Πρόγραμμα Μεταπτυχιακών Σπουδών
στη Φυσική
Τμήμα Φυσικής
Σχολή Θετικών Επιστημών
Πανεπιστήμιο Ιωαννίνων

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ (Μ.Δ.Ε)

ΥΒΡΙΔΙΚΟΣ ΠΛΗΘΩΡΙΣΜΟΣ ΣΤΗ ΘΕΩΡΙΑ ΧΟΡΔΩΝ

ΔΕΣΠΟΙΝΑ ΤΟΤΟΛΟΥ
Αριθμός μητρώου: 787

Επιβλέπουσα καθηγήτρια: Καντή Παναγιώτα

Ιωάννινα
2023

University of Ioannina
Physics Department

MSC THESIS

On the implementation of Hybrid Inflation in String Theory

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Ioannina
2023

Acknowledgements

I would like to thank the Professor Panagiota Kanti of Physics Department of University of Ioannina, for her support in beggining and completing my master's degree studies, and the Emeritus Professor of University of Ioannina George Leontaris for his support, patience and guidance in completing my master's thesis. I would also like to thank the Professor Leandros Perivolaropoulos for participating in the three-examination committee of this master's thesis.

Thank you to my family, especially my uncle Michalis for inspiring me, and to my friend Andreas for practically supporting me during the master's degree studies.

Περίληψη

Η παρούσα Μεταπτυχιακή Διπλωματική Εργασία έχει ως στόχο την μελέτη του Υβριδικού Πληθωριστικού Σεναρίου της Κοσμολογίας, στα πλαίσια της Θεωρίας Χορδών.

Κοσμολογία είναι ο κλάδος της Αστροφυσικής που ασχολείται με τη μελέτη του σύμπαντος στην ολότητά του. Συγκεκριμένα, εξετάζει τη δομή και τη δυναμική εξέλιξη του σύμπαντος, στη μεγαλύτερη δυνατή κλίμακα μήκους. Η εξέταση αυτή πραγματοποιείται μέσω των κοσμολογικών παρατηρήσεων, καθώς και των κοσμολογικών μοντέλων, τα οποία αποτελούν το μαθηματικό υπόβαθρο που αποσκοπεί στην περιγραφή των χαρακτηριστικών του σύμπαντος, όπως αυτά γίνονται γνωστά μέσω των κοσμολογικών παρατηρησιακών δεδομένων.

Το επικρατέστερο μοντέλο για την περιγραφή του σύμπαντος, είναι το Καθιερωμένο Πρότυπο της Κοσμολογίας (Standard Model of Cosmology), το οποίο βασίζεται στο Καθιερωμένο Πρότυπο των σωματιδίων (Standard Model), τη Γενική Θεωρία της Σχετικότητας, και την Κοσμολογική Αρχή (Cosmological Principle). Σύμφωνα με την Κοσμολογική Αρχή, σε επαρκώς μεγάλη κλίμακα, οι ιδιότητες του σύμπαντος είναι ίδιες για οποιονδήποτε παρατηρητή σε οποιαδήποτε τοποθεσία του σύμπαντος. Οι δύο συνέπειες της αρχής αυτής ως προς τη δομή του σύμπαντος, είναι η ομοιογένεια και η ισοτροπία. Ομοιογένεια σημαίνει ότι τα διαθέσιμα παρατηρήσιμα στοιχεία είναι ίδια για παρατηρητές οι οποίοι βρίσκονται σε διαφορετικές περιοχές του σύμπαντος, ενώ ισοτροπία ότι είναι ίδια προς οποιαδήποτε κατεύθυνση εξετάζεται το σύμπαν. Τα χαρακτηριστικά του ΚΠΚ όσον αφορά την εξέλιξη, και τις μορφές ύλης και ενέργειας του σύμπαντος, είναι η διαστολή, και η ύπαρξη ορατής ύλης, σκοτεινή ύλης, δηλαδή ύλης που αλληλεπιδρά πολύ ασθενώς με τη συνηθισμένη ύλη ώστε να μην είναι άμεσα παρατηρήσιμη, και σκοτεινής ενέργειας, η οποία είναι μια άγνωστη μορφή ενέργειας, ομοιόμορφα κατανεμημένη σε όλο το σύμπαν, και στην οποία ευθύνεται η παρατηρούμενη επιταχυνόμενη διαστολή του. Συγκεκριμένα, η σκοτεινή ενέργεια αποδίδεται στην ύπαρξη μιας θετικής κοσμολογικής σταθεράς (cosmological constant) ή αλλιώς, μιας ενέργειας κενού (vacuum energy) με θετική τιμή.

Το Καθιερωμένο Κοσμολογικό Πρότυπο έρχεται να συμπληρώσει μία πολύ σημαντική υποτιθέμενη εποχή από την οποία πέρασε το σύμπαν στις πολύ νεαρές στιγμές του, η οποία ονομάζεται κοσμολογικός πληθωρισμός (Cosmological Inflation). Κατά τη φάση του πληθωρισμού, το σύμπαν υφίσταται μια επιταχυνόμενη διαστολή, η οποία διαρκεί αρκετά ώστε να οδηγηθούμε τελικά στην παρατηρούμενη σημερινή δομή του σύμπαντος (σμήνη και υπερσμήνη γαλαξιών), τον παρατηρούμενο επίπεδο χώρο, την ισοτροπία της Κοσμικής Ακτινοβολίας Υποβάθρου (Cosmic Microwave Background), και την απουσία μαγνητικών μονοπόλων. Υπάρχει μία μεγάλη ποικιλία μοντέλων που περιγράφουν τον Κοσμολογικό Πληθωρισμό, με διαφορετικό θεωρητικό ενδιαφέρον και με διαφορετικές παρατηρησιακές προβλέψεις. Στις πιο απλές περιπτώσεις, αυτό που προκαλεί την επιταχυνόμενη διαστολή είναι ένα βαθμωτό πεδίο, το λεγόμενο inflaton πεδίο, που κυλάει αργά στο δυναμικό του, και όταν οι συνθήκες αυτής της αργής κύλησης παύουν να ικανοποιούνται, τότε ο πληθωρισμός σταματά. Στα πλαίσια των διαφόρων θεωριών υψηλών ενεργειών, όπως η Υπερβαρύτητα (Supergravity), και η Θεωρία Χορδών (String Theory), είναι σύνηθες να μελετάται το σενάριο του Υβριδικού Πληθωρισμού, το οποίο ανήκει στην κατηγορία των πληθωριστικών μοντέλων όπου παραπάνω από ένα πεδία συμμετέχουν στη δυναμική του πληθωρισμού, καθώς το μοντέλο αυτό είναι εύκολο να ενταχθεί στη δομή αυτών των θεωριών.

Οι Ενεργές Κβαντικές Θεωρίες Πεδίου (Effective Quantum Field Theories), οι οποίες προσπαθούν να περιγράψουν τη φυσική χαμηλών ενεργειών και να κάνουν κοσμολογικές προβλέψεις, πρέπει να περιλαμβάνουν μια θετική κοσμολογική σταθερά με τη σημερινή παρατηρούμενη πολύ μικρή τιμή της. Στη Θεωρία Χορδών με 10 ή 11 διαστάσεις, η οποία είναι η μόνη μέχρι στιγμής συνεπής κβαντική θεωρία για την βαρύτητα, δεν είναι βέβαιο ότι στο ενεργό δυναμικό, που προκύπτει από τη συμπαγοποίηση των έξτρα διαστάσεων, υπάρχουν θετικά κενά (de-Sitter vacua), δηλαδή ενέργειες κενού με θετική τιμή που αντιστοιχούν σε μια θετική κοσμολογική σταθερά. Μάλιστα, υπάρχουν Ενεργές Θεωρίες Πεδίου, που δεν μπορούν να είναι συμβατές με τη Κβαντική Βαρύτητα, αποτελώντας το λεγόμενο σύνολο Swampland, πράγμα που οφείλεται και στην αγνόηση των κβαντικών διορθώσεων στο προαναφερθέν εξαγόμενο ενεργό δυναμικό, έχοντας τελικά διάφορα κοσμολογικά αντίκτυπα, όπως στην υπόθεση ότι η κοσμολογική σταθερά αντιπροσωπεύει την σκοτεινή ενέργεια.

Σε αυτή την εργασία, μελετάται το προσφάτως προτεινόμενο μοντέλο του Υβριδικού Πληθωρισμού στο πλαίσιο της Ενεργούς Θεωρίας Χορδών τύπου IIB, παρουσία τεμνόμενων $D7$ βρανών, στο οποίο επιφέρονται λογαριθμικές διορθώσεις στο ενεργό βαθμωτό δυναμικό. Το όρισμα του εσωτερικού όγκου μεταφράζεται ως το inflaton πεδίο, το οποίο πέφτει σε ένα de-Sitter κενό, που όμως δεν μπορεί να είναι αληθινό, αφού η τιμή του είναι πολύ πιο

ψηλά από την παρατηρούμενη μικρή τιμή της κοσμολογικής σταθεράς. Τα λεγόμενα waterfall πεδία του Υβριδικού Πληθωρισμού, που στη συγκεκριμένη θεωρία αντιστοιχούν σε κατάλληλες διεγέρσεις των ανοιχτών χορδών οι οποίες καταλήγουν στις μαγνητικές $D7$ βράνες, είναι υπεύθυνα για το τέλος της πληθωριστικής φάσης, αλλά και για την οδήγηση της θεωρίας στο αληθινό κενό, αφού μπορεί να καθοριστεί η αρνητική συνεισφορά τους στην τιμή του κενού της θεωρίας, ώστε αυτή τελικά να μπορεί να πάρει την παρατηρούμενη μικρή τιμή της κοσμολογικής σταθεράς.

Abstract

In this thesis, we study the model of Hybrid Inflation recently proposed, in a framework of type IIB effective string theory constructions and in the presence of intersecting $D7$-brane stacks. The inflaton is identified with the internal volume modulus and falls down to a de-Sitter vacuum, (collecting most of the e-folds around it), which turns out to be a false one, with a value high above the one of the observed cosmological constant. The waterfall fields, which correspond to excitations of open strings that end on the magnetised $D7$-brane stacks, introduce new low energy physics at a saddle point around the aforementioned minimum, driving both the inflationary stage to an end and the system from the false vacuum to the true one, as one can control their negative contributions to the effective scalar potential so that its vacuum reaches the tiny value of the observed cosmological constant.

Contents

Chapter 1

Standard Model of Cosmology

1.1 Robertson-Walker metric and Friedmann equations

According to the General Theory of Relativity, the infinitesimal seperation between two events in spacetime is given by the invariant interval or line element or metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{1.1}$$

where $g_{\mu\nu}$ is the metric tensor or just metric [1], which contains all the information needed for the description of the geometry of spacetime, x^μ are coordinates of spacetime, and μ, ν, are the indices that take the values $0, 1, 2, 3$. In general, the metric $g_{\mu\nu}$ is a function of the spacetime coordinates x^μ, but taking into consideration the Cosmological Principle, this dependence is simplified significantly.

By considering the Cosmological Principle, we accept that the universe is spatially homogeneous and isotropic, while according to observations, we consider that it is time evolving. In General Relativity, these assumptions mean that the universe can be thought that it consists of three-dimensional (spacelike) slices, each of these being a maximally symmetric space (because of the assumed spatial homegeneity and isotropy) that corresponds to a particular moment of time in the universe. Thus, we consider that the universe is described by the spacetime $\mathbf{R} \times \Sigma$, where $\mathbf{R}$ is the time direction and Σ is a maximally symmetric three-manifold, and the metric takes the form

$$ds^2 = -dt^2 + a^2(t)d\sigma^2 \tag{1.2}$$

where $t = x^0$ is the timelike coordinate or cosmic time, $a(t)$ is a function of the timelike coordinate, which is also named as cosmological scale factor, and $d\sigma^2$ is the the 3-manifold Σ metric, which is defined as

$$d\sigma^2 = \gamma_{ij}(u)du^i du^j \tag{1.3}$$

where u^i with $i = 1, 2, 3$ are the coordinates of Σ and γ_{ij} is a maximally symmetric 3-dimensional metric, which is a function of the coordinates u^i only. The time dependence of the 3-dimensional space of the whole (spacetime) metric (1.2), is included in the scale factor $a(t)$.

Because of the maximal symmetry of the thee-fold Σ, its metric, $d\sigma^2$, can be expanded using spherical symmetry. Thus, it takes the general form that a static, sperically symmetric, 3-dimensional metric, can be written in [1, 2]

$$d\sigma^2 = e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{1.4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the 2-sphere (S^2) metric, r is the radial coordinate, and $\beta(r)$ is a function of the radial coordinate. Because of the radial dependence of the above rr-coefficient in (1.4), Σ is considered in general to be a curved space. The total spacetime metric has the form

$$ds^2 = -dt^2 + a^2(t)\left[e^{2\beta(r)}dr^2 + r^2 d\Omega^2\right] \tag{1.5}$$

in which the function $\beta(r)$ and the scale factor $a(t)$, are to be determined.

Firstly, we are focusing on the metric of the 3-fold Σ, (1.4), whose unknown function is the $\beta(r)$, and as we have a general expression for the components of the metric, we can calculate the Christoffel connection coefficients (or symbols), which are expressed as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{1.6}$$

and for the metric (1.4), where $g \to \gamma$, are

$$\begin{aligned}
\tilde{\Gamma}^{r}_{rr} &= \partial_r \beta \\
\tilde{\Gamma}^{r}_{\theta\theta} &= -re^{-2\beta} \\
\tilde{\Gamma}^{r}_{\phi\phi} &= -re^{-2\beta} \sin^2\theta \\
\tilde{\Gamma}^{\theta}_{r\theta} &= r^{-1} \\
\tilde{\Gamma}^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\
\tilde{\Gamma}^{\phi}_{r\phi} &= r^{-1} \\
\tilde{\Gamma}^{\phi}_{\theta\phi} &= \cot\theta
\end{aligned} \tag{1.7}$$

where all all objects with a tilde, $\sim$, will refer to the quantities related to the γ_{ij} metric, and the r, θ, ϕ indices correspond to the values $1, 2, 3$. Moreover, the Riemann tensor (or curvature tensor) components that are given by the following expression in terms of the connection coefficients

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{1.8}$$

for the (1.4) are found to be

$$\begin{aligned}
\tilde{R}^{r}{}_{\theta r\theta} &= re^{-2\beta}\partial_r\beta \\
\tilde{R}^{r}{}_{\phi r\phi} &= re^{-2\beta}\partial_r\beta \sin^2\theta \\
\tilde{R}^{\theta}{}_{\phi\theta\phi} &= (1 - e^{-2\beta})\sin^2\theta
\end{aligned} \tag{1.9}$$

and the Ricci tensor components, which are given by the contraction of two indices in the Riemann tensor,

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \tag{1.10}$$

for the (1.4) metric are

$$\begin{aligned}
\tilde{R}_{rr} &= 2r^{-1}\partial_r\beta \\
\tilde{R}_{\theta\theta} &= e^{-2\beta}(r\partial_r\beta - 1) + 1 \\
\tilde{R}_{\phi\phi} &= \sin^2\theta \tilde{R}_{\theta\theta}
\end{aligned} \tag{1.11}$$

As the 3-dimensional metric γ_{ij} is considered to be maximally symmetric, its corresponding Riemann tensor satisfies the following condition

$$\tilde{R}_{ijkl} = \frac{\tilde{R}}{n(n-1)}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$$

$$= k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \tag{1.12}$$

where $k = \tilde{R}/6$ is a constant, which is referred as curvature parameter, and $\tilde{R} = \tilde{R}^i{}_i$, is the corresponding Ricci scalar (or curvature scalar), that is generally the trace of the Ricci tensor,

$$R = R^{\mu}{}_{\mu} \tag{1.13}$$

From the condition (1.12), it follows that the corresponding Ricci tensor takes the form

$$\tilde{R}_{jl} = \tilde{R}^m{}_{jml} = k(\delta^m{}_m\gamma_{jl} - \delta^m{}_l\gamma_{jm})$$

$$\Rightarrow \tilde{R}_{jl} = 2k\gamma_{jl} \tag{1.14}$$

Equating the rr-components of the Ricci tensor, as given in the relations (1.11) and (1.14), we can solve for $\beta(r)$

$$e^{-2\beta}d\beta = krdr$$

$$\Rightarrow e^{2\beta(r)} = \frac{1}{1-kr^2}$$

$$\Rightarrow \beta(r) = -\frac{1}{2}\ln(1-kr^2) \tag{1.15}$$

Thus, the three-manifold Σ metric (1.4) becomes

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \tag{1.16}$$

while the spacetime metric (1.5) has the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{1.17}$$

The invariant inerval of the form (1.17), is the Robertson-Walker (RW) metric, and describes a universe that every moment of time is a generally curved 3-dimensional space, which is characterized by homogeneity and isotropy. This space has the ability to expand or contract in time, depending on whether the scale factor increases or decreases with time, while the exact form of the scale factor describes the particular way of this evolution. The coordinates (t, r, θ, ϕ), in which the metric has the form (1.17), where the coefficient g_{tt} does not depend on the spatial coordinates u^i, and there are no cross-terms of the type $dtdu^i$, are called comoving coordinates [1, 2]. An observer who is in constant spatial coordinates u^i is called comoving observer, its proper time is the cosmic time t, and thinks that the space looks isotropic. Also, the scale factor $a(t)$ is independent of the u^i, so different observers in the universe will measure the same scale factor, that is the same way of evolution, as it is required by homogeneity. The physical distance between two comoving points (whose spatial coordinates are not changing), is evolving as $a(t)$. The Robertson-Walker metric is the most general form of a metric, for the description of a universe which agrees with the Cosmological Principle.

Concerning the curvature of the 3-dimensional space of the spacetime metric (1.17) (RW) every particular moment of time, this comes from the curvature of the 3-manifold Σ (which comes from the radial function $1/(1-kr^2)$), since the metric of the whole 3-dimensional space is the $d\sigma^2$ metric with a factor $a^2(t)$ which is given at the particular moment and is constant. The value of the curvature parameter k determines the curvature of Σ, and of the total 3-space as well, through the relation $k = \tilde{R}/6$, and it is a constant value, as the curvature scalar $\tilde{R}$ is constant everywhere, something that is required from the maximal symmetry. Depending on the sign of $\tilde{R}$, or k, there are three types of curvature: when $\tilde{R} > 0$ the space is positively curved and is called closed, when $\tilde{R} = 0$ space is not curved (the curvature is zero) and is called flat, and when $\tilde{R} < 0$ the space is negatively curved and is called open. All these cases correspond to the following types of space (all with constant curvature): the sphere S^3, the flat space $\mathbb{R}^3$ and the hyperboloid H^3. As the spacetime metric (1.17) remains invariant under the transformations

$$k \to k/|k|$$

$$r \to \sqrt{|k|}r$$

$$a \to a/\sqrt{|k|} \tag{1.18}$$

we can normalize the curvature parameter to take the discrete values $\{k : +1, 0, -1\}$. Each of these values corresponds to a hypersurface Σ [1, 2]: $k = +1$ corresponds to a Σ with positive curvature and the metric of S^3, $k = 0$ corresponds to a Σ with zero curvature and the Euclidean metric in $\mathbb{R}^3$, and $k = -1$ corresponds to a Σ with negative curvature and the metric of H^3. In the above normalization of the curvature parameter, the scale factor $a(t)$ has units of lenght ($[length]^1$), while the radial coordinate (and the curvature parameter k) are unitless. Otherwise, we can choose the scale factor to be dimensionless, and normalize its value conveniently, for example at the present time to be $a_0 = a(t_0) = 1$, where t_0 is the current era. In the last normalization, the radial coordinate r has units of length, while the curvature parameter k has units $[length]^{-2}$. Here, k is a continuous parameter, so one distinguishes the following cases: $k > 0$, $k = 0$ or $k < 0$ (which correspond to positive, zero, and negative curvature).

Moving on, we can calculate the Christoffel coefficients of the whole (spacetime) metric (1.17), in which the only unknown function will be that of the scale factor $a(t)$. From the relation (1.6), they are

$$\Gamma^t_{rr} = \frac{a\dot{a}}{1 - kr^2}$$

$$\Gamma^t_{\theta\theta} = a\dot{a}r^2$$

$$\Gamma^t_{\phi\phi} = a\dot{a}r^2 \sin^2\theta$$

$$\Gamma^r_{tr} = \frac{\dot{a}}{a}$$

$$\Gamma^r_{rr} = \frac{kr}{1 - kr^2}$$

$$\Gamma^r_{\theta\theta} = r(kr^2 - 1)$$

$$\Gamma^r_{\phi\phi} = r\sin\theta(kr^2 - 1)$$

$$\Gamma^\theta_{t\theta} = \frac{\dot{a}}{a}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{t\phi} = \frac{\dot{a}}{a}$$

$$\Gamma^{\phi}_{r\phi} = \frac{1}{r}$$

$$\Gamma^{\phi}_{\theta\phi} = \cot\theta \tag{1.19}$$

where $\dot{a} = da/dt$ is the derivative of the scale factor with respect to the cosmic time t, and the indices t, r, θ, ϕ correspond to the values $0, 1, 2, 3$. We can also find the Riemann tensor components, which from (1.8), are

$$R^{t}{}_{rtr} = \frac{a\ddot{a}}{1-kr^2}$$

$$R^{t}{}_{\theta t\theta} = a\ddot{a}r^2$$

$$R^{t}{}_{\phi t\phi} = a\ddot{a}r^2\sin^2\theta$$

$$R^{r}{}_{ttr} = \frac{\ddot{a}}{a}$$

$$R^{r}{}_{\theta\theta r} = -(\dot{a}^2+k)r^2$$

$$R^{r}{}_{\phi\phi r} = -(\dot{a}^2+k)r^2\sin^2\theta$$

$$R^{\theta}{}_{tt\theta} = \frac{\ddot{a}}{a}$$

$$R^{\theta}{}_{r\theta r} = \frac{\dot{a}^2+k}{1-kr^2}$$

$$R^{\theta}{}_{\phi\phi\theta} = -(\dot{a}^2+k)r^2\sin^2\theta$$

$$R^{\phi}{}_{tt\phi} = \frac{\ddot{a}}{a}$$

$$R^{\phi}{}_{r\phi r} = \frac{\dot{a}^2+k}{1-kr^2}$$

$$R^{\phi}{}_{\theta\phi\theta} = (\dot{a}^2+k)r^2 \tag{1.20}$$

and the Ricci tensor components, which from (1.10) are

$$R_{tt} = -\frac{3\ddot{a}}{a}$$

$$R_{rr} = \frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2}$$

$$R_{\theta\theta} = (\ddot{a}a + 2\dot{a}^2 + 2k)r^2$$

$$R_{\phi\phi} = (\ddot{a}a + 2\dot{a}^2 + 2k)r^2\sin^2\theta \tag{1.21}$$

The corresponding Ricci scalar, from the relation (1.13), is

$$R = \frac{6(\ddot{a}a + \dot{a}^2 + k)}{a^2} \tag{1.22}$$

In order to determine the behavior of the scale factor $a(t)$, we have to study the Einstein's field equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.23}$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu}$ is the Enstein tensor, $T_{\mu\nu}$ is the energy momentum tensor, and G is the Newton's gravitational constant. Einstein's field equation (1.23) can also be written as

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{1.24}$$

where $T = T^{\mu}_{\ \mu} = -(1/8\pi G)R$ is the Einstein's tensor trace. As Einstein's equation determines the reaction of the metric in the presence of energy and momentum ($T_{\mu\nu}$), we have to assume a model for the matter and energy of the universe: we consider that they behave as a perfect fluid, for simplicity, and because of the fact that it is consistent with much observed about the universe [5, 2]. The energy-momentum tensor of a perfect fluid in General Relativity is defined as

$$T_{\mu\nu} = (\rho + p)U_{\mu}U_{\nu} + pg_{\mu\nu} \tag{1.25}$$

where ρ and p are the energy density and the pressure of the fluid respectively, and U_{μ} is the 4-velocity of the fluid. We choose the fluid elements to be in their rest frame in comoving coordinates of the RW metric, and normalizing the timelike coordinate of the 4-velocity to unit, we have $U_{\mu} = (1, 0, 0, 0)$. After these, the energy-momentum tensor (1.25) has the from

$$T^{\mu}_{\ \nu} = \mathrm{diag}(-\rho, p, p, p) \tag{1.26}$$

while its trace is

$$T = T^{\mu}_{\ \mu} = -\rho + 3p \tag{1.27}$$

The form (1.26) of the energy-momentum tensor of the perfect fluid in its rest frame, results also from the symmetries of the RW metric (as a consequence of the Einstein's field equations) [2].

Finally, substituting the Ricci tensor components (1.21) of the RW metric, together with (1.26) and (1.27), into the Einstein's field equation of the form (1.24), we get the following equations: the 00-equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.28}$$

and the ii-equation, which is only one because of the spatial isotropy

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \tag{1.29}$$

Puting (1.28) into (1.29), the latter becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.30}$$

Equations (1.28) and (1.29) are known as Friedmann equations and their solution for the scale factor describes the dynamical evolution of the universe. Equation (1.30) is also known as Friedmann equation and determines the change of the cosmological scale factor with time, in terms of the energy density ρ and the curvature parameter k of the universe, while equation (1.28) is referred as acceleration equation. The models in which the universe is described by a RW metric, that obeys the Friedmanns equations, are named as Friedmann-Robertson-Walker (FRW) models of Cosmology.

Through the cosmological scale factor $a(t)$, the following parameter is defined

$$H \equiv \frac{\dot{a}}{a} \tag{1.31}$$

which is known as Hubble parameter, and is the rate of change of the scale factor. The Friedmann equation (1.30), using the definiton (1.31), takes the form

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.32}$$

where we see that the Friedmann equations relates the rate of change of the scale factor, as it is defined through the Hubble parameter, and the total energy density ρ of the universe.

1.2 Cosmic expansion

From the RW metric we saw that the physical distance d_{ph} between two comoving points in the universe depends on the scale factor as

$$d_{ph}(t) = a(t)d_{com} \tag{1.33}$$

where d_{com} is the comoving distance between the two points, which does not change as the universe expands or contracts with time. Differentiating (1.33) with respect to the cosmic time, we have the velocity

$$v = \dot{d}_{ph} = \dot{a}d_{com} = Hd_{ph} \tag{1.34}$$

In general, the Hubble parameter is a function of time, but in the present era t_0 is considered to be a constant, $H_0 = H(t_0)$, named as Hubble constant. The Hubble constant, because of the uncertainty [20] of its value, is usually parametrized as

$$H_0 = 100h\ km\ s^{-1}Mpc^{-1} \tag{1.35}$$

where h is the reduced (dimensionless) Hubble parameter, which according to the most recent estimates of the Planck mission is [15]

$$h = 0.6736 \pm 0.0054 \tag{1.36}$$

The value of the Hubble constant H_0 is positive, as it is required from the definiton of the Hubble parameter (1.31) for an expanding universe, that is the scale factor is increasing with time. The recession velocity between two comoving points (e.g. galaxies) in the universe due to expansion, is given by (1.34) at present time

$$v = H_0 d_{ph} \tag{1.37}$$

for those comoving points that are relatively close to Earth ($z \ll 1$ [2]). The relation (1.37) is also known as Hubble law: the observed recession velocity of the not too far away galaxies, is proportional to their physical distance, with H_0 being the proportionality constant.

Through the Hubble constant H_0, two quantities are defined in terms of which the cosmological scales of time and distance are expressed: the Hubble time

$$t_H = \frac{1}{H_0} = 3.09 \times 10^{17}h^{-1}s = 9.78h^{-1} \text{ billion years}$$

and the Hubble length or radius

$$d_H = \frac{1}{H_0}c = 9.26 \times 10^{27}h^{-1}cm = 3 \times 10^3 h^{-1}Mpc$$

where c is the speed of light.

1.3 Energy density

To determine the scale factor $a(t)$ through the solution of the Friedmann equation (1.30), we need to know the energy density dependence on the scale factor, $\rho(a)$. The energy density in (1.30) corresponds to the total energy density of the universe, which consists of the different energy density contributions of the universe constituents

$$H^2 = \frac{8\pi G}{3} \sum_i \rho_i - \frac{k}{a^2} \tag{1.38}$$

where the intex i corresponds to all the possible constituents of the universe.

We have considered above that the matter and energy in the universe are described by the perfect fluid. Spatial homogeneity implies that the energy density and pressure of the perfect fluid are only functions of time, $\rho = \rho(t)$ and $p = p(t)$. Then, by conservation of the energy-momentum tensor in General Relativity (the vanishing of the covariant derivative of the tensor),

$$\nabla_\mu T^\mu_{\ \nu} = 0 \tag{1.39}$$

taking the $\nu = 0$ component ($\nabla_\mu T^\mu_{\ 0} = 0$), results in the 'continuity equation'

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{1.40}$$

which essentially describes the evolution of the energy density ρ with respect to the expansion of the universe, as the latter is defined by the Hubble parameter. Only the $\nu = 0$ component of (1.39) leads to a non-trivial equation, as the $\nu = i$ equation is satisfied automatically due to spatial isotropy. Equation (1.40) can also be derived from the Friedmann equations, if we differentiate (1.30) with respect to cosmic time and then substitute (1.30) and (1.28) into it.

We consider further an equation of state for the perfect fluid, that relates the energy density and the pressure of the fluid. The simplest equation of state is of the form $p(\rho)$, where the pressure of the fluid is just a function of its energy density. More specifically, we can assume the convenient linear relation

$$p = w\rho \tag{1.41}$$

where w is the equation of state parameter, and we will consider that it is constant, as the perfect fluids relevant to Cosmology often obey that simple type of equation of state. Accepting (1.40), we can solve (1.40) for $\rho(a)$

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(1+w)\rho = 0$$

$$\Rightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\Rightarrow \rho(a) = \frac{C_0}{a(t)^{3(1+w)}}, \quad C_0 = \rho_0 a_0^{3(1+w)} = \text{const.} \tag{1.42}$$

The (constant) value of the equation of state parameter w in (1.41) characterizes the type of the cosmological fluid and its energy density in terms of the scale factor, from the relation (1.42). The w-values that correspond to the most useful types of fluids, are:

1. The $w = 0$, corresponding to matter, or dust, that consists of non-interacting, non-relativistic particles, such as the baryonic matter and dark matter. They have zero or negligible pressure, while their energy density is evolving as

$$\rho_m \propto \frac{1}{a^3} \tag{1.43}$$

2. The $w = 1/3$, corresponding to the radiation, which consists of relativistic particles, e.g. photon, or ulta-relativistic particles. Their pressure is the 1/3 of their energy density, which is evolving as

$$\rho_r \propto \frac{1}{a^4} \tag{1.44}$$

3. The $w = -1$, corresponding to the cosmological constant Λ, as the latter can be considered as a cosmological perfect fluid type with the following energy density contribution

$$\rho_\Lambda \propto \frac{1}{a^0} \tag{1.45}$$

In particular, by the introduction of a cosmological constant in the Einstein equation (1.23), we have

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \\ &= 8\pi G\Big(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}\Big) \\ \Rightarrow G_{\mu\nu} &= 8\pi G\big(T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}\big) \end{aligned} \tag{1.46}$$

with

$$T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \tag{1.47}$$

From (1.46) we observe that the addition of Λ is equivalent to the addition of an energy-momentum tensor of the form (1.47), that describes a perfect fluid with energy density

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \tag{1.48}$$

and pressure

$$p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G} \tag{1.49}$$

and thus according to (1.41), the equation of state parameter is equal to -1. The energy-momentum tensor $T^{(\Lambda)}_{\mu\nu}$ (1.47) corresponds to a vacuum energy (energy in the absence of matter and radiation) with energy density contribution given by (1.48), and thus the cosmological constant Λ is also called vacuum energy. We have not considered from the begining the general Enstein's equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ where Λ is treated as the vacuum energy, as it is equlavent to treat it as another different type of energy density in the universe.

4. The $w = -1/3$, corresponding to the non-vanishing curvature parameter k, which we are also able to treat as another type of energy density, though fictitious, with contribution

$$\rho_k \propto \frac{1}{a^2} \tag{1.50}$$

The particular corresponding energy density term in the Friedmann equation is

$$\rho_k = -\frac{3k}{8\pi G a^2} \tag{1.51}$$

while the pressure term is

$$p_k = \frac{k}{8\pi G a^2} \tag{1.52}$$

so, according to (1.41), the equation of state parameter equals -1/3.

The different constituents of the universe belong to different types of energy, all of which contribute to the total energy density. It is useful to write the Friedmann equation as

$$H^2 = \frac{8\pi G}{3}\left(\rho - \frac{3k}{8\pi G a^2}\right)$$

$$\Rightarrow H^2 = \frac{8\pi G}{3}(\rho + \rho_k)$$

$$\Rightarrow 1 = \frac{8\pi G}{3H^2}(\rho + \rho_k) \tag{1.53}$$

where ρ is the contribution of the real energy sources, and ρ_k is the curvature parameter contribution, given by (1.51). We define the density parameter Ω, as

$$\Omega = \frac{8\pi G}{3H^2}\rho \tag{1.54}$$

and thus, any contribution in the energy density is expressed as a contribution to (1.54), given by

$$\Omega_i = \frac{8\pi G}{3H^2}\rho_i \tag{1.55}$$

where i denotes all the types of energy, including the curvature parameters's fictitious energy, which is

$$\Omega_k = \frac{8\pi G}{3H^2}\rho_k = -\frac{k}{H^2 a^2} \tag{1.56}$$

Using (1.56), the Friedmann equation (1.53) is written as

$$\Omega - 1 = -\Omega_k = \frac{k}{H^2 a^2} \tag{1.57}$$

where Ω corresponds to the total energy density of the universe, coming from the real energy components. We conclude from (1.57) that the space curvature can be determined through the comparison of the aforementioned parameter Ω and the unit. We observe that for a critical density ρ_c of the universe, we have a unit Ω parameter, which happens when the space curvature is zero. From the definition (1.54) for $\Omega = 1$, we find that the critical density is

$$\rho = \rho_c = \frac{3H^2}{8\pi G} \tag{1.58}$$

Through (1.58), the contributions (1.55) are given by the useful relation

$$\Omega_i = \frac{\rho_i}{\rho_c} \tag{1.59}$$

In conclusion, depending on the type of curvature of the 3-dimensional space of the RW metric, we have the following cases for the values of Ω and ρ parameters of the total energy density of the universe:

1. Closed space: $k > 0$: $\Omega > 1$: $\rho > \rho_c$
2. Flat space: $k = 0$: $\Omega = 1$: $\rho = \rho_c$
2. Open space: $k < 0$: $\Omega < 1$: $\rho < \rho_c$

According to observations, the Ω the parameter is very close to 1 in the current era [15, 16, 18], so we can consider the universe to be spatially flat.

1.4 Evolution of the scale factor

From the solution (1.42) for the energy density $\rho(a)$, we see that the different types of energy are evolving at different rates during the evolution of the universe,

$$\rho_\Lambda \propto a^0, \qquad \rho_k \propto a^{-2}, \qquad \rho_m \propto a^{-3}, \qquad \rho_r \propto a^{-4} \tag{1.60}$$

$$\Rightarrow \rho_\Lambda \propto \rho_k a^2 \propto \rho_m a^3 \propto \rho_r a^4 \tag{1.61}$$

or through the Ω parameter,

$$\Omega_\Lambda \propto \Omega_k a^2 \propto \Omega_m a^3 \propto \Omega_r a^4 \tag{1.62}$$

The above means that for long periods, one kind of energy will dominate the energy density. In a model of the universe such as the Standard Cosmological Model, which includes all the above kinds of energy, and is expanding (a is increasing with time), as we go back at small values of a (and t), the radiation will dominate in the energy density among other sources, so we have the radiation-dominated era, and, as it increases, sometime matter will dominate, in the so called matter-dominated era, then the curvature parameter, and for large values of it, the cosmological constant, in the cosmological constant-dominated era, since the latter is considered either to not change or change little with time [33].

With the domination in the energy density of only one type of energy among others, at different eras, the Friedmann equations are simplified significantly. Below, we study all these seperate cases in a universe with flat 3-dimensional space ($k = 0$), considering an expanding universe, that is $da/dt > 0$: In the first two cases, the universe is dominated by either matter ($w = 0$) or radiation ($w = 1/3$), and the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} C_0 a^{-3(1+w)} \tag{1.63}$$

$$\Rightarrow \frac{da}{a} a^{\frac{3}{2}(1+w)} = \sqrt{\frac{8\pi G C_0}{3}} dt \tag{1.64}$$

which is solved by

$$a(t) \propto t^{\frac{2}{3(1+w)}} = \begin{cases} t^{2/3}, & w = 0 \\ t^{1/2}, & w = 1/3 \end{cases} \tag{1.65}$$

The model (1.65) of the universe with the matter domination ($w = 0$), is called Einstein-de Sitter model. In the last case, the universe is dominated by the cosmological constant ($w = -1$), and the Friedmann equation becomes (substituting (1.48))

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_\Lambda = \frac{\Lambda}{3} = H_0^2 \tag{1.66}$$

$$\Rightarrow \frac{da}{a} = H_0 dt \tag{1.67}$$

where $\Lambda > 0$ and $H_0 = \sqrt{\Lambda/3}$, and is solved by

$$a(t) \propto e^{H_0 t} = e^{\sqrt{\frac{\Lambda}{3}} t} \tag{1.68}$$

The corresponding spacetime of (1.68) is called de-Sitter (dS) space, and is related to the current era of the universe and the inflationary era, which is mentioned below later.

The Friedmann equations are also solved in some other cases, such as those in which the curvature parameter is non-zero ($k \neq 0$) [5], so the 3-dimensional space is considered to be curved in general, and the universe is dominated by one kind of energy source or none. The calculations in these cases are simplified defining the conformal time τ through

$$dt = a(\tau)d\tau \tag{1.69}$$

and then the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{1}{a^2}\left(\frac{da}{d\tau}\frac{1}{a(\tau)}\right)^2 = \frac{1}{a^2}\left(\frac{a'}{a}\right)^2 = \frac{(a')^2}{a^4} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.70}$$

$$\Rightarrow \left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho a^2 - k \tag{1.71}$$

where $a' = da/d\tau$ is the derivative of the scale factor with respect to the conformal time. For $w = 0$ in (1.71) we set

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\frac{C_0}{a} - k = \xi^2 \tag{1.72}$$

and the acceleration equation takes the following form

$$\frac{\ddot{a}}{a} = \frac{1}{a}\frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{a'}{a}\right) = \frac{\xi'}{a^2} = -\frac{4\pi G}{3}\frac{C_0}{a^3} \tag{1.73}$$

$$\Rightarrow \frac{\xi'}{a^2} = -\frac{1}{2a^2}(\xi^2 + k) \tag{1.74}$$

$$\Rightarrow \frac{d\xi}{\xi^2 + k} = -\frac{1}{2}d\tau \tag{1.75}$$

When $k = +1$, we set $\xi = \cot z$ and finally we have

$$-dz = -\frac{1}{2}d\tau \tag{1.76}$$

$$\Rightarrow z = \cot^{-1}\xi = \frac{\tau}{2} \tag{1.77}$$

$$\Rightarrow \frac{da}{a} = \cot\frac{\tau}{2}d\tau \tag{1.78}$$

$$\Rightarrow a(\tau) \propto \sin^2\frac{\tau}{2} = 1 - \cos\tau \tag{1.79}$$

When $k = 0$, we have

$$\frac{d\xi}{\xi^2} = -\frac{1}{2}d\tau \tag{1.80}$$

$$\Rightarrow \xi^{-1} = \frac{\tau}{2} \tag{1.81}$$

$$\Rightarrow \frac{da}{a} = \frac{2}{\tau}d\tau \tag{1.82}$$

$$\Rightarrow a(\tau) \propto \tau^2 \tag{1.83}$$

where we recover the Einstein-de Sitter solution. When $k = -1$, we set $\xi = \coth z$ and finally have

$$-dz = -\frac{1}{2}d\tau \tag{1.84}$$

$$\Rightarrow z = \coth^{-1}\xi = \frac{\tau}{2} \tag{1.85}$$

$$\Rightarrow \frac{da}{a} = \coth\frac{\tau}{2}d\tau \tag{1.86}$$

$$\Rightarrow a(\tau) \propto \sinh^2\frac{\tau}{2} = \cosh\tau - 1 \tag{1.87}$$

For $w = 1/3$ in (1.71), we set

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\frac{C_0}{a^2} - k = \xi^2 \tag{1.88}$$

and the acceleration equation takes the form

$$\frac{\xi'}{a^2} = -\frac{4\pi G}{3}\frac{2C_0}{a^4} \tag{1.89}$$

$$\Rightarrow \frac{\xi'}{a^2} = -\frac{1}{a^2}(\xi^2 + k) \tag{1.90}$$

$$\Rightarrow \frac{d\xi}{\xi^2 + k} = -d\tau \tag{1.91}$$

When $k = +1$, we set $\xi = \cot z$ and have

$$z = \cot^{-1}\xi = \tau \tag{1.92}$$

$$\Rightarrow a(\tau) \propto \sin\tau \tag{1.93}$$

When $k = 0$, we have

$$\xi^{-1} = \tau \tag{1.94}$$

$$\Rightarrow a(\tau) \propto \tau \tag{1.95}$$

where we recover the flat-space solution, and when $k = -1$, we set $\xi = \coth z$ and finally have

$$z = \cot^{-1}\xi = \tau \tag{1.96}$$

$$\Rightarrow a(\tau) \propto \sinh\tau \tag{1.97}$$

When $w = -1$, using the cosmic time t, the Friedmann equation for $\Lambda > 0$, is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2} \tag{1.98}$$

$$\Rightarrow \frac{da}{\sqrt{A^2a^2 - k}} = dt \tag{1.99}$$

where $A = \sqrt{\Lambda/3}$. When $k = +1$, we set $Aa = \cosh z$ and have

$$dz = Adt \tag{1.100}$$

$$\Rightarrow z = \cosh^{-1}(Aa) = At \tag{1.101}$$

$$\Rightarrow a(t) \propto \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right) \tag{1.102}$$

When $k = 0$ we have

$$\frac{da}{Aa} = dt \tag{1.103}$$

$$\Rightarrow a(t) \propto e^{\sqrt{\frac{\Lambda}{3}}t} \tag{1.104}$$

where we recover the de-Sitter solution, and when $k = -1$ we set $Aa = \sinh z$, and finally have

$$dz = Adt \tag{1.105}$$

$$\Rightarrow z = \sinh^{-1}(Aa) = At \tag{1.106}$$

$$\Rightarrow a(t) \propto \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right) \tag{1.107}$$

The solutions (1.102), (1.104) and (1.107) all represent essentially the same spacetime, the de-Sitter space, in different coordinate systems and in particular, the $k = 0$ and $k = -1$ solutions are coordinate patches that only cover part of the de-Sitter space. For the case of a negative cosmological constant, $\Lambda = -|\Lambda| < 0$, the Friedmann equation (1.98) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{|\Lambda|}{3} - \frac{k}{a^2} \tag{1.108}$$

$$\Rightarrow \frac{da}{\sqrt{-|A|^2a^2 - k}} = dt \tag{1.109}$$

where $|A| = \sqrt{|\Lambda|/3}$. The above equation is only solved for $k = -1$,

$$a(t) \propto \sin\left(\sqrt{\frac{|\Lambda|}{3}}t\right) \tag{1.110}$$

The spacetime in which (1.110) corresponds is called Anti-de Sitter (AdS) space, and again this solution does not cover all of the Anti-de Sitter space [39].

