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Black Hole solutions in Einstein-Gauss-Bonnet Theory with a self-interacting scalar field

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Λύσεις Μελανών Οπών στην θεωρία **Einstein-Gauss-Bonnet** με ένα αυτο-αλληλεπιδρών βαθμωτό πεδίο

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Abstract

Gravity is one of the fundamental interactions in nature. The first description of gravity was provided by Newton's theory of gravity. In the early 20 th century Albert Einstein formulated a new theory for gravity, the **General Theory of Relativity**, which included Newtonian theory in the limit of the weak gravitational field. It has been experimentally proven that General Relativity describes the gravitational interactions with high precision and, in addition, predicts new gravitational objects such as *Black Holes*.

However, being a classical theory, General Relativity is not considered to be the final theory for gravity. Due to its failure to describe the gravitational phenomena at high energies it is necessary to formulate a quantum theory of gravity. Since the 80's the so-called generalized theories of gravity, namely superstring effective theories at low energies, Lovelock's theory and Horndeski's scalar-tensor theories, have emerged. These theories can embed in their framework the gravitational degrees of freedom of General Relativity with scalars, fermions and gauge fields as well as higher-order curvature terms. In the context of these theories, new black-hole solutions can be found which differentiate from GR's solutions and violate the traditional No-Hair Theorem.

In this thesis, we study the Einstein-scalar-Gauss-Bonnet theories which comprise a subclass of generalized gravitational theories. In their action, they include the usual Einstein-Hilbert term, the quadratic gravitational Gauss-Bonnet term and a non minimally coupled self-interacting scalar field. We investigate the existence of new black hole solutions for different coupling functions between the scalar field and the Gauss-Bonnet term and different forms of the scalar potential. Especially we emphasize on the Higgs potential, the Coleman-Weinberg potential and the Starobinsky type potential. We look for asymptotically flat, de-Sitter (dS) or Anti-de-Sitter (AdS) black-hole solutions, and discuss their domain of existence and physical features in each case.

Contents

Chapter 1

Introduction to mathematical formulation

1.1 Geometrical tools for General Theory of Relativity

The General theory of Relativity is a theory that describes the gravitational interaction. As distinct from Newton's theory, Einstein's theory is a *geometrical* theory. In the limit of the weak field the theory leads to Newton's gravity. Furthermore, in General Relativity the gravitational interaction is connected to the properties of the *four dimensional spacetime*. In actual fact, gravity is the curvature of the spacetime and this curvature is induced by distribution of energy and mass. In addition, this description gives us the ability to explicate the behaviour of gravity in stronger than Earth's field regimes, such as Sun's gravitational field etc. Moreover, many experiments have confirmed the validity of General Theory of Relativity, such as observations of the Mercury's orbit around the Sun, gravitational lensing, gravitational waves detection etc. Ultimately, Einstein's theory is a rich theory because describes successfully the gravitational interaction and provides new types of gravitational objects like *Black Holes* or *Wormholes*.

Before discussing the field equations, we will briefly present useful geometrical tools for a curved space. Hence, we will discuss elements of *differential geometry*, a mathematical framework which generalizes the *Euclidean* geometry. In General Relativity we are interested in the geometry of the spacetime *manifold* $(\mathcal{M}_4, g)$. This information is contained in the metric tensor $g_{\mu\nu}$, which is a $(0,2)$ and symmetric tensor. This occurs to the known *line element*,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.1.1}$$

where $x^\mu = (ct, \vec{x})$. On the other hand in spacetime of *Special Relativity* we encountered the flat metric $\eta_{\mu\nu}$ and the corresponding line element,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{1.1.2}$$

For completeness, we will mention some useful properties of the metric tensor. We are interested in the inverse of metric tensor. In order to be able to define the inverse, it must be true that the determinant $g = \det g_{\mu\nu}$ is non zero. We must add here that this is not always the case. This allows us to define the inverse metric tensor via the following relation,

$$g^{\lambda\mu} g_{\lambda\sigma} = \delta^\mu_\sigma. \tag{1.1.3}$$

A useful characterization of the metric is obtained through the *signature*. Signature is the number of both positive and negative eigenvalues. For example the metric tensor of the Minkowski spacetime has a signature in the form of $(-,+,+,+)$. If the metric has a zero eigenvalue then the inverse will not exist. If all eigenvalues have positive sign, the metric is called *Euclidean* or *Riemannian*, while if an eigenvalue has a negative sign is called *Lorentzian* or *pseudo-Riemannian*.

Mathematical objects in General Relativity are generally tensors. By mathematical definition, a tensor is an object which is transformed under coordinate transformations as follows,

$$A'^{\alpha\beta\ldots}{}_{\mu\nu\ldots} = \frac{\partial x'^\alpha}{\partial x^k} \frac{\partial x'^\beta}{\partial x^l} \ldots \frac{\partial x^p}{\partial x'^\mu} \frac{\partial x^q}{\partial x'^\nu} \ldots A^{kl\ldots}{}_{pq\ldots} \tag{1.1.4}$$

As we know there are also objects which have indices but they are not tensors. Two examples of them are the so called Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ and Kronecker's symbol δ^μ_ν. These symbols are defined by

$$\delta^\mu_\nu = \begin{cases} 1 & \text{If } \mu = \nu \\ 0 & \text{If } \mu \neq \nu \end{cases} \quad , \qquad \epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{If } \mu\nu\rho\sigma \text{ is an even permutation of 0123} \\ -1 & \text{If } \mu\nu\rho\sigma \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise} \end{cases} . \tag{1.1.5}$$

Bearing in mind the definition of the tensor, it's interesting to study how the differential of a tensor is transformed. It should be stressed that in the generality the differential of a tensor $dT^{\kappa\lambda\ldots}{}_{\mu\nu\ldots}$ is not a tensor. This is because it is defined as the difference of two tensors evaluated at different points of a given manifold. Hence, the partial derivative ∂_μ of a vector field V^ρ is transformed as follows,

$$\partial_\lambda V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\rho}{\partial x'^\lambda} \frac{\partial V^\nu}{\partial x^\rho} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho} \frac{\partial x^\rho}{\partial x'^\lambda} V^\nu. \tag{1.1.6}$$

It is obvious from the above relation that the partial derivative of a vector field is not transformed as a tensor. In General Relativity and more specifically in the curved

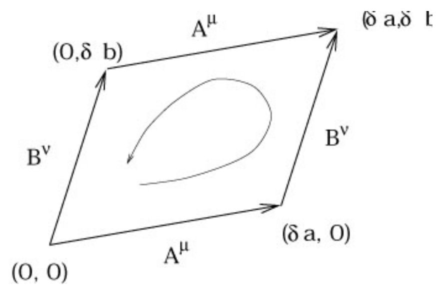


Figure 1.1.1: The displacement of a given vector V^{ρ} on a manifold $(\mathcal{M}_4, g)$. The variation of V^{ρ} gives us the curvature. [1]

spaces we are interested in derivatives which are governed by a tensorial behaviour. From the geometrical point of view this means that any vector on a curved space displaced in parallel. This is why we introduce a new type of derivative, the so called *covariant derivative*. For a given vector V^{ρ} the covariant derivative is defined by the following relation,

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\rho} V^{\rho} \tag{1.1.7}$$

where $\Gamma^{\nu}_{\mu\rho}$ are the *connection coefficients* having the following transformation,

$$\Gamma'^{\nu}_{\lambda\mu} = \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \Gamma^{\beta}_{\rho\alpha} - \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\rho}}. \tag{1.1.8}$$

It becomes immediately clear that these coefficients do not obey to a tensorial transformation. If we now require the compatibility of the metric, namely $\nabla_{\rho} g_{\mu\nu} = 0$ the coefficients are given by,

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_{\mu} g_{\beta\nu} + \partial_{\nu} g_{\mu\beta} - \partial_{\beta} g_{\mu\nu} \right) \tag{1.1.9}$$

This is the expression of the metric connection we will find in General Relativity known in the bibliography as *Christofel's symbols*.

We have already mentioned that the gravitational interaction is connected with the curvature of the spacetime. Also all the information about the geometry of the manifold is encoded to metric tensor, so we need to define a tensor about curvature. Assuming the Fig.1.1.1 we expect the expression of δV^{ρ} when the vector V^{ρ} is parallel transported around the loop, is propotional to a $(1,3)$ tensor. Hence we can write,

$$\delta V^{\rho} = R^{\rho}{}_{\sigma\mu\nu} V^{\sigma} A^{\mu} B^{\nu}. \tag{1.1.10}$$

Where $R^{\rho}{}_{\sigma\mu\nu}$ is the well known *Riemann tensor* or simply *curvature tensor*. The definition of Riemann's tensor is as follows,

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu} \Gamma^{\rho}_{\nu\sigma} - \partial_{\nu} \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\alpha} \Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\rho}_{\nu\alpha} \Gamma^{\alpha}_{\mu\sigma}, \tag{1.1.11}$$

in terms of derivatives of Christoffel's symbols and their products. As we will see below, the field equations include another tensor known as *Ricci tensor* which can be extracted from Riemann tensor. We take a contraction $R^{\rho}{}_{\sigma\mu\nu}$ and we obtain,

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \partial_{\rho}\Gamma^{\rho}_{\mu\nu} - \partial_{\mu}\Gamma^{\rho}_{\rho\nu} + \Gamma^{\rho}_{\rho\alpha}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\rho}_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu}, \tag{1.1.12}$$

where $R_{\mu\nu}$ is the Ricci tensor which is a $(0, 2)$ and symmetric tensor. Also, we can define the trace of $R_{\mu\nu}$ as

$$R = g^{\mu\nu}R_{\mu\nu} \tag{1.1.13}$$

This trace is the *Ricci scalar* and also appears in field equations. Let us now note some useful properties of these tensors. From eq. (1.1.11) we assume that the Riemann tensor is antisymmetric in its last two and first indices,

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \qquad R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \tag{1.1.14}$$

and it is invariant under interchange of the first pair of indices with the second,

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \tag{1.1.15}$$

From eq. (1.1.14)-(1.1.15) we can see that the sum of cyclic permutations of the last three indices has to be zero,

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0. \tag{1.1.16}$$

On the other hand, from eq. (1.1.12) we notice that Ricci tensor is a symmetric one in two indices,

$$R_{\mu\nu} = R_{\nu\mu} \tag{1.1.17}$$

A very useful identity is the well known *Bianchi identity*. For any coordinate system we can write it as,

$$\nabla_{\lambda}R_{\rho\sigma\mu\nu} + \nabla_{\rho}R_{\sigma\lambda\mu\nu} + \nabla_{\sigma}R_{\lambda\rho\mu\nu} = 0. \tag{1.1.18}$$

By contracting twice on eq. (1.1.18) to export an useful identity,

$$\begin{aligned} 0 &= g^{\nu\sigma}g^{\mu\lambda}(\nabla_{\lambda}R_{\rho\sigma\mu\nu} + \nabla_{\rho}R_{\sigma\lambda\mu\nu} + \nabla_{\sigma}R_{\lambda\rho\mu\nu}) \\ &= \nabla^{\mu}R_{\rho\mu} - \nabla_{\rho}R + \nabla^{\nu}R_{\rho\nu} \\ &\Rightarrow \nabla^{\mu}R_{\rho\mu} = \frac{1}{2}\nabla_{\rho}R. \end{aligned} \tag{1.1.19}$$

We define now the *Einstein tensor*,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{1.1.20}$$

and we see from contracting the Bianchi identity that the covariant derivative on Einstein's tensor vanishes,

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{1.1.21}$$

This is an important property because it is connected to the conservation of energy and momentum as we shall see next.

It is reasonable to ask how an observer moves in a gravitational background. The answer to this question is not trivial as in Newtonian gravity, where it is enough to solve the equation of the second law of mechanics. We want to find an equation that governs the motion of an observer in curved spacetime. In Euclidean spaces for a free moving observer the equation which describes the motion has to be,

$$\frac{d^2 x^{\mu}(\lambda)}{d\lambda^2} = 0, \tag{1.1.22}$$

corresponding to a parametric straight line. The generalization of this kind of line on a manifold $(\mathcal{M}_4, g)$ is called *geodesic curve.* By the definition a geodesic curve $x^{\mu}(\lambda)$ is one which along with the tangent vector $\frac{dx^{\mu}(\lambda)}{d\lambda}$ is parallel-transported. Hence we can write a condition for the parallel transportation as,

$$\frac{D}{d\lambda}\frac{dx^{\mu}}{d\lambda} = 0, \tag{1.1.23}$$

where

$$\frac{D}{d\lambda} = \frac{dx^{\mu}}{d\lambda}\nabla_{\mu}, \tag{1.1.24}$$

is the so called *directional covariant derivative*. Acting on the tangent vector we finally get,

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0. \tag{1.1.25}$$

This is the *geodesic equation* and reproduces straight lines if and only if $\Gamma^{\mu}_{\rho\sigma}$ vanishes. An alternative method to derive the geodesic equation is to consider the functional,

$$S = \int \sqrt{g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}} d\lambda, \tag{1.1.26}$$

where the integral is over the path. We demand the functional has an extremum, hence the variation δS has to be zero. The result accordingly to the Principle of Least Action, is the same with eq. (1.1.25).

1.2 General theory of Relativity-Field equations

Having the geometrical tools, we are now ready to mathematically establish General Relativity and its basic idea of the origin of gravitational interaction. The brilliant idea conceived by Albert Einstein is that *the gravitational field influences the behavior of energy and mass* and *the energy and mass shapes the gravitational field*. Newtonian gravity consists of two basic equations. The first one is an expression for the acceleration of a body in a *gravitational potential* Φ

$$\vec{a} = -\vec{\nabla}\Phi, \tag{1.2.1}$$

where $\vec{a}$ is the acceleration acquired by a body due to the gravitational potential. The second one is Poisson's differential equation which connects the density of mass $\rho(\vec{r})$ and the gravitational potential namely,

$$\nabla^2\Phi = 4\pi G\rho(\vec{r}), \tag{1.2.2}$$

where G is Newton's gravitational constant. Our ambition is to find in General Relativity an equation that incorporates analogous arguments for the nature of gravitational interaction i.e how energy and mass influence spacetime to create curvature. For this purpose we introduce the famous *Einstein's Equivalence principle (EEP)* which states that, *"In small enough regions of spacetime, the laws of physics reduce to those of special relativity, it is impossible to detect the existence of a gravitational field by means of local experiments"* and that's because the gravitational interaction is *universal*. This is exactly the reason led Einstein to think that, what perceive as gravitational interaction is the manifestation of the curvature of spacetime. We can now write a prescription for generalizing laws of physics in curved spacetime.

- Write a law in inertial coordinates in flat spacetime
- Write it in tensorial form
- Assume that the resulting law remains true in curved spacetime.

We must mention that we should be particularly careful in the context of this prescription. First of all we must replace the Minkowski $\eta_{\mu\nu}$ by a more general metric $g_{\mu\nu}$. Secondly, we must exchange partial derivatives ∂_μ with covariant derivative ∇_μ. These are all true if we want to generalize the law of motion of a free falling particle. Recall that this law is written in the form as in the eq. (1.1.22). Therefore, we can use the chain rule,

$$\frac{d^2x^\mu(\lambda)}{d\lambda^2} = \frac{dx^\nu(\lambda)}{d\lambda}\partial_\nu\frac{dx^\mu(\lambda)}{d\lambda} \Rightarrow \frac{dx^\nu(\lambda)}{d\lambda}\nabla_\nu\frac{dx^\mu(\lambda)}{d\lambda} = 0, \tag{1.2.3}$$

where we have replaced the partial derivative with a covariant one. This immediately leads to eq.(1.1.25).

Just as any field theory consists of equations that relate fields behavior with physical causes, as well as Einstein's field equation relates how the metric responds to energy and mass. In the previous subsection we discussed in detail the tensor that gives us the information about the curvature and we extracted the Einstein's tensor, which has the important property namely, $\nabla^{\mu} G_{\mu\nu} = 0$. As we know the energy-momentum tensor which describes the energy and mass distribution obeys also to the law of conservation $\nabla_{\mu} T^{\mu\nu} = 0$. The field equations should lead to differential equations with derivatives up to second order to avoid instabilities. Recall that the Riemann tensor is constructed by Christoffel symbols and their derivatives, and the Christoffel symbols are constructed from the metric and its derivatives. Hence $R^{\rho}{}_{\sigma\mu\nu}$ constructed by second derivatives of $g_{\mu\nu}$. But Riemann's tensor doesn't have the same number of indices as the energy-momentum tensor. So, we can contract it to form the Ricci tensor which is a symmetric. We could now claim that the field equations are given by the relation,

$$R_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.2.4}$$

where κ is a constant. Nevertheless there is a problem, with the conservation of energy. If we want to satisfy the conservation of energy-momentum tensor,

$$\nabla_{\mu} T^{\mu\nu} = 0$$

by eq.(1.2.4) we will have,

$$\nabla^{\mu} R_{\mu\nu} = 0. \tag{1.2.5}$$

This is not true in an arbitrary geometry as we can see from Bianchi identity namely,

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R. \tag{1.2.6}$$

In addition the eq.(1.2.4) implies that $R = \kappa g^{\mu\nu} T_{\mu\nu} = \kappa T$, where T is the trace of energy-momentum tensor. So, taking these together we have,

$$\nabla_{\mu} T = 0. \tag{1.2.7}$$

By the definition the covariant derivative when acting on a scalar quantity is just the partial derivative. Hence the eq.(1.2.7) tells us that T is a constant throughout spacetime. This is inconsistent, since $T = 0$ in vacuum while $T \neq 0$ in matter. This problem is solved if we replace the Einstein tensor in the first part of eq.(1.2.4) which always obeys conservation law. Therefore we take the field equation for the metric,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}. \tag{1.2.8}$$

The constant κ will be determined by the limit of the weak field. In this limit General Relativity should be reduce to Newtonian theory, i.e we should derive the Poisson equation like eq.(1.2.2). Suppose that the energy-momentum tensor of a perfect-fluid gravitational source is given by,

$$T_{\mu\nu} = (\rho + p)v_\mu v_\nu + p g_{\mu\nu} \tag{1.2.9}$$

where v_μ is the fluid four-velocity and ρ and p are the rest frame energy and momentum densities. We consider now the energy momentum tensor of dust which has $p = 0$, therefore

$$T_{\mu\nu} = \rho v_\mu v_\nu, \tag{1.2.10}$$

and we will work in the rest frame for a massive body where $v^\mu = (v^0, 0, 0, 0)$. The weakness of gravitational field means that we can decompose the metric field into the Minkowski metric plus a small perturbation namely,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1 \tag{1.2.11}$$

We must mention that the timelike component is fixed by the normalization condition of the four velocity $g_{\mu\nu}v^\mu v^\nu = -1$. This condition leaving us in the first order of $h_{\mu\nu}$ with,

$$v^0 = 1 + \frac{1}{2}h_{00} \tag{1.2.12}$$

The 00 component of gravitational field equation gives us,

$$R_{00} = \kappa T_{00} = \frac{1}{2}\kappa\rho, \tag{1.2.13}$$

where we have used the eq.(1.2.10) for T_{00}. After a small calculation of R_{00} from eq.(1.1.12) in first order of $h_{\mu\nu}$ we get,

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00}. \tag{1.2.14}$$

So the eq.(1.2.13) leads to the result,

$$\nabla^2 h_{00} = -\kappa\rho. \tag{1.2.15}$$

The h_{00} component can be identified by geodesic equation in the weak limit to be $h_{00} = -2\Phi$ where Φ is the Newtonian gravitational potential. Comparing eq.(1.2.15) to Poisson's equation we find that $\kappa = 8\pi G$. Now the Einstein's equation for General theory of Relativity can be presented,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{1.2.16}$$

where G is Newton's gravitational constant. This equation reflects the basic principle of Einstein's theory that the curvature of spacetime reacts to the presence of energy and mass. If the right-hand vanishes i.e $T_{\mu\nu} = 0$ therefore we get the Einstein's equation in vacuum. This is approximately true for the spacetime where it may be considered empty of energy and mass particles. This doesn't lead to the absence of gravitational field due to the vanishing of Ricci tensor. We know that the vanishing of Ricci tensor doesn't imply the flatness of spacetime as well as Ricci tensor is constructed by a contraction on Riemann's tensor.

1.3 Variational principle in General theory of Relativity

In 1915 David Hilbert formulated the so called Einstein-Hilbert action. In fact by applying the action minimization method he managed to derive the Einstein's equation. Before we give the expression for Einstein-Hilbert action, it is worth considering how we can construct it by principles. Although we don't have a rigorous method for constructing an action, every action of a physical theory ought to obey certain principles. The ones for General Relativity are the following,

- Action must be invariant under Lorentz transformations
- It should be invariant under diffeomorfisms. This principle leads to the conservation of energy-momentum tensor.
- It should lead to differential equations with derivatives up to second order
- It should be limited to the four dimensions.
- No other field will be included in the action other than metric.

The simplest action we can write that incorporates these principles is the following,

$$S_{EH} = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} R \tag{1.3.1}$$

where $\sqrt{-g}$ is the square root of determinant of metric, $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar. Furthermore we can add in eq.(1.3.1) a part corresponding to the distribution of energy and mass namely,

$$S_{EH} = \frac{1}{16\pi} \Big[\int_{\mathcal{M}} d^4x \sqrt{-g} R + \mathcal{L}_m \Big], \tag{1.3.2}$$

where $\mathcal{L}_m$ represents the energy-mass Lagrangian. Varying the action with respect to the metric we get

$$\delta S = \frac{1}{16\pi}\Big[\int_{\mathcal{M}} d^4x\ \delta(\sqrt{-g})R + \sqrt{-g}(\delta R) + \delta\mathcal{L}_m\Big], \tag{1.3.3}$$

but

$$\begin{aligned}\delta(\sqrt{-g}) &= -\frac{\delta g}{2\sqrt{-g}} \\ &= \frac{1}{2}\frac{g}{\sqrt{-g}} g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}.\end{aligned} \tag{1.3.4}$$

The variation on Ricci can be written as,

$$\delta R = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{1.3.5}$$

Using Palatini's identity the variation on Ricci tensor leaving us with

$$\delta R_{\mu\nu} = \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) - \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}). \tag{1.3.6}$$

Using the fact that,

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}A^\rho), \tag{1.3.7}$$

we turn to eq.(1.3.3) and obtain,

$$\delta S_{EH} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}\Big[(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} + \partial_\nu(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}) - \partial_\lambda(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) + \delta S_m\Big]. \tag{1.3.8}$$

By the definition the variation with respect to the metric of energy-mass action gives us the energy momentum tensor namely,

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{1.3.9}$$

Hence, we demand that $\delta_g S = 0$ but the terms of total derivatives are surface terms which are evaluated on boundary of spacetime. Therefore if we assume that $\delta g^{\mu\nu}|_{\partial\mathcal{M}} = 0$ the surface terms has to be vanished and we get Einstein's field equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.3.10}$$

1.4 The Gibbons-Hawking-York boundary term

Let us to return to the second term of eq.(1.3.3). The variation on Christoffel symbols can be written in the form

$$\delta\Gamma^{\mu}_{\sigma\alpha} = \frac{1}{2}g^{\mu\rho}\Big[\nabla_{\alpha}(\delta g_{\sigma\rho}) + \nabla_{\sigma}(\delta g_{\alpha\rho}) - \nabla_{\rho}(\delta g_{\sigma\alpha})\Big] \tag{1.4.1}$$

After some manipulation of indices and using the condition of compatibility of the metric it is straightforward to prove that

$$\int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_{\mu}(g^{\mu\nu}\delta\Gamma^{\rho}_{\rho\nu} - g^{\nu\sigma}\delta\Gamma^{\mu}_{\nu\sigma}) = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_{\mu}V^{\mu}, \tag{1.4.2}$$

where V^{μ} is a vector field on spacetime manifold. We use now the Stokes theorem to express the resulting integral as

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_{\mu}V^{\mu} = \oint_{\partial\mathcal{M}} d\Sigma_{\mu}V^{\mu} = \oint_{\partial\mathcal{M}} d^3x\sqrt{|h|}\epsilon V^{\mu}n_{\mu}, \tag{1.4.3}$$

where n_{μ} is a normalized and unit vector on $\partial\mathcal{M}$. Note that $g_{\mu\nu} = \epsilon n_{\mu}n_{\nu} + h_{\mu\nu}$, and h is the determinant of induced metric on boundary. Also $\epsilon = n^{\mu}n_{\mu} = \pm 1$ where -1 if $\partial\mathcal{M}$ is timelike and 1 if $\partial\mathcal{M}$ is spacelike. Now we assume that the variation of the metric on the boundary has to be zero i.e $\delta g^{\mu\nu}|_{\partial\mathcal{M}} = 0$. After some tedious calculation we can prove that the above integral becomes,

$$\oint_{\partial\mathcal{M}} d^3x\sqrt{|h|}\epsilon V^{\mu}n_{\mu} = -\oint_{\partial\mathcal{M}} d^3x\sqrt{|h|}\epsilon h^{\mu\nu}(\partial_{\rho}\delta g_{\mu\nu})n^{\rho} \tag{1.4.4}$$

When varying the action surface terms appear which must be vanished on the boundary. These surface terms contain the variation of metric $\delta g^{\mu\nu}$ and variations of derivatives of the metric namely $\delta(\partial_{\rho}g_{\mu\nu})$. Setting $\delta g^{\mu\nu}|_{\partial\mathcal{M}} = 0$ is not sufficient to cancel all surface contributions. Therefore, to be precise in the definition of the variation of the Einstein-Hilbert action, the boundary term known as Gibbons-Hawking-York term has to be added [4, 5] such that surface contributions are exactly canceled. The desired boundary term reads,

$$S_{GYH} = \oint_{\partial\mathcal{M}} d^3x\sqrt{|h|}\epsilon K = \oint_{\partial\mathcal{M}} d^3x\sqrt{|h|}\epsilon\nabla_{\mu}n^{\mu}, \tag{1.4.5}$$

where K is the trace of the extrinsic curvature which is given by the tensor $K_{\mu\nu} = \nabla_{\mu}n_{\nu} - \epsilon n_{\mu}a_{\nu}$ and the vector a^{μ} is given by $a^{\mu} = n^{\nu}\nabla_{\nu}n^{\mu}$. Note that the extrinsic curvature is also orthogonal to the normal direction i.e $n^{\mu}K_{\mu\nu} = 0$. We can prove that the variation of the GHY boundary term gives us the opposite contribution in relation to eq.(1.4.4). Hence we can write for variation of Einstein-Hilbert action,

$$16\pi\ \delta(S_{EH} + S_{GYH}) = \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} = 0 \qquad (1.4.6)$$

Although the addition of the Gibbons-Hawking-York boundary term has an important physical significance such as in AdS/CFT correspondence etc, we will not be concerned with it in this work. It was mentioned merely for completeness. It should be stressed that when we add the Gauss-Bonnet term, we will again assume a GYH term to take care of unneeded total derivatives.

