UNIVERSITY OF IOANNINA

Department of Mathematics

Konstantinos Liampis

LOCALIZATION THEORY in HIGHER HOMOLOGICAL ALGEBRA and TRIANGULATED CATEGORIES

DOCTORAL THESIS

Ioannina, 2023

This research is co-financed by Greece and the European Union (European Social Fund- ESF) through the Operational Programme "Human Resources Development, Education and Lifelong Learning" in the context of the project "Strengthening Human Resources Research Potential via Doctorate Research - 2nd Cycle" (MIS-5000432), implemented by the State Scholarships Foundation (IKY).

Ευρωπαϊκή Ένωση
European Social Fund

Operational Programme
Human Resources Development, Education and Lifelong Learning
Co-financed by Greece and the European Union

ΕΣΠΑ 2014-2020
ανάπτυξη - εργασία - αλληλεγγύη

Dedicated to my Professor,

Prof. Apostolos Beligiannis

The present dissertation was carried out under the Ph.D. program of the Department of Mathematics of the University of Ioannina in order to obtain the degree of Doctor of Philosophy.

Accepted by the three members of the advisory committee on 14 September 2023:

1. Apostolos Beligiannis (Supervisor, Professor, University of Ioannina)
2. Apostolos Thoma (Emeritus Professor, University of Ioannina)
3. Stavros Papadakis (Associate Professor, University of Ioannina)

Accepted by the seven members of the evaluation committee on 14 September 2023:

1. Apostolos Beligiannis (Supervisor, Professor, University of Ioannina)
2. Apostolos Thoma (Emeritus Professor, University of Ioannina)
3. Stavros Papadakis (Associate Professor, University of Ioannina)
4. Epaminondas Kechagias (Professor, University of Ioannina)
5. Ioannis Emmanouil (Professor, National and Kapodistrian University of Athens)
6. Hara-Myrto-Agapi Charalambous (Professor, Aristotle University of Thessaloniki)
7. Chrysostomos Psaroudakis (Associate Professor, Aristotle University of Thessaloniki)

Statutory Declaration

“I lawfully declare here with statutorily that the present dissertation was carried out under the international ethical and academical rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis."

Konstantinos Liampis

Acknowledgements

First, I would like to express my sincere gratitude to my supervisor Professor Apostolos Beligiannis for his invaluable guidance throughout the years of my research. I am deeply grateful for his support and encouragement as well as his invaluable advice and insights. It has been an honour to have studied under Professor Beligiannis' guidance.

Moreover, I would like to extend my cordial thanks to the other two members of the advisory committee, Professor Stavros Papadakis and Professor Apostolos Thoma whose contribution both to this research and my general academic journey has been exceptional. I am deeply grateful for the opportunity I had to learn from them and the vital insights they have offered me throughout these years.

I would also like to thank the other members of the evaluation committee: Professor Epaminondas Kechagias from University of Ioannina, Professor Ioannis Emmanouil from National and Kapodistrian University of Athens, Professor Hara-Myrto-Agapi Charalambous from Aristotle University of Thessaloniki and Professor Chrysostomos Psaroudakis from Aristotle University of Thessaloniki for their contribution. Especially, I would like to thank Professor Emmanouil and Professor Psaroudakis for their helpful comments and constructive remarks.

In addition, I would like to thank my fellow colleagues and friends: Theodoros Kasioumis, Vasiliki Petrotou, Sofia Tsouri, Dimitris Michailidis, Sokratis Zikas, Christos Onti, Kleanthis Polymerakis, as well as the other graduate students for their friendship and our fruitful conversations.

Further more, I would like to express my deepest gratitude to my parents, Lampros Liampis and Moustakli Agni as well as my brother Alexandros, for their unwavering support throughout my academic journey.

Finally, I would like to thank the State Scholarships Foundation (IKY) for their generous financial support through the following scholarship programme:

This research is co-financed by Greece and the European Union (European Social Fund- ESF) through the Operational Programme "Human Resources Development, Education and Lifelong Learning" in the context of the project "Strengthening Human Resources Research Potential via Doctorate Research - 2nd Cycle" (MIS-5000432), implemented by the State Scholarships Foundation (IKY).

Konstantinos Liampis
September 2023

Abstract

The aim of this thesis is to develop a localization theory in the higher homological algebra of n-abelian and n-angulated categories. Analogously to the classical cases of an abelian or a triangulated category, we start with a class of morphisms S which satisfies the natural conditions of a bicalculable system of morphisms. In the higher homological setting of an n-abelian category $\mathcal{M}$ or n-angulated category $\mathcal{C}$, we construct in a universal way a localized n-abelian category $\mathcal{M}[S^{-1}]$ or n-angulated category $\mathcal{C}[S^{-1}]$, respectively where the morphisms in S have been inverted. This solves satisfactory the problem of localizing an n-abelian or n-angulated category.

The thesis is divided in four chapters.

Chapter 1 consists of a brief introduction to Higher Homological Algebra and to general localization theory of categories in the form of calculus of fractions, while simultaneously aiming at fixing notation. The chapter is divided into three sections, with the first two containing definitions, preliminary notions and results concerning n-abelian and n-angulated categories respectively. In the third section we provide a brief overview of the Localization Theory in a general category, construct the category of fractions and state some results which are needed in the rest of the thesis.

The main purpose of Chapter 2 is to develop the tools which will be used in the proof of the main result concerning the localization of n-abelian categories. Due to the delicate nature of the axiom of idempotent completeness of an n-abelian category with respect to localization, the construction of the localized category will be completed in two steps. To this end, we define a pre-n-abelian category to be an additive category which satisfies all axioms of an n-abelian category except that the axiom of idempotent completeness is not necessarily satisfied. Then, utilizing a result from Jasso in [28], we provide a necessary and sufficient condition for a category to be pre-n-abelian, based

on the exactness properties of a diagram which we will call an n-diagram and captures all desired information.

In Chapter 3, we prove our first main result of the thesis, constructing in two steps the localization of an n-abelian category with respect to a bicalculable system of morphisms. In the first step, utilizing the n-diagram, we show that the localization of a pre-n-abelian category with respect to a bicalculable system of morphisms is also a pre-n-abelian category and the localization functor is n-exact. In the second step, we prove that the idempotent completion of a pre-n-abelian category is an n-abelian category. Finally, combining the above, for any n-abelian category $\mathcal{M}$ and any bicalculable system of morphisms S in $\mathcal{M}$ we consider first the pre-n-abelian category $\mathcal{M}[S^{-1}]$ and then its idempotent completion, $\widetilde{\mathcal{M}[S^{-1}]}$ which is n-abelian. Then, the composite functor $\mathcal{M} \longrightarrow \mathcal{M}[S^{-1}] \longrightarrow \widetilde{\mathcal{M}[S^{-1}]}$, where the second functor denotes idempotent completion, is universal among all S-inverting n-exact functors out of $\mathcal{M}$ to an n-abelian category.

In Chapter 4, we present an analogous result for n-angulated categories, where $n \geqslant 3$. In this setting we need a bicalculable class of morphisms S which, as in the classical case $n = 3$ of triangulated categories, satisfies a compatibility condition with respect to the n-angulation. We define a class N_S of n-angles in $\mathcal{C}[S^{-1}]$ as following: An n-Σ-sequence $X^\bullet$ in $\mathcal{C}[S^{-1}]$ is in N_S, if there exists an n-Σ-sequence $M^\bullet$ and an isomorphism of n-Σ-sequences $\phi^\bullet \colon A^\bullet \xrightarrow{\sim} X^\bullet \oplus M^\bullet$, where $A^\bullet$ is an n-angle in $\mathcal{C}$. Then, we prove our second main result of the thesis, constructing the localization of an n-angulated category $\mathcal{C}$ which is also an n-angulated category that satisfies the analogous universal property.

Contents

Introduction

Higher Homological Algebra

Maurice Auslander in his famous monograph [2] proved the existence of a natural 1-1 correspondence between equivalence classes (in the sense of Morita), of representation finite Artin algebras Λ and Artin algebras Γ with $\mathsf{gl.dim}\,\Gamma \leqslant 2 \leqslant \mathsf{dom.dim}\,\Gamma$. In this case, the category of projective Γ-modules is abelian, equivalent to the category of Λ-modules and any Artin algebra Γ for which the category of projective Γ-modules is abelian, satisfies the aforementioned homological condition. This result is known today as "Auslander Correspondence" and the Artin algebras Γ with $\mathsf{gl.dim}\,\Gamma \leqslant 2 \leqslant \mathsf{dom.dim}\,\Gamma$ are called Auslander algebras. Auslander's Correspondence played a key role in Representation Theory, since it offered a homological criterion to the classification of representation finite algebras.

In 2007, Iyama in [22] generalized Auslander's correspondence, proving that for any $n \geqslant 0$, there exists a bijection between the set of Morita-equivalence classes of Artin algebras Γ with $\mathsf{gl.dim}\,\Gamma \leqslant n+2 \leqslant \mathsf{dom.dim}\,\Gamma$ and equivalence classes of $(n+2)$-cluster tilting Λ-modules over Artin algebras Λ. This equivalence, known today as the Higher Auslander Correspondence or Auslander-Iyama Correspondence has been generalized by A. Beligiannis in [7] in the setting of abelian categories. Independently and during the same time, A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov in [10], introduced the notion of cluster-tilting modules or subcategories in the setting of triangulated categories in order to study the structure of cluster algebras, see [15], [16]. This lead to the study of cluster-tilting objects in the setting of categories of modules and more generally, in the setting of abelian and triangulated categories, see [27], [32], [33]. This theory has many important applications in various contemporary research fields in mathematics and mathematical physics: in Representation Theory, in Homological

Algebra, in (commutative and non-commutative) Algebraic Geometry in the theory of Teichmuller spaces, in the Theory of Calabi-Yau algebras, in Poisson geometry, in combinatorics etc. For more details we refer again the reader to [32].

Recently, the highly developing field of cluster tilting theory has led to the introduction of higher homological algebra where one of the main objects of interest is the study of n-abelian, for $n \geqslant 1$, and n-angulated categories, for $n \geqslant 3$. In the setting of triangulated categories, Geiss, Keller and Oppermann in their fundamental paper [20] introduced the notion of an n-angulated category which captures the exactness conditions of an $(n-2)$-cluster-tilting subcategory of a triangulated category and has important applications in Representation Theory, Algebraic Geometry, String Theory and other fields. More specifically, Geiss, Keller and Oppermann in [20] prove that any $(n-2)$-cluster-tilting subcategory of a triangulated category is an n-angulated category. On the other hand, G. Jasso in [28] in order to axiomatize the exactness properties of an n-cluster-tilting subcategory of an abelian category, introduced the notion of an n-abelian category. Especially, Jasso proved that any n-cluster-tilting subcategory of an abelian category is an n-abelian category while conversely, Kvamme and, independently, Ebrahimi and Nasr-Isfahani proved recently that any n-abelian category is equivalent to an n-cluster tilting subcategory of an abelian category, see [34], [12]. In this connection, for $n = 1$ or $n = 3$, respectively, we recover the classical notions of abelian or triangulated categories.

Higher Homological Algebra is nowadays in its prime, with connections to the generalized cluster-tilting Theory and to the Theory of Auslander-Iyama Correspondence, along with many interesting applications in Representation Theory, Algebraic Geometry, Combinatorics and other research fields.

Localization

Localization Theory is well-known to be an important tool for the study of several problems in algebra, topology and geometry. Originally developed for commutative rings, it has undergone gradual extensions, with many important applications to non-commutative rings and later to localization of spaces in topology and schemes in algebraic geometry. In several contexts, the appropriate framework for the study of the various types of localizations is the setting of abelian and triangulated categories. In this connection, Gabriel and Verdier in the early sixties studied the localization of abelian and triangulated categories respectively, laying the foundations for further developments, extensions and powerful applications.

More specifically, Gabriel in his pivotal paper [18] during the 1960, utilizing ideas first introduced by Serre, developed the localization theory of abelian categories proving that for any abelian category $\mathcal{A}$ and any multiplicative system of morphisms S in $\mathcal{A}$, the localization category $\mathcal{A}[S^{-1}]$ is also an abelian category. On the other hand, a few years later, Verdier in his influential paper [44] studied the localization of triangulated categories and proved that the localization of any triangulated category with respect to a multiplicative system of morphisms which is compatible with the triangulation, is also a triangulated category. The above results provided powerful tools for studying a wide range of problems and phenomena with one of the many applications being the construction of new abelian and triangulated categories.

Motivation and Aim of the Thesis

The importance of localization techniques in abelian and triangulated categories on the one hand, and the remarkable development of higher homological algebra in various thematic areas over the last decade on the other hand, provide strong motivation for the study of the localization theory in the setting of n-abelian and n-triangulated categories. In this thesis, starting from a class of morphisms S in an n-abelian or n-triangulated category satisfying some natural conditions, and using calculus of fractions in the sense of Gabriel-Zisman, we construct in a universal way a localized n-abelian or n-triangulated category respectively, where the morphisms in S have been inverted. This provides a satisfactory solution to the problem of localizing an n-abelian or n-triangulated category. When $n = 1$ or $n = 3$, respectively, we recover the classical results of the localization of an abelian or triangulated category.

Structure of the Thesis

The thesis is divided in four chapters.

Chapter 1 contains preliminary notions on Higher Homological Algebra and Localization Theory. The chapter is divided into three sections: We begin by defining n-cokernels, n-kernels and n-exact sequences, which leads to the definition, along with some basic properties, of an n-abelian category $\mathcal{M}$ as introduced by Jasso in [28]. In the second section, similarly, we present the definition of an n-angulated category as introduced by Geiss, Keller and Oppermann in [20], along with some related concepts. Finally, the third section consists of an introduction to general localization theory in the form of calculus of fractions, following Gabriel-Zisman in [19]. Starting with a category $\mathcal{C}$ and a bicalculable system of morphisms S in $\mathcal{C}$, we describe objects and morphisms in the localized category $\mathcal{C}[S^{-1}]$ as well as the localization functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$.

The main purpose of Chapter 2 is to develop the tools which will be used in the proof of the main result concerning the localization of n-abelian categories. Due to the delicate nature of the axiom of idempotent completeness of an n-abelian category with respect to localization, the construction of the localized category will be completed in two steps. In this connection, we define a pre-n-abelian category to be an additive category which satisfies all axioms of an n-abelian category except that the axiom of idempotent completeness is not necessarily satisfied. Then, based on the exactness properties of a diagram which captures all desired information and utilizing a result of Jasso from [28], we provide a necessary and sufficient condition for a category to be pre-n-abelian, see Proposition 2.3.2. This diagram, which we call a *compatible n-diagram* acts in our setting as a replacement for the image-coimage isomorphism appearing in the classic case of an abelian category.

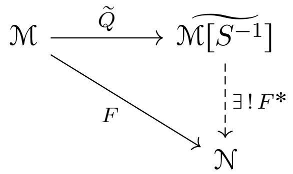
In Chapter 3, we prove our first main result of the thesis, constructing in two steps the localization of an n-abelian category with respect to a bicalculable system of morphisms. In more detail, let $\mathcal{M}$ be an n-abelian category and S a bicalculable system of morphisms in $\mathcal{M}$. We denote by $\mathcal{M}[S^{-1}]$ the localization of $\mathcal{M}$ at S in the sense of Gabriel-Zisman [19], (see also [41]).

Theorem. *There exists an n-abelian category $\widetilde{\mathcal{M}[S^{-1}]}$ and an n-exact functor*

$$\widetilde{Q} : \mathcal{M} \longrightarrow \widetilde{\mathcal{M}[S^{-1}]}$$

such that $\widetilde{Q}(s)$ is invertible, for any $s \in S$, satisfying the following universal property:

- *for any n-abelian category $\mathcal{N}$ and any n-exact functor $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ such that $F(s)$ is invertible, for any $s \in S$, there exists a unique n-exact functor $F^* \colon \widetilde{\mathcal{M}[S^{-1}]} \longrightarrow \mathcal{N}$ such that the following diagram commutes*



In the first step, utilizing the n-diagram, we show that the localization of a pre-n-abelian category with respect to a bicalculable system of morphisms is also a pre-n-abelian category and the localization functor is n-exact. In the second step, we prove that the idempotent completion $\widetilde{\mathcal{M}}$ of a pre-n-abelian category $\mathcal{M}$ is an n-abelian category. Finally, combining the above, for any n-abelian category $\mathcal{M}$ and any bicalculable system of morphisms S in $\mathcal{M}$ we consider first the pre-n-abelian category $\mathcal{M}[S^{-1}]$ and then its idempotent completion, $\widetilde{\mathcal{M}[S^{-1}]}$ which is n-abelian. Then, the composite functor $\widetilde{Q} = \iota \circ Q$:

$$\widetilde{Q}\colon \mathcal{M} \xrightarrow{Q} \mathcal{M}[S^{-1}] \xrightarrow{\iota} \widetilde{\mathcal{M}[S^{-1}]}$$

where Q is the localization functor and ι is the inclusion functor in the idempotent completion, is universal among all S-inverting n-exact functors out of $\mathcal{M}$ to an n-abelian category.

In Chapter 4, we present an analogous result for n-angulated categories, where $n \geqslant 3$. In this setting, we need a bicalculable class of morphisms S which, as in the classic case $n = 3$ of triangulated categories, satisfies a compatibility condition with respect to the n-angulation. We define a class N_S of n-angles in $\mathcal{C}[S^{-1}]$ as following:

• An n-Σ-sequence $X^\bullet$ in $\mathcal{C}[S^{-1}]$ is in N_S, if there exists an n-Σ-sequence $M^\bullet$ and an isomorphism of n-Σ-sequences:

$$\phi^\bullet\colon A^\bullet \xrightarrow{\sim} X^\bullet \oplus M^\bullet$$

where $A^\bullet$ is an n-angle in $\mathcal{C}$.

Then, we prove our second main result of the thesis, constructing the localization of an n-angulated category $\mathcal{C}$ which is also an n-angulated category that satisfies the desired universal property:

Theorem. *Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be an n-angulated category and S a bicalculable class of morphisms in $\mathcal{C}$ which is compatible with n-angulation. Then the localization $\mathcal{C}[S^{-1}]$ of $\mathcal{C}$ at S carries a natural n-angulated structure and the localization functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ is n-exact and satisfies the following universal property:*

• *for any n-angulated category $\mathcal{D}$ and any n-exact functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that $F(s)$ is invertible, for any $s \in S$, there exists a unique n-exact functor $F^*\colon \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{Q} & \mathcal{C}[S^{-1}] \\ & \searrow^{F} & \downarrow{\scriptstyle \exists !\, F^*} \\ & & \mathcal{D} \end{array}$$

CHAPTER 1

Preliminary notions

The aim of this chapter is to briefly introduce the reader to Higher Homological Algebra and to the localization theory of general categories in the form of calculus of fractions. For a detailed introduction to elementary elements of Category Theory, we refer the reader to the book [31] by M. Kashiwara and P. Shapira, to the book [46] by Weibel, as well as the books [9] by Bland and [14] by Enochs-Jenda.

The chapter is divided into three sections. The first two sections contain definitions, preliminary notions and results concerning n-abelian and n-angulated categories respectively. In the third section we provide a brief overview of localization theory, construct the category of fractions and mention some results which are needed in the rest of the thesis.

Notation

We begin by fixing notation and introducing some general notions used throughout the thesis:

Composition of morphisms: If $\mathcal{C}$ is a category and $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ two composable morphisms in $\mathcal{C}$, we denote their composition as $fg\colon X \longrightarrow Z$, following the diagrammatic order. Consequently, a morphism: $X \longrightarrow Y \oplus Z$ is denoted by a matrix: $(f,\, g)$ where $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Z$. Dually a morphism: $X \oplus Y \longrightarrow Z$ is denoted by a matrix: $\binom{f}{g}$ or ${}^{t}(f, g)$, where $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$.

Complexes: Let $\mathcal{C}$ be an additive category. A (cochain) complex in $\mathcal{C}$ is denoted by $\mathcal{C}^{\bullet}$ and a morphism between two complexes $\mathcal{C}^{\bullet}, \mathcal{D}^{\bullet}$ is denoted by $f^{\bullet}\colon \mathcal{C}^{\bullet} \longrightarrow \mathcal{D}^{\bullet}$. The category of (cochain) complexes in $\mathcal{C}$ is denoted by $\mathsf{Ch}(\mathcal{C})$. The category of all non-negative complexes ($C^i = 0$, $\forall i < 0$) is denoted by $\mathsf{Ch}^{\geqslant 0}(\mathcal{C})$. Finally, we denote by

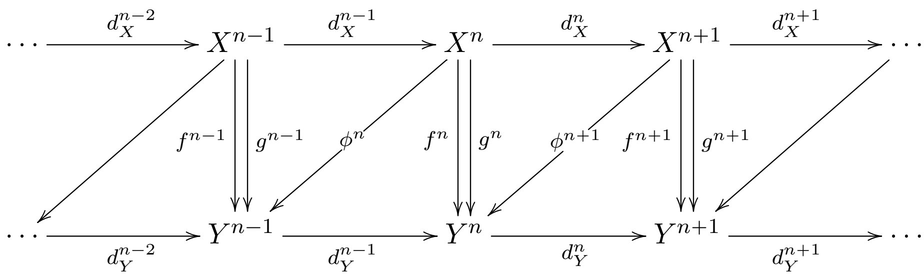
$\mathsf{Ch}^{\geqslant n}(\mathcal{C})$ the category of complexes which are concentrated in degrees $0, 1, \cdots, n+1$:

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1}$$

If $f^\bullet, g^\bullet \colon X^\bullet \longrightarrow Y^\bullet$ are two morphisms of complexes, an *homotopy*

$$\phi \colon f^\bullet \longrightarrow g^\bullet$$

is a collection of maps $\{\phi^n \colon X^n \longrightarrow Y^{n-1}\}_{n \in \mathbb{Z}}$:



such that:

$$f^n - g^n = \phi^n d_Y^{n-1} + d_X^n \phi^{n+1}$$

for all $n \in \mathbb{Z}$. In this case we say that $f^\bullet$ and $g^\bullet$ are *homotopic.*

An *homotopy equivalence* between $X^\bullet$ and $Y^\bullet$ is a pair of two morphisms of complexes $f^\bullet \colon X^\bullet \longrightarrow Y^\bullet$ and $g^\bullet \colon Y^\bullet \longrightarrow X^\bullet$ such that the composition $f^\bullet g^\bullet$ is homotopic to the identity morphism $1_{X^\bullet}$ and the composition $g^\bullet f^\bullet$ is homotopic to the identity morphism $1_{Y^\bullet}$.

Since further elaboration on the theory of complexes and homotopy equivalence is beyond the scope of the thesis, for additional details concerning the above as well as the homotopy category $\mathsf{H}(\mathcal{C})$ we refer the reader to [46, Chapter 1].

1.1 n-Abelian categories

The first aim of this section is to recall several preliminary notions mainly from [28], which are required in order to present Jasso's definition of an n-abelian category. Additionally, we will state some known auxiliary results which will be used in the proofs of our main results concerning the localization of n-abelian categories.

1.1.1 n-Cokernels and n-kernels

We begin by defining weak cokernels and weak kernels.

Definition 1.1.1. *Let $\mathcal{C}$ be an additive category and $f\colon X \longrightarrow Y$ a morphism in $\mathcal{C}$. A* **weak cokernel** *of f is a morphism $c\colon Y \longrightarrow C$ in $\mathcal{C}$, such that for all $C' \in \mathcal{C}$, the induced sequence of abelian groups:*

$$\mathcal{C}(C, C') \longrightarrow \mathcal{C}(Y, C') \longrightarrow \mathcal{C}(X, C')$$

is exact. In other words, c is a weak cokernel of f, if $fc = 0$ and for any morphism $c'\colon Y \longrightarrow C'$ such that $fc' = 0$, there exists a (not necessarily unique) morphism $u\colon C \longrightarrow C'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{c} & C \\ & & {\scriptstyle c'}\big\downarrow & \swarrow{\scriptstyle u} & \\ & & C' & & \end{array}$$

Remark 1.1.2. In the above definition, if the sequence:

$$0 \longrightarrow \mathcal{C}(C, C') \longrightarrow \mathcal{C}(Y, C') \longrightarrow \mathcal{C}(X, C')$$

is exact, or in other words, the induced morphism $u\colon C \longrightarrow C'$ is unique, c is called a *cokernel* of f.

The notion of a weak kernel is defined dually:

Definition 1.1.3. *Let $\mathcal{C}$ be an additive category and $f\colon X \longrightarrow Y$ a morphism in $\mathcal{C}$. A* **weak kernel** *of f is a morphism $k\colon K \longrightarrow X$ in $\mathcal{C}$, such that for all $K' \in \mathcal{C}$, the induced sequence of abelian groups:*

$$\mathcal{C}(K', K) \longrightarrow \mathcal{C}(K', X) \longrightarrow \mathcal{C}(K', Y)$$

is exact. In other words, k is a weak kernel of f, if $kf = 0$ and for any morphism $k'\colon K' \longrightarrow X$ such that $k'f = 0$, there exists a (not necessarily unique) morphism $u\colon K' \longrightarrow K$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & K' & & \\ & \swarrow{\scriptstyle u} & \big\downarrow{\scriptstyle k'} & & \\ K & \xrightarrow{k} & X & \xrightarrow{f} & Y \end{array}$$

Remark 1.1.4. As previously, if the sequence:

$$0 \longrightarrow \mathcal{C}(K', K) \longrightarrow \mathcal{C}(K', X) \longrightarrow \mathcal{C}(K', Y)$$

is exact, or in other words, the induced morphism $u\colon K' \longrightarrow K$ is unique, k is called a *kernel* of f.

We can now define n-kernels, n-cokernels and n-exact sequences:

Definition 1.1.5. *([28, Definition 2.2]) Let $\mathcal{C}$ be an additive category and $f\colon X \longrightarrow Y$ a morphism in $\mathcal{C}$. An n-**cokernel** of f is a sequence*

$$(c^1, c^2, \cdots, c^n)\colon \qquad Y \xrightarrow{c^1} C^1 \xrightarrow{c^2} \cdots \longrightarrow C^{n-1} \xrightarrow{c^n} C^n$$

of objects and morphisms in $\mathcal{C}$, such that for all $C' \in \mathcal{C}$ the induced sequence of abelian groups:

$$0 \longrightarrow \mathcal{C}(C^n, C') \longrightarrow \mathcal{C}(C^{n-1}, C') \longrightarrow \cdots \longrightarrow \mathcal{C}(Y, C') \longrightarrow \mathcal{C}(X, C')$$

is exact. In other words, for all $1 \leqslant i \leqslant n-1$ the morphism c^i is a weak cokernel of c^{i-1} (where we set $c^0 := f$), and the morphism c^n is a cokernel of c^{n-1}.

In this case, the sequence $(f, c^1, \cdots, c^n)$ is called a **right n-exact sequence***, see [37, Definition 2.4].*

The notions of *n-kernel* and *left n-exact sequence* are defined dually:

Definition 1.1.6. *([28, Definition 2.2]) Let $\mathcal{C}$ be an additive category and $f\colon X \longrightarrow Y$ a morphism in $\mathcal{C}$. An n-**kernel** of f is a sequence*

$$(k^n, k^{n-1}, \cdots, k^1)\colon \qquad K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{k^{n-1}} \cdots \longrightarrow K^1 \xrightarrow{k^1} X$$

of objects and morphisms in $\mathcal{C}$, such that for all $K' \in \mathcal{C}$ the induced sequence of abelian groups:

$$0 \longrightarrow \mathcal{C}(K', K^n) \longrightarrow \mathcal{C}(K', K^{n-1}) \longrightarrow \cdots \longrightarrow \mathcal{C}(K', X) \longrightarrow \mathcal{C}(K', Y)$$

is exact. In other words, for all $1 \leqslant i \leqslant n-1$ the morphism k^i is a weak kernel of k^{i-1} (where we set $k^0 := f$), and the morphism k^n is a kernel of k^{n-1}.

In this case, the sequence $(k^n, \cdots, k^1, f)$ is called a **left n-exact sequence***, see [37, Definition 2.4].*

Finally, n-kernels and n-cokernels lead to the definition of n-exact sequences:

Definition 1.1.7. *A sequence*

$$(f^0, f^1, \cdots, f^n): \qquad X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1}$$

is called an n-**exact sequence** *if it is both a left n-exact sequence and a right n-exact sequence.*

We finish this section with the following two lemmas from [28] which will be useful later.

Lemma 1.1.8. *([28, Comparison Lemma 2.1]) Let $\mathcal{C}$ be an additive category and $X^\bullet \in \mathsf{Ch}^{\geqslant 0}(\mathcal{C})$ a complex such that d_X^{i+1} is a weak cokernel of d_X^i for any $i \geqslant 0$. If $f^\bullet\colon X^\bullet \longrightarrow Y^\bullet$ and $g^\bullet\colon Y^\bullet \longrightarrow X^\bullet$ are morphisms of complexes in $\mathsf{Ch}^{\geqslant 0}(\mathcal{C})$ such that $f^0 = g^0$, then there exists a homotopy $h\colon f^\bullet \longrightarrow g^\bullet$ such that $h^1 = 0$.*

Lemma 1.1.9. *([28, Proposition 2.5]) Let $\mathcal{C}$ be an additive category, $X^\bullet, Y^\bullet$ two complexes in $\mathsf{Ch}^n(\mathcal{C})$ and $f^\bullet\colon Y^\bullet \longrightarrow X^\bullet$, $g^\bullet\colon X^\bullet \longrightarrow Y^\bullet$ two morphisms of complexes.*

$$\begin{array}{ccccccccccc}
Y^\bullet & & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \longrightarrow \cdots \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} \\
\downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} \\
X^\bullet & & X^0 & \xrightarrow{d_X^0} & X^1 & \longrightarrow \cdots \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
\downarrow{\scriptstyle g^\bullet} & & \downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g^1} & & \downarrow{\scriptstyle g^n} & & \downarrow{\scriptstyle g^{n+1}} \\
Y^\bullet & & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \longrightarrow \cdots \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1}
\end{array}$$

Then the following hold:

1. *If there exists an homotopy $\phi\colon f^\bullet g^\bullet \longrightarrow 1_{Y^\bullet}$, and $X^\bullet$ is a right (left) n-exact sequence, then $Y^\bullet$ is a right (left) n-exact sequence.*

2. *If there exists an homotopy $\psi\colon g^\bullet f^\bullet \longrightarrow 1_{X^\bullet}$, and $Y^\bullet$ is a right (left) n-exact sequence, then $X^\bullet$ is a right (left) n-exact sequence.*

3. *If the above diagram is an homotopy equivalence of complexes, then $X^\bullet$ is an n-exact sequence iff $Y^\bullet$ is an n-exact sequence.*

Remark 1.1.10. It should be noted that, while in [28, Proposition 2.5] only property (3) of the above lemma is stated, in the proof of the proposition it is explicitly demonstrated that property (1) also holds. In Lemma 1.1.9 we have also stated properties (1) and (2) separately since they will be useful later. For the proof, we direct the reader to the aforementioned source.

1.1.2 n-Pushout and n-pullback diagrams

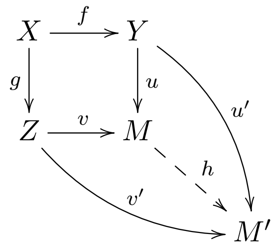
As we will see later, an important property of an n-abelian category is that it guarantees the existence of n-pushout and n-pullback diagrams, which are higher analogues of pushout and pullback diagrams. For the sake of completeness, we remind the reader of the classical definition of a pushout diagram:

Definition 1.1.11. *Let $\mathcal{C}$ be an additive category. A diagram of the form:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{v} & M \end{array}$$

is called a **weak pushout** *diagram, if it is commutative and satisfies the following property:*

- *for any other object M' and morphisms $u' \colon Y \longrightarrow M'$, $v' \colon Z \longrightarrow M'$ with $fu' = gv'$, there exists a (not necessarily unique) morphism $h \colon M \longrightarrow M'$ such that $u' = uh$ and $v' = vh$:*



Moreover, if the morphism h is unique, the diagram is called a **pushout** *diagram.*

The notions of a **weak pullback** *diagram and a* **pullback** *diagram are defined dually.*

Now we can define n-pushout and n-pullback diagrams:

Definition 1.1.12. *[28, Definition 2.11] Let $\mathcal{C}$ be an additive category, $X^\bullet$ a complex in $\mathsf{Ch}^{n-1}(\mathcal{C})$ and $f^0 \colon X^0 \longrightarrow Y^0$ a morphism in $\mathcal{C}$. An n-***pushout diagram** *of $X^\bullet$ along f^0 is a morphism of complexes:*

$$\begin{array}{ccccccccccc} X^\bullet & \quad & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n \\ \downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} \\ Y^\bullet & & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n \end{array}$$

such that the mapping cone $\mathsf{C} = \mathsf{C}(f^\bullet)$:

$$X^0 \xrightarrow{d_{\mathsf{C}}^{-1}} X^1 \oplus Y^0 \xrightarrow{d_{\mathsf{C}}^{0}} \cdots \xrightarrow{d_{\mathsf{C}}^{n-2}} X^n \oplus Y^{n-1} \xrightarrow{d_{\mathsf{C}}^{n-1}} Y^n$$

is right n-exact, where:

$$d_{\mathsf{C}}^{-1} = (-d_X^0,\, f^0), \quad d_{\mathsf{C}}^{i} = \begin{pmatrix} -d_X^{i+1} & f^{i+1} \\ 0 & d_Y^i \end{pmatrix} \text{ for } 0 \leqslant i \leqslant n-2, \quad d_{\mathsf{C}}^{n-1} = {}^t(f^n,\, d_Y^{n-1})$$

Dually, the above diagram is called an n-**pullback diagram** *of* $Y^\bullet$ *along* f^n *if the mapping cone* $\mathsf{C}(f^\bullet)$ *is left n-exact.*

The subsequent two lemmas will also be used in the proof of our main results. The proofs will be omitted and we refer the reader to the respective sources for more details.

Lemma 1.1.13. *([28, Proposition 2.12]) Let $\mathcal{C}$ be an additive category and:*

$$\begin{array}{ccccccccccc}
X^\bullet & & X^0 & \xrightarrow{d_X^0} & X^1 & \longrightarrow \cdots \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} \\
Y^\bullet & & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \longrightarrow \cdots \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n
\end{array}$$

be an n-pushout diagram. If d_Y^{i+1} is a weak cokernel of d_Y^i, then d_X^{i+1} is a weak cokernel of d_X^i, for $i \in \{0, 1, ..., n-2\}$.

The next lemma is the dual version of [37, Lemma 2.8.]:

Lemma 1.1.14. *Let $\mathcal{C}$ be an additive category and:*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} \cdots \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
 & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \| \\
0 & \longrightarrow & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} \cdots \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & X^{n+1}
\end{array}$$

be a commutative diagram of left n-exact sequences. Then the diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} \cdots \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n \\
 & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} \\
0 & \longrightarrow & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} \cdots \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n
\end{array}$$

is an n-pullback diagram.

Finally, we will need the following observation which seems to be well-known but we were unable to find it in the literature in a form suitable for our working setting. Note that similar arguments have been used in the study of n-exact sequences, for instance in Lemma 1.1.9 and Lemma 1.1.13 to which we refer for some examples.

Lemma 1.1.15. *Let $\mathcal{C}$ be an additive category, and $f^\bullet \colon X^\bullet \longrightarrow Y^\bullet$:*

$$\begin{array}{ccccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & \swarrow{\scriptstyle \phi^n} & \downarrow{\scriptstyle f^{n+1}} \\
Y^0 & \xrightarrow[d_Y^0]{} & Y^1 & \xrightarrow[d_Y^1]{} & \cdots & \longrightarrow & Y^{n-1} & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & Y^{n+1}
\end{array}$$

be a morphism of complexes in $\mathsf{Ch}^{n-1}(\mathcal{C})$ such that the following hold:

1. *For $0 \leqslant i \leqslant n-1$, the morphism d_Y^i is a weak kernel of d_Y^{i+1}.*
2. *There exists a morphism $\phi^n \colon X^{n+1} \longrightarrow Y^n$ such that $\phi^n d_Y^n = f^{n+1}$.*
3. *The sequence:*

$$X^0 \xrightarrow{(-d_X^0,\, f^0)} X^1 \oplus Y^0 \xrightarrow{\begin{pmatrix} -d_X^1 & f^1 \\ 0 & d_Y^0 \end{pmatrix}} \cdots \longrightarrow X^n \oplus Y^{n-1} \xrightarrow{{}^t(f^n,\, d_Y^{n-1})} Y^n \tag{1.1.1}$$

is right n-exact.

Then, for $0 \leqslant i \leqslant n-1$, the morphism d_X^{i+1} is a weak cokernel of d_X^i.

Proof. For $0 \leqslant i \leqslant n-1$, since the morphism d_Y^i is a weak kernel of d_Y^{i+1}, by Lemma 1.1.8 there exist morphisms $\phi^i \colon X^{i+1} \longrightarrow Y^i$, such that $f^{i+1} = \phi^i d_Y^i + d_X^{i+1}\phi^{i+1}$.

Clearly, $d_X^{n-1} d_X^n = 0$. Let $u \colon X^n \longrightarrow M$ such that $d_X^{n-1} u = 0$. From the sequence (1.1.1), we obtain a morphism $u' \colon Y^n \longrightarrow M$ such that $d_Y^{n-1} u' = 0$ and:

$$u = f^n u' = (\phi^{n-1} d_Y^{n-1} + d_X^n \phi^n) u' = d_X^n \phi^n u'$$

Thus, d_X^n is a weak cokernel of d_X^{n-1}.