Furthermore, in the case of an empty space with non-zero curvature parameter, the Friedmann equation becomes

$$(\dot{a})^2 = -k \tag{1.111}$$

and is only solved for $k = \{0, -1\}$, by $a = const$ and

$$a(t) \propto t \tag{1.112}$$

respectively. The spacetime in which (1.112) ($k = -1$) corresponds is called Milne universe.

In the above solutions for the universe, we observe that we come across an anomaly when the scale factor becomes zero, while the energy density ρ of the universe becomes infinite. In the cases where we the universe starts with the anomaly, which it can be considered to be at $t = 0$, we call it Big Bang, while in the cases where the universe ends with the anomaly, we call it Big Crunch. We have the following cases for a matter or radiation dominated universe: when the space is flat or open, it starts with the Big Bang and continues to expand forever, and when the space is closed, it starts with the Big Bang and results in the Big Crunch. We also see that a de Sitter universe ($w = -1$, $\Lambda > 0$) expands exponentally in the limit $t \to \infty$, regardless its spatial curvature.

Exact solutions of the Friedmann equations can also be found in some other useful cases [7, 13], such as the spatially flat, radiation and matter dominated era, and the specially flat, matter and cosmological constant dominated era.

1.5 Λ-CDM model

In the current era, the universe, which according to measurements is spatially flat, seems to have negligible contributions in its energy density from radiation, while there are important contributions from matter and dark energy, with the latter to greatly dominate. Thus, if it is to describe the real world now, we will have to consider a model which includes radiation, matter and the cosmological constant, in which the dark energy is considered to correspond to. In this case, the Friedmann equation takes the following form

$$H^2 = H_0^2\big(\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}\big) \tag{1.113}$$

where we normalized the today's scale factor a_0 to 1, and used the relations

$$\Omega_{i,0} = \frac{8\pi G}{3H_0^2}\rho_{i,0}, \quad \rho_i = \frac{\rho_{i,0}}{a^{3(1+w)}} \tag{1.114}$$

where $i = r, m, k, \Lambda$ which correspond to radiation ($w = 1/3$), matter ($w = 0$), curvature parameter ($w = -1/3$) and cosmological constant ($w = -1$), respectively. Evaluating (1.113) today ($t \to t_0$, $a \to a_0 = 1$, $H \to H_0$), we obtain the consistency relation

$$\Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0} = 1 \tag{1.115}$$

The most possible scenario is that radiation is the contribution of photons but it is not clear yet, and we have $\Omega_{r,0} \sim 10^{-4}$ [15, 33]. Also, most contemporary methods for calculating the mass of the matter give $\Omega_{m,0} \sim 0.31$, while best current estimates for the ordinary matter give $\Omega_{b,0} \sim 0.05$ [15, 33]. By ordinary matter we mean anything made of atoms and their constituents, and we call it baryonic, too (made of baryons, that is strongly interacting particles). According to the contribution of the total matter, we see that there is a remaing kind of matter $\Omega_{dm,0} = \Omega_{m,0} - \Omega_{b,0}$, which is estimated to be $\Omega_{dm,0} \sim 0.26$ [15, 33], which is non-baryonic, known as dark matter [22, 75, 76, 77], which we know that it must be cold (CDM) and very weakly-interacting with ordinary matter, so that is has not been directly detected yet. Furthemore, analysis of the fine structure of the anisotropies of the Cosmic Mikrowave Background (CMB) give $\Omega_{k,0} \sim 0$ [18, 15, 33], which means that the total energy density filling the universe is very close to the critical one. Finally, from observations of redshifts of type IA supernovae in distant galaxies, it is concluded that the universe is in an accelerating phase [5, 33], which is well explained if we consider that the dark energy responsible for this acceleration is due to the existence of a positive cosmological constant, with contribution $\Omega_{\Lambda,0} \sim 0.69$ [15, 33] (this corresponds to $\rho_\Lambda \sim 10^{-8} erg/cm^3$). The latter is in consistency with the CMB data. In general, dark energy is called whatever is responsible for the accelerated expansion of the universe, either it is dynamical [33, 23] or a cosmological constant. What we know is that it is relatively smoothly distributed through space, and that it is evolving slowly with time.

The Friedmann equation that takes into consideration all the non-zero contributions, the small amount of radiation, the matter (baryonic and non-baryonic) and the cosmological constant, can be solved numerically, though we can assume the very close approximation of existence only of matter and dark energy (since radiation is not comparable to them), and get an analytic solution [2, 7, 13]. Considering $\Omega_{k,0} \sim 0$, $\Omega_{\Lambda,0} > \Omega_{m,0}$ and $\Omega_{r,0} \ll \Omega_{m,0}$ in (1.113), we have

$$H^2 = H_0^2\big(\Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}\big)$$

$$\to H_0 dt = \frac{a^{1/2}da}{\sqrt{\Omega_{m,0}}\sqrt{1 + (\Omega_{\Lambda,0}/\Omega_{m,0})a^3}} \tag{1.116}$$

Setting $u^2 = \Omega_{\Lambda,0}/\Omega_{m,0}a^3$, (1.116) becomes

$$H_0 dt = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}}\frac{du}{\sqrt{1+u^2}} \tag{1.117}$$

and integrating, we obtain

$$H_0 t = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \sinh^{-1} u = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \sinh^{-1}\left(\sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}} a^{3/2}\right) \tag{1.118}$$

or

$$a(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t\right) \tag{1.119}$$

We observe that for early times (small t) we recover the Enstein de-Sitter model with the Big Bang, while for late times we recover the de-Sitter solution with the exponential expansion. From (1.118) we can calculate the age of the universe in this model for $a \to a_0 = 1$,

$$t = \frac{2}{3H_0\sqrt{\Omega_{\Lambda,0}}} \sinh^{-1}\left(\sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}}\right) \tag{1.120}$$

where $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$, so for the estimated values of $\Omega_{m,0}$ and H_0 ((1.35) and (1.36) for h), we find

$$t_0 \approx 13.9 \times 10^9 yrs \tag{1.121}$$

which is consinsent with the age of the oldest observed objects in the universe. The above cosmological model, known as Λ-CDM (because of the cosmological constant and cold dark matter), is the current best description of our universe, as it is able to fit a number of independent observations.

1.6 Redshift and Horizons

In General Relativity, for comoving observers with 4-velocity $U^\mu = (1, 0, 0, 0)$, there is the Killing tensor $K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu)$, since the latter satisfies $\nabla_{(\sigma} K_{\mu\nu)} = 0$ [1]. Thus, for a particle that has 4-velocity $V^\mu = dx^\mu/d\lambda$, the following quantity will be constant

$$K^2 = K_{\mu\nu} V^\mu V^\nu = a^2\big(V_\mu V^\mu + (UV)^2\big) = \text{const} \tag{1.122}$$

For massless particles, such as photons, we have $V_\mu V^\mu = 0$, and thus (1.122) gives

$$K^2 = a^2 (UV)^2$$

$$\to U_\mu V^\mu = \frac{K}{a} \tag{1.123}$$

But, $\omega = -U_\mu V^\mu$ is the frequency of the photon as measured by a comoving observer, so for the frequency that the photon emitted and the frequency that the photon is observed, we have

$$\frac{\omega_o}{\omega_e} = \frac{a_e}{a_o} < 1 \tag{1.124}$$

The expansion of the universe results in a shift to a longer wavelength λ of the propagating photons in it, which is called redshift and is defined as $z_e = (\lambda_o - \lambda_e)/\lambda_e$. Thus, from (1.124) we have

$$z_e = \frac{a_o}{a_e} - 1$$

$$\to a_e = \frac{1}{1 + z_e} \tag{1.125}$$

if the photon is observed today, $a_o = a_0 = 1$. So, from the redshift of an object we are able to know the scale factor when it was emitted, and often the redshift $z_e = z$ is used in place of the scale factor. Redshift also explains the relation (1.44) for radiation: the energy density of a fixed number of photons in a fixed comoving volume decreases by an extra a^{-1} factor to the already a^{-3} (as for matter), so it scales as a^{-4}, because the expansion of the universe streches the wavelengths of light.

One more important aspect of light propagation in the FRW cosmological models, is the existence of cosmological horizons. Considering for simplicity, a flat 3-dimensional space ($k = 0$), for lightrays (with $d\theta = d\phi = 0$) emitted at r_e and t_e that are coming towards us and we observe them now at $r_o = 0$ and t_o, from (1.17) we have

$$ds^2 = 0 \to dt = a(t)dr^2$$

$$\to r = \int_{t_e}^{t_o} \frac{dt}{a(t)}$$

$$\to r = \tau_o - \tau_e \tag{1.126}$$

using the definition of the conformal time, (1.69). If the time of emittion is bounded from below due to the Big Bang, according to (1.126) there is a maximum distance to which the observer can see, and with distances further than this the observer could not have had any causal contact until today. This maximum distance is called particle horizon distance, and is given by

$$r_{p.h.}(\tau_o) = \tau_o - \tau_{e,b} \tag{1.127}$$

On the other hand, if the time of the observation is bounded from above, then there is a maximum distance in which the lightrays (emitter) can influence the spacetime events and thus, there are regions of spacetime from which the observer never had any information. This maximum distance is called event horizon distance and is given by

$$r_{e.h.}(\tau_e) = \tau_{o,b} - \tau_e \tag{1.128}$$

One can also find the physical (proper) size of the particle and event horizon distances, through the relation

$$r_{p/e}(t) = a(t) \int_{t_e/t}^{t/t_o} \frac{dt'}{a(t')} \tag{1.129}$$

1.7 Early universe

We know that the universe is expanding, so going backwards in time means it is contracting. In particular, the more we go back, the more it contracts, and the temperature and energy density become very large, so that many particle species were kept in approximate thermal equilibrium by rapid interactions [5, 33, 22].

In general, the various particles in the early Universe can be characterized by whether they are in thermal equilibrium or not, whether they are bosonic or fermionic, and whether they are relativistic (hot) or non-relativistic (cold). A particle species is in thermal equilibrium with the thermal bath as soon as its interaction rate is larger than the expansion rate of the universe, $\Gamma_{int} \gg H$. Particles are squeezed together, they interact so often that they are not influenced by the expansion, and any perturbation in their energy density is smoothed out rapidly and equilibrium is achieved. In thermal equilibrium, the products of a reaction have the possibility to recombine in the reverse reaction. When, $\Gamma_{int} \ll H$, the particle species decouples from the rest of the plasma, or it "freezes out". This happens because the number density of the particles became so low due to expansion, interactions happen infrequently and cannot keep them in equilibrium.

In the very early universe, the expansion is so quick that particles cannot be in thermal equilibrium. However, as the expansion rate is decreasing, equilibrium becomes possible. Finally, the number density of the particles will become so low that thermal equilibrium could not be maintained any more. In the current universe, no particle species is in thermal equilibrium with the background plasma, which are the photons of CMB. In the case of the slow expansion, for particles in equilibrium, statistics says that their number density is

$$n_i = \frac{g_i}{(2\pi)^3} \int f_i(\vec{p}) d^3p \tag{1.130}$$

where f is the destribution function

$$f_i(\vec{p}) = \frac{1}{e^{(E_i-\mu_i)/T} \pm 1} \tag{1.131}$$

which in general is a function also of the position $\vec{x}$, but not here as we assumed homogeneity, $\pm$ corresponds to fermions and bosons respectively, $E_i^2(\vec{p}) = m_i^2 + |\vec{p}|^2$ is the energy, T is the temperature, μ_i is the chemical potential and g_i is the number of spin states of the particles. Integrating for relativistic ($T \gg m$) and for non-relativistic particles ($T \ll m$), and ignoring the chemical potential, we obtain

$$n_i = \begin{cases} \frac{\zeta(3)}{\pi^2} g_i T^3 \quad \text{(bosons)}, & T \gg m \\ \frac{3}{4}\frac{\zeta(3)}{\pi^2} g_i T^3 \quad \text{(fermions)}, & T \gg m \\ g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}, & T \ll m \end{cases} \tag{1.132}$$

where ζ is the Riemann zeta function with $\zeta(3) \approx 1.202$ [56]. We observe that in thermal equilibrium, the relativistic particles, whether bosons or fermions, remain in approximately equal abundances. This happens because annihilations are balanced from pair products, for $T \gg m$. When they become non-relativistic though, their abundance drops rapidly, as production of particle-antiparticle pairs becomes harder for $T \ll m$.

The energy density of the particles is

$$\begin{aligned} \rho_i &= \frac{g_i}{(2\pi)^3} \int E_i(\vec{p}) f_i(\vec{p}) d^3p \\ &= \begin{cases} \frac{\pi^2}{30} g_i T^4 \quad \text{(bosons)}, & T \gg m \\ \frac{7}{8}\frac{\pi^2}{30} g_i T^4 \quad \text{(fermions)}, & T \gg m \\ m_i n_i, & T \ll m \end{cases} \end{aligned} \tag{1.133}$$

We define the effective number of relativistic degrees of freedom for the energy as

$$g_{eff} = \sum_{i=bos} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=ferm} g_i \left(\frac{T_i}{T}\right)^4 \tag{1.134}$$

in which species that are not in thermal equilibrium with the rest of the plasma, $T_i \neq T$, are also included. Thus, the total energy density of all the relativistic particles (1.133), using (1.134) can be expressed as

$$\begin{aligned} \rho &= \sum_{i=bos} \rho + \sum_{i=ferm} \rho \\ &= \sum_{i=bos} g_i \frac{\pi^2}{30} T_i^4 + \sum_{i=ferm} \frac{7}{8} g_i \frac{\pi^2}{30} T_i^4 \\ &= \frac{\pi^2}{30} g_{eff} T^4 \end{aligned} \tag{1.135}$$

Also, the pressure is

$$p_i = \frac{g_i}{(2\pi)^3} \int \frac{p^2}{3E_i(\vec{p})} f_i(\vec{p}) d^3p \tag{1.136}$$

which for the relativistic particles is $p_i = (1/3)\rho_i$. The rest-frame entropy density is

$$s = \frac{\rho + p}{T} \tag{1.137}$$

and defining the effective number of the relativistic degrees of freedom for the entropy as

$$g_{eff,s} = \sum_{i=bos} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=ferm} g_i \left(\frac{T_i}{T}\right)^3 \tag{1.138}$$

the entropy density of the relativistic species is

$$s = \frac{2\pi^2}{45} g_{eff,s} T^3 \tag{1.139}$$

As the universe expands, the number and energy density of the particles are decreasing as $n \propto a^{-3}$ and $\rho \propto a^{-4}$ (radiation-dominated era). From (1.132) and (1.133), we have $n \propto T^3$ and $\rho \propto T^4$, so we see that $T \propto a^{-1}$. Thus, as the universe expands, the temperature is decreasing. A better approximation for the evolution of temperature comes from the fact that the comoving entropy density $S = sa^3$ is conserved under all forms of adiabatic evolution

$$s \propto a^{-3}$$

$$\to T \propto g_{eff,s}^{-1/3} a^{-1} \tag{1.140}$$

From (1.140) we see that the temperature falls under adiabatic evolution in an expanding universe, but with a lower rate when the effective number of relativistic degrees of freedom decreases.

When particles decouple from the plasma, they are either relativistic or non-relativistic and stay that way afterwards, or relativistic and become non-relativistic sometime later. When particles freeze out, they obtain a fixed abundance, which continues to decrease as $n \propto a^{-3}$ if they are stable, and we observe their relic abundance today [33, 25]. Also, it is possible that there are significant relic abundances for particles that were never in thermal equilibrium. We can calculate the abundace for hot and cold species at the time of decoupling, with the latter being harder as their abundance is changing rapidly with respect to the background plasma. Primordial abundances of the light elements are an important piece of evidence for the Standard Cosmological Model.

Below, we describe shortly some important events and eras of the universe, that finally lead to the universe that we see today:

Planck scale: This is an energy scale of 10^{19} GeV, which is considered to be the upper limit that classical theory of gravity holds. In scales larger than this, quantum gravity is expected to be important, and as there is no such accepted theory yet, we know nothing about the very early eras of the universe.

Inflation: It is an accelerated expansion phase that the universe is considered to have undergone, in which we will refer later in detail, at around 10^{16} GeV. At first, the inflationary phase was not included in the Stardand Model of Cosmology, but now is part of it as it solves some important problems of the latter, making up the extented version of it. At the inflationary era, the initial matter perturbations are formed and then stretched by the accelerated expansion [30, 33, 11].

Baryogenesis: This is considered to be the process which generated the observed very specific baryon asymmetry in the universe [78], that is the imbalance of matter (baryons) and antimatter, as the universe seems to be composed almost entirely of matter with little or no primordial antimatter. The exact mechanism behind baryogenesis is not known yet.

Electroweak phase transition: This phase transition corresponds to the breaking of the electroweak symmetry to the electromagnetic one, $SU(2)_L \times U(1)_Y \to U(1)_{EM}$, and happens at around 100 GeV. Through the Higgs mechanism, elementary fermions (quarks and leptons) and weak interaction bosons obtain their masses.

QCD phase transition: Below 150 MeV, strong interactions become important and quark and gluons, which were free particles, are bound into composite particles, the hadrons, that is baryons and mesons. Until 130 MeV all hadrons are formed.

Neutrino decoupling: It is the decoupling of neutrinos from the plasma, which happens below 1 MeV, while they are still relativistic, because the weak interaction rate becomes smaller than the expansion rate of the universe. Since they are still hot, they form their own (relativistic) background, with temperature at the time of decoupling and its expected evolution afterwards, the same as the plasma's.

Electron-positron annihilation: Approximately at the same temperature as above, electrons and positrons become non-relativistic and the pair production ($e^- + e^+ \leftarrow \gamma + \gamma$) is not possible any more, only annihilation ($e^- + e^+ \rightarrow \gamma + \gamma$), through which energy (and entropy) is released into the plasma, and as a result there is a difference between the final photon background temperature and the neutrino background one, $T_\gamma > T_\nu$. We can see this from (1.140), while annililation of electron/positron pairs is one of the events that change the effective number of relativistic degrees of freedom, an thus, the temperature falls with a smaller rate.

Nucleosynthesis: Big Bang nucleosynthesis happens, at around 0.1 MeV (3 minutes), where nucleons form the nuclei of the light elements, mostly ^4He, but there are also traces of $D,^3He,^7Li$. Nucleosynthesis did not happen earlier at the energy scale of binding energy of nucleons because of the large number of photons per nucleon, but under 0.1 MeV the photon energy is not enough to break the nuclear binding energy. Heavier nuclei are formed later in the universe, from supernova explosions.

Matter-Radiation equality: This is the era when the energy density of radiation becomes equal to the one of matter, at around 1 eV, and then, matter starts to dominate.

Recombination: At around 0.3 eV, the electrons combine gradually with nuclei and neutral atoms are formed, mostly hydrogen. This did not happen earlier at the scale of the binding energy of the hydrogen, due to the large number of photons per baryon, but here the formation of atoms is able to happen as the photon energy is not enough to ionise them any more.

Photon decoupling: It is the photon decoupling from the plasma at around 0.3 eV, since the number density of the free electrons drops after the formation of hydrogen, and Thomson scattering ($e^- + \gamma \rightarrow e^- + \gamma$) is inadequate. The scale factor at that time defines the last scattering surface, from which the photons that we observe today as CMB (Cosmic Microwave Background) come from.

Dark ages: It is the long period that universe passes by after recombination until the present era, in which galaxies are formed by highly complicated and non lear processes that are not yet well understood [1].

Current Era: After all the above significant eras that universe passes by, it finally becomes the present universe that we observe, with the little amount of radiation, and both important contributions of matter, baryonic and dark, and dark energy, where the latter essentially dominates, and the large scale structure, that is, clusters and superclusters of galaxies.

1.8 GR formulation of field theory

Consider a field theory in which the dynamical variables are a set of fields $\Phi_i(x^\mu)$ [1]. The Action S of the theory, which is expressed as the integral over space of a Lagrange density $\mathcal{L}$, in curved spacetime and n-dimensions, is

$$S = \int d^n x \mathcal{L}(\Phi_i, \nabla_\mu \Phi_i) \tag{1.141}$$

We write

$$\mathcal{L} = \sqrt{-g}\tilde{\mathcal{L}} \tag{1.142}$$

where $\tilde{\mathcal{L}}$ is a scalar, and $g = \det g_{\mu\nu}$ is the determinant of the metric tensor. Varying with respect to the Φ_i, we have

$$\begin{aligned}\delta S &= \int d^n x\sqrt{-g}\delta\tilde{\mathcal{L}} \\ &= \int d^n x\sqrt{-g}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\Phi_i} + \frac{\partial\tilde{\mathcal{L}}}{\partial(\nabla_\mu\Phi_i)}\delta\nabla_\mu\Phi_i\right)\end{aligned} \tag{1.143}$$

and considering the commutation of $\delta\nabla_\mu$, $\nabla_\mu\delta$, the above is written as

$$\delta S = \int d^n x\sqrt{-g}\left\{\frac{\partial\tilde{\mathcal{L}}}{\partial\Phi_i} + \nabla_\mu\left(\frac{\partial\tilde{\mathcal{L}}}{\partial(\nabla_\mu\Phi_i)}\delta\Phi_i\right) - \nabla_\mu\left(\frac{\partial\tilde{\mathcal{L}}}{\partial(\nabla_\mu\Phi_i)}\right)\delta\Phi_i\right\} \tag{1.144}$$

From the Stokes theorem

$$\int_\Sigma \nabla_\mu V^\mu\sqrt{|g|}d^n x = \int_{\partial\Sigma} n_\mu V^\mu\sqrt{|\gamma|}d^{n-1}x, \tag{1.145}$$

(1.144) becomes

$$\delta S = \int_{\partial\Sigma} d^{n-1}x\sqrt{|\gamma|}n_\mu\frac{\partial\tilde{\mathcal{L}}}{\partial(\nabla_\mu\Phi_i)}\delta\Phi_i + \int d^n\sqrt{-g}\left\{\frac{\partial\tilde{\mathcal{L}}}{\partial\Phi_i} - \nabla_\mu\left(\frac{\partial\tilde{\mathcal{L}}}{\partial(\nabla_\mu\Phi_i)}\right)\right\}\delta\Phi_i \tag{1.146}$$

But it is $\delta\Phi_i = 0$ at the boundary surface $\partial\Sigma$ (infinity), so the surface term vanishes. The classical solutions of the theory are the critical points of the action S,

$$\begin{aligned}\delta S &= \int \frac{\delta S}{\delta\Phi_i}\delta\Phi_i d^n x \\ \rightarrow \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\Phi_i} &= \frac{\partial\tilde{\mathcal{L}}}{\partial\Phi_i} - \nabla_\mu\left(\frac{\partial\tilde{\mathcal{L}}}{\partial(\nabla_\mu\Phi_i)}\right) = 0 \\ \rightarrow \frac{\partial\tilde{\mathcal{L}}}{\partial\Phi_i} &= \nabla_\mu\left(\frac{\partial\tilde{\mathcal{L}}}{\partial(\nabla_\mu\Phi_i)}\right)\end{aligned} \tag{1.147}$$

from which we finally found the associated Euler-Lagrange equations for Φ_i.

In the General theory of Relativity, the dynamical parameter is the metric tensor $g_{\mu\nu}$. The action is

$$S = \frac{1}{16\pi G}S_H + S_M \tag{1.148}$$

where the S_M corresponds to the description of matter, and S_H is the Einstein-Hilbert action corresponding to the vacuum

$$S_H = \int d^n x\sqrt{-g}(R - 2\Lambda) \tag{1.149}$$

where R is the Ricci scalar and Λ is the cosmological constant. Varying with respect to $g_{\mu\nu}$, for the Hilbert action we have

$$\delta S_H = \int d^n x\big(\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + R\delta\sqrt{-g} - 2\delta\sqrt{-g}\Lambda\big) \tag{1.150}$$

which after calculation of the variations of each term (A), leads to

$$\delta S_H = \int d^n x\sqrt{-g}\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\Big)\delta g^{\mu\nu} \tag{1.151}$$

and finally for the total action we have

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\Big) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \tag{1.152}$$

Defining

$$T_{\mu\nu,M} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{1.153}$$

then from (1.152) we have

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu,M} \tag{1.154}$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu}$ is the Einstein tensor, mentioned before. Equation (1.154) is the general Einstein equation previously discussed, and going down to the 4-dimensions we can find the associated Friedmann equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu,M} \to R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + \Lambda g_{\mu\nu} :$$

$$00 : \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

$$\to \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\big(\rho + \rho_\Lambda + 3(p + p_\Lambda)\big) \quad \text{and} \tag{1.155}$$

$$ij : \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} = 4\pi G(\rho - p) + \Lambda$$

$$\to \frac{\dot{a}^2}{a^2} + = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{k}{a^2} \tag{1.156}$$

which as we see consider that we already have included the cosmological type of energy, (1.48) and (1.49). The cosmological models in which the universe is governed from the Friedmann equations that already consider Λ, are called Friedmann-Lemaitre-Robertson-Walker (FLRW) models.

Chapter 2

Cosmological Inflation

While the conventional Standard Model of Cosmology is able to describe to great accuracy the physical processes that lead to the present day universe, there remain some very important cosmological issues to be solved and described. Most of them led to an era in the early universe, in which an accelerated expansion is considered to have taken place, named inflation [27, 28, 29], and was caused by a nearly constant energy density. Inflation now is included in the Standard Cosmological Model, as a number of inflationary model predictions have been confirmed by observations.

In this chapter, we study the inflationary scenario, firstly stating the more important problems of the conventional Standard Cosmology and showing how they are solved if one considers an early period of accelerated expansion in the universe, and then the dynamics of inflation, and finally, the outline of the various models that have been built to describe it.

2.1 Inflationary solutions

In this section, we will try to put simply the most significant issues of the SMC and the solution that the accelerated expansion, inflation, provided to them.

2.1.1 Flatness issue

In the previous chapter we showed that the Friedmann equation can be written as

$$\Omega - 1 = \frac{k}{H^2 a^2} \tag{2.1}$$

Differentiating with respect to the scale factor, we have

$$\frac{d\Omega}{da} = -\frac{2k}{H^3 a^3}\left(\frac{dH}{da}a + H\right) \tag{2.2}$$

From the acceleration equation, for $p = w\rho$, we have

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\rho(1+3w) \\ \rightarrow 1 + \frac{\dot{H}}{H^2} &= -\frac{1}{2}\frac{8\pi G}{3H^2}\rho(1+3w) = -\frac{\Omega}{2}(1+3w) \\ \rightarrow \frac{dH}{da} &= \frac{1}{\dot{a}}\left(-\frac{\Omega}{2}(1+3w) - 1\right)H^2 \end{aligned} \tag{2.3}$$

where we used the definition of the Ω parameter (1.54). Substituting into (2.2), we get

$$\frac{d\Omega}{da} = (1+3w)\frac{\Omega(\Omega-1)}{a} \tag{2.4}$$

The value $\Omega = 1$ that we observe today, which is a fixed point of this differential equation, is a repeller (unstable fixed point), if one considers gravitational matter ($\rho > 0$) and $1+3w > 0 \rightarrow w > -1/3$ (which holds for matter, and radiation). In order to observe $\Omega_0 \sim 1$, a very tiny value of $\Omega - 1$ is required in the early universe. In particular, the deviation of parameter Ω from unity, from (2.1) can be expressed as

$$\frac{\Omega-1}{\Omega_0-1} = \frac{H_0^2 a_0^2}{H^2 a^2} = \left(\frac{\dot{a}_0}{\dot{a}}\right)^2$$

$$\rightarrow \Omega - 1 = (\Omega_0 - 1)\left(\frac{\dot{a}_0}{\dot{a}}\right)^2 \tag{2.5}$$

where Ω reffers to some earlier era, t. Using the fact that in the matter-radiation equality era, the scale factor is either $a_{eq} \propto t^{1/2}$ or $a_{eq} \propto t^{2/3}$, we have that

$$\frac{\dot{a}_0}{\dot{a}} = \frac{\dot{a}_{eq}}{\dot{a}}\frac{\dot{a}_0}{\dot{a}_{eq}} = \left(\frac{t_{eq}}{t}\right)^{-1/2}\left(\frac{t_0}{t_{eq}}\right)^{-1/3}$$

$$\rightarrow \Omega - 1 = (\Omega_0 - 1)\frac{t}{t_{eq}}\left(\frac{t_{eq}}{t_0}\right)^{2/3} \tag{2.6}$$

Calculating (2.6) for the BBN, which is the earliest era that we know Standard Model is true for sure [33], gives

$$\Omega_{BBN} - 1 \lesssim 10^{-19} \tag{2.7}$$

or, for the Planck era, which is the earliest era that we can go back according to the classical theory [33], (2.6) gives

$$\Omega_{Planck} - 1 \lesssim 10^{-63} \tag{2.8}$$

In the above it is considered that $t_{BBN} \sim 3$ min, $t_{eq} \sim 10^{12}$ sec, and $t_0 \sim 13.9 \times 10^9$ yrs. We see that (2.7) and even more (2.8), is a high degree of precision, that is high degree of fine tuning, and there is no such mechanism to describe it in the conventional Standard Cosmology. This is called the Flatness Problem. All these if one does not impose this particular high degree of precision as initial condition in the universe, but searches for a dynamical explanation.

However, we observe that the fixed point $\Omega = 1$ of (2.4) becomes an attractor for $1 + 3w < 0 \rightarrow w < -1/3$, and this is precisely the condition for accelerating expansion. More specifically, we see the latter from the acceleration equation, which requires $1+3w < 0$ for $\ddot{a} > 0$. Considering for example $w = -1$, which corresponds to an exponential expansion of the universe, $a \propto e^{H\Delta t}$, as we saw, from the Friedmann equation (2.1) we have that

$$\Omega - 1 \propto a^{-2H\Delta t} \tag{2.9}$$

which means that if $H\Delta t$ is big enough, that is if exponential expansion lasts long enough, the deviation of Ω from unity will reach the aforementined tiny value, even if it starts with an arbirtary value, and today we will observe $\Omega_0 = 1$.

2.1.2 Horizon issue

For a flat space, from (1.129) the particle horizon physical dinstance is

$$r_p(t) = a(t)\int_{t_i}^{t}\frac{dt'}{a(t')} \tag{2.10}$$

For either matter or radiation, we have $a(t) \propto t^{2/3(1+w)}$ (1.65), so from (2.10) we have

$$r_p(t) = t^{\frac{2}{3(1+w)}} \int_{t_i}^{t} \frac{dt'}{t'^{\frac{2}{3(1+w)}}}$$

$$= \frac{3(1+w)}{1+3w}\left(t - t_i\left(\frac{t}{t_i}\right)^{\frac{2}{3(1+w)}}\right) \tag{2.11}$$

From the Big Bang ($t_i = 0$) almost until the last scattering surface ($t = t_{ls}$), when radiation ($w = 1/3$) dominated, (2.11) gives

$$r_p(t_{ls}) = 2t_{ls} \tag{2.12}$$

which expresses the maximum physical distance that the photons may have traveled during this period. From $t = t_{ls}$ until today $t = t_0$, in which matter dominates ($w = 0$), (2.11) gives

$$r_p = 3t_0^{2/3}(t_0^{1/3} - t_{ls}^{1/3})$$

$$\rightarrow r_p(t_0) \approx t_0 \tag{2.13}$$

ignoring $t_{ls} \sim 10^5$ yrs in comparison to $t_0 \sim 10^{10}$ yrs, which also shows the distance over which the photons could have traveled during this period. However, during the above matter dominated era, a causal patch of initial size $r_{p,c}(t_{ls})$ has grown due to expansion today to be

$$\frac{r_{p,c}(t_0)}{r_{p,c}(t_{ls})} = \frac{a(t_0)}{a(t_{ls})}$$

$$\rightarrow r_{p,c}(t_0) \approx t_0^{2/3} t_{ls}^{1/3} \tag{2.14}$$

in which we used that the particle horizon distance is only a function of the scale factor, and (2.12). Comparing the region of the last scattering surface from which we receive the CMB photons today, (2.13), to the causal region (2.14), we see that

$$\frac{r_p(t_0)}{r_{p,c}(t_0)} = \left(\frac{t_0}{t_{ls}}\right)^{1/3} \approx 10^{5/3} \tag{2.15}$$

or for the corresponding volumes of the universe,

$$\frac{V}{V_c} = \left(\frac{r_p(t_0)}{r_{p,c}(t_0)}\right)^3 \approx 10^5 \tag{2.16}$$

This means that the universe that we observe today with its isotropic CMB, actually comes from 10^5 different regions that have never been in causal contact, and therefore the high degree of isotropy in CMB can not be explained. In order for the seperate regions to know its others temperature, the causal structure of the conventional FRW cosmologies must be modified. The problem becomes even larger the more we go back in time, as the observed universe today comes from more and more regions that were never in causal contact. The above issue is known as Horizon Problem.

Once again, an accelerated phase in the early universe before $t = t_{ls}$, could enlarge a causal patch from $t = t_i$ until $t = t_{ls}$, enough, so that the observed isotropy of CMB today would be explained: We have showed that $r_p(t_0) \gg r_{p,c}(t_0)$, which means that $r_p(t_{ls}) \gg r_{p,c}(t_{ls})$. In order for the problem to be solved, we should have that $r_p(t_{ls}) \leq r_{p,c}(t_{ls})$. If ones assumes $a(t_f) = a(t_i)e^{H\Delta t}$, with $\Delta t = t_f - t_i \gg 0$, a causal patch from t_i to t_f would have become

$$r_{p,c}(t_f) = \frac{a(t_f)}{a(t_i)} r_{p,c}(t_i) = r_{p,c}(t_i)e^{H\Delta t} \tag{2.17}$$

which means that it could be $r_{p,c}(t_f = t_{ls}) \geq r_p(t_{ls})$, if inflation lasts long enough.

Both the Flatness Problem and Horizon Problem solutions can be put together as follows: From the Friedmann equation (2.1),

$$\Omega - 1 = k(aH)^{-2} \tag{2.18}$$

we observe that Ω is driven to unity if $(aH)^{-1}$ decreases. Moreover, from the particle horizon distance, starting from the BB at $t_i = 0$, the universe in causal contact is

$$\tau = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{H(a')a'^2} = \int_0^a d\ln a'(a'H)^{-1} \tag{2.19}$$

An increasing τ ($(aH)^{-1}$) means that the comoving scales that enter the horizon today where out of it at the time, for expample, of the photon decoupling, while a decreasing τ ($(aH)^{-1}$) means they were in the horizon before inflation and therefore in causal contact [8, 11]. The important quantity, $(aH)^{-1}$, which we would like to decrease, is called comoving Hubble radius, and it decreases when the universe is accelerating:

$$\frac{d}{dt}(aH)^{-1} = -(aH)^{-2}\left(\dot{H}a + H\dot{a}\right) = -\frac{\ddot{a}}{(aH)^2}$$

$$\rightarrow \frac{d}{dt}(aH)^{-1} < 0 \rightarrow \ddot{a} > 0 \tag{2.20}$$

since $(aH)^{-2} > 0$, in contrast to what it happens in the conventional Stardard Cosmology as there it increases, which we can see from the Friedmann equation (1.63), using (1.54) and (1.42) for $a_0 = 1$,

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)} \tag{2.21}$$

for matter or radiation dominated universe, since $H_0^{-1} > 0$.

Furthermore, inflation solves the predicted big relic abundance of monopoles in the Grand Unification Theories (GUT's), as their density is diluted by the accelerated expansion (if the GUT phase transition happens before inflation), and explains the origin of the initial fluctuations of matter density through the gravitational instability of which, the largest scale structures in the universe today have been formed: quantum fluctuations in the inflaton field during inflation are streched and become classical fluctuations, that we observe today [30, 33, 11].

2.2 Physics of Inflation

In this section, we firstly describe the basic idea of accelerated expansion through classical field theory in which acceleration is caused by a real scalar field, and secondly, we define more formally the physical conditions required in order to have an inflationary phase.

2.2.1 General idea

Consider again a field theory with a classical set of fields Φ_i as in Section 1.8, and $k = 0$ for simplicity. For a spacetime variation and variations of the fields,

$$\delta x^\mu = x'^\mu - x^\mu$$

$$\delta\Phi_i = \Phi_i'(x') - \Phi_i(x), \quad \delta_o\Phi_i = \Phi_i'(x) - \Phi_i(x) \tag{2.22}$$

the variation of the action is

$$\delta S = \int \left(\delta(d^4x)\mathcal{L} + d^4x\delta\mathcal{L}\right) \tag{2.23}$$

But

$$\delta(d^4x) = d^4x' - d^4x = \big(\det(\partial_\nu x'^\mu) - 1\big)d^4x \tag{2.24}$$

with $\partial_\nu x'^\mu = \delta^\mu_\nu + \partial_\nu \delta x^\mu$, and we can say that

$$J = \mathbb{I} + \Xi \approx e^\Xi \to \det J \approx e^{\mathrm{Tr}\Xi} = 1 + \mathrm{Tr}\Xi + ...$$

$$\to \det J - 1 \approx \mathrm{Tr}\Xi \tag{2.25}$$

for two matrices J and Ξ, so (2.24) becomes

$$\delta(d^4x) \approx d^4x \mathrm{Tr}\partial_\nu \delta x^\mu = d^4x \partial_\mu \delta x^\mu \tag{2.26}$$

Also,

$$\delta\Phi_i = \Phi'_i(x') - \Phi_i(x) = \Phi'_i(x + \delta x) - \Phi_i(x) = \Phi'_i(x) + \delta x^\mu \partial_\mu \Phi'_i(x) + \mathcal{O}(\delta x^2) - \Phi_i(x)$$

$$\to \delta\Phi_i = \delta_o \Phi_i + \delta x^\mu \partial_\mu \Phi_i(x) \tag{2.27}$$

in first order approximation. Using (2.26) and (2.27), (2.23) becomes

$$\delta S = \int d^4x \big(\partial_\mu \delta x^\mu \mathcal{L} + \delta_o \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L}\big)$$

$$= \int d^4x \left(\partial_\mu \left(\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \delta_o \Phi_i \right) + \left[\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \right) \right] \delta_o \Phi_i \right) \tag{2.28}$$

Imposing the Euler-Lagrange equations (1.147) for a flat 3-space,

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \tag{2.29}$$

and using (2.27), we have

$$\delta S = \int d^4x \partial_\mu \left(\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \big(\delta \Phi_i - \delta x^\nu \partial_\nu \Phi_i\big) \right)$$

$$= \int d^4x \partial_\mu \left(\delta x^\nu \left(\delta^\mu_\nu \mathcal{L} - \partial_\nu \Phi_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \right) + \delta \Phi_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \right)$$

$$= \int d^4x \partial_\mu j^\mu \tag{2.30}$$

where

$$j^\mu = \delta x^\nu \left(\delta^\mu_\nu \mathcal{L} - \partial_\nu \Phi_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \right) + \delta \Phi_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \tag{2.31}$$

For an invariant action ($\delta S = 0$), there is a conserved current j^μ given by (2.31), $\partial_\mu j^\mu = 0$, the Noether current. For the spacetime translation

$$\delta x^\mu = \epsilon^\mu = \text{const}, \quad \delta \Phi_i = 0 \tag{2.32}$$

the Noether current is

$$j^\mu = \epsilon^\nu \left(\delta^\mu_\nu \mathcal{L} - \partial_\nu \Phi_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \right)$$

$$= \epsilon^{\nu} T^{\mu}_{\ \nu} \tag{2.33}$$

where

$$T^{\mu}_{\ \nu} = \delta^{\mu}_{\nu}\mathcal{L} - \partial_{\nu}\Phi_i \frac{\partial \mathcal{L}}{\partial \partial_{\mu}\Phi_i} \tag{2.34}$$

is the energy-momentum tensor, which is conserved for a conserved Noether current, $\delta S = 0 \to \partial_{\mu} j^{\mu} = 0 \to \partial_{\mu} T^{\mu}_{\ \nu} = 0$, since $\epsilon^{\mu} = const.$

In the most simple case, inflation is considered to be caused by a real scalar field ϕ, with a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - V(\phi) \tag{2.35}$$

with a canonical kinetic term and the potential $V(\phi)$. For this Lagrangian, the energy-momentum tensor (2.33) is

$$T_{\mu\nu} = \eta_{\mu\rho}T^{\rho}_{\ \nu} = \eta_{\mu\rho}\left(\mathcal{L}\delta^{\rho}_{\nu} - \partial_{\nu}\phi\frac{\partial\mathcal{L}}{\partial\partial_{\rho}\phi}\right)$$

$$= -\eta_{\mu\nu}\left(\frac{1}{2}\partial_{\kappa}\phi\partial^{\kappa}\phi + V(\phi)\right) + \partial_{\nu}\phi\partial_{\mu}\phi \tag{2.36}$$

replacing the Lagrangian and its partial derivatives. The energy density of the field ϕ is

$$T_{00} = \rho_{\phi} = V(\phi) + \frac{1}{2}\eta^{ii}(\partial_i\phi)^2 + \frac{\dot{\phi}^2}{2} \tag{2.37}$$

for the metric $ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2)$, and for an almost homogeneous field during inflation [83] becomes

$$\rho_{\phi} = V(\phi) + \frac{\dot{\phi}^2}{2} \tag{2.38}$$

The pressure is found by

$$T_{ii} = \eta_{ii}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) = \eta_{ii}p$$

$$\to p_{\phi} = \frac{\dot{\phi}^2}{2} - V(\phi) \tag{2.39}$$

and thus, considering $p = w\rho$, the equation of state parameter corresponding to the field is

$$w = \frac{p_{\phi}}{\rho_{\phi}} = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)} \tag{2.40}$$

From (2.39) we observe that if $V > \dot{\phi}^2$ we are able to reach $w < 0$, and in some cases even $w < -1/3$ which corresponds to an accelerated expansion, for expample, especially if $V \gg \dot{\phi}^2$, which means that the field is evolving very slowly and so is the potential, $V(\phi) \approx V(\phi_0) = V_0$, we are led to $w = -1$ approximately, which corresponds to the particular type of exponential accelerated expansion. In the latter case, the almost constant potential plays the role of a cosmological constant, and thus from the Friedmann equation, as in (1.66), we have

$$H_0^2 \approx \frac{8\pi G}{3}V_0 \to a(t) \propto e^{H_0 t} \tag{2.41}$$

with $H_0 = \sqrt{(8\pi G/3)V_0} \approx const.$

2.2.2 Dynamics

The dynamics of a single scalar field ϕ minimally coupled to gravity is governed by the action

$$S = \int d^4x\sqrt{-g}\bigg(\frac{R}{16\pi G} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\bigg) = S_H + S_\phi \tag{2.42}$$

where S_H is the Einstein-Hilbert action and S_ϕ is action of the scalar field,

$$S_\phi = \int d^4x\sqrt{-g}\bigg(- \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\bigg) = \int d^x\sqrt{-g}\tilde{\mathcal{L}}_\phi \tag{2.43}$$

To find the energy-momentum tensor we use the simple type (1.153), as the expression (2.33) is not always possible to be generalized to curved space,

$$\begin{aligned} T^{(\phi)}_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} \\ &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\bigg(\frac{1}{2}\partial_\rho\phi\partial^\rho + V(\phi)\bigg) \end{aligned} \tag{2.44}$$

In the above we used the $\tilde{\mathcal{L}}_\phi$ definition of (2.42), and $\delta\sqrt{-g}/\delta g^{\mu\nu} = -(1/2)\sqrt{-g}g_{\mu\nu}$ (A.19). We observe that (2.44) leads to the same ρ_ϕ (2.38), p_ϕ (2.39) and w (2.40) as the (2.36) tensor (again with assumed homogeneity).