Chapter 2

Summary of Black Holes theory

2.1 Introduction

Black Holes are the most popular prediction of the General theory of Relativity. The confirmation of their observation was recently completed when the *Event Horizon Telescope* presented two pictures of the shadow from two of them [6, 7]. On the other hand the confirmation from *LIGO experiment* of the first gravitational radiation signal showed that it came from a merger of two supermassive black holes [8]. These observational data provide strong evidence for the existence of black holes in the universe which is full of these objects. Black holes are regions of spacetime where the gravitational field is so strong that even photons cannot escape. The astrophysical ones are created by the *gravitational collapse* of a star. It is not necessary that gravitational collapse leads to the formation of a black hole. It was proved by Chandrasekhar [9] that for a star to collapse in a *white dwarf* it should be true that $M > 1.4M_{\odot}$. This inequality is known as *Chandrasekhar's limit*. For stars with masses $M \simeq 0.75M_{\odot}$, Landau, Baade and Zwicky proposed the existence of the *neutron star* as a remnant of gravitational collapse [10, 11]. More recent calculations have shown that the limit for neutron stars is $M \simeq 3M_{\odot}$. In 1965-1966 Wheeler and his colleagues in United States and Zel'dovich with Novikov in Soviet Union studied the gravitational collapse for objects with mass $M > 3M_{\odot}$ [12, 13]. These publications together with the discoveries of powerful galactic radio sources, neutron stars and quasars, have given new impetus to research on black holes. Although they had been predicted by Schwarzschild in 1916, [14] the scientific community had not dealt with them with enthusiasm considering that they are not realistic objects. The purpose of this chapter is to study aspects of black holes theory and how we can predict them in the General Relativity's framework.

2.2 The Schwarzschild solution

In the previous chapter we mentioned that in the vacuum the energy-momentum tensor vanishes. So, the field equation takes the simple form

$$R_{\mu\nu} = 0, \tag{2.2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor. Now, we choose a spherically symmetric spacetime which is described by the following line element

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.2.2}$$

where $\alpha(r)$, $\beta(r)$ are unknown functions. All we have to do is to determine these functions and thus construct the gravitational field configuration in vacuum around a spherical source. Making a straightforward calculation of the components of Ricci's tensor, we can find the following equation

$$0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta) \Rightarrow \alpha(r) = -\beta(r). \tag{2.2.3}$$

We also need a second equation which arises from $(\theta\theta)$ component of Ricci's tensor. Hence we have

$$R_{\theta\theta} = 0 \Rightarrow \quad e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \Rightarrow \quad \partial_r(re^{2\alpha}) = 1. \tag{2.2.4}$$

We can easily integrate the above differential equation which leaves us with

$$g_{tt} \equiv e^{2\alpha} = 1 - \frac{C}{r} \tag{2.2.5}$$

with C being an integration constant. Fixing the integration constant from weak field approximation i.e, $g_{tt}(r \to \infty) = -(1 + 2\Phi)$ with $\Phi = -GM/r$ we get finally

$$e^{2\alpha(r)} = \left(1 - \frac{2GM}{r}\right) \tag{2.2.6}$$

where $R_S = 2GM$ is a characteristic distance the so called *Schwarzschild's radius*. Below we will see that this distance corresponds to the *event horizon's* surface of a spherically symmetric black hole. Since it is true that $\alpha(r) = -\beta(r)$ the metric takes the form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.2.7}$$

This is the well known *Schwarzschild metric* which describes very precisely the gravitational field around a spherical symmetric object like a star etc.

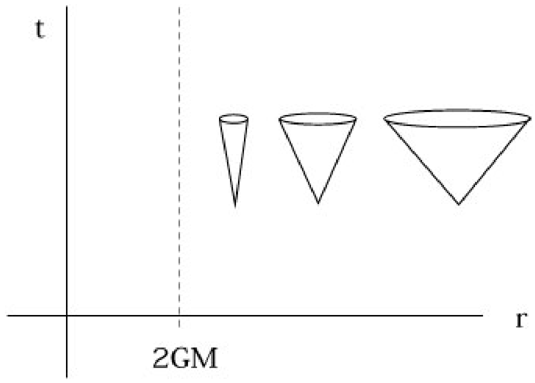


Figure 2.3.1: The slope of light cones in Schwarzschild geometry [1]

We will now make some comments on this solution. First of all, the question arises whether this solution is the unique solution in vacuum or not. This question is answered by Birkhoff's theorem [15]. After an extensive proof which we will omit here the theorem is stated as follows:

Theorem 2.2.1 *Birkhoff's theorem: "Any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat."*

So we conclude that for the vacuum, the only solution we can produce from field equation is the Schwarzschild one. It should be noted that when the radial distance vanishes a singularity point appears. The same happens at distance $r = 2GM$. It is not obvious whether these are real singularity points or not. In the next subsection we will discuss this problem.

2.3 Schwarzschild black holes

In order to be able to study the geometry of the Schwarzschild spacetime we will use geodesics to understand the causal structure of this type of spacetime. Therefore for a null geodesics, those for which θ and ϕ are constant we have

$$ds^2 = 0 \Rightarrow \quad -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = 0 \tag{2.3.1}$$

from which we can see that

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1}. \tag{2.3.2}$$

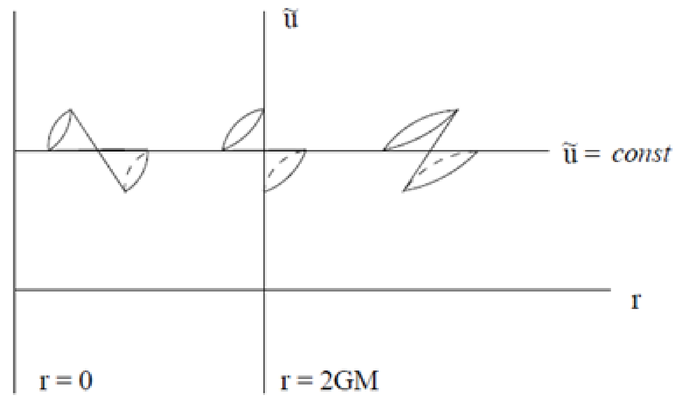


Figure 2.3.2: Light cones for Schwarzschild geometry in $(\tilde{u}, r)$ coordinates in which an observer can follow future-directed timelike trajectories and beyond $r = 2GM$ [1]

and this equation describes the slope of light cones on the $t - r$ plane. As we can see both from eq.(2.3.2) and Fig.(2.3.1) the slope of light cones closes up as $r \to 2GM$ and $dt/dr \to 1$ as $r \to \infty$. The latter is indeed to be expected since we get back the slope of Minkowski's spacetime.

We will now study more closely the limit of $r \to 2GM$ and assume that we have two observers. One of them stands at an infinite distance while the other is radially approaching the surface $r = 2GM$. If the second observer emits light towards the first observer the latter would simply see the signals reach him with a greater delay. As a result we can prove that this continues forever and then the first observer will never see the second observer cross the surface $r = 2GM$. He will just see him move more and more slowly forever. The fact that the asymptotic observer never see the infalling observer reach $r = 2GM$ is a meaningful statement, but the fact that their trajectory in the $t - r$ plane never reaches there is not. This is a highly dependent phenomenon of the coordinate system which we have chosen. Hence the only way to avoid such pathologies in the theory is to change coordinates. We transform the line element of eq.(2.2.7) so that it is better behaved at $r = 2GM$. Now we define the new coordinate transformations

$$\tilde{u} = t + r^* \quad v = t - r^*, \tag{2.3.3}$$

where r^* is the known *tortoise coordinate* which is defined by the following relation,

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \tag{2.3.4}$$

These are known as *Eddington-Finkelstein coordinates* where $\tilde{u}$ = constant is for a radial infalling null geodesic while v = constant characterises outgoing ones.

In terms of these coordinates we replace the timelike coordinate t with the new $\tilde{u}$ so we write the line element as follows

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) d\tilde{u}^2 + (d\tilde{u}dr + drd\tilde{u}) + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{2.3.5}$$

It is obvious that the metric is regular at $r = 2GM$ since the determinant $g = -r^4 \sin^2\theta$ doesn't vanish while $g_{\tilde{u}\tilde{u}} = 0$. Now we demand that $ds^2 = 0$ and we extract a similar condition to eq.(2.3.2) for null geodesics

$$\frac{d\tilde{u}}{dr} = \begin{cases} 0 & \text{infalling} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{outgoing.} \end{cases} \tag{2.3.6}$$

As we can see from eq.(2.3.6) and Fig.(2.3.2) the light cones are well defined at $r = 2GM$ for infalling particles. In addition the light cones in this coordinate system don't close up but they tilt over. Hence, the only trajectories which we can follow are the future-directed ones. Note that from the surface with $r = 2GM$ there is no return as we cannot move in the direction of increasing r. So, this is a good criterion to define which surface is an *event horizon*. We claim that an event horizon is a surface from which particles can never escape to infinity.

Turning now to the definition of $(\tilde{u}, v)$ coordinates we note the following. If we keep $\tilde{u}$ constant and decrease r it should be true that $t \to \infty$, while if we keep v constant it should be true that $t \to -\infty$. This behaviour allows us to follow trajectories in the direction of increasing r. Now we choose the v coordinate instead of $\tilde{u}$ and we write the metric in terms of (v, r) namely

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv - (dvdr + drdv) + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.3.7}$$

We can see from Fig.(2.3.3) that the surface of $r = 2GM$ has the reverse role as opposed to the usual event horizon. Now the surface at $r = 2GM$ does not allow the particles to enter the area $r < 2GM$. In addition we should note that if we follow the trajectories in this case we arrive in different parts of spacetime compared to the first case. Overall we can say that spacetime has been extended in two different directions, towards the future and towards the past.

Our next step is to investigate whether there are more regions of spacetime not covered by eq.(. Now, all we have to do is to use a new coordinate system which is defined in terms of the original coordinates (t, r) by the following relations

$$v' = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{\frac{(r+t)}{4GM}} \qquad u' = -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{\frac{(r-t)}{4GM}}. \tag{2.3.8}$$

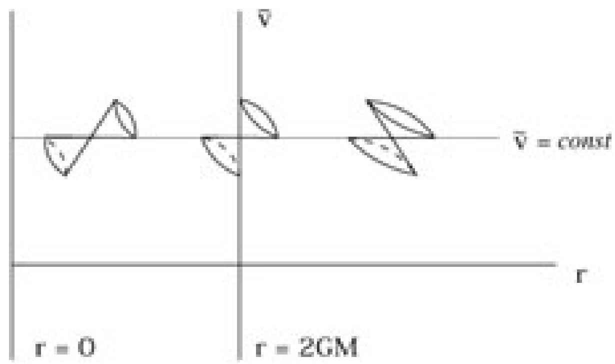


Figure 2.3.3: Light cones for Schwarzschild geometry in (v, r) coordinates in which an observer can follow past-directed timelike trajectories. [1]

In the $(v', u', \theta, \varphi)$ coordinate system the line element of the Schwarzschild solution is,

$$ds^2 = -\frac{26G^3M^3}{r}e^{-\frac{r}{2GM}}(dv'du' + du'dv') + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (2.3.9)$$

Note that if we choose $r = 2GM$ the line element has a well defined behaviour. Both u' and v' are null coordinates because their corresponding partial derivatives are null vectors. Even though there is no problem with the coordinate system $(v', u', \theta, \varphi)$ it is more convenient working in a system where we have timelike and spacelike coordinates. We therefore define

$$\begin{aligned} T &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{\frac{r}{4GM}} \sinh\left(\frac{t}{4GM}\right) \\ X &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{\frac{r}{4GM}} \cosh\left(\frac{t}{4GM}\right), \end{aligned} \qquad (2.3.10)$$

in terms of which the line element becomes,

$$ds^2 = \frac{32G^3M^3}{r}e^{-\frac{r}{2GM}}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (2.3.11)$$

This is the so called *Kruskal-Szekeres* coordinate system where, $-\infty \leq X \leq \infty, -\infty \leq T \leq \infty, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$. Furthermore the Kruskal coordinates have some useful properties. One of them is the following

$$\frac{T}{X} = \tanh\left(\frac{t}{4GM}\right), \qquad (2.3.12)$$

which defines surfaces with constant t. On the other hand surfaces with $r =$ constant are defined by

$$T^2 - X^2 = \text{constant}. \qquad (2.3.13)$$

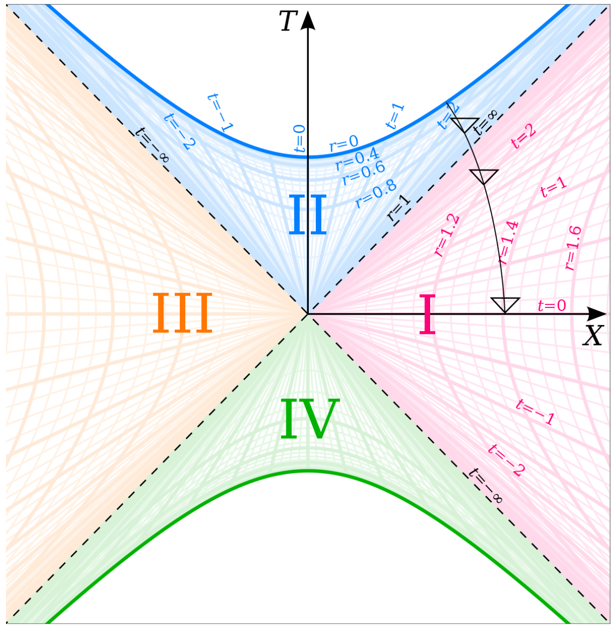


Figure 2.3.4: The Schwarzschild solution in Kruskal-Szekres coordinates. The two dashed lines correspond to the surface of event horizon [19]

From Fig.(2.3.4) it is obvious that the entire spacetime manifold has now been revealed. Now we can safely describe the Schwarzschild background. The region (I) corresponds to the exterior region of a spherically symmetric black hole. Assume now an observer in region (I). If he performs a free fall towards the object he will pass the surface $r = 2GM$ and will be in region (II). The observer now cannot escape to the previous region due to the existence of event horizon. He has only one choice to move in the direction of decreasing r. On the other hand an observer of region (III) cannot move in region (IV) because the surface $r = 2GM$ have the exact opposite properties from the corresponding surface at region (I). Finally let us note that regions (III) and (I) are not the same but different parts of the spacetime manifold.

We will close the discussion for the Schwarzschild geometry by studying the nature of the point $r = 0$. With a first glance we see that the (tt) component of the metric has a singularity. However, from this information only we can't decide whether there is a real singularity or simply the coordinate system is not well defined. To identify the real singularities of spacetime we need to use the Riemann curvature tensor because as we have mentioned in previous chapter, the Riemann tensor encodes the whole information about the curvature of spacetime. However as we know the components of Riemann's tensor are coordinate dependent. Hence the only geometrical tool that is coordinate independent is the *scalar curvature* which we can construct by combining curvature tensors and the metric tensor. The most trusted scalar curvature quantity is the *Kretschmann* scalar which has the form

$$\mathcal{K} = R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu} \tag{2.3.14}$$

where $R_{\rho\sigma\mu\nu}$ is the Riemann tensor. For the Schwarzschild metric this is

$$R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu} = \frac{48G^2M^2}{r^6}. \tag{2.3.15}$$

As we can see the Kretschmann scalar diverges at $r = 0$. Hence we can safely infer that the point $r = 0$ is a real singularity of the spacetime. Now a clear criterion has been established for the identification of real spacetime singularities. *If any of the curvature scalars goes to infinity as we approach some point, we regard that point as a real singularity of spacetime*. It's the right place to say that in nature singularities are hidden behind event horizons. This belief is encompassed by Penrose who first formulated the cosmic censorship hypothesis [16].

Theorem 2.3.1 *Cosmic censorship conjecture: Naked singularities cannot form from gravitational collapse from generic, initially nonsingular states in an astrophysically flat spacetime obeying the dominant energy condition,*

where dominant energy condition is given by the requirement $\rho \geq |p|$. Hence the energy density must be non negative and greater than or equal the magnitude of pressure. So if a star turns into a black hole through gravitational collapse then it's necessary for the singularity to be covered by an event horizon.

2.4 More solutions in General Theory of Relativity

The Schwarzschild solution that we studied in the previous subsection is not the only black hole solution that General Relativity predicts. If we want to study real astrophysical black holes, we must allow the existence of matter or energy to contribute to the energy momentum tensor. These new solutions have a number of new properties which are different from vacuum's solution. It's therefore reasonable to wonder how many of these solutions can exist. Let's recall here that Birkhoff's theorem ensures that Schwarzschild solution is the only spherically symmetric vacuum solution to General Relativity. It has been proven that only a small number of stationary black hole solutions exist in the framework of General Relativity. Also a small and specific set of parameters are necessary to describe the new solutions. Therefore the *no-hair* theorem indicates the family of black hole solutions that we can construct. This theorem states that:

Theorem 2.4.1 *No-Hair theorem: Stationary, asymptotically flat black hole solutions to General Relativity coupled to electromagnetism that are nonsingular outside the event horizon are fully characterized by the parameters of mass M, electric charge Q, and angular momentum J.*

Although these additional solutions are of greater interest, due to the variety of phenomena underlying them, we will make a brief introduction and omit the details.

We will first study the solutions representing charged black holes. It should be mentioned that we don't expect such solutions to describe realistic objects because in astrophysical environment a charged black hole would be neutralized by interactions with matter. Nevertheless, we are interested in studying them as a more general solution and understanding their features. We consider again spherical symmetry as in the Schwarzschild solution while the energy momentum tensor for electromagnetism is given by

$$T^{(em)}_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\ \rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{2.4.1}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor which is of the form $F_{rt} = E_r$ and all other components are zero. The equations of motion for gravitational and electromagnetic field are of the form,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= T^{(em)}_{\mu\nu} \\ \nabla_{\mu}F^{\mu\nu} &= 0 \\ \nabla_{[\mu}F_{\nu\rho]} &= 0. \end{aligned} \tag{2.4.2}$$

We note that the set of field equations are coupled, since the electromagnetic field strength tensor appears in the gravitational equation while the metric enters explicitly in the last two equations. The last two equations are the well known Maxwell's equations written in curved spacetime. The analytical solution of the eq.(2.4.2) is the following

$$\begin{aligned} ds^2 &= -\Delta dt^2 + \Delta^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ \Delta &= 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}, \end{aligned} \tag{2.4.3}$$

where M is the mass of the black hole and Q is the total electric charge. The above solution is known as the *Reissner-Nordstrom* metric [20, 21]. By computing the Kretschmann scalar we can check that the metric has a real singularity at $r = 0$. On the other hand the event horizon can be found from $g^{rr} = 0$. Hence we take the following expression

$$r_{\pm} = GM \pm \sqrt{G^2M^2 - GQ^2}. \tag{2.4.4}$$

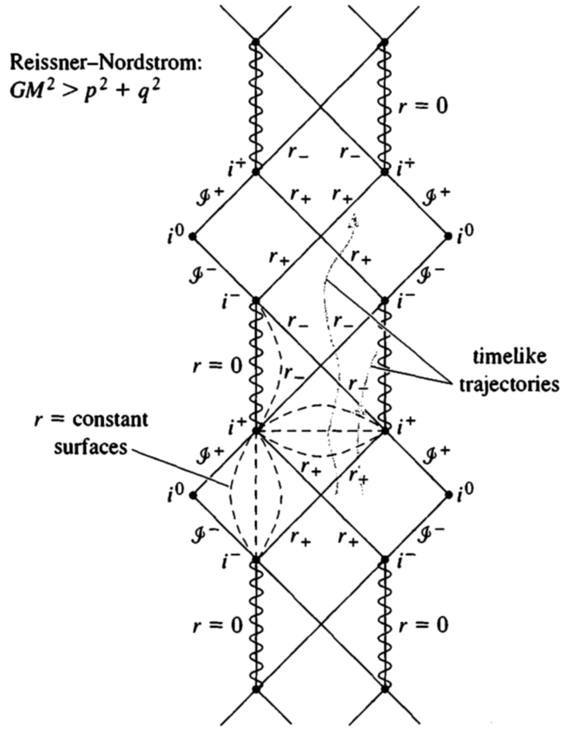


Figure 2.4.1: Conformal diagram for Reissner-Nordstrom metric in the case of $GM^2 > Q^2$. We easily see that there are infinite number of copies of the region outside of the black hole. They are not the same regions of spacetime but different of them. [1]

We easily see that eq.(2.4.4) constitutes two, one or zero solutions, depending on the relative values of GM^2 and Q^2. Here, we focus on the case of $GM^2 > Q^2$ because is the most physically correct case compared to the other two. As we can see in Fig.(2.4.1) the metric has two null surfaces defined by $r = r_\pm$ and they are both event horizons. We note that the singularity $r = 0$ is a timelike line, not a spacelike surface as in Schwarzschild black hole. If we are asymptotic observers the phenomena we will see outside the black hole are exactly the same as in the uncharged case. The only difference is that an infalling observer is not obliged to move to the singularity, but he can avoid it because the singularity is a timelike line and therefore not necessarily in the future of the observer. We note that the g_{rr} component is positive definite everywhere but in the area $r_+ < r < r_-$ is negative definite. This metric behaviour forces the observer to move along the shown timelike trajectories in the Fig.(2.4.1). As mentioned the other two cases $GM^2 < Q^2$ and $GM^2 = Q^2$ will not be studied here because they are characterized by a naked singularity (first case) and instabilities under any mass accretion (second case).

We will now refer to another black hole solution. This one is a rotating black hole with angular momentum J and mass M. To describe the geometry of a rotating black hole's spacetime we have given up on spherical symmetry. In fact we need an axially symmetric spacetime for this purpose. To find the exact solution in this case is a much more difficult process. At the end, the solution is found to be the following line element

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GM\alpha r \sin^2\theta}{\rho^2}(dtd\varphi + d\varphi dt) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+\alpha^2)^2 - \alpha^2\Delta\sin^2\theta\right] d\varphi^2 \tag{2.4.5}$$

where

$$\begin{aligned} \Delta &= r^2 - 2GMr + \alpha^2 \\ \rho^2(r,\theta) &= r^2 + \alpha^2\cos^2\theta, \end{aligned} \tag{2.4.6}$$

We define now $\alpha = J/M$ where J is the angular momentum of the black hole. Hence the parameter α is the angular momentum per unit mass. The eq.(2.4.5) is well known as the *Kerr metric* which is a solution for a rotating black hole found by R. Kerr in 1963 [17, 18]. The event horizon occurs at those fixed values of r for which $g^{rr} = 0$. Then we find two event horizons given by,

$$r_{\pm} = GM \pm \sqrt{G^2M^2 - \alpha^2} \tag{2.4.7}$$

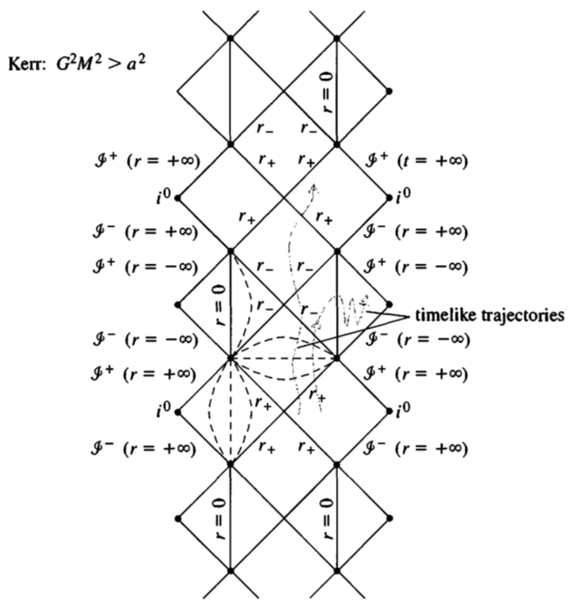
The Kerr's metric has also a real spacetime singularity. To find this one we calculate the curvature invariant

$$\mathcal{K} = R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu} = \frac{1}{(\alpha^2\cos 2\theta + \alpha^2 + 2r^2)^6}\Big[96G^2M^2(\alpha^6\cos 6\theta + 10\alpha^6 - 180\alpha^4 r^2 + 240\alpha^2 r^4 + 6\alpha^4(\alpha^2 - 10r^2)\cos 4\theta + 15\alpha^2(\alpha^4 - 16\alpha^2 r^2 + 16r^4)\cos 2\theta - 32r^6)\Big], \tag{2.4.8}$$

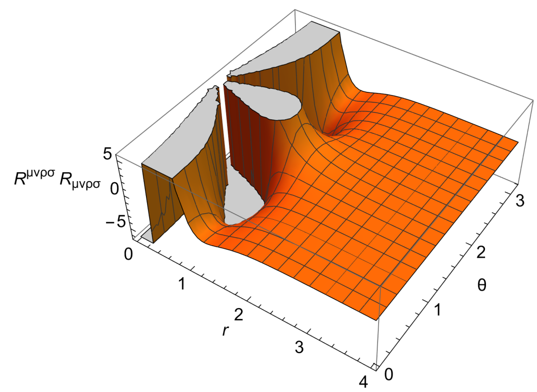
which diverges at $\alpha^2\cos 2\theta + \alpha^2 + 2r^2 = 0$ as shown in the Fig.(2.4.2). This quantity can only vanish when $r = 0$ and $\cos 2\theta = -1$ or,

$$r = 0, \quad \theta = \frac{\pi}{2}.$$

The conclusion is that $r = 0$ is not a point in space but the set of the points $r = 0$ and $\theta = \frac{\pi}{2}$ is actually a *ring singularity* at the edge of the disk. We can claim that the rotation has "softened" the Schwarzschild singularity, spreading it out over a ring. In this sense



(a)



(b)

Figure 2.4.2: (a) Conformal diagram for Kerr geometry in case of $G^2M^2 > \alpha^2$. This is an analogous diagram to the one for Reissner-Nordstrom solution. There is an infinite number of copies of the region outside the black hole not the same regions of spacetime but different of them. [1] (b) The Kretschmann invariant with $M = \alpha = 1$ for Kerr's geometry

we can think of the Schwarzschild solution as the limit of zero angular momentum. By an analytic continuation of Kerr's geometry we can make a conformal diagram as the Fig.(2.4.2). This diagram is much like the Reissner-Nordström one with the only difference being that now we can travel through singularity. For more details on the interesting phenomena associated with Kerr metric, one can study the references [1, 3] of the bibliography.