For $0 \leqslant i \leqslant n-2$, let $u \colon X^{i+1} \longrightarrow M$ be a morphism such that $d_X^i u = 0$. Again, from (1.1.1) we obtain a morphism:

$${}^t(u',\, w) \colon X^{i+2} \oplus Y^{i+1} \longrightarrow M$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} X^i \oplus Y^{i-1} & \xrightarrow{\begin{pmatrix} -d_X^i & f^i \\ 0 & d_Y^{i-1} \end{pmatrix}} & X^{i+1} \oplus Y^i & \xrightarrow{\begin{pmatrix} -d_X^{i+1} & f^{i+1} \\ 0 & d_Y^{i} \end{pmatrix}} & X^{i+2} \oplus Y^{i+1} \\ & & \Big\downarrow {\scriptstyle {}^t(u,0)} & \swarrow {\scriptstyle {}^t(u',w)} & \\ & & M & & \end{array}$$

Then, $d_Y^i w = 0$ and:

$$u = -d_X^{i+1}u' + f^{i+1}w = -d_X^{i+1}u' + (\phi^i d_Y^i + d_X^{i+1}\phi^{i+1})w = d_X^{i+1}(-u' + \phi^{i+1}w)$$

Thus, d_X^{i+1} is a weak cokernel of d_X^i. ■

We also state the dual lemma:

Lemma 1.1.16. *Let $\mathcal{C}$ be an additive category, and $f^\bullet \colon X^\bullet \longrightarrow Y^\bullet$:*

$$\begin{array}{ccccccccc} X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \longrightarrow \cdots \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} \\ {\scriptstyle f^0}\Big\downarrow & \swarrow^{\phi^0} & \Big\downarrow{\scriptstyle f^1} & & \Big\downarrow{\scriptstyle f^2} & & \Big\downarrow{\scriptstyle f^n} & & \Big\downarrow{\scriptstyle f^{n+1}} \\ Y^0 & \xrightarrow[d_Y^0]{} & Y^1 & \xrightarrow[d_Y^1]{} & Y^2 & \longrightarrow \cdots \longrightarrow & Y^n & \xrightarrow[d_Y^n]{} & Y^{n+1} \end{array}$$

be a morphism of complexes in $\mathsf{Ch}^{n-1}(\mathcal{C})$ such that the following hold:

1. *For $0 \leqslant i \leqslant n-1$, the morphism d_X^{i+1} is a weak cokernel of d_X^i.*
2. *There exists a morphism $\phi^0 \colon X^1 \longrightarrow Y^0$ such that $f^0 = d_X^0\phi^0$.*
3. *The sequence:*

$$X^1 \xrightarrow{(d_X^1, f^1)} X^2 \oplus Y^1 \xrightarrow{\begin{pmatrix} -d_X^2 & f^2 \\ 0 & d_Y^1 \end{pmatrix}} \cdots \longrightarrow X^{n+1} \oplus Y^n \xrightarrow{{}^t(f^{n+1}, d_Y^n)} Y^{n+1}$$

   is left n-exact.

Then, for $0 \leqslant i \leqslant n-1$, the morphism d_Y^i is a weak kernel of d_Y^{i+1}.

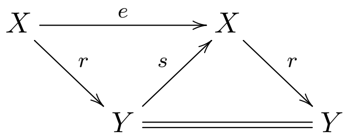
1.1.3 Definition and basic properties

The notion of an n-abelian category ($n \geqslant 1$) was introduced by Jasso in [28] in order to axiomatize the exactness properties of an n-cluster-tilting subcategory of an abelian category. In this section we provide Jasso's definition along with some basic properties of n-abelian categories.

First, we will need the following notion:

Definition 1.1.17. *If $\mathcal{C}$ is a category and X an object in $\mathcal{C}$, a morphism $e\colon X \longrightarrow X$ is called* **idempotent** *if $e^2 = e$.*

The category $\mathcal{C}$ is called **idempotent complete** *if every idempotent in $\mathcal{C}$ splits, i.e. for every idempotent e in $\mathcal{C}$ there exist an object $Y \in \mathcal{C}$ and morphisms $r\colon X \longrightarrow Y$ and $s\colon Y \longrightarrow X$ such that $rs = e$ and $sr = 1_Y$, as seen in the following commutative diagram:*



We continue with the definition of an n-abelian category:

Definition 1.1.18. *([28, Definition 3.1]) Let n be a positive integer. An* **n-abelian category** *is an additive category $\mathcal{M}$ which satisfies the following axioms:*

(A0) The category $\mathcal{M}$ is idempotent complete.
(A1) Every morphism in $\mathcal{M}$ has an n-kernel and an n-cokernel.
(A2) For every monomorphism $f\colon X \longrightarrow Y$ in $\mathcal{M}$ and for every n-cokernel $(c^1, .., c^n)$ of f, the sequence:

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{c^1} \cdots \xrightarrow{c^{n-1}} C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0$$

is n-exact.
(A3) For every epimorphism $f\colon X \longrightarrow Y$ in $\mathcal{M}$ and for every n-kernel $(k^n, .., k^1)$ of f, the sequence:

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^1} X \xrightarrow{f} Y \longrightarrow 0$$

is n-exact.

Remark 1.1.19. As stated by Jasso in [28, Remark 3.2], we can replace axioms (A2) and (A3) respectively with the following weaker axioms:

(A2') For every monomorphism $f\colon X \longrightarrow Y$ in $\mathcal{M}$ there exists an n-exact sequence:

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{c^1} \cdots \xrightarrow{c^{n-1}} C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0$$

(A3') For every epimorphism $f\colon X \longrightarrow Y$ in $\mathcal{M}$ there exists an n-exact sequence:

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^1} X \xrightarrow{f} Y \longrightarrow 0$$

Remark 1.1.20. The notion of an n-abelian category generalizes the classic notion of an abelian category which we obtain from Definition 1.1.18 by setting $n = 1$.

As mentioned, n-abelian categories encompass the exactness properties of an n-cluster-tilting subcategory of an abelian category, as proven by Jasso. Even though this result lies outside the scope of the thesis we will briefly reference it, aiming to provide a more comprehensive overview of this theory and emphasize the broader applications of our research.

In order to define the notion of n-cluster tilting subcategory, we need to remind the reader the notions of co(ntra)variantly finite and (co)generating subcategory. For more details, see [3].

Definition 1.1.21. *Let $\mathcal{C}$ be an additive category and $\mathcal{M} \subseteq \mathcal{C}$ a (full) subcategory of $\mathcal{C}$. $\mathcal{M}$ is called* **covariantly finite** *in $\mathcal{C}$, if for any object $C \in \mathcal{C}$ there exists an object $M \in \mathcal{M}$ and a morphism $f : C \longrightarrow M$ such that for any $M' \in \mathcal{M}$ the induced sequence of abelian groups:*

$$\mathcal{C}(M, M') \longrightarrow \mathcal{C}(C, M') \longrightarrow 0$$

is exact. In other words, for any object $M' \in M$ and morphism $g : C \longrightarrow M'$, there exists a morphism $u : M \longrightarrow M'$ such that the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{f} & M \\ & \searrow^{g} & \downarrow{\scriptstyle u} \\ & & M' \end{array}$$

Dually, the notion of a **contravariantly finite** *subcategory is defined. Finally, $\mathcal{M}$ is called* **functorially finite** *in $\mathcal{C}$ if $\mathcal{M}$ is covariantly and contravariantly finite in $\mathcal{C}$.*

Definition 1.1.22. *Let $\mathcal{A}$ be an abelian category and $\mathcal{M} \subseteq \mathcal{A}$ a (full) subcategory of $\mathcal{A}$. $\mathcal{M}$ is called a* **cogenerating** *subcategory of $\mathcal{A}$ if for any object $A \in \mathcal{A}$ there exists an object $M \in \mathcal{M}$ and a monomorphism: $f : A \longrightarrow M$. Dually the notion of a* **generating** *subcategory is defined.*

Now we provide the definition of an n-cluster tilting subcategory of an abelian category:

Definition 1.1.23. *([28, Definition 3.14]) Let $\mathcal{A}$ be an abelian category and $\mathcal{M} \subseteq \mathcal{A}$ a (full) subcategory of $\mathcal{A}$. $\mathcal{M}$ is called an* **n-cluster tilting subcategory** *of $\mathcal{A}$ if the following hold:*

1. *$\mathcal{M}$ is a generating-cogenerating subcategory of $\mathcal{A}$.*

2. $\mathcal{M}$ *is a functorially finite subcategory of* $\mathcal{A}$*.*

3. $\mathcal{M}$ *satisfies the following:*

$$\begin{aligned}\mathcal{M} &= \{X \in \mathcal{A} \mid \mathsf{Ext}^i_{\mathcal{A}}(X, \mathcal{M}) = 0,\ \forall i \in \{1, \cdots, n-1\}\} \\ &= \{X \in \mathcal{A} \mid \mathsf{Ext}^i_{\mathcal{A}}(\mathcal{M}, X) = 0,\ \forall i \in \{1, \cdots, n-1\}\}\end{aligned}$$

Examples of n-abelian categories can be created by utilizing Jasso's main theorem:

Proposition 1.1.24. *([28, Theorem 3.16]) Let* $\mathcal{A}$ *be an abelian category and* $\mathcal{M}$ *an* n*-cluster-tilting subcategory of* $\mathcal{A}$*. Then,* $\mathcal{M}$ *is an* n*-abelian category.*

Recently, Kvamme and, independently, Ebrahimi and Nasr-Isfahani proved that any n-abelian category is equivalent to an n-cluster tilting subcategory of an abelian category, see [34], [12].

The idempotent completeness axiom leads to the existence of n-pushout and n-pullback diagrams in an n-abelian category. More specifically, the following holds:

Proposition 1.1.25. *([28, Theorem 3.8]) Let* $\mathcal{C}$ *be an additive category which satisfies axioms (A0) and (A1). For any complex* $X^\bullet$ *in* $\mathsf{Ch}^{n-1}(\mathcal{C})$ *and morphism* $f^0 \colon X^0 \longrightarrow Y^0$*, there exists an* n*-pushout diagram:*

$$\begin{array}{ccccccccccc} X^\bullet & & X^0 & \xrightarrow{d_X^0} & X^1 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} \\ Y^\bullet & & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n \end{array}$$

In the case of $\mathcal{C}$ *being an* n*-abelian category, if* d_X^0 *is a monorphism then* d_Y^0 *is also a monomorphism.*

The dual also holds for n*-pullbacks.*

In our general setting, axiom (A0) of idempotent completeness displays a delicate behaviour with respect to localization. For this reason, in the next section we will define a pre-n-abelian category to be an n-abelian category which satisfies all axioms of an n-abelian category except that axiom (A0) is not necessarily satisfied.

Thus, in the later we will need a version of the following result of Jasso in a pre-n-abelian category which is not necessarily idempotent complete. Note that Jasso in the following result utilizes the notion of a good n-pushout diagram, see [28, Definition-Proposition 2.14], which describes an n-pushout diagram that satisfies an additional property. Given that this notion is not essential for the proofs presented in this thesis and will not be further used, we direct the reader to the aforementioned source for an in-depth definition.

Proposition 1.1.26. *([28, Proposition 3.13]) Let $\mathcal{C}$ be an n-abelian category, $f^0\colon X^0 \longrightarrow X^1$ a morphism in $\mathcal{C}$ and $(f^k\colon X^k \longrightarrow X^{k+1} | 1 \leqslant k \leqslant n)$ an n-cokernel of f^0. Then, for every $k \in \{0, 1, ..., n\}$ and every $l \in \{1, ..., n\}$ there exist morphisms $g_k^l\colon Y_k^l \longrightarrow Y_k^{l-1}$ (with $Y_K^0 \colon = X^k$) and $p_k^{l-1}\colon Y_k^{l-1} \longrightarrow Y_{k+1}^l$ satisfying the following properties:*

1. *For every $k \in \{0, 1, ..., n\}$ the diagram:*

$$\begin{array}{ccccccccccc} Y_k^n & \xrightarrow{g_k^n} & Y_k^{n-1} & \xrightarrow{g_k^{n-1}} & \cdots & \xrightarrow{g_k^2} & Y_k^1 & \xrightarrow{g_k^1} & X^k & \xrightarrow{f^k} & X^{k+1} \\ \downarrow & & \downarrow{\scriptstyle p_k^{n-1}} & & & & \downarrow{\scriptstyle p_k^1} & & \downarrow{\scriptstyle p_k^0} & \nearrow{\scriptstyle g_{k+1}^1} & \\ 0 & \longrightarrow & Y_{k+1}^n & \xrightarrow{g_{k+1}^n} & \cdots & \xrightarrow{g_{k+1}^3} & Y_{k+1}^2 & \xrightarrow{g_{k+1}^2} & Y_{k+1}^1 & & \end{array} \tag{1.1.2}$$

   commutes.
2. *The sequence $(g_k^n, ..., g_k^1)$ is an n-kernel of f^k.*
3. *The diagram:*

$$\begin{array}{ccccccccc} Y_k^n & \xrightarrow{g_k^n} & Y_k^{n-1} & \xrightarrow{g_k^{n-1}} & \cdots & \xrightarrow{g_k^2} & Y_k^1 & \xrightarrow{g_k^1} & X^k \\ \downarrow & & \downarrow{\scriptstyle p_k^{n-1}} & & & & \downarrow{\scriptstyle p_k^1} & & \downarrow{\scriptstyle p_k^0} \\ 0 & \longrightarrow & Y_{k+1}^n & \xrightarrow{g_{k+1}^n} & \cdots & \xrightarrow{g_{k+1}^3} & Y_{k+1}^2 & \xrightarrow{g_{k+1}^2} & Y_{k+1}^1 \end{array}$$

   is both an n-pullback diagram and a good n-pushout diagram. In particular, the morphism

$${}^t(p_k^0,\ g_{k+1}^2)\colon X^k \oplus Y_{k+1}^2 \longrightarrow Y_{k+1}^1$$

   is an epimorphism.
4. *If $k \neq 0$, the sequence $(g_k^{k-1}, ..., g_k^1, f^k, ..., f^n)$ is an n-cokernel of the morphism g_k^k.*

Remark 1.1.27. Note that Ebrahimi and Nasr-Isfahani in [13, Proposition 2.7] prove, in the setting of n-abelian categories, an analogous result related to the above proposition which we will discuss later providing a more general version of [13, Diagram (2.2)].

If $\mathcal{M}$ is an n-abelian category and $\mathcal{A}$ an abelian category, the notion of an n-exact functor $F\colon \mathcal{M} \longrightarrow \mathcal{A}$ is defined in [38, §4.1]. Analogously, concerning the case of a covariant additive functor between two n-abelian categories, we will use the following notion:

Definition 1.1.28. *Let $\mathcal{M}$, $\mathcal{N}$ be two n-abelian categories and $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ a covariant additive functor.*

(i) F is called **left n-exact**, *if for any left n-exact sequence:*

$$0 \longrightarrow K^n \xrightarrow{k^n} \cdots \longrightarrow K^1 \xrightarrow{k^1} X \xrightarrow{f} Y$$

in $\mathcal{M}$, the sequence:

$$0 \longrightarrow F(K^n) \xrightarrow{F(k^n)} \cdots \longrightarrow F(K^1) \xrightarrow{F(k^1)} F(X) \xrightarrow{F(f)} F(Y)$$

is left n-exact in $\mathcal{N}$.

(ii) F is called **right n-exact**, *if for any right n-exact sequence:*

$$X \xrightarrow{f} Y \xrightarrow{c^1} C^1 \longrightarrow \cdots \xrightarrow{c^n} C^n \longrightarrow 0$$

in $\mathcal{M}$, the sequence:

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(c^1)} F(C^1) \longrightarrow \cdots \xrightarrow{F(c^n)} F(C^n) \longrightarrow 0$$

is right n-exact in $\mathcal{N}$.

(iii) F is called **n-exact**, *if for any n-exact sequence:*

$$0 \longrightarrow X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1} \longrightarrow 0$$

in $\mathcal{M}$, the sequence:

$$0 \longrightarrow F(X^0) \xrightarrow{F(f^0)} F(X^1) \xrightarrow{F(f^1)} \cdots \longrightarrow F(X^n) \xrightarrow{F(f^n)} F(X^{n+1}) \longrightarrow 0$$

is n-exact in $\mathcal{N}$.

1.2 n-Angulated categories

Geiss, Keller and Oppermann, in order to axiomatize the exactness properties of an n-cluster-tilting subcategory of a triangulated category, introduced in [20] the notion of an n-angulated category.

We begin by stating the definition of an n-angulated category in [20] and we fix some notation. Throughout this section, let $n \geqslant 3$.

The structure of an n-angulated category is based on the following type of sequences:

Definition 1.2.1. *Let $\mathcal{C}$ be an additive category and $\Sigma\colon \mathcal{C} \longrightarrow \mathcal{C}$ an automorphism in $\mathcal{C}$. A diagram $X^\bullet$ of objects and morphisms in $\mathcal{C}$:*

$$X^\bullet\colon \qquad X^1 \xrightarrow{a^1} X^2 \xrightarrow{a^2} \cdots \xrightarrow{a^{n-1}} X^n \xrightarrow{a^n} \Sigma X^1$$

is called an **n-Σ-sequence** *in $\mathcal{C}$.*

The automorphism Σ is also called the **suspension functor**.

Remark 1.2.2. We do not assume that the composition of two consecutive morphisms in an n-Σ-sequence is necessarily zero, thus n-Σ-sequences are not necessarily complexes.

Morphisms between two n-Σ-sequences are defined in the following way:

Definition 1.2.3. *A* **morphism of n-Σ-sequences** $f^\bullet\colon X^\bullet \longrightarrow Y^\bullet$ *is a sequence of morphisms $(f^1, f^2, \cdots, f^n)$ such that the following diagram commutes:*

$$\begin{array}{ccccccccccc}
X^\bullet & & X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1 \\
\downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle \Sigma f^1} \\
Y^\bullet & & Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & \cdots & \longrightarrow & Y^n & \xrightarrow{b^n} & \Sigma Y^1
\end{array}$$

The morphism $f^\bullet$ is called a **weak isomorphism** *if for some $1 \leqslant i \leqslant n$, f^i and f^{i+1} are isomorphisms, where $f^{n+1} := \Sigma f^1$.*

Two n-Σ-sequences $X^\bullet$ and $Y^\bullet$ are **weakly isomorphic** *if they are connected by a finite zigzag of weak isomorphisms, i.e. there exist weak isomorphisms $f_0^\bullet, f_1^\bullet, \cdots, f_i^\bullet$, as seen in the following diagram:*

$$\begin{array}{ccccccccccccc}
 & & M_1^\bullet & & & & \cdots & & & & M_i^\bullet & & \\
 & {\scriptstyle f_0^\bullet}\swarrow & & \searrow{\scriptstyle f_1^\bullet} & & \swarrow & & \searrow & & {\scriptstyle f_{i-1}^\bullet}\swarrow & & \searrow{\scriptstyle f_i^\bullet} & \\
X^\bullet & & & & M_2^\bullet & & \cdots & & M_{i-1}^\bullet & & & & Y^\bullet
\end{array}$$

Since a morphism between n-Σ-sequences is a morphism of complexes, we can define its mapping cone as follows:

Definition 1.2.4. *The* **mapping cone** *of $f^\bullet$, denoted by $\mathsf{C}(f^\bullet)$, is the n-Σ-sequence:*

$$X^2 \oplus Y^1 \xrightarrow{d_{\mathsf{C}}^1} X^3 \oplus Y^2 \xrightarrow{d_{\mathsf{C}}^2} \cdots \longrightarrow \Sigma X^1 \oplus Y^n \xrightarrow{d_{\mathsf{C}}^n} \Sigma X^2 \oplus \Sigma Y^1$$

where:

$$d_C^i = \begin{pmatrix} -a^{i+1} & f^{i+1} \\ 0 & b^i \end{pmatrix} \text{ for } 1 \leqslant i \leqslant n-1 \text{ and } d_C^n = \begin{pmatrix} -\Sigma a^1 & \Sigma f^1 \\ 0 & b^n \end{pmatrix}$$

Finally, we state the definition of an n-angulated category:

Definition 1.2.5. *([20, Definition 1.1]) Let $\mathcal{C}$ be an additive category, $\Sigma\colon \mathcal{C} \longrightarrow \mathcal{C}$ an automorphism of $\mathcal{C}$ and $n \geqslant 3$. A* **pre-n-angulation** *of $(\mathcal{C}, \Sigma)$ is a class $\mathcal{N}$ of n-Σ-sequences, whose elements are called* **n-angles**, *satisfying the following axioms:*

(F1) (a) The class $\mathcal{N}$ is closed under finite direct sums and direct summands of n-Σ-sequences. In other words, if

$$X^\bullet: \qquad X^1 \xrightarrow{a^1} X^2 \xrightarrow{a^2} \cdots \longrightarrow X^n \xrightarrow{a^n} \Sigma X^1$$

and

$$Y^\bullet: \qquad Y^1 \xrightarrow{b^1} Y^2 \xrightarrow{b^2} \cdots \longrightarrow Y^n \xrightarrow{b^n} \Sigma Y^1$$

are two n-Σ-sequences in $\mathcal{C}$, then the n-Σ-sequence $X^\bullet \oplus Y^\bullet$:

$$X^1 \oplus Y^1 \xrightarrow{a^1 \oplus b^1} X^2 \oplus Y^2 \xrightarrow{a^2 \oplus b^2} \cdots \longrightarrow X^n \oplus Y^n \xrightarrow{a^n \oplus b^n} \Sigma X^1 \oplus \Sigma Y^1$$

is in $\mathcal{N}$ iff the n-Σ-sequences $X^\bullet$ and $Y^\bullet$ are both in $\mathcal{N}$.

(b) For any $X \in \mathcal{C}$ the sequence:

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X$$

is in $\mathcal{N}$.

(c) For any morphism $a^1\colon X^1 \longrightarrow X^2$ in $\mathcal{C}$, there exists an n-angle of the form:

$$X^1 \xrightarrow{a^1} X^2 \xrightarrow{a^2} \cdots \longrightarrow X^n \xrightarrow{a^n} \Sigma X^1$$

(F2) The n-Σ-sequence

$$X^\bullet: \qquad X^1 \xrightarrow{a^1} X^2 \xrightarrow{a^2} \cdots \xrightarrow{a^{n-1}} X^n \xrightarrow{a^n} \Sigma X^1$$

is a n-angle if and only if the n-Σ-sequence:

$$X^2 \xrightarrow{a^2} X^3 \xrightarrow{a^3} \cdots \xrightarrow{a^n} \Sigma X^1 \xrightarrow{(-1)^n \Sigma a^1} \Sigma X^2$$

which is called the left rotation of $X^\bullet$, is also an n-angle.

(F3) For any commutative diagram of the form:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1 \\
\downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & & & & & & & \downarrow{\scriptstyle \Sigma f^1} \\
Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \longrightarrow & \cdots & \longrightarrow & Y^n & \xrightarrow{b^n} & \Sigma Y^1
\end{array}$$

whose rows are n-angles, there exist morphisms $f^3, \cdots, f^n$ that complete the diagram to a morphism of n-Σ-sequences $f^\bullet \colon X^\bullet \longrightarrow Y^\bullet$:

$$\begin{array}{ccccccccccc} X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1 \\ \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^3} & & & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle \Sigma f^1} \\ Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \longrightarrow & \cdots & \longrightarrow & Y^n & \xrightarrow{b^n} & \Sigma Y^1 \end{array}$$

If $\mathcal{N}$ is a pre-n-angulation of $(\mathcal{C}, \Sigma)$, then the triple $(\mathcal{C}, \Sigma, \mathcal{N})$ is called a **pre-n-angulated category** .

Moreover, $\mathcal{N}$ is called an **n-angulation** *of $(\mathcal{C}, \Sigma)$ if the following axiom is satisfied:*

(F4) In the axiom $(F3)$, the morphisms $f^3, \cdots, f^n$ can be chosen such that the cone

$$\mathsf{C}(f^\bullet)\colon \qquad X^2 \oplus Y^1 \xrightarrow{d^1_{\mathsf{C}}} X^3 \oplus Y^2 \xrightarrow{d^2_{\mathsf{C}}} \cdots \longrightarrow \Sigma X^1 \oplus Y^n \xrightarrow{d^n_{\mathsf{C}}} \Sigma X^2 \oplus \Sigma Y^1$$

is in $\mathcal{N}$.

If $\mathcal{N}$ is a n-angulation of $(\mathcal{C}, \Sigma)$, then the triple $(\mathcal{C}, \Sigma, \mathcal{N})$ is called an **n-angulated category**.

Remark 1.2.6. The notion of an n-angulated category generalizes the classic notion of a triangulated category which we obtain from Definition 1.2.5 by setting $n = 3$.

Similarly to the classic case of triangulated categories, the following holds:

Lemma 1.2.7. *([20, Proposition 1.5]) If $(\mathcal{C}, \Sigma, \mathcal{N})$ is a pre-n-angulated category, then every n-angle in $\mathcal{C}$ is exact, i.e. for every n-angle*

$$X^\bullet\colon \qquad X^1 \xrightarrow{a^1} X^2 \xrightarrow{a^2} \cdots \xrightarrow{a^{n-1}} X^n \xrightarrow{a^n} \Sigma X^1$$

in $\mathcal{C}$, the induced sequence $\mathcal{C}(-, X^\bullet)$:

$$\cdots \longrightarrow \mathcal{C}(-, X^1) \longrightarrow \mathcal{C}(-, X^2) \longrightarrow \cdots \longrightarrow \mathcal{C}(-, X^n) \longrightarrow \mathcal{C}(-, \Sigma X^1) \longrightarrow \cdots$$

and the induced sequence $\mathcal{C}(X^\bullet, -)$:

$$\cdots \longrightarrow \mathcal{C}(\Sigma X^1, -) \longrightarrow \mathcal{C}(X^n, -) \longrightarrow \cdots \longrightarrow \mathcal{C}(X^2, -) \longrightarrow \mathcal{C}(X^1, -) \longrightarrow \cdots$$

of representable functors are exact, see [20, Definition 1.1].

Remark 1.2.8. As mentioned in [8, p.2], in the setting of a pre-n-angulated category, Lemma 1.2.7 implies that the composition of two consecutive morphisms in an n-angle is zero. Thus, similarly to the case of triangulated categories, an n-angle in a pre-n-angulated category is a complex.

Similarly to the n-abelian case and in compliance with the classic case of a triangulated category, the notion of an n-exact functor between two n-angulated categories is defined as follows:

Definition 1.2.9. *If* $(\mathcal{C}, \Sigma, \mathcal{N})$ *and* $(\mathcal{C}', \Sigma', \mathcal{N}')$ *are two* n*-angulated categories, an additive functor* $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ *is called* n**-exact**, *if for any* $X \in \mathcal{C}$ *there exist natural isomorphisms* $\phi_X\colon F(\Sigma X) \xrightarrow{\sim} \Sigma'(F(X))$ *such that for any* n*-angle*

$$X^1 \xrightarrow{a^1} X^2 \xrightarrow{a^2} \cdots \xrightarrow{a^{n-1}} X^n \xrightarrow{a^n} \Sigma X^1$$

in $\mathcal{N}$*, the* n*-*Σ*-sequence:*

$$F(X^1) \xrightarrow{F(a^1)} F(X^2) \xrightarrow{F(a^2)} \cdots \xrightarrow{F(a^{n-1})} F(X^n) \xrightarrow{F(a^n)\phi_X} \Sigma' F(X^1)$$

is an n*-angle in* $\mathcal{N}'$*.*

Geiss, Keller and Oppermann in [20] prove that in a pre-n-angulated category, the class $\mathcal{N}$ of n-angles is closed under weak isomorphisms. In the following, we state their result which will be used later, while for the proof we refer the reader to the above source.

Lemma 1.2.10. *([20, Lemma 1.4]) Let* $(\mathcal{C}, \Sigma, \mathcal{N})$ *be a pre-*n*-angulated category. If* $X^\bullet$ *and* $Y^\bullet$ *are two weakly isomorphic exact* n*-*Σ*-sequences, then* $X^\bullet$ *is an* n*-angle if and only if* $Y^\bullet$ *is also an* n*-angle.*

As mentioned before, n-angulated categories encompass the exactness properties of an $(n-2)$-cluster tilting subcategory of a triangulated category. Examples of n-angulated categories can be constructed as n-cluster tilting subcategories of triangulated categories, utilizing the main result of Geiss, Keller and Oppermann which we will mention briefly. First, in the n-angulated setting, the notion of an n-cluster tilting subcategory is defined as follows:

Definition 1.2.11. *([20, Definition 3.1]) Let* $\mathcal{T}$ *be a triangulated category with suspension functor* Σ *and let* $\mathcal{C} \subseteq \mathcal{T}$ *be a full subcategory of* $\mathcal{T}$*.* $\mathcal{C}$ *is called an* n**-cluster tilting subcategory** *of* $\mathcal{T}$ *if the following hold:*

1. $\mathcal{C}$ *is functorially finite in* $\mathcal{T}$.

2. $\mathcal{C}$ *satisfies the following:*

$$\begin{aligned}\mathcal{C} &= \{X \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(X, \Sigma^i \mathcal{C}) = 0,\ \forall i \in \{1, \cdots, n-1\}\} \\ &= \{X \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(\mathcal{C}, \Sigma^i X) = 0,\ \forall i \in \{1, \cdots, n-1\}\}\end{aligned}$$

Proposition 1.2.12. *([20, Theorem 1]) Let* $\mathcal{T}$ *be a triangulated category with suspension functor* Σ *and* $\mathcal{C}$ *an* $(n-2)$*-cluster tilting subcategory of* $\mathcal{T}$ *closed under* Σ^{n-2}. *Then* $\mathcal{C}$ *is an* n*-angulated category with suspension functor* Σ^{n-2}.

For a detailed construction of n-angles in $\mathcal{C}$ we refer the reader to the above source.

1.3 Localization

In the third part of this introductory chapter, our aim is to remind the reader of some basic localization theory of a general category in the form of the calculus of fractions, introduced by Gabriel-Zisman [19], see also [41]. The terminology used follows Popescu in [41, §4.1] and we refer the reader to that source for more details concerning the localization of a category in a general setting.

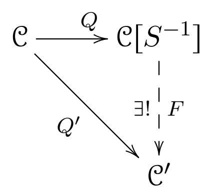
1.3.1 Definition and basic properties

The general problem behind the localization is, given a category $\mathcal{C}$ and a class of morphisms S in $\mathcal{C}$, the construction of a category $\mathcal{C}[S^{-1}]$, which is universal with respect to the property that any element of S becomes invertible.

Definition 1.3.1. *Let* $\mathcal{C}$ *be a category and* S *a class of morphisms in* $\mathcal{C}$. *The* **localization** *of* $\mathcal{C}$ *with respect to* S *is a couple* $(\mathcal{C}[S^{-1}], Q)$, *where* $\mathcal{C}[S^{-1}]$ *is a category and* $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ *is a covariant functor such that the following are satisfied:*

(i) The morphism $Q(s)$ *is an isomorphism for any* $s \in S$.

(ii) If $\mathcal{C}'$ *is a category and* $Q'\colon \mathcal{C} \longrightarrow \mathcal{C}'$ *is a covariant functor such that* $Q'(s)$ *is an isomorphism for any* $s \in S$, *then a unique functor* $F\colon \mathcal{C}[S^{-1}] \longrightarrow \mathcal{C}'$ *exists such that the following diagram of categories and functors is commutative:*



The functor Q is called the **localization functor**. *If $\mathcal{C}$ is an additive category, we demand that everything in sight is additive, in particular the localization functor is an additive functor.*

The following is obtained directly from the universal property of localization.

Corollary 1.3.2. *The localization of a category $\mathcal{C}$ with respect to a class of morphisms S, if it exists, is unique up to equivalence of categories.*

As in the classic case of an abelian or a triangulated category, we will focus on a class of morphisms S possessing properties similar to those of a multiplicative subset of a (commutative) ring. For this purpose we will need the following definition:

Definition 1.3.3. *Let $\mathcal{C}$ be a category and S a system of morphisms in $\mathcal{C}$. S is called a* **bicalculable system** *if the following conditions are satisfied:*

(F1) S is **multiplicative**, *i.e. all identity morphisms in $\mathcal{C}$ are in S and for any two composable morphisms $s_1, s_2 \in S$, the composition $s_1 s_2$ is also in S.*

(F2) S is **left permutable**, *i.e. any diagram of the form:*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle s}\downarrow & & \\ Y & & \end{array}$$

where $s \in S$, can be completed to a commutative square:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ Y & \dashrightarrow & Y' \end{array}$$

where $s' \in S$.

(F2') S is **right permutable**, *i.e. the dual of (F2) is satisfied.*

(F3) S is **left simplifiable**, *i.e. for any morphisms $f, g \colon X \longrightarrow Y$ and $s \colon K \longrightarrow X$ with $s \in S$ and $sf = sg$, there exists a morphism $s' \colon Y \longrightarrow C$ with $s' \in S$ and $fs' = gs'$.*

$$K \xrightarrow{s} X \underset{g}{\overset{f}{\rightrightarrows}} Y \overset{s'}{\dashrightarrow} C$$

(F3') S is **right simplifiable**, *i.e. the dual of (F3) is satisfied.*

If conditions $(F1)$*,* $(F2)$ *and* $(F3)$ *are satisfied, then* S *is called* **left calculable** *and dually, if conditions* $(F1)$*,* $(F2')$ *and* $(F3')$ *are satisfied, then* S *is called* **right calculable***.*

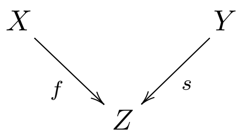
Remark 1.3.4. A natural question that arises at this point is whether, for any category $\mathcal{C}$ and any class of morphisms S, the category $\mathcal{C}[S^{-1}]$ exists. If $\mathcal{C}$ is a small category, or when the class S is a set, the localization always exists, see [46, 10.3.3], while set-theoretic problems appear mainly when the class S is not a set.

Moreover, Gabriel and Zisman in [19] prove that for any category $\mathcal{C}$ and bicalculable system S, under a mild set-theoretic condition for S, the localization $\mathcal{C}[S^{-1}]$ always exists, see also [41, Chapter 4, Theorem 1.4].

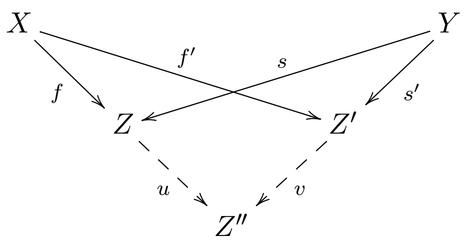
As in the standard references concerning the study of localization (see for example [19], [44]), for the rest of the thesis, we will assume that the class S is bicalculable and the necessary conditions are satisfied such that the localization exists. For more details we refer the reader to the above sources.

If S is a left calculable system of morphisms, then S admits a *calculus of fractions* in the sense of [19], i.e. morphisms in the localization can be described with diagrams of the following form:

Definition 1.3.5. *Let* $\mathcal{C}$ *be an (additive) category and* S *a left calculable system of morphisms in* $\mathcal{C}$*. A diagram of the form:*



where $X, Y, Z \in \mathcal{C}$ *and* $s \in S$ *is called a* **left (additive) fraction** *and is denoted by* (s/f)*. Two left fractions* $(s/f), (s'/f')$ *as seen in the following diagram:*

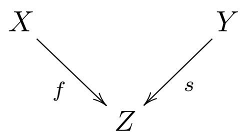


are called **equivalent** *if there exist an object* $Z'' \in \mathcal{C}$ *and morphisms* $u\colon Z \longrightarrow Z''$ *and* $v\colon Z' \longrightarrow Z''$ *such that the above diagram is commutative and* $su = s'v \in S$*.*

It is easy to check that this relation is an equivalence relation. For simplicity, and in case that no confusion arises, the equivalence class of a fraction (s/f) is still denoted by its representative (s/f). We will now describe briefly the localized category $\mathcal{C}[S^{-1}]$ and the localization functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$, arising from a left calculable system of morphisms S in an (additive) category $\mathcal{C}$. For more details we refer again the reader to [41, Chapter 4, Theorem 1.4].

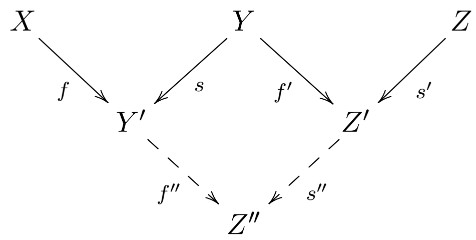
The localization $\mathcal{C}[S^{-1}]$ of an additive category $\mathcal{C}$ with respect to a left calculable system of morphisms S can be described as follows:

- The objects of $\mathcal{C}[S^{-1}]$ are the objects of $\mathcal{C}$.
- A morphism $(s/f)\colon X \longrightarrow Y$ in $\mathcal{C}[S^{-1}]$ is the equivalence class of a left fraction:



- The composition in $\mathcal{C}[S^{-1}]$ is defined in the following way:

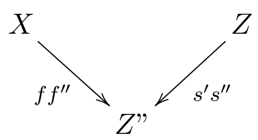
  Let $(s/f)\colon X \longrightarrow Y$, $(s'/f')\colon Y \longrightarrow Z$ be two morphisms in $\mathcal{C}[S^{-1}]$. Applying condition (F2) of Definition 1.3.3, there exist $Z'' \in \mathcal{C}$ and morphisms $f''\colon Z' \longrightarrow Z''$ and $s''\colon Y' \longrightarrow Z''$, with $s'' \in S$, such that the following diagram is commutative:



  Using the equivalence relation of the left fractions, it is easy to show that the composition:

$$(s/f)(s'/f') = (s's''/ff'')$$

  depicted by the following diagram:



  is well-defined and does not depend on the representatives of the used elements.

Finally, the localization functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ is defined as follows:

(i) For any object $X \in \mathcal{C}$, set $Q(X) = X$.
(ii) For any morphism f in $\mathcal{C}$, set $Q(f) = (1_Y/f)$.

Using this definition, any morphism $(s/f)\colon X \longrightarrow Y$ in $\mathcal{C}[S^{-1}]$ can be written as:

$$(s/f) = Q(f)Q(s)^{-1}$$

1.3.2 Properties of the localization

We finish this introductory chapter by recalling some basic well-known properties of the localization of an (additive) category which will be useful later. Additionally, we will state the fundamental results of Gabriel and Verdier on the localization of abelian and triangulated categories.

To begin with, localization preserves the additive structure in the following sense:

Lemma 1.3.6. *([41, Chapter 4, Corollary 1.5]) Let $\mathcal{C}$ be an additive category and let S be a left calculable system of morphisms in $\mathcal{C}$. Then $\mathcal{C}[S^{-1}]$ is also an additive category and the localization functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ is additive.*

Remark 1.3.7. If S is a right calculable system we can describe the category $\mathcal{C}[S^{-1}]$ in a dual manner, and if S is bicalculable, both descriptions agree up to a unique equivalence of localized categories which commutes with the localization functors.

We continue by reminding the reader of the following property regarding the kernel of the localization functor of an additive category.

