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Banach Lattices

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which was handed to me from the Department of Mathematics of the University of Ioannina.

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Statutory Declaration

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Grigorios Gousis

Dedicated to Christos, Vasiliki and Marika

Abstract in English

The main goal of this thesis is to find conditions, under which, Banach Lattices are isomorphic either to $C(K)$, where K is compact, or to $L^1(\mu)$, with respect to the measure μ.

In Chapter 1, we give some basic notations and definitions for vector lattices. Also, we provide an algebraic view of ideal and band theory, which will end up to the Riesz Decomposition Theorem. Concluding this chapter, our main focus turns to maximal and minimal ideals and the Archimedean vector lattices of finite and infinite dimension.

In Chapter 2, we make use of basic notions of Functional Analysis and discuss more about the duals of vector lattices. Firstly, we present Nakano's Theorem. Equipping a vector lattice with a norm turns our work to the study of normed vector lattices. We state properties of normed vector lattices and dive more into the topological and order properties, which will help us determine the structure of normed vector lattices. Lastly, the introduction of quasi interior points helps us to extract useful conclusions about whether or not a normed vector lattice has ideals.

In Chapter 3, we study the abstract M-spaces and the abstract L-spaces. The space of all continuous real functions on a compact space K and the space of all integrable functions are important classes of Banach Lattices. They are thoroughly discussed, as well as their duality. Using topological arguments, we restate well known theorems of Functional Analysis and Measure Theory from a lattice point of view. We conclude the chapter by mentioning some extension and representation theorems of AL and AM spaces.

Abstract in Greek

Στόχος αυτής της διατριβής είναι να βρούμε κατάλληλες προϋποθέσεις κάτω από τις οποίες ένα πλέγμα Banach είναι ισόμορφο με τον $C(K)$, για κάποιο συμπαγές K, ή ισόμορφο με τον $L^1(\mu)$, ως προς το μέτρο μ.

Στο Κεφάλαιο 1 παραθέτουμε κάποιες βασικές έννοιες και ορισμούς σχετικά με τα διανυσματικά πλέγματα. Στη συνέχεια, ασχολούμαστε με τη θεωρία ιδεωδών και λωρίδων, το Θεώρημα Αναπαράστασης του Riesz και τα μεγιστοτικά και ελαχιστικά ιδεώδη, καθώς και τα διανυσματικά πλέγματα με την Αρχιμήδεια ιδιότητα, πεπερασμένης ή μη διάστασης.

Στο Κεφάλαιο 2, χρησιμοποιώντας βασικές έννοιες της Συναρτησιακής Ανάλυσης, αναλύουμε τους δυικούς ενός διανυσματικού πλέγματος V, ξεκινώντας με το θεώρημα του Nakano. Μελετάμε διανυσματικά πλέγματα τα οποία έχουμε εφοδιάσει με νόρμα, παραθέτοντας ιδιότητες που βοηθούνε στον καθορισμό της δομής τους. Το τελευταίο μέρος του Κεφαλαίου πραγματεύεται τα οιονεί-εσωτερικά σημεία με στόχο να εξάγουμε χρήσιμα συμπεράσματα σχετικά με το εάν ή όχι ένα νορμοποιημένο διανυσματικό πλέγμα έχει ιδεώδη.

Στο Κεφάλαιο 3 μελετάμε τους αφηρημένους M και L χώρους. Ο χώρος όλων των συνεχών συναρτήσεων πάνω από ένα συμπαγές σύνολο K, καθώς και ο χώρος όλων των ολοκληρώσιμων συναρτήσεων, αποτελούν σημαντικές κατηγορίες Banach πλεγμάτων. Αυτές και η δυικότητα τους μελετώνται στο πρώτο και δεύτερο μέρος τους κεφαλαίου, αντίστοιχα. Επίσης, χρησιμοποιώντας τοπολογικά επιχειρήματα επαναδιατυπώνουμε, από την σκοπιά των πλεγμάτων, γνωστά θεωρήματα από τη Συναρτησιακή Ανάλυση και τη Θεωρία Μέτρου. Αυτή η συζήτηση ολοκληρώνεται με θεωρήματα επέκτασης και αναπαράστασης των αφηρημένων M και L χώρων.

Contents

Chapter 1

Basic Aspects of Vector Lattices

1.1 Real Vector Lattices

To introduce the concept of lattices firstly we discuss ordered sets. In this paragraph we will define lattices, as well as vector lattices, and discuss some basic properties. Also we will talk more about convergence and completeness with respect to the given order.

Remark 1.1.1. We denote by g.c.d the greatest common divider and by l.c.m the least common multiplier of any two natural numbers.

Firstly we need to define ordered sets.

Definition 1.1.2. A binary relation $\preccurlyeq$ which satisfies the following :

(i) $x \preccurlyeq x$ holds for every $x \in A$ (reflexive)

(ii) $x \preccurlyeq y$ and $y \preccurlyeq x$ implies $x = y$, for $x, y \in A$ (antisymmetric)

(iii) $x \preccurlyeq y$ and $y \preccurlyeq z$ implies $x \preccurlyeq z$, for $x, y, z \in A$ (transitive)

is called a *partial order*.

Definition 1.1.3. A set A endowed with a partial order is called a *partially ordered set* and is denoted by $(A, \preccurlyeq)$.

Definition 1.1.4. A binary relation $\preccurlyeq$ which satisfies the following :

(i) $x \preccurlyeq x$ holds for every $x \in A$

(ii) $x \preccurlyeq y$ and $y \preccurlyeq z$ implies $x \preccurlyeq z$, for all $x, y, z \in A$

is called a *partial preorder*.

Definition 1.1.5. A set A endowed with a partial preorder is called a *partially preordered* set and is denoted by $(A, \preccurlyeq)$.

The following examples of orders will be in use from now on.

Example 1.1.6. Let $\mathbb{R}^n$, where $n \in \mathbb{N}$. The binary relation $\preccurlyeq$ defined as follows:

$$a \preccurlyeq b \Leftrightarrow a_i \leqslant b_i \quad \text{for all } i = 1, 2, \ldots, n$$

for all $a, b \in X$, $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, is called the *lexicographic ordering*. Since $x \preccurlyeq x$ implies that $x_i \leq x_i$, for all $i = 1, 2, \ldots, n$, which holds for all x_i, we have that $\preccurlyeq$ is reflexive. Moreover, suppose $x, y, z \in X$, with $x \preccurlyeq y$ and $x \preccurlyeq z$. This implies $x_i \leq y_i$ and $y_i \leq z_i$ for all $i = 1, 2, \ldots, n$. Thus $x_i \leq z_i$, for all $i = 1, 2, \ldots, n$. Therefore $x \preccurlyeq z$ holds.

Furthermore, we examine if the antisymmetric property holds, in order to determine if this partial preorder is a partial order. Let $x, y \in X$ such that $x \preccurlyeq y$ and $y \preccurlyeq x$, for all $i = 1, 2, \ldots, n$. Then $x_i \leq y_i$ and $y_i \leq x_i$ holds for all $i = 1, 2, \ldots, n$. Therefore $x_i = y_i$, for all $i = 1, 2, \ldots, n$, hence $x = y$. Thus the lexicographic ordering is a partial order.

Example 1.1.7. Let $K \subseteq \mathbb{R}$. Denote by $C(K)$ the space of all real continuous functions on K. The binary relation " $\preccurlyeq$ ", called the *canonical ordering*,

$$f \preccurlyeq g \Leftrightarrow f(t) \leqslant g(t), \quad \text{for all } t \in K,$$

is a partial order.

Indeed, for some $f \in C(K)$, obviously $f(t) \leq f(t)$ for all $t \in K$ so $f \preccurlyeq f$ holds. Let $f, g, h \in C(K)$ such that $f \preccurlyeq g$ and $g \preccurlyeq h$ for all $t \in K$. This implies $f(t) \leq g(t)$ and $g(t) \leq h(t)$ for all t. Thus $f(t) \leq h(t)$ and therefore $f \preccurlyeq h$. Lastly let $f, g \in C(K)$ such that $f \preccurlyeq g$ and $g \preccurlyeq f$. Then $f(t) \leq g(t)$ and $g(t) \leq f(t)$ hold for all $t \in K$ and therefore $f(t) = g(t)$ for all $t \in K$. Hence $f = g$ and the ordering is antisymmetric.

Example 1.1.8. The binary relation "$\preccurlyeq$" defined as follows:

$$m \preccurlyeq n \Leftrightarrow m|n, \quad \text{for } n, m \in \mathbb{N}$$

is a partial order called the *divisibility relation*.

Let $x \in J$, then obviously $x|x$ and hence "$\preccurlyeq$" is reflexive. Now let $x, y, z \in J$ such that $x \preccurlyeq y$ and $y \preccurlyeq z$ or equivalently $x \backslash y$ and $y|x$. Thus there exist $m, k \in \mathbb{Z}$ such that $y = mx$ and $z = ky$. To prove the anti-symmetry of the ordering let $x, y \in J$ such that $x|y$ and $y|x$ and $x \neq y$. Thus, there exist $m, k \in \mathbb{Z}$ such that $y = mx$ and $x = ky$. We easily obtain that $x = mkx$. Thus $m = k = 1$. Therefore the divisibility relation is a partial order.

Example 1.1.9. Let X be a set and $P(X)$ be the power set of X. Then, the binary relation "$\preccurlyeq$" defined as follows:

$$U \preccurlyeq V \Leftrightarrow U \subset V$$

for all $U, V \in P(X)$ is called *set inclusion* and is a partial preorder. Let $U \in P(X)$, then obviously $U \subset U$ and hence $U \preccurlyeq U$. For $U, V, W \in P(X)$ such that $U \preccurlyeq V$ and $V \preccurlyeq W$ we obtain that $U \subset V$ and $V \subset W$. Therefore $U \subset V \subset W$ and $U \subset W$. For $U, W \in P(X)$ such that $U \preccurlyeq W$ and $W \preccurlyeq U$ we obtain that $U \subset W$ and $W \subset U$. Therefore $U = W$, so set inclusion is a partial order.

Definition 1.1.10. A partially ordered set $(A, \preccurlyeq)$ is called *totally ordered* if every two elements of A are related with respect to $\preccurlyeq$ i.e. for every $x, y \in A$ it holds that either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Remark 1.1.11. A set A is called *ordered* if it is partially ordered.

Example 1.1.12. Let $\mathbb{N}$ endowed with the divisibility relation. Suppose $x, y \in \mathbb{N}$ and $x \neq y$. It is obvious that it is not generally true that $x \backslash y$ or $y \backslash x$ and hence that either $x \preccurlyeq y$ or $y \preccurlyeq x$. Therefore the divisibility relation is not a total order. However, $\mathbb{N}$ endowed with the usual ordering '$\leq$' is totally ordered.

Example 1.1.13. Let A be a set containing at least two elements. Then the powerset $\mathcal{P}(A)$ is not totally ordered, with respect to set inclusion.

Definition 1.1.14. Let $(A, \preccurlyeq)$ be an ordered set and $x, y \in A$. The set of all $z \in A$ such that $x \preccurlyeq z \preccurlyeq y$ is called an *order interval* in A and is denoted by $[x, y]$.

The following discussion is about the least and greatest elements of an arbitrary ordered set.

Definition 1.1.15. Let $(A, \preccurlyeq)$ be an ordered set and $[x, y] \subset A$ for some $x, y \in A$. A non empty set C $\subset [x, y]$ is called *order bounded.*

Definition 1.1.16. Let $(A, \preccurlyeq)$ be an ordered set. The set $B \subset A$ is called *majorized* (or *upper bounded*) if there exists $M \in A$ such that $b \preccurlyeq M$ holds for every $b \in B$. In this case M is called a *majorant* or an *upper bound* of B. We denote by B_M the set of all majorants of B.

Definition 1.1.17. Let $(A, \preccurlyeq)$ be an ordered set. The set $B \subset A$ is called *minorized* (or *lower bounded*) if there exists $m \in A$ such that $m \preccurlyeq b$ holds for every $b \in B$. In this case, m is called a *minorant* or a *lower bound* of B. We denote by B_m the set of all minorants of B.

Definition 1.1.18. Let $(A, \preccurlyeq)$ be a partially ordered set and $B \neq \emptyset$ a subset of A. If there exists $a \in B_M$ such that $a \preccurlyeq M$ for each M in B_M then a is called *least upper bound* or *supremum* of B and we denote it by $\sup B$.

Definition 1.1.19. Let $(A, \preccurlyeq)$ be a partially ordered set and $B \neq \emptyset$ a subset of A. If there exists $a \in B_m$ such that $m \preccurlyeq a$ for each m in B_m then a is called *greatest lower bound* or *infimum* of B and we denote it by $\inf B$.

Remark 1.1.20. Let $(A, \preccurlyeq)$ be a directed set and $B \subseteq A$. If $\sup B$ and $\inf B$ exist, then they are unique.

Proof. Let a, a' be suprema of A. Since a is the supremum of A, a is a majorant of A. But a' is supremum, hence $a \preccurlyeq a'$. Similarly a' is supremum of A thus a' is a majorant of A. But a is supremum, so $a' \preccurlyeq a$. Because "$\preccurlyeq$" is a partial order, by the anti-symmetry of the order we obtain $a' = a$. Therefore the supremum is unique.

To prove that the infimum of a directed set is unique we work analogously. □

Example 1.1.21. Let $P(X)$ be the power set of X endowed with the set inclusion. It is obvious that $\emptyset$ is the infimum of $P(X)$ and that X is the supremum of $P(X)$.

Definition 1.1.22. A partially preordered set A is called *directed upward* if for any two elements $a, b \in A$ there exists $c \in A$ such that $c \succcurlyeq a$ and $c \succcurlyeq b$.

Definition 1.1.23. A partially preordered set A is called *directed downward* if for any two elements $a, b \in A$ there exists $c \in A$ such that $c \preccurlyeq a$ and $c \preccurlyeq b$.

Remark 1.1.24. Directed upward or directed downward sets are simply called directed.

Example 1.1.25. Let $S = \{1, 2, 3\}$. The relation $\{\{1, 1\}, \{2, 2\}, \{3, 3\}, \{1, 3\}, \{2, 3\}\}$ is a partial order on S. Endowed with this partial order, S is directed upward but not directed downward.

Definition 1.1.26. Let $(S, \preccurlyeq)$ be a directed set and $x \in S$. The set of all $z \in S$ such that $z \succcurlyeq x$ is called the *section* of S for x and is denoted by S_x.

Example 1.1.27. Let $S = \{1, 2, 3\}$. The relation $\{\{1, 1\}, \{2, 2\}, \{3, 3\}, \{1, 3\}, \{2, 3\}\}$ is a partial order on S. It is easy to see that the section of S for 1 is the set $\{1, 3\}$.

Definition 1.1.28. Let A be a set. A set F of subsets of A satisfying the following:

(i) $F \neq \emptyset$ and $\emptyset \notin F$

(ii) $U \in F$ and $U \subset G \subset A$ implies that $G \in F$

(iii) $U \in F$ and $G \in F$ implies $U \cap G \in F$

is called a *filter* on A.

Example 1.1.29. Let X be a topological space and fix a $x_0 \in X$. Denote by $\mathcal{N}_{x_0}$ the set of all neighborhoods of x_0. Then $\mathcal{N}_{x_0}$ is a filter in X.

- Since every neighborhood of x_0 must contain x_0, we have that $\emptyset \notin \mathcal{N}_{x_0}$.
- If $U \in \mathcal{N}_{x_0}$ and $W \supset U$, then $x_0 \in \text{int}(W)$, so $W \in \mathcal{N}_{x_0}$.
- Suppose $U \in \mathcal{N}_{x_0}$ and $W \in \mathcal{N}_{x_0}$. Then, obviously, $U \cap W \in \mathcal{N}_{x_0}$.

Now we are ready to state the definition of lattice.

Definition 1.1.30. A partially ordered set $(L, \preccurlyeq)$ is called a *lattice* if the elements $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ both exist in L, for each pair $(x, y) \in L \times L$. Moreover, the mappings $(x, y) \mapsto x \vee y$ and $(x, y) \mapsto x \wedge y$ are called the *lattice operations.*

Example 1.1.31. Let X be a topological space. We denote by $\mathbf{O}$ the set of all open subsets of X, i.e. the topology of X. Then, $(\mathbf{O}, \subset)$ endowed with the following lattice operations

$$(U, V) \mapsto U \wedge V = U \cap V, \quad \text{for every } U, V \in \mathbf{O}$$

and

$$(U, V) \mapsto U \vee V = U \cup V, \quad \text{for every } U, V \in \mathbf{O}$$

is a lattice. We remark that the union and the intersection of any two open sets are open, hence both sup and inf exist in $\mathbf{O}$.

Example 1.1.32. Let X be a topological space. We denote by $\mathbf{C}$ the set of all closed subsets of X. Then $(\mathbf{C}, \subset)$ endowed with the following lattice operations is a lattice.

$$(U, V) \mapsto U \wedge V = U \cap V \quad \text{for every } U, V \in \mathbf{C}$$

and

$$(U, V) \mapsto U \vee V = U \cup V \quad \text{for every } U, V \in \mathbf{C}.$$

The union and the intersection of any two closed sets are open, hence both sup and inf exist in $\mathbf{C}$.

We proceed to prove some basic properties regarding lattice operations.

Definition 1.1.33. $(L, \preccurlyeq)$ is called a *distributive lattice* if L is a lattice and

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

holds for every $x, y, z \in L$.

Proposition 1.1.34. *Let $(L, \preccurlyeq)$ be a lattice. The lattice operations satisfy all of the following:*

(i) They are idempotent.

(ii) They are commutative.

(iii) They are associative.

(iv) It holds that $x \wedge (x \vee y) = x$, for all $x, y \in L$.

(v) It holds that $x \vee (x \wedge y) = x$, for all $x, y \in L$.

Proposition 1.1.35. *Let $(L, \preccurlyeq)$ be a lattice.Then the following properties are equivalent:*

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad$ *for every $x, y, z \in L$.*

(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad$ *for every $x, y, z \in L$.*

Proof. (i) Let $x, y, z \in L$. Suppose that (i) holds, then

$$(x \vee y) \wedge (x \vee z) = \big[(x \vee y) \wedge x\big] \vee \big[(x \vee y) \wedge z\big]$$

By (v) of Proposition 1.1.34 it holds that $x \vee (x \wedge y) = x$. Also, by the commutativity of the infimum it holds that $(x \vee y) \wedge z = z \wedge (x \vee y)$. Thus, taking into account the assumption, we get

$$z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y) \ .$$

Hence, we obtain

$$\begin{aligned}\big[(x \vee y) \wedge x\big] \vee \big[(x \vee y) \wedge z\big] &= x \vee \big[(z \wedge x) \vee (z \wedge y)\\ &= x \vee \big[(x \wedge z) \vee (y \wedge z)\\ &= \big[x \vee (x \wedge z)\big] \vee \big[x \vee (y \wedge z)\big]\end{aligned}$$

by the distributivity of the supremum. So, by (iv) from Proposition 1.1.34

$$\big[x \vee (x \wedge z)\big] \vee \big[x \vee (y \wedge z)\big] = x \vee \big[x \vee (y \wedge z)\big] \ = x \vee (y \wedge z) \tag{1.1}$$

Therefore the assertion is proven.

(ii) We work analogously.

□

Definition 1.1.36. Let $(L, \preccurlyeq)$ be a lattice. If $\sup L$ and $\inf L$ exist in L, then $\sup L$ is called the *greatest element* of L, whereas $\inf L$ is called the *smallest element* of L.

Example 1.1.37. Let X be a set and denote by $P(X)$ the power set of X. Then $\inf P(X) = \emptyset$ and $\sup P(X) = X$ in view of Example 1.1.21. Since $\emptyset, X \in P(X)$, they are the smallest and the greatest element of $P(X)$, respectively.

Example 1.1.38. Let $D = \{(x, y) \in \mathbb{R}^2 : \ x \geq 0 \text{ and } y \geq 0\}$ and $(D, \preccurlyeq)$, where "$\preccurlyeq$" is the lexicographic order. Then $\inf D = (0, 0)$. Since $(0, 0) \in D$, it follows that $(0, 0)$ is the smallest element of D.

Definition 1.1.39. Let $(L, \preccurlyeq)$ be a lattice and $x \in L$. If there exist $y \in L$ such that $y \vee x = \sup L$ and $y \wedge x = \inf L$ then y is called *complement* of x and is denoted by x^c.

Theorem 1.1.40. *Let $(L, \preccurlyeq)$ be a distributive lattice such that $\sup L$ and $\inf L$ exist in L. Let x be an element of L such that x^c exists in L. Then x^c is unique.*

Proof. Let $x' \in L$ be another complement of x. Then $x' = x' \vee \inf L$ or equivalently $x' = x' \vee (x \wedge x^c)$ and, by distributivity, we obtain that $x' = (x' \vee x) \wedge (x' \vee x^c)$. The last equality yields the following

$$x' = \sup L \wedge (x' \vee x^c) = x' \wedge x^c.$$

Hence $x' = x' \vee x^c$. We work similarly for x^c and we end up to $x^c = x^c \vee x'$. Therefore $x^c = x'$. □

Example 1.1.41. Denote by D_{125} the set of all dividers of 125, i.e. $D = \{1, 5, 25, 125\}$. Then D_{125} with the divisibility relation is a totally ordered set. Moreover, it is easy to see that the distributive laws hold for all elements of D_{125}. But $25 \wedge 5 = 5$ and $25 \vee 5 = 25$, so, since the complement is unique, this implies that 25 and 5 have no complements. This is because 125 and 1 complement each other as $125 \vee 1 = 125 = \sup D_{125}$ and $125 \wedge 1 = 1 = \inf D_{125}$.

Definition 1.1.42. Let $(L, \preccurlyeq)$ be a distributive lattice. If $\inf L$ and $\sup L$ exist in L and every x has a complement $x^c \in L$, then $(L, \preccurlyeq)$ is called a *Boolean Algebra*.

Definition 1.1.43. Let $(L, \preccurlyeq)$ be a lattice and L_0 a subset of L. We call L_0 *sublattice* of L if $x \wedge y \in L_0$ and $x \vee y \in L_0$, for every $x, y \in L_0$.

Example 1.1.44. Let X be a topological space and denote by $P(X)$ the power set of X and by $\mathbf{O}$ the set of all open sets of $P(X)$. Then $(\mathbf{O}, \subset)$ is a sublattice of $P(X)$. Indeed, for any $U, V \in \mathbf{O}$ it holds that $U \wedge V \in \mathbf{O}$ and $U \vee V \in \mathbf{O}$.

Example 1.1.45. Let X be a topological space and denote by $P(X)$ the power set of X and by $\mathbf{C}$ the set of all closed sets of $P(X)$. Then $(\mathbf{C}, \subset)$ is a sublattice of $P(X)$. Indeed, for any $U, V \in \mathbf{C}$ it holds that $U \wedge V \in \mathbf{C}$ and $U \vee V \in \mathbf{C}$.

Definition 1.1.46. Let $(L, \preccurlyeq)$ be a lattice. L is said to be *(countably) complete* if every (countable) subset of L has infimum and supremum. A sublattice L_0 of a complete lattice L is called a *complete sublattice* of L if, for every subset A of L_0, the elements $\sup A$ and $\inf A$ both belong in L_0.

— CHECKED UNTIL THIS POINT —

Remark 1.1.47. A subset L_0 of a countably complete or complete lattice L can be a countably complete or complete lattice, respectively, despite not being a sublattice of L, under the inherited order by L.

Definition 1.1.48. Let V be a vector space over $\mathbb{R}$ and $\preccurlyeq$ an order. V is called an *ordered vector space* if the following are satisfied:

(i) If $x \preccurlyeq y$, then $x + z \preccurlyeq y + z$ holds for every $x, y, z \in V$.

(ii) If $x \preccurlyeq y$, then $\lambda x \preccurlyeq \lambda y$ for every $x, y \in V$ and $\lambda \in \mathbb{R}_+$.

An ordered vector space is denoted by $(V, \preccurlyeq)$.

Definition 1.1.49. Let $(V, \preccurlyeq)$ be an ordered vector space. V is called a *vector lattice* (also called a *Riesz space* or a *linear lattice*), if $x \wedge y$ and $x \vee y$ both exist and belong in V for every $x, y \in V$.

Example 1.1.50. The one dimensional vector space $\mathbb{R}^n$ over $\mathbb{R}$ endowed with the lexicographic order is a vector lattice. Let $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}$ such that $\bar{x} \preccurlyeq \bar{y}$. Then obviously $\bar{x} + \bar{z} \preccurlyeq \bar{y} + \bar{z}$ holds for all $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^n$. Moreover for every $\lambda \in \mathbb{R}^+$ and $\bar{x}, \bar{y} \in \mathbb{R}^n$, such that $\bar{x} \preccurlyeq \bar{y}$, iy holds that $\lambda\bar{x} \preccurlyeq \lambda\bar{y}$. Thus $\mathbb{R}^n$ is an ordered vector space. To validate that $\mathbb{R}^n$ is a vector lattice, we need to prove $\bar{x} \wedge \bar{y}$ and $\bar{x} \vee \bar{y}$ both exist in $\mathbb{R}^n$, for every $\bar{x}, \bar{y} \in \mathbb{R}^n$. Now we define $\bar{x} \wedge \bar{y} = min(x_i, y_i)$ for all $i = 1, \ldots, n$ and $x \vee y = max(x_i, y_i)$ for all $i = 1, \ldots, n$. Both $\bar{x} \wedge \bar{y}$ and $\bar{x} \vee \bar{y}$ exist in $\mathbb{R}$. Hence

$$\bar{x} \wedge \bar{y} = \big(min(x_1, y_1), min(x_2, y_2), \ldots, min(x_n, y_n)\big)$$

and

$$\bar{x} \vee \bar{y} = \big(max(x_1, y_1), max(x_2, y_2), \ldots, max(x_n, y_n)\big)$$

exist in $\mathbb{R}^n$. Therefore $\mathbb{R}^n$, with the lexicographic order, is a vector lattice.

Proposition 1.1.51. *Let V_1, V_2 be ordered vector spaces. and $T : V_1 \mapsto V_2$ a linear bijective map. Then $x \preccurlyeq_{V_1} y$ if and only if $Tx \preccurlyeq_{V_2} Ty$ defines a partial order satisfying (i) and (ii) from 1.1.48.*

Proof. Firstly we need to prove that this is actually a partial order.

- Let $x \in V_1$. Then $x \preccurlyeq_{V_1} x$ holds since V_1 is a vector space and thus $Tx \preccurlyeq_{V_2} Tx$ holds, since T is linear.

- Let $x, y \in V$ such that $x \preccurlyeq_{V_1} y$ and $y \preccurlyeq_{V_1} x$ hold. This implies that $Tx \preccurlyeq_{V_2} Ty$ and $Ty \preccurlyeq_{V_2} Tx$ hold. Since V_1 and V_2 are vector spaces, it holds that $x = y$ and $Tx = Ty$.

- Let $x, y, z \in V$ such that $x \preccurlyeq_{V_1} y$ and $y \preccurlyeq_{V_1} z$. Then, it is imminent that $Tx \preccurlyeq Tz$.

Now we need to show that this partial order satisfies (i) and (ii) from 1.1.48.

(i) Let $x, y \in V_1$ such that $x_1 \preccurlyeq_{V_1} y_1$. Since V_1 is a vector lattice it holds that $x + z \preccurlyeq_{V_1} y + z$ for all $z \in V_1$. Since T is linear and V a vector lattice it holds that $T(x + z) \preccurlyeq_{V_2} T(y + z)$.

(ii) Let $x, y \in V$ and $\lambda \in \mathbb{R}_+$ such that $x \preccurlyeq_{V_1} y$. Since V_1 is a vector lattice, it holds that $\lambda x \preccurlyeq_{V_1} \lambda y$. Since T is linear and V_2 a vector lattice, it holds that $\lambda Tx \preccurlyeq_{V_2} \lambda Ty$.

Therefore the proof is complete. □

Definition 1.1.52. Let V_1, V_2 be ordered vector spaces. A linear bijective map $T : V_1 \mapsto V_2$ such that $x \preccurlyeq_{V_1} y$ if and only if $Tx \preccurlyeq_{V_2} Ty$ is called an *isomorphism* of ordered vector spaces or simply an *order isomorphism*.

Remark 1.1.53. The range of an order isomorphism of a vector lattice V_1 into a vector lattice V_2 is not, necessarily, a sublattice of V_2.

Example 1.1.54. Let X be a topological space and denote by $(\mathbf{O}, \subset)$ the lattice of all open subsets of X. Let $p : (2^X, \subset) \mapsto (\mathbf{O}, \subset)$ be an order isomorphism. Then, by Example **??** it holds that $p(2^X)$ is not a sublattice of $(\mathbf{O}, \subset)$.

Definition 1.1.55. Let $(V, \preccurlyeq)$ be an ordered vector space. We denote by V_+ the set of all positive elements $x \in V$. Then, V_+ is called the *positive cone* of V and all $x \in V_+$ are called *positive*.

Definition 1.1.56. Let B be a Boolean Algebra. A real function $\mu : B \mapsto \mathbb{R}$ is called *finitely additive* if it satisfies:

$$\mu(x \vee y) = \mu(x) + \mu(y)$$

when $x \wedge y = 0$ for all $x, y \in B$.

Remark 1.1.57. Let B be a Boolean Algebra. Denote by $ba(B)$ the space of all bounded finitely additive functions with values from B. Then $ba(B)$ endowed with the canonical ordering is a vector lattice. The lattice operations are given as follows:

$$\mu_1 \vee \mu_2(x) = \sup_{y \preccurlyeq x}\{\mu_1(x) + \mu_2(x \wedge y^c)\}$$

and

$$\mu_1 \wedge \mu_2(x) = \inf_{y \preccurlyeq x}\{\mu_1(x) + \mu_2(x \wedge y^c)\}$$

hold for all $x, y \in V$.

Proposition 1.1.58. *Let $\{V_a\}_{\alpha \in A}$ be a family of vector lattices. Then the Cartesian product $\prod_\alpha V_a$ is a vector lattice if the vector and lattice operations are defined "coordinate-wise".*

Proof. It is obvious that the Cartesian product of a family of vector spaces is a vector space if the vector operations are defined coordinate-wise. Now let $\bar{x}, \bar{y} \in \{V_a\}$. Then

$$\bar{x} = (x_1.x_2, \ldots, x_a) \quad \text{and} \quad \bar{y} = (y_1, y_2, \ldots, y_a).$$

Since each V_a is a vector lattice for each $\alpha \in A$ it holds that $x_a \wedge y_a \in V_a$ and $x_a \vee y_a \in V_a$. Hence, the lattice operations defined as follows

$$\bar{x} \wedge \bar{y} = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_a \wedge y_a)$$

and

$$\bar{x} \vee \bar{y} = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_a \vee y_a)$$

are well defined and $\prod_\alpha V_a$ is indeed a vector lattice. □

Definition 1.1.59. Let $\{V_a\}_{\alpha \in A}$ be a family of vector lattices. The ordering of $\prod_\alpha V_a$ so obtained is called *canonical* and is a generalization of the canonical ordering defined in Example 1.1.7.

Definition 1.1.60. Let $\{V_a\}_{\alpha \in A}$ be a family of vector lattices. Then the direct sum of the family $\{V_a\}$, denoted by $\bigoplus_\alpha \{V_a\}$ is understood to be the vector sublattice of $\prod_\alpha V_a$ containing exactly all finitely non-zero families $(x_\alpha)_{\alpha \in A}$.

Proposition 1.1.61. *Suppose $(V, \preccurlyeq)$ is a vector lattice and A is a non-empty subset of V endowed with the ordering of V. Then the following are valid:*

(i) If $\sup A$ exists then $x + \sup A = \sup(x + A)$.

(ii) If $\inf A$ exists then $x + \inf A = \inf(x + A)$.

(iii) $\sup A = -\inf(-A)$.

Proof. We recall that if A is a subset of V then it holds that $x + A = \{x + y : y \in A\}$.

(i) Let $B =: \{x + y : y \in A\}$. We want to prove that $x + \sup A = \sup(x + A)$. We will show that both $x + \sup A$ and $\sup(x + A)$ are equal to $\sup B$. Obviously $\sup B = \sup(x + A)$. Moreover suppose that there exists an other upper bound M for B. Then $x + y \preccurlyeq M$ for all $y \in A$. Since $\sup A$ exists, this also holds for the supremum of A. Hence, $x + \sup(A) \preccurlyeq M$ and thus $x + \sup(A) = \sup B$. Since the supremum of B is unique we have that $x + \sup A = \sup(x + A)$.

(ii) Similarly for $\inf B$. Let $B =: \{x + y : y \in A\}$. Obviously $\inf B = \inf(x + A)$. Suppose there exist another lower bound m for B. Then $m \preccurlyeq x + y$ for all $y \in A$. Since $\inf B$ exists, this holds for the infimum of A also. Hence $m \preccurlyeq x + \inf(A)$ and thus $x + \inf(A) = \inf(B)$. Since the infimum of B is unique, then $x + \inf A = \inf(x + A)$.

(iii) Recall that if A is a subset of V, then $-A =: \{-x : x \in A\}$. Note that for any $-x \in -A$ we have that $\inf(-A) \preccurlyeq -x$. Thus $-\inf(-A) \succcurlyeq x$. Hence $-\sup(-A)$ is greater or equal to x . Suppose M is another upper bound of $-A$. Then $M \succcurlyeq x$, for all $x \in A$. This implies $-M \preccurlyeq -x$ and thus $-M$ is a lower bound of $-A$. Hence $-M \preccurlyeq inf(-A)$ or, equivalently, $M \succcurlyeq -\inf(-A)$. So $-\inf(-A)$ is the least upper bound of $-A$. Therefore $-\sup(-A) = -\inf(-A)$. But $\sup(kA) = k \sup A$ and hence $\sup A = -\inf(-A)$.

□

Definition 1.1.62. Let $(L, \preccurlyeq)$ be a lattice. For every $x \in L$, we define the *positive part*, the *negative part* and the *absolute value* or the *modulus* of x by $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| = x \vee (-x)$, respectively.

Example 1.1.63. Let X be a topological space. Denote by $C(X)$ the space of all real valued, continuous functions on X. Take $f(x) = x$. Then

$$f^+ = f(x),\ x \succcurlyeq 0$$

and

$$f^- = f(x)\ ,\ x \preccurlyeq 0$$

and

$$|f|(x) = \begin{cases} x & x \succcurlyeq 0 \\ -x & x \preccurlyeq 0. \end{cases}$$

Definition 1.1.64. Let $(L, \preccurlyeq)$ be a lattice. If $|x| \wedge |y| = 0$ holds for every pair of elements $x, y \in L$ then x, y are called *orthogonal* or *disjoint* or *lattice disjoint.* We denote such elements by $x \perp y$.

Definition 1.1.65. Let $(L, \preccurlyeq)$ be a lattice and $A, B \subset V$. Then A and B are called *orthogonal* if $x \perp y$ holds for every pair $(x, y) \in A \times B$.

Definition 1.1.66. Let $(V, \preccurlyeq)$ be an ordered vector space and $U \subset V_+$ be a non-empty set. If $0 \notin U$ and $u \wedge v = 0$ holds for every $u, v \in U$ such that $u \neq v$, then U is called an *orthogonal system*.

Remark 1.1.67. Let $(V, \preccurlyeq)$ be a vector lattice and $\emptyset \neq A \subset V$. We denote

$$A^{\perp} = \{x \in V : x \perp y, \ \textit{for all}\ y \in A\}.$$

The following provide some primary properties of the absolute value.

Proposition 1.1.68. *Let $(V, \preccurlyeq)$ be a vector lattice. Then the following properties hold for all $x \in V$:*

(i) $x = x^+ - x^-$.

(ii) $|x| = x^+ + x^-$.

(iii) $|x| = 0 \Leftrightarrow x = 0$, $|\lambda x| = |\lambda||x|$, $|x + y| \preccurlyeq |x| + |y|$. *(triangle inequality)*

(iv) $x + y = x \wedge y + x \vee y$.

(v) $|x - y| = x \wedge y - x \vee y$.

(vi) $|x \vee y - x_1 \vee y_1| \preccurlyeq |x - x_1| + |y - y_1|$.

(vii) $|x \wedge y - x_1 \wedge y_1| \preccurlyeq |x - x_1| + |y - y_1|$.

Proof. (i) First, we need to validate that $x = x^+ - x^-$ for all $x \in V$. It suffices to observe that, for $y = 0$ by (iv) we obtain

$$x + 0 = x \wedge 0 + x \vee 0.$$

By Definition 1.1.62 the assertion follows.

(ii) We need to verify that $|x| = x^+ + x^-$. Let $x \in V$ then $x^+ + x^-$ is equal to $x^+ - x^- + 2x^-$. By (i) this is equal to $x + 2x^-$.Thus

$$\begin{aligned} x^+ + x^- = 2x^- + x &= x + (-2x) \vee 0 \\ &= (-2x + x) \vee (0 + x) \\ &= (-x) \vee x \\ &= x \wedge (-x). \end{aligned}$$

- First we need to check that $|x| = 0$ or equivalently that $x = 0$. Suppose $x = 0$ then $|x| = x \vee (-x)$ by (ii). Thus $|0| = 0 \vee (-0) = 0$ which implies that $|x| = 0$. Conversely let $x \in V$ and suppose $|x| = 0$. Therefore $x \vee (-x) = 0$. By the anti-symmetry of the ordering, we obtain that if $x \vee (-x) \preccurlyeq 0$ and $x \vee (-x) \succcurlyeq 0$ hold then $x \vee -x = 0$. This implies $x \preccurlyeq 0$ and $x \succcurlyeq 0$ which yields that $x = 0$, by the anti-symmetric property.

- To validate that $|\lambda x| = |\lambda||x|$, let $x \in V$ and $\lambda \in R^+$. Since V is a vector space, $x \preccurlyeq |x|$ implies that $\lambda x \preccurlyeq \lambda|x|$ for all $x \in V$ and $\lambda \in R^+$. Hence $|\lambda x| \preccurlyeq \lambda|x|$, since every element is lower or equal to its absolute value. This implies $\lambda|x| \preccurlyeq |\lambda||x|$. Hence $|\lambda x| \preccurlyeq |\lambda||x|$.

  Now if $x \preccurlyeq x$ then $|x| \preccurlyeq |x|$ which is equivalent to $|x| \preccurlyeq |\frac{1}{\lambda} \cdot \lambda \cdot x|$ for $\lambda \in \mathbb{R}^+$. This leads to $|x| \preccurlyeq \frac{1}{|\lambda|} \cdot |\lambda x|$ or equivalently $|\lambda||x| \preccurlyeq |\lambda x|$. Since $|\lambda x| \preccurlyeq |\lambda||x|$ holds it follows that $|\lambda| \cdot |x| = |\lambda x|$.

- Now the last property is the triangle inequality. Let $x, y \in V$. Then $x^+ \preccurlyeq |x|$ and $x^- \preccurlyeq |x|$, $y^+ \preccurlyeq |y|$ and $y^- \preccurlyeq |y|$ hold for every $x, y \in V$. This also implies that $(x+y)^+. \preccurlyeq |x| + |y|$ and $(x+y)^- \preccurlyeq |x| + |y|$. By adding by members we obtain that $(x+y)^+ + (x+y)^- \preccurlyeq |x| + |y|$. But by ii) $(x+y)^+ + (x+y)^- = |x+y|$ and therefore $|x+y| \preccurlyeq |x| + |y|$.

(iii) We need to verify that $x+y = x \vee y + x \wedge y$ holds for all $x, y \in V$. By translation invariance from Proposition 1.1.61 we obtain that

$$\begin{aligned} x_1 - (x \wedge y) + y_1 &= x_1 + (-x) \vee (-y) + y_1 \\ &= (x_1 - x + y) \vee (x_1 - y + y_1). \end{aligned}$$

Let $x = x_1$ and $y = y_1$. Then

$$\begin{aligned} x - (x \wedge y) + y &= (x - x + y) \vee (x - y + y) \\ x + y &= y \vee x + x \wedge y \\ x + y &= x \vee y + x \wedge y. \end{aligned}$$

(iv) Now $x \vee y = x + (y - x) \vee 0 = x + (y - x)^+$ by the translation invariance. Then $x \wedge y = x + (y - x) \wedge 0$ or equivalently $x \wedge y = x - (x - y) \vee 0$ which is also equivalent to $x \wedge y = x - (x - y)^+$. Substracting by members, $x \vee y = x + (y - x) \vee 0 = x + (y - x)^+$ and $x \wedge y = x - (x - y)^+$ we obtain

$$\begin{aligned} x \vee y - x \wedge y &= x + (y - x)^+ - x + (x - y)^+ \\ &= (y - x)^+ + (x - y)^+ \\ &= (y - x)^+ + (y - x)^-. \end{aligned}$$

Thus $x \vee y - x \wedge y = |y - x|$ by (ii). Since $|x - y| = |y - x|$, we have that $x \vee y - x \wedge y = |x - y|$ holds for all $x, y \in V$.

(v) To check that $|x \vee y - x_1 \vee y_1| \preccurlyeq |x - x_1| + |y - y_1|$, let $x, y, x_1, y_1 \in V$. We observe that

$$\begin{aligned} x \vee y - x_1 \vee y_1 &= x \vee y + x_1 \vee y - x_1 \vee y - x_1 \vee y_1 \\ &= (x \vee y - x_1 \vee y) + (x_1 \vee y - x_1 \vee y_1) \end{aligned}$$

For the first additional of the last equality we have that

$$\begin{aligned} x \vee y - x_1 \vee y &= y + (x - y) \vee 0 - y - (x_1 - y) \vee 0 \\ &= (x - y)^+ - (x_1 - y)^+ \end{aligned}$$

which is less or equal than $|x - y| - |x_1 - y|$ with respect to the ordering. By reverse triangle inequality we get

$$|x - y| - |x_1 - y| \preccurlyeq |x - y - x_1 + y| = |x - x_1|.$$

Thus $x \vee y - x_1 \vee y \preccurlyeq |x - x_1|$. Hence $(x_1 \vee y - x_1 \vee y_1) \preccurlyeq |y - y_1|$ and therefore

$$x \vee y - x_1 \vee y_1 = (x \vee y - x_1 \vee y) + (x_1 \vee y - x_1 \vee y) \preccurlyeq |x - x_1| + |y - y_1|,$$

and

$$\begin{aligned} |x \vee y - x_1 \vee y_1| &= |x \vee y - x_1 \vee y + x_1 \vee y - x_1 \vee y| \\ &\preccurlyeq |x \vee y - x_1 \vee y| + |x_1 \vee y - x_1 \vee y| \\ &\preccurlyeq |x - x_1| + |y - y_1| \end{aligned}$$

In that case $|x \vee y - x_1 \vee y_1| \preccurlyeq |x - x_1| + |y - y_1|$.

(vi) To prove the last property we work similarly as in (vi) by replacing x, x_1, y_1, y by $-x, -y, -y_1, -x_1$ and using mainly (iii) from Proposition 1.1.61.

□

Corollary 1.1.69. *Let $(V, \preccurlyeq)$ be a vector lattice. The following relations hold for every $x, y \in V$:*

(i) $x \vee y = \frac{1}{2}(x + y + |x + y|)$.

(ii) $x \wedge y = \frac{1}{2}(x + y - |x - y|)|$.

(iii) $|x| \vee |y| = \frac{1}{2}(|x + y| + |x - y|)$.

(iv) $|x| \wedge |y| = \frac{1}{2}\big||x + y| - |x - y|\big|$.

In particular $x \perp y$ is equivalent to $|x + y| = |x - y|$.

Proof. (i) We already proved in Proposition 1.1.68 that $x + y = x \wedge y + x \vee y$ and $|x - y| = x \vee y + x \wedge y$ hold for every $x, y \in V$. Now since $x \wedge y = x + y - x \vee y$ we have that $|x - y| = x \vee y - (x + y) + x \vee y$. Equivalently $2(x \vee y) - (x + y) = |x - y|$ and hence $x \vee y = \frac{1}{2}(x + y + |x - y|)$.

(ii) Moreover we also seen that $x + y - x \wedge y = x \vee y$ holds for every $x, y \in V$ in Proposition 1.1.68. Thus $|x - y| = x \vee y - x \wedge y$ and this is equivalent to $|x - y| = x + y - x \wedge y - x \wedge y$. Therefore $2(x \wedge y) = x + y - |x - y|$ and hence $x \wedge y = \frac{1}{2}(x + y - |x - y|)$.

(iii) We will now verify that $|x| \vee |y| = \frac{1}{2}(|x+y| + |x-y|)$. By definition of the absolute values, $|x| \vee |y| = (x \vee -x) \vee (y \vee -y)$ holds for all $x, y \in V$. This is equal to $(x \vee -y) \vee (y \vee -x)$ by distributivity of the supremum. Now by direct direct computation and (i) we obtain that

$$\begin{aligned}
(x \vee -y) \vee (y \vee -x) &= \frac{1}{2}\big(x - y - |x - (-y)|\big) \vee \frac{1}{2}\big(y - x + |-x-y|\big) \\
&= \frac{1}{2}\big(x - y - |x - (-y)|\big) \vee \big(y - x + |-x-y|\big) \\
&= \frac{1}{2}|x+y| + \frac{1}{2}[(x-y) \vee (y-x)] \\
&= \frac{1}{2}|x+y| + \frac{1}{2}|x-y| \\
&= \frac{1}{2}(|x+y| + |x-y|).
\end{aligned}$$

Hence, $|x| \vee |y| = \frac{1}{2}(|x+y| + |x-y|)$ holds for all $x, y \in V$.

(iv) By applying (iv) from Proposition 1.1.68 we obtain that

$$\begin{aligned}
|x| \wedge |y| + |x| \vee |y| = |x| + |y| &\Leftrightarrow \\
|x| \wedge |y| &= |x| + |y| - |x| \vee |y| \\
&= |x| + |y| - \frac{1}{2}|x+y| - \frac{1}{2}|x-y|
\end{aligned}$$

from (iii). Let $x = u + v$ and $y = u - v$. Then

$$\begin{aligned}
|x| + |y| - \frac{1}{2}|x+y| - \frac{1}{2}|x-y| &= |u+v| + |u-v| - \frac{1}{2}|u+v+y-v| - \frac{1}{2}|u+v-u+v| \\
&= |u+v| + |u-v| - \frac{1}{2}|2u| - \frac{1}{2}|2v| \\
&= 2|u| \vee |v| - |u| - |v| \\
&= \frac{2}{2}\big(|u| + |v| + \big||u| - |v|\big|\big) - |u| - |v| \\
&= \big||u| - |v|\big| = \big|\frac{1}{2}|x+y| = |x-y|\big|.
\end{aligned}$$

Thus, $|x| \wedge |y| = \frac{1}{2}\big||x+y| - |x-y|\big|$ holds for every $x, y \in V$.

To prove the last assertion it suffices to observe that $x \perp y$ or equivalently that $|x| \wedge |y| = 0$. Hence by (iv) we obtain that

$$0 = \frac{1}{2}\big||x+y| - |x-y|\big|$$

or equivalently

$$|x+y| = |x-y|.$$

□

Corollary 1.1.70. *Let $(V, \preccurlyeq)$ be a vector lattice. Then the following hold for every $x, y \in V$:*

(i) $x \preccurlyeq y$ implies $x^+ \preccurlyeq y^+$ and $x^- \preccurlyeq y^-$ and conversely.

(ii) $x \perp y$ if and only if $|x| \vee |y| = |x| + |y|$.

(iii) $x \perp y$ implies $(x+y)^+ = x^+ + y^+$ and $|x+y| = |x| + |y|$.

Proof. (i) If $x \preccurlyeq y$ then $x^+ \preccurlyeq y^+$ and $x^- \preccurlyeq y^-$ hold for every $x, y \in V$, by Proposition 1.1.68. Conversely if $x^+ \preccurlyeq y^+$ and $x^- \preccurlyeq y^-$ hold then we need to prove that $x \preccurlyeq y$. Then $x^+ \preccurlyeq y^+$ implies that $x + x^- \preccurlyeq y + y^-$ and $x^- \preccurlyeq y^-$ implies that $x^+ - x \preccurlyeq y^+ - y$. Adding by members, we obtain $x^+ + x^- \preccurlyeq y^+ + y^-$ or equivalently $|x| \preccurlyeq |y|$. Since $x \preccurlyeq |x|$ and $y \preccurlyeq |y|$, then

$$x \preccurlyeq |x| \preccurlyeq |y|$$

and hence $x \preccurlyeq y$.