To sum up, General theory of Relativity is an accurate theory that describes successfully the gravitational interaction. In this framework gravitational interaction is the curvature of spacetime. As we have seen, it also predicts the existence of, new objects like black holes. Also the theory predicts the existence of more gravitational objects which they aren't stable under spacetime perturbations. The No-Hair theorem works as a strong constraint on the existence of new solutions therefore limiting the theory considerably. This reason together with additional theoretical and observational forces force us to go beyond General theory of Relativity and look for new solutions of black holes with more physical features. This is exactly the subject of the present thesis as we will see below.

Chapter 3

Generalised Theories of Gravity. Evasion of the Novel-No-scalar-Hair theorem and scalarized black holes

3.1 Introduction

In the previous chapters we have hopefully convinced the reader that General Relativity is an accurate theory for describing gravitational interaction. However, Einstein's theory, as we will see in the present chapter is not the final theory for gravity. The missing piece of the description puzzle is the behaviour of gravity at the high energy scale. One of the ambitions of modern theoretical research is to formulate a *quantum theory of gravity*, a theoretical framework that describes gravity at high energies. It should be stressed here that the energy scale where General Relativity breaks down and has to be replaced by a quantum theory, is the Planck's scale which is given in the following table.

It is obvious from table (3.1.1) that Planck's energy scale is much higher than the scale we have achieved at CERN's hadronic collider (LHC). This energy gap between theory and experiment is large enough to present us an obstacle on how to formulate a fundamental theory directly at high energies. But there has been progress in this direction since the 80s and 90s. At that time theoretical physicists were looking for solutions to problems of the *Standard Model of elementary particles* by formulating a new theory known as *Superstring Theory*. Unlike an ordinary Quantum Field Theory, superstring theory is a theory of extended objects (strings) in extra spatial dimensions, which managed to unify the four known fundamental interactions i.e gravity with electroweak and strong interaction, Despite its success, superstring theory is not considered to be

Name	Dimension	Expression	Value (SI units)
Planck Length	length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}}$	$1.616 \times 10^{-35} m$
Planck Mass	mass (M)	$M_P = \sqrt{\frac{\hbar c}{G}}$	$2.176 \times 10^{-8} kg$
Plank time	time (T)	$t_P = \sqrt{\frac{\hbar G}{c^5}}$	$5.391 \times 10^{-44} sec$
Planck temperature	temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}}$	$1.416 \times 10^{32} K$

Table 3.1.1: Planck's energy scale. All expressions of the scale are related to known physical constants.

the quantum theory of gravity since it most important of problems are the prediction of extra spatial dimensions, new particle states beyond Standard Model, no vacuum state in Superstring Theory to support a positive cosmological constant etc. At low energies and four dimensions it is expected that Superstring Theory takes the form of an *effective field theory*. In fact it has been proven that this effective theory is a *generalized theory of gravity* which contains new higher curvature terms and new fields. These types of theories have been extensively studied in the literature in terms of black hole solutions, and other gravitational solutions that these predict [22, 23, 24, 25]. The study of the implications of such fundamental theories at low energies is known as the *top-down* approach.

As we mentioned, establishing a theory directly at high energies has many phenomenological problems because there is no data from the experiment to inspire us to write down the correct theory. However, bearing in mind the problems of General Relativity, we can go beyond it by selectively ignoring some of the constraints, we had placed on the gravitational action in section 1.3. Therefore, new higher curvature terms and new fields coupled to gravity can emerge. To write down a generalized theory of gravity at low energies we have to be careful that it makes physical sense. For example, it must not be characterized by instabilities, i.e. the equations of motion of the theory must be up to second order. Therefore these theories bring us closer and closer to the fundamental theory of gravity. This reverse process to the previous one we mentioned is called the *bottom-up* approach.

In this chapter we will focus on presenting the generalized theories of gravity through the two approaches mentioned above. Subsequently, we will study how we can evade the No-scalar-Hair theorem to find black hole solutions with a non trivial scalar field outside the horizon.

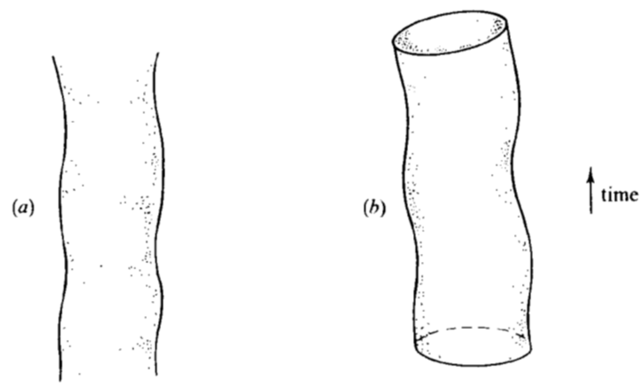


Figure 3.2.1: (a) An open string propagates in spacetime. (b) A closed string propagates in spacetime [30].

3.2 Effective string theory at low energies

String theory is a good attempt to unify all known fundamental interactions. It's a theory of extended objects called strings and was originally proposed to describe strong interactions but without success. According to string theory the fundamental constituents of our world are the strings. Each excited state of the string or otherwise its mode of oscillation is interpreted as a different particle. Since string theory aspires to unify gravity with the other three interactions, it is reasonable to produce an effective theory to study its implications at low energies. Here we will not go into details about the long process of generating an effective theory. We will just present some results from the literature. According to Gross and Sloan [28] an effective heterotic superstring theory[1] is in the form of,

$$\begin{aligned} S_{het} = \int d^{10}x\sqrt{-g}\Bigg[&\frac{R}{2\kappa^2} - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{6}e^{2\gamma\Phi}H_{\mu\nu\rho}H^{\mu\nu\rho} \\ &+ \frac{\alpha'}{8g^2}e^{\gamma\Phi}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + R_{\mu\nu}R^{\mu\nu} + R^2 - \mathrm{Tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + c_2(\partial\Phi)^4\right) + \dots\Bigg] \end{aligned} \tag{3.2.1}$$

where $\kappa = -\sqrt{2}\gamma = \sqrt{8\pi G}$ is the 10-dimensional gravitational constant, g is a coupling constant of the string and $F_{\mu\nu} = t^{\alpha}F^{\alpha}_{\mu\nu}$ is the generalization of electromagnetic tensor in non-Abelian theories. Here $H_{\mu\nu\rho}$ is a $(0,3)$ tensor which is given by the fol-

[1]The heterotic superstring theory is a hybrid combination of a supersymmetric right-handed sector and a bosonic left-handed sector. Right-handed and left-handed refers to the way the degrees of freedom are propagate over the string.

lowing equation,

$$H_{\mu\nu\rho} = \partial_{[\rho} B_{\mu\nu]} + \frac{\alpha'}{8\kappa}(\Omega_{3L\mu\nu\rho} - \Omega_{3Y\mu\nu\rho}) \tag{3.2.2}$$

where

$$\begin{aligned} \Omega_{3L\mu\nu\rho} &= \frac{1}{2}\,\mathrm{Tr}\left(\omega_{[\rho} R_{\mu\nu]} - \frac{2}{3}\omega_{[\mu}\omega_{[\nu}\omega_{\rho]]}\right), \\ \Omega_{3Y\mu\nu\rho} &= \frac{1}{2}\,\mathrm{Tr}\left(A_{[\rho} F_{\mu\nu]} - \frac{2}{3}A_{[\mu}A_{[\nu}A_{\rho]]}\right), \end{aligned} \tag{3.2.3}$$

are the Lorentz and Yang-Mills Chern-Simons terms. Note that in the above expressions $\omega_{\mu}^{ab} = -e^{ba}e^{a}_{\alpha;\mu}$ is the so called *spin connection* which is necessary for fermions in curved spacetime. The comma in the definition of spin connection represents the covariant derivative. A_μ is the Yang-Mills field. It has been shown by Green and Schwartz [29] that the expression of $H_{\mu\nu\rho}$ cancels all gravitational anomalies in the theory making it even more interesting and robust.

To see now the implications of the effective theory in four dimensions we could calculate the four dimensional effective action. The method used for this purpose is the method of *compactification* of extra dimensions. Omitting again the details we can write down the four dimensional effective string action

$$\begin{aligned} S_{eff} = \int d^4x\sqrt{-g}\Bigg[&\frac{R}{2} + \frac{1}{4}(\partial_\mu\phi)^2 + \frac{1}{4}e^{-2\phi}(\partial_\mu a)^2 + \frac{3}{4}(\partial_\mu\sigma)^2 + \frac{3}{4}e^{-2\sigma}(\partial_\mu b)^2 \\ &+ \alpha'\left(\frac{e^\phi}{8g^2} + \Delta\right)\mathcal{R}^2_{GB} + \alpha'\left(\frac{a}{8g^2} + \Theta\right)\mathcal{R}\tilde{\mathcal{R}} \\ &+ \left(-\frac{e^\phi}{8g^2} + \hat{\Delta}\right)F^{\mu\nu}F_{\mu\nu} + \alpha'\left(-\frac{a}{8g^2} + \hat{\Theta}\right)F\tilde{F}\Bigg], \end{aligned} \tag{3.2.4}$$

where ϕ is the so called dilaton field and σ, a, b are scalar fields which are called modulus and axions respectively. Moreover, $\Delta, \Theta, \hat{\Delta}, \hat{\Theta}$ are functions which depend only on the fields σ and b. Also note the following,

$$\begin{aligned} \mathcal{R}^2_{GB} &\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \\ \mathcal{R}\tilde{\mathcal{R}} &\equiv \eta^{\mu\nu\rho\sigma}R^{\kappa\lambda}_{\ \ \mu\nu}R_{\rho\sigma\kappa\lambda}, \\ F\tilde{F} &\equiv \eta^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \end{aligned} \tag{3.2.5}$$

where $\eta^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$ and we have set $\kappa = \sqrt{8\pi G} = 1$. In our work, when we study solutions inspired by string theory we will focus only on the contributions of the well known Gauss-Bonnet term $\mathcal{R}^2_{GB}$ and dilaton field ϕ. The $R\tilde{R}$ and $F\tilde{F}$ are the

Chern-Simons terms. It should be mentioned that the Gauss-Bonnet term is not only a consequence of superstring theory, but can also be constructed in the context of scalar tensor theories, as we will see below. Finally from the form of the four dimensional string effective action we conclude that the superstring theory at low energies reduces to a generalized theory of gravity. This is an important result because new black hole solutions, predicted in this framework, may pave the way for the experimental confirmation of the superstring theory. On the other hand, we see that in the bosonic sector of the theory, new scalar particles are predicted, unlike in particle physics Standard Model where the only known scalar particle the so called Higgs particle, is the one associated with the Higgs field.

3.3 Scalar-tensor gravitational theories

In the previous section we have seen how we can extract a generalized theory of gravity as a consequence of a fundamental theory at low energies. In this section we will study the reverse process , i.e. we will see how we can construct a generalized theory of gravity directly in four dimensions. Although General Theory of Relativity is a well tested theory, it is known that many problems arise in the construction of the theory. As we know the Standard Model of Cosmology has many open problems like the initial singularity problem, the nature of dark matter and dark energy. On the other hand it has been proven that General Relativity is a non renormalizable theory, hence we can't quantize it. As a result, the only choice we have in order to solve these problems is to study modifications of General Relativity. The bottom-up approach has an advantage over the approach of superstring theory (top-down). It is not necessary to include in the theory experimentally unknown fields such as moduli, additional gauge fields etc. In recent years many Generalized Theories of Gravity have been proposed. The best known type of such theories are the so called scalar-tensor theories. In 1974 Gregory Horndeski wrote down [31] the most general scalar-tensor theory, the well known Horndeski theory. It has been shown that a theory which has second order equations of motion, doesn't contain instabilities or problematic ghost states. So the functional action of the Horndeski theory is in the form of

$$S = \int d^4x \sqrt{-g} \left[\sum_{i=2}^{5} \mathcal{L}_i(g_{\mu\nu}, \phi) + \mathcal{L}_M(g_{\mu\nu}, \psi_i) \right]. \tag{3.3.1}$$

Here we assume that all matter fields ψ_i are minimally coupled to gravity. The four Lagrangians in the sum are the following

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X) \\
\mathcal{L}_3 &= G_3(\phi, X)\nabla^2\phi \\
\mathcal{L}_4 &= G_4(\phi, X)R + G_{4,X}[(\nabla^2\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2] \\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}[(\nabla^2\phi)^3 - 3(\nabla^2\phi)(\nabla_\mu\nabla_\nu\phi)^2 \\
&\quad + 2(\nabla_\nu\nabla^\mu\phi)(\nabla_\alpha\nabla^\nu\phi)(\nabla_\mu\nabla^\alpha\phi)]
\end{aligned} \tag{3.3.2}$$

where G_i are functions of ϕ and $X = \frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi$ and $G_{i,X} = \partial G_i/\partial X$. If we choose $G_4 = 1, G_2 = G_3 = G_4 = G_5 = 0$ the theory leads to General Relativity. Moreover we can obtain the $f(R)$-Brans-Dicke theories ($G_2 = \omega X/\phi, G_4 = \phi, G_3 = G_5 = 0$) as well as many other scalar tensor theories. Also, as we will see below Horndeski theory includes the Einstein-scalar-Gauss-Bonnet (EsGB) theory.

3.4 Lovelock's gravitational theory

In 1971 David Lovelock introduced [32] a generalization of General Theory of Relativity in arbitrary spacetime dimensions D. It is the most general metric theory in which it is not necessary to include new fields, such as new scalars or new gauge bosons. Lovelock's gravitational theory leads to equations of motion up to second order in D-dimensional spacetime, so the theory is ghost-free and there are no instabilities. Motivated by the idea of showing the uniqueness of Einstein's equations, Lovelock searched the appropriate set of tensors $A^{\mu\nu}$ to satisfy the following conditions,

- $A^{\mu\nu} = A^{\mu\nu}(g_{\rho\sigma}; g_{\rho\sigma,\gamma}; g_{\rho\sigma,\gamma\kappa})$
- $\nabla_\mu A^{\mu\nu} = 0$
- $A^{\mu\nu} = A^{\nu\mu}$

This problem had been partially answered by Weyl and Cartan [33, 34] who showed that if $A^{\mu\nu}$ is linear with respect to $g_{\rho\sigma,\gamma\kappa}$ then $A^{\mu\nu}$ will be necessarily a combination of Einstein tensor and cosmological constant. By dropping the requirement of linearity, Lovelock showed that there was a more general class of solutions to the problem, each of which could serve as a suitable left-hand side of the field equations in a geometric theory of gravity, without introducing any extra fundamental degrees of freedom, beyond those that exist in General Relativity. Hence, Lovelock's theory is given by the following action functional

$$S = \int d^D x \sqrt{-g} \sum_n^q \alpha_n \mathcal{R}^n \tag{3.4.1}$$

where,

$$\mathcal{R}^n \equiv \frac{1}{2^n} \delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n} \prod_{i=1}^{n} R^{\alpha_i\beta_i}_{\quad\mu_i\nu_i}, \qquad \delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n} \equiv n! \delta^{\mu_1}_{[\alpha_1} \delta^{\nu_1}_{\beta_1} \ldots \delta^{\mu_j}_{\alpha_j} \delta^{\nu_j}_{\beta_j]} \tag{3.4.2}$$

the a_j are a set of arbitrary constants, $R^{\mu\nu}_{\rho\sigma}$ are the standard components of Riemann's tensor and δ^{μ}_{ν} is the Kronecker's symbol. Note here that it must be the case that for even number of dimensions it is true that $D = 2q + 2$ and $D = 2q + 1$ for odd number of dimensions. This means that in $D = 1, 2$ dimensions the Lagrangian in the action is given by a constant. In $D = 3, 4$ dimensions the Lagrangian has the standard form of General Relativity with a cosmological constant. In $D = 5, 6$ dimensions the theory is different, and the Lovelock's expansion has the following form

$$S = \int d^5 x \sqrt{-g} \left[\alpha_0 + \alpha_1 R + \alpha_2 \mathcal{R}^2_{GB} \right] \tag{3.4.3}$$

where[2]

$$\mathcal{R}^2_{GB} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \tag{3.4.4}$$

is again the Gauss-Bonnet term. In $D < 5$ dimensions the Gauss-Bonnet term is a total derivative so the integration on spacetime manifold vanishes, hence there are no contributions to the equations of motion due to the presence of this term. Although in four dimensions there is no generalization to General Relativity through Lovelock theory, we have referred to it as a geometric generalization in arbitrary spacetime dimensions, where higher curvature terms are necessary only in $D > 4$. On the other hand note that we can use Lovelock theory to construct a four-dimensional effective theory in the presence of the Gauss-Bonnet term coupled to a scalar field. The method used in this process is called Kaluza-Klein reduction [35].

3.5 The Einstein-scalar-Gauss-Bonnet theory

In our work we will focus on a scalar tensor theory which is known as Einstein-scalar-Gauss-Bonnet theory (EsGB). Unlike Lovelock theory in EsGB theory the Gauss Bonnet term contributes to the equations of motion due to its coupling with a scalar field

[2] The action has the same form for $D = 6$ dimensions

through an arbitrary coupling function $f(\phi)$. The gravitational action is given by the following action functional

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + f(\phi)\mathcal{R}^2_{GB}\right], \tag{3.5.1}$$

where the Gauss-Bonnet term given in eq.(3.4.4) is a combination of Riemann's and Ricci's tensor and Ricci scalar. From the action we easily see that if $f(\phi)$ is a constant or zero the Gauss-Bonnet term is a total derivative and there are no contributions to the equations of motion. It has been proved that EsGB theory belongs to the family of Horndeski theories. After a non trivial calculation it may be shown that the Horndeski theory with the following coupling functions

$$\begin{aligned} G_2 &= -X + 8f^{(4)}X^2(3 - \ln X), \\ G_3 &= 4f^{(3)}X(7 - 3\ln X), \\ G_4 &= 1 + 4\ddot{f}X(2 - \ln X), \\ G_5 &= -4\dot{f}\ln X, \end{aligned} \tag{3.5.2}$$

and the EsGB lead to the exact same set of equations [36, 37]. The dots and numbers in the parenthesis symbolizes differentiation with respect to the scalar field. We must mention that string inspired theories can be recovered [22] through a special choice of the coupling function i.e $f(\phi) = \alpha e^{\lambda\phi}$ with $\lambda = \pm 1$. Also the exponential coupling is part of effective theories constructed by Lovelock theory in four dimensions [35]. The case of shift symmetric Galileon theories corresponds to the choice of coupling function $f(\phi) = \alpha\phi$ and the action is invariant under the symmetry $\phi \rightarrow \phi + c$.

In our work we will focus on black hole solutions in this theoretical framework. In order to have acceptable solutions the No-scalar-Hair theorem which was formulated by Bekenstein [38] must be violated. The first derivation of this violation was formulated in the framework of Einstein-dilaton-Gauss-Bonnet effective string theory in low energies [22]. It was later proven that the theorem is also violated for a general form of coupling function between the scalar field and the Gauss-Bonnet term [43], as we will see below. Many black hole solutions in EsGB theory have appeared in the literature; for example: (a) rotating solutions [39, 40], (b) solutions in the presence of electromagnetic field [41, 42] and (c) solutions with a cosmological constant [44, 45].

3.6 Evasion of Novel-No-scalar-Hair Theorem-Scalarized Black Holes

As we have seen, black holes in General Relativity are characterized only by three physical quantities (mass, electric charge and angular momentum) hence, according to No-Hair theorem we can construct only three families of black hole solutions. We consider now the following general scalar tensor theory,

$$S = \int d^4x \sqrt{-g} \Big[R + \mathcal{E}(\mathcal{J}, \mathcal{F}, \mathcal{K}) \Big] \tag{3.6.1}$$

where,

$$\mathcal{J} = \partial_\mu \chi \partial^\mu \chi \quad \mathcal{F} = \partial_\mu \phi \partial^\mu \phi \quad \mathcal{K} = \partial_\mu \chi \partial^\mu \phi, \tag{3.6.2}$$

and ϕ and χ are real scalar fields. So in this context Bekenstein in 1995 formulated a general No-scalar-Hair theorem [38]. To prove his theorem Bekenstein relied on the behaviour of energy-momentum tensor both near the horizon and at asymptotic infinity. The energy-momentum tensor of the theory is given by the following equation,

$$T^\nu_\mu = \mathcal{E} \delta^\nu_\mu + 2 \frac{\partial \mathcal{E}}{\partial \mathcal{J}} \partial_\mu \phi \partial^\nu \phi + 2 \frac{\partial \mathcal{E}}{\partial \mathcal{F}} \partial_\mu \chi \partial^\nu \chi + \frac{\partial \mathcal{E}}{\partial \mathcal{K}} (\partial_\mu \phi \partial^\nu \chi + \partial_\mu \chi \partial^\nu \phi). \tag{3.6.3}$$

He assumed that an asymptotically flat and spherically symmetric black hole solution exists in the theory and the scalar field has the same symmetry with spacetime, so we can write that $\phi = \phi(r)$. Also he assumed that the quantity $\mathcal{E}$ is identified with the local energy density as observed by an observer on a geodesic curve, therefore it should be positive. By focusing on the behaviour of the T^r_r component of the energy-momentum tensor and using its first derivative $(T^r_r)'$ from the conservation law $\nabla_\mu T^{\mu\nu} = 0$ he found that

$$T^r_r < 0 \quad , \quad (T^r_r)' < 0 \quad , \quad r \to r_h \, , \tag{3.6.4}$$

$$T^r_r > 0 \quad , \quad (T^r_r)' < 0 \quad , \quad r \to \infty \, . \tag{3.6.5}$$

Due to this behaviour there is a region of radial coordinate $[r_a, r_b]$ between $[r_h, \infty)$ in which T^r_r component changes sign with $(T^r_r)'$. However, Bekenstein by using the equations of motion showed that both T^r_r and $(T^r_r)'$ remain negative in the $[r_h, \infty)$ region. Therefore the two asymptotic regions cannot be smoothly connected except in the case where the scalar field is a constant or zero. Thus in this framework the only acceptable solution is the Schwarzschild one.