Corollary 1.3.8. *([41, Chapter 4, Corollary 1.8]) Let $\mathcal{C}$ be an additive category and S a left calculable system of morphisms in $\mathcal{C}$. For a morphism $f\colon X \longrightarrow Y$ in $\mathcal{C}$, the following are equivalent:*

(i) $Q(f) = 0$;
(ii) There exists a morphism $u\colon Y \longrightarrow Z \in S$ *such that* $fu = 0$.

Dually, we have the following statement for a right calculable system of morphisms:

Corollary 1.3.9. *([41, Chapter 4, Corollary 1.8]) Let $\mathcal{C}$ be an additive category and S a right calculable system of morphisms in $\mathcal{C}$. For a morphism $f\colon X \longrightarrow Y$ in $\mathcal{C}$, the following are equivalent:*

(i) $Q(f) = 0$;
(ii) There exists a morphism $u\colon Z \longrightarrow X \in S$ *such that* $uf = 0$.

In order to prove our main result regarding the localization of an n-angulated category in Chapter 4, we will need the following lemma.

Lemma 1.3.10. *([44, Lemma 2.2.7.]) Let $\mathcal{C}$ be an additive category, S a bicalculable system of morphisms in $\mathcal{C}$, and $\mathcal{C}[S^{-1}]$ the localization of $\mathcal{C}$ with respect to S. If*

$$\begin{array}{ccc} A^1 & \xrightarrow{Q(f)} & A^2 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ B^1 & \xrightarrow{Q(g)} & B^2 \end{array} \tag{1.3.1}$$

is a commutative diagram in $\mathcal{C}[S^{-1}]$, then there exists a commutative diagram

$$\begin{array}{ccc} A^1 & \xrightarrow{f} & A^2 \\ {\scriptstyle u^1}\downarrow & & \downarrow{\scriptstyle u^2} \\ C^1 & \overset{h}{\dashrightarrow} & C^2 \\ {\scriptstyle s^1}\uparrow & & \uparrow{\scriptstyle s^2} \\ B^1 & \xrightarrow{g} & B^2 \end{array} \tag{1.3.2}$$

in $\mathcal{C}$, where the morphisms s^1, s^2 lie in S and $\alpha = (s^1/u^1)$ and $\beta = (s^2/u^2)$.

Finally, we state Gabriel's and Verdier's results on the localization of abelian and triangulated categories respectively. Specifically, Gabriel in his famous paper [18] proved the following:

Theorem 1.3.11. *Let $\mathcal{C}$ be an abelian category and S a bicalculable system. Then the category $\mathcal{C}[S^{-1}]$ is an abelian category, the localization functor Q is exact, and Q is universal for exact functors inverting the elements of S, out of $\mathcal{C}$ to abelian categories.*

Proof. For a detailed proof we refer the reader to [41, Chapter 4, Corollary 1.7]. ■

On the other hand, localization theory for triangulated categories was introduced and studied by Verdier in [44]. In this case the bicalculable system S satisfies a compatibility with the triangulated structure:

Definition 1.3.12. *([44, p. 112]) Let $\mathcal{C}$ be a triangulated and S a bicalculable system in $\mathcal{C}$. The class S is called* **compatible with the triangulation** *if it satisfies the following conditions:*

(SM5) For any morphism s in $\mathcal{C}$, $s \in S$ if and only if $\Sigma s \in S$.

(SM6) For any commutative diagram of the form:

$$\begin{array}{ccccccc} X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \xrightarrow{a^3} & \Sigma X^1 \\ \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & \downarrow{\scriptstyle s^3} & & \downarrow{\scriptstyle \Sigma s^1} \\ Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \xrightarrow{b^3} & \Sigma Y^1 \end{array}$$

whose rows are 3*-angles, which are called* **distinguished triangles** *in this setting, there exists a morphism* $s^3 \colon X^3 \longrightarrow Y^3$ *lying in* S*, that completes the diagram to a morphism of distinguished triangles.*

Then the triangulated analog of theorem of Theorem 1.3.11 holds:

Theorem 1.3.13. *([44, Theorem 2.2.6]) Let* $\mathcal{C}$ *be a triangulated category and* S *a bicalculable system in* $\mathcal{C}$ *which is compatible with the triangulation. Then the localization* $\mathcal{C}[S^{-1}]$ *is also a triangulated category, the localization functor* Q *is exact, and* Q *is universal for exact functors inverting the elements of* S*, out of* $\mathcal{C}$ *to triangulated categories.*

CHAPTER 2

PRE-n-ABELIAN CATEGORIES

The aim of this chapter is to develop the tools which will be used in the proof of the main result concerning the localization of n-abelian categories.

Due to the delicate behaviour of one specific axiom of an n-abelian category with respect to localization, namely the axiom (A0) of idempotent completeness, the construction of the localized category will be completed in two steps. In this chapter, we define a pre-n-abelian category to be an additive category which satisfies all axioms of an n-abelian category except that the axiom of idempotent completeness is not necessarily satisfied. Then, utilizing a result from Jasso in [28], we will provide a necessary and sufficient condition for a category to be pre-n-abelian, based on the exactness properties of a diagram which we will call an n-diagram.

2.1 Definition

In this section we study a slightly more general version of an n-abelian category in the sense that idempotent completeness is not necessarily satisfied. As mentioned before, this will be useful later in the localization of a genuine n-abelian category. We begin by defining a pre-n-abelian category and we provide a lemma which will be used later.

Definition 2.1.1. *Let n be a positive integer. An additive category $\mathcal{M}$ that satisfies axioms (A1), (A2), (A3) of an n-abelian category, see Definition 1.1.18, will be called a* **pre-n-abelian category**.

Remark 2.1.2. As in Remark 1.1.19, in the definition of a pre-n-abelian category, we may replace axioms (A2) and (A3) by axioms (A2') and (A3'), respectively.

Since our category is not necessarily idempotent complete, we will need the following lemma which will be used instead of Proposition 1.1.25, and relies on the fact that our category has n-cokernels. The general idea behind the proof is also similar to that of the previously mentioned proposition.

Lemma 2.1.3. *Let $\mathcal{M}$ be a pre-n-abelian category and $f: X \longrightarrow Y$, $g: X \longrightarrow Z$ two morphisms in $\mathcal{M}$. Then, there exist an object M in $\mathcal{M}$ and morphisms $u: Y \longrightarrow M$, $v: Z \longrightarrow M$ such that the following diagram is a weak pushout:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{v} & M \end{array}$$

Moreover, if f is a monomorphism, then v can be chosen to be a monomorphism.

Proof. Since M is a pre-n-abelian category, the morphism $(f, -g): X \longrightarrow Y \oplus Z$ has an n-cokernel:

$$X \xrightarrow{(f,-g)} Y \oplus Z \xrightarrow{{}^t(c^1_\alpha, c^1_\beta)} C^1 \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0 \qquad (2.1.1)$$

and as a result, the following square is a weak pushout:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle c^1_\alpha} \\ Z & \xrightarrow{c^1_\beta} & C^1 \end{array}$$

If f is an monomorphism, the morphism $(f, -g)$ is also a monomorphism and since $\mathcal{M}$ is a pre-n-abelian category, the sequence (2.1.1) is n-exact. Let $u: U \longrightarrow Z$ be a morphism such that $uc^1_\beta = 0$. Since (2.1.1) is n-exact, there exists a morphism $u': M \longrightarrow X$ such that $u = -u'g$ and $u'f = 0$. Finally, since f is a monomorphism, $u' = 0$ and then $u = 0$, thus c^1_β is a monomorphism. ■

Dually we have the following:

Lemma 2.1.4. *Let $\mathcal{M}$ be a pre-n-abelian category and $f: X \longrightarrow Z$, $g: Y \longrightarrow Z$ two morphisms in $\mathcal{M}$. Then, there exist an object M in $\mathcal{M}$ and morphisms $u: M \longrightarrow Y$, $v: M \longrightarrow X$ such that the following diagram is a weak pullback:*

$$\begin{array}{ccc} M & \xrightarrow{u} & Y \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

Moreover, if f is an epimorphism, then u can be chosen to be an epimorphism.

2.2 The n-diagram

Our first main result of the thesis is to show that the localization of a pre-n-abelian category $\mathcal{M}$ with respect to a bicalculable system of morphisms S is also a pre-n-abelian category.

For this reason we will provide some necessary and sufficient conditions for an additive category to be pre-n-abelian, based on the exactness properties of a diagram which we will construct by applying Jasso's result, Proposition 1.1.26. Note that Jasso in his analysis uses drastically idempotent completeness (axiom (A0)) in order to show that any category that satisfies axioms (A0) and (A1), has n-pullbacks and n-pushouts, see Proposition 1.1.25, whereas in a pre-n-abelian category axiom (A0) is not required.

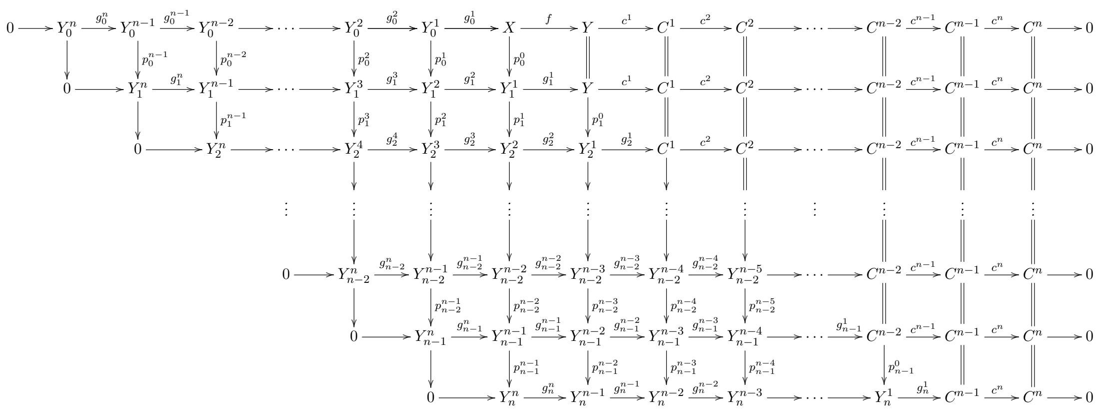
Thus, in what follows we will need a version of Proposition 1.1.26 in a pre-n-abelian category which is not necessarily idempotent complete. In this connection, below we prove this more general version in our setting verifying the points in the proof that differ.

Finally note that Ebrahimi and Nasr-Isfahani in [13, Proposition 2.7] prove, again in the setting of n-abelian categories, an analogous result related to Proposition 1.1.26 which we will discuss later providing a more general version of [13, Diagram (2.2)] in our setting.

Proposition 2.2.1. *Let $\mathcal{M}$ be a pre-n-abelian category, $f\colon X \longrightarrow Y$ a morphism in $\mathcal{M}$ and $(c^1, c^2, \cdots, c^n)$ an n-cokernel of f:*

$$X \xrightarrow{f} Y \xrightarrow{c^1} C^1 \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0$$

Then there exists a commutative diagram:



which satisfies the following properties:

1. *For $0 \leqslant i \leqslant n$, the sequence $(g_i^n, g_i^{n-1}, \cdots, g_i^1)$ is an n-kernel of c^i (where $c^0 = f$).*
2. *For $1 \leqslant i \leqslant n$, the (cone) sequence:*

$$0 \longrightarrow Y_{i-1}^n \xrightarrow{-g_{i-1}^n} Y_{i-1}^{n-1} \xrightarrow{(-g_{i-1}^{n-1},\, p_{i-1}^{n-1})} Y_{i-1}^{n-2} \oplus Y_i^n \xrightarrow{\begin{pmatrix} -g_{i-1}^{n-2} & p_{i-1}^{n-2} \\ 0 & g_i^n \end{pmatrix}} Y_{i-1}^{n-3} \oplus Y_i^{n-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow C^{i-2} \oplus Y_i^2 \xrightarrow{{}^t(p_{i-1}^0,\, g_i^2)} Y_i^1 \longrightarrow 0$$

(where $C^0 = Y$ and $C^{-1} = X$) is n-exact.
3. *For $1 \leqslant i \leqslant n$, the sequence $(g_i^{i-1}, \cdots, g_i^1, c^i, .., c^n)$ is an n-cokernel of g_i^i.*

Proof. As in the proof of [28, Proposition 3.13], we will construct the required diagram inductively.

- Let $(g_n^n, \cdots, g_n^1)$ be an n-kernel of c^n. Since c^n is an epimorphism and $\mathcal{M}$ is pre-n-abelian, the sequence $(g_n^n, \cdots, g_n^1, c^n)$ is n-exact. If $(g_{n-1}^n, \cdots, g_{n-1}^1)$ is an n-kernel of c^{n-1}, then we obtain a commutative diagram:

$$\begin{array}{ccccccccccccccccccc}
0 \to & Y_{n-1}^n & \xrightarrow{g_{n-1}^n} & Y_{n-1}^{n-1} & \xrightarrow{g_{n-1}^{n-1}} & Y_{n-1}^{n-2} & \xrightarrow{g_{n-1}^{n-2}} & Y_{n-1}^{n-3} & \xrightarrow{g_{n-1}^{n-3}} & Y_{n-1}^{n-4} & \to \cdots \to & Y_{n-1}^1 & \xrightarrow{g_{n-1}^1} & C^{n-2} & \xrightarrow{c^{n-1}} & C^{n-1} & \xrightarrow{c^n} & C^n & \to 0 \\
& \downarrow & & \downarrow{\scriptstyle p_{n-1}^{n-1}} & & \downarrow{\scriptstyle p_{n-1}^{n-2}} & & \downarrow{\scriptstyle p_{n-1}^{n-3}} & & \downarrow{\scriptstyle p_{n-1}^{n-4}} & & \downarrow{\scriptstyle p_{n-1}^1} & & \downarrow{\scriptstyle p_{n-1}^0} & & \| & & \| & \\
& 0 & \longrightarrow & Y_n^n & \xrightarrow{g_n^n} & Y_n^{n-1} & \xrightarrow{g_n^{n-1}} & Y_n^{n-2} & \xrightarrow{g_n^{n-2}} & Y_n^{n-3} & \to \cdots \to & Y_n^2 & \xrightarrow{g_n^2} & Y_n^1 & \xrightarrow{g_n^1} & C^{n-1} & \xrightarrow{c^n} & C^n & \to 0
\end{array}$$

where the existence of the vertical morphisms follows from the weak kernel property. Applying Lemma 1.1.14, it follows that we have constructed an n-pullback of the morphism p_{n-1}^0 along the sequence $(0, g_n^n, \cdots, g_n^2)$, i.e. the sequence:

$$0 \longrightarrow Y_{n-1}^n \xrightarrow{-g_{n-1}^n} Y_{n-1}^{n-1} \xrightarrow{(-g_{n-1}^{n-1},\, p_{n-1}^{n-1})} Y_{n-1}^{n-2} \oplus Y_n^n \xrightarrow{\begin{pmatrix} -g_{n-1}^{n-2} & p_{n-1}^{n-2} \\ 0 & g_n^n \end{pmatrix}} Y_{n-1}^{n-3} \oplus Y_n^{n-1} \longrightarrow \cdots \longrightarrow C^{n-2} \oplus Y_n^2 \xrightarrow{{}^t(p_{n-1}^0,\, g_n^2)} Y_n^1$$

is left n-exact. In order to prove that the sequence is n-exact, since $\mathcal{M}$ is pre-n-abelian, it is enough to show that the morphism ${}^t(p_{n-1}^0,\ g_n^2)$ is an epimorphism. Jasso's proof of this statement and the rest of the properties concerning this stage of the induction can be applied in our setting while for the reader's convenience we continue by mentioning the proof. Let $u \colon Y_n^1 \longrightarrow M$ be a morphism such that $p_{n-1}^0 u = 0$ and $g_n^2 u = 0$. Since g_n^1 is a weak cokernel of g_n^2, there exists a morphism $u' \colon C^{n-1} \longrightarrow M$ such that $u = g_n^1 u'$. Then:

$$c^{n-1} u' = p_{n-1}^0 g_n^1 u' = p_{n-1}^0 u = 0$$

and similarly, since c^n is a weak cokernel of c^{n-1}, there exists a morphism $u'' \colon C^n \longrightarrow M$ such that $u' = c^n u''$. Thus:

$$u = g_n^1 u' = g_n^1 c^n u'' = 0$$

which proves that ${}^t(p^0_{n-1},\, g^2_n)$ is an epimorphism and the above diagram is also an n-pushout diagram.

Since for $2 \leqslant i \leqslant n-1$ the morphism g^i_n is a weak cokernel of g^{i+1}_n, it follows from Lemma 1.1.13 that for $1 \leqslant i \leqslant n-2$, the morphism g^i_{n-1} is a weak cokernel of the morphism g^{i+1}_{n-1}.

Finally, we show that c^{n-1} is a weak cokernel of g^1_{n-1}. Let $u\colon C^{n-2} \longrightarrow M$ be such that $g^1_{n-1}u = 0$. Then, there exists a morphism $u'\colon Y^1_n \longrightarrow M$ such that $g^2_n u' = 0$ and $u = p^0_{n-1}u'$, as seen in the following commutative diagram:

$$\begin{array}{ccccccc}
Y^1_{n-1} \oplus Y^3_n & \xrightarrow{\begin{pmatrix} -g^1_{n-1} & p^1_{n-1} \\ 0 & g^3_n \end{pmatrix}} & C^{n-2} \oplus Y^2_n & \xrightarrow{{}^t(p^0_{n-1},\, g^2_n)} & Y^1_n & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle {}^t(u,0)} & \swarrow {\scriptstyle u'} & & & \\
 & & M & & & &
\end{array}$$

Again, since g^1_n is a weak cokernel of g^2_n, there exists a morphism $u''\colon C^{n-1} \longrightarrow U$ such that $u' = g^1_n u''$ and then:

$$u = p^0_{n-1}u' = p^0_{n-1}g^1_n u'' = c^{n-1}u''$$

• Inductively, for $2 \leqslant i \leqslant n-1$, we assume that for any $i \leqslant j \leqslant n$ we have constructed a diagram with the required properties. Let $(g^n_{i-1}, g^{n-1}_{i-1}, \cdots, g^1_{i-1})$ be an n-kernel of c^{i-1}:

$$\begin{array}{ccccccccccccccccccccccccc}
0 & \longrightarrow & Y^n_{i-1} & \xrightarrow{g^n_{i-1}} & Y^{n-1}_{i-1} & \xrightarrow{g^{n-1}_{i-1}} & Y^{n-2}_{i-1} & \xrightarrow{g^{n-2}_{i-1}} & Y^{n-3}_{i-1} & \longrightarrow & \cdots & \longrightarrow & Y^1_{i-1} & \xrightarrow{g^1_{i-1}} & C^{i-2} & \xrightarrow{c^{i-1}} & C^{i-1} & \xrightarrow{c^i} & C^i & \longrightarrow & \cdots & \longrightarrow & C^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle p^{n-1}_{i-1}} & & \downarrow {\scriptstyle p^{n-2}_{i-1}} & & \downarrow {\scriptstyle p^{n-3}_{i-1}} & & & & \downarrow {\scriptstyle p^1_{i-1}} & & \downarrow {\scriptstyle p^0_{i-1}} & & \| & & \| & & & & \| & & \\
 & & 0 & \longrightarrow & Y^n_i & \xrightarrow{g^n_i} & Y^{n-1}_i & \xrightarrow{g^{n-1}_i} & Y^{n-2}_i & \longrightarrow & \cdots & \longrightarrow & Y^2_i & \xrightarrow{g^2_i} & Y^1_i & \xrightarrow{g^1_i} & C^{i-1} & \xrightarrow{c^i} & C^i & \longrightarrow & \cdots & \longrightarrow & C^n & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow {\scriptstyle p^{n-1}_i} & & \downarrow {\scriptstyle p^{n-2}_i} & & & & \downarrow {\scriptstyle p^2_i} & & \downarrow {\scriptstyle p^1_i} & & \downarrow {\scriptstyle p^0_i} & & \| & & & & \| & & \\
 & & & & 0 & \longrightarrow & Y^n_{i+1} & \xrightarrow{g^n_{i+1}} & Y^{n-1}_{i+1} & \longrightarrow & \cdots & \longrightarrow & Y^3_{i+1} & \xrightarrow{g^3_{i+1}} & Y^2_{i+1} & \xrightarrow{g^2_{i+1}} & Y^1_{i+1} & \xrightarrow{g^1_{i+1}} & C^i & \longrightarrow & \cdots & \longrightarrow & C^n & \longrightarrow & 0
\end{array}$$

By Lemma 1.1.14 we have constructed in this way an n-pullback diagram of the morphism p^0_{i-1} along the sequence $(0, g^n_i, \cdots, g^2_i)$, i.e. the sequence:

$$0 \longrightarrow Y^n_{i-1} \xrightarrow{-g^n_{i-1}} Y^{n-1}_{i-1} \xrightarrow{(-g^{n-1}_{i-1},\, p^{n-1}_{i-1})} Y^{n-2}_{i-1} \oplus Y^n_i \xrightarrow{\begin{pmatrix} -g^{n-2}_{i-1} & p^{n-2}_{i-1} \\ 0 & g^n_i \end{pmatrix}} Y^{n-3}_{i-1} \oplus Y^{n-1}_i \longrightarrow \cdots \longrightarrow C^{i-2} \oplus Y^2_i \xrightarrow{{}^t(p^0_{i-1},\, g^2_i)} Y^1_i$$

is left n-exact. In order to prove that the above sequence is n-exact, since $\mathcal{M}$ is pre-n-abelian, it is enough to show that the morphism ${}^t(p^0_{i-1},\, g^2_i)$ is an epimorphism. As in the previous inductive step, Jasso's proof of this statement applies in our setting: Let $u\colon Y^1_i \longrightarrow M$ be such that $p^0_{i-1}u = 0$ and $g^2_i u = 0$. Since by the induction hypothesis

g_i^1 is a weak cokernel of g_i^2, there exists a morphism $u' \colon C^{i-1} \longrightarrow M$ such that $u = g_i^1 u'$. Then:

$$c^{i-1}u' = p_{i-1}^0 g_i^1 u' = p_{i-1}^0 u = 0$$

and similarly, since c^i is a weak cokernel of c^{i-1}, there exists a morphism $u'' \colon C^i \longrightarrow M$ such that $u' = c^i u''$. Thus:

$$u = g_i^1 u' = g_i^1 c^i u'' = 0$$

which proves that ${}^t(p_{i-1}^0,\, g_i^2)$ is an epimorphism and the above diagram is also an n-pushout diagram.

By the induction hypothesis, for $0 \leqslant k \leqslant i-1$, the morphism g_i^k is a weak cokernel of g_i^{k+1} and applying Lemma 1.1.13, it follows that for $1 \leqslant k \leqslant i-2$, the morphism g_{i-1}^k is a weak cokernel of g_{i-1}^{k+1}.

Finally, we show that c^{i-1} is a weak cokernel of g_{i-1}^1. Let $u \colon C^{i-2} \longrightarrow M$ be such that $g_{i-1}^1 u = 0$. Then, there exists a morphism $u' \colon Y_i^1 \longrightarrow M$ such that $g_i^2 u' = 0$ and $u = p_{i-1}^0 u'$, as seen in the following commutative diagram:

$$\begin{array}{ccccccc}
Y_{i-1}^1 \oplus Y_i^3 & \xrightarrow{\begin{pmatrix} -g_{i-1}^1 & 0 \\ p_{i-1}^1 & g_i^3 \end{pmatrix}} & C^{i-2} \oplus Y_i^2 & \xrightarrow{{}^t(p_{i-1}^0,\, g_i^2)} & Y_i^1 & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle {}^t(u,0)} & \swarrow {\scriptstyle u'} & & & \\
 & & M & & & &
\end{array}$$

Since g_i^1 is a weak cokernel of g_i^2, there exists a morphism $u'' \colon C^{i-1} \longrightarrow M$ such that $u' = g_i^1 u''$ and then:

$$u = p_{i-1}^0 u' = p_{i-1}^0 g_i^1 u'' = c^{i-1} u''$$

- Finally, let $(g_0^n, g_0^{n-1}, \cdots, g_0^1)$ be an n-kernel of f.

$$\begin{array}{cccccccccccccccccccccccccc}
0 & \to & Y_0^n & \xrightarrow{g_0^n} & Y_0^{n-1} & \xrightarrow{g_0^{n-1}} & Y_0^{n-2} & \xrightarrow{g_0^{n-2}} & Y_0^{n-3} & \to \cdots \to & Y_0^2 & \xrightarrow{g_0^2} & Y_0^1 & \xrightarrow{g_0^1} & X & \xrightarrow{f} & Y & \xrightarrow{c^1} & C^1 & \xrightarrow{c^2} & C^2 & \to \cdots \\
 & & \downarrow & & \downarrow {\scriptstyle p_0^{n-1}} & & \downarrow {\scriptstyle p_0^{n-2}} & & \downarrow {\scriptstyle p_0^{n-3}} & & \downarrow {\scriptstyle p_0^2} & & \downarrow {\scriptstyle p_0^1} & & \downarrow {\scriptstyle p_0^0} & & \| & & \| & & \| & \\
 & & 0 & \to & Y_1^n & \xrightarrow{g_1^n} & Y_1^{n-1} & \xrightarrow{g_1^{n-1}} & Y_1^{n-2} & \to \cdots \to & Y_1^3 & \xrightarrow{g_1^3} & Y_1^2 & \xrightarrow{g_1^2} & Y_1^1 & \xrightarrow{g_1^1} & Y & \xrightarrow{c^1} & C^1 & \xrightarrow{c^2} & C^2 & \to \cdots \\
 & & & & \downarrow & & \downarrow {\scriptstyle p_1^{n-1}} & & \downarrow {\scriptstyle p_1^{n-2}} & & \downarrow {\scriptstyle p_1^3} & & \downarrow {\scriptstyle p_1^2} & & \downarrow {\scriptstyle p_1^1} & & \downarrow {\scriptstyle p_1^0} & & \| & & \| & \\
 & & & & 0 & \to & Y_2^n & \xrightarrow{g_2^n} & Y_2^{n-1} & \to \cdots \to & Y_2^4 & \xrightarrow{g_2^4} & Y_2^3 & \xrightarrow{g_2^3} & Y_2^2 & \xrightarrow{g_2^2} & Y_2^1 & \xrightarrow{g_2^1} & C^1 & \xrightarrow{c^2} & C^2 & \to \cdots
\end{array}$$

As before, by Lemma 1.1.14 we have constructed an n-pullback diagram of the morphism p_0^0 along the complex $(0, g_1^n, \cdots, g_1^2)$, i.e. the sequence:

$$0 \longrightarrow Y_0^n \xrightarrow{-g_0^n} Y_0^{n-1} \xrightarrow{(-g_0^{n-1},\, p_0^{n-1})} Y_0^{n-2} \oplus Y_1^n \xrightarrow{\begin{pmatrix} -g_0^{n-2} & p_0^{n-2} \\ 0 & g_1^n \end{pmatrix}} Y_0^{n-3} \oplus Y_1^{n-1} \longrightarrow \cdots \longrightarrow X \oplus Y_1^2 \xrightarrow{{}^t(p_0^0,\, g_1^2)} Y_1^1$$

is left n-exact. It remains to show that the morphism ${}^{t}(p_0^0,\, g_1^2)$ is an epimorphism. Let $u\colon Y_1^1 \longrightarrow M$ be such that $p_0^0 u = 0$ and $g_1^2 u = 0$. We claim that u is of the form

$$u = -g_1^1 a^{n-1} + p_1^1 p_2^2 \cdots p_{n-1}^{n-1} u^{n-1}$$

where $a^{n-1}\colon Y \longrightarrow M$ and $u^{n-1}\colon Y_n^n \longrightarrow M$. To prove this, we proceed with induction using the sequences:

$$0 \longrightarrow Y_{i-1}^{n} \xrightarrow{-g_{i-1}^{n}} Y_{i-1}^{n-1} \xrightarrow{(-g_{i-1}^{n-1},\, p_{i-1}^{n-1})} Y_{i-1}^{n-2} \oplus Y_i^n \xrightarrow{\begin{pmatrix} -g_{i-1}^{n-2} & p_{i-1}^{n-2} \\ 0 & g_i^n \end{pmatrix}} Y_{i-1}^{n-3} \oplus Y_i^{n-1} \longrightarrow \cdots \longrightarrow C^{i-2} \oplus Y_i^2 \xrightarrow{{}^{t}(p_{i-1}^0,\, g_i^2)} Y_i^1 \longrightarrow 0 \tag{2.2.1}$$

for $2 \leqslant i \leqslant n$, which by the previous steps are proven to be n-exact.

(a) For $i = 2$, since $g_1^2 u = 0$, from the n-exact sequences (2.2.1), we obtain a commutative diagram of the form:

$$\begin{array}{ccccc} Y_1^2 \oplus Y_2^4 & \xrightarrow{\begin{pmatrix} -g_1^2 & p_1^2 \\ 0 & g_2^4 \end{pmatrix}} & Y_1^1 \oplus Y_2^3 & \xrightarrow{\begin{pmatrix} -g_1^1 & p_1^1 \\ 0 & g_2^3 \end{pmatrix}} & Y \oplus Y_2^2 \\ & & \Big\downarrow {\scriptstyle {}^{t}(u,0)} & \swarrow {\scriptstyle {}^{t}(a^1,u^1)} & \\ & & M & & \end{array}$$

and then $u = -g_1^1 \alpha^1 + p_1^1 u^1$ while $g_2^3 u^1 = 0$.

(b) For $2 \leqslant i \leqslant n-2$, we assume that:

$$u = -g_1^1 a^{i-1} + p_1^1 p_2^2 \cdots p_{i-1}^{i-1} u^{i-1}$$

and $g_i^{i+1} u^{i-1} = 0$ where $a^{i-1}\colon Y \longrightarrow M$ and $u^{i-1}\colon Y_i^i \longrightarrow M$. Then, from the n-exact sequences (2.2.1), we obtain a diagram of the form:

$$\begin{array}{ccccc} Y_i^{i+1} \oplus Y_{i+1}^{i+3} & \xrightarrow{\begin{pmatrix} -g_i^{i+1} & p_i^{i+1} \\ 0 & g_{i+1}^{i+3} \end{pmatrix}} & Y_i^i \oplus Y_{i+1}^{i+2} & \xrightarrow{\begin{pmatrix} -g_i^i & p_i^i \\ 0 & g_{i+1}^{i+2} \end{pmatrix}} & Y_i^{i-1} \oplus Y_{i+1}^{i+1} \\ & & \Big\downarrow {\scriptstyle {}^{t}(u^{i-1},0)} & \swarrow {\scriptstyle {}^{t}(b^i,u^i)} & \\ & & M & & \end{array}$$

and then $u^{i-1} = -g_i^i b^i + p_i^i u^i$ while $g_{i+1}^{i+2} u^i = 0$. We conclude that

$$\begin{aligned} u &= -g_1^1 a^{i-1} + p_1^1 p_2^2 \cdots p_{i-1}^{i-1}(-g_i^i b^i + p_i^i u^i) \\ &= -g_1^1 a^{i-1} - p_1^1 \cdots p_{i-1}^{i-1} g_i^i b^i + p_1^1 \cdots p_i^i u^i \\ &= -g_1^1 a^{i-1} - g_1^1 p_1^0 \cdots p_{i-1}^{i-2} b^i + p_1^1 \cdots p_i^i u^i \\ &= -g_1^1 (a^{i-1} + p_1^0 \cdots p_{i-1}^{i-2} b^i) + p_1^1 \cdots p_i^i u^i \\ &= -g_1^1 a^i + p_1^1 \cdots p_i^i u^i \end{aligned}$$

where $a^i\colon Y \longrightarrow M$.

(c) Finally, we assume that

$$u = -g_1^1 a^{n-2} + p_1^1 p_2^2 ... p_{n-2}^{n-2} u^{n-2}$$

and $g^n_{n-1} u^{n-2} = 0$, where $a^{n-2} \colon Y \longrightarrow M$ and $u^{n-2} \colon Y^{n-1}_{n-1} \longrightarrow M$. Again, from the n-exact sequences (2.2.1),we obtain a commutative diagram:

$$\begin{array}{ccccc} Y^n_{n-1} & \xrightarrow{-g^n_{n-1}} & Y^{n-1}_{n-1} & \xrightarrow{(-g^{n-1}_{n-1},\, p^{n-1}_{n-1})} & Y^{n-2}_{n-1} \oplus Y^n_n \\ & & {\scriptstyle u^{n-2}} \downarrow & \swarrow {\scriptstyle {}^t(b^{n-1},\, u^{n-1})} & \\ & & M & & \end{array}$$

and then $u^{n-2} = -g^{n-1}_{n-1} b^{n-1} + p^{n-1}_{n-1} u^{n-1}$. We conclude that

$$\begin{aligned} u &= -g_1^1 a^{n-2} + p_1^1 p_2^2 \cdots p_{n-2}^{n-2}(-g^{n-1}_{n-1} b^{n-1} + p^{n-1}_{n-1} u^{n-1}) \\ &= -g_1^1 a^{n-2} - p_1^1 \cdots p_{n-2}^{n-2} g^{n-1}_{n-1} b^{n-1} + p_1^1 \cdots p^{n-1}_{n-1} u^{n-1} \\ &= -g_1^1 a^{n-2} - g_1^1 p_1^0 \cdots p^{n-2}_{n-1} b^{n-1} + p_1^1 \cdots p^{n-1}_{n-1} u^{n-1} \\ &= -g_1^1 (a^{n-2} + p_1^0 \cdots p^{n-2}_{n-1} b^{n-1}) + p_1^1 \cdots p^{n-1}_{n-1} u^{n-1} \\ &= -g_1^1 \alpha^{n-1} + p_1^1 \cdots p^{n-1}_{n-1} u^{n-1} \end{aligned}$$

where $a^{n-1} \colon Y \longrightarrow M$, which proves our claim.

In [28], using the same inductive process as above, u is shown to be of the form:

$$u = p_1^1 p_2^2 \cdots p^{n-1}_{n-1} u^{n-1}$$

with the difference being due to the fact that in our setting we don't use the notion of a good pushout diagram, see [28, Definition-Proposition 2.14].

We have verified our claim that:

$$u = -g_1^1 a^{n-1} + p_1^1 p_2^2 \cdots p^{n-1}_{n-1} u^{n-1}$$

where $a^{n-1} \colon Y \longrightarrow M$ and $u^{n-1} \colon Y^n_n \longrightarrow M$. Since $\mathcal{M}$ is a pre-n-abelian category, applying Lemma 2.1.3, there exist morphisms $w \colon M \longrightarrow M'$ and $v^{n-1} \colon Y^{n-1}_n \longrightarrow M'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} Y^n_{n-1} & \xrightarrow{g^n_{n-1}} & Y^{n-1}_{n-1} & \xrightarrow{g^{n-1}_{n-1}} & Y^{n-2}_{n-1} \\ \downarrow & & \downarrow {\scriptstyle p^{n-1}_{n-1}} & & \downarrow {\scriptstyle p^{n-2}_{n-1}} \\ 0 & \longrightarrow & Y^n_n & \xrightarrow{g^n_n} & Y^{n-1}_n \\ & & \downarrow {\scriptstyle u^{n-1}} & & \downarrow {\scriptstyle v^{n-1}} \\ 0 & \longrightarrow & M & \xrightarrow{w} & M' \end{array}$$

and since g_n^n is a monomorphism, the morphism $w\colon M \longrightarrow M'$ can be chosen to be a monomorphism. Then, similarly to [28, Proposition 3.13]:

$$\begin{aligned} fp_1^0 \cdots p_{n-1}^{n-2} v^{n-1} &= p_0^0 p_1^1 \cdots p_{n-1}^{n-1} u^{n-1} w \\ &= p_0^0 (u + g_1^1 a^{n-1}) w \\ &= p_0^0 uw + p_0^0 g_1^1 a^{n-1} w \\ &= f a^{n-1} w \end{aligned}$$

Thus:

$$f(p_1^0 \cdots p_{n-1}^{n-2} v^{n-1} - a^{n-1} w) = 0$$

and since c^1 is a weak cokernel of f there exists a morphism $w'\colon C^1 \longrightarrow M$ such that:

$$p_1^0 \cdots p_{n-1}^{n-2} v^{n-1} - a^{n-1} w = c^1 w'$$

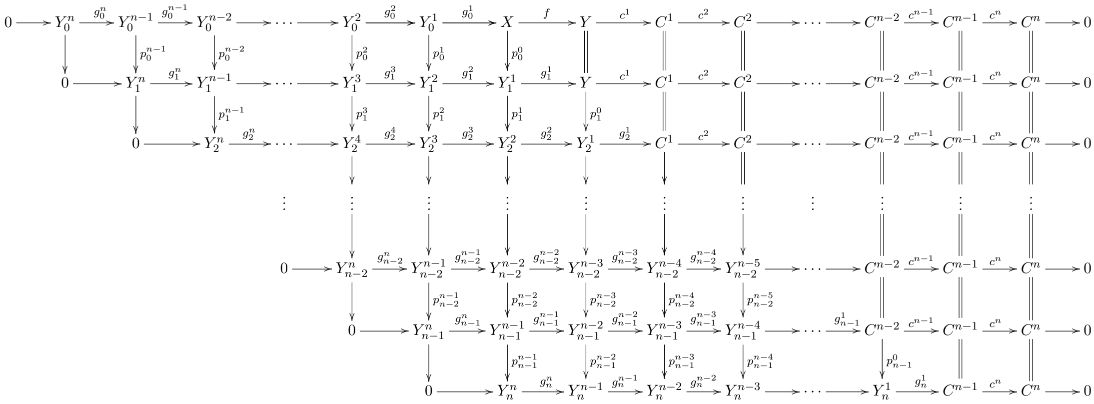
Finally,

$$\begin{aligned} uw &= (-g_1^1 a^{n-1} + p_1^1 p_2^2 \cdots p_{n-1}^{n-1} u^{n-1}) w \\ &= -g_1^1 a^{n-1} w + p_1^1 p_2^2 \cdots p_{n-1}^{n-1} u^{n-1} w \\ &= -g_1^1 a^{n-1} w + g_1^1 p_1^0 \cdots p_{n-1}^{n-2} v^{n-1} \\ &= g_1^1 (-a^{n-1} w + p_1^0 \cdots p_{n-1}^{n-2} v^{n-1}) = g_1^1 c^1 w' = 0 \end{aligned}$$

and since w is a monomorphism, $u = 0$. ■

Ebrahimi and Nasr-Isfahani proved in [13, Proposition 2.7] that in the construction of the diagram of Proposition 2.2.1 for an n-abelian category, the sequence $(c^1, \cdots, c^n)$ is an n-cokernel of f if and only if the morphisms ${}^t(p_{i-1}^0, g_i^2)\colon C^{i-2} \oplus Y_i^2 \longrightarrow Y_i^1$ are epimorphisms for $1 \leqslant i \leqslant n$. We state this additional result in our setting in a form that will be useful later and for the sake of completeness, we provide a proof which is an adaptation of the proof of [13, Proposition 2.7].

