth map. The map f induces a new vector bundle of rank k over Σ. Indeed:

- Take as total space the set
$$f^*E=\big\{(x,\xi): x\in\Sigma \text{ and } \xi\in E_{f(x)}\big\},$$
and as projection the map $\pi_f: f^*E\to\Sigma$ given by
$$\pi_f(x,\xi)=x.$$
The space f^*E contains all sections of E with base point at $f(\Sigma)$.

- Let ∇^E be a linear connection on E. Suppose that $\{\varphi_1,\dots,\varphi_k\}$ is a frame field of E defined in an open neighborhood of $f(x)\in N$. Then, any section $\sigma\in\Gamma\big(f^*E\big)$ can be written in the form
$$\sigma(x)=\sum_{\alpha=1}^{k}\sigma_\alpha(x)(\varphi_\alpha\circ f)(x),$$
where σ_α, $\alpha\in\{1,\dots,k\}$, are the components of σ with respect to the given frame field. These functions are defined in a neighborhood of Σ and they are smooth. Define now,
$$\nabla^f_X\sigma=\sum_{\alpha=1}^{k}(X\sigma_\alpha)(\varphi_\alpha\circ f)+\sum_{\alpha=1}^{k}\sigma_\alpha\nabla^E_{df(X)}\varphi_\alpha,$$
for each $X\in\mathfrak{X}(\Sigma)$. One can easily verify that the above definition of the pull-back connection is independent of the choice of the frame field.

- The curvature tensor R^f of the pull-back bundle is given by
$$R^f(X,Y)\sigma=R^E(df(X),df(Y))\sigma,$$
for each $X,Y\in T_xM$ and $\sigma\in\Gamma(f^*E)$.

- In the case $E=TN$ and ∇^N is a torsion-free connection, i.e.
$$\nabla^N_XY-\nabla^N_YX=[X,Y],$$
then the following formula holds
$$\nabla^f_X df(Y)-\nabla^f_Y df(X)=df([X,Y]),$$
for each $X,Y\in\mathfrak{X}(M)$.

1.4 Symmetric and anti-symmetric tensors

Since Riemannian geometry is written in a tensorial language, it is important to study the space of tensorial maps between vector bundles. Let (E, π_1, M) be a vector bundle of rank k and (V, π_2, M) a vector bundle of rank l over the manifold M endowed with linear connections ∇^E and ∇^V, respectively. The space $\mathrm{Hom}(E^r; V)$, of r-copies $E^r = E \times \cdots \times E$ of E to V, becomes a vector bundle with total space

$$\mathrm{Hom}(E^r; V) = \cup_{x \in M} \mathrm{Hom}(E^r_x; \mathbb{R}^l),$$

and projection map

$$\pi(x, \sigma) = x.$$

This particular bundle is called the *homomorphism bundle*. A natural connection ∇^{H} on the homomorphism bundle is given by

$$(\nabla^H_X \varphi)(\sigma_1, \dots, \sigma_r) = \nabla^V_X\{\varphi(\sigma_1, \dots, \sigma_r)\} - \varphi(\nabla^E_X \sigma_1, \dots, \sigma_r) - \cdots - \varphi(\sigma_1, \dots, \nabla^E_X \sigma_r)$$

where $X \in \mathfrak{X}(M)$, $\varphi \in \Gamma(\mathrm{Hom}(E^r; V))$ and $\sigma_1, \dots, \sigma_r \in \Gamma(E)$. There is also a natural way to construct Riemannian metrics on the homomorphism bundle. Let g_E and g_V be Riemannian metrics which are compatible with the connections ∇^E and ∇^V. Then a natural metric on the homomorphism bundle Hom that is compatible with ∇^H is given by

$$g_H(\varphi_x, \vartheta_x) = \sum_{i_1, \dots, i_r = 1}^{k} g_V\big(\varphi(\sigma_{i_1}, \dots, \sigma_{i_r}), \vartheta(\sigma_{i_1}, \dots, \sigma_{i_r})\big),$$

where $\{\sigma_1, \dots, \sigma_k\}$ is an orthonormal basis at the point x with respect to g_E. Sections of the homomorphism bundle $\mathrm{Hom}(E^r; \mathbb{R})$ are often called $(r, 0)$-*tensors* or simply r-tensors. There are two interesting types of tensors, the symmetric and the alternative ones. More precisely:

Definition 1.4.1. *A section* $\varphi \in \mathrm{Hom}(E^r; \mathbb{R})$ *is called* symmetric multilinear tensor of degree r *if*

$$\varphi(\sigma_1, \dots, \sigma_i, \dots, \sigma_j, \dots, \sigma_r) = \varphi(\sigma_1, \dots, \sigma_j, \dots, \sigma_i, \dots, \sigma_r),$$

and alternative multilinear tensor of degree r *if*

$$\varphi(\sigma_1, \dots, \sigma_i, \dots, \sigma_j, \dots, \sigma_r) = -\varphi(\sigma_1, \dots, \sigma_j, \dots, \sigma_i, \dots, \sigma_r),$$

for each $i, j \in \{1, \dots, r\}$. *We denote the space of symmetric multilinear tensors of degree* r *by the symbol* $\mathcal{S}(E^r)$. *Often we refer to elements of* $\mathcal{S}(E^2)$ *as* symmetric 2-tensors. *The space of alternative bilinear tensors of degree* r *is denoted by the letter* $\Omega(E^r)$. *Elements of* $\Omega(E^2)$ *are called* anti-symmetric 2-tensors.

1.5 Exterior powers of vector bundles

Denote by $\Lambda^r(\mathbb{R}^k)$ the dual space of all alternative multilinear forms of degree r. Elements of $\Lambda^r(\mathbb{R}^k)$ are called r-*vectors*. Given vectors $v_1, \dots, v_r$ on the Euclidean space $\mathbb{R}^k$, the *exterior product* $v_1 \wedge \cdots \wedge v_r$ is the linear map which on an alternating tensor ω of degree r takes the value

$$(v_1 \wedge \cdots \wedge v_r)(\omega) = \omega(v_1, \dots, v_r).$$

The exterior product is linear in each variable separately. Interchanging two elements the sign of the product changes and if two variables are the same the exterior product vanishes. An r-vector ξ is called *simple* or *decomposable* if it can be written as a single wedge product of vectors, that is

$$\xi = v_1 \wedge \cdots \wedge v_r.$$

Note that there are r-vectors which are not simple. For example, let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$. Then, the 2-vector

$$\xi = e_1 \wedge e_2 + e_3 \wedge e_4$$

is not simple. Using standard techniques from Linear Algebra one can verify that the exterior product $v_1 \wedge \cdots \wedge v_r$ is zero if and only if the vectors are linearly dependent. Moreover, if $\{e_1, \dots, e_k\}$ consists a basis for $\mathbb{R}^k$, then the r-vectors

$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} : 1 \leq i_1 < \cdots < i_r \leq k\}$$

consists a basis of $\Lambda^r(\mathbb{R}^k)$. Therefore, the dimension of the vector space of r-vectors is

$$\dim \Lambda^r(\mathbb{R}^k) = \binom{k}{r} = \frac{n!}{r!(k-r)!}.$$

Each simple vector represents a unique r-dimensional subspace of $\mathbb{R}^k$. We can equip $\Lambda^r(\mathbb{R}^k)$ with a natural inner product, which we denote by $(\cdot\,,\cdot)$. Indeed, define

$$(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) = \det\big(\langle v_i, w_j\rangle\big)_{1 \leq i,j \leq r}, \tag{1.5}$$

on simple r-vectors and then extend linearly. Moreover, if $\{e_1, \dots, e_k\}$ is an orthonormal basis of $\mathbb{R}^k$ then, the r-vectors

$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} : 1 \leq i_1 < \cdots < i_r \leq k\}$$

consist an orthonormal basis for the exterior power $\Lambda^r(\mathbb{R}^k)$. Now if E is a vector bundle of rank k over a manifold M, then we can form the exterior power $\Lambda^r(E)$ of E by gluing together all the spaces $\Lambda^r(E)$, i.e.

$$\Lambda^r(E) = \cup_{x \in M} \Lambda^r(E_x).$$

1.6 The Levi-Civita connection

Let g be a metric on a m-dimensional manifold M. Then there is a unique compatible with g connection ∇, referred as the *Levi-Civita connection*, given by the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X), \end{aligned} \tag{1.6}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. The Levi-Civita also satisfy

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

for each $X, Y \in \mathfrak{X}(M)$. Denote by R the associated with ∇ curvature tensor. Combining R with g we obtain a $(4,0)$-tensor, which by abuse of notation, we denote it again by letter R, i.e.

$$R(X, Y, Z, W) = -g(R(X, Y, Z), W),$$

for each $X, Y, Z, W \in \mathfrak{X}(M)$. The curvature tensor of a Riemannian manifold satisfies the following important identities:

(1) **Symmetries of the curvature tensor:**

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Z, W, X, Y). \tag{1.7}$$

(2) 1^{st} **Bianchi identity of the curvature:**

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0. \tag{1.8}$$

(3) 2^{nd} **Bianchi identity of the curvature:**

$$(\nabla_X R)(Y, Z, W) + (\nabla_Y R)(Z, X, W) + (\nabla_Z R)(X, Y, W) = 0. \tag{1.9}$$

The Riemannian curvature tensor is a very complicated object and for its better understanding we may consider various by-products of this quantity. If $X, Y \in T_x M$ are linearly independent vectors, then

$$\sec(X, Y) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2},$$

is called the *sectional curvature* of the plane Π spanned by the vectors X and Y. As a matter of fact, the sectional curvature depends on the plane Π and not on the generating vectors X and Y. So the sectional curvature of a Riemannian manifold at a point x can be regarded as a function defined on $\Lambda^2(T_x M)$. One important relationships between the Riemannian and the sectional curvature is the following algebraic result by Riemann; see [26, Proposition 3.1.3].

Theorem 1.6.1. *Let M be a Riemannian manifold and $x \in M$. Then the following two properties are equivalent:*

(1) *For each $\Pi \in \Lambda^2(T_xM)$, we have* $\sec(\Pi) = k$.

(2) *The Riemann curvature tensor at x is given by*

$$R(X, Y, Z, W) = k\big\{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\big\},$$

for each $X, Y, Z, W \in T_xM$.

Definition 1.6.2. *A Riemannian manifold M that satisfies either of these two conditions for all $x \in M$ and the same $k \in \mathbb{R}$ for all $x \in M$ is said to have* constant curvature k. *Such Riemannian manifolds are shortly called* space forms *of curvature k.*

The Euclidean space $\mathbb{R}^m$ equipped with the inner product g_R, the sphere $\mathbb{S}^m(r) \subset \mathbb{R}^{m+1}$ of radius $r > 0$ with metric g_S the induced one from the Euclidean space and the hyperbolic space $\mathbb{H}^m(r)$ modelled by

$$\mathbb{H}^m(r) = \{(x_0, x_1, \ldots, x_{m-1}) \in \mathbb{R}^m : x_0 > 0\} \text{ and Riemannian metric } g_H = r^2 x_0^{-2} g_R,$$

consist examples of space forms. According to a classical theorem in Riemannian geometry, the following result holds:

Theorem 1.6.3. *Let M be a simply connected, m-dimensional Riemannian manifold with constant sectional curvature k. Then, M is isometric to $\mathbb{R}^m$ if $k = 0$, to $\mathbb{S}^m_r$ if $k = r^{-2}$ and to $\mathbb{H}^m_r$ if $k = -r^{-2}$.*

By contracting the operator R with the metric g we obtain the *Ricci operator* Ric and *scalar curvature* S, i.e.

$$Ric(X, Y) = tr_g\big(R(X, \cdot\,, Y \cdot)\big) \quad \text{and} \quad S = tr_g(Ric),$$

where $X, Y \in \mathfrak{X}(M)$. Finally, we define the *trace-free Ricci tensor* or the *Einstein tensor* by

$$E = Ric - (S/m)g.$$

Definition 1.6.4. *A Riemannian manifold for which the Einstein tensor is identically zero is called* Einstein manifold.

It is a well-known fact that an Einstein manifold of dimension greater than two, has constant scalar curvature; see for example [26, Proposition 3.1.5 & Corollary 3.1.6.]. The classification of Einstein manifolds in dimensions greater than three, is still a wide open problem. For more details we refer to the book of Besse [3].

The symmetries given in (1.7) allow us to regard the curvature tensor of a Riemannian manifold as a symmetric bilinear form on the exterior power of 2-vectors. More precisely, we define the *curvature operator* $\mathcal{R} : \Lambda^2(TM) \times \Lambda^2(TM) \to C^\infty(M)$ by

$$\mathcal{R}(X \wedge Y, Z \wedge W) = R(X, Y, Z, W),$$

for each $X, Y, Z, W \in \mathfrak{X}(M)$. We now investigate the tensors satisfying the same algebraic identities as the curvature tensor of a Riemannian manifold at one point. There is a natural way to construct tensors satisfying the conditions (1.7) and (1.8) of the curvature tensor. First we need the following definition:

Definition 1.6.5. *Let V be a m-dimensional vector space and $\mathcal{S}(V)$ the space of symmetric bilinear forms of V. Given $\varphi, \vartheta \in \mathcal{S}(V)$, the multi-linear map*

$$\varphi \owedge \vartheta \colon V \times V \times V \times V \to \mathbb{R}$$

given by

$$\begin{aligned}(\varphi \owedge \vartheta)(v_1, v_2, v_3, v_4) = \varphi(v_1, v_3)\vartheta(v_2, v_4) + \varphi(v_2, v_4)\vartheta(v_1, v_3) \\ - \varphi(v_1, v_4)\vartheta(v_2, v_3) - \varphi(v_2, v_3)\vartheta(v_1, v_4),\end{aligned}$$

is called the Kulkarni-Nomizu product *of φ and ϑ.*

The above product appeared for the first time in papers of Kulkarni [21] and Nomizu [25] and for that reason is called by their names. Just by straight-forward computations one can check the validity of the following:

Lemma 1.6.6. *Let (M, g) be a Riemannian manifold. The following properties hold true:*

(1) *For each symmetric 2-tensors φ and ϑ, it holds $\varphi \owedge \vartheta = \vartheta \owedge \varphi$.*

(2) *The $(4, 0)$-tensor $\varphi \owedge \vartheta$ satisfies the properties (1.7) and (1.8).*

(3) *For each symmetric 2-tensors φ and ϑ, the following formulas hold*

$$\nabla_X(\varphi \owedge \vartheta) = (\nabla_X\varphi) \owedge \vartheta + \varphi \owedge (\nabla_X\vartheta), \qquad X \in \mathfrak{X}(M),$$

and

$$\Delta(\varphi \owedge \vartheta) = (\Delta\varphi) \owedge \vartheta + 2(\nabla_X\varphi) \owedge (\nabla_X\vartheta) + \varphi \owedge (\Delta\vartheta).$$

In the sequel we introduce another important tensor in Riemannian geometry, the so-called Weyl tensor. Roughly speaking, the Weyl tensor measures how far is a Riemannian metric g from being locally conformal to a flat one; locally conformally flat means that around each point there exists an open neighborhood U and a function $\varphi \in C^\infty(U)$ such that

$$\overline{g} = e^{2\varphi} g,$$

has zero Riemann curvature operator. The precise definition is the following:

Definition 1.6.7. *Let (M, g) be a Riemannian manifold of dimension $m \geq 3$. The tensor $\mathcal{W}$ given by*

$$\mathcal{W} = \mathcal{R} - \frac{S}{2m(m-1)} g \owedge g - \frac{1}{m-2} E \owedge g. \tag{1.10}$$

is called the Weyl tensor.

As we will see in the next theorem, the Weyl tensor provides information only in dimension greater than three. Furthermore, we will see that simply connected Einstein 3-manifolds are fully classified.

Theorem 1.6.8. *The Weyl tensor of a 3-dimensional Riemannian manifold M is identically zero. Therefore, the Riemannian curvature tensor of a 3-dimensional Riemannian manifold is fully determined by its Ricci tensor. In particular, an Einstein 3-manifold is isometric with a space form.*

Proof. At first observe that due to the results of Lemma 1.6.6, the Weyl tensor has the identities (1.7) and (1.8) of the Riemannian curvature tensor R. Moreover,

$$tr_g(\mathcal{W}(X, \cdot, Y, \cdot)) = 0, \quad \text{for all } \ X, Y \in \mathfrak{X}(M).$$

Consider a local orthonormal frame $\{e_1, e_2, e_3\}$ on the manifold M. Then,

$$\begin{aligned} \mathcal{W}(e_1, e_2, e_1, e_2) + \mathcal{W}(e_1, e_3, e_1, e_3) &= 0, \\ \mathcal{W}(e_2, e_1, e_2, e_1) + \mathcal{W}(e_2, e_3, e_2, e_3) &= 0, \\ \mathcal{W}(e_3, e_1, e_3, e_1) + \mathcal{W}(e_3, e_2, e_3, e_2) &= 0, \end{aligned}$$

from where it follows that

$$\mathcal{W}(e_1, e_2, e_1, e_2) = \mathcal{W}(e_1, e_3, e_1, e_3) = \mathcal{W}(e_2, e_3, e_2, e_3) = 0.$$

Moreover,

$$\mathcal{W}(e_1, e_2, e_1, e_3) = -\mathcal{W}(e_2, e_2, e_2, e_3) - \mathcal{W}(e_3, e_2, e_3, e_3) = 0.$$

Hence, in general,

$$\mathcal{W}(e_i, e_j, e_k, e_l) = 0, \quad \text{unless the indices } i, j, k \text{ are all distinct.}$$

But in dimension 3 there are only three possible choices for the indices, and so the Weyl tensor must vanish identically. Let us assume now that our 3-manifold is Einstein. Then the scalar curvature S is constant (see [26, Proposition 3.1.5 & Corollary 3.1.6.]) and from the equation (1.10) we deduce that the Riemannian manifold M is a space form. This completes the proof of the theorem. ☐

We conclude this section with a lemma which provides us with some information about the algebraic structure of the curvature operator of 3-dimensional Riemannian manifolds.

Lemma 1.6.9. *Let M be a 3-dimensional Riemannian manifold. Then, there exists a local orthonormal frame $\{e_1, e_2, e_3\}$ satisfying the following properties:*

(1) *The curvature operator is diagonalized in the form*

$$\mathcal{R}(e_2 \wedge e_3, e_2 \wedge e_3) = \lambda_1, \quad \mathcal{R}(e_1 \wedge e_3, e_1 \wedge e_3) = \lambda_2, \quad \mathcal{R}(e_1 \wedge e_2, e_1 \wedge e_2) = \lambda_3,$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are continuous functions.

(2) *The Ricci tensor, with respect to the frame $\{e_1, e_2, e_3\}$, takes the form*

$$Ric = \begin{pmatrix} \lambda_2 + \lambda_3 & 0 & 0 \\ 0 & \lambda_1 + \lambda_3 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 \end{pmatrix}.$$

(3) *The scalar curvature has the form*

$$S = 2(\lambda_1 + \lambda_2 + \lambda_3).$$

Proof. The existence of the local frame which diagonalizes $\mathcal{R}$ follows from the observation in Theorem 1.6.8 that $\mathcal{R}$ is fully determined by Ric. Note that the bi-vectors

$$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$$

form an orthonormal frame of $\Lambda^2(TM)$ with respect to the inner product $(\cdot\,,\cdot)$ given in (1.5). With respect to this frame we have that

$$\begin{aligned} Ric(e_1, e_1) &= \mathcal{R}(e_1 \wedge e_2, e_1 \wedge e_2) + \mathcal{R}(e_1 \wedge e_3, e_1 \wedge e_3) = \lambda_3 + \lambda_2, \\ Ric(e_2, e_2) &= \mathcal{R}(e_2 \wedge e_1, e_2 \wedge e_1) + \mathcal{R}(e_2 \wedge e_3, e_2 \wedge e_3) = \lambda_3 + \lambda_1, \\ Ric(e_3, e_3) &= \mathcal{R}(e_3 \wedge e_1, e_3 \wedge e_1) + \mathcal{R}(e_3 \wedge e_2, e_3 \wedge e_2) = \lambda_2 + \lambda_1, \end{aligned}$$

and the other elements of Ric are equal to zero. Combining the above results, we see that

$$\begin{aligned} S &= Ric(e_1, e_1) + Ric(e_2, e_2) + Ric(e_3, e_3) \\ &= 2(\lambda_1 + \lambda_2 + \lambda_3). \end{aligned}$$

This completes the proof. □

1.7 Lie derivatives

We discuss here another notion differentiation of vector fields which generalizes the directional derivative.

Definition 1.7.1. *Let M be a smooth m-dimensional manifold and $X \in \mathfrak{X}(M)$.*

(1) *The* Lie derivative *of $f \in C^\infty(M)$ in the direction of X is $\mathcal{L}_X f = df(X)$.*

(2) *The* Lie derivative *of a vector field $Y \in \mathfrak{X}(M)$ in the direction of X is $\mathcal{L}_X Y = [X, Y]$.*

We would like to define the notion of the Lie derivative of a tensor in the direction of a vector field. Before giving the precise definition we need some preliminaries. Let X be a vector field on M and let $\varphi : M \times I \to M$, $I \subset \mathbb{R}$, be the map satisfying

$$\begin{cases} d\varphi_{(x,t)}(\partial_t) = X_x, \\ \varphi(x, 0) = x, \end{cases} \tag{1.11}$$

for each $(x, t) \in M \times I$. Then, for each $t \in I$, the map given by $x \mapsto \varphi_t(x) = \varphi(x, t)$ is a local diffeomorphism.

Definition 1.7.2. *Let M be a smooth m-dimensional manifold and $X \in \mathfrak{X}(M)$ and let T be a r-tensor on the tangent bundle of M. The* Lie derivative *of T in the direction X is defined by*

$$(\mathcal{L}_X T)(Y_1, \ldots, Y_r)|_x = \lim_{t \to 0} \frac{(\varphi_t^* T - T)(Y_1, \ldots, Y_r)|_x}{t}, \quad x \in M.$$

It turns out that the Lie derivative of a r-tensor satisfies the property described in the following proposition; see [26, Appendix 1, page 376].

Proposition 1.7.3. *Let T be a r-tensor and X a vector field on M. Then*

$$(\mathcal{L}_X T)(Y_1, \ldots, Y_r) = X(T(Y_1, \ldots, Y_r)) - T(\mathcal{L}_X Y_1, \ldots, Y_r) - \cdots - T(Y_1, \ldots, \mathcal{L}_X Y_r).$$

Corollary 1.7.4. *Let (M, g) be a Riemannian manifold. Then,*

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y),$$

where ∇ the Levi-Civita connection and $X \in \mathfrak{X}(M)$.

Proof. Using Proposition 1.7.3 and properties of the Levi-Civita connection we have

$$\begin{aligned}(\mathcal{L}_X g)(Y, Z) &= X(g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= g(\nabla_Y X, Z) + g(\nabla_Z X, Y),\end{aligned}$$

for each $X, Y, Z \in \mathfrak{X}(M)$. This completes the proof. □

1.8 Point-wise conformal metrics

For later use, let us consider here the simplest deformations of a Riemannian metric, namely the conformal ones. They are obtained by changing at each point the lengths of all vectors by a scaling factor (depending on the point) without changing the angles.

Definition 1.8.1. *Two Riemannian metrics g and $\overline{g}$ on a manifold M are said to be* (point-wise) conformal *if there exists a smooth function φ on M such that $\overline{g} = e^{2\varphi} g$.*

In the following, we compute the various invariants of the metric $\overline{g}$ in terms of those of g and the derivatives of f with respect to the Levi-Civita connection ∇ of g. More precisely, the following result holds:

Theorem 1.8.2. *Let (M, g) be a Riemannian m-dimensional manifold, $\varphi \in C^\infty(M)$ a smooth function on M and $\overline{g}$ the metric given by $\overline{g} = e^{2\varphi} g$. Then:*

(1) *The Levi-Civita connections $\overline{\nabla}$ and ∇ of the metrics $\overline{g}$ and g, respectively, are related by the formula*

$$\overline{\nabla}_X Y = \nabla_X Y + (X\varphi)Y + (Y\varphi)X - g(X,Y)\nabla\varphi, \ \text{for all } \ X, Y \in \mathfrak{X}(M).$$

(2) *The volume forms $\overline{V}$ and V of the metrics $\overline{g}$ and g, respectively, are related by*

$$\overline{V} = e^{m\varphi} V.$$

(3) *The gradients of a function $f \in C^\infty(M)$ with respect the metrics $\overline{g}$ and g are related by*

$$\overline{\nabla} f = e^{-2\varphi} \nabla f.$$

(4) *The Laplacians of a function $f \in C^\infty(M)$ with respect to $\overline{g}$ and g are related by*

$$\overline{\Delta} f = e^{-2\varphi}\big\{\Delta f + (m-2) g(\nabla\varphi, \nabla f)\big\}.$$

(5) *The curvature operators $\overline{\mathcal{R}}$ and $\mathcal{R}$ of the conformal metrics $\overline{g}$ and g are related by*

$$\overline{\mathcal{R}} = e^{2\varphi}\big\{\mathcal{R} + g \owedge \big(\nabla^2\varphi - d\varphi \otimes d\varphi - \tfrac{1}{2}|\nabla\varphi|^2 g\big)\big\}.$$

(6) *The Ricci tensors $\overline{Ric}$ and Ric of the metrics $\overline{g}$ and g, respectively, are related by*

$$\overline{Ric} = Ric - (m-2)\big(\nabla^2\varphi - d\varphi \otimes d\varphi\big) - \big(\Delta\varphi - (m-2)|\nabla\varphi|^2\big) g.$$

(7) *The scalar curvatures $\overline{S}$ and S of the metrics $\overline{g}$ and g, respectively, are related by*

$$\overline{S} = e^{-2\varphi}\big\{S - 2(m-1)\Delta\varphi - (m-2)(m-1)|\nabla\varphi|^2\big\}.$$

Proof. The proofs of the formulas follow by long but straight-forward computations; see for example [3, Theorem 1.159]. □

1.9 Global Riemannian geometry

Suppose that $\gamma : [a, b] \to M$ is a regular curve in a manifold M equipped with a Riemannian metric g. The *length* of γ is given by

$$L(\gamma) = \int_a^b |\gamma'(t)| dt$$

The *distance* between two points $x, y \in M$ can now be defined as

$$\operatorname{dist}(x, y) = \inf \Big\{ \text{ length of all curves joining the points } x, y \Big\}.$$

Note that the above mentioned set of curves is always non-empty since manifolds are assumed to be connected. It turns out that dist is a topological metric on M. In fact the topology on M induced by this metric coincides with the original topology of M. We say that the Riemannian manifold M is *complete* if and only if (M, dist) is a complete metric space. The *diameter* $\operatorname{diam}(M)$ of a compact Riemannian manifold M is given by the formula

$$\operatorname{diam}(M) = \max\{\operatorname{dist}(x, y) : x, y \in M\}.$$

In order to find the curve with the smallest length joining two points x and y, we have to minimize the length functional. The Euler-Lagrange equation for the variation of the length leads to the following:

$$\nabla^{\gamma}_{\gamma'} \gamma' = 0. \tag{1.12}$$

Solutions of (1.12) are called *geodesic curves*. From the basic theory of ODEs we obtain the following important result; for the proof see for example [26].