Bearing in mind Bekenstein's argument, we will examine the validity of the No-scalar-Hair theorem in the context of EsGB theory [43]. We consider the EsGB theory which is given by the following action functional

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + f(\phi)\mathcal{R}^2_{GB}\right] \tag{3.6.6}$$

where $\mathcal{R}^2_{GB}$ is the Gauss-Bonnet term which is a quadratic curvature term and we choose the coupling function $f(\phi)$ to be arbitrary. The variation of the action (3.6.6) with respect to the metric tensor $g_{\mu\nu}$ and scalar field ϕ leads to Einstein's equations and the equation for the scalar field respectively. We can write them in the following form,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}(\partial_\rho\phi)^2 \\ - \frac{1}{2}(g_{\lambda\mu}g_{\rho\nu} + g_{\rho\mu}g_{\lambda\nu})\nabla_\gamma[\tilde{R}^{\rho\gamma}{}_{\alpha\beta}\eta^{\kappa\lambda\alpha\beta}\nabla_\kappa f(\phi)], \end{aligned} \tag{3.6.7}$$

$$\nabla^2\phi + \dot{f}(\phi)\mathcal{R}^2_{GB} = 0, \tag{3.6.8}$$

where the dot in the second equation means the first derivative of the $f(\phi)$ with respect to the scalar field ϕ. Note that $\tilde{R}^{\rho\gamma}{}_{\alpha\beta} = \eta^{\rho\gamma\sigma\tau}R_{\sigma\tau\alpha\beta} = \epsilon^{\rho\gamma\sigma\tau}R_{\sigma\tau\alpha\beta}/\sqrt{-g}$. To find black hole solutions in the context of EsGB theory we start with a line element of the form,

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{3.6.9}$$

which describes a spacetime with spherical symmetry. Also it is assumed that scalar field has the same symmetries with spacetime, so we can write $\phi = \phi(r)$. Using both the above line element and eq.(3.6.7) we take the explicit form of the gravitational equations

$$4e^B(e^B + rB' - 1) = \phi'^2[r^2e^B + 16\ddot{f}(e^B - 1)] - 8\dot{f}[B'\phi'(e^B - 3) - 2\phi''(e^B - 1)], \tag{3.6.10}$$

$$4e^B(e^B - rA' - 1) = -\phi'^2r^2e^B + 8(e^B - 3)\dot{f}A'\phi', \tag{3.6.11}$$

$$\begin{aligned} e^B[rA'^2 - 2B' + A'(2 - rB') + 2rA''] = -\phi'^2re^B + 8\phi'^2\ddot{f}A' \\ + 4\dot{f}[\phi'(A'^2 + 2A'') + A'(2\phi'' - 3B'\phi')], \end{aligned} \tag{3.6.12}$$

while the equation of the scalar field takes the following form

$$2r\phi'' + (4 + rA' - rB')\phi' + \frac{4\dot{f}e^{-B}}{r}[(e^B - 3)A'B' - (e^B - 1)(2A'' + A'^2)] = 0. \tag{3.6.13}$$

In the above, we have calculated the explicit form of Gauss-Bonnet term for spherical symmetry which is given by the following equation

$$\mathcal{R}^2_{GB} = \frac{2e^{-2B}}{r^2}[(e^B - 3)A'B' - (e^B - 1)(2A'' + A'^2)]. \tag{3.6.14}$$

As we can see the eq.(3.6.11) can be written in the form of a polynomial equation i.e $e^{2B} + \beta e^B + \gamma = 0$ which could be solved in terms of e^B to give

$$e^B = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2} \tag{3.6.15}$$

where

$$\beta = \frac{r^2\phi'^2}{4} - (2\dot{f}\phi' + r)A' - 1, \quad \gamma = 6\dot{f}\phi' A'. \tag{3.6.16}$$

It is easily to derive from the above equation the first derivative of $B(r)$ with respect to the radial coordinate. As a result we get,

$$B' = \frac{\gamma' + \beta' e^B}{2e^{2B} + \beta e^B} \tag{3.6.17}$$

By using eq.(3.6.15) and eq.(3.6) we can eliminate e^B thus obtaining a coupled system of differential equations with two independent equations. Then we get an ordinary and second order system of differential equations for $A(r)$ and $\phi(r)$:

$$A'' = \frac{d_1}{d}, \qquad \phi'' = \frac{d_2}{d}, \tag{3.6.18}$$

where d_1, d_2, d are functions of $(r, \phi', A', \dot{f}, \ddot{f})$. As we will see we can construct black hole solutions if and only if $f(\phi)$ and ϕ satisfy certain conditions. Approximately near the horizon for spherically symmetric spacetime it must be valid that $e^A \to 0$, as $r \to r_h$ or equivalently $A' \to \infty$. For acceptable solutions the horizon must be regular so ϕ, ϕ', ϕ'' are finite quantities in the limit $r \to r_h$. We will therefore assume $A' \to \infty$ while ϕ' is finite on the horizon. Then eq.(3.6.18) takes the following form,

$$A'' = -\frac{r^4 + 4r^3\phi'\dot{f} + 4r^2\phi'^2\dot{f}^2 - 24\dot{f}^2}{r^4 + 2r^3\phi'\dot{f} - 48\dot{f}^2} A'^2 + \dots \tag{3.6.19}$$

$$\phi'' = -\frac{(2\phi'\dot{f} + r)(r^3\phi' + 12\dot{f} + 2r^2\phi'\dot{f})}{r^4 + 2r^3\phi'\dot{f} - 48\dot{f}^2} A' + \dots \tag{3.6.20}$$

From the second equation we will see that ϕ'' diverges on the horizon if $f(\phi)$ is zero or unconstrained. On the other hand if we assume that $(2\phi'\dot{f} + r) = 0$ near the horizon, then the equation for ϕ'' will give us, $\phi'' \simeq \sqrt{A'}/\dot{f}$ and therefore ϕ'' is not regular in the limit $r \to r_h$. The only way for ϕ'' to remain regular in the limit $r \to r_h$ is to consider the second choice $r^3\phi' + 12\dot{f} + 2r^2\phi'^2\dot{f} = 0$. This equation can be solved in terms of ϕ' to give,

$$\phi'_h = \frac{r_h}{4\dot{f}_h}\left(-1 \pm \sqrt{1 - \frac{96\dot{f}_h^2}{r_h^4}}\right). \tag{3.6.21}$$

where with h we symbolize the value of any quantity on the horizon. As a first conclusion we can safely say that we can construct regular black holes for any choice of $f(\phi)$. But if we want to guarantee that ϕ'_h is also real then $f(\phi)$ must satisfy the following condition

$$\dot{f}_h^2 < \frac{r_h^4}{96}. \tag{3.6.22}$$

We will discuss now the asymptotic behaviour of the metric in the limits $r \to r_h$ and $r \to \infty$. We will see that for any $f(\phi)$ an asymptotically flat spacetime can be always constructed. Now, we can expand in the limit $r \to \infty$ the metric and scalar field in terms of power-law,

$$e^A = 1 + \frac{A_1}{r} + \frac{A_2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{3.6.23}$$

$$e^B = 1 + \frac{B_1}{r} + \frac{B_2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{3.6.24}$$

$$\phi(r) = \phi_\infty + \frac{C_1}{r} + \frac{C_2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{3.6.25}$$

We can identify the coefficients A_1 and B_1 with the ADM mass M of black hole and the scalar charge D which is associated with the scalar field. Employing this power-law expansions to the eq.(3.6.10-3.6.12) we get the following results

$$e^A = 1 - \frac{2M}{r} + \frac{MD^2}{12r^3} + \frac{24MD\dot{f} + M^2D^2}{6r^4} + \ldots, \tag{3.6.26}$$

$$e^B = 1 + \frac{2M}{r} + \frac{16M^2 - D^2}{4r^2} + \frac{32M^3 - 5MD^2}{4r^3} + \ldots, \tag{3.6.27}$$

$$\phi(r) = \phi_\infty + \frac{D}{r} + \frac{MD}{r^2} + \frac{32M^2D - D^3}{24r^3} + \frac{12M^3D - 24M^2\dot{f} - MD^3}{6r^4} + \ldots. \tag{3.6.28}$$

As we can see the asymptotic flatness of spacetime is not affected by the form of coupling function because it doesn't appear before $\mathcal{O}\left(\frac{1}{r^4}\right)$. This behaviour is consistent because at very long distances the Gauss-Bonnet term is negligible due to the small values of the curvature. On the other hand near the horizon the metric and scalar field have a different behaviour. By using the constraint eq.(3.6.21) and eq.(3.6.19) we get,

$$A'' = -A'^2 + \mathcal{O}(A'), \tag{3.6.29}$$

which can be easily integrated to leave us with,

$$A' = \frac{1}{r - r_h} + \mathcal{O}(1). \tag{3.6.30}$$

We integrate once more the above equation so the expansions near the horizon take the form

$$e^{A} = a_1(r-r_h) + a_2(r-r_h)^2 + \dots, \tag{3.6.31}$$

$$e^{-B} = b_1(r-r_h) + b_2(r-r_h)^2 + \dots, \tag{3.6.32}$$

$$\phi(r) = \phi_h + \phi_h'(r-r_h) + \phi_h''(r-r_h)^2 + \dots, \tag{3.6.33}$$

Note that we cannot yet claim with certainty that the two regions $r \to r_h$ and $r \to \infty$ can be smoothly connected. In order to see this we need to examine the No-scalar-Hair theorem.

Let us now return to No-scalar-Hair theorem and examine whether Bekenstein's argument is valid in EsGB theory [43]. We begin from conservation of energy momentum tensor $\nabla_\mu T^\mu_\nu = 0$. Its (rr) component takes the explicit form

$$(T^r_{\ r})' = \frac{A'}{2}(T^t_{\ t} - T^r_{\ r}) + \frac{2}{r}(T^\theta_{\ \theta} - T^r_{\ r}) \tag{3.6.34}$$

where we have used $T^\theta_\theta = T^\varphi_\varphi$ because of spherical symmetry. The components of the energy-momentum tensor in the presence of the Gauss-Bonnet term are:

$$T^t_{\ t} = -\frac{e^{-2B}}{4r^2}\Big\{\phi'^2[r^2e^B + 16\ddot{f}(e^B-1)] - 8\dot{f}[(B'\phi'(e^B-3) - 2\phi''(e^B-1)]\Big\}, \tag{3.6.35}$$

$$T^r_{\ r} = \frac{e^{-B}\phi'}{4}\left[\phi' - \frac{8e^{-B}(e^B-3)\dot{f}A'}{r^2}\right], \tag{3.6.36}$$

$$T^\theta_{\ \theta} = -\frac{e^{-2B}}{4r}\Big\{\phi'^2(re^B - 8\ddot{f}A') - 4\dot{f}[\phi'(A'^2 + 2A'') + A'(2\phi'' - 3B'\phi')]\Big\}. \tag{3.6.37}$$

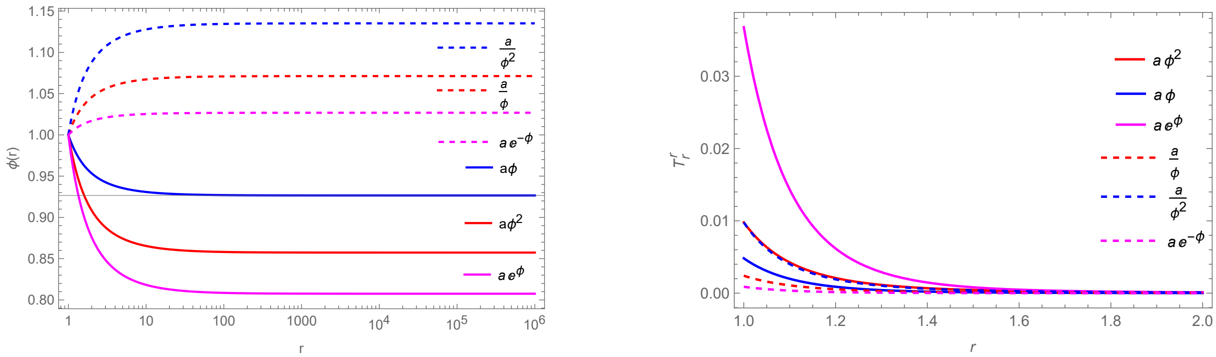
First we will use the asymptotic behaviour at infinity, therefore we find that,

$$T^t_{\ t} \simeq -\frac{1}{4}\phi'^2 + \mathcal{O}\left(\frac{1}{r^6}\right), \quad T^r_{\ r} \simeq \frac{1}{4}\phi'^2 + \mathcal{O}\left(\frac{1}{r^6}\right), \quad T^\theta_{\ \theta} \simeq -\frac{1}{4}\phi'^2 + \mathcal{O}\left(\frac{1}{r^6}\right). \tag{3.6.38}$$

In the above where we have used that far away from horizon of the black hole, the metric function has the borderline behaviour $e^A \to 1$. Moreover the dominant contributions to the $(T^r_{\ r})'$ is,

$$(T^r_{\ r})' \simeq \frac{2}{r}(T^\theta_{\ \theta} - T^r_{\ r}) \simeq -\frac{1}{r}\phi'^2. \tag{3.6.39}$$

From this behaviour we conclude that the radial component of the energy-momentum tensor at infinity is positive definite and decreasing. This result is the same as in Bekenstein's calculations for Novel-No-scalar-Hair theorem since the Gauss-Bonnet term is insignificant in this area. We continue now with the limit near the horizon of the black



(a) (b)

Figure 3.6.1: (left) The scalar field dependence on radial coordinate r for different options of coupling functions $f(\phi)$. (right) The T^r_r component of energy momentum tensor in terms of the radial coordinate r. It has been chosen $\phi_h = 1$ and $\alpha = 0.01$.

hole. Using the asymptotic behaviour of the metric in the limit $r \to r_h$ the T^r_r takes the following form,

$$T^r_r = -\frac{2e^{-B}}{r^2} A'\phi'\dot{f} + \mathcal{O}(r - r_h). \tag{3.6.40}$$

We see that the dominant contribution is due to the Gauss-Bonnet term because the curvature of spacetime near the horizon is very large. We note here that in the Novel-No-scalar-Hair theorem T^r_r near the horizon is strictly negative definite. On the other hand in EsGB theory T^r_r can be positive under the assumption that near the horizon of the black hole it holds that $\dot{f}\phi' < 0$. From the eq.(3.6.21) we see that this condition is always satisfied therefore T^r_r is positive definite. We conclude that one of the two Bekenstein's requirements can be evaded.

We will examine now the behaviour of $(T^r_r)'$ near the horizon. Using the eq.(3.6.35-3.6.36) and eq.(3.6.34) we obtain the following expression in the limit $r \to r_h$,

$$(T^r_r)' = e^{-B}A'\Big[-\frac{r\phi'^2}{4Z} - \frac{2(\ddot{f}\phi'^2 + \dot{f}\phi'')}{rZ} + \frac{4\dot{f}\phi'}{r^2}\left(\frac{1}{r} - e^{-B}B'\right)\Big] + \mathcal{O}(r - r_h) \tag{3.6.41}$$

where $Z \equiv r + 2\dot{f}\phi'$. Near the horizon we see that $(T^r_r)'$ is negative due to $A' > 0$ and $B' < 0$ while $Z > 0$ and $\dot{f}\phi' < 0$. However it is necessary to add an additional constraint namely $\ddot{f}\phi'^2 + \dot{f}\phi' > 0$. In conclusion, for the EsGB theory we have the following behaviour of the T^r_r and $(T^r_r)'$,

$$T^r_r > 0 \quad , \quad (T^r_r)' < 0 \quad , \quad r \to r_h\,, \tag{3.6.42}$$

$$T^r_r > 0 \quad , \quad (T^r_r)' < 0 \quad , \quad r \to \infty\,. \tag{3.6.43}$$

which is contrasting with Bekenstein's two requirements, thus leading to the evasion of the No-scalar-Hair theorem.

Let us now briefly give some results of the numerical integration of Einstein equations. Choosing the form of $f(\phi)$ and the value $\phi_h = 1$ while $r_h = 1$ the results of the integration are given in the Fig.(3.6.1). Three coupling functions $f(\phi) = (\alpha e^{\phi}, \alpha\phi, \alpha\phi^2)$ were chosen which lead to $\dot{f}_h > 0$ and therefore $\phi'_h < 0$. On the other hand the choice $f(\phi) = (\alpha e^{-\phi}, \alpha\phi^{-1}, \alpha\phi^{-2})$ leads to $\dot{f}_h < 0$ and thus $\phi'_h > 0$. Note that the value of the parameter α is $\alpha = 0.01$. Finally, it can be easily observed that the behaviour of the T^r_r in Fig.(3.6.1) is in agreement with the evasion of the Novel-No-scalar-Hair theorem. Here, we will refrain from presenting details about the properties of the black hole solutions. In the next chapter we will discuss the properties of the solutions, having introduced a self interacting potential for the scalar field, in EsGB theory.

3.7 Spontaneously Scalarized Black Holes in EsGB Theory

The evasion of the Novel-No-scalar-Hair theorem and numerical integration of the Einstein equations are not enough criteria for ensuring the existence of black holes with scalar hair. To make sure that the black holes that are theoretically predicted even in General Relativity are real objects in the universe, they must be stable under spacetime perturbations. It should be mentioned that the problem of stability of black holes is not a trivial problem and has concerned both theoretical physicists and mathematicians. In this sense the following questions are posed: Is the Schwarzschild black hole as a solution of the EsGB theory with $V(\phi)$ stable or not? Can the Schwarzschild solution turn into a black hole with scalar hair? We will prove that Schwarzschild black hole is unstable for some values of mass in the framework of the EsGB theory with $V(\phi)$ and thus a scalarized black hole lapses from the Schwarzschild one. This effect is the so called spontaneous scalarization mechanism and has been extensively studied in the literature.

We begin our analysis from the standard line element for the Schwarzschild geometry,

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{3.7.1}$$

where $h(r) = 1 - \frac{2M}{r}$. Substituting in the equations of motion of EsGB theory with $V(\phi)$, we extract that the Schwarzschild solution is a solution of EsGB theory with

$V(\phi)$ if it is true that,

$$\phi(r) = \phi_0 \quad \text{or} \quad \phi(r) = 0 \quad \text{and} \quad V(\phi_0) = 0 \tag{3.7.2}$$

$$\dot{f}(\phi_0) = 0 \quad \text{or} \quad \dot{f}(0) = 0 \quad \text{and} \quad \dot{V}(\phi_0) = 0 \tag{3.7.3}$$

with ϕ_0 being a constant. In order to examine the stability of the system we need small spacetime perturbations around the above solution. According to Doneva and Yazadjiev [46] the equations governing the perturbations of the metric are decoupled from the perturbative equation for the scalar field under of constraints of the eq.(3.7.2-3.7.3). Focusing on scalar field time-dependent perturbations we find that the corresponding equation takes the form

$$\begin{aligned} &2r\left(1-\frac{2M}{r}\right)^2\delta\phi'' + 4\left(1-\frac{2M}{r}\right)\left(1-\frac{M}{r}\right)\delta\phi' - 2r\delta\ddot{\phi} + 192\frac{M^2}{r^7}\left(1-\frac{2M}{r}\right)^3 \times \\ &\left[\frac{d^2f}{d\phi^2}\delta\phi - \frac{df}{d\phi}\delta B\right] + \frac{4}{r}\frac{df}{d\phi}\Bigg[-\frac{4M^2}{r^4}\delta B + \frac{2M}{r^2}\left(1-\frac{2M}{r^2}\right)\left(1+\frac{3M}{r}\right)\delta A' \\ &-\frac{2M}{r^2}\left(1-\frac{2M}{r}\right)\left(1+\frac{3M}{r}\right)\delta B' + \left(\frac{4M}{r^3}-\frac{4M^2}{r^4}\right)\delta B - \frac{2M}{r}\Bigg[2\left(1-\frac{2M}{r}\right)^2\delta A'' \\ &+\frac{4M}{r^2}\left(1-\frac{2M}{r}\right)\delta A'\Bigg] + \frac{4M}{r}\left(1-\frac{2M}{r}\right)^2\delta\ddot{B}\Bigg] - 4r\left(1-\frac{2M}{r}\right)^2\Lambda\frac{d^2V}{d\phi^2}\delta\phi = 0, \end{aligned} \tag{3.7.4}$$

where $\delta\phi$, δB and δA represents the small perturbations of the scalar field and metric's functions, while $V(\phi)$ is a general scalar potential. We assume that the perturbations have the following form,

$$\delta\phi = \frac{u(r)}{r}e^{i\sigma t}Y_m^l(\theta,\varphi) \tag{3.7.5}$$

with $Y_m^l(\theta,\varphi)$ being the standard spherical harmonic functions and σ is a constant. From eqs.(3.7.3), eq.(3.7.4) and eq.(3.7.5) the radial part of the equation for the scalar field perturbations is

$$\frac{h(r)}{r}\frac{d}{dr}\left(r^2h(r)\frac{d}{dr}\left(\frac{u(r)}{r}\right)\right) + \left(\sigma^2 + h(r)\frac{48M^2}{r^6}\ddot{f}(\phi_0)\right)u(r) = 0, \tag{3.7.6}$$

where $R_{GB(0)} = \frac{48M^2}{r^6}$ and we study only the s-wave and thus set $l = 0$. By changing the coordinate system via the tortoise coordinate $dr^* = dr/h(r)$, we can bring eq.(3.7.6) into a Schrodinger form by eliminating the first derivative of the $u(r)$. Hence, we finally find that

$$\frac{d^2u(r)}{dr_*^2} + (\sigma^2 - V_{eff}(r))u(r) = 0, \tag{3.7.7}$$

where the effective potential is of the form

$$V_{eff}(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} - \frac{48M^2}{r^6}\ddot{f}(\phi_0)\right) \tag{3.7.8}$$

In order to find an unstable mode we use the following condition

$$\int_{-\infty}^{+\infty} V_{eff}(r_*)dr_* = \int_{2M}^{+\infty} \frac{V_{eff}(r)}{h(r)}dr = \frac{5M^2 - 6\ddot{f}(\phi_0)}{20M^3} < 0 \tag{3.7.9}$$

Therefore the only way the above fraction can be negative is to assume that $\ddot{f}(\phi_0) > 0$ while M is always positive. Finally we can find a constraint for the mass which is given by,

$$M^2 < \frac{6\ddot{f}(\phi_0)}{5} \tag{3.7.10}$$

If the mass of the Schwarzschild black hole is lighter than the critical value $M_c = \frac{6\ddot{f}(\phi_0)}{5}$ then it becomes unstable and a black hole with scalar hair arises.

It is worth noting at this point that spontaneous scalarization mechanism is not an existence theorem but gives an explanation of how black holes with scalar hair are created. In addition spontaneous scalarization is a restrictive mechanism and applies only to forms of the coupling function as seen in eq.(3.7.2)-(3.7.3). Note also that spontaneous scalarized solutions are not different from the solutions we can find in the more general case in EsGB theory. In fact, the spontaneous scalarized solutions are a subgroup of the more general solutions. In the following sections we will refer to this mechanism again.

Chapter 4

Black holes with a self interacting scalar field

4.1 Introduction

In the previous chapter we presented black hole solutions in the EsGB theory. It is also known that in the presence of a negative cosmological constant Λ, we can find AdS black hole solutions as well as asymptotically-flat solutions. On the other hand in the presence of a positive cosmological constant de Sitter solutions have not been found as a regular cosmological horizon cannot be formed [44]. In our analysis we study a more realistic scenario where the cosmological constant is replaced by a scalar field potential. In this context, we ask what kind of solutions may arise in this theory. According to Bakopoulos, Kanti and Pappas [47], we can combine various forms of coupling functions $f(\phi)$ with a variety of polynomial forms of the scalar potential to find black hole solutions. Particle physics and cosmology give us specific scalar potentials for example Higgs potential, Coleman-Weinberg, Starobinsky-type etc. For such potentials we need to investigate the type of regular solutions they support in the presence of Gauss-Bonnet gravitational term.

In the first section we give the theoretical framework and in the next sections we study three well known potentials from particle physics and cosmology. Also, we present numerical results for a variety of black hole solutions and their features.

4.2 The theoretical Framework

We will study a general family of higher-curvature gravitational theories where we have introduced the quadratic Gauss-Bonnet term and a scalar-field potential. Hence the theory has the following form

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + f(\phi)\mathcal{R}^2_{GB} - 2\Lambda V(\phi)\right] \tag{4.2.1}$$

where $f(\phi)$ is the coupling function between the scalar field and the Gauss-Bonnet term, $V(\phi)$ is the scalar potential and Λ is a constant the role of which is to represent the sign of the scalar potential contribution. By setting $V(\phi) = 1$, we obtain the EsGB theory with a cosmological constant.

The variation of the action with respect to the metric $g_{\mu\nu}$ and the scalar field ϕ leaves us with the gravitational equations and the scalar field equation respectively. Therefore the equations are the following

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}(\partial_\rho\phi)^2 \\ - \frac{1}{2}(g_{\lambda\mu}g_{\rho\nu} + g_{\rho\mu}g_{\lambda\nu})\nabla_\gamma[\tilde{R}^{\rho\gamma}{}_{\alpha\beta}\eta^{\kappa\lambda\alpha\beta}\nabla_\kappa f(\phi)] - \Lambda V(\phi), \end{aligned} \tag{4.2.2}$$

$$\nabla^2\phi + \dot{f}(\phi)\mathcal{R}^2_{GB} - 2\Lambda\dot{V}(\phi) = 0. \tag{4.2.3}$$

In the second equation the dot over the coupling function and scalar potential symbolizes the derivative with respect to the scalar field. For simplicity we have chosen that $c = G = 1$. We study regular, static, and spherically-symmetric black hole solutions with scalar hair. In what follows, we will use a strikingly different form of the metric which is

$$ds^2 = -e^{2\delta(r)}N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta)d\varphi \tag{4.2.4}$$

with

$$N(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda}{3}r^2. \tag{4.2.5}$$

The reason we use this metric is because of the numerical integration [45]. We noticed that the numerical integration code can extract solutions in the case $\Lambda > 0$, while the previous one can not. Obviously we can define $e^{A(r)} = e^{2\delta(r)}N(r)$ and $e^{B(r)} = \frac{1}{N(r)}$. Therefore we can get the line element of eq.(3.6.9) Additionally we assume that scalar field shares the same symmetries with the metric and therefore, $\phi = \phi(r)$. Replacing the line element eq.(**??**) to the equations of motion and using the symmetries, we get the following system of the equations of motion,

$$
\begin{aligned}
&4\left[\frac{-3(-2r+\Lambda r^3+6m)\dot{f}(\Lambda r^3-3m+3rm')\phi'}{9r^5}\right.\\
&\left.-\frac{r(\Lambda r^3+6m)(-3r+\Lambda r^3+6m)(\phi'^2\ddot{f}+\dot{f}\phi'')}{9r^5}\right]\\
&-\frac{2m'}{r^2}-\frac{(-3r+\Lambda r^3+6m)\phi'^2}{12r}-\Lambda-\Lambda V(\phi)=0,
\end{aligned}
\tag{4.2.6}
$$

$$
\begin{aligned}
&-\frac{1}{12}\left[\frac{8(\Lambda r^3-3m-r(-3r+\Lambda r^3+6m)\delta'+3rm')(r^2+2(-2r+\Lambda r^3+6m))\dot{f}\phi'}{r^5}\right.\\
&\left.+\frac{6m(-4+r^2\phi'^2)}{r^3}\right]+\frac{1}{12}\left[(-3+\Lambda r^2)\phi'^2-4\Lambda+12\Lambda V(\phi)\right]=0,
\end{aligned}
\tag{4.2.7}
$$

$$
\begin{aligned}
&\frac{(6m(-1+r\delta')+r(6-4r^2\Lambda+r(-3+\Lambda r^2)\delta'-6m'))\phi'-r(-3r+\Lambda r^3+6m)\phi''}{3r^2}\\
&+\frac{2\dot{f}}{9r^5(3r^2+4(-3r+\Lambda r^3+6m)\dot{f}\phi'}\Big[36\Lambda r^4(\Lambda r^3+6m)V(\phi)\\
&-3r(\Lambda r^3+6m)(4(\Lambda r^3-3m+3rm')+r^2(-3r+\Lambda r^3+6m)\phi'^2)\\
&+4(\Lambda r^3-3m-r(-3r+\Lambda r^3+6m)\delta'+3rm')(6(\Lambda r^3-3m+3rm')(r-4\dot{f}\phi')\\
&-(\Lambda r^3+6m)(3r+4(-3r+\Lambda r^3+6m)(\phi'^2\ddot{f}+\dot{f}\phi'')))\Big]-2\Lambda\dot{V}(\phi)=0
\end{aligned}
\tag{4.2.8}
$$

where $m = m(r)$ and $\delta = \delta(r)$,. The eqs.(4.2.6-4.2.7) are the gravitational equations while the last one is the equation of motion of the scalar field.

