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Non Supersymmetric Unified Models From String Theory

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Μη Υπερσυμμετρικά Ενοποιημένα Πρότυπα από τη Θεωρία Χορδών

Κωνσταντίνος Βιολάρης-Γκουντώνης

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Dedicated to my family

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"The path to progress is built on selfless acts.
Through human interaction, through help and understanding,
through knowledge that brings wisdom shared as common truth,
to guide our path through dark, inhuman times."

Abstract

The subject of this doctoral dissertation is the construction and phenomenological analysis of four-dimensional non supersymmetric unified models in the framework of heterotic string theory. These models are analysed using a dual approach, in which we employ both the free fermionic formulation as well as the equivalent description in terms of orbifold compactifications in order to identify models whose low energy phenomenology is consistent with a set of constraints. These include the absence of physical tachyons, as well as the presence of chiral matter and scalar Higgs bosons needed to induce the required gauge symmetry breaking.

In this context, supersymmetry is spontaneously broken via a stringy realisation of the Scherk–Schwarz mechanism, implemented as a freely-acting $\mathbb{Z}_2$ orbifold which includes an order-two shift along a compact dimension coupled to the spacetime fermion parity. When supplemented with sufficient super no-scale conditions, these models may exhibit a low scale of supersymmetry breaking of the order of a few TeV, with a simultaneous exponential suppression of the cosmological constant in the large volume limit.

We present a detailed analysis of a class of models with a Pati–Salam gauge symmetry, in which the worldsheet fermions parametrising the compactified dimensions are complex. We exhaustively scan the parameter space of possible models, subjecting them to the aforementioned phenomenological conditions, and explicitly calculate their one-loop effective potential using numerical methods. We then provide a classification for those that satisfy all conditions based on their partition function and one-loop effective potential. We present some specific models as examples. These include models with a positive effective potential, as well as a model in which gauge symmetry enhancement can be utilised to confine all fractionally charged exotic fermions.

We then proceed with the analysis of a more realistic class of Pati–Salam vacua, in which we consider all possible shifts in the internal directions, parametrised by real worldsheet fermions, which allows for the construction of models with three generations of chiral matter. We perform a random two-stage scan over a large part of the parameter space of models, identifying those with semi-realistic phenomenology and classifying them based on their effective potential. In the analysis of the one-loop effective potential of these models, we show that the Bose–Fermi degeneracy in the massless spectrum employed in the literature as the super no-scale condition is necessary but not sufficient for the exponential suppression of the cosmological constant. We present a three-generation model in which explicit calculations of the effective potential highlight the power-law behaviour in the large volume regime.

We then provide a second condition to be imposed in parallel with massless Bose–Fermi degeneracy, which we interpret as an additional degeneracy of massive bosons and fermions that become asymptotically massless in the large volume regime. We provide an explicit example of a three generation model with a positive semi-definite potential exhibiting the desired exponential suppression.

Εκτεταμένη Περίληψη

Η σύγχρονη φυσική έχει πραγματοποιήσει αλματώδη βήματα προόδου, τόσο στο θεωρητικό, όσο και στο πειραματικό επίπεδο, με αποτέλεσμα την κατασκευή θεωριών οι οποίες μπορούν να περιγράψουν με μεγάλη ακρίβεια τα φυσικά φαινόμενα από την υποατομική κλίμακα, ως τα όρια του παρατηρήσιμου σύμπαντος. Μέσω της Κβαντικής Θεωρίας Πεδίου και της Γενικής Θεωρίας της Σχετικότητας μπορούμε να μελετήσουμε τις τέσσερις θεμελιώδεις δυνάμεις που διέπουν το σύμπαν.

Το Καθιερωμένο Πρότυπο αποτελεί την ακριβέστερη πειραματικά επιβεβαιωμένη θεωρία για την περιγραφή των χαρακτηριστικών και των αλληλεπιδράσεων των θεμελιωδών σωματιδίων στο υποατομικό επίπεδο κάτω από την Ισχυρή, Ασθενή και Ηλεκτρομαγνητική Δύναμη. Με την ανακάλυψη του μποζονίου Higgs στον Μεγάλο Επιταχυντή Αδρονίων (LHC) του CERN επιβεβαιώθηκε και η τελευταία εναπομένουσα πρόβλεψή του σχετικά με το αυθόρμητο σπάσιμο της Ηλεκτρασθενούς συμμετρίας και την προέλευση της μάζας των φερμιονίων. Παράλληλα, η πρόσφατη ανακάλυψη των βαρυτικών κυμάτων σε πλήρη συμφωνία με τις θεωρητικές προβλέψεις, επιβεβαίωσε για άλλη μια φορά πως η Γενική Θεωρία της Σχετικότητας είναι κατάλληλη για την ακριβή περιγραφή των βαρυτικών αλληλεπιδράσεων σε μεγάλες κλίμακες.

Παρά τις επιτυχίες τους όμως, οι δύο αυτές θεωρίες παραμένουν ασυμβίβαστες. Η Γενική Θεωρία της Σχετικότητας αποτελεί μια Κλασική Θεωρία Πεδίου, η κβάντωση της οποίας κατά τις συνήθεις μεθόδους απαιτεί την εισαγγή ενός άπειρου αριθμού όρων για την εξάλειψη των απειρισμών στις κβαντικές διορθώσεις, καθιστώντας την μη επανακανονικοποιήσιμη. Το Καθιερωμένο Πρότυπο αγνοεί τις βαρυτικές αλληλεπιδράσεις, καθώς αυτές αναμένονται να παίξουν σημαντικό ρόλο σε πολύ υψηλότερες ενέργειες από ότι είναι διαθέσιμες πειραματικά, κοντά στη κλίμακα Planck. Η ασυμβατότητα μεταξύ των δύο αυτών θεωριών και η ανάγκη εύρεσης ενός νέου θεωρητικού πλαισίου υπό το οποίο θα μπορούσαν να ενοποιηθούν γίνεται εμφανής από το «πρόβλημα της Κοσμολογικής Σταθεράς»: Ο υπολογισμός της ενέργειας του κενού, στην οποία οφείλεται η επιταχυνόμενη διαστολή του σύμπαντος, με τις μεθόδους της Κβαντικής Θεωρίας Πεδίου, οδηγεί σε αποτελέσματα που ξεπερνούν τις παρατηρήσεις κατά πολλές τάξεις μεγέθους.

Το Καθιερωμένο Πρότυπο αφήνει μια σειρά από σημαντικά ερωτήματα αναπάντητα, γεγονός που υποδεικνύει την ανάγκη επέκτασης του. Καμία εξήγηση για την προέλευση της Σκοτεινής Ύλης, η ύπαρξη της οποίας είναι αναγκαία για την εξήγηση των καμπυλών περιστροφής των γαλαξιών και των φαινομένων βαρυτικών φακών (gravitational lensing) δεν περιλαμβάνεται στο Καθιερωμένο Πρότυπο. Σημαντικά ερωτήματα επίσης εγείρει η μεγάλη διαφορά μεταξύ της κλίμακας Planck και της κλίμακας στην οποία σπάει η ηλεκτρασθενής συμμετρία, το επονομαζόμενο «πρόβλημα της ιεραρχίας». Επιπρόσθετα, η εξέλιξη των σταθερών σύζευξης των αλληλεπιδράσεων βαθμίδας υποδηλώνει την πιθανότητα ύπαρξης μιας ενεργειακής κλίμακας στην οποία οι σταθερές ενοποιούνται στα πλαίσια μιας ανώτερης συμμετρίας βαθμίδας. Η παρατήρηση ταλαντώσεων νετρίνων αποτελεί σαφή πειραματική απόδειξη πως το Καθιερωμένο Πρότυπο δεν προσφέρει μια πλήρη εικόνα των σωματιδιακών αλληλεπιδράσεων, καθώς συνεπάγεται μη μηδενικές μάζες για τα νετρίνα, σε αντίθεση με τη θεωρητική πρόβλεψη.

Η ενοποίηση της Ασθενούς και της Ηλεκτρομαγνητικής αλληλεπίδρασης στο πλαίσιο του Καθιερωμένου Προτύπου, η οποία παραβιάζεται σε χαμηλές ενέργειες, έδωσε το κίνητρο για την αναζήτηση επεκτάσεών του οι οποίες σε μεγάλες ενεργειακές κλίμακες της τάξης των 10^{16} GeV διέπονται από διερυμένες θεωρίες βαθμίδας. Σε ορισμένες περιπτώσεις, οι συμμετρίες αυτές μπορούν να οδηγήσουν σε ενοποίηση της Ισχυρής και της Ηλεκτρασθενούς αλληλεπίδρασης στα πλαίσια μιας Μεγαλοενοποιημένης Θεωρίας (Grand Unified Theory, GUT). Οι θεωρίες αυτές μπορούν να παρέχουν απαντήσεις σε ανοικτά ερωτήματα του Καθιερωμένου Προτύπου, όπως η κβάντωση του ηλεκτρικού φορτίου. Παρότι αποτελεί βασικό στόχο σύγχρονων πειραμάτων, μέχρι σήμερα δεν έχουμε ισχυρές ενδείξεις για την ύπαρξη «Φυσικής πέρα από το Καθιερωμένο Πρότυπο».

Ως μια πιθανή λύση του προβλήματος της ιεραρχίας έχει προταθεί η υπερσυμμετρία (supersymmetry, SUSY), η οποία αποτελεί επέκταση της συμμετρίας Poincaré με την εισαγωγή φερμιονικών βαθμών ελευθερίας. Η υπερσυμμετρία προβλέπει τη ύπαρξη ζευγών μποζονίων-φερμιονίων τα οποία εμφανίζουν τους ίδιους κβαντικούς αριθμούς. Οι συνεισφορές τους σε κβαντικές διορθώσεις βρόχων αλληλοαναιρούνται, εξασφαλίζοντας την ευστάθεια της θεωρίας χωρίς την ανάγκη λεπτομερούς ρύθμισης των παραμέτρων. Δεδομένης όμως της απουσίας της επιβεβαίωσής της στα σύγχρονα πειράματα, εάν η υπερσυμμετρία είναι πράγματι μια συμμετρία του σύμπαντος, θα πρέπει να είναι μια σπασμένη συμμετρία. Εάν η κλίμακα του σπασίματος της υπερσυμμετρίας είναι της τάξης των μερικών TeV, τότε μπορεί επίσης να οδηγήσει στην ενοποίηση των σταθερών σύζευξης των αλληλεπιδράσεων βαθμίδας. Ο ακριβής μηχανισμός μέσω του οποίου θα μπορούσε να επιτευχθεί κάτι τέτοιο αποτελεί σηματικό πεδίο έρευνας. Ο μηχανισμός Scherk–Schwarz αποτελεί μια ελκυστική επιλογή καθώς επιτυγχάνει το αυθόρμητο σπάσιμο της υπερσυμμετρίας εκμεταλλευόμενος την ύπαρξη επιπρόσθετων διαστάσεων. Στα πρότυπα τα οποία προκύπτουν, η κλίμακα σπασίματος της υπερσυμμετρίας σχετίζεται με μια ακτίνα του εσωτερικού χώρου.

Η Θεωρία Χορδών παρέχει ένα ενοποιημένο πλαίσιο για τη συμπλήρωση του Καθιερωμένου Προτύπου σε υψηλότερες ενέργειες, συμπεριλαμβάνοντας παράλληλα και τη βαρύτητα. Βασίζεται στη υπόθεση ότι το θεμελιώδες συστατικό του σύμπαντος δεν είναι κάποιο σωματίδιο, αλλά ένα εκτεταμένο μονοδιάστατο αντικείμενο, η χορδή, οι διάφοροι επιτρεπόμενοι τρόποι ταλάντωσης της οποίας γεννούν τα παρατηρήσιμα σωματίδια και αλληλεπιδράσεις. Η εκτεταμένη φύση της χορδής παρέχει ένα φυσικό κατώφλι που εμποδίζει τις αποκλίσεις στο υπεριώδες. Πέρα από την ύπαρξη ενός άμαζου βαρυτονίου (graviton) στο φάσμα της, παρέχοντας το πλαίσιο για τη μελέτη της βαρύτητας στο κβαντικό επίπεδο, η θεωρία χορδών επιδεικνύει ελκυστικά χαρακτηριστικά καθώς μπορεί να συμπεριλάβει την υπερσυμμετρία, ενώ η ανάλυση των ετερωτικών χορδών οδηγεί σε θεωρίες με εκτεταμένη θεωρία βαθμίδας, οι οποίες μπορούν να συμπεριλάβουν το Καθιερωμένο Πρότυπο ή γνωστές Μεγαλοενοποιημένες Θεωρίες.

Η μαθηματική συνέπεια της θεωρίας χορδών προϋποθέτει την ύπαρξη επιπρόσθετων διαστάσεων, πέρα των τεσσάρων που έχουμε παρατηρήσει, οι οποίες θεωρούνται μικρές και συμπαγοποιημένες, ώστε να είναι συμβατές με τις παρατηρήσεις. Παρότι στις 10 διαστάσεις υπάρχουν ελάχιστες συνεπείς θεωρίες χορδών, οι οποίες συνδέονται μέσω συμμετριών δυισμού (dualities), η συμπαγοποίηση στις τέσσερις διαστάσεις οδηγεί σε έναν τεράστιο αριθμό πιθανών λύσεων. Οι λύσεις αυτές εμφανίζουν διαφορετικά χαρακτηριστικά τα οποία εξαρτώνται από την ακριβή γεωμετρία του συμπαγοποιημένου χώρου, γεγονός που δυσχεραίνει την εξαγωγή συγκεκριμένων θεωρητικών προβλέψεων. Το γεγονός αυτό δίνει σημαντικό κίνητρο για τη συστηματική ανάλυση μεγάλων οικογενειών λύσεων, με σκοπό την ταξινόμησή τους και τον εντοπισμό προτύπων τα χαρακτηριστικά των οποίων είναι συμβατά με το Καθιερωμένο Πρότυπο ή κάποια γνωστή Μεγαλοενοποιημένη Θεωρία.

Ειδικότερα οι τετραδιάστατες θεωρίες που προκύπτουν από συμπαγοποιήσεις της ετερωτικής χορδής, μπορούν να οδηγήσουν σε πρότυπα τα οποία περιέχουν τη συμμετρία βαθμίδας του Καθιερωμένου Προτύπου, στα οποία επιτυγχάνεται επίσης η ενοποίηση των σταθερών σύζευξης των αλληλεπιδράσεων βαθμίδας. Τα πρότυπα αυτά διαθέτουν επίσης χαρακτηριστικά, όπως επιπρόσθετα μποζόνια βαθμίδας και εξωτικά σωματίδια, τα οποία θα μπορούσαν να ανιχνευτούν σε τρέχοντα ή μελλοντικά πειράματα. Επιπλέον, κάθε συμπαγοποίηση χορδών οδηγεί σε έναν «σκοτεινό τομέα», τα χαρακτηριστικά του οποίου θα μπορούσαν να εξηγήσουν την ύπαρξη της σκοτεινής ύλης.

Η υπερσυμμετρία στο χωρόχρονο δεν είναι απαραίτητη για τη συνέπεια της θεωρίας χορδών. Παρά το γεγονός αυτό, η φαινομενολογία της θεωρίας χορδών είχε μέχρι πρόσφατα επικεντρωθεί στις υπερσυμμετρικές χορδές, παρότι είχαν εντοπιστεί μη υπερσυμμετρικές θεωρίες που απορρέουν από την ετερωτική χορδή, καθώς η μελέτη των τελευταίων έρχεται αντιμέτωπη με μια σειρά από προβλήματα. Η απουσία όμως πειραματικής επιβεβαίωσης της υπερσυμμετρίας στο LHC καθιστά εύλογη τη μελέτη μη υπερσυμμετρικών προτύπων και την προσπάθεια επίλυσης των προβλημάτων αυτών. Η αντιμετώπιση του σπασίματος της υπερσυμμετρίας στο πλαίσιο της θεωρίας χορδών καθίσταται αναγκαία για τον υπολογισμό κβαντικών διορθώσεων, οι οποίες δέχονται έναν άπειρο αριθμό συνεισφορών, που συμπεριλαμβάνουν καταστάσεις χωρίς αντίστοιχη περιγραφή στη θεωρία πεδίου, όπως οι καταστάσεις περιτύλιξης (winding modes).

Ένα βασικό πρόβλημα που εμφανίζεται στις μη υπερσυμμετρικές θεωρίες χορδών είναι η αστάθεια του κενού. Το πρόβλημα αυτό οφείλεται στη μη μηδενική τιμή της κοσμολογικής σταθεράς στο επίπεδο του ενός βρόχου, η οποία οδηγεί σε μεγάλες διορθώσεις που σχετίζονται με το πεδίο του dilaton. Παρότι ο μηδενισμός της κοσμολογικής σταθεράς σε μη υπερσυμμετρικά πρότυπα δεν έχει επιτευχθεί, το πρόβλημα της αστάθειας του κενού μπορεί να αντιμετωπιστεί σε μια ομάδα μη υπερσυμμετρικών συμπαγοποιήσεων οι οποίες είναι συμβατές με το μηχανισμό Scherk–Schwarz, προσαρμοσμένο στα πλαίσια της θεωρίας χορδών. Ειδικότερα στα πρότυπα που εμφανίζουν ίσο αριθμό άμαζων μποζονικών και φερμιονικών βαθμών ελευθερίας, τα επονομαζόμενα "super no-scale" πρότυπα, επιτυγχάνεται η εκθετική μείωση της κοσμολογικής σταθεράς για μεγάλες τιμές της ακτίνας συμπαγοποίησης.

Αντικείμενο μελέτης της παρούσας διατριβής αποτελούν τα μη υπερσυμμετρικά ενοποιημένα πρότυπα που απορρέουν από την ετερωτική θεωρία χορδών. Αφού αναφερθούμε επιγραμματικά στα βασικά στοιχεία του Καθιερωμένου Προτύπου καθώς και πιθανές επεκτάσεις του, όπως οι Μεγαλοενοποιημένες Θεωρίες $SO(10)$ και Pati–Salam, παρουσιάζουμε τον μηχανισμό Scherk–Schwarz. Στη συνέχεια εισάγουμε τις βασικές έννοιες της μποζονικής θεωρίας χορδών και των θεωριών υπερχορδών, ο υβριδισμός των οποίων οδηγεί στην ετερωτική χορδή. Παρουσιάζουμε επίσης αναλυτικά την κατασκευή της συνάρτησης επιμερισμού, η οποία αποτελεί σημαντικό εργαλείο για την ανάλυση μας.

Στη συνέχεια αναλύουμε την ελεύθερη φερμιονική θεμελίωση (Free Fermionic Formulation, FFF), παρουσιάζοντας κάποιες από τις δυνατότητές της μέσω της κατηγοριοποίησης των ετερωτικών θεωριών στις δέκα διαστάσεις. Εισάγουμε παράλληλα τον φορμαλισμό των orbifolds τα οποία περιγράφουν τις συμπαγοποιήσεις στις οποίες επικεντρωνόμαστε. Η ισοδυναμία των δυο φορμαλισμών παρουσιάζεται χρησιμοποιώντας ως παράδειγμα μια κατηγορία συμπαγοποιήσεων με συμμετρία βαθμίδας $SO(10)$, για την οποία αναλύουμε και τα βασικά φαινομενολογικά χαρακτηριστικά τα οποία μελετάμε.

Χρησιμοποιώντας ταυτόχρονα τεχνικές της ελεύθερης φερμιονικής θεμελίωσης και των orbifolds αναζητούμε πρότυπα με ημι-ρεαλιστική φαινομενολογία. Πιο συγκεκριμένα, κατασκευάζουμε και αναλύουμε μια ομάδα $\sim 10^{13}$ προτύπων με συμμετρία βαθμίδας Pati–Salam. Στα πρότυπα αυτά επιβάλλουμε μια σειρά περιορισμών με σκοπό την εξασφάλιση ιδιοτήτων όπως η απουσία ταχυονίων, η ύπαρξη στο άμαζο φάσμα τόσο χειρόμορφων (chiral) φερμιονίων, όσο και των κατάλληλων μποζονίων Higgs για το αυθόρμητο σπάσιμο της συμμετρίας βαθμίδας, με την παράλληλη απουσία εξωτικών φερμιονίων με κλασματικό ηλεκτρικό φορτίο σε χαμηλές ενέργειες. Απαιτούμε επίσης το αυθόρμητο σπάσιμο της υπερσυμμετρίας μέσω του μηχανισμού Scherk–Schwarz και την ταυτόχρονη εκθετική μείωση της κοσμολογικής σταθεράς για μεγάλες τιμές της ακτίνας του εσωτερικού χώρου. Μελετάμε διεξοδικά τη συνάρτηση επιμερισμού και το ενεργό δυναμικό στο επίπεδο ενός βρόχου για όλες τις συμπαγοποιήσεις που ικανοποιούν τα παραπάνω, και παρουσιαζουμε μια κατηγοριοποίηση με βάση τα φαινομενολογικά τους χαρακτηριστικά. Παρουσιάζουμε αναλυτικά συγκεκριμένα πρότυπα τα οποία εμφανίζουν επιθυμητά χαρακτηριστικά, όπως θετικό δυναμικό στο επίπεδο του ενός βρόχου. Παρουσιάζουμε επίσης το πρώτο παράδειγμα στο οποίο επιτυγχάνεται ο περιορισμός (confinement) των εξωτικών φερμιονίων σε πρότυπο Pati–Salam.

Τέλος, επικεντρωνόμαστε σε μια ομάδα $\sim 10^{23}$ συμπαγοποιήσεων η οποία επιπρόσθετα από τις προαναφερθείσες συνθήκες μπορεί να εμφανίσει τρεις γενιές φερμιονίων. Μελετώντας το ενεργό δυναμικό στο επίπεδο ενός βρόχου παρατηρούμε ότι η συνήθης συνθήκη εκφυλισμού μεταξύ μποζονίων και φερμιονίων στο άμαζο φάσμα της θεωρίας είναι αναγκαία αλλά όχι ικανή για την εκθετική καταστολή της κοσμολογικής σταθεράς. Εισάγουμε μια νέα συνθήκη η οποία πρέπει να ικανοποιείται παράλληλα ώστε τα πρότυπα να εμφανίζουν super no-scale χαρακτηριστικά. Πραγματοποιούμε μια τυχαία αναζήτηση στον παραμετρικό χώρο των προαναφερθέντων συμπαγοποιήσεων και ταξινομούμε τα πρότυπα που ικανοποιούν όλους τους περιορισμούς με βάση το ενεργό δυναμικό τους. Τέλος, παρουσιάζουμε αναλυτικά ένα συγκεκριμένο πρότυπο με τρεις γενιές φερμιονίων, θετικό ενεργό δυναμικό και εκθετική καταστολή της κοσμολογικής σταθεράς για μεγάλες τιμές της ακτίνας συμπαγοποίησης.

List of Publications

The main results in this thesis are based on the following peer-reviewed publications presented in chronological order. The authors are listed alphabetically according to particle physics convention.