From the Euler-Lagrange equation in curved space (1.147), we have

$$\begin{aligned} &\frac{\partial\tilde{\mathcal{L}}_\phi}{\partial\phi} - \nabla_\mu\bigg(\frac{\partial\tilde{\mathcal{L}}_\phi}{\partial\partial_\mu\phi}\bigg) = 0 \\ &\to -\frac{\partial V(\phi)}{\partial\phi} + \nabla_\mu\partial^\mu\phi = 0 \\ &-\frac{\partial V(\phi)}{\partial\phi} + \partial_\mu\partial^\mu\phi + \Gamma^\mu_{\mu\lambda}\partial^\lambda\phi = 0 \end{aligned} \tag{2.45}$$

where we used the definition of the covariant derivative, $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda}A^\lambda$, where A^μ is a vector field. But, from the definition of the Christoffel symbols (1.6), we have

$$\begin{aligned} \Gamma^\mu_{\mu\lambda} &= \frac{1}{2}g^{\mu\rho}\big(\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\lambda}\big) \\ &= \frac{1}{2}g^{\mu\rho}\partial_\lambda g_{\rho\mu} \end{aligned} \tag{2.46}$$

Also,

$$\begin{aligned} \frac{1}{\sqrt{-g}}\partial_\lambda\sqrt{-g} &= \frac{1}{\sqrt{-g}}\bigg(- \frac{1}{2\sqrt{-g}}\bigg)\partial_\lambda g \\ &= \frac{1}{2}g^{\mu\nu}\partial_\lambda g_{\mu\nu} = \Gamma^\mu_{\mu\lambda} \end{aligned} \tag{2.47}$$

where we used $\partial_\lambda g = gg^{\mu\nu}\partial_\lambda g_{\mu\nu}$. Substituting (2.47) into (2.45), we have

$$\begin{aligned} &-V'(\phi) + \partial_\mu\partial^\mu\phi + \frac{1}{\sqrt{-g}}\partial_\lambda\sqrt{-g}\partial^\lambda\phi = 0 \\ &\to \ddot{\phi} + \frac{1}{\sqrt{-g}}\partial_t\sqrt{-g}\dot{\phi} + V'(\phi) = 0 \end{aligned} \tag{2.48}$$

in which we assumed homogeneity. The factor of $\dot{\phi}$ in (2.48) is

$$\frac{1}{\sqrt{-g}}\partial_t\sqrt{-g} = \frac{1}{2}g^{ii}\partial_t g_{ii} = \frac{1}{2}6\frac{\dot{a}}{a} = 3H \tag{2.49}$$

for the RW metric (1.17) (assuming homogeneity), and finally the equation of motion for the field ϕ (2.48) is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{2.50}$$

The equation (2.50) is the same equation of motion in Minkowski space ($k = 0$) but with a friction term $3H\dot{\phi}$ due to expansion, which essentially slows down the evolution of ϕ.

Any matter and radiation contributions in the energy density, decrease fast in an accelerated phase (a^{-3}, a^{-4}), and soon become negligible, so that the Friedmann equation is

$$H^2 = \frac{8\pi G}{3}\rho_\phi = \frac{8\pi G}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) \tag{2.51}$$

The acceleration equation can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_\phi + 3p_\phi) = -\frac{4\pi G}{3}\rho_\phi(1+3w)$$

$$= -\frac{H^2}{2}(1+3w) = H^2(1-\epsilon_H) \tag{2.52}$$

(assuming $p = w\rho$), where

$$\epsilon_H = \frac{3}{2}(1+w) \tag{2.53}$$

and since w is given by (2.40) again, the parameter (2.53) is also expressed as

$$\epsilon_H = 4\pi G\frac{\dot{\phi}^2}{H^2} \tag{2.54}$$

The parameter ϵ_H is called the first Hubble slow roll parameter, and we can also find its relation with the evolution of the Hubble parameter,

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2(1-\epsilon_H)$$

$$\rightarrow \epsilon_H = -\frac{\dot{H}}{H^2} \tag{2.55}$$

In order to have an accelerating expansion, we need to have $\ddot{a} > 0$, which from the above gives

$$\frac{\ddot{a}}{a} > 0 \rightarrow \dot{H} > -H^2 \rightarrow \epsilon_H < 1 \tag{2.56}$$

while especially for exponential expansion ($w = -1$) gives (from (2.52))

$$\epsilon_H \rightarrow 0 \tag{2.57}$$

which is the "de-Sitter limit". In this limit the potential dominates the kinetic energy, as we saw before, $V \gg \dot{\phi}^2$.

But the condition for an accelerated expansion, $\epsilon_H < 1$, has to also last long enough for the problems of conventional SMC to be solved, so we need the second time derivative of the field ϕ to be small enough. We define the second Hubble slow roll parameter

$$\eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{2.58}$$

which according to the above assumption, has to be small enough. We can check how the first Hubble slow roll parameter changes with time: From (2.55),

$$\dot{\epsilon}_H = -\frac{\ddot{H}}{H^2} + 2H\epsilon_H^2 \tag{2.59}$$

Also, from the acceleration equation, for the ρ_ϕ (2.38) and p_ϕ (2.39),we have

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{8\pi G}{3}(\dot{\phi}^2 - V)$$

$$\rightarrow \dot{H} = -4\pi G\dot{\phi}^2 \tag{2.60}$$

$$\rightarrow \ddot{H} = 2\frac{\ddot{\phi}}{\dot{\phi}}\dot{H} \tag{2.61}$$

so, finally (2.59) becomes

$$\dot{\epsilon}_H = 2H\epsilon_H\left(\epsilon_H - \frac{\ddot{\phi}}{\dot{\phi}H}\right) = 2H\epsilon_H\left(\epsilon_H + \eta_H\right) \tag{2.62}$$

from which we see that the smallness of the second Hubble slow roll parameter η_H, quarantees the slow variation of the first Hubble slow roll parameter ϵ_H.

2.2.3 Slow-roll approximation

The Friedmann equation (2.51) and the equation of motion (2.50) can be solved either by numerical integration or within an approximation scheme. The most widely used approximation is the so called slow-roll approximation, in which all the dynamical characteristics of the universe change little,

$$\dot{\phi}^2 \ll V(\phi)$$

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V'(\phi)| \tag{2.63}$$

The necessary and sufficient conditions for this to happen (from (2.54) and (2.58)), are

$$\epsilon_H, |\eta_H| \ll 1 \tag{2.64}$$

which are called slow-roll conditions. The first ensures that we can neglect the $\dot{\phi}^2$ term in the Friedmann equation, and the second that we can neglect the $\ddot{\phi}$ term in the equation of motion, so that we finally have

$$H_I^2 \approx \frac{8\pi G}{3}V \sim \text{const}, \quad \dot{\phi} \approx -\frac{V'}{3H_I} \tag{2.65}$$

$$\rightarrow d\ln a = H_I dt \rightarrow a(t_f) = a(t_i)e^{H_I\Delta t} \tag{2.66}$$

with $\Delta t = t_f - t_i$. We can also define the potential slow-roll (PSR) parameters, which depend on the shape on the potential,

$$\epsilon_V(\phi) \equiv \frac{M_p^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

$$\eta_V(\phi) \equiv M_p^2\frac{V''(\phi)}{V(\phi)} \tag{2.67}$$

where $M_p^2 = 1/8\pi G$, which in the slow-roll approximation satisfy $\epsilon_V, |\eta_V| \ll 1$ (These conditions alone are necessary but not sufficient for the slow-rolling of the field ϕ). The relation between the HSR and PSR parameters, in the slow-roll approximation, is: using (2.54), (2.65), and (2.58), (2.65) (for $\ddot{\phi}$), respectively we find

$$\epsilon_H \approx \epsilon_V$$

$$\eta_H \approx \eta_V - \epsilon_H \approx \eta_V - \epsilon_V \tag{2.68}$$

2.2.4 Hamilton-Jacobi equations

The most familiar form of equations that describe a homogenious scalar field ϕ that is evolving in a potential $V(\phi)$ are (2.50) and (2.51). An alternative and more convenient form is: considering $\dot{\phi} > 0$ without loss of generality, from (2.60) we have

$$H'(\phi) = -4\pi G\dot{\phi} \to \dot{\phi} = -\frac{2H'(\phi)}{8\pi G} \tag{2.69}$$

and plugging this into (2.51) we have

$$H'(\phi) = \frac{1}{M_p^2}\frac{3}{2}H^2 - \frac{1}{2}\frac{1}{M_p^4}V \tag{2.70}$$

The equations (2.69) and (2.70) are called Hamilton-Jacobi equations [36, 98]. We can also express the HSR parameters using the HJ equations, as

$$\epsilon_H = -\frac{\dot{H}}{H^2} = -\frac{\dot{\phi}H'}{H^2} = 2M_p^2\left(\frac{H'}{H}\right)^2$$

$$\eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} = 2M_p^2\frac{H''}{H} \tag{2.71}$$

From (2.71) we see that the HSR parameters [30] are posing conditions with respect to the evolution of the Hubble parameter during inflation, and from (2.67) we see that the PSR parameters are posing conditions with respect to the shape of the potential. Thus, the slow-roll approximation with the PSR parameters (PSRA) is suitable for studying inflation in which a particular potential has been defined, while slow-roll approximation with HSR parameters (HSRA) can be used in the general case where the potential of the theory is not determined. We can find the exact relation between the HSR and PSR parameters: From (2.71),

$$\frac{\epsilon_V}{\epsilon_H} = \frac{1}{4}\frac{(V'/V)^2}{(H'/H)^2} \tag{2.72}$$

But, from (2.70), we have

$$V = 3M_p^2H^2 - 2M_p^4H'^2$$

$$\to V' = 6M_p^2HH' - 4M_p^4H'H'' \tag{2.73}$$

so,

$$\frac{V'}{V} = \frac{H'}{H}\frac{6H - 4M_p^2H''}{3H - 2M_p^2H'^2H^{-1}} \tag{2.74}$$

and finally, (2.72) gives

$$\frac{\epsilon_V}{\epsilon_H} = \left(\frac{3-\eta_H}{3-\epsilon_H}\right)^2 \tag{2.75}$$

Also, from (2.73),

$$V'' = 6M_p^2(H'^2 + HH'') - 4M_p^4(H''^2 + H'H''') \tag{2.76}$$

so,

$$\frac{V''}{V} = -\frac{2H'(2M_p^2H'''/H)}{3H - 2M_p^2H'^2/H} + \frac{6(H'^2 + HH'') - 4M_p^2H''^2}{3H^2 - 2M_p^2H'^2} \pm \frac{2H'(-2M_p^2H''H'/H^2)}{3H - 2M_p^2H'^2/H} \tag{2.77}$$

But from (2.71), we have

$$\eta'_H = 2M_p^2\Big(- \frac{H''}{H^2}H' + \frac{H'''}{H}\Big) \tag{2.78}$$

so (2.77) becomes

$$\frac{V''}{V} = -\frac{2\sqrt{\epsilon_H}\eta'_H}{\sqrt{2}M_p(3-\epsilon_H)} + \frac{(3-\eta_H)(\epsilon_H+\eta_H)}{M_p^2(3-\epsilon_H)} \tag{2.79}$$

if we use the expressions (2.71), too. Finally, from (2.67) we have

$$\eta_V = \sqrt{2M_p^2\epsilon_H}\frac{\eta'_H}{(3-\epsilon_H)} + \left(\frac{3-\eta_H}{3-\epsilon_H}\right)(\epsilon_H+\eta_H) \tag{2.80}$$

2.2.5 Number of e-folds

From definition, the inflationary phase ends whenever the first HSR parameter reaches to unity, $\epsilon_H(\phi_{end}) = 1$, or in the slow-roll approximation when $\epsilon_V(\phi_{end}) \simeq 1$. An expression that describes how long inflation lasted is the number of e-folds, N, and is defined from the Friedmann equation (2.66) as

$$H(t_{end} - t) = \ln\frac{a(t_{end})}{a(t)} \equiv N(t) \tag{2.81}$$

We can calculate the number of e-folds from the shape of the potential $V(\phi)$ and the value of $\phi(t)$ at t, since

$$\begin{aligned} N(t) &= \int_t^{t_{end}} Hdt = \int_\phi^{\phi_{end}} \frac{H}{\dot{\phi}}d\phi = N(\phi) \\ \to N(\phi) &\approx -\int_\phi^{\phi_{end}} \frac{3H^2}{V'}d\phi = \frac{1}{M_p^2}\int_{\phi_{end}}^{\phi}\frac{V}{V'}d\phi \end{aligned} \tag{2.82}$$

(in which we used (2.65)), from a given value of ϕ untill its value at the end of inflation. We can also express the number of e-folds using (2.67)

$$N(\phi) \approx \frac{1}{M_p}\int_{\phi_{end}}^{\phi}\frac{d\phi}{\sqrt{2\epsilon_V}} \tag{2.83}$$

It is found that the Flatness and Horizon problems are solved for an inflation that lasts around 60 e-folds. In closing, the the first slow-roll parameter can be written in terms of number of e-folds as

$$\epsilon_H = -\frac{d\ln H}{dN} \tag{2.84}$$

using $dN = Hdt$.

The field ϕ that is responsible for the inflation in the early universe, is called the inflaton. The inflationary stage ends whenever the slow-roll conditions are not satisfied any more, and in most models, the inflaton beggins to fall towards the minimum of its potential. It oscillates at the bottom of the potential, and may decay into other particles, such as radiation or massive particles, both fermionic and bosonic [81, 82], and through a complicated process called reheating [86], finally the Standard Cosmology arises, and the universe continues to evolve according to this (Figure 2.1).

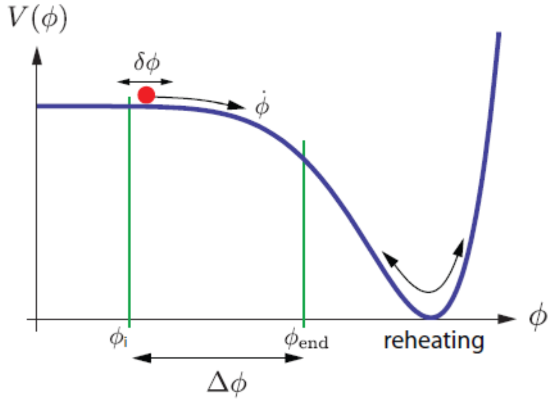

Figure 2.1: The inflaton field potential, where the inflationary stage is when its potential energy dominates its kinetic energy, and reaches to an end with the inflaton falling down at the minimum, where through reheating, its energy density is converted into radiation [11].

2.3 Models of Inflation

A wide variety of inflationary models has been proposed [101], with different theoretical motivations and observational predictions. In general, an inflationary model is defined by the specification of the kinetic term and the potential in the action, and its coupling to gravity, as well.

In the simplest case among the existing models, a single-field slow-roll inflation is considered, in which different choices of the potential define different inflationary models. The different models can be classified in a useful way depending on whether the inflaton is moving over a small or large distance during inflation, measured in Planck units: when the field is moving within a small, that is sub-Planckian, distance, $\Delta\phi \equiv \phi_i - \phi_{end} < M_p$, inflation is called small-field, and when it is moving over a large, super-Planckian distance, $\Delta\phi > M_p$, inflation is called large field. In general, the potential in small-field inflation models can be written approximately as

$$V(\phi) = V_0\left(1 - \left(\frac{\phi}{\mu}\right)^p\right) + ... \tag{2.85}$$

with higher order terms becoming important near the end of inflation and during reheating. An example of potential in small-field inflation, is the Higgs-like potential

$$V(\phi) = V_0\left(1 - \left(\frac{\phi}{\mu}\right)^2\right)^2 \tag{2.86}$$

and the famous Coleman-Weinberg potential [28, 29]

$$V(\phi) = V_0\left(\left(\frac{\phi}{\mu}\right)^4\left[\ln\left(\frac{\phi}{\mu}\right) - \frac{1}{4}\right] + \frac{1}{4}\right) \tag{2.87}$$

which arises as the potential for symmetry breaking in electroweak and grand unified theories. The prototype large-field inflationary model is the chaotic inflation model, with potential

$$V(\phi) = \lambda_p \phi^p \tag{2.88}$$

where, when q is an integer, it is called monomial inflation. Another case is the natural inflation with the following potential

$$V(\phi) = V_0\left(\cos\left(\frac{\phi}{f}\right) + 1\right) \tag{2.89}$$

which, depending on the parameter f, corresponds to a small or large-field inflation, it is more attractive however to be considered for large field variations. The potentials for small-field inflation arise in mechanisms of spontaneous symmetry breaking, where the inflaton field falls from an unstable minimum to a displaced vacuum. The small field models predict that the amplitude of the gravitational waves which are produced during inflation is too small to be detected, while super-Planckian field models predict that gravitational waves from inflation should be observed in the near future [11].

There are more complicated inflationary models that have been proposed, built in the following ways [50, 103, 104, 37]: With a non-minimal coupling to gravity, that is, between the inflaton and the metric, with modified gravity, in which the Einstein-Hilbert action is modified in high energies, with a non-canonical kinetic term, which is a function of the inflaton and its derivatives, and with more than one field being dynamically relevant during inflation [43].

2.3.1 Hybrid Inflation

We are particularly interesting in hybrid models of inflation [44, 46, 79], which belong to a class of multi-field models of inflation. The hybrid class of models is very promising as hybrid models are easily embedded in various high energy frameworks like Supersymmetry and Supergravity [102], GUT's and extra-dimensional theories. Below we present briefly the original hybrid model, which is the Hybrid Inflation proposed by A. Linde [46]. In this model, the effective potential is of the form

$$V(\phi,\sigma) = \frac{1}{4\lambda}(M^2 - \lambda\sigma^2)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}g^2\phi^2\sigma^2 \tag{2.90}$$

where ϕ is the inflaton scalar field, σ is a second scalar field included in the theory, m is the mass of the inflaton field, M is another scale mass, λ is the effective coupling constant that corresponds to σ, and g is the coupling constant which parametrizes the strength of the interactions between ϕ and σ. The local extrema of the potential (2.90) with respect to the scalar field σ, are:

$$\frac{dV}{d\sigma} = 0 \rightarrow$$

$$\sigma = 0, \qquad \text{and}$$

$$\sigma = \pm\sqrt{\frac{1}{\lambda}(M^2 - g^2\phi^2)} = \pm\sqrt{-\frac{m^2_{\sigma,eff}}{\lambda}} \qquad \text{or}$$

$$\sigma = \pm\sqrt{\frac{g^2}{\lambda}\left(\frac{M^2}{g^2} - \phi^2\right)} = \pm\sqrt{\frac{g^2}{\lambda}\left(\phi_c^2 - \phi^2\right)} \tag{2.91}$$

where we define $\phi_c \equiv M/g$ as a critical value of ϕ, and express $m^2_{\sigma,eff}(\phi) = -M^2 + g^2\phi^2$ as an effective squared-mass for σ. We observe that in this potential there are two different phases concerning the σ-direction:

• For values of ϕ above the critical, $\phi > \phi_c$, or for a positive effective squared-mass of σ, $m^2_{\sigma,eff}(\phi) > 0$, the minimum in the σ-direction is only the $\sigma = 0$ (Figure 2.2) and the potential has the symmetry $\sigma \leftrightarrow -\sigma$. From the form of the effective potential (2.90), we see that its curvature is larger in the σ-direction than in the ϕ-direction, so, for large values of ϕ, the σ field rolls down to its minimum, $\sigma = 0$, while ϕ is still large. Thus, we consider that the inflationary stage is at large values of ϕ and $\sigma = 0$ (this is the reason for the above definition of $m^2_{\sigma,eff}$), and the only responsible field for this stage is the inflaton field, with potential

$$V(\phi,0) = \frac{M^2}{4\lambda} + \frac{1}{2}m^2\phi^2 \tag{2.92}$$

which corresponds to a chaotic type of inflation. It is considered that $m^2\phi_c^2 \ll M^4/\lambda$ [46], or $M^2 \gg \lambda m^2/g^2$, and $m^2 \ll H^2$, and the equation of motion of the field ϕ during inflation is

$$3H\dot{\phi} + V'(\phi, 0) = 0 \rightarrow 3H\dot{\phi} = -m^2\phi \tag{2.93}$$

From (2.92) we see that inflation at its last stages is driven by the constant term

$$V(0,0) = \frac{M^2}{4\lambda} \tag{2.94}$$

• At the moment when ϕ becomes lower than its critical value, $\phi < \phi_c$, or when the effective squared-mass of σ becomes negative, $m^2_{\sigma,eff}(\phi) < 0$, a phase transition occurs by the symmetry breaking (Figure 2.2), and the σ-direction aquires the non vanishing minimum, $\sigma \neq 0$, given by (2.91). Exaclty when the value of the inflaton becomes equal to its critical, we have

$$V(\phi_c, 0) = \frac{M^2}{4\lambda} + \frac{1}{2}m^2\phi_c^2 \tag{2.95}$$

and considering $m^2\phi_c^2 \ll M^4/\lambda$, this becomes

$$H^2 = \frac{2\pi M^4}{3\lambda \tilde{M}_p^2} \tag{2.96}$$

where $\tilde{M}_p^2 = 1/G$ is the reduced Planck mass, which also gives that $m^2 \ll H^2 \rightarrow M^2 \gg \sqrt{(3\lambda/2\pi)}m\tilde{M}_p^2$. The time of the phase transition is expressed as $\Delta t = H^{-1} = \sqrt{(3\lambda/2\pi)}\tilde{M}_p/M^2$, and the change of the field ϕ is given by $\Delta\phi = (V'/3H)\Delta t = \lambda m^2\tilde{M}_p^2/2\pi g M^3$. Furthemore, the absolute value of the negative effective squared-mass of σ is

$$\left|m^2_{\sigma,eff}(\phi)\right| = \frac{\lambda m^2 \tilde{M}_p^2}{\pi M^2} = \frac{2}{3}\frac{m^2 M^2}{H^2} \tag{2.97}$$

and we have

$$\frac{\left|m^2_{\sigma,eff}(\phi)\right|}{H^2} = \frac{2}{3\pi^2}\left(\frac{\lambda m \tilde{M}_p}{M^3}\right)^2 \tag{2.98}$$

which means that $\left|m^2_{\sigma,eff}(\phi)\right| \gg H^2$ when $M^3 \ll \lambda m\tilde{M}_p^2$. In this case the scalar field σ falls down to its non-vanishing minimum within the time $\Delta t = H^{-1}$. The inflaton field falls to its minimum in a time much smaller than the latter if $M^3 \ll \sqrt{\lambda} g m \tilde{M}_p^2$ [46], so inflation ends almost instantly when ϕ reaches its critical value (Figure 2.2). This is why the conditions $M^3 \ll \lambda m\tilde{M}_p^2$ and $M^3 \ll \sqrt{\lambda} g m\tilde{M}_p^2$ are called waterfall conditions, and the responsible field for the fast phase transition and thus the immediate ending of inflation, σ, is called the waterfall field.

In general, in the models of inflation there is a problem according to how the inflationary stage ends. In Hybrid Inflation, there is the waterfall field, which is a second additional field to the already existing inflaton field, whose potential becomes steep and thus, drives infllation to an end. Moreover, the energy scale of hybrid inflation can be low, so that one does not need super-Planckian field values. Finally, this scenario is also interesting from the point of view of microwave background anisotropies and large scale structure, as it provides natural models for tilted primordial spectra of density perturbations [105]. Extensions of the hybrid inflation scenario are quite popular in the context of Supergravity and String Cosmology [11].

2.3.2 Inflation in String Theory

Below, we describe in brief the main interest for the study of Cosmological Inflation in String Theory, and the most important feautures that a successful effective field theory must have. (More details can be found in [61, 110, 111] and references therein).

String Theory in 10 or 11 dimensions is for now the only consistent quantum theory that unifies the four fundamental interactions, that is, a theory for Quantum Gravity, so the only possible UV completion of Effective

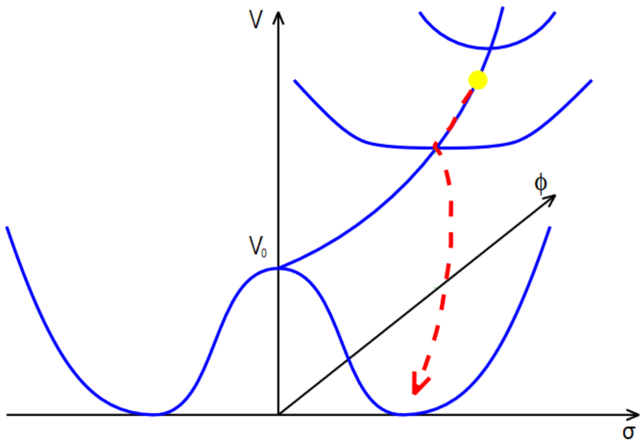

Figure 2.2: The hybrid potential, with firstly one minimum in the σ-deriection, an then the two new vacua after the phase transition. The inflationary pase in the ϕ-direction ends almost immediately when the phase transition happens [35].

Quantum Field Theories that describe the low-energy physics phenomena. The more than four extra dimensions in String Theory must be compactified in order to attain a 4-dimensional effective field theory, and EQFTs in general, should contain the observed positive tiny value for the cosmological constant or vacuum energy.

However, from deformations of the compactification manifold, massless scalar fields appear in the spectrum, know as moduli, which may have some cosmological consequences. Also, it is not clear whether the resulting effective potential from compactification of additional dimensions, has any de-Sitter vacua, that is, a positive cosmological constant. Moreover, it is possible that not all consistent EQFTs can be embeded in the String Theory landscape [112], which is the set of solutions of String Theory, making up the out of it set known as Swampland [106, 107, 108, 113]. The latter may come from the fact that quantum string corrections are not considered in the resulting effective potential after compactification. The Swampland criteria are inconsistent with the cosmological constant representing the dark energy, and with slow roll inflation, as well [109]. To summarize, in order to obtain a consistent and realistic effective field theory, one must at first ensure positive square-masses for all the moduli fields, which is called moduli stabilization, and a de-Sitter minimum of the resulting effective potential. This thesis focuses on the study of implementation of the Hybrid Inflation scenario of Cosmology, in the context of type IIB Effective String Theory constructions.

Chapter 3

Hybrid Inflation in String Theory from D7-branes

In this work we are focusing on the model of hybrid inflation through the study of the proposed inflationary scenario in [62], which is realized within type IIB effective string theory constructions [94, 66] and a geometric configuration of intersecting $D7$ branes [63, 67].

The geometric set up in [62], consists of three magnetised $D7$ brane stacks mutually orthogonal in the internal 6-dimensional space, with the magnetic fields turned on along U(1) directions on their internal worldvolumes [63]. The effects of a new 4-dimensional Einstein-Hilbert term are considered, which is localized in the internal space and is generated from higher derivative terms in the 10-dimensional string effective action [64, 65]. Turns out that logarithmic dependend corrections are induced to the effective scalar potential [60, 63, 67, 99], and slow-roll inflation is realized with the inflaton field being proportional to the logarithm of the internal volume modulus (compactification volume) [60, 96]. Moreover, charged matter fields coming from the magnetic fluxes and are located at an intersection of the $D7$ branes stacks [68, 69], play the role of waterfall fields, ending the inflationary phase, and lowering the vacuum of the theory so that it finally can be tuned to have the observed value of the cosmological constant.

In this chapter, we firstly study the effective scalar potential of the inflaton field seperately, in the Large Volume Scenario [66], which is one of the moduli stabilization schemes, and then we are concentrating in the prosposed hybrid potential with the inflaton and one waterfall field only, in the LVS, in order to study the new vacuum of the theory, making use of the program Mathematica.

3.1 Inflaton scalar potential

In the configuration considered, there are the three mutually orthogonal $D7$ brane stacks, denoted as $D7_i$, and the three Kähler moduli $\mathcal{T}_i$ [72], with $i = 1, 2, 3$. Each $D7$ brane stack spans four compact dimensions while it is localised at the remaining two [73, 74] (Table 3.1). The two basic ingredients at effective field theory level are the superpotential of the moduli fields $\mathcal{W}$ and the Kähler potential $\mathcal{K}$. In this section, we minimise the

$D7_i$	4d Minkowski				6 Compact Dimensions					
	0	1	2	3	4	5	6	7	8	9
$D7_1$		$*$	$*$	$*$	.	.	$*$	$*$	$*$	$*$
$D7_2$		$*$	$*$	$*$	$*$	$*$	.	.	$*$	$*$
$D7_3$		$*$	$*$	$*$	$*$	$*$	$*$	$*$	.	.

Table 3.1: D7-brane configuration of the model. For expample, $D7_1$ stack resides in '6''7''8''9' internal (compact) dimensions and intersects with $D7_3$ along '6''7' and with $D7_2$ along '8''9'.

resulting effective scalar potential of the inflaton field in the large volume limit to fix the ratios of the internal wolrdvolumes of the three D7-brane stacks, τ_i (in string units), which are the imaginary parts of $\mathcal{T}_i$ (and become interchangable with $\mathcal{T}_i$). In the first subsection, we study the behavior of the local extrema of the model and of the potential values at them, and investigate the existence of de-Sitter vacua. In the second subsection, we study the potential with the required parameter values inserted for agreement with observations.

After the incorporation of the new type of radiative corrections in the Kähler potential, the latter takes the form [62, 63, 67, 72]

$$\begin{aligned}\mathcal{K}(\tau_i) &= -2\ln\left(\sqrt{\tau_1\tau_2\tau_3} + \xi + \sum_{i=1}^{3}\gamma_i \ln\tau_i\right) \\ &= -2\ln(\mathcal{V} + \xi + \gamma\ln\mathcal{V}) \end{aligned} \tag{3.1}$$

(working in $\hbar = c = 1$) where $\mathcal{V} = \sqrt{\tau_1\tau_2\tau_3}$ is the 6-dimensional internal volume, $\gamma_i \equiv \gamma/2$ (as the same tension $T_i \equiv T = e^{-\Phi}T_0$ for all the brane stacks is assumed for simplicity [62]), and γ and ξ are parameters given by [67, 65]

$$\gamma = -\frac{1}{2}g_s T_0 \xi, \tag{3.2}$$

$$\xi = -\frac{\chi}{4}\times\begin{cases}\frac{\pi^2}{3}g_s^2 & \text{for orbifolds}\\ \zeta(3) & \text{for smooth Calabi-Yau threefold}\end{cases} \tag{3.3}$$

where χ is the Euler characteristic and g_s is the string coupling.

These new corrections induce a non zero F-term in the effective scalar potential, which also recieves contributions from D-terms associated with universal $U(1)$ factors of the D7-brane stacks [60], and thus has the form

$$V_{eff}(\mathcal{V}) = V_F + V_D \tag{3.4}$$

The above effective potentials V_F and V_D in large volume limit, are [60, 95]

$$V_F \approx \frac{3\mathcal{W}_o^2}{2\kappa^4\mathcal{V}^3}(2\gamma(\ln\mathcal{V} - 4) + \xi) \tag{3.5}$$

and

$$V_D \approx \frac{d_1}{\kappa^4\tau_1^3} + \frac{d_2}{\kappa^4\tau_2^3} + \frac{d_3}{\kappa^4\tau_3^3} \tag{3.6}$$

where $\mathcal{W}_o$ is a flux-dependent constant in which the superpotential has been reduced to [70, 71], $\kappa = \sqrt{8\pi G_N}$ is the reduced Planck length, and d_i for $i = 1, 2, 3$ are model-dependent constants related to $U(1)$ Fayet–Iliopoulos (FI) terms [60].

Using $\mathcal{V} = \sqrt{\tau_1\tau_2\tau_3}$, we express (3.6) as

$$V_D \approx \frac{d_1}{\kappa^4\tau_1^3} + \frac{d_2}{\kappa^4\tau_2^3} + \frac{d_3\tau_1^3\tau_2^3}{\kappa^4\mathcal{V}^6} \tag{3.7}$$

and minimising with respect to τ_i, we get

$$\begin{aligned}\frac{dV_D}{d\tau_1} &= 0 \Rightarrow \tau_1^3 = \left(\frac{d_1}{d_3}\right)^{1/2}\frac{\mathcal{V}^3}{\tau_2^{3/2}}, \\ \frac{dV_D}{d\tau_2} &= 0 \Rightarrow \tau_2^3 = \left(\frac{d_2}{d_3}\right)^{1/2}\frac{\mathcal{V}^3}{\tau_1^{3/2}}\end{aligned} \tag{3.8}$$

Combining these two, we have

$$\tau_1^3 = \left(\frac{d_1^2}{d_2 d_3}\right)^{1/3} \mathcal{V}^2,$$

$$\tau_2^3 = \left(\frac{d_2^2}{d_1 d_3}\right)^{1/3} \mathcal{V}^2 \tag{3.9}$$

and plugging (3.9) into (3.7), we have

$$V_D \simeq 3\frac{(d_1 d_2 d_3)^{1/3}}{\kappa^4 \mathcal{V}^2} = \frac{d}{\kappa^4 \mathcal{V}^2} \tag{3.10}$$

with $d = 3(d_1 d_2 d_3)^{1/3}$. Finally, the effective scalar potential (3.4) in the large volume limit, after the minimization of τ_i, is

$$\begin{aligned} V_{eff}(\mathcal{V}) &= V_F + V_D \\ &\simeq \frac{3\mathcal{W}_o^2}{2\kappa^4\mathcal{V}^3}(2\gamma(\ln\mathcal{V} - 4) + \xi) + \frac{d}{\kappa^4\mathcal{V}^2} \\ &= -\frac{3\mathcal{W}_o^2\gamma}{\kappa^4}\left(-\frac{\ln\mathcal{V} - 4}{\mathcal{V}^3} - \frac{\xi}{2\gamma\mathcal{V}^3} - \frac{d}{3\mathcal{W}_o^2\gamma\mathcal{V}^2}\right) \\ &= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right) \end{aligned} \tag{3.11}$$

with

$$C \equiv -3\mathcal{W}_o^2\gamma, \quad q \equiv \frac{\xi}{2\gamma}, \quad \sigma \equiv \frac{2d}{9\mathcal{W}_o^2\gamma} = -\frac{2d}{3C} \tag{3.12}$$

As we mentioned in the begining of this chapter, the role of the inflation field is being played by a proportional to the logarithm of the internal volume $\mathcal{V}$, quantity. The exact canonically normalized one is [62]

$$\frac{\phi}{\kappa} \equiv \frac{\sqrt{6}}{3\kappa}\ln\mathcal{V} \tag{3.13}$$

so replacing $\mathcal{V} = e^{\sqrt{\frac{3}{2}}\phi}$ in (3.11) results in the expression of the scalar potential in terms of the canonically normalized field ϕ,

$$V(\phi) \simeq -\frac{C}{\kappa^4}e^{-3\sqrt{\frac{3}{2}}\phi}\left(\sqrt{\frac{3}{2}}\phi - 4 + q + \frac{3}{2}\sigma e^{\sqrt{\frac{3}{2}}\phi}\right) \tag{3.14}$$

To ensure a de-Sitter vacuum of this potential [60, 94, 96], that is a positive potential minimum value, the parameter γ must be negative, so as we see from the relations (3.12), the constant C is positive, and the parameter q is negative. Also, the parameter d is always positive [61], so the parameter σ is negative and hence is expected to uplift the minimum of this potential so that it becomes positive [95].