(ii) Now if $x \perp y$ then by definition $|x| \wedge |y| = 0$. Thus by (iv) from 1.1.68 we obtain

$$|x| + |y| = |x| \wedge |y| + |x| \vee |y|$$

or equivalently

$$|x| + |y| = |x| \vee |y|.$$

(iii) Let $x, y \in V$ such that $x \perp y$ and assume that $(x+y) \wedge z \preccurlyeq x \wedge z + y \wedge z$ hold for every $x, y, z \in V_+$. We need to prove that $(x^+ + y^+) \wedge (x^- + y^-) = 0$. First, we need to prove that $x^+ \perp x^-$, where $x = x^+ - x^-$. Hence we need to verify that $x^+ \wedge x^- = 0$. Indeed

$$\begin{aligned} x^+ \wedge x^- &= x^- - x^- + x^+ \wedge x^- \\ &= x^- + (x^+ - x^-) \wedge 0 \\ &= x^-(-(x^+ + x^-) \vee 0) \\ &= x^-(-x) \vee 0 \\ &= x^- - x^- = 0. \end{aligned}$$

Now we observe that $|x+y| = (x+y)^+ + (x+y)^-$. Since the decomposition is unique, we have

$$x^+ + x^- + y^+ + y^- = (x+y) = (x+y)^+ + (x+y)^-.$$

So it suffices to prove that $(x+y)^+ \perp (x+y)^-$. Indeed

$$\begin{aligned} (x+y)^+ \wedge (x+y)^- &\preccurlyeq x^+ \wedge (x^- + y^-) + y^+ \wedge (x^- + y^-) \\ &\preccurlyeq x^+ \wedge x^- + x^+ \wedge y^- + y^+ \wedge x^- + y^+ \wedge y^- = 0. \end{aligned}$$

Since $x^+ \wedge y^- \preccurlyeq |x| \wedge |y| = 0$ and $y^+ \wedge x^- \preccurlyeq |x| \wedge |y| = 0$ it follows that $(x+y)^+ = x^+ + y^+$ and $|x+y| = |x| + |y|$.

□

We conclude this paragraph with results regarding the infinite distributivity of lattices.

Proposition 1.1.71. *Let $(V, \preccurlyeq)$ be a vector lattice and $(x_j)_{j\in J}$, $(y_j)_{j\in J}$ be families of elements in V. If $\sup_j(x_j) =: x$ and $\inf_j(y_j) =: y$ exist in V, then*

$$x \wedge z := \sup_j (x_j \wedge z)$$

and

$$x \vee y := \inf_j (y_j \vee z)$$

hold for every $z \in V$.

Proof. Since $x \succcurlyeq x_j$, for each $j \in J$ we have that $x \wedge z \succcurlyeq x_j \wedge z$ holds for arbitrary $z \in V$. Suppose there exists another upper bound $u \in V$ such that $u \succcurlyeq x_j \wedge z$ for all j. Then by using (iv) from Proposition 1.1.68 we obtain $u \succcurlyeq x_j + z = x_j \vee z$. Moreover $x \vee z \succcurlyeq x_j \vee z$ holds for every $z \in V$ since $x \succcurlyeq x_j$ from hypothesis. Then by adding u in both sides we obtain $u + x \vee z \succcurlyeq u + x_j \vee z$. By (iv) again we get $u + x \vee z \succcurlyeq x_j + z$, for all $j \in J$. Thus it holds for the $\sup_j x_j$. So $u + x \vee z \succcurlyeq x + z$. Hence $u \succcurlyeq (x + z - z \vee z) = x \wedge z$. Therefore $u \succcurlyeq x \wedge z$ for a random upper bound. As a consequence

$$x \wedge z = \sup_j (x_j \wedge z)$$

The proof of the other inequality is similar. □

Corollary 1.1.72. *Let $(V, \preccurlyeq)$ be a vector lattice and $A \subset V$. Then $A^\perp$ is a vector subspace of V and contains the suprema and infima of all of its subsets.*

Proof. Fix $z \in V$ such that $x, y \in \{z\}^\perp$. To verify that $A^\perp$ is a vector subspace of V it suffices to prove that $|\alpha x + \beta y| \wedge |z| \preccurlyeq 0$ holds for every $x, y \in A^\perp$ and $\alpha, \beta \in \mathbb{R}$, i.e $A^\perp$ is closed under orthogonality. By (iii) from Corollary 1.1.69 we obtain

$$|\alpha x| \vee |\beta y| = \frac{1}{2}\big(|\alpha x + \beta y| + |\alpha x - \beta y|\big)$$

or equivalently

$$2\big(|\alpha x| \vee |\beta y|\big) = \big(|\alpha x + \beta y| + |\alpha x - \beta y|\big),$$

which implies that

$$|\alpha x + \beta y| \preccurlyeq 2\big(|\alpha x| \vee |\beta y|\big) \Leftrightarrow |\alpha x + \beta y| \preccurlyeq 2\big(|\alpha||x| \vee |\beta||y|\big)$$

or

$$|\alpha x + \beta y| \wedge z \preccurlyeq \big(|\alpha||x| \vee |\beta||y|\big) \Leftrightarrow |\alpha x + \beta y| \preccurlyeq 2\big((|\alpha||x| \wedge z) \vee (|\beta||y| \wedge z)\big).$$

Since $x, y \in \{z\}^\perp$, it holds that $|x| \wedge z = 0$ and $|y| \wedge z = 0$. Thus

$$|\alpha x + \beta y| \wedge z \preccurlyeq 0.$$

Therefore $\{z\}^\perp$ is a vector subspace of V. Now, if $B \subset \{z\}^\perp$ and $\sup B$ or $\inf B$ exists in B, then $supB \in \{z\}^\perp$. This is true as $x \wedge z = sup(x_j \wedge z) = 0$, where $x = supB$ and $x_j \in B$. Lastly, since $\{z\}^\perp$ is a vector subspace then the intersection of all vector subspaces of the form $\{z\}^\perp$ is a vector subspace. Therefore $A^\perp = \bigcap_{z\in A} \{z\}^\perp$. □

Corollary 1.1.73. *Any subset A of V consisting of elements, which are pairwise orthogonal, is linearly independent.*

Proof. Let $u_1, \dots, u_n, n \in \mathbb{N}$ be non zero elements of V such that $u_i \perp u_j$ for each i, j with $i \neq j$. Suppose that V is linearly dependent. Then there exist an integer $1 \leqslant k \leqslant n$ and numbers $b_j \in \mathbb{R}$ such that

$$u_k = u_1 b_1 + u_2 b_2 + \cdots + u_{k-1} b_{k-1} + u_{k+1} b_{k+1} + \dots u_n b_j.$$

Now let $A = (u_1, \dots, u_n)$ be the set of the Corollary 1.1.72. Since $u_k = \sum_{j \neq k} b_j u_j$ then $u_k = \sup A$ and by Corollary 1.1.72 we have $u_k \in A^{\perp}$ which leads to $u_k \perp u_k$ since u_k also belongs in A. Hence $u_k = 0$ which is a contradiction. Thus V is linearly independent. □

Proposition 1.1.74. *Let $(L, \preccurlyeq)$ be any lattice and $U_j \subset L$, $j \in J$ such that $x_j = \sup U_j$ for each j. If $x := \sup_j x_j$ exists, then*

$$x = \sup \bigcup_j U_j.$$

Proof. Let $U_j \subset L$, $j \in J$. We want to prove that x is the least upper bound of $\bigcup_j U_j$. Suppose there exist an other upper bound $u \in L$ of $\bigcup_j U_j$. Then $u \succcurlyeq x_j \quad \forall j$. This implies $u \succcurlyeq \sup_j x_j$. Hence $u \succcurlyeq x \quad \forall j$. Therefore

$$x = \sup \bigcup_j U_j.$$

□

Proposition 1.1.75. *Let $(V, \preccurlyeq)$ be a vector lattice. If x, y are positive elements in V then*

$$[0, x + y] = [0, x] + [0, y] \tag{DC}$$

holds for the corresponding order intervals. Equivalently if x_i and y_j belong in V_+ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that $\sum x_i = \sum y_j$ then there exist $z_{ij} \in V$ which satisfy $x_i = \sum_i z_{ij}$ and $y_j = \sum_j z_{ij}$ for all i, j.

Proof. To prove the first form of the decomposition property we need to prove that $[0, x + y] \subset [0, x] + [0.y]$ and the reverse inclusion. Since V is an ordered space, the first inclusion of the corresponding intervals holds for all $x, y \in V_+$. To verify the converse inclusion, let $z \in [0, x + y]$ define $u := z \wedge x$ and $v := z - u$, where $z = u + v$. Since $u \in [0, x]$ by definition we need to show that $v \in [0, y]$. Hence by translation invariance and Proposition 1.1.61 we get

$$v = z - z \wedge x = z - x \wedge z = z + (-x \vee -z) = (z - x) \vee 0.$$

Thus $v \preccurlyeq (x + y - x) \vee 0$, which implies that $v \preccurlyeq y \vee 0$. Therefore $v \succcurlyeq 0$ and $v \preccurlyeq y$. So $v \in [0, y]$. □

Corollary 1.1.76. *For every $x, y, z \in V_+$ it holds that $(x+y) \wedge z \preccurlyeq x \wedge z + y \wedge z$.*

Proof. Let $w \in V_+$ such that $w := (x+y) \wedge z$. Then $0 \preccurlyeq w \preccurlyeq x+y$ which implies that $= w_1 + w_2$ by DC , where $w_1 \in [0,x]$ and $w_2 \in [0,y]$. Then $w_1 \preccurlyeq w \preccurlyeq z$ and$w_2 \preccurlyeq w \preccurlyeq z$ hold. Thus $w_1 \preccurlyeq x \wedge z$ and $w_2 \preccurlyeq y \wedge z$. Therefore $w = w_1 + w_2 = (x+y) \wedge z \preccurlyeq x \wedge z + y \preccurlyeq z$. □

Definition 1.1.77. Let $(D, \preccurlyeq)$ be an ordered set and $F \subset D$ is a filter. Then F is said to *order converge* to $x \in D$ if F contains a family of order intervals with intersection $\{x\}$.

Definition 1.1.78. Let $(D, \preccurlyeq)$ be an ordered set and $(x_n)_{n \in \mathbb{N}}$ be a sequence in D. Then (x_n) is said to *order converge* to $x \in D$ if its section filter contains a sequence of order intervals with intersection $\{x\}$.

Definition 1.1.79. Let $(D, \preccurlyeq)$ be an ordered set and $U_0 \subset D$ a non void set. U_0 is called *order dense* if for every $x \in D$ there exists a filter $F \subset D$ such that its section filter belongs in U_0 and F order converges to x.

Definition 1.1.80. Let $(V, \preccurlyeq)$ be a vector space and $A \subset V$ a non empty set. A family $(u_i)_{i \in I}$ of subsets of A is called *directed* if for every i, j there exists k such that $u_i \subset u_k$ and $u_j \subset u_k$.

Example 1.1.81. Let $(V, \preccurlyeq)$ be a vector space. Then the section filters of upward or downward directed families $(u_a)_{\alpha \in A}$ with supremum or infimum respectively are the prime examples of order convergent filters in V.

Remark 1.1.82. We use the notations $u_a \uparrow u$ for upward directed families and $u_a \downarrow u$ for downward directed families are often used.

Proposition 1.1.83. *Let $(V, \preccurlyeq)$ be a vector lattice and F a filter on V. Then F is order convergent to $x \in V$ if and only if there exists a family $(u_\alpha)_{\alpha \in A}$ in V such that $u_a \downarrow 0$ and the sets $F_\alpha := \{u \in V : |u - x| \preccurlyeq u_\alpha\}$ belong to F.*

Proof. (i) Suppose that F is order convergent to $x \in V$. This implies that F contains a family of order intervals with intersection $\{x\}$. Denote this family by $(u_\alpha)_{\alpha \in A}$. Thus $\cap_\alpha u_\alpha = \{x\}$. This implies that there exists a subfamily (z_α) of order intervals such that for all $v \in V$ it holds $|v - x| \preccurlyeq z_a$ for all $\alpha \in A$. Hence it holds that $z_a \downarrow x$. Since z_a is a subfamily of order intervals $z_a \downarrow 0$ must hold by hypothesis. Set $F_\alpha = \{v \in V : |v - x| \preccurlyeq z_a\}$ and the proof is complete.

(ii) Reversely suppose that there exists a family $(u_\alpha)_{\alpha \in A}$ in V such that $F_\alpha := \{u \in V : |u - x| \preccurlyeq u_\alpha\}$ belong to F for each α . We want to show that there exists a family of order intervals with intersection $\{x\}$. The $(u_\alpha)_{\alpha \in A}$ is this family. Since $F_\alpha \in V$ for each α and $u_a \downarrow 0$, it is imminent that $\cap_\alpha u_\alpha = \{x\}$.

□

Proposition 1.1.84. *Let $(V, \preccurlyeq)$ be a vector lattice and $(x_n)_{n \in \mathbb{N}}$ a sequence in V. Then (x_n) is order convergent to $x \in V$ if and only if there exists a sequence $u_a \downarrow 0$ such that $|x_m - x| \preccurlyeq u_n$ for all $m \geqslant n$ and $n \in \mathbb{N}$.*

Definition 1.1.85. Let $(V, \preccurlyeq)$ be a vector lattice.

(i) V is said to be *Archimedean* if for all $x, y \in V$ and for all $n \in \mathbb{N}$, $nx \preccurlyeq y$ implies $x \preccurlyeq 0$.

(ii) V is said to be *l^1-relatively complete* if $0 \preccurlyeq x_n \preccurlyeq \lambda_n x$ holds for $x_n, x \in V$ and for some $(\lambda_n)_{n\in\mathbb{N}} \in l^1$ implies that the series $\sum_{i=1}^{\infty} x_n$ is order convergent. Equivalently, if the preceding holds then $\sup \sum_{\nu=1}^{n} x_\nu$ exists.

(iii) V is said to be *countably order complete* if $\sup B$ exists in V for every non-empty countable majorized subset B of V.

(iv) V is said to be *order complete* if $\sup B$ exists in V for every non-empty majorized subset B of V.

Remark 1.1.86. In the following, (ii) from Definition 1.1.85, will be referred to as *(OS)*.

Remark 1.1.87. In terms of Riesz, spaces a countably order complete Riesz space is called *Dedekind σ-complete* and an order complete Riesz space is called *Dedekind complete*.

Example 1.1.88. The vector lattice $\mathbb{R}$ with the lexicographic order is Archimedean.

Remark 1.1.89. Each of the axioms defined in Definition 1.1.85 implies the preceding. The reverse does not always hold.

Example 1.1.90. Let K be a compact space and denote by $C(K)$ the space of all continuous real valued functions on K. Then $C(K)$ is Archimedean but not every sublattice of $C(K)$ is l^1-relative complete. Consider the vector lattice of all piecewise linear, continuous real functions on $K = [0, 1]$. Take

$$x_n = \begin{cases} nx, & 0 \leqslant x \leqslant \frac{1}{2} \\ -nx - n & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

and $\lambda_n = \frac{1}{n^2}$.

Definition 1.1.91. Let $(V, \preccurlyeq)$ be a vector lattice. If V_+ contains an element u such that $\{u\}$ is a maximal orthogonal system i.e it is not contained in any other orthogonal system, then u is called a *weak order unit.*

Proposition 1.1.92. *Let $(V, \preccurlyeq)$ be an Archimedean vector lattice and $U \subset V$ a maximal orthogonal system. Then the directed family (x_n, H) such that $n \in \mathbb{N}$ and H is any finite subset of U, order converges to $x \in V_+$ where*

$$x_{n,H} = \sum_{u\in H} (x \wedge nu) \quad \textit{and} \quad x := \sup_{n,H} x_{n,H}.$$

Proof. Since $x := \sup_{n,H} x_{n,H}$, we have $x \succcurlyeq x_{n,H}$ for any $H \subset U$ and $n \in \mathbb{N}$. Suppose $z \succcurlyeq x_{n,H}$ for every $n \in \mathbb{N}$ and arbitrary H. Then it suffices to show that $z \succcurlyeq z$. Let $u \in U$ be fixed. Then by distributivity and translation invariance we obtain

$$\begin{aligned} 0 \preccurlyeq z - (x \wedge nu) &\Leftrightarrow 0 \preccurlyeq (x - z) \wedge (nu - z) \\ &\Leftrightarrow 0 \preccurlyeq (z - x) \vee (z - nu). \end{aligned}$$

Hence

$$\begin{aligned} 0 = (z - x) \vee (z - nu) \wedge 0 &\Leftrightarrow 0 = \big[((z - x) \wedge 0\big] \vee \big[(z - nu) \wedge 0\big] \\ &\Leftrightarrow 0 = (z - x)^+ \vee (z - nu)^+ \\ &\Leftrightarrow 0 = (z - x)^- \wedge (z - nu)^-. \end{aligned}$$

Since V is Archimedean $\sup_n \left(u - \dfrac{1}{n} z\right) = u$ holds for every $n \in \mathbb{N}$ and since $u \succcurlyeq 0$ we get $\sup_n \left(u - \dfrac{1}{n} z\right)^+ = u$. Thus

$$0 = (z - x)^- \wedge (z - nu)^-$$

or equivalently

$$0 = (z - x)^- \wedge (u - n^{-1} z)^+.$$

Using the distributive law, we have the following:

$$\begin{aligned} 0 &= \sup_n \big[(z - x)^- \wedge (u - n^{-1} z)^+\big] \\ &= \sup_n (z - x)^- \wedge sup_n (u - n^{-1} z)^+ \\ &= \sup_n (z - x)^- \wedge 0. \end{aligned}$$

Since $u \in U$ is positive and U is maximal it follows that $(z - x)^- = 0$. Therefore, $z - x \succcurlyeq 0$ or, equivalently, $z \succcurlyeq x$.

□

The following corollary comes as a consequence.

Corollary 1.1.93. *If V is an Archimedean vector lattice and u a weak order unit in V, then $x = \sup_n (x \wedge nu)$ for all $x \in V_+$.*

Proof. Since $\{u\}$ is a maximal orthogonal system, we can apply Proposition 1.1.92 to $\{u\}$. Therefore, since the only finite subset of $\{u\}$ is u itself we obtain

$$x_{n,u} = \sum_{t \in u} (x \wedge nu) \quad \text{and} \quad x := \sup_{n,t} x_{n,t}$$

and the assertion is imminent. □

The Dedekind completion of any vector lattice V will be useful in the next paragraphs.

Proposition 1.1.94. *For every Archimedean vector lattice V, there exist a Dedekind completion vector lattice $\overline{V}$ such that $V \subset \overline{V}$ and V is a sublattice of $\overline{V}$. Moreover*

$$\bar{x} = \sup\{x \in V : x \preccurlyeq \bar{x}\} = \inf\{x \in V : x \succcurlyeq \bar{x}\}$$

holds, for every $\bar{x} \in \overline{V}$ and $\overline{V}$ is determined uniquely to within isomorphism by the preceding properties.

It follows that V is embedded in its Dedekind completion with respect to arbitrary infima and suprema.

1.2 Bands and band projections

In this chapter we will mainly discuss about bands and band projections. Before stating the definition of band, we will focus on some basic aspects of ideal theory and some properties of the set of all ideals denoted by $\mathbf{I}(V)$, where V is a vector lattice.

Definition 1.2.1. Let $(V. \preccurlyeq)$ be an ordered vector space and A be a non-empty subset of V. Then V is called *saturated* if $x, y \in A$ implies $[x, y] \in V$.

Definition 1.2.2. Let $(V, \preccurlyeq)$ be a vector lattice and A be a non-void subset of V. If for every $x \in A$ and $y \in V$, such that $|y| \preccurlyeq |x|$, it holds that $y \in A$, then A is called *solid* or *absolute convex*.

Solidness is a stronger notion than saturancy, mainly and commonly used to define ideals.

Definition 1.2.3. Let $(V. \preccurlyeq)$ be a vector lattice and $I \subset V$. If I is a solid vector subspace of V then I is called an *ideal* or *lattice ideal*.

Remark 1.2.4. Let V be a vector lattice. The set of all ideals of V is denoted by $\mathbf{I}(V)$.

Proposition 1.2.5. *Let V be a vector lattice. Then each ideal I of V is a vector sublattice of V and conversely each saturated vector sublattice is an ideal.*

Proof. Let I be an ideal in V. Then, by definition, I is a vector subspace of V. Hence, it is closed under the lattice operations and I is a sublattice of V. Moreover, since I is solid, the existence of $x \wedge y$ and $x \vee y$ in V implies that both the supremum and the infimum of any $x, y \in I$ is in I. Hence I is a vector sublattice.

Conversely, let A be a saturated vector sublattice of V. To prove that A is an ideal, we need to show that A is solid and a vector subspace of V. Since A is a vector sublattice of V, we have that A is closed under the lattice operations, which implies that A is a vector subspace of V. Furthermore, it holds that $[x, y] \in A$ whenever $x, y \in A$. Let $z \in V$ and $x_0 \in [x, y] \subset A$ for some $x, y \in A$ such that $|z| \preccurlyeq |x_0|$. By the decomposition property, we get $[0, z] \subset [0, x_0]$. Therefore $[0.z] \subset [x, y] \subset A$ and hence $z \in A$. Consequently A is solid. □

Proposition 1.2.6. *Let V be a vector lattice. Then the following properties hold:*

(i) Let $I_a \in \boldsymbol{I}(V)$, $a \in A$. Then $\bigcap_a I_a$ is an ideal.

(ii) The intersection of saturated spaces is also saturated.

Proof. (i) Let I_a be ideals of V. Then take $x \in \bigcap_a I_a$ and $y \in V$ such that $|y| \preccurlyeq |x|$. By the decomposition property this implies that $[0, y] \subset [0, x] \subset I_a$ for some $a \in A$. Hence $y \in I_a$ or equivalently $y \in \bigcap_a I_a$.

(ii) Let V_a be saturated ordered spaces and $A \subset \bigcap_a V_a$. Let $x, y \in A$. Thus $x, y \in A \subset V_a$ for some $a \in A$. Since each V_A is saturated, we get $[x, y] \in A \subset V_a$. Hence $[x, y] \subset \bigcap_a V_a$.

□

Proposition 1.2.7. *Let V be a vector lattice and $U \subset V$. Then any subset $U$is contained in the smallest solid subset that contains U and in the smallest ideal of V containing B.*

Proof. (i) Denote by $A := \bigcap_i \{B_i \subset V \mid B_i$ are solid and $B_i \supset U\}$. We will prove that this set is solid and is the smallest solid set containing U. First we observe that A is solid as the intersection of solid sets is also solid. Moreover suppose there exists another set $W \subset V$ smaller than A containing U. If $x \in \bigcap_i \{B_i \subset V \mid B_i$ are solid and $B_i \supset U\}$ then $x \in B_i$ for some $B_i \supset U$. This implies that there exists i such that $B_i = W$. Hence $x \in W$ and therefore $A = W$.

(ii) Denote by $J := \bigcap_i \{B_i \in \mathbf{I}(V) : B_i \supset U\}$. We will prove that J is the smallest ideal containing U. We observe that the intersection of ideals is also an ideal by the previous proposition. Let $I \in \mathbf{I}(V)$ containing U smaller than J. Then for every $x \in \bigcap_i \{B_i \in \mathbf{I}(V) : B_i \supset U\}$ there exists i such that $x \in B_i$. Thus $x \in B_i \cap I$ and $x \in I$. Therefore $I = J$.

□

Definition 1.2.8. Let V be a vector lattice. The set A defined in the previous proposition is called the *solid hull* or *solid cover* of U and is denoted by $S(U)$.

Definition 1.2.9. Let V be a vector lattice. The ideal I generated by the singleton $\{u\}$ is called a *principal ideal* and is denoted by V_u. We can assume that u is positive.

Definition 1.2.10. Let V be a vector lattice. The element $u \in V_+$ is called a *strong order unit* if $V = V_u$

Example 1.2.11. For any vector lattice V, the sets $\{0\}$ and V itself are ideals. Indeed, let $y \in V$ such that $|y| \preccurlyeq |x|$ where $x \in \{0\}$. This implies $|y| \preccurlyeq 0$ or equivalently $y^+ + y^- \preccurlyeq 0$. Thus $y^+ \preccurlyeq 0$ and $y^- \preccurlyeq 0$, but 0 is the only element in V and thus $y^+ =$ and $y^- = 0$. Therefore $\{0\}$ is an ideal. The proof that V is an ideal is trivial.

Remark 1.2.12. Let V be the vector lattice of real functions on a non-empty set X endowed with the canonical ordering. Denote by $L(F, V)$ the space of all positive linear maps from F to V, where F, V are ordered vector spaces.

Example 1.2.13. Let V be any vector lattice and $x \in V_+$. Then the symmetric order interval $[-x, x]$ is solid. Let $y \in V$ and $x_0 \in [-x, x]$ such that $|y| \preccurlyeq |x_0|$. This implies $y^+ + y^- \preccurlyeq (x_0)^+ + (x_0)^-$. By the uniqueness of the representation of both y and x_0, we obtain $y^+ \preccurlyeq (x_0)^+$ and $y^- \preccurlyeq (x_0)^-$. The decomposition property yields that $y^+ \in [0, (x_0)^+]$ and $y^- \in [0, (x_0)^-]$. Equivalently $y^+ - y^- \in [0, (x_0)^+] - [0, (x_0)^-]$. Thus $y \in [0, x_0] \subset [0, x]$. Therefore $y \in [-x, x]$ and this validates our initial claim. Furthermore, the vector subspace $\bigcup_1^\infty n[-x, x]$ is the principal ideal V_x. Now suppose A is a directed subset of V_+. Then $\bigcup \{n[-x, x] : n \in \mathbb{N}, x \in A\}$ is the ideal generated by A. Any subset B of V is solid if and only if $B = \bigcup \{[-|x|, |x|] : x \in B\}$.

Proposition 1.2.14. *Let V_a be a family of vector lattices. Then each V_a can be identified by an ideal of $\prod_a V_a$ and of $\bigoplus_a V_a$.*

Proof. (i) Let $I \in \mathbf{I}(\prod_a V_a)$. Then I is of the following form

$$I = I_1 \times I_2 \times \cdots \times I_a,$$

where I_a are ideals of V_a for each a. Suppose that $I_a = \{1\}$ for all a except in the j-th position. Hence

$$I = \{1\} \times \{1\} \cdots \times I_j \times \ldots \{1\}.$$

Suppose that $I_j = V_J$ and therefore V_j can be identified with an ideal I in $\prod_a V_a$ for all j.

(ii) Let $I \in \mathbf{I}(\bigoplus V_a)$ then each $x \in I$ has a components and is of the following form

$$x = x_1 + x_2 + \ldots x_a.$$

Suppose that $x_a = 0$ for all a except in the j-th position. Hence $I \ni x = x_j$ for all $x_j \in V_J$. Hence V_j can be identified with an ideal I of $I \in \mathbf{I}(\bigoplus V_a)$. This holds for all j.

□

Proposition 1.2.15. *The mapping $B \mapsto S(B)$ is monotone and idempotent.*

Proof. Denote by π the mapping $B \mapsto S(B)$. Then

$$\pi(\pi(B)) = \pi(S(B)) = S(B) = \pi(B)$$

since the smallest solid subset containing $S(B)$ is itself. Thus π is idempotent. Moreover, suppose that there exist B_1, B_2 subsets of V such that $B_1 \subset B_2$. Suppose that $S(B_1) \not\succcurlyeq S(B_2)$ then this implies that there exist $U \in S(B_1)$ and $V \in S(B_2)$ such that $B_2 \subset V \subset B_1 \subset V$ which is a contradiction. Hence $S(B_1) \subset S(B_2)$ and therefore the mapping is monotone with respect to set inclusion. □

Definition 1.2.16. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and A a non-empty subset of V. If $|\lambda| \leqslant 1$, $\lambda \in \mathbb{R}$ implies $\lambda A \subset A$ then A is called *circled.*

Proposition 1.2.17. *Let $(V. \preccurlyeq)$ be any vector lattice and $A \subset V$. Denote by $S(A)$ the set of all $x \in V$ such that there exist $y \in A$ satisfying $|y| \preccurlyeq |x|$. Then*

(i) If A is solid then A is circled.

(ii) The convex hull of A is solid.

(iii) Any fixed sum of solid sets is solid.

Proof. First we need to check that $S(A)$ is solid. Clearly $S(A) \subset B$ for any $B \subset V$ where B is any superset of A. Now let $u \in V$ such that $|z| \preccurlyeq |y|$ for any $y \in S(A)$. This implies $|z| \preccurlyeq |x|$ for some $x \in A$. Therefore by definition of $S(A)$, $u \in S(A)$ and hence $S(A)$ is solid.

(i) To validate that solid sets are circled we need to prove that $\lambda A \subset A$ holds for any $\lambda \in \mathbb{R}$, $|\lambda| \leqslant 1$. Take $w \in \lambda A$. Since A is solid, we have that $\frac{1}{\lambda} w \in \lambda A$. For arbitrary $y \in V$ if $|y| \preccurlyeq |\frac{1}{\lambda} w|$ then $y \in A$ as A is solid. Now $|y| \preccurlyeq |\frac{1}{\lambda} w|$ or equivalently $|\lambda||y| \preccurlyeq |w|$ which implies $|\lambda y| \preccurlyeq |w|$ and conversely , by using (iv) from Proposition 1.1.68. Thus $\lambda y \in A$ for any $\lambda \in \mathbb{R}$ such that $|\lambda| \leqslant 1$. Hence A is circled.

(ii) Recall that $convA = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \sum_{i=1}^{n} \lambda_i = 1, x_i \in A \right\}$. Let A be a solid subset of V and $y \in V$. We need to prove that $|y| \preccurlyeq \sum_{i=1}^{n} \lambda_i x_i$ implies $y \in convA$ for a convex combination of x_i. Now $|y| \preccurlyeq \sum_{i=1}^{n} \lambda_i x_i$ implies $y^+ \preccurlyeq \sum_{i=1}^{n} \lambda_i |x_i|$ and $y^- \preccurlyeq \sum_{i=1}^{n} \lambda_i |x_i|$. By using the decomposition property there exist $b_i, c_i \in [0, |x_i|]$ such that $y^+ = \sum_{i=1}^{n} \lambda_i b_i$ and $y^- = \sum_{i=1}^{n} \lambda_i c_i$ for every for every $i = 1, \ldots, n$. Then $b_i - c_i \in \big[-|x_i|, |x_i|\big] \subset A$ since A is solid. Moreover, since $y = y^+ - y^-$ we obtain $y = \sum_{i=1}^{n} \lambda_i b_i - \sum_{i=1}^{n} \lambda_i c_i = \sum_{i=1}^{n} \lambda_i (b_i - c_i)$. So y is convex combination of elements of A and therefore $y \in convA$. Therefore $convA$ is solid.

(iii) Let $\mathcal{F}_i = \big\{A_i : A_i \text{ are solid for all } i\big\}$ be a family of subsets of V and denote by $\sum_i A_i$ be the sum of A_i. Let $y \in V$ and $x \in \sum_i A_i$ such that $|y| \preccurlyeq |x|$. This implies $|y| \preccurlyeq |\sum_i x_i|$ and equivalently $y^+ + y^- \preccurlyeq \sum_i |x_i|$. By the decomposition property, there exists z_i, w_i such that $y^+ = \sum_i w_i, w_i \in [0, |x_i|]$ and $y^- = \sum_i z_i, z_i \in [0, |z_i|]$ for any i. Since A_i are solid $w_i - z_i \in [-|x_i|, |x_i|]$ which is a subset of $\sum_i A_i$. Moreover, since $y = y^+ - y^-$ we obtain $y = \sum_i w_i - sum_i z_i = \sum_i (w_i - z_i)$ and this implies $y \in \sum A_i$. Therefore $\sum_i A_i$ is solid.

□

Recall that by $\mathbf{I}(V)$ we have denoted the set of all ideals of a vector lattice V. We proceed now to prove that $\mathbf{I}(V)$ is a distributive lattice. First, we need to show that $\mathbf{I}(V)$ is a lattice.

Proposition 1.2.18. *Let $(V. \preccurlyeq)$ be any vector lattice. Then $\mathbf{I}(V)$ endowed with set inclusion is a lattice. The lattice operations are defined as follows:*

(i) $I \wedge J = I \cap J$ for all $I, J \in \mathbf{I}(V)$.

(ii) $I \vee J = I + J$ for all $I, J \in \mathbf{I}(V)$.

Proof. We discussed earlier that "$\subset$" is a partial order, hence $(\mathbf{I}(V), \subset)$ is an ordered set. Obviously $I + J$ and $I \cap J$ are vector subspaces of $\mathbf{I}(V)$, so we need to verify that $I + J$ and $I \cap J$ are solid in order to validate that both lattice operations are well defined.

(i) Let $y \in V$ and $x \in I \cap J$ such that $|y| \preccurlyeq |x|$ for some $x \in I \cap J$. Since $x \in I \cap J$ then $x \in I$ and $x \in J$ thus $|y| \preccurlyeq |x|$ implies that $y \in I$ and $y \in J$ since I, J are ideals. So $y \in I \cap J$ and therefore $I \cap J$ is solid.

(ii) The sum of ideals is indeed solid by Proposition 1.2.17 for $n = 2$. Hence $I + J$ is solid

Consequently $\mathbf{I}(V)$ is a lattice. □

Proposition 1.2.19. *Let $(V. \preccurlyeq)$ be any vector lattice. Then $\big(\boldsymbol{I}(V), \subset\big)$ is a distributive lattice.*

Proof. Since we proved the equivalence of the two distributive laws in Proposition 1.1.35, it suffices to prove that one of them holds.

For arbitrary $I, J, K \in \mathbf{I}(V)$ we will prove that $(I + J) \cap K = (I \cap K) + (J \cap K)$. Obviously $(I \cap K) + (J \cap K) \subset (I + J) \cap K$. To prove the reverse let $z \in (I + J) \cap K$. Then $z = x + y$ such that $x \in I$ and $y \in J$ by the decomposition property. This implies $|z| = |x + y|$ which is less or equal to $|x| + |y|$ with respect to the ordering. Thus there exist $u \in [0, |x|]$ and $v \in [0, |y|]$ such that $|z| = u + v$. So $u \in I \cap K$ and $v \in J \cap K$. Thus $|z| \in I \cap K + J \cap K$ and therefore $z \in I \cap K + J \cap K$ since the sum is solid. Hence $(I + J) \cap K = (I \cap K) + (J \cap K)$ holds for any $I, J, K \in \mathbf{I}(V)$. □

Remark 1.2.20. It is easily shown that $\mathbf{I}(V)$ is a complete lattice.

Proposition 1.2.21. *Let $(V. \preccurlyeq)$ be any vector lattice. Then $\boldsymbol{I}(V)$ is a complete lattice.*

Proof. We proved already that $\mathbf{I}(V)$ endowed with set inclusion is a vector lattice. So it is enough to show that every subset of $\mathbf{I}(V)$ has a least upper bound and a greatest lower bound. The sets $\{0\}, V$ are ideals and hence belong in $\mathbf{I}(V)$. Moreover, for every $J \subset \mathbf{I}(V)$, it holds that $\{0\} \subset J \subset V$. Hence $\{0\}$ is the least upper bound and V is the greatest lower bound of $\mathbf{I}(V)$. □

Before moving further we need to turn our focus to linear maps.

Definition 1.2.22. Let V, E be ordered vector spaces and $T : V \mapsto E$ be a linear operator.

(i) If $Tx \succcurlyeq 0$ holds for all $x \in V$ then T is *positive*. The operator T is called *strictly positive* if $Tx \succ 0$ stands for all $x \in V_+$ except $\{\emptyset\}$.

(ii) The operator T is called a *lattice homomorphism* if T satisfies the following:

$$T(x \vee y) = Tx \vee Ty$$

and

$$T(x \wedge y) = Tx \wedge Ty$$

for all $x, y \in V$, when V, E are vector lattices.

(iii) The operator T is called *order continuous* if for every order convergent filter F in V the filter with base $T(F)$ is order convergent in E i.e.

$$F \xrightarrow{\preccurlyeq_V} x \text{ implies } T(F) \xrightarrow{\preccurlyeq_E} Tx.$$

(iv) The operator T is called *sequentially order continuous* if for every sequence $(x_n)_{n\in\mathbb{N}} \in V$ with order limit x, $T(x_n)$ order converges to Tx in F i.e

$$x_n \xrightarrow{\preccurlyeq_V} x \text{ implies } T(x_n) \xrightarrow{\preccurlyeq_E} Tx.$$

We will state an example of a lattice homomorphism from an algebraic aspect. To assist the reader, we will use Hasse Diagrams, which obviously are lattices. In this example, we will make use of D_6 and D_{30}.

Example 1.2.23. Let D_6 be the set of all the divisors of 6 and D_{30} be the set of all divisors of 30. Let $T : D_6 \mapsto D_{30}$ be the following linear map:

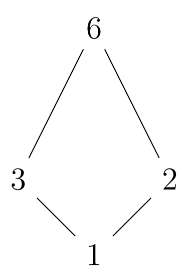
$$T(1) = 1 \quad T(2) = 6 \quad T(3) = 15 \quad T(6) = 30.$$

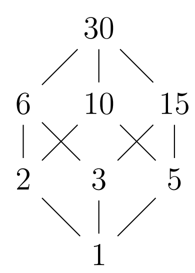
To verify that T is a lattice homomorphism we must check that $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ hold for all $x, y \in D_6$ and $Tx, Ty \in D_{30}$.

- Let $x = 1$ and $y = 2$ then $T(1 \vee 2) = T(1) \vee T(2)$ or equivalently $T(2) = 1 \vee 6$ or equivalently $6 = 6$ which is true.
- Let $x = 1$ and $y = 3$ then $T(1 \vee 3)T(1) \vee T(3)$ or equivalently $T(3) = 1 \vee 15$ or equivalently $15 = 15$ which is true.
- Let $x = 2$ and $y = 6$ then $T(2 \vee 6) = T(2) \vee T(6)$ or equivalently $T(6) = 6 \vee 30$ or equivalently $30 = 30$ which is true.
- Let $x = 2$ and $y = 3$ then $T(2 \vee 3) = T(2) \vee T(3)$ or equivalently $T(3) = 6 \vee 15$ or equivalently $15 = 15$ which is true.

By repeating the same process for the remaining pairs we validate that $T(x \vee y)Tx \vee Ty$ holds for all $x, y \in D_6$ and $Tx, Ty \in D_{30}$. Similarly we can verify that $T(x \wedge y) = Tx \wedge Ty$ is a similar process.

Hence $T(x \wedge y) = Tx \wedge Ty$ also holds for all $x, y \in D_6$ and $Tx, Ty \in D_{30}$. Therefore T is lattice homomorphism.





Proposition 1.2.24. *Let V, F be ordered vector spaces. Then the set $K \subset L(V, F)$ of all positive linear maps satisfies the following:*

(i) $K + K \subset K$

(ii) $\lambda K \subset K$, $\lambda \leqslant 0$

Proof. (i) Let $f, g \in K$. Then $f + g$ is positive. This holds for all f, g and therefore the sum of $K + K$ is equal to K.

(ii) Let $f \in K$ and $\lambda \in \mathbb{R}_+$. Then λf is a positive linear map and thus $\lambda f \in K$. Therefore $\lambda K \subset K$.

□

Definition 1.2.25. Let V, F be ordered vector spaces. Then the set $K \subset L(V, F)$ of all positive linear maps satisfying (i) and (ii) from Proposition 1.2.24 is called a *wedge* or a *proper cone*

Definition 1.2.26. Let V, F be ordered vector spaces. If in addition the set K of all positive linear maps satisfies $K \cap -K = \{0\}$ then K is the positive cone of an ordering called the *canonical ordering* of $L(V, F)$

Proposition 1.2.27. *Let V, F be ordered vector spaces and denote by $L(V, F)$ the set of all linear maps and K the subset of all positive linear maps. If $F \neq \{0\}$ then $K \cap -K = \{0\}$ if and only if $V = V_+ - V_+$.*

Remark 1.2.28. If $F = \{0\}$, then we may have $K \cap -K \neq \{0\}$ since for example, the linear functions $f(x) = x + 4$ and $-f(x) = -(x + 4)$ map all $x \in V$ to 0.

Proposition 1.2.29. *Let V be a vector lattice and $F \neq \{0\}$. Then it holds that $K \cap -K = \{0\}$.*

Proof. Recall than K is the set of all positive linear maps from V to F. Then $K \cap -K = \{f \in \wedge - f \in K\} = \{0\}$. □

Proposition 1.2.30. *Let V, F be ordered vector spaces and $\phi : V_+ \mapsto F_+$ be an additive, positive homogeneous map. Then there exists a unique positive linear map $T : (V_+ - V_+) \mapsto F$ extending ϕ.*

Proof. Let $x \in (V_+ - V_+)$. Then $x = z - y$ where $z, y \in V_+$. We define $Tx = \phi z - \phi y$. Suppose there exist an other decomposition of x such that $x = u - v$ where both $u, v \in V_+$. Then $Tx = Tx$ or equivalently $\phi z - \phi y = \phi u - \phi v$. Since ϕ is additive and the representation is unique it follows that $z = u$ and $y = v$. Hence the value Tx is independent of the representation of x. Moreover let $w, x \in (V_+ - V_+)$ such that $w = w_1 + w_2$ and $x = x_1 + x_2$. Then

$$\begin{aligned} T(w + x) &= \phi(w_1 + x_1) - \phi(w_2 + x_2) \\ &= \phi w_1 + \phi x_1 - \phi w_2 + \phi x_2 \\ &= (\phi w_1 - \phi w_2) + (\phi x_1 - \phi x_2) \\ &= Tw + Tx. \end{aligned}$$

Hence T is linear. Moreover since ϕ is positive so is T, as T is an extension of ϕ. □

Corollary 1.2.31. *Every lattice homomorphism T between vector lattices is positive. Moreover if $T : V \mapsto F$ is positive then $\big|T(x)\big| \preccurlyeq T|x|$ for all $x \in V$.*

Proof. Let $x \in V_+$. Then $Tx = Tx^+ = (Tx)^+ \succcurlyeq 0$. Thus T is positive. Moreover suppose that T is positive, Then for each $x \in V$ it holds $x^+ \preccurlyeq |x|$ and $x^- \preccurlyeq |x|$. Hence $Tx^+ \preccurlyeq T|x|$ and $Tx^- \preccurlyeq T|x|$. Since $Tx \preccurlyeq \big||Tx|\big|$ for every $Tx \in F$ we obtain $|Tx| \preccurlyeq T|x|$ for all $x \in V$. □

Proposition 1.2.32. *Let V, F be vector lattices and $T : V \mapsto F$ be a linear operator. Then the following are equivalent:*

(i) T is a lattice homomorphism.

(ii) $|Tx| = T|x|$, for all $x \in V$.

(iii) $Tx^+ \wedge Tx^- = 0$, for all $x \in V$.

Moreover if T is a surjective lattice homomorphism between V and F then $T(A)$ is solid in F, for any solid subset A in V.

Proof. (i) Let $T : V \mapsto F$ be a lattice homomorphism. We will show that $Tx^+ \wedge Tx^- = 0$ holds for all $x \in V$. Let $x \in V$, then $Tx^+ \wedge Tx^- = T(x^+ \wedge x^-)$. Since $x^+ \wedge x^- = 0$ we obtain that $T(x^+ \wedge x^-) = T(0) = 0$.

(ii) Suppose that $T(x^+ \wedge x^-) = 0$ holds for all $x \in V$. Since T is linear and $x = x^+ - x^-$ by the decomposition property, we have that $Tx = T(x^+ - x^-) = Tx^+ - Tx^-$. By the uniqueness of the representation, $Tx = (Tx)^+ = (Tx)^-$. Hence we obtain that $Tx^+ = (Tx)^+$ and $Tx^- = (Tx)^-$. Thus

$$|Tx| = (Tx)^+ + (Tx)^- = Tx^+ + Tx^- = T(x^+ + x^-) = T|x|$$

holds for all $x \in V$.

(iii) To conclude suppose that $|Tx| = T|x|$ holds for all $x \in V$. In order to prove that T is a lattice homomorphism, we need to show that $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ hold for all $x, y \in V$. Let $x, y \in V$ and $x \succcurlyeq 0$, then $|x| = x$ and this implies $T|x| = Tx \succcurlyeq 0$. Thus $(Tx)^+ \succcurlyeq T(x^+)$ and $(Tx)^- \succcurlyeq T(x^-)$ since T is positive. By the uniqueness of the representation, linearity of T and translation invariance we obtain

$$\begin{aligned} T(x \vee y) &= Tx + T[0 \vee (y - x)] \\ &= Tx + T(y - x)^+ \\ &= Tx + [0 \vee T(y - x)] \\ &= Tx \vee [T(y - x) + Tx] \\ &= Tx \vee [Ty - Tx + Tx] \\ &= Tx \vee Ty. \end{aligned}$$

Hence $T(x \vee y) = Tx \vee Ty$. We work similarly for $T(x \wedge y)$.

Suppose T is a surjective lattice homomorphism from V to $I \subset f$ and $A \subset V$ a non-empty solid set. To prove the last assertion we need to show that if $x \in A$ and $y \in V$ such that $|Ty| \preccurlyeq |Tx|$ holds then $Ty \in T(A)$. Suppose that $|Ty| \preccurlyeq |Tx|$ holds for $x \in A$ and $y \in V$. Then $Ty^+ = (Ty)^+ \preccurlyeq |Tx|$ and $Ty^- = (Ty)^- \preccurlyeq |Tx| = T|x|$. Since A is solid $y^+ \preccurlyeq |x|$ and this implies $y^+ = y^+ \wedge |x|$. Hence $Ty^+ = T(y^+ \wedge |x|)$ which is equal to $Ty^+ \wedge T|x|$. Similarly for y^- we obtain that $Ty^- = Ty^- \wedge T|x|$. By substracting Ty^+ and Ty^- we obtain $T(|x| \wedge y^+) - T(|x| \wedge y^-)$ which is equal to $T(|x| \wedge y^+) - (|x| \wedge y^-)$, since T is linear. Now denote by z the element $|x| \wedge y^+) - (|x| \wedge y^-)$. It suffices to show that $z \in A$. Since $y \in A$ we have that $y^+ - y^- \in A$ which implies that $|x| \wedge y^+) - (|x| \wedge y^-) = z \in A$. Hence $Ty = Tz \in A$. Thus $T(A)$ is solid. □

Remark 1.2.33. Let $T : V \mapsto F$ be a lattice homomorphism. Then $T\big(T^{-1}(x)\big) = x$ holds for all $x \in V$.

Proposition 1.2.34. *Let V, F be vector lattices and $T : V \mapsto F$ is a positive linear map. The set $\{x \in V : T|x| = 0\}$ is an ideal in V.*

Proof. We need to prove that $\{x \in V : T|x| = 0\}$ is a solid vector subspace. Let $x, y \in V$ then if $x, y \in \{x \in V : T|x| = 0\}$ it holds that $T|x| = 0$ and $T|y| = 0$. Since T is linear it holds that $T(|x| + |y|) = 0$ which implies by Corollary 1.1.70 that $T(|x + y|) = 0$. Hence $x + y \in \{x \in V : T|x| = 0\}$. Moreover let $\lambda \in \mathbb{R}$, then $T(|\lambda x|) = |\lambda| T|x| = 0$ which implies that $\lambda x \in \{x \in V : T|x| = 0\}$. Hence $\{x \in V : T|x| = 0\}$ is a vector subspace of V. To prove that $\{x \in V : T|x| = 0\}$ is solid let $x \in \{x \in V : T|x| = 0\}$ and $y \in V$ such that $|y| \preccurlyeq |x|$. By the linearity of T this implies that $T|y| \preccurlyeq T|x| = 0$. Hence $y \in \{x \in V : T|x| = 0\}$. Therefore $\{x \in V : T|x| = 0\}$ is an ideal in V. □

Definition 1.2.35. The set $\{x \in V : T|x| = 0\}$ is called the *absolute kernel* or the *null ideal* of T and is denoted by $\ker T$.