[1]. *On Non-supersymmetric Heterotic Pati-Salam Models,*
I. Florakis, J. Rizos and K. Violaris-Gountonis,
Ann. U. Craiova Phys. 30 (2020) no.2, 140-149

[2]. *Super No-Scale Models with Pati-Salam Gauge Group,*
I. Florakis, J. Rizos and K. Violaris-Gountonis,
Nucl. Phys. B 976 (2022), 115689

[3] *Three-Generation Super No-Scale Models in Heterotic Superstrings*
I. Florakis, J. Rizos and K. Violaris-Gountonis,
Phys. Let. B 833 (2022), 137311

[4]. *On nonsupersymmetric Pati–Salam string models,*
I. Florakis, J. Rizos and K. Violaris-Gountonis,
PoS, vol. CORFU2021, p. 061, 2022

Contents

Chapter 1

Introduction

In this introductory chapter we will briefly outline the Standard Model of particle physics, which provides a high precision description of physics at the quantum regime, subject to decades of experimental scrutiny. We will introduce its most important features and highlight some of the open questions that it leaves unanswered, motivating the search for new physics. We will then present some proposed theories of physics "Beyond the Standard Model". Within the context of Grand Unification, we will outline the $SO(10)$ and Pati–Salam schemes, the investigation of which in the framework of heterotic string theory will be the main focus of this thesis. We will also introduce supersymmetry and present the Scherk–Schwarz mechanism which can be used to spontaneously break it. Finally, we will provide the motivation for the development of string theory, setting the stage for a more detailed presentation in the following chapter.

1.1 The Standard Model of Particle Physics

The Standard Model of particle physics (SM) [5–7] is the product of decades of theoretical and experimental efforts to understand the natural phenomena that take place at the sub-atomic level. In a theoretical framework encompassing Quantum Mechanics and Special Relativity, it provides a description of all known elementary particles and determines the rules which govern their interactions under the Strong, Weak and Electromagnetic forces. It has been tested to very high accuracy in various experiments over the years [8], with the discovery [9,10] of the Higgs boson [11–14] at CERN confirming its last remaining prediction regarding electroweak symmetry breaking.

The SM is based on the framework of a renormalisable Quantum Field Theory exhibiting an

$$SU(3)_C \times SU(2)_L \times U(1)_Y \tag{1.1}$$

gauge symmetry. In this framework, the elementary constituents of matter are organised into representations, based on their properties under gauge transformations. The $SU(3)_C$ symmetry encompasses Quantum Chromodynamics, the theory that describes the Strong force. This force, mediated by 8 massless gluon fields, divides fermions into quarks, which carry colour charge and transform as $SU(3)$ triplets, and leptons, which do not interact via the strong force, transforming as singlets. Gluons also carry colour charge and therefore interact with one another. The complex pattern of interactions among quarks and gluons gives rise to protons and neutrons, the basic building blocks of nuclei.

The $SU(2)_L \times U(1)_Y$ symmetry, formulated in terms of two quantum numbers termed weak isospin and hypercharge, is mediated by four massless vector bosons, $W^{1,2,3}$ and B. This symmetry is responsible for the description of the Weak and Electromagnetic forces, which are unified in the framework of the SM. Quarks and leptons come in left-handed and right-handed pairs. Left-handed fermions subject to Weak interactions transform as $SU(2)_L$ doublets, while right-handed fields transform as singlets. Each fermion also carries a hypercharge, which is fixed up to an overall scale factor by the requirement of anomaly cancellation [15]. The electric charge is obtained by the linear combination of the third component of weak isospin and hypercharge:

$$Q = T_3 + Y \ . \tag{1.2}$$

Electroweak unification is only achieved at energy scales above $\sim$ 100 GeV, below which spontaneous symmetry breaking via the Higgs mechanism [11–14] occurs, leaving an unbroken $U(1)_{em}$ symmetry corresponding to Quantum Electrodynamics. As a result of this symmetry breaking, three of the electroweak gauge bosons acquire mass, providing the Weak gauge bosons $W^{\pm}$ and Z, while one massless degree of freedom, the photon, remains. In addition to the left-handed quarks, the weak interactions mediated by $W^{\pm}$ and Z also involve the left-handed leptons, while right-handed fermions do not interact via the Weak force. The theory of weak interactions thus only requires the existence of left-handed neutrinos. This asymmetrical nature of the weak interactions, called chirality, is one of the defining features of the SM.

A remarkable feature of the SM is that each fermion representation has a triple degeneracy; quarks and leptons come in three copies, called "generations". All particles of a given representation exhibit identical quantum numbers. The only exception to this is mass, with each subsequet generation being more massive. Each particle is accompanied by an antimatter counterpart with equal mass but opposite electric charge. The particle content of the SM is outlined in table 1.1.

	Particles		Representation
spin-1	Gluons:	G^{α}_{μ}	$G(\mathbf{8},\mathbf{1},0)$
	W bosons:	W^{α}_{μ}	$W(\mathbf{1},\mathbf{3},0)$
	B boson:	B_{μ}	$B_{\mu}(\mathbf{1},\mathbf{1},0)$
spin-1/2	Quarks:	$\binom{u}{d}, \binom{c}{s}, \binom{t}{b}$	$Q(\mathbf{3},\mathbf{2},1/6)$
	Anti-quarks:	u^c, c^c, t^c	$u^c(\bar{\mathbf{3}},\mathbf{1},-2/3)$
		d^c, s^c, b^c	$d^c(\bar{\mathbf{3}},\mathbf{1},1/3)$
	Leptons:	$\binom{e}{\nu_e}, \binom{\mu}{\nu_\mu}, \binom{\tau}{\nu_\tau}$	$L(\mathbf{1},\mathbf{2},-1/2)$
	Anti-leptons:	e^c, μ^c, τ^c	$e^c(\mathbf{1},\mathbf{1},1)$
spin-0	Higgs boson:	H	$H(\mathbf{1},\mathbf{2},-1/2)$

Table 1.1: *Representations of all elementary particles under the Standard Model $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry.*

While gauge invariance prohibits the addition of any fermion mass term, this can be elegantly circumvented by the introduction of Yukawa couplings, describing the strength of the interaction between fermions and the Higgs. The mass of each fermion is then proportional to the corresponding Yukawa coupling, which can be determined experimentally. An exception to this are neutrinos. Since the Weak interactions do not require right-handed neutrinos, the SM does not admit neutrino Yukawa couplings and therefore predicts that they should be massless.

The Lagrangian describing the SM can be famously summarised in the compact form[1]:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ i\bar{\psi}\partial\!\!\!/\psi + h.c. \\ &+ \psi_i y_{ij}\psi_j H + h.c. \\ &+ |D_\mu H|^2 - V(H)\ .\end{aligned} \tag{1.3}$$

The first line corresponds to the kinetic terms of the gauge bosons. If we reintroduce all the suppressed indices and express it explicitly in terms of the field strength tensors it takes the form:

$$-\frac{1}{4}\sum_{A=1}^{8} G^{A,\mu\nu}G_{A,\mu\nu} - \frac{1}{4}\sum_{a=1}^{3} W^{a,\mu\nu}W_{a,\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}\ , \tag{1.4}$$

[1]The SM Lagrangian also includes gauge-fixing and ghost terms, which are omitted here for simplicity.

where $G^{A,\mu\nu}$, $W^{a,\mu\nu}$, and $B^{\mu\nu}$ correspond to the $SU(3)$, $SU(2)$ and $U(1)$ factors respectively:

$$\begin{aligned} G^A_{\mu\nu} &= \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g_3 f^{ABC} G^B_\mu G^C_\nu \ , \\ W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g_2 \epsilon^{abc} W^b_\mu W^c_\nu \ , \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \ . \end{aligned} \tag{1.5}$$

Here, g_3 and g_2 are the gauge couplings of the strong and weak interactions, while f^{ABC} and ϵ^{abc} are the totally antisymmetric structure constants of $SU(3)$ and $SU(2)$.

The second line comprises the fermionic fields, which obey the massless Dirac equation. The full covariant derivative, having restored all relevant indices reads:

$$D_\mu = \partial_\mu - ig_3\theta_S G^A_\mu T^A - ig_2\theta_W W^a_\mu T^a - ig_Y Y B_\mu \ , \tag{1.6}$$

where g_Y is the gauge coupling of the hypercharge component and T^A, T^a denote the generators of the non-abelian gauge factors. The constants $\theta_{S,W}$ vanish when a state is a singlet under the corresponding gauge group and are equal to unity when the transformation is non-trivial.

The third line consists of the allowed Yukawa couplings mixing left-handed and right-handed fermions with the Higgs field, which notably does not involve neutrinos:

$$-\sum_{i,j=1}^{3} Y^u_{ij} Q^\dagger_i (i\sigma_2 H^*) u^c_j - \sum_{i,j=1}^{3} Y^d_{ij} Q^\dagger_i H^* d^c_j - \sum_{i,j=1}^{3} Y^e_{ij} L^\dagger_i H^* e^c_j + h.c. \ , \tag{1.7}$$

where $Y^{u,d,e}_{ij}$ are 3×3 matrices which are related to the fermion masses.

Finally, the last line contains the kinetic term of the Higgs, as well as its potential which takes the form:

$$V(H) = -\mu^2 H^\dagger H + \frac{\lambda}{2}(H^\dagger H)^2 \ , \tag{1.8}$$

illustrated in figure 1.1, where μ is the Higgs mass and λ is the Higgs self-coupling. This potential is invariant under the $SU(2)_L \times U(1)_Y$ transformations, but its ground state is not. The Higgs potential exhibits a set of minima, which lead to the spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y \to U(1)_{em}$ when the Higgs field acquires a non-vanishing vacuum expectation value (vev):

$$\langle 0 | H | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon \end{pmatrix} \ , \quad \upsilon = \sqrt{\frac{\mu^2}{\lambda}} \ . \tag{1.9}$$

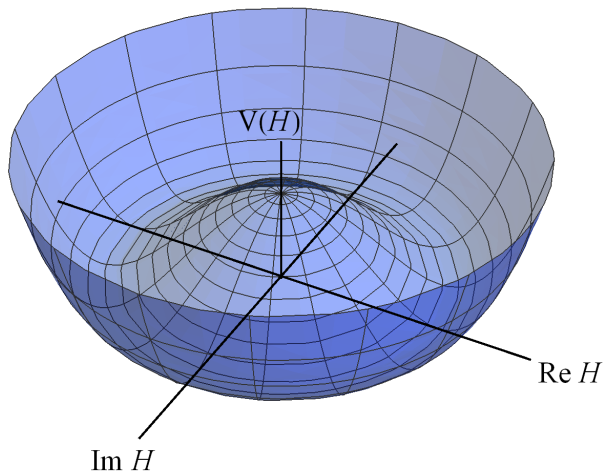

Figure 1.1: Illustration of the Higgs potential leading to spontaneous symmetry breaking.

1.2 Physics Beyond the Standard Model

Despite its remarkable theoretical success and steadfast consistency with experimental data, the Standard Model is ultimately unsatisfying as a fundamental theory of nature, due to arguments arising from experimental, observational, theoretical and philosophical points of view. First of all, while the SM makes outstanding predictions about the interactions of elementary particles, these only cover three out of the four fundamental forces of nature. The first and foremost issue therefore is that the SM completely ignores the gravitational force.

The most successful gravitational theory to date on the other hand, General Relativity, is equally successful at describing gravity at large scales, under the assumption that other forces are negligible. It is a classical theory whose quantisation using the standard methods is rendered impossible due to its non-renormalisable nature. The quantisation of gravity is one of the most important open problems physics is faced with and the main hurdle in our attempts to derive a unified theory of all interactions.

This apparent fundamental incompatibility between the SM and GR has spawned one of the most infamous discrepancies in theoretical physics calculations, regarding the value of the cosmological constant responsible for the accelerated expansion of the universe. Calculating the vacuum-to-vacuum amplitude using the standard techniques of QFT results in a value that overshoots the upper bound set by observations [16, 17] by as many as 120 orders of magnitude [18–21]. This "cosmological constant problem" highlights the need for a new theoretical framework, in which the SM and GR can be reconciled.

Moreover, the SM fails to provide an explanation for the existence of Dark Matter, needed to account for observed cosmological phenomena such as galactic rotational curves [22–25] and gravitational lensing [26–29]. Unable to incorporate Dark Matter and Dark Energy within its framework, the SM is thus a highly accurate theory which cannot, however, account for $\sim 95\%$ of the energy content of our universe [30]. Even within the $\sim 5\%$ that it does describe, however, certain issues have appeared. As mentioned previously, the SM predicts massless left-handed neutrinos with no right-handed counterparts. The observational confirmation of neutrino oscillations [31–38], a process in which neutrinos seemingly swap their flavour eigenstates, implies that they do have mass, resulting in the first experimentally verified indication that "Physics Beyond the Standard Model" is needed. One possible theory that generates masses for the neutrinos is the see-saw mechanism [39–42], in which a sterile neutrino is introduced. Adding right-handed neutrinos (ν^c) then allows for gauge invariant neutrino Yukawa couplings, similar to those of the other fermions. A Majorana term can also be added to the Lagrangian, setting the scale for the right-handed neutrino mass:

$$-\sum_{i,j=1}^{3} Y_{ij}^{\nu} L_i^{\dagger}(i\sigma_2 H^*)\nu_j^c + \frac{1}{2}\nu_i^c M_R \nu_j^c + h.c. \tag{1.10}$$

Another unexplained phenomenon within the SM is the apparent non-violation of CP symmetry by the strong interactions. The most general QCD Lagrangian includes the term

$$\theta \frac{g_3^2}{32\pi^2} \sum_{a=1}^{8} G_{\mu\nu}^{a} \tilde{G}^{a,\mu\nu} , \tag{1.11}$$

which violates the CP symmetry [43,44]. Experimental measurements of the electric dipole moment of the neutron [45], which can be related to θ [46], pose stringent constraints which require θ to be very close, if not equal, to zero. The SM lacks a mechanism that would account for this without fine-tuning, leading to the "strong CP problem", which has important cosmological consequences, as CP violation is needed to explain the matter-antimatter asymmetry of the universe [47–49]. A possible solution is the introduction of an additional global $U(1)$ symmetry, termed Peccei–Quinn symmetry [50, 51]. In this framework, θ is promoted to a dynamical field, whose potential is minimized when $\theta = 0$. This additional symmetry involves pseudoscalar Axion particles [52, 53], which also function as Dark Matter candidates [54, 55].

Furthermore, the SM does not offer an explanation for the apparent quantisation of electric charge. Given that the proton and electron have charges of equal absolute value, $Q_p = -Q_e = e$, while the neutron is electrically neutral, the charges of the up and down quarks must be $q_d = -\frac{1}{2}q_u = -\frac{1}{3}e$. In this case, the absolute values of the electric charges of all known elementary particles are integer multiples of the down quark charge, though this appears to be accidental within the framework of the SM.

An additional limitation of the SM is the dependency on many ad-hoc parameters which cannot be determined theoretically. These parameters are the gauge couplings, the Yukawa couplings which result in fermion masses, the values of the CKM matrix [56, 57], θ, and the Higgs potential. Once experimentally measured, incorporating the values of these parameters in SM calculations does lead to a very accurate description of sub-atomic phenomena, but the mechanism by which they obtain their values remains unknown. In this light, while the addition of new particles and interactions can solve some of the open problems of the SM, it also introduces additional parameters which need to be experimentally determined.

The existence of physics beyond the SM is also strongly suggested by the "running of the gauge couplings". Gauge couplings are sensitive to the energy scale at which they are measured and the extrapolation of their behaviour to high energies of order $\sim 10^{16}\, GeV$ hints at the possibility of gauge coupling unification, which is however narrowly avoided in the SM, as depicted in figure 1.2. Modifications of the SM in which the gauge couplings are unified are then highly motivated, as they could lead to a unified description of the Strong and Electroweak interactions, analogous to the unification of the Weak and Electromagnetic forces within the SM itself.

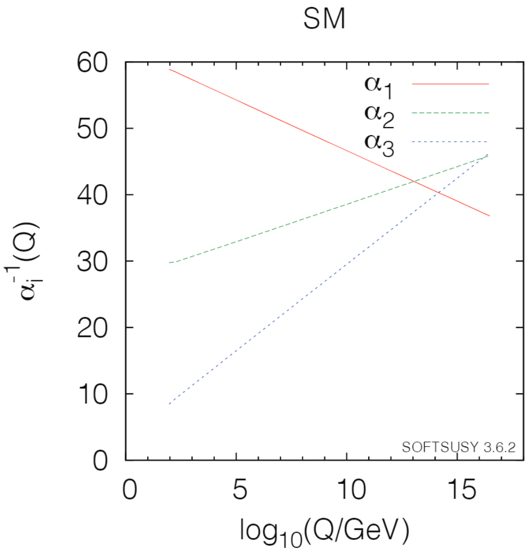

Figure 1.2: Running of the SM gauge couplings. Figure adapted from [8].

Additionally, the hierarchy problem [58–60] emerging from the existence of two independent energy scales within the SM, them being the Planck scale, M_p, and the electroweak scale, M_{EW}, which differ by many orders of magnitude, also needs to be addressed. The Higgs mass, in particular, is not protected by any symmetry and is therefore sensitive to radiative corrections which should drive its mass to high energy scales. This calls for a mechanism taming the quantum corrections, in order to avoid fine-tuning.

The self-consistency of the SM also appears to come under question by the results of recent experiments concerning the anomalous magnetic moment of the muon [61,62] and the mass of the W boson [63]. Should these deviations from the theoretical prediction persist, they are an indication that the SM is incomplete and in need of revision.

As a final motivating factor towards searches for new physics, we resort to some philosophical musings regarding the gauge symmetry and fermion generations. The SM provides no insight as to why our universe appears to be governed by an $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry rather than a simpler one at low energies. Given the existence of the appropriately labeled "exceptional" symmetries, the fact that subatomic physics involve the infinite family of $SU(N)$ for no apparent reason seems unsatisfying. Likewise, the SM includes three generations of fermions, with no explanation as to why this specific number is chosen.

In light of the aforementioned shortcomings, the modern view of the SM is to interpret it as an Effective Field Theory: an approximate description of a more fundamental underlying theory. This approximation is valid up to an energy scale Λ, beyond which the SM breaks down and the full fundamental theory is needed for an accurate description of physics. What this underlying theory is exactly is not yet clear, with many possible avenues being considered, attempting to resolve the deficiencies outlined above by considering new particles, new interactions, or more unified frameworks.

1.2.1 Grand Unified Theories

Grand Unified Theories (GUT) are rooted in the observation that low energy physics need not respect the symmetries present at higher energy scales. This is already apparent in the SM, where at energies below the electroweak scale the $SU(2)_L \times U(1)_Y$ gauge symmetry is broken. Extrapolating from this, an argument can be made that at higher energy scales the gauge symmetry is enlarged, potentially allowing for the unification of the strong and electroweak forces. This unification may offer better insight on the hypercharge assignment, charge quantisation, as well as the apparent asymmetries inherent in the SM, by the presence of a more fundamental symmetry giving rise to the SM at lower energies. Such attempts were first considered shortly after the formulation of the SM and over the years many different unification schemes have been proposed [64–72]. Unfortunately, no experimental observation has yet uncovered evidence of such new physics, and the search for a successful GUT continues. In this section, we will briefly present the two GUT symmetries that will be relevant to this thesis: $SO(10)$ and Pati–Salam models.

Pati–Salam Models

The Pati–Salam model [64] was historically the first attempt at unification beyond the SM. While the underlying symmetry is still a product of three gauge groups, thus not realising gauge coupling unification, it does achieve a unification among quarks and leptons, while also restoring the left-right symmetry which was explicitly broken in the SM. The unification of quarks and leptons is accomplished by reinterpreting the lepton number not as an additional quantum number, but rather as a fourth colour charge, promoting the colour symmetry of QCD to $SU(4)$. In addition to this, the introduction of a right-handed isospin symmetry, $SU(2)_R$, which incorporates right-handed neutrinos, naturally predicts neutrino mass, in line with observations. This framework, governed by the gauge symmetry $SU(4) \times SU(2)_L \times SU(2)_R$, is considered to be valid at high energy scales, with the SM recovered at low energies by the spontaneous symmetry breaking of $SU(4)$ to $SU(3)_C \times U(1)_{B-L}$, concurrent with the $SU(2)_R \times U(1)_{B-L} \to U(1)_Y$ breaking which must occur at some energy scale sufficiently higher than the electroweak scale. The spontaneous breaking of the $SU(2)_L \times SU(2)_R$ symmetry then offers an explanation for parity violation which the SM lacks. The SM hypercharge is identified as a linear combination of the $(B-L)$ quantum number and the diagonal generator T_{3R} of $SU(2)_R$.

In this scheme, all SM fermions with the addition of right-handed neutrinos can be accommodated in two representations, transforming as colour (anti-)quadruplets, and as doublets under $SU(2)_{L/R}$:

$$\begin{aligned} &F_L(\mathbf{4},\mathbf{2},\mathbf{1}) = Q(\mathbf{3},\mathbf{2},1/6) + L(\mathbf{1},\mathbf{2},-1/2)\ , \\ &\overline{F}_R(\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) = u^c(\bar{\mathbf{3}},\mathbf{1},-2/3) + d^c(\bar{\mathbf{3}},\mathbf{1},1/3) + e^c(\mathbf{1},\mathbf{1},1) + \nu^c(\mathbf{1},\mathbf{1},0)\ . \end{aligned} \tag{1.12}$$

The SM Higgs boson is accommodated in a bi-doublet representation

$$h(\mathbf{1},\mathbf{2},\mathbf{2}) = H_u(\mathbf{1},\mathbf{2},1/2) + H_d(\mathbf{1},\mathbf{2},-1/2) \tag{1.13}$$

and the coupling $F_L\overline{F}_R h$ generates fermion masses.

In order to realise the spontaneous symmetry breaking needed to recover the SM at low energy scales, an additional "heavy" Higgs boson is required. In the original construction by Pati and Salam [64], spontaneous symmetry breaking was achieved by giving a vev to the neutral component of a scalar in the adjoint representation. An alternative choice [73] employs the representation $H(\mathbf{4},\mathbf{1},\mathbf{2})$. This modification of the original construction is what Pati–Salam models formulated in heterotic string theory employ, as the level $k=1$ Kač–Moody algebra of the string worldsheet does not contain scalars in the adjoint representation [74].

$SO(10)$ Models

The Pati–Salam $SU(4)\times SU(2)_L\times SU(2)_R$ symmetry is isomorphic to $SO(6)\times SO(4)$, which can naturally be embedded in $SO(10)$ [66,75]. This symmetry allows for the unification among matter particles to go one step further: $SO(10)$ is the simplest gauge symmetry that can accommodate an entire fermion generation, including right-handed neutrinos, in a single representation: the spinorial **16**. In addition to this, all representations of $SO(10)$ in $d=4$ dimensions are anomaly free, which could provide an explanation for the anomaly cancellation inherent in each generation of the SM. The SM Higgs boson is placed in the vectorial **10** representation, accompanied by additional SM triplets:

$$\mathbf{10} = (\mathbf{1},\mathbf{2},1/2) + (\mathbf{1},\mathbf{2},-1/2) + (\mathbf{3},\mathbf{1},-1/3) + (\bar{\mathbf{3}},\mathbf{1},1/3)\ . \tag{1.14}$$

The $SO(10)$ symmetry can be broken to the SM in various ways, by implementing a combination of scalars in either **45** or **54** representations, along with a pair of either **16** and $\overline{\mathbf{16}}$ or **126** and $\overline{\mathbf{126}}$ [66]. A non-zero vev of a Higgs in the **54** breaks $SO(10)$ to the Pati–Salam subgroup, $SU(4)\times SU(2)_L\times SU(2)_R$. The spinorial representations **16** and $\overline{\mathbf{16}}$ can be seen to decompose to the Pati–Salam representations:

$$\begin{aligned} \mathbf{16} &= F_L(\mathbf{4},\mathbf{2},\mathbf{1}) + \overline{F}_R(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})\ , \\ \overline{\mathbf{16}} &= \overline{F}_L(\bar{\mathbf{4}},\mathbf{2},\mathbf{1}) + F_R(\mathbf{4},\mathbf{1},\mathbf{2})\ , \end{aligned} \tag{1.15}$$

while the SM Higgs boson in the vectorial **10** representation, produces a bi-doublet and a colour sextet:

$$\mathbf{10} = h(\mathbf{1}, \mathbf{2}, \mathbf{2}) + D(\mathbf{6}, \mathbf{1}, \mathbf{1}) \; . \tag{1.16}$$

Choosing instead the **45**, breaks $SO(10)$ to either $SU(5)$, flipped $SU(5)$, $SU(4) \times SU(2) \times U(1)$, $SU(3) \times SU(2)_L \times SU(2)_R$ or $SU(3) \times SU(2)_L \times U(1)^2$. Giving a vev to either **16** and $\overline{\mathbf{16}}$ or **126** and $\overline{\mathbf{126}}$ breaks $SO(10)$ to $SU(5)$, while a combination of both $\mathbf{45}/\mathbf{54}$ and $\mathbf{16} + \overline{\mathbf{16}}/\mathbf{126} + \overline{\mathbf{126}}$ breaks $SO(10)$ to the SM.