Theorem 1.9.1 (Existence & uniqueness)**.** *Let M be a Riemannian manifold. Then:*

(1) *For each fixed point $x \in M$ and each $v \in T_xM$, there is a neighborhood U_x in M around the point x, an open ball $B(0, \delta_x)$ in T_xM around v such that for each $y \in U_x$ and $w \in B(0, \delta_x)$, there is a geodesic $\gamma_{y,w} : (-2, 2) \to U$ such that*

$$\gamma_{y,w}(0) = y \quad \text{and} \quad \gamma'_{y,w}(0) = w.$$

Moreover, the mapping $F : U_x \times B(0, \delta_x) \times (-2, 2) \to M$ given by

$$F(y, w, t) = \gamma_{y,w}(t)$$

is smooth.

(2) *Let I_1 and I_2 be two open intervals with $t_0 \in I_1 \cap I_2$ and $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are geodesics with $\gamma_1(t_0) = \gamma_2(t_0)$ and $\gamma_1'(t_0) = \gamma_2'(t_0)$. Then, $\gamma_1(I_1 \cap I_2) = \gamma_2(I_1 \cap I_2)$.*

Remark 1.9.2. Let us make some comments regarding the last theorem:

(1) As one can easily see any geodesic curve has speed of constant length. Therefore, in the sequel, we will always assume that geodesics are parametrized with respect to the arc length.

(2) Geodesics are locally length minimizing. However, this property is not true in general as already the sphere consists a counterexample.

Definition 1.9.3. *Let $x \in M$ and consider an open ball $B(0, \delta_x) \subset T_xM$ as in Theorem 1.9.1. The smooth map* $\exp_x : B(0, \delta_x) \to M$ *given by*

$$\exp_x(v) = F(x, v, 1),$$

is called the exponential map.

Let us collect in the next theorem the most important properties of the exponential map. The reader can find the proofs in any classical book of differential geometry; for example see [26].

Theorem 1.9.4. *Fix a point x in a Riemannian manifold M. Then, the following statements hold true:*

(1) *For each $v \in T_0(T_xM)$, it holds*

$$d\exp_x|_0(v) = v,$$

which means that the exponential map when restricted in a small neighborhood of the origin of T_xM is a diffeomorphism.

(2) *Let δ a sufficiently small number such that $\exp_x : B(0, \delta) \to M$ is a diffeomorphism. Then each geodesic starting from x meets orthogonally the boundary of $\exp_x(B(0, \delta))$.*

(3) *If the distance function $d_x : M \to \mathbb{R}$ given by*

$$d_x(y) = \text{dist}(x, y)$$

is smooth at y, then

$$(\nabla d_x)(y) = \gamma_0'(b),$$

where $\gamma_0 : [0, b] \to M$ is the unique geodesic that is connecting the points x and y,

(4) *The Riemannian manifold M is complete if and only if, for each $x \in M$, the exponential map $\exp_x$ is defined in all of T_xM. In this case, for each pair of points $x, y \in M$ there exists at least one geodesic curve joining these two given points.*

The second part of the above theorem is known in the literature as the Gauß Lemma. The third part is due to Hopf and Rinow. Notice that compact Riemannian manifolds are always complete.

Definition 1.9.5. *Fix a point x in a complete Riemannian manifold M.*

(1) *The* cut locus *of x in T_xM is defined to be the set of all vectors $v \in T_xM$ such that*

$$\gamma(t) = \exp_x(tv)$$

is a length minimizing geodesic for all times $t \in [0,1]$ but fails to be minimizing for $t \in [1, 1+\varepsilon)$ for each positive number $\varepsilon > 0$.

(2) *The cut locus* $\mathrm{Cut}(x)$ *of x in M is defined to be the image of the cut locus of x in T_xM under the exponential map.*

(3) *The least distance from x to cut locus* $\mathrm{Cut}(x)$ *is called the* injectivity radius $\mathrm{inj}(x)$ *of x. The* injectivity radius $\mathrm{inj}(M)$ *of a Riemannian manifold is the infimum of the injectivity radii at all points.*

Fix a point $x \in M$ and consider the closed set

$$D_x = \left\{ v \in T_xM : \mathrm{dist}(\exp_x(v), x) = |v| \right\} \subset T_xM.$$

It turns out that the boundary ∂D_x of D_x is exactly the cut locus of x in T_xM and that

$$\mathrm{Cut}(x) = \exp_x(\partial D_x).$$

Moreover, the map

$$\exp_x : \mathrm{int}(D_x) \to M - \mathrm{Cut}(x)$$

is a diffeomorphism. From Sard's Theorem it follows that the set $\mathrm{Cut}(x)$ has measure zero.

Another interesting fact is that the injectivity radius $\mathrm{inj}(x)$ of x can be defined equivalently as the supremum of all $r > 0$ such that the exponential map $\exp_x : B(0,r) \to M$ is diffeomorphism.

We conclude this section with three results which show how the geometry affects the topology of the manifold and vise versa; for the proofs we refer to [8] and [26].

Theorem 1.9.6 (Cheng-Bonnet-Myers)**.** *Let (M,g) be a complete m-dimensional Riemannian manifold such that*

$$Ric \geq (m-1)kg,$$

where k is a positive constant. Then, the following facts hold:

(1) *The manifold M is compact with finite fundamental group.*

(2) *The diameter of M can be estimated from above by* $\mathrm{diam}(M) \leq \pi/\sqrt{k}$*.*

(3) *If* $\mathrm{diam}(M) = \pi/\sqrt{k}$*, then M is isometric to $\mathbb{S}^m_k$.*

Theorem 1.9.7 (Bishop-Cheeger-Gromov)**.** *Suppose that (M, g) is a Riemannian manifold of dimension m such that $Ric \geq (m-1)kg$, where k is a real constant. Then, for each $x \in M$, we have that*

$$V(B(x,r)) \leq V_m^k(r).$$

where $V_m^k(r)$ the volume of the ball of radius r in the simply-connected m-dimensional space form of curvature k. As a matter of fact, the function

$$r \mapsto \frac{V(B(x,r))}{V_m^k(r)},$$

is non-increasing and its limit as $r \to 0$ is 1.

Theorem 1.9.8 (Klingenberg)**.** *Let M be a Riemannian manifold. The following facts hold:*

(1) *If M is compact, then the injectivity radius* $\mathrm{inj}(M)$ *is always positive.*

(2) *If M is compact and all the sectional curvatures are bounded from above by $k > 0$, then*

$$\mathrm{inj}(M) \geq \min\left\{\pi/\sqrt{k}\,, (1/2)\cdot \textit{length of the shortest closed geodesic}\right\}.$$

(3) *If M is complete and simply connected whose sectional curvatures are pinched between k and $k/4$ for some positive constant k, then*

$$\mathrm{inj}(M) \geq \pi/\sqrt{k}.$$

(4) *If M is compact, even-dimensional, orientable whose sectional curvatures are positive and bounded from above by a positive constant k, then*

$$\mathrm{inj}(M) \geq \pi/\sqrt{k}.$$

1.10 Index notation

In the following chapters we will perform computations involving tensors with respect to local coordinates or orthonormal frames. Let us briefly discuss the conventions that we will use, following the exposition in [26]. Suppose that E is a m-dimensional real vector space. We use subscripts to denote vectors in E. Therefore, a basis of E will be denoted by $\{e_1, \ldots, e_m\}$. Given a vector $v \in V$, we then write it as a linear combination of this basis as follows

$$v = \sum_{i=1}^{m} v^i e_i = v^i e_i.$$

Note that we use superscripts for the coefficients of v and then automatically sum over indices that are repeated as both subscripts and superscripts. Let us consider now the basis $\{e^1, \ldots, e^m\}$ of dual space V^* given by $e^i(e_j) = \delta^i_j$. This basis of the dual space is called *associated* with $\{e_1, \ldots, e_m\}$. Then we see that

$$v^i = e^i(v).$$

Hence, we decide to use superscripts for dual basis in V^*. If $\varphi : V \to V$ is a linear map then we denote the components of the matrix of φ with respect to the basis $\{e_1, \ldots, e_m\}$ by φ^j_i, namely

$$\varphi(e_i) = \varphi^j_i e_j.$$

If $\varphi : V \times V \to \mathbb{R}$ is a bilinear form, then we may represent the coefficients of its matrix with respect to the basis $\{e_1, \ldots, e_m\}$ by φ_{ij}, that is $\varphi_{ij} = \varphi(e_i, e_j)$. Consequently, we may write φ in the form

$$\varphi = \varphi_{ij}\, e^i \otimes e^j,$$

where $\otimes$ is the multiplication on 1-forms, i.e. if $\omega_1, \omega_2 \in V^*$ and $X, Y \in V$, then

$$\omega_1 \otimes \omega_2(X, Y) = \omega_1(X) \cdot \omega_2(Y).$$

On the other hand, if $\varphi : V \times V \to V$ is a vector valued bilinear form, then its coefficients with respect to the basis $\{e_1, \ldots, e_m\}$ will be denoted by φ^k_{ij}, that is

$$\varphi(e_i, e_j) = \varphi^k_{ij} e_k.$$

Suppose that $(x^1, \ldots, x^m)$ is a coordinate system in a Riemannian manifold M. Then we denote the corresponding basic vector fields by $\{\partial_1, \ldots, \partial_m\}$. With respect to this basis, we have

(1) **Components of the metric and of its inverse:** $g(\partial_i, \partial_j) = g_{ij}$ and $(g^{ij}) = (g_{ij})^{-1}$.

(2) **Components of the Riemannian 3-tensor:** $R(\partial_i, \partial_j, \partial_k) = R^l_{ijk}\partial_l$.

(3) **Components of the Riemannian 4-tensor:** $R(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl} = g_{hk}R^h_{ijl}$.

(4) **Components of the Ricci curvature:** $Ric(\partial_i, \partial_j) = R_{ij} = g^{kl}R_{ikjl}$.

(5) **Scalar curvature in local coordinates:** $S = g^{ij}R_{ij} = g^{ij}g^{kl}R_{ikjl}$.

CHAPTER 2

The Ricci flow equation

In this chapter, we introduce the Ricci flow as an evolution equation, give examples of special solutions to the Ricci flow and compute the evolution equations of various geometric quantities.

2.1 Motivation

The concept of the Ricci flow was introduced in 1982 in the seminal paper of Hamilton [18]. The Ricci flow is an evolution equation which deforms a Riemannian metric in the direction of its Ricci curvature. In local coordinates, we can describe the Ricci flow by the equation

$$g'_{ij}(t) = -2R_{ij}(t), \qquad t \in (0,T), \tag{2.1}$$

where $g_{ij}(t)$ denotes the components of a time-dependent Riemannian metric and $R_{ij}(t)$ the components of the Ricci curvature of the corresponding metric at time $t \in (0,T)$. Hamilton was inspired by the work of Eells and Sampson [14] on the harmonic heat map flow, where under certain conditions they succeeded to deform a smooth map between Riemannian manifolds into a harmonic one. Hamilton's main idea was to try to deform a given Riemannian metric on manifold by a heat-type equation. He was led to consider the equation (2.1) due to this fact: If M is a Riemannian m-dimensional manifold, then around each point there exists a coordinate system $(x^1,\dots,x^m)$ where each coordinate function $x^i : M \to \mathbb{R}$, $1 \le i \le m$, is harmonic with respect to the Riemannian metric of M; see [26, page 409]. In such a coordinate system, the components of the Ricci tensor satisfy

$$-2R_{ij} = \Delta g_{ij} + (\text{lower order terms}).$$

However, let us mention here that, the property of a coordinate system to be harmonic is not preserved under (2.1). Moreover, it turns out that the equation (2.1) is not parabolic, so the existence for short time of the initial value problem

$$\begin{cases} g'_{ij}(t) &= -2R_{ij}(t), \\ g_{ij}(0) &= \ g_{ij}, \end{cases} \tag{2.2}$$

is not guarantied from the standard theory of parabolic PDEs.

2.2 Definitions and examples

Suppose that $(0, T)$ is an open interval of the real line and let $\{g_t\}_{t\in(0,T)}$ be an arbitrary smooth family of Riemannian metrics on a manifold M. This means that for each $(x, t) \in M \times (0, T)$ we have an inner product $g_{(x,t)}$ on the tangent space T_xM. Then, we can regard $\{g_t\}_{t\in(0,T)}$ as a metric g acting on the *spatial tangent bundle* $\mathcal{H}$, defined by

$$\mathcal{H} = \{X \in T(M \times (0, T)) : d\pi_2(X) = 0\},$$

where $\pi_2 : M \times (0, T) \to (0, T)$ is the natural projection map given by

$$\pi_2(x, t) = t.$$

Observe that each g_t is a Riemannian metric on $\mathcal{H}$ since $\mathcal{H}_{(x,t)}$ is isomorphic to T_xM via π_2. We can even extend naturally g into a Riemannian metric on $M \times (0, T)$, with respect to which we have the orthogonal decomposition

$$T(M \times (0, T)) = \mathcal{H} \oplus \mathbb{R}\partial_t.$$

Since $\mathcal{H}$ is a vector subbundle of $T(M \times (0, T))$, every section of $\mathcal{H}$ is also a section of the tangent bundle $T(M \times (0, T))$. We call the elements of $\Gamma(\mathcal{H})$ *spatial vector fields.* There is a natural connection ∇ on $M \times (0, T)$. As a matter of fact, define ∇ by

$$\nabla_X Y = \nabla^t_X Y, \quad \nabla_X \partial_t = 0, \quad \nabla_{\partial_t}\partial_t = 0 \quad \text{and} \quad \nabla_{\partial_t} X = [\partial_t, X], \tag{2.3}$$

for each spatial vector fields X and Y, where ∇^t stands for the Levi-Civita connection of g_t. One can readily check that ∇ is compatible with g, i.e.

$$Xg(Y_1, Y_2) = g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2),$$

for each $X \in T(M\times(0, T))$ and spatial vector fields $Y_1, Y_2 \in \Gamma(\mathcal{H})$. Moreover, the connection ∇ is spatially symmetric, that is

$$\nabla_{Y_1} Y_2 - \nabla_{Y_2} Y_1 = [Y_1, Y_2],$$

for each $Y_1, Y_2 \in \Gamma(\mathcal{H})$. Let us give now the formal definition of the Ricci flow.

Definition 2.2.1. *Let M be a m-dimensional manifold and $\{g_t\}_{t\in(0,T)}$ be a one-parameter family of Riemannian metrics on M. We say that $\{g_t\}_{t\in(0,T)}$ is a solution of the* Ricci flow *if*

$$\nabla_{\partial_t} g_t = -2Ric_{g_t},$$

where Ric_g is the Ricci curvature of the metric g.

Example 2.2.2. Let us give some examples of metrics evolving under the Ricci flow.

(1) **Spheres.** Let us denote by $(\mathbb{S}^m, g_0)$ the m-dimensional unit sphere lying in $\mathbb{R}^{m+1}$. Then,

$$Ric_{g_0} = (m-1)g_0.$$

Consider now the family of metrics $\{g_t\}_{t\in(0,T)}$ given by

$$g_t = \big\{1 - 2(m-1)t\big\}g_0 \quad \text{where} \quad T = \frac{1}{2(m-1)}.$$

From the formulas of Theorem 1.8.2, we have that

$$Ric_{g_t} = Ric_{g_0} = (m-1)g_0,$$

for each $t \in (0,T)$. Moreover, for each time-independent vector fields X, Y on M, we obtain

$$\begin{aligned}\big\{\nabla_{\partial_t} g_t\big\}(X,Y) &= \partial_t\big\{g_t(X,Y)\big\} = \partial_t\big\{g_0(X,Y) - 2(m-1)tg_0(X,Y)\big\}\\ &= -2(m-1)g_0(X,Y)\\ &= -2Ric_{g_t}(X,Y).\end{aligned}$$

Hence $\{g_t\}_{t\in(0,T)}$ is a solution to the Ricci flow for every $t < T$. This shows that the sphere evolves by shrinking homothetically and at T it collapses to a point.

(2) **Hyperbolic spaces.** Let $(\mathbb{H}^m, g_0)$ be the m-dimensional hyperbolic space of constant sectional curvature -1. Then, $Ric_{g_0} = -(m-1)g_0$. A similar computation as above shows that the metrics $\{g_t\}_{t\in(0,\infty)}$ given by

$$g_t = \big\{1 + 2(m-1)t\big\}g_0, \quad \text{where} \quad t > 0,$$

consist a solution to the Ricci flow. Hence the hyperbolic space expands homothetically to infinity.

(3) **Einstein manifolds.** Let (M, g_0) be an Einstein manifold. Then, $Ric_{g_0} = \lambda\, g_0$, where $\lambda \in \mathbb{R}$ is a constant. Consider the family of metrics $\{g_t\}_{t\in(0,T)}$ given by

$$g_t = (1 - 2\lambda t)g_0, \ \text{ for } t \text{ such that } 1 - 2\lambda t > 0.$$

Similar computations as in the previous examples show that g_t is a solution to the Ricci flow. Hence, if $\lambda > 0$ the flow exists up to time $1/(2\lambda)$ and if $\lambda \le 0$ the flow exists for all positive times.

(4) **Product manifolds** Let $(M_1, g_1(t))$ and $(M_2, g_2(t))$ be solutions to the Ricci flow defined in a common time interval $(0,T)$. Then, the family of metrics $\{g_t\}_{t\in(0,T)}$ given by

$$g_t = g_1(t) \times g_2(t),$$

consists a solution to the Ricci flow on $M_1 \times M_2$.

2.3 Evolution equations

In this section we will see how various geometric quantities evolve under the Ricci flow. With abuse of notation, we will denote all connections by the same letter ∇.

Lemma 2.3.1 (Uhlenbeck's trick)**.** *Let $\{g_t\}_{t\in[0,T)}$ be a solution of the Ricci flow. Then,*

(1) *There exists a local smooth time-dependent tangent orthonormal frame field $\{e_1,\dots,e_m\}$ with respect to g_t satisfying*

$$\nabla_{\partial_t} e_i = R_{ij} e_j, \quad i \in \{1,\dots,m\},$$

for each $t \in [0,T)$.

(2) *The induced volume form $d\mu_t$ on (M, g_t) evolves according to the equation*

$$\nabla_{\partial_t} d\mu_t = -S d\mu_t.$$

Moreover, the volume V_t of the evolved metrics satisfy

$$\partial_t V = -\int S d\mu_t.$$

Proof. Denote by $P_t : TM \to TM$ the (time-dependent) adjoint operator associated with the Ricci curvature, i.e.

$$Ric(X,Y) = g_t(P_t X, Y) = g_t(X, P_t Y), \quad X, Y \in \mathfrak{X}(M).$$

Consider now the time-dependent family of bundle isomorphisms $\varphi_t : TM \to TM$ given by

$$\begin{cases} \nabla_{\partial_t} \varphi_t = P_t \circ \varphi_t, & t \in (0,T), \\ \varphi_0 = I. \end{cases}$$

We claim that $\varphi_t^* g_t = g_0$, for every $t \in [0,T)$. Indeed, consider a local coordinate chart $\{\partial_1,\dots,\partial_m\}$ in a neighborhood of φ_t. Using from (2.3) the fact that

$$[\partial_t, \partial_i] = \nabla_{\partial_t} \partial_i = 0, \quad i \in \{1,\dots,m\},$$

we obtain

$$\begin{aligned} \partial_t\{\varphi_t^* g_t(\partial_i,\partial_j)\} &= \partial_t\{g_t(\varphi_t(\partial_i),\varphi_t(\partial_j))\} = (\nabla_{\partial_t} g_t)(\varphi_t(\partial_i),\varphi_t(\partial_j)) \\ &\quad + g_t(\nabla_{\partial_t}\varphi_t(\partial_i),\varphi_t(\partial_j)) + g_t(\varphi_t(\partial_i),\nabla_{\partial_t}\varphi_t(\partial_j)) \\ &= -2Ric(\varphi_t(\partial_i),\varphi_t(\partial_j)) \\ &\quad + g_t((P_t\circ\varphi_t)(\partial_i),\varphi_t(\partial_j))) + g_t(\varphi_t(\partial_i),(P_t\circ\varphi_t)(\partial_j))) \\ &= 0. \end{aligned}$$

Hence, if $\{v_1, \ldots, v_m\}$ is a local orthonormal frame with respect to the metric g_0, then

$$\{e_1 = \varphi_t(v_1), \ldots, e_m = \varphi_t(v_m)\}$$

is a local time-dependent orthonormal frame with respect to g_t, for each $t \in [0, T)$. Moreover, this frame satisfies

$$\nabla_{\partial_t} e_i = R_{ij} e_j.$$

Denote by $\{\omega_1, \ldots, \omega_m\}$ the corresponding dual frame of $\{e_1, \ldots, e_m\}$. Then,

$$\nabla_{\partial_t} \omega_i = -R_{ij} \omega_j,$$

for each $i \in \{1, \ldots, m\}$. Hence,

$$\nabla_{\partial_t} d\mu_t = \nabla_{\partial_t}(\omega_1 \wedge \cdots \wedge \omega_m) = -S\omega_1 \wedge \cdots \wedge \omega_m = -S d\mu_t.$$

By integrating we get

$$\partial_t V = -\int S d\mu_t,$$

and the proof is completed. □

Our next goal is to compute how the Riemann curvature tensor evolves under the Ricci flow. We start with some auxiliary results that we will frequently use in the sequel. First let us denote by $\mathcal{C}_B(TM)$ the space of all $(4, 0)$-tensors satisfying the properties (1.7), (1.8) and (1.9) of the Riemannian curvature tensor. Define now the bundle map $Q : \mathcal{C}_B(TM) \to C^\infty(M)$ given by

$$Q(R)_{ijkl} = R_{ijab} R_{klab} + 2R_{iakb} R_{jalb} - 2R_{ialb} R_{jakb}, \tag{2.4}$$

where we use Einstein's summation convention and the components are regarded with respect to a local orthonormal frame. Another thing that we will use often in our computations are the Ricci identities that we state in the following lemma.

Lemma 2.3.2 (Ricci identities)**.** *Let φ be a $(r, 0)$-tensor on a Riemannian manifold. Then, the following formula holds:*

$$\begin{aligned}(\nabla^2_{X,Y}\varphi - \nabla^2_{Y,X}\varphi)(Z_1, Z_2, \ldots, Z_r) \\ = -\varphi(R(X, Y, Z_1), Z_2, \ldots, Z_r) - \cdots - \varphi(Z_1, Z_2, \ldots, R(X, Y, Z_r)),\end{aligned} \tag{2.5}$$

where $X, Y, Z_1, \ldots, Z_r \in \mathfrak{X}(M)$.

Proof. The proof follows by direct computations and for that reason we omit it. □

Lemma 2.3.3. *The following formula holds:*

$$\begin{aligned}(\Delta R + Q(R))_{ijkl} \;=\; & (\nabla^2_{ik}Ric)_{jl} - (\nabla^2_{il}Ric)_{jk} - (\nabla^2_{jk}Ric)_{il} + (\nabla^2_{jl}Ric)_{ik} \\ & + R_{ia}R_{ajkl} + R_{ja}R_{iakl},\end{aligned} \tag{2.6}$$

where the indices are regarded with respect to a local orthonormal frame.