Proposition 1.2.36. *Let V be a vector lattice and $I \subset V$ an ideal. Define the canonical map q from V to V/I. If V/I is endowed with the finest ordering that makes q positive then q is a lattice homomorphism of V onto V/I and V/I is a vector lattice.*

Proof. Let $I \subset V$ be an ideal and $q : V \mapsto V/I$ be the canonical map. The finest ordering that makes q positive is defined as follows:

$$q(x) \preccurlyeq q(y) \text{ if and only if there exists } x_1 \in x + I \text{ and } y_1 \in y + I \text{ such that } x_1 \preccurlyeq y_1.$$

We need to prove that this ordering is indeed a partial one satisfying the axioms needed in order for V/I to be a vector lattice.

(i) Let $x_1 \in V$ such that $x_1 \in x + I$. Then obviously $x_1 \preccurlyeq x_1$ and equivalently $q(x_1) \preccurlyeq q(x_1)$. Thus the ordering is reflexive.

(ii) To prove that "$\preccurlyeq$" is transitive we need to prove that for all $x_1, y_1, z_1 \in V$ such that $x_1 \preccurlyeq y_1$ and $y_1 \preccurlyeq z_1$, it holds that $x_1 \preccurlyeq z_1$ holds. Let $x_1, y_1, z_1 \in V$ such that $x_1 \preccurlyeq y_1$ and $y_1 \preccurlyeq z_1$. If $x_1 \in x + I, y_1 \in y + I, z_1 \in z + I$ then $x + I \subset y + I$ and $y + I \subset z + I$. Since the subset relation is transitive we obtain that $x+I \subset y+I \subset z+I$ holds for all $x_1, y_1, z_1 \in V$. Hence $x+I \subset z+I$ holds for all $x, y \in V$ and therefore $x_1 \preccurlyeq z_1$ or equivalently $q(x_1) \preccurlyeq q(z_1)$.

(iii) We need to verify that if $x_1, y_1 \in V$ such that $q(x) \preccurlyeq q(y)$ and $q(y) \preccurlyeq q(x)$ hold, then $q(x_1) = q(x_2)$. Let $x_1, y_1 \in V$ such that $q(x) \preccurlyeq q(y)$ and $q(y) \preccurlyeq q(x)$. This implies that there exist $x_1 \in x + I$ and $y_1 \in y + I$ respectively such that $x_1 \preccurlyeq y_1$ and $y_1 \preccurlyeq x_1$, or equivalently $x + I \subset y + I$ and $y + I \subset x + I$ holds respectively. Since q is positive, we obtain that $x+I = y+I$ holds for all x_1, y_1 and hence $q(x) = q(y)$.

Moreover q is a linear map, since it maps a vector space V to the quotient of V. We need to show that "$\preccurlyeq$" satisfies both axioms of vector lattices.

(i) Let $x, y, z \in V$ such that $q(x) \preccurlyeq q(y)$. Since $q(z) \preccurlyeq q(z)$, adding by members we obtain $q(x) + q(z) \preccurlyeq q(y) + q(z)$. Equivalently $q(x + z) \preccurlyeq q(y + z)$ holds. This implies $x + z \preccurlyeq y + z$, by definition of the ordering.

(ii) Now let $\lambda \in \mathbb{R}^+$ and $x, y \in V$. If $x \preccurlyeq y$ holds then $q(x) \preccurlyeq q(y)$. We multiply both members of the last inequality by λ and we obtain $\lambda q(x) \preccurlyeq \lambda q(y)$. Thus we have $q(\lambda x) \preccurlyeq q(\lambda y)$ and so $\lambda x \preccurlyeq \lambda y$.

We proceed to proof that q is a lattice homomorphism and V/I is a vector lattice. We need to check that $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for all $x, y \in V/I$. Specifically we need to show that for any given $x, y \in V$ the least upper bound of the element $\{q(x), q(y)\}$ is the element $q(x \vee y)$. We observe that $q(x \vee y)$ is a majorant of $q(x)$ and $q(y)$ as q is positive. Hence $q(x \vee y) \succcurlyeq q(x) \vee q(y)$. We work analogously for the proof of the greatest upper bound. Now, to prove that q is a lattice homomorphism, let $z \in V$ such that $q(z) \succcurlyeq q(x)$ and $q(z) \succcurlyeq q(y)$ for some $x, y \in V$. This implies that there exist $z_1 \in z + I$ and $z_2 \in z + I$ such that $z_1 \succcurlyeq x$ and $z_2 \succcurlyeq y$. Since I is an ideal it follows that $z_1 - z_2 \in I$. Thus $|z_1 - z_2| \in I$. So there exists an element $w \in I$ such that $w = z_2 + |z_1 - z_2|$. It follows that $q(w) = q(z) \succcurlyeq q(x \vee y)$ since $w \succcurlyeq x \vee y$. Hence $q(x \vee y) \succcurlyeq q(x) \vee q(y)$ and the proof is complete. Now the proof of $q(x \wedge y) = q(x) \wedge q(y)$ comes as a consequence from (v) of Proposition 1.1.68, since q is linear. □

An immediate corollary is the following.

Corollary 1.2.37. *Let V, F be vector lattices and $T : V \mapsto F$ a linear map satisfying $T(V_+) = F_+$. Then the following are equivalent:*

(i) T is a lattice homomorphism.

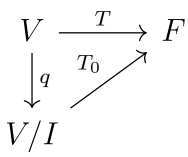
(ii) $T^{-1}(B)$ is solid for each solid set $B \subset F$.

(iii) $T^{-1}(0)$ is an ideal in V.

Proof. (i) Let $T : V \mapsto F$ such that T is a lattice homomorphism. Let $x \in T^{-1}(B)$, where B is a solid subset of F, and $y \in V$ such that $|y| \preccurlyeq |x|$. Since T is linear it holds that $|Ty| = T|y|$, which leads to $|Ty| \preccurlyeq |Tx| \in B$, for $Ty \in B$. Hence $y \in T^{-1}(B)$ and therefore $T^{-1}(B)$ is solid.

(ii) Assume that $T^{-1}(B)$ is solid for some $B \subset F$. We need to prove that $T^{-1}(0)$ is an ideal in V or equivalently that $T^{-1}(0)$ is a solid vector subspace of V. We know that $\{0\}$ is a solid subset of F and thus by assumption $T^{-1}(0)$ is solid. Since T is linear and $T(V_+) = F_+$ we have that $T^{-1}(0)$ is a vector subspace of V. Therefore $T^{-1}(0)$ is an ideal in V.

(iii) Let $I = T^{-1}(0)$ be an ideal, $T_0 : V/I \mapsto F$ a bijection map and $q : V \mapsto V/I$ the canonical map. We know that $\ker T_0 = \{x \in V : T_0(x + I)\} = 0$. Equivalently, $\ker T_0$ contains all $x \in V$ such that $T(x) = 0$. Since T is a linear bijection, we have that $x \in T^{-1}(0)$. But $T(V_+) = F_+$ and $T^{-1}(0)$ is an ideal, thus $x = 0$ and $\ker T_0 = \{0\}$. By the First Theorem of isomorphisms it holds that $V/I \cong F$. It holds that $T = T_0 \circ q$ as the following commutative diagram implies.



Since q is a lattice homomorphism so is T.

□

Remark 1.2.38. The space V/I does not inherit any of the properties stated in the Definition 1.1.85 and the canonical map q is not always order continuous.

Example 1.2.39. Let X be a non empty set and V be the vector lattice of all real functions on X under the canonical ordering. Set $F = \mathbb{R}^{X_0}$. If $V = \mathbb{R}^X$ then the restriction map $f \mapsto f_0$ where $f \in V$ and $f_0 = f|_{X_0}$ is lattice homomorphism and eventually order continuous. Suppose X is a normal topological space and X_0 is nowhere dense subset of X. Consider the family of all $f \in V_+$ such that $f(X_0) = \{1\}$. Then this family is directed downward and order converges to $0 \in V_+$ since X_0 is a nowhere dense. Thus the restriction of a lattice homomorphism is not order continuous. Set $I = \{f \in V : f(X_0) = \{0\}\}$. Then I is an ideal and V/I can be identified as the vector sublattice of F whose elements are restrictions to X_0 of functions $f \in V_+$. Thus the restriction map $f \mapsto f_0$ is exactly the canonical map $q : V \to V/I$. Thus it follows that q is not order continuous.

Definition 1.2.40. Let V be a vector lattice and $I, J \in \mathbf{I}(V)$. Then I, J are said to be *complementary ideals* of V if $I \cap J = \{0\}$ and $I + J = V$.

Example 1.2.41. Let V, F be ordered spaces and denote by $L(V, F)$ the space of all linear maps from V to F. Let $A = \{f \in L(V, F) : f(x) = 0\}$ and $B = \{f \in L(V, F) : f(x) \neq 0\}$. Then $A \cap B = \{0\}$ and $A + B = \{f \in L(V, F) : f(x) = 0 \text{ and } f(x) \neq 0\} = L(F, V)$. Hence A, B are complementary.

Proposition 1.2.42. *Let V be any vector lattice and $I, J \in \boldsymbol{I}(V)$ such that I, J are complementary. Then the projection $p : V \mapsto I$ is positive and $\ker p = J$. Moreover $I = J^{\perp}$.*

Proof. First, we need to show that the projection p is positive. Since $V = I + J$, for every $x \in V$, we have that $x = y + w$ such that $x \in I$ and $y \in J$, by the decomposition property. Suppose that $x \succcurlyeq 0$. Then $x^+ = y^+ + w^+$ and $x^- = y^- + w^-$. This implies $0 \preccurlyeq y^+ - y^- + w^+ + w^-$. Thus in order to validate that x is positive we need to verify that $y^- = 0$ and $w^- = 0$. We claim that $y^- \wedge (y^+ + w^+) = 0$. Indeed $y^- \perp y^+ + w^+$ holds because $y^- \wedge y^+ = 0$ and $y^- \wedge w^+ \in I \cap J = \{0\}$. By Corollary 1.1.76 we have $y^- \wedge (y^+ + w^+) \preccurlyeq (y^- \wedge y^+) + y^- \wedge w^+ = 0 + 0 = 0$. Hence our claim is true and $y^- = 0$, so $y^+ + w^+ \succcurlyeq 0$. Analogously we prove that $w^- \wedge (y^+ + w^+) = 0$ and we conclude that $w^- = 0$. Hence p is positive.

Now we will determine the kernel of p.

$$\begin{aligned}\ker p = \{x \in V : p(x) = 0\} &= \{x \in V : p(y + w) = 0 : y \in I \text{ and } w \in J\}\\ &= \{y + w : p(y + w) = 0\}\\ &= \{y + w : y = 0\}\\ &= \{w \in J\}.\end{aligned}$$

Hence $\ker p = J$.

To prove the last assertion we need to verify that $I \subset J^{\perp}$ and $J^{\perp} \subset J$. Since I, J are complementary it holds that $I \subset J^{\perp}$ $\forall z \in J^{\perp}$. Suppose $x \in J^{\perp}$ then $|x| \in J^{\perp}$ since x is positive. Then $|x| = u + v$ for appropriate u, v such that $u \in I_+$ and $v \in J_+$. Hence $v \preccurlyeq |x| \in J^{\perp}$ or equivalently $v \in J^{\perp} \cup J \supset I \cup J = \{0\}$. Thus $v = 0$. So $|x| = u \in I_+$ implies $|x| \in I$ and hence $x \in I$ since $x \succcurlyeq 0$ and I is an ideal. Therefore $J^{\perp} \subset I$ for every x and hence $I = J^{\perp}$. By symmetry we obtain that $I = I^{\perp\perp}$. □

We are ready to define band and band projections.

Definition 1.2.43. Let V be any vector lattice and $I \subset V$ an ideal. If $A \subset I$ and $\sup A = x \in V$ implies that $x \in I$ then I is called a *band*. A complemented ideal I of V is called a *projection band*. The corresponding projection map $V \to I$ with kernel $I^{\perp}$ is called a *band projection*.

Example 1.2.44. Let F, V be vector spaces such that $V = [-2, 2] \times [-2, 2]$ and denote by $L(F, V)$ the space of all linear maps and $K \subset L(F, V)$ the set of all positive linear maps. Then K is a band.

Example 1.2.45. Let $X = \mathbb{R}^2$ endowed with the lexicographic order. The sets $S_x = \{(x, 0) : x \in \mathbb{R}\}$ and $S_y = \{(0, y) : y \in \mathbb{R}\}$ are bands in $\mathbb{R}^2$. It follows that S_x and S_y are complemented ideals and projection bands.

Proposition 1.2.46. *For any subset A of a vector lattice V it holds that $A^{\perp}$ is a band.*

Proof. First we need to show that $A^{\perp}$ is solid. Let $x \in V$ and $z \in A^{\perp}$ such that $|x| \preccurlyeq |z|$. Because $z \in A^{\perp}$, it holds that $|z| \wedge |y| = 0$ for all $y \in A$. By the decomposition property it follows that $|x| \wedge |y| \preccurlyeq |z| \wedge |y| = 0$. Hence $x \in A^{\perp}$. Now from Corollary 1.1.72 we obtain that $A^{\perp}$ contains all suprema and infima of any of its subsets. Therefore $A^{\perp}$ is a band. □

Proposition 1.2.47. *Let V be an Archimedean vector lattice. A subset B in V is a band if and only if B is of the form $B = B^{\perp\perp}$*

Proof. Let $B \subset V$ such that $B = B^{\perp\perp}$. By Corollary 1.1.72 , since B is the orthogonal of a subset of V it follows that B contains all infima and suprema of any of its subsets. Hence it is a band by Proposition 1.2.46 . Conversely suppose that B is band. Obviously $B^{\perp\perp} \subset B$. To complete the proof we need to show the reverse allocation. But $B \subset B^{\perp\perp}$ is obvious. □

Definition 1.2.48. Let V be any vector lattice and $I \subset V$ an ideal. If, for any countable subset A of I, $\sup A = x \in V$ implies that $x \in I$ then I is called a *σ -ideal.*

Remark 1.2.49. Denote by $B(V)$ the set of all projection bands and by $P(V)$ the set of all band projections.

Remark 1.2.50. Observe that $B(V)$ is not empty as $\{\emptyset\}$ and V are complementary ideals and thus belong in $B(V)$. At the same time $P(V)$ is not empty also.

Theorem 1.2.51. *Let V be a vector lattice. Then $B(V)$ is a sublattice of $\mathbf{I}(V)$ and a Boolean Algebra. Moreover, let p be an idempotent endomorphism. Then p is a band projection if and only if $p \succcurlyeq 0$ and $1_V - p \succcurlyeq 0$ where 1_V is the identity mapping of V. Lastly, every band projection is an order continuous lattice homomorphism.*

Proof.

- First we will prove that $B(V)$ is a sublattice of $\mathbf{I}(V)$ and a Boolean Algebra. The lattice operations of $B(V)$ are inherited by $\mathbf{I}(V)$, thus it holds

  $$A \wedge B = A \cap B \qquad \text{and} \qquad A \vee B = A + B,$$

  where A, B are bands. Thus it suffices to show that $B(V)$ is closed under these operations i.e $A \cap B \in B(V)$ and $A + B \in B(V)$. Since A, B are bands it holds that $V = A + A^{\perp}$ and $V = B + B^{\perp}$. Equivalently we obtain that $V = V \cap V = A + A^{\perp} \cap B + B^{\perp}$. Hence by distributivity we obtain the following:

  $$V = A \cap B + A \cap B^{\perp} + A^{\perp} \cap B + A^{\perp} \cap B^{\perp}.$$

  In view of the definition of band and Proposition 1.2.42, it suffices to prove that $A \cap B^{\perp} + A^{\perp} \cap B + A^{\perp} \cap B^{\perp} \perp A \cap B$. This is imminent as any decomposition of $x \in A \cap B$ and $y \in A \cap B^{\perp} + A^{\perp} \cap B + A^{\perp} \cap B^{\perp}$ is $\{\emptyset\}$. Thus $A \cap B \in B(V)$. Now we observe that $A \cap V = A$ and $B \cap V = B$ or equivalently $A = A \cap (B + B^{\perp})$ and $B = B \cap (A + A^{\perp})$. Then distributivity implies that $A = A \cap B + A \cap B^{\perp}$ and $B = B \cap A + B \cap A^{\perp}$. Now $A + B = A \cap B + A \cap B^{\perp} + B \cap A + B \cap A^{\perp}$ which is equal to $A \cap B + B \cap A^{\perp} + A \cap B^{\perp}$ since $A + B$ is an ideal. We observe that $A \cap B + B \cap A^{\perp} + A \cap B^{\perp} \perp A^{\perp} \cap B^{\perp}$ and thus $A + B \in B(V)$. Therefore $B(V)$ is a sublattice of $\mathbf{I}(V)$. Moreover $\{0\}$ and V are the infimum and the supremum of $B(V)$ respectively and hence $B(V)$ is a Boolean Algebra.

- Now let $A \in B(V)$ and $p \in P(V)$ be an idempotent endomorphism where p_A is the corresponding band projection with kernel $A^{\perp}$. By Proposition 1.2.42 p is positive. Also let $1_V - p : V \mapsto V - A$ be the associated projection of $A^{\perp}$ with kernel A. Similarly by Proposition 1.2.42 $1_V - p$ is positive. Hence $1_V - p \succcurlyeq 0$. Conversely, let $p \in L(V, V)$ be an idempotent endomorphism and $1_V - p \succcurlyeq 0$. Then $1_V \succcurlyeq 0$. This implies that there exist $x \in V$ such that $0 \preccurlyeq px^+ \preccurlyeq x^+$ and $0 \preccurlyeq px^- \preccurlyeq x^-$, or equivalently $0 \preccurlyeq px^+ \wedge px^- \preccurlyeq x^+ \wedge x^- = 0$. Thus $px^+ \wedge px^- = 0$ and by Proposition 1.2.32 p is a lattice homomorphism. Hence $p^{-1}(0)$ and $(1_V - p)^{-1}(0)$ are ideals by corollary 1.2.37. Thus p is a band projection, since $p^{-1}(0)$ and $(1_V - p)^{-1}(0)$ are complementary.

- Furthermore if p is a band projection then for every x it holds that $x = x_1 + x_2$ and $y = y_1 + y_2$ are the corresponding decompositions of $x, y \in V$ into components of p and p^{-1}. Thus $[x, y] = [x_1 + y_1, x_2 + y_2]$ holds for the order interval which shows that every order convergent filter F of V then pF order converges to px. Hence p is order continuous.

□

In the next theorem, we will establish an isomorphism between $P(V)$ and $B(V)$. By that means, since $B(V)$ is a Boolean Algebra, we expect that $P(V)$ is too.

Theorem 1.2.52. *Let V be a vector lattice. Then $P(V)$ is a Boolean Algebra under the following lattice operations:*

$$F \vee G = F + G - FG$$

and

$$F \wedge G = FG$$

for all $F, G \in P(V)$, and the mapping $P \mapsto PV$ is an isomorphism from $P(V)$ to $B(V)$. Furthermore, every pair of band projections commutes.

Proof. It is clear that the mapping $P \mapsto PV$ is a bijection from $P(V)$ to $B(V)$. To validate the remaining assumptions it suffices to prove that $P_A \wedge P_B$ and $P_A \vee P_B$ are band projections with ranges $A \cap B$ and $A + B$ respectively. In the previous theorem we provided a decomposition of the space V as follows:

$$V = A \cap B + A \cap B^{\perp} + A^{\perp} \cap B + A^{\perp} \cap B^{\perp}.$$

Now let $x \in V$ such that $x \in A \cap B^{\perp} + A^{\perp} \cap B + A^{\perp} \cap B^{\perp}$. Thus $P_A P_B(x)$ vanishes in $A \cap B^{\perp} + A^{\perp} \cap B + A^{\perp} \cap B^{\perp}$ and for all $y \in A \cap B$ it holds that $P_A P_B(y) = y$ and hence $P_A P_B$ is the associated band projection of $A \cap B$ with range $A \cap B$. It is immediate that $P_B P_A$ is also a band projection with range $A \cap B$. Hence $P_A P_B = P_B P_A$. Now, let $x \in V$ such that $x \in A + B$ for all $A, B \in B(V)$. By the previous decomposition it follows that $x \in A \cap B^{\perp} + A^{\perp} \cap B + A \cap B$. Thus $P_A \vee P_B = (P_A + P_B - P_A P_B)(x)$ or equivalently $P_A \vee P_B = (P_A + P_B)(x) = x$ leaving its element x fixed. Moreover $P_A \vee P_B(y) = (P_A + P_B - P_A P_B)(y)$ which vanishes in $A^{\perp} \cap B^{\perp}$ for all $y \in A^{\perp} \cap B^{\perp}$. Hence $P_A \vee P_B$ is the band projection associated with the band $A + B$ with range $A + B$. □

The last main theorem proven in this chapter decomposes V into the direct sum of orthogonal bands.

Theorem 1.2.53 (Riesz Decomposition Theorem)**.** *Let V be an order complete vector lattice and A be a non empty subset of V. Then V is the direct sum of the band generated by A and the band $A^{\perp}$ i.e. $V = B_A + A^{\perp}$. Specifically, each band is a band projection.*

Proof. Suppose A is a non-void subset of V. Denote by B_A the band generated by A and $A^{\perp}$ the complement of A. We need to prove that B_A and $A^{\perp}$ are complementary ideals. Equivalently we need to show that $y \wedge w =$ for all $y \in B_A$ and $w \in A^{\perp}$. For an arbitrary $a \in A$ it holds that $a \in B_A$. Let $y \in A^{\perp}$. Then $y \in A^{\perp}$ for all $x \in A$. Take $x = a$ and hence $y \perp a$. Therefore, $B_A \perp A^{\perp}$ for all a, y. Since B_A and $A^{\perp}$ are bands, it holds that $B_A \cap A^{\perp} = \{0\}$. Therefore, $V = B_A + A^{\perp}$ as the direct sum of vector subspaces. By definition, for any band A of V, $B(A)$ is a projection band.

To finish the proof, we need to prove that every x in V is of the form $x = y + z$, where $y \in B_A$ and $z \in A^{\perp}$. Take $y =: \sup[0, x] \cap B_A$. This y exists in B_A, as V is order complete by hypothesis, for all $x \in V_+$. Now, suppose that $x = y + z$. It suffices to prove that $z \in A^{\perp}$. Suppose that $u \in A$. Then $|u| \in A$, since A is solid. Let $w := z \wedge |u|$. Since $w \in B_A$ it holds that $w \preccurlyeq x - y$ or equivalently $w + y \preccurlyeq x$. Hence $w + y \preccurlyeq y$. So $w \preccurlyeq 0$ and hence $w = 0$ since $x \in V_+$. By the definition of w, we obtain that $z \wedge |u| = 0$. This holds of all $u \in A$, thus $z \in A^{\perp}$. □

The following example marks the fact that not all ideals are bands.

Example 1.2.54. Let X be the Cantor set and V the vector sublattice of $\mathbb{R}^X$ containing the constant one function with countable support. Denote by I the set of all $f \in V$ such that $f(\mathbf{F}) = \{0\}$ where $\mathbf{F}$ is the filter of all substets of X with countable complement. Then I is a vector subspace of X and solid, hence an ideal. It holds that I is a σ-ideal but not a band.

Example 1.2.55. Let $X = [0, 1]$ under its standard topology and let $V = C(X)$ denote the vector lattice of all continuous functions on X under the canonical ordering. Set $B_p = \{f \in V : f(t) = 0 \text{ for all } t \geqslant p\}$ for all $0 \leqslant p \leqslant 1$. We know that any continuous function on a compact has a supremum and an infimum and since $f(t) = 0$, it follows that B_p is a band but not a projection band. Denote by e the constant function equal to one. If B_p was a band, then $e = e_1 + e_2$ is the unique decomposition according to Theorem 1.2.53. Hence e_1 and e_2 are continuous functions only with values 0 and 1 or, equivalently, the characteristic functions of a pair of closed-and-open subsets of X. Any compact space of the form of X is connected and thus it follows that $e_1 = 0$ or $e_1 = e$ and, therefore, B_p is not a projection band.

Remark 1.2.56. Example 1.2.55 states that not every band is a projection band in a vector lattice V.

The following proposition helps us determine when the band generated by A, where A is a subset of V, is a projection band.

Proposition 1.2.57. *Let V be a vector lattice and $A \subset V$. Then the band, generated by A, is a projection band if and only if*

$$x_A := \sup_{n,H} \big(x \wedge n \sum_{y \in H} |y|\big)$$

exists for each $x \in V_+$ and $n \in \mathbb{N}$ and H a finite subset of A. Furthermore, if B_A is a projection band, the corresponding band projection is given by $x \mapsto (x_A)^+ - (x_A)^+$.

Proof. Let A be a majorized subset of V and denote by B_A the band generated by A. Let $F_A \in P(V)$, such that F_A is the band projection associated with B_A and take A, such that $y := \sup[0,x] \cap B_A$ exists in B_A. Thus, by 1.2.53, B_A is a projection band if and only if $F(x) := \sup[0,x] \cap B_A$ exists, for all $x \in V_+$ and if so $F_A(x) = x \mapsto F_A x^+ = F_A x^-$. Now denote by I_A the ideal generated by A, containing $z \in V$, such that $|z| \preccurlyeq n \sum_{y \in H} |y|$ for suitable n and $H \in A$, where H is finite. Such ideal exists, since A is arbitrary. Now, denote by J_z the vector subspace of all $z \in V_+$, such that $z = \sup C$, where $(C, \preccurlyeq)$ is a directed subset of A_+. To show that J_z is an ideal we need to prove that it is solid. Let $z \in J$ and $x \in V_+$, such that $|x| \preccurlyeq |z|$. Since $|z| \in A$, this implies that $|x| \in A$. It also implies that there exist $C \subset A$, such that $x = \sup C$. Therefore, $x \in J_z$ and J_z is solid. Hence J_z is an ideal. □

Definition 1.2.58. Let V be a vector lattice. The band generated by the singleton $\{u\}$ is called a *principal band* and is denoted by $B(u)$. If each principal band of V is a projection band, then V is said to have the *principal projection property* (PPP).

Proposition 1.2.59. *Let V be a vector lattice having the principal projection property. Then V is Archimedean.*

Proof. Let $u \in V$ such that B_u is the principal band generated by the singleton $\{u\}$. We need to prove that if $nx \preccurlyeq y$, for all $y \in V$, then $x \preccurlyeq 0$. Equivalently, we need to show that $\{nx\}$ is majorized, when $x \preccurlyeq 0$ and $n \in \mathbb{N}$. Suppose that $x \succcurlyeq 0$ and $\{nx\}$ is majorized. Since B_u is a band, $\{nx\}$ has a supremum in B_u. Thus $\mathbb{N}$ has a supremum, which is a contradiction. Hence $x \preccurlyeq 0$ and therefore V is Archimedean. □

Corollary 1.2.60. *Let V be any vector lattice and $B(u) \subset V$. Then $B(u)$ is a projection band if and only if* $\sup\big(x \wedge n|u|\big)$ *exists in V for all $x \in V_+$.*

Proof. Obviously, if B_u is projection band, then $\sup\big(x \wedge n|u|\big)$ exists in V for all $x \in V_+$. Conversely, suppose that, for all $x \in V_+$, it holds that $\sup\big(x \wedge n|u|\big)$ exists in V. Thus, by proposition 1.2.57, this supremum exists for all finite subsets of any subset A. Take $\{u\} \subset A$. Obviously, $\{u\}$ is of finite dimension and thus the band generated by u is a projection band, by Proposition 1.2.57. □

Corollary 1.2.61. *V has the principal projection property if and only if* $\sup\big(x \wedge ny\big)$ *exists for each pair $x, y \in V_+$. Moreover, if V is countably order complete, then V has the principal projection property.*

Proof. Suppose that for every pair of $x, y \in V_+$ the sup $\big(x \wedge ny\big)$ exists. Then, by Corollary 1.2.60, the band generated by y is a principal band and hence a projection band. Therefore, V has the principal projection property.

Conversely, suppose that V has the principal projection property. This means that every principal band is a projection band. It follows that for every y the band generated by y is a projection band, hence by Corollary 1.2.60, it follows that sup $\big(x \wedge ny\big)$ exists for all pairs of $x \in V_+$. So it exists for every pair of x, y and the proposition is complete. $\square$

Theorem 1.2.62 (Main Inclusion Theorem)**.** *Let V be any vector lattice. Then the following implications hold on any vector lattice V:*

$$\begin{array}{ccccc} & & (COC) & & \\ & \nearrow & & \searrow & \\ (OC) & & & & (PPP) \Longrightarrow (A) \\ & \searrow & & \nearrow & \\ & & (PP) & & \end{array}$$

Proof. We observe that $(PP) \to (PPP)$ is imminent. By Definition 1.1.85, it holds that any order complete vector lattice V is countably order complete i.e. $(OC) \to (COC)$. From Proposition 1.2.59, we obtained the last implication i.e. $(PPP) \to (A)$ and by Corollary 1.2.61, we obtained that $(COC) \to (PPP)$. By Theorem 1.2.53, we have that for any vector lattice V satisfying the axiom (OC) each band is a projection band, hence V has the (PP) property. $\square$

Remark 1.2.63. None of the implications can be reversed yet.

The following result will be in later use but allows us, for now, to describe $P(V)$ as an ordered subset of V, under satisfactory assumptions.

Proposition 1.2.64. *Let V be a vector lattice having the principal projection property, containing weak order units. The Boolean Algebra $P(V)$ is mapped isomorphically onto the set of all weak order units $u \in V$, satisfying $v \wedge (u - v) = 0$, $v \in V$ by $P \mapsto Pu$.*

Proof. Let $p : P(V) \mapsto \{$the set of all weak order units $u \in V : v \wedge (v - u) = 0\}$. Since V has the principal projection property, then $u_p = p_u$ is a weak order unit in pV, by Corollary 1.1.93, and the order continuity of p. Now, let U be a fixed order unit in V. Then $0 \preccurlyeq u_p \preccurlyeq u$, by Theorem 1.2.51 and thus $0 \preccurlyeq u_p - u$. This implies $u_p \wedge (u - u_p) = 0$, for all $p \in P(V)$. If $p \preccurlyeq q$, then it holds $u_p \preccurlyeq u_q$, since both p, q are positive. Conversely, if $u_p \preccurlyeq u_q$ holds, then $pV \subset qV$, which implies that $p \preccurlyeq q$. Hence, for arbitrary $p, q \in P(V)$ the map $p \mapsto p_u$ is an isomorphism from $P(V)$ onto the set of all $v \in [0, u]$, such that $v \wedge (u - v) = 0$. Since $P(V)$ is a Boolean Algebra, so is this set. Now, we need to validate that the range of the isomorphism is all of $v \in V$, such that $v \wedge (u - v) = 0$. Since V has the principal projection property, every band B_u is a projection band and thus $()u - v)^{\perp} \in B(u)^{\perp}$. If pV is the associated projection, then by the decomposition property, it holds that $u = v + (u - v)$ is the unique decomposition by Theorem 1.2.53. Thus $v = p_V u$. Therefore, $p \mapsto p_u$ maps $P(V)$ onto all of $v \in V$, such that $v \wedge (u - v) = 0$. □

To conclude this paragraph we will prove a proposition regarding the set of characteristic elements of an order interval.

Definition 1.2.65. Let V be a vector lattice. The set $\{x \in V : x \wedge (v - x) = 0\}$ is sometimes called the set of *characteristic elements* of the order interval $[0, v]$ and is denoted by B_v.

Example 1.2.66. Let X be a non void set and $V = \mathbb{R}^X$ under the canonical ordering and v is the constant one function. Then $B_v = \{f \in V : f \wedge (v - f) = 0\}$

Remark 1.2.67. Let X be a non-void set. If $V = \mathbb{R}^X$ under the canonical ordering, where v is the constant one function. Then $x \in B_v$ if and only if x is the characteristic function of a subset of X

Definition 1.2.68. Let V be a vector lattice and K a non-void subset of V. The element $x \in V$ is an *extreme point* of K if there not exist $y, z \in K$, such that $y \neq z$ and $0 \leqslant \lambda \leqslant 1$ such that $x = \lambda y + (1 - \lambda)z$. The set of all extreme points of V is called the *extreme boundary* of V.

Proposition 1.2.69. *Let V be any vector lattice and $v \in V_+$. Then $(B_v, \preccurlyeq)$ is a Boolean Algebra and identical with the extreme boundary of the order interval $[0, v]$.*

Proof. Since "$\preccurlyeq$" is induced by V, to prove that B_v is a Boolean Algebra it suffices to prove that $x \wedge y \in B_v$ and $x \vee y \in B_v$ for all $x, y \in B_v$. We observe that if $x \in B_v$ then $v - x$ is the complement of $x \in B_v$. Let $x, y \in B_v$ and denote $w := x \vee y$. We need to show that $w \wedge (w - v) = 0$. Hence by translation invariance and distributivity we obtain that

$$\begin{aligned} v - z &= v - x \vee y \\ &= v + (-x) \wedge (-y) \\ &= (v - x) \wedge (v - y) \end{aligned}$$

and thus by associativity of the supremum and the infimum and distributivity again we obtain

$$\begin{aligned} w \wedge (v - w) &= w \wedge [(v - x) \wedge (v - y)] \\ &= x \vee y \wedge [(v - x) \wedge (v - y)] \\ &= \big(x \wedge [(v - x) \wedge (v - y)]\big) \vee \big(y \wedge [(v - x) \wedge (v - y)]\big) \\ &= \big((x \wedge (v - x) \wedge (v - y)\big) \vee \big(y \wedge (v - y) \wedge (v - x)\big). \end{aligned}$$

Since $x, y \in B_v$, we have that $w \wedge (v - w) = 0$, thus $z := x \vee y \in B_v$. We work, similarly, for $x \wedge y$. Therefore B_v is a Boolean Algebra. □

1.3 Simple and Semi-Simple Vector Lattices

In this paragraph we will discuss more about ideal theory. Archimedean vector lattices of finite dimension will play a major role in this discussion. The main target is to prove that any vector lattice V is the sum of orthogonal ideals satisfying properties we will discuss in the meantime.

Proposition 1.3.1. *Let V be any vector lattice and $I \subset \boldsymbol{I}(V)$ be an ideal. Denote by q the canonical map from $V \mapsto V/I$. Then q is a homomorphism of $\boldsymbol{I}(V)$ onto $\boldsymbol{I}(V/I)$. Moreover denote by $S_V(I)$ the sublattice of all ideals superset of I. Then the restriction map from $S_V(I)$ is an isomorphism onto $\boldsymbol{I}(V/I)$.*

Proof. Let $J \in \mathbf{I}(V)$. Then $q(J)$ is a vector subspace of V/I. Since q is a lattice homomorphism between vector lattices, q maps J onto an ideal of $\mathbf{I}(V/I)$. Indeed, by Proposition 1.2.32, $q(J)$ is solid in V/I. To prove that $J \mapsto q(J)$ is a lattice homomorphism of the vector lattice $\mathbf{I}(V)$ onto $\mathbf{I}(V/I)$, we need to validate that $q(J+K) = q(J) + q(K)$ and $q(J \cap K) = q(J) \cap q(K)$ for any given ideal $I, J \in \mathbf{I}(V)$.

Obviously, $q(J \vee K) = q(J+K)$, since q is a positive linear map with respect to the sum of ideals. We observe that $q(J \cap K) \subset q(J) \cap q(K)$ holds, for any $J, K \in \mathbf{I}(V)$. Moreover, it holds that $J \cap K \subset K$ and $J \cap K \subset J$. Thus $q(J \cap K) \subset q(K)$ and $q(J \cap K) \subset q(J)$ or, equivalently, $q(J \cap K) \subset q(K) \wedge q(J)$, since V/I is a vector lattice. So we only need to prove that $q(J) \cap q(K) \subset q(J \cap K)$. Let $z \in q(J) \cap q(K)$ Then there exist $x \in J$ and $y \in K$, such that $z = q(x) = q(y)$. Now, $|z| = |q(x) \wedge q(y)|$ and since q is a lattice homomorphism, it follows that $|z| = |q(x)| \wedge |q(y)|$ and by Proposition 1.2.32, $|z| = q(|x|) \wedge q(|y|)$ and thus $|z| = q(|x| \wedge |y|)$. So $|z| \in q(K \cap J)$ and since $K \cap J$ is solid, it follows that $z \in q(I \cap J)$. Hence $q(J) \cap q(K) \subset q(J \cap K)$ holds for any $J, K \in \mathbf{I}(V)$. Thus $J \mapsto q(J)$ is a lattice homomorphism of the vector lattice $\mathbf{I}(V)$ into $\mathbf{I}(V/I)$.

We need to prove that q is surjective. We observe that for any $U \in V/I$, since q is linear and there exist a one-to-one correspondence between ideals of V and V/I, it follows that there exists $J \in V$, such that $q^{-1}(U) \in \mathbf{I}(V)$ and therefore $U = q(q^{-1}(U)$. □

Definition 1.3.2. Let V be a vector lattice and $I \in \mathbf{I}(V)$. Then I is called a *maximal* ideal, if V is the only ideal properly containing I.

Example 1.3.3. Let X be a compact topological vector space and denote by V the vector lattice of real, continuous functions with values from X. Then, the sets $I_x = \{f \in C(X) : f(x) = 0\ , x \in V\}$ are maximal ideals in V. Indeed, it is easy to prove than the only proper ideal containing I_x is $C(X)$. Set $J = C(X)$, then suppose $J \supset I_x$. Then there exists a continuous function $f \in J$, such that $f \notin I_x$. This implies that $f \neq 0$. Hence J contains all zero functions and all the non-zero functions. Therefore $J = C(X)$ and I_x are maximal ideals.

Example 1.3.4. Let V be the set of all integers endowed with the divisibility relation. Then $p\mathbb{Z}$, p is a prime integer, are maximal ideals. Suppose there exist another maximal ideal, such that $J \supset p\mathbb{Z}$. Then there exists an integer $k \in J$, such that k/p. It follows that $k = p$ and hence the only ideal properly containing $p\mathbb{Z}$ is $\mathbb{Z} = V$.

Definition 1.3.5. Let V be a vector lattice and $I \in \mathbf{I}(V)$. Then I is called a *minimal* ideal if $\{0\}$ is the only ideal properly contained in I.

Definition 1.3.6. Let V be a vector lattice. The intersection of all maximal ideals of V is called the *radical* of V and is denoted by R_V.

Definition 1.3.7. Let V be a vector lattice. Then V is called *simple* if $\mathbf{I}(V) = \{0, V\}$ and *semi-simple* if $R_V = \{0\}$.

Remark 1.3.8. If V is simple then $\{0\}$ is a maximal ideal in V.

Example 1.3.9. Then n-dimensional vector lattice $\mathbb{R}^n$ endowed with the lexicogr-pahic ordering is totally ordered and hence $\mathbb{R}^n$ has one maximal ideal

$$J = \{\bar{x} : x_1 = 0\}$$

and one minimal ideal

$$I = \{\bar{x} : x_i = 0 \text{ for } i = 1, \dots, n-2\}.$$

Remark 1.3.10. Maximal and minimal ideals does not always exist in any vector lattice V.

Proposition 1.3.11. *Let V be any vector lattice. Then V/R_V is semi-simple and Archimedean.*

Proof. We observe that R_V is an ideal by the corrseponding definition. To prove that V/R_V is semi-simple, we need to validate that $R_{V/R_V} = \{0\}$. Suppose $R_V = J$, where J is a non trivial ideal. Hence there exist $I \in V/R_V$ such that $q^{-1}(I) = J$, where $J \subset \bigcap K$ and $K \in S_V(R_V)$. By Proposition 1.3.1, q^{-1} is a bijection onto the set of all maximal ideals of V. Thus J is maximal, which is a contradiction. Hence $R_V = \{0\}$ and therefore R_V is semi-simple. □

Corollary 1.3.12. *Suppose V is a semi-simple vector lattice. Then V is Archimedean.*

Proof. Let I be a maximal ideal in V and $x, y \in V$ such that $nx \preccurlyeq y$. If $y \in I$ then it follows that $x \in I$. Otherwise, the ideal generated by $I \cup \{x\}$ is proper and hence $I \subset I \cup \{x\}$, which is a contradiction as H is maximal. Thus $x \in H$ or, equivalently, $x \in R_V$ and since V is semi-simple, it follows that $x = 0$. Hence V is Archimedean. □

Proposition 1.3.13. *Let V be a non-trivial vector lattice. Then the following assertions are equivalent:*

(i) V is isomorphic to $\mathbb{R}_0$.

(ii) V is simple.

(iii) V is totally ordered and Archimedean.

Proof. (i) Let $J \in \mathbf{I}(\mathbb{R}_0)$ be a non-empty maximal ideal. Then the order interval $[-x, x]$ is contained in J for some $x \in \mathbb{R}_0$. This holds for every $x \in \mathbb{R}_0$, otherwise J would not be maximal. Therefore $\mathbb{R}_0 \subset J$ which is a contradiction. Thus $\mathbb{R}_0$ is simple and by hypothesis so is V.

(ii) We must prove that for each pair $(x, y) \in V \times V$ it holds either $x \preccurlyeq y$ or $x \succcurlyeq y$. Suppose that $x \succcurlyeq 0$ and $x \preccurlyeq 0$ holds for any given $x \in V$. This implies $x^+ \succ 0$ and $x^- \succ 0$. Thus the ideal generated by x^+ would be non empty and a subset of V, which is a contradiction as V is simple. Thus $x^+ \succ 0$ or $x^- \succ 0$. This proves our first assertion. By Corollary 1.3.12, since V is simple, it is semi-simple and hence Archimedean.

(iii) Let $f : V \to \mathbb{R}_0$ be a linear map such that $x \mapsto \lambda e$ where e is any positive element in V such that $x = \lambda e$. Since f is linear we want to validate that there is only one $\lambda \in \mathbb{R}$ such that $x = \lambda e$. We will make use of the fact that $\mathbb{R}$ is connected. Since V is totally ordered, let $C_1 = \{\lambda \in \mathbb{R} : x \preccurlyeq \lambda e\}$ and $C_2 = \{\lambda \in \mathbb{R} : x \succcurlyeq \lambda e\}$. By the decomposition property, any convex combination $x = \mu y + (1-\mu)z$, such that $y \neq z$ and $\mu \leq 1$, is contained either in C_1 or C_2. Hence C_1 and C_2 are convex. Let $y_n = n^{-1}e$. Since e is positive and V Archimedean, it follows that $\inf(n^{-1}e) = 0$ and $0 \in C_1$. Thus C_1 is closed. Similarly for C_2. It is easy to observe that $\emptyset \neq C_1$ and $\emptyset \neq C_2$. Since $\mathbb{R}$ is connected, it follows that $C_1 \cap C_2$ is non empty but can not correspond to more than one λ per x. Therefore, there is only one $\lambda \in \mathbb{R}$, such that $x = \lambda e$ and thus f is an isomorphism.

□

Corollary 1.3.14. *Let V be any vector and $I \in \mathbf{I}(V)$. If I is maximal, then $V/I \cong \mathbb{R}_0$. If I is minimal, then $I \cong \mathbb{R}_0$.*

Proof. Let $I \in \mathbf{I}(V)$ such that I is maximal. We observe, by Proposition 1.3.1 that $q(I) \in V/I$. But, if there exists $J \supset I$ such that $j \in V$ then $q(J) \supset q(I) \in V/I$, which is a contradiction in view of Proposition 1.3.11. Thus $R_{V/I} = \{0, V/I\}$ and hence V/I is simple. Now, by Proposition 1.3.13, it follows that $V/I \cong \mathbb{R}_0$. If I is minimal then $\{0\}$ is the only ideal contained properly in I by definition. Thus $\mathbf{I}(I) = \{0, I\}$ and hence I is simple. Therefore, by Proposition 1.3.13, $I \cong \mathbb{R}_0$. □

Proposition 1.3.15. *Let V be an Archimedean vector lattice and $I \in \mathbf{I}(V)$. If I is minimal, then $I^\perp$ is maximal and $V = I + I^\perp$. If I is maximal, then $I^\perp$ is minimal if and only if $I^\perp$ is a projection band.*

Proof. Let I be a minimal ideal of V. By Corollary 1.3.14, it follows that $I \cong \mathbb{R}_0$. To prove that $V = I + I^\perp$ we need to show that I is a projection band. Denote by $x_I := \sup_{n,H}(x \wedge n \sum |y|)$, $x \in V_+, n \in \mathbb{N}$ and H is a finite subset of A. Obviously, x_I exists since $\mathbb{R}_0$ is simple and, by Proposition 1.2.57, I is a projection band. Thus $V = I + I^\perp$. This also implies that I has linear dimension 1. Hence $I^\perp$ is maximal. If not, there would exist $y \in J \supset I^\perp$ but $J^\perp \supset I^{\perp\perp} = I$ as V is Archimedean, and thus $y \in I$, which comes to a contradiction of the dimension of I. If I is maximal, then I is a projection band if and only if $V = I + I^{\perp}$. By Corollary 1.3.14, it follows that $V/I \cong \mathbb{R}_0$ thus $I^{\perp}$ has at most linear dimension of 1. If $I^{\perp}$has linear dimension of 1, then $I^{\perp} \cong \mathbb{R}_0$ and by 1.3.14 $I^{\perp}$ is minimal. Therefore, $V = I + I^{\perp}$ implies that $I^{\perp}$ and conversely. □

Definition 1.3.16. Let V be a vector lattice. An element $u \in V$ is called an *atom* if the principal ideal V_u is totally ordered.

Proposition 1.3.17. *Let V be an Archimedean vector lattice. Then the following are equivalent:*

(i) u is an atom in V.

(ii) V_u is minimal.

(iii) $V_u^{\perp}$ is maximal.

Proof. (i) Suppose u is an atom. It follows that the principal ideal V_u generated by u is totally ordered. This implies that $x \preccurlyeq y$ or $y \preccurlyeq x$ for all $x, y \in V_u$. We need to prove that the only ideal properly contained in V_u is $\{0\}$. Now let $x, y \in V_u$ such that $x, y \in I \subset V_u$. Then the ideal generated by u is of the following form:

$$V_u = \bigcup_{\infty}^{n=1} -n[-u, u]$$

for all $n \in \mathbb{N}$. Suppose $x \preccurlyeq y$. Since V is Archimedean it follows that if $nx \preccurlyeq y$ holds then $x \preccurlyeq 0$ for all $n \in \mathbb{N}$. Take $n = 1$ and thus $x \preccurlyeq 0$. Therefore $x \in I$ implies that $x \preccurlyeq 0$ and thus $I = \{0\}$.

(ii) Since V is Archimedean by Proposition 1.3.15 it is imminent that $V_u^{\perp}$ is maximal.

(iii) Let u be an order unit and V_u is the principal ideal generated by u. We need to prove that u is an atom in V or, equivalently, that the princiapal ideal V_u is totally ordered. By hypothesis, $V_u^{\perp}$ is maximal. Since V_u is principal, it follows that V_u is a minimal ideal in V. By Proposition 1.3.13 and Corollary 1.3.14, it follows that V_u is totally ordered and Archimedean. Therefore u is an atom. □

Proposition 1.3.18. *Suppose V is of finite dimension. Then maximal and minimal ideal exist and $\mathbf{I}(V)$ is finite.*

Proof. Let V be a finite dimensional vector lattice and suppose $\mathbf{I}(V)$ is not finite. It follows that there exist ideals $I \in \mathbf{I}(V)$ of infinite dimension. This implies that there exists a chain of ideals, such that there exist $x \in I_{j+1}$, where $x \notin I_j$ for all $j \in \mathbb{N}$. By hypothesis, there would exist infinite such x. Therefore V has infinite elements, which is a contradiction. Hence $\mathbf{I}(V)$ is finite. Moreover, since $\mathbf{I}(V)$ is a vector lattice endowed with set inclusion the existence of maximal and minimal ideal is obvious □

Before moving to finite dimensional vector lattices, we need to recall some topological properties and definitions.

Definition 1.3.19. Let V be a vector lattice and $A \neq \{\emptyset\}$ a linearly independent subset of V that is maximal. Then A is called a *Hamel basis*.