Corollary 2.2.2. *Let $\mathcal{M}$ be an additive category, $f\colon X \longrightarrow Y$ a morphism in $\mathcal{M}$ and a* *commutative diagram:*



in $\mathcal{M}$ whose rows are complexes and satisfies the following properties:

1. *For $1 \leqslant i \leqslant n$, the (cone) sequence:*

$$0 \longrightarrow Y_{i-1}^{n} \xrightarrow{-g_{i-1}^{n}} Y_{i-1}^{n-1} \xrightarrow{(-g_{i-1}^{n-1}, p_{i-1}^{n-1})} Y_{i-1}^{n-2} \oplus Y_i^n \xrightarrow{\begin{pmatrix} -g_{i-1}^{n-2} & p_{i-1}^{n-2} \\ 0 & g_i^n \end{pmatrix}} Y_{i-1}^{n-3} \oplus Y_i^{n-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow C^{i-2} \oplus Y_i^2 \xrightarrow{{}^t(p_{i-1}^0, g_i^2)} Y_i^1 \longrightarrow 0$$

   (where $C^0 = Y$ and $C^{-1} = X$) is n-exact.

2. *The sequence*

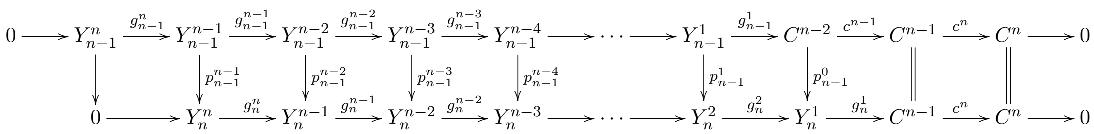
$$0 \longrightarrow Y_n^n \xrightarrow{g_n^n} Y_n^{n-1} \longrightarrow \cdots \longrightarrow Y_n^1 \xrightarrow{g_n^1} C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0$$

   is n-exact.

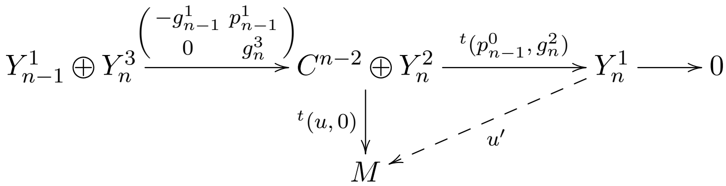
Then, for $1 \leqslant i \leqslant n$, the sequence $(g_i^{i-1}, \cdots, g_i^1, c^i, \cdots, c^n)$ is an n-cokernel of g_i^i and the sequence $(c^1, \cdots, c^n)$ is an n-cokernel of f.

Proof. With the above conditions, it has already been shown in the proof of Proposition 2.2.1 that for $1 \leqslant i \leqslant n$, in the sequence $(g_i^{i-1}, \cdots, g_i^1, c^i, \cdots, c^n)$ the morphism g_i^j is a weak cokernel of g_i^{j+1}, where $0 \leqslant j \leqslant i-1$ and $g_i^0 = c^i$.

- Clearly, by (2), the sequence $(g_n^{n-1}, \cdots, g_n^1, c^n)$ is an n-cokernel of g_n^n.



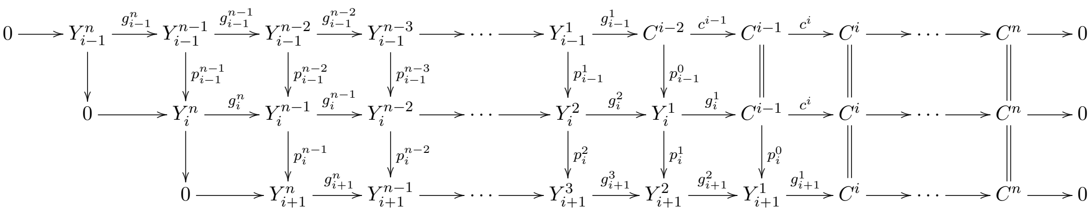
Applying Lemma 1.1.13, since for $2 \leqslant i \leqslant n-1$ the morphism g_n^i is a weak cokernel of g_n^{i+1}, it follows that for $1 \leqslant i \leqslant n-2$, the morphism g_{n-1}^i is a weak cokernel of the morphism g_{n-1}^{i+1}. Finally, we show that c^{n-1} is a weak cokernel of g_{n-1}^1. Let $u\colon C^{n-2} \longrightarrow M$ be such that $g_{n-1}^1 u = 0$. Then, there exists a morphism $u'\colon Y_n^1 \longrightarrow M$ such that $g_n^2 u' = 0$ and $u = p_{n-1}^0 u'$, as seen in the following commutative diagram:



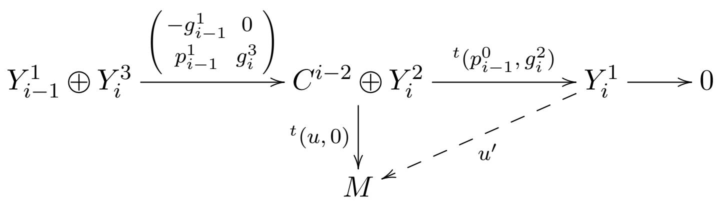
Again, since g_n^1 is a weak cokernel of g_n^2, there exists a morphism $u''\colon C^{n-1} \longrightarrow U$ such that $u' = g_n^1 u''$ and then:

$$u = p_{n-1}^0 u' = p_{n-1}^0 g_n^1 u'' = c^{n-1} u''$$

• Inductively, for $2 \leqslant i \leqslant n-1$, we assume that our claim holds for any $j \geqslant i$ and we will prove that it also holds for $j = i-1$:



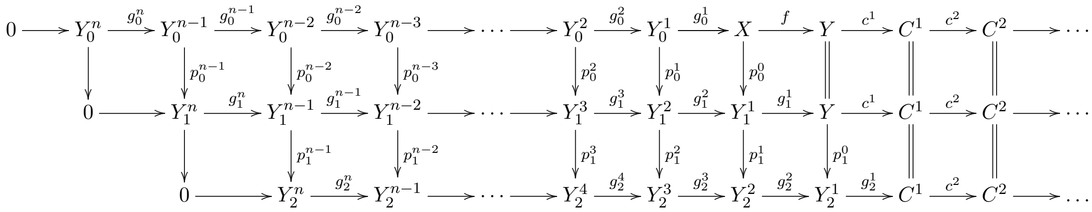
Applying Lemma 1.1.13, since by the induction hypothesis, for $2 \leqslant j \leqslant i-1$, the morphism g_i^j is a weak cokernel of g_i^{j+1}, it follows that for $1 \leqslant j \leqslant i-2$, the morphism g_{i-1}^j is a weak cokernel of g_{i-1}^{j+1}. Finally, we show that c^{i-1} is a weak cokernel of g_{i-1}^1. Let $u\colon C^{i-2} \longrightarrow M$ be such that $g_{i-1}^1 u = 0$. Then, there exists a morphism $u'\colon Y_i^1 \longrightarrow M$ such that $g_i^2 u' = 0$ and $u = p_{i-1}^0 u'$, as seen in the following commutative diagram:



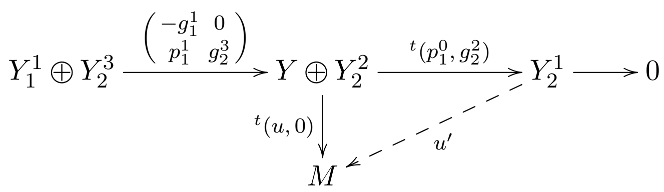
Since g_i^1 is a weak cokernel of g_i^2, there exists a morphism $u''\colon C^{i-1} \longrightarrow M$ such that $u' = g_k^1 u''$ and then:

$$u = p_{i-1}^0 u' = p_{i-1}^0 g_i^1 u'' = c^{i-1} u''$$

• Finally, we assume that in the sequence $(g_2^1, c^2, \cdots, c^n)$, g_2^1 is a weak cokernel of g_2^2 and c^2 is a weak cokernel of g_2^1. We will prove that c^1 is a weak cokernel of g_1^1.



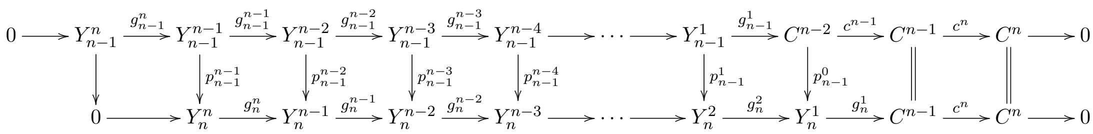
Let $u\colon Y_1^1 \longrightarrow M$ be such that $g_1^1 u = 0$. Then, there exists a morphism $u'\colon Y_2^1 \longrightarrow M$ such that $g_2^2 u' = 0$ and $u = p_1^0 u'$, as seen in the following commutative diagram:



Since g_2^1 is a weak cokernel of g_2^2, there exists a morphism $u''\colon C^1 \longrightarrow M$ such that $u' = g_2^1 u''$ and then:

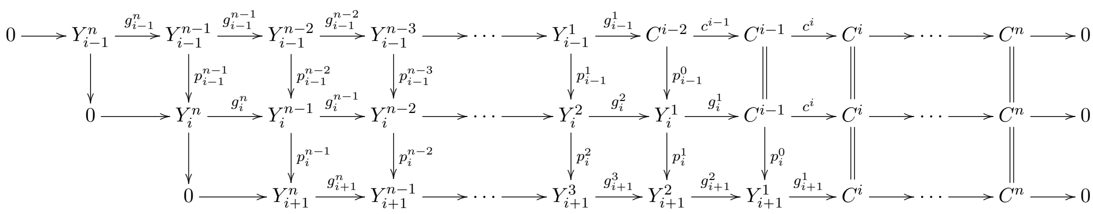
$$u = p_1^0 u' = p_1^0 g_2^1 u'' = c^1 u''$$

It remains to show that $(c^1, \cdots, c^n)$ is an n-cokernel of f.



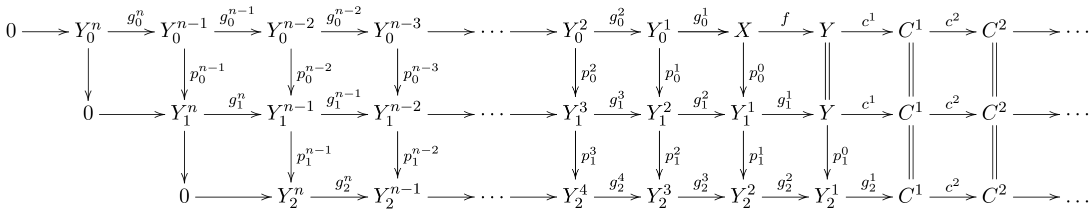
• We begin by proving that c^n is a cokernel of c^{n-1}. Clearly, c^n is an epimorphism and let $u\colon C^{n-1} \longrightarrow M$ be a morphism such that $c^{n-1}u = 0$. Then, ${}^t(p_{n-1}^0,\, g_n^2) g_n^1 u = 0$ and since ${}^t(p_{n-1}^0,\, g_n^2)$ is an epimorphism, $g_n^1 u = 0$. Since c^n is a cokernel of g_n^1, there exists a morphism $u'\colon C^n \longrightarrow M$ such that $u = c^n u'$.

• Inductively, for $2 \leqslant i \leqslant n-1$, we will show that c^i is a weak cokernel of c^{i-1}.



Let $u\colon C^{i-1} \longrightarrow M$ be a morphism such that $c^{i-1}u = 0$. Then, ${}^t(p^0_{i-1},\, g^2_i)g^1_i u = 0$ and since ${}^t(p^0_{i-1},\, g^2_i)$ is an epimorphism, $g^1_i u = 0$. Since c^i is a weak cokernel of g^1_i, there exists a morphism $u'\colon C^i \longrightarrow M$ such that $u = c^i u'$.

- Finally, we will show that c^1 is a weak cokernel of f.



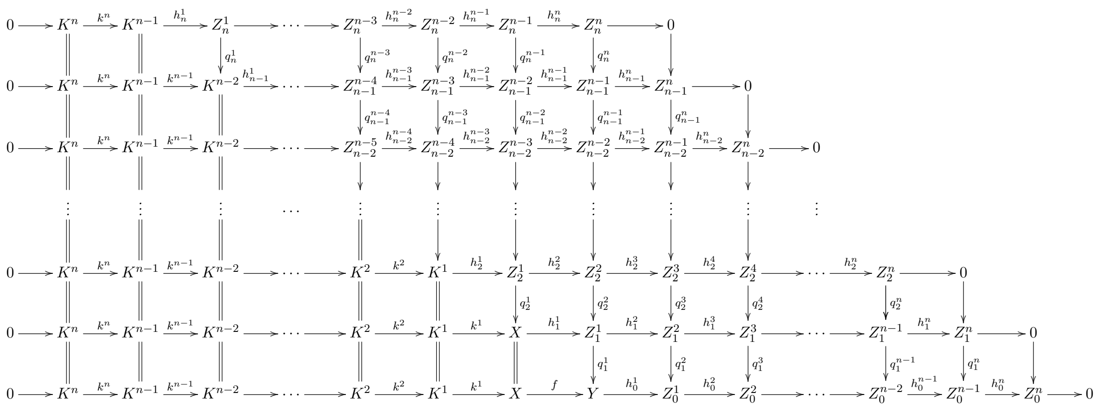
Let $u\colon Y \longrightarrow M$ be a morphism such that $fu = 0$. Then, ${}^t(p^0_0,\, g^2_1)g^1_1 u = 0$ and since ${}^t(p^0_0,\, g^2_1)$ is an epimorphism, $g^1_1 u = 0$. Since c^1 is a weak cokernel of g^1_1, there exists a morphism $u'\colon C^1 \longrightarrow M$ such that $u = c^1 u'$. ■

For the sake of completeness, we state the dual versions of Proposition 2.2.1 and Corollary 2.2.2.

Proposition 2.2.3. *Let $\mathcal{M}$ be a pre-n-abelian category, $f\colon X \longrightarrow Y$ a morphism in $\mathcal{M}$ and $(k^n, k^{n-1}, \cdots, k^1)$ an n-kernel of f:*

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^2} K^1 \xrightarrow{k^1} X \xrightarrow{f} Y$$

Then, there exists a commutative diagram:



which satisfies the following properties:

1. *For $0 \leqslant i \leqslant n$, the sequence $(h^1_i, h^2_i, \cdots, h^n_i)$ is an n-cokernel of k^i (where $k^0 = f$).*

2. For $1 \leqslant i \leqslant n$, the (cone) sequence:

$$0 \longrightarrow Z_i^1 \xrightarrow{(-h_i^2,\, q_i^1)} Z_i^2 \oplus K^{i-2} \xrightarrow{\begin{pmatrix} -h_i^3 & q_i^2 \\ 0 & h_{i-1}^1 \end{pmatrix}} Z_i^3 \oplus Z_{i-1}^1 \longrightarrow \cdots$$

$$\cdots \longrightarrow Z_i^n \oplus Z_{i-1}^{i-2} \xrightarrow{{}^t(q_i^n,\, h_{i-1}^{n-1})} Z_{i-1}^{n-1} \xrightarrow{h_{i-1}^n} Z_{i-1}^n \longrightarrow 0$$

(where $K^0 = X$ and $K^{-1} = Y$) is n-exact.

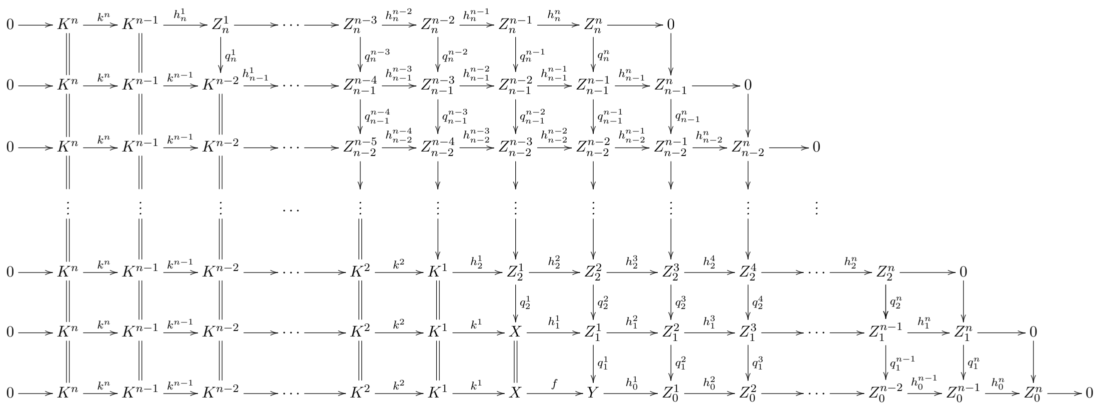
3. For $1 \leqslant i \leqslant n$, the sequence $(k^n, \cdots, k^i, h_i^1, \cdots, h_i^{i-1})$ is an n-kernel of h_i^i.

Proof. The proof is dual to the proof of Proposition 2.2.1. ■

Corollary 2.2.4. *Let $\mathcal{M}$ be an additive category, $f\colon X \longrightarrow Y$ a morphism in $\mathcal{M}$ and a commutative diagram:*

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^2} K^1 \xrightarrow{k^1} X \xrightarrow{f} Y$$

Then, there exists a commutative diagram:



in $\mathcal{M}$ whose rows are complexes and satisfies the following properties:

(i) For $1 \leqslant i \leqslant n$, the (cone) sequence:

$$0 \longrightarrow Z_i^1 \xrightarrow{(-h_i^2,\, q_i^1)} Z_i^2 \oplus K^{i-2} \xrightarrow{\begin{pmatrix} -h_i^3 & q_i^2 \\ 0 & h_{i-1}^1 \end{pmatrix}} Z_i^3 \oplus Z_{i-1}^1 \longrightarrow \cdots$$

$$\cdots \longrightarrow Z_i^n \oplus Z_{i-1}^{i-2} \xrightarrow{{}^t(q_i^n,\, h_{i-1}^{n-1})} Z_{i-1}^{n-1} \xrightarrow{h_{i-1}^n} Z_{i-1}^n \longrightarrow 0$$

(where $K^0 = X$ and $K^{-1} = Y$) is n-exact.

(ii) The sequence

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{h_n^1} Z_n^1 \cdots \longrightarrow Z_n^{n-1} \xrightarrow{h_n^n} Z_n^n \longrightarrow 0$$

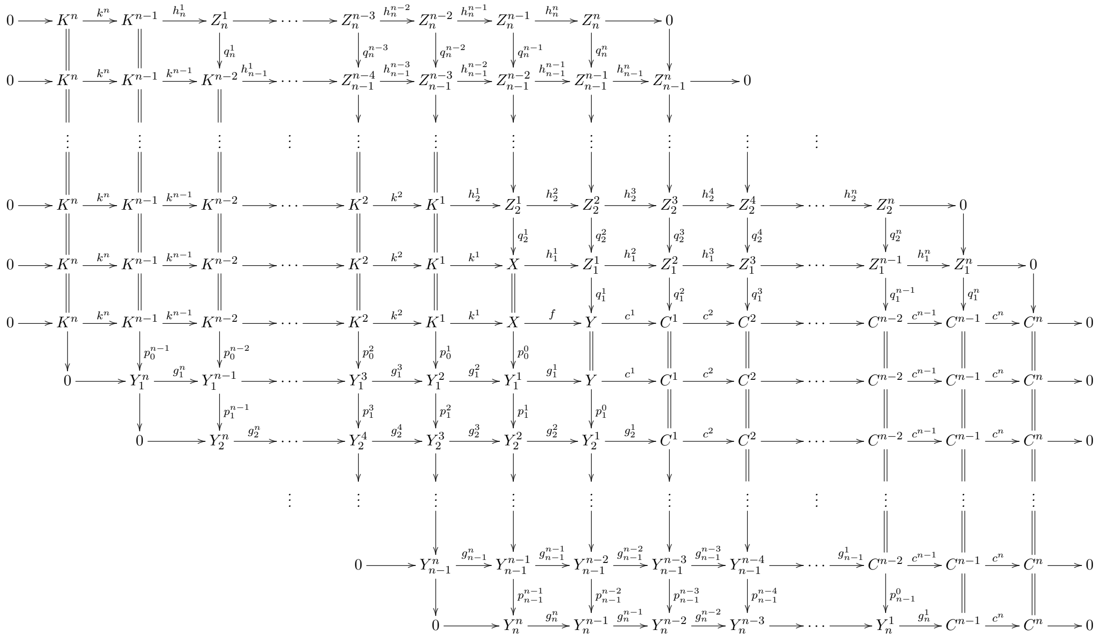
is n-exact.

Then, for $1 \leqslant i \leqslant n$, the sequence $(k^n, \cdots, k^i, h_i^1, \cdots, h_i^{i-1})$ is an n-kernel of h_i^i and the sequence $(k^n, \cdots, k^1)$ is an n-kernel of f.

Proof. Dual to the proof of Corollary 2.2.2. ■

Combining the above we are led to the following definition:

Definition 2.2.5. *Let $\mathcal{M}$ be an additive category and $f\colon X \longrightarrow Y$ a morphism in $\mathcal{M}$. A diagram of the form*



(2.2.2)

*is called an n-**diagram** of f in $\mathcal{M}$ if the following conditions hold:*

1. *The sequence $(c^1, \cdots, c^n)$ is an n-cokernel of f and the sequence $(k^n, \cdots, k^1)$ is an n-kernel of f.*
2. *For each $1 \leqslant i \leqslant n$, the sequence $(h_i^1, \cdots, h_i^n)$ is a n-cokernel of k^i.*
3. *For each $1 \leqslant i \leqslant n$, the sequence $(g_i^n, \cdots, g_i^1)$ is a n-kernel of c^i.*

Corollary 2.2.6. *In an additive category $\mathcal{M}$ with n-kernels and n-cokernels, any morphism has an n-diagram.*

2.3 Compatible n-diagrams

Refining the notion of an n-diagram we arrive at the following notion which will be used to characterize when an additive category is pre-n-abelian.

Definition 2.3.1. *Let $\mathcal{M}$ be an additive category and $f\colon X \longrightarrow Y$ a morphism in $\mathcal{M}$. We will call an n-diagram of f in $\mathcal{M}$, a* **compatible n-diagram** *if for $1 \leqslant i \leqslant n$, the mapping cone sequences:*

$$0 \longrightarrow Z_i^1 \longrightarrow Z_i^2 \oplus K^{i-2} \longrightarrow Z_i^3 \oplus Z_{i-1}^1 \longrightarrow \cdots \longrightarrow Z_i^n \oplus Z_{i-1}^{n-2} \longrightarrow Z_{i-1}^{n-1} \longrightarrow Z_{i-1}^n \longrightarrow 0$$

where $K^0\colon = X$, $K^{-1}\colon = Y$ and $Z_0^i\colon = C^i$, and

$$0 \longrightarrow Y_{i-1}^n \longrightarrow Y_{i-1}^{n-1} \longrightarrow Y_{i-1}^{n-2} \oplus Y_i^n \longrightarrow \cdots \longrightarrow Y_{i-1}^1 \oplus Y_k^3 \longrightarrow C^{i-2} \oplus Y_i^2 \longrightarrow Y_i^1 \longrightarrow 0$$

where $C^0\colon = Y$, $C^{-1}\colon = X$ and $Y_0^i\colon = K^i$, are n-exact.

Now we can prove the main result of this section:

Proposition 2.3.2. *For an additive category $\mathcal{M}$, the following are equivalent:*

1. *$\mathcal{M}$ is a pre-n-abelian category.*
2. *Any morphism f in $\mathcal{M}$ admits a compatible n-diagram.*

Proof. ($1 \Longrightarrow 2$) If $\mathcal{M}$ is pre-n-abelian, then by combining Propositions 2.2.1 and 2.2.3 it follows that every morphism f in $\mathcal{M}$ admits a compatible n-diagram.

($2 \Longrightarrow 1$) Let $f\colon X \longrightarrow Y$ be a morphism in $\mathcal{M}$ and assume that a compatible n-diagram of f exists. Then f has an n-cokernel $(c^1, \cdots, c^n)$ and an n-kernel $(k^n, \cdots, k^1)$, thus every morphism in $\mathcal{M}$ has an n-kernel and an n-cokernel.

Assuming that the morphism $f\colon X \longrightarrow Y$ is an epimorphism, we will prove that the sequence $(k^n, .., k^1, f)$ is n-exact isolating the following part of the compatible n-diagram (2.2.2):

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & K^n & \xrightarrow{k^n} & K^{n-1} & \xrightarrow{k^{n-1}} & K^{n-2} & \longrightarrow \cdots \longrightarrow & K^1 & \xrightarrow{k^1} & X & \xrightarrow{f} & Y & \xrightarrow{c^1} & C^1 \\
 & & \downarrow & & \downarrow{\scriptstyle p_0^{n-1}} & & \downarrow{\scriptstyle p_0^{n-2}} & & \downarrow{\scriptstyle p_0^{1}} & & \downarrow{\scriptstyle p_0^{0}} & \swarrow{\scriptstyle \phi^0} & \| & & \| \\
 & & 0 & \longrightarrow & Y_1^n & \xrightarrow{g_1^n} & Y_1^{n-1} & \longrightarrow \cdots \longrightarrow & Y_1^2 & \xrightarrow{g_1^2} & Y_1^1 & \xrightarrow{g_1^1} & Y & \xrightarrow{c^1} & C^1
\end{array}$$

of f, where $(g_1^n, \cdots, g_1^1)$ is an n-kernel of c^1. Since f is an epimorphism and $fc^1 = 0$ it follows that $c^1 = 0$, and since g_1^1 is a weak kernel of c^1, there exists a morphism $\phi^0 \colon Y \longrightarrow Y_1^1$ such that $\phi^0 g_1^1 = 1_Y$. Since by assumption the sequence:

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{(-k^{n-1},\, p_0^{n-1})} K^{n-2} \oplus Y_1^n \longrightarrow \cdots \longrightarrow K^1 \oplus Y_1^3 \xrightarrow{\begin{pmatrix} -k^1 & p_0^1 \\ 0 & g_1^3 \end{pmatrix}} X \oplus Y_1^2 \xrightarrow{{}^t(p_0^0,\, g_1^2)} Y_1^1 \longrightarrow 0$$

is n-exact, applying Lemma 1.1.15, it follows that for $1 \leqslant i \leqslant n$, the morphism k^{i-1} is a weak cokernel of the morphism k^i, where $k^0 = f$. Thus, the sequence $(k^n, \cdots, k^1, f)$ is n-exact.

In a dual manner, if $f \colon X \longrightarrow Y$ is a monomorphism, we will show that the sequence $(f, c^1, .., c^n)$ is n-exact. Since f is a monomorphism and $k^1 f = 0$, then $k^1 = 0$ and since h_1^1 is a weak cokernel of k^1, there exists a morphism $\psi_0 \colon Z_1^1 \longrightarrow X$ such that $h_1^1 \psi^0 = 1_X$, as seen in the following commutative diagram which is part of the compatible n-diagram (2.2.2):

$$\begin{array}{ccccccccccccccccc}
K^1 & \xrightarrow{k^1} & X & \xrightarrow{h_1^1} & Z_1^1 & \xrightarrow{h_1^2} & Z_1^2 & \xrightarrow{h_1^3} & \cdots & \longrightarrow & Z_1^{n-1} & \xrightarrow{h_1^n} & Z_1^n & \longrightarrow & 0 & & \\
\| & & \| & \overset{\psi^0}{\swarrow} & \downarrow{\scriptstyle q_1^1} & & \downarrow{\scriptstyle q_1^2} & & & & \downarrow{\scriptstyle q_1^{n-1}} & & \downarrow{\scriptstyle q_1^n} & & \downarrow & & \\
K^1 & \xrightarrow{k^1} & X & \xrightarrow{f} & Y & \xrightarrow{c^1} & C^1 & \xrightarrow{c^2} & \cdots & \longrightarrow & C^{n-2} & \xrightarrow{c^{n-1}} & C^{n-1} & \xrightarrow{c^n} & C^n & \longrightarrow & 0
\end{array}$$

Since by assumption the sequence

$$0 \longrightarrow Z_1^1 \xrightarrow{(-h_1^2,\, q_1^1)} Z_1^2 \oplus Y \xrightarrow{\begin{pmatrix} -h_1^3 & q_1^2 \\ 0 & c^1 \end{pmatrix}} Z_1^3 \oplus C^1 \longrightarrow \cdots \longrightarrow Z_1^n \oplus C^{n-2} \xrightarrow{{}^t(q_1^n,\, c^{n-1})} C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0$$

is n-exact, applying Lemma 1.1.16, it follows that for $1 \leqslant i \leqslant n$, the morphism c^{i-1} is a weak kernel of the morphism c^i, where $c^0 = f$. ■

Remark 2.3.3. In case $n = 1$, we deduce that an additive category $\mathcal{A}$ is abelian iff any morphism f in $\mathcal{A}$ admits a compatible 1-diagram which then is of the following shape:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K^1 & \xrightarrow{k^1} & X & \xrightarrow{h_1^1} & Z_1^1 & \longrightarrow & 0 & & \\
 & & \| & & \| & & \downarrow{\scriptstyle q_1^1} & & \downarrow & & \\
0 & \longrightarrow & K^1 & \xrightarrow{k^1} & X & \xrightarrow{f} & Y & \xrightarrow{c^1} & C^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle p_0^0} & & \| & & \| & & \\
 & & 0 & \longrightarrow & Y_1^1 & \xrightarrow{g_1^1} & Y & \xrightarrow{c^1} & C^1 & \longrightarrow & 0
\end{array}$$

Since in a compatible 1-diagram, the top row and the sequence:

$$0 \longrightarrow K^1 \xrightarrow{-k^1} X \xrightarrow{p_0^0} Y_1^1 \longrightarrow 0$$

are exact, we obtain an isomorphism $Z_1^1 \xrightarrow{\sim} Y_1^1$, thus we recover the definition of an abelian category, see [43, IV.4].

In case $n = 2$ a compatible 2-diagram is of the following shape:

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & K^2 & \xrightarrow{k^2} & K^1 & \xrightarrow{h_2^1} & Z_2^1 & \xrightarrow{h_2^2} & Z_2^2 & \longrightarrow & 0 & & & & \\
 & & \| & & \| & & \downarrow{\scriptstyle q_2^1} & & \downarrow{\scriptstyle q_2^2} & & \downarrow & & & & \\
0 & \longrightarrow & K^2 & \xrightarrow{k^2} & K^1 & \xrightarrow{k^1} & X & \xrightarrow{h_1^1} & Z_1^1 & \xrightarrow{h_1^2} & Z_1^2 & \longrightarrow & 0 & & \\
 & & \| & & \| & & \| & & \downarrow{\scriptstyle q_1^1} & & \downarrow{\scriptstyle q_1^2} & & \downarrow & & \\
0 & \longrightarrow & K^2 & \xrightarrow{k^2} & K^1 & \xrightarrow{k^1} & X & \xrightarrow{f} & Y & \xrightarrow{c^1} & C^1 & \xrightarrow{c^2} & C^2 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p_0^1} & & \downarrow{\scriptstyle p_0^0} & & \| & & \| & & \| & & \\
 & & 0 & \longrightarrow & Y_1^2 & \xrightarrow{g_1^2} & Y_1^1 & \xrightarrow{g_1^1} & Y & \xrightarrow{c^1} & C^1 & \xrightarrow{c^2} & C^2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle p_1^2} & & \downarrow{\scriptstyle p_1^1} & & \| & & \| & & \\
 & & & & 0 & \longrightarrow & Y_2^2 & \xrightarrow{g_2^2} & Y_2^1 & \xrightarrow{g_2^1} & C^1 & \xrightarrow{c^2} & C^2 & \longrightarrow & 0
\end{array}$$

If $\mathcal{M}$ is an n-abelian category and $\mathcal{A}$ is an abelian category, a functor $F \colon \mathcal{M} \longrightarrow \mathcal{A}$ is n-exact iff it is left and right n-exact, see [13, Proposition 3.2]. In our setting, if $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ is a functor between pre-n-abelian categories, the analogous result also holds. For the sake of completeness, we provide a short proof which follows the proof of [13, Proposition 3.2].

Corollary 2.3.4. *An additive functor $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ between pre-n-abelian categories is n-exact if and only if F is left-n-exact and right-n-exact.*

Proof. Clearly, if F is left-n-exact and right-n-exact, then F is n-exact. Now assume that F is n-exact and let

$$X \xrightarrow{f} Y \xrightarrow{c^1} C^1 \xrightarrow{c^2} \cdots \longrightarrow C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0$$

be a right n-exact sequence in $\mathcal{M}$. We have seen that we can complete this sequence, to a compatible n-diagram of f in $\mathcal{M}$ that satisfies the properties of Corollary 2.2.2. Applying the functor F to this n-diagram, we obtain an induced diagram in $\mathcal{N}$ and since F is n-exact, the n-exact sequences of Corollary 2.2.2, induce n-exact sequences in $\mathcal{M}$. Applying Corollary 2.2.2, it follows that the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(c^1)} F(C^1) \xrightarrow{F(c^2)} \cdots \longrightarrow F(C^{n-1}) \xrightarrow{F(c^n)} F(C^n) \longrightarrow 0$$

is right n-exact in $\mathcal{N}$. Dually, if

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{k^{n-1}} \cdots \longrightarrow K^1 \xrightarrow{k^1} X \xrightarrow{f} Y$$

is a left n-exact sequence in $\mathcal{M}$, applying Corollary 2.2.4 to the induced n-diagram of f in $\mathcal{N}$, it follows that the sequence

$$0 \longrightarrow F(K^n) \xrightarrow{F(k^n)} F(K^{n-1}) \xrightarrow{F(k^{n-1})} F(K^{n-2}) \longrightarrow \cdots \xrightarrow{F(k^1)} F(X) \xrightarrow{F(f)} F(Y)$$

is left n-exact in $\mathcal{N}$. We conclude that the functor F is also left n-exact. ■

CHAPTER 3

Localization of n-abelian categories

In this chapter we present our first main result of the thesis, constructing in two steps the localization of an n-abelian category with respect to a bicalculable system of morphisms. In the first step, utilizing the notion of the n-diagram (Definition 2.2.5), we show that the localization of a pre-n-abelian category with respect to a bicalculable system of morphisms is also a pre-n-abelian category and the localization functor is n-exact. In the second step, we prove that the idempotent completion of a pre-n-abelian category is an n-abelian category. Finally, we will combine the above to construct the localized category which is n-abelian and satisfies the desired universal property. For $n = 1$ we recover Gabriel's classic result on the localization of an abelian category, see [18].

3.1 Localization of pre-n-abelian categories

In the sequel, it will be necessary to verify that the localization functor Q preserves n-kernels, n-cokernels, and consequently n-exact sequences. In this connection, we analyze the structure of n-cokernels and n-kernels in the localization.

Throughout we fix an additive category $\mathcal{C}$ equipped with a (left and/or right) bicalculable system of morphisms S. As before we denote by $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ the localization functor.

3.1.1 n-Cokernels and n-kernels in localized categories

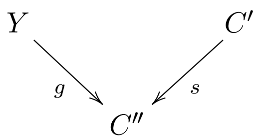
It is well known that if $\mathcal{C}$ is an additive category with kernels and cokernels then the localization functor $\mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$, where S is a bicalculable class of morphisms in $\mathcal{C}$, preserves kernels and cokernels, see for example [41, Chapter 4, Corollary 1.6]. Using this, we obtain directly the following Lemma and the subsequent Corollaries 3.1.2 and 3.1.3. For the sake of completeness we provide a short proof.

Lemma 3.1.1. *Let $\mathcal{C}$ be an additive category and $f\colon X \longrightarrow Y$ a morphism in $\mathcal{C}$. Then the following hold:*

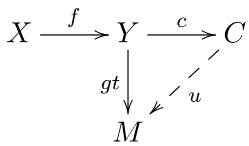
1. *If S is a left calculable system and $\mathcal{C}$ is a category with weak cokernels, then $\mathcal{C}[S^{-1}]$ is also a category with weak cokernels and the localization functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ preserves weak cokernels.*
2. *If S is a right calculable system and $\mathcal{C}$ is a category with weak kernels, then $\mathcal{C}[S^{-1}]$ is also a category with weak kernels and the localization functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ preserves weak kernels.*

Proof. We will only prove the first statement, since the proof of the second is dual. Clearly, $Q(f)Q(c) = Q(fc) = 0$.

Now, let $(s/g)\colon Y \longrightarrow C'$ be a morphism in $\mathcal{C}[S^{-1}]$:

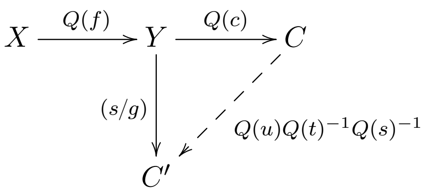


such that $Q(f)(s/g) = 0$. Then, $Q(f)Q(g)Q(s)^{-1} = 0$ and thus $Q(fg) = 0$. Applying Corollary 1.3.8, there exists a morphism $t\colon C'' \longrightarrow M$ in S such that $fgt = 0$. Since c is a weak cokernel of f in $\mathcal{C}$, there exists a morphism $u\colon C \longrightarrow M$ such that $gt = cu$, as seen in the following commutative diagram:



It follows that

$$(s/g) = Q(g)Q(s)^{-1} = Q(c)Q(u)Q(t)^{-1}Q(s)^{-1}$$



and we conclude that $Q(c)$ is a weak cokernel of $Q(f)$. ■

Immediately we have the subsequent corollary concerning n-cokernels, n-kernels and n-exact sequences:

Corollary 3.1.2. *Let $\mathcal{C}$ be an additive category and $f\colon X \longrightarrow Y$ a morphism in $\mathcal{C}$. Then the following hold:*

1. *If S a left calculable system and $(c^1, \cdots, c^n)$ is an n-cokernel of f, then $(Q(c^1), \cdots, Q(c^n))$ is an n-cokernel of $Q(f)$ in $\mathcal{C}[S^{-1}]$.*
2. *If S is a right calculable system and $(k^n, \cdots, k^1)$ is an n-kernel of f, then $(Q(k^n), \cdots, Q(k^1))$ is an n-kernel of $Q(f)$ in $\mathcal{C}[S^{-1}]$.*
3. *If S is a bicalculable system and $(f^0, \cdots, f^n)$ is an n-exact sequence in $\mathcal{C}$, then $(Q(f^0), \cdots, Q(f^n))$ is an n-exact sequence in $\mathcal{C}[S^{-1}]$.*

Proof. Follows directly from Lemma 3.1.1. ■

The proof of the following consequence is an application of the ideas of [41, Chapter 4, Corollary 1.6] in our setting.