In order to find a complete solution which describes a regular black hole with scalar hair, it is necessary to determine the unknown three functions $m = m(r)$, $\delta = \delta(r)$ and $\phi = \phi(r)$. To be able to solve the system of differential equations we need boundary conditions. In our work we use as boundary conditions the near horizon expansions of the metric and scalar field. Using the near horizon behaviour from eqs.(3.6.33) we find the expressions for the new metric functions,

$$m(r) = \frac{r_h}{2} - \frac{r_h^3 \Lambda}{6} + m_1(r - r_h) + \mathcal{O}((r - r_h)^2), \tag{4.2.9}$$

$$\delta(r) = \delta_0 + \delta_1(r - r_h) + \mathcal{O}((r - r_h)^2), \tag{4.2.10}$$

$$\phi(r) = \phi_h + \phi_h'(r - r_h) + \mathcal{O}((r - r_h)^2) \tag{4.2.11}$$

Substituting the above expressions in the equations of motion we extract the following

$$\begin{aligned}
m_1 &= \frac{(-1 + V(\phi_h))r_h^3\Lambda - 2(-1 + r_h^2\Lambda)\phi_h'\dot{f}(\phi_h)}{2(r_h + 2\phi_h'\dot{f}(\phi_h))}, \\
\delta_1 &= \frac{2r_h^3\Lambda + 4(\phi_h' + 2r_h^2\Lambda\phi_h')\dot{f}(\phi_h) + 4r_h^4\Lambda^2\phi_h'(V(\phi_h))^2\dot{f}(\phi_h)}{3r_h(-1 + r_h^2\Lambda V(\phi_h))(r_h + 2\phi_h'\dot{f}(\phi_h))} \\
&\quad + \frac{8r_h\Lambda\phi_h'(\dot{f}(\phi_h))^2 - 2r_h\Lambda V(\phi_h)(r_h^2 + 8r_h\phi_h'\dot{f}(\phi_h) + 4(\phi_h'\dot{f}(\phi))^2)}{3r_h(-1 + r_h^2\Lambda V(\phi_h))(r_h + 2\phi_h'\dot{f}(\phi_h))}, \\
\phi_h' &= \frac{-r_h^3 + r_h^5\Lambda V(\phi_h) - 48r_h\Lambda V(\phi_h)(\dot{f}(\phi_h))^2 + 16r_h^3(\Lambda V(\phi_h)\dot{f}(\phi_h))^2}{4\dot{f}(\phi_h)(-\Lambda V(\phi_h)(r_h^4 - 16\dot{f}(\phi_h)) + r_h^2(1 - 4\Lambda\dot{f}(\phi_h)\dot{V}(\phi_h))} \\
&\quad + \frac{8r_h^3\Lambda\dot{f}(\phi_h)\dot{V}(\phi_h) \pm (1 - r_h^2\Lambda V(\phi_h))\sqrt{\mathcal{F}}}{4\dot{f}(\phi_h)(-\Lambda V(\phi_h)(r_h^4 - 16\dot{f}(\phi_h)) + r_h^2(1 - 4\Lambda\dot{f}(\phi_h)\dot{V}(\phi_h))}
\end{aligned} \tag{4.2.12}$$

where

$$\begin{aligned}
\mathcal{F} = r_h^6 &+ 32r_h^2(-3 + 2r_h^2\Lambda V(\phi_h))(\dot{f}(\phi_h))^2 \\
&+ 256\Lambda V(\phi_h)(-6 + r_h^2\Lambda V(\phi_h))(\dot{f}(\phi_h))^4 + 384r_h^2\Lambda(\dot{f}(\phi_h))^3\dot{V}(\phi_h).
\end{aligned}$$

The quantity $\mathcal{F}$ should always be positive definite as it appears under a square root. Furthermore for specific scalar potentials $V(\phi)$ and coupling functions $f(\phi)$ the constraint of the first derivative of the scalar field imposes bounds on the value of the event horizon r_h. We can easily see that the only free parameters are ϕ_h, Λ, r_h. The parameter δ_0 is free since the function $\delta(r)$ does not appear in the equations of motion but only its first derivative. Now it is possible to solve numerically the eqs.(4.2.12).

4.3 The Higgs Potential

Before we proceed to present our results we will briefly discuss the Higgs mechanism. In 2012, experiments at CERN proved the existence of the Higgs particle with mass near to 125 GeV [48]. Therefore, we can claim that the Higgs field is responsible for the masses of the heavy bosons and fermions except neutrinos. We call heavy bosons the carriers of weak interactions and denote them by $W^{\pm}$ and Z^0. The Higgs mechanism is described as a case of *spontaneous symmetry breaking*. In terms of Quantum Field Theory we call spontaneous symmetry breaking or *Goldstone realization of the symmetry* the phenomenon in which the vacuum state does not obey the symmetry of the Lagrangian. Here, we will discuss this mechanism through a simple example of Quantum Field Theory.

Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4, \tag{4.3.1}$$

where μ and λ are constants. We can easily see that the Lagrangian is invariant under the transformation $\phi \to -\phi$. The scalar potential is

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4. \tag{4.3.2}$$

The field configuration which corresponds to the lowest energy is called vacuum state and is given by $\phi_0 = v$. The minimization of the potential gives us two minima,

$$v = \pm\sqrt{\frac{\mu^2}{\lambda}}. \tag{4.3.3}$$

The system has to choose between the two vacuua $\pm v$ and therefore the symmetry $\phi \to -\phi$ is broken in the ground state. Note that in order for symmetry breaking to occur the mass term must have the wrong sign. On the other hand by choosing $\mu^2 > 0$ in the original Lagrangian we can prove that the vacuum expectation value would be zero and thus ground state would respect the symmetry. We can always expand the scalar field around the vacuum state as

$$\phi(x) = v + \sigma(x), \tag{4.3.4}$$

splitting it into a classical part v and quantum fluctuations $\sigma(x)$. Therefore we can express the Lagrangian in terms of the new field $\sigma(x)$ as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \mu\sqrt{\lambda}\sigma^3 - \frac{\lambda}{4}\sigma^4 + \frac{\mu^4}{2\lambda}. \tag{4.3.5}$$

Note that despite the wrong sign of μ^2 in the original Lagrangian the new one describes a massive scalar field which has a physical mass $m_\sigma = \sqrt{2}\mu > 0$. In this work we choose the scalar potential in the gravitational action of EsGB theory to be of the form

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2, \tag{4.3.6}$$

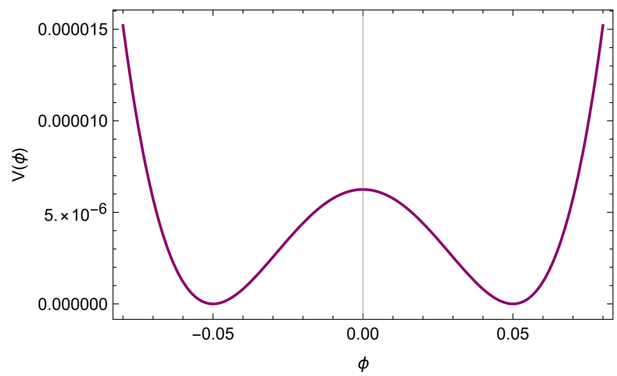


Figure 4.3.1: The Higgs potential for $v = 0.05$ and $\lambda = 1$.

which spontaneously breaks $\mathbb{Z}_2$ symmetry. In the following we will study the black hole solutions emerging in the presence of the above scalar potential and we will discuss their features.

4.3.1 Black Hole solutions in the case $\Lambda < 0$

By choosing the values of the parameters we can proceed to the numerical integration of the system. The integration starts from $r = r_h + \mathcal{O}(10^{-7})$ and stops when the metric reaches an asymptotic flat solution. In our analysis we have chosen $r_h = 1$. We study the case where we have set $f(\phi) = (\alpha e^{\phi}, \alpha\phi, \alpha\phi^2, \alpha\phi^3)$, $\Lambda = -1$, $\lambda = 0.001$, $\alpha = 0.01$ and $v = 0.05$. The scalar potential has the form of the eq.(4.3.6). The boundary conditions are the set of eq.(4.2.12) so we must determine the free parameter ϕ_h. Here we choose

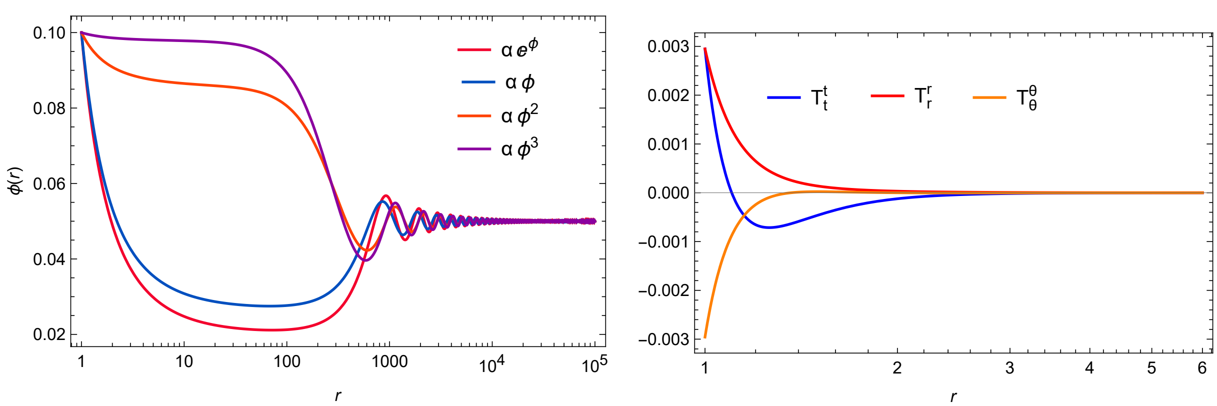


Figure 4.3.2: (a) The scalar field dependence on radial coordinate r for different options of coupling functions $f(\phi)$. (b) The components of energy momentum tensor for $f(\phi) = \alpha e^{\phi}$, in terms of the radial coordinate r. It has been chosen $\phi_h = 1$ and $\alpha = 0.01$.

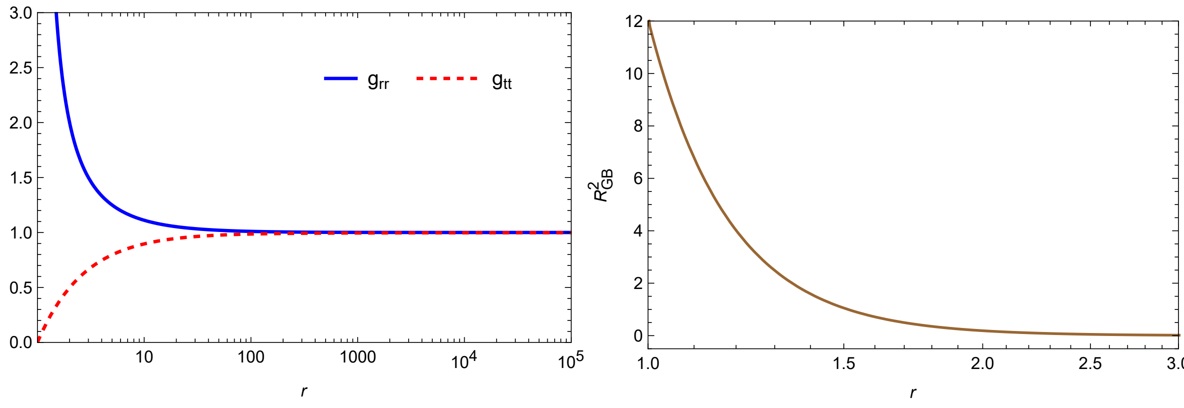


Figure 4.3.3: (a) The components of the metric tensor in terms of radial coordinate r, for $f(\phi) = \alpha e^{\phi}$. (b) The Gauss-Bonnet gravitational term in dependence of radial coordinate r, for $f(\phi) = \alpha e^{\phi}$

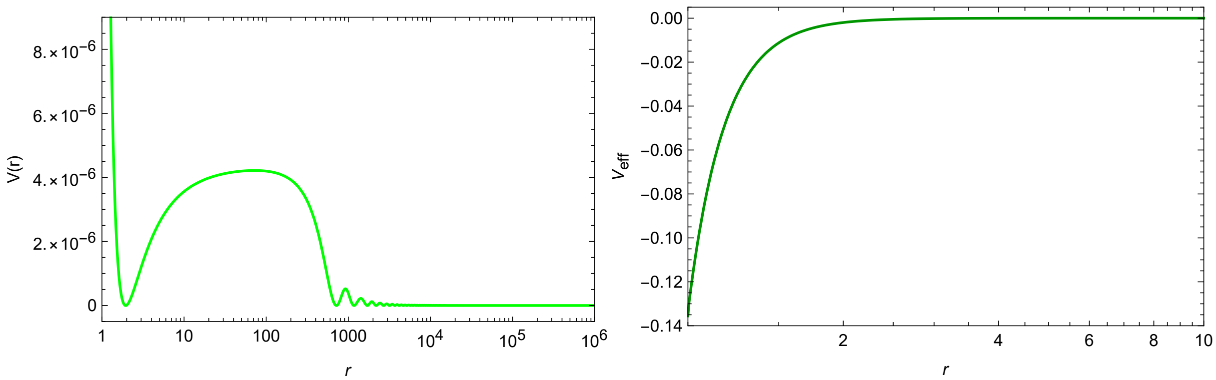
for simplicity the value $\phi_h = 0.1$. As we have mentioned before, there is another free parameter of the system which is δ_0. Its value will be determined by the requirement $g_{tt} \to -1$ as $r \to \infty$ therefore $\delta_0 = 0.005$.

In Fig.(4.3.2) we see the profile of the scalar field in dependence of the radial coordinate r. We can observe that the scalar field is regular near the event horizon independently of the form of the coupling function $f(\phi)$. Although we have chosen the wrong sign for the scalar potential $V(\phi)$ the scalar field remains finite in all the radial regime. Asymptotically the scalar field takes the vacuum expectation value (vev) for all the forms of the coupling functions. From this behaviour of the scalar field we expect the solution to be asymptotically flat due to the fact that the scalar potential vanishes.

The energy-momentum tensor which receives contributions from the scalar field and the Gauss-Bonnet term is regular in the all radial regime. It is clear that the T^r_r which describes the radial pressure is positive definite near the horizon and decreases as the radial coordinate r increases. On the other hand the energy density $\rho = -T^t_t$ is negative definite. Both these features of the energy momentum tensor are the reason that the No-scalar-Hair theorems are violated [43].

We can easily see from Fig.(4.3.3) that the spacetime remains regular everywhere. This is in accordance to the fact that both the energy-momentum tensor as well as the gravitational Gauss-Bonnet term, as we may see from Fig.(4.3.3), are also finite.

Let us now discuss the asymptotic flatness of the solution. We saw that the scalar field takes asymptotically its vev value. As a result, its potential also vanishes asymp-



(a) (b)

Figure 4.3.4: (a) The effective potential and scalar potential $V(r)$ in terms of coordinate r for $f(\phi) = \alpha e^{\phi}$.

totically as we see in Fig.(4.3.4a). The same holds also for the effective potential which is given by

$$V_{\text{eff}} = -f(\phi)\mathcal{R}^2_{GB} + 2\Lambda V(\phi) \tag{4.3.7}$$

as we see in Fig.(4.3.4b). Thus we can safely claim there is no a constant energy remnant at infinity to act as a cosmological constant. Asymptotic flatness is also reflected in the fact that the energy-momentum tensor is vanishing in the limit $r \to \infty$. For all the forms of $f(\phi)$ considered the effective potential has the same qualitative behaviour.

4.3.2 Black Hole solutions in the case $\Lambda > 0$

According to Bakopoulos, Antoniou and Kanti, finding solutions with positive cosmological constant is impossible in the context of EsGB theory using the metric ansatz of eq.(3.6.9) [44]. Here we will present de Sitter Black Hole solutions employing the new metric ansatz of the eq.(4.2.4). We have selected the parameters to be $\Lambda = 1$, $\lambda = 0.001$, $\alpha = 0.01$, $v = 0.05$ and $f(\phi) = (\alpha e^{\phi}, \alpha\phi)$. It should be stressed that no other coupling functions support regular solutions. We start again the integration from $r = r_h + \mathcal{O}(10^{-7})$ and we proceed outwards until the cosmological horizon emerges. We have fixed the boundary conditions as before, i.e $\phi_h = 0.1$ and $r_h = 1$. Note that here it is not necessary to give a specific value for the parameter δ_0 nevertheless we keep the same value $\delta_0 = 0.005$.

From Fig.(4.3.5) we observe that the scalar field decreases in the near horizon regime while it decays oscillatory to zero far away from the horizon. From the form of the

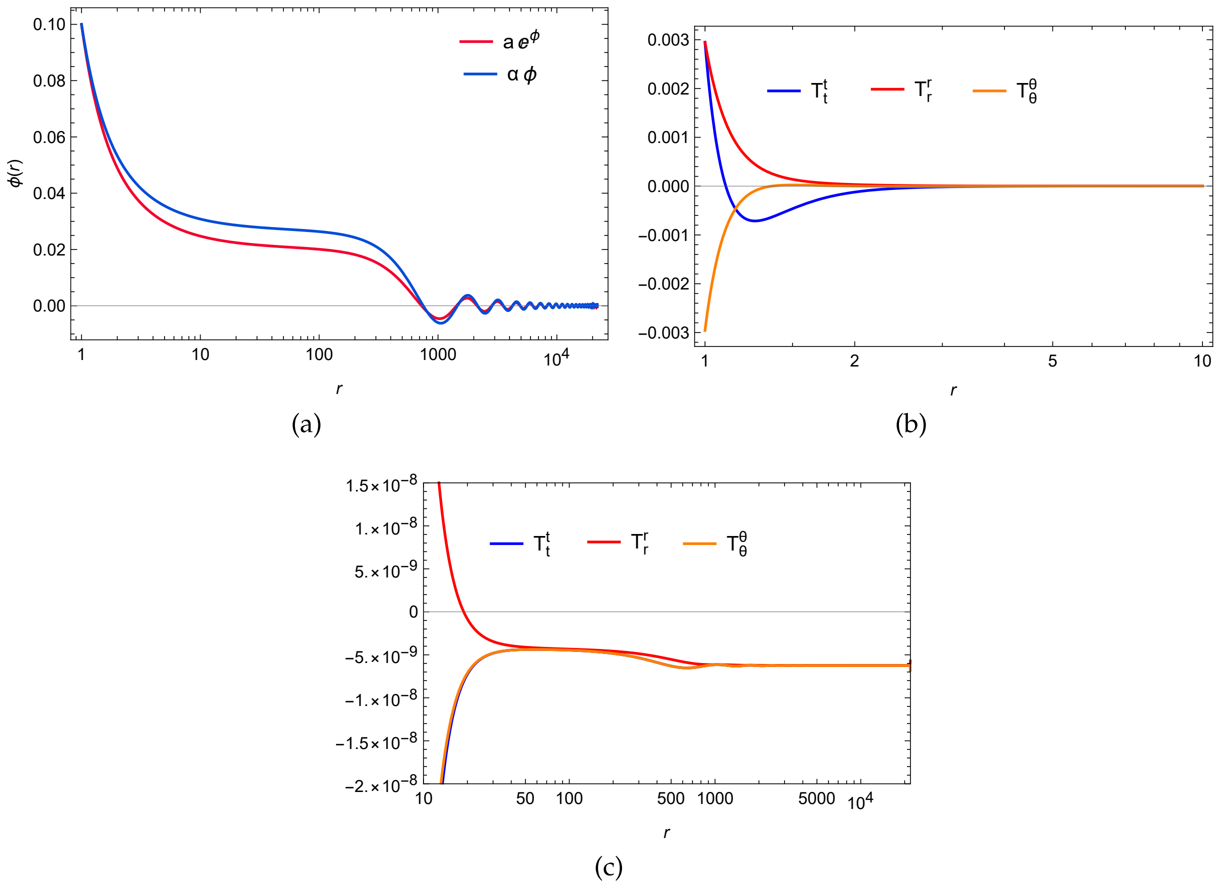
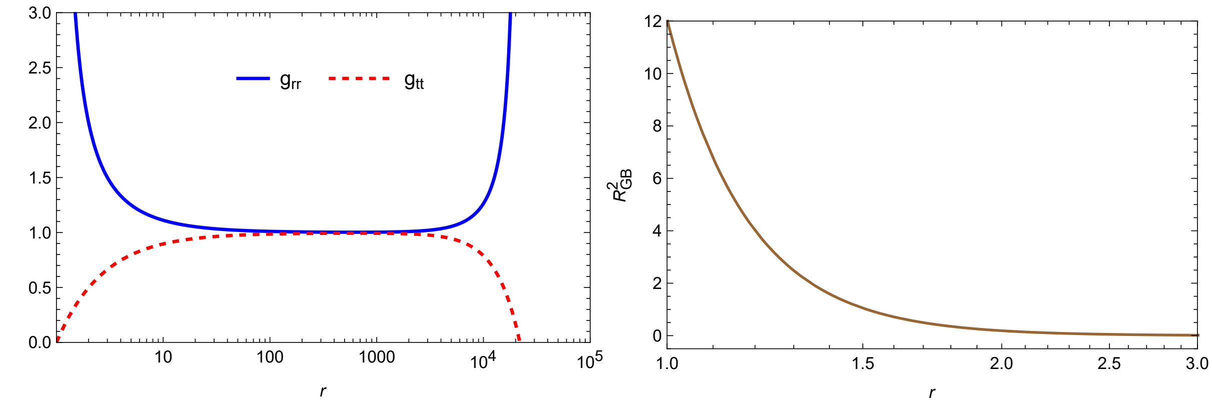


Figure 4.3.5: (a) The scalar field dependence on the radial coordinate r for different options of the coupling function $f(\phi)$. (b) The components of energy momentum tensor for $f(\phi) = \alpha e^{\phi}$, in terms of the radial coordinate r. We have chosen $\phi_h = 0.1$ and $\alpha = 0.01$. (c) The behaviour of energy-momentum tensor near the cosmological horizon.

scalar potential we see that a non-vanishing contribution to the energy-momentum tensor will remain asymptotically in that case. All components of the energy-momentum tensor reduce asymptotically to a negative constant. We have already seen that the energy density is given by $\rho = -T_t^t$ thus in our case we have asymptotically a positive energy density which acts as a positive cosmological constant. Note here that asymptotically the radial component $p = T_r^r$ is negative definite. The spacetime remains regular as both the energy-momentum tensor and the Gauss-Bonnet term are finite while the components of the metric have a smooth behaviour near the event and cosmological horizon. This behaviour is illustrated in the Fig.(4.3.6). The cosmological horizon is located at a distance r_c which has been numerically calculated to be $r_c = 21908.3988$ for $f(\phi) = \alpha e^{\phi}$ and $r_c = 21908.4019$ for $f(\phi) = \alpha\phi$.