Remark 1.3.20. The existence of such basis in topological vector spaces is evident due to Zorn's Lemma. Indeed., let V be a topological vector space and $\{A_i : A_i \subset V \text{ , for all } i\}$ be a family of V. Set

$$\begin{aligned}
&A_1 = \{x_1, x_2 \in V : x_1 \perp x_2\}\\
&A_2 = \{x_1, x_2, x_3 \in V : x_1 \perp x_2,\ x_2 \perp x_3,\ x_3 \perp x_1\}\\
&\quad\vdots\\
&A_j = \{x_i \in V : x_1 \perp x_j,\ x_j \perp x_{j+1} \text{ for all } j = 2, \dots k-1\}.
\end{aligned}$$

We observe that $A_1 \subset A_2 \subset \cdots \subset A_j$ and hence A_j is an upper bound for this chain. Therefore by Zorn's Lemma A_j is maximal and by Definition 1.3.19 it follows that A_j is a Hamel basis.

Definition 1.3.21. Let V be a vector space over a field K. The set of all $x \in V$ satisfying $f(x) = a$, where f is a map and $a \in V$, is called a *hyperplane* and is denoted by H.

Definition 1.3.22. Let V be a vector space over a fields K and H a hyperplane. The following sets are uniquely determined by H:

(i) $O_a = \{x \in V : f(x) < a\}$.

(ii) $O^a = \{x \in V : f(x) > a\}$.

(iii) $C_a = \{x \in V : f(x) \leq a\}$.

(iv) $C^a = \{x \in V : f(x) \geq a\}$.

These sets are called *semi-spaces.*

Remark 1.3.23. All semi-spaces are convex sets.

Example 1.3.24. In view of Proposition 1.3.13, C_1 is a C_a semi-space and C_2 is a C^a semi-space and H is the hyperplane of all $x \in V$, where $f(x) = \lambda e$.

Definition 1.3.25. Let V be a vector space over K and $A \subset V$. Then A is called *absorbing* if for every $x \in A$ there exists $\lambda \in \mathbb{R}^+$ such that $x \in \lambda A$.

Example 1.3.26. Let V be a vector lattice. Then the order interval $[-u, u]$, for any u in V, is absorbing for any $n \in \mathbb{N}$.

Recall that a vector space V endowed with a topology $\mathcal{T}$ is a topological vector space if the operations of addition and multiplication are continuous.

Definition 1.3.27. Let V be any vector space endowed with a a topology $\mathcal{T}$ and $x \in V$. A subset U of V is called a *neighborhood* of x if there exists $B \subset U$ such that $x \in B$. The system of all neighborhoods of any given element x is denoted by $\mathcal{N}_x$.

We state the following separation theorem from [1], as we will make use of in the following proposition.

Theorem 1.3.28. *Let V be a topological vector space V and U a convex subset of V, such that $U^o \neq \emptyset$. Let $C \subset V$ be convex set, such that $U^o \cap C = \emptyset$. Then there exists a closed real hyperplane H separating U and C. Moreover, if U, C are also open, then H separates U, C strictly.*

Proposition 1.3.29. *If V is a totally ordered vector lattice of finite dimension, then V is isomorphic to $\mathbb{R}^n$ endowed with the lexicographic order.*

Proof. We observe that $\mathbb{R}^n$ is a totally ordered space endowed with the lexicographic ordering defined as follows: if $x_n = (x_1, x_2, \ldots, x_n)$ and $y_n = (y_1, y_2, \ldots y_n)$ then $x_n \succcurlyeq y_n$ if and only if $x_i \succcurlyeq y_i$ for the first ordinal $i < n$ such that $x_j \neq y_j$. First, we need to check that V has the same 0-neighborhoods as $\mathbb{R}^n$. It is easy to observe that for an order unit $u \in V$ the order interval is absorbing, convex and obviously a zero neighborhood when V is a topological vector space endowed with the unique Hausdorff Topology. Suppose that $n \geqslant 1$ is the dimension of V. Since V is a vector lattice then V_+ contains elements in its interior as the decomposition property yields that $[0, 2u] = u + [-u, u]$, where u is an order unit. Because V is totally ordered, it follows that V is the union of the positive cone V_+ and the $V_+/\{0\}$. By the decomposition property, both sets are convex and obviously one of them with non empty interior. Hence, by Theorem 1.3.28, there exists a hyperplane H strictly separating V_+^o from $(-V_+/\{0\})^o$. This implies $0 \in H$ and each $x \in V$ has the following representation: $x = \lambda u + y$ such that $y \in H$ and $\lambda \in \mathbb{R}$. Moreover, it follows that V_+ is a C_a semi-space obviously containing u. Hence $x \succcurlyeq 0$ if and only if either $\lambda > 0$ or $\lambda = 0$ and $y \succcurlyeq 0$. Therefore, V is isomorphic to the Cartesian product $\mathbb{R} \times H$ using the map $x \mapsto \lambda u + y$ ordered by the relation $(\lambda, y) \succcurlyeq 0$ if and only if $\lambda > 0$ or $\lambda = 0$ and $y \succcurlyeq 0$. By induction to n, the proof is complete. The dimension of a hyperplane is always lower from the space V. Actually H is totally ordered vector sublattice and the induction hypothesis yields that $H \cong \mathbb{R}^{n-1}$. Therefore $V \cong \mathbb{R} \times H \cong \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$. □

Remark 1.3.30. If V is any vector lattice, then the real space $\mathbb{R} \times M$ endowed with the following ordering:

$$(\lambda, y) \succcurlyeq 0 \text{ if and only if } \lambda > 0 \text{ or } y \succcurlyeq 0 \text{ and } \lambda = 0$$

forms a vector lattice.

Definition 1.3.31. If V is any vector lattice, then the real space $\mathbb{R} \times M$ is called the *lexicographic union* of $\mathbb{R}$ with M and is denoted by $\mathbb{R} \circ M$.

We analyze the lexicographic union more in the following proposition.

Proposition 1.3.32. *Let V be a vector lattice and $M \subset V$ be a proper ideal containing every other proper ideal of V. Then V is isomorphic to $\mathbb{R} \circ M$ and V has no other projection bands other than $\{0\}$ and V itself.*

Proof. Obviously M is a maximal ideal and it contains all proper ideals by hypothesis. To prove the isomorphism let $x \in V_+$ such that $x \notin M$. Denote by $V_1(x)$ the linear vector subspace generated by x. Then $V = V_1(x) + M$ since $dim V_1(x) = 1$. We need to validate that $z > 0$ and $\lambda > 0$, where $z = \lambda x + y$, $\lambda \in \mathbb{R}$ and $y \in M$. If $\lambda > 0$ then the ideal generated by x is all of V as $I_z \nsubseteq M$ and the order interval $[-x.x]$ is contained in V. Hence $V = I_{z^+} + I_{z^-}$, where I_{z^+}, I_{z^-} are complementary ideals thus the projection bands. Suppose that $I \in \mathbf{I}(V)$ is a projection band such that $V = I + I^+$. This implies that $\{0\} \neq I \neq V$ and similarly for $I^\perp$. Hence $I \subset M$ and $I^\perp \subset M$ since M is a maximal ideal. This implies $I + I^\perp \subset M$ or equivalently $V \subset M$ which is a contradiction. Thus there are no other projection bands rather than $\{0\}$ and V itself. Therefore $I_{z^+} = \{0\}$ or $I_{z^-} = \{0\}$. But $I_{z^+} = \{0\}$ is impossible since its image through the canonical map q is always positive. Thus $\lambda > 0$ and $z^- = 0$ or equivalently $\lambda > 0$ and $z > 0$ and the proof is complete as the map $x \mapsto \lambda x + y$ is an isomorphism by an inspection of the proof from Proposition 1.3.29. □

Remark 1.3.33. In comparison of the proof of 1.3.29 M plays the role of the hyperplane H and linear space $V_1(x)$ is obviously isomorphic to $\mathbb{R}_0$.

Before stating the main theorem we shall prove the following lemma.

Lemma 1.3.34. *Let V be any vector lattice and $I \in \boldsymbol{I}(V)$. Denote by q the canonical map from V to V/I. Then for any finite orthogonal system $\{y_i : i = 1, \dots, n\}$ of V/I there exists an orthogonal system $\{x_i : i = 1, \dots, n\}$ of V such that $y_i = q(x_i)$ for all i.*

Proof. Let $\{y_i : i = 1, \dots, n\}$ be an orthogonal system of V/I. We will use induction on n. Obviously, for $n = 1$ the assertion is true. We want to construct an orthogonal system $\{x_i\} \in V$ and prove that $y_i = q(x_i)$, for all i. By the induction hypothesis there exist $x_i \in V_+$ such that $y_i = q(x_i)$ holds for all indexes up to $n-1$. Let $x_n > 0$ be an element such that $q(x_k) = y_k$. We define $x'_i = x_i - x_i \wedge x_k$, $i = 1, \dots, n-1$ and $x'_n = x_n - x_n \wedge (x_1 + \dots x_{n-1})$. By the hypothesis $q(x_i \wedge x_k) = q(x_i) \wedge q(x_k) = u_i \wedge u_k = 0$ and hence $x_i \wedge x_k \in I$. By the distributivity of $\mathbf{I}(V/I)$ we obtain

$$\begin{aligned} q(x'_n) &= q(x_n - x_n \wedge (x_1 + \dots x_n)) \\ &= q(x_n) - q(x_n) \wedge q(x_1 + \dots x_n) \\ &= q(x_n) - q(x_n) \wedge [q(x_1) + \dots + q(x_{n-1})]. \end{aligned}$$

Hence $q(x_n) \wedge q(x_i) = u_n \wedge u_i = 0$ for all i up to $n-1$ and hence $x_n \wedge (x_1 + \dots x_{n-1}) \in I$. Thus $q(x'_n) \in V/I$ and $q(x'_i) = q(x_i) = u_i$. Lastly, we need to verify that $\{x'_i\}$ is an orthogonal system. To validate that, it suffices to prove that for indexes i, j such that $i \neq j$ and $q(x'_i) \wedge q(x_j) = 0$ where q is the canonical map. Let $x'_i = x_i - x_i \wedge x_n$ and $x'_j = x_j - x_j \wedge x_n$. Then

$$q(x'_i \wedge x'_j) = q\big[(x_i - x_i \wedge x_n) \wedge (x_j - x_j \wedge x_n)\big].$$

By distributivity this equals to

$$\big[q(x_i) - q(x_i \wedge x_n)\big] \wedge \big[q(x_j) - q(x_j \wedge x_n)\big].$$

Therefore

$$q(x_i' \wedge x_j') = (u_i - 0) \wedge (u_n - 0) = u_i \wedge u_n = 0.$$

Hence $\{x_i\}$ is an orthogonal system in V. □

Theorem 1.3.35. *Let V be a finite vector lattice. Denote by r the dimension of R_V. Then $V = \sum_{j=1}^{n-r} I_j$, where I_j are orthogonal ideals of the form of the lexicographic union $\mathbb{R}$ and M_j, where M_j is a unique maximal ideal. This decomposition is unique except of a variation of indexes.*

Proof. The radical of V is denoted by R_V. From Proposition 1.3.11, it follows that V/R_V is Archimedean. Let $dimV = n$, thus $dimV/R_V = n - r$ where r is the dimension of R_V. By Corollary 1.3.14 each minimal ideal is isomorphic to $\mathbb{R}_0$ and hence their linear dimension is 1. By Proposition 1.3.15 each minimal ideal is a projection band. Hence by induction to k we obtain that $V/R_V = \sum^k I_j$ where I_j are minimal ideals, orthogonal by pairs. Therefore there exist $y_j \in I_j$ for each j such that $\{y_j : j = 1, \dots, n\}$ is an orthogonal system. Now Lemma 1.3.34 implies that there exists an orthogonal system $\{x_j : j = 1, \dots, n\}$ in V, where $q(x_j') = y_j \in V/R_V$. Denote by V_1 the vector subspace of V generated by $\{x_j\}$. Obviously, $\{y_j\}$ is a Hamel basis in V/R_V hence $V = V_1 + R_V$. We claim that $\mathbf{I}(V_1) = V$. If not then $\mathbf{I}(V_1) \not\supseteq R_V$ and since $\mathbf{I}(V_1)$ is proper it follows that $\mathbf{I}(V_1)$ is not maximal. Hence there exists a maximal ideal J superset of $\mathbf{I}(V_1)$ such that $J \not\supseteq R_V$ which is a contradiction to the definition of R_V. Hence, if I_j denotes the ideal generated by $\{x_j\}$ then V is the sum of k mutually orthogonal ideals I_j. Now we want to prove that M_j is maximal and thus $I_j = \mathbb{R} \circ M_J$. We claim that M_j is the unique maximal ideal of I_j where $M_j = I_j \cap R_V$ for all j. Since I_j is a linear subspace of V of dimension 1 then M_j has codimension 1. Suppose that there is another maximal ideal K of codimension one different from M_j. Denote by K' the ideal of codimension 1 such that $K' = K + \sum_{i \neq j} I_i$. By the decomposition property K' is maximal and clearly a superset of R_V. Thus $K = K' \cap I_j \supset R_V \cap I_j = M_j$ which is a contradiction. Hence $K' = M_j$. This implies also, by Proposition 1.3.32, that M_j contains every proper ideal of I_j and therefore $I_j = \mathbb{R} \circ M_j$. Hence $V = \sum\limits_{i=1}^{k} \mathbb{R} \circ M_J$. Lastly, we need to check that this decomposition is unique except a permutation of indices. We need to prove that if J is a projection band of the form $\mathbb{R} \circ N$ then $J = I_j$ for some j up to k. We know that $I_j = V \cap I_j = (J + J^\perp) \cap I_j$, By the distributivity of $\mathbf{I}(V)$ it follows that $I_j = J \cap I_j + J^\perp \cap I_j$. Since V has no projection bands, by Proposition 1.3.32, it follows that $J \cap I_j = \{0\}$ or $I_j \cap J = I_j$ for at least one index j. We observe that $J^\perp \supset I_j^\perp$, since I_j is a projection band, thus $J^\perp \cap I_j = \{0\}$. Now if $I_j \cap J = \{0\}$, this implies that $J \subset I_j^\perp$ and thus J is a non trivial projection band, which is a contradiction to Proposition 1.3.32. Therefore $J \cap I_j = I_j$ and the proof is complete. □

Corollary 1.3.36. *Denote by n the dimension of V. For any finite dimensional vector lattice V such that $n \geqslant 1$ the following are equivalent:*

(i) V is semi-simple.

(ii) V is isomorphic to $\mathbb{R}^n$.

(iii) V is Archimedean.

Proof. (i) By Theorem 1.3.35, V is the sum of orthogonal ideals isomorphic to $\mathbb{R}_0$. Hence $V \cong \mathbb{R}^n$.

(ii) Suppose V is isomorphic to $\mathbb{R}^n$. By Proposition 1.3.13, $\mathbb{R}$ is totally ordered and Archimedean. Hence V, by induction to n, is simple. Therefore, any intersection of proper maximal ideals is actually the set $\{0\}$/ Thus $R_V = \{0\}$ and hence V is semi-simple.

(iii) Suppose V is Archimedean and $R_V \neq \{0\}$. Then there exist a non-void ideal $J \in R_V$, such that $J \subset M$ for every maximal ideal $M \in \mathbf{I}(V)$. Thus I is minimal in V. By Proposition 1.3.15, this implies that $I^\perp$ is maximal and this leads to R_V being a subset of $I^\perp$, which is a contradiction, by the definition of R_V. Thus $R_V = \{0\}$ and therefore V is semi-simple.

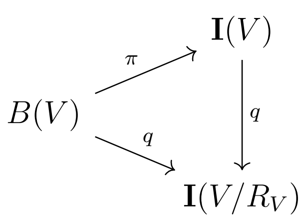
□

Summarizing the previous discussion we state the next remark.

Remark 1.3.37. $\mathbb{R}^n$ endowed with the canonical ordering corresponds to radical $R_{\mathbb{R}^n} = \{0\}$, whereas $R_{\mathbb{R}^n}$ is a totally ordered maximal ideal when $\mathbb{R}^n$ is endowed with the lexicographic ordering.

Corollary 1.3.38. *The Boolean Algebra $B(V)$ is isomorphic to the set of all ideals of V/R_V when V is a finite dimensional vector lattice.*

Proof. We observe that Theorem 1.3.35 implies that V is the sum of orthogonal ideals i.e $V = \sum_k^{j=1} I_j$. Thus any band in V is the sum of ideals such that $V = \sum_{i \in I} I_j$ where I is a finite subset of indexes and conversely. Now by proposition 1.3.1 ti follows that $B \mapsto q(B)$ is a lattice homomorphism since $B(V)$ is a sublattice of $\mathbf{I}(V)$ by Theorem 1.2.51. Hence the canonical map q maps $B(V)$ isomorphically to $\mathbf{I}(V/R_V)$ as the following diagram shows in view of 1.3.1.



□

Corollary 1.3.39. *If M is a unique maximal ideal of V and V is of finite dimension with no projection bands except $\{\emptyset\}$ and V itself, then V is the lexicographic union $\mathbb{R} \circ M$.*

Proof. V/R_V is isomorphic to $\mathbb{R}_0$, thus, by Corollary 1.3.14, V/R_V has dimension 1 and R_V is the only maximal ideal. Otherwise, there would exist $J \supset R_V$, such that $J^\perp$ is minimal and hence, by Proposition 1.3.15, it follows that $V = I + I^\perp$ or, equivalently, J is a projection band which is a contradiction. □

Remark 1.3.40. Corollary 1.3.39 states a converse of Proposition 1.3.32. We note that this converse fails in the infinite dimensional case.

Example 1.3.41. Let $X = [0.1]$ be the real unit interval and denote by V the vector lattice of real valued continuous functions with values from X. Then Example 1.2.55 and Example 1.3.3, indicate that $M = B_p = \{f \in V : f(t) = 0 \text{ for all } t \geqslant p\}$, for all $0 \leqslant p \leqslant 1$, is the unique maximal ideal but V has no projection bands at all.

Definition 1.3.42. Let V be a vector lattice. An element $v \in V$ is called *infinitely small* if $n|v| \preccurlyeq x$ holds for some $x \in V$ and for all $n \in \mathbb{N}$.

Corollary 1.3.43. *The set of all infinitely small elements of V is an ideal.*

Proof. The decomposition property implies that any addition of infinitely small elements and any scalar multiplication by $\lambda \geq 0$ are infinitely small elements. Moreover, by definition, for any $y \in V$, a infinitely small element x, such that $|y| \preccurlyeq |x|$, it holds that y is infinitely small. Therefore, the proof is complete. □

Corollary 1.3.44. *If V is Archimedean then the set of all infinitely small elements is the zero ideal and conversely.*

Proof. Since V is Archimedean for any infinitely small element u it holds that if $n|u| \preccurlyeq v$, $n \in \mathbb{N}$ then $|u| = 0$. Since the set of all infinitely small elements is an ideal this implies that $u = 0$. Thus the set is the zero ideal. □

Proposition 1.3.45. *Let V be a vector lattice and u is an order unit in V. Then V is Archimedean, if and only if V is semi-simple. Moreover R_V is the ideal of all infinitely small elements of V.*

Proof. If u is an order unit in V by definition $V = V_u$ where V_u is the ideal generated by $\{u\}$. Hence, every proper ideal in V is contained in a maximal ideal. Now, if $v \in V$ is an infinitely small element, then it follows that $n|v| \preccurlyeq u$, for all $n \in \mathbb{N}$. Thus by the decomposition property, the order interval $n[-u, u]$ is contained in $[-u.u]$. Hence by 1.3.13 v is contained in any maximal ideal. Hence $v \in R_V$. Suppose that $v \in R_V$ is a positive element not infinitely small. Hence u is not a majorant of nv, for some $n \in \mathbb{N}$. Then denote by $y := (nv - u)^+$, which is positive and is also equal to $(u - nv)^-$. Since $(nv - u)^+$ is not an order unit of V, then $(nv - u)^+$ is contained in a maximal ideal J of V. Because $0 \preccurlyeq (u - nv)^+ \preccurlyeq nv$ and $v \in I$, it follows that $(u - nv)^+ - (u - nv)^- \in I$ or equivalently $u - nv \in I$ and hence $u \in I$, which is a contradiction. Thus V is an infinitely small element. Now, if V is Archimedean, then R_V is the zero ideal, by Corollary 1.3.44 and the previous discussion. Hence V is semi-simple. The reverse has proven in Proposition 1.3.13. □

An immediate corollary is the following.

Corollary 1.3.46. *Let V be a non-trivial totally ordered vector lattice with order unit $u \in V$. Then V is the lexicographic union of $\mathbb{R}$ with the ideal of all infinitely small elements.*

Proof. We know that $\mathbf{I}(V)$ is totally ordered under set inclusion. By Proposition 1.3.45 it follows that $R_V = I_{fs}$, where I_{fs} is the ideal of all infinitely small elements and eventually a maximal ideal in view of Proposition 1.3.45. Hence by Proposition 1.3.15, since $I_{fs}^{\perp}$ is a projection band, which is a contradiction to the statement of Proposition 1.3.32. Since $V \cong \mathbb{R} \circ M$, by Corollary 1.3.39, where M is a maximal ideal, it follows that $V = \mathbb{R} \circ I_{fs}$. □

Chapter 2

Duality and Normed Lattices

2.1 Dual Spaces of Vector Lattices

In this section we will talk about the structure of the space $V^\star$, which is the vector space of all order bounded linear forms. The main target is to prove Nakano's Theorem and in order to do this we will use functional analysis.

Definition 2.1.1. Let V be a vector space. A linear map $f : V \to \mathbb{R}$ is called a *linear form*.

Example 2.1.2. Let $X = [0, 1]$ and denote by $C(X)$ the space of all continuous functions with domain the compact space X endowed with the canonical ordering. The function $f(x) = \int_0^1 g(x)dx$ is a linear form. It is clear that f is positive homogeneous and additive.

Definition 2.1.3. For any vector lattice V, the vector space of all linear forms on V is denoted by V^* and is called the *algebraic dual* of V.

Definition 2.1.4. Let V be any vector lattice. If for every interval $[x, y] \subset V$ the set $f([x, y])$ is order bounded then $f \in V^\star$ is called *order bounded.* The vector space of all order bounded linear forms of V is called the *order dual* of V and is denoted by $V^\star$.

Definition 2.1.5. Let V be a vector lattice. A linear form $f \in V^*$ is called *order continuous* or *sequentially order continuous* if f is zero convergent for every filter or sequence respectively, that order converges to 0 in V.

Remark 2.1.6. The space of all order continuous linear forms is denoted by $V_0^\star$ whereas the space of all sequentially order continuous linear forms is denoted by $V_{00}^\star$. Moreover it is easy to observe that $V_{00}^\star \subset V_0^\star \subset V^\star$.

Remark 2.1.7. Continuity at $0 \in V$ implies that f is continuous in all $x \in V$, since f is linear.

Remark 2.1.8. On a non-Archimedean vector lattice V, a linear form convergent on order convergent filters is not necessary order bounded.

Remark 2.1.9. The canonical order of $V^\star$ is determined as shown in Example 1.1.7. Moreover since $f \in V^\star$ is uniquely determined by its values on V_+ we can view $V^\star$ as a subspace of $\mathbb{R}^V$.

Proposition 2.1.10. *Let V be a vector lattice. Then $V^\star$ endowed with the canonical ordering is an Archimedean and order complete vector lattice. The lattice operations are defined as follows:*

(i) $f \vee g(x) = \sup\{f(y) + g(z) : y \succcurlyeq 0 , z \succcurlyeq 0 \text{ and } y + z = x\}$.

(ii) $f \wedge g(x) = \inf\{f(y) + g(z) : y \succcurlyeq 0 , z \succcurlyeq 0 \text{ and } y + z = x\}$.

such that $f, g \in V^\star$ and $x \in V_+$. Furthermore if $A \subset V^\star$ is any directed, majorized set, then at $x \in V_+$ the supremum $f_0 = supA$ is given by

$$f_0(x) = \sup_{f \in A} f(x).$$

Proof. (i) Let $h : V_+ \to \mathbb{R}$ defined by $h(x) = sup\{f(y) + g(x - y) : 0 \preccurlyeq y \preccurlyeq x\}$. We observe that $h(x) \preccurlyeq k(x) \in V^\star$ for all $k(x)$ that majorizes f and g. Thus by Proposition 1.2.30 it suffices to show that h is positive homogeneous and additive and that the linear form $\tilde{h}$ defined by h is order bounded. It is evident by the definition of h that is positive homogeneous for all $\lambda \in \mathbb{R}_+$. Let $x \in V$ such that $x = x_1 + x_2$, where $x_1, x_2 \in V_+$. Thus, by the decomposition property, we obtain $[0.x] = [0, x_1] + [0.x_2]$ and therefore

$$h(x_1 + x_2) = sup\{f(y_1) + f(y_2) + g(x_1 - y_1) + g(x_2 - y_2)\} = h(x_1) + h(x_2),$$

where the supremum is taken over all $y_1 \in [0, x_1]$ and $y_2 \in [0, y_1]$. Lastly, we need to verify that $\tilde{h}$ is order bounded. By the definition of h, it is enough to observe that $h([0, x])$ is order bounded in $\mathbb{R}$, for each $x \in V_+$. This is true since for every majorized $k(x) \in V^\star$, such that $h(x) \preccurlyeq k(x)$, it follows that $h([x, y]) \subset k([x, y])$ for all intervals. Since $k(x)$ is positive, it follows that h is order bounded and hence $\tilde{h}$ is order bounded.

(ii) To prove the last assertion let A be a directed and majorized subset of $V^\star$. We set $r(x) = sup\{f(y) : f \in A\}$, where $r : V_+ \to \mathbb{R}$. Obviously r is positive homogeneous and additive by definition. Thus the positive linear form defined by r, denoted by f_0, is order bounded since f_0 is majorized and minorized by elements in $V^\star$. Therefore $f_0 = \sup A$ in $V^\star$.

□

Corollary 2.1.11. *Let V be a vector lattice. Then for every $f, g \in V^\star$ and $x \in V_+$ it holds that $f^+ = f \vee 0$, $f^- = (-f) \vee 0$ and $|f| = f \vee (-f)$, where*

(i) $f^+(x) = \sup\{f(y) : 0 \preccurlyeq y \preccurlyeq x\}$.

(ii) $f^-(x) = -\inf\{f(y) : 0 \preccurlyeq y \preccurlyeq x\}$.

(iii) $|f|(x) = \sup\{|f(z)| : 0 \preccurlyeq |z| \preccurlyeq x\}$.

In addition, for every majorized subset $\{f_k : k \in A\}$ *of* $V^\star$,$f_o := \sup_k f_k$ *then at* $x \in V_+$, f_0 *is given by*

$$f_0(x) = \sup\{f_{k_1}(x_1) + \cdots + f_{k_n}(x_n)\},$$

where $\{k_i\}$, *for all* i *up to* n, *runs over all non-empty finite subsets of* A *and* $\{x_i\}$ *runs over all finite decomposition of* $x = x_1 + \cdots + x_n$ *into positive summands.*

Proof. The positive and the negative part of each $f \in V^\star$ is an immediate consequence of (i) by substituting g with the zero function. The unique representation of any element as the difference of two positive elements implies that if $z = u - v$, $u \succcurlyeq 0$ and $v \succcurlyeq 0$ then $\{u + v \preccurlyeq x\} = \{z : |z| \preccurlyeq x\}$ for $x \in V_+$. Thus

$$|f|(x) = (-f) \vee f = \sup\{f(z) : |z| \preccurlyeq |x|\},$$

which is equal to $\sup\{|f(z)| : z \preccurlyeq x\}$. To prove the last assertion, let $\{k_1, \ldots k_n\}$ be a finite non-empty subset of A and $x \in V_+$ where $x = x_1 + x_2 + \cdots + x_n$ such that $x_i \succcurlyeq 0$ $\forall i$. Hence we obtain

$$(sup_i f_{k_i})(x) = \sup\{f_{k_1}(x_1) + \cdots + f_{k_n}(x_n)\},$$

where the supremum is taken over all positive decompositions of x, as a result of multiple application of (i) and the associativity of supremum. We observe that the set of all suprema is directed in $V^\star$. Indeed, if there exist $\{f_i : i \in A\}$ finite subsets of $V^\star$, then $sup_i f_i \in \{f_i\}$ since $V^\star$ is order complete and since $V^\star$ is Archimedean, the assertion is imminent. Hence there exist $f_0 \in V^\star$, such that $f_0 = (sup_i f_{k_i}(x))$, Proposition 2.1.10. Therefore

$$f_0 = \sup\{f_{k_1}(x_1) + \cdots + f_{k_n}(x_n)\}.$$

□

Corollary 2.1.12. *Let* V *be a vector lattice. Any* $f \in V^*$ *also belongs in* $V^\star$ *if and only if it is the difference of two positive linear forms.*

Proof. Suppose $f \in V^*$ is the difference of two positive linear forms. By the uniqueness of the decomposition and Corollary 2.1.11 it holds that

$$f = f^+ - f^- = sup\{f(y) : 0 \preccurlyeq y \preccurlyeq x\} - inf\{f(y) : 0 \preccurlyeq y \preccurlyeq x\}.$$

Hence there exist $y_1 \preccurlyeq x$ and $y_2 \preccurlyeq x$ such that $f(y_1) = sup\{f(y)\}$ and $f(y_2) = inf\{f(y)\}$. Thus we obtain

$$0 < f(y_1) - f(y_2) = f(y_1 - y_2) \preccurlyeq f(x - y_1) \preccurlyeq f(x).$$

It follows that $f([y_2, y_1]) \subset f([y_2, x])$ for an arbitrary interval $[x, y]$. Therefore f is order bounded.

Conversely suppose f is order bounded. Hence by Corollary 2.1.11 and the uniqueness of the representation it follows that f is the difference of two positive functions such that

$$f = f^+ - f^-.$$

□

Corollary 2.1.13. *Let V be a vector lattice and $f, g \in V^\star$. Two functions f, g are called lattice orthogonal if and only if there exists a positive decomposition of $x = y + z$ such that $|f|(y) < \epsilon$ and $|g|(z) < \epsilon$, where $\epsilon > 0$ and $x \in V_+$.*

Proof. By Definition 1.1.64, two elements are lattice orthogonal if the infimum of their absolute values is zero. Since $f, g \in V^\star$ then $|f| \wedge |g| = \inf\{|f|(y) + |g|(z)\}$, where $y \succcurlyeq 0$ and $z \succcurlyeq 0$ and $x = y + z$. If there exists $\epsilon > 0$ such that $|f|(y) < \epsilon$ and $|g|(z) < \epsilon$ then $|f| \wedge |g| = inf\{|f|(y) + |g|(z)\} \preccurlyeq \epsilon + \epsilon = 2\epsilon$. Since ϵ is arbitrary we can choose ϵ so close as we want to zero. Hence $inf\{|f|(y) + |g|(z)\} = 0$ and therefore $|f| \wedge |g| = 0$. The converse is evident. □

Proposition 2.1.14. *Let V be a vector lattice and $F \subset V$ a filter possessing a base of symmetric order intervals $\{[-x, x] : x \in D\}$. Then the set B_F of all $f \in V^\star$ such that $\lim_F f(x) = 0$ is a band in $V^\star$.*

Proof. First we need to verify that B_F is an ideal. We observe that B_F is a vector subspace of $V^\star$. Thus we need to validate that B_F is a solid vector subspace. Let $f, g \in V^\star$. Then $f \wedge g(x) = \inf\{f(z) + g(y) : x = y + z, z \succcurlyeq 0 \text{ and } y \succcurlyeq 0\}$. Since $\lim_F f(x) = 0$ and $\lim_F g(x) = 0$, for an appropriate F satisfying the hypothesis, it follows that $f \wedge g \in V^\star$. We work similarly for $f \vee g$ for $x \in V_+$. Thus B_F is a vector subspace of $V^\star$. To prove that B_F is solid, let $f \in B_F$ and $g \in V^\star$, such that $|g|(z) \preccurlyeq |f|(x)$, where $|z| \preccurlyeq x$ or, equivalently, $z \in [-x, x]$. Since x belongs in a directed set it follows that there exists $\epsilon > 0 \in \mathbb{R}$, such that $|f(z)| < \epsilon$, $|z| \preccurlyeq x$. Hence by (iii) it follows that

$$\big||g|(z)\big| \preccurlyeq \big||f|(z)\big| \preccurlyeq |f|(z) \preccurlyeq \epsilon,$$

where $z \in [-x, x]$. Hence $|g| \in B_F$ and thus $g \in B_F$. Therefore, B_F is an ideal. To prove that B_F is a band, let $\{(g_\beta) : \beta \in B\}$ be a directed family of B_F and denote by $g := \sup_s g_s$ where $g \in V^\star$. Then there exists $x_i \in D$ for some i and $\beta_0 \in B$ such that $(g - g_{\beta_0})(x_i) \preccurlyeq \epsilon$. Since D is directed there exists $x_1 \in D$ such that $x_1 \preccurlyeq x_i$ and $|g_{\beta_0}|(x - 1) \preccurlyeq \epsilon$. Hence we obtain

$$|g(z) - g_{\beta_0}(z)| \preccurlyeq (g - g_{\beta_0})(x_1) \preccurlyeq (g - g_{\beta_0})(x_i) \preccurlyeq \epsilon$$

for all $z \in [-x, x]$. Therefore $|g(z)| \preccurlyeq 2\epsilon$ by the triangle inequality and the relation above. Thus $g \in B_F$, since B_F is an ideal. □

Corollary 2.1.15. *Let V be a vector lattice. Then $V_0^\star$ and $V_{00}^\star$ are projection bands in $V^\star$.*

Proof. We know that $V_{00}^\star \subset V_0^\star \subset V^\star$. Let

$$\mathcal{F}_0 := \{F_i \subset V : F_i \text{ is order convergent} \\ \text{and they have a base of symmetric order intervals}\}.$$

Respectively we define $\mathcal{F}_{00}$ the family of order convergent filters with countable base of symmetric order intervals. By Proposition 2.1.14, $B_{\mathcal{F}_0}$ and $B_{\mathcal{F}_{00}}$ are bands. Since all bands are intersection invariant and $V_0^\star$, $V_{00}^\star$ are vector spaces, it follows that $V_0^\star = \bigcap_{F \in \mathcal{F}_0} B_F$ and $V_{00}^\star = \bigcap_{F \in \mathcal{F}_{00}} B_F$. Therefore by Theorem 1.2.53 each band is a projection band and the proof is complete. □

Proposition 2.1.16. *Let V be a vector lattice and $f \neq 0$ be a linear form on V. The following are equivalent:*

(i) f is a lattice homomorphism of V onto $\mathbb{R}$.

(ii) $f(x^+) \wedge f(x^-) = 0$ for all $x \in V$.

(iii) f is positive $f^{-1}(0)$ is maximal ideal in V.

(iv) f is positive and I_f is minimal in $V^\star$.

Proof. *i)* $\rightarrow$ *ii)* Since f is a linear mapping the equivalences from (i) to (ii) are obtained by Proposition 1.2.32.

ii) $\rightarrow$ *iii)* Now since f is a lattice homomorphism it follows that f is positive and by Corollary 1.2.37 $f^{-1}(0)$ is an ideal. Thus by Example 1.3.3 $f^{-1}(0)$ is maximal.

iii) $\rightarrow$ *iv)* Let $g \in V^\star$ and denote by $I_f^\star$ the ideal generated by f such that $|g| \preccurlyeq cf$ for some $c \in \mathbb{R}_+$ and $g \in I_f^\star$. Then $x \in f^{-1}(0)$ implies that $|g(x)| \preccurlyeq cf(x) = 0$ since $f^{-1}(0)$ is an ideal in V and f a lattice homomorphism. Hence $f^{-1}(0) \subset g^{-1}(0)$ and it follows that $g = 0$ or $g = \mu f$, $\mu \in \mathbb{R}$. This implies that g is a scalar multiple of f for some $\mu \in \mathbb{R}$. Hence $I_f^\star$ is one dimensional and isomorphic to $\mathbb{R}_0$. Therefore $I_f^\star$ is minimal by Corollary 1.3.14.

iv) $\rightarrow$ *i)* Let $f \in V$ and $x \in V$. Suppose $f(x^+) > 0$. Denote by τ the mapping from $V_+ \rightarrow \mathbb{R}_+$ where

$$\tau(v) = sup\{f(z) : z \in [0, v] \cap P\},$$

and $P = \bigcup^{\infty} n[0, x]$. By the decomposition property it follows that $[0, v] = [0, v_1] + [0, v_2]$ when $v = v_1 + v_2$. We observe that τ is additive and positive homogeneous. Hence there exist a positive extension $\tilde{\tau} : V \rightarrow \mathbb{R}$ defined by τ such that $0 \preccurlyeq \tilde{\tau} \preccurlyeq f$. Since $I_f^\star$ is minimal it follows that $\tilde{\tau} = \mu f$ for some $\mu \in \mathbb{R}_+$. Because $\tilde{\tau}(x^-) = f(x^-) > 0$ it follows that $\mu = 1$. Now if $\tau(x^+) = 0$ then $f(x^+) = 0$ and therefore $f(x^+) \wedge f(x^-) = 0$ holds.

□

Remark 2.1.17. In view of Definition 1.3.16 and Proposition 1.3.17 we observe that every lattice homomorphism $f \in V^\star$ is a positive atom.

Definition 2.1.18. Let V be a vector lattice. The band $V_a^\star$ generated by the set of all atoms a is called the *atomic part* of $V^\star$.

Example 2.1.19. Consider the space $V = \mathbb{R}^n$. Then $(\mathbb{R}^n)^\star = \mathbb{R}^n$, if $\mathbb{R}^n$ is finite dimensional, endowed with the lexicographic ordering. By Corollary 1.3.36, $\mathbb{R}^n$ is Archimedean and thus is countably order complete. Hence, there exists subsets of $\mathbb{R}^n$, such that any positive linear form is bounded, by Proposition 2.1.43. Moreover, finite dimensional $\mathbb{R}^n$ is super Dedekind Complete .

Example 2.1.20. Let $V = \mathbb{R} \circ M$ be the lexicographic union of $\mathbb{R}$ and the vector lattice M. By definition of V it follows that M is order bounded in V and hence for each $f \in V^\star$ we obtain that $f(M) = 0$. Hence the set $\{f \in V^\star : f(M) = 0\}$ is an ideal and it follows that $V^\star$ is isomorphic with $(V/M)^\star$. Moreover, by definition of V, it follows that $V/M \cong \mathbb{R}$. Since $dim\mathbb{R} = 1$, it follows that $dimV^\star = 1$.

A more general version of the previous example is the following.

Example 2.1.21. In view of Theorem 1.3.35, if V is finite dimensional it follows that $V^\star$ is isomorphic to $(V/R_V)^\star$ as $\{f \in V^\star : f(M_j) = 0\ \forall j\}$ is an ideal and thus $dimV^\star = n - r$.

Proposition 2.1.22. *Let V be a vector lattice. Then $V^\star$ is isomorphic to V if and only if V is Archimedean.*

Proof. (i) Suppose $V^\star$ is isomorphic to V. By Theorem 1.3.35 and Example 2.1.21, it follows that $R_V = \{0\}$. Hence, by Corollary 1.3.36, it follows that V is Archimedean.

(ii) Conversely, suppose V is Archimedean. Then, by Corollary 1.3.36, it follows that V is semi-simple or, equivalently, $R_V = \{0\}$. Hence by Example 2.1.21, $V^\star \cong (V/R_V)^\star$ holds and hence $V^\star \cong V$.

□

Example 2.1.23. Let V be a vector lattice. We have seen that there exist non-Archimedean vector lattices of arbitrary linear dimension such that $dimV^\star = 1$. Sometimes, even if V is Archimedean, it can occur that $V^\star = \{0\}$. An example of such space is the vector lattice $L^p(\mu)$ of all finite Lebesgue measurable function on $\mathbb{R}$ modulo μ-null functions, where μ is a Lebesgue measure on $\mathbb{R}$.

In general, if V is any vector lattice, the spaces $V^\star, V_0^\star$ and$V_{00}^\star$ are distinct.

Example 2.1.24. Let $V = C([0,1])$. Then $V_0^\star = \{0\}$. Let $\mathcal{F} = \frac{1}{n}[0,x]$ such that $x \in C([0,1])$. Then by definition, each sequence of linear function on $\mathcal{F}$ order converges to 0. Hence for every $g \in V_0^\star$ it follows that $g \xrightarrow{\preceq} 0$. Since g is positive it follows that $g = 0$ and therefore $V_0^\star = \{0\}$.

Example 2.1.25. Let B be a Boolean Algebra and denote by K_B the Stone Representation Space. Denote by V_B the vector lattice of real function on K_B which is the linear hull of the characteristic functions of all open-and-closed subsets of K. The vector lattice M of all bounded, finitely additive real functions on B can be identified with the order dual of $V_B^\star$ and in fact with the order dual of $C(K_B)$.

Example 2.1.26. Let X be a locally compact space. Denote by $\mathcal{K}(X)$ the vector lattice of all real valued continuous functions with compact support on X. The order dual $\mathcal{M}(X)$ of $\mathcal{K}(X)$ is an order complete vector lattice whose elements are called Radon Measures on X.

Remark 2.1.27. Sometimes an immediate question arises if a linear form defined on a vector lattice V or ideal I can be extended to V with preservation of order related properties. In general the answer is negative.

Example 2.1.28. Let $V = \mathbb{R} \circ M$. Then no non-zero, order bounded linear form has an order bounded extension to V.

Before moving further, we need to remind the reader of the Hahn-Banach Theorem as we will make use of it in the next proposition.

Theorem 2.1.29 (Hahn-Banach Theorem)**.** *Let V be a topological vector space, L an affine hyperplane in V and A a non-void, convex, open subset of V, such that $A \cap L = \emptyset$. Then there exists a closed hyperplane H, which contains L and not intersecting A.*

Proposition 2.1.30. *Let V be a vector lattice and M a vector subspace of V. A function $f \in M^\star$ has a positive extension F in $V^\star$ if and only if there exists an appropriate convex, absorbing subset of V such that f is bounded above on $M \cap (U - V_+)$.*

Proof. Suppose that f has a positive extension. Then $U = \{x \in V : f(x) < 1\}$ is convex and absorbing. Any convex combination of $x = \lambda y + (1-\lambda)z$, $\lambda \in \mathbb{R}$, $y \neq z \in V$ implies $f(\lambda y + (1 - \lambda z)) = f(\lambda y) + f(1-\lambda)z$. Since f is a linear form it follows that

$$f(\lambda y + (1 - \lambda z)) = \lambda f(x) + (1-\lambda)f(x) = \lambda f(x) + f(x) - \lambda f(x) < 1.$$

Moreover, if $x \in U$, there exists $\mu \in \mathbb{R}_+$ such that $f\left(\frac{1}{\mu}x\right) = \frac{1}{\mu}f(x) < 1$. Hence $f(x) < \mu \cdot 1$. Thus $x \in \mu U$. Now obviously f is bounded above on $M \cap (U - V_+)$. Conversely, suppose U is a convex, absorbing subset of V, such that f is bounded above on $M \cap (U - V_+)$. Let $c \in \mathbb{R}$ and $y \in V$, such that $y \in M \cap (U - V_+)$ and $f(y) < c$. We observe that the set of all $x \in M$, such that $f(x) = c$ is a hyperplane in M and given that f is non-zero it follows that the set $\{x : f(x) = c\}$ is an affine hyperplane denoted by H_c. By the definition of H_c it follows that $H_c \cap (U - V_+) = \emptyset$. Moreover, U is a 0-neighborhood for the local convex topology and by translation so is $(U - V_+)$. Hence, by Theorem 2.1.29, there exists $H \subset V$ such that $H_c \subset H$ and $H \cap (U - V_+) = \emptyset$. This implies that $0 \notin H$ hence H is defined by $H = \{x \in V : F(x)\}$ for suitable $F \in V^\star$. Since H_c is a proper maximal subspace of M and $0 \notin H$, then $M \cap H = \emptyset$ must holds. Hence F is a positive extension of f. Indeed, $F(x) < c$ for all $x \in ()U - V_+)$. Therefore, $F \succcurlyeq 0$, for all $x \in V_+$. □

Definition 2.1.31. The mapping $p : V \to \mathbb{R}$, where V is a vector space, is called a *semi-norm*, if all of the following hold:

- $p(x + y) \preccurlyeq p(x) + p(y)$ for every $x, y \in V$.
- $p(\lambda x) = |\lambda| p(x)$ for every $x \in V$ and $\lambda \in \mathbb{R}$.

Proposition 2.1.32. *Let V be a vector lattice and $M \subset V$ a vector subspace such that $I_M = V$. Then each $f \in M^\star$ has an order bounded extension on V.*

Proof. We will apply Proposition 2.1.30. Due to Proposition 2.1.10, it is sufficient to show that any positive linear form on M has a positive extension. For that reason, let $\tau : V \to \mathbb{R}_+$ defined by

$$\tau(v) = \inf\{f(y) : y \in M : y \succcurlyeq |x|\}, \quad x \in V.$$

Since I_M is an ideal and $I_M = V$, we obtain that τ is well defined as $\tau(v)$ exists. In addition τ is a semi-norm. Indeed it is obvious that τ is positive homogeneous. Let $v \in V$, such that $v = z + w$ then

$$\tau(z + w) = \inf\{f(y) : y \in M : y \succcurlyeq |z + w|\}.$$

By the decomposition property, it follows that $y \notin [-(z+w), (z+w)]$ or, equivalently, $y \notin [-z, z] + [-w.w]$. Hence we obtain

$$inf\{f(y) : y \in M : y \succcurlyeq |z + w|\} \preccurlyeq inf\{f(y) : y \succcurlyeq |z|\} + inf\{f(y) : y \succcurlyeq |w|\},$$

by distributivity. Therefore, $\tau(x + y) \preccurlyeq \tau(x) + \tau(y)$, for all $x, y \in V_+$. Now, we denote by U the set of all $x \in V$ such that $\tau(x) < 1$. In view of 2.1.30 and since τ is a semi-norm is obvious that U is convex and absorbing. In order to apply 2.1.30we need to show that $f(x) < 1$, for all $y \in M \cap (U - V_+)$. Now let $u = y + v$, where $p(v) < 1$, $v \in V_+$ and $y \in M$. Obviously $p(y) \preccurlyeq p(u)$. Equivalently, $f(y) \preccurlyeq f(u)$ and thus $0 \preccurlyeq y \preccurlyeq u$. Hence $0 \preccurlyeq y^+ \preccurlyeq u + y^-$. Therefore we obtain

$$\begin{aligned} f(y) = p(y^+) - p(y^-) &\preccurlyeq p(u + y^-) - p(y^-) \\ &\preccurlyeq p(u) + p(y^-) - p(y^-) \\ &= p(u) < 1. \end{aligned}$$

Hence $f(y) < 1$ holds, for all $x \in V$. And thus by Proposition 2.1.30 there exists an extension F of f, which is evident order bounded. □

Corollary 2.1.33. *If V is an Archimedean vector lattice then every $f \in V^\star$ has an order bounded extension F to its Dedekind completion.*

Proof. We know that V can be identified as a sublattice of $\bar{V}$. Moreover since V is Archimedean, $\bar{V}$ is also Archimedean. Thus it follows, from Corollary 1.3.36, that $\bar{V}$ is Semi-Simple. Hence the ideal generated by V is maximal in $\bar{V}$ and $I_V = \bar{V}$. Therefore, by Proposition 2.1.32, each f has an order extension to $\bar{V}$. □

Remark 2.1.34. Let V be any vector lattice and denote by $V_s^\star$ a vector sublattice of $V^\star$. We shall write $\langle V, V_s^\star \rangle$ for the pair $(V, V_s^\star)$ endowed with the evaluation map defined as follows:

$$(x, f) \mapsto \langle x, f \rangle = f(x) \text{ on } V \times V_s^\star.$$

Unless the contrary is clearly stated, we do not assume that $V_s^\star$ separates V.

Definition 2.1.35. Let V be a vector lattice and $A \subset V$. The set A^p is called the *polar* of A and is defined as follows:

$$A^p = \{x \in F : \langle x, f\rangle \preccurlyeq 1, \text{ for all } x \in v\}.$$

Definition 2.1.36. Let V be a vector lattice and $A \subset V$. If A is a subspace of V, then the set A^p is called the *annihilator* of A.