Corollary 3.1.3. *Let $\mathcal{M}$ be a pre-n-abelian category, S a bicalculable system of morphisms in $\mathcal{M}$ and $(s/f) = Q(f)Q(s)^{-1}\colon X \longrightarrow Y$ a morphism in $\mathcal{M}[S^{-1}]$ represented by a diagram of the form:*

$$\begin{array}{ccccc} X & & & & Y \\ & \overset{f}{\searrow} & & \overset{s}{\swarrow} & \\ & & Z & & \end{array}$$

Then the following hold:

1. *The morphism (s/f) has an n-cokernel:*

$$X \xrightarrow{(s/f)} Y \xrightarrow{Q(s)Q(c^1)} C^1 \xrightarrow{Q(c^2)} \cdots \longrightarrow C^{n-1} \xrightarrow{Q(c^n)} C^n \longrightarrow 0$$

in $\mathcal{M}[S^{-1}]$, where the following sequence is an n-cokernel of f in $\mathcal{M}$:

$$X \xrightarrow{f} Z \xrightarrow{c^1} C^1 \xrightarrow{c^2} \cdots \longrightarrow C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0$$

2. *The morphism (s/f) has an n-kernel:*

$$0 \longrightarrow K^n \xrightarrow{Q(k^n)} K^{n-1} \xrightarrow{Q(k^{n-1})} \cdots \longrightarrow K^1 \xrightarrow{Q(k^1)} X \xrightarrow{(s/f)} Y$$

in $\mathcal{M}[S^{-1}]$, where the following sequence is an n-kernel of f in $\mathcal{M}$:

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{k^{n-1}} \cdots \longrightarrow K^1 \xrightarrow{k^1} X \xrightarrow{f} Z$$

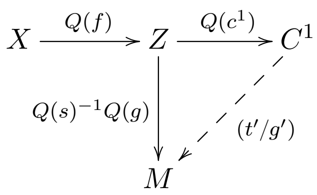
Proof. 1. If $(c^1, \cdots, c^n)$ is an n-cokernel of f in $\mathcal{C}$, applying Corollary 3.1.2 the sequence $(Q(c^1), Q(c^2), \cdots, Q(c^n))$ is an n-cokernel of $Q(f)$ in $\mathcal{M}[S^{-1}]$. It remains to show that $Q(s)Q(c^1)$ is a weak cokernel of $Q(f)Q(s)^{-1}$. Let $(t/g)\colon Y \longrightarrow M$ be a morphism $\mathcal{M}[S^{-1}]$ such that $(s/f)(t/g) = 0$. Then:

$$Q(f)Q(s)^{-1}Q(g) = 0$$

and since $Q(c^1)$ is a weak cokernel of $Q(f)$, there exists a morphism

$$(t'/g')\colon C^1 \longrightarrow M$$

such that the following diagram commutes:



Then:

$$(t/g) = Q(g)Q(t)^{-1} = Q(s)Q(s)^{-1}Q(g)Q(t)^{-1} = Q(s)Q(c^1)(t'/g')Q(t)^{-1}$$

We conclude that $Q(s)Q(c^1)$ is a weak cokernel of (s/f) in $\mathcal{M}[S^{-1}]$.

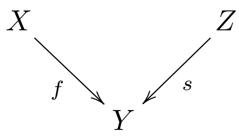
2. Again, applying Corollary 3.1.2, the sequence $(Q(k^n), Q(k^{n-1}), ..., Q(k^1))$ is an n-kernel of $Q(f)$ in $\mathcal{M}[S^{-1}]$. We can easily see that this sequence is also an n-kernel of the morphism $(s/f) = Q(f)Q(s)^{-1}$.
Let $(t/g)\colon M \longrightarrow X$ be a morphism in $\mathcal{M}[S^{-1}]$ such that $(t/g)(s/f) = 0$. Since $Q(s)$ is an isomorphism, $(t/g)Q(f) = 0$ and since $Q(k)^1$ is a weak kernel of $Q(f)$, there exists a morphism $(t'/g')\colon M \longrightarrow K^1$ in $\mathcal{M}[S^{-1}]$ such that $(t/g) = (t'/g')Q(k^1)$. We conclude that $Q(k^1)$ is a weak kernel of (s/f). ■

3.1.2 Main result

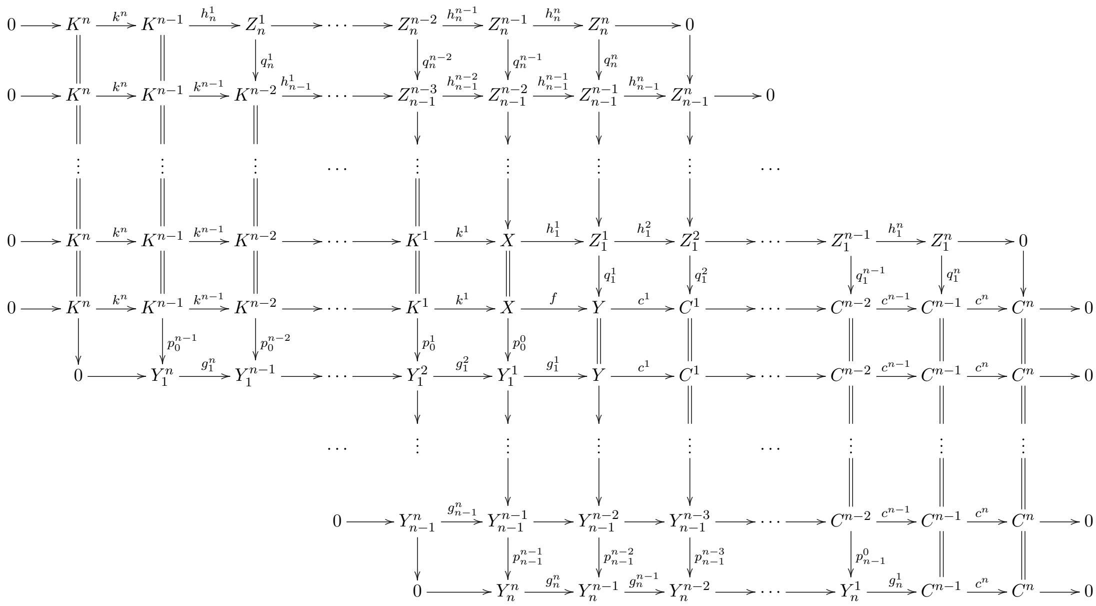
We can now prove our main result of this section:

Proposition 3.1.4. *Let $\mathcal{M}$ be a pre-n-abelian category, S a bicalculable system of morphisms in $\mathcal{M}$ and $\mathcal{M}[S^{-1}]$ the localization of $\mathcal{M}$ with respect to S. Then $\mathcal{M}[S^{-1}]$ is a pre-n-abelian category.*

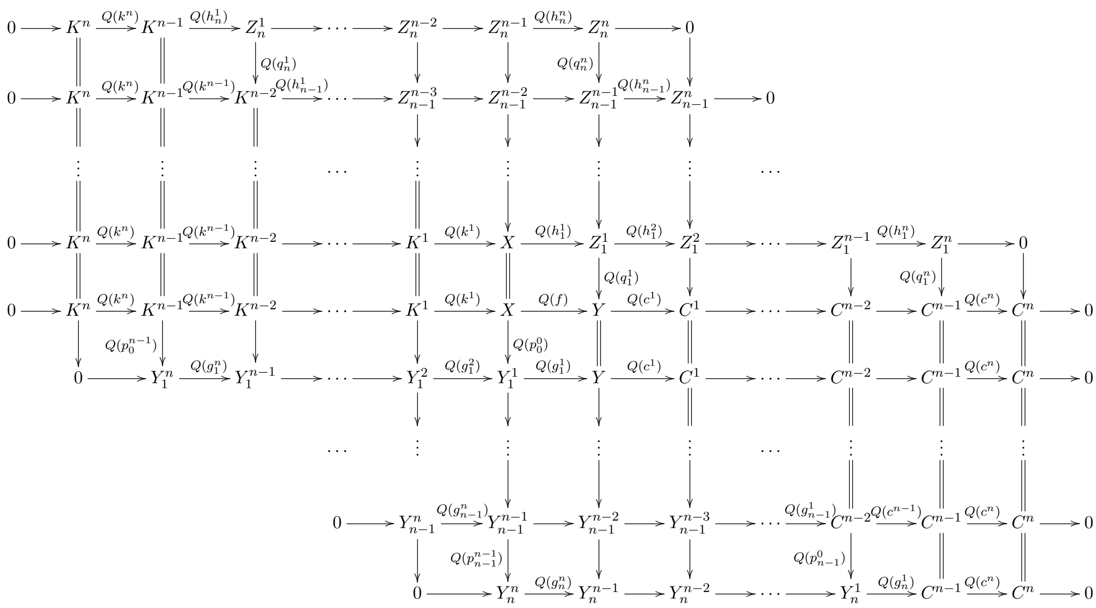
Proof. Let $(s/f) = Q(f)Q(s)^{-1} \colon X \longrightarrow Z$ be a morphism in $\mathcal{M}[S^{-1}]$ represented by a diagram:



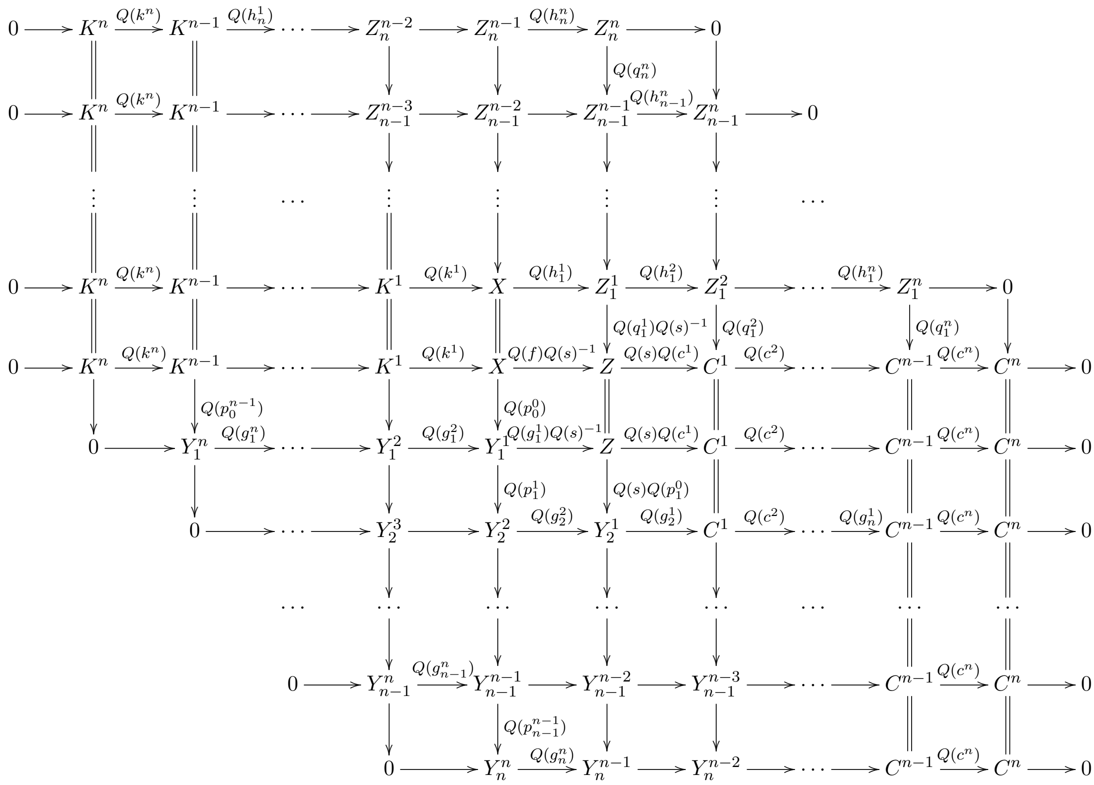
Let $(c^1, .., c^n)$ be an n-cokernel and $(k^n, \cdots, k^1)$ be an n-kernel of f in $\mathcal{M}$. Since $\mathcal{M}$ is a pre-n-abelian, we can construct a compatible n-diagram of f in $\mathcal{M}$:



Applying the localization functor Q, and using that the functor Q preserves n-kernels, n-cokernels and n-exact sequences, see Corollary 3.1.2, we obtain a compatible n-diagram of $Q(f)$ in $\mathcal{M}[S^{-1}]$:



Applying Corollary 3.1.3 and using that $Q(s)$ is an isomorphism in $\mathcal{M}[S^{-1}]$, it follows that the following diagram is a compatible n-diagram of (s/f) in $\mathcal{M}[S^{-1}]$:



By Proposition 2.3.2, we infer that $\mathcal{M}[S^{-1}]$ is a pre-n-abelian category. ■

Our next result ensures that the localized category $\mathcal{M}[S^{-1}]$ satisfies the desired universal property:

Theorem 3.1.5. *Let $\mathcal{M}$ be a pre-n-abelian category, S a bicalculable system of morphisms in $\mathcal{M}$, and $\mathcal{M}[S^{-1}]$ the localization of $\mathcal{M}$ with respect to S. Then $\mathcal{M}[S^{-1}]$ is a pre-n-abelian category, such that $Q(s)$ is invertible, for any $s \in S$, and the localization functor $Q\colon \mathcal{M} \longrightarrow \mathcal{M}[S^{-1}]$ is n-exact.*

Moreover, Q satisfies the following universal property:

- *for any pre-n-abelian category $\mathcal{N}$ and any n-exact functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ such that $F(s)$ is invertible for any $s \in S$, there exists a unique n-exact functor $F^*\colon \mathcal{M}[S^{-1}] \longrightarrow \mathcal{N}$*

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad Q \quad} & \mathcal{M}[S^{-1}] \\ & \searrow^{F} & \Big\downarrow{\scriptstyle \exists\,!\,F^*} \\ & & \mathcal{N} \end{array}$$

Proof. We have already seen that $\mathcal{M}[S^{-1}]$ is a pre-n-abelian category and that the functor Q is n-exact. The existence and the uniqueness of the additive functor F^* follow from the universal property of the localization in a general setting and its definition on objects and morphisms is well-known from classical localization theory:

- For any object $X \in \mathcal{M}[S^{-1}]$, set $F^*(X) = F(X)$.
- For any morphism (s/f) in $\mathcal{M}[S^{-1}]$, set $F^*(s/f) = F(f)F(s)^{-1}$.

For more details about the construction of F^* we refer the reader to [41, §4.1].

We now assume that F is n-exact and let $(\phi, \gamma^1, \cdots, \gamma^n)$ be a right n-exact sequence in $\mathcal{M}[S^{-1}]$, where $\phi = (s/f)\colon X \longrightarrow Y$. Since $\mathcal{M}$ is pre-n-abelian, the morphism f has an n-cokernel $(c^1, \cdots, c^n)$ in $\mathcal{M}$ and since Q is right n-exact, $(Q(c^1), \cdots, Q(c^n))$ is an n-cokernel of $Q(f)$ in $\mathcal{M}[S^{-1}]$. In this way we obtain a commutative diagram:

$$\begin{array}{ccccccccccccc} X & \xrightarrow{\phi} & Y & \xrightarrow{\gamma^1} & M^1 & \xrightarrow{\gamma^2} & M^2 & \longrightarrow \cdots \longrightarrow & M^{n-1} & \xrightarrow{\gamma^n} & M^n & \longrightarrow & 0 \\ \| & & \Big\downarrow{\scriptstyle Q(s)} & & \Big\downarrow{\scriptstyle \phi^1} & & \Big\downarrow{\scriptstyle \phi^2} & & \Big\downarrow{\scriptstyle \phi^{n-1}} & & \Big\downarrow{\scriptstyle \phi^n} & & \\ X & \xrightarrow{Q(f)} & Z & \xrightarrow{Q(c^1)} & C^1 & \xrightarrow{Q(c^2)} & C^2 & \longrightarrow \cdots \longrightarrow & C^{n-1} & \xrightarrow{Q(c^n)} & C^n & \longrightarrow & 0 \\ \| & & \Big\downarrow{\scriptstyle Q(s)^{-1}} & & \Big\downarrow{\scriptstyle \psi^1} & & \Big\downarrow{\scriptstyle \psi^2} & & \Big\downarrow{\scriptstyle \psi^{n-1}} & & \Big\downarrow{\scriptstyle \psi^n} & & \\ X & \xrightarrow{\phi} & Y & \xrightarrow{\gamma^1} & M^1 & \xrightarrow{\gamma^2} & M^{n-1} & \longrightarrow \cdots \longrightarrow & M^2 & \xrightarrow{\gamma^n} & M^n & \longrightarrow & 0 \end{array}$$

in $\mathcal{M}[S^{-1}]$, where the existence of the morphisms $\phi^1, \cdots, \phi^n$ and $\psi^1, \cdots, \psi^n$ follows from the weak cokernel property. It readily follows (see Lemma 1.1.8), that the above diagram is a homotopy equivalence of complexes, i.e. for $1 \leqslant i \leqslant n-1$, there exist morphisms $u^i\colon M^{i+1} \longrightarrow M^i$ and $v^i\colon C^{i+1} \longrightarrow C^i$, such that:

$$1 - \phi^{i+1}\psi^{i+1} = u^i\gamma^{i+1} + \gamma^{i+2}u^{i+1} \text{ and } 1 - \psi^{i+1}\phi^{i+1} = v^iQ(c^{i+1}) + Q(c^{i+2})v^{i+1}$$

Applying F^* to the above diagram, we obtain in $\mathcal{N}$ an induced homotopy equivalence of complexes:

$$\begin{array}{ccccccccccccc}
F^*(X) & \xrightarrow{F^*(\phi)} & F^*(Y) & \xrightarrow{F^*(\gamma^1)} & F^*(M^1) & \longrightarrow & \cdots & \longrightarrow & F^*(M^{n-1}) & \xrightarrow{F^*(\gamma^n)} & F^*(M^n) & & \\
\| & & \downarrow{\scriptstyle F(s)} & & \downarrow{\scriptstyle F^*(\phi^1)} & & & & \downarrow{\scriptstyle F^*(\phi^{n-1})} & & \downarrow{\scriptstyle F^*(\phi^n)} & & \\
F(X) & \xrightarrow{F(f)} & F(Z) & \xrightarrow{F(c^1)} & F(C^1) & \longrightarrow & \cdots & \longrightarrow & F(C^{n-1}) & \xrightarrow{F(c^n)} & F(C^n) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle F(s)^{-1}} & & \downarrow{\scriptstyle F^*(\psi^1)} & & & & \downarrow{\scriptstyle F^*(\psi^{n-1})} & & \downarrow{\scriptstyle F^*(\psi^n)} & & \\
F^*(X) & \xrightarrow{F^*(\phi)} & F^*(Y) & \xrightarrow{F^*(\gamma^1)} & F^*(M^1) & \longrightarrow & \cdots & \longrightarrow & F^*(M^{n-1}) & \xrightarrow{F^*(\gamma^n)} & F^*(M^n) & &
\end{array}$$

Note that by construction of F^*, for any object $X \in \mathcal{M}$, $F^*(X) = F^*(Q(X)) = F(X)$. Since F is n-exact, applying Corollary 2.3.4, F is right n-exact, thus $(F(c^1), \cdots, F(c^n))$ is an n-cokernel of $F(f)$ in $\mathcal{N}$. Applying Lemma 1.1.9 it follows that $(F^*(\gamma^1), \cdots, F^*(\gamma^n))$ is an n-cokernel of $F^*(\phi)$ in $\mathcal{N}$ which proves that F^* is right n-exact. Dually, we can show that F^* is left n-exact. Hence F^* is n-exact, completing the proof. ■

Remark 3.1.6. In the universal property of the above theorem, if the functor F^* is n-exact, then $F = F^* \circ Q$ is also n-exact. Thus, F is n-exact iff F^* is n-exact.

3.2 Idempotent completion of pre-n-abelian categories

In this section we prove that the idempotent completion of a pre-n-abelian category $\mathcal{M}$ is pre-n-abelian and therefore is an n-abelian category.

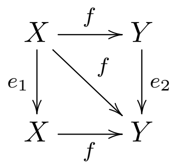
The construction of the idempotent completion of a category is classical, see [17, Chapter 2, Exercise B, page 61], and has been studied in different other settings, for example in the setting of triangulated categories by Balmer and Schlichting in [4], in the setting of n-angulated categories by Lin in [36], and in the setting of extriangulated categories by Msapato in [39] and Wang, Wei, Zhang, and Zhao in [45]. Our setting differs from the above and the tools developed in the previous sections will be applied.

3.2.1 Definition

We start by recalling the definition of the idempotent completion of a category in a general setting, and for more details we refer the reader to [4].

Definition 3.2.1. *The* **idempotent completion** *of an additive category $\mathcal{C}$, is a category $\widetilde{\mathcal{C}}$ defined as follows:*

- *An object of* $\widetilde{\mathcal{C}}$ *is a pair* (X, e) *where* X *is an object in* $\mathcal{C}$ *and* $e\colon X \longrightarrow X$ *is a morphism in* $\mathcal{C}$.
- *A morphism* $f\colon (X, e_1) \longrightarrow (Y, e_2)$ *in* $\widetilde{\mathcal{C}}$ *is a morphism* $f\colon X \longrightarrow Y$ *of* $\mathcal{C}$ *such that*
$$e_1 f = f = f e_2$$
i.e. the following diagram is commutative:



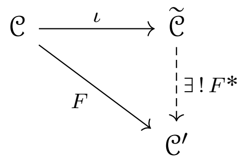
Remark 3.2.2. By definition, for a morphism $f\colon (X, e_1) \longrightarrow (Y, e_2)$ in $\widetilde{\mathcal{C}}$ the following hold:

1. If $f\colon X \longrightarrow Y$ is a monomorphism in $\mathcal{C}$, then $f\colon (X, e_1) \longrightarrow (Y, e_2)$ is also a monomorphism in $\widetilde{\mathcal{C}}$.
2. If $f\colon X \longrightarrow Y$ is an epimorphism in $\mathcal{C}$, then $f\colon (X, e_1) \longrightarrow (Y, e_2)$ is also an epimorphism in $\widetilde{\mathcal{C}}$.

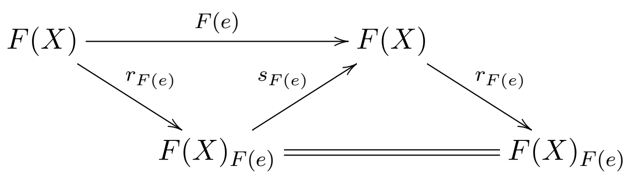
By construction, the idempotent completion of an additive category satisfies the following:

Proposition 3.2.3. *([4, Proposition 1.3]) Let* $\mathcal{C}$ *be an additive category and* $\widetilde{\mathcal{C}}$ *the idempotent completion of* $\mathcal{C}$. *Then* $\widetilde{\mathcal{C}}$ *is additive and the assignment* $C \longmapsto (C, 1)$ *induces a fully faithful additive functor* $\iota\colon \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ *satisfying the following universal property:*

- *For any idempotent complete additive category* $\mathcal{C}'$ *and any additive functor* $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$, *there exists a unique additive functor* $F^*\colon \widetilde{\mathcal{C}} \longrightarrow \mathcal{C}'$ *such that the following diagram commutes*



Remark 3.2.4. Note that the functor $F^*\colon \widetilde{\mathcal{C}} \longrightarrow \mathcal{C}'$ is defined as follows: for every idempotent $e\colon X \longrightarrow X$ in $\mathcal{C}$, we choose a splitting



in $\mathcal{C}'$ for the idempotent $F(e)$. Then:

- For every object (X, e) in $\widetilde{\mathcal{C}}$, set $F^*(X, e) = F(X)_{F(e)}$
- For every morphism $f\colon (X, e_1) \longrightarrow (Y, e_2)$ in $\widetilde{\mathcal{C}}$, set

$$F^*(f)\colon F(X)_{F(e_1)} \longrightarrow F(Y)_{F(e_2)}$$

to be the following composition in $\mathcal{C}'$:

$$F(X)_{F(e_1)} \xrightarrow{s_{F(e_1)}} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{r_{F(e_2)}} F(Y)_{F(e_2)}$$

Notation. From now on, we will denote an object of the form $(X, 1_X)$ in $\widetilde{\mathcal{C}}$ simply by $(X, 1)$.

3.2.2 n-Cokernels and n-kernels in idempotent completion

We begin by characterizing weak kernels and weak cokernels in $\widetilde{\mathcal{C}}$. For some similar characterizations and arguments, for example in the case of an extriangulated category, see [39, Lemma 3.9]. Our setting differs from the above, thus the lemma presented below, which will be needed later, does not appear in the literature in this form.

Lemma 3.2.5. *Let $\mathcal{C}$ be an additive category, $\widetilde{\mathcal{C}}$ the idempotent completion of $\mathcal{C}$ and*

$$f\colon (X, e_1) \longrightarrow (Y, e_2)$$

a morphism in $\widetilde{\mathcal{C}}$.

A morphism $k\colon (K, e_K) \longrightarrow (X, e_1)$ is a weak kernel of f in $\widetilde{\mathcal{C}}$ iff the following conditions are satisfied:

1. *The morphism $k\colon K \longrightarrow X$ is a weak kernel of the morphism:*

$$(f,\, 1 - e_1)\colon X \longrightarrow Y \oplus X$$

in $\mathcal{C}$.

2. *$e_K k = k$.*

Proof. We assume that $k\colon (K, e_K) \longrightarrow (X, e_1)$ is a weak kernel of f in $\widetilde{\mathcal{C}}$, thus $e_K k = k = k e_1$.

Now let $u\colon M \longrightarrow X$ be a morphism in $\mathcal{C}$ such that $u(f,\, 1-e_1)=0$. Since $u = ue_1$, the morphism $u\colon (M,1) \longrightarrow (X,e_1)$ is in $\widetilde{\mathcal{C}}$ and since $uf = 0$ in $\widetilde{\mathcal{C}}$, there exists a morphism $u'\colon (M,1) \longrightarrow (K,e_K)$ such that $u = u'k$.

$$\begin{array}{ccccc} & & (M,1) & & \\ & \swarrow^{u'} & \downarrow{\scriptstyle u} & & \\ (K,e_K) & \xrightarrow{k} & (X,e_1) & \xrightarrow{f} & (Y,e_2) \end{array}$$

Since $u\colon M \longrightarrow K$ is a morphism in $\mathcal{C}$, this proves our claim.

We now assume that k is a weak kernel of the morphism $(f,\, 1-e_1)$ and $e_Kk = k$. It follows that $kf = 0$ and $k = ke_1$, thus the morphism $k\colon (K,e_K) \longrightarrow (X,e_1)$ is well defined in $\widetilde{\mathcal{C}}$. Let $u\colon (M,e_M) \longrightarrow (X,e_1)$ be a morphism in $\widetilde{\mathcal{C}}$ such that $uf = 0$. Since u is a morphism in $\widetilde{\mathcal{C}}$, we have: $e_Mu = u = ue_1$, thus $u(f,\, 1-e_1) = 0$. Since k is a weak kernel of $(f,\, 1-e_1)$ in $\mathcal{C}$, there exists a morphism $u'\colon M \longrightarrow K$ in $\mathcal{C}$ such that $u = u'k$. Since the following diagram is commutative,

$$\begin{array}{ccccc} M & \xrightarrow{e_Mu'e_K} & K & \xrightarrow{k} & X \\ \downarrow{\scriptstyle e_M} & & \downarrow{\scriptstyle e_K} & & \downarrow{\scriptstyle e_1} \\ M & \xrightarrow{e_Mu'e_K} & K & \xrightarrow{k} & X \end{array}$$

in $\mathcal{C}$, it readily follows that

$$e_Mu'e_K\colon (M,e_M) \longrightarrow (K,e_K)$$

is a morphism in $\widetilde{\mathcal{C}}$ and $e_Mu'e_Kk = e_Mu'k = e_Mu = u$. We conclude that the following diagram commutes in $\widetilde{\mathcal{C}}$:

$$\begin{array}{ccccc} (K,e_K) & \xrightarrow{k} & (X,e_1) & \xrightarrow{f} & (Y,e_2) \\ & \nwarrow_{e_Mu'e_K} & \uparrow{\scriptstyle u} & & \\ & & (M,e_M) & & \end{array}$$

and as a result k is a weak kernel of f in $\widetilde{\mathcal{C}}$. ■

Dually, we obtain the following:

Lemma 3.2.6. *Let $\mathcal{C}$ be an additive category, $\widetilde{\mathcal{C}}$ the idempotent completion of $\mathcal{C}$ and*

$$f\colon (X,e_1) \longrightarrow (Y,e_2)$$

a morphism in $\widetilde{\mathcal{C}}$.

A morphism $c\colon (Y, e_2) \longrightarrow (C, e_C)$ *is a weak cokernel of* f *in* $\widetilde{\mathcal{C}}$ *iff the following conditions are satisfied:*

1. *The morphism* $c\colon Y \longrightarrow C$ *is a weak cokernel of the morphism*

$$^t(f,\, 1 - e_2)\colon X \oplus Y \longrightarrow Y$$

   in $\mathcal{C}$.
2. $ce_C = c$.

Proof. We assume that $c\colon (Y, e_2) \longrightarrow (C, e_C)$ is a weak cokernel of f in $\widetilde{\mathcal{C}}$, thus $e_2c = c = ce_C$. Now let $v\colon Y \longrightarrow M$ be a morphism in $\mathcal{C}$ such that $^t(f,\, 1 - e_2)v = 0$. Since $v = e_2v$, the morphism $v\colon (Y, e_2) \longrightarrow (M, 1)$ is in $\widetilde{\mathcal{C}}$ and since $fv = 0$ in $\widetilde{\mathcal{C}}$, there exists a morphism $v'\colon (C, e_C) \longrightarrow (M, 1)$ such that $v = cv'$.

$$\begin{array}{ccccc}
(X, e_1) & \xrightarrow{f} & (Y, e_2) & \xrightarrow{c} & (C, e_C) \\
 & & \downarrow{\scriptstyle v} & \swarrow{\scriptstyle v'} & \\
 & & (M, 1) & &
\end{array}$$

Since $v'\colon C \longrightarrow M$ is a morphism in $\mathcal{C}$, this proves our claim.

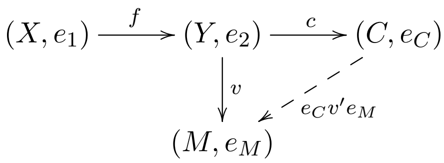
We now assume that c is a weak cokernel of the morphism $^t(f,\, 1 - e_2)$ and $ce_C = c$. It follows that $fc = 0$ and $c = e_2c$, thus the morphism $c\colon (Y, e_2) \longrightarrow (C, e_C)$ is well defined in $\widetilde{\mathcal{C}}$. Let $v\colon (Y, e_2) \longrightarrow (M, e_M)$ be a morphism in $\widetilde{\mathcal{C}}$ such that $fv = 0$. Since v is a morphism in $\widetilde{\mathcal{C}}$, we have: $e_2v = v = ve_M$, thus $^t(f,\, 1 - e_2)v = 0$. Since c is a weak cokernel of $^t(f,\, 1 - e_2)$ in $\mathcal{C}$, there exists a morphism $v'\colon C \longrightarrow M$ in $\mathcal{C}$ such that $v = cv'$. Since the following diagram is commutative,

$$\begin{array}{ccccc}
Y & \xrightarrow{c} & C & \xrightarrow{e_Cv'e_M} & M \\
\downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_C} & & \downarrow{\scriptstyle e_M} \\
Y & \xrightarrow{c} & C & \xrightarrow{e_Cv'e_M} & M
\end{array}$$

in $\mathcal{C}$, it readily follows that

$$e_Cv'e_M\colon (C, e_C) \longrightarrow (M, e_M)$$

is a morphism in $\widetilde{\mathcal{C}}$ and $ce_Cv'e_M = cv'e_M = ve_M = v$. We conclude that the following diagram commutes in $\widetilde{\mathcal{C}}$:



and as a result c is a weak cokernel of f in $\widetilde{\mathcal{C}}$. ■

The following direct consequences will be useful in the sequel.

Corollary 3.2.7. *Let $\mathcal{C}$ be an additive category, $\widetilde{\mathcal{C}}$ the idempotent completion of $\mathcal{C}$ and $f\colon (X,1) \longrightarrow (Y,e_2)$ a morphism in $\widetilde{\mathcal{C}}$. If $k\colon K \longrightarrow X$ is a weak kernel of $f\colon X \longrightarrow Y$ in $\mathcal{C}$, then the following hold:*

1. *If $e_K\colon K \longrightarrow K$ is an idempotent in $\mathcal{C}$ such that $e_K k = k$, then the morphism $k\colon (K,e_K) \longrightarrow (X,1)$ is a weak kernel of f in $\widetilde{\mathcal{C}}$.*
2. *The morphism $k\colon (K,1) \longrightarrow (X,1)$ is a weak kernel of $f\colon (X,1) \longrightarrow (Y,e_2)$ in $\widetilde{\mathcal{C}}$. Moreover, if k is a kernel of f in $\mathcal{C}$, then $k\colon (K,1) \longrightarrow (X,1)$ is also a kernel of f in $\widetilde{\mathcal{C}}$.*

Proof. Since k is a weak kernel of f, k is also a weak kernel of the morphism $(f,\ 0)\colon X \longrightarrow Y \oplus X$. Then, applying Proposition 3.2.5 for $e_1 = 1$ we obtain the first statement, while the second statement follows directly from the first by setting $e_K = 1$. Finally, if k is a monomorphism in $\mathcal{C}$, $k\colon (K,1) \longrightarrow (X,1)$ is also a monomorphism in $\widetilde{\mathcal{C}}$ and consequently, a kernel of f in $\widetilde{\mathcal{C}}$. ■

Next, we also present the dual statement:

Corollary 3.2.8. *Let $\mathcal{C}$ be an additive category, $\widetilde{\mathcal{C}}$ the idempotent completion of $\mathcal{C}$ and $f\colon (X,e_1) \longrightarrow (Y,1)$ a morphism in $\widetilde{\mathcal{C}}$. If $c\colon Y \longrightarrow C$ is a weak cokernel of $f\colon X \longrightarrow Y$ in $\mathcal{C}$, then the following hold:*

1. *If $e_C\colon C \longrightarrow C$ an idempotent in $\mathcal{C}$ such that $ce_C = c$, then the morphism $c\colon (Y,1) \longrightarrow (C,e_C)$ is a weak cokernel of f in $\widetilde{\mathcal{C}}$.*
2. *The morphism $c\colon (Y,1) \longrightarrow (C,1)$ is a weak cokernel of $f\colon (X,e_1) \longrightarrow (Y,1)$ in $\widetilde{\mathcal{C}}$. Moreover, if c is a cokernel of f in $\mathcal{C}$, then $c\colon (Y,1) \longrightarrow (C,1)$ is a cokernel of f in $\widetilde{\mathcal{C}}$.*

Proof. The proof is dual to the proof of Corollary 3.2.7. ■

The above results lead to the existence of an n-kernel and an n-cokernel for any morphism in the idempotent completion of a pre-n-abelian category:

Proposition 3.2.9. *Let $\mathcal{M}$ be a pre-n-abelian category, $\widetilde{\mathcal{M}}$ the idempotent completion of $\mathcal{M}$, and $f\colon (X, e_1) \longrightarrow (Y, e_2)$ a morphism in $\widetilde{\mathcal{M}}$. Then the following hold:*

1. The morphism f has an n-kernel $(k^n, \cdots, k^1)$:

$$0 \longrightarrow (K^n, 1) \xrightarrow{k^n} (K^{n-1}, 1) \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^2} (K^1, 1) \xrightarrow{k^1} (X, e_1) \xrightarrow{f} (Y, e_2)$$

in $\widetilde{\mathcal{M}}$, where $(k^n, \cdots, k^1)$ is an n-kernel of the morphism

$$(f,\, 1 - e_1)\colon X \longrightarrow Y \oplus X$$

in $\mathcal{M}$:

$$0 \longrightarrow K^n \xrightarrow{k^n} K^{n-1} \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^2} K^1 \xrightarrow{k^1} X \xrightarrow{(f,\, 1-e_1)} Y \oplus X$$

2. The morphism f has an n-cokernel $(c^1, \cdots, c^n)$:

$$(X, e_1) \xrightarrow{f} (Y, e_2) \xrightarrow{c^1} (C^1, 1) \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} (C^{n-1}, 1) \xrightarrow{c^n} (C^n, 1) \longrightarrow 0$$

in $\widetilde{\mathcal{M}}$, where $(c^1, \cdots, c^n)$ is an n-cokernel of the morphism

$$^{t}(f,\, 1 - e_2)\colon X \oplus Y \longrightarrow Y$$

in $\mathcal{M}$:

$$X \oplus Y \xrightarrow{^{t}(f,\, 1-e_2)} Y \xrightarrow{c^1} C^1 \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0$$

Proof. 1. Let $(k^n, \cdots, k^1)$ be an n-kernel of the morphism $(f,\, 1-e_1)$ in $\mathcal{M}$. By Lemma 3.2.5, the morphism $k^1\colon (K^1, 1) \longrightarrow (X, e_1)$ is a weak kernel of $f\colon (X, e_1) \longrightarrow (Y, e_2)$ in $\widetilde{\mathcal{M}}$. Moreover, by Corollary 3.2.7 we have the following in $\widetilde{\mathcal{M}}$:

- The morphism $k^2\colon (K^2, 1) \longrightarrow (K^1, 1)$ is a weak kernel of the morphism $k^1\colon (K^1, 1) \longrightarrow (X, e_1)$.
- For $2 < i \leqslant n-1$, the morphism $k^i\colon (K^i, 1) \longrightarrow (K^{i-1}, 1)$ is a weak kernel of the morphism $k^{i-1}\colon (K^{i-1}, 1) \longrightarrow (K^{i-2}, 1)$.
- The morphism $k^n\colon (K^n, 1) \longrightarrow (K^{n-1}, 1)$ is the kernel of $k^{n-1}\colon (K^{n-1}, 1) \longrightarrow (K^{n-2}, 1)$.