Proof. Using the second Bianchi identity (1.9) we obtain

$$\begin{aligned}(\nabla_{e_i}R)(X,Y,Z,W) &= -(\nabla_X R)(Y,e_i,Z,W) - (\nabla_Y R)(e_i,X,Z,W) \\ &= (\nabla_X R)(e_i,Y,Z,W) - (\nabla_Y R)(e_i,X,Z,W).\end{aligned}$$

Since $\{e_i\}$ is an orthonormal frame we have

$$(\nabla^2_{e_i,X}R)(e_i,Y,Z,W) = \nabla_{e_i}\nabla_X R(e_i,Y,Z,W)$$

and

$$(\nabla^2_{e_i,Y}R)(e_i,X,Z,W) = \nabla_{e_i}\nabla_Y R(e_i,X,Z,W).$$

Thus, we can write the Laplacian operator as

$$\begin{aligned}(\Delta R)(X,Y,Z,W) &= \sum_{i=1}^m (\nabla^2_{e_i,e_i}R)(X,Y,Z,W) \\ &= \sum_{i=1}^m (\nabla_{e_i}((\nabla_X R)(e_i,Y,Z,W) - (\nabla_Y R)(e_i,X,Z,W)) \\ &= \sum_{i=1}^m \big((\nabla^2_{e_i,X}R)(e_i,Y,Z,W) - (\nabla^2_{e_i,Y}R)(e_i,X,Z,W)\big).\end{aligned} \tag{2.7}$$

Using the Ricci identity (2.5) we deduce

$$\begin{aligned}&\sum_{i=1}^m \big(\nabla^2_{X,e_i}R)(e_i,Y,Z,W) - (\nabla^2_{e_i,X}R)(e_i,Y,Z,W)\big) = \\ &\sum_{i,j=1}^m R(X,e_i,e_i,e_j)R(e_j,Y,Z,W) + \sum_{i,j=1}^m R(X,e_i,Y,e_j)R(e_i,e_j,Z,W) \\ &+ \sum_{i,j=1}^m R(X,e_i,Z,e_j)R(e_i,Y,e_j,W) + \sum_{i,j=1}^m R(X,e_i,W,e_j)R(e_i,Y,Z,e_j).\end{aligned} \tag{2.8}$$

By interchanging X with Y we obtain

$$\begin{aligned}
&\sum_{i=1}^{m} \big((\nabla^2_{Y,e_i} R)(e_i, X, Z, W) - (\nabla^2_{e_i,Y} R)(e_i, X, Z, W) \big) \\
&= \sum_{i,j=1}^{m} R(Y, e_i, e_i, e_j) R(e_j, X, Z, W) + \sum_{i,j=1}^{m} R(Y, e_i, X, e_j) R(e_i, e_j, Z, W) \\
&+ \sum_{i,j=1}^{m} R(Y, e_i, Z, e_j) R(e_i, X, e_j, W) + \sum_{i,j=1}^{m} R(Y, e_i, W, e_j) R(e_i, X, Z, e_j).
\end{aligned} \tag{2.9}$$

Then, we subtract (2.9) by (2.8),

$$\begin{aligned}
&\sum_{i=1}^{m} \big((\nabla^2_{X,e_i} R)(e_i, Y, Z, W) - (\nabla^2_{Y,e_i} R)(e_i, X, Z, W) \big) \\
&- \sum_{i=1}^{m} \big((\nabla^2_{e_i,X} R)(e_i, Y, Z, W) + (\nabla^2_{e_i,Y} R)(e_i, X, Z, W) \big) \\
&= \sum_{i,j=1}^{m} \big(R(Y, e_i, X, e_j) - R(X, e_i, Y, e_j) \big) R(e_i, e_j, Z, W) \\
&+2 \sum_{i,j=1}^{m} R(X, e_i, Z, e_j) R(e_i, Y, e_j, W) \\
&-2 \sum_{i,j=1}^{m} R(X, e_i, W, e_j) R(Y, e_i, Z, e_j) \\
&- \sum_{i,j=1}^{m} \big(Ric(X, e_j) R(e_j, Y, Z, W) - Ric(Y, e_j) R(e_j, X, Z, W) \big).
\end{aligned}$$

We use the first Bianchi identity (1.8) to get

$$R(Y, e_i, X, e_j) - R(X, e_i, Y, e_j) = R(X, Y, e_i, e_j).$$

Then, by the definition of $Q(R)$

$$\begin{aligned}
&\sum_{i=1}^{m}\big((\nabla^2_{X,e_i}R)(e_i,Y,Z,W)-(\nabla^2_{Y,e_i}R)(e_i,X,Z,W)\big)\\
&-\sum_{i=1}^{m}\big((\nabla^2_{e_i,X}R)(e_i,Y,Z,W)+(\nabla^2_{e_i,Y}R)(e_i,X,Z,W)\big)\\
&=Q(R)(X,Y,Z,W)\\
&-\sum_{j=1}^{m}(Ric(X,e_j)R(e_j,Y,Z,W)-Ric(Y,e_j)R(e_j,X,Z,W)).
\end{aligned}$$

Hence, by (2.7)

$$\begin{aligned}
(\Delta R+Q(R))(X,Y,Z,W) \;=\; &\sum_{i=1}^{m}(\nabla^2_{X,e_i}R)(e_i,Y,Z,W)\\
&-\sum_{i=1}^{m}(\nabla^2_{Y,e_i}R)(e_i,X,Z,W)\\
&+\sum_{i=1}^{m}Ric(X,e_i)R(e_i,Y,Z,W)\\
&+\sum_{i=1}^{m}Ric(Y,e_i)R(X,e_i,Z,W).
\end{aligned} \tag{2.10}$$

Then, using the second Bianchi identity (1.9),

$$\nabla_{e_k}R(e_k,Y,Z,W)=\nabla_{e_k}R(e_k,Y,e_k,W)-\nabla_W R(e_k,Y,e_k,Z).$$

As a consequence,

$$\begin{aligned}
\sum_{i=1}^{m}(\nabla^2_{X,e_i}R)(e_i,Y,Z,W) &= \sum_{i=1}^{m}(\nabla^2_{X,Z}R)(e_i,Y,e_i,W)\\
&\quad-\sum_{i=1}^{m}(\nabla^2_{X,W}R)(e_i,Y,e_i,Z)\\
&=(\nabla^2_{X,Z}Ric)(Y,W)-(\nabla^2_{X,W}Ric)(Y,Z).
\end{aligned}$$

Interchanging X with Y we get the following expression

$$\begin{aligned}\sum_{i=1}^{m}(\nabla^2_{Y,e_i}R)(e_i,X,Z,W) &= \sum_{i=1}^{m}(\nabla^2_{Y,Z}R)(e_i,X,e_i,W)\\ &\quad -\sum_{i=1}^{m}(\nabla^2_{Y,W}R)(e_i,X,e_i,Z)\\ &= (\nabla^2_{Y,Z}Ric)(X,W)-(\nabla^2_{Y,W}Ric)(X,Z).\end{aligned}$$

By replacing the last two equations in (2.10) the proof is completed. □

Lemma 2.3.4. *Let $\{g_t\}_{t\in[0,T)}$ be a solution of the Ricci flow equation and $\{e_1,\ldots,e_m\}$ an orthonormal frame as in Lemma 2.3.1. Then, with respect to this frame, the following holds:*

(1) *The Riemannian curvature tensor evolves according to*

$$(\nabla_{\partial_t}R)_{ijkl} = (\Delta R + Q(R))_{ijkl}. \tag{2.11}$$

(2) *The Ricci tensor evolves according to*

$$(\nabla_{\partial_t}Ric)_{ij} = (\Delta Ric)_{ij} + 2R_{iajb}R_{ab}. \tag{2.12}$$

(3) *The scalar curvature evolves according to*

$$\partial_t S = \Delta S + |Ric|^2. \tag{2.13}$$

Proof. The proof follows using Lemma 2.3.1 and 2.3.3 and straightforward computations. □

CHAPTER 3

SHORT-TIME EXISTENCE

In this chapter, we prove that the Ricci flow equation, with initial data a compact Riemannian manifold, has always a smooth solution for a short time. It turns out that the Ricci flow equation is a weakly parabolic system and the existence of a short-time solution is not guarantied from the standard theory of PDEs. We will present the so-called DeTurck's trick, presented in [13]. DeTurck's idea was to modify the Ricci flow equation in a way that it becomes strictly parabolic. Then he shows that these two equations are equivalent and the existence and uniqueness of the solution follows from the theory of parabolic PDEs.

3.1 Nature of the Ricci flow

3.1.1 The symbol

Firstly, we introduce the concept of parabolicity of differential operators on vector bundles. Let M be a smooth manifold with a Riemannian metric g associated with the Levi-Civita connection ∇^M. Let E and F be vector bundles over M with E equipped with a Riemannian metric h which is compatible with the connection ∇^E. Using the connections ∇^M and ∇^E we can construct ∇^n, the $n-$th iterated covariant derivative of a section $\psi \in \Gamma(E)$.

Definition 3.1.1. *A differential operator $\mathcal{L}\colon \Gamma(E) \to \Gamma(F)$ of the form*

$$\mathcal{L}(\psi)(x) = Q(x, \nabla\psi(x), \dots, \nabla^n\psi(x)) \in F_x,$$

where Q is smooth in all its variables, is called differential operator of order n. *If $\mathcal{L}$ is $\mathbb{R}$-linear in ψ then we say that $\mathcal{L}$ is a* linear differential operator. *Otherwise, $\mathcal{L}$ is called a* non-linear differential operator.

Let $\mathcal{L}\colon \Gamma(E) \to \Gamma(F)$ be a linear differential operator of order n. Then, in index notation $\mathcal{L}$ can be written as

$$\mathcal{L}(\psi) = \sum_{i_1,\dots,i_n} A^n(\nabla^n_{\partial_{i_1}\dots\partial_{i_n}}\psi) + \dots + \sum_{i_1} A^1(\nabla_{\partial_{i_1}}\psi) + A^0(\psi),$$

where for each $x \in M$,

$$A^0(x), \ldots, A^n(x) \colon E_x \to F_x,$$

are linear maps. These maps are called the *coefficients of the linear operator* $\mathcal{L}$.

Definition 3.1.2. *Let* $\mathcal{L} \colon \Gamma(E) \to \Gamma(F)$ *be a linear differential operator of order* n. *Also, let* $x \in M$ *and* $\xi = \sum_{i=1}^{m} \xi_i \partial_i \in T_x M$. *The linear map* $\sigma_\xi(\mathcal{L}, x) \colon (E)_x \to (F)_x$ *given by*

$$\sigma_\xi(\mathcal{L}, x)\psi = \sum_{i_1,\ldots,i_n} \xi_{i_1} \ldots \xi_{i_k} A^n(\psi(x)),$$

is called the principal symbol of the operator $\mathcal{L}$ at the point x in the direction ξ.

Definition 3.1.3. *The operator* $\mathcal{L}$ *is called* elliptic operator *if* $\sigma[\mathcal{L}](\xi)(\psi)$ *is a bundle isomorphism of the fiber for every non-zero* $\xi \in \mathfrak{X}(M)$ *or equivalently, if there exists* $c > 0$ *such that for all* ξ *and* ψ *we have*

$$\langle \sigma[\mathcal{L}](\xi)(\psi), \psi \rangle \geq c|\xi|^2|\psi|^2. \tag{3.1}$$

We are interested in the case where the manifold M is equipped with a one-parameter family of smooth metrics $\{g_t\}_{t\in[0,T)}$. We denote by $\{\nabla^{g_t}\}_{t\in[0,T)}$ the corresponding Levi-Civita connections. Let E and F be vector bundles over M where E is equipped with a fixed metric h and connections $\{\nabla^t\}_{t\in[0,T)}$ which are compatible with h, that is,

$$vh(\psi_1, \psi_2) = h({\nabla^t}_v\psi_1, \psi_2) + h(\psi_1, {\nabla^t}_v\psi_2),$$

for each tangent vector v, sections $\psi_1, \psi_2 \in \Gamma(E)$ and $t \in [0, T)$. One can use the connections ∇^t and ∇^{g_t} to construct $(\nabla^t)^n$ acting on sections of E. Let $\{\psi(t)\}_{t\in[0,T)}$ be a smooth time-dependent family of sections of E, where smooth means for each $(x, t) \in M \times [0, T)$ the time-derivative

$$\frac{\partial \psi}{\partial t} = \lim_{h\to 0} \frac{\psi(x, t + h) - \psi(x, t)}{h},$$

exists. Hence, $\{\partial_t\psi\}_{t\in[0,T)}$ is another family of sections on E. We consider the equation

$$\partial_t\psi(x, t) = (\mathcal{L}\psi)(x, t) = Q(x, t, (\nabla^t)\psi(x, t), \ldots, (\nabla^t)^n\psi(x, t)) \tag{3.2}$$

where $\mathcal{L} \colon \Gamma(E) \to \Gamma(F)$ is a time-dependent differential operator of order n. If for each fixed t the operator $\mathcal{L}$ is linear elliptic then we say that (3.2) is a *linear parabolic differential equation*.

Theorem 3.1.4. *Let* $\mathcal{L}$ *be a parabolic differential operator at* $\psi_0 \in \Gamma(E)$. *Then, there exists* $T > 0$ *and a smooth family* $\psi(t) \in \Gamma(E), t \in [0, T]$ *such that there exists a unique smooth solution for the initial value problem*

$$\begin{cases} \partial_t\psi = \mathcal{L}\psi, \\ \psi(0) = \psi_0 \end{cases} \tag{3.3}$$

for $t \in [0, T]$ *where* T *depends on the initial data* ψ_0.

3.1.2 Linearization of the Ricci tensor

We need to explain what parabolicity means when $\mathcal{L}$ is a non-linear operator $\mathcal{L}\colon \Gamma(E) \to \Gamma(F)$.

Definition 3.1.5. *The* linearization $D\mathcal{L}$ of $\mathcal{L}$ at ψ_0*, in case that it exists, is defined to be the linear map* $D\mathcal{L}_{\varphi_0}\colon \Gamma(E) \to \Gamma(F)$ *is given by*

$$D\mathcal{L}_{\varphi_0}(\psi) = \lim_{s\to 0} \frac{\mathcal{L}(\varphi_0 + s\psi) - \mathcal{L}(\varphi_0)}{s} = \frac{\partial \mathcal{L}(\varphi(s))}{\partial s}\Big|_{s=0},$$

where $\varphi\colon [0,1] \to \Gamma(E)$ *is a one-parameter family of sections with* $\varphi(0) = \varphi_0$ *and* $\varphi'(0) = \partial_s\varphi(0) = \psi$.

Definition 3.1.6. *We say that the equation* (3.2) *is* strictly (or strong) parabolic *when the equation*

$$\frac{\partial \psi}{\partial t} = D\mathcal{L}_{\varphi_0}(\psi),$$

is parabolic for every $\varphi_0 \in \Gamma(E)$.

We focus now on the Ricci flow equation where the operator is

$$\mathcal{L} = -2Ric\colon \Gamma(\mathrm{Sym}(T^*M \times T^*M)) \to \Gamma(\mathrm{Sym}(T^*M \times T^*M)).$$

Using Lemma 2.3.4 with $\partial_t g = h$ the linearization of $-2Ric$ is given by

$$\begin{aligned} D(-2Ric) &= -2D(Ric)(h)_{kl} \\ &= \sum_{i=1}^{m} (\nabla^2_{i,l}h)_{ki} - (\nabla^2_{k,l}h)_{ii} + (\nabla^2_{k,i}h)_{li} - (\nabla^2_{i,i}h)_{kl}. \end{aligned}$$

Then, the principal symbol is

$$\sigma[-2D(Ric)](\xi)(h)_{kl} = \sum_{i=1}^{m} \xi_i\xi_l h_{ki} - \xi_k\xi_l h_{ii} + \xi_k\xi_i h_{li} - \xi_i\xi_i h_{kl}.$$

We can choose $h_{kl} = \xi_k\xi_l$ and then $\sigma[-2D(Ric)](\xi)(h)_{kl} = 0$ and thus an inequality such (3.1) can not hold. Hence, Ricci flow is not a strictly parabolic equation.

3.2 Ricci-DeTurck flow

Firstly, we introduce some definitions that we will use. Let $f\colon M \to N$ be a smooth map between two Riemannian manifolds (M, g) and (N, h) of dimension m and n respectively. We know that the derivative df of f is viewed as a section of the vector bundle $TM^* \otimes f^*(TN)$. This vector bundle is endowed with the induced connection ∇. We denote with ∇^M the Levi-Civita connection on TM,∇^N the Levi-Civita connection on TN and ∇^f the pull-back connection on $f^*(TN)$. Then the connection ∇ is given by

$$(\nabla_X df)(Y) = \nabla^f_X df(Y) - df(\nabla^M_X Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

Definition 3.2.1. *The* harmonic map Laplacian of f with respect to the metrics g and h *is defined by*

$$\Delta_{g,h} f = \sum_{i=1}^{n} (\nabla_{e_i} df)(e_i),$$

where $\{e_i\}$ *is a local orthonormal frame on* (M, g).

Note that $\Delta_{g,h} f$ is a section of the vector bundle $f^*(TN)$. Let $\{x^i\}$ and $\{y^a\}$ be local coordinates around the points $x \in M$ and $f(x) \in N$ respectively. Then, the harmonic map Laplacian of f can be written in the form

$$\Delta_{g,h} f = \sum_{a=1}^{n} \Big(\Delta_M f^a + \sum_{i,j=1}^{m} \sum_{a,b=1}^{n} g^{ij} (\Gamma_h)^a_{bc} \frac{\partial f^b}{\partial x^i} \frac{\partial f^c}{\partial x^j} \Big) \frac{\partial}{\partial y^a} \tag{3.4}$$

where

$$\Delta_M f^a = \sum_{i,j,k=1}^{m} g^{ij} \left(\frac{\partial^2 f^a}{\partial x^i \partial x^j} - (\Gamma_g)^k_{ij} \frac{\partial f^a}{\partial x^k} \right),$$

is the Laplacian operator of f^a. Also f^a are the components of f and Γ_g, Γ_h the Christoffel symbols with respect to the connections ∇^M and ∇^N respectively.
The next Lemma states that the harmonic map Laplacian of a map $f \colon M \to N$ is unchanged under the action of a diffeomorphism on M.

Lemma 3.2.2. *Let* $f \colon M \to N$ *be a smooth map between two Riemannian manifolds* (M, g) *and* (N, h)*. Also, let* $\varphi \colon M \to M$ *be a diffeomorphism. Then, it holds*

$$(\Delta_{\varphi^*(g),h}(f \circ \varphi))(x) = (\Delta_{g,h} f)(\varphi(x)) \in T_{f(\varphi(x))} N,$$

for all $x \in M$.

Proof. Fix $x \in M$. Let $\{x^i\}$ be local coordinates around $\varphi(x) \in M$. Then, we can induce local coordinates $\{y^i\}$ around x by $y^i = x^i \circ \varphi$. Also, fix local coordinates $\{z^a\}$ around

$f \circ \varphi(x) \in N$. Then, in these coordinates we have that $\frac{\partial (f\circ\varphi)^a}{\partial y^i}(x) = \frac{\partial f^a}{\partial x^i}(\varphi(x))$, $(\varphi^* g)^{ij} = g^{ij}$ and $(\Gamma_{\varphi^* g})^k_{ij} = (\Gamma_g)^k_{ij}$. We compute,

$$\begin{aligned}
(\Delta_{\varphi^*(g),h}(f\circ\varphi))(x) &= \sum_{a=1}^{n}\Bigg(\sum_{i,j,k=1}^{m}(\varphi^* g)^{ij}\left(\frac{\partial^2 (f\circ\varphi)^a}{\partial y^i \partial y^j} - (\Gamma_{\varphi^* g})^k_{ij}\frac{\partial (f\circ\varphi)^a}{\partial y^k}\right) \\
&\quad + \sum_{i,j=1}^{m}\sum_{a,b=1}^{n}(\varphi^* g)^{ij}(\Gamma_h)^a_{bc}\frac{\partial (f\circ\varphi)^b}{\partial y^i}\frac{\partial (f\circ\varphi)^c}{\partial y^j}\Bigg)\frac{\partial}{\partial z^a} \\
&= \sum_{a=1}^{n}\Bigg(\sum_{i,j,k=1}^{m} g^{ij}\left(\frac{\partial^2 f^a}{\partial x^i \partial x^j} - (\Gamma_g)^k_{ij}\frac{\partial f^a}{\partial x^k}\right) \\
&\quad + \sum_{i,j=1}^{m}\sum_{a,b=1}^{n} g^{ij}(\Gamma_h)^a_{bc}\frac{\partial f^b}{\partial x^i}\frac{\partial f^c}{\partial x^j}\Bigg)\frac{\partial}{\partial z^a} \\
&= (\Delta_{g,h} f)(\varphi(x)).
\end{aligned}$$

This completes the proof. ☐

Now, we can introduce the Ricci-DeTurck flow.

Definition 3.2.3. *Let M be a compact Riemannian manifold endowed with a fixed metric h. Also, let $\tilde{g}(t) = \tilde{g}_t, t \in [0,T)$ be a one-parameter family of Riemannian metrics on M. The metric $\{\tilde{g}_t\}_{t\in[0,T)}$ is a* solution to the Ricci-DeTurck's flow on M *if it suffices the following equation*

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2Ric_{\tilde{g}(t)} - \mathcal{L}_{\xi_t}\tilde{g}(t),$$

where $\xi_t = \Delta_{\tilde{g}_t,h} I$.

We will show that the Ricci-DeTurck flow is strictly parabolic and as a consequence the parabolic theory of PDEs implies that there exists a unique solution.

Proposition 3.2.4. *Let M be a compact Riemannian manifold endowed with a fixed metric h. Given any initial metric g_0, there exists $T > 0$ and a smooth one-parameter family of Riemannian metrics $\{\tilde{g}_t\}_{t\in[0,T)}$, such that $\{\tilde{g}_t\}_{t\in[0,T)}$ is a solution to the Ricci-DeTurck flow with $\tilde{g}(0) = g_0$. Moreover, the solution $\{\tilde{g}_t\}_{t\in[0,T)}$ is unique.*

Proof. In local coordinates $\{x_i\}$, the Ricci tensor of $\tilde{g}$ takes the form

$$R_{jl} = \frac{1}{2}\sum_{i,j,k,l,p=1}^{m}\tilde{g}^{ik}(\partial_i\partial_l\tilde{g}_{jk} + \partial_j\partial_k\tilde{g}_{il} - \partial_j\partial_l\tilde{g}_{ik} - \partial_i\partial_k\tilde{g}_{jl}) + (\Gamma_{\tilde{g}})^p_{jl}(\Gamma_{\tilde{g}})a^i_{ip} - (\Gamma_{\tilde{g}})^p_{il}(\Gamma_{\tilde{g}})^i_{jp}. \quad (3.5)$$

We denote with $\Gamma_{\tilde{g}}$ and Γ_h the Christoffel symbols associated with the metrics g and h, respectively. Then, the vector field $\xi = \Delta_{\tilde{g},h} I$ can be written locally, according to (3.4), in the form

$$\xi = \sum_{i,k,l=1}^{m} \tilde{g}^{ik}\big((\Gamma_h)^l_{ik} - (\Gamma_{\tilde{g}})^l_{ik}\big)\partial_l.$$

By the definition of the Christoffel symbols, this implies

$$\xi = \frac{1}{2}\sum_{i,k,l=1}^{m} \tilde{g}^{ik}(h^{jl}(\partial_i h_{jk} + \partial_k h_{ij} - \partial_j h_{ik}) - \tilde{g}^{jl}(\partial_i \tilde{g}_{jk} + \partial_k \tilde{g}_{ij} - \partial_j \tilde{g}_{ik}))\partial_l.$$

By Corollary (1.7.4) we obtain

$$(\mathcal{L}_\xi \tilde{g})_{jl} = \sum_{i,j,k,l=1}^{n} \tilde{g}^{ik}(\partial_i\partial_j \tilde{g}_{jk} + \partial_j\partial_k \tilde{g}_{il} - \partial_j\partial_l \tilde{g}_{ik}) + (\text{lower order terms}). \tag{3.6}$$

Finally, combining (3.5) and (3.6) we deduce

$$-2R_{jl} - (\mathcal{L}_\xi \tilde{g})_{jl} = \sum_{i,j,k,l=1} \tilde{g}^{ik}\partial_i\partial_k \tilde{g}_{jl} + (\text{lower order terms}).$$

The above form of the Ricci-DeTurck shows that it is strictly parabolic. The theory of parabolic PDEs implies that there exists a unique solution on a short time interval $[0, T)$ where T is a positive real number. This completes the proof. $\square$

Proposition 3.2.5. *Let M be a compact Riemannian manifold endowed with a fixed metric h. Also, let $\{\tilde{g}_t\}_{t\in[0,T)}$ be a family of Riemannian metrics on M satisfying the Ricci-DeTurck flow, that is,*

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2Ric_{\tilde{g}(t)} - \mathcal{L}_{\xi_t}\tilde{g}(t),$$

where $\xi_t = \Delta_{\tilde{g}_t,h} I$. Moreover, assume that $\{\varphi_t\}_{t\in[0,T)}$ is a one-parameter family of diffeomorphisms such that

$$\frac{\partial}{\partial t}\varphi_t(x) = \xi_t(\varphi_t(x)),$$

for all $(x,t) \in M \times [0,T)$. Then, the family of metrics $\{g_t\}_{t\in[0,T)}$ defined by $g_t = \varphi_t^* \circ \tilde{g}_t$ forms a solution to the Ricci flow.

Proof. Since $g(t) = \varphi_t^*(\tilde{g}(t)), t \in [0, T)$ we have

$$
\begin{aligned}
\frac{\partial}{\partial t} g(t) &= \partial_t(\varphi^*(\tilde{g}(t))) = \lim_{s \to 0} \frac{\varphi_{t+s}^*(\tilde{g}(t+s)) - \varphi_t^*(\tilde{g}(t))}{s} \\
&= \lim_{s \to 0} \frac{\varphi_{t+s}^*(\tilde{g}(t+s)) - \varphi_t^*(\tilde{g}(t+s))}{s} + \lim_{s \to 0} \frac{\varphi_t^*(\tilde{g}(t+s)) - \varphi_t^*(\tilde{g}(t))}{s} \\
&= \varphi_t^* \left(\lim_{s \to 0} \frac{(\varphi_t^*)^{-1} \circ \varphi_{t+s}^* - I}{s} \right) (\tilde{g}(t+s)) + \varphi_t^*(\partial_t \tilde{g}(t)) \\
&= \varphi_t^*(\mathcal{L}_X \tilde{g}(t)) + \varphi_t^*(\partial_t \tilde{g}(t)) \qquad (3.7) \\
&= \varphi_t^*(\mathcal{L}_X \tilde{g}(t)) + \varphi_t^*(-2Ric_{\tilde{g}(t)} - \mathcal{L}_X \tilde{g}(t)) \\
&= \varphi_t^*(-2Ric_{\tilde{g}(t)}) = -2Ric_{g_t}.
\end{aligned}
$$

Indeed, the metrics $\{g_t\}_{t \in [0,T)}$ is a solution to the Ricci flow. This completes the proof. □

Conversely, we can show that if we have a solution to the Ricci flow then we can construct a solution of the Ricci-DeTurck flow.