The stringy realisations of $SO(10)$ we introduce in chapter 4 and employ as intermediate setups in later chapters are limited to the case of unbroken $SO(10)$, as the level $k = 1$ Kač–Moody algebra does not admit massless scalars in the **45**, **54**, **126** or $\overline{\mathbf{126}}$ representations. Nevertheless, these models are useful both as a way of introducing the necessary notation and defining the plan of our analysis in a simplified framework, as well as identifying some key phenomenological features of the more realistic Pati–Salam models which will be our main focus in chapters 5 and 6.

1.2.2 Supersymmetry

In addition to considering enlarged gauge symmetries in which it is embedded, another way of expanding the SM is by investigating extensions of the Poincaré symmetry via the introduction of fermionic degrees of freedom [76]. The resulting spacetime symmetry that arises, supersymmetry, provides a mapping between particles whose spin differs by a half-integer. This provides a connection between bosons and fermions which are paired up into supermultiplets, essentially yielding a unification of matter and forces. Supersymmetry implies the existence of a partner for each SM particle, identical in all of its properties with the exception of spin. The additional particle content dictated by supersymmetry is precisely the kind of "new physics" needed to realise gauge coupling unification, as figure 1.3 shows.

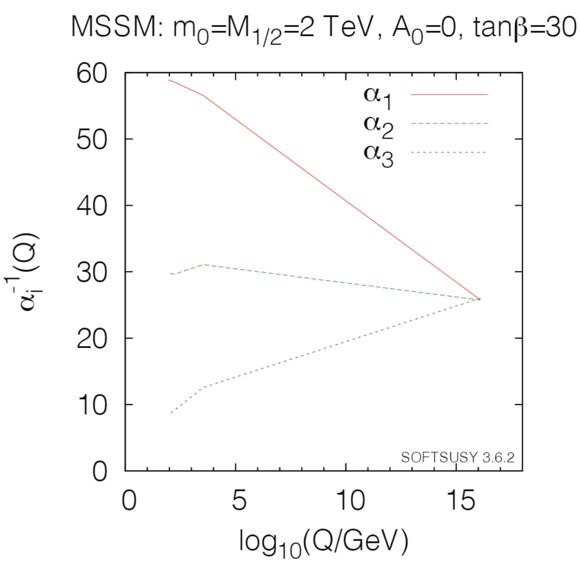

Figure 1.3: Running of the gauge couplings in the Minimal Supersymmetric Standard Model (MSSM) [77]. Figure adapted from [8].

In the presence of supersymmetry, fermionic and bosonic contributions to quantum corrections cancel against one another, stabilising the Higgs mass and thus offering a solution to the hierarchy problem by protecting the vacuum from large radiative corrections. This cancellation also applies to the vacuum-to-vacuum amplitude, predicting an exactly vanishing value for the cosmological constant. Furthermore, while supersymmetry can function as a global symmetry, gauged supersymmetry has even more appealing characteristics, as it provides the framework for the unification of particle physics and gravity in the framework of supergravity theories [78–80]. By implementing supersymmetry within the framework of GUT we can therefore obtain models in which the hierarchy and cosmological constant problems are naturally resolved, gauge coupling unification can be achieved and Dark Matter candidates are present in the spectrum.

Despite being a very appealing option due to the elegant way in which it resolves many of the issues inherent in the SM, the main argument against supersymmetry is the complete absence of experimental evidence supporting it, despite it being the focus of many experimental searches [81–83]. If supersymmetry were a property of nature, then the masses of bosons and their fermion partners should be degenerate, which is clearly not what we observe. Since we have not detected any superpartners, supersymmetry, if present at all, must be a broken symmetry. Furthermore, in order to preserve the solutions to the hierarchy and cosmological constant problems, supersymmetry must be broken at a low energy scale of the order of a few TeV [84–86].

Identifying the precise mechanism driving supersymmetry breaking is a complicated task, which is beyond the scope of this thesis. For the purposes of this work, we will restrict the discussion to the spontaneous breaking of supersymmetry via the Scherk–Schwarz mechanism [87, 88], which takes advantage of compactified dimensions. In its simplest field theory form, the Scherk–Schwarz mechanism can be interpreted as a generalisation of Kaluza–Klein compactifications [89, 90]. Consider the simplest case of a $D+1$ dimensional theory containing a scalar field ϕ described by a Lagrangian which is invariant with respect to a global symmetry Q. For the sake of simplicity, we will assume that Q is diagonal and the charge of a state under Q is q. By compactifying the extra dimension into a circle of radius R, we obtain a theory living on an $M^D \times S^1$ manifold. The scalar ϕ will then be periodic along the compactified direction, up to a phase factor related to Q:

$$\phi(x^\mu, y + 2\pi R) = e^{i\pi q}\phi(x^\mu, y) \ , \tag{1.17}$$

implying that ϕ can be expanded in terms of Fourier modes:

$$\phi(x^\mu, y) = e^{iqy/2R} \sum_{n\in\mathbb{Z}} e^{iny/R}\phi_n(x^\mu) \ . \tag{1.18}$$

Consider now the kinetic term in the Lagrangian:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\partial^M\phi\partial_M\phi, \tag{1.19}$$

where $M = 0, 1, \ldots, D$. If we substitute the Fourier expansion of ϕ we obtain the following:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\partial^M\phi\partial_M\phi = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi + \partial^y\phi\partial_y\phi\right) = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\sum_{n\in\mathbb{Z}}\frac{1}{R^2}\left(n + \frac{q}{2}\right)^2\phi_n^2 \ . \tag{1.20}$$

From a D-dimensional point of view, the degrees of freedom associated with the extra dimension appear as an infinite number of massive particles. The mass of this tower of states is given by:

$$m_n^2 = \frac{1}{R^2}\left(n + \frac{q}{2}\right)^2 \ . \tag{1.21}$$

This tower of states is shifted compared to the standard Kaluza–Klein form $m_{KK} = n/R$, by a factor proportional to the charge under Q. Taking Q to be the identity matrix, or equivalently setting $q = 0$ recovers the Kaluza–Klein theory.

Now consider a more general case, in which the Lagrangian is a function of multiple fields ϕ_i that have an $SO(N)$ rotational symmetry. Assuming that all fields have the same charge under the global symmetry Q, we obtain the mass formula of equation (1.21) for all of them and the $SO(N)$ symmetry remains. We can generalize this even further by assuming that the charges under Q differ for each field ϕ, so the boundary conditions read:

$$\phi_i(x^\mu, y + 2\pi R) = e^{i\pi q_i}\phi_i(x^\mu, y) \ . \tag{1.22}$$

This results in a very elegant way of spontaneously breaking the $SO(N)$ symmetry: each tower of KK modes (including their zero modes) has different mass levels, dictated by the charges q_i. As the mass shifts differ among the fields, the $SO(N)$ rotations no longer leave the Lagrangian invariant.

The ability of Scherk–Schwarz compactifications to spontaneously break symmetries can also be exploited to break supersymmetry. This can be achieved by identifying the spacetime fermion number as the operator Q. Different boundary conditions are thus assigned to states within a supermultiplet which no longer have equal masses, as the mass shifts now differ between superpartners. In such constructions, the supersymmetry breaking scale is naturally set at $m_{3/2} \sim 1/R$ and can in principle be dynamically determined by the theory [2, 3, 91]. We note that the spectrum of the theory is non supersymmetric at all energy scales. These constructions therefore differ from theories predicting supersymmetry at high energy scales, as there is no continuous transformation that transitions between a supersymmetric and non supersymmetric phase. Supersymmetry is only restored in the higher-dimensional theory obtained when the compactification radius becomes infinite.

The Scherk–Schwarz mechanism can be generalised in the framework of string theory, in the form of coordinate dependent compactifications [92–96]. This method of spontaneous supersymmetry breaking will be employed throughout this thesis to obtain non supersymmetric constructions, in which the operator $(-1)^F$, where F is the spacetime fermion number is coupled to a freely-acting orbifold on an internal circle. The exact implementation of these momentum shifts in the internal lattice is presented in chapter 3 and showcased in the context of four-dimensional $SO(10)$ toy models in chapter 4, while concrete applications in four-dimensional heterotic Pati–Salam models are the focus of chapters 5 and 6.

1.3 String Theory

The need for a unified framework in which the SM and GR can be combined without inconsistencies has brought string theory to the forefront. String theory is based on the hypothesis that the fundamental building blocks of our universe are not point-like, but rather extended one-dimensional objects called strings, whose different vibration modes produce the observed spectrum of particles and interactions. In this framework, we can obtain consistent theories with only one parameter: the string tension, which can be associated with the Planck scale.

While originally intended as a theory that would explain the interactions between hadrons, string theory quickly fell out of favour when QCD emerged as a successful framework for studying the Strong force, only to resurface as a much more ambitious theory potentially unifying all interactions. Seminal results in early string theory research include the discovery of a massless graviton in its spectrum [97, 98], the incorporation of supersymmetry eliminating all tachyons [99], the cancellation of all anomalies [100], and the discovery of heterotic strings [101–103], with gauge symmetries that could accommodate the SM within well-studied GUT frameworks. The unification of the five supersymmetric string theories under the umbrella of M-theory [104] further implied a common background for all model-building attempts, in contrast with the infinite possibilities within field theory.

There is, however, one significant problem in string model building. In order to be consistent, the superstring theories require 10 spacetime dimensions, which is in conflict with the $(3, 1)$-dimensional signature of the observable universe. In order to overcome this problem, the six extra dimensions must be compactified. In going from the critical 10 dimensions to four, the uniqueness of the superstring theories gives way to the string landscape: an immense number of possible vacua arise [105], with wildly different characteristics that depend on the exact geometry of the compactified space.

Spacetime supersymmetry is not a necessary ingredient for consistent string theory models, as shown in the $SO(16) \times SO(16)$ heterotic theory [106, 107]. String phenomenology, however, tended to focus on supersymmetric vacua, under the assumption that supersymmetry breaking can be tackled at a later stage, in the framework of a low-energy EFT. In the context of heterotic string theory, various $\mathcal{N} = 1$ supersymmetric vacua showing promising phenomenology were identified, including models with flipped-$SU(5)$ [108–110], Pati–Salam [111, 112], and SM-like gauge symmetries [113]. Given the fact that a comprehensive study of all solutions is impossible, numerous classification attempts have been carried out [114–120], seeking to scan large regions of the parameter space of possible vacua and identify those whose low energy phenomenology resembles that of the SM.

The ongoing failure to detect evidence supporting supersymmetry in modern experiments, however, motivates the investigation of non supersymmetric strings. The investigation of supersymmetry breaking within the full string theory context is made necessary by the need to accurately account for quantum corrections, which are sensitive to the entire tower of string states, including those with no field theory analogue such as winding modes.

Non supersymmetric strings have received attention over the years [93–96, 121–133], though their detailed analysis has been hindered by a series of issues that arise in addition to the ones supersymmetric strings are already facing. Besides the possibility of tachyonic excitations, non supersymmetric constructions typically predict large values for the vacuum energy, thus reintroducing the cosmological constant problem, as well as leading to dilaton tadpoles signaling instabilities [134, 135]. While a non-vanishing one-loop cosmological constant appears to be inevitable [136], conditions ensuring it can be exponentially suppressed have been identified [137–140] and used in the construction of models with interesting implications [2, 91, 141–146].

In addition to the above, in the absence of supersymmetry the theory is faced with the decompactification problem: the large volume limit of the theory either predicts very small values for the gauge couplings, or a divergence growing linearly with the compactification volume. In the latter case, the theory exhibits strong coupling, rendering the perturbative approach invalid. Possible ways to avoid this have been considered [147, 148], while recent research proposes the reinterpretation of decompactification as a selection rule for realistic vacua rather than a problem [143].

In this thesis, we will focus on the analysis and classification of a specific family of non supersymmetric string vacua with Pati–Salam gauge symmetry realised as orbifold compactifications of the heterotic string [2]. These models will be subjected to a series of phenomenological conditions ensuring that they lead to semi-realistic predictions of low energy phenomena including chirality, spontaneous symmetry breaking and an exponentially suppressed cosmological constant. We will then concentrate on a particular class of vacua capable of generating three fermion generations [3] and explore the moduli dependence of the vacuum energy at the one-loop level, identifying models in which the large volume regime exhibits exponentially suppressed values.

1.4 Thesis Outline

The structure of the thesis is as follows: In chapter 2, we present a brief introduction to the framework of string theory. We summarise the main features of the bosonic string, as well as the superstring, the two ingredients needed for the construction of 10 dimensional heterotic strings. Chapter 3 focuses on the generalisation of heterotic string constructions to $D \leq 10$ dimensions. We present the framework of the free fermionic formulation, highlighting its features in the classification of 10-dimensional heterotic string theories. We also briefly remark on the finiteness properties of string theory in the absence of supersymmetry. We conclude the chapter by introducing orbifold compactifications and provide explicit formulas for the toroidal lattices parametrising the compactified space. In chapter 4, we highlight the process by which we construct our models using a class of vacua with $SO(10)$ gauge symmetry as an example. We introduce some of the phenomenological aspects of these vacua that will be relevant to our further analysis. Chapter 5, based on [2], is focused on the detailed analysis of models with a Pati–Salam gauge symmetry. We detail the massless particle spectrum of these models, as well as all possible tachyonic states. We then apply a series of constraints to reveal phenomenologically appealing solutions and classify them based on the behaviour of their one-loop effective potential. We conclude the chapter by presenting some explicit model constructions with interesting phenomenological characteristics. In chapter 6 we generalise the previous models by allowing for all possible shifts in the internal directions. This allows us to search for solutions with more appealing phenomenology, at the cost of losing the ability to exhaustively scan the parameter space of solutions. The requirements imposed in order to obtain models with three fermion generations reveal the need for more strict conditions for the exponential suppression of the cosmological constant [3], which we derive and apply in our search for semi-realistic vacua. We present the results of our analysis, as well as an example of a super no-scale model with three fermion generations. Finally, in chapter 7, we summarise our work, discuss the results, and present options for possible avenues of further analysis in the future.

Chapter 2

A Brief Introduction to String Theory

In this chapter, we will introduce the basic ideas behind string theory, establishing the theoretical background upon which the rest of the thesis will rest. This is, of course a vast topic whose detailed presentation has been the scope of numerous textbooks [149–155]. In this work, however, we will be very concise, only covering the most crucial elements. Throughout this thesis, we will focus on closed strings, leaving aside the possibilities offered by open string constructions, as heterotic string theories which are the main focus of this work are described purely in terms of closed string modes.

We will first introduce the bosonic string, highlighting its most salient features and discuss the issues which arise when the theory is quantised. We will then present the supersymmetric RNS string, in which some of these deficiencies are alleviated. Subsequently, we will outline the hybrid approach in which these two theories are combined into the heterotic string which appears promising from a phenomenological perspective. Finally, we will focus on the various components entering the string partition function, a tool which will be of great use to our analysis in the following chapters.

2.1 The Bosonic String

Our starting point is the simplest theory describing strings: the bosonic string. The basic principle behind string theory is the promotion of the fundamental building blocks of nature from particles to extended objects, strings, which propagate in a D-dimensional spacetime. In a straightforward generalisation of a particle's worldline, a string covers a two-dimensional worldsheet as it propagates, as depicted in figure 2.1. The embedding of this worldsheet in the target spacetime is then described by a set of scalar fields $X^\mu(\sigma,\tau)$, parametrised by two coordinates $\sigma^0=\sigma$, $\sigma^1=\tau$ which can be interpreted as the continuous set of all points along the string (σ) and the proper time for each such point (τ) respectively. These parameters do not correspond to any observable and therefore any consistent string theory must be invariant under the reparametrisations:

$$\sigma^a \to f^a(\sigma)=\sigma'^a \;\text{ and }\; h_{\alpha\beta}(\sigma)=\frac{\partial f^\gamma}{\partial\sigma^\alpha}\frac{\partial f^\delta}{\partial\sigma^\beta}h_{\gamma\delta}(\sigma')\ . \tag{2.1}$$

Closed strings are then defined by the boundary conditions:

$$X^\mu(\sigma,\tau)=X^\mu(\sigma+2\pi,\tau)\ ,\ \sigma\in(0,2\pi]\ . \tag{2.2}$$

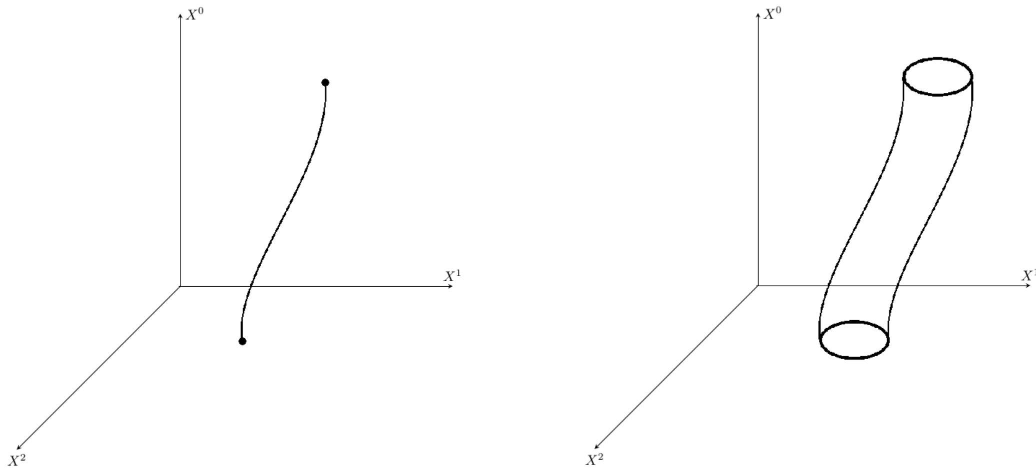

Figure 2.1: Left: the worldline of a particle. Right: the cylindrical worldsheet of a propagating closed string.

2.1.1 Action and Equations of Motion

By analogy with a relativistic particle moving along a geodesic, the Nambu–Goto action [156, 157] describing a string will extremise the area of its worldsheet. Defining the induced metric

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu} \tag{2.3}$$

describing the pullback of the flat metric on Minkowski space, the action is given by:

$$S = -T\int_\Sigma d^2\sigma\sqrt{-\det\gamma}\ , \tag{2.4}$$

where T is the string tension, which is related to the string length and Regge slope parameter:

$$T = \frac{1}{2\pi a'},\ \ell_s = \sqrt{2a'}. \tag{2.5}$$

The Nambu–Goto action can alternatively be written in a more explicit form:

$$S = -T\int_\Sigma d^2\sigma\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}\ , \tag{2.6}$$

where

$$\dot{X} = \frac{\partial X}{\partial \tau} \ \text{ and } \ X' = \frac{\partial X}{\partial \sigma} \tag{2.7}$$

and the scalar product is defined as

$$\dot{X}\cdot X' = \eta_{\mu\nu}\dot{X}^\mu X'^\nu\ . \tag{2.8}$$

While the Nambu–Goto action has a clear physical interpretation, describing the area of the worldsheet, the square root present in the action is problematic when quantisation of the theory is considered and an alternative description is required. Taking advantage of their equivalence at the classical level, we can reformulate the theory in terms of the string sigma model, or Polyakov action [158–160]:

$$S = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}. \tag{2.9}$$

In addition to the reparametrisation invariance which, as mentioned, is necessary for the consistency of the theory, the Polyakov action is also invariant under the Poincaré transformations:

$$\delta X^\mu = \omega^\mu_\nu X^\nu + a^\mu \ \text{ and } \ \delta h^{\alpha\beta} = 0, \tag{2.10}$$

as well as Weyl transformations:

$$h_{\alpha\beta} \to e^{\Phi} h_{\alpha\beta} \quad \text{and} \quad \delta X^{\mu} = 0 \,. \tag{2.11}$$

Reparametrisations and Weyl transformations are local symmetries and can be utilised in order to simplify the Polyakov action. In the conformal gauge, $h_{\alpha\beta}$ can be reduced to the flat worldsheet metric $\eta_{\alpha\beta}$, in which case the action simplifies to:

$$S = -\frac{T}{2} \int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu} \,, \tag{2.12}$$

where the two equations of motion amount to a free wave equation for X^μ:

$$\partial_\alpha \partial^\alpha X^\mu = 0 \,, \tag{2.13}$$

and the vanishing of the stress-energy tensor, given by:

$$T_{\alpha\beta} = -\frac{2}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} = \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} \eta_{\alpha\beta} \eta^{\gamma\delta} \eta_{\mu\nu} \partial_\gamma X^\mu \partial_\delta X^\nu \,, \tag{2.14}$$

whose components are given by:

$$\begin{aligned} T_{01} &= T_{10} = \dot{X} \cdot X' = 0 \,, \\ T_{00} &= T_{11} = \frac{1}{2} \left(\dot{X}^2 + X'^2 \right) = 0 \,. \end{aligned} \tag{2.15}$$

In order to solve the equations of motion, it is convenient to introduce the light-cone coordinates on the worldsheet: $\sigma^\pm = \tau \pm \sigma$, $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. The wave-equation in light-cone coordinates then takes the form: $\partial_+ \partial_- X^\mu = 0$ and the general solution can be decomposed into a sum of left-moving and right-moving waves:

$$X^\mu(\sigma, \tau) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma) \,. \tag{2.16}$$

In the case of the closed string, these waves can expanded in Fourier modes as follows:

$$\begin{aligned} X^\mu_R &= \frac{1}{2} x^\mu + \frac{1}{2} l_s^2 p^\mu (\tau - \sigma) + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-2in(\tau - \sigma)} \,, \\ X^\mu_L &= \frac{1}{2} x^\mu + \frac{1}{2} l_s^2 p^\mu (\tau + \sigma) + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-2in(\tau + \sigma)} \,, \end{aligned} \tag{2.17}$$

where x^μ and p^μ are the position and momentum of the string's center of mass. The requirement that X_L and X_R are real implies that x^μ and p^μ must also be real, while positive and negative modes must be conjugate to each other:

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^* \quad \text{and} \quad \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^*. \tag{2.18}$$

The vanishing of the stress-energy tensor then implies:

$$\begin{aligned} T_{++} &= \partial_+ X^\mu \partial_+ X_\mu = 0 \,, \\ T_{--} &= \partial_- X^\mu \partial_- X_\mu = 0 \,, \\ T_{+-} &= T_{-+} = 0 \,. \end{aligned} \tag{2.19}$$

After expanding the stress-energy tensor in terms of Fourier modes, we obtain:

$$\begin{aligned} T_{--} &= 2\ell_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau - \sigma)} \,, \\ T_{++} &= 2\ell_s^2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-2im(\tau + \sigma)} \,, \end{aligned} \tag{2.20}$$

where the Fourier coefficients are given by the Virasoro generators:

$$\begin{aligned} L_m &= \frac{1}{2}\sum_{n=-\infty}^{\infty} a_{m-n}\cdot a_n \ , \\ \tilde{L}_m &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{a}_{m-n}\cdot \tilde{a}_n \ , \end{aligned} \tag{2.21}$$

and in the classical theory satisfy the condition

$$L_m = \tilde{L}_m = 0 \quad \forall m \in \mathbb{Z} \ . \tag{2.22}$$

2.1.2 Quantisation

In order to extract physical predictions from the theory, we need to quantise it and analyse the physical states it contains. There are three different approaches to the quantisation of the string: Canonical (or Covariant) quantisation, light-cone quantisation and BRST quantisation [161–164].