2. Dually, let $(c^1, \cdots, c^n)$ be an n-cokernel of the morphism ${}^t(f,\, 1-e_2)$ in $\mathcal{M}$. By Lemma 3.2.6, the morphism $c^1\colon (Y, e_2) \longrightarrow (C_1, 1)$ is a weak cokernel of $f\colon (X, e_1) \longrightarrow (Y, e_2)$ in $\widetilde{\mathcal{M}}$. Moreover, by Corollary 3.2.8 we have the following in $\widetilde{\mathcal{M}}$:
   - The morphism $c^2\colon (C^1, 1) \longrightarrow (C^2, 1)$ is a weak cokernel of the morphism $c^1\colon (Y, e_2) \longrightarrow (C^1, 1)$.
   - For $2 < i \leqslant n-1$, the morphism $c^i\colon (C^{i-1}, 1) \longrightarrow (C^i, 1)$ is a weak cokernel of $c^{i-1}\colon (C^{i-2}, 1) \longrightarrow (C^{i-1}, 1)$.
   - The morphism $c^n\colon (C^{n-1}, 1) \longrightarrow (C^n, 1)$ is a cokernel of $c^{n-1}\colon (C^{n-2}, 1) \longrightarrow (C^{n-1}, 1)$. ■

Remark 3.2.10. By setting $e_1 = e_2 = 1$ in the above proposition, it follows that if $(k^n, \cdots, k^1)$ is an n-kernel of f in $\mathcal{M}$, then $(\iota(k^n), \cdots, \iota(k^1))$ is an n-kernel of $\iota(f)$ in $\widetilde{\mathcal{M}}$. Dually, if $(c^1, \cdots, c^n)$ is an n-cokernel of g in $\mathcal{M}$, then $(\iota(c^1), \cdots, \iota(c^n))$ is an n-cokernel of $\iota(g)$ in $\widetilde{\mathcal{M}}$. Thus the inclusion functor $\iota\colon \mathcal{M} \longrightarrow \widetilde{\mathcal{M}}$ preserves n-kernels and n-cokernels.

3.2.3 Axioms (A2) and (A3)

It remains to verify that $\widetilde{\mathcal{M}}$ satisfies axioms $(A2)$ and $(A3)$.

Proposition 3.2.11. *Let $\mathcal{M}$ be a pre-n-abelian category, $\widetilde{\mathcal{M}}$ the idempotent completion of $\mathcal{M}$ and $f\colon (X, e_1) \longrightarrow (Y, e_2)$ a morphism in $\widetilde{\mathcal{M}}$. If f is an epimorphism in $\widetilde{\mathcal{M}}$, then there exists an n-exact sequence of the form:*

$$0 \longrightarrow (K^n, 1) \xrightarrow{k^n} (K^{n-1}, 1) \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^2} (K^1, 1) \xrightarrow{k^1} (X, e_1) \xrightarrow{f} (Y, e_2) \longrightarrow 0$$

in $\widetilde{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ is pre-n-abelian, there exists a compatible n-diagram (see Definition 2.3.1) of the morphism $(f,\, 1-e_1)\colon X \longrightarrow Y \oplus X$ in $\mathcal{M}$. We isolate the following part of such a diagram:

$$\begin{array}{ccccccccccccccc}
K^n & \xrightarrow{k^n} & K^{n-1} & \xrightarrow{k^{n-1}} & \cdots & \longrightarrow & K^1 & \xrightarrow{k^1} & X & \xrightarrow{(f,\,1-e_1)} & Y \oplus X & \xrightarrow{{}^t(c^1_\alpha,\, c^1_\beta)} & C^1 & \xrightarrow{c^2} & \cdots \\
\downarrow & & \downarrow{\scriptstyle p^{n-1}} & & & & \downarrow{\scriptstyle p^1} & & \downarrow{\scriptstyle p^0} & & \| & & \| & & \\
0 & \longrightarrow & Y^n & \xrightarrow{y^n} & \cdots & \longrightarrow & Y^2 & \xrightarrow{y^2} & Y^1 & \xrightarrow{(y^1_\alpha,\, y^1_\beta)} & Y \oplus X & \xrightarrow{{}^t(c^1_\alpha,\, c^1_\beta)} & C^1 & \xrightarrow{c^2} & \cdots
\end{array}$$

where the sequence $(k^n, k^{n-1}, \cdots, k^1)$ is an n-kernel of $(f,\, 1-e_1)$, the sequence $({}^t(c^1_\alpha,\, c^1_\beta), c^2, \cdots, c^n)$ is an n-cokernel of $(f,\, 1-e_1)$, and the sequence $(y^n, \cdots, y^2, (y^1_\alpha,\, y^1_\beta))$ is an n-kernel of ${}^t(c^1_\alpha,\, c^1_\beta)$. Moreover, the sequence:

$$0 \longrightarrow K^n \xrightarrow{-k^n} K^{n-1} \xrightarrow{(-k^{n-1},\, p^{n-1})} K^{n-2} \oplus Y^n \xrightarrow{\begin{pmatrix} -k^{n-2} & p^{n-2} \\ 0 & y^n \end{pmatrix}} \cdots \longrightarrow X \oplus Y^2 \xrightarrow{{}^t(p^0,\, y^2)} Y^1 \longrightarrow 0$$

is n-exact. Applying Proposition 3.2.9, the sequence:

$$0 \longrightarrow (K^n, 1) \xrightarrow{k^n} (K^{n-1}, 1) \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^2} (K^1, 1) \xrightarrow{k^1} (X, e_1)$$

is an n-kernel of f in $\widetilde{\mathcal{M}}$. Since ${}^t(c^1_\alpha,\, c^1_\beta)$ is a weak cokernel of $(f,\, 1-e_1)$, it follows that $fc^1_\alpha + (1-e_1)c^1_\beta = 0$, thus $e_1 f c^1_\alpha + e_1(1-e_1)c^1_\beta = 0$ which implies that $fe_2c^1_\alpha = fc^1_\alpha = 0$ and consequently $(1-e_1)c^1_\beta = 0$. The composition:

$$(X, e_1) \xrightarrow{f} (Y, e_2) \xrightarrow{e_2 c^1_a} (C^1, 1)$$

is well defined in $\widetilde{\mathcal{M}}$, as seen in the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{e_2 c^1_a} & C^1 \\ \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{e_2 c^1_a} & C^1 \end{array}$$

and vanishes. Since f is an epimorphism in $\widetilde{\mathcal{M}}$, $e_2 c^1_\alpha = 0$. Moreover, since

$$(f,\, 1-e_1)\begin{pmatrix} e_2 & 0 \\ 0 & 1-e_1 \end{pmatrix} = (fe_2,\, (1-e_1)(1-e_1)) = (f,\, 1-e_1)$$

it follows that the diagram:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & K^n & \xrightarrow{k^n} & K^{n-1} & \xrightarrow{k^{n-1}} & \cdots & \longrightarrow & K^1 & \xrightarrow{k^1} & X & \xrightarrow{(f,1-e_1)} & Y \oplus X \\ & & \downarrow & & \downarrow{\scriptstyle p^{n-1}} & & & & \downarrow{\scriptstyle p^1} & & \downarrow{\scriptstyle p^0} & & \downarrow{\scriptstyle \begin{pmatrix} e_2 & 0 \\ 0 & 1-e_1 \end{pmatrix}} \\ & & 0 & \longrightarrow & Y^n & \xrightarrow{y^n} & \cdots & \longrightarrow & Y^2 & \xrightarrow{y^2} & Y^1 & \xrightarrow{(y^1_\alpha,\, y^1_\beta)} & Y \oplus X \end{array}$$

is also commutative and since:

$$\begin{pmatrix} e_2 & 0 \\ 0 & 1-e_1 \end{pmatrix} {}^t(c^1_\alpha,\, c^1_\beta) = {}^t(e_2 c^1_a,\, (1-e_1)c^1_\beta) = 0$$

there exists a morphism ${}^t(\phi^1,\, \phi^2)\colon Y \oplus X \longrightarrow Y^1$ such that:

$$ {}^t(\phi^1,\, \phi^2)(y^1_\alpha,\, y^1_\beta) = \begin{pmatrix} e_2 & 0 \\ 0 & 1-e_1 \end{pmatrix} $$

We have shown that the above diagram satisfies the conditions of Lemma 1.1.15, and as a result, for $1 \leqslant i \leqslant n-1$, the morphism k^i is a weak cokernel of k^{i+1} and the morphism $(f,\, 1-e_1)$ is a weak cokernel of k^1. Applying Corollary 3.2.8 it follows that for $1 \leqslant i \leqslant n-1$, the morphism k^i is a also weak cokernel of k^{i+1} in $\widetilde{\mathcal{M}}$.

$$ 0 \longrightarrow (K^n,1) \xrightarrow{k^n} (K^{n-1},1) \xrightarrow{k^{n-1}} \cdots \xrightarrow{k^2} (K^1,1) \xrightarrow{k^1} (X,e_1) \xrightarrow{f} (Y,e_2) \longrightarrow 0 $$

It remains to show that f is a weak cokernel of k^1 in $\widetilde{\mathcal{M}}$. We claim that f is a weak cokernel of the morphism ${}^t(k^1,\, 1-e_1)\colon K^1 \oplus X \longrightarrow X$ in $\mathcal{M}$. Clearly, $k^1 f = 0$ and $(1-e_1)f = 0$. Let $u\colon X \longrightarrow M$ such that $k^1 u = 0$ and $u = e_1 u$. Since $(f,\, 1-e_1)$ is a weak cokernel of k^1, there exists a morphism ${}^t(u_1,\, u_2)\colon Y \oplus X \longrightarrow M$ such that $u = (f,\, 1-e_1)\,{}^t(u_1,\, u_2)$. Then:

$$ u = e_1 u = e_1 f u_1 + e_1(1-e_1)u_2 = e_1 f u_1 = f u_1 $$

which proves our claim. Applying Lemma 3.2.6 we conclude that f is a weak cokernel of k^1 in $\widetilde{\mathcal{M}}$, and consequently a cokernel, which completes the proof. ■

For completeness we also prove the dual proposition:

Proposition 3.2.12. *Let $\mathcal{M}$ be a pre-n-abelian category, $\widetilde{\mathcal{M}}$ the idempotent completion of $\mathcal{M}$, and $f\colon (X,e_1) \longrightarrow (Y,e_2)$ a morphism in $\widetilde{\mathcal{M}}$. If f is a monomorphism in $\widetilde{\mathcal{M}}$, then there exists an n-exact sequence of the form:*

$$ 0 \longrightarrow (X,e_1) \xrightarrow{f} (Y,e_2) \xrightarrow{c^1} (C^1,1) \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} (C^{n-1},1) \xrightarrow{c^n} (C^n,1) \longrightarrow 0 $$

in $\widetilde{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ is pre-n-abelian, there exists a compatible n-diagram (see Definition 2.3.1) of the morphism ${}^t(f,\, 1-e_2)\colon X \oplus Y \longrightarrow Y$ in $\mathcal{M}$. We isolate the following part of such a diagram:

$$ \begin{array}{ccccccccccccc} \cdots \xrightarrow{k^2} & K^1 & \xrightarrow{(k^1_\alpha,\, k^1_\beta)} & X \oplus Y & \xrightarrow{{}^t(x^1_\alpha,\, x^1_\beta)} & X^1 & \xrightarrow{x^2} & X^2 & \longrightarrow \cdots \xrightarrow{x^n} & X^n & \longrightarrow & 0 \\ & \| & & \| & & \downarrow q^0 & & \downarrow q^1 & & \downarrow q^{n-1} & & \downarrow \\ \cdots \xrightarrow{k^2} & K^1 & \xrightarrow{(k^1_\alpha,\, k^1_\beta)} & X \oplus Y & \xrightarrow{{}^t(f,\, 1-e_2)} & Y & \xrightarrow{c^1} & C^1 & \longrightarrow \cdots \xrightarrow{c^{n-1}} & C^{n-1} & \xrightarrow{c^n} & C^n \end{array} $$

where the sequence $(c^1, c^2, \cdots, c^n)$ is an n-cokernel of ${}^t(f,\, 1-e_2)$, the sequence $(k^n, \cdots, k^2, (k^1_\alpha,\, k^1_\beta))$ is an n-kernel of ${}^t(f,\, 1-e_2)$ and the sequence $({}^t(x^1_\alpha,\, x^1_\beta), x^2, \cdots, x^n)$ is an n-cokernel of $(k^1_\alpha,\, k^1_\beta)$. Moreover, the sequence:

$$0 \longrightarrow X^1 \xrightarrow{(-x^2,\, q^0)} X^2 \oplus Y \xrightarrow{\begin{pmatrix} -x^3 & q^1 \\ 0 & c^1 \end{pmatrix}} \cdots \longrightarrow X^n \oplus C^{n-2} \xrightarrow{{}^t(q^{n-1},\, c^{n-1})} C^n \xrightarrow{c^n} C^n \longrightarrow 0$$

is n-exact. Applying Proposition 3.2.9, the sequence:

$$(Y, e_2) \xrightarrow{c^1} (C^1, 1) \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} (C^{n-1}, 1) \xrightarrow{c^n} (C^n, 1) \longrightarrow 0$$

is an n-cokernel of f in $\widetilde{\mathcal{M}}$. Since $(k^1_\alpha,\, k^1_\beta)$ is a weak kernel of ${}^t(f,\, 1-e_2)$, it follows that $k^1_\alpha f + k^1_\beta(1-e_2) = 0$, thus $k^1_\alpha f e_2 + k^1_\beta(1-e_2)e_2 = 0$ which implies that $k^1_\alpha e_1 f = k^1_\alpha f = 0$ and consequently $k^1_\beta(1-e_2) = 0$. The composition:

$$(K^1, 1) \xrightarrow{k^1_\alpha e_1} (X, e_1) \xrightarrow{f} (Y, e_2)$$

is well defined in $\widetilde{\mathcal{M}}$, as seen in the following commutative diagram:

$$\begin{array}{ccccc} K^1 & \xrightarrow{k^1_\alpha e_1} & X & \xrightarrow{f} & Y \\ \| & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} \\ K^1 & \xrightarrow{k^1_\alpha e_1} & X & \xrightarrow{f} & Y \end{array}$$

and vanishes. Since f is an monomorphism in $\widetilde{\mathcal{M}}$, $k^1_\alpha e_1 = 0$. Moreover, since

$$\begin{pmatrix} e_1 & 0 \\ 0 & 1-e_2 \end{pmatrix} {}^t(f,\, 1-e_2) = {}^t(e_1 f,\, (1-e_2)(1-e_2)) = {}^t(f,\, 1-e_2)$$

it follows that the diagram:

$$\begin{array}{ccccccccccc} X \oplus Y & \xrightarrow{{}^t(x^1_\alpha,\, x^1_\beta)} & X^1 & \xrightarrow{x^2} & X^2 & \longrightarrow & \cdots & \xrightarrow{x^n} & X^n & \longrightarrow & 0 \\ \downarrow{\scriptstyle \begin{pmatrix} e_1 & 0 \\ 0 & 1-e_2 \end{pmatrix}} & & \downarrow{\scriptstyle q^0} & & \downarrow{\scriptstyle q^1} & & & & \downarrow{\scriptstyle q^{n-1}} & & \downarrow \\ X \oplus Y & \xrightarrow{{}^t(f,\, 1-e_2)} & Y & \xrightarrow{c^1} & C^1 & \longrightarrow & \cdots & \xrightarrow{c^{n-1}} & C^{n-1} & \xrightarrow{c^n} & C^n \longrightarrow 0 \end{array}$$

is also commutative and since:

$$(k^1_\alpha,\, k^1_\beta) \begin{pmatrix} e_1 & 0 \\ 0 & 1-e_2 \end{pmatrix} = (k^1_\alpha e_1,\, k^1_\beta(1-e_2)) = 0$$

there exists a morphism $(\phi^1,\, \phi^2)\colon X^1 \longrightarrow X \oplus Y$ such that:

$$ {}^t(x^1_\alpha,\, x^1_\beta)(\phi^1,\, \phi^2) = \begin{pmatrix} e_1 & 0 \\ 0 & 1-e_2 \end{pmatrix} $$

We have shown that the above diagram satisfies the conditions of Lemma 1.1.16, and as a result, for $1 \leqslant i < n$, the morphism c^i is a weak kernel of c^{i+1} and the morphism ${}^t(f,\, 1-e_2)$ is a weak kernel of c^1. Applying Corollary 3.2.7 it follows that for $1 \leqslant i \leqslant n-1$, the morphism c^i is a weak kernel of c^{i+1} in $\widetilde{\mathcal{M}}$.

$$ 0 \longrightarrow (X, e_1) \xrightarrow{f} (Y, e_2) \xrightarrow{c^1} (C^1, 1) \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} (C^{n-1}, 1) \xrightarrow{c^n} (C^n, 1) \longrightarrow 0 $$

It remains to show that f is a weak kernel of c^1 in $\widetilde{\mathcal{M}}$. We claim that f is a weak kernel of the morphism $(c^1,\, 1-e_2)\colon Y \longrightarrow C^1 \oplus Y$ in $\mathcal{M}$. Clearly, $fc^1 = 0$ and $f(1-e_2) = 0$. Let $u\colon M \longrightarrow Y$ such that $uc^1 = 0$ and $u = ue_2$. Since ${}^t(f,\, 1-e_2)$ is a weak kernel of c^1, there exists a morphism $(u_1,\, u_2)\colon M \longrightarrow X \oplus Y$ such that $u = (u_1,\, u_2)^t(f,\, 1-e_2)$. Then:

$$ u = ue_2 = u_1 f e_2 + u_2(1-e_2)e_2 = u_1 f e_2 = u_1 f $$

which proves our claim. Applying Lemma 3.2.5 we conclude that f is a weak kernel of k^1 in $\widetilde{\mathcal{M}}$, and consequently a kernel, which completes the proof. ■

3.2.4 Main result

The above results are combined in the following proposition:

Proposition 3.2.13. *If $\mathcal{M}$ is a pre-n-abelian category, then the idempotent completion $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is an n-abelian category, and the inclusion functor $\iota\colon \mathcal{M} \longrightarrow \widetilde{\mathcal{M}}$ is full, faithful and n-exact. Moreover, $\widetilde{\mathcal{M}}$ satisfies the following universal property: for any n-abelian category $\mathcal{N}$ and any n-exact functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$, there exists a unique n-exact functor $F^*\colon \widetilde{\mathcal{M}} \longrightarrow \mathcal{N}$ such that the following diagram commutes:*

$$ \begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \iota\ } & \widetilde{\mathcal{M}} \\ & {}_{F}\searrow & \Big\downarrow{\scriptstyle \exists !\, F^*} \\ & & \mathcal{N} \end{array} $$

Proof. Applying Propositions 3.2.9, 3.2.11 and 3.2.12, it follows directly that $\widetilde{\mathcal{M}}$ is an n-abelian category and the inclusion functor is n-exact. Since an n-abelian category is idempotent complete, the existence and the uniqueness of the additive functor

$F^*\colon \mathcal{M} \longrightarrow \mathcal{N}$ in the above commutative diagram, follow from the universal property of the idempotent completion, see Remark 3.2.4.

We now assume that F is n-exact and let

$$(X,e_X) \xrightarrow{f} (Y,e_Y) \xrightarrow{\gamma^1} (M^1,e_1) \longrightarrow \cdots \xrightarrow{\gamma^n} (M^n,e_n) \longrightarrow 0 \tag{3.2.1}$$

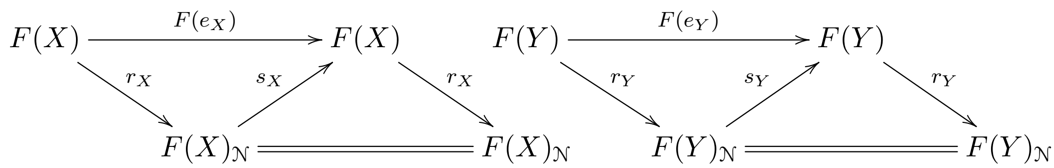
be a right n-exact sequence in $\widetilde{\mathcal{M}}$. Applying Proposition 3.2.9, the morphism f has an n-cokernel $(c^1,\cdots,c^n)$:

$$(X,e_X) \xrightarrow{f} (Y,e_Y) \xrightarrow{c^1} (C^1,1) \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} (C^{n-1},1) \xrightarrow{c^n} (C^n,1) \longrightarrow 0 \tag{3.2.2}$$

in $\widetilde{\mathcal{M}}$, where $(c^1,\cdots,c^n)$ is an n-cokernel of the morphism ${}^t(f,\, 1-e_Y)$ in $\mathcal{M}$:

$$X\oplus Y \xrightarrow{{}^t(f,\,1-e_Y)} Y \xrightarrow{c^1} C^1 \xrightarrow{c^2} \cdots \xrightarrow{c^{n-1}} C^{n-1} \xrightarrow{c^n} C^n \longrightarrow 0 \tag{3.2.3}$$

Now let:



be splittings in $\mathcal{N}$ for the idempotents $F(e_X)$ and $F(e_Y)$, respectively. Applying the induced functor F^* to the sequence (3.2.2) and the functor $F = F^*\circ\iota$ to the sequence (3.2.3) we obtain the following commutative diagram in $\mathcal{N}$:

$$\begin{array}{ccccccccccc}
F(X)_{\mathcal{N}} & \xrightarrow{s_XF(f)r_Y} & F(Y)_{\mathcal{N}} & \xrightarrow{s_YF(c^1)} & F(C^1) & \xrightarrow{F(c^2)} & \cdots & \xrightarrow{F(c^n)} & F(C^n) & \longrightarrow & 0 \\
\downarrow{\scriptstyle (s_X,0)} & & \downarrow{\scriptstyle s_Y} & & \| & & & & \| & & \\
F(X)\oplus F(Y) & \xrightarrow{{}^t(F(f),1-F(e_Y))} & F(Y) & \xrightarrow{F(c^1)} & F(C^1) & \xrightarrow{F(c^2)} & \cdots & \xrightarrow{F(c^n)} & F(C^n) & \longrightarrow & 0 \\
\downarrow{\scriptstyle {}^t(r_X,0)} & & \downarrow{\scriptstyle r_Y} & & \| & & & & \| & & \\
F(X)_{\mathcal{N}} & \xrightarrow{s_XF(f)r_Y} & F(Y)_{\mathcal{N}} & \xrightarrow{s_YF(c^1)} & F(C^1) & \xrightarrow{F(c^2)} & \cdots & \xrightarrow{F(c^n)} & F(C^n) & \longrightarrow & 0
\end{array} \tag{3.2.4}$$

Note that the top row of the diagram is the image of (3.2.2) through the functor F^* in $\mathcal{N}$. Since F is n-exact, applying Corollary 2.3.4, F is right n-exact thus the middle row is a right n-exact sequence in $\mathcal{N}$. We claim that the top row is also a right n-exact sequence: Let $u\colon F(Y)_{\mathcal{N}} \longrightarrow M$ be a morphism such that $s_XF(f)r_Yu = 0$. From the above diagrams, we obtain:

$${}^t(F(f),\, 1-F(e_Y))r_Yu = {}^t(r_X,0)s_XF(f)r_Yu = 0$$

Thus, there exists a morphism $u'\colon F(Y) \longrightarrow M$ such that $r_Y u = F(c^1)u'$. It follows that $u = s_Y r_Y u = s_Y F(c^1)u'$ and $s_Y F(c^1)$ is a weak cokernel of $s_X F(f) r_Y$. We only need to show that $F(c^1)$ is also a weak cokernel of $s_Y F(c^1)$: Let $v\colon F(C^1) \longrightarrow M$ be such that $s_Y F(c^1)v = 0$. Then, $F(c^1)v = r_Y s_Y F(c^1)v = 0$, thus there exists a morphism $v'\colon F(C^1) \longrightarrow M$ such that $v = F(c^1)v'$ which completes the proof of our claim.

Finally, the right n-exact sequences (3.2.1) and (3.2.2) induce a homotopy equivalence of complexes in $\mathcal{M}$. Applying the functor F^*, we obtain a homotopy equivalence of complexes in $\mathcal{N}$:

$$\begin{array}{ccccccccccc}
F^*(X,e_X) & \xrightarrow{F^*(f)} & F^*(Y,e_Y) & \xrightarrow{F^*(\gamma^1)} & F^*(M^1,e_1) & \longrightarrow & \cdots & \xrightarrow{F^*(\gamma^n)} & F^*(M^n,e_n) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & & & \downarrow & & \\
F^*(X,e_X) & \xrightarrow{F^*(f)} & F^*(Y,e_Y) & \xrightarrow{F^*(c^1)} & F^*(C^1,1) & \longrightarrow & \cdots & \xrightarrow{F^*(c^n)} & F^*(C^n,1) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & & & \downarrow & & \\
F^*(X,e_X) & \xrightarrow{F^*(f)} & F^*(Y,e_Y) & \xrightarrow{F^*(\gamma^1)} & F^*(M^1,e_1) & \longrightarrow & \cdots & \xrightarrow{F^*(\gamma^n)} & F^*(M^n,e_n) & \longrightarrow & 0
\end{array}$$

Since the middle row of this diagram, which is exactly the top row of diagram (3.2.4), is right n-exact, similarly to Theorem 3.1.5, applying Corollary 2.3.4 the top row is also right n-exact, thus F^* is a right n-exact functor between two n-abelian categories. Dually, we can show that F^* is left n-exact, and as a result n-exact, completing the proof. ■

Combining Theorem 3.1.5 and Proposition 3.2.13, we obtain the first main result of the thesis which shows that localizations of n-abelian categories (at bicalculable systems of morphisms) are n-abelian and satisfy the corresponding universal property.

Theorem 3.2.14. *Let $\mathcal{M}$ be an n-abelian category, S a bicalculable system of morphisms in $\mathcal{M}$ and $\mathcal{M}[S^{-1}]$ the localization of $\mathcal{M}$ with respect to S. Let $\widetilde{\mathcal{M}[S^{-1}]}$ denote the idempotent completion of $\mathcal{M}[S^{-1}]$ and $\widetilde{Q}\colon \mathcal{M} \longrightarrow \widetilde{\mathcal{M}[S^{-1}]}$ be the functor $\widetilde{Q} = \iota \circ Q$, where Q is the localization functor and ι is the inclusion functor in the idempotent completion:*

$$\widetilde{Q}\colon \mathcal{M} \xrightarrow{Q} \mathcal{M}[S^{-1}] \xrightarrow{\iota} \widetilde{\mathcal{M}[S^{-1}]}$$

Then $\widetilde{\mathcal{M}[S^{-1}]}$ is an n-abelian category and the functor $\widetilde{Q}$ is n-exact.

Moreover, Q satisfies the following universal property:

• *for any n-abelian category $\mathcal{N}$ and any n-exact functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ such that $F(s)$ is invertible for any $s \in S$, there exists a unique n-exact functor $F^*\colon \widetilde{\mathcal{M}[S^{-1}]} \longrightarrow \mathcal{N}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \widetilde{Q}\ } & \widetilde{\mathcal{M}[S^{-1}]} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle \exists!\,F^*} \\ & & \mathcal{N} \end{array}$$

Proof. The proof follows directly from Theorem 3.1.5 and Proposition 3.2.13. More specifically, $\mathcal{M}[S^{-1}]$ is a pre-n-abelian category, $\widetilde{\mathcal{M}[S^{-1}]}$ is an n-abelian category and the composition $\widetilde{Q} = \iota \circ Q$ is an n-exact functor between n-abelian categories.

The existence, the uniqueness and the exactness properties of the additive functors F' and F^* follow from the universal property of the localization of $\mathcal{M}$ and the universal property of the idempotent completion of $\mathcal{M}[S^{-1}]$, as shown in Theorem 3.1.5 and Proposition 3.2.13, respectively. For any n-abelian category $\mathcal{N}$ and any additive functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ such that $F(s)$ is invertible, for any $s \in S$, there exist unique additive functors $F'\colon \mathcal{M}[S^{-1}] \longrightarrow \mathcal{N}$ and then $F^*\colon \widetilde{\mathcal{M}[S^{-1}]} \longrightarrow \mathcal{N}$ such that the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{\ Q\ } & \mathcal{M}[S^{-1}] & \xrightarrow{\ \iota\ } & \widetilde{\mathcal{M}[S^{-1}]} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle \exists!\ F'} & \swarrow{\scriptstyle \exists!\,F^*} & \\ & & \mathcal{N} & & \end{array}$$

Finally, if F is n-exact, then F' is n-exact and consequently F^* is also n-exact. ■

Remark 3.2.15. In the universal property of Theorem 3.2.14, since the functors Q and ι are n-exact, it readily follows that F is n-exact iff F' is n-exact iff F^* is n-exact.

Remark 3.2.16. Setting $n = 1$ in Theorem 3.2.14, we retrieve Gabriel's Theorem 1.3.11 on the localization of an abelian category. Note that in this case, the intermediate step of the idempotent completion is not necessary, since abelian categories are idempotent complete, thus the functor ι is in this setting an equivalence of categories.

Remark 3.2.17. Let $\mathcal{M}$ be an n-abelian category realized as an n-cluster tilting subcategory of an abelian category $\mathcal{A}$ and let S be a bicalculable system of morphisms in $\mathcal{A}$. If the restriction $S_{\mathcal{M}} = S \cap \mathcal{M}$ of S to $\mathcal{M}$ is a bicalculable system in $\mathcal{M}$, it is well known that under some mild conditions, see [46, Lemma 10.3.13], the natural functor $\iota\colon \mathcal{M}[S_{\mathcal{M}}^{-1}] \longrightarrow \mathcal{A}[S^{-1}]$ is fully faithful, so the localization $\mathcal{M}[S_{\mathcal{M}}^{-1}]$ is realized as a full subcategory of the abelian category $\mathcal{A}[S^{-1}]$. On the other hand, applying the main results from [34], [12], the n-abelian category $\mathcal{M}[S_{\mathcal{M}}^{-1}]$ can also be realized as an n-cluster tilting subcategory of an abelian category. In this way, our main result creates, via localization, new cluster tilting subcategories of abelian categories.

Remark 3.2.18. Recently, Jian He, Jing He, and Pan Zhou in [29] study independently localizations of n-exangulated categories in the following setting: starting with an n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and an additive full subcategory $\mathcal{N}_F \subseteq \mathcal{C}$ which is associated to a system of morphisms F, let $\bar{F}$ be a set of morphisms in the ideal quotient $\bar{\mathcal{C}} = \mathcal{C}/[\mathcal{N}_F]$ obtained from F. Assuming that $\bar{F}$ satisfies some set of conditions, namely conditions (MR1)-(MR3) in [29, §3], and that the localization $\tilde{\mathcal{C}}$ satisfies an additional condition, namely condition (C4) of [29, Definition 2.8], it is shown that $\tilde{\mathcal{C}}$ is an n-exangulated category if and only if any n-exangle in $\mathcal{C}$ induces a weak kernel-cokernel sequence in $\tilde{\mathcal{C}}$, see [29, Theorem 3.4].

There are significant differences between our approach and the approach of the paper [29] concerning the setting, the imposed conditions, and the methods used. For example, we do not assume that S satisfies condition (MR3) of [29, §3], which in our case trivializes the verification of axioms (A2) and (A3) for $\mathcal{C}[S^{-1}]$. Moreover, we have not imposed any conditions on $\mathcal{C}[S^{-1}]$ that ensure that it is idempotent complete. Finally, we do not use any part of the theory of n-exangulated categories and the external structure that they provide. Our approach is different being as close as possible to the classic case of localizations of abelian categories, and our goal is to prove, similarly to the classic case, that the localization of an n-abelian category at a class of morphisms S satisfying the same properties as in the classic case, is n-abelian and satisfies the expected universal property. As a final comment, we notice that, even though every n-abelian category is an n-exangulated category, our result cannot be recovered from [29], and vice versa.

CHAPTER 4

Localization of n-angulated categories

In this chapter we study the localization of an n-angulated category where $n \geqslant 3$ and we present an analogous result for n-angulated categories. Similarly to the classic case of triangulated categories for $n = 3$, we need a bicalculable class of morphisms S which satisfies a compatibility condition with respect to the n-angulation. After defining a suitable class N_S of n-angles in $\mathcal{C}[S^{-1}]$ we prove that the localized category is an n-angulated category. Finally, we show that the functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ is universal among all S-inverting n-exact functors out of $\mathcal{C}$ to an n-angulated category.

Throughout this chapter, let $(\mathcal{C}, \Sigma, \mathcal{N})$ be an n-angulated category, S a bicalculable class of morphisms in $\mathcal{C}$ and $\mathcal{C}[S^{-1}]$ the localization of $\mathcal{C}$ with respect to S.

4.1 n-Angles in localization

As in the classic case of the localization of a triangulated category, in order to show that the localization $\mathcal{C}[S^{-1}]$ of an n-angulated category $\mathcal{C}$ at a bicalculable system of morphisms S in $\mathcal{C}$, is n-angulated, a compatibility of S with the n-angulation is necessary. Thus we are led to the following definition.

Definition 4.1.1. *Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be a pre-n-angulated category. A bicalculable class S in $\mathcal{C}$ will be called* **compatible with the n-angulation** *if it satisfies the following conditions:*

(CT1) For any morphism s in $\mathcal{C}$, $s \in S$ if and only if $\Sigma s \in S$.

(CT2) For any commutative diagram of the form:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1 \\
\downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & & & & & & & \\
Y^1 & \xrightarrow{b^1} & Y^2 & & & & & & & &
\end{array}$$

where the top row is an n-angle and $s^1, s^2 \in S$, there exists a commutative diagram:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1 \\
\downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & \downarrow{\scriptstyle s^3} & & & & \downarrow{\scriptstyle s^n} & & \downarrow{\scriptstyle \Sigma s^1} \\
Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \dashrightarrow & \cdots & \dashrightarrow & Y^n & \xrightarrow{b^n} & \Sigma Y^1
\end{array}$$

where the bottom row is an n-angle and $s^3, \cdots, s^n \in S$.

Remark 4.1.2. If $\mathcal{C}$ is a triangulated category, condition (CT2) is equivalent to the condition (SM6) of Definition 1.3.12, which generally appears in the localization of triangulated categories:

(SM6): For any commutative diagram of the form:

$$\begin{array}{ccccccc}
X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \xrightarrow{a^3} & \Sigma X^1 \\
\downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & \downarrow{\scriptstyle s^3} & & \downarrow{\scriptstyle \Sigma s^1} \\
Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \xrightarrow{b^3} & \Sigma Y^1
\end{array}$$

whose rows are 3-angles, which are called *distinguished triangles* in this setting, there exists a morphism $s^3 \colon X^3 \longrightarrow Y^3$ lying in S, that completes the diagram to a morphism of distinguished triangles.

It follows readily that since any morphism f can be embedded in a triangle, if S satisfies (SM6), then S also satisfies the corresponding axiom (CT2) in the triangulated setting. We assume now that S satisfies the triangulated (CT2) and let:

$$\begin{array}{ccccccc}
X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \xrightarrow{a^3} & \Sigma X^1 \\
\downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & & & \downarrow{\scriptstyle \Sigma s^1} \\
Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \xrightarrow{b^3} & \Sigma Y^1
\end{array}$$

be a commutative diagram whose rows are distinguished triangles. Applying (CT2), we obtain a commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc}
X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \xrightarrow{a^3} & \Sigma X^1 \\
\downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & \downarrow{\scriptstyle s^3} & & \downarrow{\scriptstyle \Sigma s^1} \\
Y^1 & \xrightarrow{b^1} & Y^2 & \dashrightarrow^{c^2} & C^3 & \dashrightarrow^{c^3} & \Sigma Y^1 \\
\| & & \| & & \wr\downarrow{\scriptstyle \phi} & & \| \\
Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \xrightarrow{b^3} & \Sigma Y^1
\end{array}$$

Since $\mathcal{C}$ is triangulated, there exists a morphism $\phi\colon C^3 \longrightarrow Y^3$ that completes the two bottom rows of the above diagram into a morphism of distinguished triangles, and in this case it is well known that ϕ is an isomorphism in $\mathcal{C}$. Thus, $s^3\phi \in S$ which proves that S satisfies (SM6).

Remark 4.1.3. As mentioned in the above remark, in the classic case of a triangulated category, if two of the three vertical morphisms in a morphism of distinguished triangles is an isomorphism, the third is also an isomorphism. However, in an n-angulated category where $n > 3$, in a given morphism $(s^1, s^2, \cdots, s^n)$ between two n-angles, if s^1 and s^2 are isomorphisms, then it does not necessarily hold that $s^3, \cdots, s^n$ are isomorphisms. Still, if we replace (CT2) with the following, slightly restrictive condition, the results presented in this section can be verified without any change:

(CT2') For any commutative diagram of the form:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1 \\
\downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & & & & & & & \downarrow{\scriptstyle \Sigma s^1} \\
Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \longrightarrow & \cdots & \longrightarrow & Y^n & \xrightarrow{b^n} & \Sigma Y^1
\end{array}$$

in which both rows are n-angles and $s^1, s^2 \in S$, there exist morphisms $s^3, \cdots, s^n \in S$ that complete the diagram to a morphism of n-angles:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1 \\
\downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & \downarrow{\scriptstyle s^3} & & & & \downarrow{\scriptstyle s^n} & & \downarrow{\scriptstyle \Sigma s^1} \\
Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \longrightarrow & \cdots & \longrightarrow & Y^n & \xrightarrow{b^n} & \Sigma Y^1
\end{array}$$

This condition also appears in the setting of n-exangulated categories in [29, (MR3)] however we will continue using (CT2) since it is slightly more general.