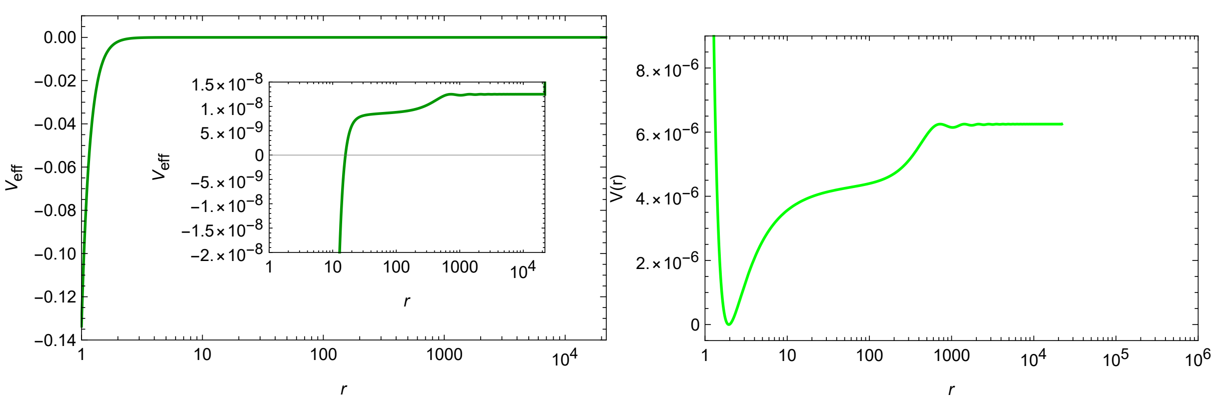


(a) (b)

Figure 4.3.6: (a) The components of $g_{\mu\nu}$ and (b) the Gauss-Bonnet term dependence on radial coordinate r, for $f(\phi) = \alpha e^{\phi}$.

As mentioned above, the solution is characterised as asymptotically de-Sitter. By observing both the effective and the radial potential, we can confirm the designation de-Sitter. We see from Fig.(4.3.7) that asymptotically the effective potential V_{eff} and $V(r)$ have a finite positive value. As we expect the maximum values of these potentials are located near the event horizon where, the contributions from the Gauss-Bonnet term are significant.

It should be stressed here that the plots of the quantities we have discussed have no differences if we set $f(\phi) = \alpha\phi$ thus we can omit them.



(a) (b)

Figure 4.3.7: (a) The plots of the effective potential $V_{\text{eff}}(\phi)$ and (b) the potential $V(r)$ with respect to the radial coordinate r for $f(\phi) = \alpha e^{\phi}$ and $\alpha = 0.01$.

4.4 The Coleman-Weinberg Potential

In the 1970s, Erick Weinberg and Sidney Coleman investigated the possibility that radiative corrections cause spontaneous symmetry breaking [49]. The so called *Coleman-Weinberg mechanism* has various applications in Particle Physics and Cosmology. In this work we are interested in presenting some information on how we can construct such an *effective potential* which is associated with spontaneous symmetry breaking. Subsequently, we will study the black hole solutions in the context of EsGB theory.

We know from Quantum Field Theory that the information of a quantum theory is encoded in the so called *generating functional* $Z[J] = e^{iW[J]}$ or equivalently in the *effective action* which is defined by

$$\Gamma = \sum_n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \Gamma^{(n)}(x_1 \ldots x_n) \phi_c(x_1) \ldots \phi_c(x_n) \tag{4.4.1}$$

and

$$\phi_c = \langle 0| \phi |0\rangle = \frac{\delta W[J]}{\delta J(x)}, \tag{4.4.2}$$

where $\Gamma^{(n)}(\ldots)$ are called *1-Particle Irreducible (1PI) Green's Functions* and $\Gamma^{(n)}(\ldots)$ is given by the sum of all 1PI Feynman graphs with n external lines. It can be shown that the quantum action can also be written as an expansion in terms of the derivatives of the classical field ϕ_c,

$$\Gamma = \int d^4x \, [-V(\phi_c) + \frac{1}{2} Z(\phi_c) \partial_\mu \phi_c \partial^\mu \phi_c + \ldots]. \tag{4.4.3}$$

In the above, $V(\phi_c)$ is the so called effective potential that we will use in our work. At tree level it coincides with the classical potential of the theory.

Consider a simple model of field theory for a real massless scalar field ϕ which is given by the following Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4!} \phi^4. \tag{4.4.4}$$

The one-loop effective potential of the theory is calculated by using the minimal subtraction scheme (MS) and has the form

$$V(\phi) = \frac{\lambda}{4!} \phi^4 + \frac{\lambda^2}{256\pi^2} \phi^4 \left(\ln \left(\frac{\phi^2}{M^2} \right) - \frac{25}{6} \right) \tag{4.4.5}$$

where we have also used the renormalization conditions

$$\left. \frac{\partial^2 V}{\partial \phi^2} \right|_{\phi=0} = 0 \tag{4.4.6}$$

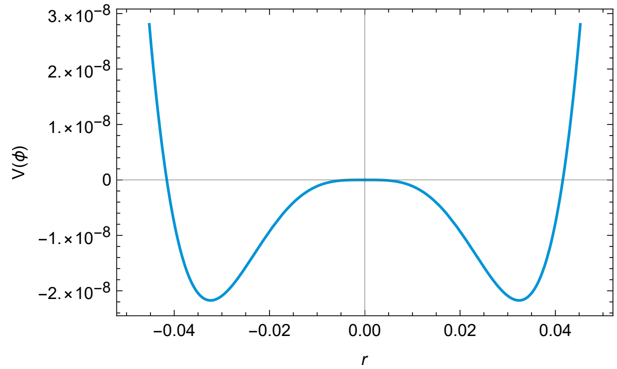


Figure 4.4.1: The scalar potential of the eq.(4.4.5) for $\lambda = 10$ and $M = 1$

meaning that the mass is still zero at the quantum level. The coupling λ is defined by the renormalization condition

$$\left.\frac{\partial^4 V}{\partial\phi^4}\right|_{\phi=M} = \lambda \tag{4.4.7}$$

which introduces the mass scale M. Note that the effective potential possesses two minima at a non-zero value $\langle\phi\rangle = v$ given by

$$\lambda \ln\left(\frac{\langle\phi\rangle^2}{M^2}\right) = \frac{11}{3} - \frac{32\pi^2}{3} \tag{4.4.8}$$

Hence, we see that the minima do not satisfy the conditions of the perturbation theory $\lambda \ln E \ll 1$. As a conclusion we cannot safely claim that the symmetry breaking occurs at the quantum level. However we will use the effective potential of eq.4.4.5 as a toy model to study black hole solutions.

4.4.1 Black Hole solutions in the case $\Lambda < 0$

We start the numerical integration from $r = r_h + \mathcal{O}(10^{-7})$, where we have again selected $r_h = 1$ for simplicity. For all solutions we present we choose $\phi_h = 0.1$. The coupling functions between the scalar field and $\mathcal{R}^2_{GB}$ are of the form $f(\phi) = (\alpha e^{\phi}, \alpha\phi, \alpha\phi^2, \alpha\phi^4, \alpha e^{-\phi})$ where $\alpha = 0.01$. We look for black hole solutions in the case $\Lambda = -1$. The parameters of the potential have been chosen to be $\lambda = 10$ and $M = 1$. As we will see below the solutions are classified as asymptotically de-Sitter (dS) solutions, therefore the integration stops near the cosmological horizon.

Let us first discuss the profile of the scalar field which is depicted in the Fig.(4.4.2a). We observe that in the case $\Lambda < 0$ the scalar field reduces asymptotically at the value where the potential is minimum. So we can easily see that this behaviour leaves us

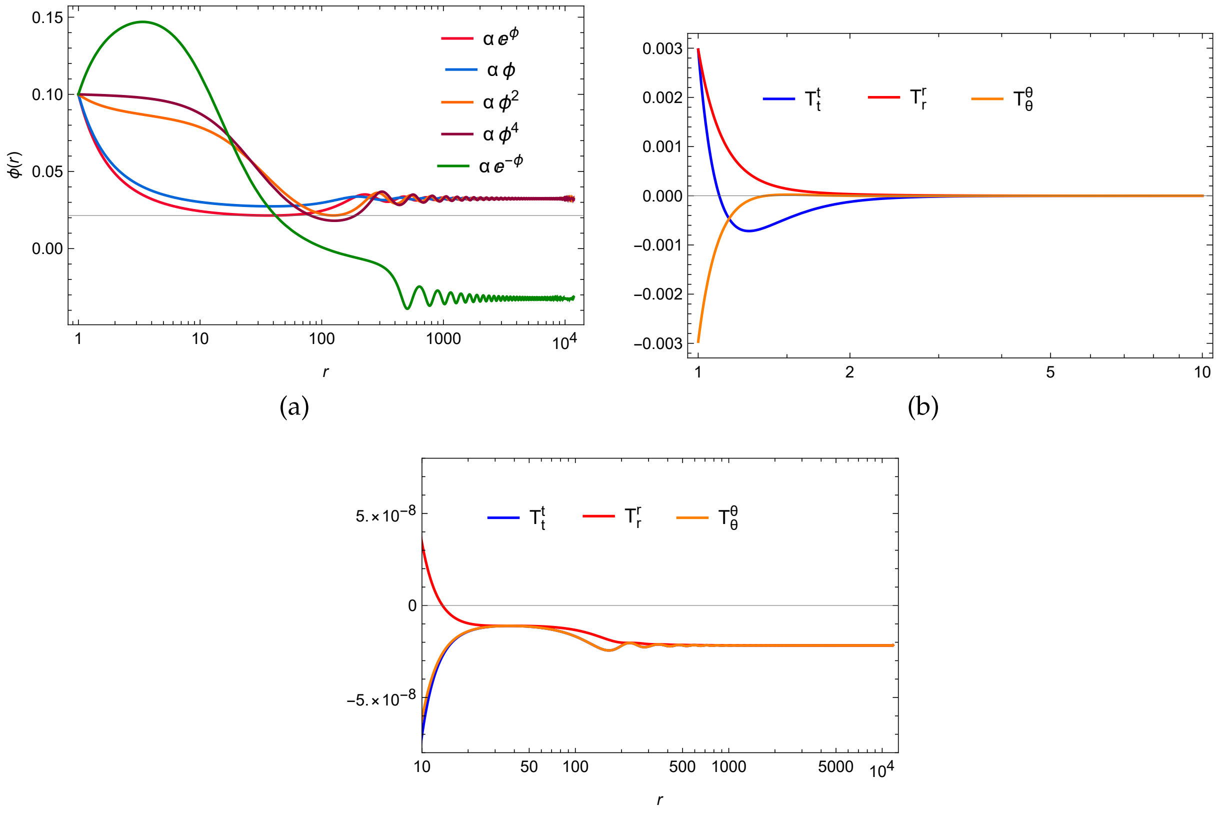
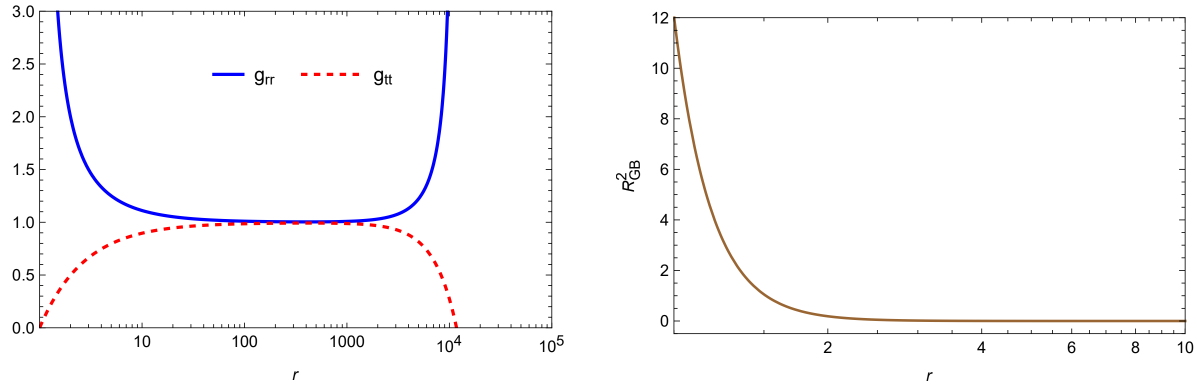


Figure 4.4.2: (a) The scalar field dependence on the radial coordinate r for different choices of coupling function $f(\phi)$. (b) The components of energy momentum tensor for $f(\phi) = \alpha e^{\phi}$, in terms of the radial coordinate r. We have chosen $\phi_h = 0.1$ and $\alpha = 0.01$. (c) The behaviour of the energy-momentum tensor near the cosmological horizon.

with a contribution of energy to the energy-momentum tensor as we present in the Fig.(4.4.2c). It should be emphasized that the Coleman-Weinberg potential has negative definite minima hence, it makes sense that this behaviour is similar to the Higgs case for $\Lambda > 0$.

The energy-momentum tensor is finite over the whole regime $[r_h, \infty)$ thus justifying the regularity of the spacetime. The solution for the components of the metric is presented in the Fig.(4.4.3a) and is smoothly behaved. We have calculated that the cosmological horizon is located at the distance $r_c = 11748.4523$ for $f(\phi) = \alpha e^{\phi}$. The distance r_c differs from the aforementioned value differentiates only in decimal digits for every other case of the coupling function $f(\phi)$. On the other hand the Gauss-Bonnet term has a maximum finite value on the event horizon and decreases to zero far away from it. We can safely claim that a regular black hole has emerged due to the regularity



(a) (b)

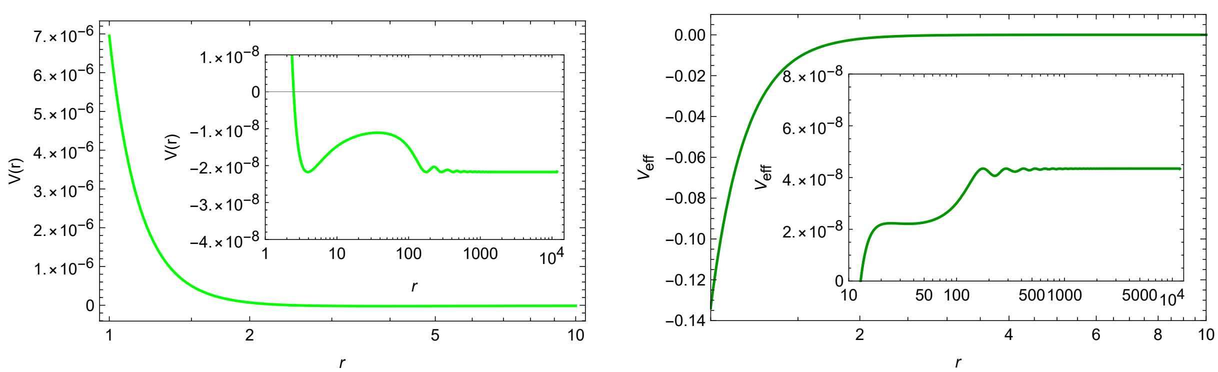
Figure 4.4.3: (a) The components of the metric and (b) Gauss-Bonnet term in terms of radial coordinate r for $f(\phi) = \alpha e^{\phi}$.

of space-time in the interval $[r_h, \infty)$.

The effective potential which is shown in the Fig.(4.4.4b), as expected, has a maximum value near r_h and decays asymptotically to a constant value as also does the $V(r)$ potential, depicted in Fig.(4.4.4a).

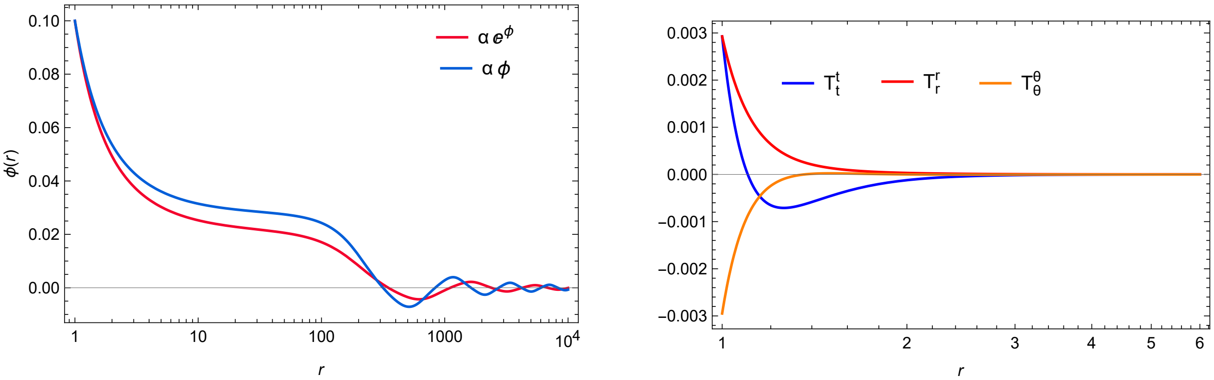
4.4.2 Black Holes in the case $\Lambda > 0$, asymptotically flat solutions.

In order to find black hole solutions in the case where $\Lambda > 0$ we have chosen the coupling functions $f(\phi) = (\alpha e^{\phi}, \alpha\phi)$ and the indicative values, $\alpha = 0.01$, $\phi_h = 0.1$, $\lambda = 10$, $M = 1$ and $\Lambda = 1$. As we have already mentioned we start the numerical



(a) (b)

Figure 4.4.4: (a) The effective potential V_{eff} and (b) the radial-dependent potential in terms of radial coordinate r.



(a) (b)

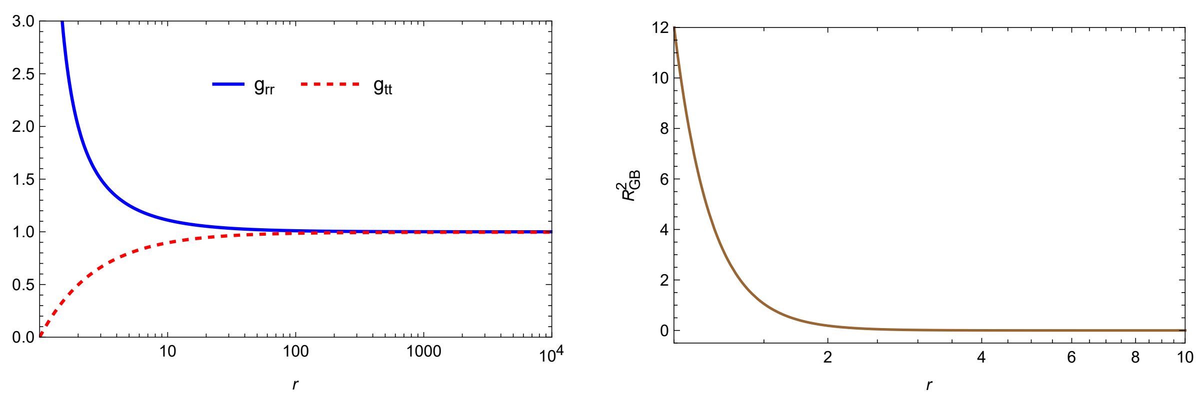
Figure 4.4.5: (a) The solutions of the scalar field in dependence on the radial coordinate for different forms of the coupling function (b) The energy momentum tensor in terms of r for $f(\phi) = \alpha e^{\phi}$.

integration near the horizon and we stop when a flat solution has emerges.

In contrast to the previous case, the scalar field vanishes asymptotically as we can see in the Fig.(4.4.5a). Therefore, the potential of eq.(4.4.5) also vanishes asymptotically and we expect the solution to be asymptotically flat. This indeed from Fig.(4.4.6a) where the components of the metric tensor approach the Minkowski spacetime.

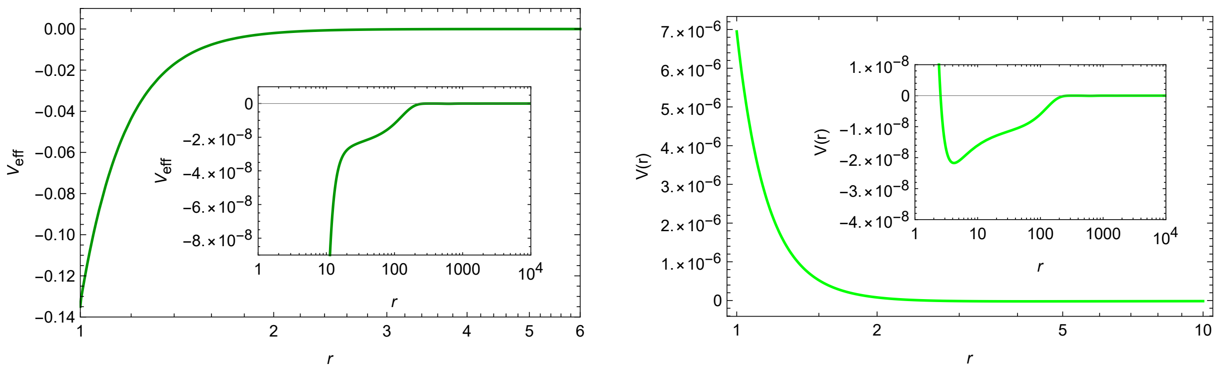
The effective potential also vanishes at the asymptotic limit which confirms the asymptotically flatness of the solutions.

Our conclusion is that we can construct regular black hole solutions in the case



(a) (b)

Figure 4.4.6: (a) The solutions of metric's components and (b) the Gauss-Bonnet term in terms of r for $f(\phi) = \alpha e^{\phi}$.



(a) (b)

Figure 4.4.7: (a) The effective potential and (b) the radial potential in terms of r for $f(\phi) = \alpha e^{\phi}$.

of $\Lambda > 0$ due to the smooth behaviour of the energy-momentum tensor, the Gauss-Bonnet term and the metric, near black hole's horizon. We should to mention that the behaviour of the quantities shown (except the metric) is similar to the corresponding case of the Higgs potential.

4.5 The Starobinsky scalar potential

In 1980 Alexei Starobinsky emphasised the role of quantum corrections to general relativity in the early universe. His model describes the early universe phase called *inflation* [50]. The presence of quantum corrections in the early universe motivated Starobinsky to extend General Relativity by adding a squared term of curvature. It should be stressed that the Starobinsky model is not a quantum theory of gravity but an effective theory described by the following action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(R + \frac{R^2}{6M^2}\right), \tag{4.5.1}$$

where R is the Ricci scalar. Making a conformal transformation of the metric $\tilde{g}_{\mu\nu} = (1+\phi/3M^2)g_{\mu\nu}$ along with the field redefinition, $\phi' = \sqrt{\frac{2}{3}}\ln\left(1+\frac{\phi}{3M^2}\right)$ we get

$$S = \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} + (\partial_\mu\phi')^2 - \frac{3}{2}M^2(1-e^{-\sqrt{\frac{2}{3}}\phi'})^2\right). \tag{4.5.2}$$

We identify the scalar potential as

$$V(\phi') = \frac{3M^2}{2}\left(1-e^{-\sqrt{\frac{2}{3}}\phi'}\right)^2 \tag{4.5.3}$$

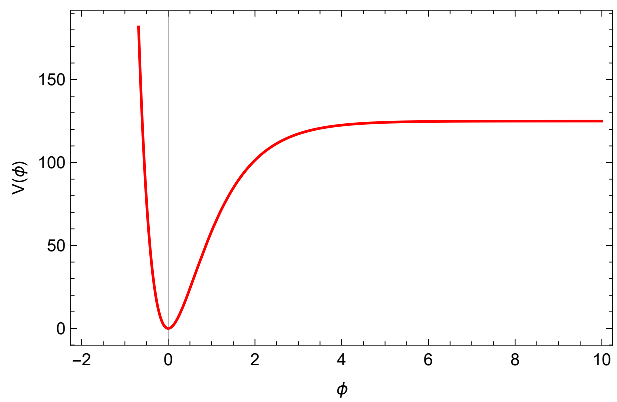


Figure 4.5.1: The Starobinsky potential for $M = 1$ and $\alpha = 0.01$.

hence we can claim that inflation occurs due to the contributions of the squared term of curvature.

However, it was soon realized that the potential of eq.(4.5.3) can be constructed in the framework of theories which describe elementary particles for example the supersymmetry [53]. Having the motivation from such theories we ask whether the Starobinsky potential can support black hole solutions in the framework of the EsGB theory. We have to mention that we will use the following form of the potential

$$V(\phi) = \frac{M^2}{8a}\left(1 - e^{-\frac{2\phi}{\sqrt{3}M}}\right)^2, \tag{4.5.4}$$

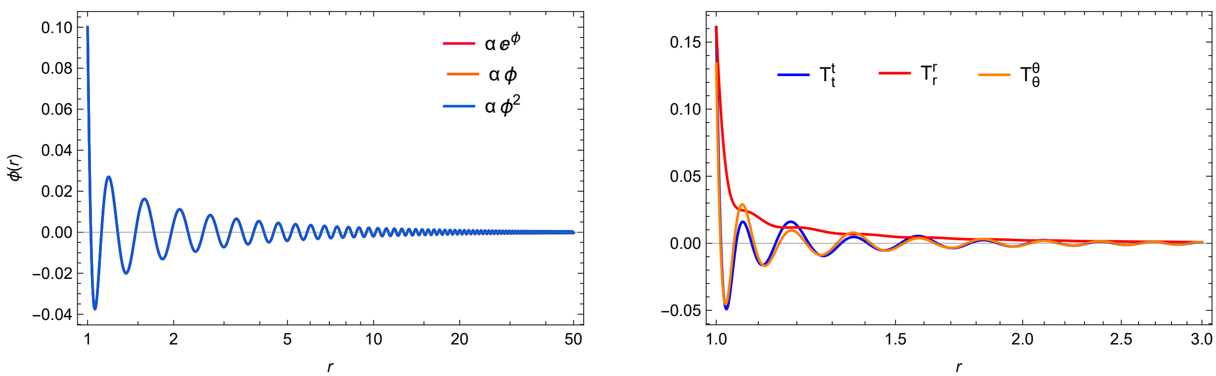
where M is an energy scale and a is a parameter.

4.5.1 Black Hole solutions in the case $\Lambda < 0$

In this subsection we discuss black hole solutions for $\Lambda = -1$. We choose the parameters to be $M = 1$, $a = 0.01$, $f(\phi) = (\alpha e^{\phi}, \alpha\phi, \alpha\phi^2)$, $\alpha = 0.001$ and $\phi_h = 0.1$. We start again the numerical integration near the horizon and we stop when the metric components approach the Minkowski spacetime.