Proposition 3.2.6. *Let M be a compact Riemannian manifold endowed with a fixed metric h. Also, let the family of Riemannian metrics $\{g_t\}_{t \in [0,T)}$ be a solution to the Ricci flow. Moreover, suppose that $\{\varphi_t\}_{t \in [0,T)}$ is a family of diffeomorphisms on M evolving under the harmonic heat map flow, that is,*

$$\frac{\partial}{\partial t} \varphi_t = \Delta_{g_t, h} \varphi_t.$$

Then, the family of metrics $\{\tilde{g}_t\}_{t \in [0,T)}$ defined by $\varphi_t^* \circ \tilde{g}_t = g_t$ forms a solution to the Ricci-DeTurck flow. Furthermore,

$$\frac{\partial}{\partial t} \varphi_t(x) = \xi_t(\varphi_t(x)),$$

for all $(x, t) \in M \times [0, T)$, where $\xi_t = \Delta_{\tilde{g}_t, h} I$.

Proof. According to the equation (3.7), since $\varphi_t^*(\tilde{g}(t)) = g(t), t \in [0, T)$ we have

$$\varphi_t^*(\mathcal{L}_X \tilde{g}(t)) + \varphi_t^*(\partial_t \tilde{g}(t)) = \partial_t g(t).$$

We have assumed that the family of metrics $\{g_t\}_{t \in [0,T)}$ is a solution to the Ricci flow. Thus,

$$\varphi_t^*(\mathcal{L}_X \tilde{g}(t)) + \varphi_t^*(\partial_t \tilde{g}(t)) = -2Ric_{g_t}$$

and as a consequence,

$$\varphi_t^*(\mathcal{L}_X \tilde{g}(t) + \partial_t \tilde{g}(t) + 2Ric_{\tilde{g}}(t)) = 0.$$

This implies that the family of metrics $\{\tilde{g}_t\}_{t\in[0,T)}$ forms a solution to the Ricci-DeTurck flow. Finally, by Lemma 3.2.2 we have,

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_t(x) &= (\Delta_{g_t,h}\varphi_t)(x) = (\Delta_{\varphi_t^*(\tilde{g}_t),h}\varphi_t)(x)\\ &= (\Delta_{\tilde{g}_t,h}I)(\varphi_t(x)) = \xi_t(\varphi_t(x)),\end{aligned}$$

for all $(x,t)\in M\times[0,T)$. This completes the proof. □

Theorem 3.2.7. *Let M be a compact Riemannian manifold and let g_0 be a smooth metric on M. Then, there exist a real number $T>0$ and a smooth family of metrics $\{g_t\}_{t\in[0,T)}$ such that $\{g_t\}_{t\in[0,T)}$ is a solution to the Ricci flow and $g(0)=g_0$. Moreover, the solution $\{g_t\}_{t\in[0,T)}$ is unique.*

Proof. We have already shown in Proposition 3.2.4 that there exist $T>0$ and a smooth one-parameter family of metrics $\{g_t\}_{t\in[0,T)}$, such that $\{g_t\}_{t\in[0,T)}$ is a solution to the Ricci-DeTurck flow with $\tilde{g}(0)=g_0$. Hence,

$$\partial_t\tilde{g}(t) = -2Ric_{\tilde{g}(t)} - \mathcal{L}_{\xi_t}\tilde{g}(t),$$

where $\xi_t=\Delta_{\tilde{g}_t,h}I$. For each point $x\in M$ we denote by $\phi_t(x)$ the solution of the ODE

$$\frac{\partial}{\partial t}\varphi_t(x) = \xi_t(\varphi_t(x)),$$

with initial condition $\varphi_0=I$. By Proposition 3.2.5 we know that the metrics

$$g(t) = \varphi_t^*(\partial_t\tilde{g}(t)), \quad t\in[0,T),$$

form a solution to the Ricci flow with $g(0)=g_0$.

In the next step, we prove the uniqueness statement. Let $g_1(t)$ and $g_2(t)$, $t\in[0,T)$ be two solutions to the Ricci flow with $g_1(0)=g_2(0)$. We want to show that

$$g_1(t)=g_2(t)$$

for all $t\in[0,T)$. We will argue by contradiction. Suppose that $g_1(t)\neq g_2(t)$ for some $t\in[0,T)$. We define $\tau\in\mathbb{R}$ by

$$\tau = \inf\{t\in[0,T)\mid g_1(t)\neq g_2(t)\}.$$

Then, it is true that $g_1(\tau)=g_2(\tau)$. Also, let φ_t^1 be solution to the harmonic map heat flow

$$\begin{cases}\frac{\partial}{\partial t}\varphi_t^1 = \Delta_{g_1(t),h}\varphi_t, \\ \varphi_\tau^1 = I\end{cases} \tag{3.8}$$

and similarly, let ϕ_t^2 be solution to the harmonic map heat flow

$$\begin{cases} \frac{\partial}{\partial t}\varphi_t^2 = \Delta_{g_2(t),h}\varphi_t \\ \varphi_\tau^2 = I. \end{cases} \tag{3.9}$$

The harmonic map heat flow is a parabolic equation so there exists a unique solution on a short time interval $[\tau, \tau+\varepsilon)$, where $\varepsilon > 0$. Furthermore, if we choose ε to be small enough, we have that φ_t^1 and φ_t^2 are diffeomorphisms for all $t \in [0, T)$. Then, for each $t \in [\tau, \tau+\varepsilon)$ we define two Riemannian metrics $\tilde{g}_1$ and $\tilde{g}_1$ on M by

$$g_1(t) = (\varphi_t^1)^*(\tilde{g}_1(t)) \qquad \text{and} \qquad g_2(t) = (\varphi_t^2)^*(\tilde{g}_2(t)).$$

Using Proposition 3.2.6 we obtain that the metrics $\tilde{g}_1$ and $\tilde{g}_2$ are solutions to the Ricci-DeTurck flow on $t \in [\tau, \tau+\varepsilon)$. Hence, by Proposition 3.2.4 and due to the fact that $\tilde{g}_1(\tau) = \tilde{g}_2(\tau)$ we conclude that $\tilde{g}_1(t) = \tilde{g}_2(t)$ for all $t \in [\tau, \tau+\varepsilon)$. For each $t \in [\tau, \tau+\varepsilon)$, we also define a vector field ξ_t on M by

$$\Delta_{\tilde{g}_1(t),h} I = \xi_t = \Delta_{\tilde{g}_2(t),h} I.$$

Then, using Proposition 3.2.6 we obtain

$$\frac{\partial}{\partial t}\varphi_t^1(x) = \xi_t(\varphi_t^1(x)) \qquad \text{and} \qquad \frac{\partial}{\partial t}\varphi_t^2(x) = \xi_t(\varphi_t^2(x)),$$

for all $(x, t) \in M \times [\tau, \tau+\varepsilon)$. By the way that we have define φ_t^1 and φ_t^2 in (3.8) and (3.9) respectively, it holds $\varphi_\tau^1 = I = \varphi_\tau^2$. Hence, we have that

$$\varphi_t^1 = \varphi_t^2,$$

for all $t \in [\tau, \tau+\varepsilon)$. Finally, we can conclude that

$$g_1(t) = (\varphi_t^1)^*(\tilde{g}_1(t)) = (\varphi_t^2)^*(\tilde{g}_2(t)) = g_2(t),$$

for every $t \in [\tau, \tau+\varepsilon)$. This contradicts the definition of τ and the proof is competed. ☐

3.3 Curvature blow-up at finite time

In this section, we show that as we approach the finite maximal time of existence of the flow T the Riemannian curvature tensor explodes.

3.3.1 Derivative estimates for the curvature tensor

We denote with$\nabla^n A$ the $n-$th iterated covariant derivative of the tensor A and with $A * B$ any linear combination of contractions or metric contractions of $A \otimes B$ with coefficients that do not depend on A or B. For example, if $A = A_{ijkl}$ and $B = B_{pqr}$ then $A * B$ may represent

$$2A_{ijkl}B_{jql}, \quad \text{or} \quad A_{ijkl}g^{lm}B_{mqr} \quad \text{or} \quad \sum_{s=1}^{3} A_{ijk_sl}g^{lm}B_{k_1mk_3}.$$

The $*$-notation is very abstract but we use it in order to take bounds and avoid complicated combinations of tensors. The most useful property obtained by the Cauchy-Schwartz inequality is

$$|A * B| \leq C|A||B|,$$

where $C > 0$ is a constant.

We denote $(A^*)^k$ any k-fold product $A * \ldots * A$. Let A be a n-tensor. We use the following equations

$$\begin{aligned}[\nabla_k, \Delta]A = \nabla_k\Delta A - \Delta\nabla_k A &= g^{ij}(\nabla_k\nabla_i\nabla_j A - \nabla_i\nabla_j\nabla_k A) \\ &= g^{ij}([\nabla_k, \nabla_i]\nabla_j A + \nabla_i\nabla_k\nabla_j A - \nabla_i\nabla_j\nabla_k A) \\ &= g^{ij}([\nabla_k, \nabla_i]\nabla_j A + \nabla_i[\nabla_k, \nabla_j]A).\end{aligned}$$

Since

$$[\nabla_k, \nabla_j]A_{i_1\ldots i_n} = -\sum_{s=1}^{n} R^m_{kji_s}A_{i_1\ldots m\ldots i_n} = \sum_{s=1}^{n} R_{kji_sl}g^{lm}A_{i_1\ldots m\ldots i_n} = R * A,$$

we get

$$[\nabla, \Delta]A = R * \nabla A + \nabla(R * A) = R * \nabla A + \nabla R * A.$$

Then, we can use the second Bianchi identity (1.9) and obtain [see [9], page 227]

$$[\nabla, \Delta]A = R * \nabla A + \nabla Ric * A. \tag{3.10}$$

It also holds

$$\Delta|A|^2 = 2\langle\Delta A, A\rangle + 2|\nabla A|^2. \tag{3.11}$$

Let A be a tensor field satisfying the evolution equation

$$\nabla_{\partial_t} A = \Delta A + B.$$

where B is a tensor of same type as A. Then, we can derive the following equations:

Using (3.11) we obtain

$$\begin{aligned}\partial_t|A|^2 &= \nabla_{\partial_t} g_t(A,A) = 2g_t(\nabla_{\partial_t}A, A) + \nabla_{\partial_t} g_t(A,A) \\ &= 2g_t(\nabla_{\partial_t}A, A) + Ric * A * A \\ &= 2g_t(\Delta A + B, A) + Ric * A * A \\ &= \Delta|A|^2 - 2|\nabla A|^2 + B * A + Ric * (A^*)^2.\end{aligned} \tag{3.12}$$

We would like to know how the covariant derivative of A, $\nabla A = \partial A + A * \Gamma$ evolves. The evolution equation of the Christoffel symbols

$$\nabla_{\partial_t}\Gamma_{ij}^k = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} + \nabla_l R_{ij})$$

can be written in the form

$$\nabla_{\partial_t}\Gamma = g^{-1} * \nabla Ric.$$

Then, by (3.10)

$$\begin{aligned}\nabla_{\partial_t}\nabla A &= \partial_t(\partial A) + \nabla_{\partial_t}A * \Gamma + A * \nabla_{\partial_t}\Gamma \\ &= \partial\nabla_{\partial_t}A + \nabla_{\partial_t}A * \Gamma + A * \nabla_{\partial_t}\Gamma \\ &= \nabla(\partial_t A) + A * \nabla Ric \\ &= \nabla(\Delta A + B) + A * \nabla Ric \\ &= \Delta\nabla A + R * \nabla A + \nabla Ric * A + \nabla B + A * \nabla Ric \\ &= \Delta\nabla A + R * \nabla A + \nabla Ric * A + \nabla B.\end{aligned} \tag{3.13}$$

We observe that ∇A satisfies an evolution equation of the type

$$\nabla_{\partial_t}\nabla A = \Delta(\nabla A) + C, \quad \text{where } C = R * \nabla A + \nabla Ric * A + \nabla B.$$

Thus, using (3.12) we obtain

$$\partial_t|\nabla A|^2 = \Delta|\nabla A|^2 - 2|\nabla\nabla A|^2 + (R*\nabla A + \nabla Ric * A + \nabla B) * \nabla A + Ric * ((\nabla A)^*)^2. \tag{3.14}$$

We use the above formulas to compute quantities concerning the Riemannian curvature tensor whose evolution equation (2.11) can be rewritten with the $*$-notation as:

$$\nabla_{\partial_t}R = \Delta(R) + R * R. \tag{3.15}$$

Then, using (3.12) we get

$$\partial_t(|R|^2) = \Delta|R|^2 - 2|\nabla R|^2 + (R)^{*3} \tag{3.16}$$

and by (3.14) we have

$$\partial_t|\nabla R|^2 = \Delta|\nabla R|^2 - 2|\nabla\nabla R|^2 + R * (\nabla R)^{*2}. \tag{3.17}$$

Proposition 3.3.1. *Let M be a m-dimensional compact Riemannian manifold and let g_t be a solution to the Ricci flow on M. Then, given $a > 0$ and integer n, there exists a constant C, depending only on m, n and* $\max\{a, 1\}$ *such that if*

$$|R(x,t)|_{g_t} \leq K,$$

for all $(x,t) \in M \times [0, \frac{a}{K}]$ then,

$$|\nabla^n R(x,t)|_{g_t} \leq CK \frac{1}{t^{n/2}},$$

for all $(x,t) \in M \times (0, \frac{a}{K}]$.

Proof. We work by induction on n. Let $n = 1$. Then, by (3.17) we have

$$\partial_t |\nabla R|^2 = \Delta |\nabla R|^2 - 2|\nabla\nabla R|^2 + R * (\nabla R)^{*2}. \tag{3.18}$$

We define

$$F(x,t) = t|\nabla R|^2 + \beta |R|^2,$$

where β a constant that we will choose later. Note that at $t = 0$ we have

$$F = \beta |R|^2 \leq \beta K^2.$$

Then, we differentiate F and get

$$\partial_t F = |\nabla R|^2 + t\partial_t(|\nabla R|^2) + \beta \partial_t |R|^2.$$

Thus, using (3.16) and (3.17) we have

$$\begin{aligned}
\partial_t F &= |\nabla R|^2 + t(\Delta |\nabla R|^2 - 2|\nabla^2 R|^2 + R * (\nabla R)^{*2}) \\
&\quad + \beta(\Delta |R|^2 - 2|\nabla R|^2 + (R)^{*3}) \\
&= \Delta F + (1 - 2\beta)|\nabla R|^2 + \beta (R)^{*3} + tR * (\nabla R)^{*2} - 2t|\nabla^2 R|^2 \\
&\leq \Delta F + (1 - 2\beta)|\nabla R|^2 + \beta C_2 |R|^3 + tC_1 |R||\nabla R|^2 \\
&= \Delta F + |\nabla R|^2 (1 - 2\beta + tC_1 |R|) + \beta C_2 |R|^3
\end{aligned}$$

where C_1, C_2 are constants depending on m. By assumption we have that $|R| \leq K$ for all $t \in [0, \frac{a}{K}]$. Thus,

$$\partial_t F \leq \Delta F + |\nabla R|^2 (1 - 2\beta + C_1 K) + \beta C_2 K^3.$$

If we choose β such that $\beta \geq \frac{C_1 a + 1}{2}$ we obtain

$$\partial_t F \leq \Delta F + \beta C_2 K^3,$$

for all $t \in [0, \frac{a}{K}]$. We can apply the scalar maximum principle and solve the associated ODE

$$\frac{du}{dt} = C_2 \beta K^3, \quad u(0) = \beta K^2.$$

Hence,

$$\sup_{x \in M} F(x,t) \leq C_2 \beta K^3 t + \beta K^2 \leq C_2 \beta a K^2 + \beta K^2 = (aC_2 + 1)\beta K^2 \leq C^2 K^2,$$

for all $t \in [0, \frac{a}{K}]$, where C is a constant depending only on m and a. Concluding, by the definition of F we have

$$|\nabla R| \leq \frac{CK}{\sqrt{t}},$$

for all $t \in (0, \frac{a}{K}]$. This proves the case where $n = 1$. Similarly, we work on the case where $n > 1$ by using the identities

$$[\nabla^n, \Delta]A = \nabla^n \Delta A - \Delta \nabla^n A = \sum_{i=1}^{n} \nabla^i R * \nabla^{n-i} A \tag{3.19}$$

and calculate

$$\nabla_{\partial_t}(\nabla^n R) = \Delta \nabla^n R + \sum_{i=1}^{n} \nabla^i R * \nabla^{n-i} R.$$

Also, the following holds

$$\partial_t |\nabla^n R|^2 = \Delta |\nabla^n R|^2 - 2|\nabla^{n+1} R|^2 + \sum_{i=1}^{n} \nabla^i R * \nabla^{n-i} R * \nabla^n R.$$

In order to control $|\nabla^n R|^2$ we consider the quantity

$$G = t^n |\nabla^n R|^2 + b_m \sum_{i=1}^{n} c_{n,i} t^{n-i} |\nabla^{n-i} R|^2$$

and apply the scalar maximum principle. This completes the proof. □

3.3.2 Convergence of smooth metrics

We need to define a notion of convergence of sequences of metrics, or more general convergence of sections of a certain vector bundle.

Definition 3.3.2. *Let E be a vector bundle over a Riemannian manifold M with Riemannian metric g and connection ∇ on E. Let $U \subset M$ be an open set with $\overline{U}$ a compact set in M and let (ξ_i) be a sequence of sections of E. For each $p \geq 0$ we say that ξ_k* converges in C^p to $\xi \in \Gamma(E|_{\overline{U}})$ *if for every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ such that*

$$\sup_{0\leq a\leq p} \sup_{x\in\overline{U}} |\nabla^a(\xi_k - \xi)|_g < \varepsilon.$$

whenever $k > k_0$. Moreover, we say that ξ_k converges in C^∞ to ξ on $\overline{U}$ *if ξ_k converges in C^p to ξ on $\overline{U}$ for every $p \in \mathbb{N}$.*

We write $g_2 \geq g_1$ when $g_2 - g_1$ is non-negative definite.

Proposition 3.3.3. *Let M be a compact manifold and $\{g_t\}_{t\in(0,T]}$ be a smooth one-parameter family of metrics on M. If there exists a constant $K < \infty$ such that*

$$\int_0^T \Big|\frac{\partial}{\partial t} g(x,t)\Big|_{g_t} dt \leq K,$$

for all $x \in M$, then

$$e^{-C} g_{(x,0)} \leq g_{(x,t)} \leq e^C g_{(x,0)},$$

for all $(x,t) \in M \times [0,T)$. Moreover, the metrics g_t converge uniformly to a continuous metric g_T as $t \to T$.

Proof : Fix $(x,t) \in M \times [0,T)$ and $X \in T_xM$. Then,

$$\begin{aligned}
\Big|\log\Big(\frac{g_{(x,t)}(X,X)}{g_{(x,0)}(X,X)}\Big)dt\Big| &= \Big|\int_0^t \partial_t(\log g_{(x,t)}(X,X))dt\Big| \\
&= \Big|\int_0^t \frac{1}{g_{(x,t)}(X,X)}\partial_t g_{(x,t)}(X,X)dt\Big| \\
&= \Big|\int_0^t \frac{1}{|X|^2_{g(x,t)}}\partial_t g_{(x,t)}(X,X)dt\Big| \\
&\leq \int_0^t \Big|\partial_t g_{(x,t)}\Big(\frac{X}{|X|_{g_{(x,t)}}}, \frac{X}{|X|_{g_{(x,t)}}}\Big)\Big|dt \\
&\leq \int_0^t \Big|\partial_t g_{(x,t)}\Big|_{g(x,t)} \\
&\leq K,
\end{aligned}$$

since

$$|T(X,X)| \leq |T|_g = \sup_{\substack{Y,Z\in T_xM \\ |Y|_g=|Z|_g=1}} |T(Y,Z)|,$$

for all unit vectors $X \in T_xM$ and 2-tensors T. By considering the exponential of the inequality we obtain

$$e^{-C} g_{(x,0)}(X,X) \leq g_{(x,t)}(X,X) \leq e^{C} g_{(x,0)}(X,X).$$

Since X is arbitrary we have

$$e^{-C} g_{(x,0)} \leq g_{(x,t)} \leq e^{C} g_{(x,0)}. \tag{3.20}$$

This shows that the metrics $\{g_t\}_{t\in[0,T)}$ are equivalent. Hence,

$$\int_0^T \Big|\frac{\partial}{\partial t} g(x,t)\Big|_{g_0} dt \leq K', \tag{3.21}$$

where K' a positive constant. Note that now the norm is taken with respect to a constant metric g_0 rather that the time-dependent g_t. Define now

$$g(x,T) = g(x,0) + \int_0^T \frac{\partial}{\partial t} g(x,t) dt.$$

The definition is well since by (3.21) $\frac{\partial}{\partial t} g(x,t)$ is absolutely integrable with respect to the metric g_0. Hence, using the fundamental theorem of calculus we get

$$|g(x,T) - g(x,t)|_{g_0} = \Big| \int_t^T \frac{\partial}{\partial t} g(x,t) dt \Big|_{g_0} \leq \int_t^T \Big| \frac{\partial}{\partial t} g(x,t) \Big|_{g_0} dt.$$

Then,

$$\lim_{t\to T} |g(x,T) - g(x,t)|_{g_0} \leq \lim_{t\to T} \int_t^T \Big| \frac{\partial}{\partial t} g(x,t) \Big|_{g_0} dt = 0,$$

for all $x \in M$. Since M is compact the convergence is uniform. Hence, $g_t \to g_T$ uniformly as $t \to T$. Since g_t are continuous we can conclude that g_T is continuous. By taking the limit of the equation (3.20) as $t \to T$ we conclude that g_T is positive definite. Thus, the metrics g_t converge to a continuous Riemannian metric g_T. This completes the proof. □

Lemma 3.3.4. *Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection associated with the metric g. Also, let $\widetilde{\nabla}$ be a torsion free connection on M. Then,*

$$\nabla_X Y - \widetilde{\nabla}_X Y = \Gamma(X,Y),$$

where

$$2g(\Gamma(X,Y), Z) = (\widetilde{\nabla}_X g)(Y,Z) + (\widetilde{\nabla}_Y g)(X,Z) - (\widetilde{\nabla}_Z g)(X,Y).$$

Moreover, it holds

$$\nabla(\partial_t g_t) - \widetilde{\nabla}(\partial_t g_t) = \widetilde{\nabla} g_t * \partial_t g_t.$$

Proof. By Koszul's formula (1.6) and since $\widetilde{\nabla}$ is torsion-free we have

$$\begin{aligned}2g\big(\nabla_Y X, Z\big) &= X\big(g(Y,Z)\big) + Y\big(g(X,Z)\big) - Z\big(g(X,Y)\big)\\ &\quad + g\big([X,Y],Z\big) - g\big([X,Z],Y\big) - g\big([Y,Z],X\big)\\ &= (\widetilde{\nabla}_X g)(Y,Z) + (\widetilde{\nabla}_Y g)(X,Z) - (\widetilde{\nabla}_Z g)(X,Y)\\ &\quad + 2g(\widetilde{\nabla}_X Y, Z)\end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. This completes the proof. $\square$

Lemma 3.3.5. *Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection associated with the metric g. Also, let $\widetilde{\nabla}$ be a fixed background connection on M which is torsion free. Then, given an integer $n \geq 1$ we have,*

$$\nabla^n \partial_t g_t - \widetilde{\nabla}^n \partial_t g_t = \sum_{l=0}^{m-1} \sum_{i_1+\ldots+i_q=m-l} \widetilde{\nabla}^{i_1} g_t * \cdots * \widetilde{\nabla}^{i_q} g_t * \widetilde{\nabla}^l \partial_t g_t.$$

Proof. We work by induction on n. When $n = 1$ we have by Lemma 3.3.4

$$\nabla(\partial_t g_t) - \widetilde{\nabla}(\partial_t g_t) = \widetilde{\nabla} g_t * \partial_t g_t.$$

Suppose that $n \geq 2$ and

$$\begin{aligned}&\nabla^{n-1}(\partial_t g_t) - \widetilde{\nabla}^{n-1}(\partial_t g_t)\\ &= \sum_{l=0}^{m-2} \sum_{i_1+\ldots+i_q=m-l-1} \widetilde{\nabla}^{i_1} g_t * \cdots * \widetilde{\nabla}^{i_q} g_t * \widetilde{\nabla}^l \partial_t g_t.\end{aligned} \tag{3.22}$$

Hence,

$$\begin{aligned}&\nabla\nabla^{n-1}(\partial_t g_t) - \nabla\widetilde{\nabla}^{n-1}(\partial_t g_t)\\ &= \sum_{l=0}^{m-2} \sum_{i_1+\ldots+i_q=m-l-1} \widetilde{\nabla}^{i_1} g_t * \cdots * \widetilde{\nabla}^{i_q} g_t * \nabla\widetilde{\nabla}^l \partial_t g_t\\ &+ \sum_{l=0}^{m-2} \sum_{i_1+\ldots+i_q=m-l-1} \widetilde{\nabla}^{i_1} g_t * \cdots * \nabla\widetilde{\nabla}^{i_q} g_t * \widetilde{\nabla}^l \partial_t g_t.\end{aligned} \tag{3.23}$$

By Lemma 3.3.4 we can deduce

$$\nabla\widetilde{\nabla}^l \partial_t g_t = \widetilde{\nabla}\widetilde{\nabla}^l \partial_t g_t + \widetilde{\nabla} g_t * \widetilde{\nabla}^l \partial_t g(t)$$

and

$$\nabla\widetilde{\nabla}^j g_t = \widetilde{\nabla}\widetilde{\nabla}^j g_t + \widetilde{\nabla} g_t * \widetilde{\nabla}^j g_t.$$

Therefore, (3.23) becomes

$$\nabla^n \partial_t g - \widetilde{\nabla}^n \partial_t g = \sum_{l=0}^{n-1} \sum_{i_1+\cdots+i_q=m-l} \widetilde{\nabla}^{i_1} g_t * \cdots * \widetilde{\nabla}^{i_q} g_t * \widetilde{\nabla}^l \partial_t g_t.$$

This completes the proof. □

Let $\widetilde{\nabla}$ be the Levi-Civita connection associated with the metric g_0. For every integer n we define the continuous functions

$$u_n\colon [0,T) \to \mathbb{R} \quad \text{given by} \quad u_n(t) = \sup_{x\in M} |\nabla^n \partial_t g(x,t)|_{g(x,t)}$$

and

$$\tilde{u}_n\colon [0,T) \to \mathbb{R} \quad \text{given by} \quad \tilde{u}_n(t) = \sup_{x\in M} |\widetilde{\nabla}^n \partial_t g(x,t)|_{g_0},$$

for each $t \in [0,T)$. Since M is compact we can write

$$\widetilde{\nabla}^n \int_0^t \partial_\tau g(\tau) d\tau = \int_0^t \widetilde{\nabla}^n \partial_\tau g(\tau) d\tau,$$

for all $t \in [0,T)$ and hence,

$$\sup_{x\in M} |\widetilde{\nabla}^n g_t|_{g_0} \leq \int_0^t \tilde{u}_n(\tau) d\tau \tag{3.24}$$

for all $t \in [0,T)$.