3.1.1 Local de-Sitter Minimum

Now we can study the effective scalar potential of the theory in order to search for any de-Sitter vacua. The first and second derivatives of the potential (3.14) with respect to the canonically normalized field ϕ are

$$V'(\phi) \simeq -3\sqrt{\frac{3}{2}}\frac{C}{\kappa^4}e^{-3\sqrt{\frac{3}{2}}\phi}\left(-\sqrt{\frac{3}{2}}\phi + \frac{13}{3} - q - \sigma e^{\sqrt{\frac{3}{2}}\phi}\right) \tag{3.15}$$

$$V''(\phi) \simeq -\frac{27}{2}\frac{C}{\kappa^4}e^{-3\sqrt{\frac{3}{2}}\phi}\left(+\sqrt{\frac{3}{2}}\phi - \frac{14}{3} + q + \frac{2}{3}\sigma e^{\sqrt{\frac{3}{2}}\phi}\right) \tag{3.16}$$

Taking (3.15) to equal zero, we have

$$e^{-\sqrt{\frac{3}{2}}\phi}\left(\sqrt{\frac{3}{2}}\phi+q-\frac{13}{3}\right)=-\sigma \tag{3.17}$$

Setting the quantity of the above parenthesis as

$$\sqrt{\frac{3}{2}}\phi+q-\frac{13}{3}\equiv m\rightarrow e^{-\sqrt{\frac{3}{2}}\phi}=e^{-m+q-\frac{13}{3}} \tag{3.18}$$

equation (3.17) can be written as

$$me^{-m+q-\frac{13}{3}}=-\sigma$$

$$\Rightarrow -me^{-m}=\sigma e^{-q+\frac{13}{3}}$$

$$\Rightarrow -me^{-m}=-e^{\ln(-\sigma)}e^{-q+\frac{13}{3}} \tag{3.19}$$

Equation (3.19) is of the form $ye^y=z$ which for real y and z is only solved if $z\geq-\frac{1}{e}$; we have $y=W_0(z)$ for $z\geq 0$, and $y=W_0(z)$ and $y=W_{-1}(z)$ for $-\frac{1}{e}\leq z<0$, where $W_{0,-1}$ are the two branches of the Lambert W function (product logarithm) [57, 58, 59]. Defining

$$x\equiv q-\frac{16}{3}-\ln(-\sigma)\rightarrow\ln(-\sigma)-q+\frac{13}{3}=-1-x\rightarrow\sigma=-e^{q-\frac{16}{3}-x} \tag{3.20}$$

equation (3.19) becomes

$$-me^{-m}=-e^{-x-1} \tag{3.21}$$

and as $-e^{-1}\leq -e^{-x-1}<0$, is solved by

$$-m=W_0(-e^{-x-1})$$

$$-m=W_{-1}(-e^{-x-1}) \tag{3.22}$$

From (3.22), inserting (3.18), we find that the local extrema of the potential are at

$$\phi_-=-\sqrt{\frac{2}{3}}\Big(q-\frac{13}{3}+W_0(-e^{-x-1})\Big)$$

$$\phi_+=-\sqrt{\frac{2}{3}}\Big(q-\frac{13}{3}+W_{-1}(-e^{-x-1})\Big) \tag{3.23}$$

where ϕ_- and ϕ_+ are the local minimum and local maximum, respectively, with $\phi_-<\phi_+$,:

$$V''(\phi_-)=\frac{9C}{2\kappa^4}e^{3q-13+3W_0(-e^{-x-1})}\Big(1+W_0(-e^{-x-1})\Big)\geq 0$$

$$V''(\phi_+)=\frac{9C}{2\kappa^4}e^{3q-13+3W_{-1}(-e^{-x-1})}\Big(1+W_{-1}(-e^{-x-1})\Big)\leq 0 \tag{3.24}$$

In the above, we put (3.23) into (3.15) and (3.16). Equalities hold for $x=0$, and in order to find the second derivative of the potential at the local extrema we also used the identity [55]

$$e^{-W_k(z)}=\frac{W_k(z)}{z}\quad\text{for } k=0,-1 \tag{3.25}$$

From the relations (3.23) we observe that varying the parameter q while x is kept constant, just shifts the local extrema $\phi_{-,+}$. In particular, by increasing the value of q, $\phi_{-,+}$ are driven towards lower values. Additionally,

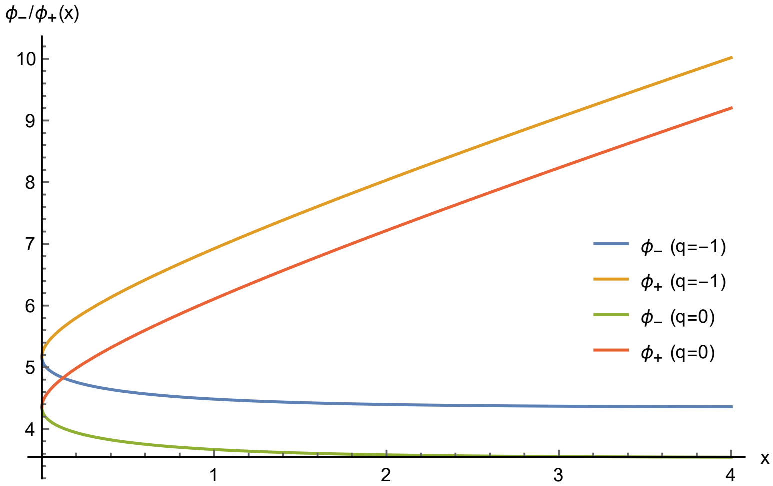

Figure 3.1: Local extrema (3.23) of the effective scalar potential in terms of x, for $q = 0$ and $q = -1$. We observe that the minimum is in the W_0-branch and the maximum in the W_{-1}-branch, of the Lambert W function.

because of the fact that particularly $-\phi_-$ and $-\phi_+$ lies in the W_0 and W_{-1}-branch, respectively, of the W function, we see that an increasing value of the parameter x moves the local minimum and maximum, at lower and higher values, respectively (Figure 3.1). Finally, from (3.14), for different values of the parameter q (and x=constant) the potential is decreasing at different rates: for smaller q-values the potential becomes steeper. In conclusion, the only true parameter of the model is the parameter x, while variation of the parameter q changes the origin of the field, and constant C rescales the potential.

From (3.23) we can also find the useful expression of the local extrema through the volume $\mathcal{V}$

$$\mathcal{V}_- = e^{\sqrt{\frac{3}{2}}\phi_-} = e^{-q} e^{\frac{13}{3} - W_0(-e^{-x-1})}$$

$$\mathcal{V}_+ = e^{\sqrt{\frac{3}{2}}\phi_+} = e^{-q} e^{\frac{13}{3} - W_{-1}(-e^{-x-1})} \tag{3.26}$$

from which it is also observed that for large negative values of the parameter q, large volumes are reached, for a given value of x, and thus, from (3.2) into (3.12), the weak coupling and the large volume limit are related in a natural way [60, 62].

Moving on, to the study of the potential at its local extrema, with the help of (3.25) again, we plug (3.23) into (3.14), and have

$$V(\phi_-) = -\frac{C}{6\kappa^4} e^{3q-13+3W_0(-e^{-x-1})} \left(2 + 3W_0(-e^{-x-1})\right)$$

$$V(\phi_+) = -\frac{C}{6\kappa^4} e^{3q-13+3W_{-1}(-e^{-x-1})} \left(2 + 3W_{-1}(-e^{-x-1})\right) \tag{3.27}$$

From variation of the parameter x (and different values of the q), we find that the minimum value of the potential, (3.27), while x is increasing, starts positive, then becomes zero and continues towards negative values (Figure 3.2). The critical value of the parameter x where the minimum value of the potential is zero (same critical value regardless the value of the q), using the program Mathematica, is found to be

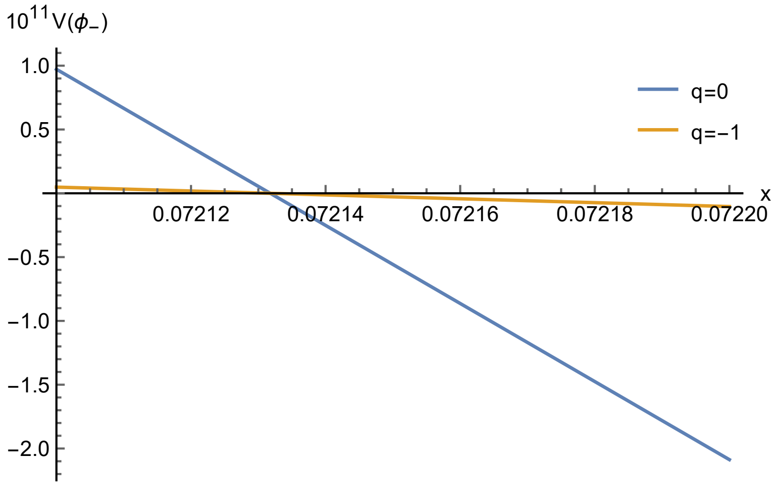

Figure 3.2: Minimum value (3.27) of the effective scalar potential in terms of x, for $q = 0$ and $q = -1$.

$$V(\phi_-) = 0 \to x_c \simeq 0.0721318 \tag{3.28}$$

Taking for simplicity $q = 0$, since the parameter q does not affect the properties of the inflationary stage, we can plot (in Mathematica) the behavior of the function $V(\phi)$ (3.14) in defferent regions of the parameter x. We indeed see that for positive values of the x below the critical, $0 < x < x_c$, the potential has a positive minumum value and thus the theory has a de-Sitter vacuum, which corresponds to a positive cosmological constant (vacuum energy). At the critical value $x = x_c$, the minimum of the potential is zero, so this case corresponds to a model with a Minkowski vacuum (zero cosmological constant), and for values bigger than the critical, $x > x_c$, the potential has a negative minimum, so this case corresponds to an Anti-de-Sitter vacuum (negative cosmological constant). For negative values, $x \leq 0$, the potential exhibits no local extrema (the two branches of the Lambert function join). All these cases are shown in Figure 3.3. Concerning the maximum value of the potential, (3.27), it keeps decreasing while x is increasing.

3.1.2 Consistency with observations

Below, we find some useful and important quantities of the inflationary phase: the distance between the positions of the local extrema, from (3.23), is

$$\phi_+ - \phi_- = \sqrt{\frac{2}{3}}\Big(W_0(-e^{-x-1}) - W_{-1}(-e^{-x-1})\Big) \tag{3.29}$$

The potential slow roll parameter η (with $M_p^2 = 1$) at the local extrema, using (3.24) and (3.27), is

$$\eta_- = \frac{V''(\phi_-)}{V(\phi_-)} = -9\frac{1 + W_0(-e^{-x-1})}{\frac{2}{3} + W_0(-e^{-x-1})}$$

$$\eta_+ = \frac{V''(\phi_+)}{V(\phi_+)} = -9\frac{1 + W_{-1}(-e^{-x-1})}{\frac{2}{3} + W_{-1}(-e^{-x-1})} \tag{3.30}$$

and the ratio of the potential value at the maximum towards the one at the minimum, using (3.27), and the help of (3.25), is

$$\frac{V(\phi_+)}{V(\phi_-)} = \frac{2 + 3W_{-1}(-e^{-x-1})}{2 + 3W_0(-e^{-x-1})}\left(\frac{W_0(-e^{-x-1})}{W_{-1}(-e^{-x-1})}\right)^3 \tag{3.31}$$

We observe that the above three quantities are all expressed in terms of the parameter x only [60].

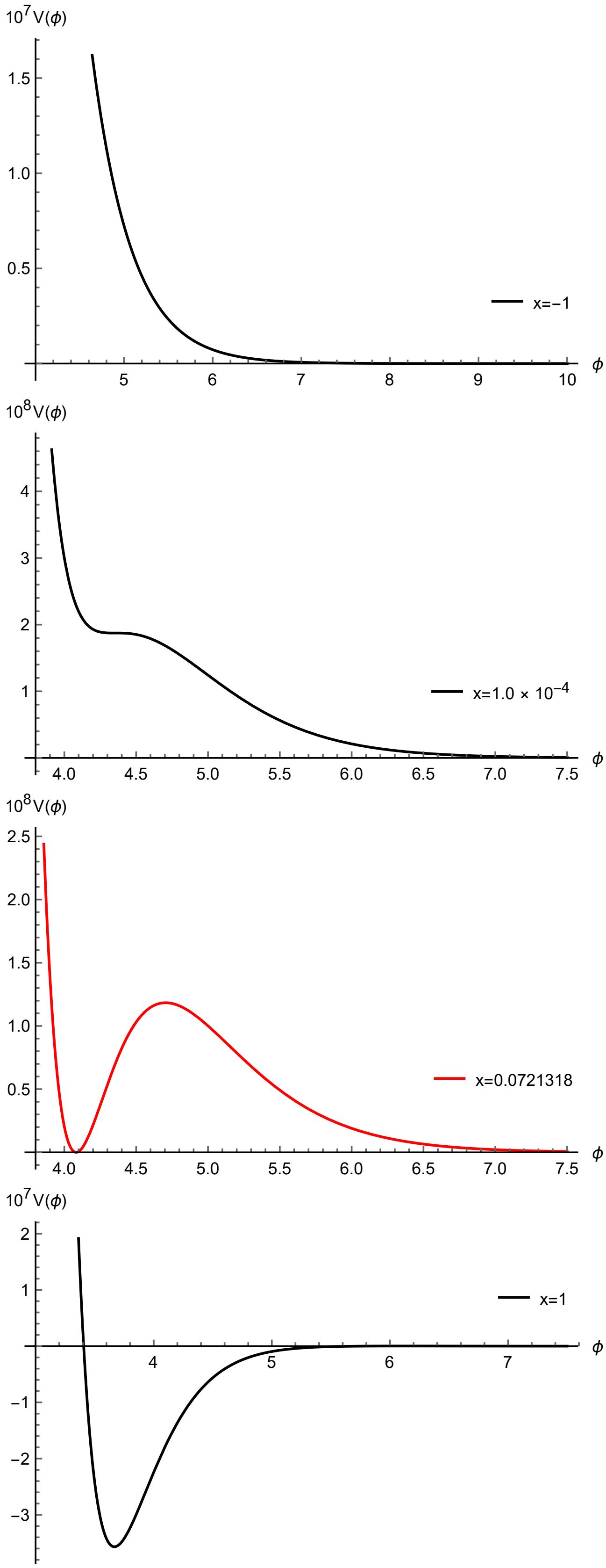

Figure 3.3: Effective scalar potential (3.14) of the inflaton field in terms of ϕ, for different values of x.

It has been shown that some well known inflationary scenarios, cannot be realized in this model [60]. More general inflationary scenarios have been studied, scaning the space of the parameter x: for a given value of x, evolution equation for the Hubble parameter is solved and slow-roll parameters and number of e-folds are computed. A new scenario appears, in which the inflaton field starts rolling down the potential from a point near the maximum with $\eta(\phi_+) < -0.02$, and no initial speed (if one considers that this maximum has to do with a symmetry restoration [60, 62]). It passes from the horizon exit point ϕ^*, where $\eta(\phi^*) = -0.02$ in order to to agree with the data [60], before the inflection point (where the second derivative of the potential changes sign). Potential slow roll parameters satisfy $\epsilon \ll |\eta|$ in the field space $[\phi_-, \phi_+]$ [60]. The number of e-folds are obtained from horizon exit point to the minimum of the potential, while most of them are obtained around the minimum [60], in contrast with other inflationary scenarios. The required $N \simeq 60$ e-folds are achieved for $x \simeq 3.3 \times 10^{-4}$ [60]. Futhermore, the distance between the two extrema is $\phi_+ - \phi_- \simeq 0.042$ and the inflaton field displacement is $\Delta\phi = \phi^* - \phi_- \simeq 0.02$ [60] which corresponds to a small field inflation and therefore is compatible with the validity of the effective field theory. The most significant problem that arises is that the value of the potential at the minimum is of the same order of the inflation scale, $V(\phi_-) \simeq V(\phi^*)$ and therefore is very shallow, and thus the aquired value of the vacuum of the theory [60, 61, 62] is high above the observed value of the cosmological constant.

About this work, for the theory to be consistent with a de-Sitter vacuum, first of all we require $0 < x < x_c$, as we showed before, which, using (3.20), gives us

$$0 < x < x_c \tag{3.32}$$

$$\rightarrow -\frac{3}{2}e^{-\frac{16}{3}} < \frac{3}{2}\sigma < -\frac{3}{2}e^{-\frac{16}{3}-x_c}$$

$$\rightarrow -0.00724192 < s < -0.00673795 \tag{3.33}$$

if we assume $q = 0$. The above s is a useful parameter defined by

$$s \equiv \frac{3}{2}\sigma \tag{3.34}$$

Then, we are mostly interested in the value $x \simeq 3.3 \times 10^{-4}$, in order to be consistent with slow roll inflation which is compatible with observations, as well. This particular value of x corresponds to $s \simeq -0.00723954$. Also, the value of the constant C is fixed by observational constraints to be $C \simeq e^{-3q}7.81 \times 10^{-4}$ [60, 62]. Taking the simple expression of the potential in terms of the volume $\mathcal{V}$, (3.11),

$$\begin{aligned} V(\mathcal{V}) &= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right) \\ &= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2}\right) \end{aligned} \tag{3.35}$$

we can study its behavior for the above particular values of x and C. From the program Mathematica, taking $q = 0$ and $\kappa^2 = 1$, we find that its local extrema are at (Figure 3.4)

$$V' = 0 \rightarrow \quad \mathcal{V}_- \simeq 201.923 \quad (\phi_- \simeq 4.33387),$$

$$\mathcal{V}_+ \simeq 212.559 \quad (\phi_+ \simeq 4.37578) \tag{3.36}$$

which correspond to the following potential values

$$V(\mathcal{V}_-) \simeq 1.46034 \times 10^{-11}, \qquad V(\mathcal{V}_+) \simeq 1.46064 \times 10^{-11} \tag{3.37}$$

The inflection point is at

$$V'' = 0 \rightarrow \quad \mathcal{V}_{infl} \simeq 206.923 \quad (\phi_{infl} \simeq 4.35384): \quad V(\mathcal{V}_{infl}) \simeq 1.46048 \times 10^{-11} \tag{3.38}$$

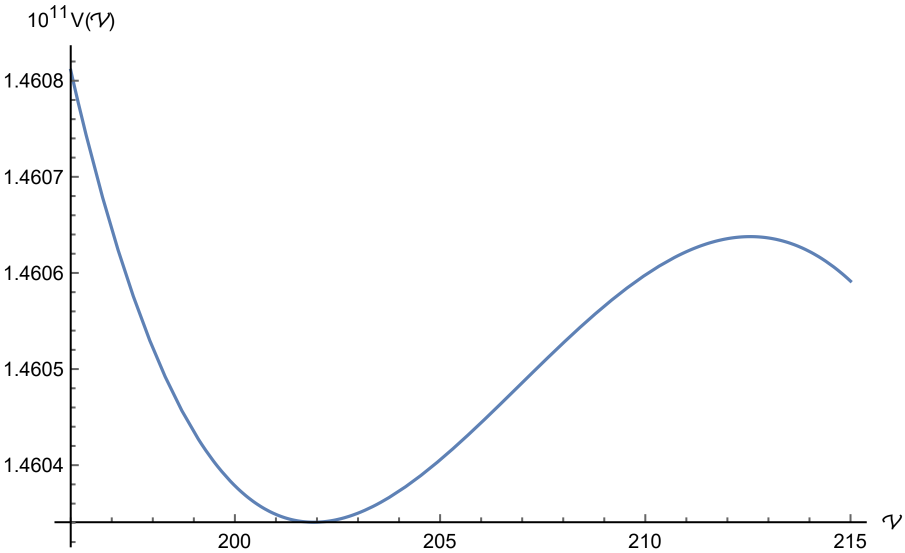

Figure 3.4: Effective scalar potential (3.11) of the inflaton field in terms of $\mathcal{V}$, when $q = 0$, for the desired values $x = 3.3 \times 10^{-4}$ and $C = 7.81 \times 10^{-4}$.

and the distance between the extrema (3.36) is

$$\Delta\mathcal{V} = \mathcal{V}_+ - \mathcal{V}_- \simeq 10.641 \quad (\text{or } \ \Delta\phi = \phi_+ - \phi_- \simeq 0.04191) \tag{3.39}$$

(the value of $\Delta\phi = \phi_+ - \phi_-$ through the relation (3.29) is $\Delta\phi =\simeq 0.0419531$, for $x \simeq 3.3 \times 10^{-4}$). Also, the ratio of the potential extrema values (3.37) is

$$\frac{V(\mathcal{V}_+)}{V(\mathcal{V}_-)} \simeq 1.0002 \tag{3.40}$$

(the same through (3.31), for $x \simeq 3.3 \times 10^{-4}$). From the above, we see that the minimum value of the inflaton potential is of the same order of its maximum value and not low enough to agree with the observed value of the cosmological constant, which has to be

$$\Lambda \approx 10^{-120} M_{Pl}^4 = 10^{-120} \frac{1}{\kappa^4} \tag{3.41}$$

Also, the parameters η are

$$\eta(\phi_-) \simeq 0.743867, \quad \eta(\phi_+) \simeq -0.648478 < 0$$

(and $\eta(\phi_-) \simeq 0.744613$, $\eta(\phi_+) \simeq -0.649136$, if we find them directly from (3.30), for $x \simeq 3.3 \times 10^{-4}$).

3.2 Hybrid potential

The proposed solution [62] to the above high value of the potential minimum is through the model of Hybrid Inflation, which includes the posibility of having a second field in the theory, the waterfall field, that drives inflation to an end, while it falls to another, lower than the inflaton's potential minimum, and therefore may help the vacuum to reach the value (3.41).

Possible candidates for the waterfall field in this theory are the aforementioned charged matter fields coming from the introduction of magnetic fields on the $D7$ brane stacks. They correspond to excitations of open strings that end on the $D7$-brane stacks or their intersections. The (squared) mass of a charged open string scalar field can have two types of contributions, that may be able to differ in their signs and dependence on the internal volume $\mathcal{V}$; this means that tachyonic fields are possible to appear, in different values of the volume $\mathcal{V}$. In the first constructed model in [62], the masses of all charged open string states are positive in the large volume limit, and only one charged open string scalar, coming from the $D7_2$ brane stack, becomes tachyonic below a critical value of $\mathcal{V}$, which can be chosen to be around the minimum of the inflaton potential.

In this section, we minimise the resulting effective scalar potential of the inflaton and the waterfall field (in the large volume limit) to fix the ratios of the internal areas moduli $\mathcal{A}_i$, which are equivalent to the Kähler moduli τ_i, and then, using the program Mathematica, we try to find the contributions of the two fields for which the new vaccum takes the lower possible value.

3.2.1 Stabilization

In [62], a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold on a factorized 6-torus into 2-tori $T^6 = T^2 \times T^2 \times T^2$ is considered. $\mathcal{A}_i$ is defined as the unit cell area of the i-th torus T_i^2. The quantised magnetic fields $H_a^{(i)}$ on the $D7_a$ brane stack in the i-th internal plane, are described through the following D-term [62] in the effective scalar potential

$$V_D = \sum_a \frac{g^2_{U(1)_a}}{2} \left(\xi_a + \sum_n q_a^n \, |\varphi_a^n|^2 \right)^2 + ... \tag{3.42}$$

where $g^2_{U(1)_a}$ are the gauge couplings, ξ_a the Fayet-Iliopoulos parameters, φ_a the charged scalar matter fields, and q_a are their charges. As we would like to study the waterfall direction, in the above we keep only the canonically normalised tachyonic field φ_- and its charge conjugate φ_+, which have charges $q_a = \pm 2$ respectively, and put the other massive scalar fields to zero

$$\begin{aligned} V_D &= \sum_{a=1,3} \frac{g^2_{U(1)_a}}{2} \xi_a^2 + \frac{g^2_{U(1)_2}}{2} \left(\xi_2 + 2\,|\varphi_+|^2 - 2\,|\varphi_-|^2 + ... \right)^2 + ... \\ &= \sum_{a=1,3} \frac{g^2_{U(1)_a}}{2} \xi_a^2 + \frac{g^2_{U(1)_2}}{2} \xi_2^2 + 2 g^2_{U(1)_2} \xi_2 \big(|\varphi_+|^2 - |\varphi_-|^2 \big) + 2 g^2_{U(1)_2} \big(|\varphi_+|^2 - |\varphi_-|^2 \big)^2 + ... \end{aligned} \tag{3.43}$$

The magnetic field contribution to the mass of the matter fields in the large volume (small magnetic field) limit is [62]

$$m^2_{H_2} \equiv 2 g^2_{U(1)_2} \xi_2 = \frac{2 \big| \zeta_2^{(3)} \big|}{\alpha'} \approx \frac{2 \big| k_2^{(3)} \big|}{\pi \alpha'} \frac{\alpha'}{\mathcal{A}_3} \approx \frac{2 \big| k_2^{(3)} \big|}{\pi} \frac{g_s^2}{\kappa^2 \mathcal{V}} \frac{\alpha'}{\mathcal{A}_3} \tag{3.44}$$

where $\zeta_2^{(3)}$ is the oscillator shift, α' is the string Regge slope [65], $\mathcal{A}_3$ is the unit cell area of the 3rd torus T_3^2, and $k_2^{(3)} = n_2^{(3)} / m_2^{(3)}$ is the magnetic flux which is defined as the ratio of the flux number $n_2^{(3)}$ towards the wrapping number $m_2^{(3)}$ of the $D7_2$ brane on the 3rd torus T_3^2 and comes from the quantisation of the magnetic field. In the above, it is used that $\zeta_2^{(3)}$ in the large volume limit is [62]

$$\zeta_2^{(3)} \approx \frac{\alpha' k_2^{(3)}}{\pi \mathcal{A}_3} \tag{3.45}$$

and also that [62]

$$\frac{1}{\kappa^2} = \frac{\mathcal{V}}{\alpha' g_s^2} \tag{3.46}$$

The gauge couplings in the large volume limit are [62]

$$\frac{1}{g^2_{U(1)_a}} \approx \big| m_a^{(j)} m_a^{(k)} \big| \frac{\mathcal{A}_j \mathcal{A}_k}{g_s \alpha'^2} = \big| m_a^{(j)} m_a^{(k)} \big| \frac{\mathcal{V}}{g_s} \frac{\alpha'}{\mathcal{A}_a}, \quad \text{with} \quad \alpha \neq j \neq k \neq \alpha \tag{3.47}$$

Using the expression (3.44) and (3.47), the Fayet-Iliopoulos parameter ξ_2 in terms of the volume $\mathcal{V}$, is

$$\xi_2 = \frac{m_{H_2}^2}{2g_{U(1)_2}^2} \approx \frac{\left|k_2^{(3)}\right| g_s \alpha'^2 \left|m_2^{(1)} m_2^{(3)}\right|}{\pi \kappa^2 \mathcal{A}_2 \mathcal{A}_3}$$

$$= \left|m_2^{(1)} m_2^{(3)}\right| \frac{g_s \left|k_2^{(3)}\right|}{\pi \kappa^2 \mathcal{V}} \frac{\mathcal{A}_1}{\alpha'} \tag{3.48}$$

where in the last line we used [62]

$$\mathcal{V} = \sqrt{\tau_1 \tau_2 \tau_3} = \frac{\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3}{\alpha'^3} \tag{3.49}$$

We also have

$$m_{H_1}^2 \equiv 2g_{U(1)_1}^2 \xi_1 \approx \frac{2\left|k_1^{(2)}\right|}{\pi} \frac{g_s^2}{\kappa^2 \mathcal{V}} \frac{\alpha'}{\mathcal{A}_2}$$

$$m_{H_3}^2 \equiv 2g_{U(1)_3}^2 \xi_3 \approx \frac{2\left|k_3^{(1)}\right|}{\pi} \frac{g_s^2}{\kappa^2 \mathcal{V}} \frac{\alpha'}{\mathcal{A}_1} \tag{3.50}$$

which, with the use of the expressions for $g_{U(1)_1}$ and $g_{U(1)_3}$ from (3.47), lead to

$$\xi_1 = \frac{m_{H_1}^2}{2g_{U(1)_1}^2} \approx \left|m_1^{(2)} m_1^{(3)}\right| \frac{g_s \left|k_1^{(2)}\right|}{\pi \kappa^2 \mathcal{V}} \frac{\mathcal{A}_3}{\alpha'}$$

$$\xi_3 = \frac{m_{H_3}^2}{2g_{U(1)_3}^2} \approx \left|m_3^{(1)} m_3^{(2)}\right| \frac{g_s \left|k_3^{(1)}\right|}{\pi \kappa^2 \mathcal{V}} \frac{\mathcal{A}_2}{\alpha'} \tag{3.51}$$

in an analogous way with the derivation of ξ_2. Taking the D-part (3.43) and inserting (3.48), (3.51), (3.44) and (3.50) into it, we have

$$V_D = \frac{g_{U(1)_1}^2}{2}\xi_1^2 + \frac{g_{U(1)_3}^2}{2}\xi_3^2 + \frac{g_{U(1)_2}^2}{2}\xi_2^2 + m_{H_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g_{U(1)_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2 + \ldots$$

$$= \frac{1}{2}\xi_1 \frac{m_{H_1}^2}{2} + \frac{1}{2}\xi_3 \frac{m_{H_3}^2}{2} + \frac{1}{2}\xi_2 \frac{m_{H_2}^2}{2} + m_{H_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g_{U(1)_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2 + \ldots$$

$$\approx \frac{1}{2}\frac{g_s^3\left|k_1^{(2)}\right|^2\left|m_1^{(2)} m_1^{(3)}\right|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_3}{\mathcal{A}_2} + \frac{1}{2}\frac{g_s^3\left|k_3^{(1)}\right|^2\left|m_3^{(1)} m_3^{(2)}\right|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_2}{\mathcal{A}_1} + \frac{1}{2}\frac{g_s^3\left|k_2^{(3)}\right|^2\left|m_2^{(1)} m_2^{(3)}\right|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_1}{\mathcal{A}_3}$$

$$+ m_{H_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g_{U(1)_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2$$

$$= \frac{1}{\kappa^4\mathcal{V}^2}\left(d_1\frac{\mathcal{A}_3}{\mathcal{A}_2} + d_3\frac{\mathcal{A}_2}{\mathcal{A}_1} + d_2\frac{\mathcal{A}_1}{\mathcal{A}_3}\right) + m_{H_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g_{U(1)_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2 \tag{3.52}$$

where the following Kähler moduli D-term parameters are defined

$$d_a \equiv \frac{g_{U(1)_a}^2}{2}\xi_a^2 = \frac{1}{2}g_s^3\left|m_a^{(j)} m_a^{(k)}\right|\left(\frac{k_a^{(j)}}{\pi}\right)^2 \tag{3.53}$$

When the canonically normalized scalar field φ_2, which is associated with the $D7_2$ brane position x_2, aquires a non-zero vacuum expectation value (VEV), a mass is given to the tachyonic scalar fields φ_- and φ_+, through the F-term [62]

$$V_F \ni m_{x_2}^2\left(|\varphi_+|^2 + |\varphi_-|^2\right) \tag{3.54}$$

where $m_{x_2}^2$ is the physical mass coming from the brane position x_2 [62]

$$m_{x_2}^2 = y(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\mathcal{A}_2}{\alpha'} \tag{3.55}$$

where $y(U_2)$ is the torus complex structure modulus dependence which is positive [62]. Furthermore, the leading quartic contribution in the F-term scalar potential with regard to the tachyonic field is [62]

$$V_F \ni \kappa^2 m_{x_2}^2 |\varphi_-|^4 \tag{3.56}$$

In addition, the F-term scalar potential for the volume modulus $\mathcal{V}$ is

$$V_F \approx \frac{3\mathcal{W}_o^2}{2\kappa^4\mathcal{V}^3}(2\gamma(\ln\mathcal{V}-4)+\xi) \tag{3.57}$$

Taking the sum of (3.57), (3.52), (3.54) and (3.56), we have the approximate new effective scalar potential for the volume modulus $\mathcal{V}$, the tori moduli $\mathcal{A}_i$, and the tachyonic scalar fields φ_-, φ_+

$$\begin{aligned} V(\mathcal{A}_i,\varphi_\pm) \approx\ & \frac{3\mathcal{W}_o^2}{2\kappa^4\mathcal{V}^3}(2\gamma(\ln\mathcal{V}-4)+\xi) + \frac{1}{\kappa^4\mathcal{V}^2}\left(d_1\frac{\mathcal{A}_3}{\mathcal{A}_2} + d_3\frac{\mathcal{A}_2}{\mathcal{A}_1} + d_2\frac{\mathcal{A}_1}{\mathcal{A}_3}\right) \\ & + m_{H_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g_{U(1)_2}^2\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2 \\ & + m_{x_2}^2\left(|\varphi_+|^2 + |\varphi_-|^2\right) + \kappa^2 m_{x_2}^2|\varphi_-|^4 \end{aligned} \tag{3.58}$$

We define the ratios of the internal tori areas $\mathcal{A}_i$ as

$$u \equiv \frac{\mathcal{A}_3}{\mathcal{A}_2}, \quad \upsilon \equiv \frac{\mathcal{A}_1}{\mathcal{A}_3}, \quad \frac{1}{u\upsilon} = \frac{\mathcal{A}_2}{\mathcal{A}_1} \tag{3.59}$$

and minimise the D-part of the effective scalar potential (3.58) with respect to them, neglecting the matter fields: First of all, with the above definitions, this D-part takes the form

$$V_D(\mathcal{A}_i) = V_D(\mathcal{V},u,\upsilon) = \frac{1}{\kappa^4\mathcal{V}^2}\left(d_1 u + d_2\upsilon + d_3\frac{1}{u\upsilon}\right) \tag{3.60}$$

Its derivatives with respect to u and υ are

$$\begin{aligned} \frac{dV_D}{du} &= 0 \rightarrow \quad u^2 = \frac{d_3}{d_1\upsilon} \\ \frac{dV_D}{du} &= 0 \rightarrow \quad \upsilon^2 = \frac{d_3}{d_2 u} \end{aligned} \tag{3.61}$$

Combining these two we find that

$$\begin{aligned} u = u_0 &\equiv \left(\frac{d_3 d_2}{d_1^2}\right)^{1/3} \\ \upsilon = \upsilon_0 &\equiv \left(\frac{d_3 d_1}{d_2^2}\right)^{1/3} \end{aligned} \tag{3.62}$$

The minimised potential is then

$$V_D(\mathcal{V},u_0,\upsilon_0) = \frac{1}{\kappa^4\mathcal{V}^2}\left(3(d_1d_2d_3)^{1/3}\right) = \frac{d}{\kappa^4\mathcal{V}^2} \tag{3.63}$$

with $d \equiv 3(d_1d_2d_3)^{1/3}$, which now, from (3.53), reads

$$d \equiv 3(d_1 d_2 d_3)^{1/3} = \frac{3g_s^3}{2\pi}\left|\frac{m_1^{(3)}m_2^{(1)}m_3^{(2)}}{m_1^{(2)}m_2^{(3)}m_3^{(1)}}\right|^{1/3}\left(n_1^{(2)}n_2^{(3)}n_3^{(1)}\right)^{2/3} \tag{3.64}$$

For the derivation of (3.64) we used the relation $k_a^{(i)} = n_a^{(i)}/m_a^{(i)}$. In [62], is checked that the masses of the canonically normalized fields which correspond to $\mathcal{V}$, $1/(uv)$ and u/v (proportional to the logarithm of $\mathcal{V}$, $1/(uv)$ and u/v, in this setup), m_ϕ^2, m_U^2 and m_V^2, respectively, satisfy $m_\phi^2 \ll m_U^2, m_V^2$ (masses are larger than the volume modulus mass) in the region $[\phi_-, \phi_+]$ where inflation happens, so, the minimisation procedure is consistent).

After the stabilization of the moduli ratios u and v, we can find the tori moduli $\mathcal{A}_i$ in terms of the parameters d_a and the internal volume $\mathcal{V}$: From (3.59), using (3.49), we have

$$\begin{aligned}
u_0 &= \frac{\mathcal{A}_3}{\mathcal{A}_2} = \frac{\alpha'^3\mathcal{V}}{\mathcal{A}_1\mathcal{A}_2^2} \rightarrow \quad \mathcal{A}_1 = \frac{\alpha'^3\mathcal{V}}{u_0\mathcal{A}_2^2} \\
v_0 &= \frac{\mathcal{A}_1}{\mathcal{A}_3} = \frac{\alpha'^3\mathcal{V}}{\mathcal{A}_2\mathcal{A}_3^2} \rightarrow \quad \mathcal{A}_2 = \frac{\alpha'^3\mathcal{V}}{v_0\mathcal{A}_3^2} \\
\frac{1}{u_0 v_0} &= \frac{\mathcal{A}_2}{\mathcal{A}_1} = \frac{\alpha'^3\mathcal{V}}{\mathcal{A}_1^2\mathcal{A}_3} \rightarrow \quad \mathcal{A}_3 = \frac{\alpha'^3\mathcal{V}}{\mathcal{A}_1^2}u_0 v_0
\end{aligned} \tag{3.65}$$

By combination of the above derived relations (replacing the $\mathcal{A}$'s twice in each), we find that

$$\begin{aligned}
\mathcal{A}_1 &= \alpha'(u_0 v_0^2)^{1/3}\mathcal{V}^{1/3} \\
\mathcal{A}_2 &= \alpha'\frac{1}{(u_0^2 v_0)^{1/3}}\mathcal{V}^{1/3} \\
\mathcal{A}_3 &= \alpha'\left(\frac{u_0}{v_0}\right)^{1/3}\mathcal{V}^{1/3}
\end{aligned} \tag{3.66}$$

From (3.62) we have

$$\begin{aligned}
u_0 v_0^2 &= \frac{d_3}{d_2} \\
u_0^2 v_0 &= \frac{d_3}{d_1} \\
\frac{u_0}{v_0} &= \frac{d_2}{d_1}
\end{aligned} \tag{3.67}$$

Finally, plugging (3.67) in (3.66), we get the following expression for the tori moduli $\mathcal{A}_i$

$$\begin{aligned}
\mathcal{A}_1 &= \alpha'\left(\frac{d_3}{d_2}\right)^{1/3}\mathcal{V}^{1/3} \\
\mathcal{A}_2 &= \alpha'\left(\frac{d_1}{d_3}\right)^{1/3}\mathcal{V}^{1/3} \\
\mathcal{A}_3 &= \alpha'\left(\frac{d_2}{d_1}\right)^{1/3}\mathcal{V}^{1/3}
\end{aligned} \tag{3.68}$$

Further, from the condition for the elimination of other tachyons from different brane intersections, $|\zeta_1^{(2)}| = |\zeta_2^{(3)}| = |\zeta_3^{(1)}|$ [62], we find that

$$\left|\zeta_1^{(2)}\right| = \left|\zeta_2^{(3)}\right| = \left|\zeta_3^{(1)}\right| \tag{3.69}$$

$$\rightarrow \frac{\alpha'|k_1^{(2)}|}{\mathcal{A}_2} = \frac{\alpha'|k_2^{(3)}|}{\mathcal{A}_3} = \frac{\alpha'|k_3^{(1)}|}{\mathcal{A}_1}$$

$$\rightarrow |k_1^{(2)}|\left(\frac{d_3}{d_1}\right)^{1/3} = |k_2^{(3)}|\left(\frac{d_1}{d_2}\right)^{1/3} = |k_3^{(1)}|\left(\frac{d_2}{d_3}\right)^{1/3}$$

$$\rightarrow |k_1^{(2)}||k_3^{(1)}|^2\left|\frac{m_3^{(1)}m_3^{(2)}}{m_1^{(2)}m_1^{(3)}}\right| = |k_2^{(3)}||k_1^{(2)}|^2\left|\frac{m_1^{(2)}m_1^{(3)}}{m_2^{(1)}m_2^{(3)}}\right| = |k_3^{(1)}||k_2^{(3)}|^2\left|\frac{m_2^{(1)}m_2^{(3)}}{m_3^{(1)}m_3^{(2)}}\right|$$

$$\rightarrow A|k_1^{(2)}||k_3^{(1)}|^2 = B|k_2^{(3)}||k_1^{(2)}|^2 = \Gamma|k_3^{(1)}||k_2^{(3)}|^2 \tag{3.70}$$

with

$$A \equiv \left|\frac{m_3^{(1)}m_3^{(2)}}{m_1^{(2)}m_1^{(3)}}\right|, \quad B \equiv \left|\frac{m_1^{(2)}m_1^{(3)}}{m_2^{(1)}m_2^{(3)}}\right|, \quad \Gamma \equiv \left|\frac{m_2^{(1)}m_2^{(3)}}{m_3^{(1)}m_3^{(2)}}\right| \tag{3.71}$$

In derivation of (3.70), we used (3.45) and the analogous relations for the other $\zeta_a^{(i)}$ ($\zeta_1^{(2)} \approx \alpha' k_1^{(2)}/\pi\mathcal{A}_2$ and $\zeta_3^{(1)} \approx \alpha' k_3^{(1)}/\pi\mathcal{A}_1$), the moduli stabilization condition (3.68), and (3.53). Setting $|k_1^{(2)}| = x$, $|k_3^{(1)}| = z$ and $|k_2^{(3)}| = y$, (3.70) becomes $Axz^2 = Byx^2 = \Gamma zy^2$, from which we have $x^3 = (\Gamma^2/AB)y^3$ and $z^3 = (B^2/A\Gamma)x^3$, that is

$$\begin{aligned} |k_1^{(2)}| &= \left|\frac{m_2^{(1)}m_2^{(3)}}{m_3^{(1)}m_3^{(2)}}\right||k_2^{(3)}| \rightarrow \quad n_2^{(3)} = \left|\frac{m_3^{(1)}m_3^{(2)}}{m_2^{(1)}m_1^{(2)}}\right|n_1^{(2)} \\ |k_3^{(1)}| &= \left|\frac{m_1^{(2)}m_1^{(3)}}{m_2^{(1)}m_2^{(3)}}\right||k_1^{(2)}| \rightarrow \quad n_3^{(1)} = \left|\frac{m_3^{(1)}m_1^{(3)}}{m_2^{(1)}m_2^{(3)}}\right|n_1^{(2)} \end{aligned} \tag{3.72}$$

if we use the relation $k_a^{(i)} = n_a^{(i)}/m_a^{(i)}$. Under the conditions (3.72), the parameter (3.64) simplifies to

$$d = \frac{3}{2}g_s^3\left(\frac{k}{\pi}\right)^2 \tag{3.73}$$

with

$$k = n_1^{(2)}\left|\frac{m_1^{(3)}m_3^{(1)}m_3^{(2)}}{m_1^{(2)}m_2^{(1)}m_2^{(3)}}\right|^{1/2} \tag{3.74}$$

We can express the masses (3.44), (3.55), and the coupling (3.47), using (3.68), as

$$m_{H_2}^2 = 2\sqrt{2}|m_2^{(1)}m_2^{(3)}|^{-1/2}\frac{\sqrt{g_s}}{\kappa^2\mathcal{V}^{4/3}}(d_1^2 d_2)^{1/6} \tag{3.75}$$

$$g_{U(1)_2}^2 = |m_2^{(1)}m_2^{(3)}|^{-1}\frac{g_s}{\mathcal{V}^{2/3}}\left(\frac{d_1}{d_3}\right)^{1/3} \tag{3.76}$$

$$m_{x_2}^2 = \frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}y(U_2)\left(\frac{d_1}{d_3}\right)^{1/3} \tag{3.77}$$

where for the derivation of (3.75), we also used (3.53). Then, we can express the masses and the coupling in terms of the parameters g_s, $y(U_2)$, $m_a^{(i)}$, $n_1^{(2)}$ and $\mathcal{V}$: Using (3.53), and (3.72) to replace the $|k_2^{(3)}|$, (3.75) becomes

$$m_{H_2}^2 = 2\frac{g_s^2 k}{\pi\kappa^2\mathcal{V}^{4/3}}\left|\frac{1}{\prod_{a\neq i} m_a^{(i)}}\right|^{1/6} \tag{3.78}$$

Using (3.53) again, and (3.72) to replace the $|k_3^{(1)}|$, (3.76) becomes

$$g^2_{U(1)_2} = \frac{g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{\prod_{a\neq i} m_a^{(i)}} \right|^{1/3} \tag{3.79}$$

and, finally, using (3.53) and (3.72) to replace the $|k_3^{(1)}|$, (3.77) becomes

$$m^2_{x_2} = \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} y(U_2) \left| \frac{{m_2^{(1)}}^2 {m_2^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \tag{3.80}$$

The above effective scalar potential after minimization of the ratios of the internal areas moduli $\mathcal{A}_i$, and neglecting the massive charge conjugate of the tachyonic field, φ_+, becomes

$$V(\mathcal{V}, \varphi_-) = \frac{3\mathcal{W}_o^2}{2\kappa^4 \mathcal{V}^3} (2\gamma(\ln \mathcal{V} - 4) + \xi) + \frac{d}{\kappa^4 \mathcal{V}^2}$$

$$-m^2_{H_2} |\varphi_-|^2 + 2g^2_{U(1)_2} |\varphi_-|^4 + m^2_{x_2} |\varphi_-|^2 + \kappa^2 m^2_{x_2} |\varphi_-|^4 \tag{3.81}$$

The first line takes the form of (3.11) or (3.35), so we have

$$V(\mathcal{V}, \varphi_-) = \frac{C}{\kappa^4} \left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) + \frac{1}{2} m_Y^2(\mathcal{V}) |\varphi_-|^2 + \frac{1}{4} \lambda_Y(\mathcal{V}) |\varphi_-|^4 \tag{3.82}$$

with

$$C = -3\mathcal{W}_o^2 \gamma, \quad q = \frac{\xi}{2\gamma}, \quad s = \frac{3}{2}\sigma = \frac{d}{3\mathcal{W}_o^2 \gamma} = -\frac{d}{C} = \frac{3}{2C} g_s^3 \left(\frac{k}{\pi}\right)^2,$$

$$m_Y^2(\mathcal{V}) = 2\big(m^2_{x_2} - m^2_{H_2}\big), \quad \lambda_Y(\mathcal{V}) = 4\big(2g^2_{U(1)_2} + \kappa^2 m^2_{x_2}\big) \tag{3.83}$$

From the relations (3.80) and (3.78), we find that the mass of the tachyonic scalar field φ_-, expressed through the parameters g_s, $y(U_2)$, $m_a^{(i)}$, $n_1^{(2)}$ and the volume $\mathcal{V}$, is

$$m_Y^2(\mathcal{V}) = 2\big(m^2_{x_2} - m^2_{H_2}\big)$$

$$= 2\frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} y(U_2) \left| \frac{{m_2^{(1)}}^2 {m_2^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \left(1 - \frac{1}{\mathcal{V}^{2/3}} \frac{2k}{\pi y(U_2)} \left| \frac{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}{{m_2^{(1)}}^5 {m_2^{(3)}}^5} \right|^{1/6} \right)$$

$$= 2\frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} y(U_2) \left| \frac{{m_2^{(1)}}^2 {m_2^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \left(1 - \left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{2/3} \right) \tag{3.84}$$

with

$$\mathcal{V}_{c2} \equiv \left(\frac{2k}{\pi y(U_2)}\right)^{3/2} \left| \frac{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}{{m_2^{(1)}}^5 {m_2^{(3)}}^5} \right|^{1/4} \tag{3.85}$$

which is the critical value of the volume $\mathcal{V}$ that φ_- becomes tachyonic, and k is given by (3.74). From the relations (3.79) and (3.80), we find that the coupling is expressed through the parameters g_s, $y(U_2)$, $m_a^{(i)}$ and the volume $\mathcal{V}$ as

$$\lambda_Y(\mathcal{V}) = 4\big(2g^2_{U(1)_2} + \kappa^2 m^2_{x_2}\big)$$

$$= \frac{4g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \left(2 + g_s y(U_2) |m_2^{(1)} m_2^{(3)}| \right)$$

$$= \frac{4g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{\prod_{a \neq i} m_a^{(i)}} \right|^{1/3} \left(2 + g_s y(U_2) |m_2^{(1)} m_2^{(3)}| \right) \tag{3.86}$$

We observe that the effective scalar potential coresponding to the inflaton and the tachyonic field φ_-, (3.82), has taken the form of the hybrid potential (Section 2.3.1), with φ_- playing the role of the waterfall field. Both the squared-mass of the waterfall field and the coupling depend on the volume $\mathcal{V}$. Especially its squared-mass, becomes negative below a critical value of the volume which is defined as $\mathcal{V}_{c2}$. As we see from (3.85) and (3.74), this critical value depends on the parameters $m_a^{(i)}$, $n_1^{(2)}$ and on the $D7_2$ brane position through $y(U_2)$, so it could be possible, with an appropriate choice of the values of these parameters, that it has the value of local minimum of the inflaton-part in the scalar potential.