Proposition 2.1.37. *Let V be a vector lattice and $V_s^\star$ be a vector sublattice of $V^\star$ and $A \subset V$. If A is solid in V, then A^p is solid in $V_s^\star$. Actually the polar of any ideal in V is an ideal in $V_s^\star$.*

Proof. Let $x \in A$ and $f \in A^p$, $g \in V_s^\star$. Then, in view of Corollary 2.1.11, it holds $|f|(x) = sup\{|f(z)| : |z| \preccurlyeq |x|\} \preccurlyeq 1$ and since A is solid, it follows that $|f| \in A^p$. Thus, if there exists $g \in V_s^\star$, such that $|g|(x) \preccurlyeq |f|(x)$, then it implies $|g(x)| \preccurlyeq |f(x)| \preccurlyeq |f|(|x|) \preccurlyeq 1$ as seen above. Hence $g \in A^p$. The last assertion is imminent as I^p is a vector subspace of $V_s^\star$. □

Remark 2.1.38. We can not change the roles of V and $V_s^\star$ and expect the same results. The polar J^p of an ideal in $V_s^\star$ is not necessarily an ideal in V. This follows from the fact that the evaluation map $V \to (V_s^\star)^\star$ is not necessarily a lattice homomorphism despite the fact that each element on V can define an order bounded linear form on $V_s^\star$.

The situation can be improved if $V_s^\star$ is assumed to be an ideal of $V^\star$.

Proposition 2.1.39. *Let V be a vector lattice and J an ideal of $V^\star$. Then the evaluation map $\tilde{q} : V \to J^\star$ where $x \mapsto \tilde{x}$ and $\tilde{x}(f) = \langle x, f\rangle$, such that $x \in V$ and $f \in J$ is a lattice homomorphism of V into $J^\star$.*

Proof. It is obvious that the evaluation map is linear. In view of Proposition 1.2.32 and since each lattice homomorphism is positive, we need to prove the uniqueness of the representation of x or, equivalently, $(\tilde{q})^+$ and $(q^+)^\sim$ agree on $J^\star$. We know that $(f(x))^+ \preccurlyeq f(x^+)$. So we need to verify the reverse allocation. For that purpose, let $\tau : V_+ \to \mathbb{R}_+$, such that

$$\tau_g(x) = \sup\{g(z) : z \in [0, y] \cap P\},$$

where $P = \bigcup_{\infty}^{n=1} n[0, x^+]$, which is a subset of V. By Proposition 2.1.32, τ is positive homogeneous and additive, hence τ_g extends to a linear form $T_g \in V^\star$. We observe that $0 \preccurlyeq T_g \preccurlyeq f$ by definition. Since J is an ideal in $V^\star$, it follows that $T_g(x^-) = 0$ and hence $T_g \in J$. So, we obtain

$$f(x^+) = T_g(x^+) \preccurlyeq \sup_{0 \preccurlyeq h \preccurlyeq f} = (f(x))^+.$$

This is true for all $f \in J^\star$, hence we obtain

$$(\tilde{q})^+ \succcurlyeq (x^+)^\sim$$

and the proof is complete. □

Corollary 2.1.40. *Let V be a vector lattice and F an ideal of $V^\star$. The annihilator $F^p \subset V$ is an ideal in V and V/F^p is identified as a vector sublattice of $F^\star$.*

Proof. We showed that $\tilde{q}$ is a lattice homomorphism and by definition

$$F^p = \{f \in V^\star : \langle x, f\rangle \preccurlyeq 1 \ , \ x \in F\}.$$

It is clear that $\ker \tilde{q} \subset F^p$. By Corollary1.2.37, it follows that $\tilde{q}^{-1}(0)$ is an ideal in V and thus $\emptyset \neq F^p \cap \ker \tilde{q}$. Hence F^p is the kernel of $\tilde{q}$. Since $F^p \subset V$, then $F^p \mapsto V/F^p$ can be identified as a vector sublattice of $F^\star$. □

Remark 2.1.41. It is worth mentioning that the evaluation map $V \to F^\star$ is not sequentially order continuous, either not order continuous, unless $F \subset V^\star_{00}$. If f is order continuous, then it follows that q is sequentially order continuous as $V^\star_{00} \subset V^\star_0$. We will see later that if V is super Dedekind Complete, then $V^\star_{00} = V^\star_0$.

Definition 2.1.42. Let V be a vector lattice. The mapping $r \mapsto x_r$, $x_r \in V$ is called a *monotone transfinite sequence*, where x_r maps the set of all ordinals $r < q$ into V, such that if $r_1 < r_2 < q$ holds, then $x_{r_1} \preccurlyeq x_{r_2}$.

Recall that a function ϕ is monotone with respect to the ordering when $x \preccurlyeq y$ implies $\phi(x) \preccurlyeq \phi(y)$ for all $x, y \in V_+$.

Proposition 2.1.43. *Let V be a countable order complete vector lattice. If there exists a strictly monotone map $\phi : V_+ \to \mathbb{R}$ in V, then V is order complete. Furthermore if $\emptyset \neq A \subset V$ such that each subset of A is majorized. Then $\sup A$ exists and there exists a countable $C_0 \subset A$ such that $\sup C_0 = \sup A$.*

Proof. To prove the first assertion consider a transfinite sequence $r \mapsto x_r$. The construction of $\mathbb{R}$ implies that there are no uncountable sets $B \subset \mathbb{R}$, where B is an ordered set and the ordering is induced by $\mathbb{R}$. Hence the existence of a strictly positive monotone function implies that any transfinite sequence in V is countable in V_+. Therefore, for an arbitrary $B \subset \mathbb{R}$ a transfinite sequence validates the existence of $\sup B$ which is in $\mathbb{R}$. Hence V is order complete.

Suppose that $\emptyset \neq A$ is a subset of V. Also every countable subset C of A is majorized. Let $v_c = \sup C$ and the set of all v_c is denoted by C_M. Since V is order complete, we can construct a sequence $(r_q)_q < \epsilon$ by transfinite recursion where q is an ordinal and belongs to the set of all countable ordinals up to ϵ such that

$$r_q = r_{q+1} \text{ if and only if } r_q \text{ is the greatest element of } C_M.$$

We observe that r_q can not be strictly monotone for each countable ordinal q. This implies that v_c is the supremum of A and the proof is complete.

□

Remark 2.1.44. In Remark 1.1.89 we stated that a countably order complete vector lattice V is order complete. Then Proposition 2.1.43 implies the reverse, if there exists a strictly monotone function.

Definition 2.1.45. Let V be an order complete vector lattice. If every majorized subset A of V has a countable subset C such that $supC = supA$ then V is called *super Dedekind Complete.*

Proposition 2.1.46. *Let V be a super Dedekind Complete vector lattice. Then* $V_{00}^{\star} = V_0^{\star}$.

Proof. We know that $V_{00}^{\star} \subset V_0^{\star}$. Hence to prove the equality we need to validate the reverse inclusion. For that reason, let $f \in V_0^{\star}$ and $\mathcal{F} \in V$ a filter. Moreover, let $A \subset V$ be a majorized subset of V and A_0 a countable subset of A. By hypothesis, since V is super Dedekind Complete, it follows that $supA_0 = supA$ and thus $inf A_0$ exists as A_0 is a directed subset of A. Now Proposition 1.1.84, implies that there exists a family $(z_n) \downarrow 0$ and and a $(x_n)_{n\in\mathbb{N}}$ such that $|x_m - 0| \preccurlyeq z_n$, for all $m \geqslant n$. Let $F_j = \{z \in V : |z - 0| \preccurlyeq z_a\}$ $(j \in J)$. These sets belong to the filter $\mathcal{F}$ and since V is super Dedekind Complete, J is countable and therefore by Proposition 1.1.83, it follows that $\mathcal{F}$ is order convergent to 0. Thus $f \in V_{00}^{\star}$. □

Corollary 2.1.47. *Let V be a countably order complete vector lattice. If there exists a strictly positive linear form on V, then V is super Dedekind complete.*

Proof. We can define a strictly positive monotone mapping on V_+ as a strictly positive linear form and we follow the previous proof of Proposition 2.1.43. □

Definition 2.1.48. Let V be a vector lattice and $0 \preccurlyeq f \in V^{\star}$. Then the set of all $x \in V$ such that $f(|x|) = 0$ is called the *absolute kernel* and is denoted by $N(f)$.

Remark 2.1.49. Let V be a vector lattice and $0 \preccurlyeq f \in V^{\star}$. Then $N(f)$ is an ideal of V. Moreover if $f \in V_{00}^{\star}$ then $N(f)$ is a band.

The following proposition provides conditions sufficient to make $N(f)$ a projection band whenever $f \in V_{00}^{\star}$.

Proposition 2.1.50. *Let V be a countably order complete vector lattice and $f \in V_{00}^{\star}$. Then $V = N(f) + N(f)^{\perp}$ provided that f is a positive order continuous linear form and $N(f)^{\perp}$ is super Dedekind Complete.*

Proof. Let $f \in V_{00}^{\star}$ be a strictly positive map in $N(f)^{\perp}$. Since $N(f)^{\perp}$ is the orthogonal of a band it is a band itself. By the Corollary 2.1.47, since V is a countably order complete, it follows that $N(f)^{\perp}$ is super Dedekind Complete. Because V is countably order complete, the set $[0, v] \cap N(f)^{\perp}$ is directed and each subset of it is majorized. Thus $\sup[0, v] \cap N(f)^{\perp}$ exists in $N(f)^{\perp}$ by Proposition 2.1.43, and therefore $N(f)^{\perp}$ is a projection band, as a result of Theorem 1.2.53. Hence $V = N(f)^{\perp} + N(f)^{\perp\perp}$. Moreover, we know that $N(f) \cap N(f)^{\perp} = \emptyset$ and thus $N(f) \subset (N(f))^{\perp}$. Now suppose that M is a maximal system in $N(f)$. Then there exists a maximal system $S \in V$, such that $S \supset M$. This implies $S/M \subset N(f)^{\perp}$. Therefore, if $x \in N(f)^{\perp\perp}$ by Proposition 1.1.92, it follows that $x \in N(f)$, as $N(f)$ is a band. Hence $N(f) = N(f)^{\perp\perp}$ by Proposition 1.2.47, since V is Archimedean. □

Definition 2.1.51. Let V be a vector lattice and $0 \preccurlyeq f \in V^{\star}$. For every f the band $N(f)^{\perp} \subset V$ is called the *band of strict positivity* of f, denoted by $P(f)$.

Proposition 2.1.52. *Let V be a countably order complete vector lattice. Then $N(f \wedge g) = N(f) + N(g)$, such that $f, g \in V_0^\star$. Actually if $f \perp g$, then $V = N(f) + N(g)$.*

Proof. We know that $N(f \wedge g) = \{x \in V : f \wedge g(|x|) = 0\}$. By Proposition 2.1.10, it follows that $N(f \wedge g) = \big\{x \in V : inf\{f(y) + g(z) : y \succcurlyeq 0 \text{ and } z \succcurlyeq 0 : \ x = y + z\}\big\}$. So in order to prove that $N(f \wedge g) \subset N(f) + N(g)$ we need to prove that $f(y) = 0$ and $g(z) = 0$ for $y, z \in V_+$. Let $x \in N(f \wedge g)$, $x \succcurlyeq 0$. Then there exists a positive decomposition of x such that $x = y_z + z_n$, where $f(y_n) + g(z_n) < \dfrac{1}{2^{n+1}}$, $n \in \mathbb{N}$. Now we set $\bar{z_n} = \sup\limits_{k \geqslant n} z_k$ and $\bar{y_n} = x - \bar{z_n}$. By hypothesis, $supz_k$ exists for all countable majorized subsets of $N(f \wedge g)$. Thus it follows that $g(\bar{z_n}) \preccurlyeq \sum\limits_{k \geqslant n}^{\infty} g(z_k)$, because g is order continuous. Therefore $g(\bar{z_n}) \preccurlyeq \dfrac{1}{2^n}$, for all $n \in \mathbb{N}$. Thus $g(\bar{z_n}) \to 0$. Moreover, we observe that $\bar{y_n} \preccurlyeq y_k$ as $\bar{y_n} \preccurlyeq x - z_k \preccurlyeq y_k$. Hence $f(\bar{y_n}) - 0$, since $f(\bar{y_k}) < \dfrac{1}{w^{k+1}}$, for all $n \in \mathbb{N}$. Now, if $x = y + z$, where $z = \inf_n \bar{z_n}$ and $y = \sup_n \bar{y_n}$, then $z \in N(g)$ as $g(z) = 0$ and $y \in N(f)$, as $f(y) = 0$, since $f \in V_0^\star$. Hence, $x \in N(f) + N(g)$.

The reverse inclusion is obtained by using Proposition 2.1.10.

Now if $f \perp g$, it follows that $|f| \wedge |g| = 0$ and hence $f \wedge g = 0$. This implies $f = 0$ or $g = 0$ If $f \in V_0^\star$ and $f = 0$, it holds that $N(f) = N(0)$. Because $N(f)$ is a band in V, it follows that $N(f)^\perp = N(g)$ as $N(g)$ contains all non-zero functions than maps the absolute value of x to 0. Moreover, $N(f)$ is a projection band by Proposition 2.1.50 and therefore $V = N(f) + N(f)^\perp$ or, equivalently, $V = N(f) + N(g)$. □

Corollary 2.1.53. *Let V be a countably order complete vector lattice. Then for all $f, g \in V_0^\star$, it holds that*

(i) $N(f \vee g) = N(f) \cap N(g)$.

(ii) $N(f \wedge g) = N(f) + N(g)$.

(iii) $P(f \wedge g) = P(f) \cap P(g)$.

(iv) $P(f \vee g) = P(f) + P(g)$.

Proof. If $0 \preccurlyeq f \in V_{00}^\star$ then $P(f)$ is a projection band by Proposition 2.1.50 and hence $V = P(f) + P(f)^\perp$. By the definition of $P(f)$, it follows that $P(f)^\perp = (N(f)^\perp)^\perp$. Since V is countably order complete, it follows that V is Archimedean and hence by Proposition 1.2.47 and Theorem 1.2.53, it follows that $V = P(f) + N(f)$. Therefore, it follows that $P(f) \in B(V)$ by Theorem 1.2.51 and hence $P(f) \in I(V)$. Thus $P(f \wedge g) = P(f) \cap P(g)$ and $P(f \vee g) = P(f) + P(g)$ hold for all pairs of $f, g \in V_{00}^\star$. We proved in Proposition 2.1.52, that $N(f \wedge g) = N(f) + N(g)$ holds. Hence we only need to show that $N(f \vee g) = N(f) \cap N(g)$. It is obvious by Proposition 2.1.10 that $N(f) \cap N(g) \subset N(f \vee g)$. To prove the reverse, let $0 \preccurlyeq x \in N(f \vee g)$. This implies that $f \vee g(x) = 0$ or equivalently by Proposition 2.1.10, that

$$\sup\{f(y) + g(z) : y \succcurlyeq 0 \text{ and } z \succcurlyeq 0 : x = y + z\}.$$

Suppose $y \in N(f)$ and $z \in N(g)$. They are both positive and we assume that $x = y + z$. Hence $f(y) = f(x) = 0 \in N(f)$ and $g(x) = g(z) = 0 \in N(g)$. Therefore $f(x) = 0$ and $g(x) = 0$. Hence $x \in N(f)$ and $x \in N(g)$ or, equivalently, $x \in N(f) \cap N(g)$ and the proof is complete. □

Example 2.1.54. Let μ be a signed measure on a $(X, \mathbf{S})$ measurable space and denote by V the order complete vector lattice of all μ-integrable real functions on X. Let N be the ideal of μ-null integrable functions. Since μ defines a strictly positive linear form on V, by definition, it follows that V/N is super Dedekind Complete by 2.1.47, since V is countably order complete.

Example 2.1.55. With the previous conditions applied for the measure μ and the space V we can apply Proposition 2.1.52, where $\mu^+ = f$ and $\mu^- = g$. Hence we obtain that $V = N(\mu^+) + N(\mu^-)$, because $\mu^+ \perp \mu^-$. This implies that there exists a decomposition of X into components X_1, X_2, where obviously $X_1 \cap X_2$ is $|\mu| - null$ as $N(\mu^+) \cap N(\mu^-) = \{0\}$ and the restriction of μ to $\{X_1 \cap S : S \in \mathbf{S}\}$ is positive and to $\{X_2 \cap S : S \in \mathbf{S}\}$ is negative. Such a decomposition is called a *Hahn Decomposition* of X with respect to μ.

Theorem 2.1.56 (Nakano). *Let V be a vector lattice that satisfies the (OC) property and $F \subset V_{00}^\star$ such that F is an ideal. Denote by q the evaluation map from V to $F^\star$. Then $q(V)$ is an order dense ideal of $F_{00}^\star$.*

Proof. The proof is divided into four parts. Before proceeding to the first suppose that F separates V. By Corollary 2.1.40 we obtain that F^p is an ideal in V and since V is order complete, F^p is a band. This is a contradiction to the separation of V from F. Therefore, we can construct $(F^p)^\perp$ and by Theorem 1.2.53, it follows that $V = F^p + (F^p)^\perp$ and thus V/F^p is isomorphic to $(F^p)^\perp$, since q is the kernel of the evaluation map. Hence we can assume that q is injective and V can be identified as the order complete vector sublattice of $q(V)$ it $F_{00}^\star$.

(i) In the first part, we want to decompose both V and F into the sum of projection bands. Denote by V_x the band generated by $x \in V_+$. Let $N(x) = \{f \in F < x, |f| >= 0\}$ and $P(x) = N(f)^\perp$. Since $f \in V_{00}^\star$, it follows that $N(x)$ is a band and hence, by Proposition 2.1.50 :

$$F = N(x) + P(x) \quad \text{and} \quad V = V_x + \big(V_x\big)^\perp$$

We observe that these decompositions are dual in the sense that $N(x) = \big(V_x\big)^p$ and $P(x) = \big((V_x)^\perp\big)^p$. Since $N(x), P(x)$ are bands it follows that F is the sum of ideals such that

$$F = (V_x)^p + \big((V_x)^\perp\big)^p.$$

Thus it is imminent that $N(x) = (V_x)^p$, since each f is order continuous. Lastly, $P(x) = \big((V_x)^\perp\big)^p$ as both ideals are complementary to $N(x)$ and, by Proposition 1.2.42 and Definition 1.2.43, they must agree.

(ii) In this part we will find a decomposition of $F^\star_{00}$ into bands $B(u_i)$ and a decomposition of bands $V_i \subset V$, subsets of $B(u_i)$. Firstly, let $0 \preccurlyeq u \in V$. We want to show that if $u = u_1 + u_2$, where $u_1 \wedge u_2 = 0$ and $u_1, u_2 \in F^\star_{00}$, then it follows that $u_1, u_2 \in V$. Now denote by $P(u_i) \subset F$ the band of strict positivity of u_i ,$i = 1, 2$. By Proposition 2.1.50, it follows that $P(u) = P(u_1) + P(u_2)$. Hence in view of the first part we obtain the following decomposition

$$
\begin{aligned}
F^\star_{00} &= B(u_1) + B(u_2) + B(u)^\perp \\
V &= \ V_u \qquad\qquad\quad + (V_u)^\perp \\
F &= P(u_1) + P(u_2) + N(u),
\end{aligned}
$$

where $B(u_i)$ are bands generated by u_1 and u_2 and V_u is band generated by u. For the purpose of the second part, denote by V_i the band generated by all bands $N(f)^\perp \subset V$, such that f is positive and $N(f)$ is the absolute kernel. It follows from Corollary 2.1.53, that $V_1 \cap V_2 = \{\emptyset\}$, and since $P(u) = \big((V_u)^\perp\big)^p$ from the first part, it holds that $V_i \subset V$. Suppose that $xe \in V$ is orthogonal to $V_1 + V_2$. Then $\langle |x|, f\rangle = 0$ for all strictly positive function $f \in P(u_1) + P(u_2)$. Therefore $x \in (V_u)^\perp$ since $(V_x)^\perp = P(x)^p$. Hence $V_1 + V_2$ is a band in V and by Theorem 1.2.53, if follows that $V = V_1 + V_2$. But we know that $f \in P(u_i)$ are positive and thus $N(f)^\perp \subset B(u_i)$. So $V_i \subset B(u_i)$ for $i = 1, 2$. The decomposition of u in the beginning is unique. Otherwise, let $v_1, v_2 \in F^\star_{00}$, such that $u = v_1 + v_2$, where $v_i \in V_i \subset B(u_i)$. Then $u_i = v_i$ must hold for $i = 1, 2$, since $B(u_1) \perp B(u_2)$ and $B(u_i)$ are bands.

(iii) We will show that V is an ideal in $F^\star_{00}$. Since V is identified with an order complete sublattice of $F^\star_{00}$, it suffices to show that V is solid in $F^\star_{00}$. For that reason, let $u \in V$ and $v \in F^\star_{00}$, such that $0 \preccurlyeq v \preccurlyeq u$. Now we set $w := sup[0, v] \cap V$, which exists in V and is taken as element of $F^\star_{00}$. If we prove that $w = v$ then this part is complete. Assume that $v = w + y$. If we show that $y = 0$ then we get the desired result. Suppose that $y \neq 0$, then there exists $\mu \in \mathbb{R}_+$, such that $z = y - \mu u$ and $(y - \mu u)^+ > 0$. If Pz is the band projection of $F^\star_{00}$ onto Bz^+, where Bz^+ is the band generated by z^+, then

$$Pz(y - \mu u) = Pz(y) - Pz(\mu u) = z^+ > 0.$$

Since $Pz \succcurlyeq 0$, then $0 \preccurlyeq \mu Pz(u) \preccurlyeq Pz(y) \preccurlyeq y$, by Theorem 1.2.51. But in view of Theorem 1.2.53, it follows that $u = Pz(u) + (1_V - Pz)(u)$ is a decomposition of $u \in V$. Hence $Pz(u) \in V$ by the second part. This is a contradiction though as $w + \mu Pz(u) \preccurlyeq w + y = v$ and $w + \mu Pz(u) \in V$. Therefore $y = 0$ necessarily and V is an ideal in $F^\star_{00}$.

(iv) We proceed to the last part in order to validate that V is order dense in $F^\star_{00}$ or, equivalently, in view of Definition 1.1.79 and Definition 2.1.5, that the band generated by V in $F^\star$ is $F^\star_{00}$. We want to show that if f is a positive order continuous linear form on F and $f \perp V$, then $f = 0$. If $f \perp V$, then it follows that $P(f) \perp P(u)$ for all $u \in V_+$, by Corollary 2.1.53. Thus the orthogonality implies that $P(f) \subset N(u)$, for all $u \in V_+$, since $P(u) = N(u)^\perp$. Therefore, from the first part it follows that $P(f) \subset V^p = \{0\}$ and hence $f = 0$.

□

2.2 Normed Vector Lattices

In this section we will discuss vector lattices endowed with a semi-norm or a norm. We will define Banach Lattices and discuss thoroughly properties of these spaces.

Before stating the main definitions we will remind the reader of some basic topological relations.

Definition 2.2.1. Let V be a topological vector space over $\mathbb{R}$. If for every $x, y \in V$, $x \neq y$, there exist U_x and U_y open subsets of V, such that $x \in U_x$ and $y \in U_y$ and $U_x \cap U_y = \emptyset$, then V is called *Hausdorff* or T_2.

Definition 2.2.2. Let V be a topological vector space. If V is a vector lattice and a Hausdorff topological vector space over $\mathbb{R}$, which has a base of solid 0-neighborhoods, then V is called a *topological vector lattice.*

Example 2.2.3. Let $\mathbb{R}_0$ be the vector lattice over $\mathbb{R}$. Then $\mathbb{R}_0$ endowed with the lexicographic ordering is a Hausdorff space, as $\mathbb{R}$ itself is. Moreover, let U_0 be a 0-neighborhood given as the union of symmetrical order intervals, for some $x_i \in V$ i.e $U = \{\bigcup[-x_i, x_i] : x_i \in V \text{ for some } i\}$. In Example 1.2.13, we proved that the symmetric order interval is solid and we also know that solidness is union invariant. Hence $\mathbb{R}_0$ is a topological vector lattice.

Definition 2.2.4. Let V be a topological vector lattice and $\mathcal{T}$ is the topology endowed with V. If $\mathcal{T}$ is locally convex, then V is called a *locally convex vector lattice.*

Remark 2.2.5. Definition 2.2.4 has meaning, since locally convex vector lattices have a 0-neighborhood base of solid sets, as the convex hull of a solid set is also solid, as shown in Proposition 1.2.17.

Remark 2.2.6. Recall that a semi-norm is a norm if $p(x) = 0$ implies $x = 0$.

Definition 2.2.7. Let V be a topological vector lattice and $M \subset V$ an absorbing set. Then the non negative, real function p_M defined as follows:

$$p_M : V \to \mathbb{R} \text{ such that } p_M(x) = \inf\{\lambda > 0 : x \in \lambda M\}$$

is called the *gauge* or *Minkowski functional.*

Definition 2.2.8. Let V be a topological vector space. The Minkowski functional of an absorbing, convex and solid subset of V is called a *lattice semi-norm.*

Definition 2.2.9. Let V be a vector lattice and $p : V \to \mathbb{R}$ a semi-norm (norm). If $|x| \preccurlyeq |y|$ implies $p(x) \leqslant p(y)$, then p is a *lattice semi norm (lattice norm).*

Definition 2.2.10. Let V be a vector lattice and p a lattice norm. The pair (V, p) is called a normed vector lattice. If in addition (V, p) is complete with respect to the norm, then V is called a *Banach Lattice.*

Remark 2.2.11. If p is a function on a topological vector space V, then p is a lattice semi-norm if p is a semi-norm, such that $0 \preccurlyeq x \preccurlyeq y$, and also implies that $p(x) \preccurlyeq p(y)$ and $p(x) = p(|x|)$, for all $x, y \in V$. In this case, lattices semi-norms(norms), often called *monotone* semi-norms(norms).

Proposition 2.2.12. *Let (V, p) be a normed vector lattice. Then V is necessarily Archimedean.*

Proof. Suppose V is not Archimedean. Then, there would exist a sequence $(y_n)_{n\in\mathbb{N}} \in V$, such that $\inf(n^{-1}y) \neq 0$. Let $v := \inf(n^{-1}y)$. Then, it holds that $0 \prec v \preccurlyeq n^{-1}y_n$. Set $y_n = \dfrac{n^2+1}{n^4}$. Since p is a semi norm, it holds that $p(v) \preccurlyeq \dfrac{n^2+1}{n^4}$ for $n \in \mathbb{N}$. Hence, $p(v) \preccurlyeq 0$. By the anti-symmetric property of the ordering, this implies that $p(v) = 0$. Because p is positive, it follows that $v = 0$, which contradicts the definition of a semi-norm. □

Proposition 2.2.13. *Let V be a vector lattice and U is the unit ball of V. Then V is a normed vector lattice if and only if U is solid in V.*

Proof. Suppose V is a normed vector lattice. Then V is necessarily Archimedean by Proposition 2.2.12. We want to validate that U is solid or, equivalently, that for every $x \in V$ and $y \in U$, such that $|x| \preccurlyeq |y|$, it follows that $x \in U$. Let $y \in U$, then $p(y) \leqslant 1$. It follows by the decomposition property that $\dfrac{1}{n} \cdot y \in U$. Thus, let $x \in V$, such that $|x| \preccurlyeq |\dfrac{1}{n} \cdot y|$. Since V is Archimedean, it follows that $x = 0$ and since $0 \in U$, it follows that $x \in U$. Therefore U is solid.

Let V be a vector lattice endowed with a positive norm p and U is the unit ball. We suppose that U is solid in V. In view of Definition 2.2.9, we want to prove that for every $x, y \in V$, such that $|x| \preccurlyeq |y|$, it follows that $p(x) \leqslant p(y)$. Let $x \in V$ and $y \in U \subset V$, such that $|x| \preccurlyeq |y|$. It follows that $x \in U$, since U is solid. Therefore, $p(x) \leqslant p(y)$. Otherwise, $p(x) \geqslant p(y') \geqslant 0$, since p is positive for some $y' \in U$. This implies that $x \succcurlyeq 0$. Let $y' = \dfrac{1}{n}y$, such that $|x| \preccurlyeq |\dfrac{1}{n}y|$. Since V is necessarily Archimedean, in order to be a vector lattice it follows that $x = 0$, which is a contradiction. Hence $p(x) \leqslant p(y)$, which validates that p is a lattice norm and (V, p) is a normed vector lattice.

□

Example 2.2.14. Let V be a vector lattice and $f \in V^\star$ a positive function. Then the mapping $x \mapsto f(|x|)$ is a lattice semi-norm. First we need to check that this mapping is a semi-norm.

- The triangle inequality holds for all $x, y \in V$ with respect to the ordering. Hence, we obtain $|x+y| \preccurlyeq |x| + |y|$. Since f is linear and positive this implies

$$f(|x+y|) \preccurlyeq f(|x| + |y|) \Leftrightarrow f(|x+y|) \preccurlyeq f(|x|) + f(|y|).$$

Hence the triangle inequality holds for all x, y and $p(x), p(y)$.

- By definition of p for $\lambda \in \mathbb{R}$ we obtain that

  $$p(\lambda x) = f(|\lambda x|) = f(|\lambda||x|)$$

  by Proposition 1.1.68. Now, $f(|\lambda||x|) = |\lambda| f(|x|)$, since $f \in V^\star$. Hence $p(\lambda x) = |\lambda| p(x)$ holds, for all $x \in V$ and $\lambda \in \mathbb{R}_+^*$. Thus p is a semi-norm. Moreover, if $|x| \preccurlyeq |y|$ holds, then $f(|x|) \preccurlyeq f(|y|)$, which implies that p is a lattice semi-norm.

Example 2.2.15. Let (X, Σ, μ) be a measure space. Then $\|.\|_{L^1}$ is lattice semi-norm. We know that

$$L^1 = \{f \in V^\star : \int |f| d\mu < \infty\}.$$

We know that L^1 is a vector lattice endowed with the canonical ordering. Now we know that

$$\begin{aligned} \|f + g\|_{L^1} &= \int |f + g| d\mu \\ &\preccurlyeq \int |f| d\mu + \int |g| d\mu \end{aligned}$$

and

$$\|\lambda f\|_{L^1} = \int |\lambda f| d\mu = \int |\lambda||f| d\mu = |\lambda| \int |f| d\mu.$$

Hence $\|.\|_{L^1}$ is a semi-norm. Moreover, if $|f| \preccurlyeq |g|$ then

$$\int |f| d\mu \preccurlyeq \int |g| d\mu$$

holds, because the integral is monotone with respect to the ordering we use here. Hence, $(L^1, \|.\|_{L^1}, \preccurlyeq)$ is a normed vector lattice. Moreover, we know that $(L^1, \|.\|_{L^1})$ is a Banach space and therefore $(L^1, \|.\|_{L^1}, \preccurlyeq)$ is Banach lattice.

Example 2.2.16. Let $K = [a, b]$ be a compact interval. We denote by $C([a, b])$ the space of all continuous, real functions on $[a, b]$. The space $C([a, b])$ endowed with the supremum norm is a Banach lattice with order unit e the constant function equal to one. The supremum norm is defined as follows:

$$\| \cdot \|_{C([a,b])} = \sup_{f \in C([a,b])} \|f\|.$$

We know that the supremum norm is indeed a norm. To validate that it is a lattice norm, we need to verify that for every $f, g \in C([a, b])$, such that $|f| \preccurlyeq |g|$, it follows that $\sup \|f\| \leqslant \sup \|g\|$. We know that $C([a, b])$ is a vector lattice endowed with the canonical ordering. This implies that $f \preccurlyeq g$ if and only if $f(t) \preccurlyeq g(t)$, for all $t \in [a, b]$. Therefore it is imminent that $\sup \|f\| \leqslant \sup \|g\|$. Moreover, we know that the supremum norm is complete. Hence $\big(C([a, b]), \| \cdot \|_{C([a,b])}\big)$ is a Banach lattice.

Example 2.2.17. The vector lattice M_B of all bounded, finitely additive, real functions on a Boolean Algebra B, is a Banach lattice under the norm:

$$\mu \mapsto \|\mu\| := \sup_{b \in B} |\mu|(b).$$

By this definition, $\|\mu\|$ is called the *total variation* of μ. Let μ_1 and μ_2 be two bounded finitely additive, real function on B, such that $|\mu_1| \preccurlyeq |\mu_2|$. Since both functions are bounded, it follows that there exist $A, B \in \mathbb{R}$, such that $|\mu_1| \preccurlyeq A \preccurlyeq |\mu_2| \preccurlyeq B$. Hence $\sup_{b\in B} |\mu_1|(b) \leqslant \sup_{b\in B} |\mu_2|(b)$. So $\|\mu\|$ is a lattice norm and therefore $\big(M_B, \|\mu\|\big)$ is a normed vector lattice. Let $(\mu_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of functions $\mu \in M_B$. Then

$$\text{for all}, \epsilon > 0, \text{there exists } n_0 \in \mathbb{N}, \text{for all } k, l \geqslant n_0, \text{ such that } \|\mu_k - \mu_l\| < \epsilon.$$

This implies $\sup\limits_{b\in B} |\mu_k - \mu_l| < \epsilon$, by definition of the norm. Since μ_k and μ_l are bounded, there exist upper bounds for both A and B respectively. Set $B = \mu \in V$. Then

$$\text{for all } k, \text{for all } \epsilon', \text{there exists } n_0^{'} \in \mathbb{N}, \text{for all } k \geqslant n_0^{'}, \text{ such that } \sup_{b\in B} |\mu_k - \mu| < \epsilon'.$$

Therefore $\sup\limits_{b\in B} |\mu_k - \mu| < \epsilon' + \epsilon$. The quantity $\epsilon + \epsilon'$ can be as close to zero as we want, hence $\sup |\mu_k - \mu| \to 0$ or, equivalently, $\mu_k \to \mu$. Therefore the sequence is convergent and the norm is complete. So $\big(M_B, \|\mu\|\big)$ is a Banach lattice.

Example 2.2.18. Let V be the vector lattice of all rapidly decreasing sequences $a = (a_1, a_2, \dots)$. The lattice norms

$$\rho_k(a) = \sum_{n=1}^{\infty} n^k |a_n| \quad (k = 1, 2, \dots)$$

define a topology, such that V is a complete metrizable locally convex vector lattice. The fact that V is complete is an easy verification, as all sequence are rapidly decreasing, which implies that every Cauchy sequence has the desired behavior. Moreover, it is an easy verification that ρ_k are norms and lattice norms as $a \preccurlyeq b$ implies

$$|a| \preccurlyeq |b| \Leftrightarrow |a_n| \preccurlyeq |b_n|,\ \forall n \in \mathbb{N} \Leftrightarrow n^k|a_n| \preccurlyeq n^k|b_n| \Leftrightarrow \sum_{n=1}^{\infty} n^k|a_n| \leqslant \sum_{n=1}^{\infty} n^k|b_n|.$$

Therefore $\rho_k(a) \leqslant \rho_k(b)$. Lastly, the fact that V is metrizable comes as a consequence of the local convexity of the topology.

Proposition 2.2.19. *Let V be a normed vector lattice. Then the following hold for all $x \in V$:*

(i) The mappings $x \mapsto x^+$, $x \mapsto x^-$, $x \mapsto |x|$, $(x, y) \mapsto x \wedge y$ and $(x, y) \mapsto x \vee y$ are uniformly continuous.

(ii) V_+ is closed and V is Archimedean.

(iii) *If S is a solid subset of V then S is closed.*

(iv) *If B is a band in V then B is closed in V. This holds for σ-ideals too.*

(v) *Let B be a band in V and denote by p_B the associated band projection. Then p_B is continuous.*

Proof. (i) Let p be a lattice norm. By Remark 2.2.11, p is monotone. In Proposition 1.1.68 we proved that $|x \vee y - x_1 \vee y_1| \preccurlyeq |x - x_1| + |y - y_1|$ and $|w \wedge y - x_1 \wedge y_1| \preccurlyeq |x - x_1| + |y - y_1|$ holds, for all $x, y \in V$. Thus the lattice operations are uniformly continuous, as the difference of the values (x, y) and (x_1, y_1) can be as less than a positive $\epsilon > 0$ as we want. Now, if we take $y = 0$ we observe that $(x, 0) \mapsto x \vee 0 = x^+$ and $(x, 0) \mapsto x \wedge 0 = x^-$ and therefore are uniformly continuous. Lastly, $x \mapsto \|x\|$ is the sum of uniformly continuous functions and as a result uniformly continuous itself.

(ii) We define the positive cone in V as all $x \in V$ such that the negative part is equal to 0. Since $x \mapsto x^-$ is uniformly continuous, it follows that that $x \mapsto x^-$ is continuous. Thus $V_+ = (x^-)^{-1}(0)$, where $\{0\}$ is closed in V. Therefore V_+ is closed as the inverse image of a closed set via continuous map.

(iii) We denote the closure of a subset S by $\overline{S}$. Suppose that S is solid. We want to validate that $\overline{S}$ is solid. We need to prove that for $x \in \overline{S}$ and $y \in V$, such that $|y| \preccurlyeq |x|$ then $y \in \overline{S}$. For that reason, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence to $x \in \overline{S}$. Then $|x_n| \in S$, since S is solid. Thus we define (y_n) , $n \in \mathbb{N}$, such that $y_n^+ = y^+ \wedge |x_n|$ and $y_n^- = |x_n| \wedge y$, $y_n \in S$, since $|y_n| \preccurlyeq |x_n|$. Now we obtain that $\lim_n y_n = y$, since the lattice operation of infimum is uniformly continuous. Therefore $y \in \overline{S}$.

(iv) Let B be a band in V and (x_n) , $n \in \mathbb{N}$ be an increasing sequence in B. If $x_n \to x \in V$ then $x = \sup_n x_n$, since V_+ is closed. Because B is a band, it follows that $\sup_n x_n = x \in B$. Moreover, let I be a σ-ideal. Let (k_n) be a sequence in I convergent to $k \in V$. Set $u_n = |k_n| \wedge |k|$. Then (i), implies that $u_n \to k$. Furthermore, let (x_n) be an increasing sequence in I, such that $x_n := \sup_{0 \preccurlyeq z \preccurlyeq n} u_z$, $n \in \mathbb{N}$. Hence, $0 \preccurlyeq u_n \preccurlyeq x_n \preccurlyeq |k|$. Therefore, $\|x_n - |k|\| \preccurlyeq \|u_n - |k|\|$. This verifies that x_n is convergent to $|k| \in V$. It follows that $|k| = \sup_n x_n$ and since I is a σ-ideal, we have $|k| \in I$. So $k \in I$, since I is solid. Therefore I is closed.

(v) Let B be a band in V and denote by p_B the associated band projection. Then $|p_B x| \preccurlyeq p_B|x| \preccurlyeq |x|$ holds, by Theorem 1.2.51, which implies that $\|p_B x\| \preccurlyeq \|x\|$, for all $x \in V$. Hence p_B is continuous, as it is linear and bounded with norm equal or less than 1.

□

Corollary 2.2.20. *Let V be a vector lattice and V_0 a vector sublattice of V. Then $\overline{V_0}$ is a sublattice in V. This also holds for ideals respectively.*

Proof. The closure of a vector subspace is also a vector subspace. Moreover, since the lattice operations are uniform continuous by Proposition 2.2.19, we obtain that $\overline{V_0}$ is closed in V. Furthermore, by (iii) from Proposition 2.2.19, it follows that the closure of a solid set is solid. Hence, for any ideal $I \in \mathbf{I}(V)$, we have that $\bar{I}$ is an ideal. □

Corollary 2.2.21. *If V is a normed vector lattice then the completion of V is a Banach Lattice with respect to the unique extensions of vector and lattice operations, and also of the norm.*

Proof. Let V be a Banach lattice. By definition, V is a vector sublattice of its completion. The closure of V_+ is the positive cone of $\tilde{V}$, since V is Archimedean. Moreover, by (ii) of Proposition 2.2.19, it follows that $(\tilde{V_+})$ is closed in $\tilde{V}$. Hence $\tilde{V}$ is also Archimedean, and both the lattice operations and the norm are uniformly continuous as extensions. Therefore $\tilde{V}$ is a Banach lattice as the rest properties of Proposition 2.2.19 follow from the norm of V. □

In order to prove the last implication of the following Corollary 2.2.23 we will make use of the following lemma.

Lemma 2.2.22. *Let V be a topological vector space and $H \subset V$ be a hyperplane such that $H = \{x : f(x) = a\}$. Then f is continuous if and only if H is closed.*

Corollary 2.2.23. *Let V be a normed vector lattice and $f : V \to \mathbb{R}$ be a function in $V_0^\star$. If f is lattice homomorphism then f is norm-continuous.*

Proof. Since f is a lattice homomorphism, the inverse image of $\{0\}$ via f is a σ-ideal in V by Proposition 2.2.19. Moreover, as f is positive and $f^{-1}(0)$ determines uniquely the semi spaces in Definition 1.3.22, it follows that $f^{-1}(0)$ is a hyperplane in V. Thus f is continuous by Lemma 2.2.22. □

The following Theorem results in stronger assertions about continuity given that (V, p) is norm complete.

Theorem 2.2.24. *Let V, F be normed vector lattices such that V is a Banach lattice. If $T : V \to F$ is a positive linear map, then T is continuous. This holds for all T.*

Proof. Let $T : V \to F$ be a positive linear map. Since the unit ball U is an absorbing, convex and solid subset of V, then the lattice norm in V can be the gauge of U. Now, suppose T is not continuous. This means that T is unbounded in the unit ball U. Since U contains the symmetric order interval $[-x, x]$, such that $\|x\| \preccurlyeq 1$, it follows that T is unbounded on $U \cap V_+$. Hence, there exists a sequence $(v_n)_{n\in\mathbb{N}} \in V$, such that $\|T(v_n)\| \succcurlyeq n^3$, for all $n \in \mathbb{N}$. Because V is complete with respect to the norm we know that $\sum_{j=1}^{\infty} n^{-2}v_n$ exists in V and it holds that $z \succcurlyeq n^{-2}v_n$, since V_+ is closed in V by Proposition 2.2.19, where $z = \sum_{j=1}^{\infty} n^{-2}v_n$. This implies that $Tz \succcurlyeq n^{-2}T_{v_n} \succcurlyeq 0$ or, equivalently, $\|Tz\| \succcurlyeq n^{-2}\|T_{v_n}\| \succcurlyeq n$, for all $n \in \mathbb{N}$. This is a contradiction to the norm completion in V. Therefore T is continuous. □

Remark 2.2.25. V_+ is norm complete, since V_+ is closed in V and V is norm complete.

To proceed further, we will need the following notations.

Definition 2.2.26. Let V be a Banach lattice and F a normed lattice. A linear map $f : V \to F$ is called *absolutely majorized*, if there exists a linear map $T : V \to F$ satisfying $|fx| \preccurlyeq Tx$, for all $x \in V_+$.

Example 2.2.27. Let $V = \mathbb{R}$ and $F = \mathbb{R}$ endowed with the lexicographic ordering and the absolute value. Consider the map $f : \mathbb{R} \to \mathbb{R}$, where $r \mapsto 2r$, and $T : \mathbb{R} \to \mathbb{R}$, where $r \mapsto 3r$, $r \in \mathbb{R}_+$. It is easy to observe that $|fx| \preccurlyeq Tx$ holds for all $x \in \mathbb{R}$. Hence f is absolutely majorized.

Corollary 2.2.28. *Let V be a Banach Lattice and F a normed vector lattice. Any absolutely majorized linear map is continuous.*

Proof. Let $S : V \to F$ be an absolutely majorized linear map. If T is a positive linear map, then $|Sx| \preccurlyeq Tx$ holds, for all $x \in V_+$. Hence $S = T - (T - S)$, which is the difference of two positive linear maps. Thus S is positive, by Corollary 2.1.12, and, by Theorem 2.2.24, it is also continuous. □

For the following corollaries, assume that V is a Banach Lattice and F is a normed vector lattice.

Corollary 2.2.29. *Any positive linear form is continuous.*

Proof. By Corollary 2.1.12, any positive linear form is the difference of two positive linear forms. By Proposition 2.2.19, both f^+ and f^- are uniformly continuous and hence continuous. □

Corollary 2.2.30. *Any maximal ideal $M \subset V$ is closed.*

Proof. By Corollary 1.3.14, any maximal ideal is the kernel of a positive linear map. Since any positive linear map is continuous, and the kernel is the inverse image of $\{0\}$, which is a closed set via a continuous map, it follows that M is closed. □

Corollary 2.2.31. *Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are different norms for the same Banach Lattice V. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*

Proof. Let V be a Banach lattice and U be the unit ball of V. Then V is necessarily Archimedean by Proposition 2.2.12. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two different norms for V. We want to prove that they are equivalent or, that there exists positive fixed numbers k_1 and k_2 such that

$$k_1\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant k_2\|\cdot\|_1.$$

By Theorem 2.2.24, it follows that the identity mapping is positive and continuous in both directions and since V is Archimedean the respective open subsets contain each other. Hence, we can choose $k_1 = \min\limits_{\|u\|=1} u$ and $k_2 = \max\limits_{\|u\|=1} u$, where $u \in U$ and the proof is complete. □

Proposition 2.2.32. *Let V be a normed vector lattice and $I \subset V$ a closed ideal. Under the canonical ordering and the norm induced by V, V/I is a normed vector lattice. In addition, if V is norm complete, then V/I is norm complete.*

Proof. If V is complete, then V/I is norm complete with respect to the norm given by $x \mapsto p(x+I) = \inf\{p(x) : x \in x+I\}$. To verify that the quotient norm is a lattice norm, we need to validate that $q(U)$ is solid in V/I, where U is the unit ball and q the quotient map. This is obvious, as q is a lattice homomorphism, by Proposition 1.2.32 and Corollary 1.2.37, and thus q maps U to a solid $q(U)$. □

Definition 2.2.33. Let V be a locally convex space V. If each solid absorbing and convex subset of V is a 0-neighborhood, then V is called *barreled.*

We will need the following Theorem due to Banach and Steinhaus.

Definition 2.2.34. Let V be a topological space and $H \subset \mathbb{R}^V$. Then H is called *equicontinuous* at t_0, if for each neighborhood N, there exists a neighborhood U_{t_0} of t_0, such that $[f(t), f(t_0)] \in N$, whenever $t_0 \in U_0$ and $f \in H$.

Theorem 2.2.35. *Let V be a barreled locally convex space. Then every simply bounded subset in V is equicontinuous.*

Definition 2.2.36. The space of all continuous linear forms is called the *strong dual* of V and is denoted by V'.

Proposition 2.2.37. *Let V be a normed vector lattice. Then $V' \subset V^\star$ and V' is a Banach Lattice, which is order complete under the ordering generated by $V^\star$ and its dual norm. Moreover, V' is an ideal in $V^\star$ and if V is barreled, then V' is a band in $V^\star$. If V is a Banach Lattice itself, then $V' = V^\star$.*

Proof. We know that f is a continuous linear form thus, by Proposition 2.1.10, it follows that $f = f^+ - f^-$. This implies that $f^- \in V'$ and $f^+ \in V'$, by Corollary 2.2.29. Hence $V' \subset V^\star$.

Moreover, by (i) of Proposition 2.1.10, it follows that V' is closed under the lattice operations and hence V' is a sublattice of $V^\star$. Proposition 2.1.37 indicates that the polar of the unit ball is solid in $V^\star$ and thus V' is an ideal in $V^\star$. This also implies that the dual norm is a lattice norm and, since V is a normed vector lattice, the dual norm is complete due to the uniform continuity of the lattice operations by Proposition 2.2.19. Therefore V' is a Banach lattice. Since $V^\star$ is order complete, by Proposition 2.1.10, it follows that V' is order complete.

Now we want to validate that the barreldness of V implies that V' is a band in $V^\star$. For that reason. let $\{f_a\}$ be a directed family in V', such that $f_0 = \sup_a f_a$ exists in $V^\star$. By (i) of Proposition 2.1.10, we obtain that $\lim_a f_a(x) = f_0(x)$, for all $x \in V$. Since V' is a sublattice of $V^\star$, we can verify that there exists a filter or, more specific, a U-neighborhood for each $x \in V$, such that $[f_a(x), f_a(x_0)] \subset U$, for some $x_0 \in U$. Hence V' is bounded for all $x \in V$ and, by the Banach-Steinhaus Theorem, V' is equicontinuous. Therefore $f_0 \in V'$ and V' is a band in $V^\star$. The last assertion comes from Corollary 2.2.21. Indeed, let $f \in V^\star$. Since f is a positive linear form, it follows that f is continuous and since V is a Banach lattice, it follows that $f \in V'$. □

Definition 2.2.38. Let $\langle x, f\rangle$ be the evaluation map from V to $\mathbb{R}$. The coarsest topology that makes $\langle x, f\rangle$ continuous, for all $f \in V'$, is called the *weak topology* and is denoted by $\sigma(V, V')$.