Proposition 3.3.6. *Assume that $\int_0^T u_n(t)dt < \infty$ for $m \in \mathbb{N}$. Then, $\int_0^T \tilde{u}_n(t)dt < \infty$ for $n \geq 1$.*

Proof. We work by induction on n. Fix $n \geq 1$ and suppose that $\int_0^T \tilde{u}_l(t)dt < \infty$ for $1 \leq l \leq n-1$. By (3.24) we have

$$\sup_{(x,t)\in M\times[0,T)} |\widetilde{\nabla}^l g_t|_{g_0} < \infty \tag{3.25}$$

for $1 \leq l \leq n-1$. We have already shown in Proposition 3.3.3 that the metrics g_0 and g_t are uniformly equivalent for all $t \in [0,T)$. Using Lemma 3.3.5 we have

$$\begin{aligned} &|\widetilde{\nabla}^n \partial_t g_t|_{g_0} - |\nabla^n \partial_t g_t|_{g_0} \\ &\leq C_1 \sum_{l=0}^{n-1} \sum_{i_1+\ldots+i_1=n-l} |\widetilde{\nabla}^{i_1} g_t|_{g_0} \cdots |\widetilde{\nabla}^{i_q} g_t|_{g_0} |\widetilde{\nabla}^l \partial_t g_t|_{g_0}, \end{aligned}$$

where C_1 is a positive constant. Then, due to (3.25) we obtain

$$|\widetilde{\nabla}^n \partial_t g_t|_{g_0} \leq |\nabla^n \partial_t g_t|_{g_0} + C_2 \sum_{l=1}^{n-1} |\widetilde{\nabla}^l \partial_t g_t|_{g_0} \\ + C_2(1 + |\widetilde{\nabla}^n g_t|_{g_0})|\partial_t g_t|_{g_0}$$

and again by the equivalence of the metrics we get

$$|\widetilde{\nabla}^n \partial_t g_t|_{g_0} \leq C_3 |\nabla^n \partial_t g_t|_{g_t} + C_2 \sum_{l=1}^{n-1} |\widetilde{\nabla}^l \partial_t g_t|_{g_0} \\ + C_2 C_3 (1 + |\widetilde{\nabla}^n g_t|_{g_0})|\partial_t g_t|_{g_t}.$$

By (3.24) we have

$$\tilde{u}_n(t) \leq C_3 u_n(t) + C_2 \sum_{l=1}^{n-1} \tilde{u}_l(t) + C_2 C_3 \Big(1 + \int_0^t \tilde{u}_n(\tau) d\tau\Big) u_0(t),$$

for all $t \in [0, T)$. Then, we can conclude that

$$\frac{d}{dt} \log \Big(1 + \int_0^t \tilde{u}_n(\tau) d\tau\Big) \leq C_3 u_n(t) \frac{1}{1 + \int_0^t \tilde{u}_n(\tau) d\tau} \\ + C_2 \frac{1}{1 + \int_0^t \tilde{u}_n(\tau) d\tau} \sum_{l=1}^{n-1} \tilde{u}_l(t) + C_2 C_3 u_0(t) \\ \leq C_3 u_n(t) + C_2 \sum_{l=1}^{n-1} \tilde{u}_l(t) + C_2 C_3 u_0(t),$$

for all $t \in [0, T)$. By assumption we have $\int_0^T u_n(t) dt < \infty$ for every $n \in \mathbb{N}$ and by the induction hypothesis we have that $\int_0^T \tilde{u}_l(t) dt < \infty$ for every $1 \leq l \leq n-1$. Thus,

$$\int_0^t \tilde{u}_n(\tau) d\tau < 0.$$

This completes the proof. □

Proposition 3.3.7. *The metric g_t in Proposition 3.3.3 is smooth and the metrics g_t converge in C^∞ to g_t as t $\to T$.*

Proof. By Proposition 3.3.6 we have that $\int_0^T \tilde{u}_n(t)dt$ hence we define

$$g(x,T) = g(x,t) - \int_t^T \partial_t g(x,\tau)d\tau$$

and we have

$$|\nabla^n g(x,T) - \nabla^n g(x,t)| \leq \left| \int_t^T \tilde{u}_n(\tau)d\tau \right| \to 0, \tag{3.26}$$

for all $x \in M$ as $t \to T$. This completes the proof. □

3.3.3 Curvature blow up at finite time

Theorem 3.3.8. *Let M be a m-dimensional compact Riemannian manifold and let $\{g_t\}_{t\in[0,T)}$ be the maximal solution to the Ricci flow on M. Furthermore, suppose that $T < \infty$. Then,*

$$\lim_{t\to T}(\sup_{x\in M} |R(x,t)|) = \infty.$$

Proof. Suppose by contradiction that there is a constant $K > 0$ such that $|R(x,t)| \leq K$. Then, it follows by Proposition 3.3.7 that g_t converges uniformly in C^∞ to a smooth metric g_T. Since g_T is smooth by Theorem 3.2.7 we know that there exists a unique solution to the Ricci flow $\bar{g}(t)$ with $\bar{g}(0) = g(T)$ on $[0,\bar{\varepsilon}], \bar{\varepsilon} > 0$. Thus, define

$$\tilde{g}(t) = \begin{cases} g(t), & t \in [0,T), \\ \bar{g}(t-T), & t \in [T, T+\bar{\varepsilon}) \end{cases}$$

and then $\tilde{g}_t$ is a solution to the Ricci flow with $\tilde{g}(0) = g_0$. This contradicts the definition of T and completes the proof. □

CHAPTER 4

THE VECTORIAL MAXIMUM PRINCIPLE

The maximum principle is one of the most useful tools in geometric analysis. In this chapter, we present the maximum principle for scalar smooth functions on a manifold M and in the second part we generalize the principle for sections of a vector bundle. This generalization was firstly made by Hamilton in [17]. In particular, we present a generalized version of Hamilton's maximum principle, based on [4].

4.1 Scalar maximum principle

In this section, we assume that M is a compact manifold endowed with a continuous time-dependent family of Riemannian metrics $\{g_t\}_{t\in[0,T)}$. Consider the parabolic semi-linear operator $\mathcal{L}$, given by

$$\mathcal{L}u = u_t - \Delta u - g(X, \nabla u) - \Psi(u,t),$$

where $u\colon M\times[0,T)\to\mathbb{R}$ is a smooth function, X is a time-dependent continuous vector field, Δ is the Laplacian with respect to g_t and $\Psi(x,t)\colon \mathbb{R}\times[0,T)\to\mathbb{R}$ is a map locally Lipschitz in the first variable and continuous in the second.

Proposition 4.1.1. *Let $u\colon M\times[0,T)\to\mathbb{R}$ be a smooth function, satisfying*

$$\partial_t u - \Delta u \geq 0. \tag{4.1}$$

If $u(x,0)\geq c$ for all $x\in M$ for some $c\in\mathbb{R}$ then $u(x,t)\geq c$ for all $(x,t)\in M\times[0,T)$.

Proof. Fix $\varepsilon>0$. Define an auxiliary function $u_\varepsilon\colon M\times[0,T)\to\mathbb{R}$ given by

$$u_\varepsilon(x,t) = u(x,t) + \varepsilon(1+t).$$

By hypothesis,

$$u_\varepsilon(x,0) = u(x,0)+\varepsilon \geq c+\varepsilon > c.$$

Suppose that there exists an $\varepsilon>0$ such that

$$u_\varepsilon(x,t)\leq c,$$

for all $(x, t) \in M \times [0, T)$. Since M is compact, there exists a point $(x', t') \in M \times [0, T)$ such that

$$u_\varepsilon(x', t') = c \qquad \text{and} \qquad u_\varepsilon(x, t) \leq c,$$

for all $(x, t) \in M \times [0, t']$. Thus, at (x', t') it holds

$$\partial_t u_\varepsilon(x', t') \leq 0 \qquad \text{and} \qquad \Delta u_\varepsilon(x', t') \geq 0.$$

Therefore, using (4.1)

$$0 \geq \partial_t u_\varepsilon - \Delta u_\varepsilon = \partial_t u + \varepsilon - \Delta u \geq \varepsilon > 0,$$

which is a contradiction. Hence, $u_\varepsilon(x, t) > c$ for all $(x, t) \in M \times [0, T)$ and since $\varepsilon > 0$ is arbitrary the result follows. ☐

Proposition 4.1.2. *Let* $u, v \colon M \times [0, T) \to \mathbb{R}$ *be smooth functions, satisfying the differential inequality*

$$\mathcal{L}v \leq \mathcal{L}u,$$

on $M \times [0, T)$ *and* $v(x, 0) \leq u(x, 0)$ *for all* $x \in M$. *Then,*

$$v(x, t) \leq u(x, t),$$

for all $(x, t) \in M \times [0, T)$.

Proof. Consider the smooth function $w \colon M \times [0, T)$ given by $w = u - v$. Then, by hypothesis we have

$$0 \leq \partial_t w - \Delta w - g(X, \nabla w) - \big(\Psi(u, t) - \Psi(v, t)\big).$$

Since Ψ is locally Lipschitz in the first variable, there exists a constant $c > 0$ such that

$$\big|\Psi(u(x, t), t) - \Psi(v(x, t), t)\big| \leq c|u(x, t) - v(x, t)| = c|w(x, t)|,$$

for all $(x, t) \in M \times [0, t_1]$ where $t_1 < T$. Let $\varepsilon > 0$ and define the auxiliary function w_ε by

$$w_\varepsilon(x, t) = w(x, t) + \varepsilon e^{2ct}.$$

Note that at $(x, 0)$ for all $x \in M$ it holds

$$w_\varepsilon(x, 0) = w(x, 0) + \varepsilon = u(x, 0) - v(x, 0) + \varepsilon > 0.$$

On the other hand,

$$\mathcal{L}w_\varepsilon = \mathcal{L}w + \varepsilon \mathcal{L}(e^{2ct}) = \mathcal{L}w + 2c\varepsilon e^{2ct} \geq 2c\varepsilon e^{2ct}.$$

Hence,

$$\partial_t w_\varepsilon \geq \Delta w + g(X, \nabla w) - c|w| + 2c\varepsilon e^{2Ct}.$$

Let $(x', t') \in M \times (0, T)$ be the first point and time that $w_\varepsilon(x', t') = 0$. Then,

$$w(x', t') = -\varepsilon e^{2ct'}.$$

Moreover, since this is a local minimum for the spatial derivatives it holds

$$\nabla w_\varepsilon(x', t') = \nabla w(x', t') = 0 \qquad \text{and} \qquad \Delta w_\varepsilon(x', t') = \Delta w(x', t') \geq 0.$$

On the other hand, the time derivative

$$\partial_t w(x', t') + 2c\varepsilon e^{2ct'} = \partial_t w_\varepsilon(x', t') \leq 0.$$

Thus, at $(x', t') \in M \times [0, T)$ we obtain

$$0 \geq \partial_t w_\varepsilon \geq \Delta w + g(X, \nabla w) - c\varepsilon e^{2ct'} + 2c\varepsilon e^{2ct'} \geq c\varepsilon e^{2ct'} > 0,$$

which is a contradiction. Hence, for every $\varepsilon > 0$ we have $w_\varepsilon > 0$ which implies $w \geq 0$ on $M \times [0, t_1]$. Since $t_1 \in (0, T)$ is arbitrary we have $w = u - v \geq 0$ on $M \times [0, T)$. This completes the proof. □

Theorem 4.1.3 (Comparison Principle)**.** *Let $u\colon M \times [0, T) \to \mathbb{R}$ be a smooth function which satisfies the differential inequality*

$$\partial_t u - \Delta u \leq g(X, \nabla u) + \Psi(u, t).$$

Let ϕ be the solution to the associated ODE

$$\begin{cases} \phi'(t) = \Psi(\phi(t), t), \\ \phi(0) = \max_{x \in M} u(x, 0). \end{cases} \tag{4.2}$$

Then, the solution u of the partial inequality is bounded from below by the solution ϕ of the ODE, that is

$$u(x, t) \leq \phi(t),$$

for all $(x, t) \in M \times [0, T)$.

Proof. The proof is an immediate consequence of the Proposition 4.1.2. □

4.2 Hamilton's maximum principle

In this section, let (E, π, M) be a vector bundle of rank d over a compact smooth m-dimensional manifold M equipped with a fixed metric k on the fibers $E_x = \pi^{-1}(x), x \in M$. Let $\{g_t\}_{t\in[0,T)}$ be a family of time-dependent Riemannian metrics on M and let $\{\nabla^{g_t}\}_{t\in[0,T)}$ the corresponding Levi-Civita connections on M. Furthermore, let $\{\nabla^t\}_{t\in[0,T)}$ denote a family of connections on E compatible with k. For a section $\phi\colon M \to E$ of the vector bundle we can define a new section $\Delta\phi\colon M \to E$ using the connections ∇^{g_t} and ∇^t. Suppose that a time-dependent section $\phi(\cdot, t) \in \Gamma(E)$ satisfies the parabolic equation

$$\partial_t \phi(x,t) = (\Delta\phi)(x,t) + f(\phi(x,t)) \tag{4.3}$$

where $f\colon E \to E$ a locally Lipschitz map, mapping each fiber E_x to itself.

The Hamilton's maximum principle provides us, roughly speaking, that the behavior of the PDE (4.3) can be described by the behavior of the ODE

$$\frac{d}{dt}\phi(x,t) = f(\phi(x,t)) \tag{4.4}$$

in the fibers $E_x, x \in M$.

Definition 4.2.1. *A subset $C \subset \mathbb{R}^n$ is called* convex *if for each pair of points the segment that connects them lies within the set, that is, for every $x, y \in C$ we have,*

$$[x,y] = \{(1-t)x + ty \mid 0 \le t \le 1\} \subset C.$$

The set C is called strictly convex *if for every $x, y \in C$ the above segment is contained in the interior of C.*

Definition 4.2.2. *Let $C \subset \mathbb{R}^n$ be a closed and convex set.*

(1) *A* supporting half-space of the set C *is a half-space of* $\mathbb{R}^n$ *which contains C and has points of C arbitrarily close to its boundary.*

(2) *A* supporting hyperplane to C *is a hyperplane which is the boundary of a supporting half-space to C.*

(3) *The* tangent cone $T_{x_0}C$ of C at $x_0 \in \partial C$ *is the intersection of the supporting half-spaces of C that are arbitrarily close to x_0.*

Definition 4.2.3. *Let $C \subset \mathbb{R}^n$ be a closed and convex set and $x_0 \in \partial C$. Then,*

(1) *A non-zero vector ξ is called* normal vector of ∂C at x_0, *if ξ is normal to a supporting hyperplane of C passing through x_0. This normal vector is called* inward normal *if it points into the half-space containing C.*

(2) *A non-zero vector η is called* inward pointing at x_0, *if*

$$\langle \xi, \eta \rangle \geq 0,$$

for each inward normal vector ξ at x_0.

Definition 4.2.4. *Suppose that (E, π, M) is a vector bundle and C a closed subset of E.*

(1) *The set C is called* fiber-convex (or convex in the fiber) *if for every $x \in M$ the set $C_x = C \cap E_x$ is a convex subset of the fiber E_x.*

(2) *The set C is called* invariant under parallel transport by the connection ∇_t *if for every curve $c\colon [0,b] \to M \times \mathbb{R}$ and any vector $V_0 \in C_{c(0)}$, the unique parallel section $V(t) \in E_{c(t)}, t \in [0,b]$ along $c(t)$ with $V(0) = V_0$ is contained in C.*

In order to prove the Hamilton's maximum principle we need the following result.

Lemma 4.2.5. *Let $C \subset E$ be a closed, convex in the fiber and invariant under parallel transport with respect to ∇^t subset of E. If φ is a smooth section mapped in C then, for all $x \in M$ and $v \in T_xM$, the Hessian*

$$\nabla^2_{v,v}\varphi = \nabla_v \nabla_v \varphi - \nabla_{\nabla_v v}\varphi,$$

belongs into the tangent cone T_xC_x of C_x at the point $\varphi(x)$.

Proof. It suffices to prove the result in the case where there exists a point x_0 which is mapped via φ is in the boundary of C, since otherwise the result is trivially true.
Consider a unit vector $v \in T_{x_0}M$ and a normal coordinate system $\{x_i\}$ in an open neighborhood U around a point x_0 such that $\partial_{x_1}|_{x_0} = v$. Moreover, choose a local basis $\{\varphi_1(x_0), \ldots, \varphi_k(x_0)\}$ of E_{x_0} and extend it into a local geodesic orthonormal frame field. Then,

$$\varphi = \sum_{i=1}^{k} u_i \varphi_i \ ,$$

where the components $u_i\colon U \to \mathbb{R}$, $i \in \{1\,, \ldots, k\}$, are smooth functions. We calculate,

$$\begin{aligned}
\nabla^2_{v,v}\varphi(x_0) &= \nabla_{\partial_{x_1}}\nabla_{\partial_{x_1}}\varphi(x_0) - \nabla_{\nabla_{\partial_{x_1}}\partial_{x_1}}\varphi(x_0) \\
&= \nabla_{\partial_{x_1}}\sum_{i=1}^{k}\nabla_{\partial_{x_1}}u_i(x_0)\varphi_i(x_0) \\
&= \sum_{i=1}^{k}\nabla_{\partial_{x_1}}(\partial_{x_1}u_i(x_0))\varphi_i(x_0) \\
&= \sum_{i=1}^{k}(\partial_{x_1}\partial_{x_1}u_i)(x_0)\varphi_i(x_0) \\
&= \sum_{i=1}^{k}(u_i\circ\gamma)''(0)\varphi_i(x_0),
\end{aligned}$$

where $\gamma\colon(-\varepsilon,\varepsilon)\to U\times\mathbb{R}$ is a length minimizing geodesic such that

$$\gamma(0) = x_0 \quad \text{and} \quad \gamma'(0) = \partial_{x_1}x_0.$$

Define now the set

$$\mathcal{C} = \Big\{(y_1\,, \ldots, y_k) \in \mathbb{R}^k : \sum\nolimits_{i=1}^{k} y_i\varphi_i|_{x_0} \in C_{x_0}\Big\}.$$

Clearly $\mathcal{C}$ is a closed and convex subset of $\mathbb{R}^k$. Since $\varphi\in C$ and C is invariant under parallel transport, we deduce that the curve $\sigma\colon(-\varepsilon,\varepsilon)\to\mathbb{R}^k$, given by

$$\sigma = (u_1\circ\gamma\,, \ldots, u_k\circ\gamma),$$

lies in $\mathcal{C}$. It suffices to prove that $\sigma''(0)$ points into $\mathcal{C}$. Indeed, because $\mathcal{C}$ is convex, for each unit inward pointing normal ξ of $\mathcal{C}$ at $\sigma(0)$, we have

$$g(s) = \langle\xi, \sigma(s)-\sigma(0)\rangle \geq 0,$$

for all $s\in(-\varepsilon,\varepsilon)$. Because g attains its minimum at $s=0$, from standard calculus we get that $g''(0)\geq 0$, which implies $\langle\sigma''(0),\xi\rangle\geq 0$. This completes the proof. □

Definition 4.2.6. *Consider the family of closed and convex sets $\{C(t)\}_{t\geq 0}\subset E$.*

(1) *We say that the sets $C(t)$* depend continuously on t *if* $\lim_{t\to t_0} C(t) = C(t_0)$ *with respect to the pointed Hausdorff topology.*

(2) *We say that the family* $\{C(t)\}$ *is* invariant under the ODE (4.4) *if for every* $t_0 \geq 0, x \in M$ *and* $\phi_0 \in C_x(t_0)$ *the solution* $\phi(x,t)$ *of* (4.4) *with* $\phi(x,t_0) = \phi_0$ *satisfies* $\phi(x,t) \in C_x(t)$ *for all* $t \geq t_0$*, for which the solution* $\phi(x,t)$ *exists.*

Theorem 4.2.7 (Hamilton's vectorial maximum principle)**.** *For* $t \in [0,\delta], \delta > 0$ *let* $C(t) \subset E$ *be a closed subset depending continuously on* t*. Suppose that each of the sets* $C(t)$ *is invariant under parallel transport, convex in the fiber and that the family* $\{C(t)\}_{t\in[0,\delta]}$ *is invariant under the ODE* (4.4)*. Then, for every solution* $\phi(x,t) \in \Gamma(E)$ *on* $M \times [0,\delta]$ *of the parabolic equation* (4.3) *with* $\phi(x,t) \in C(0)$ *for all* $x \in M$ *we have* $\phi(x,t) \in C(t)$ *for all* $(x,t) \in M \times [0,\delta]$*.*

Proof. For each $S \in E_y, y \in M$ we let

$$r_t(S) = d_k(S, C_t(y)),$$

denote the distance between S and the convex set $C_y(t)$ in the fiber V_y. For each solution $\phi(x,t)$ to the parabolic equation (4.3), defined on $M \times [0,\delta]$ we consider the maximal distance to $C(t)$,

$$s(t) = \sup_{x\in M} r_t(\phi(x,t)).$$

The function s is not differentiable but we can define the upper converse Dini derivative

$$s'(t_0) = \limsup_{h\searrow 0} \frac{s(t_0) - s(t_0 - h)}{h}.$$

Let r_0 denote the maximum of s on $[0,\delta]$. Since f is locally Lipschitz, we can find a constant $L > 0$ such that the restriction of f to the ball $B_{2r_0}(\phi(y,t))$ is $L/2$-Lipschitz continuous for all $(y,t) \in M \times [0,\delta]$. Our goal is to show that

$$s'(t) \leq Ls(t),$$

for all $t \in [0,\delta]$. Then, if we define

$$g(t) = s(t)e^{-Lt},$$

by (4.2) we get that

$$g'(t) = e^{-Lt}(s'(t) - Ls(t)) \leq 0,$$

for all $t \in [0,\delta]$. Since $s(0) = 0$ we deduce $g(0) = 0$ which implies $g(t) \leq 0$ for all $t \in [0,\delta]$ and hence $s(t) = 0$ for all $t \in [0,\delta]$. This proves the theorem.