Finally, as in the classic case of a triangulated category, we obtain an induced automorphism in the localization category as follows:

Remark 4.1.4. Let $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ be the localization functor of $\mathcal{C}$ with respect to S. Since S is closed under Σ, condition (CT1), ensures that the functor $(Q \circ \Sigma)$ inverts the elements of S. By the universal property of localization, the functor $Q \circ \Sigma$ factors uniquely through Q making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{Q} & \mathcal{C}[S^{-1}] \\ \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Sigma_S} \\ \mathcal{C} & \xrightarrow{Q} & \mathcal{C}[S^{-1}] \end{array}$$

From now on we will denote the automorphism $\Sigma_S\colon \mathcal{C}[S^{-1}] \longrightarrow \mathcal{C}[S^{-1}]$ simply by Σ.

Notation. If $A^\bullet$ is an n-Σ-sequence:

$$A^1 \xrightarrow{a^1} A^2 \xrightarrow{a^2} \cdots \longrightarrow A^n \xrightarrow{a^n} \Sigma A^1$$

in $\mathcal{C}$, we will denote by $Q(A^\bullet)$ the induced n-Σ-sequence:

$$A^1 \xrightarrow{Q(a^1)} A^2 \xrightarrow{Q(a^2)} \cdots \longrightarrow A^n \xrightarrow{Q(a^n)} \Sigma A^1$$

in $\mathcal{C}[S^{-1}]$.

We can now define a class $\mathcal{N}_S$ of n-angles in the localized category.

Definition 4.1.5. *Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be a pre-n-angulated category and S a bicalculable class of morphisms in $\mathcal{C}$. We define a class $\mathcal{N}_S$ of n-Σ-sequences in $\mathcal{C}[S^{-1}]$ in the following way:*

An n-Σ-sequence $X^\bullet$:

$$X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\alpha^2} \cdots \longrightarrow X^n \xrightarrow{\alpha^n} \Sigma X^1$$

in $\mathcal{C}[S^{-1}]$ is in $\mathcal{N}_S$, if there exists an n-Σ-sequence $M^\bullet$:

$$M^1 \xrightarrow{\mu^1} M^2 \xrightarrow{\mu^2} \cdots \longrightarrow M^n \xrightarrow{\mu^n} \Sigma M^1$$

in $\mathcal{C}[S^{-1}]$ and an isomorphism of n-Σ-sequences $\phi^\bullet\colon A^\bullet \xrightarrow{\sim} X^\bullet \oplus M^\bullet$:

$$\begin{array}{ccccccccc} A^1 & \xrightarrow{Q(a^1)} & A^2 & \xrightarrow{Q(a^2)} & A^3 & \longrightarrow \cdots \longrightarrow & A^n & \xrightarrow{Q(a^n)} & \Sigma A^1 \\ \wr\downarrow{\scriptstyle \phi^1} & & \wr\downarrow{\scriptstyle \phi^2} & & \wr\downarrow{\scriptstyle \phi^3} & & \wr\downarrow{\scriptstyle \phi^n} & & \wr\downarrow{\scriptstyle \Sigma\phi^1} \\ X^1 \oplus M^1 & \xrightarrow{\alpha^1 \oplus \mu^1} & X^2 \oplus M^2 & \xrightarrow{\alpha^2 \oplus \mu^2} & X^3 \oplus M^3 & \longrightarrow \cdots \longrightarrow & X^n \oplus M^n & \xrightarrow{\alpha^n \oplus \mu^n} & \Sigma X^1 \oplus \Sigma M^1 \end{array} \tag{4.1.1}$$

in $\mathcal{C}[S^{-1}]$*, where* $A^\bullet$*:*

$$A^1 \xrightarrow{a^1} A^2 \xrightarrow{a^2} \cdots \longrightarrow A^n \xrightarrow{a^n} \Sigma A^1$$

is an n*-angle in* $\mathcal{C}$*. The elements of the class* $\mathcal{N}_S$ *will be called* n**-angles** *in* $\mathcal{C}[S^{-1}]$*.*

Remark 4.1.6. The reason that n-angles are defined in $\mathcal{C}[S^{-1}]$ in the above way, is to ensure closure under direct summands, a problem which in the classic case of triangulated categories has an easy solution. In fact, we will see that in the triangulated case, each 3-angle in $\mathcal{C}[S^{-1}]$ in the above sense is isomorphic to the image of a triangle of $\mathcal{C}$ through the localization functor Q. In this way we recover the classic definition of distinguished triangles in the localization of a triangulated category, see [44, 2.1.7].

From Definition 4.1.5, we obtain directly the following consequence:

Corollary 4.1.7. *Let* $(\mathcal{C}, \Sigma, \mathcal{N})$ *be a pre-*n*-angulated category and* S *a bicalculable class of morphisms in* $\mathcal{C}$*. If* $A^\bullet$ *is an* n*-angle in* $\mathcal{C}$*, then* $Q(A^\bullet)$ *is an* n*-angle in* $\mathcal{C}[S^{-1}]$*.*

Proof. Follows directly from Definition 4.1.5 by setting $A^\bullet = X^\bullet$ and $M^\bullet = 0$. ■

Corollary 4.1.8. *The class* $\mathcal{N}_S$ *of* n*-angles in* $\mathcal{C}[S^{-1}]$ *is closed under isomorphisms.*

Proof. Let $\phi^\bullet \colon X^\bullet \xrightarrow{\sim} Y^\bullet$ be an isomorphism of n-Σ-sequences where $X^\bullet$ is an n-angle in $\mathcal{C}[S^{-1}]$. By definition, there exists an n-Σ-sequence $M^\bullet$ and an isomorphism

$$\psi^\bullet \colon Q(A^\bullet) \xrightarrow{\sim} X^\bullet \oplus M^\bullet$$

in $\mathcal{C}[S^{-1}]$, where $A^\bullet$ is an n-angle in $\mathcal{C}$. Then, the composition:

$$Q(A^\bullet) \xrightarrow{\psi^\bullet} X^\bullet \oplus M^\bullet \xrightarrow{\phi^\bullet \oplus 1} Y^\bullet \oplus M^\bullet$$

is an isomorphism in $\mathcal{C}[S^{-1}]$, thus $Y^\bullet$ is also an n-angle in $\mathcal{C}[S^{-1}]$. ■

4.2 Axioms (F1) and (F2)

Our next goal is to prove our main result concerning the localization of n-angulated categories. Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be a pre-n-angulated category and S a bicalculable system of morphisms in $\mathcal{C}$. In the next three results, we will verify that $(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ satisfies axioms (F1) and (F2) of a pre-n-angulated category.

Proposition 4.2.1. *The category* $(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ *satisfies axiom (F1)(a) of a pre-n-angulated category, i.e. it is closed under direct sums and direct summands.*

Proof. By construction, the class $\mathcal{N}_S$ of n-angles in $\mathcal{C}[S^{-1}]$ is closed under direct summands. Now, let:

$$X^\bullet: \qquad X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\alpha^2} \cdots \longrightarrow X^n \xrightarrow{\alpha^n} \Sigma X^1$$

and

$$Y^\bullet: \qquad Y^1 \xrightarrow{\beta^1} Y^2 \xrightarrow{\beta^2} \cdots \longrightarrow Y^n \xrightarrow{\beta^n} \Sigma Y^1$$

be two n-angles in $\mathcal{C}[S^{-1}]$. By definition, there exist isomorphisms:

$$\begin{array}{ccccccccc}
A^1 & \xrightarrow{Q(a^1)} & A^2 & \xrightarrow{Q(a^2)} & A^3 & \longrightarrow \cdots \longrightarrow & A^n & \xrightarrow{Q(a^n)} & \Sigma A^1 \\
\wr\downarrow \phi^1 & & \wr\downarrow \phi^2 & & \wr\downarrow \phi^3 & & \wr\downarrow \phi^n & & \wr\downarrow \Sigma\phi^1 \\
X^1 \oplus M^1 & \xrightarrow{\alpha^1 \oplus \mu^1} & X^2 \oplus M^2 & \xrightarrow{\alpha^2 \oplus \mu^2} & X^3 \oplus M^3 & \longrightarrow \cdots \longrightarrow & X^n \oplus M^n & \xrightarrow{\alpha^n \oplus \mu^n} & \Sigma X^1 \oplus \Sigma M^1
\end{array}$$

$$\begin{array}{ccccccccc}
B^1 & \xrightarrow{Q(b^1)} & B^2 & \xrightarrow{Q(b^2)} & B^3 & \longrightarrow \cdots \longrightarrow & B^n & \xrightarrow{Q(b^n)} & \Sigma B^1 \\
\wr\downarrow \psi^1 & & \wr\downarrow \psi^2 & & \wr\downarrow \psi^3 & & \wr\downarrow \psi^n & & \wr\downarrow \Sigma\psi^1 \\
Y^1 \oplus N^1 & \xrightarrow{\beta^1 \oplus \nu^1} & Y^2 \oplus N^2 & \xrightarrow{\beta^2 \oplus \nu^2} & Y^3 \oplus N^3 & \longrightarrow \cdots \longrightarrow & Y^n \oplus N^n & \xrightarrow{\beta^n \oplus \nu^n} & \Sigma Y^1 \oplus \Sigma N^1
\end{array}$$

of n-angles in $\mathcal{C}[S^{-1}]$, where $A^\bullet$ and $B^\bullet$ are n-angles in $\mathcal{C}$. It follows that there exists an isomorphism:

$$\begin{array}{ccccc}
A^1 \oplus B^1 & \xrightarrow{Q(a^1 \oplus b^1)} & A^2 \oplus B^2 & \longrightarrow & \cdots \\
\wr\downarrow & & \wr\downarrow & & \\
X^1 \oplus Y^1 \oplus M^1 \oplus N^1 & \xrightarrow{\alpha^1 \oplus \beta^1 \oplus \mu^1 \oplus \nu^1} & X^2 \oplus Y^2 \oplus M^2 \oplus N^2 & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \longrightarrow & A^n \oplus B^n & \xrightarrow{Q(a^n \oplus b^n)} & \Sigma A^1 \oplus \Sigma B^1 \\
& & \wr\downarrow & & \wr\downarrow \\
\cdots & \longrightarrow & X^n \oplus Y^n \oplus M^n \oplus N^n & \xrightarrow{\alpha^n \oplus \beta^n \oplus \mu^n \oplus \nu^n} & \Sigma X^1 \oplus \Sigma Y^1 \oplus \Sigma M^1 \oplus \Sigma N^1
\end{array}$$

of n-Σ-sequences in $\mathcal{C}[S^{-1}]$. Since $\mathcal{C}$ is pre-n-angulated, the n-Σ-sequence $A^\bullet \oplus B^\bullet$ is an n-angle in $\mathcal{C}$, thus we conclude that $X^\bullet \oplus Y^\bullet$ is an n-angle in $\mathcal{C}[S^{-1}]$. ■

We continue with axioms (F1)(b) and (F1)(c):

Proposition 4.2.2. *The category* $(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ *satisfies axioms (F1)(b) and (F1)(c) of a pre-n-angulated category.*

Proof. Since for any object $X \in \mathcal{C}[S^{-1}]$, the n-Σ-sequence:

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow \Sigma X$$

is an n-angle in $\mathcal{C}$, applying the localization functor Q, we obtain an n-angle

$$X \xrightarrow{Q(1_X)} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow \Sigma X$$

in $\mathcal{C}[S^{-1}]$, thus axiom (F1)(b) is satisfied.

Now let $(s/f)\colon X^1 \longrightarrow X^2$ be a morphism in $\mathcal{C}[S^{-1}]$ represented as a fraction:

$$X^1 \xrightarrow{f} Y \xleftarrow{s} X^2$$

Since $\mathcal{C}$ is pre-n-angulated, we can complete the morphism f to an n-angle:

$$X^1 \xrightarrow{f} Y \xrightarrow{f^2} X^3 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{f^n} \Sigma X^1$$

in $\mathcal{C}$. The diagram:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{Q(f)Q(s)^{-1}} & X^2 & \xrightarrow{Q(s)Q(f^2)} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{Q(f^n)} & \Sigma X^1 \\
\| & & \wr\downarrow Q(s) & & \| & & & & \| & & \| \\
X^1 & \xrightarrow{Q(f)} & Y & \xrightarrow{Q(f^2)} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{Q(f^n)} & \Sigma X^1
\end{array}$$

is an isomorphism of n-Σ-sequences in $\mathcal{C}[S^{-1}]$ and since the bottom row is an n-angle, the top row is also an n-angle which completes the proof. ■

Axiom (F2) can be verified as follows:

Proposition 4.2.3. *The category* $(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ *satisfies axiom (F2) of a pre-n-angulated category.*

Proof. Let $X^\bullet$ be an n-angle in $\mathcal{C}[S^{-1}]$ and let $\phi^\bullet\colon A^\bullet \xrightarrow{\sim} X^\bullet \oplus M^\bullet$:

$$\begin{array}{ccccccccccc}
A^1 & \xrightarrow{Q(a^1)} & A^2 & \xrightarrow{Q(a^2)} & A^3 & \longrightarrow \cdots \longrightarrow & A^n & \xrightarrow{Q(a^n)} & \Sigma A^1 \\
\wr\downarrow{\scriptstyle \phi^1} & & \wr\downarrow{\scriptstyle \phi^2} & & \wr\downarrow{\scriptstyle \phi^3} & & \wr\downarrow{\scriptstyle \phi^n} & & \wr\downarrow{\scriptstyle \Sigma\phi^1} \\
X^1 \oplus M^1 & \xrightarrow{\alpha^1\oplus\mu^1} & X^2 \oplus M^2 & \xrightarrow{\alpha^2\oplus\mu^2} & X^3 \oplus M^3 & \longrightarrow \cdots \longrightarrow & X^n \oplus M^n & \xrightarrow{\alpha^n\oplus\mu^n} & \Sigma X^1 \oplus \Sigma M^1
\end{array} \tag{4.2.1}$$

be an isomorphism n-Σ-sequences. We will prove that the left rotation:

$$X^2 \xrightarrow{\alpha^2} X^3 \xrightarrow{\alpha^3} \cdots \xrightarrow{\alpha^n} \Sigma X^1 \xrightarrow{(-1)^n\Sigma\alpha^1} \Sigma X^2$$

of $X^\bullet$ is also an n-angle. The morphism:

$$\begin{array}{ccccccccc}
A^2 & \xrightarrow{Q(a^2)} & A^3 & \longrightarrow \cdots \longrightarrow & A^n & \xrightarrow{Q(a^n)} & \Sigma A^1 & \xrightarrow{(-1)^n\Sigma Q(a^1)} & \Sigma A^2 \\
{\scriptstyle \phi^2}\downarrow\wr & & {\scriptstyle \phi^3}\downarrow\wr & & {\scriptstyle \phi^n}\downarrow\wr & & {\scriptstyle \Sigma\phi^1}\downarrow\wr & & {\scriptstyle \Sigma\phi^2}\downarrow\wr \\
X^2 \oplus M^2 & \xrightarrow{\alpha^2\oplus\mu^2} & X^3 \oplus M^3 & \longrightarrow \cdots \longrightarrow & X^n \oplus M^n & \xrightarrow{\alpha^n\oplus\mu^n} & \Sigma X^1 \oplus \Sigma M^1 & \xrightarrow{(-1)^n\Sigma(\alpha^1\oplus\mu^1)} & \Sigma X^2 \oplus \Sigma M^2
\end{array}$$

is an isomorphism of n-Σ-sequences in $\mathcal{C}[S^{-1}]$ and since $A^\bullet$ is an n-angle in $\mathcal{C}$, its left rotation is also an n-angle. We conclude that the left rotation of $X^\bullet$ is also an n-angle.

Dually if the left rotation of $X^\bullet$ is an n-angle in $\mathcal{C}[S^{-1}]$, there exists an n-Σ-sequence $N^\bullet$ and an isomorphism of n-Σ-sequences:

$$\begin{array}{ccccccccc}
B^1 & \xrightarrow{Q(b^1)} & B^2 & \longrightarrow \cdots \longrightarrow & B^{n-1} & \xrightarrow{Q(b^{n-1})} & B^n & \xrightarrow{Q(b^n)} & \Sigma B^1 \\
{\scriptstyle \psi^1}\downarrow\wr & & {\scriptstyle \psi^2}\downarrow\wr & & {\scriptstyle \psi^{n-1}}\downarrow\wr & & {\scriptstyle \psi^n}\downarrow\wr & & {\scriptstyle \Sigma\psi^1}\downarrow\wr \\
X^2 \oplus N^1 & \xrightarrow{\alpha^2\oplus\nu^1} & X^3 \oplus N^2 & \longrightarrow \cdots \longrightarrow & X^n \oplus N^{n-1} & \xrightarrow{\alpha^n\oplus\nu^{n-1}} & \Sigma X^1 \oplus N^n & \xrightarrow{(-1)^n\Sigma\alpha^1\oplus\nu^n} & \Sigma X^2 \oplus \Sigma N^1
\end{array}$$

is in $\mathcal{C}[S^{-1}]$ where $B^\bullet$ is an n-angle in $\mathcal{C}$. Then, the morphism:

$$\begin{array}{ccccccc}
\Sigma^{-1}B^n & \xrightarrow{(-1)^n\Sigma^{-1}Q(b^n)} & B^1 & \longrightarrow \cdots \longrightarrow & B^{n-1} & \xrightarrow{Q(b^{n-1})} & B^n \\
{\scriptstyle \Sigma^{-1}\psi^n}\downarrow\wr & & {\scriptstyle \psi^1}\downarrow\wr & & {\scriptstyle \psi^{n-1}}\downarrow\wr & & {\scriptstyle \psi^n}\downarrow\wr \\
X^1 \oplus \Sigma^{-1}N^n & \xrightarrow{\alpha^1\oplus(-1)^n\Sigma^{-1}\nu^n} & X^2 \oplus N^1 & \longrightarrow \cdots \longrightarrow & X^n \oplus N^{n-1} & \xrightarrow{\alpha^n\oplus\nu^{n-1}} & \Sigma X^1 \oplus N^n
\end{array}$$

is also an isomorphism in $\mathcal{C}[S^{-1}]$. However, the n-angle $B^\bullet$ is the left rotation of the n-Σ-sequence:

$$\Sigma^{-1}B^n \xrightarrow{(-1)^n\Sigma^{-1}b^n} B^1 \longrightarrow \cdots \longrightarrow B^{n-1} \xrightarrow{b^{n-1}} B^n$$

Since $\mathcal{C}$ is a pre-n-angulated category, the above n-Σ-sequence is also an n-angle in $\mathcal{C}$. We infer that $X^\bullet$ is an n-angle in $\mathcal{C}[S^{-1}]$. ■

4.3 Axioms (F3) and (F4)

In the classic case of a triangulated category $\mathcal{C}$, any 3-angle (in other words any distinguished triangle using standard notation e.g. from [44]) in $\mathcal{C}[S^{-1}]$ is isomorphic to the image of a 3-angle in $\mathcal{C}$. In our case, a much weaker statement holds which we will prove in order to verify axioms (F3) and (F4).

Lemma 4.3.1. *Let* $(\mathcal{C}, \Sigma, \mathcal{N})$ *be a pre-*n*-angulated category and* S *a bicalculable class of morphisms in* $\mathcal{C}$ *compatible with the* n*-angulation. If*

$$X^\bullet\colon \qquad X^1 \xrightarrow{Q(f)} X^2 \xrightarrow{\alpha^2} \cdots \longrightarrow X^n \xrightarrow{\alpha^n} \Sigma X^1$$

is an n*-angle in* $\mathcal{C}[S^{-1}]$ *where* f *is a morphism in* $\mathcal{C}$*, then there exists a diagram of the form:*

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1 \\
\| & & \| & & \downarrow{\scriptstyle \gamma^3} & & & & \downarrow{\scriptstyle \gamma^n} & & \| \\
X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(f^2)} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{Q(f^n)} & \Sigma X^1 \\
\| & & \| & & \downarrow{\scriptstyle \delta^3} & & & & \downarrow{\scriptstyle \delta^n} & & \| \\
X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1
\end{array} \tag{4.3.1}$$

in $\mathcal{C}[S^{-1}]$*, where the following diagram is an* n*-angle in* $\mathcal{C}$*:*

$$C^\bullet\colon\ X^1 \xrightarrow{f} X^2 \xrightarrow{f^2} C^3 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{f^n} \Sigma X^1$$

Proof. Since $\mathcal{C}$ is pre-n-angulated, there exists an n-angle

$$C^\bullet\colon\ X^1 \xrightarrow{f} X^2 \xrightarrow{f^2} C^3 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{f^n} \Sigma X^1$$

in $\mathcal{C}$. Moreover, since $X^\bullet$ is an n-angle in $\mathcal{C}[S^{-1}]$, there exists an isomorphism of n-angles $\phi^\bullet = (\phi^\bullet_\alpha, \phi^\bullet_\beta)\colon A^\bullet \longrightarrow X^\bullet \oplus M^\bullet$:

$$\begin{array}{ccccccccccc}
A^1 & \xrightarrow{Q(a^1)} & A^2 & \xrightarrow{Q(a^2)} & A^3 & \longrightarrow & \cdots & \longrightarrow & A^n & \xrightarrow{Q(a^n)} & \Sigma A^1 \\
\wr\downarrow{\scriptstyle (\phi^1_\alpha, \phi^1_\beta)} & & \wr\downarrow{\scriptstyle (\phi^2_\alpha, \phi^2_\beta)} & & \wr\downarrow{\scriptstyle (\phi^3_\alpha, \phi^3_\beta)} & & & & {\scriptstyle (\phi^n_\alpha, \phi^n_\beta)}\downarrow\wr & & {\scriptstyle (\Sigma\phi^1_\alpha, \Sigma\phi^1_\beta)}\downarrow\wr \\
X^1 \oplus M^1 & \xrightarrow{\alpha^1 \oplus \mu^1} & X^2 \oplus M^2 & \xrightarrow{\alpha^2 \oplus \mu^2} & X^3 \oplus M^3 & \longrightarrow & \cdots & \longrightarrow & X^n \oplus M^n & \xrightarrow{\alpha^n \oplus \mu^n} & \Sigma X^1 \oplus \Sigma M^1
\end{array}$$

and thus we obtain in $\mathcal{C}[S^{-1}]$ a commutative diagram:

$$\begin{array}{ccccccccccc} A^1 & \xrightarrow{Q(a^1)} & A^2 & \xrightarrow{Q(a^2)} & A^3 & \longrightarrow & \cdots & \longrightarrow & A^n & \xrightarrow{Q(a^n)} & \Sigma A^1 \\ \downarrow{\scriptstyle \phi^1_\alpha} & & \downarrow{\scriptstyle \phi^2_\alpha} & & & & & & & & \downarrow{\scriptstyle \Sigma\phi^1_\alpha} \\ X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(f^2)} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{Q(f^n)} & \Sigma X^1 \end{array} \tag{4.3.2}$$

Since both rows are n-angles in $\mathcal{C}$, as in [44] we can complete this diagram to a morphism of n-angles in $\mathcal{C}[S^{-1}]$: Applying Lemma 1.3.10, we obtain a commutative diagram

$$\begin{array}{ccccccccccc} A^1 & \xrightarrow{a^1} & A^2 & \xrightarrow{a^2} & A^3 & \longrightarrow & \cdots & \longrightarrow & A^n & \xrightarrow{a^n} & \Sigma A^1 \\ \downarrow{\scriptstyle u^1} & & \downarrow{\scriptstyle u^2} & & & & & & & & \downarrow{\scriptstyle \Sigma f^1} \\ B^1 & \xrightarrow{b} & B^2 & & & & & & & & \Sigma B^1 \\ \uparrow{\scriptstyle s^1} & & \uparrow{\scriptstyle s^2} & & & & & & & & \uparrow{\scriptstyle \Sigma s^1} \\ X^1 & \xrightarrow{f} & X^2 & \xrightarrow{f^2} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{f^n} & \Sigma X^1 \end{array}$$

in $\mathcal{C}$ where $\phi^1_\alpha = (s^1/u^1)$ and $\phi^2_\alpha = (s^2/u^2)$. Since S satisfies condition (CT2) of Definition 4.1.1, there exists an n-angle $B^\bullet$ in $\mathcal{C}$ and morphisms $s^3, \cdots, s^n$ such that $s^\bullet\colon C^\bullet \longrightarrow B^\bullet$ is a morphism of n-angles in $\mathcal{C}$:

$$\begin{array}{ccccccccccc} A^1 & \xrightarrow{a^1} & A^2 & \xrightarrow{a^2} & A^3 & \longrightarrow & \cdots & \longrightarrow & A^n & \xrightarrow{a^n} & \Sigma A^1 \\ \downarrow{\scriptstyle u^1} & & \downarrow{\scriptstyle u^2} & & \vdots{\scriptstyle u^3} & & & & \vdots{\scriptstyle u^n} & & \downarrow{\scriptstyle \Sigma u^1} \\ B^1 & \xrightarrow{b} & B^2 & \overset{b^2}{\dashrightarrow} & B^3 & \dashrightarrow & \cdots & \dashrightarrow & B^n & \overset{b^n}{\dashrightarrow} & \Sigma B^1 \\ \uparrow{\scriptstyle s^1} & & \uparrow{\scriptstyle s^2} & & \vdots{\scriptstyle s^3} & & & & \vdots{\scriptstyle s^n} & & \uparrow{\scriptstyle \Sigma s^1} \\ X^1 & \xrightarrow{f} & X^2 & \xrightarrow{f^2} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{f^n} & \Sigma X^1 \end{array}$$

Moreover, since $\mathcal{C}$ is n-angulated, there exist morphisms $u^3, \cdots, u^n$ such that $u^\bullet\colon A^\bullet \longrightarrow B^\bullet$ is also a morphism of n-angles in $\mathcal{C}$. In this way, we have completed diagram (4.3.2) to a morphism of n-angles $\chi^\bullet\colon A^\bullet \longrightarrow C^\bullet$, where $\chi^i = (s^i/u^i)$ for $3 \leqslant i \leqslant n$. Now, from the commutative diagram:

$$\begin{array}{ccccccccccc} A^1 & \xrightarrow{Q(a^1)} & A^2 & \xrightarrow{Q(a^2)} & A^3 & \longrightarrow & \cdots & \longrightarrow & A^n & \xrightarrow{Q(a^n)} & \Sigma A^1 \\ \downarrow{\scriptstyle \phi^1}\wr & & \downarrow{\scriptstyle \phi^2}\wr & & \downarrow{\scriptstyle \phi^3}\wr \;\; {\scriptstyle \chi^3} & & & & {\scriptstyle \chi^n}\;\; \downarrow{\scriptstyle \phi^n}\wr & & \downarrow{\scriptstyle \Sigma\phi^1}\wr \\ X^1 \oplus M^1 & \xrightarrow{\left(\begin{smallmatrix} Q(f) & 0 \\ 0 & \mu^1 \end{smallmatrix}\right)} & X^2 \oplus M^2 & \xrightarrow{\left(\begin{smallmatrix} \alpha^2 & 0 \\ 0 & \mu^2 \end{smallmatrix}\right)} & X^3 \oplus M^3 & \longrightarrow & \cdots & \longrightarrow & X^n \oplus M^n & \xrightarrow{\left(\begin{smallmatrix} \alpha^n & 0 \\ 0 & \mu^n \end{smallmatrix}\right)} & \Sigma X^1 \oplus \Sigma M^1 \\ \downarrow{\scriptstyle {}^t(1,0)} & & \downarrow{\scriptstyle {}^t(1,0)} & & \vdots{\scriptstyle {}^t(\gamma^3_\alpha, \gamma^3_\beta)} & & & & \vdots{\scriptstyle {}^t(\gamma^n_\alpha, \gamma^n_\beta)} & & \downarrow{\scriptstyle {}^t(1,0)} \\ X^1 & \xrightarrow[Q(f)]{} & X^2 & \xrightarrow[Q(f^2)]{} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow[Q(f^n)]{} & \Sigma X^1 \end{array}$$

in $\mathcal{C}[S^{-1}]$, where ${}^t(\gamma_\alpha^i,\, \gamma_\beta^i) = (\phi^i)^{-1}\chi^i$ for $3 \leqslant i \leqslant n$, we obtain a morphism: $X^\bullet \longrightarrow C^\bullet$ of n-angles in $\mathcal{C}[S^{-1}]$:

$$\begin{array}{ccccccccccc} X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1 \\ \| & & \| & & \downarrow{\scriptstyle \gamma_\alpha^3} & & & & \downarrow{\scriptstyle \gamma_\alpha^n} & & \| \\ X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(f^2)} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{Q(f^n)} & \Sigma X^1 \end{array} \tag{4.3.3}$$

Since $\phi^\bullet$ is an isomorphism, dually we can obtain a morphism: $C^\bullet \longrightarrow X^\bullet$ of n-angles in $\mathcal{C}[S^{-1}]$: For convenience, we set $\psi^\bullet\colon = (\phi^\bullet)^{-1}$ and consider the morphism

$$\psi^\bullet = {}^t(\psi_\alpha^\bullet,\, \psi_\beta^\bullet)\colon X^\bullet \oplus M^\bullet \longrightarrow A^\bullet$$

as seen in the following commutative diagram:

$$\begin{array}{ccccccccccc} X^1 \oplus M^1 & \xrightarrow{\alpha^1 \oplus \mu^1} & X^2 \oplus M^2 & \xrightarrow{\alpha^2 \oplus \mu^2} & X^3 \oplus M^3 & \longrightarrow & \cdots & \longrightarrow & X^n \oplus M^n & \xrightarrow{\alpha^n \oplus \mu^n} & \Sigma X^1 \oplus \Sigma M^1 \\ \wr\downarrow{\scriptstyle {}^t(\psi_\alpha^1, \psi_\beta^1)} & & \wr\downarrow{\scriptstyle {}^t(\psi_\alpha^2, \psi_\beta^2)} & & \wr\downarrow{\scriptstyle {}^t(\psi_\alpha^3, \psi_\beta^3)} & & & & {\scriptstyle {}^t(\psi_\alpha^n, \psi_\beta^n)}\downarrow\wr & & {\scriptstyle {}^t(\Sigma\psi_\alpha^1, \Sigma\psi_\beta^1)}\downarrow\wr \\ A^1 & \xrightarrow[Q(a^1)]{} & A^2 & \xrightarrow[Q(a^2)]{} & A^3 & \longrightarrow & \cdots & \longrightarrow & A^n & \xrightarrow[Q(a^n)]{} & \Sigma A^1 \end{array}$$

Thus we obtain in $\mathcal{C}[S^{-1}]$ a commutative diagram:

$$\begin{array}{ccccccccccc} X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(f^2)} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{Q(f^n)} & \Sigma X^1 \\ \downarrow{\scriptstyle \psi_\alpha^1} & & \downarrow{\scriptstyle \psi_\alpha^2} & & & & & & & & \downarrow{\scriptstyle \Sigma\psi_\alpha^1} \\ A^1 & \xrightarrow{Q(a^1)} & A^2 & \xrightarrow{Q(a^2)} & A^3 & \longrightarrow & \cdots & \longrightarrow & A^n & \xrightarrow{Q(a^n)} & \Sigma A^1 \end{array} \tag{4.3.4}$$

Since both rows are n-angles in $\mathcal{C}$, as previously, we obtain a commutative diagram:

$$\begin{array}{ccccccccccc} X^1 & \xrightarrow{f} & X^2 & \xrightarrow{f^2} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{f^n} & \Sigma X^1 \\ \downarrow{\scriptstyle v^1} & & \downarrow{\scriptstyle v^2} & & \downarrow{\scriptstyle v^3} & & & & \downarrow{\scriptstyle v^n} & & \downarrow{\scriptstyle \Sigma v^1} \\ D^1 & \xrightarrow{d} & D^2 & \xrightarrow{d^2} & D^3 & \dashrightarrow & \cdots & \dashrightarrow & D^n & \xrightarrow{d^n} & \Sigma D^1 \\ \uparrow{\scriptstyle t^1} & & \uparrow{\scriptstyle t^2} & & \uparrow{\scriptstyle t^3} & & & & \uparrow{\scriptstyle t^n} & & \uparrow{\scriptstyle \Sigma t^1} \\ A^1 & \xrightarrow{a^1} & A^2 & \xrightarrow{a^2} & A^3 & \longrightarrow & \cdots & \longrightarrow & A^n & \xrightarrow{a^n} & \Sigma A^1 \end{array}$$

in $\mathcal{C}$ where $\psi_\alpha^1 = (t^1/v^1)$ and $\psi_\alpha^2 = (t^2/v^2)$. In this way, we have completed diagram (4.3.4) to a morphism of n-angles $\chi\colon C^\bullet \longrightarrow A^\bullet$ where $\chi^i = (t^i/v^i)$ for $3 \leqslant i \leqslant n$. Now, from the commutative diagram:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(f^2)} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{Q(f^n)} & \Sigma X^1 \\
\downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle {}^t(\delta^3_\alpha,\,\delta^3_\beta)} \; \searrow{\scriptstyle \chi^3} & & & & {\scriptstyle \chi^n}\swarrow \; \downarrow{\scriptstyle {}^t(\delta^n_\alpha,\,\delta^n_\beta)} & & \downarrow{\scriptstyle (1,0)} \\
X^1 \oplus M^1 & \xrightarrow{\left(\begin{smallmatrix} Q(f) & 0 \\ 0 & \mu^1 \end{smallmatrix}\right)} & X^2 \oplus M^2 & \xrightarrow{\left(\begin{smallmatrix} \alpha^2 & 0 \\ 0 & \mu^2 \end{smallmatrix}\right)} & X^3 \oplus M^3 & \longrightarrow & \cdots & \longrightarrow & X^n \oplus M^n & \xrightarrow{\left(\begin{smallmatrix} \alpha^n & 0 \\ 0 & \mu^n \end{smallmatrix}\right)} & \Sigma X^1 \oplus \Sigma M^1 \\
{\scriptstyle \psi^1}\downarrow \wr & & {\scriptstyle \psi^2}\downarrow \wr & & {\scriptstyle \psi^3}\downarrow \wr \; \swarrow & & & & \searrow \; \wr \downarrow{\scriptstyle \psi^n} & & \wr \downarrow{\scriptstyle \Sigma\psi^1} \\
A^1 & \xrightarrow[Q(a^1)]{} & A^2 & \xrightarrow[Q(a^2)]{} & A^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow[Q(a^n)]{} & \Sigma A^1
\end{array}$$

in $\mathcal{C}[S^{-1}]$, where ${}^t(\delta^i_\alpha,\, \delta^i_\beta) = \chi^i(\psi^i)^{-1}$ for $3 \leqslant i \leqslant n$, we obtain a morphism $C^\bullet \longrightarrow X^\bullet$ of n-angles in $\mathcal{C}[S^{-1}]$:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(f^2)} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{Q(f)^n} & \Sigma X^1 \\
\| & & \| & & \downarrow{\scriptstyle \delta^3_\alpha} & & & & \downarrow{\scriptstyle \delta^n_\alpha} & & \| \\
X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1
\end{array} \tag{4.3.5}$$

■

Remark 4.3.2. For any n-angle:

$$X^\bullet\colon\; X^1 \xrightarrow{\alpha^1} X^2 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{\alpha^n} \Sigma X^1$$

in $\mathcal{C}[S^{-1}]$, where $\alpha^1 = (s/f)$ is a morphism:

$$\begin{array}{ccc}
X^1 & & X^2 \\
& {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle s} & \\
& Y &
\end{array}$$

there exists in $\mathcal{C}[S^{-1}]$ an isomorphism of n-angles:

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{\alpha^1} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1 \\
\| & & \wr\downarrow{\scriptstyle Q(s)} & & \| & & & & \| & & \| \\
X^1 & \xrightarrow{Q(f)} & Y & \xrightarrow{Q(s)^{-1}\alpha^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1
\end{array}$$

In the sequel we will verify axioms (F3) and (F4) under the assumption that the morphisms a^1 and b^1 of (F3) in Definition 1.2.5 are of the form $Q(f)$ and $Q(g)$ respectively. Using the above isomorphism, one can easily then show that the required properties hold for any n-angle.

We proceed with the proof of axioms (F3) and (F4).