3.2.2 The new vacuum

We can now minimise the obtained effective scalar potential (3.82), in which the waterfall field contributes too, to find the new minimum. In this form of potential we have two seperate phases for the waterfall field, depending on the sign of its effective squared-mass:

- For $m_Y^2(\mathcal{V}) > 0$, which holds for $\mathcal{V} > \mathcal{V}_{c2}$ as we see from (3.84) (since $y(U_2) > 0$), the system is in its symmetric phase and the waterfall field sits at its minimum which is vanishing, $\langle \varphi_- \rangle = 0$, possessing a large mass [62]. The only contribution to the scalar potential is that of the inflaton field and thus, the inflationary phase is equivalent to that with only one field in the model.

- For $m_Y^2(\mathcal{V}) < 0$, which holds for $\mathcal{V} < \mathcal{V}_{c2}$ as we see from (3.84) again, the system has undergone a phase transition when the value of the volume became lower than the critical, somewhere near the minimum of the inflaton scalar potential, and the waterfall field started rolling down to its new obtained non-vanishing VEV, $\langle \varphi_- \rangle = \pm \upsilon_2$. If this waterfall direction is steep enough so that $\epsilon > 1$, the inflationary phase reaches to an end.

We are interested in what happens in the waterfall direction, so we are focusing in the second above case of the system. After the change in the sign of the squared-mass, the non-zero VEV that the waterfall field rolls down to, is (Section 2.3.1)

$$\langle \varphi_- \rangle = \pm \upsilon_2 = \pm \frac{|m_Y|}{\sqrt{\lambda_Y}} \tag{3.87}$$

The effective scalar potential (3.82) at this new minimum, substituting the mass (3.84) and the coupling (3.86), is expressed in terms of the volume $\mathcal{V}$ as

$$V(\mathcal{V}, \upsilon_2) = \frac{C}{\kappa^4} \left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - \frac{1}{4} \frac{m_Y^4(\mathcal{V})}{\lambda_Y(\mathcal{V})}$$

$$= \frac{C}{\kappa^4} \left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - \frac{1}{4} \frac{g_s^3 y^2(U_2)}{\kappa^4 \mathcal{V}^{2/3}} \left| \frac{{m_2^{(1)}}^5 {m_2^{(3)}}^5}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \frac{\left(1 - (\mathcal{V}_{c2}/\mathcal{V})^{2/3}\right)^2}{2 + g_s y(U_2) |m_2^{(1)} m_2^{(3)}|}$$

$$= \frac{C}{\kappa^4} \left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - \frac{C_2}{\kappa^4 \mathcal{V}^{2/3}} \left(1 - \left(\frac{\mathcal{V}_{c2}}{\mathcal{V}} \right)^{2/3} \right)^2 \tag{3.88}$$

with

$$C_2 \equiv \frac{1}{4}\frac{g_s^3 y^2(U_2)}{2+g_s y(U_2)\left|m_2^{(1)}m_2^{(3)}\right|}\left|\frac{{m_2^{(1)}}^5 {m_2^{(3)}}^5}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3} \tag{3.89}$$

The coefficient (3.89) can be written in terms of the d and V_{c2} as

$$C_2 = \beta_2 \frac{d}{3\mathcal{V}_{c2}^{4/3}} \tag{3.90}$$

with the definition

$$\beta_2 \equiv \frac{2}{2+g_s y(U_2)\left|m_2^{(1)}m_2^{(3)}\right|}, \tag{3.91}$$

We have the following values for the above including parameters: as we mentioned in the previous section, for $q=0$, C and s should have the values 7.81×10^{-4} and $\simeq -0.00723954$, respectively, from the observational data. Further, from (3.83), we see that d must be $\simeq 5.65408\times10^{-6}$. For $q=0$ also, in the previous section, the local minimum of inflaton potential $\mathcal{V}_-$ has been found to be $\simeq 201.9$. By numerical computation in [62], it is found that the global minimum of the potential in (3.88) has an almost vanishing value for a certain value of the coefficient C_2, which finally from (3.90), gives the following value for β_2

$$\beta_2(\Lambda\sim 0)\simeq 3.228 \tag{3.92}$$

However, since $y(U_2)>0$, we see from (3.91) that the minimum value of the parameter β_2 is 0, while its maximum value is 1, the maximal tachyonic field contribution being that for $\beta_2=1$. Thus, an almost vanishing value of the vacuum cannot be achieved in this model, as the $\beta_2\in[0,1]$ of this theory is not consistent with (3.92).

Below, we study the effective scalar potential (3.82) at its new non-vanishing VEV, with the help of the program Mathematica:

We insert the form of the effective scalar potential (3.82) with

$$\kappa^2=1,\quad C=7.81\times10^{-4},\quad q=0,$$

$$m_Y^2 = 2\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}y(U_2)\left|\frac{{m_2^{(1)}}^2{m_2^{(3)}}^2}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(1-\left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{2/3}\right),$$

$$\mathcal{V}_{c2} = \left(\frac{2k}{\pi y(U_2)}\right)^{3/2}\left|\frac{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}{{m_2^{(1)}}^5{m_2^{(3)}}^5}\right|^{1/4},$$

$$k = n_1^{(2)}\left|\frac{m_1^{(3)}m_3^{(1)}m_3^{(2)}}{m_1^{(2)}m_2^{(1)}m_2^{(3)}}\right|^{1/2},$$

$$s=\frac{3}{2}\sigma = -\frac{d}{C} = -\frac{3}{2C}g_s^3\left(\frac{k}{\pi}\right)^2,$$

$$\lambda_Y = \frac{4g_s}{\mathcal{V}^{2/3}}\left|\frac{1}{m_1^{(2)}m_1^{(3)}m_2^{(1)}m_2^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(2+g_s y(U_2)\left|m_2^{(1)}m_2^{(3)}\right|\right) \tag{3.93}$$

We minimize the potential with respect to φ_-, and find its expression in terms of the internal volume $\mathcal{V}$. To begin our investigation, we choose some values for g_s and $y(U_2)$, so that the product $g_s y(U_2)$ is small and therefore corresponds to $\beta_2 \sim 1$. For example, at first, we choose $g_s = 2.596 \times 10^{-3}$ and $y(U_2) = 3.85$ (the values given in [62]).

With these parameters fixed in general, we search for the combinations of a positive value of $\mathcal{V}$ and positive integer values of $m_a^{(i)}$, $n_1^{(2)}$, for which the potential minimum vanishes, $V(\mathcal{V}, \upsilon_2) = 0$, and for which it owns a positive value, $0 < V(\mathcal{V}, \upsilon_2) \lesssim 10^{-11}$. In both cases, the procedure is as follows: From the combinations that give the required value of the potential, we choose those that give a resonable $\mathcal{V}_{c2}$ ($\simeq 201.9$), and among the the latter, we choose those that give a critical value bigger than the potential minimum $\mathcal{V}$, $\mathcal{V}_{c2} \geq \mathcal{V}$. Finally, we calculate the parameter s for them. Combinations of $\{\mathcal{V}, m_a^{(i)}, n_1^{(2)}\}$ that give a value of s that satisfies the constraint (3.33), $-0.00724192 < s < -0.00673795$, are studied in detail, modifying properly the values of g_s and $y(U_2)$ to aquire the exact required s (-0.00723954) and $\mathcal{V}_{c2}$ (201.9), respectively.

For the fixed g_s and $y(U_2)$ chosen in the begining, among the combinations $\{\mathcal{V}, m_a^{(i)}, n_1^{(2)}\}$ found, neither from the two above cases, gives the appropriate value for $\mathcal{V}_{c2}$, besides they give a tiny value for it and $\mathcal{V}_{c2} < \mathcal{V}$. Reducing the value of the parameter $y(U_2)$, for instance $y = 0.09$, with the expectation to uplift the value of $\mathcal{V}_{c2}$ (as we see from (3.93)), we repeat the above procedure for different values of g_s. We indeed find bigger $\mathcal{V}_{c2}$'s in both cases, and $\mathcal{V}_{c2} > \mathcal{V}$, while accepted values for s (that correspond to a dS minimum-(3.33)) are found only in the second case (where we search for a positive minimum value, not vanishing). Moreover, the value for s closest to the desired -0.00723954, is found for $g_s \sim 2.751 \times 10^{-3}$.

Lastly, with the appropriate modification mentioned before of the above g_s (and $y(U_2)$), we find the combination

$$g_s = 2.770653 \times 10^{-3}, \quad y(U_2) = 0.0908187, \quad n_1^{(2)} = 17,$$

$$m_1^{(2)} = 29, \quad m_1^{(3)} = 31, \quad m_2^{(1)} = 42, \quad m_2^{(3)} = 5, \quad m_3^{(1)} = 41, \quad m_3^{(2)} = 29 \tag{3.94}$$

which gives $s \simeq -0.00723954$, $\mathcal{V}_{c2} \simeq 201.9$ and $\beta_2 \sim 1$, with a potential minimum at

$$\mathcal{V}_{min} \simeq 163.235 \rightarrow \quad V(\mathcal{V}_{min}, \upsilon_2) \simeq 1.43381 \times 10^{-11} \tag{3.95}$$

The pontential minimum value in terms of the volume $\mathcal{V}$ with the (3.94) parameters incorporated is shown in the Figure 3.5. The resulting value of the potential at the minimum, (3.95), is in agreement with [62]. We can see seperately the contribution of the inflaton and the waterfall field to the potential (3.88), for the combination (3.94) (and $\kappa^2 = 1$, $C = 7.81 \times 10^{-4}$ and $q = 0$),

$$V(\mathcal{V}, \upsilon_2) = 1.55413 \times 10^{-11} - 1.2032 \times 10^{-12} \tag{3.96}$$

from which we observe that the waterfall contribution is not big enough to cancel the inflaton contribution so that it gives the desired almost vanishing value of V. This was expected from above, where we saw through the parameter β_2 that the tachyonic contribution in this model is not enough for the vacuum to almost vanish. In Figure 3.6 though, we see that a waterfall direction indeed appears, with a maximum near $\simeq 201.9$. This waterfall direction increases the slow roll parameters and hence, ends the inflationary phase [62]. Finally, in Figure 3.7 we see the 3-dimensional version of the effective potential for the (3.94) parameter values, which has the form of a hybrid potential (2.2).

To conclude, from (3.93) we have

$$\frac{m_Y^2}{\lambda_Y} = \frac{1}{2\kappa^2} \frac{g_s y(U_2) |m_2^{(1)} m_2^{(3)}|}{2 + g_s y(U_2) |m_2^{(1)} m_2^{(3)}|} \left(1 - \left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{2/3}\right)$$

$$\rightarrow \langle \varphi_- \rangle = \pm \upsilon_2 = \pm \frac{|m_Y|}{\sqrt{\lambda_Y}} = \pm \frac{1}{\sqrt{2}\kappa} \sqrt{\frac{g_s y(U_2) |m_2^{(1)} m_2^{(3)}|}{2 + g_s y(U_2) |m_2^{(1)} m_2^{(3)}|}} \left|1 - \left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{2/3}\right|^{1/2} \tag{3.97}$$

from which we see that the VEV depends on the values of the product $g_s y(U_2)$, on the critical value $\mathcal{V}_{c2}$, and the integer wrapping numbers $m_2^{(i)}$ as well. Thus, according to the combination (3.94) found, the range of the VEV in κ units is

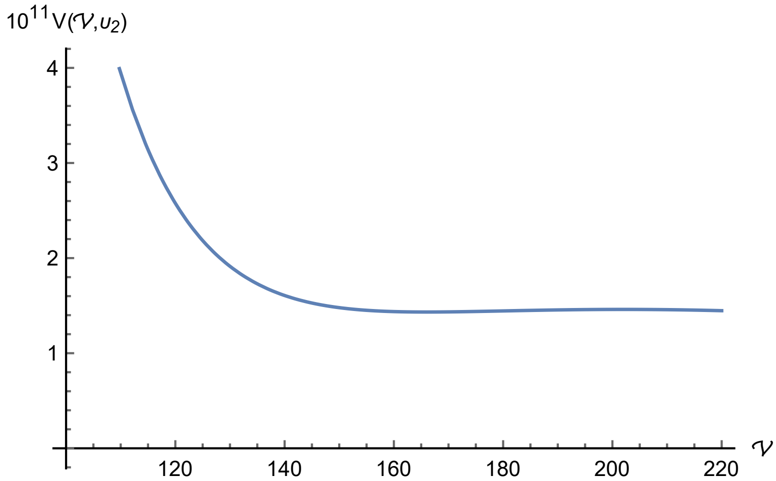

Figure 3.5: Minimum value of the effective scalar potential of the inflaton and the waterfall field in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (3.94).

$$\kappa \upsilon_2 \in [0, \simeq 0.0457201] \tag{3.98}$$

which is small, so the quartic expansion in (3.43) holds. It has to be stated also, that the parameters $m_a^{(i)}$ and $n_1^{(2)}$ of the model are constrained as they are subject to tadpole cancellation conditions [62], but this is beyond the purpose of this work.

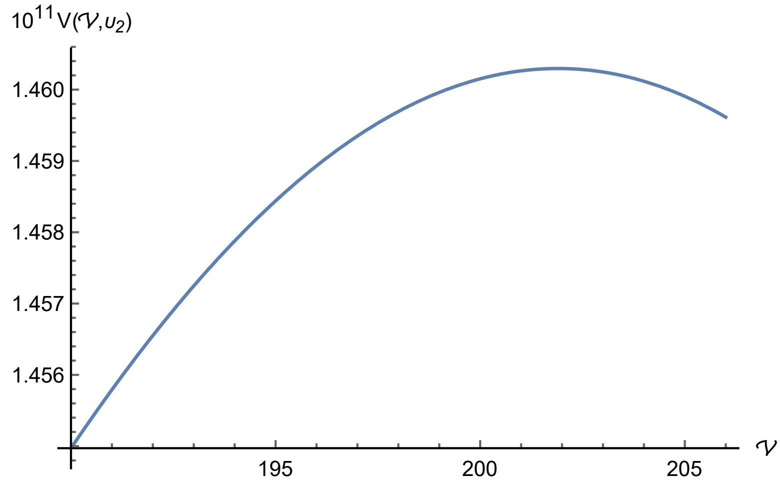

Figure 3.6: Waterfall direction in the effective scalar potential of the inflaton and the waterfall field in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (3.94).

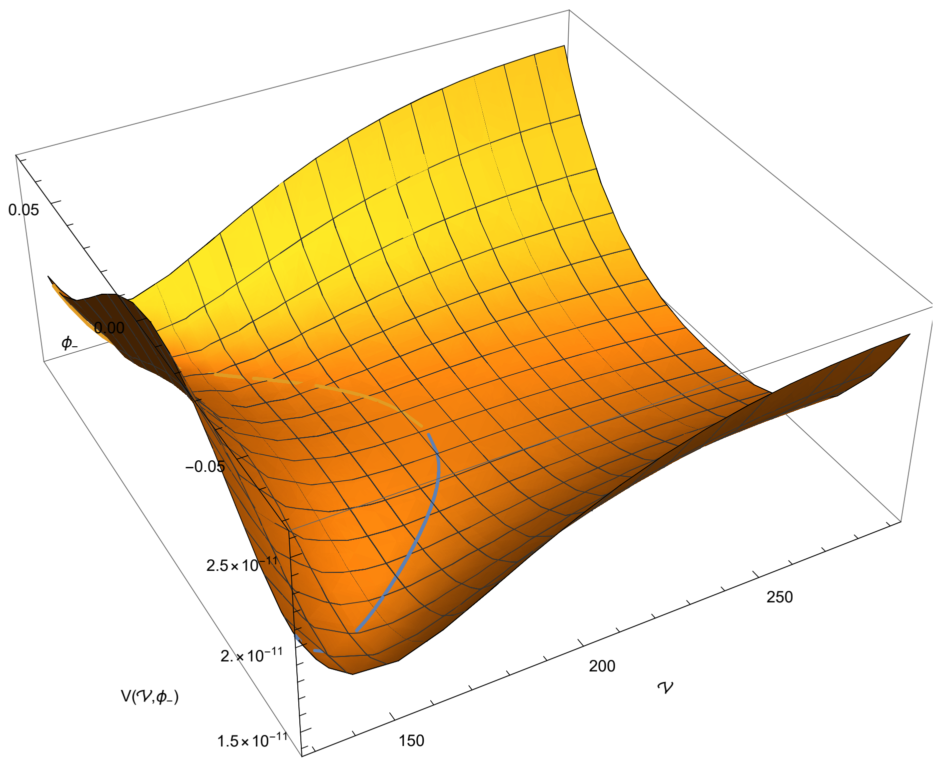

Figure 3.7: Effective scalar potential with the inflaton-direction ($\mathcal{V}$) and the waterfall-direction (here $\phi_- = \varphi_-$) (orange, blue), when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (3.94).

Chapter 4

Additional waterfall fields

As we mentioned in the previous chapter, the value of the global minimum cannot almost-vanish, neither in the case of the inflaton only nor in the case with the introduction of a waterfall field. In this chapter, we study the case proposed in [62] with more than one waterfall fields in the theory, that are possible to deepen further the vacuum of the theory.

Before the addition of other waterfall fields, a possible solution by the parameters q, γ and ξ of the potential, is studied in [62]: As it turns out, the potential minimum value (3.89), does not depend explicitly on q so this parameter cannot help in its lowering, and quantum corrections in the squared mass and quartic term coming from γ and ξ factors previously neglected, stay small in the large volume limit, so they cannot contribute significanlty in its reduction. The proposed extra waterfall (tachyonic) fields are coming from the other $D7$ brane stacks ($D7_3$, $D7_1$) and are expected to open new waterfall directions which along with the previous one will provide bigger negative contribution to the minimum of the potential. Their squared masses also depend on the volume $\mathcal{V}$ and under certain critical values of it they become tachyonic, while this can be chosen to happen successively.

In the first section of this chapter, we add a second tachyonic field to the already existing one, coming from the $D7_3$ brane stack, and study the total negative waterfall contribution to the vacuum. In the second section, we do the same thing by adding a third tachyonic field coming from the last one $D7$ brane stack, $D7_1$.

4.1 Second waterfall field

In this theory, we have one more tachyonic field that it is not eliminated [62] any more, the one coming from the $D7_3$ brane, and is denoted as ψ_-. Below, we firstly derive the new effective scalar potential of all three interested fields, the inflaton field, the waterfall field φ_-, and the second tachyonic field ψ_-, in the large volume limit, and secondly, using the program Mathematica, we investigate their contributions to the potential minimum.

The D-term part in the effective scalar potential that describes the magnetic fields, now, is

$$
\begin{aligned}
V_D = & \frac{g^2_{U(1)_1}}{2}\xi_1^2 + \frac{g^2_{U(1)_3}}{2}\left(\xi_3 + 2\left|\psi_+\right|^2 - 2\left|\psi_-\right|^2 + ...\right)^2 + \frac{g^2_{U(1)_2}}{2}\left(\xi_2 + 2\left|\varphi_+\right|^2 - 2\left|\varphi_-\right|^2 + ...\right)^2 + ... \\
= & \frac{g^2_{U(1)_1}}{2}\xi_1^2 + \frac{g^2_{U(1)_3}}{2}\xi_3^2 + 2g^2_{U(1)_3}\xi_3\left(\left|\psi_+\right|^2 - \left|\psi_-\right|^2\right) + 2g^2_{U(1)_3}\left(\left|\psi_+\right|^2 - \left|\psi_-\right|^2\right)^2 \\
& + \frac{g^2_{U(1)_2}}{2}\xi_2^2 + 2g^2_{U(1)_2}\xi_2\left(\left|\varphi_+\right|^2 - \left|\varphi_-\right|^2\right) + 2g^2_{U(1)_2}\left(\left|\varphi_+\right|^2 - \left|\varphi_-\right|^2\right)^2 + ... \qquad (4.1)
\end{aligned}
$$

if we keep only the canonically normalised tachyonic fields φ_-, ψ_- (with charge $q_- = +2$) and their charge conjugates φ_+, ψ_+ (with charge $q_+ = -2$). The magnetic field contribution to the mass of the second tachyon, $m^2_{H_3}$, in the large volume limit, is given by (3.50). Also, the gauge coupling $g_{U(1)_3}$ in the large volume limit is given by (3.47). Substituting the expressions for all magnetic field mass contributions, $m^2_{H_1}$, $m^2_{H_2}$, $m^2_{H_3}$, and Fayet-Iliopoulos parameters, ξ_1, ξ_2, ξ_3, given by (3.44),(3.50), (3.48) and (3.51), in (4.1), we have

$$\begin{aligned}
V_D = &\frac{g^2_{U(1)_1}}{2}\xi_1^2 + \frac{g^2_{U(1)_3}}{2}\xi_3^2 + m^2_{H_3}\left(|\psi_+|^2 - |\psi_-|^2\right) + 2g^2_{U(1)_3}\left(|\psi_+|^2 - |\psi_-|^2\right)^2 \\
&+\frac{g^2_{U(1)_2}}{2}\xi_2^2 + m^2_{H_2}\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g^2_{U(1)_2}\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2 + \ldots \\
= &\frac{1}{2}\xi_1\frac{m^2_{H_1}}{2} + \frac{1}{2}\xi_3\frac{m^2_{H_3}}{2} + \frac{1}{2}\xi_2\frac{m^2_{H_2}}{2} + m^2_{H_3}\left(|\psi_+|^2 - |\psi_-|^2\right) + 2g^2_{U(1)_3}\left(|\psi_+|^2 - |\psi_-|^2\right)^2 \\
&+m^2_{H_2}\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g^2_{U(1)_2}\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2 + \ldots \\
\approx &\frac{1}{2}\frac{g_s^3|k_1^{(2)}|^2|m_1^{(2)}m_1^{(3)}|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_3}{\mathcal{A}_2} + \frac{1}{2}\frac{g_s^3|k_3^{(1)}|^2|m_3^{(1)}m_3^{(2)}|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_2}{\mathcal{A}_1} + \frac{1}{2}\frac{g_s^3|k_2^{(3)}|^2|m_2^{(1)}m_2^{(3)}|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_1}{\mathcal{A}_3} \\
&+m^2_{H_3}\left(|\psi_+|^2 - |\psi_-|^2\right) + 2g^2_{U(1)_3}\left(|\psi_+|^2 - |\psi_-|^2\right)^2 \\
&+m^2_{H_2}\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g^2_{U(1)_2}\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2 \\
= &\frac{1}{\kappa^4\mathcal{V}^2}\left(d_1\frac{\mathcal{A}_3}{\mathcal{A}_2} + d_3\frac{\mathcal{A}_2}{\mathcal{A}_1} + d_2\frac{\mathcal{A}_1}{\mathcal{A}_3}\right) + m^2_{H_3}\left(|\psi_+|^2 - |\psi_-|^2\right) + 2g^2_{U(1)_3}\left(|\psi_+|^2 - |\psi_-|^2\right)^2 \\
&+m^2_{H_2}\left(|\varphi_+|^2 - |\varphi_-|^2\right) + 2g^2_{U(1)_2}\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2
\end{aligned} \tag{4.2}$$

with

$$d_a \equiv \frac{g^2_{U(1)_a}}{2}\xi_a^2 = \frac{1}{2}g_s^3|m_a^{(j)}m_a^{(k)}|\left(\frac{k_a^{(j)}}{\pi}\right)^2 \tag{4.3}$$

a definition which is also made in the previous case.

The D-term (4.2) without the matter fields is exactly of the form (3.60), and after minimisation of the moduli ratios, it becomes (3.63). Using the moduli stabilization condition (3.68), the magnetic field contribution to the mass of the tachyon ψ_- becomes

$$m^2_{H_3} \equiv 2g^2_{U(1)_3}\xi_3 \approx \frac{2|k_3^{(1)}|}{\pi}\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\alpha'}{\mathcal{A}_1} = 2\sqrt{2}|m_3^{(1)}m_3^{(2)}|^{-1/2}\frac{\sqrt{g_s}}{\kappa^2\mathcal{V}^{4/3}}(d_2^2d_3)^{1/6} \tag{4.4}$$

(where we also use (4.3) to replace the flux $k_3^{(1)}$), and the gauge coupling becomes

$$g^2_{U(1)_3} \approx |m_3^{(1)}m_3^{(2)}|^{-1}\frac{g_s}{\mathcal{V}}\frac{\mathcal{A}_3}{\alpha'} = |m_3^{(1)}m_3^{(2)}|^{-1}\frac{g_s}{\mathcal{V}^{2/3}}\left(\frac{d_2}{d_1}\right)^{1/3} \tag{4.5}$$

We also have the following F-term contributions in the scalar potential, as in the previous case, with regard to the tachyonic field ψ_-: The mass contribution

$$V_F \ni m^2_{x_3}\left(|\psi_+|^2 + |\psi_-|^2\right) \tag{4.6}$$

where $m^2_{x_3}$ is the physical mass coming from the brane position x_3

$$m^2_{x_3} = z(U_3)\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\mathcal{A}_3}{\alpha'} \tag{4.7}$$

with $z(U_3)$ being the analogous quantity to $y(U_2)$ of the previous case, and the leading quartic contribution

$$V_F \ni \kappa^2 m_{x_3}^2 |\psi_-|^4 \tag{4.8}$$

The mass contribution (4.7), again using (3.68), becomes

$$m_{x_3}^2 = \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} z(U_3) \left(\frac{d_2}{d_1} \right)^{1/3} \tag{4.9}$$

We express $m_{H_3}^2$, $g_{U(1)_3}^2$ and $m_{x_3}^2$ in terms of the parameters g_s, $z(U_3)$, $m_a^{(i)}$, $n_1^{(2)}$ and $\mathcal{V}$, as before: Using (4.3), and (3.72) (condition for the elimination of tachyons from different brane interstections*) to replace the appearing fluxes $k_2^{(3)}$ and $k_3^{(1)}$, (4.4) becomes

$$m_{H_3}^2 = m_{H_2}^2 = 2 \frac{g_s^2 k}{\pi \kappa^2 \mathcal{V}^{4/3}} \left| \frac{1}{\prod_{a \neq i} m_a^{(i)}} \right|^{1/6} \tag{4.10}$$

Using (4.3) again, and (3.72) to replace $k_2^{(3)}$, (4.5) becomes

$$g_{U(1)_3}^2 = g_{U(1)_2}^2 = \frac{g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{\prod_{a \neq i} m_a^{(i)}} \right|^{1/3} \tag{4.11}$$

Finally, using (4.3) and (3.72) to replace $k_2^{(3)}$, (4.9) becomes

$$m_{x_3}^2 = \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} z(U_3) \left| \frac{m_3^{(1)^2} m_3^{(2)^2}}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)}} \right|^{1/3} \tag{4.12}$$

expression similar to the one of $m_{x_2}^2$ in (3.80), but with the corresponding quantities for the $D7_3$ brane.

The approximate new effective scalar potential for the volume modulus $\mathcal{V}$ and the tachyonic scalar fields φ_-, ψ_-, is the sum of the F-part for the volume modulus (3.57), the D-part (4.2) and the F-parts (3.54), (3.56), (4.6) and (4.8), negleting their charge conjugates,

$$\begin{aligned} V(\mathcal{V}, \varphi_-, \psi_-) = {} & \frac{C}{\kappa^4} \left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - m_{H_3}^2 |\psi_-|^2 + 2 g_{U(1)_3}^2 |\psi_-|^4 + m_{x_3}^2 |\psi_-|^2 + \kappa^2 m_{x_3}^2 |\psi_-|^4 \\ & - m_{H_2}^2 |\varphi_-|^2 + 2 g_{U(1)_2}^2 |\varphi_-|^4 + m_{x_2}^2 |\varphi_-|^2 + \kappa^2 m_{x_2}^2 |\varphi_-|^4 \\ = {} & \frac{C}{\kappa^4} \left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) + \frac{1}{2} m_Y^2(\mathcal{V}) |\varphi_-|^2 + \frac{1}{4} \lambda_Y(\mathcal{V}) |\varphi_-|^4 \\ & + \frac{1}{2} m_Z^2(\mathcal{V}) |\psi_-|^2 + \frac{1}{4} \lambda_Z(\mathcal{V}) |\psi_-|^4 \end{aligned} \tag{4.13}$$

with

$$\begin{aligned} & C = -3 \mathcal{W}_o^2 \gamma, \quad q = \frac{\xi}{2\gamma}, \quad s = \frac{3}{2}\sigma = \frac{d}{3 \mathcal{W}_o^2 \gamma} = -\frac{d}{C} = \frac{3}{2C} g_s^3 \left(\frac{k}{\pi} \right)^2, \\ & m_Y^2(\mathcal{V}) = 2 \big(m_{x_2}^2 - m_{H_2}^2 \big), \quad \lambda_Y(\mathcal{V}) = 4 \big(2 g_{U(1)_2}^2 + \kappa^2 m_{x_2}^2 \big), \\ & m_Z^2(\mathcal{V}) = 2 \big(m_{x_3}^2 - m_{H_3}^2 \big), \quad \lambda_Z(\mathcal{V}) = 4 \big(2 g_{U(1)_3}^2 + \kappa^2 m_{x_3}^2 \big) \end{aligned} \tag{4.14}$$

The expressions for the mass and the coupling of the tachyonic field φ_-, with the explicit dependence on $\mathcal{V}$, (4.14), have already been found in the previous chapter and are given by (3.84), (3.85) and (3.86). The mass of the tachyonic field ψ_- in terms of g_s, $z(U_3)$, $m_a^{(i)}$, $n_1^{(2)}$ and the volume $\mathcal{V}$, is, using (4.12) and (4.10),

$$m_Z^2(\mathcal{V}) = 2\big(m_{x_3}^2 - m_{H_3}^2\big)$$

$$= 2\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}z(U_3)\left|\frac{{m_3^{(1)}}^2{m_3^{(2)}}^2}{m_1^{(2)}m_1^{(3)}m_2^{(1)}m_2^{(3)}}\right|^{1/3}\left(1-\frac{1}{\mathcal{V}^{2/3}}\frac{2k}{\pi z(U_3)}\left|\frac{m_1^{(2)}m_1^{(3)}m_2^{(1)}m_2^{(3)}}{{m_3^{(1)}}^5{m_3^{(2)}}^5}\right|^{1/6}\right)$$

$$= 2\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}z(U_3)\left|\frac{{m_3^{(1)}}^2{m_3^{(2)}}^2}{m_1^{(2)}m_1^{(3)}m_2^{(1)}m_2^{(3)}}\right|^{1/3}\left(1-\left(\frac{\mathcal{V}_{c3}}{\mathcal{V}}\right)^{2/3}\right) \tag{4.15}$$

with

$$\mathcal{V}_{c3} \equiv \left(\frac{2k}{\pi z(U_3)}\right)^{3/2}\left|\frac{m_1^{(2)}m_1^{(3)}m_2^{(1)}m_2^{(3)}}{{m_3^{(1)}}^5{m_3^{(2)}}^5}\right|^{1/4} \tag{4.16}$$

which is the critical value of the volume $\mathcal{V}$ that ψ_- becomes tachyonic, and k is given by (3.74). Also, from the relations (4.11) and (4.12), we find that its coupling is expressed through the parameters g_s, $z(U_3)$, $m_a^{(i)}$ and the volume $\mathcal{V}$, as

$$\lambda_Z(\mathcal{V}) = 4\big(2g_{U(1)_3}^2 + \kappa^2 m_{x_3}^2\big)$$

$$= \frac{4g_s}{\mathcal{V}^{2/3}}\left|\frac{1}{m_1^{(2)}m_1^{(3)}m_2^{(1)}m_2^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(2+g_s z(U_3)\big|m_3^{(1)}m_3^{(2)}\big|\right)$$

$$= \frac{4g_s}{\mathcal{V}^{2/3}}\left|\frac{1}{\prod_{a\neq i} m_a^{(i)}}\right|^{1/3}\left(2+g_s z(U_3)\big|m_3^{(1)}m_3^{(2)}\big|\right) \tag{4.17}$$

From the above, we see that the expressions for the mass and the coupling of the second tachyonic field ψ_-, are similar to ones for the first tachyonic field φ_-. We observe that now, the effective scalar potential (4.13), has taken the form of a hybrid potential with two waterfall fields: the φ_- and ψ_-. The squared masses of these two waterfall fields become negative below a critical value of the volume, which is defined as $\mathcal{V}_{c2}$ and $\mathcal{V}_{c3}$, respectively. As mentioned in the previous case, the critical value $\mathcal{V}_{c2}$ can be chosen to have the value of the minimum of the inflation potential, with an appropriate combination of its parameter values $\{m_a^{(i)}, n_1^{(2)}, y(U_2)\}$. Furthermore, as we see from (4.16) and (3.74), the critical value $\mathcal{V}_{c3}$ depends on the parameters $m_a^{(i)}$, $n_1^{(2)}$ and $z(U_3)$, so with an appropriate choice of their values, $\mathcal{V}_{c3}$ can have a value around and somewhat lower than the $\mathcal{V}_{c2}$ (which is something we want, as explained later). This means that exaclty after φ_- becomes tachyonic at the local minimum of the inflaton potential, ψ_- becomes tachyonic, too.

Thus, in the potential of the form (4.13), we have the following phases:

• For $\mathcal{V} > \mathcal{V}_{c2}$, where we have $m_Y^2(\mathcal{V}) > 0$ (as $y(U_2) > 0$) and $m_Z^2(\mathcal{V}) > 0$ (as $z(U_3) > 0$ and $\mathcal{V}_{c3} < \mathcal{V}_{c2} < \mathcal{V}$), both waterfall fields sit at their vanishing minima, $\langle\varphi_-\rangle = \langle\psi_-\rangle = 0$. Only the inflaton field contributes to the scalar potential, as in the first phase of the previous model.

• For $\mathcal{V}_{c3} < \mathcal{V} \lesssim \mathcal{V}_{c2}$, where we have $m_Y^2(\mathcal{V}) < 0$ (as $y(U_2) > 0$) and $m_Z^2(\mathcal{V}) > 0$ (as $z(U_3) > 0$), the first phase transition occurs and the waterfall field φ_- falls to its new non-vanishing VEV, $\langle\varphi_-\rangle = \pm\upsilon_2$, while the second waterfall field, ψ_-, still sits at $\langle\psi_-\rangle = 0$. The system is in the previously studied case, with the scalar potential rechieving contributions from the inflaton and one waterfall field only, and its minimum is at $\pm\upsilon_2$, given by (3.87). The first waterfall field, φ_-, is also responsible for the end of the inflationary stage again.

• For $\mathcal{V} \lesssim \mathcal{V}_{c3}$, where we have $m_Y^2(\mathcal{V}) < 0$ (as $y(U_2) > 0$) and $m_Z^2(\mathcal{V}) < 0$ (as $z(U_3) > 0$), the second phase transition occurs and the waterfall field ψ_- also falls to its new non-vanishing VEV, $\langle \psi_- \rangle = \pm \upsilon_3$. Now, there is the inflaton and two waterfall fields in the theory, and the potential minimum is at $\pm\upsilon_2$, $\pm\upsilon_3$, and aquires a lower value than in the phase above.

We are concentrating on the last phase above, where both waterfall directions are included in the theory. The first waterfall field is already at $\langle \varphi_- \rangle = \pm \upsilon_2 = \pm |m_Y|/\sqrt{\lambda_Y}$ and now the second waterfall field falls to its new VEV, which is of the same form,

$$\langle \psi_- \rangle = \pm \upsilon_3 = \pm \frac{|m_Z|}{\sqrt{\lambda_Z}} \tag{4.18}$$

The potential (4.13) at the new minimum, substituting the masses and the couplings of the waterfall fields, (3.84), (3.86), (4.15) and (4.17), is

$$\begin{aligned}
V(\mathcal{V}, \upsilon_2, \upsilon_3) &= \frac{C}{\kappa^4}\left(- \frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - \frac{1}{4}\frac{m_Y^4(\mathcal{V})}{\lambda_Y(\mathcal{V})} - \frac{1}{4}\frac{m_Z^4(\mathcal{V})}{\lambda_Z(\mathcal{V})} \\
&= \frac{C}{\kappa^4}\left(- \frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - \frac{1}{4}\frac{g_s^3 y^2(U_2)}{\kappa^4 \mathcal{V}^{2/3}} \left| \frac{{m_2^{(1)}}^5 {m_2^{(3)}}^5}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \frac{\left(1 - (\mathcal{V}_{c2}/\mathcal{V})^{2/3}\right)^2}{2 + g_s y(U_2) |m_2^{(1)} m_2^{(3)}|} \\
&\quad - \frac{1}{4}\frac{g_s^3 z^2(U_3)}{\kappa^4 \mathcal{V}^{2/3}} \left| \frac{{m_3^{(1)}}^5 {m_3^{(2)}}^5}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)}} \right|^{1/3} \frac{\left(1 - (\mathcal{V}_{c3}/\mathcal{V})^{2/3}\right)^2}{2 + g_s z(U_3) |m_3^{(1)} m_3^{(2)}|} \\
&= \frac{C}{\kappa^4}\left(- \frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - \sum_{a=2}^{3} \frac{C_a}{\kappa^4 \mathcal{V}^{2/3}} \left(1 - \left(\frac{\mathcal{V}_{ca}}{\mathcal{V}} \right)^{2/3} \right)^2
\end{aligned} \tag{4.19}$$

with

$$\begin{aligned}
C_2 &\equiv \frac{1}{4} \frac{g_s^3 y^2(U_2)}{2 + g_s y(U_2) |m_2^{(1)} m_2^{(3)}|} \left| \frac{{m_2^{(1)}}^5 {m_2^{(3)}}^5}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3}, \\
C_3 &\equiv \frac{1}{4} \frac{g_s^3 z^2(U_3)}{2 + g_s z(U_3) |m_3^{(1)} m_3^{(2)}|} \left| \frac{{m_3^{(1)}}^5 {m_3^{(2)}}^5}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)}} \right|^{1/3}
\end{aligned} \tag{4.20}$$

The coefficients (4.20) are written in terms of the parameter d and their respective critical volume values, as

$$\begin{aligned}
C_2 &= \beta_2 \frac{d}{3\mathcal{V}_{c2}^{4/3}}, \\
C_3 &= \beta_3 \frac{d}{3\mathcal{V}_{c3}^{4/3}}
\end{aligned} \tag{4.21}$$

with the following definitions

$$\begin{aligned}
\beta_2 &\equiv \frac{2}{2 + g_s y(U_2) |m_2^{(1)} m_2^{(3)}|}, \\
\beta_3 &\equiv \frac{2}{2 + g_s z(U_3) |m_3^{(1)} m_3^{(2)}|}
\end{aligned} \tag{4.22}$$

We observe that the parameter β_3 is in the region $\beta_3 \in [0, 1]$ too, as the parameter β_2, so the maximum negative contribution of the tachyonic fields to the potential minimum value (4.19), with respect to the β parameters, is for $\beta_2 = \beta_3 = 1$. Also, the largest possible contribution of the second tachyon, that is not restricted to happen at small volumes only (because if we increase the value of the coefficient C_3, the critical volume $\mathcal{V}_{c3}$ is driven towards lower values), is for $\mathcal{V}_{c3} \approx \mathcal{V}_{c2}$ (that is $C_3 \approx C_2$ from (4.21)). Then, if we imagine the double tachyonic contribution in (4.19) as a single contribution with effective tachyonic coeficient $C_{tot} = C_2 + C_3$, the corresponding effective β parameter would be $\beta_{tot} = \beta_2 + \beta_3 = 2$, always smaller than the required value (3.92) ($\simeq 3.228$) for an almost vanishing vacuum of the theory, that is, it cannot be low enough again.