Thus, it remains to compute $s'(t)$. For $t_0 \in [0,\delta]$ there exists $x_0 \in M$ with $s(t_0) = r_{t_0}(\phi(x_0,t_0))$. We assume that $s(t_0) > 0$. For $h > 0$ by the definition of s we have

$$s(t_0 - h) \geq r_{t_0-h}(\phi(x_0, t_0 - h)).$$

Hence, we compute

$$\begin{aligned} s'(t_0) &= \limsup_{h\searrow 0} \frac{s(t_0)-s(t_0-h)}{h} \\ &\leq \limsup_{h\searrow 0} \frac{r_{t_0}(\phi(x_0,t_0)) - r_{t_0-h}(\phi(x_0,t_0-h))}{h} \\ &= \limsup_{h\searrow 0} \frac{r_{t_0}(\phi(x_0,t_0)) - r_{t_0-h}(\phi(x_0,t_0) - h\Delta\phi(x_0,t_0) - hf(\phi(x_0,t_0)))}{h}. \end{aligned}$$

The last equality follows as we write $\phi(x_0, t_0 - h) = \phi(x_0, t_0) - h \cdot \frac{d}{dt}\phi(x_0, t) + o(h)$ and use the fact that r_{t_0-h} are uniformly Lipschitz continuous functions as distance functions.
One can prove that for each t the function r_t is C^1 on $E \setminus C(t)$ since $C(t)$ is convex. We observe that the closed neighborhood $r^{-1}([0, s(t_0)])$ of C_{t_0} is convex. Also, by construction $\phi(x, t_0) \in E \setminus C(t)$ for all $x \in M$. Then, $\nabla r_{t_0}(\phi(x_0, t_0))$ is equal to the unit outward normal to $r^{-1}([0, s(t_0)])$ at $\phi(x_0, t_0)$. Thus by Lemma 4.2.5 we deduce that

$$k(\Delta\phi(x_0,t_0), \nabla r_{t_0}(\phi(x_0,t_0))) \leq 0.$$

Thus,

$$\begin{aligned} s'(t_0) &\leq \limsup_{h\searrow 0} \frac{r_{t_0}(\phi(x_0,t_0)) - r_{t_0-h}(\phi(x_0,t_0) - h\Delta\phi(x_0,t_0) - hf(\phi(x_0,t_0)))}{h} \\ &\leq \limsup_{h\searrow 0} \frac{r_{t_0}(\phi(x_0,t_0)) - r_{t_0-h}(\phi(x_0,t_0) - hf(\phi(x_0,t_0)))}{h}. \end{aligned}$$

Since $C(t)$ is continuous with respect to the pointed Hausdorff topology, we can see that ∇r_t is also continuous with respect to t. Thus, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$k(\Delta\phi(x_0,t_0), \nabla r_{t-h}(\phi(x_0,t_0))) \leq \varepsilon,$$

for all (ϕ, h) with $|\phi(x_0, t_0) - \phi| + |h| \leq \delta$. We know that for every convex function it holds

$$r_{t_0-h}(\phi(x_0,t_0) - hf(\phi(x_0,t_0)) - h\Delta\phi(x_0,t_0)) \geq -\varepsilon h + r_{t_0-h}(\phi(x_0,t_0) - hf(\phi(x_0,t_0))),$$

for small $h > 0$. Since ε is arbitrary we have

$$r_{t_0-h}(\phi(x_0,t_0) - hf(\phi(x_0,t_0)) - h\Delta\phi(x_0,t_0)) \geq r_{t_0-h}(\phi(x_0,t_0) - hf(\phi(x_0,t_0))).$$

For $h > 0$ we choose a unique $\phi_h \in C_{x_0}(t_0 - h)$ with

$$r_{t_0-h}(\phi(x_0,t_0) - hf(\phi(x_0,t_0))) = |\phi_h + hf(\phi(x_0,t_0)) - \phi(x_0,t_0)|.$$

Using the triangle inequality and the fact that f is $L/2$-Lipschitz continuous on $B_{2s(t_0)}(\phi(x_0,t_0))$ we have

$$
\begin{aligned}
& r_{t_0-h}(\phi(x_0,t_0) - hf(\phi(x_0,t_0))) - r_{t_0}(\phi(x_0,t_0)) \\
&\quad = |\phi_h + hf(\phi(x_0,t_0)) - \phi(x_0,t_0)| - r_{t_0}(\phi(x_0,t_0)) \\
&\quad \geq |\phi_h + hf(\phi_h) - \phi(x_0,t_0)| - h|f(\phi(x_0,t_0)) - f(\phi_h)| - d_k(\phi(x_0,t_0), C_{t_0}) \\
&\quad \geq d_k(\phi_h + hf(\phi_h), \phi(x_0,t_0)) - h\frac{L}{2}|\phi(x_0,t_0) - \phi_h| - d_k(\phi(x_0,t_0), C_{t_0}) \\
&\quad \geq d_k(\phi_h + hf(\phi_h), \phi(x_0,t_0)) - hLs(t_0) - d_k(\phi(x_0,t_0), C_{t_0}) \\
&\quad \geq -d_k(\phi_h + hf(\phi_h), C(t_0)) - hLs(t_0).
\end{aligned}
\tag{4.5}
$$

The term $\phi_h + hf(\phi_h)$ approximates the solution γ_{ϕ_h} with $\gamma_{\phi_h}(t_0 - h) = \phi_h \in C_{x_0}(t_0 - h)$ up to first order. Since the family $C(t)$ is invariant under the ODE (4.4) we have

$$\gamma_{\phi_h}(t) \in C_{x_0}(t)$$

for all $t \geq t_0 - h$. Hence, we conclude that $d_k(C(t_0), \phi) = o(h)$ and obtain by (4.5)

$$s'(t_0) = Ls(t_0).$$

This completes the proof. □

4.3 The maximum principle for 2-tensors

Lemma 4.3.1. *Let E be a vector bundle over a Riemannian manifold M and let K be the set of non-negative definite symmetric 2-tensors,*

$$K = \{\theta \in \mathrm{Sym}(E^* \otimes E^*) : \theta \geq 0\}.$$

Then, K is invariant under parallel transport.

Proof. Let $\gamma : [0,1] \to M$ be a geodesic, P_s the parallel transport operator of vectors along γ and Π_s the parallel transport operator of 2-tensors along the curve γ. Consider $\vartheta \in K_{\gamma(0)}$. Then, for each $v \in T_{\gamma(0)}M$, we have

$$\partial_s\big\{\big(\Pi_s\vartheta\big)(P_s v, P_s v)\big\} \;=\; \left(\nabla_{\partial_s}\Pi_s\vartheta\right)(P_s v, P_s v) + 2\Pi_s\theta\left(\nabla_{\partial_s}P_s v, P_s v\right) = 0.$$

Therefore, for each vector $v \in T_{\gamma(0)}M$, it holds $\big(\Pi_s\vartheta\big)(P_s v, P_s v) = \vartheta(v,v)$. Consequently, for each $w \in T_{\gamma(s)}M$, we obtain that

$$\big(\Pi_s\vartheta\big)(w,w) = \vartheta(P_s^{-1}w, P_s^{-1}w) \geq 0.$$

This completes the proof. □

CHAPTER 5

3-MANIFOLDS WITH POSITIVE RICCI CURVATURE

5.1 Statement of the main result

Hamilton's first major achievement using the Ricci flow method was the following result proved in [18]:

Main Theorem: *Let M^3 be an oriented compact 3-dimensional manifold which admits a smooth Riemannian metric with strictly positive Ricci curvature. Then, M^3 also admits a smooth Riemannian metric of constant positive curvature. In particular, if M^3 is simply connected then it is diffeomorphic to $\mathbb{S}^3$.*

Hamilton proved this result starting the Ricci flow process from a metric with strictly positive Ricci curvature. From the maximum principle, it follows that this property is preserved under the flow. The maximal time of existence of the flow is finite and as time is approaching its maximal value, the volume of the manifold decreases to zero and the shape of the evolved manifold becomes spherical. One idea to conclude the proof would be to pass to a limit, but the fact that the manifold is shrinking to a point is preventing us to use this idea. Hamilton overcomes this problem by rescaling properly the Riemannian metric and the time to ensure that the volumes of the evolved manifolds remain constant. Let us describe the main steps of the proof of Hamilton.

Step 1: The Ricci flow exists for finite time and the Riemannian curvature operator of the evolved metrics explodes as time is approaching is maximal time of existence.

Step 2: The positivity of the Ricci curvature is preserved under the Ricci flow process.

Step 3: The sectional curvatures of the evolved metrics get close to each other as time is approaching its maximal time of existence.

Step 4: Rescale time and metric to obtain a solution to the volume preserving Ricci flow

$$\nabla_{\partial_t} g = -2Ric + \frac{2}{3}\frac{\int S\,dM}{\int dM}\cdot g. \tag{5.1}$$

Step 5: The solution of the (5.1) exists for all times and converges to a Riemannian metric of constant sectional curvature.

5.2 Curvature quantities in three dimensions

Let us investigate here the structure of the curvature operator of a three dimensional compact manifold M^3. At this point let us recall a well-known result due to Stiefel [30], which says that each such manifold is parallelizable, that is there exists a smooth globally defined frame field. Recall now that the Riemann curvature tensor is fully determined by the Ricci tensor, i.e. in local coordinates we have

$$R_{ijkl} = g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk} + g_{ik}R_{jl} - \frac{g_{ik}g_{jl} - g_{il}g_{jk}}{2}S, \tag{5.2}$$

where g_{ij}, R_{ijkl} and R_{ij} are the components of the metric, the Riemann curvature tensor and of the Ricci curvature, respectively, with respect to a fixed orthonormal frame. Suppose that $\{e_1, e_2, e_3\}$ is an orthonormal basis at $x \in M^3$ such that $\{e_2 \wedge e_3, e_1 \wedge e_3, e_1 \wedge e_2\}$ diagonalizes the curvature operator $\mathcal{R}$ and denote by $\kappa_1 \geq \kappa_2 \geq \kappa_3$ the corresponding eigenvalues. Then at the point x we have that

$$Ric = \begin{pmatrix} \kappa_2 + \kappa_3 & 0 & 0 \\ 0 & \kappa_1 + \kappa_3 & 0 \\ 0 & 0 & \kappa_1 + \kappa_2 \end{pmatrix}.$$

and $S = 2(\kappa_1 + \kappa_2 + \kappa_3)$, see Lemma 1.6.9. By replacing

$$\kappa_i = \frac{1}{2}\lambda_i, \quad i \in \{1, 2, 3\}$$

we get the following

$$Ric = \frac{1}{2}\begin{pmatrix} \lambda_2 + \lambda_3 & 0 & 0 \\ 0 & \lambda_1 + \lambda_3 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 \end{pmatrix}, \tag{5.3}$$

and $S = \lambda_1 + \lambda_2 + \lambda_3$. Observe that $\lambda_1 + \lambda_2 \geq \lambda_1 + \lambda_3 \geq \lambda_2 + \lambda_3$. Moreover, we define

$$\mathcal{R}^2(X \wedge Y, Z \wedge W) = \sum_{i,j=1}^{m} R(X, Y, e_i, e_j)R(Z, W, e_i, e_j)$$

and

$$\begin{aligned} \mathcal{R}^\#(X \wedge Y, Z \wedge W) = 2 &\sum_{i,j=1}^{m} R(X, e_i, Z, e_j)R(Y, e_i, W, e_j) \\ - 2 &\sum_{i,j=1}^{m} R(X, e_i, W, e_j)R(Y, e_i, Z, e_j). \end{aligned}$$

Note that

$$\mathcal{R}^2 + \mathcal{R}^\# = Q(R),$$

where $Q(R)$ is given in (2.4).

Hence, with respect to the frame $\{e_2 \wedge e_3, e_1 \wedge e_3, e_1 \wedge e_2\}$, the tensors $\mathcal{R}^2$ and $\mathcal{R}^\#$ have the representations

$$\mathcal{R}^2 = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{and} \quad \mathcal{R}^\# = \begin{pmatrix} \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_1\lambda_2 \end{pmatrix}$$

and consequently

$$\mathcal{R}^2 + \mathcal{R}^\# = \begin{pmatrix} \lambda_1^2 + \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_2^2 + \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_3^2 + \lambda_1\lambda_2 \end{pmatrix}. \tag{5.4}$$

In general, [see [1] Section 12.2.1]

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}^\# = \begin{pmatrix} df - e^2 & ce - bf & be - cd \\ ce - bf & af - c^2 & bc - ae \\ be - cd & bc - ae & ad - b^2 \end{pmatrix}. \tag{5.5}$$

Let us investigate now the behavior of the initial value problem

$$\begin{cases} S'(t) &= S^2(t) + S^\#(t), \quad t \in [0, T), \\ S(0) &= S_0, \end{cases} \tag{5.6}$$

on various subsets of the space of symmetric matrices with real coefficients.

Lemma 5.2.1. *Let $\{S = S(t)\} \subset \mathcal{C}_B(\mathbb{R}^3)$ be a solution of the ODE given in (5.6). Then, the components s_{ij} of S satisfy*

$$s'_{ij} = 2s_{ij}^2 - tr(S)s_{ij} - \frac{1}{2}\big(|S|^2 - tr(S)^2\big)\delta_{ij}. \tag{5.7}$$

Consequently, if $S(0)$ is diagonal, then $S(t)$ remains diagonal for all $t \in (0, T)$.

Proof. The expression for s_{ij} follows directly from (5.5). The second statement of the lemma is a consequence of the uniqueness of solutions of initial value problems for ODEs. □

Lemma 5.2.2. *Let $\{S = S(t)\} \subset \mathcal{C}_B(\mathbb{R}^3)$ be a solution of the ODE given in (5.6). Then, the eigenvalues of A are smooth and satisfy*

$$\mu_1' = \mu_1^2 + \mu_2\mu_3, \quad \mu_2' = \mu_2^2 + \mu_1\mu_3 \quad \text{and} \quad \mu_3' = \mu_3^2 + \mu_1\mu_2, \tag{5.8}$$

for every $t \in [0, T)$.

Proof. Assume that $S(0)$ is diagonal and suppose that

$$s_{11}(0) \geq s_{22}(0) \geq s_{33}(0). \tag{5.9}$$

As we have already proved in the last lemma, A remains diagonal for all $t \in [0, T)$. Moreover, according to (5.7) we have

$$s_{11}' = s_{11}^2 + s_{22}s_{33}, \quad s_{22}' = s_{22}^2 + s_{11}s_{33} \quad \text{and} \quad s_{33}' = s_{33}^2 + s_{11}s_{22}.$$

Subtracting, we deduce that

$$(s_{11} - s_{22})' = (s_{11} - s_{22})(s_{11} + s_{22} - s_{33})$$

and

$$(s_{22} - s_{33})' = (s_{22} - s_{33})(s_{22} + s_{33} - s_{11}).$$

From (5.9), we conclude that

$$s_{11}(t) \geq s_{22}(t) \geq s_{33}(t), \quad \text{for all} \quad t \in [0, T).$$

Consequently, we have $\mu_1 = s_{11}$, $\mu_2 = s_{22}$ and $\mu_3 = s_{33}$, from where the assertion follows. □

Lemma 5.2.3. *Consider the eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$ of an element in $\mathcal{C}_B(\mathbb{R}^3)$ as real-valued functions on $\mathcal{C}_B(\mathbb{R}^3)$. Then, μ_1 is a convex function while μ_3 and $\mu_2 + \mu_3$ are concave functions.*

Proof. Fix $S, T \in \mathcal{C}_B(\mathbb{R}^3)$ and let $t \in [0, 1]$. Then,

$$\begin{aligned} \mu_1(tS + (1-t)T) &= \max_{v \in \mathbb{S}^2}(tS(v,v) + (1-t)T(v,v)) \\ &\leq \max_{v \in \mathbb{S}^2}(tS(v,v)) + \max_{v \in \mathbb{S}^2}((1-t)T(v,v)) \\ &= t\mu_1(S) + (1-t)\mu_1(T). \end{aligned}$$

Hence, μ_1 is convex. Now using the fact

$$\mu_3(S) = \min_{v \in \mathbb{S}^2} S(v,v) = -\mu_1(-S),$$

it follows that μ_3 is concave. Moreover, from

$$\mu_2 + \mu_3 = \min_{v,w \in \mathbb{S}^2; v \perp w}(S(v,v) + S(w,w)),$$

it follows that $\mu_2 + \mu_3$ is concave. This completes the proof. □

Lemma 5.2.4. *Fix a real number $\varepsilon > 0$. Then, the set*

$$\mathcal{K} = \big\{S \in \mathcal{C}_B(\mathbb{R}^3)\colon \mu_2(S) + \mu_3(S) \geq \varepsilon\big\},$$

where $\mu_1(S) \geq \mu_2(S) \geq \mu_3(S)$ are the eigenvalues of S, is convex and invariant under the ODE (5.6).

Proof. Let us show at first the invariance of our set under the ODE. Consider the function $f : [0, T) \to \mathbb{R}$ given by $f = \mu_2 + \mu_3$. From (5.8), we have that

$$f' = \mu_2^2 + \mu_3^2 + \mu_1\mu_3 + \mu_1\mu_2 \geq \mu_1 f. \tag{5.10}$$

From our assumptions we have that

$$f(0) = \mu_2(S) + \mu_3(S) \geq \varepsilon \quad \text{and} \quad 2\mu_1(S) \geq \mu_2(S) + \mu_3(S) > 0. \tag{5.11}$$

First we show that f stays positive on $[0, T)$. To show this, suppose to the contrary that there exists a first time $t_0 \in (0, T)$ such that $f(t) > 0$ for each $t \in [0, t_0)$ and $f(t_0) = 0$. By integrating (5.11) we see that

$$f(t_0) \geq f(0)e^{\int_0^{t_0} \mu_1(s)ds} > 0, \tag{5.12}$$

which leads to a contradiction. Hence f is everywhere positive and from the second inequality of (5.11) the function μ_1 is everywhere positive. Going back to (5.12) we see that $f \geq f(0) \geq \varepsilon$.
To show convexity, we consider the function $g : \mathcal{C}_B(\mathbb{R}^3) \to \mathbb{R}$ given by

$$g(S) = \mu_2(S) + \mu_3(S) = \min_{v,w\in\mathbb{S}^2\ \&\ v\perp w}\big\{S(v, v) + S(w, w)\big\}.$$

Then one can readily check that g is concave which implies that our set is also convex; see also Lemma 5.2.3. This completes the proof. ☐

Lemma 5.2.5. *Fix a real number $\varepsilon \in (0, 1)$. Then, the set*

$$\mathcal{K} = \big\{S \in \mathcal{C}_B(\mathbb{R}^3) : \mu_2(S) + \mu_3(S) \geq 2\varepsilon\mu_1(S)\big\},$$

where $\mu_1(S) \geq \mu_2(S) \geq \mu_3(S)$ are the eigenvalues of S, is convex and invariant under the ODE (5.6).

Proof. Let us show at first the invariance of our set under the ODE. Consider the function $f : [0, T) \to \mathbb{R}$ given by $f = \mu_2 + \mu_3 - 2\varepsilon\mu_1$. Note that since $\varepsilon \in (0, 1)$ we have that $\mu_1 \geq 0$. From (5.8), we have that

$$\begin{aligned}
f' &= (\mu_2 + \mu_3 - 2\mu\lambda_1)' \\
&= \mu_2^2 + \mu_3^2 - 2\varepsilon\mu_2\mu_3 + (\mu_2 + \mu_3 - 2\varepsilon\mu_1)\mu_1 \\
&\geq (1 - \varepsilon)(\mu_2^2 + \mu_3^2) + \varepsilon(\mu_3^2 - 2\mu_2\mu_3 + \lambda_2^2) + (\mu_2 + \mu_3 - 2\varepsilon\mu_1)\mu_1 \\
&\geq f\mu_1.
\end{aligned}$$

Using the same arguments as in the previous lemma, we deduce that the function f stays positive in $[0, T)$ and so the inequality

$$\mu_2 + \mu_3 \geq 2\varepsilon\mu_1 \geq 0$$

is preserved in time. To show convexity consider the function $g : \mathcal{C}_B(\mathbb{R}^3) \to \mathbb{R}$ given by

$$\begin{aligned} g(S) &= 2\varepsilon\mu_1(S) - (\mu_2(S) + \mu_3(S)) \\ &= 2\varepsilon \max_{v\in\mathbb{S}^2} S(v,v) + \max_{v,w\in\mathbb{S}^2\,\&\,v\perp w}\big\{ - S(v,v) - S(w,w)\big\}. \end{aligned}$$

Then one can readily check that g is convex which implies that our set is also convex. $\square$

Lemma 5.2.6. *For every $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that, for each $c > 0$, the set*

$$\mathcal{K} = \big\{S \in \mathcal{C}_B(\mathbb{R}^3) : \mu_2(S) + \mu_3(S) \geq 2\varepsilon\mu_1(S) \textit{ and } \mu_1(S) - \mu_3(S) \leq c\, tr(S)^{1-\delta}\big\},$$

where $\mu_1(S) \geq \mu_2(S) \geq \mu_3(S)$ are the eigenvalues of S, is convex and invariant under the ODE (5.6).

Proof. We already know by Lemma 5.2.5, that the inequality $\mu_2 + \mu_3 \geq 2\varepsilon\mu_1$ is preserved in time by the ODE (5.6). If $\mu_1(S) = \mu_3(S)$, then $\mu_1(t) = \mu_3(t)$ for all $t \in [0, T)$ and in this case we have nothing to prove. So let us suppose that $\mu_1(S) > \mu_3(S)$. This condition will be preserved under the ODE and moreover, $2\mu_1 > \mu_2 + \mu_3 \geq 2\varepsilon\mu_1$. Since, $\varepsilon \in (0, 1)$, it follows that $\mu_1 > 0$. So we may assume that the solution of the ODE satisfies $\mu_2 + \mu_3 \geq 2\varepsilon\mu_1 > 0$. Using (5.8) we get that

$$\big(\log(\mu_1 - \mu_3)\big)' = \frac{1}{\mu_1 - \mu_3}(\mu_1^2 + \mu_2\mu_3 - \mu_3^2 - \mu_1\mu_2) = \mu_1 + \mu_3 - \mu_2. \tag{5.13}$$

Moreover,

$$\begin{aligned} \big(\log(\mu_1 + \mu_2 + \mu_3)\big)' &= \frac{1}{\mu_1 + \mu_2 + \mu_3}(\mu_1^2 + \mu_2\mu_3 + \mu_2^2 + \mu_1\mu_3 + \mu_3^2 + \mu_1\mu_2) \\ &= \frac{\mu_1(\mu_2 - \mu_3) + \mu_2(\mu_2 + \mu_3) + (\mu_1 + \mu_3)^2}{\mu_1 + \mu_2 + \mu_3}. \end{aligned} \tag{5.14}$$

Combining (5.13) and (5.14), we conclude that

$$\begin{aligned}
\Big(\log\frac{\mu_1-\mu_3}{(\mu_1+\mu_2+\mu_3)^{1-\delta}}\Big)' &= \Big(\log(\mu_1-\mu_3)-(1-\delta)\log(\mu_1+\mu_2+\mu_3)\Big)' \\
&= \delta(\mu_1+\mu_3-\mu_2)+(1-\delta)(\mu_1+\mu_3-\mu_2) \\
&\quad -(1-\delta)\frac{\mu_1(\mu_2-\mu_3)+\mu_2(\mu_2+\mu_3)+(\mu_1+\mu_3)^2}{\mu_1+\mu_2+\mu_3} \\
&= \delta(\mu_1+\mu_3-\mu_2)-(1-\delta)\frac{\mu_1(\mu_2-\mu_3)+\mu_2(\mu_2+\mu_3)+\mu_2^2}{\mu_1+\mu_2+\mu_3} \\
&= \delta(\mu_1+\mu_3-\mu_2)-(1-\delta)\frac{\mu_2^2}{\mu_1+\mu_2+\mu_3}.
\end{aligned}$$

Due to $2\varepsilon\mu_1\le\mu_2+\mu_3\le 2\mu_2$ it follows that $\varepsilon(\mu_1+\mu_3-\mu_2)\le\mu_2$. Moreover, since

$$\mu_1+\mu_2+\mu_3\le 3\mu_1 \quad \text{and} \quad 2\mu_2\ge\mu_2+\mu_3,$$

we deduce that

$$\frac{\mu_2^2}{\mu_1+\mu_2+\mu_3}\ge\frac{\mu_2(\mu_2+\mu_3)}{6\mu_1}\ge\frac{2\varepsilon\mu_2\mu_1}{6\mu_1}=\frac{\varepsilon\mu_2}{3}.$$

Putting everything together, we deduce that for $\delta\le\varepsilon^2/(3+\varepsilon^2)$, we have that

$$\Big(\log\frac{\mu_1-\mu_3}{(\mu_1+\mu_2+\mu_3)^{1-\delta}}\Big)'\le\mu_2\Big(\frac{\delta}{\varepsilon}-\frac{(1-\delta)\varepsilon}{3}\Big)\le 0.$$

Convexity follows from the convexity of $g:\mathcal{C}_B(\mathbb{R}^3)\to\mathbb{R}$ given by

$$g(S)=\mu_1(S)-\mu_3(S)-c\big(\mu_1(S)+\mu_2(S)+\mu_3(S)\big)^{1-\delta}.$$

This completes the proof. □

5.3 Finite-time explosion of the curvature

We already know that the Ricci flow has a unique (up to diffeomorphisms) solution on a maximal time interval $[0,T)$, where $T\le\infty$. We claim that if the Ricci curvature is strictly positive, then T is finite. More precisely, the following more general result holds:

Theorem 5.3.1. *Let $g=g_t$ be a solution of the Ricci flow on a compact m-dimensional manifold M^m defined on a maximal time interval $[0,T)$. If the metric $g(0)$ has positive scalar curvature, then T is finite.*

Proof. Because M^m is compact and the scalar curvature initially is strictly positive, there exists a positive constant $\varrho > 0$ such that the minimum of the scalar curvature at time zero is equal to ϱ. From Proposition 2.13, we obtain that

$$\partial_t S = \Delta S + 2|Ric|^2 \geq \Delta S + \frac{2}{m} S^2.$$

From the comparison maximum principle 4.1.3, we immediately see that

$$S \geq \frac{m\varrho}{m - 2t\varrho}.$$

Since the right hand side diverges to ∞, the scalar curvature becomes singular in finite time. □

Corollary 5.3.2. *Let $g = g_t$ be a solution of the Ricci flow on a compact 3-dimensional manifold M^3 defined in a maximal time interval $[0, T)$. If $g(0)$ has strictly positive scalar curvature, then*

$$\limsup_{t \to T} \left\{ \max_{x \in M^3} |Ric|(x, t) \right\} = \infty.$$

Proof. Since the maximal time T is finite, the Riemann curvature tensor must explode as time approaches its maximal value. Recall from (5.2) that in the 3-dimensional case we have that

$$R_{ijkl} = g_{jl} R_{ik} - g_{jk} R_{il} - g_{il} R_{jk} + g_{ik} R_{jl} - \frac{g_{ik} g_{jl} - g_{il} g_{jk}}{2} S,$$

where g_{ij}, R_{ijkl} and R_{ij} are the components of the metric, Riemann curvature tensor and of the Ricci curvature, respectively, with respect to an orthonormal frame. Consequently, there exists a positive constant c such that $|\mathrm{Rm}| \leq c\,|Ric|$. Since the left hand side explodes as time tends to its maximal value, we obtain our result. □

Corollary 5.3.3. *Let $g = g_t$ be a solution of the Ricci flow on a compact 3-dimensional manifold M^3 defined in a finite maximal time interval $[0, T)$. If the Ricci curvature of the initial metric is positive, then the scalar curvature explodes as time approaches its maximal value T. More precisely,*

$$\limsup_{t \to T} \left\{ \max_{x \in M^3} S(x, t) \right\} = \infty.$$

Proof. From Corollary 5.3.2 and the fact that

$$4|Ric|^2 = (\lambda_1 + \lambda_2)^2 + (\lambda_1 + \lambda_3)^2 + (\lambda_2 + \lambda_3)^2 \leq 4(\lambda_1 + \lambda_2 + \lambda_3)^2 = 4S^2,$$

we immediately obtain the result. □

5.4 Evolution of the Ricci curvature

Suppose that $g = g_t$ is a solution of the Ricci flow on a compact 3-dimensional manifold M^3 defined in a finite maximal time interval $[0, T)$. From now on we work only on the bundle (V, h) obtained via the Uhlenbeck's Trick in Section 2.3.