The profile of the scalar field is shown in the Fig.(4.5.2a) which is characterized by a strong damping oscillatory behaviour.Asymptotically the scalar field vanishes. Note that the $\phi = 0$ is the minimum of the potential, which we can easily prove by studying the minimization of the potential in the eq.(4.5.4).

The energy momentum tensor exhibits also an oscillatory behaviour. Hence we can claim numerically that the solution violate Bekenstein's No-scalar-Hair theorem. We easily see from Fig.(4.5.2b) that all components of the energy-momentum tensor remain



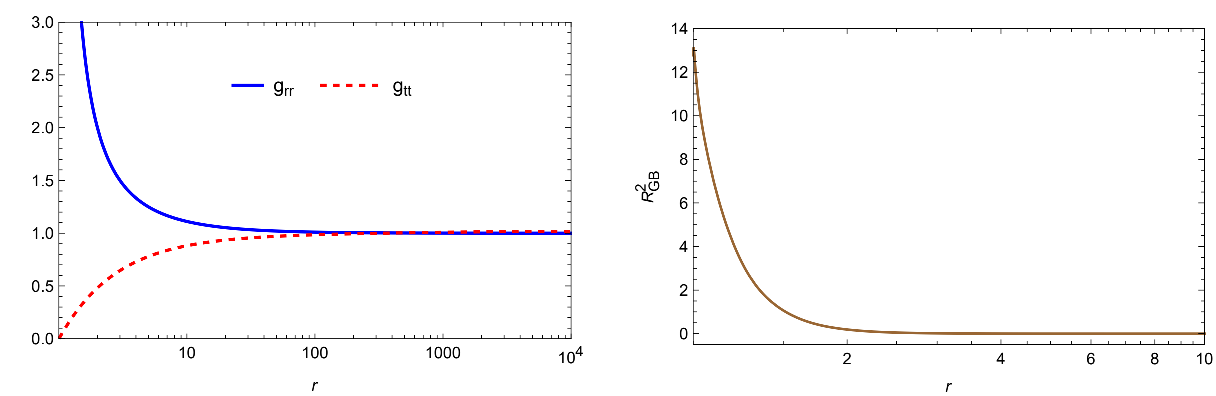
(a) (b)

Figure 4.5.2: (a) The profile of scalar field and (b) the energy momentum tensor in terms of r.

finite in the all spacetime regime $[r_h, \infty)$ and quickly decrease to zero. Furthermore the Gauss-Bonnet term has a maximum value near the event horizon and behaves smoothly in the whole area. In conclusion we have demonstrated that we can construct a regular black hole.

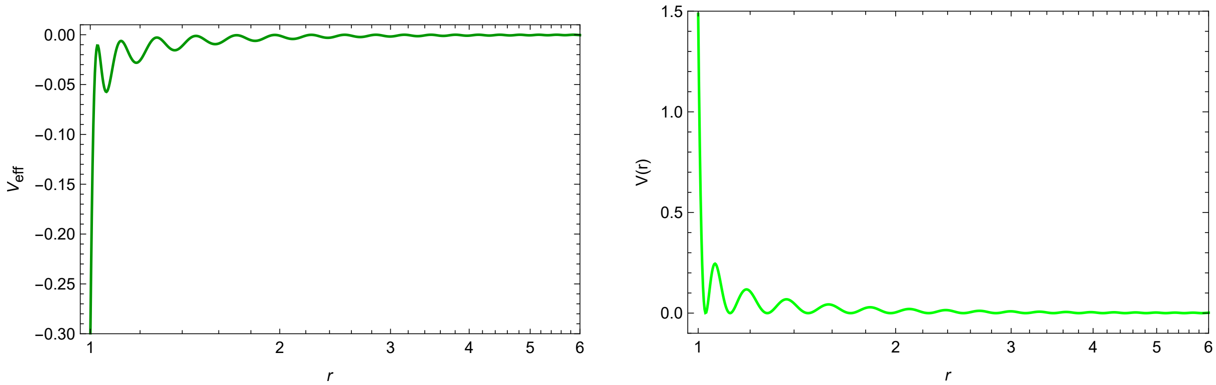
The effective and radial potentials have their maximum value near the horizon due to the Gauss-Bonnet contributions and also decay oscillatory at zero as we see in the Fig.(4.5.4a-4.5.4b).

According to our numerical analysis, regular black hole solutions with $\Lambda > 0$ can not emerge for arbitrary forms of the coupling function. Furthermore, we observed



(a) (b)

Figure 4.5.3: (a) The g_{rr} and g_{tt} dependence on r. (b) The Gauss-Bonnet term in terms of r. We have choose $f(\phi) = \alpha e^{\phi}$.



(a) (b)

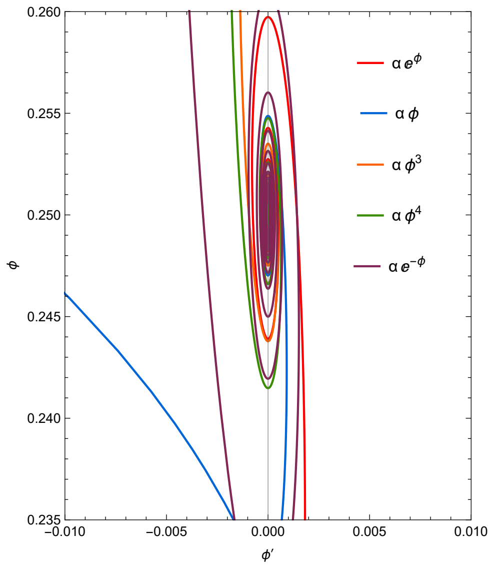
Figure 4.5.4: (a) The effective potential dependence on r. (b) The scalar potential in terms of r. We have choose $f(\phi) = \alpha e^{\phi}$.

that in the case $\Lambda < 0$ there are only three solutions which we have aforementioned.

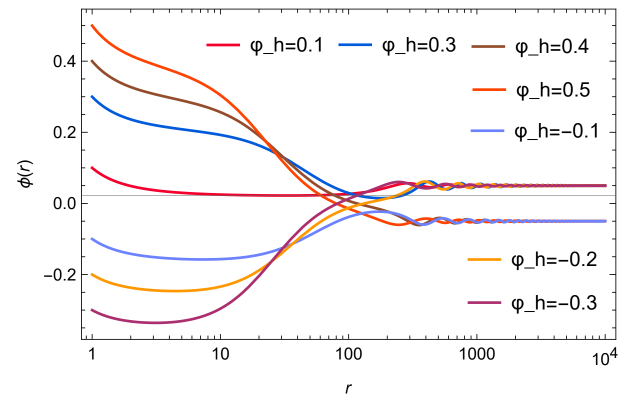
Chapter 5

Conclusions and Outlook

In this thesis we have studied black hole solutions in the Einstein-scalar-Gauss-Bonnet (EsGB) theoretical framework. Having introduced self-interactions through the scalar potential $V(\phi)$ we were inspired from Particle Physics and Cosmology and we have considered the forms of three basic potentials. However, the study is not closely related to particle physics and cosmology as we borrowed only the forms of the scalar potentials. Each potential was studied in two main cases, $\Lambda > 0$ and $\Lambda < 0$ signifying the total sign of the potential. In the presence of the well known Higgs, Coleman-Weinberg and Starobinsky potential we obtain black hole solutions for special forms of the coupling function $f(\phi)$ but, it was impossible to find solutions for inverse forms of $f(\phi)$, except the case where $f(\phi) = \alpha e^{-\phi}$. In the case of the Higgs potential with $\Lambda < 0$ the scalar field picks the vacuum expectation value (v.e.v) asymptotically resulting in asymptotically flat solutions. In the opposite case where $\Lambda > 0$ the scalar field oscillates around zero at large values of the radial coordinate r therefore, a de-Sitter black hole emerges. We have checked this behaviour for all forms of the coupling function $f(\phi)$ allowed by the theory, hence we claim that it is a general feature of the theory that deserves further study. We obtain the same behaviour for Starobinsky potential. In the case where $\Lambda < 0$ the scalar field takes the zero value asymptotically. As a result, we can construct asymptotically flat solutions. However, we cannot find any type of regular black hole for $\Lambda > 0$. In the case of Coleman-Weinberg potential the scalar field oscillates around the v.e.v when $\Lambda < 0$, and thus we find de-Sitter black holes, whereas it oscillates around zero for $\Lambda > 0$ thus, obtaining asymptotically flat solutions. We have numerically confirmed that for the Higgs and Coleman-Weinberg potentials for $\Lambda < 0$ many black hole solutions emerged compared to the case where $\Lambda > 0$. On the other hand, it seems difficult to obtain black hole solutions for the Starobinsky potential in the case with $\Lambda < 0$ because we did not find any smooth



(a)



(b)

Figure 5.0.1: (a) The attractor behaviour of the Higgs potential for $\Lambda = -1$, $\lambda = 0.01$, $v = 0.25$ and different choices for coupling function $f(\phi)$ while we have set $\alpha = 0.01$. (b) Solutions for the Higgs potential with $f(\phi) = \alpha e^{\phi}$, $\alpha = 0.01$, $\Lambda = -1$, $\lambda = 0.01$, $v = 0.05$ and different choices of the ϕ_h.

solutions except those presented in section 4.5.

According to Bakopoulos, Kanti and Pappas [47], the potentials that they studied had a minimum which was the trivial one and they obtained only asymptotically-flat solutions. In our work it is possible to obtain de-Sitter black holes because the scalar field takes asymptotically a non-trivial minimum. As a result there is asymptotically a constant energy remnant which acts as a cosmological constant. On the other hand, we did not find asymptotically AdS solutions, since the asymptotic behaviour of the system in the presence of the potentials we considered did not lead to a negative cosmological constant.

Let us now discuss the asymptotic behaviour of the scalar field. It is sufficiently confirmed that the scalar field asymptotically always takes values corresponding to an extremum of the potential. We believe that the scalar field chooses these values due to the attractor behaviour of the potentials seen in Fig.(5.0.1a). Furthermore, we observed that if we keep all parameters constant and change the value of the scalar field at the event horizon, then the choice of the extremum value at radial infinity is affected as shown in Fig.(5.0.1b). We need to study more deeply this behaviour in the future.

All the above solutions violate the traditional No-scalar-Hair theorem as we have shown numerically by the form of the energy-momentum tensor. Therefore we can easily claim that black holes are characterized by *scalar hair of second type*. The term second type refers to the fact that no new independent parameter can be found to characterize the black hole. The only new feature of the black hole compared to those of General Relativity is the non-trivial profile of the scalar field near the horizon. We found that the spacetime is regular in the whole area $[r_h, \infty)$ as it is confirmed by the results of numerical analysis for the form of energy-momentum tensor, metric and Gauss-Bonnet gravitational term.

In conclusion we can claim that by enriching the EsGB theory with sophisticated forms of scalar potential black hole solutions with interesting features emerge while it seems that the alternative forms of coupling functions are limited. For example, we can not find black hole solutions with inverse polynomial forms of the coupling function. Note that there are at least three common solutions for all of the potentials we have considered and these correspond to the coupling functions $f(\phi) = (\alpha e^{\phi}, \alpha\phi, \alpha\phi^2)$. Additionally, we found new de-Sitter black hole solutions and we hope to study them more in a future work. Finally, an interesting aspect of our solutions is their stability behaviour the study of which is still pending.

Appendix A

Equations of Motion of EsGB Theory

A.1 Variation with respect to the metric tensor

In this section we discuss the variation of the action of EsGB theory. According to variational principle we derive the covariant equations of motion. Consider the following action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + f(\phi) \mathcal{R}^2_{GB} \right) \tag{A.1.1}$$

where R is scalar Ricci, ϕ is a scalar field and $f(\phi)$ is a coupling function between the scalar field and $\mathcal{R}^2_{GB}$ is a quadratic curvature term . This is the so called "Gauss-Bonnet" gravitational term. As we will see later this term is a *topological invariant* because the integration on spacetime manifold has to be vanish. The variation of the action with respect to the metric tensor is,

$$\delta_g S = \frac{1}{16\pi} \int d^4x \overbrace{\delta(\sqrt{-g}R)}^{\text{A}} - \frac{1}{2} \overbrace{\delta(\sqrt{-g}\partial_\mu \phi \partial^\mu \phi)}^{\text{B}} + \overbrace{\delta(\sqrt{-g} f(\phi) \mathcal{R}^2_{GB})}^{\text{C}} \tag{A.1.2}$$

The term A gives the Einstein tensor $G_{\mu\nu}$, while the term B gives

$$
\begin{aligned}
B &= -\frac{1}{2}\delta(\sqrt{-g}\partial_\mu\phi\partial^\mu\phi) \\
&= \frac{1}{2}\delta(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) \\
&= -\frac{1}{2}\delta(\sqrt{-g})g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\
&= \frac{1}{4}g_{\mu\nu}(\partial_\rho\phi)^2\delta g^{\mu\nu} - \frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu} \\
&= \left[\frac{1}{4}g_{\mu\nu}(\partial_\rho\phi)^2 - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi\right]\sqrt{-g}\delta g^{\mu\nu}
\end{aligned}
\tag{A.1.3}
$$

Finally we get

$$
\boxed{B = \left[\frac{1}{4}g_{\mu\nu}(\partial_\rho\phi)^2 - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi\right]\sqrt{-g}\delta g^{\mu\nu}}
\tag{A.1.4}
$$

The Gauss-Bonnet term has the following form

$$
\mathcal{R}^2_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2.
\tag{A.1.5}
$$

Obviously this is a quadratic combination of Riemann tensor, Ricci tensor and Ricci scalar. We can write the Gauss-Bonnet term in the form of

$$
\mathcal{R}^2_{GB} = \frac{1}{4}R_{\lambda\sigma}{}^{\alpha\beta}R^{\lambda'\sigma'}{}_{\alpha'\beta'}\eta_{\lambda'\sigma'\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}
\tag{A.1.6}
$$

where $\eta^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$. We use the identity of Levi-Civita symbol

$$
\epsilon_{\lambda'\sigma'\alpha\beta}\epsilon^{\lambda\sigma\alpha'\beta'} = -\begin{vmatrix}
\delta^\lambda_{\lambda'} & \delta^\lambda_{\sigma'} & \delta^\lambda_\alpha & \delta^\lambda_\beta \\
\delta^\sigma_{\lambda'} & \delta^\sigma_{\sigma'} & \delta^\sigma_\alpha & \delta^\sigma_\beta \\
\delta^{\alpha'}_{\lambda'} & \delta^{\alpha'}_{\sigma'} & \delta^{\alpha'}_\alpha & \delta^{\alpha'}_\beta \\
\delta^{\beta'}_{\lambda'} & \delta^{\beta'}_{\sigma'} & \delta^{\beta'}_\alpha & \delta^{\beta'}_\beta
\end{vmatrix}
\tag{A.1.7}
$$

Then

$$
\mathcal{R}^2_{GB} = R_{\lambda\sigma}{}^{\alpha\beta}R^{\lambda'\sigma'}{}_{\alpha'\beta'}\begin{vmatrix}
\delta^\lambda_{\lambda'} & \delta^\lambda_{\sigma'} & \delta^\lambda_\alpha & \delta^\lambda_\beta \\
\delta^\sigma_{\lambda'} & \delta^\sigma_{\sigma'} & \delta^\sigma_\alpha & \delta^\sigma_\beta \\
\delta^{\alpha'}_{\lambda'} & \delta^{\alpha'}_{\sigma'} & \delta^{\alpha'}_\alpha & \delta^{\alpha'}_\beta \\
\delta^{\beta'}_{\lambda'} & \delta^{\beta'}_{\sigma'} & \delta^{\beta'}_\alpha & \delta^{\beta'}_\beta
\end{vmatrix} = 4(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2)
\tag{A.1.8}
$$

Going back to the C term we get

$$C = \frac{1}{4}\delta(\sqrt{-g}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta})$$
$$= \frac{1}{4}\delta(\sqrt{-g})R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} + \frac{1}{4}\sqrt{-g}(\delta R_{\lambda\sigma\alpha\beta})R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}$$
$$+ \frac{1}{4}\sqrt{-g}R_{\lambda'\sigma'\alpha'\beta'}(\delta R_{\lambda\sigma\alpha\beta})\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} + \frac{1}{4}\sqrt{-g}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}(\delta\eta^{\lambda\sigma\alpha'\beta'})\eta^{\lambda'\sigma'\alpha\beta}$$
$$+ \frac{1}{4}\sqrt{-g}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}(\delta\eta^{\lambda'\sigma'\alpha\beta}) \tag{A.1.9}$$

We redefine the indices in the above expression and we get the following result

$$C = \frac{1}{4}\delta(\sqrt{-g})R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} + \frac{1}{2}\sqrt{-g}R_{\lambda'\sigma'\alpha'\beta'}(\delta R_{\lambda\sigma\alpha\beta})\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}$$
$$+ \frac{1}{2}\sqrt{-g}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}(\delta\eta^{\lambda'\sigma'\alpha\beta}) \tag{A.1.10}$$

From the identity

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

the C goes to

$$C = -\frac{1}{8}\sqrt{-g}\delta g^{\mu\nu}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}$$
$$+ \underbrace{\frac{1}{2}[(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta} - (\delta\Gamma^{\mu}_{\sigma\beta})_{;\alpha}]g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}}_{1} + \frac{1}{2}\sqrt{-g}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}(\delta\eta^{\lambda'\sigma'\alpha\beta}) \tag{A.1.11}$$

and 1 goes

$$\frac{1}{2}[(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta} - (\delta\Gamma^{\mu}_{\sigma\beta})_{;\alpha}]g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}$$
$$= \frac{1}{2}\sqrt{-g}g_{\mu\lambda}(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} \tag{A.1.12}$$

$$C = -\frac{1}{8}\sqrt{-g}\delta g^{\mu\nu}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} + \frac{1}{2}\sqrt{-g}g_{\mu\lambda}(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}$$
$$+ \frac{1}{2}\sqrt{-g}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}(\delta\eta^{\lambda'\sigma'\alpha\beta}) \tag{A.1.13}$$

We need to calculate the variation of Levi-Civita symbol in curved spacetime so we get

$$\begin{aligned}
\delta\eta^{\lambda\sigma\alpha'\beta'} &= \delta\left(\frac{\epsilon^{\lambda\sigma\alpha'\beta'}}{\sqrt{-g}}\right) = \epsilon^{\lambda\sigma\alpha'\beta'}\delta\left(\frac{1}{\sqrt{-g}}\right) \\
&= \epsilon^{\lambda\sigma\alpha'\beta'}\delta((-g)^{1/2}) = \epsilon^{\lambda\sigma\alpha'\beta'}\left(-\frac{1}{2}(-g)^{-3/2}\delta g\right) \\
&= \epsilon^{\lambda\sigma\alpha'\beta'}\left(-\frac{1}{2}(-g)^{3/2}(-g g_{\mu\nu}\delta g^{\mu\nu})\right) \\
&= \frac{1}{2}\epsilon^{\lambda\sigma\alpha'\beta'}(-g)^{1/2}g_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{2}\frac{\epsilon^{\lambda\sigma\alpha'\beta'}}{\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{2}\eta^{\lambda\sigma\alpha'\beta'}g_{\mu\nu}\delta g^{\mu\nu}
\end{aligned} \tag{A.1.14}$$

Therefore

$$\boxed{\delta\eta^{\lambda\sigma\alpha'\beta'} = \frac{1}{2}\eta^{\lambda\sigma\alpha'\beta'}g_{\mu\nu}\delta g^{\mu\nu}} \tag{A.1.15}$$

Finally for the C quantity we have

$$\begin{aligned}
C = &-\frac{1}{8}\sqrt{-g}\delta g^{\mu\nu}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} + \frac{1}{2}\sqrt{-g}g_{\mu\lambda}(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta}R_{\lambda'\sigma'\alpha'\beta'}R_{\lambda\sigma\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} \\
&+ \frac{1}{4}\sqrt{-g}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\kappa\sigma\alpha\beta}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}\eta^{\lambda\sigma\alpha'\beta'}.
\end{aligned} \tag{A.1.16}$$

Now we use the very useful identity

$$\underbrace{\eta^{\lambda\sigma\alpha'\beta'}g^{\mu\kappa}}_{1}\underbrace{-\eta^{\kappa\sigma\alpha'\beta'}g^{\lambda\mu}}_{2}\underbrace{-\eta^{\lambda\kappa\alpha'\beta'}g^{\mu\sigma}}_{3}\underbrace{-\eta^{\lambda\sigma\kappa\beta'}g^{\mu\alpha}}_{4}\underbrace{+\eta^{\lambda\sigma\alpha'\kappa}g^{\mu\beta}}_{5} = 0. \tag{A.1.17}$$

We multiply this identity with $\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta}$ we get

$$\begin{aligned}
&\boxed{1} : \eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} \\
&\boxed{2} : \eta^{\kappa\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\lambda\mu}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} \\
&\boxed{3} : \eta^{\lambda\kappa\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\sigma}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} \\
&\boxed{4} : \eta^{\lambda\sigma\kappa\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\alpha}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} \\
&\boxed{5} : \eta^{\lambda\sigma\alpha'\kappa}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\beta'}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta}
\end{aligned} \tag{A.1.18}$$

From the above expressions we have

$$\begin{aligned}
\boxed{4}+\boxed{5} &= -\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\alpha'\sigma\alpha\beta}R_{\lambda'\sigma'\kappa\beta'} + \eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\alpha'\sigma\alpha\beta}R_{\lambda'\sigma'\kappa\beta'} = 0 \\
\boxed{1}+\boxed{3} &= \eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} - \eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\sigma\kappa\alpha\beta} \\
&= 2\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} \\
\boxed{1}+\boxed{2}+\boxed{3} &= 2\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} - \eta^{\kappa\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\lambda\mu}\delta g_{\lambda\mu}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} \\
&= 0.
\end{aligned} \tag{A.1.19}$$

We derive the following identity

$$\boxed{\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\kappa}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} = \frac{1}{2}\eta^{\kappa\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\lambda\mu}\delta g_{\lambda\mu}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta}} \quad \text{(A.1.20)}$$

and after the use of the above identity C becomes,

$$\begin{aligned} C = & -\frac{1}{8}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}R_{\lambda\sigma\alpha\beta}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} + \sqrt{-g}g_{\mu\lambda}(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} \\ & + \frac{1}{8}\sqrt{-g}\eta^{\kappa\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}g^{\mu\lambda}\delta g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}R_{\kappa\sigma\alpha\beta} \\ & = \sqrt{-g}g_{\mu\lambda}(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} \end{aligned} \quad \text{(A.1.21)}$$

Then,

$$\boxed{C = \sqrt{-g}g_{\mu\lambda}(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}} \quad \text{(A.1.22)}$$

Thanks to the following

$$(\delta\Gamma^{\mu}_{\sigma\alpha})_{;\beta}R_{\lambda'\sigma'\alpha'\beta'} = (\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'})_{;\beta} - (\delta\Gamma^{\mu}_{\sigma\alpha})(R_{\lambda'\sigma'\alpha'\beta'})_{;\beta} \quad \text{(A.1.23)}$$

the C quantity becomes

$$C = \sqrt{-g}g_{\mu\lambda}(\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'})_{;\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} - \sqrt{-g}g_{\mu\lambda}\delta\Gamma^{\mu}_{\sigma\alpha}(R_{\lambda'\sigma'\alpha'\beta'})_{;\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta} \quad \text{(A.1.24)}$$

In addition, because of the synergy of the Bianchi identity and Levi-Civita symbol the second term has to be exactly zero.

$$\begin{aligned} \eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\lambda'\sigma'})_{;\beta} + \eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\sigma'\beta})_{;\lambda'} + \eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\beta\lambda'})_{;\sigma'} &= 0 \\ \eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\lambda'\sigma'})_{;\beta} + \eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\lambda'\beta})_{;\sigma'} + \eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\beta\lambda'})_{;\sigma'} &= 0 \\ \eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\lambda'\sigma'})_{;\beta} + 2\eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\beta\lambda'})_{;\sigma'} &= 0 \\ \Rightarrow 3\eta^{\lambda'\sigma'\alpha\beta}(R_{\alpha'\beta'\lambda'\sigma'})_{;\beta} &= 0 \end{aligned} \quad \text{(A.1.25)}$$

Finally,

$$\boxed{C = \sqrt{-g}g_{\mu\lambda}(\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'})_{;\beta}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}} \quad \text{(A.1.26)}$$