Although it is expected that the vacuum cannot almost vanish, we study the effective scalar potential (4.13) at its new non-vanishing VEV, using the program Mathematica, aiming to find how deep these two waterfall directions together, can be:

We insert the effective scalar potential (4.13) with the relations (3.93) found in the previous chapter, plus the mass and the coupling that corresponds to the second tachyon of this section

$$m_Z^2 = 2\frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} z(U_3) \left| \frac{{m_3^{(1)}}^2 {m_3^{(2)}}^2}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)}} \right|^{1/3} \left(1 - \left(\frac{\mathcal{V}_{c3}}{\mathcal{V}}\right)^{2/3}\right),$$

$$\mathcal{V}_{c3} = \left(\frac{2k}{\pi z(U_3)}\right)^{3/2} \left| \frac{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)}}{{m_3^{(1)}}^5 {m_3^{(2)}}^5} \right|^{1/4},$$

$$\lambda_Z = \frac{4 g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \left(2 + g_s z(U_3) \left| m_3^{(1)} m_3^{(2)} \right| \right) \tag{4.23}$$

Then, we minimise the pontential with respect to φ_- and ψ_-, and repeat the procedure of the previous chapter: for different values of g_s, $y(U_2)$ and $z(U_3)$ that always give small products $g_s y(U_2)$ and $g_s z(U_3)$ (so that $\beta_2, \beta_3 \approx 1$), we search for the combinations $\{\mathcal{V}, m_a^{(i)}, n_1^{(2)}\}$ that give firstly a vanishing minimum, $V(\mathcal{V}, \upsilon_2, \upsilon_3) = 0$, and secondly, a positive minimum, $0 < V(\mathcal{V}, \upsilon_2, \upsilon_3) \lesssim 10^{-11}$. From these we choose those combinations that give a reasonable $\mathcal{V}_{c2}$ ($\simeq 201.9$), and resonable $\mathcal{V}_{c3}$, that is $\mathcal{V}_{c3} \lesssim \mathcal{V}_{c2}$ and $\mathcal{V}_{c3} \geq \mathcal{V}$, and investigate the parameter s for them.

It is found that in the case of the vanishing minimum, the parameter s generally does not approach the required range (3.33) for a de-Sitter minimum. On the other hand, positive minimum values of order $\sim 10^{-12}$ and 10^{-11} give values of s closer to the region (3.33). In the case of these positive minima, values of $\{g_s, y(U_2), z(U_3)\}$ that finally give an s closer to the range (3.33), are studied in detail by modifying them properly in order to get the desired values of s, $\mathcal{V}_{c2}$ and $\mathcal{V}_{c3}$.

We find the combination

$$g_s = 3.92611 \times 10^{-3}, \quad y(U_2) = 0.0771281, \quad z(U_3) = 0.107312, \quad n_1^{(2)} = 56,$$

$$m_1^{(2)} = 44, \quad m_1^{(3)} = 12, \quad m_2^{(1)} = 8, \quad m_2^{(3)} = 8, \quad m_3^{(1)} = 23, \quad m_3^{(2)} = 2 \tag{4.24}$$

which gives $s \simeq -0.00723953$, $\mathcal{V}_{c2} \simeq 201.9$, $\mathcal{V}_{c3} \simeq 201.89$ and $\beta_2, \beta_3 \sim 1$, with a potential minimum at

$$\mathcal{V}_{min} \simeq 137.5 \rightarrow \quad V(\mathcal{V}_{min}, \upsilon_2, \upsilon_3) \simeq 1.14958 \times 10^{-11} \tag{4.25}$$

The potential minimum value in terms of the volume $\mathcal{V}$, with the (4.24) parameter values incorporated, is shown in the Figure 4.1. The seperate contributions of the inflaton and of the two waterfal fields to the potential (4.19) does not cancel enough in this model either

$$V(\mathcal{V}, \upsilon_2, \upsilon_3) = 2.15743 \times 10^{-11} - 1.00785 \times 10^{-11}$$

and as it is also explained above, the resulting value of the vacuum is not close to the desired almost vanishing one. Besides that, we observe that two waterfall fields provide a bigger negative contribution and thus, the minimum of the potential in this model is lower than the one of the previous case with one waterfall field, (3.95). In Figure 4.2, we can see the two waterfall contributions together in terms of $\mathcal{V}$, with a maximum near $\simeq 201.9$.

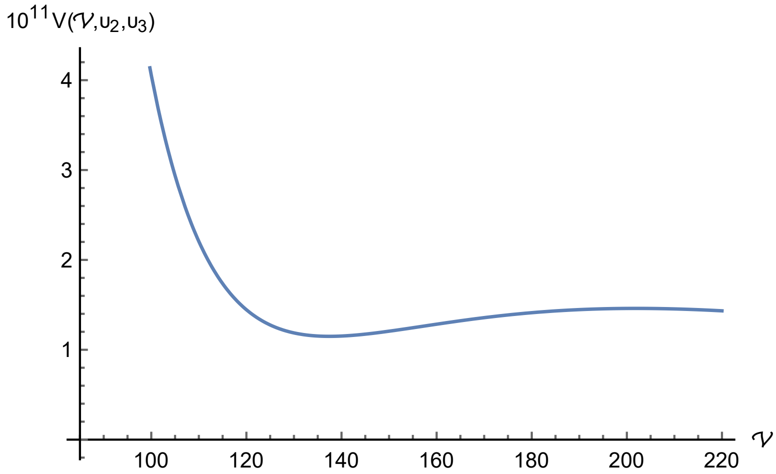

Figure 4.1: Minimum value of the effective scalar potential of the inflaton and the two waterfall fields in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (4.24).

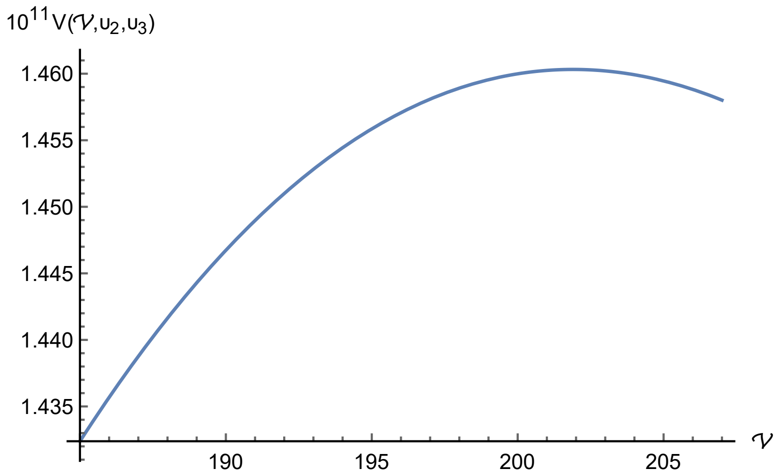

Figure 4.2: Waterfall direction in the effective scalar potential of the inflaton and the two waterfall fields in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (4.24).

Moreover, from (4.23) we have

$$\frac{m_Z^2}{\lambda_Z} = \frac{1}{2\kappa^2}\frac{g_s z(U_3)\left|m_3^{(1)}m_3^{(2)}\right|}{2+g_s z(U_3)\left|m_3^{(1)}m_3^{(2)}\right|}\left(1-\left(\frac{\mathcal{V}_{c3}}{\mathcal{V}}\right)^{2/3}\right)$$

$$\rightarrow \langle\psi_-\rangle = \pm\upsilon_3 = \pm\frac{|m_Z|}{\sqrt{\lambda_Z}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s z(U_3)\left|m_3^{(1)}m_3^{(2)}\right|}{2+g_s z(U_3)\left|m_3^{(1)}m_3^{(2)}\right|}}\left|1-\left(\frac{\mathcal{V}_{c3}}{\mathcal{V}}\right)^{2/3}\right|^{1/2} \tag{4.26}$$

from which we see that the υ_3 VEV depends on the values of the product $g_s z(U_3)$, the critical value $\mathcal{V}_{c3}$ and the integer values $m_3^{(i)}$ as well. From the combination (4.24) found, the range of this VEV in κ units is

$$\kappa\upsilon_3 \in [0, \simeq 0.0374224] \tag{4.27}$$

Also, the expression for the first VEV (3.96)

$$\langle\varphi_-\rangle = \pm\upsilon_2 = \pm\frac{|m_Y|}{\sqrt{\lambda_Y}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s y(U_2)\left|m_2^{(1)}m_2^{(3)}\right|}{2+g_s y(U_2)\left|m_2^{(1)}m_2^{(3)}\right|}}\left|1-\left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{2/3}\right|^{1/2}$$

gives for the combination (4.24), the following range (in κ units),

$$\kappa\upsilon_2 \in [0, \simeq 0.0374245] \tag{4.28}$$

so the two VEV's of the theory stay small, and the quartic expansions in (4.1) hold.

4.2 Third waterfall field

We now have another one field in the theory that becomes tachyonic, the one coming from the last one brane, $D7_1$, and is denoted as χ_-. We firstly find the effective potential of the four fields in the large volume limit: the inflaton field and the three tachyonic fields φ_-, ψ_-, χ_-. Then, we investigate again their contributions to the potential minimum, using Mathematica.

Exaclty as for $\left(g_{U(1)_3}^2\xi_3^2\right)/2$ in the previous section, the term $\left(g_{U(1)_1}^2\xi_1^2\right)/2$ of the D-term part in the effective scalar potential (4.1), is now replaced by the term

$$\frac{g_{U(1)_1}^2}{2}\left(\xi_1 + 2\left|\chi_+\right|^2 - 2\left|\chi_-\right|^2 + ...\right)^2$$

$$= \frac{g_{U(1)_1}^2}{2}\xi_1^2 + 2g_{U(1)_1}^2\xi_1\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right) + 2g_{U(1)_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right)^2 \tag{4.29}$$

where χ_+ is the charge conjugate of χ_-. Substituting again all the mass contributions m_{H_1}, m_{H_2}, m_{H_3} as given in the relations (3.44), (3.50), and the Fayet-Iliopoulos parameters ξ_1, ξ_2, ξ_3 as given in (3.48), (3.51) into the D-term part, the $D7_1$-contribution (4.29) will be

$$\frac{g_{U(1)_1}^2}{2}\xi_1^2 + m_{H_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right) + 2g_{U(1)_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right)^2$$

$$= \frac{1}{2}\xi_1\frac{m_{H_1}^2}{2} + m_{H_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right) + 2g_{U(1)_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right)^2$$

$$= \frac{1}{2}\frac{g_s^3\left|k_1^{(2)}\right|^2\left|m_1^{(2)}m_1^{(3)}\right|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_3}{\mathcal{A}_2} + m_{H_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right) + 2g_{U(1)_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right)^2$$

$$= \frac{1}{\kappa^4\mathcal{V}^2}d_1\frac{\mathcal{A}_3}{\mathcal{A}_2} + m_{H_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right) + 2g_{U(1)_1}^2\left(\left|\chi_+\right|^2 - \left|\chi_-\right|^2\right)^2 \tag{4.30}$$

with the definition (4.3) for d_1, so the D-term part is exaclty as before, (4.2), but with the extra last two terms of (4.30).

After the minimisation of the moduli ratios, using the moduli stabilization condition (3.68), the magnetic field contribution to the mass of the tachyon χ_-, becomes

$$m_{H_1}^2 \equiv 2g_{U(1)_1}^2 \xi_1 \approx \frac{2|k_1^{(2)}|}{\pi} \frac{g_s^2}{\kappa^2 \mathcal{V}} \frac{\alpha'}{\mathcal{A}_2} = 2\sqrt{2}|m_1^{(2)} m_1^{(3)}|^{-1/2} \frac{\sqrt{g_s}}{\kappa^2 \mathcal{V}^{4/3}} (d_3^2 d_1)^{1/6} \tag{4.31}$$

(where we also use (4.3) to replace the flux $k_1^{(2)}$), and the gauge coupling becomes

$$g_{U(1)_1}^2 \approx |m_1^{(2)} m_1^{(3)}|^{-1} \frac{g_s}{\mathcal{V}} \frac{\mathcal{A}_1}{\alpha'} = |m_1^{(2)} m_1^{(3)}|^{-1} \frac{g_s}{\mathcal{V}^{2/3}} \left(\frac{d_3}{d_2}\right)^{1/3} \tag{4.32}$$

We also have the following F-term contributions as before, with regard to the tachyonic field χ_-: The mass contribution

$$V_F \ni m_{x_1}^2 \left(|\chi_+|^2 + |\chi_-|^2 \right) \tag{4.33}$$

where $m_{x_1}^2$ is the physical mass coming from the brane position x_1

$$m_{x_1}^2 = x(U_1) \frac{g_s^2}{\kappa^2 \mathcal{V}} \frac{\mathcal{A}_1}{\alpha'} \tag{4.34}$$

with $x(U_1)$ being the analogous quantity to $y(U_2)$ and $z(U_3)$, and the leading quartic contribution

$$V_F \ni \kappa^2 m_{x_1}^2 |\chi_-|^4 \tag{4.35}$$

The mass contribution (4.34), using (3.68), becomes

$$m_{x_1}^2 = \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} x(U_1) \left(\frac{d_3}{d_2}\right)^{1/3} \tag{4.36}$$

As before, we express $m_{H_1}^2$, $g_{U(1)_1}^2$ and $m_{x_1}^2$ in terms of the parameters g_s, $x(U_1)$, $m_a^{(i)}$, $n_1^{(2)}$ and $\mathcal{V}$: Using (4.3), and (3.72) (condition for the elimination of tachyons from different brane interstections*) to replace the appearing flux $k_3^{(1)}$, (4.31) becomes

$$m_{H_1}^2 = m_{H_3}^2 = m_{H_2}^2 = 2 \frac{g_s^2 k}{\pi \kappa^2 \mathcal{V}^{4/3}} \left| \frac{1}{\prod_{a \neq i} m_a^{(i)}} \right|^{1/6} \tag{4.37}$$

Using (4.3) again, and (3.72) to replace $k_3^{(1)}$ and $k_2^{(3)}$, (4.32) becomes

$$g_{U(1)_1}^2 = g_{U(1)_3}^2 = g_{U(1)_2}^2 = \frac{g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{\prod_{a \neq i} m_a^{(i)}} \right|^{1/3} \tag{4.38}$$

Finally, using (4.3), and (3.72) to replace $k_3^{(1)}$ and $k_2^{(3)}$, (4.36) becomes

$$m_{x_1}^2 = \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} x(U_1) \left| \frac{{m_1^{(2)}}^2 {m_1^{(3)}}^2}{m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \tag{4.39}$$

expression similar to the ones for $m_{x_3}^2$ and $m_{x_2}^2$, but with the corresponding quantities for the $D7_1$ brane.

The approximate new effective scalar potential of the volume modulus $\mathcal{V}$ and the tachyonic scalar fields φ_-, ψ_-, χ_- is the sum of the F-part for the volume modulus (3.57), the D-part in the previous section with the extra last two terms of (4.30), and the F-parts (3.54), (3.56), (4.6), (4.8), (4.33) and (4.35), negleting their charge conjugates,

$$V(\mathcal{V},\varphi_-,\psi_-,\chi_-)$$

$$= \frac{C}{\kappa^4}\bigg(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3}-\frac{s}{\mathcal{V}^2}\bigg) - m_{H_1}^2\,|\chi_-|^2 + 2g_{U(1)_1}^2\,|\chi_-|^4 + m_{x_1}^2\,|\chi_-|^2 + \kappa^2 m_{x_1}^2|\chi_-|^4$$
$$-m_{H_3}^2\,|\psi_-|^2 + 2g_{U(1)_3}^2\,|\psi_-|^4 + m_{x_3}^2\,|\psi_-|^2 + \kappa^2 m_{x_3}^2|\psi_-|^4$$
$$-m_{H_2}^2\,|\varphi_-|^2 + 2g_{U(1)_2}^2\,|\varphi_-|^4 + m_{x_2}^2\,|\varphi_-|^2 + \kappa^2 m_{x_2}^2|\varphi_-|^4$$
$$= \frac{C}{\kappa^4}\bigg(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3}-\frac{s}{\mathcal{V}^2}\bigg) + \frac{1}{2}m_Y^2(\mathcal{V})\,|\varphi_-|^2 + \frac{1}{4}\lambda_Y(\mathcal{V})\,|\varphi_-|^4$$
$$+\frac{1}{2}m_Z^2(\mathcal{V})\,|\psi_-|^2 + \frac{1}{4}\lambda_Z(\mathcal{V})\,|\psi_-|^4 + \frac{1}{2}m_X^2(\mathcal{V})\,|\chi_-|^2 + \frac{1}{4}\lambda_X(\mathcal{V})\,|\chi_-|^4 \tag{4.40}$$

with

$$C = -3\mathcal{W}_o^2\gamma, \quad q = \frac{\xi}{2\gamma}, \quad s = \frac{3}{2}\sigma = \frac{d}{3\mathcal{W}_o^2\gamma} = -\frac{d}{C} = \frac{3}{2C}g_s^3\bigg(\frac{k}{\pi}\bigg)^2,$$
$$m_Y^2(\mathcal{V}) = 2\big(m_{x_2}^2 - m_{H_2}^2\big), \quad \lambda_Y(\mathcal{V}) = 4\big(2g_{U(1)_2}^2 + \kappa^2 m_{x_2}^2\big),$$
$$m_Z^2(\mathcal{V}) = 2\big(m_{x_3}^2 - m_{H_3}^2\big), \quad \lambda_Z(\mathcal{V}) = 4\big(2g_{U(1)_3}^2 + \kappa^2 m_{x_3}^2\big)$$
$$m_X^2(\mathcal{V}) = 2\big(m_{x_1}^2 - m_{H_1}^2\big), \quad \lambda_X(\mathcal{V}) = 4\big(2g_{U(1)_1}^2 + \kappa^2 m_{x_1}^2\big) \tag{4.41}$$

The expressions for the mass and the coupling of the tachyonic fields φ_-, ψ_-, with the explicit dependence on $\mathcal{V}$, (4.41), are given by (3.84), (3.85) and (3.86), and (4.15), (4.16) and (4.17), respectively. The mass of the tachyonic field χ_- in terms of g_s, $x(U_1)$, $m_a^{(i)}$, $n_1^{(2)}$ and the volume $\mathcal{V}$, is, using (4.37) and (4.39),

$$m_X^2(\mathcal{V}) = 2\big(m_{x_1}^2 - m_{H_1}\big)$$

$$= 2\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}x(U_1)\left|\frac{{m_1^{(2)}}^2{m_1^{(3)}}^2}{m_2^{(1)}m_2^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(1-\frac{1}{\mathcal{V}^{2/3}}\frac{2k}{\pi x(U_1)}\left|\frac{m_2^{(1)}m_2^{(3)}m_3^{(1)}m_3^{(2)}}{{m_1^{(2)}}^5{m_1^{(3)}}^5}\right|^{1/6}\right)$$
$$= 2\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}x(U_1)\left|\frac{{m_1^{(2)}}^2{m_1^{(3)}}^2}{m_2^{(1)}m_2^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(1-\left(\frac{\mathcal{V}_{c1}}{\mathcal{V}}\right)^{2/3}\right) \tag{4.42}$$

with

$$\mathcal{V}_{c1} \equiv \left(\frac{2k}{\pi x(U_1)}\right)^{3/2}\left|\frac{m_2^{(1)}m_2^{(3)}m_3^{(1)}m_3^{(2)}}{{m_1^{(2)}}^5{m_1^{(3)}}^5}\right|^{1/4} \tag{4.43}$$

which is the critical value of the volume $\mathcal{V}$ that χ_- becomes tachyonic, and k is given by (3.74). Also, from the relations (4.38) and (4.39), we find that its coupling is expressed through the parameters g_s, $x(U_1)$, $m_a^{(i)}$ and the volume $\mathcal{V}$, as

$$\lambda_X(\mathcal{V}) = 4\big(2g_{U(1)_1}^2 + \kappa^2 m_{x_1}^2\big)$$

$$= \frac{4g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \left(2 + g_s x(U_1) \big| m_1^{(2)} m_1^{(3)} \big| \right)$$

$$= \frac{4g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{\prod_{a \neq i} m_a^{(i)}} \right|^{1/3} \left(2 + g_s x(U_1) \big| m_1^{(2)} m_1^{(3)} \big| \right) \tag{4.44}$$

The above expressions for the mass of the tachyon χ_- and the coupling, are similar to the ones for the two previous tachyons. The effective scalar potential (4.40) has taken the form of a hybrid potential with three waterfall fields: the φ_-, the ψ_- and the χ_-. The squared mass of the third tachyon also becomes tachyonic below a critical value of $\mathcal{V}$ which is defined as $\mathcal{V}_{c1}$, and depends on the parameters $m_a^{(i)}$, $n_1^{(2)}$ and $x(U_1)$, so with an appropriate choice of these parameter values the critical volume $\mathcal{V}_{c1}$ can have a value near and below $\mathcal{V}_{c3}$, so that the third tachyon χ_- becomes tachyonic exactly after the second tachyon ψ_- becomes tachyonic.

Thus, in the potential of the form (4.40), we have the following phases:

- For $\mathcal{V} > \mathcal{V}_{c2}$, where we have $m_Y^2(\mathcal{V}) > 0$, $m_Z^2(\mathcal{V}) > 0$ and $m_X^2(\mathcal{V}) > 0$ (as $y(U_2) > 0$, $z(U_3) > 0$, $x(U_1) > 0$ and $\mathcal{V}_{c1} < \mathcal{V}_{c3} < \mathcal{V}_{c2} < \mathcal{V}$), the three waterfall fields sit at their vanishing minima, $\langle \varphi_- \rangle = \langle \psi_- \rangle = \langle \chi_- \rangle = 0$, and only the inflaton field contributes to the scalar potential.

- For $\mathcal{V}_{c3} < \mathcal{V} \lesssim \mathcal{V}_{c2}$, where we have $m_Y^2(\mathcal{V}) < 0$, $m_Z^2(\mathcal{V}) > 0$ and $m_X^2(\mathcal{V}) > 0$ (as $y(U_2) > 0$, $z(U_3) > 0$ and $x(U_1) > 0$), a phase transition occurs and the waterfall field φ_- falls to its new non-vanishing VEV, $\langle \varphi_- \rangle = \pm \upsilon_2$, ending the inflationary stage, while the other waterfall fields still sit at $\langle \psi_- \rangle = \langle \chi_- \rangle = 0$. The system is in the case studied in the previous chapter, where the scalar potential gets contributions from the inflaton and one waterfall field only, and its minimum is at $\pm \upsilon_2$, given by (3.87).

- For $\mathcal{V}_{c1} < \mathcal{V} \lesssim \mathcal{V}_{c3}$, where we have $m_Y^2(\mathcal{V}) < 0$, $m_Z^2(\mathcal{V}) < 0$ and $m_X^2(\mathcal{V}) > 0$ (as $y(U_2) > 0$, $z(U_3) > 0$ and $x(U_1) > 0$), another phase transition occurs and the waterfall field ψ_- also falls to its new non-vanishing VEV, $\langle \psi_- \rangle = \pm \upsilon_3$, while the third field is still at $\langle \chi_- \rangle = 0$. The system is in the case studied in the previous section, with the inflaton field and two waterfall fields in the theory. The potential minimum is at $\pm \upsilon_2$, $\pm \upsilon_3$, and aquires a lower value than in the phase above.

- For $\mathcal{V} \lesssim \mathcal{V}_{c1}$, where we have $m_Y^2(\mathcal{V}) < 0$, $m_Z^2(\mathcal{V}) < 0$ and $m_X^2(\mathcal{V}) < 0$ (as $y(U_2) > 0$, $z(U_3) > 0$ and $x(U_1) > 0$), the third phase transition occurs and the waterfall field χ_- is driven to its new non-vanishing VEV, $\langle \chi_- \rangle = \pm \upsilon_1$. Now, there is the inflaton field and three waterfall fields in the theory, and the potential minimum is at $\pm \upsilon_2$, $\pm \upsilon_3$, $\pm \upsilon_1$ aquiring a lower value than before.

We are concentrating again on the last phase above, where now three waterfall directions are included in the theory. The first and second waterfall fields are already at $\langle \varphi_- \rangle = \pm \upsilon_2 = \pm |m_Y| / \sqrt{\lambda_Y}$ and $\langle \psi_- \rangle = \pm \upsilon_3 = \pm |m_Z| / \sqrt{\lambda_Z}$, respectively, and now the third waterfall field falls to its new VEV, which is

$$\langle \chi_- \rangle = \pm \upsilon_1 = \pm \frac{|m_X|}{\sqrt{\lambda_X}} \tag{4.45}$$

The potential (4.40) at the new minimum, substituting the masses and the couplings of the waterfall fields, (3.84), (3.86), (4.15), (4.17), (4.42) and (4.44), is

$$V(\mathcal{V}, \upsilon_2, \upsilon_3, \upsilon_1)$$

$$= \frac{C}{\kappa^4} \left(- \frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - \frac{1}{4} \frac{m_Y^4(\mathcal{V})}{\lambda_Y(\mathcal{V})} - \frac{1}{4} \frac{m_Z^4(\mathcal{V})}{\lambda_Z(\mathcal{V})} - \frac{1}{4} \frac{m_X^4(\mathcal{V})}{\lambda_X(\mathcal{V})}$$

$$= \frac{C}{\kappa^4}\left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2}\right) - \frac{1}{4}\frac{g_s^3 y^2(U_2)}{\kappa^4 \mathcal{V}^{2/3}}\left|\frac{{m_2^{(1)}}^5 {m_2^{(3)}}^5}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3} \frac{\left(1 - (\mathcal{V}_{c2}/\mathcal{V})^{2/3}\right)^2}{2 + g_s y(U_2)\left|m_2^{(1)} m_2^{(3)}\right|}$$

$$- \frac{1}{4}\frac{g_s^3 z^2(U_3)}{\kappa^4 \mathcal{V}^{2/3}}\left|\frac{{m_3^{(1)}}^5 {m_3^{(2)}}^5}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)}}\right|^{1/3} \frac{\left(1 - (\mathcal{V}_{c3}/\mathcal{V})^{2/3}\right)^2}{2 + g_s z(U_3)\left|m_3^{(1)} m_3^{(2)}\right|}$$

$$- \frac{1}{4}\frac{g_s^3 x^2(U_1)}{\kappa^4 \mathcal{V}^{2/3}}\left|\frac{{m_1^{(2)}}^5 {m_1^{(3)}}^5}{m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3} \frac{\left(1 - (\mathcal{V}_{c1}/\mathcal{V})^{2/3}\right)^2}{2 + g_s x(U_1)\left|m_1^{(2)} m_1^{(3)}\right|}$$

$$= \frac{C}{\kappa^4}\left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2}\right) - \sum_{a=1}^{3} \frac{C_a}{\kappa^4 \mathcal{V}^{2/3}}\left(1 - \left(\frac{\mathcal{V}_{ca}}{\mathcal{V}}\right)^{2/3}\right)^2 \tag{4.46}$$

with

$$C_2 \equiv \frac{1}{4}\frac{g_s^3 y^2(U_2)}{2 + g_s y(U_2)\left|m_2^{(1)} m_2^{(3)}\right|}\left|\frac{{m_2^{(1)}}^5 {m_2^{(3)}}^5}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3},$$

$$C_3 \equiv \frac{1}{4}\frac{g_s^3 z^2(U_3)}{2 + g_s z(U_3)\left|m_3^{(1)} m_3^{(2)}\right|}\left|\frac{{m_3^{(1)}}^5 {m_3^{(2)}}^5}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)}}\right|^{1/3},$$

$$C_1 \equiv \frac{1}{4}\frac{g_s^3 x^2(U_1)}{2 + g_s x(U_1)\left|m_1^{(2)} m_1^{(3)}\right|}\left|\frac{{m_1^{(2)}}^5 {m_1^{(3)}}^5}{m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3} \tag{4.47}$$

The coefficients (4.47) are written in terms of the parameter d and their respective critical volume values, as

$$C_2 = \beta_2 \frac{d}{3\mathcal{V}_{c2}^{4/3}},$$

$$C_3 = \beta_3 \frac{d}{3\mathcal{V}_{c3}^{4/3}},$$

$$C_1 = \beta_1 \frac{d}{3\mathcal{V}_{c1}^{4/3}} \tag{4.48}$$

with the following definitions

$$\beta_2 \equiv \frac{2}{2 + g_s y(U_2)\left|m_2^{(1)} m_2^{(3)}\right|},$$

$$\beta_3 \equiv \frac{2}{2 + g_s z(U_3)\left|m_3^{(1)} m_3^{(2)}\right|},$$

$$\beta_1 \equiv \frac{2}{2 + g_s x(U_1)\left|m_1^{(2)} m_1^{(3)}\right|} \tag{4.49}$$

We observe that the parameter β_1 is in the region $\beta_1 \in [0, 1]$ too, as the parameters β_2 and β_3, so the maximum negative contribution of the tachyonic fields to the potential minimum value (4.46), with respect to the β parameters, is for $\beta_2 = \beta_3 = \beta_1 = 1$. Also, the largest possible contribution of the third tachyon, that is not restricted to happen at small volumes only, is for $\mathcal{V}_{c1} \approx \mathcal{V}_{c3} \approx \mathcal{V}_{c2}$ (that is $C_1 \approx C_3 \approx C_2$ from (4.48)). Then, if we imagine the three tachyonic contributions in (4.46) as a single contribution with effective tachyonic coeficient

$C_{tot} = C_2 + C_3 + C_1$, the corresponding effective β parameter would be $\beta_{tot} = \beta_2 + \beta_3 + \beta_1 = 3$, always smaller than the required value (3.92) ($\simeq 3.228$) for an almost vanishing vacuum of the theory.

We study again the derived effective scalar potential (4.40) at its new non-vanishing VEV, using the program Mathematica:

We instert the potential (4.40) with the relations (3.93) and (4.23), plus the mass and the coupling that corresponds to the third tachyon

$$m_X^2 = 2\frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} x(U_1) \left| \frac{{m_1^{(2)}}^2 {m_1^{(3)}}^2}{m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \left(1 - \left(\frac{\mathcal{V}_{c1}}{\mathcal{V}}\right)^{2/3}\right),$$

$$\mathcal{V}_{c1} = \left(\frac{2k}{\pi x(U_1)}\right)^{3/2} \left| \frac{m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}}{{m_1^{(2)}}^5 {m_1^{(3)}}^5} \right|^{1/4},$$

$$\lambda_X = \frac{4 g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \left(2 + g_s x(U_1) |m_1^{(2)} m_1^{(3)}|\right) \tag{4.50}$$

Then, we minimise the pontential with respect to φ_-, ψ_- and χ_-, and repeat the procedure of the previous section: for different values of g_s, $y(U_2)$, $z(U_3)$ and $x(U_1)$, that always give small products $g_s y(U_2)$, $g_s z(U_3)$ and $g_s x(U_1)$ (so that $\beta_2, \beta_3, \beta_1 \approx 1$), we search for the combinations $\{\mathcal{V}, m_a^{(i)}, n_1^{(2)}\}$ that give firstly a vanishing minimum, $V(\mathcal{V}, \upsilon_2, \upsilon_3, \upsilon_1) = 0$, and secondly, a positive minimum, $0 < V(\mathcal{V}, \upsilon_2, \upsilon_3, \upsilon_1) \lesssim 10^{-11}$. From these combinations we would like to choose those that give a reasonable $\mathcal{V}_{c2}$ ($\simeq 201.9$), $\mathcal{V}_{c1} \lesssim \mathcal{V}_{c3} \lesssim \mathcal{V}_{c2}$ and $\mathcal{V}_{c1} \geq \mathcal{V}$, and then investigate the parameter s for them.

However, this procedure did not find any solutions $\{\mathcal{V}, m_a^{(i)}, n_1^{(2)}\}$ in either case that satisfy all the above constraints, so for different values of g_s, $y(U_2)$, $z(U_3)$ and $x(U_1)$, we only searched for vanishing and positive minimum solutions in general, and concentrated on the s parameter that they give, regardless the values of their critical volumes. It is found that for positive minimum values of order $\sim 10^{-12}$ and 10^{-11}, the $g_s \sim 3.9 \times 10^{-3}$ gives an s closer to the range (3.33) for a dS minimum. These values of g_s among with their $y(U_2)$'s, $z(U_3)$'s and $x(U_1)'s$, are modified properly in order to get the desired values of s, $\mathcal{V}_{c2}$, $\mathcal{V}_{c3}$ and $\mathcal{V}_{c1}$.

We find the combination

$$g_s = 3.91452 \times 10^{-3}, \quad y(U_2) = 0.013852, \quad z(U_3) = 0.0218498, \quad x(U_1) = 0.0153587, \quad n_1^{(2)} = 25,$$

$$m_1^{(2)} = 23, \quad m_1^{(3)} = 36, \quad m_2^{(1)} = 18, \quad m_2^{(3)} = 51, \quad m_3^{(1)} = 97, \quad m_3^{(2)} = 6 \tag{4.51}$$

which gives $s \simeq -0.00723954$, $\mathcal{V}_{c2} \simeq 201.901$, $\mathcal{V}_{c3} \simeq 201.89$, $\mathcal{V}_{c1} \simeq 201.88$ and $\beta_2, \beta_3, \beta_1 \sim 1$, with a potential minimum at

$$\mathcal{V}_{min} \simeq 119.44 \rightarrow \quad V(\mathcal{V}_{min}, \upsilon_2, \upsilon_3, \upsilon_1) \simeq 3.81356 \times 10^{-12} \tag{4.52}$$

The potential minimum value in terms of the volume $\mathcal{V}$ with the (4.51) parameter values incorporated, is shown in the Figure 4.3. The seperate contributions of the inflaton and of the three waterfall fields to the potential (4.46) are

$$V(\mathcal{V}, \upsilon_2, \upsilon_3, \upsilon_1) = 3.75285 \times 10^{-11} - 3.3715 \times 10^{-11}$$

and as it is also explained above through the β_i parameters, the value of the vacuum in this model cannot almost vanish as required, even though it is closer to, than the previously studied cases. In Figure 4.4, we see the three waterfall directions together in terms of $\mathcal{V}$, with a maximum near $\simeq 201.9$ again. Finally, in Figure 4.5, we also show the potential minimum value in terms of $\mathcal{V}$, in the three already studied cases, together. We observe that each time we add a waterfall field indeed the vacuum gets a lower value, which is displaced towards lower values of the volume $\mathcal{V}$.

From (4.50) we have

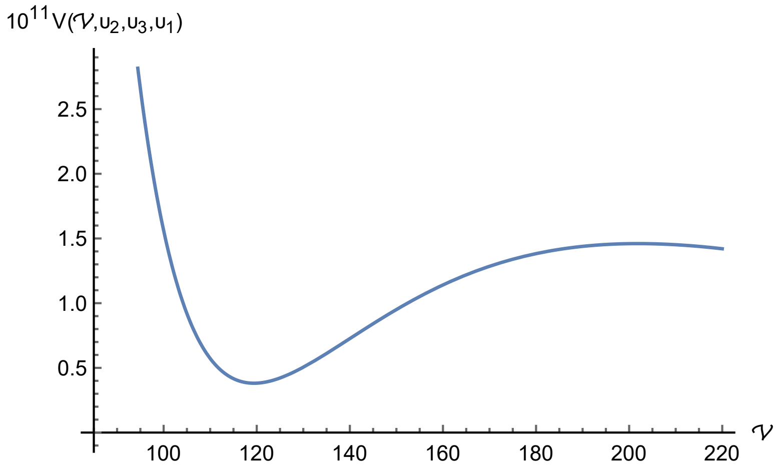

Figure 4.3: Minimum value of the effective scalar potential of the inflaton and the three waterfall fields in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (4.51).

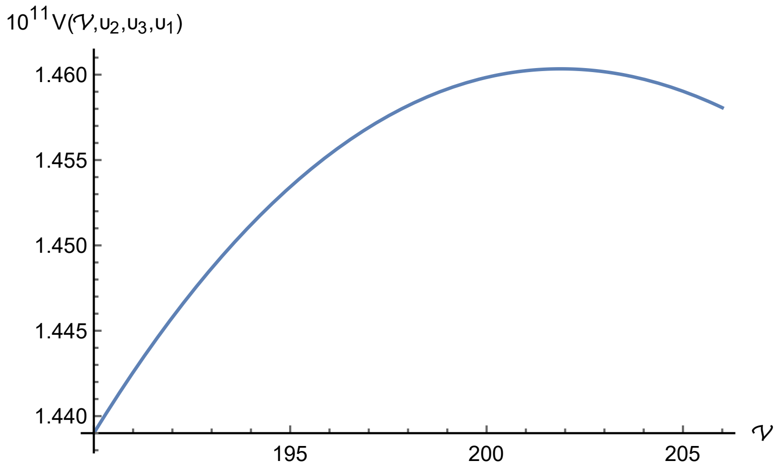

Figure 4.4: Waterfall direction in the effective scalar potential of the inflaton and the three waterfall fields in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (4.51).

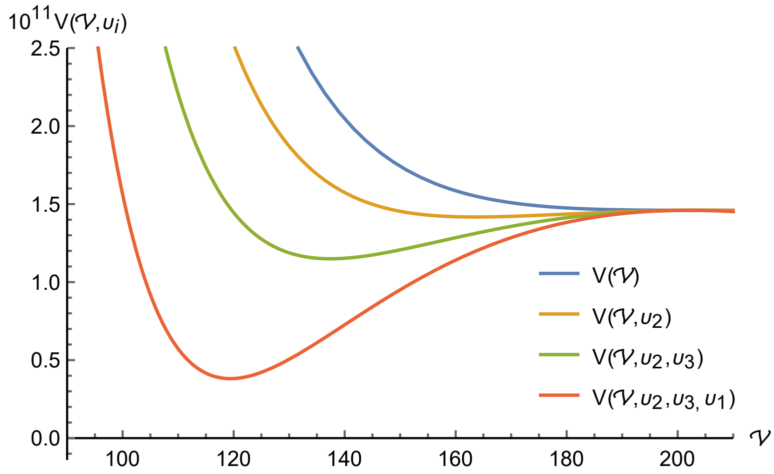

Figure 4.5: Minimum value of the effective scalar potential of the inflaton field only (blue), and of the inflaton with one (orange), two (green), and three waterfall fields (red), in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the different combinations of parameter values found in each case.

$$\frac{m_X^2}{\lambda_X} = \frac{1}{2\kappa^2}\frac{g_s x(U_1)\big|m_1^{(2)}m_1^{(3)}\big|}{2+g_s x(U_1)\big|m_1^{(2)}m_1^{(3)}\big|}\left(1-\left(\frac{\mathcal{V}_{c1}}{\mathcal{V}}\right)^{2/3}\right)$$

$$\to \langle\chi_-\rangle = \pm\upsilon_1 = \pm\frac{|m_X|}{\sqrt{\lambda_X}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s x(U_1)\big|m_1^{(2)}m_1^{(3)}\big|}{2+g_s x(U_1)\big|m_1^{(2)}m_1^{(3)}\big|}}\left|1-\left(\frac{\mathcal{V}_{c1}}{\mathcal{V}}\right)^{2/3}\right|^{1/2} \tag{4.53}$$

from which we see that the υ_1 VEV depends on the values of the product $g_s x(U_1)$, the critical value $\mathcal{V}_{c1}$ and the integer values of $m_1^{(i)}$. From the combination (4.51) found, the range of this VEV in κ units is

$$\kappa\upsilon_1 \in [0, \simeq 0.071324] \tag{4.54}$$

Also, from (3.96)

$$\langle\varphi_-\rangle = \pm\upsilon_2 = \pm\frac{|m_Y|}{\sqrt{\lambda_Y}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s y(U_2)\big|m_2^{(1)}m_2^{(3)}\big|}{2+g_s y(U_2)\big|m_2^{(1)}m_2^{(3)}\big|}}\left|1-\left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{2/3}\right|^{1/2}$$

for the combination (4.51), the range of the υ_2 VEV in κ units, is

$$\kappa\upsilon_2 \in [0, \simeq 0.07133] \tag{4.55}$$

and from (4.26)

$$\langle\psi_-\rangle = \pm\upsilon_3 = \pm\frac{|m_Z|}{\sqrt{\lambda_Z}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s z(U_3)\big|m_3^{(1)}m_3^{(2)}\big|}{2+g_s z(U_3)\big|m_3^{(1)}m_3^{(2)}\big|}}\left|1-\left(\frac{\mathcal{V}_{c3}}{\mathcal{V}}\right)^{2/3}\right|^{1/2}$$

for the combination (4.51), the range of the υ_3 VEV in κ units, is

$$\kappa\upsilon_3 \in [0, \simeq 0.0713269] \tag{4.56}$$

so, the three VEV's of the theory stay small and the quartic expansions in (4.1) and (4.29) hold.