Theorem 5.4.1. *Let $g = g_t$ be a solution of the Ricci flow on a compact 3-dimensional manifold defined in a finite maximal time interval $[0, T)$. If the there exists a point where the Ricci curvature of the initial metric is strictly positive, then the evolved Riemannian metrics have strictly positive Ricci curvature everywhere.*

Proof. Recall from (5.2) that in the 3-dimensional case we have that

$$R_{ijkl} = g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk} + g_{ik}R_{jl} - \frac{g_{ik}g_{jl} - g_{il}g_{jk}}{2}S.$$

Hence, from the evolution equation of the curvature tensor, we deduce that

$$\nabla_{\partial_t} Ric = \Delta Ric + 3tr(Ric)Ric - 4Ric^{(2)} + \big(2|Ric|^2 - tr(Ric)^2\big)g.$$

Define now the map $\Psi : \mathrm{Sym}(V^* \otimes V^*) \to \mathrm{Sym}(V^* \otimes V^*)$ given by

$$\Psi(S) = 3tr(S)S - 4S^{(2)} + \big(2|S|^2 - tr(S)^2\big)g.$$

Thus, Ψ satisfies the null-eigenvector condition and the result follows from Hamilton's tensorial maximum principle. $\square$

Lemma 5.4.2. *Let $g = g_t$ be a solution of the Ricci flow on a compact 3-dimensional manifold M^3 defined in a finite maximal time interval $[0, T)$. If the Ricci curvature of the initial metric is positive, then there exist constants $c > 0$ and $\varepsilon > 0$ depending only on the initial metric such that*

$$\frac{\lambda_1(x,t)}{\lambda_2(x,t) + \lambda_3(x,t)} \leq c,$$

where $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$, are the eigenvalues of the curvature operator of the metric g_t at $x \in M^3$ given in (5.3) and $Ric \geq \varepsilon\, g$.

Proof. Note that such constants c and ε exist at time 0 by the compactness of M^3 and the positivity of the Ricci tensor. As a matter of fact, since the Ricci curvature is initially positive, the continuous function $\lambda_2(\cdot\,,0) + \lambda_3(\cdot\,,0)$ is positive and bounded from below. Hence, there exists a constant $c > 0$ such that

$$\frac{\lambda_1(\cdot\,,0)}{\lambda_2(\cdot\,,0) + \lambda_3(\cdot\,,0)} \leq c.$$

It remains to show that these bounds are preserved under the flow. To achieve this goal, consider the subset K of $\mathcal{S}(V)$ given by

$$\mathcal{K} = \{S \in \mathcal{S}(V) : \lambda_1(S) \leq c(\lambda_2(S) + \lambda_3(S)) \ \& \ \lambda_2(S) + \lambda_3(S) \geq 2\varepsilon\}.$$

From the Lemmas 5.2.4,5.2.5 and 4.3.1 it follows that the set $\mathcal{K}$ is fiber-wise convex, invariant under parallel transport and that the fiber-wise map $Q : \mathcal{S}(V) \to \mathcal{S}(V)$ given by

$$Q(S) = S^2 + S^{\#},$$

points into $\mathcal{K}$. Now the result follows from Hamilton's tensorial maximum principle. □

Lemma 5.4.3. *Let $g = g_t$ be a solution of the Ricci flow on a compact 3-dimensional manifold M^3 defined in a finite maximal time interval $[0, T)$. If the Ricci curvature of the initial metric is positive, then there exists a positive time-independent constant β such that*

$$Ric \geq \beta S\, g.$$

Proof. From Theorem 5.4.2, we have that there exists a time-independent constant $c > 0$ such that

$$Ric \geq \frac{\lambda_2 + \lambda_3}{2} g \geq \frac{\lambda_1}{2c} g \geq \frac{\lambda_1 + \lambda_2 + \lambda_3}{6c} g = \frac{S}{6c} g.$$

as long the flow exists. Setting $\beta = 1/6c > 0$, we deduce the result. □

In the following theorem, we show that the pinching of the eigenvalues becomes better as the scalar curvature tends to infinity. As a matter of fact, we show the following result.

Lemma 5.4.4. *Let $g = g_t$ be a solution of the Ricci flow on a compact 3-dimensional manifold M^3 defined in a finite maximal time interval $[0, T)$ and suppose that the Ricci curvature of the initial metric is positive. There exist positive constants $\delta \in (0, 1)$ and c depending only on the initial metric, such that*

$$\frac{\lambda_1(x,t) - \lambda_3(x,t)}{\lambda_1(x,t) + \lambda_2(x,t) + \lambda_3(x,t)} \leq \frac{c}{(\lambda_1(x,t) + \lambda_2(x,t) + \lambda_3(x,t))^{\delta}},$$

where $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$, are the eigenvalues of the curvature operator of the metric g_t at $x \in M^3$ given in (5.3).

Proof. The proof follows as a direct consequence of Hamilton's tensorial maximum principle and Lemma 5.2.6. □

Theorem 5.4.5. *Let $g = g_t$ be a solution of the Ricci flow on a compact 3-dimensional manifold M^3 defined in a finite maximal time interval $[0, T)$. If $g(0)$ has strictly positive*

Ricci curvature then, there exist positive constants c and δ, depending only on the initial metric g_0, such that

$$\frac{|E|^2}{S^2} \leq cS^{-2\delta}.$$

Proof. From

$$E = \frac{1}{6}\begin{pmatrix} \lambda_2+\lambda_3-2\lambda_1 & 0 & 0 \\ 0 & \lambda_1+\lambda_3-2\lambda_2 & 0 \\ 0 & 0 & \lambda_1+\lambda_2-2\lambda_3 \end{pmatrix},$$

we have that

$$\begin{aligned} \frac{|E|^2}{S^2} &= \frac{(\lambda_2+\lambda_3-2\lambda_1)^2+(\lambda_1+\lambda_3-2\lambda_2)^2+(\lambda_1+\lambda_2-2\lambda_3)^2}{36(\lambda_1+\lambda_2+\lambda_3)^2} \\ &= \frac{6(\lambda_1^2+\lambda_2^2+\lambda_3^2)-6(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)}{36(\lambda_1+\lambda_2+\lambda_3)^2} \\ &= \frac{(\lambda_1-\lambda_2)^2+(\lambda_2-\lambda_3)^2+(\lambda_3-\lambda_1)^2}{12(\lambda_1+\lambda_2+\lambda_3)^2} \\ &\leq \frac{3(\lambda_1-\lambda_3)^2}{12(\lambda_1+\lambda_2+\lambda_3)^2}. \end{aligned}$$

Now the result follows from the estimate in Lemma 5.4.4. ☐

5.5 The gradient estimate for the scalar curvature

The above estimates are results that concern a point of the manifold M. We need a global result. Since the dimension of the manifold is 3 and we want to show that the metric is getting close to an Einstein metric, it is reasonable to expect that the scalar curvature is getting close to being constant (see Section 1.6) and hence a bound on $|\nabla S|$. This section is about finding a way to bound the quantity $|\nabla S|$ that will help us pass to global results.

Consider the functions S_{max} and S_{min} given by

$$S_{\text{max}}(t) = \text{max}_{x\in M^3} S(x,t) \quad \text{and} \quad S_{\text{min}}(t) = \text{min}_{x\in M^3} S(x,t).$$

The purpose of this section is to show that these functions pinch together as time approaches its maximum value. The proof rely heavily in a gradient estimate for the scalar curvature. In order to achieve this estimate we need to obtain several evolution equations.

Firstly, we will use the following general result. Let A be a $m \times m$ matrix then it holds

$$|A|^2 \geq \frac{1}{m}(\text{trace } A)^2. \tag{5.15}$$

Recall that in Corollary 5.3.3 we proved that

$$|Ric|^2 \leq S^2. \tag{5.16}$$

Proposition 5.5.1. *Let M^m be an m-dimensional manifold and $\{g_t\}_{t\in[0,T)}$ a solution to the Ricci flow. Then,*

$$\partial_t \left(\frac{|\nabla S|^2}{S}\right) = \Delta \left(\frac{|\nabla S|^2}{S}\right) - 2S \left|\nabla \left(\frac{\nabla S}{S}\right)\right|^2 - 2\frac{|\nabla S|^2}{S^2}|Ric|^2 + \frac{4}{S}\langle \nabla S, \nabla |Ric|^2\rangle. \tag{5.17}$$

Proof. We calculate using the evolution equation of the scalar curvature (2.13)

$$\partial_t |\nabla S|^2 = \Delta |\nabla S|^2 - 2|\nabla\nabla S|^2 + 4\langle \nabla S, \nabla |Ric|^2\rangle. \tag{5.18}$$

We know that $S > 0$ is preserved, thus we compute

$$\partial_t \left(\frac{|\nabla S|^2}{S}\right) = \frac{1}{S}\big(\Delta |\nabla S|^2 - 2|\nabla\nabla S|^2 + 4\langle \nabla S, \nabla |Ric|^2\rangle\big) - \frac{|\nabla S|^2}{S^2}(\Delta S + 2|Ric|^2).$$

Since $|\nabla S|^2$ and S are smooth functions we use the fact that

$$\Delta \left(\frac{|\nabla S|^2}{S}\right) = \frac{\Delta(|\nabla S|^2)}{S} - \frac{|\nabla S|^2 \Delta S}{S^2} - \frac{2}{S^2}\langle \nabla |\nabla S|^2, \nabla S\rangle + \frac{|\nabla S|^2}{S^3}|\nabla S|^2.$$

Then,

$$\begin{aligned}
\partial_t \left(\frac{|\nabla S|^2}{S}\right) &= \Delta \left(\frac{|\nabla S|^2}{S}\right) - \frac{2|\nabla S|^4}{S^3} + \frac{2\langle \nabla |\nabla S|^2, \nabla S\rangle}{S^2} - \frac{2|\nabla\nabla S|^2}{S} \\
&\quad - \frac{2}{S^2}|\nabla S|^2|Ric|^2 + \frac{4}{S}\langle \nabla S, \nabla |Ric|^2\rangle \\
&= \Delta \left(\frac{|\nabla S|^2}{S}\right) - 2S\left(\frac{|\nabla S|^4}{S^4} + \frac{\langle \nabla |\nabla S|^2, \nabla S\rangle}{S^3} - \frac{|\nabla\nabla S|^2}{S^2}\right) \\
&\quad - \frac{2}{S^2}|\nabla S|^2|Ric|^2 + \frac{4}{S}\langle \nabla S, \nabla |Ric|^2\rangle \\
&= \Delta \left(\frac{|\nabla S|^2}{S}\right) - 2S\left|\nabla \left(\frac{\nabla S}{S}\right)\right|^2 - \frac{2}{S^2}|\nabla S|^2|Ric|^2 + \frac{4}{S}\langle \nabla S, \nabla |Ric|^2\rangle.
\end{aligned}$$

This completes the proof. □

Corollary 5.5.2. *Let M^m be an m-dimensional manifold and $\{g_t\}_{t\in[0,T)}$ a solution to the Ricci flow. Then,*

$$\partial_t \left(\frac{|\nabla S|^2}{S}\right) \leq \Delta\left(\frac{|\nabla S|^2}{S}\right) - 2S\left|\nabla\left(\frac{\nabla S}{S}\right)\right|^2 - 2\frac{|\nabla S|^2}{S^2}|Ric|^2 + 8\sqrt{3}|\nabla Ric|^2. \quad (5.19)$$

Proof. It suffices to show that

$$\frac{4}{S}\langle \nabla S, \nabla |Ric|^2\rangle \leq 8\sqrt{3}|\nabla Ric|^2.$$

Note that

$$|\nabla |Ric|^2| \leq 2|\nabla Ric||Ric|$$

and using (5.15)

$$|\nabla S|^2 \leq 3|\nabla Ric|^2.$$

Thus, by the Cauchy-Schwartz inequality and (5.16)

$$\begin{aligned}\frac{4}{S}\langle \nabla S, \nabla |Ric|^2\rangle &\leq \frac{4}{S}|S||\nabla |Ric|^2| \\ &\leq 8|\nabla S||\nabla Ric|\frac{|Ric|}{S} \\ &\leq 8\sqrt{3}|\nabla Ric|^2.\end{aligned}$$

This completes the proof. □

Note that the reaction terms in equation (5.19) are negative except the last one. Thus, we need to control the term $|\nabla Ric|^2$.

Lemma 5.5.3. *On a 3-dimensional manifold it holds*

$$|\nabla Ric|^2 - \frac{1}{3}|\nabla S|^2 \geq \frac{1}{37}|\nabla Ric|^2. \quad (5.20)$$

Proof. Note that

$$|\nabla E|^2 = |\nabla Ric|^2 - \frac{1}{3}|\nabla S|^2.$$

Using equation (5.15) we have

$$|\nabla_k E|^2 \geq \frac{1}{3}\sum_{i=1}^{3}|\nabla_i Ric_{ij} - \frac{1}{3}\nabla_i S g_{ij}|^2 = \frac{1}{3}\left|\frac{1}{2}\nabla_j S - \frac{1}{3}\nabla_j S\right|^2 = \frac{1}{108}|\nabla S|^2.$$

This completes the proof. □

Lemma 5.5.4. *Let M^3 be a 3-dimensional manifold and $g_t, t \in [0, T)$ a solution to the Ricci flow. Then,*

$$\partial_t |E|^2 = \Delta(|E|^2) - 2|\nabla E|^2 - 8\,\mathrm{tr}_g(Ric^3) + \frac{26}{3} S|Ric|^2 - 2S^3.$$

Proof. We compute using the evolution equation of the scalar curvature (2.13)

$$\partial_t S^2 = 2S(\Delta S + |Ric|^2) = \Delta S^2 - 2|\nabla S|^2 + 4S|Ric|^2. \tag{5.21}$$

By the evolution equation of the Ricci tensor 2.3.4 we obtain

$$\partial_t |Ric|^2 = \partial_t R_{ij} R_{ij} = \Delta |Ric|^2 - 2|\nabla Ric|^2 - 4R_{ipjq} R_{pq} R_{ij}. \tag{5.22}$$

Note that the above equations hold in any dimension m. Since $m = 3$ we can use (5.2) and then (5.22) becomes

$$\partial_t |Ric|^2 = \Delta |Ric|^2 - 2|\nabla Ric|^2 - 2S^3 - 8\,\mathrm{tr}_g(Ric^3) + 10S|Ric|^2.$$

Combining this with (5.21) the result follows. □

We define

$$W = \frac{26}{3} S|Ric|^2 - 8\,\mathrm{tr}(Ric^3) - 2S^3$$

and

$$V = \frac{|\nabla S|^2}{S} + \frac{37}{2}(8\sqrt{3} + 1)|E|^2.$$

Using Corollary 5.19 and Lemma 5.5.4 we compute

$$\partial_t V \leq \Delta\left(\frac{|\nabla S|^2}{S}\right) - 2S\left|\nabla\left(\frac{\nabla S}{S}\right)\right|^2 - 2\frac{|\nabla S|^2}{S^2}|Ric|^2 + 8\sqrt{3}|\nabla Ric|^2 \tag{5.23}$$

$$+ \frac{37}{2}(8\sqrt{3} + 1)(\Delta(|E|^2) - 2|\nabla E|^2 + W) \tag{5.24}$$

Thus, by Lemma 5.5.3 we obtain

$$\partial_t V \leq \Delta V - |\nabla Ric|^2 + \frac{37}{2}(8\sqrt{3} + 1)W. \tag{5.25}$$

Then, since we have diagonalize Ric and E we can prove algebraically that

$$W \leq \frac{50}{3} S|E|^2. \tag{5.26}$$

As a consequence, equation (5.25) becomes

$$\partial_t V \leq \Delta V - |\nabla Ric|^2 + C_1 S|E|^2.$$

Using equation (5.15) we have

$$\frac{1}{3}|\nabla S|^2 \leq |\nabla Ric|^2. \tag{5.27}$$

Concluding by Theorem 5.4.5 we get

$$\partial_t V \leq \Delta V - \frac{1}{3}|\nabla S|^2 + C_2 S^{3-2\gamma}. \tag{5.28}$$

for some positive constants C_2 and γ.

Theorem 5.5.5. *Let $g = g_t, t \in [0,T)$ be a solution to the Ricci flow on a compact manifold M^3 with initially strictly positive Ricci curvature, then there exist positive constants $\beta > 0$ and $\delta \in (0,1)$, depending only on the initial metric g_0 such that for each $\beta_0 \in [0,\beta]$, there exists $c > 0$ depending only on β_0 and g_0 such that*

$$\frac{|\nabla S|^2}{S^3} \leq \beta_0 S^{-\delta} + cS^{-3}.$$

Proof. We compute

$$\partial_t S^{2-\gamma} = \Delta(S^{2-\gamma}) - (2-\gamma)(1-\gamma)S^{-\gamma}|\nabla S|^2 + 2(2-\gamma)S^{1-\gamma}|Ric|^2.$$

Hence, by (5.28) we get

$$\begin{aligned}\partial_t(V - \beta S^{2-\gamma}) \leq \Delta(V - \beta S^{2-\gamma}) + \big(\beta(2-\gamma)(1-\gamma)S^{-\gamma} - \frac{1}{3}\big)|\nabla S|^2 \\ + CS^{3-2\gamma} - 2\beta(2-\gamma)S^{1-\gamma}|Ric|^2.\end{aligned} \tag{5.29}$$

We proved in Lemma 5.4.2 that there exists $\varepsilon > 0$ such that $Ric \geq \varepsilon g$. Thus, $S \geq \varepsilon$ for $\varepsilon > 0$. We can choose $\bar{\beta}$ such that

$$\bar{\beta} \leq \frac{(3\varepsilon)^\gamma}{3(2-\gamma)(1-\gamma)}.$$

Then, for every $\beta \in [0,\bar{\beta}]$ the second term of equation (5.29) is non-positive.
Using $3|Ric|^2 \geq |S|^2$ we get

$$C_3 S^{3-2\gamma} - 2\beta(2-\gamma)S^{1-\gamma}|Ric|^2 \leq C_3 S^{3-2\gamma} - C_4 S^{3-\gamma},$$

where C_3, C_4 are constants. Note that when S is large enough the above term is dominated by the quantity $S^{3-\gamma}$ which is negative, hence we can get an upper bound for it. Thus, we obtain

$$\partial_t(V - \beta S^{2-\gamma}) \leq \Delta(V - \beta S^{2-\gamma}) + C_5,$$

for some constant C_5. Thus, by the scalar maximum principle we get

$$(V - \beta S^{2-\gamma}) \leq C_5 t + C_6,$$

for a constant C_6. By Theorem (5.3.1) we know that T is finite, hence

$$\frac{|\nabla S|^2}{S} \leq V \leq \beta S^{2-\gamma} + C_5 T + C_6 \leq \beta S^{2-\gamma} + C,$$

for some constant C.
This completes the proof. $\square$

5.6 Consequences of the gradient estimate

Lemma 5.6.1. *Let $g = g_t$, $t \in [0, T)$, be a solution of the Ricci flow on a 3-dimensional compact manifold M^3. If $g(0)$ has strictly positive Ricci curvature then, then there exists positive constants $c, \delta > 0$, depending only on the initial metric such that*

$$1 \geq \frac{S_{\text{min}}}{S_{\text{max}}} \geq 1 - \frac{c}{S_{\text{max}}^{\delta}}.$$

More precisely,

$$\lim_{t \to T} \frac{S_{\text{min}}}{S_{\text{max}}} = 1.$$

Proof. By Corollary 5.3.3 we have that R_{max} tends to infinity as time tends to its maximal value T. Hence, there exist constants $\beta_0, c, \delta > 0$ such that

$$|\nabla S|^2 \leq \beta_0 S_{\text{max}}^{3-\delta} + c^2,$$

from where it follows that there exists $\tau \in [0, T)$ such that

$$|\nabla S| \leq \sqrt{\beta_0}\, S_{\text{max}}^{\frac{3-\delta}{2}},$$

for all $t \in (\tau, T)$. Fix $t \in (\tau, T)$. Since M^3 is compact there exists $x_0(t) \in M^3$ such that

$$S_{\text{max}}(t) = S(x_0(t), t).$$

Fix $\varepsilon \in (0, 1)$ and consider the geodesic ball $B(x_0(t), L(\varepsilon, t))$ where

$$L(\varepsilon, t) = \frac{1}{\varepsilon \sqrt{S_{\text{max}}(t)}} < \infty.$$

Let γ be a minimizing unit length geodesic from $x_0(t)$ to $x_1(t) \in B(x_0(t), L(\varepsilon, t))$. Then,

$$S_{\text{max}} - S(x_1(t)) \leq \int_\gamma |\nabla S| ds \leq \sqrt{\beta_0}\, L(\varepsilon, t) S_{\text{max}}^{\frac{3-\delta}{2}} \leq \frac{\sqrt{\beta_0}}{\varepsilon} S_{\text{max}}^{1-\frac{\delta}{2}}.$$

This implies a lower bound for S on $B(x_0(t), L(t))$, that is

$$S \geq S_{\text{max}} \Big(1 - \frac{\sqrt{\beta_0}}{\varepsilon} \Big(\frac{1}{S_{\text{max}}} \Big)^{\frac{\delta}{2}} \Big). \tag{5.30}$$

The proof will be completed if we can choose a time-independent $\varepsilon > 0$ such that

$$B(x_0(t), L(\varepsilon, t)) \equiv M^3.$$

Since $S_{\max} \to \infty$ as $t \to T$, there exists $\bar{t} \in (\tau, T)$, depending on β_0, δ and ε, such that

$$S \geq (1-\varepsilon) S_{\max}, \tag{5.31}$$

on $B(x_0(t), L(\varepsilon, t))$ for all $t \in [\bar{t}, T)$. According to (5.30) and Theorem 5.4.3, there exists a constant $\beta > 0$, depending only on g_0 such that

$$Ric \geq \beta S\, g \geq \beta(1-\varepsilon) S_{\max}\, g.$$

Choose a time-independent $\varepsilon > 0$ such that

$$\frac{\sqrt{2}\,\pi}{\sqrt{\beta(1-\varepsilon)}} \leq \frac{1}{\varepsilon}.$$

Then, by Bonnet-Myers' Theorem 1.9.6 we have that

$$\operatorname{diam}_{g_t}(M^3) \leq \frac{\sqrt{2}\,\pi}{\sqrt{\beta(1-\varepsilon) S_{\max}(t)}} \leq \frac{1}{\varepsilon\sqrt{S_{\max}(t)}} = L(\varepsilon, t).$$

This completes the proof. □

As a consequence of the above estimate we obtain the following important corollary.

Corollary 5.6.2. *Let $g = g_t, t \in [0, T)$, be a solution of the Ricci flow on a 3-dimensional compact manifold M^3 such that $g(0)$ has strictly positive Ricci curvature. Then, for each $\varepsilon \in (0, 1)$ there exists $T_\varepsilon \in [0, T)$ such that*

$$\min_{x \in M^3} \lambda_3(x, t) \geq (1-\varepsilon)^3 \max_{y \in M^3} \lambda_1(y, t) > 0,$$

for all $t \in [T_\varepsilon, T)$, where λ_1 stands for the biggest and λ_3 for the smallest eigenvalue of the curvature operator. As a matter of fact, after some time, the metric will have strictly positive sectional curvature everywhere. Additionally, the evolved metrics are approaching an Einstein metric uniformly as time approaches its maximal value.