Differentiating with covariant derivative the Levi-Civita symbol we get

$$
\begin{aligned}
(\eta^{\lambda\sigma\alpha'\beta'})_{;\beta} &= (\eta^{\lambda\sigma\alpha'\beta'})_{,\beta} + \Gamma^{\lambda}_{\rho\beta}\eta^{\rho\sigma\alpha'\beta'} + \Gamma^{\sigma}_{\rho\beta}\eta^{\lambda\rho\alpha'\beta'} + \Gamma^{\alpha'}_{\rho\beta}\eta^{\lambda\sigma\rho\beta'} + \Gamma^{\beta'}_{\rho\beta}\eta^{\lambda\sigma\alpha'\rho} \\
\eta^{\rho\sigma\alpha'\beta'}\Gamma^{\lambda}_{\rho\beta} &= \frac{1}{2}g^{\lambda\kappa}\eta^{\rho\sigma\alpha'\beta'}\left((g_{\beta\kappa})_{,\rho} + (g_{\rho\kappa})_{,\beta} - (g_{\rho\beta})_{,\kappa}\right) \\
&= \frac{1}{2}\eta^{\kappa\sigma\alpha'\beta'}g^{\lambda\rho}\left((g_{\rho\beta})_{,\kappa} + (g_{\kappa\rho})_{,\beta} - (g_{\kappa\beta})_{,\rho}\right) \qquad \boxed{1} \\
\Gamma^{\sigma}_{\rho\beta} &= \frac{1}{2}g^{\sigma\kappa}\left((g_{\kappa\beta})_{,\rho} + (g_{\sigma\kappa})_{,\beta} - (g_{\rho\beta})_{,\kappa}\right) \\
\eta^{\lambda\rho\alpha'\beta'}\Gamma^{\sigma}_{\rho\beta} &= \frac{1}{2}\eta^{\lambda\rho\alpha'\beta'}g^{\sigma\kappa}\left((g_{\kappa\beta})_{,\rho} + (g_{\sigma\kappa})_{,\beta} - (g_{\rho\beta})_{,\kappa}\right) \\
&= \frac{1}{2}\eta^{\lambda\kappa\alpha'\beta'}g^{\sigma\rho}\left((g_{\rho\beta})_{,\kappa} + (g_{\sigma\rho})_{,\beta} - (g_{\kappa\beta})_{,\rho}\right) \qquad \boxed{2} \\
\Gamma^{\alpha'}_{\rho\beta} &= \frac{1}{2}g^{\alpha'\kappa}((g_{\beta\kappa})_{,\rho} + (g_{\rho\kappa})_{,\beta} - (g_{\rho\beta})_{,\kappa}) \\
\eta^{\lambda\sigma\rho\beta'}\Gamma^{\alpha'}_{\rho\beta} &= \frac{1}{2}\eta^{\lambda\sigma\kappa\beta'}g^{\alpha'\rho}((g_{\beta\rho})_{,\kappa} + (g_{\kappa\rho})_{,\beta} - (g_{\kappa\beta})_{,\rho}) \qquad \boxed{3} \\
\Gamma^{\beta'}_{\rho\beta} &= \frac{1}{2}g^{\beta'\kappa}((g_{\kappa\beta})_{,\rho} + (g_{\kappa\rho})_{,\beta} - (g_{\rho\beta})_{,\kappa}) \\
\eta^{\lambda\sigma\alpha'\rho}\Gamma^{\beta'}_{\rho\beta} &= \frac{1}{2}\eta^{\lambda\sigma\alpha'\kappa}g^{\beta'\rho}((g_{\rho\beta})_{,\kappa} + (g_{\rho\kappa})_{,\beta} - (g_{\kappa\beta})_{,\rho}) \qquad \boxed{4}
\end{aligned}
\tag{A.1.27}
$$

Finally we have

$$
\begin{aligned}
(\eta^{\lambda\sigma\alpha'\beta'})_{;\beta} = (\eta^{\lambda\sigma\alpha'\beta'})_{,\beta} + ((g_{\rho\beta})_{,\kappa} + (g_{\rho\kappa})_{,\beta}) - (g_{\kappa\beta})_{,\rho}) \\
\frac{1}{2}\left[\eta^{\kappa\sigma\alpha'\beta'}g^{\lambda\rho} + \eta^{\lambda\kappa\alpha'\beta'}g^{\sigma\rho} + \eta^{\lambda\sigma\kappa\beta'}g^{\alpha'\rho} + \eta^{\lambda\sigma\alpha'\kappa}g^{\beta'\rho}\right]
\end{aligned}
\tag{A.1.28}
$$

We use the known identity

$$
\eta^{\lambda\sigma\alpha'\beta'}g^{\kappa\rho} - \eta^{\kappa\sigma\alpha'\beta'}g^{\lambda\rho} - \eta^{\lambda\kappa\alpha'\beta'}g^{\rho\sigma} - \eta^{\lambda\sigma\kappa\beta'}g^{\alpha'\rho} - \eta^{\lambda\sigma\alpha'\kappa}g^{\rho\beta} = 0
$$

and the covariant derivative on Levi-Civita symbol is simplified as

$$
(\eta^{\lambda\sigma\alpha'\beta'})_{;\beta} = (\eta^{\lambda\sigma\alpha'\beta'})_{,\beta} + \frac{1}{2}\eta^{\lambda\sigma\alpha'\beta'}g^{\kappa\rho}\left[(g_{\rho\beta})_{,\kappa} + (g_{\rho\kappa})_{,\beta} - (g_{\kappa\beta})_{,\rho}\right]
\tag{A.1.29}
$$

But

$$
\begin{aligned}
(\eta^{\lambda\sigma\alpha'\beta'})_{,\beta} &= \left(\frac{\epsilon^{\lambda\sigma\alpha'\beta'}}{\sqrt{-g}}\right)_{,\beta} = \epsilon^{\lambda\sigma\alpha'\beta'}\left(\frac{1}{\sqrt{-g}}\right)_{,\beta} \\
&= \epsilon^{\lambda\sigma\alpha'\beta'}((-g)^{-1/2})_{,\beta} = \epsilon^{\lambda\sigma\alpha'\beta'}\left(-\frac{1}{2}(-g)^{-3/2}(g)_{,\beta}\right)
\end{aligned}
\tag{A.1.30}
$$

We calculate the derivative on the square root of the metric tensor

$$\partial_\beta(\log\sqrt{-g}) = \frac{\partial_\beta\sqrt{-g}}{\sqrt{-g}} = \frac{1}{2}g^{\rho\kappa}\partial_\beta g_{\rho\kappa} = \partial_\beta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\rho\kappa}\partial_\beta g_{\rho\kappa} \Rightarrow \\ \partial_\beta g = g g^{\rho\kappa}\partial_\beta g_{\rho\kappa} \tag{A.1.31}$$

Then we get

$$\boxed{\partial_\beta g = g g^{\rho\kappa}\partial_\beta g_{\rho\kappa}} \tag{A.1.32}$$

$$(\eta^{\lambda\sigma\alpha'\beta'})_{,\beta} = -\frac{1}{2}\eta^{\lambda\sigma\alpha'\beta'}g^{\rho\kappa}(g_{\rho\kappa})_{,\beta} \tag{A.1.33}$$

After some trivial calculations we have

$$(\eta^{\lambda\sigma\alpha'\beta'})_{;\beta} = \left[\frac{1}{2}\eta^{\lambda\sigma\alpha'\beta'}g^{\kappa\rho}(g_{\rho\beta})_{,\kappa} - \frac{1}{2}\underbrace{\eta^{\lambda\sigma\alpha'\beta'}g^{\kappa\rho}(g_{\kappa\beta})_{,\rho}}_{\rho\leftrightarrow\kappa}\right] = 0 \tag{A.1.34}$$

Finally, the quantity C is

$$\begin{aligned} C &= \sqrt{-g}(g_{\mu\lambda}\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta})_{;\beta} \\ &= \frac{\partial}{\partial x^\beta}\left(\sqrt{-g}g_{\mu\lambda}\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}\right) \end{aligned} \tag{A.1.35}$$

Note here that the C after integration on spacetime has to be vanish

$$\delta S_{GB} = \int d^4x\,\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}g_{\mu\lambda}\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}\right) = 0 \tag{A.1.36}$$

This is the so called *Gauss-Bonnet* theorem. Now consider a coupling function $f(\phi)$ which multiplies the Gauss-Bonnet term. Then we get

$$\begin{aligned} \delta S_{GB} &= \int d^4x\sqrt{-g}f(\phi)C \\ &= \int d^4x\, f(\phi)\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}g_{\mu\lambda}\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}\right) \\ &= \int d^4x\,\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}g_{\mu\lambda}\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}f(\phi)\right) \\ &- \int d^4x\,\left[\sqrt{-g}g_{\mu\lambda}\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}(\nabla_\beta f(\phi))\right] \\ &= \int d^4x\,\left(-\sqrt{-g}g_{\mu\lambda}\delta\Gamma^{\mu}_{\sigma\alpha}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}(\nabla_\beta f(\phi))\right) \end{aligned} \tag{A.1.37}$$

Then we will prove the useful identity

$$\boxed{\delta\Gamma^{\mu}_{\sigma\alpha} = \frac{1}{2}g^{\mu\rho}\left[(\delta g_{\sigma\rho})_{;\alpha} + (\delta g_{\alpha\rho})_{;\sigma} - (\delta g_{\sigma\alpha})_{;\rho}\right]}$$

The proof goes as follows

$$\begin{aligned}
&\bullet(\delta g_{\sigma\rho})_{;\alpha} = (\delta g_{\sigma\rho})_{,\alpha} - \Gamma^{\kappa}_{\sigma\alpha}\delta g_{\kappa\rho} - \Gamma^{\kappa}_{\rho\alpha}\delta g_{\kappa\sigma} \qquad \boxed{+}\\
&\bullet(\delta g_{\alpha\rho})_{;\sigma} = (\delta g_{\alpha\rho})_{,\sigma} - \Gamma^{\kappa}_{\sigma\alpha}\delta g_{\kappa\rho} - \Gamma^{\kappa}_{\sigma\rho}\delta g_{\kappa\alpha} \qquad \boxed{+}\\
&\bullet(\delta g_{\sigma\alpha})_{;\rho} = (\delta g_{\alpha\sigma})_{,\rho} - \Gamma^{\kappa}_{\rho\sigma}\delta g_{\kappa\alpha} - \Gamma^{\kappa}_{\rho\alpha}\delta g_{\kappa\sigma} \qquad \boxed{-}
\end{aligned} \tag{A.1.38}$$

Then

$$\begin{aligned}
&(\delta g_{\sigma\alpha})_{;\rho} + (\delta g_{\alpha\rho})_{;\sigma} - (\delta g_{\sigma\alpha})_{;\rho}\\
&= (\delta g_{\sigma\rho})_{,\alpha}) + (\delta g_{\alpha\rho})_{,\sigma} - (\delta g_{\alpha\sigma})_{,\rho} - \Gamma^{\kappa}_{\sigma\alpha}\delta g_{\kappa\rho} - \Gamma^{\kappa}_{\rho\alpha}\delta g_{\kappa\sigma} - \Gamma^{\kappa}_{\sigma\alpha}\delta g_{\kappa\rho}\\
&- \Gamma^{\kappa}_{\sigma\rho}\delta g_{\kappa\alpha} + \Gamma^{\kappa}_{\rho\sigma}\delta g_{\kappa\alpha} + \Gamma^{\kappa}_{\rho\alpha}\delta g_{\kappa\sigma}\\
&= (\delta g_{\sigma\rho})_{,\alpha} + (\delta g_{\alpha\rho})_{,\sigma} - (\delta g_{\alpha\sigma})_{,\rho} - 2\Gamma^{\kappa}_{\sigma\alpha}\delta g_{\kappa\rho}
\end{aligned} \tag{A.1.39}$$

Therefore the variation of the Christofell symbols with respect to the metric is

$$\begin{aligned}
&\frac{1}{2}g^{\mu\rho}\left[(\delta g_{\sigma\rho})_{,\alpha} + (\delta g_{\alpha\rho})_{,\sigma} - (\delta g_{\alpha\sigma})_{,\rho}\right] - g^{\mu\rho}\Gamma^{\kappa}_{\sigma\alpha}\delta g_{\kappa\rho}\\
&= -\delta g_{\kappa\rho}g^{\mu\rho}\Gamma^{\kappa}_{\sigma\alpha} + \frac{1}{2}g^{\mu\rho}\left[(\delta g_{\sigma\rho})_{,\alpha} + (\delta g_{\alpha\rho})_{,\sigma} - (\delta g_{\alpha\sigma})_{,\rho}\right] = \delta\Gamma^{\mu}_{\sigma\alpha}
\end{aligned} \tag{A.1.40}$$

The variation of the action takes the following form

$$\begin{aligned}
\delta S_{GB} = &\int d^4x\ -\frac{\sqrt{-g}}{2}g^{\mu\rho}\left[(\delta g_{\sigma\rho})_{;\alpha} + (\delta g_{\alpha\rho})_{;\sigma} - (\delta g_{\sigma\alpha})_{;\alpha}\right]\\
&g_{\mu\lambda}R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_{\beta}f(\phi)\\
&= \int d^4x - \frac{\sqrt{-g}}{2}\left[(\delta g_{\sigma\lambda})_{;\alpha} + (\delta g_{\alpha\lambda})_{;\sigma} - (\delta g_{\sigma\alpha})_{;\lambda}\right]R_{\lambda'\sigma'\alpha'\beta'}\eta^{\lambda\sigma\alpha'\beta'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_{\beta}f(\phi)
\end{aligned} \tag{A.1.41}$$

We define $\tilde{R}^{\mu\nu}{}_{\kappa\lambda} = \eta^{\mu\nu\rho\sigma}R_{\rho\sigma\kappa\lambda}$ therefore

$$\delta S_{GB} = \int d^4x - \frac{\sqrt{-g}}{2}\left[(\delta g_{\sigma\lambda})_{;\alpha} + (\delta g_{\alpha\lambda})_{;\sigma} - (\delta g_{\sigma\alpha})_{;\lambda}\right]\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_{\beta}f(\phi). \tag{A.1.42}$$

Now we integrate by parts

$$
\begin{aligned}
&\bullet \int d^4x\ -\frac{\sqrt{-g}}{2}(\delta g_{\sigma\lambda})_{;\alpha}\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\\
&= \int d^4x\ -\frac{\sqrt{-g}}{2}\left[(\delta g_{\sigma\lambda})\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\alpha}\\
&+ \int d^4x\ \frac{\sqrt{-g}}{2}\delta g_{\sigma\lambda}\left[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\alpha}\\
&\bullet \int d^4x\ -\frac{\sqrt{-g}}{2}(\delta g_{\alpha\lambda})_{;\sigma}\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\\
&= \int d^4x\ -\frac{\sqrt{-g}}{2}\left[\delta g_{\alpha\lambda}\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\sigma}\\
&+ \int d^4x\frac{\sqrt{-g}}{2}\delta g_{\alpha\lambda}\left[\tilde{R}^{\lambda\sigma}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\sigma}\\
&\bullet \int d^4x\ \frac{\sqrt{-g}}{2}\left[(\delta g_{\sigma\alpha})_{;\lambda}\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\nabla_\beta f(\phi)\right]\\
&= \int d^4x\ \frac{\sqrt{-g}}{2}\left[(\delta g_{\sigma\alpha})\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\lambda}\\
&- \int d^4x\ \delta g_{\sigma\alpha}\left[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\lambda}
\end{aligned}
\tag{A.1.43}
$$

Finally we extract the following

$$
\begin{aligned}
\delta S_{GB} = \int d^4x\ \frac{\sqrt{-g}}{2}\Big\{&\delta g_{\sigma\lambda}\left[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\alpha}\\
&+\delta g_{\alpha\lambda}\left[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\sigma} - \delta g_{\sigma\alpha}\left[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)\right]_{;\lambda}\Big\} \Rightarrow
\end{aligned}
\tag{A.1.44}
$$

$$
\delta S_{GB} = \int d^4x\ \frac{\sqrt{-g}}{2}\Big\{\delta g_{\alpha\lambda}[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)]_{;\sigma} - \delta g_{\sigma\alpha}[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)]_{;\lambda}\Big\}
\tag{A.1.45}
$$

Thanks to the identity $\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}$ the quantity δS_{GB} takes the form,

$$
\begin{aligned}
\delta S_{GB} = \int d^4x\ \frac{\sqrt{-g}}{2}\Big\{&-g_{\alpha\mu}g_{\lambda\nu}\delta g^{\mu\nu}[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)]_{;\sigma}\\
&+g_{\sigma\mu}g_{\alpha\nu}\delta g^{\mu\nu}[\tilde{R}^{\lambda\sigma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)]_{;\lambda}\Big\}
\end{aligned}
\tag{A.1.46}
$$

We exchange the indices λ and σ and we get

$$\begin{aligned}
\delta S_{GB} &= \int d^4x\ \frac{\sqrt{-g}}{2}\Big\{ - g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}[\tilde{R}^{\rho\gamma}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)]_{;\gamma} \\
&+ g_{\rho\mu}g_{\alpha\nu}\delta g^{\mu\nu}[\tilde{R}^{\gamma\rho}{}_{\lambda'\sigma'}\eta^{\lambda'\sigma'\alpha\beta}\nabla_\beta f(\phi)]_{;\sigma}\Big\} \\
&= \int d^4x\ \frac{\sqrt{-g}}{2}\Big\{ g_{\lambda\mu}g_{\rho\nu}\delta g^{\mu\nu}[\tilde{R}^{\gamma\rho}{}_{\alpha\beta}\eta^{\alpha\beta\lambda\kappa}\nabla_\kappa f(\phi)]_{;\gamma} \\
&+ g_{\rho\mu}g_{\lambda\nu}\delta g^{\mu\nu}[\tilde{R}^{\gamma\rho}{}_{\alpha\beta}\eta^{\alpha\beta\lambda\kappa}\nabla_\kappa f(\phi)]_{;\gamma}\Big\} \\
&= \int d^4x\ \frac{\sqrt{-g}}{2}\Big\{ (g_{\lambda\mu}g_{\rho\nu} + g_{\rho\mu}g_{\lambda\nu})\delta g^{\mu\nu}[\tilde{R}^{\gamma\rho}{}_{\alpha\beta}\eta^{\alpha\beta\lambda\kappa}\nabla_\kappa f(\phi)]_{;\gamma}\Big\} \\
&= \int d^4x\ \frac{\sqrt{-g}}{2}\Big\{ (g_{\lambda\mu}g_{\rho\nu} + g_{\rho\mu}g_{\lambda\nu})[\tilde{R}^{\rho\gamma}{}_{\alpha\beta}\eta^{\kappa\lambda\alpha\beta}\nabla_\kappa f(\phi)]_{;\gamma}\ \delta g^{\mu\nu}\Big\}
\end{aligned} \tag{A.1.47}$$

Now, we go back to the eq.(A.1.1),

$$\begin{aligned}
\delta_g S_{GB} &= \int d^4x\ \sqrt{-g}\Big\{ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{4}g_{\mu\nu}(\partial_\rho\phi)^2 - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi \\
&\quad + \frac{1}{2}(g_{\lambda\mu}g_{\rho\nu} + g_{\rho\mu}g_{\lambda\nu})[\tilde{R}^{\rho\gamma}{}_{\alpha\beta}\eta^{\kappa\lambda\alpha\beta}\nabla_\kappa f(\phi)]_{;\gamma}\Big\}\delta g^{\mu\nu} \quad \forall \delta g^{\mu\nu}
\end{aligned} \tag{A.1.48}$$

The equations of motion are given by the condition $\delta S = 0$,

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}(\partial_\rho\phi)^2 - \frac{1}{2}(g_{\lambda\mu}g_{\rho\nu} + g_{\rho\mu}g_{\lambda\nu})\nabla_\gamma[\tilde{R}^{\rho\gamma}{}_{\alpha\beta}\eta^{\kappa\lambda\alpha\beta}\nabla_\kappa f(\phi)]} \tag{A.1.49}$$

A.1.1 Variation with respect to scalar field ϕ

In this subsection we derive the equation of motion of the scalar field. Consider again the action,

$$S_{GB} = \frac{1}{16\pi}\int d^4x\ \sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + f(\phi)\mathcal{R}^2_{GB}\right) \tag{A.1.50}$$

The variation of the above action with respect to the scalar field ϕ is

$$\begin{aligned}
\delta_\phi S_{GB} &= \frac{1}{16\pi}\int d^4x\ \sqrt{-g}\left(-\frac{1}{2}\nabla_\mu(\delta\phi)\nabla^\mu\phi - \frac{1}{2}\nabla_\mu\phi\nabla^\mu(\delta\phi) + \frac{df(\phi)}{d\phi}\mathcal{R}^2_{GB}\delta\phi\right) \\
&= \frac{1}{16\pi}\int d^4x\ \sqrt{-g}\left(-\nabla^\mu\phi\nabla_\mu(\delta\phi) + \frac{df(\phi)}{d\phi}\mathcal{R}^2_{GB}\delta\phi\right)
\end{aligned} \tag{A.1.51}$$

We use integration by parts

$$\bullet \int d^4x \, \sqrt{-g} \left(-\nabla^\mu \phi \nabla_\mu (\delta\phi)\right) = -\underbrace{\int d^4x \sqrt{-g} \nabla_\mu (\nabla^\mu \phi \delta\phi)}_{\to 0} + \int d^4x \sqrt{-g} \nabla_\mu \nabla^\mu \phi \delta\phi \tag{A.1.52}$$

Therefore we get

$$\delta_\phi S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[\nabla_\mu \nabla^\mu \phi + \frac{df(\phi)}{d\phi} \mathcal{R}^2_{GB}\Big] \delta\phi \quad \forall \delta\phi \tag{A.1.53}$$

The above equation is true for all $\delta\phi$ so we demand that the integrated quantity vanishes. The result is the equation of motion of the scalar field, namely

$$\nabla_\mu \nabla^\mu \phi + \dot{f}(\phi) \mathcal{R}^2_{GB} = 0, \tag{A.1.54}$$

where $\dot{f}(\phi) \equiv \frac{df(\phi)}{d\phi}$.

Appendix B

Perturbative equations of EsGB theory with scalar potential

In order to obtain the the perturbative equations of motion we varied the non-perturbative equations of the time dependent theory as

$$\begin{aligned} A(r) &= A(r) + \delta A(r) \\ B(r) &= B(r) + \delta B(r) \\ \phi(r) &= \phi(r) + \delta\phi(r) \end{aligned} \tag{B.0.1}$$

where the δ refers to the perturbation. Next, we evaluate the perturbative equations in Schwarzschild geometry therefore we have the following set of equations

$$\begin{aligned} 4r^2\delta B + 4r^3\left(1-\frac{2GM}{r}\right)\delta B' &= 32GMr\left(1-\frac{2GM}{r}\right)\frac{df}{d\phi}\delta\phi'' \\ &+ 32GM\left(\frac{3GM}{r}-1\right)\frac{df}{d\phi}\delta\phi' + 8r^4\Lambda\delta B + 4r^4\Lambda\frac{dV}{d\phi}\delta\phi \end{aligned} \tag{B.0.2}$$

$$\begin{aligned} 4r^2\left(1-\frac{2GM}{r}\right)\delta B - 4r^3\delta A' &= 32GMr\frac{df}{d\phi}\delta\ddot{\phi} + 32GM\left(1-\frac{2GM}{r}\right)\left(\frac{6GM}{r}-1\right)\frac{df}{d\phi}\delta\phi' \\ &+ 8r^4\Lambda V\left(1-\frac{2GM}{r}\right)^2\delta B + 4r^4\Lambda\left(1-\frac{2GM}{r}\right)\frac{dV}{d\phi}\delta\phi \end{aligned} \tag{B.0.3}$$

$$2r^4\left(1-\frac{2GM}{r}\right)^2\delta\ddot{B} = \left(1-\frac{2GM}{r}\right)\left[-8G^2M^2\frac{df}{d\phi}\delta\dot{\phi} - 8GMr^2\left(1-\frac{2GM}{r}\right)\frac{df}{d\phi}\delta\dot{\phi}'\right] \tag{B.0.4}$$

$$\left(1-\frac{2GM}{r}\right)\left[2\delta A'\left(1-\frac{2GM}{r}\right)+2\delta B'\left(\frac{GM}{r}-1\right)+2r\left(1-\frac{2GM}{r}\right)\delta A''\right]$$
$$-2r\delta\ddot{B}=\frac{16GM}{r^2}\frac{df}{d\phi}\delta\ddot{\phi}\left(1-\frac{2GM}{r}\right)^2+4\frac{df}{d\phi}\left(1-\frac{2GM}{r}\right)\left[\left(\frac{8G^2M^2}{r^4}-\frac{4GM}{r^3}\left(1-\frac{2GM}{r}\right)\right)\delta\phi'\right.$$
$$\left.+\frac{4GM}{r^2}\left(1-\frac{2GM}{r}\right)\delta\phi''\right]-8r\left(1-\frac{2GM}{r}\right)\Lambda V\delta B-4\Lambda\frac{dV}{d\phi}\left(1-\frac{2GM}{r}\right)\delta\phi \tag{B.0.5}$$

The scalar equation is given by

$$2r\left(1-\frac{2GM}{r}\right)^2\delta\phi''+4\left(1-\frac{2GM}{r}\right)\left(1-\frac{GM}{r}\right)\delta\phi'-2r\delta\ddot{\phi}+192\frac{G^2M^2}{r^7}\left(1-\frac{2GM}{r}\right)^3\times$$
$$\left[\frac{d^2f}{d\phi^2}\delta\phi-\frac{df}{d\phi}\delta B\right]+\frac{4}{r}\frac{df}{d\phi}\left[-\frac{4G^2M^2}{r^4}\delta B+\frac{2GM}{r^2}\left(1-\frac{2GM}{r^2}\right)\left(1+\frac{3GM}{r}\right)\delta A'\right.$$
$$-\frac{2GM}{r^2}\left(1-\frac{2GM}{r}\right)\left(1+\frac{3GM}{r}\right)\delta B'+\left(\frac{4GM}{r^3}-\frac{4G^2M^2}{r^4}\right)\delta B-\frac{2GM}{r}\left[2\left(1-\frac{2GM}{r}\right)^2\delta A''\right.$$
$$\left.\left.+\frac{4GM}{r^2}\left(1-\frac{2GM}{r}\right)\delta A'\right]+\frac{4GM}{r}\left(1-\frac{2GM}{r}\right)^2\delta\ddot{B}\right]-4r\left(1-\frac{2GM}{r}\right)^2\Lambda\frac{d^2V}{d\phi^2}\delta\phi=0 \tag{B.0.6}$$

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