The covariant quantisation approach consists of promoting all fields X^μ to operators, finding the corresponding canonical momenta $P^\mu = T\dot{X}^\mu$ and imposing the commutation relations

$$\begin{aligned} &[\hat{X}^\mu(\sigma,\tau), \hat{P}^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \ , \\ &[\hat{X}^\mu(\sigma,\tau), \hat{X}^\nu(\sigma',\tau)] = [\hat{P}^\mu(\sigma,\tau), \hat{P}^\nu(\sigma',\tau)] = 0 \ , \end{aligned} \tag{2.23}$$

which then imply the following commutation relations for the Fourier modes x^μ, p^μ, a^μ_n and $\tilde{a}^\mu_n$:

$$\begin{aligned} &[x^\mu, p^\nu] = i\eta^{\mu\nu} \ , \\ &[x^\mu, x^\nu] = [p^\mu, p^\nu] = 0 \ , \\ &[a^\mu_m, a^\nu_n] = [\tilde{a}^\mu_m, \tilde{a}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \ , \\ &[a^\mu_m, \tilde{a}^\nu_n] = 0 \ . \end{aligned} \tag{2.24}$$

By redefining a^μ_m and $\tilde{a}^\nu_n$ as

$$a^\mu_m = \frac{1}{\sqrt{m}} a^\mu_m \quad \text{and} \quad a^{\mu\dagger}_m = \frac{1}{\sqrt{m}} a^\mu_{-m} \ , \tag{2.25}$$

we recover the commutation relations of the harmonic oscillator

$$[a^\mu_m, a^{\nu\dagger}_n] = [\tilde{a}^\mu_m, \tilde{a}^{\nu\dagger}_n] = \eta^{\mu\nu}\delta_{m+n,0} \ . \tag{2.26}$$

We can now define the ground state of the theory, $|0\rangle$, as the state annihilated by all lowering operators:

$$a^\mu_m |0\rangle = \tilde{a}^\mu_m |0\rangle = 0, \quad m > 0 \tag{2.27}$$

and construct the Hilbert space of states by acting on the ground state with the creation operators $a^{\mu\dagger}_m$. A generic state Φ carrying momentum k^μ is then generated by:

$$|\Phi\rangle = a^{\mu_1\dagger}_{m_1} a^{\mu_2\dagger}_{m_2} \dots a^{\mu_n\dagger}_{m_n} |0;k\rangle \ , \tag{2.28}$$

where k is the eigenvalue of the momentum operator p^μ.

Upon quantization, the Virasoro modes defined in (2.21) are promoted to operators, subject to normal ordering:

$$\begin{aligned} L_m &= \frac{1}{2}\sum_{n=-\infty}^{\infty} :a_{m-n}a_n: , \\ \tilde{L}_m &= \frac{1}{2}\sum_{n=-\infty}^{\infty} :\tilde{a}_{m-n}\tilde{a}_n: , \end{aligned} \tag{2.29}$$

where we define

$$L_0 = \frac{1}{2}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} :a_{-n}a_n: \ . \tag{2.30}$$

These can be shown to satisfy the Virasoro algebra [151]:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \; , \tag{2.31}$$

where c is the central charge or conformal anomaly, leading to the loss of conformal invariance when the last term is non-zero.

The vanishing of the stress-energy tensor can now be expressed as a series of constraints on the physical states of the theory:

$$\begin{aligned} &(L_0 - \tilde{L}_0)\,|\Phi\rangle = 0 \; , \\ &L_m\,|\Phi\rangle = \tilde{L}_m\,|\Phi\rangle = 0, \quad m > 0 \; , \\ &(L_0 - a)\,|\Phi\rangle = (\tilde{L}_0 - a)\,|\Phi\rangle = 0 \; , \end{aligned} \tag{2.32}$$

where a is a yet undetermined constant arising due to the normal ordering ambiguity of L_0 and $\tilde{L}_0$.

A critical issue that arises when trying to quantize the bosonic string in this manner is the appearance of states with negative norm, signaling a breakdown of unitarity. Consider, for example, the state $|\phi\rangle = a_m^{0\dagger}\,|0\rangle$, whose norm is:

$$\langle\phi|\phi\rangle = \langle 0|\, a_m^0 a_m^{0\dagger}\,|0\rangle = -\,\langle 0|0\rangle \, . \tag{2.33}$$

Such states can be eliminated from the spectrum in the canonical quantisation scheme when the conditions

$$a = 1 \quad \text{and} \quad D = 26 \tag{2.34}$$

are satisfied [153]. The critical bosonic string thus lives in $D = 26$ dimensions.

The residual diffeomorphism symmetries of the bosonic string theory which are present even after choosing a gauge such that the space-time metric $h^{\alpha\beta}$ becomes Minkowskian, allow for the possibility of an additional gauge choice. This presents another approach to string quantisation. By utilising the light-cone gauge, it is possible to construct a Hilbert space that is manifestly free of negative-norm states and explicitly solve the Virasoro conditions instead of imposing them as constraints. After introducing light-cone coordinates:

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \; , \tag{2.35}$$

the D space-time coordinates X^μ consist of the null coordinates $X^\pm$ and the $D-2$ transverse coordinates X^i. The space-time coordinates still satisfy the two dimensional wave equation, decomposing into a sum of left and right-movers. The light-cone gauge uses the residual freedom described above to make the choice:

$$X^+(\tilde{\sigma}, \tilde{\tau}) = x^+ + l_s^2 p^+ \tau \; , \tag{2.36}$$

which corresponds to setting $\alpha_n^+ = 0$ for all $n \neq 0$, eliminating the oscillator modes of X^+. The modes of X^- can be determined by solving the Virasoro constraints:

$$(\dot{X} \pm X')^2 = 0 \; , \tag{2.37}$$

which in the light-cone gauge take the form:

$$\dot{X}^- \pm X'^- = \frac{1}{2l_s^2 p^+}\left(\dot{X}^i \pm (X')^i\right)^2 \, . \tag{2.38}$$

This pair of equations can be used to solve for X^- in terms of X^i. The light-cone gauge therefore allows the elimination of both X^+ and X^-, except for their zero modes, and the description of the theory in terms of the transverse oscillators. In doing this, however, the manifest Lorentz invariance of the theory is lost. In this case, the conformal anomaly manifests as a Lorentz anomaly, which needs to be eliminated [153].

The commutation relations obtained in the light-cone gauge are similar to (2.24) and (2.26) and the ground state and physical states are obtained in the same manner as (2.27) and (2.28). In this case, however, the Hilbert space is positive definite and the theory is free of negative norm states. The downside to this approach is that manifest Lorentz invariance is lost and its presence must be imposed as a condition. This constrains the parameters a and D, once again leading to the critical bosonic string, in $D = 26$ dimensions, where $a = 1$ [149].

Having fixed the values of a and D we can proceed with the determination of the string spectrum. The spectrum of closed string states is made up of tensor products of the left and right-movers, whose masses satisfying the mass-shell condition $M_L^2 = M_R^2 = -p_\mu p^\mu$ can be cast in the form:

$$\alpha' M^2 = 4(N-1) = 4(\tilde{N}-1), \tag{2.39}$$

where

$$N = \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \alpha_{-n}^i \cdot \alpha_n^i \quad \text{and} \quad \tilde{N} = \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \cdot \tilde{\alpha}_n^i \ , \tag{2.40}$$

subject to the level-matching condition

$$N = \tilde{N} \ . \tag{2.41}$$

The ground state of the closed string spectrum, $|0,k\rangle$, defined by $N = \tilde{N} = 0$, is a tachyon with mass $\alpha' M^2 = -4$.

At the $N = 1$ level, there are states of the form $\left|\Omega^{ij}\right\rangle = \alpha_{-1}^i \tilde{\alpha}_{-1}^j \left|0,k\right\rangle$ corresponding to the tensor product of two massless vectors. The symmetric, traceless part of $\left|\Omega^{ij}\right\rangle$ transforms under $SO(24)$ as a massless spin-two particle, the graviton. The discovery of a massless graviton in the bosonic string spectrum was a major milestone in the development of string theory, implying that it can incorporate a quantum theory of gravity [97, 98]. The trace term $\delta_{ij}\left|\Omega^{ij}\right\rangle$ is a massless scalar called the dilaton. The antisymmetric part $\left|\Omega^{ij}\right\rangle - \left|\Omega^{ji}\right\rangle$ transforms under $SO(24)$ as an antisymmetric second rank tensor (Kalb–Ramond two-form). The rest of the string spectrum consists of bosonic states of increasing mass, obtained by taking $N > 1$.

In summary, the quantisation of the bosonic string may be approached in various ways, each with its advantages and drawbacks. In the canonical quantisation, the theory is manifestly Lorentz invariant yet contains negative norm states and identifying the physical spectrum is difficult. Light-cone quantisation on the other hand is free of negative norm states and physical states are easily identified, but at the cost of losing manifest Lorentz invariance. In addition to these, there is a third approach, BRST quantisation, in which the Polyakov action is modified by the addition of fermionic ghost and gauge fixing terms, leading to a manifestly Lorentz invariant theory with unphysical states which must be eliminated. A detailed derivation of the BRST approach can be found in [150, 151, 153].

2.2 The Superstring

The bosonic string theory described above contains no fermions in its spectrum, and thus cannot possibly provide a realistic description of nature. In addition to that, the presence of a tachyon in the physical spectrum indicates an instability of the vacuum. Both of these problems can be solved by introducing fermionic fields on the string worldsheet. At the technical level, the introduction of fermions can be achieved in two ways: the Ramond–Neveau–Schwarz formalism [165–167], which requires worldsheet supersymmetry, and the Green–Schwarz formalism [168–171], which involves spacetime supersymmetry in a ten-dimensional Minkowski background. In this work, we will make use of the RNS formalism, in which the bosonic fields $X^\mu(\tau,\sigma)$ are paired with fermionic partners $\psi^\mu(\tau,\sigma)$. These fermions are two component spinors on the string worldsheet that transform as vectors with respect to spacetime.

2.2.1 Action and Equations of Motion

After adding d free massless Majorana spinors $\psi^\mu(\tau,\sigma)$ to the bosonic theory, the action, in the conformal gauge, reads:

$$S = -\frac{1}{2\pi} \int d^2\sigma (\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) \ , \tag{2.42}$$

where $\bar{\psi}^\mu = (\psi^\mu)^\dagger i \rho^0$, and ρ^α are the 2 dimensional Dirac matrices, which can be defined as:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}. \tag{2.43}$$

In the light-cone coordinate system, the action describing the two component Majorana spinors $\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$ is:

$$S_f = \frac{i}{\pi} \int d^2\sigma (\psi^\mu_- \partial_+ \psi_{-\mu} + \psi^\mu_+ \partial_- \psi_{+\mu}) \; . \tag{2.44}$$

The resulting equation of motion is the Dirac equation, which splits into separate equations for left-moving and right-moving waves:

$$\begin{aligned} \partial_+ \psi_- &= 0 \; , \\ \partial_- \psi_+ &= 0 \; , \end{aligned} \tag{2.45}$$

where we have suppressed the spacetime indices.

The timelike spinors ψ^0 would, in similar fashion to the bosonic string, lead to states with negative norm, requiring the presence of a symmetry in order to be eliminated. Superstring theory does possess such a symmetry: the super-reparametrisation invariance of the string worldsheet. In addition to Poincaré invariance, the action of equation (2.42) is invariant under the infinitesimal transformations:

$$\begin{aligned} \delta X^\mu &= \bar{\epsilon}\psi^\mu \; , \\ \delta \psi^\mu &= \rho^\alpha \partial_\alpha X^\mu \epsilon \; , \end{aligned} \tag{2.46}$$

where $\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix}$ is a constant infinitesimal Majorana spinor. The supersymmetry transformations then take the form:

$$\begin{aligned} \delta X^\mu &= i(\epsilon_+ \psi^\mu_- - \epsilon_- \psi^\mu_+) \; , \\ \delta \psi^\mu_- &= -2\partial_- X^\mu \epsilon_+ \; , \\ \delta \bar{\psi}^\mu_+ &= 2\partial_+ X^\mu \epsilon_- \; . \end{aligned} \tag{2.47}$$

The RNS string possesses two conserved currents, corresponding to the global symmetries. The stress-energy tensor, arising from the translational symmetries, is given by:

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi^\mu + \frac{1}{4}\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi^\mu - (\text{trace}) \; , \tag{2.48}$$

while the conserved current associated with supersymmetry is:

$$J^\alpha_A = -\frac{1}{2}\left(\rho^\beta \rho^\alpha \psi_\mu\right)_A \partial_\beta X^\mu \; . \tag{2.49}$$

In light-cone coordinates we have:

$$\begin{aligned} T_{++} &= \partial_+ X_\mu \partial_+ X^\mu + \frac{i}{2}\psi^\mu_+ \partial_+ \psi_{+\mu} \; , \\ T_{--} &= \partial_- X_\mu \partial_- X^\mu + \frac{i}{2}\psi^\mu_- \partial_- \psi_{-\mu} \; , \\ T_{+-} &= T_{-+} = 0 \; , \\ J_\pm &= \psi^\mu_\pm \partial_\pm X_\mu \; , \end{aligned} \tag{2.50}$$

which satisfy the constraints: $\partial_- J_+ = \partial_+ J_- = \partial_- T_{++} = \partial_+ T_{--} = 0$. However, upon quantisation, the superconformal symmetry imposes even more strict constraints on the stress-energy tensor and supercurrent:

$$J_+ = J_- = T_{++} = T_{--} = 0 \; . \tag{2.51}$$

By considering variations of the fermionic fields $\psi_\pm$ the action can be shown to be stationary if the equations of motion are satisfied and the boundary term

$$\delta S \sim \int d\tau (\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)\Big|^{\sigma=\pi}_{\sigma=0} \tag{2.52}$$

vanishes. The periodicity conditions in this case are: $\psi(\sigma,\tau) = \pm\psi(\sigma+\pi,\tau)$ and can be imposed on the left-movers and right-movers separately. The periodic Ramond boundary conditions imply the mode expansions:

$$\begin{aligned} \psi_-^\mu &= \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \ , \\ \psi_+^\mu &= \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \ , \end{aligned} \tag{2.53}$$

for the left-moving and right-moving fermions respectively, while anti-periodic Neveu–Schwarz conditions lead to the expansions:

$$\begin{aligned} \psi_-^\mu &= \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2ir(\tau-\sigma)} \ , \\ \psi_+^\mu &= \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)} \ , \end{aligned} \tag{2.54}$$

where d_n^μ, $\tilde{d}_n^\mu$,b_r^μ and $\tilde{b}_r^\mu$ satisfy:

$$\begin{aligned} d_{-n}^\mu &= d_n^{\mu\dagger} \ , \\ \tilde{d}_{-n}^\mu &= \tilde{d}_n^{\mu\dagger} \ , \\ b_{-r}^\mu &= b_r^{\mu\dagger} \ , \\ \tilde{b}_{-r}^\mu &= \tilde{b}_r^{\mu\dagger} \ . \end{aligned} \tag{2.55}$$

The independent choice of boundary conditions for left-movers and right-movers then leads to four distinct sectors, labeled R-R, NS-NS, R-NS and NS-R.

2.2.2 Quantisation

In order to quantise the theory, we promote both X^μ and ψ^μ to operators and complement the commutation relations for the bosonic modes (2.26) with the corresponding anti-commutation relations for the fermions. These are:

$$\begin{aligned} &\{b_r^\mu, b_s^\nu\} = \{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \ , \\ &\{d_r^\mu, d_s^\nu\} = \{\tilde{d}_r^\mu, \tilde{d}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \ , \\ &\{b_r^\mu, d_s^\nu\} = \{b_r^\mu, \tilde{d}_s^\nu\} = \{\tilde{b}_r^\mu, d_s^\nu\} = \{\tilde{b}_r^\mu, \tilde{d}_s^\nu\} = 0 \ . \end{aligned} \tag{2.56}$$

We can then define the ground state of the R and NS sectors as being annihilated by both the bosonic and fermionic lowering operators:

$$\begin{aligned} &\alpha_m^\mu \left|0\right\rangle_R = d_m^\mu \left|0\right\rangle_R = 0, \quad m > 0 \ , \\ &\alpha_m^\mu \left|0\right\rangle_{NS} = b_r^\mu \left|0\right\rangle_{NS} = 0, \quad m, r > 0 \ . \end{aligned} \tag{2.57}$$

The full spectrum is obtained by acting on the ground states with the bosonic and fermionic creation operators. In the NS sector, the ground state is unique and corresponds to a spin-0 state in spacetime. All states constructed in the NS sector will then also be spacetime bosons, since all oscillators transform as spacetime vectors. The R sector ground state, on the other hand, is a spin-1/2 spacetime spinor, which is degenerate, since the d_0^μ creation operators act on the vacuum without changing its mass. Moreover, the algebra they satisfy is identical to the Dirac algebra:

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \ , \tag{2.58}$$

allowing the expression of the degenerate states as:

$$d_0^\mu \left|\alpha\right\rangle = \frac{1}{\sqrt{2}}\Gamma^\mu_{\alpha\beta}\left|\beta\right\rangle , \tag{2.59}$$

where Γ^μ is a matrix representation of d_0, with spinor indices α, β. States constructed by acting on this vacuum with oscillators will give rise to spacetime fermions.

Similarly to the bosonic string, we can expand the stress-energy tensor and supercurrent in terms of Fourier modes and obtain the super-Virasoro generators. In addition to the bosonic modes introduced in (2.29), we now also have fermionic modes, given by:

$$\begin{aligned} L_m^{(f)} &= \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2}\left(r+\frac{m}{2}\right) : b_{-r}\cdot b_{m+r} : \,, \quad m\in\mathbb{Z}\,, \\ L_m^{(f)} &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(n+\frac{m}{2}\right) : d_{-n}\cdot d_{m+n} : \,, \quad m\in\mathbb{Z}\,, \end{aligned} \tag{2.60}$$

with similar expressions for $\tilde{L}_m^{(f)}$. The Fourier modes of the supercurrent in the R and NS sectors are given by:

$$F_m = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{in\sigma} J_+ = \sum_n \alpha_{-n}\cdot d_{m+n} \tag{2.61}$$

and

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ = \sum_n \alpha_{-n}\cdot b_{r+n}\,. \tag{2.62}$$

The algebra satisfied by the Fourier modes of the stress-energy tensor and supercurrent also depend on the sector. In the R sector, the set $X\{L,\tilde{L},F,\tilde{F}\}$ forms the super-Virasoro algebra defined by the (anti-)commutation relations:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n,0}\,, \\ [L_m, F_n] &= \left(\frac{m}{2}-n\right)F_{m+n}\,, \\ \{F_m, F_n\} &= 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n,0}\,, \end{aligned} \tag{2.63}$$

while the NS sector super-Virasoro algebra is given by:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m+n,0}\,, \\ [L_m, G_r] &= \left(\frac{m}{2}-r\right)G_{m+r}\,, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2-\frac{1}{4}\right)\delta_{r+s,0}\,. \end{aligned} \tag{2.64}$$

The super-Virasoro constraints can then be expressed as conditions on the physical states. In addition to the level-matching condition

$$(L_0 - \tilde{L}_0)\,|\Phi\rangle = 0\,, \tag{2.65}$$

in the R sector, these include:

$$\begin{aligned} &F_n\,|\Phi\rangle = 0\,, \quad n\geq 0\,, \\ &L_m\,|\Phi\rangle = 0\,, \quad m>0\,, \\ &(L_0-\alpha_R)\,|\Phi\rangle = 0\,, \end{aligned} \tag{2.66}$$

with the zero-modes

$$\begin{aligned} L_0^{(f)} &= \frac{1}{2}a_0^2 + N\,, \\ \tilde{L}_0^{(f)} &= \frac{1}{2}\tilde{a}_0^2 + \tilde{N}\,, \end{aligned} \tag{2.67}$$

defined in terms of the number operators:

$$\begin{aligned} N_R &= \sum_{n=1}^{\infty} a_{-n}\cdot a_n + \sum_{n=1}^{\infty} n d_{-n}\cdot d_n\,, \\ \tilde{N}_R &= \sum_{n=1}^{\infty} \tilde{a}_{-n}\cdot \tilde{a}_n + \sum_{n=1}^{\infty} n\tilde{d}_{-n}\cdot \tilde{d}_n\,, \end{aligned} \tag{2.68}$$

The corresponding NS sector conditions are:

$$\begin{aligned} &G_r \left|\Phi\right\rangle = 0 \ , \quad r > 0 \ , \\ &L_m \left|\Phi\right\rangle = 0 \ , \quad m > 0 \ , \\ &(L_0 - \alpha_{NS}) \left|\Phi\right\rangle = 0 \ , \end{aligned} \tag{2.69}$$

with number operators:

$$\begin{aligned} N_{NS} &= \sum_{n=1}^{\infty} a_{-n} \cdot a_n + \sum_{r=1/2}^{\infty} r b_{-r} \cdot b_r \ , \\ \tilde{N}_{NS} &= \sum_{n=1}^{\infty} \tilde{a}_{-n} \cdot \tilde{a}_n + \sum_{r=1/2}^{\infty} r \tilde{b}_{-r} \cdot \tilde{b}_r \ . \end{aligned} \tag{2.70}$$

These conditions imply that the mass of a state is given by $\alpha' M^2 = \tilde{N}_{R,NS} - a_{R,NS} = N_{R,NS} - a_{R,NS}$, where $a_{R,NS}$ are constants that allow a normal-ordering ambiguity. In order for negative norm states to decouple from our theory the constants α_R and α_{NS} must take the values:

$$\alpha_R = 0 \quad \text{and} \quad \alpha_{NS} = \frac{1}{2} \tag{2.71}$$

in $D = 10$ dimensions [153].

As was the case in the bosonic theory, it is convenient to use the residual symmetries to go to the light-cone gauge, where:

$$X^+(\tilde{\sigma}, \tilde{\tau}) = x^+ + l_s^2 p^+ \tau \tag{2.72}$$

and

$$\psi^+(\sigma, \tau) = 0. \tag{2.73}$$

The coordinate X^-, as well as ψ^- are not independent degrees of freedom in the light-cone gauge, and the spectrum of physical excitations is obtained by the action of transverse bosonic and fermionic oscillators. Given that $a_R = 0$, the mass for a state in the R sector is given by:

$$\alpha' M^2 = \sum_{n=1}^{\infty} a^i_{-n} a^i_n + n d^i_{-n} d^i_n \ . \tag{2.74}$$

The ground state is defined by the action of the annihilation operators,

$$\alpha^i_n \left|0; k\right\rangle_R = d^i_n \left|0; k\right\rangle_R = 0, \quad n > 0 \ , \tag{2.75}$$

in addition to the condition

$$F_0^\mu \left|0; k\right\rangle_R = 0 \ , \tag{2.76}$$

which is equivalent to the massless Dirac equation [152]. This ground state is degenerate, as mentioned earlier, since the zero-modes satisfy the ten-dimensional Dirac algebra. The combination of the Dirac equation and the fact that worldsheet fermions are constrained by both Majorana and Weyl conditions reduce the number of independent components to 8 and the ground state transforms as a spinor of $SO(8)$. The excited states are then obtained by the action of either a bosonic or a fermionic oscillator. In either case, the states comprise a tensor product of two spacetime vectors and are therefore massive spacetime fermions.