In the same frame, we can define another topology in the bidual V'' of V.

Definition 2.2.39. Let $x \in V$ and $f \in V'$. The evaluation map $\langle x, f\rangle : V' \to \mathbb{R}$ is a linear functional. The coarsest topology that makes $\langle x, f\rangle$ continuous, is called the *weak apostrophe topology* and is denoted by $\sigma(V', V)$.

Corollary 2.2.40. *If I is a closed ideal of a normed vector lattice, then I^p is a $\sigma(V', V)$ closed band of V'. Furthermore, I' can be identified with the quotient V'/I^p, which is a Banach Lattice, and $(V/I)'$ can be identified with the normed ideal I^p of V'.*

Proof. The polar of an ideal I in V is given by

$$I^p = \{f \in V' : \langle x, f\rangle \preccurlyeq 1 \text{ , for all } x \in I\}.$$

Since I is a closed ideal and f is a continuous map, it follows that $f^{-1}(0)$ maps closed sets to closed sets. Hence I^p is closed in the $\sigma(V', V)$ weak topology, for all $x \in I$. To verify that I^p is a band we need to prove that for every subset U of I^p, if $\sup U$ exists in V, then $\sup U \in I^p$. Let $U \subset I^p$ be a directed and majorized set. Then for all $x \in I_+$, we set $f_0 := \sup U$. Hence, by Proposition 2.1.10, it follows that $f_0 = \sup\limits_{f \in I^p} f$. Therefore f_0 exists in V' and obviously $f_0 \in I^p$. Thus I^p is a band in V'.

For the last two assertions we need to show that there exists p and s order isomorphisms, such that $p : I' \to V'/I^p$ and $s : (V/I)' \to I^p$. By Corollary 2.1.40, it follows that I^p is the kernel of the evaluation map $V \mapsto V' \subset V^\star$, since V' is an ideal in $V^\star$ by Proposition 2.2.37. Hence, $e|_{I'} : I' \to V'$ and $\ker e|_{I'} = I^p$. Therefore, by the first Theorem of isomorphisms, it follows that $I' \cong V'/I^p$. Since V is a normed vector lattice and I is a closed ideal, it follows that $(V'/I') = (V/I)'$.

Lastly, since I^p is the kernel of a lattice homomorphism, it follows that I^p is a projection band by Theorem 1.2.53 and the fact that V is Archimedean, by Proposition 2.2.12. Therefore, $V'/I' = (V/I)' \cong I^p$. □

Corollary 2.2.41. *The evaluation map yields an isomorphism between a normed vector lattice V and a sublattice of its bidual V''.*

Proof. The evaluation map provides a natural embedding of V into its bidual V''. In fact, if $f \in V''$, then $\langle x, f\rangle := f(x), x \in V$ and $f \in V''$. Hence, in view of Proposition 2.1.39, we obtain that $V \mapsto V''$ is a lattice homomorphism as V' is an ideal of $V^\star$. The proof is complete, as the Hahn-Banach Theorem yields that $\langle x, f\rangle$ is a norm isomorphism. □

Proposition 2.2.42. *Let N be a normed vector sublattice of V. If $f \in N$, then there exists a positive function $F \in V'$, such that $F|_N = f$ and $\|F\| = \|f\|$.*

Proof. In view of Proposition 2.1.30, in order for f to have a positive extension it needs to be bounded above on $N \cap (U - V_+)$. Let U be the unit ball in V and $f \in N'$ such that f is positive and $f(x) \preccurlyeq 1$, for all $x \in U \cap N$. To verify our claim, let $x \in (U - V_+) \cap N'$. Thus $x \preccurlyeq u$, which implies that $x^+ \preccurlyeq u^+ \in U$. Hence, by the decomposition property, it follows that $x^+ \in U \cap N$. Moreover, $f(x) \preccurlyeq f(x^+) \preccurlyeq 1$, since f is a positive linear map. Therefore, there exists a positive extension F such that

$$\{F(x) = 1\ ,\ x \in V\} \cap U^o = \emptyset.$$

Hence $\|F\| \preccurlyeq 1$, which implies that $\|F\| = \|f\|$. □

Remark 2.2.43. Let V be a normed vector lattice with a unit ball U and f be a positive function on V. Since $|f(x)| \preccurlyeq f(|x|)$ holds, for all x, and U is solid in V, the norm of f is given as follows:

$$\|x\| = \sup\{f(x) : x \in U \cap V_+\}.$$

Remark 2.2.44. In view of Proposition 2.1.10, formula (i), the supremum of a directed family $\{f_a\}$ of V'_+ exists in V' and is denoted by $f := \sup_a f_a$. Therefore $\|f\| = \sup_a \|f_a\|$.

Remark 2.2.45. As V can be identified as a normed vector sublattice of V'', the norm of each $x \in V_+$ is given by

$$\|f\| = \sup\{\langle x, x'\rangle : x' \in U^p \cap V'_+\},$$

in view of Corollary 2.2.41. Thus V_+ can by viewed as a convex cone of continuous real functions on the $\sigma - (V', V)$-compact space $X = U^p \cap V'_+$.

Definition 2.2.46. Let V be a normed vector lattice. Suppose there exists a subset P of V_+, such that P is the unique smallest set, such that each $x \in V_+$ reaches its maximum in P. If so, P is called the *Silov boundary* of V_+.

Definition 2.2.47. Let X be a topological space and K be a non-void, closed subset of X. Then a non-void subset A of K is called an *extreme subset* of K if the following are satisfied:

(i) A is closed and convex.

(ii) If for some $x, y \in K$, there exist $0 < \lambda < 1$, such that if $\lambda x + (1 - \lambda)y \in A$, then $x, y \in A$.

Definition 2.2.48. An extreme subset of a convex set C is called a *face*.

Definition 2.2.49. Let V be a vector space and C a convex subset of V. A face F of C is called *hereditary* with respect to the ordering, if $v \in F$ and $u \in C$, such that $u \succcurlyeq v$ implies $u \in F$.

Remark 2.2.50. If V is a normed vector lattice with unit ball U, we write U^p_+ for the positive part of $U^p \cap V'_+$ of the dual unit ball.

Theorem 2.2.51. *Let A, B be non-empty, disjoint convex subsets of a local convex space, such that A is closed and B is compact. Then there exists a closed, real hyperplane in V, strictly separating A and B.*

Lemma 2.2.52. *Let H be an ordered space and G be an H-separating hyperplane of H^*. H is endowed with the weak topology $\sigma(H, G)$. If K is a convex subset of H and x_0 be a maximal extreme point of K, then there exists a positive linear f in G, for all x_0-neighborhoods such that*

$$\sup_{x \in K/U} f(x) < f(x_0).$$

Proof. The weak topology $\sigma(H, G)$ allows the existence of closed semi spaces $H_i = \{x \in H : f_i(x) \preccurlyeq a_i\}$, where $f_i \in G$, for all i, such that

$$x_0 \in H \setminus (H_1 \cup \cdots \cup H_n) \subset U.$$

Let $K_i = K \cap H_i$ and denote by K_0 the convex hull of $\bigcup\limits_1^n K_i$. It follows that K_0 is compact. It is necessary that $x_0 \neq K_0$. Otherwise, $x_0 \in \bigcup\limits_1^n K_i$, which is a contradiction by hypothesis. Since x_0 is maximal in K, it follows that $(x_0 + H_+) \cap K = \{x_0\}$ and so $(x_0 + H_+) \cap K_0 = \emptyset$. It follows from Theorem 2.2.51, that there exists a $\sigma(H, G)$-closed real hyperplane $\{x \in H : f(x) = \alpha\}$ strictly separating K_0 from $x_0 + H_+$.

Moreover, we can assume that $\sup\{f(x) : x \in K_0\} < \alpha < f(x_0)$ without loss of generality. This yields that f is positive and since it is bounded below on H_+, the assertion follows because $K \setminus U \subset K_0$. □

Theorem 2.2.53. *Let V be a normed vector lattice and U be the unit ball of V. If $P \subset V_+$, then P can be considered as the positive cone of real functions on the weak compact space U_+^p. Moreover, P is identical with the weak closure P of the set of all extreme points in U_+^p, such that all extreme points are maximal in U_+^p with respect to the canonical ordering in V'.*

Proof. We recall that a boundary Q of V_+ is a closed subset of U_+^p, such that each $x \in V_+$ takes its maximum on Q or, equivalently, its maximum with respect to the norm. The proof consists of two parts. In the first part, we will show that P is a boundary of V_+ and in the second part we will prove that P is contained in every other boundary Q.

(i) Let $x_0 \in V_+$. We will validate that there exists $x_0' \in P$, such that $\langle x_0, x_0' \rangle = \sup\{\langle x_0, x' \rangle : x' \in U_+^p = \|x_0\|\}$. By Definition 2.2.49, it follows that the family of all closed hereditary faces of U_+^p is ordered under downward inclusion and hence we can apply Zorn's lemma. Thus, we obtain that each closed hereditary face F of U_+^p contains a minimal such face F_0. Now, we set $F = \{y' \in U_+^p : \langle x_0, y' \rangle = \|x_0\|\}$. By the defintion of F, it follows that F is a hereditary face and indeed closed. Hence, it contains a minimal face F_0. Since F_0 is convex and compact, it follows that F_0 contains an extreme point $x_0^{'}$, as F_0 is a face of U_+^p, which is extreme in U_+^p. Suppose that $x_0^{'}$ were not maximal in U_+. Then there would exist $y' \in F_0$, satisfying $y' > x_+^{'}$ as F_0 is hereditary. But, we know that V_+ separates V' and this would lead to the existence of some $y \in V_+$, such that $\langle y, y' \rangle > \langle y, x_0^{'} \rangle$, which follows that the set

$$F_y := \{y' \in F_0 : \langle y, y' \rangle = \sup_{z' \in F_0} < y, z' >\}$$

would be a closed hereditary face properly contained in F_0. This contradicts the minimality of F_0. Therefore $F_0 = \{x_0^{'}\}$ and so $x_0^{'} \in P$.

(ii) Verifying that every boundary Q of V_+ contains P comes as a result of Lemma 2.2.52 by setting $H = V_\sigma^{'}, G = V$ and $K = U_+^p$.

□

The following remark provides a more general result in view of the proof of Theorem 2.2.53.

Remark 2.2.54. Let V be an ordered topological vector space with total positive cone V_+ and K be a convex weak compact subset of V. Then the Silov boundary of V_+ is the weak closure of the set of extreme points of K, which are maximal in K, under the canonical ordering induced by V'.

Lemma 2.2.55. *Suppose V is an ordered vector space and a topological vector space, such that the positive cone is closed. Let C be a directed set in V, such that its section filter is convergent to $x \in V$. Then $x = \sup A$.*

Proof. Fix a $w \in A$. Then there exist $c \in C$, such that $c \succcurlyeq w$ or, equivalently, $c - w \in V_+$. Since the section filter converges to x, it follows that $c - w \in V_+$ implies $x - w \in V_+$, as V_+ is closed. Therefore, x is a majorant in C. Moreover, let u be another majorant of C. It follows that $u - c \in V_+$, which implies that $u - x \in V_+$ as x is a majorant of C. Thus $u \succcurlyeq x$ for all $c \in C$. Therefore $x = \sup C$ and the proof is complete. □

Theorem 2.2.56. *Let V be a normed vector lattice and $\{x_a\}$ be a directed family in V. Moreover, there exists $x \in V$, such that $\lim_a \langle x_a, x' \rangle = \langle x, x' \rangle$, for all x' in the Silov boundary P of V_+. Then $x = \sup_a x_a$ and $\lim_a \|x - x_a\| = 0$.*

Proof. Let $\{x_a\}$ be a family in V, such that $x' \mapsto \langle x_a, x' \rangle$ is a family of continuous real valued functions on the weak topology of P, where P is compact. These functions are convergent pointwise to $\langle x, x' \rangle$, where $\langle x, x' \rangle$ is a continuous function. Since $\{x_a\}$ is directed, the convergence is uniform, as Dini's classical Theorem implies. Hence, $\lim_a \|x - x_a\| = 0$ follows from Theorem 2.2.53. Moreover, since V_+ is closed in V, Lemma 2.2.52 yields that $x = \sup_a x_a$. □

Remark 2.2.57. Theorem 2.2.56 is an extension of Dini's classical convergence Theorem.

Remark 2.2.58. Recall that Dini's Theorem presupposes that the family of continuous real-valued functions should be monotonically increasing. This fact is obtained as x_a is directed in $P \subset V_+$, where P is the Silov boundary.

Remark 2.2.59. We can apply the previous theorem for downward directed families.

Proposition 2.2.60. *A family satisfying the hypothesis of Theorem 2.2.56 contains a countable subfamily having the same limit and supremum.*

Proof. Let $\{x_{a_k}\}$ be a countable subfamily of $\{x_a\}$. Since $\{x_{a_k}\}$ is countable and V is Archimedean, it follows that $\{x_{a_k}\}$ is closed in $\{x_a\}$. Since $\lim_k \langle x_{a_k}, x' \rangle = \langle x, x' \rangle$, for all x' in the Silov boundary P of V_+, it follows that $x = \lim_k x_{a_k}$. Moreover, since V is normed, it follows that $\lim_k \|x - x_{a_k}\| = 0$. □

Corollary 2.2.61. *Any directed family which is weakly convergent, is also norm convergent in a normed vector lattice.*

Proof. The result follows from Lemma 2.2.55 and Theorem 2.2.56, as any $\{x_a\}$ satisfying the hypothesis of 2.2.55 is continuous on the $\sigma(V', V)$-compact space P. Hence the result of 2.2.56 implies the assertion. □

Theorem 2.2.62. *Let V be a Banach Lattice. The following assertions are equivalent:*

(i) *V is countably order complete and each decreasing sequence $y_n \in V$ such that* $\lim y_n = 0$ *norm converges to* 0.

(ii) *V is order complete and each $f \in V'$ is order continuous.*

(iii) *Any majorized directed family in V is weakly convergent.*

(iv) *Any directed family in V having infimum of zero converges to 0 with respect to the norm of V.*

(v) *The evaluation map $V \mapsto V''$ maps V onto an ideal of the Banach Lattice V''.*

(vi) *The order intervals $[x, y]$, for all $x, y \in V$, are $\sigma(V', V)$ compact.*

Proof. Suppose V is order complete and each $f \in V'$ is order continuous. We want to validate that any majorized, directed family in V is weakly convergent. Let f_n be a directed family in V, such that $f = \sup_n f_n$. Since V is order complete, this supremum exists and is well defined. Moreover, since f is order continuous it follows that $f_n \to f$ weakly.

Suppose that every majorized, directed family in V is weakly convergent. We want to prove that any directed family in V, having infimum, is zero convergent to 0 with respect to the norm. By inspection of the proofs of Lemma 2.2.55 and Theorem 2.2.56, we can assume that if $A \subset V$ and $\inf A = 0$, then the weak convergence implies that any minorized family has $\inf = 0$. Hence, by Theorem 2.2.56, we obtain that any family norm converges to 0.

Suppose that any directed family in V, having infimum, is zero convergent to 0 with respect to the norm. We want to validate that V is countably order complete and each decreasing sequence $y_n \in V$, such that $\lim y_n = 0$, norm converges to 0. Suppose that any directed family with zero infimum is norm convergent to 0. Such family can be the set of all majorants of a directed set in V. Denote this set by M_A. It follows that $B = M_A - A$ is a directed downward set, such that $\inf B = 0$. Thus, by hypothesis, A is a Cauchy Family, as M_A is countable and by (i) it follows that $\sup A$ exists in view of Lemma 2.2.55.

Suppose that V is countably order complete and each decreasing sequence $y_n \in V$, such that $\lim y_n = 0$, norm converges to 0. We need to validate that V is order complete and each $f \in V'$ is order continuous. An inspection of this proof gives us another way to prove that countable order complete vector lattices are order complete as we did in Proposition 2.1.43. Now, let $A \subset V$ be a directed set. Assume further that A contains the supremum of each of its countable subsets. Analogously as shown in Proposition 2.1.43, we claim that for a strictly positive transfinite sequence $\{x_a\}$, such that $\{x_a\}_{a<\beta} \in A$, $\beta \in B$, such that B is countable. If not, then there would exist an increasing sequence u_n of ordinals, such that $a_n < \beta$ and a real c, such that $\|x_{a_n+1} - x_{a_n}\| > \epsilon$, for all $n \in \mathbb{N}$. But (x_{a_n}) is convergent to its supremum in A, which shows that (x_{a_n}) is a Cauchy sequence, which is a contradiction. Hence, by transfinite recursion we can construct an increasing sequence $x_a \in A$, where all a are countable, such that $x_{a+1} = x_a$ if and only if $x_a = \sup A$. As a consequence, V is super Dedekind Complete and by (v), each continuous linear form on V is order continuous.

Suppose that V is order complete and each $f \in V$ is order continuous. We will prove that the evaluation map maps V onto and an ideal of V''. By Proposition 2.2.37, we can obtain that $V' = V^{\star}_{00}$ and Nakano's Theorem, shows that $q(V)$ is order dense in $V^{\star}_{00} \subset V''$. Hence by Proposition 2.2.37, as V' is an ideal, the assertion is imminent.

Suppose that the order intervals $[x, y]$ are weakly compact. We need to validate that any $f \in V'$ is order continuous. For that reason, let A is a directed subset of V and $v \in V$ is a majorant of A. Thus, the order interval $[0, v]$ is a superset of A, provided without loss of generality, that $A \subset V_+$. Therefore, A is weakly convergent as $[0, v]$ is compact, to some $x \in [0, v]$ by Theorem 2.2.56. Moreover, Theorem 2.2.56, implies that $x = \sup A$, since V_+ is closed with respect to the norm and the weak topology. Hence, any convergent linear form on V' is clearly order continuous.

□

We proceed to the following useful corollaries.

Corollary 2.2.63. *Suppose V is a Banach lattice such that any one of the allegations in Theorem 2.2.62 are satisfied. Then V is super Dedekind complete. Moreover, since $V \subset V''$, the band B_V, generated by V, is considered to be the band of all order continuous linear forms on V.*

Proof. Since V has a directed family with zero infimum, its norm is zero convergent by (iii) of Theorem 2.2.62. Moreover, the topology of V allows us to apply Urishon's Lemma and obtain the fact that V is metrizable. Hence, it is countable order complete, because (i) indicates that we can find a directed subset A of the directed family $\{f_a\}$, such that $\sup A = \sup_a f_a$. Hence, V is super Dedekind complete, as $\{f_a\}$ is norm convergent in A.

This assertion is valid as $q(V)$ is order dense in $V^{\star}_{00}$, by Nakano's Theorem. Since (i) holds from Theorem 2.2.62, it follows that B_V is the band of all order continuous linear forms on V.

□

Corollary 2.2.64. *If V satisfies any of the assertions establishes in Theorem 2.2.62, then any band projection B_V in V' is weakly apostrophe continuous. Hence any band in V' is $\sigma(V', V)$-closed.*

Proof. Endow V' with the locally convex topology. Since the Minkowski gauge of a solid, convex and absorbing set is a semi-norm, we can define a family of lattice semi-norms through evaluation, such that $f \mapsto p_x(f) = \langle x, |f|\rangle$ generating the topology. This topology is called the *topology of the uniform convergence* on order bounded subsets of V. We denote this topology by $\mathrm{o}(V', V)$. Theorem 2.2.62 (i), indicates that each interval in the weak apostrophe topology is compact and hence the uniform convergence topology is well defined and consistent with the dual system $\langle V, V''\rangle$ by Corollary 2.2.41 and the weak topology associated with $\mathrm{o}(V', V)$ is the weak topology $\sigma(V', V)$. Reversely, denote by P' a band projection in V'. It follows that, $0 \preccurlyeq P(|f|) = |Pf| \preccurlyeq |f|$ holds from Theorem 1.2.51, for all $f \in V'$. Thus the semi-norms p_x satisfy

$$p_x(Pf) \preccurlyeq p_x(f),$$

since p_x are monotone. This holds, for all $x \in V_+$ and consequently P' is $\mathrm{o}(V', V)$-continuous, as it is positive and satisfy Theorem 2.2.24. Equivalently, P' is $\sigma(V', V)$ continuous and as V' is order complete by Theorem 1.2.53, each band is a projection band and by Proposition 1.3.15, it follows that each band is closed in $\sigma(V', V)$. □

Definition 2.2.65. Let V be a Banach lattice. Then V is *reflexive*, if the canonical embedding q from V to V'' is surjective, where q is the evaluation map.

The following Theorem results in a characterization of reflexive Banach Lattices due to Ogasawara [1948].

Theorem 2.2.66. *V is a reflexive Banach Lattice if and only if the following two are satisfied:*

(i) Any norm bounded increasing sequence in V norm converges.

(ii) Each positive decreasing sequence in V' is norm convergent.

Furthermore, if V is reflexive then V is super Dedekind complete. Also, both conditions above may be satisfied with directed families instead of sequences.

Proof. We need to validate that both conditions are necessary and sufficient in order for V to be reflexive.

(i) Firstly, conditions (i) and (ii) are necessary. In order to prove that, let $f \in V$ be a norm bounden increasing sequence. Since V is reflexive we obtain that f is weak apostrophe convergent and by Thoerem 2.2.56 it follows that f is norm convergent. Moreover, any sequence satisfying (ii) is weak apostrophe convergent by Theorem 2.2.62, which implies that is norm convergent, by Theorem 2.2.56 in V', as V is reflexive.

(ii) Now we need to validate that conditions (i) and (ii) are sufficient. We observe that the first condition clearly implies (i), from Theorem 2.2.62, is valid. We know that $q(V)$ is order dense in $V_{00}^{\star}$ by Nakano's Theorem. Since V' is order complete, by Corollary 2.2.63, it follows that $V'' = (V')_{00}^{\star}$. Hence, V'' is the band generated by $q(V)$. Now let $v \in V''$, such that $v \succcurlyeq 0$. Then, there exists a directed set $A \subset V_{+}$, such that $v = \sup q(A)$, by Proposition 1.2.57. We need to verify that $v \in q(V)$. As shown above in Theorem 2.2.62, by transfinite recursion we can construct a sequence $(x_a)_{a<\beta}$ that is maximal and strictly increasing. Thus (x_a) is countable, since A is norm bounded in V. Now (i) implies that $v = \sup_a x_a$ exists in V and it is imminent that $v = \sup_a x_a = \sup q(A)$, since $q(V)$ is an ideal in V''. Furthermore, any strictly increasing transfinite sequence must be countable otherwise, as in Theorem 2.2.62, there would exist a positive $c \in \mathbb{R}$, such that $\|x_{a_{n+1}} - x_{a_n}\| \succcurlyeq c$, which is a contradiction, as every norm bounded increasing sequence is convergent in V by (i). Therefore both conditions are sufficient.

□

Example 2.2.67. Let c_o be the space of all real null sequences. The space c_o satisfies all of the implications of Theorem 2.2.62. We will validate is the fifth implication. What we know is that the dual of c_0 is l^1 and the dual of l^1 is l^∞. Hence, what we need to prove is that c_0 is an ideal through evaluation in l^∞. We notice that $c_o \subset l^\infty$. We will validate that c_0 is a solid vector subspace in l^∞. Since the zero function belongs in c_0, it is easy to observe that the sum of zero sequences is also a zero sequence, as also the scalar multiplication of a sequence. Thus, c_o is a vector subspace of l^∞. Lastly, we need to prove that c_0 is solid. Let $x \in l^\infty$ and $y \in c_0$ such that $|x| \preccurlyeq |y|$. Taking limits in both members we obtain the following

$$\lim |x| \preccurlyeq \lim |y| \to 0,$$

for all x, y. Thus $\lim |x| \to 0$ and therefore $x \in c_0$. Hence c_0 is an ideal in l^∞ and the fact that the evaluation maps c_0 onto an ideal in l^∞ is more obvious. The rest of implications follow in view of the proof of Theorem 2.2.62.

Definition 2.2.68. Let V be a reflexive Banach lattice. If V is isomorphic to an ideal in V'' through evaluation, then V is called a *KB-space*.

Remark 2.2.69. The first implication of Theorem 2.2.66 is equivalent to V be a KB-space.

Example 2.2.70. An intermediate class between reflexive Banach lattice and those, which are ideals in their biduals, are KB spaces. Examples of non-reflexive KB-spaces are the spaces l^1 and $L^1(\mu)$.

Definition 2.2.71. Let V be a normed vector lattice. The norm of V is said to be *order continuous*, if every order convergent filter in V converges with respect to the norm.

Lemma 2.2.72. *Let V be a normed vector lattice. For a couple of real numbers $c > 0$ and $\delta > 0$ we take a sequence $(u_n)_{n\in\mathbb{N}}$, such that $\|u_n\| \geqslant 1+\delta$ and $\|\sum_{v=1}^{n} u_v\| \leqslant c$, hold for all n. Suppose also that at least one of the following holds :*

(i) (u_n) is majorized.

(ii) V is order complete with order continuous norm.

Then there exists a sequence $(k(n))_{n\in\mathbb{N}}$ of natural numbers and a disjoint sequence $(x_n) \in V_+$, satisfying $\|x_n\| \geqslant 1$ and $x_n \preccurlyeq u_{k(n)}$, for all $n \in \mathbb{N}$.

Proof. We should note first that for each $t > 0$ there exists a sequence $(k(n))_{n\in\mathbb{N}} \in \mathbb{N}$ such that

$$\|(cv_{k(1)} - v_{k(n)})^-\| \geqslant 1 + \delta/2 \tag{2.1}$$

for all n. Otherwise, there would exist a subsequence (u_n) of (v_n) such that

$$\|(cu_1 - u_k)^-\| \geqslant 1 + \delta/2$$

whenever $j < k$. This implies

$$tc \geqslant t\|u_1 + \cdots + u_n\| = \|nu_{n+1} - (u_{n+1} - tu_1) - \cdots - (u_{n+1} - au_n)\|.$$

This leads to

$$tc \geqslant \|nu_{n+1} - (u_{n+1} - tu)^+ - \cdots - (u_{n+1} - tu_n)^+\|$$

or, equivalently,

$$tc \geqslant n(1+\delta) - n(1+\delta/2) = n\delta/2,$$

for all n, which is a contradiction. Secondly, there exists $\rho \in \mathbb{N}$, such that

$$\|(u_1 - u_{\rho+1} - \cdots - u_{\rho+n})^+\| \geqslant 1 + \delta/2. \tag{2.2}$$

Otherwise, for any $\rho \in \mathbb{N}$, there would exist at least one real $r(\rho)$, such that

$$\|(u_1 - u_{\rho+1} - \cdots - u_{\rho+r(\rho)})^+\| \geqslant 1 + \delta/2.$$

By setting $\rho = 1$ and $\rho_{j+1} = r(\rho_i)$ recursively, we obtain the following

$$\begin{aligned} c &\geqslant \|u_2 + \cdots + u_{\rho_{(n+1)}}\| \\ &= \|nu_1 - (u_1 - u_2 - \ldots u_{\rho_2}) - \cdots - (u_1 - u_{\rho_{n+1}}) - \cdots - u_{\rho_{n+1}}\|, \end{aligned}$$

which results to

$$c \geqslant \|nu_1 - (u_1 - u_2 - \cdots - u_{\rho_2})^+ - \cdots - (u_1 - u_{\rho_n+1} - \cdots - u_{\rho_{n+1}})^+\|.$$

So,

$$c \geqslant n(1+\delta) - n(1+\delta/2) = n\delta/2,$$

which is a contradiction. Now, we will validate that there exist $(x_1, y_1, \dots)$, such that $0 \preccurlyeq x_1 \preccurlyeq u_{k(1)}$, $\|x\| \geqslant 1$, and $0 \preccurlyeq y_n \preccurlyeq u_{k(n)}$, $\|y_n\| \geqslant 1+\delta/2$ and $x_1 \wedge y_n = 0$, for all $n \geqslant 2$, under each of the additional assumptions (i) and (ii). Then the proof will be completed by induction.

- Suppose that there exists a majorized sequence $(u_n)_{n\in\mathbb{N}} \in V$. Then, by relation 2.1, there exists a sequence $(k(n))_{n\in\mathbb{N}} \in \mathbb{N}$, such that $\left\|(3\delta^{-1}\|x\|u_{k(1)} - u_{k(n)})^{-}\right\| > 1+\delta/2$, whenever $n \geqslant 2$. By defining

  $$x_1 := \left(u_{k(1)} - \frac{\delta}{3\|x\|}x\right)^{+} \quad \text{and} \quad y_n := \left(u_{k(1)} - \frac{3\|x\|}{\delta}u_{k(1)}\right)^{+}$$

  it is clear that $x_1 \wedge y_n = 0$, for all $n \in \mathbb{N}$. Furthermore, by the decomposition property,

  $$x_1 = \left(u_{k(1)} - \frac{\delta}{3\|x\|}x\right) + (u_{k(1)} - \frac{\delta}{3\|x\|}x\Big)^{-}.$$

  Thus

  $$\begin{aligned}\|x_1\| &= \left\|\left(u_{k(1)} - \frac{\delta}{3\|x\|}x\right) + (u_{k(1)} - \frac{\delta}{3\|x\|}x\Big)^{-}\right\| \\ &\geqslant \|u_{k(1)}\| - \delta/3 - \left\|\left(u_{k(1)} - \frac{\delta}{3\|x\|}x\right)^{-}\right\| \\ &\geqslant 1+\delta/3.\end{aligned}$$

- Now suppose that V is order complete with an order continuous norm. Hence, by relation 2.1, there exists a sequence of natural numbers $(k(n))_{n\in\mathbb{N}}$, such that

  $$\|(u_{k(n)} - u_{k(1)})^{+}\| \geqslant 1+\delta/2,$$

  for all $n \geqslant 2$. By relation 2.2 there exists $\rho \in \mathbb{N}$, such that

  $$\|(u_{k(1)} - u_{k(\rho+1)} - \dots - u_{k(\rho+n)})^{+}\| \geqslant 1+\delta/2,$$

  for all n. We define the following:

  $$\begin{aligned}x_1 &:= \inf_n (u_{k(1)} - u_{k(\rho+1)} - \dots - u_{k(\rho+n)})^{+}, \\ y_1 &:= (u_{k(\rho+n-1)} - u_{k(1)})^{+} \quad \text{for } n \geqslant 2.\end{aligned}$$

  It is clear that $\|x_1\| \geqslant 1+\delta/2$, because x_1 is the infimum, $\|\cdot\|$ is order continuous and $\|y_n\| \geqslant 1+\delta/2$, by hypothesis. Consequently,

  $$0 \preccurlyeq x_1 \wedge y_1 \preccurlyeq (u_{k(1)} - u_{k(\rho+1)} - \dots - u_{k(\rho+n)})^{+} \wedge (u_{k(\rho+n-1)} - u_{k(1)})^{+}.$$

  or, equivalently, by distributivity

  $$0 \preccurlyeq x_1 \wedge y_1 \preccurlyeq (y_n)^{-} \wedge (y_n)^{+} = 0$$

  whenever $n \geqslant 2$.

The proof is complete. □

The previous lemma is vital for the remaining of the chapter.

Theorem 2.2.73. *Let V be a countably order complete Banach lattice. The following assertions are equivalent:*

(i) V has an order continuous norm.

(ii) Suppose V_0^L is a Banach sublattice of V. Then there exist no $V_0^L \in V$ such that $V_0^L \cong l^\infty$.

(iii) Any positive linear map T from l^∞ to V is weakly compact.

(iv) Let $I \in \boldsymbol{I}(V)$. Every closed ideal I in V is a band.

Proof. Suppose V has order continuous norm. Then, Corollary 2.2.31, states that every vector lattice isomorphism is continuous in both directions. Since, l^∞ has no order continuous norm, we can not find a suitable vector sublattice isomorphic to l^∞.

Suppose V has order continuous norm. It is clear that if (i) holds, then (iv) holds, by (iii) from Theorem 2.2.62. Moreover, (iii) holds, when V has order continuous norm. Indeed, any interval is mapped in V through a linear positive map. Hence, (vi) from Theorem 2.2.62, implies that any interval is weakly compact. Thus, if $[x, y]_{l^\infty}$, then $f(x, y)$ is σ-$(V, V') \in V$, and f is a positive linear map.

We suppose that there exists no Banach sublattice V_0^L isomorphic to l^∞. Now, suppose that V has no order continuous norm. We will validate the implication, by contradiction. If V has no order continuous norm, then there exists a positive $c \in \mathbb{R}$ and a sequence $(z_n)_{n\in\mathbb{N}} \in V$, such that $\|z_n - z_{n+1}\| \succcurlyeq c$, for all $n \in \mathbb{N}$. Take z_n to be a decreasing sequence. We can find a specific u_n satisfying the hypothesis of Lemma 2.2.72. Take u_n, such that $u_n = \dfrac{1}{\epsilon}(1+\delta)(z_n z_{n+1})$ for any given $\delta > 0$. It is easy to observe that $\|u_n\| \succcurlyeq 1$. Therefore, there exists a disjoint normalized sequence $(x_n)_{n\in\mathbb{N}}$, such that $x_0 = \epsilon^{-1}(1+\delta)z_1$ is the majorant of the linear hull of x_n forms a sublattice of V, and $\sum_1^k x_n \preccurlyeq x_0$, $k \in \mathbb{N}$ holds. Since V is countably order complete by Lemma 2.2.72, it follows that there exists a sequence $(a_n)_{n\in\mathbb{N}} \in l^\infty$, such that $\sum_1^n a_n x_n$ is convergent in $x \in V$. Then the mapping $a_n \mapsto x$ is continuous and there exist an one-to-one correspondence for every a_n. Therefore, $a_n \mapsto x$ is an isomorphism from l^∞ to a vector sublattice of V. The norm in V_0 is well defined as the following estimate validates that V_0 is closed in V.

$$\sup_n |a_n| \preccurlyeq \| \sum -1^\infty \|a_n x_n\| \preccurlyeq \sup_n (a_n)\|x_0.$$

Hence, we found a vector sublattice of V isomorphic to l^∞, which is a contradiction. Therefore V has order continuous norm.

Suppose that there exists a Banach sublattice, which is isomorphic to l^∞. Thus by (iii), there exists a positive linear map, mapping the unit ball to a weakly compact set in V. Thus, U_{l^∞} is weakly compact, which is a contradiction.

Suppose that all ideal in V are closed and that V has no order continuous norm. Take a sequence $(x_n)_{n\in\mathbb{N}} \in V_+$ and, by Lemma 2.2.72, we can find a disjoint, normalized sequence $(x_n)_{n\in\mathbb{N}} \in V_+$, which is majorized. Let $x := \sup_n \sum_{i=1}^{n} x_i$. Since V is countably order complete, it follows that $x \in B_{(x_n)}$, where $B_{(x_n)}$ is the band generated by x_n . If I is the ideal generated by x_n, we will show that $x \notin I$. If $x \in I$, then there exist $v \in I$, such that $0 \preccurlyeq v \preccurlyeq x$ and $\|x - v\| < \frac{1}{2}$ and $v \preccurlyeq \sum_{i=1}^{k} c_i x_i$ for appropriate $c_i \in \mathbb{R}$. If P_n is the band projection of V onto the principal ideal generated by x_n and if $y_n = P_n v$, then we obtain $0 \preccurlyeq y_n \preccurlyeq P_n x = x_n$, by Theorem 1.2.51, for all $n \in \mathbb{N}$. Recall that if V is countable order complete, then V has the principal projection property. Moreover, (x_n) forms a maximal system in I and hence x_n generates a principal ideal. As a consequence

$$\frac{1}{2} > \|x - v\| \succcurlyeq \|\sum_{i=1}^{k+1} x_i - \sum_{i=1}^{k} y_i\| \succcurlyeq \|\sum_{i=1}^{k+1} x_i - \sum_{i=1}^{k} x_i\|,$$

which is equal to $\|x_{k+1}\|$. By Lemma 2.2.72, it follows that $\|x_{k+1}\| = 1$, which is a contradiction.

□

Corollary 2.2.74. *Every separable and countable order complete Banach lattice has order continuous norm.*

Proof. Suppose V is a separable Banach lattice and $(x_n)_{n\in\mathbb{N}} \in V$ a sequence. Since V is separable, $span\{(x_n)\} = V$. This implies that $span\{(x_n)\} \cap I \neq \emptyset$, where $I \in \mathbf{I}(V)$. Since V is a Banach lattice, it is also Archimedean by Proposition 2.2.12 . Thus, it follows that there exist n_0, such that x_{n_0} is the limit of a sequence (y_n) and $x_{n_0} \in I$. Thus I is closed, since $\inf y_n = (x_{n_0}) \in I$. This implies that $\sup y_n \in I$, since V is countably order complete. Therefore, I is band and by Theorem 2.2.73, it follows that V has order continuous norm. □

Proposition 2.2.75. *Let V be a Banach lattice. The following allegations are equivalent:*

(i) The evaluation map provides an isomorphism from V to a band in V''.

(ii) Every norm bounded increasing sequence $(x_n)_{n\in\mathbb{N}} \in V$ converges.

(iii) Suppose V_0^L is a vector sublattice of V. Then $V_0^L \not\cong c_0$.

(iv) Let $T : c_0 \to V$ be a positive linear map. Then T is weakly compact.

Proof. Suppose the evaluation map provides an isomorphism from V to a band in V''. By assumption the band of all continuous linear forms denoted by B_V, by Corollary

2.2.63, is isomorphic to V. Since V is a subset of V'', we suppose that $B_V \subset V''$. Hence (i), from Theorem 2.2.62, holds and as a result, (iii) from 2.2.62 holds. So the proof is complete.

Suppose every norm bounded increasing sequence $(x_n)_{n\in\mathbb{N}} \in V$ converges. If $T : c_0 \to V$ is a positive linear map, then for every sequence $(e_n)_{n\in\mathbb{N}}$ it follows that $T_{e_n} \in V$. Denote by $z =: \sup \sum_{n=1}^{k} T_{e_n}$, where $(e_n) = (\delta_{nm}) \in c_0$. Then T maps the unit ball U of c_0 to the symmetric order interval $[-z, z]$. By (vi) from Theorem 2.2.62, every order interval is weakly compact and hence T is weakly compact.

Let $T : c_0 \to V$ be a weakly compact, linear and positive operator. Suppose there exists a closed vector sublattice of V, such that $V_0^L \cong c_0$. Then every positive linear map is weakly compact. Let $1_{c_0} : c_0 \to V_0^L$ be the identity map. If $e_n = (1, 1, 1, \ldots, 1, 0, \ldots, 0)$, then $\lim e_n = 0$. Hence $e_n \in c_0$ but $1(e_n)$ has not a weakly convergent subsequence, hence 1 is not weakly compact, which is a contradiction.

Given that no Banach sublattice of V is isomorphic to c_0, we claim that V has order continuous norm. Otherwise, it follows from Lemma 2.2.72 that there would exist a majorized normalized, orthogonal sequence or, equivalently, a Banach sublattice isomorphic to c_0, which contradicts our assumption. Therefore (v) holds, from Theorem 2.2.62, and V is an ideal in V''. Now, let $(y_n)_{n\in\mathbb{N}} \in V$ be a norm bounded increasing sequence, which is not Cauchy. We want to apply Lemma 2.2.72 for that sequence. Assume also that $y_n \subset V_+$. By Lemma 2.2.72, it follows that y_n is majorized in V'', hence there exists a constant $c > 0$ and a subsequence $(y_{k(n)})$, such that $v_n := c(y_{k(n+1)} - y_{k(n)})$, for all $n \in \mathbb{N}$. If v_n is considered as a sequence of V'', it satisfies the hypothesis of Lemma 2.2.72. Hence there exists a normalized disjoint sequence $(x_n) \subset V_+$, such that its linear hull form a vector sublattice, which is isomorphic to c_0 and that is a contradiction. Therefore, (y_n) is Cauchy and convergent in V.

□

The following theorem provides an analogous characterization as Theorem 2.2.66, not supposing that V is countably order complete as in Theorem 2.2.73.

Theorem 2.2.76. *Let V be a Banach lattice. The following allegations are equivalent:*

(i) V is reflexive.

(ii) V' and V'' have order continuous norm.

(iii) Let V_0^L be a Banach sublattice of V. Then $V_0^L \ncong c_0$.

(iv) $(V_0^L)'$ be a Banach sublattice of V'. Then $(V_0^L)' \ncong c_0$.

(v) If V_0^L is separable, then V_0^L is reflexive.

Proof. Suppose V is reflexive. Then by Nakano's Theorem, it holds that $q(V)$ is order dense in $V_{00}^{\star}$. Hence (iv) holds from Theorem 2.2.62 and as a consequence (i) also holds, from Theorem 2.2.62. Therefore the assertion is imminent.

Suppose V' and V'' have order continuous norm. Then we can apply Theorem 2.2.62 and, by Theorem 2.2.66, we obtain that V' is reflexive, hence so is V.

Suppose V is reflexive. Since the dual of Banach subspace of a reflexive Banach space is also reflexive, it follows than there exist neither Banach sublattices V_0^L isomorphic to c_0 nor Banach sublattices $(V_0^L)'$ isomorphic to c_0

Suppose V is not reflexive. Then at least one of V' and V'' does not have order continuous norm.

- Suppose V' has not order continuous norm while V' does. Then by Lemma 2.2.72, there exists a disjoint majorized, normalized sequence in V', which forms a vector lattice isomorphic to c_0, which is a contradiction, by Proposition 2.2.75.

- Suppose V'' has not order continuous norm, while V' does. Since V'' is order complete, it follows that l^∞ is contained in V'' as a closed vector subspace. Now, the band generated, in V'', by $e_n = (\delta_{nm}) \in l^\infty$, is denoted by B_{e_n}. By Corollary 2.2.64, it follows that the corresponding band projection is weakly continuous and as a result the adjoint of a band projection $V' \to B_n'$.

Let V_0^L be a Banach sublattice of V, such that V_0^L is not isomorphic to c_0. It follows that the Banach space V is reflexive if and only if V' is closed. Hence V is reflexive.

Suppose V is reflexive. The sequence $(x_n)_{n\in\mathbb{N}}$ constructed previously generates a vector sublattice of V. Hence, if V_0^L is separable, it follows from Theorem 2.2.73 that V_0^L is reflexive.

Suppose that V is not reflexive. Then V contains one closed vector sublattice isomorphic to c_0 or l^1, which are not reflexive. □

2.3 Quasi Interior Positive Elements

In this paragraph we will analyze elements of the positive cone V_+ of any vector lattice. Especially, our interest will turn to the quasi interior elements. Usefull properties and some equivalences which concern continuity would be further asserted.

Proposition 2.3.1. *Let V be an Archimedean vector lattice. A positive element $u \in V$ is a weak order unit if and only if the principal ideal V_u is order dense in V.*

Proof. It suffices to prove that u is a weak order unit or, equivalently, that if $u \wedge |x| = 0$, then $x = 0$ due to Corollary 1.1.93. Hence, by Proposition 1.2.47, it follows that V_u is order dense, if $\{v^\perp\} = \{0\}$. Therefore, $u \wedge |x| = 0$, and this implies $x = 0$. □

Remark 2.3.2. In general, order convergence does not imply topological convergence thus weak order units are of no use and we need to introduce a new notion.

Example 2.3.3. Let $V = C([0,1])$ and f_n be a sequence of functions given by $f_n = t^n$, for all $t \in [0,1]$ and $n \in \mathbb{N}$. It follows that $f_n \xrightarrow{\preceq_V} 0$, with the canonical ordering, whereas $\|f_n\| = 1$, for every $n \in \mathbb{N}$.

Definition 2.3.4. Let V be any topological vector lattice and $u \in V_+$. A positive element u is called *quasi interior point*, if the principal ideal V_u generated by u is dense in V.

We observe that there exists a connection between the notions of weak order units and quasi interior points.

Example 2.3.5. Suppose $V = \mathbb{R}$. Then V is locally convex and Hausdorff. Hence it is a topological vector lattice. Let $r \in \mathbb{R}$, such that $r = 1$. Then the ideal I_1 generated by 1, is of the following form

$$I_1 = \bigcup_{i=1}^{\infty} n[-1,1], n \in \mathbb{N}.$$

It is readily seen that I_1 is a principal ideal and $I_1 = \mathbb{R}$. Therefore, 1 is a strong unit and hence a quasi interior point of R.

For the remaining of this session, denote by W the set of all weak order units and by Q the set of all quasi interior points of V_+.

Remark 2.3.6. The set of all weak order units satisfies $W + V_+ = W$.

Proof. If W is empty then the assertion is eminent. Suppose $W \neq \emptyset$. Let $u \in W$ be a weak order unit. Hence, $u \wedge v = 0$, for all $v \in V_+$. Hence, by the decomposition property, the interval $[0,u]$ is a superset to $[0,v]$. Otherwise, there would exist $u' \in W$, such that $u \subset u'$, which is a contradiction since u is a maximal orthogonal system. Hence, $[0,u] + [0,v] = [0,u]$. This holds for all u and v and hence the assertion is proven. □

The following proposition presents a similar result for the set of all quasi interior points Q of V_+. But first we will present a separation theorem.

Theorem 2.3.7 (Second Separation Theorem)**.** *Let A, B be non-empty, disjoint, convex subsets of a locally convex space V, such that A is closed and B is compact. Then there exists a closed, real hyperplane $H \in V$, strictly separating A and B, i.e.*

$$f(A) < 0 < f(B).$$

Definition 2.3.8. Let V a topological vector space over a field. The real function $x \to |x|$ is called a *pseudo-norm* if the following are satisfied :

(i) $|\lambda| \geq 1$ implies that $|\lambda x| \geq |x|$ for each x $\in V$.

(ii) $|x| = 0$ if and only if $x = 0$.

Proposition 2.3.9. *Let V be a topological vector lattice. Then the following hold:*

(i) Q is a sublattice and a convex subcone of V_+ such that $Q + V_+ = Q$.

(ii) If $Q \neq \emptyset$ then Q is dense in $V+$ as long as V is locally convex.

(iii) If V_+ has non-empty interior then $Q = (V_+)^o$.

(iv) $Q \neq \emptyset$ if V is complete metrizable and separable or equivalently if V is a Banach lattice.

Proof. (i) We need to show that Q is a convex subcone of V_+. Let $x \in Q$ and $\lambda > 0$ in $\mathbb{R}$. Then the ideal generated by the element $\lambda x + y$, where $y \in V+$ implies that $V_{\lambda x+y} \supset V_x$. Since $x \in Q$, then $\lambda x + y \in Q$.

To prove that Q is a sublattice, we need to show that $x \wedge y$ and $x \vee y$ both belong in Q for every quasi interior point x and y. The supremum of two maximal orthogonal systems is also a maximal orthogonal system and hence the ideal generated by $x \wedge y$ is principal and as a consequence dense in V. Therefore, $x \wedge y \in Q$. Moreover, let $u \in \bar{V}_x \cap \bar{V}_y$. Since the intersection of ideals is solid, then $|u| \in \bar{V}_x \cap \bar{V}_y$. Hence, we can find two sequences one in each ideal and denote by $w := \inf(x_n, y_n) \in V_x \cap V_y$ and $\lim_n w_n = |w|$. Hence $|w| \in \overline{V_x \cap V_y}$, thus $w \in \overline{V_x \cap V_y}$. Therefore, the intersection is dense in V and since $V_{x\wedge y} = V_x \cap V_y$, then $x \wedge y \in Q$.

(ii) We need to prove that $\overline{Q} = V_+$. Suppose there exists $y \in V_+ \backslash \overline{Q}$. Then (i) implies that $y \neq 0$ and $\overline{Q}$ is compact, since it is a closed subset of a closed set and hence the previous separation theorem can be applied. Therefore, there exists a positive continuous function f, such that $f(y) < -1$ and $f(u) \geq 0$, for some $u \in \overline{Q}$. We can find an appropriate element u_0 in $\overline{Q}$, such that $f(u + u_0) < 0$. This leads to a contradiction, as $u + u_0 \in Q$ and since Q is a sublattice and a convex subcone, this implies $y \in Q$. Hence Q is dense in V_+.