Chapter 5

Fourth D7-brane stack

In the above chapters, we have shown through the program Mathematica, that the more waterfall fields we add in the theory the deeper the minimum of the potential becomes by choosing appropriate values for its parameters, we could not reach though the dark energy observational value yet. In this chapter, we study also the case proposed in [62] with an additional $D7$ brane stack included in the model, from which additional waterfall fields make their appearance, and together with the previous existing waterfall fields may be able to contribute negatively enough to the vacuum.

In fact, from the discussions in the previous cases about the parameters β_i, which essentially count the contribution of each tachyonic field in the potential minimum, we see that the addition of a fourth tachyonic field in the potential could possibly solve the problem, as the effective parameter β could reach the desired value (3.92). In [62], a fourth $D7$ brane stack is added in the theory and is parallel to the already existing $D7_2$ brane stack. This induces not only one, but two extra tachyonic fields which provide two new waterfall directions that are expected to deepen maybe more than enough the previously studied vacuum cases. We have the ability to choose the waterfall fields to become tachyonic successively, as before.

In the first section of this chapter, we describe all the D- and F-term parts in the effective scalar potential in this theory. In the second section, we derive the new relations between the fluxes using the new condition for the elimination of tachyonic fields from different brane intersections. In the third section, we study the resulting effective scalar potential of the inflaton and the tachyonic fields, and in the fourth section, the negative contribution of the latter in the potential minimum.

5.1 D- and F-term contributions

In the model we study now, we have an extra $D7$ brane stack, which is denoted as $D7_{2b}$, and is parallel to the previous $D7_2$ brane stack, which now is denoted as $D7_{2a}$. We name their corresponding tachyonic fields as y_{b-} and y_{a-}, respectively.

The new D-term contribution from the magnetic fields in the effective scalar potential, neglecting the charge conjugates of the fields, becomes [62]

$$V_D = \sum_{k=3,1} \frac{g^2_{U(1)_k}}{2} \left(\xi_k - 2|\varphi_{k-}|^2 + ...\right)^2$$

$$+\frac{g^2_{U(1)_{2a}}}{2} \left(\xi_{2a} - 2|\varphi_{2a-}|^2 - |\varphi_{2ab-}|^2 + ...\right)^2 + \frac{g^2_{U(1)_{2b}}}{2} \left(\xi_{2b} - 2|\varphi_{2b-}|^2 - |\varphi_{2ab-}|^2 + ...\right)^2 + ... \tag{5.1}$$

where $\varphi_{2a-} \equiv y_{a-}$ and $\varphi_{2b-} \equiv y_{b-}$, the term that corresponds to the $D7_3$ and $D7_1$ tachyonic fields* is of this form because they are not eliminated, as in the previous chapter, with $\varphi_{3-} \equiv \psi_-$ and $\varphi_{1-} \equiv \chi_-$, and the extra $\varphi_{ab-} \equiv y_{ab-}$ field comes from contributions that $D7_{2a} - D7_{2b}$ states receive from their relative distance, $x_{2ab} = x_{2a} - x_{2b}$ [62]. Expanding (5.1) as in the previous cases, we have

$$V_D =$$

$$= \frac{g^2_{U(1)_3}}{2}\xi_3^2 - 2g^2_{U(1)_3}\xi_3 \left|\psi_-\right|^2 + 2g^2_{U(1)_3}\left|\psi_-\right|^4 + \frac{g^2_{U(1)_1}}{2}\xi_1^2 - 2g^2_{U(1)_1}\xi_1 \left|\chi_-\right|^2 + 2g^2_{U(1)_1}\left|\chi_-\right|^4$$

$$+\frac{g^2_{U(1)_{2a}}}{2}\xi_{2a}^2 - g^2_{U(1)_{2a}}\xi_{2a}\big(2|y_{a-}|^2 + |y_{ab-}|^2\big)$$

$$+\frac{g^2_{U(1)_{2a}}}{2}\big(4|y_{a-}|^4 + 4|y_{a-}|^2|y_{ab-}|^2 + |y_{ab-}|^4\big)$$

$$+\frac{g^2_{U(1)_{2b}}}{2}\xi_{2b}^2 - g^2_{U(1)_{2b}}\xi_{2b}\big(2|y_{b-}|^2 + |y_{ab-}|^2\big)$$

$$+\frac{g^2_{U(1)_{2b}}}{2}\big(4|y_{b-}|^4 + 4|y_{b-}|^2|y_{ab-}|^2 + |y_{ab-}|^4\big) \tag{5.2}$$

The magnetic field contributions to the masses of the matter fields, in the large volume limit, have the same form as in the previous cases

$$m^2_{H_{2a}} \equiv 2g^2_{U(1)_{2a}}\xi_{2a} = \frac{2\big|\zeta^{(3)}_{2a}\big|}{\alpha'} \approx \frac{2\big|k^{(3)}_{2a}\big|}{\pi\kappa^2}\frac{g_s^2}{\mathcal{V}}\frac{\alpha'}{\mathcal{A}_3}$$

$$m^2_{H_{2b}} \equiv 2g^2_{U(1)_{2b}}\xi_{2b} = \frac{2\big|\zeta^{(3)}_{2b}\big|}{\alpha'} \approx \frac{2\big|k^{(3)}_{2b}\big|}{\pi\kappa^2}\frac{g_s^2}{\mathcal{V}}\frac{\alpha'}{\mathcal{A}_3}$$

$$m^2_{H_3} \equiv 2g^2_{U(1)_3}\xi_3 = \frac{2\big|\zeta^{(1)}_3\big|}{\alpha'} \approx \frac{2\big|k^{(1)}_3\big|}{\pi}\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\alpha'}{\mathcal{A}_1}$$

$$m^2_{H_1} \equiv 2g^2_{U(1)_1}\xi_1 = \frac{2\big|\zeta^{(2)}_1\big|}{\alpha'} \approx \frac{2\big|k^{(2)}_1\big|}{\pi}\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\alpha'}{\mathcal{A}_2} \tag{5.3}$$

with the gauge couplings in the large volume limit given by (3.47). Furthemore, the condition for eliminating the tachyonic fields from mixed-states, is [62]

$$\big|\zeta^{(2)}_1\big| = \big|\zeta^{(3)}_{2a}\big| = \big|\zeta^{(3)}_{2b}\big| = \big|\zeta^{(1)}_3\big| \tag{5.4}$$

Substituting the definitions for all magnetic field mass contributions (5.3) into (5.2) and taking for simplicity $m^{(i)}_{2a} = m^{(i)}_{2b}$, from which we have that $g^2_{U(1)_{2a}} = g^2_{U(1)_{2b}}$ ((3.47)) since the two stacks are parallel [62] and then, $\xi_{2a} = \xi_{2b}$ ((5.3) and (5.4)), we have

$$V_D =$$

$$= \frac{g^2_{U(1)_3}}{2}\xi_3^2 - m^2_{H_3}\left|\psi_-\right|^2 + 2g^2_{U(1)_3}\left|\psi_-\right|^4 + \frac{g^2_{U(1)_1}}{2}\xi_1^2 - m^2_{H_1}\left|\chi_-\right|^2 + 2g^2_{U(1)_1}\left|\chi_-\right|^4$$

$$+g^2_{U(1)_{2a}}\xi_{2a}^2 - m^2_{H_{2a}}\left|y_{a-}\right|^2 + 2g^2_{U(1)_{2a}}\left|y_{a-}\right|^4$$

$$-m^2_{H_{2b}}\left|y_{b-}\right|^2 + 2g^2_{U(1)_{2a}}\left|y_{b-}\right|^4$$

$$-m^2_{H_{2a}}\left|y_{ab-}\right|^2 + g^2_{U(1)_{2a}}\left|y_{ab-}\right|^4 \tag{5.5}$$

where we have ignored the $|y_{l-}|^2|y_{ab-}|^2$ terms, with $l = a, b$. We can substitute all the expressions for the mass contributions (5.3) and the Fayet-Iliopoulos parameters, ξ_{2a}, ξ_{2b}, ξ_3, ξ_1, which we found in the same way as (3.48) and (3.51), into (5.5), and we get

$$V_D =$$

$$= \frac{1}{2}\xi_3\frac{m^2_{H_3}}{2} + \frac{1}{2}\xi_1\frac{m^2_{H_1}}{2} + \xi_{2a}\frac{m^2_{H_{2a}}}{2} - m^2_{H_3}|\psi_-|^2 + 2g^2_{U(1)_3}|\psi_-|^4 - m^2_{H_1}|\chi_-|^2 + 2g^2_{U(1)_1}|\chi_-|^4$$

$$-m^2_{H_{2a}}|y_{a-}|^2 + 2g^2_{U(1)_{2a}}|y_{a-}|^4 - m^2_{H_{2b}}|y_{b-}|^2 + 2g^2_{U(1)_{2a}}|y_{b-}|^4$$

$$-m^2_{H_{2a}}|y_{ab-}|^2 + g^2_{U(1)_{2a}}|y_{ab-}|^4$$

$$\approx \frac{1}{2}\frac{g_s^3|k_3^{(1)}|^2|m_3^{(1)}m_3^{(2)}|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_2}{\mathcal{A}_1} + \frac{1}{2}\frac{g_s^3|k_1^{(2)}|^2|m_1^{(2)}m_1^{(3)}|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_3}{\mathcal{A}_2} + \frac{g_s^3|k_{2a}^{(3)}|^2|m_{2a}^{(1)}m_{2a}^{(3)}|}{\pi^2\kappa^4\mathcal{V}^2}\frac{\mathcal{A}_1}{\mathcal{A}_3}$$

$$-m^2_{H_3}|\psi_-|^2 + 2g^2_{U(1)_3}|\psi_-|^4 - m^2_{H_1}|\chi_-|^2 + 2g^2_{U(1)_1}|\chi_-|^4$$

$$-m^2_{H_{2a}}|y_{a-}|^2 + 2g^2_{U(1)_{2a}}|y_{a-}|^4 - m^2_{H_{2b}}|y_{b-}|^2 + 2g^2_{U(1)_{2a}}|y_{b-}|^4$$

$$-m^2_{H_{2a}}|y_{ab-}|^2 + g^2_{U(1)_{2a}}|y_{ab-}|^4$$

$$= \frac{1}{\kappa^4\mathcal{V}^2}\left(d_3\frac{\mathcal{A}_2}{\mathcal{A}_1} + d_1\frac{\mathcal{A}_3}{\mathcal{A}_2} + d_2\frac{\mathcal{A}_1}{\mathcal{A}_3}\right) - m^2_{H_3}|\psi_-|^2 + 2g^2_{U(1)_3}|\psi_-|^4 - m^2_{H_1}|\chi_-|^2 + 2g^2_{U(1)_1}|\chi_-|^4$$

$$-m^2_{H_{2a}}|y_{a-}|^2 + 2g^2_{U(1)_{2a}}|y_{a-}|^4 - m^2_{H_{2b}}|y_{b-}|^2 + 2g^2_{U(1)_{2a}}|y_{b-}|^4$$

$$-m^2_{H_{2a}}|y_{ab-}|^2 + g^2_{U(1)_{2a}}|y_{ab-}|^4 \tag{5.6}$$

with the following definitions being made

$$d_l \equiv \frac{g^2_{U(1)_l}}{2}\xi_l^2 = \frac{1}{2}g_s^3|m_l^{(j)}m_l^{(k)}|\left(\frac{k_l^{(j)}}{\pi}\right)^2, \quad \text{for } l = 3, 1,$$

$$d_2 \equiv g^2_{U(1)_{2a}}\xi^2_{2a} = g^2_{U(1)_{2b}}\xi^2_{2b} = g_s^3|m_{2a}^{(1)}m_{2a}^{(3)}|\left(\frac{k_{2a}^{(3)}}{\pi}\right)^2 = g_s^3|m_{2b}^{(1)}m_{2b}^{(3)}|\left(\frac{k_{2b}^{(3)}}{\pi}\right)^2 \tag{5.7}$$

The D-term (5.6) without the matter fields is exactly of the form (3.60) and after minimisation of the ratios becomes (3.63) again. The only difference is in the definiton of the d_i parameters as the d_2 parameter now lacks a 1/2 factor. From the latter, we also have $k_{2a}^{(3)} = k_{2b}^{(3)}$ and from this, $n_{2a}^{(3)} = n_{2b}^{(3)}$, since we chose $m_{2a}^{(3)} = m_{2b}^{(3)}$.

Beyond the D-terms contributions in the effective scalar potential, we also take into consideration the following F-term contributions again, that correspond to the tachyonic fields: the already mentioned contributions of the ϕ_- field, (4.6) and (4.8), and the ones of the χ_- field, (4.33) and (4.35), while we also have the contributions of the y_{a-}, y_{b-} fields, which, neglecting their charge conjugates, are

$$V_F \ni m^2_{x_{2a}}|y_{a-}|^2 + m^2_{x_{2b}}|y_{b-}|^2 \tag{5.8}$$

where

$$m^2_{x_{2a}} = y_a(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\mathcal{A}_2}{\alpha'},$$

$$m^2_{x_{2b}} = y_b(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\mathcal{A}_2}{\alpha'} \tag{5.9}$$

are the physical masses coming from the corresponding brane positions x_{2a} and x_{2b}, with $y_j(U_2)$ playing the same role as $z(U_3)$ and $x(U_1)$, and

$$V_F \ni \kappa^2 m^2_{x_{2a}}|y_{a-}|^4 + \kappa^2 m^2_{x_{2b}}|y_{b-}|^4 \tag{5.10}$$

Moreover, there are F-term contributions with respect to the aformentioned relative brane distance x_{2ab}, which are

$$V_F \ni m^2_{x_{2ab}}\,|y_{ab-}|^2 \tag{5.11}$$

where

$$m^2_{x_{2ab}} = y_{ab}(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\mathcal{A}_2}{\alpha'} \tag{5.12}$$

and

$$V_F \ni \kappa^2 m^2_{x_{2ab}}|y_{ab-}|^4 \tag{5.13}$$

We can now express all the above mass contributions in terms of the parameters d_i: using the stabilization condition (3.68) of the moduli ratios and (5.7) to replace the fluxes, the relations (5.3) become

$$m^2_{H_{2a}} = m^2_{H_{2b}} \approx \frac{2\left|k^{(3)}_{2a}\right|}{\pi\kappa^2}\frac{g_s^2}{\mathcal{V}}\frac{\alpha'}{\mathcal{A}_3} = 2\left|m^{(1)}_{2a}m^{(3)}_{2b}\right|^{-1/2}\frac{\sqrt{g_s}}{\kappa^2\mathcal{V}^{4/3}}(d_1^2d_2)^{1/6}$$

$$m^2_{H_3} \approx \frac{2\left|k^{(1)}_3\right|}{\pi}\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\alpha'}{\mathcal{A}_1} = 2\sqrt{2}\left|m^{(1)}_3m^{(2)}_3\right|^{-1/2}\frac{\sqrt{g_s}}{\kappa^2\mathcal{V}^{4/3}}(d_2^2d_3)^{1/6}$$

$$m^2_{H_1} \approx \frac{2\left|k^{(2)}_1\right|}{\pi}\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\alpha'}{\mathcal{A}_2} = 2\sqrt{2}\left|m^{(2)}_1m^{(3)}_1\right|^{-1/2}\frac{\sqrt{g_s}}{\kappa^2\mathcal{V}^{4/3}}(d_3^2d_1)^{1/6} \tag{5.14}$$

and using (3.68), the relations (5.9), (5.12), (4.7) and (4.34) become

$$m^2_{x_{2a}} = y_a(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}\left(\frac{d_1}{d_3}\right)^{1/3}$$

$$m^2_{x_{2b}} = y_b(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}\left(\frac{d_1}{d_3}\right)^{1/3}$$

$$m^2_{x_{2ab}} = y_{ab}(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}\left(\frac{d_1}{d_3}\right)^{1/3}$$

$$m^2_{x_3} = z(U_3)\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}\left(\frac{d_2}{d_1}\right)^{1/3}$$

$$m^2_{x_1} = x(U_1)\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}\left(\frac{d_3}{d_2}\right)^{1/3} \tag{5.15}$$

We also express the gauge couplings (3.47) in terms of the parameters d_i, using (3.68),

$$g^2_{U(1)_{2a}} = g^2_{U(1)_{2b}} = \left|m^{(1)}_{2a} m^{(3)}_{2a}\right|^{-1} \frac{g_s}{\mathcal{V}^{2/3}} \left(\frac{d_1}{d_3}\right)^{1/3}$$

$$g^2_{U(1)_3} = \left|m^{(1)}_3 m^{(2)}_3\right|^{-1} \frac{g_s}{\mathcal{V}^{2/3}} \left(\frac{d_2}{d_1}\right)^{1/3}$$

$$g^2_{U(1)_1} = \left|m^{(2)}_1 m^{(3)}_1\right|^{-1} \frac{g_s}{\mathcal{V}^{2/3}} \left(\frac{d_3}{d_2}\right)^{1/3} \tag{5.16}$$

5.2 Elimination condition

Starting from the condition for the elimination of mixed-state tachyons in this case, (5.4), we can follow the same procedure as in Section 3.2.1, and extract the new relations between the fluxes:

$$\left|\zeta^{(2)}_1\right| = \left|\zeta^{(3)}_{2a}\right| = \left|\zeta^{(3)}_{2b}\right| = \left|\zeta^{(1)}_3\right|$$

$$\rightarrow \frac{\alpha'\left|k^{(2)}_1\right|}{\mathcal{A}_2} = \frac{\alpha'\left|k^{(3)}_{2a}\right|}{\mathcal{A}_3} = \frac{\alpha'\left|k^{(3)}_{2b}\right|}{\mathcal{A}_3} = \frac{\alpha'\left|k^{(1)}_3\right|}{\mathcal{A}_1}$$

$$\rightarrow \left|k^{(2)}_1\right|\left(\frac{d_3}{d_1}\right)^{1/3} = \left|k^{(3)}_{2a}\right|\left(\frac{d_1}{d_2}\right)^{1/3} = \left|k^{(1)}_3\right|\left(\frac{d_2}{d_3}\right)^{1/3}$$

$$\rightarrow \left|k^{(2)}_1\right|\left|k^{(1)}_3\right|^2\left|\frac{m^{(1)}_3 m^{(2)}_3}{m^{(2)}_1 m^{(3)}_1}\right| = \left|k^{(3)}_{2a}\right|\left|k^{(2)}_1\right|^2\left|\frac{1}{2}\frac{m^{(2)}_1 m^{(3)}_1}{m^{(1)}_{2a} m^{(3)}_{2a}}\right| = \left|k^{(1)}_3\right|\left|k^{(3)}_{2a}\right|^2\left|2\frac{m^{(1)}_{2a} m^{(3)}_{2a}}{m^{(1)}_3 m^{(2)}_3}\right|$$

$$\rightarrow A\left|k^{(2)}_1\right|\left|k^{(1)}_3\right|^2 = B\left|k^{(3)}_{2a}\right|\left|k^{(2)}_1\right|^2 = \Gamma\left|k^{(1)}_3\right|\left|k^{(3)}_{2a}\right|^2 \tag{5.17}$$

with

$$A \equiv \left|\frac{m^{(1)}_3 m^{(2)}_3}{m^{(2)}_1 m^{(3)}_1}\right|, \quad B \equiv \left|\frac{1}{2}\frac{m^{(2)}_1 m^{(3)}_1}{m^{(1)}_{2a} m^{(3)}_{2a}}\right|, \quad \Gamma \equiv \left|2\frac{m^{(1)}_{2a} m^{(3)}_{2a}}{m^{(1)}_3 m^{(2)}_3}\right| \tag{5.18}$$

where we used the general expression of (3.45) for the $\zeta^{(i)}_j$ with $j = 1, 2a = 2b, 3$, $|k^{(3)}_{2a}| = |k^{(3)}_{2b}|$, the moduli stabilization condition (3.68), and (5.7). Setting $|k^{(2)}_1| = x$, $|k^{(1)}_3| = z$ and $|k^{(3)}_{2a}| = y$, (5.17) takes the form $Axz^2 = By_a x^2 = \Gamma z y_a^2$, from which we have $x^3 = (\Gamma^2/AB)y^3$ and $z^3 = (B^2/A\Gamma)x^3$. Replacing all the definitions in the latter, we get

$$\left|k^{(2)}_1\right| = 2\left|\frac{m^{(1)}_{2a} m^{(3)}_{2a}}{m^{(1)}_3 m^{(2)}_3}\right|\left|k^{(3)}_{2a}\right| \rightarrow \quad n^{(3)}_{2a} = \frac{1}{2}\left|\frac{m^{(1)}_3 m^{(2)}_3}{m^{(1)}_{2a} m^{(2)}_1}\right| n^{(2)}_1$$

$$\left|k^{(1)}_3\right| = \frac{1}{2}\left|\frac{m^{(2)}_1 m^{(3)}_1}{m^{(1)}_{2a} m^{(3)}_{2a}}\right|\left|k^{(2)}_1\right| \rightarrow \quad n^{(1)}_3 = \frac{1}{2}\left|\frac{m^{(1)}_3 m^{(3)}_1}{m^{(1)}_{2a} m^{(3)}_{2a}}\right| n^{(2)}_1 \tag{5.19}$$

where we used the relation $k^{(i)}_j = n^{(i)}_j / m^{(i)}_j$, too.

The parameter d, using the definitions (5.7), and the conditions (5.19) that fluxes now satisfy, is

$$d = 3\left(d_1 d_2 d_3\right)^{1/3}$$

$$= \frac{3g_s^3}{2^{2/3}\pi}\left|\frac{m^{(3)}_1 m^{(1)}_{2a} m^{(2)}_3}{m^{(2)}_1 m^{(3)}_{2a} m^{(1)}_3}\right|^{1/3}\left(n^{(2)}_1 n^{(3)}_{2a} n^{(1)}_3\right)^{2/3}$$

$$= \frac{3}{4} g_s^3 \left(\frac{k}{\pi}\right)^2 \tag{5.20}$$

with

$$k = n_1^{(2)} \left| \frac{m_1^{(3)} m_3^{(1)} m_3^{(2)}}{m_1^{(2)} m_{2a}^{(1)} m_{2a}^{(3)}} \right|^{1/2} \tag{5.21}$$

We can further express the mass contributions (5.14) and (5.15), and the gauge couplings (5.16), in terms of the integer parameters $m_j^{(i)}$ with $j = 1, 2a, 3$ and $n_1^{(2)}$, as in the previous cases, using (5.7), and (5.19) where fluxes appear,

$$m^2_{H_{2a}} = m^2_{H_{2b}} = m^2_{H_3} = m^2_{H_1} = 2^{1/3} \frac{g_s^2 k}{\pi \kappa^2 \mathcal{V}^{4/3}} \left| \frac{1}{\prod_{j\neq i} m_j^{(i)}} \right|^{1/6} \tag{5.22}$$

$$m^2_{x_{2a}} = 2^{2/3} \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} y_a(U_2) \left| \frac{{m_{2a}^{(1)}}^2 {m_{2a}^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3}$$

$$m^2_{x_{2b}} = 2^{2/3} \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} y_b(U_2) \left| \frac{{m_{2a}^{(1)}}^2 {m_{2a}^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3}$$

$$m^2_{x_{2ab}} = 2^{2/3} \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} y_{ab}(U_2) \left| \frac{{m_{2a}^{(1)}}^2 {m_{2a}^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3}$$

$$m^2_{x_3} = 2^{-1/3} \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} z(U_3) \left| \frac{{m_3^{(1)}}^2 {m_3^{(2)}}^2}{m_1^{(2)} m_1^{(3)} m_{2a}^{(1)} m_{2a}^{(3)}} \right|^{1/3}$$

$$m^2_{x_1} = 2^{-1/3} \frac{g_s^2}{\kappa^2 \mathcal{V}^{2/3}} x(U_1) \left| \frac{{m_1^{(2)}}^2 {m_1^{(3)}}^2}{m_{2a}^{(1)} m_{2b}^{(3)} m_3^{(1)} m_3^{(2)}} \right|^{1/3} \tag{5.23}$$

$$g^2_{U(1)_{2a}} = g^2_{U(1)_{2b}} = 2^{2/3} \frac{g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{\prod_{j\neq i} m_j^{(i)}} \right|^{1/3}$$

$$g^2_{U(1)_3} = g^2_{U(1)_1} = 2^{-1/3} \frac{g_s}{\mathcal{V}^{2/3}} \left| \frac{1}{\prod_{j\neq i} m_j^{(i)}} \right|^{1/3} \tag{5.24}$$

We see that the expressions for the mass and coupling contributions in the present case are similiar to the ones of the previous cases, with a small difference in their coefficients, which comes from

5.3 Hybrid potential

The effective scalar potential for the volume modulus $\mathcal{V}$, and the tachyonic fields y_{a-}, ψ_-, χ_-, y_{b-}, and y_{ab-}, after minimisation of the moduli ratios, in the large volume limit, is the sum of the F-part for $\mathcal{V}$, (3.57), the D-part (5.6) and the F-parts (4.6), (4.8), (4.33), (4.35), (5.8), (5.10), (5.11) and (5.13),

$$V(\mathcal{V}, y_{a-}, \psi_-, \chi_-, y_{b-}, y_{ab-}) =$$

$$= \frac{C}{\kappa^4} \left(- \frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2} \right) - m^2_{H_{2a}} |y_{a-}|^2 + 2 g^2_{U(1)_{2a}} |y_{a-}|^4 + m^2_{x_{2a}} |y_{a-}|^2 + \kappa^2 m^2_{x_{2a}} |y_{a-}|^4$$

$$-m^2_{H_3}\,|\psi_-|^2+2g^2_{U(1)_3}\,|\psi_-|^4+m^2_{x_3}\,|\psi_-|^2+\kappa^2 m^2_{x_3}|\psi_-|^4$$

$$-m^2_{H_1}\,|\chi_-|^2+2g^2_{U(1)_1}\,|\chi_-|^4+m^2_{x_1}\,|\chi_-|^2+\kappa^2 m^2_{x_1}|\chi_-|^4$$

$$-m^2_{H_{2a}}\,|y_{b-}|^2+2g^2_{U(1)_{2a}}\,|y_{b-}|^4+m^2_{x_{2b}}\,|y_{b-}|^2+\kappa^2 m^2_{x_{2b}}|y_{b-}|^4$$

$$-m^2_{H_{2a}}\,|y_{ab-}|^2+g^2_{U(1)_{2a}}\,|y_{ab-}|^4+m^2_{x_{2ab}}\,|y_{ab-}|^2+\kappa^2 m^2_{x_{2ab}}|y_{ab-}|^4$$

$$=\frac{C}{\kappa^4}\bigg(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3}-\frac{s}{\mathcal{V}^2}\bigg)+\frac{1}{2}m^2_{Y_a}(\mathcal{V})\,|y_{a-}|^2+\frac{1}{4}\lambda_{Y_a}(\mathcal{V})\,|y_{a-}|^4$$
$$+\frac{1}{2}m^2_Z(\mathcal{V})\,|\psi_-|^2+\frac{1}{4}\lambda_Z(\mathcal{V})\,|\psi_-|^4+\frac{1}{2}m^2_X(\mathcal{V})\,|\chi_-|^2+\frac{1}{4}\lambda_X(\mathcal{V})\,|\chi_-|^4$$
$$+\frac{1}{2}m^2_{Y_b}(\mathcal{V})\,|y_{b-}|^2+\frac{1}{4}\lambda_{Y_b}(\mathcal{V})\,|y_{b-}|^4+\frac{1}{2}m^2_{Y_{ab}}(\mathcal{V})\,|y_{ab-}|^2+\frac{1}{4}\lambda_{Y_{ab}}(\mathcal{V})\,|y_{ab-}|^4 \tag{5.25}$$

with

$$C=-3\mathcal{W}_o^2\gamma,\quad q=\frac{\xi}{2\gamma},\quad s=\frac{3}{2}\sigma=\frac{d}{3\mathcal{W}_o^2\gamma}=-\frac{d}{C}=\frac{3}{4C}g_s^3\bigg(\frac{k}{\pi}\bigg)^2,$$

$$m^2_{Y_a}(\mathcal{V})=2\big(m^2_{x_{2a}}-m^2_{H_{2a}}\big),\quad \lambda_{Y_a}(\mathcal{V})=4\big(2g^2_{U(1)_{2a}}+\kappa^2 m^2_{x_{2a}}\big),$$

$$m^2_Z(\mathcal{V})=2\big(m^2_{x_3}-m^2_{H_3}\big),\quad \lambda_Z(\mathcal{V})=4\big(2g^2_{U(1)_3}+\kappa^2 m^2_{x_3}\big),$$

$$m^2_X(\mathcal{V})=2\big(m^2_{x_1}-m^2_{H_1}\big),\quad \lambda_X(\mathcal{V})=4\big(2g^2_{U(1)_1}+\kappa^2 m^2_{x_1}\big),$$

$$m^2_{Y_b}(\mathcal{V})=2\big(m^2_{x_{2b}}-m^2_{H_{2a}}\big),\quad \lambda_{Y_b}(\mathcal{V})=4\big(2g^2_{U(1)_{2a}}+\kappa^2 m^2_{x_{2b}}\big),$$

$$m^2_{Y_{ab}}(\mathcal{V})=2\big(m^2_{x_{2ab}}-m^2_{H_{2a}}\big),\quad \lambda_{Y_{ab}}(\mathcal{V})=4\big(g^2_{U(1)_{2a}}+\kappa^2 m^2_{x_{2ab}}\big) \tag{5.26}$$

The above expressions for the masses and the couplings of the tachyonic fields in terms of the volume $\mathcal{V}$, and the parameters g_s, $y_l(U_2), l=a,b,ab$, $z(U_3)$, $x(U_1)$, $m_j^{(i)}, j=1,2a,3$ and $n_1^{(2)}$, are found to be (using (5.22)-(5.24)):

- For y_{a-}:

$$m^2_{Y_a}(\mathcal{V})=2\big(m^2_{x_{2a}}-m^2_{H_{2a}}\big)$$

$$=2^{5/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}y_a(U_2)\left|\frac{{m_{2a}^{(1)}}^2{m_{2a}^{(3)}}^2}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(1-\frac{1}{\mathcal{V}^{2/3}}\frac{k}{2^{1/3}\pi y_a(U_2)}\left|\frac{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}{{m_{2a}^{(2)}}^5{m_{2a}^{(3)}}^5}\right|^{1/6}\right)$$
$$=2^{5/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}y_a(U_2)\left|\frac{{m_{2a}^{(1)}}^2{m_{2a}^{(3)}}^2}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(1-\left(\frac{\mathcal{V}_{c2a}}{\mathcal{V}}\right)^{2/3}\right) \tag{5.27}$$

with

$$\mathcal{V}_{c2a} \equiv \frac{1}{\sqrt{2}}\left(\frac{k}{\pi y_a(U_2)}\right)^{3/2}\left|\frac{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}{m_{2a}^{(2)}{}^5\, m_{2a}^{(3)}{}^5}\right|^{1/4} \tag{5.28}$$

and

$$\lambda_{Y_a}(\mathcal{V}) = 4\big(2g^2_{U(1)_{2a}} + \kappa^2 m^2_{x_{2a}}\big)$$

$$= 2^{8/3}\frac{g_s}{\mathcal{V}^{2/3}}\left|\frac{1}{\prod_{j\neq i} m_j^{(i)}}\right|^{1/3}\left(2 + g_s y_a(U_2)\big|m_{2a}^{(1)}m_{2a}^{(3)}\big|\right) \tag{5.29}$$

- For ψ_-:

$$m_Z^2(\mathcal{V}) = 2\big(m^2_{x_3} - m^2_{H_3}\big)$$

$$= 2^{2/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}z(U_3)\left|\frac{m_3^{(1)}{}^2 m_3^{(2)}{}^2}{m_1^{(2)}m_1^{(3)}m_{2a}^{(1)}m_{2a}^{(3)}}\right|^{1/3}\left(1 - \frac{1}{\mathcal{V}^{2/3}}\frac{2^{2/3}k}{\pi z(U_3)}\left|\frac{m_1^{(2)}m_1^{(3)}m_{2a}^{(1)}m_{2a}^{(2)}}{m_3^{(1)}{}^5\, m_3^{(2)}{}^5}\right|^{1/6}\right)$$

$$= 2^{2/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}z(U_3)\left|\frac{m_3^{(1)}{}^2 m_3^{(2)}{}^2}{m_1^{(2)}m_1^{(3)}m_{2a}^{(1)}m_{2a}^{(3)}}\right|^{1/3}\left(1 - \left(\frac{\mathcal{V}_{c3}}{\mathcal{V}}\right)^{2/3}\right) \tag{5.30}$$

with

$$\mathcal{V}_{c3} \equiv 2\left(\frac{k}{\pi z(U_3)}\right)^{3/2}\left|\frac{m_1^{(2)}m_1^{(3)}m_{2a}^{(1)}m_{2a}^{(2)}}{m_3^{(1)}{}^5\, m_3^{(2)}{}^5}\right|^{1/4} \tag{5.31}$$

and

$$\lambda_Z(\mathcal{V}) = 4\big(2g^2_{U(1)_3} + \kappa^2 m^2_{x_3}\big)$$

$$= 2^{5/3}\frac{g_s}{\mathcal{V}^{2/3}}\left|\frac{1}{\prod_{j\neq i} m_j^{(i)}}\right|^{1/3}\left(2 + g_s z(U_3)\big|m_3^{(1)}m_3^{(2)}\big|\right) \tag{5.32}$$

- For χ_-:

$$m_X^2(\mathcal{V}) = 2\big(m^2_{x_1} - m^2_{H_1}\big)$$

$$= 2^{2/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}x(U_1)\left|\frac{m_1^{(2)}{}^2 m_1^{(3)}{}^2}{m_{2a}^{(1)}m_{2a}^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(1 - \frac{1}{\mathcal{V}^{2/3}}\frac{2^{2/3}k}{\pi x(U_1)}\left|\frac{m_{2a}^{(1)}m_{2a}^{(3)}m_3^{(1)}m_3^{(2)}}{m_1^{(2)}{}^5\, m_1^{(3)}{}^5}\right|^{1/6}\right)$$

$$= 2^{2/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}}x(U_1)\left|\frac{m_1^{(2)}{}^2 m_1^{(3)}{}^2}{m_{2a}^{(1)}m_{2a}^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\left(1 - \left(\frac{\mathcal{V}_{c1}}{\mathcal{V}}\right)^{2/3}\right) \tag{5.33}$$

with

$$\mathcal{V}_{c1} \equiv 2\left(\frac{k}{\pi x(U_1)}\right)^{3/2}\left|\frac{m_{2a}^{(1)}m_{2a}^{(3)}m_3^{(1)}m_3^{(2)}}{m_1^{(2)}{}^5\, m_1^{(3)}{}^5}\right|^{1/4} \tag{5.34}$$

and

$$\lambda_X(\mathcal{V}) = 4\big(2g^2_{U(1)_1} + \kappa^2 m^2_{x_1}\big)$$

$$= 2^{5/3}\frac{g_s}{\mathcal{V}^{2/3}}\left|\frac{1}{\prod_{j\neq i} m_j^{(i)}}\right|^{1/3}\left(2 + g_s x(U_1)\big|m_1^{(2)} m_1^{(3)}\big|\right) \tag{5.35}$$

- For y_{b-}:

$$m^2_{Y_b}(\mathcal{V}) = 2\big(m^2_{x_{2b}} - m^2_{H_{2a}}\big)$$

$$= 2^{5/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}} y_b(U_2)\left|\frac{{m_{2a}^{(1)}}^2 {m_{2a}^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3}\left(1 - \frac{1}{\mathcal{V}^{2/3}}\frac{k}{2^{1/3}\pi y_b(U_2)}\left|\frac{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}{{m_{2a}^{(2)}}^5 {m_{2a}^{(3)}}^5}\right|^{1/6}\right)$$

$$= 2^{5/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}} y_b(U_2)\left|\frac{{m_{2a}^{(1)}}^2 {m_{2a}^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3}\left(1 - \left(\frac{\mathcal{V}_{c2b}}{\mathcal{V}}\right)^{2/3}\right) \tag{5.36}$$

with

$$\mathcal{V}_{c2b} \equiv \frac{1}{\sqrt{2}}\left(\frac{k}{\pi y_b(U_2)}\right)^{3/2}\left|\frac{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}{{m_{2a}^{(2)}}^5 {m_{2a}^{(3)}}^5}\right|^{1/4} \tag{5.37}$$

and

$$\lambda_{Y_b}(\mathcal{V}) = 4\big(2g^2_{U(1)_{2a}} + \kappa^2 m^2_{x_{2b}}\big)$$

$$= 2^{8/3}\frac{g_s}{\mathcal{V}^{2/3}}\left|\frac{1}{\prod_{j\neq i} m_j^{(i)}}\right|^{1/3}\left(2 + g_s y_b(U_2)\big|m_{2a}^{(1)} m_{2a}^{(3)}\big|\right) \tag{5.38}$$

- For y_{ab-}:

$$m^2_{Y_{ab}}(\mathcal{V}) = 2\big(m^2_{x_{2ab}} - m^2_{H_{2a}}\big)$$

$$= 2^{5/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}} y_{ab}(U_2)\left|\frac{{m_{2a}^{(1)}}^2 {m_{2a}^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3}\left(1 - \frac{1}{\mathcal{V}^{2/3}}\frac{k}{2^{1/3}\pi y_{ab}(U_2)}\left|\frac{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}{{m_{2a}^{(2)}}^5 {m_{2a}^{(3)}}^5}\right|^{1/6}\right)$$

$$= 2^{5/3}\frac{g_s^2}{\kappa^2\mathcal{V}^{2/3}} y_{ab}(U_2)\left|\frac{{m_{2a}^{(1)}}^2 {m_{2a}^{(3)}}^2}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3}\left(1 - \left(\frac{\mathcal{V}_{c2ab}}{\mathcal{V}}\right)^{2/3}\right) \tag{5.39}$$

with

$$\mathcal{V}_{c2ab} \equiv \frac{1}{\sqrt{2}}\left(\frac{k}{\pi y_{ab}(U_2)}\right)^{3/2}\left|\frac{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}{{m_{2a}^{(2)}}^5 {m_{2a}^{(3)}}^5}\right|^{1/4} \tag{5.40}$$

and

$$\lambda_{Y_{ab}}(\mathcal{V}) = 4\big(g^2_{U(1)_{2a}} + \kappa^2 m^2_{x_{2ab}}\big)$$

$$= 2^{8/3}\frac{g_s}{\mathcal{V}^{2/3}}\left|\frac{1}{\prod_{j\neq i} m_j^{(i)}}\right|^{1/3}\left(1+g_s y_{ab}(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|\right) \tag{5.41}$$

From the above we see that the effective potential (5.25) has the form of a hybrid potential with five waterfall fields: the y_{a-} which essentially is the one studied in the first case, the ψ_- of the second case, the χ_- of the third case, the additional y_{b-} in this theory, and the extra y_{ab-} from the $D7_{2a} - D7_{2b}$ state. All squared masses become negative below their respective critical values of the volume, $\mathcal{V}_{ci}$, with $i = 2a, 3, 1, 2b, 2ab$, and according to the parameters each of these depends on, we can choose them to satisfy $\mathcal{V}_{c2a} \gtrsim \mathcal{V}_{c3} \gtrsim \mathcal{V}_{c1} \gtrsim \mathcal{V}_{c2b} \gtrsim \mathcal{V}_{c2ab}$, so that the waterfall fields become tachyonic one after the other.

5.4 The new vacuum

So, in this model, we distinguish the following phases, in which we consider that the parameters $y_l(U_2)$ with $l = a, b, ab$, $z(U_3)$ and $x(U_1)$, are all positive and that the critical volumes satisfy $\mathcal{V}_{c2a} \gtrsim \mathcal{V}_{c3} \gtrsim \mathcal{V}_{c1} \gtrsim \mathcal{V}_{c2b} \gtrsim \mathcal{V}_{c2ab}$:

• For $\mathcal{V} > \mathcal{V}_{c2a}$, where we have $m^2_{Y_a}(\mathcal{V}) > 0$, $m^2_Z(\mathcal{V}) > 0$, $m^2_X(\mathcal{V}) > 0$, $m^2_{Y_b}(\mathcal{V}) > 0$ and $m^2_{Y_{ab}}(\mathcal{V}) > 0$, the five waterfall fields sit at their vanishing minima, $\langle y_{a-}\rangle = \langle\psi_-\rangle = \langle\chi_-\rangle = \langle y_{b-}\rangle = \langle y_{ab-}\rangle = 0$, and the inflationary stage is equivalent to that of one field only (the inflaton field).