In the NS sector, $a_{NS} = 1/2$, and the mass is given by:

$$\alpha' M^2 = \sum_{n=1}^{\infty} a^i_{-n} a^i_n + \sum_{r=1/2}^{\infty} r b^i_{-r} b^i_r - \frac{1}{2} \ . \tag{2.77}$$

The ground state is obtained by solving:

$$\alpha^i_n \left|0; k\right\rangle_{NS} = b^i_r \left|0; k\right\rangle_{NS} = 0 \ , \quad n, r > 0 \ , \tag{2.78}$$

as well as

$$\alpha_0^i |0;k\rangle_{NS} = \sqrt{2\alpha'} k^\mu |0;k\rangle_{NS} \tag{2.79}$$

and its mass can be read from (2.77) to be $\alpha' M^2 = -1/2$. The theory is thus still plagued by tachyons, which we will need to eliminate from the spectrum. The first excited state can be obtained by acting on the ground state with both a left-moving and a right-moving oscillator. While we have the option of both bosonic and fermionic modes, the first excitation corresponds to the oscillator with lowest frequency, in which case the states are:

$$b^i_{-1/2}\tilde{b}^j_{-1/2} |0;k^\mu\rangle \ , \tag{2.80}$$

transforming as spacetime vectors.

2.2.3 The GSO Projection and Type II superstrings

The string spectrum outlined above leads to an inconsistent theory, since the ground state of the NS sector is tachyonic. Moreover, while fermions are now present in addition to bosons, the spacetime spectrum is not supersymmetric, as is evident by the absence of a massless gravitino. The spectrum is also inconsistent with respect to the worldsheet modular invariance and is therefore not protected against global anomalies. These issues can all be resolved by a careful truncation of the spectrum, accomplished by the application of the Gliozzi–Scherk–Olive (GSO) projection [99] on physical states, such that:

$$|\psi\rangle \to P_{GSO} |\psi\rangle \ . \tag{2.81}$$

The NS sector GSO projection is given by:

$$P_{GSO} = \frac{1}{2}\left[1-(-1)^F\right] \ , \tag{2.82}$$

where F is the worldsheet fermion number operator, counting the number of b-oscillator excitations:

$$F = \sum_{r=1/2}^{\infty} b^i_{-r}\cdot b^i_r \ . \tag{2.83}$$

In the R sector, the GSO projection is modified by introducing Γ_{11}, the ten-dimensional analogue of the Dirac γ_5 matrix:

$$P\pm_{GSO} = \frac{1}{2}\left[1\mp\Gamma_{11}(-1)^F\right] \ , \tag{2.84}$$

with the fermion number operator:

$$F = \sum_{n=1}^{\infty} d^i_{-n}\cdot d^i_n \ . \tag{2.85}$$

The GSO projection then effectively acts as spacetime chirality, with Γ_{11} projecting fermions with either positive or negative chirality:

$$\Gamma_{11}\psi^\mu = \pm\psi^\mu. \tag{2.86}$$

While this definition for the GSO projection may seem arbitrary, it is well motivated by the requirement for the theory to be invariant under modular transformations at the one-loop and two-loop level. We will return to this in more detail in a later part of this chapter where we will analyse the partition function of the string and derive the conditions that ensure modular invariance. We note here that the result of the GSO projection is the elimination of the NS sector tachyon, as well as the enforcement of spacetime supersymmetry.

The choice of G-parity in the R sector leads to two distinct theories: In type IIB theory, the left-moving and right-moving R sector ground states have the same chirality, which can be taken to be positive, while in type IIA theory they have opposite chiralities. The massless spectrum of the type IIA and IIB theories consists of the following states:

$$\begin{aligned} &|\pm\rangle_R \otimes |+\rangle_R \ , \\ &\tilde{b}^i_{-1/2}|0\rangle_{NS}\otimes b^j_{-1/2}|0\rangle_{NS} \ , \\ &|\pm\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS} \ , \\ &\tilde{b}^i_{-1/2}|0\rangle_{NS}\otimes|+\rangle_R \ , \end{aligned} \tag{2.87}$$

where $|\pm\rangle_R$ denotes the chirality of the R sector ground state.

The R-R sector contains the tensor product of two Majorana–Weyl spinors, with either the same (IIB) or opposite (IIA) chiralities. In the type IIA theory, these states correspond to a vector field and a three-form, while in IIB these are a two-form gauge field and a four-form gauge field with self-dual field strength. The NS-NS spectrum is identical in the two theories, giving rise to a rank two tensor which decomposes into the graviton, dilaton, and Kalb–Ramond two-form. Finally, each of the R-NS and NS-R sectors contains a spin-1/2 dilatino and a spin-3/2 gravitino.

2.2.4 Heterotic string theories

The introduction of fermions on the string worldsheet in conjunction with the GSO projection resulted in the elimination of the tachyon and the appearance of fermionic states in the string spectrum, alleviating the two major deficiencies of the bosonic string and leading to a consistent quantised theory which includes gravity. The type II theories, however, fail to realise a framework in which the SM can be accommodated, as they only admit a $U(1)$ gauge symmetry. In addition to that, the presence of unbroken $\mathcal{N}=2$ supersymmetry excludes the possibility of chirality.

A more careful consideration of the possibilities offered by the decoupling of bosonic and fermionic worldsheet degrees of freedom into independent left-moving and right-moving components, however, provides a framework in which both of the aforementioned issues can be tackled. A hybrid approach to the construction of supersymmetric strings can be realised, in which the left-moving degrees of freedom of the 10-dimensional superstring and the right-moving degrees of freedom of the 26-dimensional bosonic string are combined into a heterotic string [101–103]. In this case, the left-movers carry the supersymmetry charges corresponding to $\mathcal{N}=1$ supersymmetry, while the right-movers carry gauge currents. The critical dimension in this case remains $D=10$, with the 16 additional bosonic degrees of freedom on the right-moving part interpreted as internal degrees of freedom. In order to ensure the absence of tachyons, a GSO projection must also be imposed.

The heterotic string is the only pertrubative theory of closed strings which admits extended gauge symmetries, large enough to accommodate the Standard Model[1]. More specifically, the consistency conditions required by anomaly cancellation allow for 10-dimensional heterotic superstrings with either $SO(32)$ or $E_8\times E_8$ gauge symmetry [100].

There are two equivalent ways to construct the heterotic string: the fermionic and bosonic constructions. In the bosonic construction, the heterotic string is obtained by compactifying 16 dimensions of the bosonic theory on a torus described by a Lorentzian, even, self-dual lattice of dimension 16 [152]. In the fermionic construction, by taking advantage of the correspondence between bosons and fermions in the 2-dimensional conformal field theory of the string worldsheet [154], the 16 additional degrees of freedom of the bosonic string are instead interpreted as a set of 32 real fermionic fields, labeled λ_A, which propagate freely on the string worldsheet. These, together with the 10 left and right-moving bosonic degrees of freedom, as well as the 10 left-moving fermions of the superstring, can be described by the action:

$$S=\frac{1}{\pi}\int d^2\sigma\Big(2\partial_+X_\mu\partial_-X^\mu+i\psi^\mu\partial_-\psi_\mu+i\sum_{A=1}^{32}\lambda^A\partial_+\lambda^A\Big)\,. \tag{2.88}$$

For the left-moving modes, there is a world-sheet supersymmetry defined by the transformations:

$$\begin{aligned}\delta X^\mu&=i\epsilon\psi^\mu\,,\\ \delta\psi^\mu&=-2\epsilon\partial_-X^\mu\,.\end{aligned} \tag{2.89}$$

The worldsheet action of (2.88) has a manifest $SO(32)$ symmetry that rotates the λ^A fermions into each other. As long as all of these fermions have the same boundary conditions, this will give rise to a local $SO(32)$ gauge symmetry in spacetime.

The super-Virasoro conditions introduced in the previous section continue to hold for the left-moving modes of the heterotic string, which satisfy similar constraints to those of type II superstrings, giving rise to a Ramond and a Neveu–Schwarz sector. In the NS sector, the conditions:

$$G_r\,|\Phi\rangle=L_m\,|\Phi\rangle=0\quad,\quad r,m>0\,, \tag{2.90}$$

[1] Type I superstring theory, which contains both open and closed strings, admits an $SO(32)$ gauge symmetry. In this work, however, we will only consider theories of closed strings.

are imposed to obtain on-shell physical states and the mass formula is given by:

$$\left(L_0 - \frac{1}{2}\right)|\Phi\rangle = \left(\frac{p^2}{8} + N_L - \frac{1}{2}\right)|\Phi\rangle = 0 \; , \tag{2.91}$$

where

$$N_L = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1/2}^{\infty} r b_{-r} \cdot b_r \; . \tag{2.92}$$

In the R sector, on-shell states are identified by:

$$F_m |\Phi\rangle = L_m |\Phi\rangle = 0 \quad , \quad m \geq 0 \tag{2.93}$$

with the mass-shell condition:

$$L_0 |\Phi\rangle = \left(\frac{p^2}{8} + N_L\right)|\Phi\rangle = 0 \; , \tag{2.94}$$

where

$$N_L = \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + n d_{-n} \cdot d_n) \; . \tag{2.95}$$

In addition to the above, GSO projections are then imposed on both sectors.

The right-moving fermionic fields λ^A can have periodic or anti-periodic boundary conditions. These boundary conditions give rise to the P and A sectors which are analogous to the R and NS sectors of the superstring respectively. The mode expansion in the P sector is given by:

$$\lambda^A(\tau - \sigma) = \sum_{n \in \mathbb{Z}} \lambda_n^A e^{-2in(\tau - \sigma)} \tag{2.96}$$

with the anti-commutation relation defined by:

$$\{\lambda_m^A, \lambda_n^B\} = \delta^{AB} \delta_{m+n,0} \; . \tag{2.97}$$

In the A sector, the mode expansion for the fermionic fields is:

$$\lambda^A(\tau - \sigma) = \sum_{r \in \mathbb{Z}+1/2} \lambda_r^A e^{-2ir(\tau - \sigma)} \tag{2.98}$$

which satisfies:

$$\{\lambda_r^A, \lambda_s^B\} = \delta^{AB} \delta_{r+s,0} \tag{2.99}$$

The Virasoro constraints for the right-movers are:

$$\widetilde{L}_m |\Phi\rangle = (\widetilde{L}_0 - \tilde{\alpha}) |\Phi\rangle = 0 \quad , \quad m > 0 \; . \tag{2.100}$$

Once again, the P and A sectors need to be evaluated separately. In the P sector we have the condition:

$$(\widetilde{L}_0 - \tilde{\alpha}_P) |\Phi\rangle = \left(\frac{p^2}{8} + N_R - \tilde{\alpha}_P\right)|\Phi\rangle = 0 \; , \tag{2.101}$$

where

$$N_R = \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + n \lambda_{-n}^A \cdot \lambda_n^A) \; , \tag{2.102}$$

while in the A sector:

$$(\widetilde{L}_0 - \tilde{\alpha}_A) |\Phi\rangle = \left(\frac{p^2}{8} + N_R - \tilde{\alpha}_A\right)|\Phi\rangle = 0 \; , \tag{2.103}$$

with

$$N_R = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \sum_{r=1/2}^{\infty} r \lambda_{-r}^A \cdot \lambda_r^A \; . \tag{2.104}$$

The normal ordering constants are then determined to be $\tilde{\alpha}_A = 1$ and $\tilde{\alpha}_P = -1$. An easy way to verify this is by working in the light-cone gauge [153]. The mass formula can then be written as:

$$\begin{aligned} \frac{1}{8}M_P^2 &= N_L = N_R + 1 \,, \\ \frac{1}{8}M_A^2 &= N_L = N_R - 1 \,, \end{aligned} \tag{2.105}$$

for the P and A sectors respectively. In both cases, the physical spectrum is non-tachyonic since $N_{L,R} \geq 0$. Massless states can only be generated in the A sector, as tensor products of left-moving modes with $N_L = 0$ and right-movers with $N_R = 1$. After applying a GSO projection, these states can be shown to be identical to those of the NS sector of the type II theories.

We have thus far only considered the case where the same boundary conditions (A or P) were assigned to all of the right-moving fermions λ^A, respecting the $SO(32)$ symmetry of the action and obtaining an $SO(32)$ gauge symmetry in spacetime. This, however, is not the most general solution. We can consider instead the case where n of the λ^A fermions satisfy A or P boundary conditions, while the other $(32-n)$ fermions satisfy the opposite. This results in four different choices for the normal-ordering constant $\tilde{\alpha}$ and four different sectors, labeled AA, AP, PA and PP, with:

$$\begin{aligned} \tilde{\alpha}_{AA} &= 1 \,, \\ \tilde{\alpha}_{AP} &= \frac{n}{16} - 1 \,, \\ \tilde{\alpha}_{PA} &= 1 - \frac{n}{16} \,, \\ \tilde{\alpha}_{PP} &= -1 \,. \end{aligned} \tag{2.106}$$

Physical states in each sector must then obey a level-matching condition $N_L = N_R - \tilde{\alpha}$. Since N_L has integer eigenvalues, while N_R can be integer of half-integer, $\tilde{\alpha}$ must be an integer or half-integer, implying that n must be a multiple of 8.

The $n = 0$ or $n = 32$ cases trivially reproduce the $SO(32)$ theory, while $n = 8$ and $n = 24$ lead to a spectrum with gauge anomalies and must therefore be ruled out. The only remaining choice that leads to an anomaly free theory is therefore $n = 16$. In this case, the AP and PA sectors have vanishing normal-ordering constants, and can therefore contribute to the massless states, while the AA and PP sectors are identical to the A and P sectors of the $SO(32)$ theory. After imposing the GSO projection, the massless spectrum can organised in supermultiplets of $E_8 \times E_8$, as we will show in chapter 3, in the framework of the free fermionic formulation.

We conclude this section by noting that while we have so far limited the discussion of string theories to the case of critical dimensions, it is also possible to formulate the theory in $D < 10$. A more detailed discussion on this will follow in chapter 3, while chapters 4, 5, and 6 will focus on the construction and analysis of 4-dimensional models.

2.3 Modular Invariance and the String Partition Function

While the focus of our attention so far have been freely propagating strings, string theory also provides an elegant mathematical background in which string interactions can be described in a similar manner to the perturbation theory approach of Quantum Field Theory. In typical perturbative QFT, scattering amplitudes are calculated using an approximation in which the strength of the interaction is considered to be small. The amplitude is then written in the form of a perturbative expansion in terms of the coupling constant and the processes can be encoded in the form of Feynman diagrams. Figure 2.2 illustrates a tree-level example.

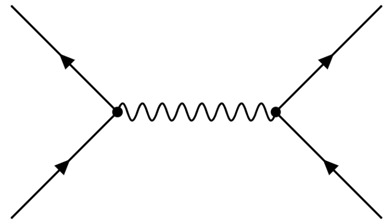

Figure 2.2: A tree-level Feynman diagram depicting the scattering of two fermions.

In the framework of string theory, any interaction can be described as a process in which strings either join or split. This process can then be represented by the combination of the worldsheets of all strings that take part in the interaction, in a string theoretical version of Feynman diagrams. All interactions can then be reduced to a description purely in terms of a single worldsheet, from which the amplitude can be calculated. An example is presented in figure 2.3. This is in contrast to the QFT framework in which multiple Feynman diagrams are needed to account for all possible interactions at a specific loop-level.

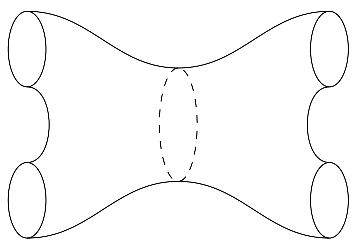

Figure 2.3: A stringy Feynman diagram representing a scattering process with two incoming and two outgoing strings.

Another notable difference concerns the properties of the interaction vertex. In QFT, this vertex corresponds to a specific point in spacetime where the interaction takes place. This singularity leads to a divergence in the UV regime which needs to be tamed. In string theory, the picture is different. Due to the extended nature of strings, the interaction vertex has a finite size. String theory smooths out the singularity by introducing a natural cutoff, the string length, and protects the theory from UV divergences.

Since string worldsheets are described by two dimensional Riemann manifolds, the perturbative expansion in the framework of string theory must involve a summation over all inequivalent manifolds. The expansion in that case essentially becomes a sum over topologies, as illustrated in figure 2.4, with higher order terms of the coupling constant corresponding to manifolds of higher genus.

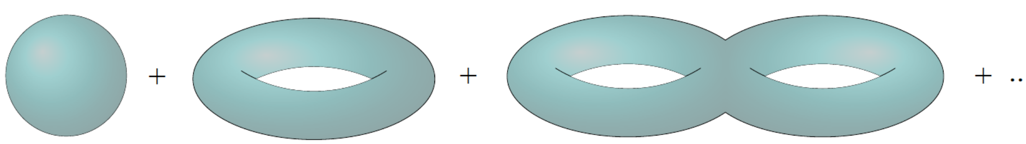

Figure 2.4: The perturbation theory expansion of an interaction vertex depicted as a sum over topologies. At tree-level, the vertex corresponds to a worldsheet with the topology of a sphere, while n-loop calculations correspond to tori of genus n.

In order to analyse string interactions while taking into account the effects of quantum corrections, we will focus our attention on the toroidal one-loop worldsheet. In principle, we also need to consider higher order calculations. While such calculations at the two loop level have been performed in specific cases [144, 172–178] this is in general a very complicated procedure, especially when supersymmetry is broken, as is the case in our work.

If we omit all external lines, the one-loop worldsheet will directly correspond to the vacuum-to-vacuum amplitude, generating a contribution to the vacuum energy of the theory. While the exact calculation of this is of high importance, in light of the cosmological constant problem of QFT, the discussion of this section will only concern the consistency of this calculation in general. Precise calculations in four dimensional constructions will be provided in chapters 5 and 6.

The genus-1 torus of the one-loop string worldsheet can be interpreted as the quotient of the complex plane modulo a lattice. This mapping can be obtained by cutting the torus into a cylinder, slicing it along its length then unfolding it into a plane. Due to the scale invariance of the string worldsheet, the resulting rectangle can be described by a single parameter, as one side can always be taken to be equal to unity. The lattice, as depicted in figure 2.5, can then be parametrised by a complex modulus $\tau = \tau_1 + i\tau_2$, which takes into account the possibility of twisting the cylinder. The unit cell which tiles the complex

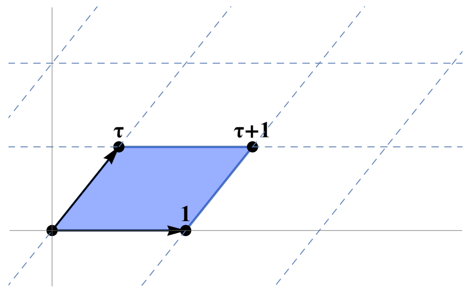

Figure 2.5: The representation of a torus as a plane modulo a two dimensional lattice, defined by the vectors "1" and "τ". The shaded region corresponds to the unit cell.

plane can be described by two basis vectors which cannot, however, be uniquely defined. In fact, the worldsheet is invariant under the modular transformations:

$$\begin{aligned} T &: \tau \to \tau + 1 \ , \\ S &: \tau \to -\frac{1}{\tau} \ . \end{aligned} \tag{2.107}$$

These transformations generate the modular group $PSL(2;\mathbb{Z}) = SL(2;\mathbb{Z})/\mathbb{Z}_2$, whose general transformations are given by:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \ . \tag{2.108}$$

Since the worldsheet is invariant under the modular transformations, all physical quantities must be invariant under these transformations as well. As we will see in the later parts of this chapter, modular invariance will be a key ingredient in the construction of consistent theories.

Having parametrised the one-loop worldsheet by the τ modulus, the integration over all moduli $\int d\Omega$ in the calculation of the one-loop amplitude then amounts to an integration over all τ: $\int d\tau d\bar{\tau} = \int d^2\tau$. Due to the modular transformations, however, the domain of integration must be restricted in order to avoid over-counting. More specifically, the region in the interior of the unit circle is mapped to the exterior and vice-versa due to the S transformation, while the T transformation identifies the two lines at $x = \pm 1/2$ with each other. The moduli space of interest then, as long as modular invariance holds, is the fundamental domain defined as:

$$\mathcal{F} = \{\tau \in \mathbb{C}^+ : |\tau| \geq 1, |\mathrm{Re}\,\tau| \leq 1/2\} \ , \tag{2.109}$$

which we illustrate in figure 2.6.

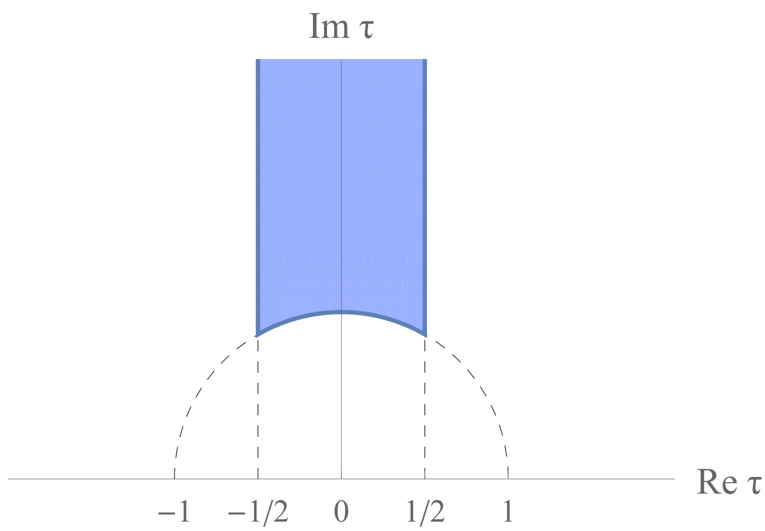

Figure 2.6: The fundamental domain of the torus, $\mathcal{F} = \{\tau \in \mathbb{C}^+ : |\tau| \geq 1, |\mathrm{Re}\,\tau| \leq 1/2\}$.

The restriction of the integration to the fundamental domain has a very important consequence: the point at 0 is excluded from the integration, implying that, as long as our theory is modular invariant, UV divergences can be avoided. The integration measure $d\tau d\bar{\tau}$ is not modular invariant, since under the modular transformations of equation (2.108):

$$d^2\tau \to d^2\tau' = |c\tau + d|^{-4} d^2\tau \; . \tag{2.110}$$

In order to restore modular invariance, it is therefore necessary to add a term to the integration measure. It is easy to check that the term needed is the square of the imaginary part of τ. Integration over all moduli then amounts to:

$$\int d\Omega = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \; . \tag{2.111}$$

The standard path integral in quantum mechanics reads:

$$\int Dq\, e^{-S[q]} = \mathrm{Tr}(e^{-\beta H}) \; , \tag{2.112}$$

where $q = q(t)$ is a coordinate, and we consider a time interval $0 \leq t \leq \beta$. In the lattice introduced above, the real axis corresponds to the σ^1 direction and the imaginary axis to the Euclidean time. If $\tau_1 = 0$, then the path integral takes the form:

$$\int DX\, e^{-S_E[X]} = \mathrm{Tr}(e^{-2\pi\tau_2 H}) \; . \tag{2.113}$$

When $\tau_1 \neq 0$, we have an additional contribution that represents the twisting of the cylinder. This contribution can be traced to the action of the momentum operator P, as a rotation by $2\pi\tau_1$ is achieved by $e^{iP(2\pi\tau_1)}$. The most general expression for the path integral therefore is:

$$\int Dq\, e^{-S[q]} = \mathrm{Tr}(e^{-2\pi\tau_2 H + 2\pi\tau_1 P}) \; . \tag{2.114}$$

The momentum and Hamiltonian operators can be expressed in terms of the zero-mode Virasoro operators:

$$\begin{aligned} H &= L_0 - 1 + \tilde{L}_0 - 1 \; , \\ P &= L_0 - \tilde{L}_0 \; . \end{aligned} \tag{2.115}$$

By substituting these expressions in equation (2.114), the path integral can be brought to the form:

$$\int Dq\, e^{-S[q]} = \mathrm{Tr}\left[e^{2\pi i\tau(L_0-1) - 2\pi i\bar{\tau}(\tilde{L}_0 - 1)} \right] \; . \tag{2.116}$$

In the heterotic string theories under consideration, the left-moving part of the string is taken to be an RNS superstring, while the right-moving part is described by the bosonic string, as previously mentioned. In order to arrive at the partition function of the full heterotic string, we need to identify the correct expressions for the contributions of bosons and fermions.