(iii) Suppose $(V_+)^o \neq \emptyset$. Let w $\in (V_+)^o$. We need to construct a principal ideal dense in V_+. The set $(-x + (V_+)^o) \cap (x + (V_+)^o)$ is a neighborhood of the zero element and as a consequence, it is contained in the symmetric interval $[-x, x]$. Thus, V_x, the ideal generated by x, contains all elements of V_+. Therefore, $V_x = V_+$ and thus, $x \in Q$. Conversely, suppose $q \in Q$ and the ideal generated by q is dense in V_+. This implies that $[0, u] \cap V_0 \neq \emptyset$, where V_0 is any subset of V_+. Therefore, $u \in V_0$, for all V_0 and thus in V_+. Consequently, $Q \subset (V_+)^o$ and the assertion is proven.

(iv) Suppose V is separable and complete metrizable. The lattice operations of V are inherited by V_+ and since they are continuous, any dense ideal in V can be mapped via a continuous map to V_+. Hence, V_+ is also separable. Let $v_n, n \in \mathbb{N}$ be a sequence of elements in U, where U is a countable dense subset of V_+. We need to find a norm convergent sequence of x_n to any element $x \in V_+$. For that reason we want to define a topology on V. Since V is metrizable, the pseudo norm ρ on V defines a metric $(x, y) \mapsto \rho(x - y)$. Since lattice operations are continuous and V is complete, it follows that ρ defines the topology of V. Thus, we can find a positive sequence of numbers $c_n, n \in \mathbb{N}$, so that $\rho(c_n, v_n) < \dfrac{1}{2^n}$. Since V is complete and V_+ is closed, the series $\sum_n c_n x_n$ is convergent to a positive element in V_+. Since U is dense in V, it follows that $x \in V_+$ and as a consequence in $(V_+)^o$. Therefore by (iii), $x \in Q$.

□

Remark 2.3.10. The following inclusion is eminent in any topological vector lattice:

$$(V_+)^o \subset Q \subset W.$$

Theorem 2.3.11. *Let V be a normed vector lattice and $u \in V_+$. The following notions are equivalent:*

(i) u is a quasi interior point of V_+.

(ii) The sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n := x \wedge nu$, for any $x \in V$, is norm convergent to x.

(iii) Let f be a positive linear form on V', then $f(u)$ is strictly positive.

Proof. Let $x \in V_+$ and a sequence (x_n), such that $x_n = x \wedge nu$. Suppose u is quasi interior to V_+. This implies that we can find a convergent sequence $(y_n)_{n\in\mathbb{N}} \in V_u$ convergent to x. Without loss of generality, suppose y_k is increasing. Since any normed lattice is Archimedean, we can find an integer-valued function of positive numbers $k \mapsto n(k)$, such that $y_k \leqslant n(k)u$. This implies $y_k \leqslant x \wedge n(k)u = x_{n(k)} \leqslant x$. Since y_k is increasing, it follows that $\lim_k ||x - x_{n(k)}|| = 0$ and hence $\lim_n ||x - x_n|| = 0$, because x_n is also increasing.

Suppose $(x_n) \to x$ with respect to the norm of V. Let f be a continuous positive map, such that $f(u) \geqslant 0$, where u is a quasi interior point of V_+. If $f(u) = 0$, this implies $f(x_n) = 0$, for all n and each $x \in V_+$. Since x_n is convergent and f is continuous, it follows that $f(x) = 0$, for all $x \in V_+$ and hence $f = 0$. Therefore, $f(u) > 0$.

Suppose f is a positive and continuous linear map and u is an element in V, such that $\overline{V_u} \neq V$. Then V_u is non empty and V_+ is closed. Hence the Hahn Banach Theorem can be applied and imply that $f(\overline{V_u}) = \emptyset$. Moreover, V' is an ideal of $V^\star$, hence, by Corollary 2.1.11, we obtain that $\sup |f(v)|$ are going to zero, where $0 \leqslant v \leqslant u$. Hence, by the continuity of f this implies $f = 0$, which is a contradiction. Therefore, u is a quasi interior point of V_+.

□

Definition 2.3.12. Let V be a topological vector space and A a non empty subset of V. Then a real hyperplane H is called a *supporting hyperplane* of A, if $A \cap H \neq \emptyset$, and if A is contained in one of the closed semi-spaces determined by H.

Corollary 2.3.13. *Suppose V is a normed vector lattice and u is a positive element in V_+. Then there exists a closed supporting hyperplane H containing u if and only if $x \notin Q \subset V_+$.*

Proof. Let H be a hyperplane and $u \in H$. Then H is of the following form $H = \{x \in V : f(x) = \alpha \mid f \in V'\}$. Since f is a positive continuous map, then α must be zero, since u is also positive. It also holds that $f \geqslant 0$. Now, by Theorem 2.3.11, the assertion is imminent as the negation of (iii) is satisfied. Therefore u is not a quasi interior point of V_+. □

Remark 2.3.14. If V is a Banach lattice then the assumption that H is closed, can be omitted.

Proof. Let H be a supporting hyperplane of V_+. By Definition 1.3.21, it holds that $f \in V^*$. Since V is a Banach lattice, it follows, from Corollary 2.2.29, that $f \in V'$. Since $\{a\}$ is closed, it follows that H is necessarily closed. □

Remark 2.3.15. If S is the set of all points of V_+, where V_+ is supported by a closed hyperplane H, then it follows that V_+ is the disjoint union of S and the quasi interior Q.

Proof. We know that V_+ contains all positive x, such that $x \succcurlyeq 0$ with respect to the ordering. Let H be a closed hyperplane supporting V_+ as a subset of V. Then V_+ is included in one of the semi-spaces determined by H. Thus, for some $x \in V_+$, it follows that $x \in V_+ \cap H$. But, if there exists $x \in Q \cap (V_+ \cap H)$, it follows that V_x is not dense, as there would exist $\epsilon > 0$, such that $B(x, \epsilon) \cap B(y, c) = \emptyset$, where $y \in V_+ \backslash H$ and $c > 0$ since H is closed in V. Hence $V = S \cup Q$. □

Corollary 2.3.16. *Let V be a vector lattice and F a sublattice of V. If x is a quasi interior of V_+, then x is a quasi interior point of F_+.*

Proof. Let f_0 be a continuous positive linear map from F to $\mathbb{R}$. By Theorem 2.3.11, it follows that f_0 is indeed positive for every quasi interior point $x \in V_+$. Denote by U the unit ball of V. It follows, from Proposition 2.2.42, that f_0 is bounded above in $F \cap (U - V_+)$ and hence we can find a positive extension F from V to $\mathbb{R}$, such that $F(x) \geqslant 0$. Therefore, x is quasi interior in F_+. □

Examples 2.3.17. (i) Let (X, Σ, μ) be a measure space. If (X, Σ, μ) is totally σ-finite, then each of the Banach Lattices $L^p(\mu)$ possesses quasi-interior positive elements.

- If $p < \infty$, then, by Definition 2.3.4, $f(t)$ must be positive in order to be a quasi interior point. Moreover, since L^p contains the equivalence class of all μ measurable functions, it follows that all $f \in L^p$ are quasi interior to L^p. This holds, as X is totally σ-finite. Otherwise, the assertion is not true in general. Moreover, in reference to Theorem 2.3.11, one can find quasi interior points, if the second assertion is valid for all $f \in L^p(\mu)$.
- If $p = \infty$, then f is a quasi interior positive element, if and only if the μ-essential infimum of f is > 0, while the strictly positive functions $f \in V$ are the weak order units of V.

Thus $Q = (V_+)^o$ for $V = L^\infty(\mu)$, while the functions $f \in V$, satisfying $f(t) > 0$ a.e (μ), are precisely the weak order units of V.

(ii) Let X be a completely regular topological space and denote by $V = C_b(X)$ the Banach Lattice of all real-valued, bounded and continuous functions on X with respect to the supremum norm. By Definition 2.3.4 and Theorem 2.3.11, it follows that the quasi interior points of V are the functions f, such that $\inf_{t \in X} f(t) > 0$. Moreover, let $f \in V$. Then f is a weak order unit, if the zero set $\{t \in X : f(t) = 0\}$, namely U, is nowhere dense. Indeed, f should be strictly positive in order for the principal ideal generated by f to be equal to V. By definition of X, we validate that the closure $\bar{U}$ has void interior. Suppose that there exists a $y \in \bar{U}^o$. Thus, there exist $\epsilon > 0$, such that $B(y, \epsilon) \subset \bar{U}$. Hence, for every $x \in U$, it follows that $y \notin \bar{U}$. Therefore, there exists a positive function g, such that $g(x) = 0$, for every $x \in U$, and $g(y) = 1$, which is a contradiction. Thus $\bar{U}^o = \emptyset$.

(iii) Let X be a locally compact, non compact space and let $V = C_0(X)$ be the Banach lattice of all real-valued, continuous function on X vanishing at infinity. Suppose $f \in V^+$. This implies that $f \succcurlyeq 0$. It follows that $\lim f$ does not tend to 0, when $n \to \infty$. Thus V^+ is void. Moreover, the quasi interior of V is non void, if and only if X is σ-compact. Equivalently, it follows that X is the countable union of compact sets. Obviously, $f \in V$ should be positive in order to generate a dense ideal by Definition 2.3.4. But, since V^+ is empty, it is necessary and sufficient that X is σ-compact. Suppose X is the countable union of compact sets such that

$$V = \bigcup_{i=1}^{n} K_n.$$

Since X is locally compact and σ-compact, we take the one-point compactification $V^\star = V \cup \{\infty\}$. It follows that K is also closed in $V^\star$. Now we define $f_n : V^\star \to [0, 1]$, such that $f_n(\infty) = 0$ and $f_n|_{K_n} = \{1\}$ by Uryshon's Lemma.

Then we define $f : V^{\star} \to \mathbb{R}$ such that

$$f(x) = \sum_{n} \frac{1}{2^n} f_n(x).$$

The function is well defined and continuous. It follows, by the definition of f, that $f(\infty) = 0$ but, if $x \in K_n$ for some n, it follows that

$$f(x) \geqslant \frac{1}{2^n} f_m(x) = \frac{1}{2^m} > 0,$$

since all f_m are positive and equal to 1 on K_m. Therefore, $f|_V$ is a strictly positive continuous real function that generate a dense ideal in V.

Remark 2.3.18. The property of a Banach lattice V to contain quasi interior points is not inherited by closed ideals of V. This can be seen taking $C(X)$ as V and $C_b(X)$ as a closed ideal. The assertion comes from the discussion in Example 2.3.17 .

Proposition 2.3.19. *Let V, F be normed vector lattices. Consider a positive continuous linear map $T : V \to F$, such that $T(V)$ is dense in F. If $x \in Q$, then Tx is quasi interior in F_+.*

Proof. Take $x \in V_+$ be a quasi interior point to V_+. Since V_+ is dense in V, then $T(V_+)$ is dense in $T(V)$, as T is continuous and positive. By hypothesis, $T(V_+) \subset T(V)$, where $\overline{T(V)} = F$. Thus for the ideal V_x, generated by x, we obtain the following

$$T(V_x) = \bigcup_n nT([-x, x]),$$

where $V_x = \bigcup_n n[-x, x]$, which is a subset of $\bigcup_n n[-Tx, Tx]$. Thus $\bigcup_n n[-Tx, Tx] = F_{T_x}$ and consequently Tx is quasi interior to F_+ □

Corollary 2.3.20. *Suppose V is a normed vector lattice and I is a closed ideal and $q : V \to V/I$ denotes the canonical map. Then q maps the quasi-interior of V_+ into the quasi-interior of $(E/I)_+$.*

Proof. By Proposition 2.2.32, V and V/I are normed vector lattices, and since q is positive and surjective, we can apply Proposition 2.3.19 and the result is imminent. □

Corollary 2.3.21. *If V_+ contains quasi-interior points and V_0 is a vector sublattice of V, which is the range of a continuous, positive projection, then V_0 contains quasi-interior points.*

Proof. Since V_+ is closed and p is continuous and positive, by Proposition 2.2.42, we can find an extension P to V. Thus, if $u \in Q_{V_+}$, then $P\big|_{Q_{V_+}} = Q_{(V_0)+}$ and applying Proposition 2.3.19 we obtain the desired result. □

Proposition 2.3.22. *Suppose V is an normed vector lattice and $q : V \to I \in \boldsymbol{I}(V'')$ is an isomorphism, where q is the evaluation map. Then $W = Q$. In other words W and Q are identical in V_+.*

Proof. We can assume that V is a Banach lattice. Then, Proposition 1.1.94 states that V is a sublattice of $\widetilde{V}$. Hence the completion $\widetilde{V}$ can be identified with the closure of V in its bidual and $\widetilde{V}$ is an ideal in V'', whenever V is. Hence, each weak order unit and quasi interior point of V_+ preserves its status under the transition from V to $\widetilde{V}$, as the evaluation map is positive and continuous. Since V is Archimedean for every weak order unit $u \in V_+$, it holds, by Corollary 1.1.93, that $x = \sup_n x \wedge nu$. Moreover (vi) from Theorem 2.2.62, holds by hypothesis, hence the $\sup_n$ is a norm limit, where $n \in \mathbb{N}$. Therefore, u is a quasi interior point to V_+, by Theorem 2.3.11. Therefore, $W \subset Q$. The reverse inclusion is obvious by the definition of Q. □

The idea behind the proof of Proposition 2.3.22 is that V is not necessarily order complete in order to apply Theorem 2.2.62, hence through isomorphism we used the completion of V.

Now we present a theorem regarding the bipolar of a polar set.

Example 2.3.23. The spaces $L^p(\mu)$ $(1 \leqslant p < \infty)$ are examples of Banach Lattices to which Proposition 2.3.22 can be applied. More specifically, if $L^1(\mu)$ is a non-reflexive KB- space, by Example 2.2.70, it follows that $W_{L^1(\mu)} = Q_{L^1(\mu)}$. More generally, Proposition 2.3.22 applies to reflexive Banach spaces.

Remark 2.3.24. Consider the space $C_0(X)$ of all real valued continuous functions vanishing on infinity. If X is the discrete space $\mathbb{N}$, then we obtain c_0. It is interesting to observe that those space with X discrete are the only ones which are ideals in their bidual.

Definition 2.3.25. Suppose V is a vector space and M is a subset of V. Then the polar of M^p is also a subset of F, denoted by M^{pp} and called the bipolar of M.

The following theorem is a consequence of Hahn-Banach theorem.

Theorem 2.3.26. *Let $\langle V, F \rangle$ be a duality. The bipolar U^{pp} of any subset U of V is the convex hull of $M \cup \{0\}$ with respect to the weak topology.*

Corollary 2.3.27. *Let $\{U_a : a \in A\}$ be a family of $\sigma(V, F)$ closed convex subsets of V, containing 0. If $M := \bigcap_\alpha M_\alpha$, then M^p is the $\sigma(V, F)$ closed convex hull of $\bigcup M_a^p$.*

Theorem 2.3.28. *Let V be a countably order complete Banach lattice. Each of the following assertion implies the next one:*

(i) V has order continuous norm and $Q \neq \emptyset$.

(ii) V' has weak order units.

(iii) V has order continuous norm.

Proof. Suppose that V has order continuous norm and Q is non empty. Let (x'_a) be a maximal orthogonal system of V'. We know that V' is a Banach lattice. Hence, we only need to validate that (x'_a) is countable. Now, if B'_a denotes the band generated by $\{u'_a\}^{\perp}$ in V', it follows, from Corollary 2.2.64, that there exists a band $B_a \in V$ with polar $B_a^p = B'_a$. Moreover, since $\bigcap_a B_a^p = \{0\}$, it follows from Corollary 2.3.27, that the convex closure of $\bigcup_a B_a = V$. This also implies that V is the band generated by $\bigcup_a B_a$. Hence, if u is quasi interior to V_+ and if u_a is the projection of u in B_a, we obtain that $u = sup_a u_a$ and $(u_a)_{a \in A}$ is a maximal orthogonal system of V. But, we know that V is Super Dedekind Complete, by Corollary 2.2.63. Therefore, so (u_a) and hence (u'_a) is countable.

Suppose that V' has weak order units and V has not order continuous norm. Then, Theorem 2.2.73 results in the existence of a vector lattice isomorphism $i : l^\infty \to V$ with range a closed vector sublattice of V. Since order convergent filters in V' are weakly convergent, it follows that the adjoint operator is an order continuous positive surjection of V' onto the strong dual of l^∞. Thus, by Proposition 2.3.19, it follows that if w is a weak order unit of V'. Then $i'(w)$ is a weak order unit in $(l^\infty)'$. It is easy to verify that the dual of l^∞ has no weak order units. Moreover, such elements should be contained in the band $l^1 \subset (l^\infty)'$ and this is a contradiction. Therefore, we obtain the desired implication.

□

Corollary 2.3.29. *Suppose V is a reflexive Banach lattice. Then V_+ has non-void quasi-interior if and only if V' does.*

Proof. Obviously, if V is reflexive and V_+ has quasi interior points, then V' also has quasi interior points. The reverse allocation results from the fact that the dual of a reflexive Banach lattice is reflexive, as V has an order continuous norm by Theorem 2.2.76. □

Chapter 3

AM-Spaces and AL-Spaces

3.1 AM-spaces

In this section we will discuss the space of all continuous real functions over a compact space K. For that reason we will present a new norm and the respective normed vector lattices will play a major role.

Definition 3.1.1. Let V be a vector lattice and $\|\cdot\|$ be a lattice norm. The lattice norm $\|\cdot\|$ is called an *M-norm*, if the following holds:

$$\|x \vee y\| = \|x\| \vee \|y\|,$$

for all $x, y \in V_+$. The space $\big(V, \|\cdot\|\big)$ is called a *M-norm space*. Furthermore, if the norm is complete, then $\big(V, \|\cdot\|\big)$ is called briefly *AM-space* (abstract M-space).

Remark 3.1.2. An M-normed space V is a normed vector lattice on whose positive cone V_+ the norm commutes with the formation of finite suprema.

Remark 3.1.3. Let V be a M-normed space with unit e. Then the unit ball is the order interval $[-e, e]$. The following proposition provides a converse statement.

Proposition 3.1.4. *Let V be an Archimedean vector lattice, where e exists in V. The gauge function of $[-e, e]$, given by*

$$\rho_e(x) = inf\{\lambda \in \mathbb{R} : x \preccurlyeq |\lambda e|\}$$

is an M-norm on (V, ρ_e), if and only if V is l^1-relatively complete.

Proof. The Minkowski gauge ρ_e is well defined, as, in view of Proposition 2.1.30 and Remark 3.1.3, the symmetric order interval is convex and absorbing. Moreover, it is imminent that ρ_e is a lattice norm and the fact that V is Archimedean yields that $x = 0$, when $\rho_e(x) = 0$. Since $x \in V_+$, then $|x| \preccurlyeq \rho_e(x)e$ comes as a consequence of the Archimedean property. Hence, for all x, y, it holds that $x \vee y \preccurlyeq \big(\rho_e(x) \vee \rho_e(y)\big)$. By definition of the gauge, the previous inequality implies that $\rho_e(x \vee y) \preccurlyeq \big(\rho_e(x) \vee \rho_e(y)\big)$. Conversely, assume that $x \vee y \preccurlyeq \lambda e$ holds, for all x, y. Then $\big(\rho_e(x) \vee \rho_e(y)\big) \preccurlyeq \lambda$, by definition. This shows that $\rho_e(x) \vee \rho_e(y) \preccurlyeq \rho_e(x \vee y)$. Therefore, ρ_e is an M-norm.

Now, we want to prove that ρ_e is an M-norm, when (OS) is satisfied. Obviously, if (V, ρ_e) is an AM-space, then for every $x_n, x \in V$ and $(\lambda_n) \in l^1$, such that $0 \preccurlyeq x_n \preccurlyeq \lambda_n x$, we have that $\sum\limits_{j=1}^{\infty}$ is order convergent. Conversely, suppose that (OS) is satisfied in V. We need to prove that for every Cauchy sequence, there exists a subsequence $(x_n)_{n\in\mathbb{N}}$, such that $\sum_n (x_{n+1} - x_n)$ converges. This is equivalent to the classic definition of norm completeness, as this utilizes the (OS) property. Let $(y_n)_{n\in\mathbb{N}} \in (V, \rho_e)$ be a Cauchy sequence. Since (OS) is satisfied, there exists a subsequence $(x_n)_{n\in\mathbb{N}}$ and a sequence of real numbers $(\lambda_n)_{n\in\mathbb{N}} \in l^1$, such that $|x_{n+1} - x_n| \preccurlyeq \lambda_n e$. Let $\lambda_n = \dfrac{1}{2^n}$ and choose a sequence v_n, such that $v_n = (x_{n+1} - x_n)^+$ and u_n, such that $u_n = (x_{n+1} - x_n)^-$. Hence $0 \preccurlyeq u_n \vee v_n \preccurlyeq \lambda_n e$. Since $\rho_e(x)$ is the norm, it suffices to prove that $\sum_n x_n$ converges. An alternative result of (OS) is that $\sup_m \sum\limits_{n=1}^{m} v_n = v$ exists in V. Thus we obtain

$$v - \sum_{n=1}^{m} v_n = \sup_k \sum_{n=m+1}^{m+k} v_n \preccurlyeq \Big(\sum_{n=m+1}^{\infty} \lambda_n \Big) e.$$

Therefore, by definition of the norm,

$$\rho_e(v \sum_{m=1}^{m} v_n) \preccurlyeq \sum_{n=m+1}^{\infty} \lambda_n$$

for all m. Since $\lambda_n = \dfrac{1}{2^n}$, we obtain the desired result. □

Corollary 3.1.5. *Let V be a Banach lattice that has a unit e. Then the principal ideal V_e, generated by e, is an AM-space and the symmetric order interval is the unit ball in V. Moreover, the canonical embedding $V_e \mapsto V$ is continuous.*

Proof. The principal ideal $V_e = \bigcup\limits_{i=1}^{\infty} n[-e, e]$ is generated by the symmetric order interval where e is a unit. Hence, we can define a gauge as in Proposition 3.1.4 and result in the fact that V_e is an AM-space, as any Banach lattice satisfies the (OS) axiom. Since the unit ball in V is the symmetric order interval under the norm endowed with V, then the embedding is continuous. □

The previous corollary gives us a brief insight in the role of AM-spaces as Banach lattices.

Examples 3.1.6. (i) Denote by $\mu(V_n)$ the Banach space of all bounded sequences $x = (x_n)$, where $x_n \in V_n$ and V_n is an AM-space with unit e_n $(n \in \mathbb{N})$. If $m(V_n)$ is endowed with the norm $x \mapsto \|x\| := \sup \|x_n\|$, then $\mu(V_n)$ is an AM-space with unit $e = e_n$ under the canonical ordering. It is easy to verify that this norm is indeed an M-norm due to the distributivity of the lattice supremum. Moreover, suppose that $V_n = \mathbb{R}$. Then $\mu(\mathbb{R})$ is denoted by l^∞, which is not separable. We can easily verify this, due to the fact that the set of all characteristic functions of arbitrary subsets of $\mathbb{N}$ form an uncountable subset of l^∞, whose elements have mutual distance 1. We notice that every real bounded sequence is convergent. Hence, the space c of all convergent real sequences is a closed vector sublattice and as a consequence, a separable AM-space with unit $e = (1, 1, 1, 1, \dots)$. Lastly, the space of all null convergent real sequences is also an AM-space without unit.

(ii) Let X be a topological space. Then the vector space $C_b(X)$ of all bounded real valued functions on X endowed with the canonical ordering is an AM-space with unit e, where $e(x) = 1$, for all $x \in X$. The supremum norm is an M-norm due to the distributivity of the lattice supremum.

(iii) Let (X, Σ, μ) be a measure space, where μ is a positive countably additive measure and Σ is a σ-algebra of subsets of X. Then the Banach lattice $\mathcal{L}^\infty(\mu)$ of all bounded Σ-measurable real functions is an AM-space with unit $e = e(x)$ for all $x \in X$ endowed with the canonical ordering and the supremum norm is an AM-space. Furthermore, the μ-null subset is a σ-ideal and hence, by Proposition 2.2.19, is closed. Hence, the quotient $\mathcal{L}^\infty(\mu)/\mathcal{N}$ is an AM-space and equal to $L^\infty(\mu)$.

Proposition 3.1.7. *Let $\big(V, \|\cdot\|\big)$ be an AM-space. The following properties hold:*

(i) Each closed vector sublattice is an AM-space.

(ii) If I is a closed ideal and V an AM-space with unit, then the quotient V/I is an AM-space.

(iii) The completion $\tilde{V}$ of an M-normed space is an AM-space.

Proof. (i) Let V_0 be a closed vector sublattice of V. Since V is norm complete, it follows that V_0 is norm complete either. Moreover, V_0 is closed with respect to the M-norm. Hence V_0 is an AM-space.

(ii) The assertion follows from Proposition 2.2.32.

(iii) The assertion follows immediately from the uniform continuity of operations, from Proposition 2.2.19 and Corollary 2.2.21.

□

Theorem 3.1.8 (Stone-Weierstrass, lattice version)**.** *Let K be a compact space and F be a vector sublattice of $C(K)$, where the constant function, equal to one, is in F. If F separates the points of K, then F is dense in $C(K)$.*

Proof. Let s, t be points of K and α, β real numbers, such that $\alpha = \beta$ if $s = t$. Suppose also that F separates the point of K. Thus there exists a function $f \in F$, such that $f(s) = \alpha$ and $f(t) = \beta$. This is clear, if $s = t$. Now, if $s \neq t$, then there exists $g \in F$, such that $g(s) \neq g(t)$. Set

$$f(t) = \alpha\frac{g - g(t)}{g(s) - g(t)} + \beta\frac{g - g(s)}{g(s) - g(t)}.$$

Then $f \in F$, $f(s) = \alpha$ and $f(t) = \beta$. Let $h \in C(K)$ and $\epsilon > 0$. Since F separates the points of K and F is a vector sublattice, we can find $g \in F$, such that $\|h - g\| < \epsilon$. For the proof, we will use the fact that K is compact. Let v be a fixed element in K. For every $t \in K$, there exists $f_t \in F$, such that $f_t(v) = h(v)$ and $f_t(t) = h(t)$. Now, we denote by U_t the set of all $r \in K$, such that $f_t(r) > h(r) - \epsilon$. Since $f_t \in F$, we can find an appropriate $c_{f(t)}$, such that $B(h(r), c_{f_t}) \subset U_t$. Hence, U_t is open for all $r \in K$. Thus $K = \bigcup_{t \in K} U_t$. Hence K is compact, which implies the existence of countable $\{t_i\}, i = 1, \dots, n$, such that $K = \bigcup\limits_{i=1}^{n} U_{t_i}$. Denote $g_v =: \sup_i f_{t_i}$. Since F is closed, g_v exists in F and we obtain that $g_v(t) > h(t) - \epsilon$, for all $t \in K$. Since ϵ is arbitrary, it follows that $g_v(v) = h(v)$.

Now, suppose that g_v is in F with the properties chosen for all $v \in K$. The set $V_v = \{r \in K : g_v(r) > h(r) + \epsilon\}$ is open and contains v. Respectively, since K is compact, we get $K = \bigcup_{v \in K} V_v$. Hence, there exists $\{v_1, \dots, v_m\} \in K$, such that $K = \bigcup\limits_{i=1}^{m} V_{v_i}$. Now, we denote $g = \inf_m g_{v_m}$. Since F is a lattice, $g \in F$. Putting all together, we obtain that $h(r) - \epsilon < g(r) < h(r) + \epsilon$, for all $r \in K$. Therefore, $\|h - g\| < \epsilon$. So we proved that there exist an h-neighborhood for every $y \in C(K)$ such that $B(h, \epsilon) \cap g \neq \emptyset$, where $g \in F$. Hence F is dense in $C(K)$. □

Theorem 3.1.9. *Suppose V is an AM-space and e is the unit in V. Let K be the $\sigma(V', V)$-compact set of real valued, lattice homomorphisms of norm 1 on V. Then there exists an isomorphism of V onto $C(K)$ through the evaluation map.*

Proof. We want to prove that the set of all extreme points of X is identical with K. It is easy to observe that the set X of all positive linear forms of norm 1 on V is identical with

$$V'_+ \cap \left[x' : \langle e, x' \rangle = 1\right] \subset U^p,$$

where e is the unit of V, by Remark 2.2.45. Since all linear forms are positive, from the previous identification of X, we can assume that all elements of X are lattice homomorphism. Recall that a point x is extreme, if there not exist discrete elements y, z, such that $x = \lambda y + (1 - \lambda) z$, for a scalar λ. Hence a point $x' \in X$ is extreme if and only if for $y' \in V'$, such that $0 \preccurlyeq y' \preccurlyeq x'$ implies that $y' = \mu x'$, for a scalar μ or, equivalently, if $I_{x'}$ is minimal. Hence each x' generates a minimal ideal and as V' is an ideal in $V^\star$, by Proposition 2.2.37, it follows from Proposition 2.1.16, that the set of all extreme points of K is identical with K. Actually K is the Silov Boundary of V_+, as Theorem 2.2.53 shows. Thus, for each $x \in V$, it follows that $\|x\| = \||x|\| = \sup\{\langle |f|, t \rangle : t \in K\} = \sup\{f(t) : t \in K\}$. Therefore, the evaluation map is well defined and is a homomorphism. Furthermore, since each lattice homeomorphism x' of K generates a minimal ideal, then $y' = \mu x'$, for some scalar μ. Thus $x \mapsto \langle x, t \rangle, t \in K$ is an isomorphism with respect to the norm. Because all $t \in K$ have norm 1, the isomorphism is an isometry. In view of Theorem 3.1.8, since K is compact and closed, it satisfies the hypothesis and separates the point of V. The unit of V is mapped onto the constant one function. Therefore the isomorphism is surjective. □

The next corollary is imminent.

Corollary 3.1.10. *Suppose K is the Silov boundary of V_+. Then every AM-space with unit is isomorphic to $C(K)$.*

Proof. An inspection of Theorem 3.1.9, indicates that each lattice homomorphism attains its maximum value with respect to the norm in K. Moreover, since K is $\sigma(V', V)$ closed, it follows that K is the Silov Boundary of V_+ due to Theorem 3.1.9. By assumption we obtain the desired assertion. □

Corollary 3.1.11. *Suppose $C(K_1)$ and $C(K_2)$ are isomorphic, for some compact space K_1 and K_2. Then K_1 and K_2 are homeomorphic.*

Proof. Since $C(K_1)$ and $C(K_2)$ are isomorphic, the respective M-norms are equivalent. We can define such norms as per Proposition 3.1.4 with order units e_1 and e_2 respectively. Moreover, $X_1 = \{x' \in V'_+ : \langle e_1, x'\rangle = 1\}$ and $X_2 = \{x' \in V'_+ : \langle e_2, x'\rangle = 1\}$. We want to find a bijection that maps X_1, homeomorphically to X_2. Let $f : X_1 \to X_2$ such that $x \mapsto \langle e_2, x'\rangle^{-1}x'$. This is a well defined map and f is a bijection of X_1 onto X_2. Since the extreme points generate minimal ideals, it comes as a consequence that f is a homeomorphism from K_1 to K_2. □

Remark 3.1.12. Corollary 3.1.11 should be compared with the following well-known Theorem of Banach and Stone.

Theorem 3.1.13 (Banach-Stone)**.** *Let X and Y be compact Hausdorff spaces and let $T : C(X) \to C(Y)$ be a surjective linear isometry. Then there exists a homeomorphism $\phi : Y \to X$ and a function $g \in C(Y)$ where $|g(y)| = 1$, for all $y \in Y$, such that*

$$(Tf)(y) = g(y)f(\phi(y))$$

for all $y \in Y$ and $f \in C(X)$.

Remark 3.1.14. From the aspect of Banach lattices, AM-spaces with unit, and spaces $C(K)$, where K is compact, are the same.

Proposition 3.1.15. *Let K be a compact space. Then the Banach space $C(K)$ is separable if and only if K is metrizable.*

Proof. Let $V = C(K)$ be separable and denote by F a total subset of V. The weak topology $\sigma(V', F)$ is metrizable, as the metric can be generated by countable many semi-norms and agrees with the dual unit ball U^p with $\sigma(V', V)$. Thus K is metrizable, as it can be identified with a subspace of U^p.

Conversely, suppose K is metrizable. Our target is to find a total subset F that is dense in $C(K)$. Now, we know that $K \times K$ is metrizable and hence the diagonal $\Delta := \{(t,t) : t \in K\}$ of $K \times K$ has a countable base $\{U_n\}$ of neighborhoods. Let $\{G_i^{(n)} : i = 1, \dots, k_n\}$ be a finite open cover of K, such that $\bigcup_{i=1}^{k_n} G_i^{(n)} \times G_i^{(n)} \subset U_n$, for each $n \in \mathbb{N}$ and let $\{f_i^{(n)} : i = 1, \dots, k_n\}$ be a partition of unity on K subordinated to $\{G_i^{(n)}\}$. The set F_n of all linear combinations of the functions $f_i^{(n)}$ with rational coefficients is countable and so is $F := \bigcup_n F_n$. Now, we claim that F is dense in $C(K)$. More specific, for a given $f \in C(K)$ and $0 < \epsilon \in \mathbb{R}$, there exists $n \in \mathbb{N}$, such that $U_n \subset \{(s,t) | f(s) - f(t)| < \epsilon/2\}$.

Now for each i up to k_n, we choose $s_i \in G_i^{(n)}$ and a rational number α_i, such that $|f(s_i)\alpha_i| < \epsilon/2$. By the definition of a partition of unity, we have that $f_i^{(n)} \succcurlyeq 0, \sum_i^{k_n} f_i^{(n)}(s) = 1$ identically in K and $f_i^{(n)}(s) = 0$, if $s \neq U_i^{(n)}$. Therefore,

$$|f(s) - \sum_{i=1}^{k_n} \alpha_i f_i^{(n)}(s)| \preccurlyeq \sum_{i=1}^{k_n} |f(s) - \alpha_i| f_i^{(n)}(s) < \epsilon,$$

for all $s \in K$. This shows F to be dense in $C(K)$.

□

Examples 3.1.16. Let X be a non void set and V be the vector lattice of bounded, real-valued functions on X containing the constant function equal to one, which is complete under the supremum norm. It follows that V is an AM-space as the norm is additive in bounded and real valued functions and, from hypothesis, V is complete with respect to the supremum norm. Hence, there exists a compact K, such that $V \cong C(K)$. Then, K(the Silov Boundary of V^+) can be viewed as the completion of the Hausdorff uniform space associated with $(X, \mathbf{U})$, where $\mathbf{U}$ is the coarsest uniformity on X, such that each $f \in V$ is uniformly continuous.

(i) Suppose X is completely regular and V is the space of all bounded continuous functions on X. Then K is homoemorphic to the Stone-Cech Compactification of X.

(ii) Suppose Σ is the σ-algebra of subsets of X and V is the space of all bounded Σ-measurable real functions. Then K is the Stone Representation space of Σ.

Definitions 3.1.17. (i) Let X be a compact space and A a subset of $C(X)$. Then A is called *relatively compact* if and only if A is equicontinuous.

(ii) A is called *relatively compact* it its closure is compact.

Definition 3.1.18. Let X be a compact space and $A \subset C(X)$. Then A is called *relatively weakly compact* if and only if each sequence $(x_n)_{n \in \mathbb{N}} \in A$ contains a subsequence converging pointwise to a function in $C(K)$.

Relative compactness allows us to acquire the following result for AM-spaces.

Proposition 3.1.19. *Suppose V is an AM-space and A is a non-void subset of V. Denote by $|A|$ the set of all absolute values of $x \in A$. If A is relatively compact, so is $|A|$. Moreover,* $\sup A$ *and* $\inf A$ *exist in V and respectively if A is relatively weakly compact, then $|A|$ is also. The mapping $x \mapsto x^+, x \mapsto x^-$ and $x \mapsto |x|$ are weakly sequentially continuous.*

Proof. Let A be a relatively compact subset. By Definition 3.1.17, it suffices to prove that $|A|$ is equicontinuous. Since A is relatively compact, it follows that there exists a finite open cover $\{V_i\}$, such that $A \subset \bigcup_i V_i$. Assume that there exists a positive ϵ and $\{U_i\}$ is a finite open cover for $|A|$, where each U_i, for all i, is given by $U_i = (V_i)_+$. It follows for all $(s,t) \in U_i \times U_i$ that $|f(s) - f(t)| < \epsilon$, by hypothesis. Therefore, $|A|$ is relatively compact.

Assume that A is relatively compact. It follows that A is equicontinuous. Thus for a non-void finite subset F of A, it follows that $|g(s) - g(t)| \leqslant \sup_{f \in F} |f(s) - f(t)|$, for all $(s, t) \in K$. Therefore, the set of all suprema of finite subsets of A exists and is equicontinuous assuming that A is relatively compact or equicontinuous. Moreover, since $\sup A$ exists, it follows that $\inf A$ exists.

Let A be a relatively weakly compact subset. This implies that there exists a subsequence for every sequence $(x_n)_{b \in \mathbb{N}}$ converging pointwise in $C(K)$. Assume that we take the positive part of all sequences in A. Thus we obtain all the sequences in $|A|$. Hence, the desired subsequences for each sequence can be obtained by taking the positive part of every subsequence. By hypothesis, it follows that they converge pointwise to a function in $C(K)$. Therefore, $|A|$ is relatively weakly compact.

It is easy to prove that the mappings, $x \mapsto x^+, x \mapsto x^-$ and $x \mapsto |x|$ are weakly sequentially continuous.

□

Definitions 3.1.20. Let X be a topological vector space. A subset A of X is called

(i) an F_σ set if it is the countable union of closed subsets of X.

(ii) an G_δ set if it is the countable intersection of open subsets of X.

Definition 3.1.21. Let X be a topological vector space. Then V is called *normal*, if for any given disjoint subsets F, G there exist neighborhoods U, V respectively such that U and V are also disjoint.

Lemma 3.1.22 (Urysohn)**.** *Let X be a normal topological vector space and U, V be closed and district to each other subsets of X. Then there exits a continuous map $f : X \to [0, 1]$, such that $f|_A = 0$ for all $x \in A$ and $f|_B = 1$ for all $x \in B$.*

Definitions 3.1.23. Let X be a topological vector space and $f : X \to \mathbb{R}$ be a map. Then f is called

(i) *lower semi-continuous*, if O^a is open or equivalently C^a is closed for all $\alpha \in \mathbb{R}$.

(ii) *upper semi-continuous*, if O_a is open or equivalently C_a is closed for all $\alpha \in \mathbb{R}$.

Remark 3.1.24. For convenience purposes we denote O^a by $[h < a]$ and O_a by $[h > a]$.

Proposition 3.1.25. *Let X be a topological vector space. Denote by $C_b(X)$ the space of all bounded, continuous functions from X to $\mathbb{R}$ and consider the following assertions:*

(i) Every open subset of X has open closure.

(ii) Every open F_σ-subset of X has open closure.

(iii) $C_b(X)$ is order complete.

(iv) $C_b(X)$ is countably order complete.

Then (i) implies (iii) and (ii) implies (iv). Moreover, if X is normal, then the respective reverses hold.

Proof. Recall that the *upper limit function* f_u of a bounded real function f on X is defined by

$$f_u := \inf_{U(s)} \sup_{t \in U(s)} h(t),$$

where $s \in X$ and $U(s)$ runs through a neighborhood base of s.

(i) Let A be a directed majorized subset of $C_b(X)$ and let g denote the numerical supremum of A, which is the function $g : X \to \mathbb{R}$, where $g(s) := \sup_{f \in A} f(s)$, for all $s \in X$. Moreover, if $k \in C_b(X)$ is any majorant of A, then clearly $f \preccurlyeq g \preccurlyeq k$ and more specifically, $f \preccurlyeq g_u \preccurlyeq k$, for all $f \in A$. Thus we validated the continuity of g_u and it follows that $g_u = \sup A$.

Now, we observe that g_u is upper semi continuous. Indeed, the set $[g_u < a]$ is open, since for $g(s_0) < a$, there exists $\epsilon > 0$ and a neighborhood U of s, such that $\sup_{t \in U} g(t) \preccurlyeq g_u(s_0) + \epsilon < a$. To validate that g_u is lower semi continuous, if (i) holds, we first observe that

$$[g > a] = \bigcup_{f \in A} [f > a] = \bigcup_{f \in A, n \in \mathbb{N}} [f \succcurlyeq an^{-1}]$$

is open (respectively an open F_σ set whenever A is countable). Furthermore, we obtain for all $\alpha \in \mathbb{R}$ that

$$[g_u > a] = \bigcup_{n \in \mathbb{N}} [g_u \succcurlyeq a + n^{-1}] = \bigcup_{n \in \mathbb{N}} [g \succcurlyeq a + n^{-1}]^- = \bigcup_{n \in \mathbb{N}} [g > a + n^{-1}]^- \quad (1)$$

The last equality holds, because $[g > a+n^{-1}] \subset [g \succcurlyeq a+m^{-1}] \subset [g > a+m^{-1}]$ for $m > n$. Thus (i) (and respectively (ii) if A is countable), implies $[g_u > a]$ to be open for all $\alpha \in \mathbb{R}$ and hence g_u to be continuous. Therefore, (ii) implies (iv) by the preceeding, supposing that A is countable.

(ii) Now, suppose X is a normal topological vector space. Suppose that $C_b(X)$ is countably order complete. If $V \neq X$ is an open non-void F_σ-subset of X, such that $V = \bigcup_n F_n$, where $(F_n)_{n \in \mathbb{N}}$ are closed, let $f : X \to [0,1]$ be a continuous function, such that $f_n(F_n) = \{1\}$, $f_n(X/V) = \{0\}$. Denote $A = \{f_n : n \in \mathbb{N}\}$. Then, it follows that A is a countable majorized subset of $C_b(X)$. Hence $f := \sup A$, exists in $C_b(X)$. It is clear that $f(V) = \{1\}$ and $f(X/\bar{V}) = \{0\}$ and since f is continuous, it follows that $f(\bar{V}) = \{1\}$, which shows that f is the characteristic function of $\bar{V}$. Hence $\bar{V}$ is open. The respective proof that (iii) implies (i) is similar, supposing that V is open.

□

Definition 3.1.26. [Tychonoff] Let X be a topological vector space. Then X is called a *completely regular* space if any points of X can be separated from closed sets via bounded, continuous real valued functions.

Definitions 3.1.27. Suppose X is a completely regular space. Then X is called:

(i) *totally disconnected*, if there exists a basis of open and closed (clopen) subsets.

(ii) *Stonian* or *extremely disconnected*, if the closure of every open set is open.

(iii) *quasi-Stonian* or *σ-Stonian*, if the closure of every open F_σ set is open.

Corollaries 3.1.28. *Let K be a compact space. Then, the AM-space $C(K)$ is*

(i) order complete, if and only if K is Stonian.

(ii) countably order complete if and only if K is quasi-Stonian.

Proof. In view of Definitions 3.1.27 and the proof of Proposition 3.1.25 the result is imminent. □

Remark 3.1.29. A Stonian space is different from a Stone representation space. Stone represenation space of a Boolean Algebra is always totally disconnected, as it possesses a base of clopen subsets, but it is not necessarily extremely disconnected.

Definition 3.1.30. The *lower limit function* g_l of $g : G \to \mathbb{R}$ is the continuous function defined by

$$g_l(s) := \sup_{U(s)} \inf_{t \in U(s) \cap G} g(t),$$

where $U(s)$ runs through a neighborhood base of $s \in X$.

The following proposition provides a twist of quasi-Stonian and Stonian spaces.

Proposition 3.1.31. *Let X be a topological vector space. If in addition, X is Stonian (respectively quasi-Stonian), then each bounded, continuous function $f : G \to \mathbb{R}$, where $G \in X$ is open and dense (respectively a dense open F_σ set) has a continuous extension $\bar{g} : X \to \mathbb{R}$.*

Proof. Now in view of Proposition 3.1.25, we can verify easily that g_l is lower semi continuous on X and obviously g_l agrees with g on G. Analogously, we can show that g_l is upper semi continuous as well. For that reason, we observe that

$$[g_l < a] = \bigcup_{n \in \mathbb{N}} [g_l \preccurlyeq a - n^{-1}] = \bigcup_{n \in \mathbb{N}} [g \preccurlyeq a - n^{-1}]^- = \bigcup_{n \in \mathbb{N}} [g < a - n^{-1}]^- \qquad (2)$$

in reference to the corresponding relations in (1). Moreover, to prove that $[g_t < a]$ is open in X, it suffices to prove, by (2), that the sets $[g < a - n^{-1}]$, as subsets of G, are open F_σ in X. It is clear, that these sets are open F_σ-sets$\in G$, since G is an open F_σ-set itself in X. Thus, it follows that each set $[g < a - n^{-1}]$ is an open $F_\sigma \in X$. Therefore, this proves that g_t is continuous on X and hence we have the desired assertion. □

Remark 3.1.32. Let X be a completely regular topological space. The Stone-Cech compactification βX of a completely regular topological space X is a compact space, densely containing X. Also βX has the property that each bounded, continuous $f : X \to \mathbb{R}$ has a continuous extension $\bar{f} : \beta X \to \mathbb{R}$.

The following corollary comes as a consequence of Proposition 3.1.31.

Corollary 3.1.33. *Let K be a Stonian (quasi-Stonian) compact space and let G be an open (an open F_σ) subspace of K. Then the closure $\bar{G} \in K$ is homeomorphic with the Stone-Cech compactification of G.*

Example 3.1.34. Suppose K to be a quasi-Stonian space. Since the order interval $[-1, 1]$ is homeomorphic to the two point compactification of $\mathbb{R}$, denoted by $\bar{\mathbb{R}} = [-\infty, \infty]$, it follows from Proposition 3.1.31 and Corollary 3.1.36, that each continuous function $g : G \to \mathbb{R}$, where G is a dense open F_σ set in K, has a continuous extension $\bar{g} : K \to \bar{\mathbb{R}}$.

Example 3.1.35. Let $C_\infty(K)$ denote the set of all continuous function from K, which are finite except on nowhere dense sets. We want to construct an open dense F_σ set and find a function than can extend via Proposition 3.1.31. Let $f_i \in C^\infty(K)$ and define the set $G = \{t \in K : |f_1(t)| + |f_2(t)| < \infty\}$. It follows that there exists $t \in K$ and $\epsilon > 0$, such that $B(f_i, e) \cap G \neq \emptyset$. Moreover, since $f_i \in C_\infty(K)$ it follows that there exist countable many closed sets U_i, such that $G = \bigcup_1^n U_i$. Therefore, G is an open dense F_σ set of K. By applying Proposition 3.1.31, it follows that the function $g : G \to \mathbb{R}$ defined as $g = af_1 + bf_2$, for some constants $a, b \in \mathbb{R}$, has a continuous extension $K \to \bar{\mathbb{R}}$.

The following corollary comes as a consequence of Corollary 3.1.28 and Example 3.1.35.

Corollary 3.1.36. *Suppose K is a quasi-Stonian (Stonian) compact space. Then $C_\infty(K)$ is a countably order complete (order complete) vector lattice containing $C(K)$ as an order dense ideal.*

Definition 3.1.37. Let X be a real space and V a vector lattice. The function $\rho : X \to V$ is called sublinear if

- $\rho(x + y) \preccurlyeq \rho(x) + \rho(y)$ for all $x, y \in X$.
- $\rho(\lambda x) = \lambda\rho(x),\ x \in X$ and $\lambda \in \mathbb{R}$.