• For $\mathcal{V}_{c3} < \mathcal{V} \lesssim \mathcal{V}_{c2a}$, where we have $m^2_{Y_a}(\mathcal{V}) < 0$, $m^2_Z(\mathcal{V}) > 0$, $m^2_X(\mathcal{V}) > 0$, $m^2_{Y_b}(\mathcal{V}) > 0$ and $m^2_{Y_{ab}}(\mathcal{V}) > 0$, a phase transition occurs and the waterfall field y_{a-} falls to its new non-vanishing VEV, $\langle y_{a-}\rangle = \pm\upsilon_{2a}$, ending the inflationary phase, while the other waterfall fields sit at $\langle\psi_-\rangle = \langle\chi_-\rangle = \langle y_{b-}\rangle = \langle y_{ab-}\rangle = 0$. The system is equivalent to the first case studied, in section 3.2, with one waterfall field only in the effective scalar potential, and its minimum is at $\pm\upsilon_{2a}$, which is lower than the above inflaton's potential minimum.

• For $\mathcal{V}_{c1} < \mathcal{V} \lesssim \mathcal{V}_{c3}$, where we have $m^2_{Y_a}(\mathcal{V}) < 0$, $m^2_Z(\mathcal{V}) < 0$, $m^2_X(\mathcal{V}) > 0$, $m^2_{Y_b}(\mathcal{V}) > 0$ and $m^2_{Y_{ab}}(\mathcal{V}) > 0$, a second phase transition occurs and the waterfall field ψ_- also falls to its new non-vanishing VEV, $\langle\psi_-\rangle = \pm\upsilon_3$, while the other fields sit at $\langle\chi_-\rangle = \langle y_{b-}\rangle = \langle y_{ab-}\rangle = 0$. The system is equivalent to the second case studied, in section 4.1, with two waterfall fields contributing to the effective scalar potential, whose minimum is at $\pm\upsilon_{2a}$, $\pm\upsilon_3$, and aquires an even lower value than in the phase above.

• For $\mathcal{V}_{c2b} < \mathcal{V} \lesssim \mathcal{V}_{c1}$, where we have $m^2_{Y_a}(\mathcal{V}) < 0$, $m^2_Z(\mathcal{V}) < 0$, $m^2_X(\mathcal{V}) < 0$, $m^2_{Y_b}(\mathcal{V}) > 0$ and $m^2_{Y_{ab}}(\mathcal{V}) > 0$, a third phase transition occurs and the waterfall field χ_- is driven to its new non-vanishing VEV, $\langle\chi_-\rangle = \pm\upsilon_1$, while $\langle y_{b-}\rangle = \langle y_{ab-}\rangle = 0$. The system is equivalent to the third case studied, in section 4.2, with three waterfall fields in the theory, and the potential minimum is at $\pm\upsilon_{2a}$, $\pm\upsilon_3$, $\pm\upsilon_1$, with an even lower value than before.

• For $\mathcal{V}_{c2ab} < \mathcal{V} \lesssim \mathcal{V}_{c2b}$, where we have $m^2_{Y_a}(\mathcal{V}) < 0$, $m^2_Z(\mathcal{V}) < 0$, $m^2_X(\mathcal{V}) < 0$, $m^2_{Y_b}(\mathcal{V}) < 0$ and $m^2_{Y_{ab}}(\mathcal{V}) > 0$, another phase transition occurs and the waterfall field y_{b-} is driven to its new non-vanishing VEV, $\langle y_{b-}\rangle = \pm\upsilon_{2b}$, while $\langle y_{ab-}\rangle = 0$. Now, there are four waterfall fields in the theory, and the potential minimum is at $\pm\upsilon_{2a}$, $\pm\upsilon_3$, $\pm\upsilon_1$, $\pm\upsilon_{2b}$, with a lower value than above.

• For $\mathcal{V} \lesssim \mathcal{V}_{c2ab}$, where we have $m^2_{Y_a}(\mathcal{V}) < 0$, $m^2_Z(\mathcal{V}) < 0$, $m^2_X(\mathcal{V}) < 0$, $m^2_{Y_b}(\mathcal{V}) < 0$ and $m^2_{Y_{ab}}(\mathcal{V}) < 0$, the fifth phase transition occurs and the waterfall field y_{ab-} is also driven to its new non-vanishing VEV, $\langle y_{ab-}\rangle = \pm\upsilon_{2ab}$. Now, the effective scalar potential recieves contributions from the inflaton and five waterfall fields, and its minimum is at $\pm\upsilon_{2a}$, $\pm\upsilon_3$, $\pm\upsilon_1$, $\pm\upsilon_{2b}$,$\pm\upsilon_{2ab}$, with an even smaller value than above.

We are focusing on the last phase above, where all the waterfall directions are included in the theory. The non-vanishing VEV's of the five tachyonic fields have the form

$$\langle y_{j-}\rangle = \pm v_{2j} = \pm\frac{|m_{Y_j}|}{\sqrt{\lambda_{Y_j}}},\ j=a,b,ab, \quad \langle\psi_-\rangle = \pm v_3 = \pm\frac{|m_Z|}{\sqrt{\lambda_Z}}, \quad \langle\chi_-\rangle = \pm v_1 = \pm\frac{|m_X|}{\sqrt{\lambda_X}} \tag{5.42}$$

The effective scalar potential (5.25) at the v_i, with $i = 2a, 3, 1, 2b, 2ab$, substituting the masses and the couplings of the waterfall fields, (5.27), (5.29), (5.30), (5.32), (5.33), (5.35), (5.36), (5.38), (5.39), (5.41), is

$$V(\mathcal{V}, v_i) =$$

$$= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2}\right) - \frac{1}{4}\frac{m^4_{Y_a}(\mathcal{V})}{\lambda_{Y_a}(\mathcal{V})} - \frac{1}{4}\frac{m^4_Z(\mathcal{V})}{\lambda_Z(\mathcal{V})} - \frac{1}{4}\frac{m^4_X(\mathcal{V})}{\lambda_X(\mathcal{V})}$$
$$-\frac{1}{4}\frac{m^4_{Y_b}(\mathcal{V})}{\lambda_{Y_b}(\mathcal{V})} - \frac{1}{4}\frac{m^4_{Y_{ab}}(\mathcal{V})}{\lambda_{Y_{ab}}(\mathcal{V})}$$

$$= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2}\right) - \frac{1}{4}\frac{2^{2/3}g_s^3y_a^2(U_2)}{\kappa^4\mathcal{V}^{2/3}}\left|\frac{m_{2a}^{(1)\,5}m_{2a}^{(3)\,5}}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\frac{\left(1-(\mathcal{V}_{c2a}/\mathcal{V})^{2/3}\right)^2}{2+g_sy_a(U_2)|m_{2a}^{(1)}m_{2a}^{(3)}|}$$
$$-\frac{1}{4}\frac{g_s^3z^2(U_3)}{2^{1/3}\kappa^4\mathcal{V}^{2/3}}\left|\frac{m_3^{(1)\,5}m_3^{(2)\,5}}{m_1^{(2)}m_1^{(3)}m_{2a}^{(1)}m_{2a}^{(3)}}\right|^{1/3}\frac{\left(1-(\mathcal{V}_{c3}/\mathcal{V})^{2/3}\right)^2}{2+g_sz(U_3)|m_3^{(1)}m_3^{(2)}|}$$
$$-\frac{1}{4}\frac{g_s^3x^2(U_1)}{2^{1/3}\kappa^4\mathcal{V}^{2/3}}\left|\frac{m_1^{(2)\,5}m_1^{(3)\,5}}{m_{2a}^{(1)}m_{2a}^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\frac{\left(1-(\mathcal{V}_{c1}/\mathcal{V})^{2/3}\right)^2}{2+g_sx(U_1)|m_1^{(2)}m_1^{(3)}|}$$
$$-\frac{1}{4}\frac{2^{2/3}g_s^3y_b^2(U_2)}{\kappa^4\mathcal{V}^{2/3}}\left|\frac{m_{2a}^{(1)\,5}m_{2a}^{(3)\,5}}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\frac{\left(1-(\mathcal{V}_{c2b}/\mathcal{V})^{2/3}\right)^2}{2+g_sy_b(U_2)|m_{2a}^{(1)}m_{2a}^{(3)}|}$$
$$-\frac{1}{4}\frac{2^{2/3}g_s^3y_{ab}^2(U_2)}{\kappa^4\mathcal{V}^{2/3}}\left|\frac{m_{2a}^{(1)\,5}m_{2a}^{(3)\,5}}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3}\frac{\left(1-(\mathcal{V}_{c2ab}/\mathcal{V})^{2/3}\right)^2}{1+g_sy_{ab}(U_2)|m_{2a}^{(1)}m_{2a}^{(3)}|}$$
$$= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3} - \frac{s}{\mathcal{V}^2}\right) - \sum_{k=1}^{5}\frac{C_k}{\kappa^4\mathcal{V}^{2/3}}\left(1-\left(\frac{\mathcal{V}_{ck}}{\mathcal{V}}\right)^{2/3}\right)^2 \tag{5.43}$$

with

$$k = 2 \equiv 2a, \quad k = 4 \equiv 2b, \quad k = 5 \equiv 2ab,$$

$$C_2 \equiv C_{2a} = \frac{1}{4}\frac{2^{2/3}g_s^3y_a^2(U_2)}{2+g_sy_a(U_2)|m_{2a}^{(1)}m_{2a}^{(3)}|}\left|\frac{m_{2a}^{(1)\,5}m_{2a}^{(3)\,5}}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3},$$

$$C_3 \equiv \frac{1}{4}\frac{1}{2^{1/3}}\frac{g_s^3z^2(U_3)}{2+g_sz(U_3)|m_3^{(1)}m_3^{(2)}|}\left|\frac{m_3^{(1)\,5}m_3^{(2)\,5}}{m_1^{(2)}m_1^{(3)}m_2^{(1)}m_2^{(3)}}\right|^{1/3},$$

$$C_1 \equiv \frac{1}{4}\frac{1}{2^{1/3}}\frac{g_s^3x^2(U_1)}{2+g_sx(U_1)|m_1^{(2)}m_1^{(3)}|}\left|\frac{m_1^{(2)\,5}m_1^{(3)\,5}}{m_2^{(1)}m_2^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{1/3},$$

$$C_4 \equiv C_{2b} = \frac{1}{4}\frac{2^{2/3} g_s^3 y_b^2(U_2)}{2 + g_s y_b(U_2)\left|m_{2a}^{(1)} m_{2a}^{(3)}\right|}\left|\frac{{m_{2a}^{(1)}}^5 {m_{2a}^{(3)}}^5}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3},$$

$$C_5 \equiv C_{2ab} = \frac{1}{4}\frac{2^{2/3} g_s^3 y_{ab}^2(U_2)}{1 + g_s y_{ab}(U_2)\left|m_{2a}^{(1)} m_{2a}^{(3)}\right|}\left|\frac{{m_{2a}^{(1)}}^5 {m_{2a}^{(3)}}^5}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{1/3} \tag{5.44}$$

All the above coefficients C_k can be written in terms of the parameter d, (5.20), and their respective critical volumes, as in the previous cases,

$$C_{2j} = \frac{1}{2}\beta_{2j}\frac{d}{3\mathcal{V}_{c2j}^{4/3}}, \quad \text{with } j = a, b,$$

$$C_l = \beta_l \frac{d}{3\mathcal{V}_{cl}^{4/3}}, \quad \text{with } l = 3, 1, 2ab \tag{5.45}$$

with the following definitions being made

$$\beta_{2a} \equiv \frac{2}{2 + g_s y_a(U_2)\left|m_{2a}^{(1)} m_{2a}^{(3)}\right|},$$

$$\beta_3 \equiv \frac{2}{2 + g_s z(U_3)\left|m_3^{(1)} m_3^{(2)}\right|},$$

$$\beta_1 \equiv \frac{2}{2 + g_s x(U_1)\left|m_1^{(2)} m_1^{(3)}\right|},$$

$$\beta_{2b} \equiv \frac{2}{2 + g_s y_b(U_2)\left|m_{2a}^{(1)} m_{2a}^{(3)}\right|},$$

$$\beta_{2ab} \equiv \frac{1}{1 + g_s y_{ab}(U_2)\left|m_{2a}^{(1)} m_{2a}^{(3)}\right|} \tag{5.46}$$

From the above, we observe that the tachyonic coefficients of the y_a and y_b fields in (5.45), aquire an extra 1/2 factor, which is coming from the addition of the parallel to $D7_{2a}$, brane stack $D7_{2b}$. Also, as we see from (5.45), this factor is missing from the tachyonic coefficient of the y_{ab-} field, because of the fact that the factor 2 is missing from the term $g^2_{U(1)_{2a}}$ in the definition of the coupling of y_{ab-} in (5.26), and we do not have to include it in (5.45) for the definition of its β parameter, β_{2ab}, as we see in (5.46). The maximum tachyonic contribution to the potential minimum (5.43), that is not restricted to happen at small volumes only, is for $\mathcal{V}_{c2ab} \approx \mathcal{V}_{c2b} \approx \mathcal{V}_{c1} \approx \mathcal{V}_{c3} \approx \mathcal{V}_{c2a}$ and $\beta_{2ab} = \beta_{2b} = \beta_1 = \beta_3 = \beta_{2a} = 1$. This means that if we count the tachyonic contributions in (5.43) as a single one, with effective tachyonic coeficient $C_{tot} = C_{2a} + C_3 + C_1 + C_{2b} + C_{2ab}$, the corresponding effective β parameter will be

$$\beta_{tot} = \frac{1}{2}\beta_{2a} + \beta_3 + \beta_1 + \frac{1}{2}\beta_{2b} + \beta_{2ab} = \frac{1}{2}1 + 1 + 1 + \frac{1}{2}1 + 1 = 4 \tag{5.47}$$

larger than the desired value (3.92) ($\simeq 3.228$), that is we expect to obtain a vacuum lower than the particular small positive one that we desire. This can be solved by [62] either lowering the β_i parameters or the critical volume values, except for the one of the first waterfall field, which is responsible for the end of the inflationary phase.

Below, we study the effective scalar potential (5.25) at the new minimum in a similar way as in the previous chapters, using the program Mathematica:

We insert the potential (5.25) with the expressions for the masses, the critical volumes and the couplings, (5.27)-(5.41), and the expression for k, (5.21), with

$$\kappa^2 = 1, \quad C = 7.81 \times 10^{-4}, \quad q = 0, \tag{5.48}$$

and also the new relation for the parameter s, since the parameter d has changed,

$$s = \frac{3}{2}\sigma = -\frac{d}{C} = -\frac{3}{4C} g_s^3 \left(\frac{k}{\pi}\right)^2 \tag{5.49}$$

We minimise the potential with respect to y_{a-}, ψ_-, χ_-, y_{b-} and y_{ab-}, and then, for different values of g_s, $y_a(U_2)$, $z(U_3)$, $x(U_1)$, $y_b(U_2)$ and $y_{ab}(U_2)$, that always give small products $g_s y_a(U_2)$, $g_s z(U_3)$, $g_s x(U_1)$, $g_s y_b(U_2)$ and $g_s y_{ab}(U_2)$ (so that $\beta_{2a}, \beta_3, \beta_1, \beta_{2b}, \beta_{2ab} \approx 1$), we search for combinations of $\{\mathcal{V}, m_j^{(i)}, n_1^{(2)}\}$ with $j = 1, 2a, 3$, that give a vanishing minimum, $V(\mathcal{V}, \upsilon_i) = 0$, and a positive minimum, $0 < V(\mathcal{V}, \upsilon_i) \lesssim 10^{-11}$, with $i = 2a, 3, 1, 2b, 2ab$, and finally, we investigate only the parameter s for the resulting cases.

It is found that positive minimum values of order $\sim 10^{-12}$, give an s closer to the required range (3.33) for a dS minimum. Among them, the parameters g_s that give the closest s values to the range (3.33), are studied in detail by modifying appropriately the values of g_s, $y_a(U_2)$, $z(U_3)$, $x(U_1)$, $y_b(U_2)$ and $y_{ab}(U_2)$ in order to get the desired values of s, $\mathcal{V}_{c2a}$, $\mathcal{V}_{c3}$, $\mathcal{V}_{c1}$, $\mathcal{V}_{c2b}$ and $\mathcal{V}_{c2ab}$. However, from the latter, only negative potential minimum values are obtained (Anti-de Sitter), as in [62]. Lowering the values of the critical volumes $\mathcal{V}_{c3}$, $\mathcal{V}_{c1}$, $\mathcal{V}_{c2b}$ and $\mathcal{V}_{c2ab}$, as proposed in [62], the potential minimum values are indeed uplifted to small positive ones (de-Sitter).

We find the combination

$$g_s = 1.52874 \times 10^{-3}, \quad y_a(U_2) = 0.00654901, \quad z(U_3) = 0.017, \quad x(U_1) = 1.15,$$

$$y_b(U_2) = 0.00663, \quad y_{ab}(U_2) = 0.0066, \quad n_1^{(2)} = 57,$$

$$m_1^{(2)} = 3, \quad m_1^{(3)} = 22, \quad m_2^{(1)} = 81, \quad m_2^{(3)} = 70, \quad m_3^{(1)} = 59, \quad m_3^{(2)} = 84 \tag{5.50}$$

which gives $s \simeq -0.00723954$, $\mathcal{V}_{c2a} \simeq 201.9$, $\mathcal{V}_{c3} \simeq 167.089$, $\mathcal{V}_{c1} \simeq 195.414$, $\mathcal{V}_{c2b} \simeq 198.212$, $\mathcal{V}_{c2ab} \simeq 199.565$ and $\beta_{2a}, \beta_3, \beta_1, \beta_{2b}, \beta_{2ab} \sim 1$, with a potential minimum at

$$\mathcal{V}_{min} \simeq 115.878 \rightarrow \quad V(\mathcal{V}_{min}, \upsilon_i) \simeq 7.13283 \times 10^{-13} \tag{5.51}$$

The potential minimum value in terms of the volume $\mathcal{V}$ with the (5.50) parameter values incorporated, is shown in Figure 5.1. The resulting value of the vacuum for the above combination of parameters is in agreement with [62]. The seperate contributions of the inflaton and the five waterfall fields to the potential (5.43), are

$$V(\mathcal{V}_{min}, \upsilon_i) \simeq 4.33497 \times 10^{-11} - 4.26365 \times 10^{-11} \tag{5.52}$$

from which we see that the two terms are closer to cancel each other than in the other cases. The total waterfall direction has a maximum near $\mathcal{V} \simeq 188$, and is shown in Figure 5.2. Also, in the combination (5.50) of the parameters, which is induced after we lower the critical volume values so that the minimum becomes positive, we see from the resulting critical values, that the waterfall fields become tachyonic with the following order: y_{a-}, y_{ab-}, y_{b-}, χ_- and ψ_-. The potential minimum value in terms of the volume $\mathcal{V}$, for the four cases studied altogether, is also shown in Figure 5.3. We observe that as before, the addition of two more waterfall fields lowered the vacuum more and shifted it towards lower values of the volume $\mathcal{V}$.

We can also calculate the effective parameter β_{tot} for the combination (5.50), considering $\mathcal{V}_{ck} \approx \mathcal{V}_{c2a} \sim 201.9$ for all the tachyonic fields, at the potential minimum $\mathcal{V} \sim 115.878$: From (5.43), we find

$$-\sum_{k=1}^{5} \frac{C_k}{\kappa^4 \mathcal{V}^{2/3}} \left(1 - \left(\frac{\mathcal{V}_{ck}}{\mathcal{V}}\right)^{2/3}\right)^2 \simeq -4.26365 \times 10^{-11}$$

$$\rightarrow -\frac{1}{\kappa^4 \mathcal{V}^{2/3}} \beta_{tot} \frac{d}{3\mathcal{V}_{ck}^{4/3}} \left(1 - \left(\frac{\mathcal{V}_{ck}}{\mathcal{V}}\right)^{2/3}\right)^2 \Bigg|_{\mathcal{V} \sim 115.878, \mathcal{V}_{ck} \sim 201.9} \simeq -4.26365 \times 10^{-11}$$

$$\rightarrow \beta_{tot} \approx 3.17607 \tag{5.53}$$

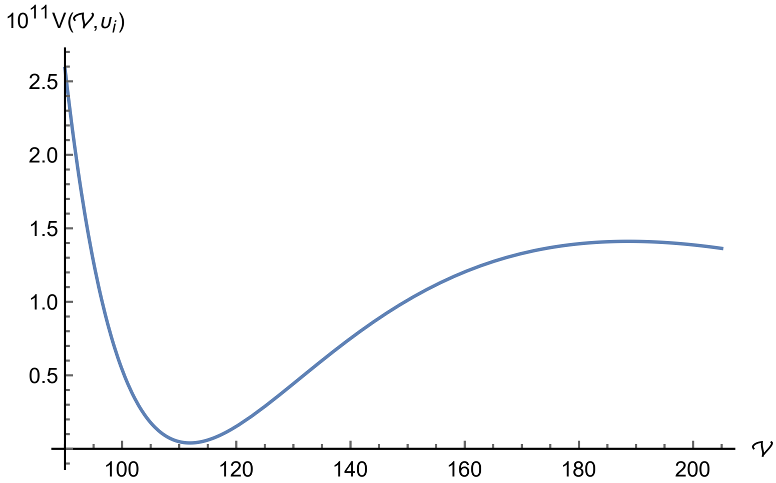

Figure 5.1: Minimum value of the effective scalar potential of the inflaton and the five waterfall fields in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (5.50).

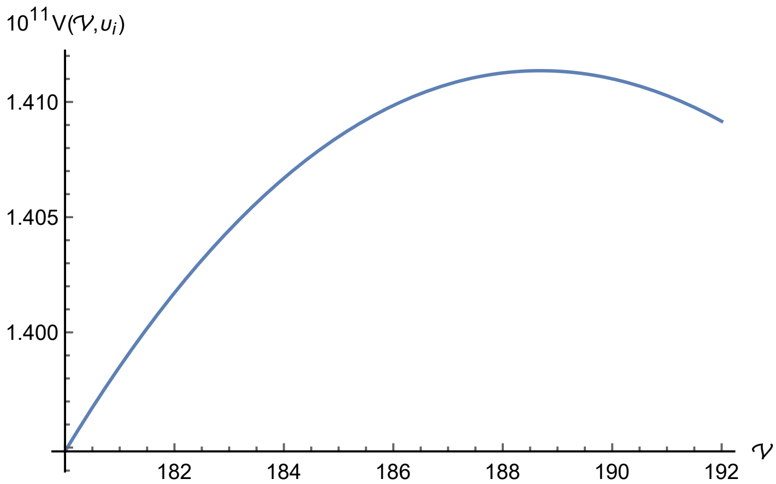

Figure 5.2: Waterfall direction in the effective scalar potential of the inflaton and the five waterfall fields in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the combination (5.50).

which we see that it is very close to the desired (3.92). Thus, by requiring lower critical volumes values of the four tachyons (expect for the first which is responsible for the end of inflation) in the theory with the additional fourth $D7$-brane stack, we can end up with a de-Sitter minimum that can reach the desired value. This could be done also by choosing lower β parameters for the five tachyons. In principle, one can tune the coefficients of the two final contributions in (5.52), or add more tachyonic fields in the theory and then lower their critical volume values or β parameters, so that the vacuum aquires the exact large accuracy of the observational value of the cosmological constant.

In closing, we can calculate the range of the values of the VEV's (in κ units): From (5.27) and (5.29), we have

$$\frac{m_{Y_a}^2}{\lambda_{Y_a}} = \frac{1}{2\kappa^2}\frac{g_s y_a(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|}{2+g_s y_a(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|}\left(1-\left(\frac{\mathcal{V}_{c2a}}{\mathcal{V}}\right)^{2/3}\right)$$

$$\rightarrow \langle y_{a-}\rangle = \pm\upsilon_{2a} = \pm\frac{\left|m_{Y_a}\right|}{\sqrt{\lambda_{Y_a}}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s y_a(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|}{2+g_s y_a(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|}}\left|1-\left(\frac{\mathcal{V}_{c2a}}{\mathcal{V}}\right)^{2/3}\right|^{1/2} \tag{5.54}$$

which for the combination (5.50), gives the range

$$\kappa\upsilon_{2a} \in [0, \simeq 0.0786247] \tag{5.55}$$

From (5.30) and (5.32), we have

$$\langle \psi_-\rangle = \pm\upsilon_3 = \pm\frac{\left|m_Z\right|}{\sqrt{\lambda_Z}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s z(U_3)\left|m_3^{(1)}m_3^{(2)}\right|}{2+g_s z(U_3)\left|m_3^{(1)}m_3^{(2)}\right|}}\left|1-\left(\frac{\mathcal{V}_{c3}}{\mathcal{V}}\right)^{2/3}\right|^{1/2} \tag{5.56}$$

which for (5.50) gives

$$\kappa\upsilon_3 \in [0, \simeq 0.0914307] \tag{5.57}$$

From (5.33) and (5.35), we get

$$\langle \chi_-\rangle = \pm\upsilon_1 = \pm\frac{\left|m_X\right|}{\sqrt{\lambda_X}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s x(U_1)\left|m_1^{(2)}m_1^{(3)}\right|}{2+g_s x(U_1)\left|m_1^{(2)}m_1^{(3)}\right|}}\left|1-\left(\frac{\mathcal{V}_{c1}}{\mathcal{V}}\right)^{2/3}\right|^{1/2} \tag{5.58}$$

which for (5.50) gives

$$\kappa\upsilon_1 \in [0, \simeq 0.106897] \tag{5.59}$$

From (5.36) and (5.38),

$$\langle y_{b-}\rangle = \pm\upsilon_{2b} = \pm\frac{\left|m_{Y_b}\right|}{\sqrt{\lambda_{Y_b}}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s y_b(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|}{2+g_s y_b(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|}}\left|1-\left(\frac{\mathcal{V}_{c2b}}{\mathcal{V}}\right)^{2/3}\right|^{1/2} \tag{5.60}$$

which for (5.50) gives

$$\kappa\upsilon_{2b} \in [0, \simeq 0.0775186] \tag{5.61}$$

and finally, from (5.39) and (5.41),

$$\langle y_{ab-}\rangle = \pm\upsilon_{2ab} = \pm\frac{\left|m_{Y_{ab}}\right|}{\sqrt{\lambda_{Y_{ab}}}} = \pm\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{g_s y_{ab}(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|}{1+g_s y_{ab}(U_2)\left|m_{2a}^{(1)}m_{2a}^{(3)}\right|}}\left|1-\left(\frac{\mathcal{V}_{c2ab}}{\mathcal{V}}\right)^{2/3}\right|^{1/2} \tag{5.62}$$

which for (5.50) gives

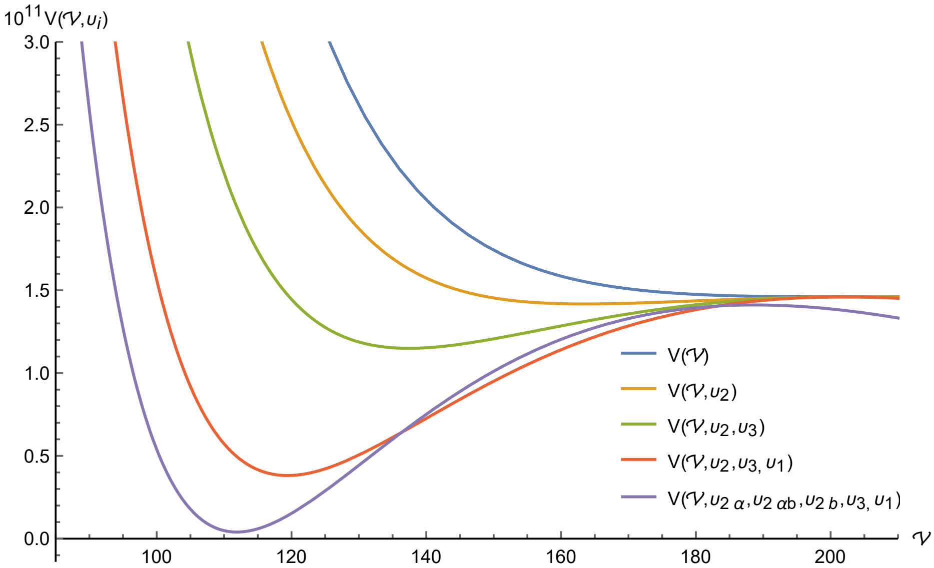

Figure 5.3: Minimum value of the effective scalar potential of the inflaton field only (blue), and of the inflaton with one (orange), two (green), three (red) and five waterfall fields (purple), in terms of $\mathcal{V}$, when $\kappa^2 = 1$, $q = 0$ and $C = 7.81 \times 10^{-4}$, for the different combinations of parameter values found in each case.

$$\kappa\upsilon_{2ab} \in [0, \simeq 0.108709] \tag{5.63}$$

From the ranges (5.55), (5.57), (5.59), (5.61) and (5.63), we see that the five VEV values stay small, so the quartic expansions in (5.1) hold.

Conclusions

In this work, we have studied the inflationary scenario proposed in [62], which is realized in the framework of type IIB String Theory considering a geometric configuration of intersecting $D7$-brane stacks. In this context, logarithmic pertubative corrections where investigated in [60], which come from the dimensional reduction of the effective 10-dimensional action, in the weak coupling and large volume limit (LVS). This set up ensures Kähler moduli stabilization and metastable de-Sitter vacua.

Concerning the inflationary phase in the above model, it can be realized by identifying a proportional to the logarithm of the internal (compactification) volume $\mathcal{V}$ quantity, as the inflaton field. It is shown that there is only one effective (free) parameter, x, that determines the shape of the resulting effective scalar potential (inflaton's potential), and hence the measurable parameters related to inflation. A de-Sitter minimum value of the effective potential, restricts the x parameter to the range $0 < x < x_c$, where $x_c \simeq 0.0721318$; below this range, the potential loses its local extrema as the two branches of the Lambert W-function join, while above this range the potential aquires an Anti-de Sitter minimum, with x_c being the critical value that corresponds to a vanishing minimum value and therefore to the Minskowki vacuum. Through the minimization of the potential, the free parameter x can be related to an equivalent, more useful parameter for the calculations which we call s, and then the range required for a de-Sitter minimum is expressed as $-0.00724192 < s < -0.00673795$.

In particular, the parameter x must be fixed to the value $x \simeq 3.3 \times 10^{-4}$ in order to realize slow-roll inflation compatible with observations and the required number of the 60 e-folds, which corresponds to $s \simeq -0.00723954$. Most of the e-folds are obtained at the vicinity of the minimum, in contrast to other inflationary scenarios, such as hiltop inflation. Demanding the above value for the free parameter in the effective scalar potential, we see that the inflaton field displacement is $\Delta\phi \simeq 0.04$, which is small compared to the Planck scale (small-field inflation) and thus compatible with the validity of the effective field theory. Also, the minimum value of the potential is of order of the inflation energy scale, which means that is very shallow. More specifically, by fixing the overall constant of the potential (which plays no role in the minimisation and in the inflationary dynamics, but is related to the observed spectral amplitude [60]) as required by observational constraints, we see that the metastable minimum is high above the observational value of the cosmological constant, and in conclusion, it cannot be the true vacuum of the theory.

Subsequently, we studied the proposed solution in [62] through the model of hybrid inflation, which can be realized if one identifies the waterfall fields as excitations of open strings with endpoints on the $D7$-brane stacks. The latter correspond to tachyonic states that can appear in the spectrum when magnetic fields are introduced on the $D7$ branes, as their squared-masses may receive negative together with their positive contributions [62]. The waterfall field squared-masses depend on the value of the inflaton, and by choosing appropriate values for the integers corresponding to magnetic fluxes and for the other string parameters that squared-masses are expressed in terms of, they become tachyonic under certain (critical) inflaton values, successively, with the first waterfall field responsible for the end of inflation becoming tachyonic at the inflaton's minimum as desired. In this way, apart from the ending of the inflationary stage, the waterfall directions generated can deepen the vacuum of the theory.

More specifically, in the first studied case, where all charged open string states are chosen to have positive squared-masses, except for one tachyonic state which corresponds to one waterfall field, with the help of the program Mathematica, we minimised the effective scalar potential and obtained a volume-depended vacuum which is expressed in terms of the aforementioned integers relevant to magnetic fluxes, and the other parameters. We found a new combination for these paramaters of the vacuum, from the one proposed in the initial work [62], which gives the lowest value that the vacuum can take in this theory. This value found is in agreement with the one given by the proposed combination in [62]. Although the waterfall direction lowered the vacuum of the theory with the appropriate combination of the parameter values, turns out that it has a much higher value than the one of the cosmological constant. In fact, there is a parameter which essentially expresses the amount of contribution of the waterfall part in the vacuum, and its specific value required for an almost vanishing value has been calculated in [62]. Indeed, one waterfall field in the effective potential does not contribute negatively enough to the vacuum according to the above parameter.

In the second and third studied cases, where practically another one and two respectively tachyonic states are not eliminated any more, so we have two and three waterfall fields respectively in the hybrid potential, we showed that one can work in a similar way as in the first case, and find a suitable combination of the vacua parameter-values, so that they possess (in each case) the lowest possible value. With the appropriate combinations found in this work, the waterfall fields indeed provide deeper total waterfall directions in the potential, but even with three waterfall fields the vacuum is not low enough to correspond to the observed cosmological constant, in agreement with [62]. Again, this can be predicted by the relevant parameter that measures the contribution of the waterfall part. All in all, we see that there are appropriate combinations of the parameter values under which indeed the more waterfall fields we add in the hybrid model, the lower the vacuum becomes, as the positive contribution of the inflaton and the negative contribution of the waterfall part of the vacuum are closer to cancel each other. Also, the vacuum is displaced towards lower values of the inflaton field, and in all three cases, the local maximum of the potential is near the minimum of the inflaton's potential.

Finally, we tried the proposed solution of the addition of a fourth $D7$-brane stack, parallel to one of the already existing $D7$-brane stacks [62], from which two more tachyonic states appear in the spectrum, so we totally have five waterfall fields in the hybrid potential. We have shown that the waterfall part in the vacuum contributes more than enough, as it was anticipated from the parameter that measures the waterfall contribution in the vacuum, and one obtains an Anti-de Sitter minimum value, as in [62]. Accepting a bit lower critical volume values for the waterfall fields, except for the one responsible for the end of inflation, we showed that the value of the vacuum can be uplifted to positive values, so it can become de-Sitter again, as in [62]. We found an appropriate new combination of the vacuum parameter-values, from the one proposed in the initial work [62], so that it aquires the lowest possible value. The value found is in agreement with the one given by the proposed combination in [62], and the parameter that measures the waterfall contribution in the vacuum is very close to the one required for the exact almost vanishing value of the minimum.

The value of the vacuum with the appropriate combination of its parameters in the above configuration, is even lower than in the other cases, as the contributions of the inflaton and the waterfall part in the vacuum are even closer to cancel each other, and also the vacuum is displaced towards lower values of the inflaton, with the maximum being very close to the minimum of the inflaton's potential, as in the other cases and as desired. We have to mention that in the cases of three and five waterfall fields in the hybrid potential, the program could not find solutions (combinations) that satisfy all the desired constraints together, and we searched firstly for the coupling constants that result in combinations which give a value for the parameter s closer to the desired by the observational restrictions, then for the best combinations found we fixed the other parameters to give the desired critical volume values (this is where we required lower critical volume values in the last case), and finally we chose the combination that gives the best positive minimum.

We see that the addition of the $D7$-brane stack is sufficient to reach the desired amount of waterfall contribution to the vacuum of the theory. In principle, one can fine tune the contributions of the inflaton and the waterfall part so that they cancel in the exact large accurary, or add more waterfall fields in the hybrid potential and then lower their critical volume values or the parameter that counts their contribution to the vacuum [62]. In fact, more phase transitions from new low-energy physics are expected to affect the scalar potential anyway. In summary, this String Theory construction offers an implementation for the scenario of Hybrid Inflation, from which we finally aquire a tunable vacuum that can reach the observational value of the cosmological constant, which is thought to account for the present observed dark energy in the universe.

Appendix A

Variation of the Hilbert Action in GR

The Einstein-Hilbert action in General Relativity is

$$S_H = \int \sqrt{-g} R d^n x = S_H = \int \sqrt{-g} g^{\mu\nu} R_{\mu\nu} d^n x \tag{A.1}$$

For variation of this action, we have

$$\delta S_H = \int d^n x \big(\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \delta \sqrt{-g} (R - 2\Lambda) \big)$$

$$\rightarrow \delta S_H = \delta S_1 + \delta S_2 + \delta S_3 \tag{A.2}$$

with

$$\delta S_1 = \int d^n x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}, \quad \delta S_2 = \int d^n x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu}, \quad \delta S_3 = \int d^n x \delta \sqrt{-g} (R - 2\Lambda) \tag{A.3}$$

For arbitrary variations, from the Christoffel coefficients we have

$$\Gamma^\rho_{\nu\mu} \rightarrow \Gamma^\rho_{\nu\mu} + \delta \Gamma^\rho_{\nu\mu} \tag{A.4}$$

and we keep in mind the covariant derivative of $\delta\Gamma^\rho_{\nu\mu}$ [1], which is

$$\nabla_\lambda (\delta \Gamma^\rho_{\nu\mu}) = \partial_\lambda (\delta \Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\lambda\sigma} \delta \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\lambda\nu} \delta \Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\lambda\mu} \delta \Gamma^\rho_{\nu\sigma} \tag{A.5}$$

The variation of the Riemann tensor (1.8) is

$$\delta R^\rho{}_{\mu\lambda\nu} = \partial_\lambda (\delta \Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\lambda\sigma} \delta \Gamma^\sigma_{\nu\mu} + \Gamma^\sigma_{\nu\mu} \delta \Gamma^\rho_{\lambda\sigma}$$

$$- \partial_\nu (\delta \Gamma^\rho_{\nu\mu}) - \Gamma^\rho_{\nu\sigma} \delta \Gamma^\sigma_{\lambda\mu} - \Gamma^\sigma_{\lambda\mu} \delta \Gamma^\rho_{\nu\sigma} \tag{A.6}$$

keeping only first order $\delta\Gamma$-terms. Using (A.5), this becomes

$$\delta R^\rho{}_{\mu\lambda\nu} = \nabla_\lambda (\delta \Gamma^\rho_{\nu\mu}) + \Gamma^\sigma_{\lambda\nu} \delta \Gamma^\rho_{\sigma\nu} - \big(\nabla_\nu (\delta \Gamma^\rho_{\lambda\mu}) + \Gamma^\sigma_{\nu\lambda} \delta \Gamma^\rho_{\sigma\mu} \big) \tag{A.7}$$

and as $\Gamma^\sigma_{\lambda\nu} = \Gamma^\sigma_{\nu\lambda}$, (A.7) is

$$\delta R^\rho{}_{\mu\lambda\nu} = \nabla_\lambda (\delta \Gamma^\rho_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\rho_{\lambda\mu}) \tag{A.8}$$

Thus, the variation of the Ricci tensor is

$$\delta R_{\mu\nu} = \delta R^\lambda{}_{\mu\lambda\nu} = \nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu}) \tag{A.9}$$

and finally δS_1 (A.3) is

$$\delta S_1 = \int d^n x\sqrt{-g}g^{\mu\nu}\big(\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\big)$$

$$= \int d^n x\sqrt{-g}\nabla_\sigma\big(g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^\lambda_{\lambda\mu})\big) \tag{A.10}$$

using metric compatibility $\nabla_\sigma g^{\mu\nu} = 0$. Then, from the definition of the Christoffel coefficients (1.6), we have

$$\delta\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\delta g^{\sigma\rho}\big(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\big)$$

$$+\frac{1}{2}g^{\sigma\rho}\big(\partial_\mu \delta g_{\nu\rho} + \partial_\nu \delta g_{\rho\mu} - \partial_\rho \delta g_{\mu\nu}\big) \tag{A.11}$$

But,

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}\big(g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu a}g_{\nu\beta}\nabla^\sigma(\delta g^{a\beta})\big)$$

$$= (A.11) \tag{A.12}$$

using $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$ and

$$\nabla_\rho(\delta g_{\mu\nu}) = \partial_\rho(\delta g_{\mu\nu}) - \Gamma^\lambda_{\rho\mu}(\delta g_{\lambda\nu}) - \Gamma^\lambda_{\rho\nu}(\delta g_{\mu\lambda}) \tag{A.13}$$

Plugging (A.12) into (A.10), we finally have

$$\delta S_1 = \int d^n x\sqrt{-g}\nabla_\sigma\big(g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\big) \tag{A.14}$$

which from the Stoke's theorem (1.145), assuming $\delta g^{\mu\nu} = 0$ at the boundary, vanishes, $\delta S_1 = 0$.

Moreover, for any square matrix M with non-vanishing determinant, it is

$$\ln(\det M) = \mathrm{Tr}(\ln M) \tag{A.15}$$

for $e^{\ln M} = M$. Varying this identity, we get

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M) \tag{A.16}$$

If $M = g_{\mu\nu}$, from (A.16) we have

$$\frac{1}{g}\delta g = \mathrm{Tr}(g^{-1}\delta g) = g^{\mu\nu}\delta g_{\mu\nu}$$

$$\to \delta g = g(g^{\mu\nu}\delta g_{\mu\nu}) \tag{A.17}$$

and using $\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$, (A.17) becomes

$$\delta g = -g(g_{\mu\nu}\delta g^{\mu\nu}) \tag{A.18}$$

From the above, we find that

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g = -\frac{1}{2}\frac{1}{\sqrt{-g}}(-g(g_{\mu\nu}\delta g^{\mu\nu}))$$

$$= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{A.19}$$

and inserting this in (A.3), we finally have

$$\delta S_3 = -\int d^n x \left(\frac{R}{2} - \Lambda \right) \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{A.20}$$

Substituting into (A.2), the variation of the action takes the form (1.151),

$$\delta S_H = \int d^n x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu} \tag{A.21}$$

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