Proof. According to Lemma 5.4.4, there exist positive uniform constants $c > 0$ and $\delta \in (0, 1)$, such that

$$\lambda_3 \geq \lambda_1 - c\, S^{1-\delta}.$$

Using the estimate

$$3\lambda_1 \geq \lambda_1 + \lambda_2 + \lambda_3 = S,$$

we deduce that, for each point in space-time it holds

$$\frac{\lambda_3}{\lambda_1} \geq 1 - \frac{c}{\lambda_1} S^{1-\delta} \geq 1 - 3c\, S^{-\delta} \geq 1 - 3c\, S_{\min}^{-\delta}.$$

Fix $x, y \in M^3$ and consider $\varepsilon \in (0,1)$. By Theorem 5.6.1 the quantity S_{min} tends to infinity as t approaches T. Hence, there exists $T_\varepsilon \in (0,T)$ such that for each $t \in (T_\varepsilon, T)$, we have that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq (1-\varepsilon)\lambda_1. \tag{5.32}$$

Consequently, keeping in mind (5.31) and (5.32), we get

$$\begin{aligned}
\lambda_3(x,t) &\geq (1-\varepsilon)\lambda_1(x,t) \geq \frac{1-\varepsilon}{3}S(x,t) \\
&\geq \frac{1-\varepsilon}{3}S_{\text{min}}(t) \geq \frac{(1-\varepsilon)^2}{3}S_{\text{max}}(t) \\
&\geq \frac{(1-\varepsilon)^2}{3}S(y,t) = \frac{(1-\varepsilon)^2}{3}\big(\lambda_1(y,t)+\lambda_2(y,t)+\lambda_3(y,t)\big) \\
&\geq (1-\varepsilon)^3\lambda_1(y,t).
\end{aligned}$$

From the last inequality we conclude the proof. □

Corollary 5.6.3. *Let $g = g_t, t \in [0,T)$, be a solution of the Ricci flow on a 3-dimensional compact manifold M^3 such that $g(0)$ has strictly positive Ricci curvature. Then,*

$$\lim_{t\to T}\int_0^T S_{\text{max}}(\tau)d\tau \geq \lim_{t\to T}\int_0^T S_{\text{min}}(\tau)d\tau = \infty.$$

Proof. By the estimates in Lemma 5.6.1, it suffices to show that

$$\lim_{t\to T}\int_0^T S_{max}(\tau)d\tau = \infty.$$

Consider the differentiable function $f : [0,T) \to (0,\infty)$ given by

$$f(t) = e^{2\int_0^t S_{max}(\tau)d\tau}S_{max}(0).$$

Moreover,

$$f' = 2S_{max}f.$$

From (2.13) and the fact that $|Ric| \leq S$, we have

$$\begin{aligned}
\partial_t(S-f) &= \Delta S + 2|Ric|^2 - 2S_{\text{max}}f \\
&\leq \Delta S + 2S^2 - 2S_{\text{max}}f \\
&\leq \Delta(S-f) + 2S_{\text{max}}S - 2S_{\text{max}}f \\
&= \Delta(S-f) + 2S_{\text{max}}(S-f).
\end{aligned}$$

Note that

$$(S - f)(0) \leq S_{\max}(0) - f(0) = 0.$$

Hence, by the comparison maximum principle 4.1.3 and Lemma 5.6.1 we obtain that there exists a time-independent constant $\varepsilon > 0$ such that

$$f(t) \geq S(x, t) \geq S_{\min}(t) \geq (1 - \varepsilon) S_{\max}(t),$$

for all points in space-time. Since $S_{\max}$ tends to infinity, then f tends to infinity as well as $t \to T$. Going back to the definition of f, we see that

$$\lim_{t \to T} \int_0^t S_{\max}(\tau) d\tau = \infty.$$

This completes the proof. □

Corollary 5.6.4. *Let $g = g_t, t \in [0, T)$, be a solution of the Ricci flow on a 3-dimensional compact manifold M^3 such that $g(0)$ has strictly positive Ricci curvature. Then, the volume of the evolved metrics tends to zero as time approaches its maximal value, i.e.*

$$\lim_{t \to T} V(t) = 0.$$

Consequently, as the curvature explodes the volume shrinks to zero.

Proof. Recall from Lemma 2.3.1 that the volume evolves in time under the Ricci flow according to the equation

$$V' = - \int S \, dM \leq -S_{\min} V.$$

By integration we deduce that

$$\log V(t) - \log V(0) \leq - \int_0^t S_{\min}(s) ds.$$

Due to Corollary 5.6.3 the right hand side of the last inequality tends to $-\infty$ as t tends to T. Consequently,

$$\lim_{t \to T} \log V(t) = -\infty,$$

or, equivalently,

$$\lim_{t \to T} V(t) = 0.$$

This completes the proof. □

5.7 The normalized Ricci flow

We have already seen that a solution $g = g_t$ of a Ricci flow in a 3-dimensional manifold exists only for finite time T, if the initial metric has positive Ricci curvature. In this case, the sectional curvatures get pinched together but the volume tends to zero. In order to avoid shrinking of the manifold, we will perform a time-depending rescaling of each metric to keep the volume of the evolved manifolds constant in time. To achieve this goal, let $\psi : [0, T) \to (0, \infty)$ be a smooth positive function with $\psi(0) = 1$ and consider the 1-parameter family of metrics given by

$$\overline{g}(t) = \psi(t)g(t), \quad t \in [0, T).$$

In the sequel we use over-line symbol to refer to quantities depending to the metric $\overline{g}$. According to the formulas of the subsection 1.8 we obtain that

$$\overline{\nabla} = \nabla, \quad \overline{\Delta} = \psi^{-1}\Delta, \quad \overline{\mathrm{Rm}} = \psi\,\mathrm{Rm} \quad \text{and} \quad \overline{Ric} = Ric. \tag{5.33}$$

Moreover,

$$\overline{R} = \psi^{-1}R, \quad \overline{d\mu} = \psi^{3/2}d\mu, \quad \overline{V} = \psi^{3/2}V \quad \text{and} \quad \overline{R}_{av} = \psi^{-1}R_{av}. \tag{5.34}$$

The volume with respect to the metric $\overline{g}$ is given by the formula

$$\overline{V} = \int \overline{d\mu} = \int \psi^{3/2}d\mu = \psi^{3/2}V.$$

Since we want to keep the volume fixed we need to choose a smooth function ψ such that $\overline{V}(t) = V(0)$, for each $t \in [0, T)$. From the last equality we deduce that

$$\psi(t) = \big(V(t)/V(0)\big)^{-2/3}, \quad \text{for each} \quad t \in [0, T). \tag{5.35}$$

Lemma 5.7.1. *Let $g = g_t, t \in [0, T)$ be a solution to the Ricci flow and let $\psi : [0, T) \to \mathbb{R}$ given by* (5.35)*. Then, the one-parameter family of the Riemannian metrics $\overline{g} = \psi\, g$, evolves according to the equation*

$$\nabla_{\partial_t}\overline{g} = \psi\big(- 2\,\overline{Ric} + \overline{\varrho}\,\overline{g}\,\big),$$

where ϱ is the time-dependent function given by $\varrho = (2/3)R_{av}$. By rescaling the time via

$$t \to s = \int_0^t \psi(\tau)d\tau,$$

the evolution equation of the family of Riemannian metrics $\overline{g}$ becomes

$$\overline{\nabla}_{\partial_s}\overline{g} = -2\,\overline{Ric} + \overline{\varrho}\,\overline{g}. \tag{5.36}$$

Equation (5.36) *is called* the normalized Ricci flow.

Proof. First observe that $\partial_t = \psi\,\partial_s$. Moreover,

$$\partial_t \psi^{-3/2}(t) = \frac{\partial_t V(t)}{V(0)} = -\frac{\int R d\mu}{\int d\mu}\frac{V(t)}{V(0)} = -R_{av}(t)\psi^{-3/2}(t) = -(3/2)\varrho(t)\psi^{-3/2}(t),$$

from where we deduce that

$$\partial_t \psi = \psi\varrho = \psi^2\overline{\rho} \quad \text{and} \quad \partial_s\psi = \psi\overline{\varrho}. \tag{5.37}$$

The first identity follows by differentiating $\overline{g}$ and using the evolution equations of the metric g and the first identity of (5.37). The second claim can be readily verified. □

We will need in the sequel to use maximum principle arguments similar to those employed in the previous sections to control the behavior of various geometric quantities under the normalized Ricci flow. For this purpose, we have to compute the corresponding evolution equations. There is a very simple way of going from the unnormalized evolution equations to the normalized ones, just by exploiting the formulas (5.33) and (5.34). More precisely, this correspondence is due to the following simple observation. If $P = P(g)$ is some tensorial quantity involving components of the metric for the Ricci flow, then the same quantity calculated for the normalized Ricci flow will be related to P by a rule of the form

$$P(\overline{g}) = \psi^k P(g),$$

where k is an integer. The number k is called the *degree* of P.

Lemma 5.7.2. *Let $P = P(g)$ be a tensorial quantity of degree k and suppose that under the Ricci flow it satisfies*

$$\nabla_{\partial_t} P = \Delta P + Q.$$

Then, $\overline{P} = P(\overline{g})$ evolves in time under the normalized Ricci flow by the equation

$$\overline{\nabla}_{\partial_s}\overline{P} = \overline{\Delta}\,\overline{P} + k\overline{\varrho}\overline{P} + \psi^{k-1}Q.$$

Proof. From the formulas in (5.33), the facts $\partial_s = \psi^{-1}\partial_t$ and $\partial_t\psi = \psi^2\overline{\varrho}$, we obtain that

$$\begin{aligned}
\overline{\nabla}_{\partial_s}\overline{P} &= \psi^{-1}\nabla_{\partial_t}(\psi^k P) = \psi^{-1}(\partial_t\psi^k)P + \psi^{k-1}\nabla_{\partial_t}P \\
&= k\psi^{k-2}(\partial_t\psi)P + \psi^{k-1}\big(\Delta P + Q\big) \\
&= k\psi^{k-2}(\psi^2\overline{\varrho})P + \psi^{k-1}\Delta P + \psi^{k-1}Q \\
&= \psi^{-1}\Delta(\psi^k P) + \overline{Q} + k\overline{\varrho}\overline{P} \\
&= \overline{\Delta}\,\overline{P} + \psi^{k-1}Q + k\overline{\varrho}\overline{P}.
\end{aligned}$$

This completes the proof. □

As for the unnormalized Ricci flow, it is convenient to perform computations for the evolution equations of various geometric quantities, with respect to orthonormal frames in space-time. It turns out that the Uhlenbeck's trick can be adapted also for the normalized Ricci flow. More precisely, consider the family $u_s : (TM, g(0)) \to (TM, \overline{g}(s))$, given as the solution of

$$\begin{cases} \overline{\nabla}_{\partial_s} u_s = \overrightarrow{\overline{Ric}}^{\flat}_{\overline{g}(s)} \circ u_s - (\varrho/2) u_s, \\ u_0 = I. \end{cases}$$

where $\overrightarrow{\overline{Ric}}^{\flat}$ is the $(1,1)$-Ricci tensor. Easily we show that

$$u_s^* \overline{g}(s) = g(0),$$

for each $s \in [0, T)$. Hence, if $\{e_1(0), \dots, e_m(0)\}$ is a local orthonormal with respect to $g(0)$, then

$$\{e_1 = u_s e_1(0), \dots, e_m = u_s e_m(0)\}$$

is orthonormal with respect to $\overline{g}(s)$. As a matter of fact,

$$\nabla_{\partial_t} e_i = \sum\nolimits_j \big(\overline{R}_{ij} - (\overline{\varrho}/2)\delta_{ij}\big) e_j.$$

In order to simply the notation, let us denote with the same letter pullbacks of tensors via u_s. Using the above observation we see that

$$\overline{\nabla}_{\partial_s} \overline{\mathcal{R}} = \Delta \overline{\mathcal{R}} + \overline{\mathcal{R}}^2 + \overline{\mathcal{R}}^{\#} - \overline{\varrho}\,\overline{\mathcal{R}}.$$

Lemma 5.7.3. *Let $\overline{g}$ be the solution of the normalized Ricci flow in a compact 3-dimensional manifold. Then, the curvature operator $\overline{\mathcal{R}}$ evolves under the normalized Ricci flow by the equation*

$$\overline{\nabla}_{\partial_s} \overline{\mathcal{R}} = \overline{\Delta}\, \overline{\mathcal{R}} + \overline{\mathcal{R}}^2 + \overline{\mathcal{R}}^{\#} - \overline{\varrho} \overline{\mathcal{R}}.$$

5.8 Convergence of the normalized Ricci flow

In this section we will show exponential convergence to a metric of constant curvature of the normalized Ricci flow. The following lemma is of crucial importance:

Lemma 5.8.1. *Let (M^3, g_0) be a compact Riemannian manifold with positive Ricci curvature and $\overline{g} = \overline{g}(s)$ the solution of the normalized Ricci flow, with initial data the Riemannian metric g_0, and denote by $\overline{T}$ the maximal time of existence of the flow. Then, there exist constants $c > 0$ and $\beta > 0$ such that the following facts hold true:*

(1) $\lim_{s\to\overline{T}} \overline{S}_{\max}(s)/\overline{S}_{\min}(s) = 1.$

(2) $\overline{Ric} \geq \beta\,\overline{S}_{\min}\,\overline{g}.$

(3) $\overline{S}_{\max} \leq c.$

(4) $\overline{T} = \infty.$

(5) $\overline{S}_{\min} \geq 1/c$ *and* $\operatorname{diam}_{\overline{g}}(M^3) \leq c.$

(6) $\left|\overline{E}_{\overline{g}(s)}\right| \leq ce^{-\beta s}, \quad s \in [0,\infty).$

Proof. Part (1) of the lemma follows from the corresponding estimates in Lemma 5.6.1 for the unnormalized Ricci flow, since the inequalities are scale-invariant.

(2) By Lemma 5.4.3 and the formulas (5.33) and (5.34) we have that there exists a positive time-independent constant $\beta > 0$ such that

$$\overline{Ric} = Ric \geq \beta S\, g \geq \beta S_{\min}\, g = \beta\, \overline{S}_{\min}\, \overline{g}.$$

(3) By Bonnet-Myers' Theorem 1.9.6, the diameter of $(M^3, \overline{g})$, is estimated from above by

$$\overline{L} = \operatorname{diam}_{\overline{g}}(M^3) \leq \frac{\sqrt{2}\,\pi}{\sqrt{\beta}\sqrt{\overline{S}_{\min}}}. \tag{5.38}$$

We can get a lower bound for the diameter from the fact that the volume of the manifold $(M^3, \overline{g})$ is constant $\overline{V}$. Because

$$\overline{Ric} = Ric \geq 0,$$

by Bishop-Gromov volume comparison theorem 1.9.7, we have that

$$\overline{V} = V\big(B(x_0, \overline{L})\big) \leq V(\mathbb{S}^3(\overline{L})) = \frac{4\pi}{3}\overline{L}^3.$$

Combining the last two inequalities, we have that

$$\left(\frac{3\overline{V}}{4\pi}\right)^{\frac{1}{3}} \leq \overline{L} \leq \frac{\sqrt{2}\,\pi}{\sqrt{\beta}\sqrt{S_{\min}}},$$

from where we obtain an upper bound for $S_{\min}$. By the estimate in Lemma 5.6.1 we obtain that there exists a positive time-independent constant ε such that

$$\frac{\overline{S}_{\min}}{\overline{S}_{\max}} = \frac{S_{\min}}{S_{\max}} \geq 1 - \varepsilon, \tag{5.39}$$

from where we obtain the desired upper bound for $S_{\max}$.
(4) By Corollary 5.6.3 we have that

$$\infty = \lim_{t \to T} \int_0^t S_{\max}(\tau) d\tau = \lim_{s \to \overline{T}} \int_0^s \psi(x) \overline{S}_{\max}(x) \psi^{-1}(x) dx = \lim_{s \to \overline{T}} \int_0^s \overline{S}_{\max}(x) dx.$$

Since $\overline{S}_{\max}$ is bounded from above, we deduce that necessarily the maximal time of existence of the normalized Ricci flow is infinite, that is

$$\overline{T} = \infty.$$

(5) Recall from Corollary 5.6.2 that

$$\lim_{t \to T} \frac{\lambda_3(x,t)}{\lambda_1(y,t)} \to 1,$$

uniformly for all $x, y \in M^3$. It follows by scaling invariance that

$$\lim_{s \to \infty} \frac{\overline{\lambda_3}(x,s)}{\overline{\lambda_1}(y,s)} \to 1,$$

uniformly for all points $x, y \in M^3$. Therefore if we wait long enough, there exists $s_0 > 0$ such that $(M^3, \overline{g}(s_0))$ is $1/4$-pinched. This means that for each $s \geq s_0$, the sectional curvatures are pinched between $K(s)$ and $K(s)/4$, for some time-dependent constant $K(s) > 0$. Observe that $K(s)$ equals to some multiple of $S_{\min}(s)$. Let us pass now to the universal covering $\mathcal{M}^3$ of M^3. Clearly each metric $\overline{g}(s)$ can be lifted in a locally isometric way to a Riemannian metric on $\mathcal{M}^3$. For simplicity, we denote the lifted metrics again by the same symbol $\overline{g}(s)$. According to Klingenberg's injectivity radius estimate 1.9.8, we have that for each $s \geq s_0$, there exists a positive time-independent constant ε_1 such that

$$\mathrm{inj}_{\overline{g}(s)}(\mathcal{M}^3) \geq \varepsilon_1 \big(\overline{S}_{\max}(s)\big)^{-1/2}. \tag{5.40}$$

On the other hand, because we have a uniform upper bound on sectional curvatures, the volume of $\mathcal{M}^3$ is at least some multiple of the cube of the injectivity radius of $\mathcal{M}^3$, by the second part of the Bishop-Gromov volume comparison theorem 1.9.7. Hence, from (5.40), we have

$$V_{\overline{g}(s)}(\mathcal{M}^3) \geq \varepsilon_2 \big(\mathrm{inj}_{\overline{g}(s)}(\mathcal{M}^3)\big)^3 \geq \varepsilon_2 \varepsilon_1^3 \big(\overline{S}_{max}(s)\big)^{-3/2}, \tag{5.41}$$

for each $s \geq s_0$. Because the Ricci tensor of $(M^3, \overline{g}(s))$ is bounded from below, by Bonnet-Myers' theorem 1.9.6 the fundamental group of M^3 is finite. Furthermore, since the volume of M^3 is constant under the normalized Ricci flow we have that

$$V_{\overline{g}(s)}(\mathcal{M}^3) = |\pi_1(M^3)|\overline{V} = \text{constant}, \tag{5.42}$$

for each $s \geq s_0$. Combining the equations (5.42) with (5.41), we obtain a lower bound for $\overline{S}_{max}$. From part (1) of the lemma, we get a uniform lower bound for $\overline{S}_{\text{min}}$ as well.
(6) It suffices to prove that

$$\left|\overline{E}_{\overline{g}(s)}\right| \leq c\,\overline{S}(s)e^{-\delta s}, \quad s \in [0, \infty),$$

since by what we already showed in (3) and (5) the scalar curvature $\overline{R}$ is uniformly bounded. From the computations in Theorem 5.4.5 we see that

$$\left|\overline{E}_{\overline{g}(s)}\right| \leq \frac{\overline{\lambda}_1 - \overline{\lambda}_3}{2}.$$

Thus, it suffices to prove that there exist time-independent constants $c, \delta > 0$ such that

$$\overline{\lambda}_1 - \overline{\lambda}_3 \leq ce^{-\delta t}(\overline{\lambda}_1 + \overline{\lambda}_2 + \overline{\lambda}_3).$$

To achieve this goal, let us consider the time-dependent convex sets

$$\begin{aligned} \mathcal{F}(s) \;=\; \big\{ S \in \mathcal{C}_B(V)\colon\, & \lambda_2(S) + \lambda_3(S) \geq \varepsilon_1, \\ & \lambda_2(S) + \lambda_3(S) \geq \varepsilon_2 \lambda_1(S), \\ & \lambda_1(S) - \lambda_3(S) \leq ce^{-\delta s}(\lambda_2(S) + \lambda_3(S))\big\}. \end{aligned}$$

where $\varepsilon_1, \varepsilon_2, c, \delta$ are positive constants. We show at first that the ODE

$$S' = S^2 + S^{\#} - \varrho S,$$

with $S(0) \in \mathcal{F}(0)$ remains inside $\mathcal{F}(s)$ for appropriately chosen $\delta > 0$. Using the equations

$$\lambda_1' = \lambda_1^2 + \lambda_2\lambda_3 - \varrho\lambda_1, \;\; \lambda_2' = \lambda_2^2 + \lambda_1\lambda_3 - \varrho\lambda_2 \;\text{ and }\; \lambda_3' = \lambda_3^2 + \lambda_1\lambda_2 - \varrho\lambda_3,$$

and following the same idea as in Lemmas 5.2.4 and 5.2.5 we deduce that the first two conditions are preserved under the ODE. Moreover,

$$\log\left(e^{\delta s}\frac{\lambda_1-\lambda_3}{\lambda_2+\lambda_3}\right)' = \delta+\lambda_3-\lambda_2-\frac{\lambda_2^2+\lambda_3^2}{\lambda_2+\lambda_3}\leq \delta-\frac{\lambda_2+\lambda_3}{2}.$$

Using the facts $\varepsilon_2\lambda_1 \leq \lambda_2+\lambda_3$ and $2\lambda_1 \geq \lambda_2+\lambda_3 \geq \varepsilon_1$, we see that

$$(3/2)\varepsilon_1 \leq \lambda_1+\lambda_2+\lambda_3 \leq \varepsilon_2(\lambda_2+\lambda_3)+\lambda_2+\lambda_3 = (1+\varepsilon_2)(\lambda_2+\lambda_3).$$

By choosing

$$\delta = \frac{3\,\varepsilon_1}{2(1+\varepsilon_2)},$$

we have $\lambda_2+\lambda_3 \geq 2\delta$ and thus

$$\log\left(e^{\delta s}\frac{\lambda_1-\lambda_3}{\lambda_2+\lambda_3}\right) \leq 0.$$

The sets $\mathcal{F}(s)$ are closed, convex and invariant under parallel translation by analogous methods to those used in the proof of Lemma 5.2.6. The result follows from the Hamilton's tensorial maximum principle. This concludes the proof of the lemma. $\square$

Theorem 5.8.2. *Let $\overline{g} = \overline{g}(s)$ be the solution to the normalized Ricci flow on a 3-dimensional manifold with initially strictly positive Ricci curvature. Then $\overline{g}$ converges in infinite time uniformly to a smooth metric $\overline{g}_\infty$ with constant positive sectional curvatures.*

Proof. By Proposition 3.3.7 (see also [5]), the family of Riemannian metrics $\overline{g}(s)$ converge smoothly, exponentially and uniformly to a smooth Riemannian metric $\overline{g}_\infty$. On the other hand,

$$|\overline{E}| = \left|\overline{Ric} - (\overline{S}/3)\overline{g}\right| \leq ce^{-\beta s},$$

where c, β are time-independent positive constant. Passing to the limit, we see that the metric $\overline{g}_\infty$ is Einstein with positive scalar curvature. By a classical theorem in Riemannian Geometry, it follows that $\overline{g}_\infty$ has positive constant sectional curvatures. This completes the proof. $\square$

Bibliography

[1] B. Andrews and C. Hopper, *The Ricci flow in Riemannian geometry*, Lecture Notes in Mathematics, vol. 2011, Springer, Heidelberg, 2011. A complete proof of the differentiable 1/4-pinching sphere theorem.

[2] W. Ballmann, *Vector bundles and connections: A short and elementary exposition of vector bundles and connections*, Lecture Notes, University of Bonn (Germany), https://people.mpim-bonn.mpg.de/hwbllmnn/notes.html (2002).

[3] A. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb., 3. Folge, vol. 10, Springer, 1987.

[4] C. Böhm and B. Wilking, *Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature*, Geom. Funct. Anal. **17** (2007), 665–681.

[5] S. Brendle, *Ricci flow and the sphere theorem*, Graduate Studies in Mathematics, vol. 111, American Mathematical Society, Providence, RI, 2010.

[6] M. Dajczer and R. Tojeiro, *Submanifold theory beyond an introduction*, Universitext, Springer, New York, 2019.

[7] M.-P. do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992.

[8] J. Cheeger and D.G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Math. Libr., vol. 9, Elsevier (North-Holland), Amsterdam, 1975.

[9] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, Mathematical Surveys and Monographs, vol. 110, American Mathematical Society, 2004.

[10] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications. Geometric aspects. Part I*, Mathematical Surveys and Monographs, vol. 135, American Mathematical Society, Providence, RI, 2007.

[11] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications. Part II*, Mathematical Surveys and Monographs, vol. 144, American Mathematical Society, Providence, RI, 2008. Analytic aspects.

[12] F. Da Lio, *Remarks on the strong maximum principle for viscosity solutions to fully nonlinear parabolic equations*, Commun. Pure Appl. Anal. **3** (2004), 395-415.

[13] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. **18** (1983), 157-162.

[14] J. Eells Jr. and J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, American Journal of Mathematics **86** (1964), no. 1, 109-160.

[15] L.-C. Evans, *A strong maximum principle for parabolic systems in a convex set with arbitrary boundary*, Proc. Amer. Math. Soc. **138** (2010), 3179-3185.

[16] R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995, pp. 7-136.

[17] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153-179.

[18] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255-306.

[19] J. Jost, *Riemannian geometry and geometric analysis*, 7th ed., Universitext, Springer, Cham, 2017.

[20] W. Kühnel, *Differential geometry curves-surfaces-manifolds*, 3rd ed., Student Mathematical Library, vol. 77, Springer Vieweg, 2015.

[21] R.S. Kulkarni, *On the Bianchi identities*, Math. Ann. **199** (1972), 175-204.

[22] J. Lee, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer-Verlag, 2013.

[23] P. Li, *Geometric analysis*, Cambridge Studies in Advanced Mathematics, vol. 134, Cambridge University Studies, 2012.

[24] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.

[25] K. Nomizu, *On the decomposition of generalized curvature tensor fields. Codazzi, Ricci, Bianchi and Weyl revisited*, Differential Geometry (in honor of Kentaro Yano), Kinokuniya, Tokyo, 1972.

[26] P. Petersen, *Riemannian geometry*, 3rd ed., Graduate Texts in Mathematics, vol. 171, Springer, Cham, 2016.

[27] M. Protter and H. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, New York, 1984.

[28] A. Savas-Halilaj and K. Smoczyk, *Bernstein theorems for length and area decreasing minimal maps*, Calc. Var. Partial Differential Equations **50** (2014), 549-577.

[29] N. Sheridan, *Hamilton's Ricci Flow*, 2006.

[30] E. Stiefel, *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten*, Comment. Math. Helv. **8** (1935), 305-353.

[31] P. Topping, *Lectures on the Ricci flow*, 2006.

[32] L.W. Tu, *Differential geometry connections, curvature, and characteristic classes*, Springer Cham, 2017.

[33] X. Wang, *A remark on strong maximum principle for parabolic and elliptic systems*, Proc. Amer. Math. Soc. **109** (1990), 343-348.

[34] H.F. Weinberger, *Invariant sets for weakly coupled parabolic and elliptic systems*, Rend. Mat. (6) **8** (1975), 295-310.