Theorem 3.1.38 (Generalized Hahn-Banach Theorem)**.** *Let X be a vector space over $\mathbb{R}$, V an order complete vector lattice and ρ a sublinear form X to V. If X_0 is a vector subspace of X, such that $f(x) \preccurlyeq \rho(x)$ for a function $f : X_0 \to V$ and for all $x \in X_0$, then there exits a linear extension $F : X \to V$ of f satisfying $F(x) \preccurlyeq \rho(x)$, for all $x \in X$.*

Proof. Assume that $X_0 \neq X$. Thus there exists a $x_0 \in X \backslash X$. Now we will validate that f has a linear extension to the linear span of $X_0 \cup \{x_0\}$. Denote this span by Z. Let $f_1 : Z \to V$ be a linear map dominated by ρ. Hence for any $u, v \in X_0$, we have $f(u) - f(v) = f(u) - f(v) + f(x_0) - f(x_0)$. Since f is linear, it follows that

$$f(u) - f(v) = f(u + x_0) + f(-x_0 - v) \preccurlyeq \rho(u + x_0) + \rho(-x_0 - v).$$

Hence we obtain that

$$f(u) - \rho(u + x_0) \preccurlyeq f(v) + \rho(-x_0 - v).$$

This shows that the element $f(u) - \rho(u + x_0)$ is majorized in V as both u, v are arbitrary and V is order complete. Moreover, denote

$$\tau := \sup_{u \in x_0} \big(f(u) - \rho(u + x_0)\big).$$

This supremum exists in V, as V is order complete and we have that $\tau \preccurlyeq f(v) - \rho(-x_0 - v)$, for all $x \in X_0$. Now, $f_1 : Z \to V$ is an extension of f, such that $f_1(x_0) = -\tau$. To verify that ρ dominates f_1 we take two cases:

- Let $z \in Z$, $z = \lambda x_0 + x$, $x \in X_0$ and $\lambda > 0$. Since τ is the supremum above, we choose $u = \lambda^{-1}x$ and because X_0 is a subspace, it follows that $u \in X_0$. Hence $f_1(z) = f_1(\lambda^{-1}x - x_0) = f_1(\lambda^{-1}x) - f_1(x_0) = f_1(\lambda^{-1}x) - u$. Therefore, $f_1(\lambda^{-1}x) - u \preccurlyeq \rho(\lambda^{-1}x + x_0) = \rho(z)$, whence $f_1(z) \preccurlyeq \rho(z)$.

- Let $z \in Z$, $z = \lambda x_0 + x$, $x \in X_0$ and $\lambda > 0$. Since $v = \lambda^{-1}x \in X_0$ and τ is the supremum cordinatewise, it follows that $-f(\lambda^{-1}x) + u \preccurlyeq \rho(-\lambda^{-1}x - x_0)$. Therefore, $f_1(z) \preccurlyeq \rho(z)$.

To complete the proof we will make use of Zorn's Lemma. Set Γ the set of all pairs (Z, f_z) where Z is of the form $X_0 \cup \{0\}$, for some $x_0 \in X \backslash X_0$ and f_z be the linear extension of f, such that $f_z(x) \preccurlyeq \rho(x)$, for all $x \in Z$. Obviously, $\Gamma \neq \emptyset$, because $(X_0, f) \in \Gamma$. We endow Γ with a partial order $\prec$, such that $(Z_1, f_{z_1}) \prec (Z.f_{z_2})$, if and only if $Z_1 \subset Z_2$ and f_{z_2} be an extension of f_{z_1}. Take $\Delta = \{(Z_i, f_{z_i}) : i \in I\}$ be a total ordered subset of Γ. If $K = \bigcup_{i \in I} Z_i$, $f_k : K \to V$, such that $f_k(x) = f_{z_i}(x)$ if $x \in Z_i$, then $K \in \Delta$ and K is an upper bound of Δ. Since V is order complete by Zorn's Lemma, it follows that Δ has a maximal element. Suppose that (H, f_H) is the maximal element. We will show that $H = X$. Suppose $H \prec X$. Since we found $f_1(z)$ to be an extension of f, we set $X_0 = H$. Then $Z = H_0 \cup \{x_0\}$ and thus $(Z, f_1) \in \Gamma$. Therefore, $(H, f_H) \prec (Z, f_1)$, which is a contradiction. Hence $H = X$ and $F = f_H$. □

Theorem 3.1.39. *Suppose V_0 is a normed vector subspace of a real Banach space V and K be a Stonian space. Then any continuous linear map from $T_0 : V_0 \to C(K)$ has a linear extension T to V, where $\|T_0\| = \|T\|$.*

Proof. We want to apply the generalized Hahn-Banach Theorem. Hence, we need to find proper T_0 satisfying the hypothesis of Theorem 3.1.38. Let $T_0 : V_0 \to C(K)$, where we assume that $\|T_0\| \leqslant 1$. Define the sublinear form f_0, such that $f_0 := \|x\|e$, for all $x \in V$, where e is the unit of V. We obtain without loss of generality that $T_0(x) \preccurlyeq \|x\|e$, for all $x \in V_0$ and hence T_0 satisfies the hypothesis of Theorem 3.1.38, as $V = C(K)$ and $C(K)$ is order complete. Therefore, there exists a positive extension $T : V \to C(K)$ satisfying $T(x) \preccurlyeq \|x\|e$. This also verifies that $\|T\| = \|T_0\| \leqslant 1$. □

Corollary 3.1.40. *Suppose G is a Banach space and G_0 a Banach subspace of G. If G_0 is isometrically isomorphic to some order complete $C(K)$, where K is compact, then there exists a contractive projection map T such that $R(T) = G_0$.*

Proof. In view of Theorem 3.1.39, let $1_{G_0} : G_0 \to C(K)$ be the identity mapping. We assume that 1_{G_0} is a continuous isomorphism and hence we apply Theorem 3.1.39 and obtain that there exists a norm preserving linear extension to G with values in G_0. □

Corollary 3.1.41. *Suppose V is an order complete AM-space with unit e. If V_0 is a closed, order complete lattice and a vector sublattice of V, then there exits a contraction T such that $R(T) = V_0$.*

Proof. Since any $C(K)$, where K is compact, can be identified with an AM-space with unit e, the existence of such contraction map is imminent, by Corollary 3.1.40. We only need to prove that T is positive. For that reason, let $v \in V_+$, such that $\|v\| \leqslant 1$, if and only if $v = e - x$ for some $x \in V$, such that $\|x\| \leqslant 1$. Since $e \in V$, we can find such v. Let $P : V \to V_0$ be a projection band. Thus $Pe = e$ since V_0 is a closed vector sublattice. Hence $\|v\| \leqslant 1$ implies that

$$Pv = P(e - x) = Pe - Px = e - Px,$$

which is positive because $\|Px\| \leqslant \|x\| \leqslant 1$. □

Corollary 3.1.42. *Let V be a Banach lattice and V_0 a Banach sublattice. Denote by E an order complete AM-space with unit. Then every positive linear map $T_0 : V_0 \to E$ has a positive extension $T : V \to E$ such that $\|T_0\| = \|T\|$.*

Proof. We notice that T_0 is necessarily continuous, by Theorem 2.2.24. Now, in order to satisfy the hypothesis of Corollary 3.1.41, we identify V with $C(K)$, for some compact Stonian space K, such that $C(K)$ is contained in l^∞. Thus, there exists a positive contractive projection $P : l^\infty \to C(K)$. Now, for each $t \in K$, the mapping $x \mapsto Tx(t)$ is a positive linear form, whose norm is less or equal to the norm of T_0 on V_0 and, as a consequence of Proposition 2.2.42, there exists a positive norm preserving extension $\rho_t \in V'$. By definition, it follows that the mapping $x \mapsto \big(\rho_t(x)\big)_{t \in K}$ is a positive linear map $\tilde{T} : G \to l^\infty$, which extends T_0 and that $\|\tilde{T}\| = \|T_0\|$. Therefore, $T := P \circ \tilde{T}$ is a norm preserving linear extension of T_0 with values in $C(K)$. □

Proposition 3.1.43. *Let K_1 and K_2 be compact spaces, where K_2 is totally disconnected. Let $T : C(K_1) \to C(K_2)$ be a positive linear map satisfying the following:*

(i) $Te_1 = e_2$ and $T[0, f] = [0, Tf]$ for all $f \in C(K_1)$, where e_i are the respective units.

(ii) The set $\{f : T|f| = 0\}$ is a projection band of $C(K_1)$.

Proof. We will use the Boolean Algebras B_1 and B_2, where $B_1 = \{f \in C(K_1) : f \wedge (e_1 - f) = 0\}$ and $B_2 = \{g \in C(K_2) : g \wedge (e_2 - g) = 0\}$. Then, Proposition 1.2.64, indicates that B_1 and B_2 can be identified with the Boolean Algebra of all open and closed subsets of K_i. Suppose T is strictly positive ($T \gg 0$), then $\ker|T|$ is a maximal ideal and hence a projection band of $C(K_1)$, since $C(K_1)$ as a Banach lattice is Archimedean. Therefore, (ii) is satisfied.

We now claim that T maps a Boolean subalgebra B_1' to B_2. If $f \in [0, e_1]$, then for a $g \in B_2$, by (i), we have that $Tf = g$. Since T is positive and linear, it follows

$$0 \preccurlyeq T(f \wedge (e - f)) \preccurlyeq g \wedge (e_2 - g) = 0.$$

Therefore, $f \in B_1$. Moreover, g is uniquely determined. Suppose that there exist $f_1, f_2 \in B_1$, such that $Tf_1 = Tf_2 = g$, $f_1 \neq f_2$. Hence for $h = \dfrac{1}{2}(f_1 \vee f_2) + \dfrac{1}{2}(f_1 \wedge f_2)$, it follows that $Th = g$. This also implies that $h \in B_1$. But h is extreme in $[]0, e_1]$, hence $f_1 = f_2$.

Now, denote by B_1' the subalgebra, where $B_1' = T^{-1}(B_2) \cap [0, e_1]$. Obviously, B_1' is well defined as $T^{-1}(B_2) \subset [0.e_1]$ or, equivalently, $T^{-1}(B_2) \supset [0, e_1]$ both belong in B_1. To prove that B_1' is closed we need to check that $g_1, g_2 \in B_2$, such that $g_1 \wedge g_2 = 0$ implies $f_1 \wedge f_2 = 0$, where $f_1, f_2 \in B_1'$. This is imminent as T is strictly positive and $g_1 \wedge g_2 = Tf_1 \wedge Tf_2$, for some $f_1, f_2 \in B_1$. Hence $T(f_1 \wedge f_2) \preccurlyeq Tf_1 \wedge Tf_2 = 0$. Therefore, $f_1 \wedge f_2 = 0$ and the restriction $\tau = T|_{B_1'}$ is an isomorphism onto B_2. Lastly, since τ is linear, it follows that τ^{-1} is also a Boolean isomorphism of B_2 onto B_1'.

Suppose that (ii) holds instead of (i) and thus T is not necessarily strictly positive. Moreover, $\ker|T| := A$ is a projection band of $C(K_1)$. The restriction $T_0 = T|_{A^\perp}$ is strictly positive and satisfies (i). Moreover, $A^\perp$ is minimal, because A is maximal and hence only $e_1 \in A^\perp$. Thus $T_0(e_1) = T|_{A^\perp}(e_1) = e_2$, $T_0[0, f] \in C(K_2)$ and $T_0[0, f] = [0, T_0]$. Now, by applying the previous construction to T_0, there exists $S : C(K_1) \to A^\perp$, such that S is a lattice isometry and $T_0 \circ S(f) = T_0(f)$, such that $f \in A^\perp$, because $A^\perp$ is minimal. Therefore, $T_0 \circ S = 1_{C(K_1)}$. Lastly, the map $j : A^\perp \to C(K_2)$ is a canonical injection and $j \circ s$ is the desired right inverse of T and thus it follows

$$T_0 \circ (j \circ S)(f) = T_0 \circ j(f) = T_0(e_2) = e_2.$$

□

3.2 AL-Spaces

To proceed further with Banach lattices it is necessary to introduce AL-spaces. As we did with AM-spaces, we will discuss some basic properties and through this paragraph we will be able to prove a similar injective property as we did with order complete AM-spaces. Furthermore we will be able to state the Theorems of Vitali, Radon and others.

Definition 3.2.1. Let V be a vector lattice and $\|\cdot\|$ be a lattice norm. The lattice norm $\|\cdot\|$ is called an *L-norm*, if the following holds:

$$\|x+y\| = \|x\| + \|y\|$$

for all $x, y \in V_+$. The space $\big(V, \|\cdot\|\big)$ is called a *L-norm space*. Furthermore, if the norm is complete then $\big(V, \|\cdot\|\big)$ is called briefly *AL-space* (abstract L-space).

Proposition 3.2.2. *Let V be an L-normed space. The following hold for every L-normed space:*

(i) Every closed vector sublattice is an L-normed space.

(ii) The completion of V is an AM-space.

(iii) The quotient of an V over a closed ideal is an L-normed space.

Proof. (i) Let V_0 be a closed vector sublattice of V. Since V is L-normed, it follows that V_0 is closed, with respect to the L-norm and the lattice operations. Moreover, if V is L-norm complete, it follows by the Proposition 3.1.7 that V_0 is an AL-space.

(ii) The assertion follows immediately due to the uniform continuity of operations from Proposition 2.2.19 and Corollary 2.2.21.

(iii) Since I is a closed ideal in V, it is a projection band, by Proposition 3.2.4. Thus it follows, from Proposition 2.2.32, that V/I is a an L-normed space.

□

Proposition 3.2.3. *Let V be an L-normed space. Then each directed norm bounded family is a Cauchy family. Furthermore, it is necessary and sufficient that every directed norm bounded family has a supremum in order for V to be an AL-space.*

Proof. Let $\{f_n\}$ be a directed family in V, such that $\|f_n\| \leqslant k$ for some constant k. Now, suppose that $\{f_n\}$ is not bounded. Then there exist a number $c > 0$ and an infinite subsequence $(x_n)_{n\in\mathbb{N}} \subset \{f_n\}$, such that $\|x_{n+1} - x_n\| > c$, for all $n \in \mathbb{N}$. By Definition 3.2.1, this implies

$$nc \leqslant \sum_{k=2}^{n+1} \|x_k - x_{k-1}\| = \|x_{n+1} - x + n\| \leqslant k,$$

for all n which is a contradiction.

The condition is necessary due to Lemma 2.2.55. Conversely, suppose that V is an L-normed space, such that every directed norm bounded sequence has a supremum. What we need to validate is that given a Cauchy sequence, a subsequence converges. For that reason, let $(y_n)_{n\in\mathbb{N}}$ be a subsequence of a given Cauchy sequence, such that $\|y_{n+1} - y_n\| < 2^{-2(n+1)}$. Now, let $v_k = (y_{k+1} - y_k)^+$, for $k \in \mathbb{N}$. We need to show that $\sum_k v_k$ converges in V. Defining $u_k = 2^k v_k$, for $(k \in \mathbb{N})$ the choice of y_n and the hypothesis imply that both $v := \sup_n \sum_{k=1}^n v_k$, $u := \sup_n \sum_{k=1}^n v_k$ exists in V. Thus we obtain

$$0 \preccurlyeq v - \sum_{k=1}^{n} v_k \preccurlyeq 2^{-n} v,$$

which implies that $\|v - \sum_{k=1}^n v_k\| \leqslant 2^{-n}\|u\|$, for $n \in \mathbb{N}$, and the proof is complete. □

Proposition 3.2.4. *Every AL-space V has the following properties:*

(i) Every directed norm bounded family in V converges.

(ii) V is order complete and $V' = V^\star = V^\star_{00}$.

(iii) Suppose $I \in \boldsymbol{I}(V)$ is a closed ideal. Then I is a projection band.

(iv) Every weak order unit of V is a quasi interior point to V_+.

(v) The evaluation map provides an isomorphism between V to the band of all order continuous linear forms on V.

Proof. (i) Let (f_n) be a directed norm bounded family. Then Proposition 3.2.3 indicates that this family is a Cauchy family necessarily having a supremum. Thus the assertion is imminent.

(ii) We have seen, by Remark 2.1.6, that $V^\star_{00} \subset V^\star \subset V'$ holds. Now in order to prove the reverse inclusion, we notice that every norm bounded family in V implies, by Theorem 2.2.62, that V is order complete and each continuous linear form on V is order continuous. Hence the reverse is ominous.

(iii) Since each norm bounded family has a supremum, it follows from (i) that each closed ideal in V is a band. Moreover, Theorem 1.2.53 implies that each band is a projection band, as (ii) implies that V is order complete.

(iv) Let u be a weak order unit in V. We need to show that the principal ideal generated by u is dense in V. It holds that $x = sup_n(x \wedge nu)$, for all $x \in V_+$ by Corollary 1.1.93. Then the sequence $(x_n) = x \wedge nu$, where u is the weak order unit norm converges to x. Thus Theorem 2.3.11 implies that u is quasi interior.

(v) Let V be an order complete AL-space. Since (ii) holds, it follows from Theorem 2.1.56 that $q(V)$ is an order dense ideal of $(V')^\star_{00}$, as V' is an ideal of V. We want to prove that the range of $q(V)$ is all of $(V')^\star_{00}$. Let (f_n) be a norm bounded family in $q(V)$, then (f_n) is convergent as (i) holds. Hence the proof is complete.

□

Example 3.2.5. Let V be any vector lattice and let f be a positive linear form on V. For convenience purposes, in order to show the desired relevance we denote f by x'. If N is the absolute kernel of the dual, it is of the form $\{x : \langle |x|, x' \rangle\}$. By Propositions 2.2.32 and 3.2.2, it follows that V/N is an $L-$*normed* space with respect to the norm $\bar{x} \mapsto \langle |x|, x' \rangle$, where $x \in \bar{x} \in V/N$.

Lemma 3.2.6. *Let V be a Banach lattice having the principal projection property with a weak order unit u. Then the representation B_u by its Stone representation space K_u generates an isomorphism of the AM-space V_u onto $C(K_u)$.*

Proof. Theorem 3.1.9 states that V_u is isomorphic to $C(K)$ for some compact space K. Since V has the principal projection property so does V_u, which implies that K is totally disconnected. Conversely, let τ induce an isomorphism between V_u and $C(K)$. We notice that τ maps B_u isomorphically onto the Boolean Algebra of characteristic functions of all closed-and-open subsets of K. Since K is totally disconnected, it follows that these sets form a base of the topology of K and the unicity of the Stone space K_u. □

Definition 3.2.7 (Radon Measure)**.** Let (X, Σ, μ) be a Hausdorff topological space. A *Radon measure* is a measure on the σ-algebra of Borel sets of X that is finite on all compact sets, outer regular on all Borel, and inner regular on open sets.

Remark 3.2.8. An equivalent definition for Radon measure is the following: If X is a compact space, then a Radon measure is every linear form, $\mu \mapsto \mu_0(f)$, on the space of all bounded, real valued and continuous functions.

Theorem 3.2.9 (Kakutani)**.** *Let V be an AL-space. Then for every V there exists a locally compact space X and a strictly positive Radon measure μ on X, such that V is isomorphic with $L^1(\mu)$. Moreover, V possesses weak order units if and only if X can be chosen to be compact, such that the isomorphism $V \to L^1(\mu)$ maps V_u onto $L^\infty(\mu)$.*

Proof. Suppose that V has a weak order unit. Now if V is isomorphic to $L^1(\mu)$, for a Radon measure μ, on the compact space X then, it follows that $e \in V$ corresponding to the constant-one function on X is a weak order unit in V. Conversely, if u is a weak order unit of V it follows from Proposition 3.2.4 that V_u is a dense ideal of V. Now Lemma 3.2.6 implies that V_u can be identified with $C(K_u)$. To define a strictly positive Radon measure on K_u, we notice that in view of $\mu(x) = \|x^+\| - \|x^-\|$, the norm of V defines a strictly positive linear form μ on V such that $\|x\| = \mu(|x|)$ and the restriction of μ to V_u defines a strictly positive Radon measure on K_u. Since $C(K_u)$ is dense in $L^1(\mu)$ it follows that the isomorphism $V_u \to C(K_u)$ extends to an isomorphism of Banach lattices $V \to L^1(\mu)$. Lastly since V_u is an ideal of V it follows that $C(K_u)$ is the ideal in L^1 generated by the constant one function and hence can be identified with $L^\infty(\mu)$.

Suppose that V has no weak order units. Let $U := \{u_a : a \in A\}$ be a maximal orthogonal system. It follows from Proposition 1.1.92 and Proposition 3.2.4, that the ideal I generated by S is dense in V. On the other hand, since S is a maximal orthogonal system, it follows that I is the algebraic direct sum $\oplus_a V_{u_a}$. Also the principal ideal V_{u_a} is isomorphic to $C(K_u)$, since V_u is an AM-space, by Corollary 3.1.5 and by Lemma 3.2.6, isomorphic to $C(K_a)$, where K_a can be identified with the Stone Representation space of the Boolean Algebra of $B_a := \{v \in V : v \wedge (u_a - v) = 0\}$. Thus I can be identified with $\mathcal{K}(X)$, where the locally compact space X is the direct sum of the family of compact spaces $(K_a)_{a \in A}$. Now, in view of $\mu(x) := \|x^+\| - \|x^-\|$, where $x \in I$, the norm of V defines a strictly positive Radon measure μ on X, such that $\|x\| = \int |x| d\mu$ and the isomorphism $I \to \mathcal{K}(X)$ can be extended to an isomorphism of Banach lattices $V \to L^1(\mu)$. □

Example 3.2.10. Let X be a locally compact space. Let $V = \mathcal{K}(X)$ be the vector lattice of all continuous functions $X \to \mathbb{R}$ with compact support. A linear form on $\mathcal{K}$ satisfies the alternate definition given for a Radon Measure in Remark 3.2.8. Hence, by Theorem 3.2.9, the AL-space (V, μ), where μ is a positive order bounded function, is isomorphic to $L^1(\mu)$.

Corollary 3.2.11. *Let V be an AL-space. Then the following are equivalent:*

(i) V is separable.

(ii) V has a weak order unit and the base B_u is separable.

Proof. (i) Suppose V is separable. It holds that V has a total and dense subset. By Lemma 3.2.6, it follows that B_u is a total subset of V. Now, by Proposition 3.2.4, it follows that each weak order unit is a quasi interior point of V. Since B_u is a Boolean Algebra, it follows that B_u is separable.

(ii) It follows from Lemma 3.2.6, that B_u is a total subset of V. Since each weak order unit is a quasi interior point of V, it follows that V is separable.

□

Theorem 3.2.12 (Egorov)**.** *If U is a measurable subset of finite measure and if $\{f_n\}$ is a sequence of a.e finite valued measurable functions, which converges a.e on V to a finite valued measurable function f, then for every $\epsilon > 0$, there exists a measurable subset F of U, such that $\mu(F) < \epsilon$, and such that the sequence $\{f_n\}$ converges to f uniformly on $U - F$.*

Remark 3.2.13. We consider the case where V has weak order unit. Then Lemma 3.2.6 indicates that $V_u \cong C(K_u)$ is a dense ideal of V. Hence. by Theorem 3.2.9, we obtain that $C(K_u) \cong L^\infty(\mu)$, for a positive Radon measure. By applying Egorov's Theorem, it follows that each class $[f] \in L^1(\mu, K_u)$ contains a finite continuous function on a dense and open subset of K_u. Therefore, it follows from Corollary 3.1.28, that each class in $L^1(\mu, K_u)$ contains a continuous extended real function f, which is unique by the strict positivity of μ.

Theorem 3.2.14 (Baire Category Theorem)**.** *Let (X, ρ) be a complete metric space and let $(G_n)_{n\in\mathbb{N}}$ be a countable family of open and dense subsets of X. Then the intersection $\bigcap_{n=1}^{\infty}$ is dense.*

Definition 3.2.15. Let B be a Boolean Algebra. A function $\mu : B \to \mathbb{R}$ is called *additive* if

$$\mu(u \vee v) + \mu(u \wedge v) = \mu(u) + \mu(v)$$

holds for all $u, v \in B$, and if $\mu(0) = 0$.

Theorem 3.2.16. *Let V be a Banach lattice, $v \in V_+$ and suppose $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of continuous, additive real functions on the Boolean algebra $B_v = \{u \in V : u \wedge (v - u) = 0\}$, such that $(\mu_n(u))$ converges in $\mathbb{R}$, for each $u \in B_v$. Then it follows that (μ_n) is uniformly equicontinuous and hence, converges to a continuous, additive function $\mu : B_v \to \mathbb{R}$. In fact, it holds that $\lim\limits_{\|u\|\to 0} \mu_n(u) = 0$ uniformly for $n \in \mathbb{N}$.*

Proof. Let $\epsilon > 0$ be a preassigned number and define

$$B_{nm} := \{u \in B_e : |\mu_n(u) - \mu_m(u)| \leqslant \epsilon\},$$

for $n, m \in \mathbb{N}$. Since the real functions μ_n are continuous, it follows that B_{nm} is a closed subset of B_e and $B_p := \bigcap_{m,n\geqslant p} B_{nm}$ is also closed. Moreover, since B_e is a complete metric space, the Baire category theorem implies that some B_p, B_q contain interior points. More specially, there exists $\delta > 0$ and $u_0 \in B_e$, such that

$$|\mu_n(u) - \mu_m(u)| \leqslant \epsilon,$$

whenever $\|u - u_0\| \leqslant \delta$ and $n, m \geqslant q$. We notice that, for $u \in B_e$, we have that $u_0 - u_0 \wedge v = (e - u_0 \wedge v) \wedge u_0 \in B_e$. Thus the additivity of μ_n implies that

$$\mu_n(v) = \mu_n(u_0 \vee v) - \mu_n(u_0 - u_0 \wedge v),$$

for all $v \in B_e$ and $n \in \mathbb{N}$ and hence

$$\mu_n(v) = \mu_q(v) + [\mu_n(u_0 \vee v) - \mu_q(u_0 \vee v)] - [\mu_n(u_0 - u_0 \wedge v) - \mu_q(u_0 - u_0 \wedge v)].$$

Since $\|v\| \leqslant \delta$, we have that $\|u_0 \vee v - u_0\| < \delta$ and $\|u_0 \wedge v\| \leqslant \delta$. Thus we obtain that

$$|\mu_n(v) - \mu_q(v)| \leqslant 2\epsilon,$$

whenever $\|v\| \leqslant \delta$ and $n \geqslant q$. So the sequence (μ_n) is equicontinuous at $0 \in B_e$. Furthermore, the symmetric difference $u\Delta u_1$ belongs in B_e and hence from the disjoint decompositions $u = u \wedge u_1 + v$ and $u_1 = u \wedge u_1 + v_1$ we obtain, by the triangle inequality and the addivity, that $|\mu_n(u) - \mu_n(u_1)| \leqslant |\mu_n(v) + \mu_n(v_1)|$ and hence

$$|\mu_n(u)\mu_n(u_1)| \leqslant 2 \sup_{v \preccurlyeq u\Delta u_1} |\mu_n(v)|.$$

This implies that

$$|\mu_n(u) - \mu_n(u_1)| \leqslant 2\epsilon + \sup_{v \preccurlyeq u\Delta u_1} |\mu_q(v)|,$$

whenever $n \geqslant q$ and $\|u - u_1\| \leqslant \delta$, where $u, u_1 \in B_e$. Therefore, the sequence (μ_n) is uniformly equicontinuous, which implies the continuity of the limits function $u \mapsto \mu(u) := \lim_n \mu_n(u)$. $\square$

Definition 3.2.17. Let X be a non void set. Then X is of *σ-finite measure*, when X can be written as the union of countably many measurable sets of finite measure.

Corollary 3.2.18 (Vitali-Hahn-Sacks)**.** *Let (X, Σ, μ) be a σ-finite measure space, and let (λ_n) be a sequence of finite (countably additive) measures on Σ, which are absolutely continuous with respect to μ. If $\lim_n \lambda_n(S) =: \lambda(S) \in \mathbb{R}$, for each $S \in \Sigma$, then λ is a measure absolutely continuous with respect to μ and the countable additivity of the measures λ_n is uniform with respect to $n \in \mathbb{N}$.*

Proof. First, we notice that $L^1(\mu)$ contains weak order units u and the Boolean Algebra B_u is isomorphic to Σ/N, where N denotes the μ-null sets in Σ. If the mapping $h : \Sigma \to \Sigma/N$ is the canonical map, then $\lambda \mapsto \lambda' \circ h$ is a bijection of the additive real function on B_u onto the set of additive real function on Σ, vanishing on each $\Sigma \in N$. Furthermore, we observe that λ is continuous on B_u with respect to the topology of V if and only if λ' is countably additive on Σ. Theorem 3.2.16 states that the family Γ of such functions on B_u is uniformly equicontinuous if and only if the additivity of $\lambda' = \lambda \circ h$ is uniform, for $\lambda \in \Lambda$. □

Definition 3.2.19. Let ν be a signed measure and μ be a positive on $(X.\Sigma)$. We say that ν is *absolutely continuous* with respect to μ, if

$$\nu(S) = 0 \text{ whenever } S \in \Sigma \text{ and } \mu(S) = 0.$$

Theorem 3.2.20 (Radon-Nikodym)**.** *Let (X, Σ, μ) be a σ-finite measure space. Then the mapping $g \mapsto \phi_g$, where*

$$\phi_g(f) = \int fg d\mu,$$

is an isomorphism of the Banach lattice $L^\infty(\mu)$ onto the dual of $L^1(\mu)$. Furthermore, suppose that u is any weak order unit of $L^1(\mu)$ and λ is an order bounded linear form on $L^\infty(\mu)$. Then the following assertions are equivalent:

(i) λ is order continuous.

(ii) λ is sequentially order continuous.

(iii) The restriction of λ to the unit ball of $L^\infty(\mu)$ is continuous for the topology of the uniform convergence on $[-u, u] \subset L^1(\mu)$.

(iv) λ is $\sigma(L^\infty, L^1)$-continuous.

Proof. Consider the measure $u.\mu$ instead of μ assuming that μ is finite and u to be the constant function e, equal to one, on X. Then, the Hilbert space $L^2(\mu)$ can be viewed as a dense ideal of $L^1(\mu)$ and hence the strong dual of $L^1(\mu)$ can be identified with a dense ideal of $L^2(\mu)'$, which is $L^2(\mu)$, where the canonical bilinear form is given by

$$(f, g) \mapsto \int_S fg d\mu.$$

Under this identification, the unit ball of V' is clearly the order interval $[-e, e]$, which shows V' to be isomorphic to $L^\infty(\mu)$.

(i) Let λ be a continuous order bounded linear form on $L^\infty(\mu)$. Since $L^1(\mu)$ can be identified under evaluation with the band of all order continuous linear forms on L^∞, by Proposition 3.2.4, it follows that λ is $\sigma(L^\infty, L^1)$-continuous.

(ii) Suppose that λ is $\sigma(L^\infty, L^1)$-continuous. In the unit ball of V namely $[-e, e] \in L^\infty$ we notice that the weak topology generated by the finite subsets of the form $[-e, e] \subset L^1(\mu)$ agrees with the $\sigma(L^\infty, L^1)$ as the former is a coarser Hausdorff topology and the unit ball is weakly compact with respect to the duality $\langle L^\infty, L^1 \rangle$. Hence (iv) implies (iii).

(iii) The uniform convergence topology $\mathbf{T}$ is the topology of $L^\infty(\mu)$ induced by L^1 as L^∞ is considered a subspace of L^1. Hence, by Proposition 3.2.4, it follows that every order convergent filter on V is $\mathbf{T}$-convergent and the implication follows.

(iv) Since μ defines a strictly positive linear form on $L^\infty(\mu)$, it follows from Corollary 2.1.47, that $L^\infty(\mu)$ is super Dedekind complete.

□

Corollary 3.2.21 (Radon-Nikodym). *Let (X, Σ, μ) be a σ-finite measure space, and let ν be a finite measure on (X, Σ), which is absolutely continuous, with respect to μ. Then there exists a unique class $[f]$ of $f \in L^1(\mu)$, such that for all $S \in \Sigma$*

$$\nu(S) = \int_S f d\mu.$$

Proof. The absolute μ-continuity of ν implies that ν defines an order bounded linear form on $L^\infty(\mu)$. Since each sequence in L^∞ converges if and only if it is bounded and converges a.e. (μ), it follows that ν is sequentially order continuous on $L^\infty(\mu)$ □

Definition 3.2.22. Let μ be a measure and $\{U_k\}_{k=1}^n$ be any countable disjoint collection of sets. Then μ is said to be *countably additive*, if it satisfies

$$\mu(\bigcup_{k=1}^\infty U_k) = \sum_{k=1}^\infty \mu(U_k).$$

Proposition 3.2.23. *Let (X, Σ, μ) be a σ-finite measure space, and let $V := L^1(\mu)$. Then the following hold:*

(i) V is weakly sequentially complete.

(ii) For a relatively weakly compact subset A of V, it is necessary and sufficient that A be bounded and that the measures $\nu_f : S \to \int_S f d\mu$, where $(S \in \Sigma)$ be uniformly countably additive for $f \in U$.

(iii) If $A \subset V$ is relatively weakly compact, then so is the solid hull $\bigcup_{f \in A} [-|f|, |f|]$.

Proof. (i) Let (f_n) be a weak Cauchy sequence in V and consider the sequence (ν_n) of finite measures on Σ defined by $\nu_n(S) := \int_S f_n d\mu$. We observe that, for each n, the measures ν_n are absolutely continuous with respect to μ. Since $\nu_n(S) = \langle f_n, \chi_S \rangle$, $S \in \Sigma$, where $\chi_S \in L^\infty(\mu)$ is the characteristic function of S. Then it follows from the Vitali-Hahn-Sacks Theorem, that $S \mapsto \nu(S) := \lim_n \nu_n(S)$ is a measure absolutely continuous, with respect to μ. Hence by the Radon-Nikodym theorem we have that $\nu(S) = \int_S f d\mu$, for a unique $f \in L^1(\mu)$. But, (f_n) defines a bounded sequence of linear forms on L^∞ convergent to f on the total subset $\{\chi_S : S \in \Sigma\}$. Therefore, $\lim_n f_n = f$ weakly in V.

(ii) Let A be bounded subset of V. If A is relatively weakly compact, it follows from Eberlein's Theorem 3.2.24, that each sequence in A contains a weakly convergent subsequence. Thus if the countable additivity of the measure ν_f were not uniform in A, it would not be uniform on a weakly convergent sequence in A, which is a contradiction to the Vitali-Hahn-Sacks Theorem 3.2.18.

Conversely suppose that ν_f are uniformly countably additive for $f \in A$. Since A is bounded, it follows that A is relatively $\sigma(V'', V')$-compact as a subset of V''. Let h be any point in $\sigma(V'', V')$-closure of $\bar{A}$. Then $\nu_h(S) = \langle h, \chi_S \rangle$ defines a measure on Σ, which is countably additive because of the uniform countable additivity of ν_f $(f \in A)$. Since ν_h is absolutely continuous with respect to μ, it follows from the Radon-Nikodym Theorem that $h \in L^1(\mu)$. Therefore, $\bar{A} \subset V$ and thus A is relatively weakly compact.

(iii) Suppose $A \subset V$ is relatively $\sigma(V', V)$-compact. To prove that the given solid hull is relatively weakly compact, it is enough to prove that $|A| := \{|f| : f \in A\}$ is relatively weakly compact. If $|A|$ is not relatively $\sigma(V', V)$-compact, then there would exist a sequence $(f_n) \in A$ such that the sequence of measures $S \mapsto \int_S |f_n| d\mu$ is not uniformly countably additive. This implies that there exists a decreasing sequence $(S_n) \in \Sigma$, such that $\cap_n S_n \neq \emptyset$ and an $c > 0$, such that $\int_S |f_n| d\mu \geqslant c$ for infinitely many $n \in \mathbb{N}$. Define $S_n^+ := \{t \in S_n : f_n(t) \succcurlyeq 0\}$ and $S_n^- := \{t \in S_n : f_n(t) \prec 0\}$. Then one of $\left(\int_{S_n^+} f_n d\mu\right)$ and $\left(\int_{S_n^-} f_n d\mu\right)$ can not be a null sequence. Indeed, we assume that $\int_{S_n^+} f_n d\mu \geqslant \epsilon/2$, for all $n \in \mathbb{N}$. Now $\lim_n(S_n) = 0$ implies $\lim(S_n^+) = 0$ and this contradicts the uniform countable additivity of measures ν_{f_n} , (f_n) being realtively $\sigma(V, V')$-compact. Therefore, $|A|$ is relatively $\sigma(V', V)$-compact, and it is now clear that the same holds for the solid hull of A.

□

Theorem 3.2.24 (Eberlein). *Let V be a Banach space. Then each weakly countably compact subset is weakly compact and weakly sequentially compact.*

Corollary 3.2.25. *Every AL-space V is weakly sequentially complete. Furthermore, suppose $U \subset V$ is relatively weakly compact, then so is the solid hull $\bigcup_{x \in A}[-|x|, |x|]$.*

Proof. We notice, by Eberlein's Theorem 3.2.24, that weak compactness of subsets of V is equivalent to weak sequential compactness. Although we observe that every sequence $(x_n) \in V$ is contained in some AL-subspace possessing a weak order unit.

For example, we can take $(\overline{V_u}), x := \sum |x_n|/2^n\|x_n\|$. Thus, Theorem 3.2.9 reduces the proof to the case $V = L^1(\mu)$, where μ is a Radon measure on a compact space and the statement is imminent. □

Proposition 3.2.26. *Let V be an AL-space. Then every order interval is weakly compact.*

Proof. Since V is an AL-space, it follows from Theorem 3.2.9 that $V \cong L^1(\mu)$ for a locally compact space X and a Radon measure μ. We denote the isomoprhism map by i. Let $x, y \in V$, such that the order interval $[-x, y] \subset V$. It follows that $i|_{[-x,y]} \in L^1(\mu)$. Thus, it suffices to prove that every order interval of the previous form is indeed weakly compact in $L^1(\mu)$.

Suppose that there exists a convergent sequence (x_n) to $x \in i|_{[-x,y]}$ and $k_n, n \in \mathbb{N}$, such that the subsequence (x_{k_n}) is not weakly convergent to a point in $i|_{[-x,y]}$. This implies that there exists a function $f \in X^\star$, such that $\|f(x_{k_n}) - f\|$ does not tend to 0. This is a contradiction, as $L^1(\mu)$ is complete with respect to the norm of $L^1(\mu)$. Therefore, $i|_{[-x,y]} \in L^1(\mu)$ is weakly compact and, as a consequence, the order interval $[-x, y] \in V$. □

Examples 3.2.27. (i) Let X be a locally compact space and μ a Radon measure on X. Then the representation space $V = L^1(\mu)$,from Theorem 3.2.9,is not in general homeomorphic to X.

(ii) Let (X, Σ, μ) be a finite measure space. Denote by N the ideal of μ-null sets in Σ. Then the Boolean Algebra $(\Sigma/N, \rho)$ is a complete metric space under the metric ρ defined as follows:

$$\rho(S_1, S_2) = \mu(S_1 \Delta S_2).$$

Suppose u is a weak order unit of $V = L^1(\mu)$, then $(\Sigma/N, \rho)$ induces a sequence of measures μ for every pair of $S_1, S_2 \in \Sigma/N$, such that $(\mu_n)_{n\in\mathbb{N}}$ satisfies the hypothesis of Theorem 3.2.16. Therefore, there exists an isomorphism from $(\Sigma/N, \rho)$ to B_u. It follows from Corollary 3.2.11, that V is separable if and only if $(\Sigma/N, \rho)$ is separable.

(iii) In Proposition 3.2.26 we showed that every order interval, in an AL-space, is weakly compact. But in general, even norm compact subsets of V are not order bounded. Suppose $V = L^1(\mu)$, where μ is a Lebesque measure on $\mathbb{R}$. Now let $(U_n)_{n\in\mathbb{N}}$ be a sequence of disjoint Borel sets, such that $\mu(U_n) = n^{-1}$. Taking the sequence of the characteristics functions over these sets (U_n), we observe that is a null sequence, which is not order bounded, as there exists not real compact interval containing the images of the characteristic functions.

3.3 Duality of AM and AL spaces

We will conclude this chapter stating representation and extension theorems of AL-spaces.

Proposition 3.3.1. *The dual of each AM-space is an AL-space and the dual of each AL-space is an AM-space with unit.*

Proof. (i) Suppose first that V is an M-normed space. Then in view of Proposition 2.2.37, it suffices to show that $\|x' + y'\| \geqslant \|x'\| + \|y'\|$, for $x', y' \in V'_+$. Now given $\epsilon > 0$, it follows, by Remark 2.2.45, that there exist elements $x, y \in V_+$, such that $\|x\| = \|y\| \leqslant 1$ and $\langle x, x'\rangle > \|x'\| - \epsilon/2$, $\langle y, y'\rangle > \|y'\| - \epsilon/2$. If $z := x \vee y$, then $\|z\| \leqslant 1$, since V is M-normed and this implies

$$\|x' + y'\| \geqslant \langle x' + y'\rangle \geqslant \langle x, x'\rangle + \langle y, y'\rangle > \|x'\| + \|y'\| - \epsilon$$

by triangle inequality. This verifies that the norm of V' is additive in V'_+.

(ii) Conversely, if V is an L-normed space and if $z = u_1 - v_1 = u_2 - v_2$ are any two decompositions of $z \in V$, into different elements, then the norm of V is additive on V_+. Furthermore, $e(z) := \|u_1\| - \|v_1\|$ or $e(z) = \|u_2\| - \|v_2\|$ yields a positive linear form e on V, for which $e(|z|) = \|z\|$, for $z \in V$. Evidently, e is a continuous linear form $f \in V'$ of norm equal or less than 1, which satisfies $|f(z)| \leqslant e(|z|)$ or, equivalently, $-e \leqslant f \leqslant e$. Now, it follows from Proposition 3.1.4, that V is an AM-space with unit e.

□

Remark 3.3.2. If V is an AL-space, then V' is an order complete AM-space with unit and hence, by Theorem 3.1.9 and Corollary 3.1.28, isomorphic to some $C(K)$, for a compact Stonian space K.

Theorem 3.3.3 (Lotz)**.** *Let V be a Banach lattice and V_0 a Banach sublattice. Denote by E an AL-space. Then every positive linear map $T_0 : V_0 \to E$, has a positive extension $T : V \to E$, such that $\|T_0\| = \|T\|$.*

Proof. By Proposition 3.3.1, it follows that V' is an order complete AM-space with unit e, where $e' \in V'$ is given by $e(x) = \|x^+\| - \|x^-\|$. In fact, the unit $[-e, e]$ is the unit ball of V'. Thus, the adjoint map $T'_0 : V' \to V'_0$ carries V' into the ideal $(V'_0)_{e_2}$, where $e_2 := T'e$ and $\|e_2\| = \|T'\ e\| = \|T'\| = \|T\|$. It follows by Proposition 2.2.42, that e_2 has a norm preserving extension in V'_+ namely e_1, such that $\|e_1\| = \|e_2\|$.

Now, consider the adjoint operator τ' of the canonical injection $\tau : V_0 \to V$. Then τ' is a positive linear map from V' onto V'_0, such that $\tau[0, x'] = [0, \tau x']$ for all $x \in V'$. Moreover, since every directed, majorized subset A of V' converges to $\sup A$ for $\sigma(V', V)$, and τ' is continuous for $\sigma(V', V)$ and $\sigma(V'_0, V_0)$, it follows that $j(\sup A) = \sup j(A)$.

The ideals $(V')_{e_2}$ and $(V'_0)_{e_2}$ are order complete AM-spaces with respective units e_1 and e_2. Thus, they can be identified with $C(K_1)$ and $C(K_2)$ respectively, for suitable compact Stonian spaces K_1 and K_2. Now, the restriction τ'_0 of τ' to $(V')_{e_1} \cong C(K_1)$

satisfies the hypothesis of Theorem 3.1.43, as the order continuity of τ' implies that the absolute kernel of τ_0' is a projection band. As a consequence, Theorem 3.1.43 implies that there exists an isometric lattice homomorphism $S : (V_0')_{e_2} \to (V')_{e_1}$, such that $Se_2 = e_1$ and $\tau_0' \circ S$ is the identity map of $(V_0')_{e_2}$. The composition $S \circ T_0'$ maps V' into G' is positive and has norm $\|S \circ T_0'\| = \|T_0'\|$. Moreover $j \circ S \circ T_0' = T_0'$. This implies $T_0'' = T_0'' \circ S' \circ j'$ and if P denotes the band projection $V'' \to V$, then the restriction T of $P \circ T_0'' \circ S'$ to G is a positive linear norm preserving extension of T_0. $\square$

Corollary 3.3.4. *Let V be a Banach lattice and V_0 a Banach sublattice of V isomorphic to an AL-space. Then there exits a contractive projection map T such that $R(T) = V_0$.*

Proof. Since V_0 is a Banach sublattice isomorphic to an AL-space, it follows that V_0 is closed in V and hence the identity mapping is a norm preserving projection. $\square$

Corollary 3.3.5. *Let V be a Banach lattice and V_0 a Banach sublattice isomorphic to an AL-space. Then every continuous linear map $T_0 : V_0 \to E$ into a Banach space E, has an extension $T : V \to E$, such that $\|T_0\| = \|T\|$.*

Proof. Suppose H is ordered and without loss of generality suppose that $T_0 \succcurlyeq 0$. Then T can be chosen positive, by Proposition 2.2.42. Therefore, by Theorem 3.3.3, it follows that there exists such extension. $\square$

Definition 3.3.6. Let K be a compact Stonian space. Then K is called *hyperstonian*, if the band N of all order continuous Radon measures (*normal measures*) on K separates $C(K)$. Equivalently, one can state that K is *hyperstonian*, if the union of the supports of all $n \in \mathbb{N}$ is dense in K.

We obtain the following Theorem of Dixmier concerning hyperstonian spaces.

Theorem 3.3.7 (Dixmier)**.** *$C(K)$ is isomoprhic to the dual of an L-space if and only if K is hyperstonian.*

Proof. Let K be a compact and Stonian space. Suppose that $V' = C(K)$. It follows from Proposition 3.3.1 and Theorem 3.1.9, that V is an L-normed space. Moreover, every element x defines, through the evaluation map, $q : V \to V''$, an order continuous Radon measure. Now, the set of all order continuous Radon measure, N is a band, by Proposition 2.1.14. Thus, by Theorem 3.1.8, it follows that N separates the points of $C(K)$ and therefore K is hyperstonian.

Let K be a hyperstonian space. Suppose that $V = N$, where N is the normal measures on K. It follows that V is a band in $C(K)$ and hence a Banach Lattice itself. Since V separates the points of $C(K)$, by hypothesis, it follows, from Proposition 3.2.4 and Nakano's Theorem 2.1.56, that $C(K)$ is a dense ideal in V'. Since the unit of $C(K)$ accounts for the unit of V', it follows that $V' = C(K)$

$\square$

Theorem 3.3.8 (Mackey-Arens)**.** *Let $\langle F, G\rangle$ be a given duality. Suppose $\mathcal{T}$ is a locally convex topology on F. Then $\mathcal{T}$ is consistent with $\langle F, G\rangle$ if and only if $\mathcal{T}$ is the $\mathcal{G}$-topology for a saturated class $\mathcal{G}$, which covers G of $\sigma(G, F)$-relatively compact subsets of G.*

Theorem 3.3.9. *Let V be any AL-space. Then there exists a compact hyperstonian space K, unique to isomorphism, such that V can be identified with the AL-space of all order continuous Radon measures on K.*

Proof. We notice, by Proposition 3.3.1, that V' can be identified with $C(K)$ for some compact Stonian space K. Moreover, Proposition 3.2.4, implies that V is mapped onto the band of all order continuous Radon measures on K, through evaluation. Now let K' be any hyperstonian space, such that there exists an isomorphism $i : V \to N(K')$ onto the AL-space $N(K')$ of order continuous Radon measures on K'. Moreover, since $\langle C(K'), N\rangle$ is a separated duality by hypothesis, it follows from the Mackey-Arens Theorem 3.3.8, that $N(K')$ is a Banach predual to $C(K')$. Thus, the adjoint i is an isomorphism of $C(K')$ to $C(K) \cong V'$, which shows that K' is homeomorphic with K in view of Corollary 3.1.11. □

Corollary 3.3.10. *Every band in the dual of an AL-space is $\sigma(V', V)$-closed.*

Proof. Let B be a band of V'. Since $V' = C(K)$ is order complete, it follows from Theorem 1.2.53 that B is a projection band. Now, if P_B denotes the associated band projection and e denotes the unit of V', then $B = P_B(V)$ and B is an AM-space with unit $e_1 := P_B e$, where e_1 is the characteristic function of an open-and-closed subset of K. It follows that the adjoint $P'_B : M(K) \to M(K)$ of P_B is mapping $\mu \mapsto e_1.\mu$ and since $e_1.\mu$ is an order continuous measure on K, whenever μ is continuous, it follows that leaves V invariant. Therefore, P_B is $\sigma(V', V)$-continuous and B is $\sigma(V', V)$-closed. □

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