2.3.1 Bosonic Partition Functions

The first step is to split the trace of equation (2.116) into a left-moving and a right-moving part. These will be functions of τ and $\bar{\tau}$ respectively. The traces we need to compute, can be cast in the form of an expansion:

$$Tr\left[e^{2\pi i\tau(L_0-1)}\right] = \sum d_n q^n \ , \tag{2.117}$$

where $q = e^{2\pi i\tau}$ and d_n counts the number of states with $L_0 - 1$ eigenvalue n. The Hilbert space of physical states is obtained by acting on the vacuum with creation operators. In the simplest case of one oscillator a_{-n}, the trace yields:

$$q^{-1} + q^{n-1} + q^{2n-1} + \cdots = q^{-1}\frac{1}{1-q^n} \ . \tag{2.118}$$

Adding more oscillators then amounts to taking a product of such expansions, since each oscillator acts independently. The full contribution of a set of D oscillators will thus be:

$$q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-D} \ . \tag{2.119}$$

This can be expressed in terms of the Dedekind eta function:

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \ . \tag{2.120}$$

If we take the number of dimensions to be $D = 26$, as is the case in critical bosonic string theory, and consider the corresponding right-moving oscillators, the partition function is given by:

$$Z_{\mathrm{B,osc}}(\tau,\bar{\tau}) = [\eta(\tau)\cdot\eta(-\bar{\tau})]^{-24} = \eta(\tau)^{-24}\cdot\bar{\eta}(\bar{\tau})^{-24} = |\eta(\tau)|^{-48} \ . \tag{2.121}$$

Having identified the contribution of bosonic oscillators, we now need to compute the contribution of the momentum excitations of the vacuum. Since these can take continuous values, instead of a trace we will now have an integral:

$$Z_{\mathrm{B,mom}}(\tau,\bar{\tau}) = \int \frac{d^{D-2}p}{(2\pi)^{D-2}} e^{-2\pi\tau_2(\frac{1}{2}p^2)} = (2\pi\sqrt{\tau_2})^{2-D} \ . \tag{2.122}$$

All things considered, the one-loop integral for the 26 dimensional bosonic string is found to be:

$$\int_{\text{1-loop}} D[\Phi]e^{-S_E[\Phi]} \sim \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}(\sqrt{\tau_2}|\eta(\tau)|^2)^{-24} \ . \tag{2.123}$$

Under modular transformations, $\eta(\tau)$ has the following properties:

$$\begin{aligned} \eta(\tau+1) &= e^{i\pi/12}\eta(\tau) \ , \\ \eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau) \ . \end{aligned} \tag{2.124}$$

It is easy to check that the combination $\sqrt{\tau_2}|\eta(\tau)|^2$ is in fact modular invariant, since:

$$\begin{aligned} |\eta(\tau+1)| &= |\eta(\tau)| \ , \\ |\eta(-1/\tau)|^2 &= |\tau||\eta(\tau)|^2 \ . \end{aligned} \tag{2.125}$$

The extra $|\tau|$ factor that appears in the S-transformation cancels against the factor coming from the transformation of the momentum contribution $\sqrt{\tau_2}$.

2.3.2 Fermionic Partition Functions

Having dealt with the bosonic string, we now turn our attention to the second main element of the heterotic string. The RNS superstring lives in $D = 10$ dimensions and its worldsheet spectrum in the light-cone gauge consists of 8 bosons and 8 fermions both on the left and the right-moving part of the string. The bosonic contribution to the partition function, as follows from the discussion above will be:

$$Z_B = (\tau_2)^{-4} |\eta(\tau)|^{-16} . \tag{2.126}$$

In the case of worldsheet fermions, the oscillators acting on the vacuum anti-commute. Each fermionic oscillator can therefore act on the NS sector ground state at most once. If we consider a single fermionic oscillator, this will contribute:

$$\text{Tr } q^{L_0} = q^a(1 + q^r) , \tag{2.127}$$

where the factor q^a accounts for the vacuum energy. This can be generalised to the case of multiple oscillators, in which case we have:

$$Z_{NS}(q) = q^{-1/48} \prod_{r=1/2}^{\infty} (1 + q^r) . \tag{2.128}$$

In the R sector, the oscillator modes take integer values. The ground state is a spinorial representation of $SO(N)$ and the contribution to the partition function is:

$$Z_R(q) = \sqrt{2} q^{1/24} \prod_{n=1}^{\infty} (1 + q^n) . \tag{2.129}$$

Since the purpose of partition function coefficients is to count the number of states, the $\sqrt{2}$ factor might appear to be problematic. This is not the case, however, since fermions will always come in pairs and expressions of the form $Z_R(q)^N$, which appear in the partition function, will not be problematic when N is even. If we also consider the effect of right-movers, then the product $Z_R(q)^N Z_R(\bar{q})^N$ will have integer coefficients for all N.

The two kinds of partition functions introduced above correspond to taking either anti-periodic (NS) or periodic (R) boundary conditions along the real axis. In addition to that, due to the topology of the torus, we are free to choose periodic or anti-periodic boundary conditions along the imaginary axis as well. The path integral corresponding to the choice of anti-periodic boundary conditions along the imaginary axis is:

$$\int_A D\psi e^{-S(\psi)} = \text{Tr } e^{-\beta H} . \tag{2.130}$$

Choosing periodic boundary conditions on the other hand, amounts to making a twist in the cylinder and is equivalent to flipping the sign of each fermion. In this case, we have:

$$\int_P D\psi e^{-S(\psi)} = \text{Tr } \left[(-1)^F e^{-\beta H}\right] , \tag{2.131}$$

where F is the fermion number operator:

$$F = \sum_r : b_r \cdot b_{-r} : \tag{2.132}$$

which has the property $(-1)^F b = -b(-1)^F$ for any fermionic oscillator b_n.

In order to compute this trace, we need to determine the effect the operator $(-1)^F$ has on the ground state. In the NS sector, it is enough to require that the fermion number of the vacuum is zero, as this naturally leads to:

$$(-1)^F |0\rangle = |0\rangle \tag{2.133}$$

and

$$Z_{NS,P} = \text{Tr } \left[(-1)^F q^{L_0 - 1/48}\right] = q^{L_0 - 1/48} \prod_{r=1/2}^{\infty} (1 - q^r) . \tag{2.134}$$

In the Ramond sector, the addition of d_0 oscillators transforms the ground state back to itself, while flipping the fermion number. The ground states then split into even and odd, depending on the eigenvalue of $(-1)^F$ and the trace vanishes:

$$Z_{R,P} = 0 \; . \tag{2.135}$$

The same can be shown to hold for all excited states.

All things considered, the fermionic partition function involves a combination of periodic and anti-periodic boundary conditions on the real and imaginary axes, corresponding to the R-R, R-NS, NS-R and NS-NS sectors. This partition function can be expressed in terms of the Jacobi theta functions with characteristics:

$$\vartheta[^a_b](\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi[(n-a/2)^2\tau-(n-a/2)b]} \; . \tag{2.136}$$

The particular cases of interest are those in which the characteristics a, b take the values $0, 1$. In order to simplify the notation, we use the standard definitions:

$$\begin{aligned} \vartheta[^1_1](\tau) &\equiv \vartheta_1(\tau) \; , \\ \vartheta[^1_0](\tau) &\equiv \vartheta_2(\tau) \; , \\ \vartheta[^0_0](\tau) &\equiv \vartheta_3(\tau) \; , \\ \vartheta[^0_1](\tau) &\equiv \vartheta_4(\tau) \; . \end{aligned} \tag{2.137}$$

The partition functions for the four sectors can then be summarised in the following table [155]:

Sector	Partition Function	
R-R	$\text{Tr}_R \; [(-1)^F q^{L_0-c/24}]$	$= (\vartheta_1/\eta)^{N/2}$
NS-NS	$\text{Tr}_{NS} \; [q^{L_0-c/24}]$	$= (\vartheta_3/\eta)^{N/2}$
R-NS	$\text{Tr}_R \; [q^{L_0-c/24}]$	$= (\vartheta_2/\eta)^{N/2}$
NS-R	$\text{Tr}_{NS} \; [(-1)^F q^{L_0-c/24}]$	$= (\vartheta_4/\eta)^{N/2}$

Table 2.1: *The four possible fermionic partition functions for N real fermions in terms of Jacobi and Dedekind functions.*

At this point, there are four possible choices for the boundary conditions leading to four different possible contributions to the partition function. In order to find out how to construct a modular invariant partition function for the fermions out of them, we need to investigate the modular transformations of the Jacobi functions. These can be determined to be:

$$\begin{aligned} \vartheta_1(\tau+1) &= e^{i\pi/4}\vartheta_1(\tau) \; , & \vartheta_1(-1/\tau) &= i\sqrt{-i\tau}\vartheta_1(\tau) \; , \\ \vartheta_2(\tau+1) &= e^{i\pi/4}\vartheta_2(\tau) \; , & \vartheta_2(-1/\tau) &= \sqrt{-i\tau}\vartheta_4(\tau) \; , \\ \vartheta_3(\tau+1) &= \vartheta_4(\tau) \; , & \vartheta_3(-1/\tau) &= \sqrt{-i\tau}\vartheta_3(\tau) \; , \\ \vartheta_4(\tau+1) &= \vartheta_3(\tau) \; , & \vartheta_4(-1/\tau) &= \sqrt{-i\tau}\vartheta_2(\tau) \; . \end{aligned} \tag{2.138}$$

The partition functions for the left-moving part of the RNS superstring corresponding to the four types of boundary conditions are:

$$P_i(\tau) = \eta(\tau)^{-8}\left(\frac{\vartheta_i(\tau)}{\eta(\tau)}\right)^4 \; , \tag{2.139}$$

which therefore have the following properties

$$\begin{aligned} P_1(\tau+1) &= P_1(\tau) \; , & P_1(-1/\tau) &= \tau^{-4}P_1(\tau) \; , \\ P_2(\tau+1) &= P_2(\tau) \; , & P_2(-1/\tau) &= \tau^{-4}P_4(\tau) \; , \\ P_3(\tau+1) &= -P_4(\tau) \; , & P_3(-1/\tau) &= \tau^{-4}P_3(\tau) \; , \\ P_4(\tau+1) &= -P_3(\tau) \; , & P_4(-1/\tau) &= \tau^{-4}P_2(\tau) \; . \end{aligned} \tag{2.140}$$

The above transformations clearly show that it is impossible to construct a modular invariant partition function by choosing only one type of boundary conditions. What we need is a specific linear combination of the four P_i which leaves the partition function invariant, the GSO projection [99]:

$$P_{GSO} = \frac{1}{2}\Big(P_3(\tau) - P_4(\tau) - P_2(\tau) \pm P_1(\tau)\Big) \ . \tag{2.141}$$

The sign difference between $P_3(\tau)$ and $P_4(\tau)$ in the GSO projection is there to cancel the sign difference arising from the transformations. In the NS sector, the combination

$$\frac{1}{2}\big(P_3(\tau) - P_4(\tau)\big) = \mathrm{Tr}_{NS}\left[\frac{1}{2}\left(1-(-1)^F\right) q^{L_0-1/2}\right] \tag{2.142}$$

acts as a projection operator to the partition function, eliminating all states that have even fermion number, including the tachyon. The coefficient of $P_1(\tau)$ is undetermined at the one-loop level, but can be fixed to ±1 by requiring that modular invariance continues to hold at two-loops. In the R sector, the combination

$$\frac{1}{2}\big(P_2(\tau) \pm P_1(\tau)\big) = \mathrm{Tr}_{R}\left[\frac{1}{2}\left(1\pm(-1)^F\right) q^{L_0-1}\right] \tag{2.143}$$

projects out half of the states, depending on their chirality. The free choice of sign leads to two different partition functions, corresponding to the two type II string theories: type IIB, when the same sign is chosen for the left-movers and right-movers and type IIA when opposite signs are chosen.

Altogether, the full partition function for the RNS superstring in $D = 10$ dimensions is:

$$Z_{\mathrm{II}}(\tau,\bar\tau) = (\tau_2)^{-4} P_{GSO}(\tau) P_{GSO}(-\bar\tau) \ . \tag{2.144}$$

2.3.3 Bosonisation / Fermionisation

The hybrid approach of heterotic string theory in $D = 10$ dimensions exhibits a number of appealing features including extended gauge symmetries, indicating a possibility of realising a consistent theory unifying all interactions. In order to make contact with low energy physics, however, the question of how to reduce this theory to four dimensions must then be addressed. In $D = 10$ dimensions, the construction of the heterotic string involves a bosonic string whose additional degrees of freedom needed to eliminate the conformal anomaly are encoded in 32 freely propagating internal fermions. An alternate way of describing the theory is to regard the construction of the heterotic string as a compactification of the bosonic string from $D = 26$ to $D = 10$ dimensions, in which case all right-moving degrees of freedom are bosonic. The construction of heterotic strings in $D < 10$ dimensions can then be seen as a further compactification which includes the left-moving sector of the theory.

The equivalence of the two descriptions [124, 179–184] is based on the idea of bosonisation / fermionisation, in which the theory is invariant under the exchange of fermionic variables with bosonic ones and vice versa [185–187]. This is a property inherent to two-dimensional field theories since all representations of the $SO(2)$ Lorentz group are one-dimensional. This allows us to alternate between a fermionic and bosonic point of view, taking advantage of the tools offered by both formulations in order to investigate large classes of models. This dual approach is instrumental in the analysis of string theoretical constructions in four dimensions, which are the focal point of this thesis, and the subject of chapters 4, 5, and 6.

In this section, we will highlight a simple example where the partition functions of bosons and fermions can be seen to match. In order to show this, we need to consider the effect compactification has on the bosonic partition function. Our starting point will be the oscillator and momentum contributions of (2.119) and (2.122). While the oscillator contribution is unaffected by the compactification, the momentum contribution needs to be modified in order to properly account for the quantisation of the momentum along the compactified direction. The discrete nature of the momentum eigenvalues then naturally leads to a description in terms of a lattice Γ, contributing a multiplicative factor of

$$Z_\Gamma(\tau,\bar\tau) = \sum_{(\vec p_L,\vec p_R)\in\Gamma} e^{i\pi\tau\vec p_L^{\,2}} e^{-i\pi\bar\tau\vec p_R^{\,2}} \tag{2.145}$$

to the mass of states, instead of the factor $(1/\sqrt{\tau_2})$ that is present when momentum takes continuous values. In order for the partition function to remain modular invariant, the lattice Γ on which the momenta lie must be Lorentzian, even, and self-dual [188].

Consider now a pair of real left-moving fermions in the NS-NS sector. Following the analysis of the previous section, their partition function will be given by:

$$\mathrm{Tr}_{NS}[q^{L_0-c/24}] = \frac{\vartheta_3}{\eta} = \frac{1}{\eta}\sum_{n\in\mathbb{Z}} q^{n^2/2} \ , \tag{2.146}$$

which clearly reproduces the oscillator and momentum contributions of a left-moving boson. This result can also be shown to hold in the NS-R, R-NS and R-R sectors, due to the properties of $\vartheta[^a_b]/\eta$, implying that a boson and a pair of real fermions are equivalent at the partition function level.

This equivalence can also be shown to hold for the linear combination

$$\frac{1}{2}\left[\left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^4 \pm \left(\frac{\vartheta_1(\tau)}{\eta(\tau)}\right)^4\right] \tag{2.147}$$

entering the GSO projection, which can be brought to the form

$$\frac{1}{\eta(\tau)}\sum_{\vec{r}\in\Lambda^{\pm}} q^{r^2}(-1)^F \ , \tag{2.148}$$

where F is the spacetime fermion number, and the lattices $\Lambda^{\pm}$ are defined by vectors of the form:

$$\begin{aligned}
&(n_1,n_2,n_3,n_4) \ , \quad n_i\in\mathbb{Z} \ , \quad \sum_i n_i = odd \ , \\
&\left(n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}, n_4+\frac{1}{2}\right) \ , \quad n_i\in\mathbb{Z} \ , \quad \sum_i n_i = \begin{cases} odd \ , & \Lambda^+ \\ even \ , & \Lambda^- \end{cases} \quad .
\end{aligned} \tag{2.149}$$

It is therefore possible to bosonise the left-moving fermions of the RNS string and obtain a description of the heterotic string purely in terms of bosons. Alternatively, by fermionising the bosonic degrees of freedom we can obtain an equivalent description purely in terms of fermions. The bosonic and fermionic points of view will be introduced in the following chapter, with the latter applied in the classification of 10-dimensional heterotic string theories. The equivalence of the two descriptions is highlighted in chapter 4, in which we construct four-dimensional models, showcasing the dual approach we employ in chapters 5 and 6 to tackle more realistic vacua.

Chapter 3

Heterotic Strings in $D \leq 10$ Dimensions

Up to this point, we have only considered string theories living in spacetimes with ten dimensions. The focus of this chapter is the introduction of the mathematical tools that allow the investigation of constructions in $D \leq 10$ dimensions. We introduce the Free Fermionic Formulation (FFF) [189–193] and proceed with the derivation and classification of all possible string theories in $D = 10$. We then discuss the finiteness of the theory in the absence of supersymmetry. We complete this chapter by discussing string compactifications to lower dimensions, introducing the orbifolds we will utilise in the following chapters.

3.1 The Free Fermionic Formulation

We begin this chapter by outlining the framework of the free fermionic formulation [189–193], which can be implemented to construct heterotic string theories directly in $D \leq 10$ dimensions. The free fermionic formulation employs fermionisation, expressing bosonic degrees of freedom in terms of worldsheet fermions. In this formalism, compactification takes place on tori whose radii are fixed to the self-dual point, termed the "fermionic point", in which case the Thirring coupling vanishes and we obtain a theory purely in terms of free fermions.

The D-dimensional left and right-moving conformal anomaly of the heterotic string is given by [149, 150]:

$$\begin{aligned} c_L &= -15 + \frac{3D_L}{2} \ , \\ c_R &= -26 + D_R \ , \end{aligned} \tag{3.1}$$

which vanish in the $D = 10$ heterotic theory. Canceling the conformal anomalies and restoring conformal invariance in $D < 10$ necessitates the introduction of additional degrees of freedom on the string worldsheet. In the general case, where $D \leq 10$, the vanishing of the conformal anomaly requires $3(10 - D)$ real Majorana–Weyl fermions on the supersymmetric (left-moving) part of the string and $2(26 - D)$ real fermions on the non-supersymmetric (right-moving) part. These are needed in addition to the $X^\mu_\pm$, ψ^μ_+ and λ^A_- which parametrise the $D = 10$ theory. All things considered, the worldsheet degrees of freedom in light-cone gauge consist of:

(i) $D - 2$ left-moving and $D - 2$ right-moving bosons: X^μ, $\bar{X}^\mu$

(ii) $D - 2$ real, left-moving fermions: ψ^μ

(iii) $3(10 - D)$ real, left-moving fermions: $\chi^{1,\dots,10-D}, y^{1,\dots,10-D}, \omega^{1,\dots,10-D}$

(iv) $2(10 - D)$ real, right-moving fermions: $\bar{y}^{1,\dots,10-D}, \bar{\omega}^{1,\dots,10-D}$

(v) 16 complex, right-moving fermions: $\bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\dots,8}$

Here the fermions are labeled using the conventional notation we will employ in four dimensional constructions. Essentially, the $D < 10$ framework boils down to taking the bosonic and RNS strings and separating the degrees of freedom corresponding to the $10-D$ compactified dimensions from those corresponding to the D extended ones. X^μ and $\bar{X}^\mu$ are the left-moving and right-moving modes of the coordinate vector in the light-cone gauge of a D-dimensional spacetime, while ψ^μ are the fermionic superpartners of X^μ, as required by supersymmetry on the left-moving part of the string.

The $10-D$ compactified dimensions would normally admit an identical description, in terms of left and right-moving bosons: $X^{1,\dots,10-D}$, $\bar{X}^{1,\dots,10-D}$, and the corresponding real left-moving fermions, labeled $\psi^{1,\dots,10-D}$ would be required to implement supersymmetry. These bosons are instead fermionised by replacing each one with a pair of fermions y, ω. The corresponding $\psi^{1,\dots,10-D}$ fermions are then relabeled to $\chi^{1,\dots,10-D}$.

The remaining right-moving complex fermions are responsible for the gauge symmetry of the D-dimensional theory and the naming scheme will become clear in later chapters where we focus on $D=4$ constructions with $SO(10)$ and Pati–Salam gauge symmetries.

The physical content of the theory can then be determined by the partition function. At the one-loop level, the toroidal topology of the string worldsheet requires the assignment of boundary conditions to the bosonic and fermionic degrees of freedom. While the boundary conditions of the bosonic fields are fixed by the worldsheet metric, the boundary conditions of fermions are restricted by conformal invariance, the worldsheet supersymmetry [192, 194], as well as invariance under both local reparametrisations and modular transformations. When transported around the non-contractible loops of the two-dimensional torus, fermions may acquire phase factors

$$f_A \to -e^{i\pi a_i(f_A)} f_A \ , \tag{3.2}$$

where $a_i(f_A) \in (-1, 1]$. Consider, for example, a theory with m left-moving and n right-moving fermions. The boundary conditions of all fermions for any non-contractible loop a can be described by an $m+n$ component vector:

$$\mathbf{a} = \{a_1(f_1^l), \dots, a_m(f_m^l) | a_{m+1}(f_1^r), \dots, a_{m+n}(f_n^r)\} \ , \tag{3.3}$$

where $a(f_i) = 1$ and $a(f_i) = 0$ correspond to periodic and anti-periodic boundary conditions respectively[1]. These boundary conditions can be independently imposed on the spatial and temporal directions along the string worldsheet. The allowed choices of boundary conditions are then determined by modular invariance conditions which define the spin structure [195].

The vast number of distinct models arising from variations in the boundary conditions of fermions are in turn defined by the choice of spin structures, which can be expressed by a set of basis vectors [193]. A set of n such vectors, labeled u_i, then builds up a finite additive group of sectors

$$\Xi = \sum_{i=1}^{n} m_i u_i \ , \quad m_i = 0, \dots, N_i - 1 \ , \tag{3.4}$$

where N_i is the smallest integer satisfying $N_i u_i = 0 \bmod 2$. This implies that Ξ is isomorphic to an additive set of $\mathbb{Z}_N$ factors $\mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_k}$, suggesting that $\Xi = 0$ if and only if $m_i = 0 \bmod N_i$ for all i, in which case the basis $\{u_1, \dots, u_n\}$ is said to be canonical.

Assuming all fermions are taken to be complex for simplicity, the one-loop partition function can then be written as:

$$Z = \int \frac{d\tau d\bar{\tau}}{\tau_2^2} \sum_{\mathbf{a},\mathbf{b} \in \Xi} c\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix} \left[\sqrt{\tau_2}\,\eta(\tau)\bar{\eta}(\bar{\tau})\right]^{-(D-2)} \prod_{f_\ell=1}^{14-D} \frac{\vartheta\begin{bmatrix}\mathbf{a}(f_\ell)\\\mathbf{b}(f_\ell)\end{bmatrix}(\tau)}{\eta(\tau)} \prod_{f_r=1}^{26-D} \frac{\overline{\vartheta}\begin{bmatrix}\mathbf{a}(f_r)\\\mathbf{b}(f_r)\end{bmatrix}(\bar{\tau})}{\bar{\eta}(\bar{\tau})}, \tag{3.5}$$

encompassing the contributions of the $D-2$ left and right-moving bosons as well as the left and right-moving fermions.