Proposition 4.3.3. *Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be a pre-n-angulated category and S a bicalculable class of morphisms in $\mathcal{C}$ compatible with the n-angulation. Then the category $(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ satisfies axiom $(F3)$ of a pre-n-angulated category.*

Proof. Let $X^\bullet$, $Y^\bullet$ be n-angles and let

$$\begin{array}{ccccccccccc}
X^1 & \xrightarrow{\alpha^1} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow \cdots \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1 \\
\downarrow{\scriptstyle \omega^1} & & \downarrow{\scriptstyle \omega^2} & & & & & & \downarrow{\scriptstyle \Sigma\omega^1} \\
Y^1 & \xrightarrow{\beta^1} & Y^2 & \xrightarrow{\beta^2} & Y^3 & \longrightarrow \cdots \longrightarrow & Y^n & \xrightarrow{\beta^n} & \Sigma Y^1
\end{array}$$

be a commutative diagram in $\mathcal{C}[S^{-1}]$. As mentioned in Remark 4.3.2, without loss of generality we may assume that $\alpha^1 = Q(f)$ and $\beta^1 = Q(g)$ where f, g are morphisms in $\mathcal{C}$. Applying Lemma 4.3.1 to $X^\bullet$ and $Y^\bullet$, we obtain a commutative diagram:

$$\begin{array}{cccccccccccc}
X^\bullet: & X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow \cdots \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1 \\
\downarrow{\scriptstyle \gamma^\bullet} & \| & & \| & & \downarrow{\scriptstyle \gamma^3} & & \downarrow{\scriptstyle \gamma^n} & & \| \\
C^\bullet: & X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(c^2)} & C^3 & \longrightarrow \cdots \longrightarrow & C^n & \xrightarrow{Q(c^n)} & \Sigma X^1 \\
 & \downarrow{\scriptstyle \omega^1} & & \downarrow{\scriptstyle \omega^2} & & & & & & \downarrow{\scriptstyle \Sigma\omega^1} \\
D^\bullet: & Y^1 & \xrightarrow{Q(g)} & Y^2 & \xrightarrow{Q(d^2)} & D^3 & \longrightarrow \cdots \longrightarrow & D^n & \xrightarrow{Q(d^n)} & \Sigma Y^1 \\
\downarrow{\scriptstyle \delta^\bullet} & \| & & \| & & \downarrow{\scriptstyle \delta^3} & & \downarrow{\scriptstyle \delta^n} & & \| \\
Y^\bullet: & Y^1 & \xrightarrow{Q(g)} & Y^2 & \xrightarrow{\beta^2} & Y^3 & \longrightarrow \cdots \longrightarrow & Y^n & \xrightarrow{\beta^n} & \Sigma Y^1
\end{array} \tag{4.3.6}$$

As in the proof of Lemma 4.3.1, since $C^\bullet$ and $D^\bullet$ are n-angles in $\mathcal{C}$, there exists in $\mathcal{C}$ a commutative diagram:

$$\begin{array}{cccccccccccc}
C^\bullet & X^1 & \xrightarrow{f} & X^2 & \xrightarrow{c^2} & C^3 & \longrightarrow \cdots \longrightarrow & C^n & \xrightarrow{c^n} & \Sigma X^1 \\
\downarrow{\scriptstyle u^\bullet} & \downarrow{\scriptstyle u^1} & & \downarrow{\scriptstyle u^2} & & \downarrow{\scriptstyle u^3} & & \downarrow{\scriptstyle u^n} & & \downarrow{\scriptstyle \Sigma u^1} \\
B^\bullet & B^1 & \xrightarrow{b} & B^2 & \xrightarrow{b^2} & B^3 & \dashrightarrow \cdots \dashrightarrow & B^n & \xrightarrow{b^n} & \Sigma B^1 \\
\uparrow{\scriptstyle s^\bullet} & \uparrow{\scriptstyle s^1} & & \uparrow{\scriptstyle s^2} & & \uparrow{\scriptstyle s^3} & & \uparrow{\scriptstyle s^n} & & \downarrow{\scriptstyle \Sigma s^1} \\
D^\bullet & Y^1 & \xrightarrow{g} & Y^2 & \xrightarrow{d^2} & D^3 & \longrightarrow \cdots \longrightarrow & D^n & \xrightarrow{d^n} & \Sigma Y^1
\end{array}$$

such that: $\omega^1 = (s^1/u^1)$, $\omega^2 = (s^2/u^2)$ and $B^\bullet$ is an n-angle in $\mathcal{C}$. By setting $\omega^i = (s^i/u^i)$ for $3 \leqslant i \leqslant n$, we obtain a morphism of n-angles $\omega^\bullet \colon C^\bullet \longrightarrow D^\bullet$ which completes diagram 4.3.6 in a morphism of complexes:

$$\begin{array}{ccccccccccccc} X^\bullet & & X^1 & \xrightarrow{\alpha^1} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1 \\ \downarrow{\scriptstyle \chi^\bullet} & & \downarrow{\scriptstyle \omega^1} & & \downarrow{\scriptstyle \omega^2} & & \downarrow{\scriptstyle \gamma^3\omega^3\delta^3} & & & & \downarrow{\scriptstyle \gamma^n\omega^n\delta^n} & & \downarrow{\scriptstyle \Sigma\omega^1} \\ Y^\bullet & & Y^1 & \xrightarrow{\beta^1} & Y^2 & \xrightarrow{\beta^2} & Y^3 & \longrightarrow & \cdots & \longrightarrow & Y^n & \xrightarrow{\beta^n} & \Sigma Y^1 \end{array} \tag{4.3.7}$$

in $\mathcal{C}[S^{-1}]$. ■

4.4 Main result

Combining the previous results we arrive at the following consequence.

Proposition 4.4.1. *Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be a pre-n-angulated category and S a bicalculable class of morphisms in $\mathcal{C}$ which is compatible with the n-angulation. Let $\mathcal{C}[S^{-1}]$ be the localization of $\mathcal{C}$ with respect to S and $\mathcal{N}_S$ the induced class of n-angles in $\mathcal{C}[S^{-1}]$. Then $(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ is a pre-n-angulated category and the localization functor Q is n-exact.*

Remark 4.4.2. Applying Lemma 1.2.10, n-angles in $\mathcal{C}[S^{-1}]$ are closed under weak isomorphisms, i.e., if $X^\bullet$, $Y^\bullet$ are two weakly isomorphic n-Σ-sequences in $\mathcal{C}[S^{-1}]$, then $X^\bullet$ is an n-angle iff $Y^\bullet$ is also an n-angle.

Remark 4.4.3. Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be a triangulated category. Setting $n = 3$ in Lemma 4.3.1, it follows that for every 3-angle in $\mathcal{C}[S^{-1}]$ of the form $(Q(f), \alpha^2, \alpha^3)$, there exists a commutative diagram:

$$\begin{array}{ccccccc} X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{\alpha^2} & X^3 & \xrightarrow{\alpha^3} & \Sigma X^1 \\ \| & & \| & & \downarrow{\scriptstyle \gamma^3} & & \| \\ X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(f^2)} & C^3 & \xrightarrow{Q(f^3)} & \Sigma X^1 \\ \| & & \| & & \downarrow{\scriptstyle \delta^3} & & \| \\ X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{\alpha^2} & X^3 & \xrightarrow{\alpha^3} & \Sigma X^1 \end{array}$$

where

$$C^\bullet : \; X^1 \xrightarrow{f} X^2 \xrightarrow{f^2} C^3 \xrightarrow{f^3} \Sigma X^1$$

is a 3-angle in $\mathcal{C}$. Applying Lemma 1.2.7, since $\mathcal{C}[S^{-1}]$ is pre-3-angulated, every 3-angle in $\mathcal{C}[S^{-1}]$ is exact, thus it follows that γ^3 and δ^3 are isomorphisms in $\mathcal{C}[S^{-1}]$.

In other words, every 3-angle in $\mathcal{C}[S^{-1}]$ of the form $(Q(f), \alpha^2, \alpha^3)$ is isomorphic in $\mathcal{C}[S^{-1}]$ to the image of a 3-angle of $\mathcal{C}$ through the localization functor Q. Taking into account Remark 4.3.2, this holds for any 3-angle in $\mathcal{C}[S^{-1}]$. In this way, we obtain Verdier's classical definition of distinguished triangles in the localization of a triangulated category in [44].

If $(\mathcal{C}, \Sigma, \mathcal{N})$ also satisfies axiom (F4), i.e. $(\mathcal{C}, \Sigma, \mathcal{N})$ is an n-angulated category, then the following holds:

Proposition 4.4.4. *Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be an n-angulated category and S a bicalculable class of morphisms in $\mathcal{C}$ which is compatible with n-angulation. Let $\mathcal{C}[S^{-1}]$ be the localization of $\mathcal{C}$ with respect to S and $\mathcal{N}_S$ the corresponding class of n-angles in $\mathcal{C}[S^{-1}]$. Then, the category $(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ is an n-angulated category and the localization functor Q is n-exact.*

Proof. We only need to show that in the proof of Proposition 4.3.3, the morphisms $\chi^i \colon X^i \longrightarrow Y^i$, where $3 \leqslant i \leqslant n$:

$$\begin{array}{ccccccccccccc}
X^\bullet & & X^1 & \xrightarrow{\alpha^1} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow \cdots \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1 \\
\downarrow{\scriptstyle \chi^\bullet} & & \downarrow{\scriptstyle \omega^1} & & \downarrow{\scriptstyle \omega^2} & & \downarrow{\scriptstyle \chi^3} & & \downarrow{\scriptstyle \chi^n} & & \downarrow{\scriptstyle \Sigma\omega^1} \\
Y^\bullet & & Y^1 & \xrightarrow{\beta^1} & Y^2 & \xrightarrow{\beta^2} & Y^3 & \longrightarrow \cdots \longrightarrow & Y^n & \xrightarrow{\beta^n} & \Sigma Y^1
\end{array}$$

can be chosen such that the cone $\mathsf{C}(\chi^\bullet)$:

$$X^2 \oplus Y^1 \xrightarrow{\left(\begin{smallmatrix} -\alpha^2 & \omega^2 \\ 0 & \beta^1 \end{smallmatrix}\right)} X^3 \oplus Y^2 \xrightarrow{\left(\begin{smallmatrix} -\alpha^3 & \chi^3 \\ 0 & \beta^2 \end{smallmatrix}\right)} \cdots \longrightarrow \Sigma X^1 \oplus Y^n \xrightarrow{\left(\begin{smallmatrix} -\Sigma\alpha^1 & \Sigma\omega^1 \\ 0 & \beta^n \end{smallmatrix}\right)} \Sigma X^2 \oplus \Sigma Y^1$$

is an n-angle. Hence, to proceed with the proof of the assertion, we may continue the proof of Proposition 4.3.3, using the same notation.

Since $\mathcal{C}$ is n-angulated, the morphisms $u^i \colon C^i \longrightarrow B^i$, where $3 \leqslant i \leqslant n$:

$$\begin{array}{ccccccccccccc}
C^\bullet & & X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{c^2} & C^3 & \longrightarrow \cdots \longrightarrow & C^n & \xrightarrow{c^n} & \Sigma X^1 \\
\downarrow{\scriptstyle u^\bullet} & & \downarrow{\scriptstyle u^1} & & \downarrow{\scriptstyle u^2} & & \downarrow{\scriptstyle u^3} & & \downarrow{\scriptstyle u^n} & & \downarrow{\scriptstyle \Sigma u^1} \\
B^\bullet & & B^1 & \xrightarrow{b} & B^2 & \xrightarrow{b^2} & B^3 & \dashrightarrow \cdots \dashrightarrow & B^n & \xrightarrow{b^n} & \Sigma B^1 \\
\uparrow{\scriptstyle s^\bullet} & & \uparrow{\scriptstyle s^1} & & \uparrow{\scriptstyle s^2} & & \uparrow{\scriptstyle s^3} & & \uparrow{\scriptstyle s^n} & & \downarrow{\scriptstyle \Sigma s^1} \\
D^\bullet & & Y^1 & \xrightarrow{g} & Y^2 & \xrightarrow{d^2} & D^3 & \longrightarrow \cdots \longrightarrow & D^n & \xrightarrow{d^n} & \Sigma Y^1
\end{array}$$

can be chosen such that the cone $\mathsf{C}(u^\bullet)$:

$$X^2 \oplus B^1 \xrightarrow{d_u^1} C^3 \oplus B^2 \xrightarrow{d_u^2} \cdots \longrightarrow \Sigma X^1 \oplus B^n \xrightarrow{d_u^n} \Sigma X^2 \oplus \Sigma B^1$$

is an n-angle in $\mathcal{C}$. As we have seen, from the above diagram we obtain a morphism of n-angles: $\omega^\bullet\colon C^\bullet \longrightarrow D^\bullet$ in $\mathcal{C}[S^{-1}]$ with mapping cone $\mathsf{C}(\omega^\bullet)$:

$$X^2 \oplus Y^1 \xrightarrow{d_\omega^1} C^3 \oplus Y^2 \xrightarrow{d_\omega^2} C^4 \oplus D^3 \longrightarrow \cdots \longrightarrow \Sigma X^1 \oplus D^n \xrightarrow{d_\omega^n} \Sigma X^2 \oplus \Sigma Y^1$$

The morphism $s^\bullet\colon B^\bullet \longrightarrow D^\bullet$ induces an isomorphism in $\mathcal{C}[S^{-1}]$, thus the diagram:

$$\begin{array}{ccccccccc}
X^2 \oplus Y^1 & \xrightarrow{\partial_\omega^1} & C^3 \oplus Y^2 & \xrightarrow{\partial_\omega^2} & C^4 \oplus D^3 & \longrightarrow \cdots \longrightarrow & \Sigma X^1 \oplus D^n & \xrightarrow{\partial_\omega^n} & \Sigma X^2 \oplus \Sigma Y^1 \\
\Big\downarrow \begin{pmatrix} 1 & 0 \\ 0 & Q(s^1) \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 1 & 0 \\ 0 & Q(s^2) \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 1 & 0 \\ 0 & Q(s^3) \end{pmatrix} & & \begin{pmatrix} 1 & 0 \\ 0 & Q(s^n) \end{pmatrix} \Big\downarrow & & \begin{pmatrix} 1 & 0 \\ 0 & \Sigma Q(s^1) \end{pmatrix} \Big\downarrow \\
X^2 \oplus B^1 & \xrightarrow[Q(d_u^1)]{} & C^3 \oplus B^2 & \xrightarrow[Q(d_u^2)]{} & C^4 \oplus B^3 & \longrightarrow \cdots \longrightarrow & \Sigma X^1 \oplus B^n & \xrightarrow[Q(d_u^n)]{} & \Sigma X^2 \oplus \Sigma B^1
\end{array}$$

is an isomorphism of n-Σ-sequences in $\mathcal{C}[S^{-1}]$. Since $\mathsf{C}(u^\bullet)$ is an n-angle in $\mathcal{C}$, it follows that $\mathsf{C}(\omega^\bullet)$ is an n-angle in $\mathcal{C}[S^{-1}]$. Moreover, the diagram:

$$\begin{array}{ccccccccc}
X^2 \oplus Y^1 & \xrightarrow{\begin{pmatrix} -Q(c^2) & \omega^2 \\ 0 & Q(g) \end{pmatrix}} & C^3 \oplus Y^2 & \xrightarrow{\begin{pmatrix} -Q(c^3) & \omega^3 \\ 0 & Q(d^2) \end{pmatrix}} & C^4 \oplus D^3 & \longrightarrow \cdots \longrightarrow & \Sigma X^1 \oplus D^n & \xrightarrow{\begin{pmatrix} -\Sigma Q(f) & \Sigma\omega^1 \\ 0 & Q(d^n) \end{pmatrix}} & \Sigma X^2 \oplus \Sigma Y^1 \\
\Big\| & & \Big\| & & \Big\downarrow \begin{pmatrix} 1 & 0 \\ 0 & \delta^3 \end{pmatrix} & & \begin{pmatrix} 1 & 0 \\ 0 & \delta^n \end{pmatrix} \Big\downarrow & & \Big\| \\
X^2 \oplus Y^1 & \xrightarrow{\begin{pmatrix} -Q(c^2) & \omega^2 \\ 0 & Q(g) \end{pmatrix}} & C^3 \oplus Y^2 & \xrightarrow{\begin{pmatrix} -Q(c^3) & \omega^3\delta^3 \\ 0 & \beta^2 \end{pmatrix}} & C^4 \oplus Y^3 & \longrightarrow \cdots \longrightarrow & \Sigma X^1 \oplus Y^n & \xrightarrow{\begin{pmatrix} -\Sigma Q(f) & \Sigma\omega^1 \\ 0 & \beta^n \end{pmatrix}} & \Sigma X^2 \oplus \Sigma Y^1
\end{array}$$

is a weak isomorphism of n-Σ-sequences. Since $\mathsf{C}(\omega^\bullet)$, the top row, is an n-angle, using Lemma 1.2.10, it follows that the bottom row is an n-angle in $\mathcal{C}[S^{-1}]$. Similarly, the diagram:

$$\begin{array}{ccccccccc}
X^2 \oplus Y^1 & \xrightarrow{\begin{pmatrix} -\alpha^2 & \omega^2 \\ 0 & Q(g) \end{pmatrix}} & X^3 \oplus Y^2 & \longrightarrow \cdots \longrightarrow & X^n \oplus Y^{n-1} & \xrightarrow{\begin{pmatrix} -\alpha^n & \chi^n \\ 0 & \beta^{n-1} \end{pmatrix}} & \Sigma X^1 \oplus Y^n & \xrightarrow{\begin{pmatrix} -\Sigma Q(f) & \Sigma\omega^1 \\ 0 & \beta^n \end{pmatrix}} & \Sigma X^2 \oplus \Sigma Y^1 \\
\Big\| & & \Big\downarrow \begin{pmatrix} \gamma^3 & 0 \\ 0 & 1 \end{pmatrix} & & \begin{pmatrix} \gamma^n & 0 \\ 0 & 1 \end{pmatrix} \Big\downarrow & & \Big\| & & \Big\| \\
X^2 \oplus Y^1 & \xrightarrow{\begin{pmatrix} -Q(c^2) & \omega^2 \\ 0 & Q(g) \end{pmatrix}} & C^3 \oplus Y^2 & \longrightarrow \cdots \longrightarrow & C^n \oplus Y^{n-1} & \xrightarrow{\begin{pmatrix} -Q(c^n) & \omega^n\delta^n \\ 0 & \beta^{n-1} \end{pmatrix}} & \Sigma X^1 \oplus Y^n & \xrightarrow{\begin{pmatrix} -\Sigma Q(f) & \Sigma\omega^1 \\ 0 & \beta^n \end{pmatrix}} & \Sigma X^2 \oplus \Sigma Y^1
\end{array}$$

is a weak isomorphism of n-Σ-sequences and since by the above diagram, the bottom row is an n-angle, the top row $\mathsf{C}(\chi^\bullet)$ is also an n-angle in $\mathcal{C}[S^{-1}]$.

Finally, using Corollary 4.1.7 it follows that the localization functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ is n-exact. This completes the proof. ∎

The main result of this section is the following theorem which describes the necessary universal property satisfied by the localization of an n-angulated category.

Theorem 4.4.5. *Let $(\mathcal{C}, \Sigma, \mathcal{N})$ be an n-angulated category and S a bicalculable system of morphisms in $\mathcal{C}$ which is compatible with n-angulation. Let $\mathcal{C}[S^{-1}]$ be the localization of $\mathcal{C}$ with respect to S and $\mathcal{N}_S$ the induced class of n-angles in $\mathcal{C}[S^{-1}]$. Then,*

$(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ *is an* n*-angulated category and the localization functor* Q *is an* n*-exact functor of* n*-angulated categories.*

Moreover, Q *satisfies the following universal property:*

• *for any* n*-angulated category* $(\mathcal{C}', \Sigma', \mathcal{N}')$ *and any* n*-exact functor* $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ *such that* $F(s)$ *is invertible for any* $s \in S$*, there exists a unique* n*-exact functor* $F^*\colon \mathcal{C}[S^{-1}] \longrightarrow \mathcal{C}'$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{Q} & \mathcal{C}[S^{-1}] \\ & {\scriptstyle F}\searrow & \Big\downarrow {\scriptstyle \exists !\, F^*} \\ & & \mathcal{C}' \end{array}$$

Proof. We already know that $(\mathcal{C}[S^{-1}], \Sigma, \mathcal{N}_S)$ is an n-angulated category and Q is an n-exact functor. The existence of F^* follows from the universal property of the localization. Assuming now that F is n-exact, let

$$X^\bullet\colon\ X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\alpha^2} X^3 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{\alpha^n} \Sigma X^1$$

be an n-angle in $\mathcal{C}[S^{-1}]$. As we have seen in Remark 4.3.2, we can assume (up to isomorphism of n-angles) that the morphism α^1 is of the form $Q(f)$, where f is a morphism in $\mathcal{C}$ and then applying Lemma 4.3.1, there exists in $\mathcal{C}[S^{-1}]$ a weak isomorphism of n-angles:

$$\begin{array}{ccccccccccc} X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{\alpha^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{\alpha^n} & \Sigma X^1 \\ \| & & \| & & \downarrow{\scriptstyle \omega^3} & & & & \downarrow{\scriptstyle \omega^n} & & \| \\ X^1 & \xrightarrow{Q(f)} & X^2 & \xrightarrow{Q(f^2)} & C^3 & \longrightarrow & \cdots & \longrightarrow & C^n & \xrightarrow{Q(f^n)} & \Sigma X^1 \end{array}$$

where $(f, f^2, \cdots, f^n)$ is an n-angle in $\mathcal{C}$. It is well known from the classic case of the localization of a triangulated category, see [44], that for any $X \in \mathcal{C}$ the natural isomorphisms: $\phi_X\colon F(\Sigma X) \xrightarrow{\sim} \Sigma' F(X)$, also induce natural isomorphisms:

$$\phi_X\colon F^*(\Sigma X) \xrightarrow{\sim} \Sigma' F^*(X)$$

(note that by the universal construction of F^*, $F(X) = F^*(Q(X)) = F^*(X)$ for any object $X \in \mathcal{C}$). Applying F^*, and composing with these isomorphisms, we obtain a weak isomorphism of n-Σ-sequences:

$$\begin{array}{ccccccccccc} F^*(X^1) & \xrightarrow{F(f)} & F^*(X^2) & \xrightarrow{F^*(\alpha^2)} & F^*(X^3) & \longrightarrow & \cdots & \longrightarrow & F^*(X^n) & \xrightarrow{F^*(\alpha^n)\phi_{X^1}} & \Sigma' F^*(X^1) \\ \| & & \| & & \downarrow{\scriptstyle F^*(\omega^3)} & & & & {\scriptstyle F^*(\omega^n)}\downarrow & & \| \\ F(X^1) & \xrightarrow{F(f)} & F(X^2) & \xrightarrow{F(f^2)} & F(C^3) & \longrightarrow & \cdots & \longrightarrow & F(C^n) & \xrightarrow{F(f^n)\phi_{X^1}} & \Sigma' F(X^1) \end{array}$$

in $\mathcal{C}'$. Since F is n-exact, the bottom row is an n-angle in $\mathcal{C}'$. Using Lemma 1.2.10, it follows that the top row is also an n-angle in $\mathcal{C}'$, thus F^* is n-exact and this completes the proof. ∎

Remark 4.4.6. In the universal property of Theorem 4.4.5, if the functor F^* is n-exact, then $F = F^* \circ Q$ is also n-exact. Thus, F is n-exact iff F^* is n-exact.

Remark 4.4.7. As mentioned in Remark 3.2.18, recently, Jian He, Jing He, and Pan Zhou in [29] study independently localizations of n-exangulated categories in the following setting: starting with an n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and an additive full subcategory $\mathcal{N}_F \subseteq \mathcal{C}$ which is associated to a system of morphisms F, let $\bar{F}$ be a set of morphisms in the ideal quotient $\bar{\mathcal{C}} = \mathcal{C}/[\mathcal{N}_F]$ obtained from F. Assuming that $\bar{F}$ satisfies some set of conditions, namely conditions (MR1)-(MR3) in [29, §3], and that the localization $\tilde{\mathcal{C}}$ satisfies an additional condition, namely condition (C4) of [29, Definition 2.8], it is shown that $\tilde{\mathcal{C}}$ is an n-exangulated category if and only if any n-exangle in $\mathcal{C}$ induces a weak kernel-cokernel sequence in $\tilde{\mathcal{C}}$, see [29, Theorem 3.4].

However, there are significant differences between our approach and the approach of the paper [29] concerning the setting, the imposed conditions, and the methods used. In particular, we have not imposed a priori any conditions on the localization category $\mathcal{C}[S^{-1}]$ and we do not use any part of the theory of n-exangulated categories and the external structure that they provide. Our approach is different being as close as possible to the classic case of localizations of triangulated categories.

In this connection, similarly to the classic case, it is necessary to impose a condition on the class of morphisms S in order to ensure a compatibility with n-angulation. In the classic case of a triangulated category $\mathcal{C}$, Verdier in [44, 2.1.2] defines the notion of a bicalculable system of morphisms that is compatible with triangulation and satisfies condition [44, (SM6)], see Definition 1.3.12.

A direct approach in defining a higher analogue of this condition is condition (CT2') mentioned in Remark 4.1.3, which also appears in [29, (MR3)]. In this setting, let $(\mathcal{C}, \mathcal{N}, \Sigma)$ be an n-angulated category and S a bicalculable system of morphisms in $\mathcal{C}$. As stated in Remark 4.1.3, for any diagram of the form:

$$\begin{array}{ccccccccccc} X^1 & \xrightarrow{a^1} & X^2 & \xrightarrow{a^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{a^n} & \Sigma X^1 \\ \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} & & \downarrow{\scriptstyle u^3} & & & & \downarrow{\scriptstyle u^n} & & \downarrow{\scriptstyle \Sigma s^1} \\ Y^1 & \xrightarrow{b^1} & Y^2 & \xrightarrow{b^2} & Y^3 & \longrightarrow & \cdots & \longrightarrow & Y^n & \xrightarrow{b^n} & \Sigma Y^1 \end{array}$$

whose rows are n-angles in $\mathcal{C}$, applying axiom (F3) of Definition 1.2.5, there exist morphisms $u^3, \cdots, u^n$ in $\mathcal{C}$ that complete the above diagram to a morphism of n-angles.

However, for $n > 3$, it does not necessarily hold that if s^1, s^2 are isomorphisms, then $u^3, \cdots, u^n$ are also isomorphisms in $\mathcal{C}$. This property is true only for $n = 3$.

For this reason, we defined a higher analogue of axiom (SM6) by introducing condition (CT2) in Definition 4.1.1. Since every morphism in an n-angulated category can be embedded in an n-angle, it is clear that if S satisfies condition (CT2') then S also satisfies condition (CT2). On the other hand, if S satisfies (CT2), for every diagram of the above form the morphisms $u^3, \cdots, u^n$ are not necessarily isomorphisms, thus S does not have to satisfy (CT2'). We conclude that a bicalculable system S may satisfy condition (CT2) while not necessarily satisfying (CT2'). This makes condition (CT2) slightly more general than [29, (MR3)] in our setting, since it can be applied in a wider range of bicalculable systems when studying localizations of n-angulated categories.

We also note that if we replace condition (CT2) with condition (CT2') the results of this section can be proved without significant difference.

An additional difference between our approach and that of [29] is that in our case we define the elements of the class $\mathcal{N}_S$ as direct summands of induced elements of $\mathcal{N}$ in $\mathcal{C}[S^{-1}]$, through the localization functor Q. This is necessary in order to ensure the closure of n-angles under direct summands, a problem which does not appear in the context of an n-exangulated category.

As a final comment, we notice that, even though any n-angulated category is an n-exangulated category, our result cannot be recovered from [29], and vice versa.

Περιληψη

Στόχος της διατριβής είναι η ανάπτυξη μίας θεωρίας τοπικοποίησης στην ανώτερη ομολογική άλγεβρα των n-αβελιανών και n-τριγωνισμένων κατηγοριών. Ανάλογα με τις κλασικές περιπτώσεις μίας αβελιανής ή μίας τριγωνισμένης κατηγορίας, ξεκινούμε με μία κλάση μορφισμών S η οποία ικανοποιεί τις φυσικές συνθήκες ενός υπολογίσιμου (bicalculable) συστήματος μορφισμών. Στο ανώτερο ομολογικό πλαίσιο μίας n-αβελιανής κατηγορίας $\mathcal{M}$ ή μίας n-τριγωνισμένης κατηγορίας $\mathcal{C}$, κατασκευάζουμε με καθολικό τρόπο μία τοπικοποιημένη n-αβελιανή κατηγορία $\mathcal{M}[S^{-1}]$ ή μία n-τριγωνισμένη κατηγορία $\mathcal{C}[S^{-1}]$, αντίστοιχα όπου οι μορφισμοί στο S έχουν αντιστραφεί. Έτσι λύνεται ικανοποιητικά το πρόβλημα της τοπικοποίησης μίας n-αβελιανής ή μίας n-τριγωνισμένης κατηγορίας.

Η διατριβή χωρίζεται σε τέσσερα κεφάλαια.

Το Κεφαλαίο 1 αποτελεί μία σύντομη εισαγωγή στην Ανώτερη Ομολογική Άλγεβρα και στη γενική Θεωρία Τοπικοποίησης κατηγοριών με τη μορφή λογισμού κλασμάτων, ενώ ταυτόχρονα στοχεύει στην εδραίωση συμβολισμού. Το κεφάλαιο χωρίζεται σε τρεις ενότητες, με τις δύο πρώτες να περιέχουν ορισμούς, βασικές έννοιες και αποτελέσματα που αφορούν τις n-αβελιανές και n-τριγωνισμένες κατηγορίες αντίστοιχα. Στην τρίτη ενότητα συμπεριλαμβάνουμε μία σύντομη σύνοψη της Θεωρίας Τοπικοποίησης σε μία γενική κατηγορία, κατασκευάζουμε την κατηγορία κλασμάτων και αναφέρουμε ορισμένα αποτελέσματα που θα είναι απαραίτητα στη συνέχεια της διατριβής.

Ο κύριος στόχος του Κεφαλαίου 2 είναι η ανάπτυξη των εργαλείων που θα χρησιμοποιηθούν στην απόδειξη του κεντρικού αποτελέσματος που αφορά την τοπικοποίηση n-αβελιανών κατηγοριών. Λόγω της ιδιαίτερης φύσης του αξιώματος ταυτοδύναμης πλήρωσης μίας n-αβελιανής κατηγορίας σε σχέση με την τοπικοποίηση, η κατασκευή της τοπικοποιημένης κατηγορίας θα ολοκληρωθεί σε δύο στάδια. Έτσι, ορίζουμε μία ημι-n-αβελιανή (pre-n-abelian) κατηγορία ως μία προσθετική κατηγορία η οποία ικανοποιεί όλα τα αξι-

ώματα μίας n-αβελιανής κατηγορίας πλην ενδεχομένως του αξιώματος της ταυτοδύναμης πλήρωσης. Τότε, χρησιμοποιώντας ένα αποτέλεσμα του Jasso, βλ. [28], αποδεικνύουμε μία ικανή και αναγκαία συνθήκη έτσι ώστε μία κατηγορία να είναι ημι-n-αβελιανή, με βάση τις ιδιότητες ακριβείας ενός διαγράμματος το οποίο καλούμε n-διάγραμμα και περιέχει όλες τις επιθυμητές πληροφορίες.

Στο Κεφάλαιο 3, αποδεικνύουμε το πρώτο κύριο αποτέλεσμα της διατριβής, κατασκευάζοντας σε δύο βήματα την τοπικοποίηση μίας n-αβελιανής κατηγορίας ως προς ένα υπολογίσιμο σύστημα μορφισμών. Στο πρώτο βήμα, χρησιμοποιώντας το n-διάγραμμα, δείχνουμε ότι η τοπικοποίηση μίας ημι-n-αβελιανής κατηγορίας ως προς ένα υπολογίσιμο σύστημα μορφισμών είναι επίσης μία ημι-n-αβελιανή κατηγορία και ο συναρτητής τοπικοποίησης είναι n-ακριβής. Στο δεύτερο βήμα, αποδεικνύουμε ότι η ταυτοδύναμη πλήρωση μίας ημι-n-αβελιανής κατηγορίας είναι μία n-αβελιανή κατηγορία. Τέλος, συνδυάζοντας τα παραπάνω, για μία τυχαία n-αβελιανή κατηγορία $\mathcal{M}$ και ένα υπολογίσιμο σύστημα μορφισμών S στην $\mathcal{M}$ θεωρούμε αρχικά την ημι-n-αβελιανή κατηγορία $\mathcal{M}[S^{-1}]$ και στη συνέχεια την ταυτοδύναμη πλήρωση αυτής, $\widetilde{\mathcal{M}[S^{-1}]}$ η οποία είναι n-αβελιανή. Τότε, ο συναρτητής ο οποίος προκύπτει από την σύνθεση των συναρτητών: $\mathcal{M} \longrightarrow \mathcal{M}[S^{-1}] \longrightarrow \widetilde{\mathcal{M}[S^{-1}]}$, όπου ο δεύτερος συναρτητής συμβολίζει την ταυτοδύναμη πλήρωση, είναι καθολικός ως προς όλους τους S-αντιστρεπτικούς n-ακριβείς συναρτητές από την $\mathcal{M}$ σε μία n-αβελιανή κατηγορία.

Στο Κεφάλαιο 4, παρουσιάζουμε ένα ανάλογο αποτέλεσμα για n-τριγωνισμένες κατηγορίες όπου $n \geqslant 3$. Σε αυτό το πλαίσιο θα χρειαστούμε ένα υπολογίσιμο σύστημα μορφισμών S το οποίο, όπως στην κλασσική περίπτωση $n = 3$ των τριγωνισμένων κατηγοριών, ικανοποιεί μία συνθήκη συμβατότητας ως προς τον n-τριγωνισμό. Ορίζουμε μία κλάση n-τριγώνων N_S στη $\mathcal{C}[S^{-1}]$ όπως ακολούθως: Μία n-Σ-ακολουθία $X^\bullet$ στη $\mathcal{C}[S^{-1}]$ ανήκει στην N_S, αν υπάρχει μία n-Σ-ακολουθία $M^\bullet$ και ένας ισομορφισμός n-Σ-ακολουθιών $\phi^\bullet\colon A^\bullet \xrightarrow{\sim} X^\bullet \oplus M^\bullet$, όπου $A^\bullet$ είναι n-τρίγωνο στη $\mathcal{C}$. Στη συνέχεια, αποδεικνύουμε το δεύτερο κύριο αποτέλεσμα της διατριβής, κατασκευάζοντας την τοπικοποίηση μίας n-τριγωνισμένης κατηγορίας $\mathcal{C}$ η οποία είναι επίσης μία n-τριγωνισμένη κατηγορία που πληροί την ανάλογη καθολική ιδιότητα.

Index

Bibliography

[1] M. Artin, A. Grothendieck and J.-L. Verdier, *S-G-A 4, Sém. Géom. Algébrique du Bois Marie(1963–64), Théorie des topos et cohomologie étale des schémas, (SGA 4)*, Lecture Notes in Math. , Springer-Verlag, Berlin, New York, (1972–1973).

[2] M. Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes, London (1971).

[3] M. Auslander, I. Reiten, and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press (1995).

[4] P. Balmer, M. Schlichting, *Idempotent Completion of Triangulated Categories*, Journal of Algebra, **236**, (2001), 819–834.

[5] A. Beligiannis, *The Homological theory of contravariantly finite subcategories: Gorenstein categories, Auslander-Buchweitz contexts and (co-)stabilization*, Communications in Algebra **28**, (2000), 4547–4596.

[6] A. Beligiannis, *Homotopy theory of modules and Gorenstein rings*, Math. Scand. **89** (2001), 5–45.

[7] A. Beligiannis, *Relative Homology, Higher Cluster Tilting Theory, and Categorified Auslander-Iyama Correspondence*, Journal of Algebra, **444**, No. 3, (2015), 367–503.

[8] P. A. Bergh M. Thaule, *The axioms for n-angulated categories*, Algebraic & Geometric Topology **13** (2013), 2405–2428.

[9] P. Bland, *Rings and their Modules*, De Gruyter, (2011).

[10] A. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.

[11] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., (1956).

[12] R. Ebrahimi and A. Nasr-Isfahani, *Higher Auslander's formula*, International Mathematics Research Notices, **22**, (2021), 18186–18203.

[13] R. Ebrahimi and A. Nasr-Isfahani, *Representation of n-abelian categories in abelian categories*, J. Algebra, **563**, (2020), 352–375.

[14] E. Enochs and M. G. Jenda, *Relative homological algebra*, De Gruyter Expositions in mathematics; **30**, Walter de Gruyter, Berlin, New York, (2000).

[15] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations* J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[16] S. Fomin and A. Zelevinsky, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121.

[17] P. Freyd, *Abelian Categories*, Harper and Row, New York, (1964).

[18] P. Gabriel, *Des catégories abéliennes*, Bulletin de la Société Mathématique de France, **90** (1962), 323–448.

[19] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, Berlin Heidelberg, New York, (1967).

[20] C. Geiss, B. Keller, S. Oppermann, *n-Angulated categories*, J. Reine Angew. Math. **675** (2013), 101–120.

[21] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, in: London Mathematical Society Lecture Note Series, vol. **119**, Cambridge University Press, Cambridge, (1988).

[22] O. Iyama, *Auslander correspondence*, Adv. Math. **210** (2007), 51–82.

[23] O. Iyama, *Higher Auslander-Reiten theory on maximal orthogonal sucategories*, Adv. Math. **210** (2007), 22-50.

[24] O. Iyama. *Cluster tilting for higher Auslander algebras*, Adv. Math., **226**, (2011), no. 1, 1–61.

[25] O. Iyama and G. Jasso, *Higher Auslander correspondence for dualizing R-varieties*, Algebra Rep. Theory, **20** (2017), no. 2, 335–354.

[26] O. Iyama and S. Oppermann, *Stable categories of higher preprojective algebras*, Adv. Math. **244** (2013), 23–68.

[27] O. Iyama and Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (2008), no. 1, 117–168.

[28] G. Jasso, *n-Abelian and n-exact categories*, Math. Z, **283**, Issue 3-4 (2016), 703–759.

[29] Jian He, Jing He, and P. Zhou, *Localization of n-exangulated categories*, arXiv:2205.07644, (2022).

[30] M. Karoubi, *K-theory: An introduction*, Grundlehren der mathematischen Wissenschaften **226**, Springer (1978).

[31] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. **332**, Springer-Verlag, Berlin, (2006).

[32] B. Keller, *Cluster algebras, quiver representations and triangulated categories*, (English summary) Triangulated categories, 76–160, London Math. Soc. Lecture Note Ser., **375**, Cambridge Univ. Press, Cambridge, (2010).

[33] B. Keller and I. Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), 123–151.

[34] S. Kvamme, *Axiomatizing subcategories of Abelian categories*, Journal of Pure and Applied Algebra, **226**, Issue 4, 106862, (2022).

[35] K. Liampis, *Localization theory in higher homological algebra*, J. Algebra, **635**, (2023), 85–136.

[36] Z. Lin, *Idempotent Completion of n-angulated Categories*, Appl. Categor. Struct., **29**, (2021), 1063–1071.

[37] Z. Lin, *Right n-angulated categories arising from covariantly finite subcategories*, Comm. Algebra, **45**, (2017), 828–840.

[38] D. Luo, *Homological algebra in n-abelian categories*, Proc. Indian Acad. Sci., **127**(4), (2017), 625–656.

[39] D. Msapato, *The Karoubi envelope and weak idempotent completion of an extriangulated category*, Appl. Categor. Struct., **30**, (2022), 499–535.

[40] H. Nakaoka, Y. Ogawa and A. Sakai., *Localization of extriangulated categories*, arXiv:2103.16907, (2021).

[41] N. Popescu, *Abelian categories with Applications to Rings and Modules*, L.M.S. Monograph No.**3**, Academic Press, London, (1973).

[42] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.

[43] B. Stenstrom, *Rings of quotients. An Introduction to Methods of Ring Theory*, Die Grundlehren der mathematischen Wissenschaften, Vol. **217** (1975).

[44] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque No.**239**, (1996).

[45] L. Wang, J. Wei, H. Zhang and T. Zhao, *Idempotent completion of extriangulated categories*, arXiv preprint, arXiv:2007.04788, (2020).

[46] C. Weibel, *An Introduction to Homological Algebra*, Cambridge studies in advanced mathematics, Cambridge University Press, Cambridge, UK, (2004).