In order to define a specific model, we then need to determine the exact form of all basis vectors u_i, $i = 1, \dots, n$ encoding the boundary conditions of fermions, and fix all corresponding $c\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}$ phases. The requirement of a modular invariant partition function, along with the modular transformation properties of the Jacobi and Dedekind functions as covered in the previous chapter, govern the behaviour of the

[1] In addition to (anti-)periodic boundary conditions, consistent theories including complex phases can also be realised. Such constructions are in fact necessary for the construction of four dimensional theories realising the SM gauge symmetry, but will not be considered in this thesis.

spin structures $c[^{\mathbf{a}}_{\mathbf{b}}]$ under modular transformations, imposing a set of conditions known as the ABK rules [189, 190], which all consistent realisations must satisfy. A one-loop level evaluation reveals the constraints:

$$c[^{\alpha}_{\beta}] = e^{i\pi(\alpha\cdot\alpha+\mathbb{1}\cdot\mathbb{1})/4} c[^{\alpha}_{\beta-\alpha+\mathbb{1}}] \tag{3.6}$$

and

$$c[^{\alpha}_{\beta}] = e^{i\pi(\alpha\cdot\beta)/2} c[^{\beta}_{-\alpha}]^* \,, \tag{3.7}$$

where we have introduced the basis vector $\mathbb{1}$ corresponding to periodic boundary conditions for all fermions, whose presence in the basis is necessitated by consistency requirements. The dot product gives half weight to real fermions and opposite signs to left and right-movers:

$$\mathbf{a}\cdot\mathbf{b} \equiv \left[\frac{1}{2}\sum_{\substack{\text{real}\\ \text{left-moving}}} + \sum_{\substack{\text{complex}\\ \text{left-moving}}} - \frac{1}{2}\sum_{\substack{\text{real}\\ \text{right-moving}}} - \sum_{\substack{\text{complex}\\ \text{right-moving}}}\right] \mathbf{a}(f)\mathbf{b}(f) \,. \tag{3.8}$$

Imposing modular invariance at the two-loop level leads to the additional condition:

$$c[^{\alpha}_{\beta}]c[^{\alpha'}_{\beta'}] = \delta_\alpha \delta_{\alpha'} e^{-i\pi\alpha\cdot\alpha'/2} c[^{\alpha}_{\beta+\alpha'}]c[^{\alpha'}_{\beta'+\alpha}] \,. \tag{3.9}$$

with the spacetime spin-statistics index δ_α defined as:

$$\delta_\alpha = \begin{cases} +1, & \alpha(\psi^\mu) = 0 \\ -1, & \alpha(\psi^\mu) = 1 \end{cases} \,. \tag{3.10}$$

Equation (3.9) further implies that $c[^{0}_{0}] = 1$, as well as $c[^{\alpha}_{0}] = \delta_\alpha$, where 0 in this case corresponds to anti-periodic conditions for all fermions. If we define the set of vectors $\Xi = \{\alpha | c[^{\alpha}_{0}] = \delta_\alpha\}$, we can see that for any two vectors $\beta, \gamma \in \Xi$, the sum $\beta + \gamma$ is also part of the set:

$$c[^{\alpha}_{\beta+\gamma}] = \delta_\alpha c[^{\alpha}_{\beta}]c[^{\alpha}_{\gamma}] \,. \tag{3.11}$$

If all the boundary conditions in the vector α are rational, Ξ is finite.

Moreover, from (3.6) we can see that

$$c[^{\alpha}_{\alpha}] = e^{i\pi(\alpha\cdot\alpha+\mathbb{1}\cdot\mathbb{1})/4} c[^{\alpha}_{\mathbb{1}}] \,, \tag{3.12}$$

which together with (3.7) implies that for any pair of basis elements u_i, u_j, the phase $c[^{u_j}_{u_i}]$ where $j \geq i$ can be determined by $c[^{u_i}_{u_j}]$, $c[^{\mathbb{1}}_{u_i}]$. The $N \times N$ matrix accommodating all phases $c[^{u_i}_{u_j}]$ corresponding to the set of N vectors will thus only have $N(N-1)/2 + 1$ independent elements, them being $c[^{\mathbb{1}}_{\mathbb{1}}]$ and the elements above the diagonal. Each of these phases can then be freely chosen, generating a set of $2^{N(N-1)/2+1}$ a priori distinct models.

The basis Ξ, defined in (3.4), must contain a vector u_1 satisfying

$$\frac{1}{2}N_1 u_1 = 0 \,, \tag{3.13}$$

while equation (3.7) also implies that given a pair of basis vectors u_i, u_j,

$$e^{i\pi N_{ij} u_i\cdot u_j/2} = (\delta_{u_i}\delta_{u_j})^{N_{ij}} \,, \tag{3.14}$$

where N_{ij} is the least common multiple between the pair of vectors. This imposes the constraints

$$N_{ij} u_i \cdot u_j = 0 \bmod 4 \quad \text{and} \quad N_i u_i^2 = 0 \bmod 8 \,, \quad N_i = 0 \bmod 2 \,. \tag{3.15}$$

Moreover, when $\frac{1}{2}N_i(u_i + 1) = 0 \bmod 2$,

$$N_i u_i^2 = N_i \mathbb{1}\cdot\mathbb{1} \bmod 16 \,. \tag{3.16}$$

Supersymmetry on the left-moving RNS string is non-linearly realised, with the supercurrent in this framework given by:

$$T_F = \psi^\mu \partial X_\mu + f_{abc}\chi^a y^b \omega^c \,, \quad a, b, c = 1, \ldots, 10 - D \,, \tag{3.17}$$

where f_{abc} are the structure constants of a Lie group of dimension $3(10 - D)$.

D	Lie group
4	$SU(2)^6$, $SU(3) \times SO(5)$, $SU(2) \times SU(4)$
5	$SU(2)^5$, $SU(4)$
6	$SU(2)^4$
7	$SU(2)^3$
8	$SU(2)^2$
9	$SU(2)$

Table 3.1: *All possible Lie groups for each given number of dimensions.*

In the four dimensional constructions analysed throughout this thesis, we will only consider the $SU(2)^6$ case, as it is the only one allowing the construction of models with $\mathcal{N}=1$ supersymmetry. The non-linear implementation of supersymmetry of (3.17) imposes the additional condition:

$$u_i(\chi^I) + u_i(y^I) + u_i(\omega^I) = u_i(\psi^\mu) \bmod 2 \tag{3.18}$$

on the basis vectors.

The Hilbert space of states for each sector is obtained by acting on the vacuum with bosonic and fermionic oscillators with frequencies ν_b, ν_f, ν_{f^*}. The one-loop partition function (3.5) can be rewritten in terms of the Hamiltonian:

$$Z = \int \frac{d\tau d\bar{\tau}}{\tau_2^2} \left[\sqrt{\tau_2}\eta(\tau)\bar{\eta}(\bar{\tau})\right]^{-(D-2)} \sum_{\mathbf{a},\mathbf{b}\in\Xi} c[^{\mathbf{a}}_{\mathbf{b}}]\, \mathrm{Tr}_{\mathcal{H}_\alpha}[q^{H_\alpha} e^{i\pi\beta\cdot F_\alpha}]\,, \tag{3.19}$$

where F_α is the fermionic number in a given sector α, defined as $F(f)=1$, $F(f^*)=-1$ for fermions and their complex conjugates respectively, and H_α is the Hamiltonian in the Hilbert space $\mathcal{H}_\alpha$. The operator $\beta\cdot F_\alpha$ follows the Lorentzian form of the dot product:

$$\beta\cdot F_\alpha \equiv \left[\frac{1}{2}\sum_{\substack{\text{real}\\\text{left-moving}}} + \sum_{\substack{\text{complex}\\\text{left-moving}}} - \frac{1}{2}\sum_{\substack{\text{real}\\\text{right-moving}}} - \sum_{\substack{\text{complex}\\\text{right-moving}}}\right]\beta(f)F(f)\,. \tag{3.20}$$

The partition function then takes the form of a sum over sectors:

$$\begin{aligned} Z = \int \frac{d\tau d\bar{\tau}}{\tau_2^2} \sum_{a\in\Xi} \delta_\alpha \mathrm{Tr}\Bigg[\prod_{u_i}\Bigg(&1+\delta_\alpha c[^{\ a}_{u_i}]e^{i\pi u_i\cdot F_a} \\ &+\left(\delta_\alpha c[^{\ a}_{u_i}]e^{i\pi u_i\cdot F_a}\right)^2 + \cdots + \left(\delta_\alpha c[^{\ a}_{u_i}]e^{i\pi u_i\cdot F_a}\right)^{N_i-1}\Bigg)e^{i\pi\tau H_a}\Bigg]\,, \end{aligned} \tag{3.21}$$

which implies that the Hilbert space of physical string states that have non-zero contribution is:

$$\mathcal{H} = \bigoplus_{a\in\Xi}\prod_{i=1}^{n}\left\{e^{i\pi u_i F_a} = \delta_a c[^{\ a}_{u_i}]^*\right\}\mathcal{H}_a\,. \tag{3.22}$$

The condition

$$e^{i\pi u_i\cdot F_a}\left|s\right\rangle_a = \delta_a c[^{\alpha}_{u_i}]^*\left|s\right\rangle_a \tag{3.23}$$

is the generalised GSO (GGSO) projection, and as such the phases $c[^{\alpha}_{u_i}]$ are also termed GGSO coefficients.

The mass of a string state in a given sector a is determined by the zero-moment Virasoro gauge conditions, which take the form:

$$M_L^2 = -\frac{1}{2} + \frac{1}{8} a_L \cdot a_L + \sum_{\substack{\text{left} \\ \text{moving}}} \nu_f \ , \qquad M_R^2 = -1 + \frac{1}{8} a_R \cdot a_R + \sum_{\substack{\text{right} \\ \text{moving}}} \nu_f \ , \tag{3.24}$$

where a_L and a_R are the components of the vector a corresponding to the left and right-movers respectively. Physical states are additionally subject to the level-matching condition $M_L^2 = M_R^2$. The summation in equation (3.24) is taken with respect to all the fermion oscillators that act on the vacuum to create the state under consideration. The frequency of the fermionic modes is determined by:

$$\nu_f = \frac{1 + \alpha(f)}{2} + F(f) \ . \tag{3.25}$$

For each complex worldsheet fermion f, there is a corresponding $U(1)$ current of the form $f^* f$ whose conserved $U(1)$ charge is given by:

$$Q(f) = \frac{a(f)}{2} + F(f). \tag{3.26}$$

The sectors that are crucial from a phenomenological point of view are those that give rise to massless states, as any state that is massive at the string level will obtain a mass comparable to the Planck scale, rendering it irrelevant to low energy physics. In addition to massless sectors, non supersymmetric models also exhibit tachyonic sectors, whose physical tachyonic states must be eliminated from the spectrum by appropriate GGSO projections in order to ensure the consistency of the theory.

The gauge symmetry of a model is determined by the boundary conditions imposed on the 16 complex fermions $\bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\dots,8}$. In general, when a group of N right-moving fermions $\bar{\varphi}^{1,\dots,N}_{-\frac{1}{2}}$ share the same boundary conditions in all sectors of the theory, an $SO(2N)$ gauge symmetry is generated by the resulting 0-sector bosons:

$$\psi^\mu_{-\frac{1}{2}} \left|0\right\rangle_L \times \begin{pmatrix} \bar{\varphi}^a_{-\frac{1}{2}} \bar{\varphi}^b_{-\frac{1}{2}} \\ \bar{\varphi}^{*a}_{-\frac{1}{2}} \bar{\varphi}^b_{-\frac{1}{2}} \\ \bar{\varphi}^{*a}_{-\frac{1}{2}} \bar{\varphi}^{*b}_{-\frac{1}{2}} \end{pmatrix} \left|0\right\rangle_R \ . \tag{3.27}$$

In special cases, however, additional gauge bosons may arise in the string spectrum, resulting in enhancements of the gauge symmetry. This can either be an inherent feature of the vector basis itself, or arise due to specific combinations of the GGSO phases, as we will see in the following section. The latter will be explicitly showcased in the context of the $SO(16) \times SO(16)$ and $E_8 \times E_8$ theories which share the same vector basis. We will revisit the topic of gauge symmetry enhancements in chapters 4, 5, and 6, in which four dimensional constructions are analysed, employing a more extensive, model-independent approach.

In addition to the gauge bosons, the 0-sector also gives rise to the state

$$\psi^\mu_{-1/2} \left|0\right\rangle_L \times \partial \bar{X}^\nu_{-1} \left|0\right\rangle_R \tag{3.28}$$

which always survives the GGSO projection, regardless of the choice of GGSO coefficients that defines a model. Thus, all models obtained in this formulation are provided with a massless graviton, dilaton, and Kalb–Ramond two-form. The existence of their fermionic partners in the massless string spectrum, however, is not guaranteed. A careful analysis of the spectrum reveals that massless gravitinos are only present in models which satisfy two conditions [189]. The first condition concerns the elements of the vector basis, necessitating the inclusion of the vector $S = \{\psi^\mu, \chi^{1,\dots,10-D}\}$, corresponding to periodic boundary conditions for the left-moving $\psi^\mu, \chi^{1,\dots,10-D}$ fermions and anti-periodic boundary conditions for the remaining left and right-movers. This provides the theory with an S-sector, capable of generating states of the form $\left|(\psi^\mu, \chi^{1,\dots,10-D})\right\rangle_L \times \partial \bar{X}^\nu_{-1} \left|0\right\rangle_R$, which include the gravitino(s), provided they survive the GGSO projection. In fact, the survival of the gravitino can be ensured if and only if the GGSO phases $c\left[{S \atop a}\right]$ for all basis vectors a satisfying $S \cap a = \emptyset$ are fixed to -1. In the following chapters, we will explicitly violate this condition in order to restrict our focus to the study of non supersymmetric vacua.

The free fermionic formulation has been widely utilised in the literature, and has aided in the construction of four-dimensional models with GUT symmetries, including flipped-$SU(5)$ [110, 196–202], Pati–Salam [111, 112], and Standard-like models [113, 203–214].

3.2 Classification of Ten-Dimensional Heterotic String Theories

The uniqueness of string theories in spacetimes with critical dimension $D = 10$ provide a motivation for categorising them. Having introduced the technical background of the fermionic formulation in the previous section, we now proceed by utilising it to derive all consistent heterotic string theories in ten dimensions [107, 215], and classify them based on their gauge symmetry, the presence of unbroken spacetime supersymmetry, their massless fermion content, as well as the number of physical tachyons in their spectra.

In the $D = 10$ light-cone gauge, the worldsheet particle content consists of 8 bosons X^μ and 8 real fermions ψ^μ on the left-moving part of the string, and 8 bosons $\bar{X}^\mu$ in addition to 16 complex fermions, which we choose to relabel $\bar{\phi}^{1,\dots,16}$ for simplicity, on the right-moving part. In the following, we will adopt the common convention of representing the basis vectors as sets of all fermions which are periodic in the corresponding sector, omitting those that satisfy anti-periodic boundary conditions.

3.2.1 Non Supersymmetric $SO(32)$

The starting point of any vector basis, as necessitated for the consistency of the theory, will be the vector

$$u_1 = \mathbb{1} = \{\psi^\mu | \bar{\phi}^{1,\dots,16}\} , \tag{3.29}$$

corresponding to periodic boundary conditions for all fermions. The trivial basis $B = \{\mathbb{1}\}$ generates a pair of sectors $\Xi = \{0, \mathbb{1}\}$:

$$\begin{aligned} 0 &= \{\emptyset\} , \\ \mathbb{1} &= \{\psi^\mu | \bar{\phi}^{1,\dots,16}\} , \end{aligned} \tag{3.30}$$

which along with the single GGSO phase $c[^{\mathbb{1}}_{\mathbb{1}}] = \pm 1$ define the model in question. The sign of the GGSO phase can be freely chosen, establishing the chirality convention of the theory. Given the mass formula (3.24), we can now investigate the spectrum of the theory. The sector $\mathbb{1}$ is not relevant to our analysis, as its level-matched mass-squared is positive definite, indicating that any physical state arising in this sector will be supermassive and will therefore decouple from the observable states at low energy scales. Massless states can be generated in the 0 sector by taking the product of left-movers with frequency $f_l = 1/2$ and right-movers with frequency $f_r = 1$. These states are:

$$|G\rangle = \psi^\mu_{-\frac{1}{2}} |0\rangle_L \times \partial \bar{X}^\nu_{-1} |0\rangle_R \tag{3.31}$$

and

$$|V\rangle = \psi^\mu_{-\frac{1}{2}} |0\rangle_L \times \begin{pmatrix} \bar{\phi}^a_{-\frac{1}{2}} \bar{\phi}^b_{-\frac{1}{2}} \\ \bar{\phi}^{*a}_{-\frac{1}{2}} \bar{\phi}^b_{-\frac{1}{2}} \\ \bar{\phi}^{*a}_{-\frac{1}{2}} \bar{\phi}^{*b}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R , \tag{3.32}$$

where $a, b = 1, \dots, 16$. It is easy to verify that all these states trivially satisfy the Hilbert space constraint (3.22) and are therefore physical. The first state, labelled $|G\rangle$, gives rise to the graviton, Kalb–Ramond field and dilaton[2], while the states labelled $|V\rangle$ correspond to a Lorentz vector, with $120 + 256 + 120 = 496$ degrees of freedom. These states comprise the gauge bosons of our theory, in the adjoint **496** representation of $SO(32)$.

In addition to the above massless states, the 0 sector spectrum also features physical tachyons with $M^2 = -1/2$, obtained by acting on the vacuum with right-moving $f_r = 1/2$ oscillators:

$$|\mathcal{T}\rangle = |0\rangle_L \times \begin{pmatrix} \bar{\phi}^{1,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,\dots,16}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R . \tag{3.33}$$

We also note that the theory under consideration is purely bosonic, with no fermionic states appearing at the massless level. The simplest ten-dimensional heterotic string theory is therefore non supersymmetric, exhibiting an $SO(32)$ gauge symmetry and 32 physical tachyons.

[2]The Neveu–Schwarz sector has a fixed GGSO projection, which along with the definition of the spin structure ensuring spacetime spin-statistics can never project out $|G\rangle$. This holds in all following models, including the four-dimensional constructions of later chapters, and will not be explicitly repeated in the following sections.

In the subsequent sections, we proceed by introducing additional basis vectors. The addition of new vectors will have a twofold effect on the string spectrum. Each new vector will introduce an additional GGSO projection, thus truncating the spectrum, while simultaneously introducing new sectors which may potentially generate massless or tachyonic states. While the presence of the vector $u_1 = \mathbb{1}$ is necessitated by consistency conditions, modular invariance constraints restrict the form of additional vectors, but allow for different possibilities [216]. More specifically, the left-moving part of the string may feature 8 periodic or anti-periodic ψ^μ fermions, while on the right-moving part vectors may independently imply anti-periodic boundary conditions for all fermions, or define groups of 4, 8, or 12 periodic fermions. In this section, we will derive the models obtained by considering all possible combinations of the above.

3.2.2 The $SO(32)$ superstring theory

Let us first consider adding the vector

$$u_2 = S = \{\psi^\mu\} \ , \tag{3.34}$$

corresponding to periodic boundary conditions for the left-moving fermions and anti-periodic boundary conditions for all right-movers.

The new vector basis, $B = \{\mathbb{1},\ S\}$, generates the set of four sectors $\Xi = \{0, \mathbb{1}, S, \mathbb{1} + S\}$, where

$$\begin{aligned} 0 &= \{\emptyset\} \ , \\ \mathbb{1} &= \{\psi^\mu | \bar{\phi}^{1,\dots,16}\} \ , \\ S &= \{\psi^\mu\} \ , \\ \mathbb{1} + S &= \{\bar{\phi}^{1,\dots,16}\} \ , \end{aligned}$$

from which the full spectrum of the theory emerges. As mentioned in the previous section, in order to define a model we must supplement the vector basis with appropriate GGSO phases. In this case, given the rules outlined above, there are two possible configurations:

$$c\big[\substack{u_i \\ u_j}\big] = \begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix} \quad \text{or} \quad c\big[\substack{u_i \\ u_j}\big] = \begin{pmatrix} +1 & -1 \\ -1 & -1 \end{pmatrix} \ , \tag{3.35}$$

where we have taken advantage of the freedom in the choice of $c\big[\substack{\mathbb{1} \\ \mathbb{1}}\big]$ to fi it to $+1$. The two configurations turn out to be equivalent, with the only difference between $c\big[\substack{\mathbb{1} \\ S}\big] = c\big[\substack{S \\ S}\big] = \pm 1$ also being an overall flip in the chirality of the right-moving fermions. Since the choice of chirality is a matter of convention, both choices lead to the same model.

The physical states arising in the 0 sector, outlined in (3.31),(3.32) and (3.33) are now subject to an additional GGSO projection associated with the vector S. It is easy to verify that this projection results in the elimination of the physical tachyons, while the graviton multiplet and $SO(32)$ gauge bosons remain unaffected.

In addition to the 0-sector, massless states can now also be encountered in the sector S. The S-sector massless states are obtained from the Ramond vacuum of the sector, containing the zero-modes of left-moving fermions, upon the action of right-moving oscillators with frequency $f_r = 1$. These states are:

$$|g\rangle = |(\psi^\mu)\rangle_L \times \partial \bar{X}^\nu_{-1} \, |0\rangle_R \tag{3.36}$$

and

$$|v\rangle = |(\psi^\mu)\rangle_L \times \begin{pmatrix} \bar{\phi}^a_{-\frac{1}{2}} \bar{\phi}^b_{-\frac{1}{2}} \\ \bar{\phi}^{*a}_{-\frac{1}{2}} \bar{\phi}^b_{-\frac{1}{2}} \\ \bar{\phi}^{*a}_{-\frac{1}{2}} \bar{\phi}^{*b}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R \ . \tag{3.37}$$

These states are the superpartners of the 0-sector bosons, as their spins clearly differ by 1/2: $|g\rangle$ will give rise to a spin-3/2 gravitino, as well as a spin-1/2 dilatino, while $|v\rangle$ introduces fermions in the adjoint **496** representation of $SO(32)$, which can be paired up with the gauge bosons into $SO(32)$ supermultiplets. The massless spectrum of the model we have obtained is therefore that of an $\mathcal{N} = 1$ theory with an $SO(32)$ gauge symmetry. By inspecting the full one-loop partition function, we can verify that this is not an accidental property of the massless states, but rather persists to all mass levels, notably including non level-matched states. Indeed, the full partition function can be factorised into:

$$Z_{SO(32)} = \frac{1}{4}\frac{1}{\eta(\tau)^{12}\bar{\eta}(\bar{\tau})^{24}}\left(\vartheta[^0_0]^4(\tau) - \vartheta[^0_1]^4(\tau) - \vartheta[^1_0]^4(\tau)\right)\left(\bar{\vartheta}[^0_0]^{16}(\bar{\tau}) + \bar{\vartheta}[^0_1]^{16}(\bar{\tau}) + \bar{\vartheta}[^1_0]^{16}(\bar{\tau})\right) , \tag{3.38}$$

which vanishes identically due to Jacobi's "abstruse" identity:

$$\vartheta[^0_0]^4(\tau) - \vartheta[^0_1]^4(\tau) - \vartheta[^1_0]^4(\tau) = 0 . \tag{3.39}$$

In deriving (3.38), we have also taken advantage of the fact that $\vartheta[^1_1](\tau) = \vartheta[^1_1](0|\tau) = 0$.

The introduction of the S sector thus clearly generates $\mathcal{N} = 1$ spacetime supersymmetry and the basis

$$B = \{\mathbb{1},\ S\} \tag{3.40}$$

is the minimal basis that results in a consistent supersymmetric theory, that of the heterotic $SO(32)$ superstring.

3.2.3 $E_8 \times SO(16)$

As previously mentioned, while the vector $\mathbb{1}$ must be the starting point of any consistent vector basis, many alternatives arise when additional vectors are considered. Instead of S, we now consider adding the vector

$$u_2 = z_1 = \{\bar{\phi}^{1,\dots,8}\} , \tag{3.41}$$

implying periodic boundary conditions for half of the right-moving fermions. The sectors introduced by the new vector basis $B = \{\mathbb{1},\ z_1\}$ are:

$$\begin{aligned} 0 &= \{\emptyset\} , \\ \mathbb{1} &= \{\psi^\mu|\bar{\phi}^{1,\dots,16}\} , \\ z_1 &= \{\bar{\phi}^{1,\dots,8}\} , \\ \mathbb{1} + z_1 &= \{\psi^\mu|\bar{\phi}^{9,\dots,16}\} . \end{aligned} \tag{3.42}$$

Since the right-moving fermions responsible for the gauge symmetry are now split into two sets of 8, we would naively expect this family of models to exhibit an $SO(16) \times SO(16)$ gauge symmetry. This basis, however, provides a textbook example of gauge symmetry enhancement, as the analysis of the massless spectrum reveals.

The 0-sector spectrum is, once again, truncated by the introduction of an additional vector. The 0-sector gauge bosons that survive the projection are:

$$|V_1\rangle = \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^a_{-\frac{1}{2}}\bar{\phi}^b_{-\frac{1}{2}} \\ \bar{\phi}^{*a}_{-\frac{1}{2}}\bar{\phi}^b_{-\frac{1}{2}} \\ \bar{\phi}^{*a}_{-\frac{1}{2}}\bar{\phi}^{*b}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R \ , \quad a,b = 1,\dots,8 , \tag{3.43}$$

and

$$|V_2\rangle = \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^a_{-\frac{1}{2}}\bar{\phi}^b_{-\frac{1}{2}} \\ \bar{\phi}^{*a}_{-\frac{1}{2}}\bar{\phi}^b_{-\frac{1}{2}} \\ \bar{\phi}^{*a}_{-\frac{1}{2}}\bar{\phi}^{*b}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R \ , \quad a,b = 9,\dots,16 , \tag{3.44}$$

with states mixing oscillators of the two groups of 8 $\bar{\phi}$ fermions eliminated from the spectrum. The gauge bosons now transform in the $(\mathbf{120},\mathbf{1})$ and $(\mathbf{1},\mathbf{120})$ representations of $SO(16) \times SO(16)$. In addition to these, the 32 tachyons of the $SO(32)$ theory are now reduced to 16 by the z_1 projection, leaving

$$|\mathcal{T}\rangle = |0\rangle_L \times \begin{pmatrix} \bar{\phi}^{9,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R \tag{3.45}$$

to be physical.

Additional contributions to the mass spectrum can now be traced to the z_1 and $\mathbb{1}+z_1$ sectors. In the z_1 sector, massless states are comprised of the degenerate vacuum containing zero-modes of the fermions with periodic boundary conditions, subject to a left-moving oscillator with frequency $f_l = 1/2$. Since the action of a zero mode on a ground state does not change its mass, we are free to act on it with any number of zero modes and still get a massless ground state of the form $|(\phi^{a_1}, \phi^{a_2}, \dots \phi^{a_n})\rangle$. While the mass of the ground state does not change by the action of the zero-modes, the result of the GGSO projection does. The GGSO projection essentially eliminates half of the possible degrees of freedom, those with either even or odd number of zero modes, leaving

$$\frac{1}{2}\sum_{k=0}^{8}\binom{8}{k} = \frac{1}{2}\cdot 2^8 = 128 \tag{3.46}$$

physical degrees of freedom, associated with new gauge bosons

$$|V_3\rangle = \psi^{\mu}_{-\frac{1}{2}}\,|0\rangle_L \times \left|(\phi^{1,\dots,8})\right\rangle_R \ . \tag{3.47}$$

These additional gauge bosons transform in the $(\mathbf{128},\mathbf{1})$ representation of $SO(16)\times SO(16)$, and form the adjoint $\mathbf{248}$ representation of E_8 in the $SO(16)$ embedding:

$$\mathbf{248} = \mathbf{120} + \mathbf{128} \tag{3.48}$$

when taken together with $|V_1\rangle$. Their presence in the spectrum therefore leads to the gauge symmetry enhancement:

$$SO(16)\times SO(16) \to E_8 \times SO(16)\ . \tag{3.49}$$

The final sector which generates massless states is $\mathbb{1}+z_1$, which is responsible for the spacetime fermions:

$$|v\rangle = |(\psi^{\mu})\rangle_L \times \left|(\phi^{9,\dots,16})\right\rangle_R \ , \tag{3.50}$$

resulting in two copies of the representation $(\mathbf{1},\mathbf{128})$.

3.2.4 $SO(24)\times SO(8)$

A third possible vector that can be added to $\mathbb{1}$ is

$$z_2 = \{\bar{\phi}^{7,8,9,10}\}\ , \tag{3.51}$$

forming the vector basis $B = \{\mathbb{1}, z_2\}$[3]. While naively we would expect this to generate an $SO(24)\times SO(8)$ theory, this basis actually recovers the non supersymmetric $SO(32)$ theory, due to symmetry enhancements similar to the one described above. While the 0-sector tachyons and gauge bosons are truncated by the z_2 projection, additional states arise in the sector z_2, enhancing the gauge symmetry and bringing the number of tachyons back to 32.

Linear combinations of the three vectors S, z_1, z_2 may also be considered. A careful analysis of the resulting models reveals that the bases $\{\mathbb{1}, S+z_1\}$ and $\{\mathbb{1}, z_1+z_2\}$ reproduce the $E_8\times SO(16)$ and non supersymmetric $SO(32)$ theories, essentially corresponding to a relabelling of the sectors. We stress here that while the 10-dimensional heterotic string theories themselves are unique, there are many alternative descriptions of them in the fermionic formulation. In the following sections, we will only introduce vector bases which generate models not previously introduced. The full table of possible bases and the theories they generate can be found in Appendix B. In contrast to $\{\mathbb{1}, S+z_1\}$ and $\{\mathbb{1}, z_1+z_2\}$, the bases $\{\mathbb{1}, S+z_2\}$ and $\{\mathbb{1}, S+z_1+z_2\}$ produce equivalent descriptions of a distinct ten-dimensional theory with an $SO(24)\times SO(8)$ gauge symmetry[4].

Let us consider the case

$$u_2 = S + z_2 = \{\psi^{\mu}|\bar{\phi}^{7,\dots,10}\}. \tag{3.52}$$

[3] Consistent realisations of z_2 can be obtained by any choice of four $\bar{\phi}$. We have chosen to introduce it with this specific choice of fermions in order to avoid relabelling it in following constructions.

[4] In addition to the aforementioned vector bases, the $SO(24)\times SO(8)$ theory can also be derived in the basis $\{\mathbb{1}, \zeta\}$, where ζ is a vector under which 12 right-movers are periodic. We choose not to present it in this notation in order to avoid unnecessarily introducing an additional vector in our classification.

The 0-sector gauge bosons that survive the GGSO projection in this case are:

$$|V_1\rangle = \psi^\mu_{-\frac{1}{2}}|0\rangle_L \times \begin{pmatrix} \bar\phi^a_{-\frac{1}{2}}\bar\phi^b_{-\frac{1}{2}} \\ \bar\phi^{*a}_{-\frac{1}{2}}\bar\phi^b_{-\frac{1}{2}} \\ \bar\phi^{*a}_{-\frac{1}{2}}\bar\phi^{*b}_{-\frac{1}{2}} \end{pmatrix}|0\rangle_R \ , \quad a,b = 1,\dots,6,11,\dots,16 \ , \tag{3.53}$$

and

$$|V_2\rangle = \psi^\mu_{-\frac{1}{2}}|0\rangle_L \times \begin{pmatrix} \bar\phi^a_{-\frac{1}{2}}\bar\phi^b_{-\frac{1}{2}} \\ \bar\phi^{*a}_{-\frac{1}{2}}\bar\phi^b_{-\frac{1}{2}} \\ \bar\phi^{*a}_{-\frac{1}{2}}\bar\phi^{*b}_{-\frac{1}{2}} \end{pmatrix}|0\rangle_R \ , \quad a,b = 7,8,9,10 \ , \tag{3.54}$$

transforming as $(\mathbf{276},\mathbf{1})$ and $(\mathbf{1},\mathbf{28})$ under $SO(24)\times SO(8)$ respectively. In addition to these, the 0 sector exhibits 8 physical tachyons:

$$|\mathcal{T}\rangle = |0\rangle_L \times \begin{pmatrix} \bar\phi^{7,8,9,10}_{-\frac{1}{2}} \\ \bar\phi^{*7,8,9,10}_{-\frac{1}{2}} \end{pmatrix}|0\rangle_R \ . \tag{3.55}$$

Massless fermions arise in the $S+z_2$ sector. These are

$$|v\rangle = |(\psi^\mu)\rangle_L \times \begin{pmatrix} \bar\phi^a_{-\frac{1}{2}} \\ \bar\phi^{*a}_{-\frac{1}{2}} \end{pmatrix}|(\bar\phi^{7,8,9,10})\rangle_R \ , \quad a = 1,\dots,6,11,\dots,16 \ , \tag{3.56}$$

yielding two copies of the $(\mathbf{24},\mathbf{8})$ representation.

Before moving on to more complicated vector bases, we stress the important role the vector S plays. As explicitly shown in the two $SO(32)$ theories, the addition of S to the vector basis introduces spacetime supersymmetry and eliminates all 0-sector tachyons. On the other hand, any vector basis that does not include S, will always lead to models which exhibit tachyons in the 0 sector. Modified versions of S of the form $S+z_a = \{\psi^\mu|\bar\phi^{n_1,\dots,n_m}\}$ obtained by adding m periodic right-moving fermions also tend to be problematic for the same reasons, indicating that supersymmetry is either absent, or explicitly broken. Consistent tachyon-free theories thus necessitate the incorporation of $u_2 = S$ to the vector basis. In general, tachyonic states may also arise from sectors other than 0, such as the z_2 sector of the $SO(24)\times SO(8)$ model. In such cases, the introduction of the S-vector is not adequate to ensure complete absence of all tachyons by itself, as additional GGSO projections are needed to eliminate the tachyons in all sectors.

3.2.5 $E_8 \times E_8$ and $SO(16)\times SO(16)$

We now enlarge the dimension of our vector bases by considering those comprised of three basis vectors. This allows for many potential bases, which however turn out to only generate a few independent models. A table of all possible bases and the resulting models can be found in Appendix B. In the following, we will only provide one example for each new theory we encounter.

Let us first consider the basis:

$$\begin{aligned} u_1 &= \mathbb{1} = \{\psi^\mu|\bar\phi^{1,\dots,16}\} \ , \\ u_2 &= S = \{\psi^\mu\} \ , \\ u_3 &= z_1 = \{\bar\phi^{1,\dots,8}\} \ . \end{aligned} \tag{3.57}$$

The resulting theory will consist of the 8 sectors:

$$\Xi = \{0,\, \mathbb{1},\, S,\, z_1,\, \mathbb{1}+S,\, \mathbb{1}+z_1,\, S+z_1,\, \mathbb{1}+S+z_1\} \ .$$

Massless states in this case may potentially arise in the 0, S, z_1, $S+z_1$, $\mathbb{1}+z_1$ and $\mathbb{1}+S+z_1$ sectors, while the splitting of the right-moving fermions into two groups of 8 hints at an $SO(16)\times SO(16)$ gauge symmetry. By expanding the vector basis, we have also introduced additional GGSO coefficients whose sign needs to be specified. These are $c[{z_1 \atop 0}] = c[{0 \atop z_1}] = \delta_{z_1} = 1$, $c[{z_1 \atop z_1}] = c[{\mathbb{1} \atop z_1}]$ and $c[{S \atop z_1}] = c[{z_1 \atop S}]$. In the previous constructions, all GGSO coefficients were determined by the consistency conditions, with the only free choices amounting to chirality conventions. In this case, however, while the phases $c[{\mathbb{1} \atop \mathbb{1}}]$, $c[{\mathbb{1} \atop S}]$ and $c[{\mathbb{1} \atop z_1}]$ do amount to overall chirality factors, the choice of sign in $c[{S \atop z_1}]$ has important consequences. Before making a choice, let us first investigate the partition function, keeping $c[^S_{z_1}]$ as a free parameter. After a bit of algebra, we can bring the partition function to the form

$$\begin{aligned}Z_{\{\mathbb{1},S,z_1\}} =& \frac{1}{8}\frac{1}{\eta(\tau)^{12}\bar{\eta}(\bar{\tau})^{24}}\Bigg\{\Big(\vartheta[^0_0]^4(\tau)-\vartheta[^0_1]^4(\tau)-\vartheta[^1_0]^4(\tau)\Big)\Big(\bar{\vartheta}[^0_0]^{16}(\bar{\tau})+\bar{\vartheta}[^0_1]^{16}(\bar{\tau})+\bar{\vartheta}[^1_0]^{16}(\bar{\tau})\Big)\\ &+\Big(\vartheta[^0_0]^4(\tau)-\vartheta[^0_1]^4(\tau)+c[^S_{z_1}]\vartheta[^1_0]^4(\tau)\Big)\bar{\vartheta}[^0_0]^8(\bar{\tau})\bar{\vartheta}[^0_1]^8(\bar{\tau})\\ &+\Big(\vartheta[^0_0]^4(\tau)+c[^S_{z_1}]\vartheta[^0_1]^4(\tau)-\vartheta[^1_0]^4(\tau)\Big)\bar{\vartheta}[^0_0]^8(\bar{\tau})\bar{\vartheta}[^1_0]^8(\bar{\tau})\\ &-\Big(c[^S_{z_1}]\vartheta[^0_0]^4(\tau)+\vartheta[^0_1]^4(\tau)+\vartheta[^1_0]^4(\tau)\Big)\bar{\vartheta}[^0_1]^8(\bar{\tau})\bar{\vartheta}[^1_0]^8(\bar{\tau})\Bigg\}\ ,\end{aligned} \tag{3.58}$$

in which all terms are multiplied by modified versions of Jacobi's abstruse identity, pending the definition of $c[^S_{z_1}]$. Expanding the Dedekind and Jacobi functions in terms of $q_r = e^{-2\pi\tau_2}$ and $q_i = e^{2\pi i\tau_1}$ then yields:

$$\begin{aligned}Z_{\{\mathbb{1},S,z_1\}} = \left(c[^S_{z_1}]+1\right)\Bigg[&4\frac{q_i}{q_r}+\left(-1056+\frac{2048}{q_i}+64q_i^2\right)+\\ &+\left(73728+\frac{122880}{q_i^2}-\frac{147312}{q_i}-16896q_i+576q_i^3\right)q_r+\mathcal{O}(q_r)^2\Bigg]\ .\end{aligned} \tag{3.59}$$

Some very critical properties of the two cases are already apparent from this expansion. First of all, neither choice of $c[^S_{z_1}]$ leads to physical tachyons in the string spectrum. Level-matched tachyons in this notation would manifest as poles in the q_r expansion with no accompanying q_i factors. An unphysical tachyon does appear, if $c[^S_{z_1}]$ is taken to be $+1$. This state is known in the literature as the "proto-graviton", since its construction follows that of the graviton itself. More specifically, while the graviton is obtained by oscillator excitations in both the left and right-moving vacua of the 0 sector, the proto-graviton corresponds to the case where the left-moving vacuum is excited, while the right-moving vacuum is not:

$$\psi^\mu_{-\frac{1}{2}}\left|0\right\rangle_L \times \left|0\right\rangle_R\ . \tag{3.60}$$

In supersymmetric constructions, this off-shell, tachyonic state is accompanied by its superpartner and their contributions to the partition function cancel against each other. However, the proto-gravitino is eliminated by any GSO projection that eliminates the gravitino, thus generating a non-vanishing leading term of

$$(D-2)\frac{q_i}{q_r} \tag{3.61}$$

in all non supersymmetric partition functions in D-dimensions.

Another important feature, is that the one-loop vacuum to vacuum amplitude vanishes to all orders of perturbation theory if we set $c[^S_{z_1}] = -1$. This is a clear indication that unbroken supersymmetry is present, ensuring that all bosonic and fermionic contributions cancel against each other to all orders.

The GGSO projections in the 0, $S+z_1$ and $\mathbb{1}+z_1$ sectors do not depend on $c[^S_{z_1}]$ and the massless spectrum of these sectors is therefore identical in both models. Massless 0-sector states are, again, constructed by taking the product of left-movers with frequency $f_l = 1/2$ and right-movers with frequency $f_r = 1$. The gauge bosons that survive the GGSO projection are:

$$\left|V_1\right\rangle = \psi^\mu_{-\frac{1}{2}}\left|0\right\rangle_L \times \begin{pmatrix}\bar{\phi}^{1,\dots,8}_{-\frac{1}{2}}\bar{\phi}^{1,\dots,8}_{-\frac{1}{2}}\\ \bar{\phi}^{*1,\dots,8}_{-\frac{1}{2}}\bar{\phi}^{1,\dots,8}_{-\frac{1}{2}}\\ \bar{\phi}^{*1,\dots,8}_{-\frac{1}{2}}\bar{\phi}^{*1,\dots,8}_{-\frac{1}{2}}\end{pmatrix}\left|0\right\rangle_R \tag{3.62}$$

and

$$\left|V_2\right\rangle = \psi^\mu_{-\frac{1}{2}}\left|0\right\rangle_L \times \begin{pmatrix}\bar{\phi}^{9,\dots,16}_{-\frac{1}{2}}\bar{\phi}^{9,\dots,16}_{-\frac{1}{2}}\\ \bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}}\bar{\phi}^{9,\dots,16}_{-\frac{1}{2}}\\ \bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}}\bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}}\end{pmatrix}\left|0\right\rangle_R\ , \tag{3.63}$$

resulting in two sets of 120 states transforming in the $(\mathbf{120},\mathbf{1})$ and $(\mathbf{1},\mathbf{120})$ representations of the $SO(16)\times SO(16)$ gauge symmetry generated by $\phi^{1,\dots,8}$ and $\phi^{9,\dots,16}$.

In the $S + z_1$ sector, the states are comprised of the degenerate vacuum containing zero-modes of the fermions with periodic boundary conditions, generating fermions in the $(\mathbf{128}, \mathbf{1})$ representation.

$$|v_3\rangle = |(\psi^\mu)\rangle_L \times \left|(\bar{\phi}^{1,\dots,8})\right\rangle_R \ . \tag{3.64}$$

Similarly, $\mathbb{1} + z_1$ generates massless fermions transforming as $(\mathbf{1}, \mathbf{128})$:

$$|v_4\rangle = |(\psi^\mu)\rangle_L \times \left|(\bar{\phi}^{9,\dots,16})\right\rangle_R \ . \tag{3.65}$$

Here, we have labeled these fermions with the indices $3, 4$ in anticipation of their gauge boson partners which will accompany them in the supersymmetric theory.

The rest of the massless spectrum is sensitive to the choice of sign for the phase $c[^S_{z_1}]$ and will have to be tackled on a case by case basis. Let us first focus on the case $c[^S_{z_1}] = -1$, which we expect to generate a supersymmetric spectrum. In this case, the S sector states that survive the GSO projection are:

$$\begin{aligned} |g\rangle &= |(\psi^\mu)\rangle_L \times \partial \bar{X}_1^\nu \, |0\rangle_R \ , \\ |v_1\rangle &= |(\psi^\mu)\rangle_L \times \begin{pmatrix} \bar{\phi}^{1,\dots,8}_{-\frac{1}{2}} \bar{\phi}^{1,\dots,8}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,\dots,8}_{-\frac{1}{2}} \bar{\phi}^{1,\dots,8}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,\dots,8}_{-\frac{1}{2}} \bar{\phi}^{*1,\dots,8}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R \ , \\ |v_2\rangle &= |(\psi^\mu)\rangle_L \times \begin{pmatrix} \bar{\phi}^{9,\dots,16}_{-\frac{1}{2}} \bar{\phi}^{9,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}} \bar{\phi}^{9,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}} \bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R \ , \end{aligned} \tag{3.66}$$

clearly providing the superpartners of the 0-sector states. The gravitino coming from $|g\rangle$ is present in the spectrum and the fermions $|v_{1,2}\rangle$ transform as $(\mathbf{120}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{120})$ under $SO(16) \times SO(16)$ as needed to form supermultiplets.

Finally, the z_1 and $\mathbb{1} + S + z_1$ sectors contribute the following states to the massless spectrum:

$$|V_3\rangle = \psi^\mu_{-\frac{1}{2}} \, |0\rangle_L \times \left|(\bar{\phi}^{1,\dots,8})\right\rangle_R \tag{3.67}$$

and

$$|V_4\rangle = \psi^\mu_{-\frac{1}{2}} \, |0\rangle_L \times \left|(\bar{\phi}^{9,\dots,16})\right\rangle_R . \tag{3.68}$$

These additional gauge bosons, transforming as $(\mathbf{128}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{128})$ form the adjoint $\mathbf{248}$ representations of E_8 when taken together with the vector bosons arising in the 0 sector. Their presence in the spectrum is therefore responsible for the enhancement of the $SO(16) \times SO(16)$ gauge symmetry to $E_8 \times E_8$. The fermions in the S, $S + z_1$ and $\mathbb{1} + z_1$ sectors are also accommodated in the adjoint of E_8 and the spectrum is, indeed, supersymmetric.

Consider now the case $c[^S_{z_1}] = +1$. The only additional states that survive the GGSO projection now are those whose right-movers mix the two groups of 8 $\bar{\phi}$ coming from the S sector:

$$|v_a\rangle = |(\psi^\mu)\rangle_L \times \begin{pmatrix} \bar{\phi}^{1,\dots,8}_{-\frac{1}{2}} \bar{\phi}^{9,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,\dots,8}_{-\frac{1}{2}} \bar{\phi}^{9,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{1,\dots,8}_{-\frac{1}{2}} \bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,\dots,8}_{-\frac{1}{2}} \bar{\phi}^{*9,\dots,16}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R \ . \tag{3.69}$$

These transform as $(\mathbf{16}, \mathbf{16})$ under the $SO(16) \times SO(16)$ gauge symmetry. All massless states from the z_1 and $\mathbb{1} + S + z_1$ sectors are eliminated. In addition to the $SO(32)$ and $E_8 \times E_8$ heterotic superstring theories, we have thus obtained a non supersymmetric heterotic string which does not exhibit physical tachyons. This $SO(16) \times SO(16)$ theory, was the first tachyon-free non supersymmetric heterotic string theory discovered [106, 107].

It is important to note that both the $E_8 \times E_8$ and the $SO(16) \times SO(16)$ models were obtained using the same basis of vectors. Gauge symmetry enhancements sensitive to the choice of specific GGSO coefficients such as this example are common in free fermionic model constructions, and as such, care needs to be taken when determining the gauge symmetry. We reiterate that in order to define a specific model both the vector basis and the GGSO coefficients must be fully determined. We note, also, that the coefficient choice responsible for SUSY breaking in the example above is $c[^S_{z_1}] = +1$, which explicitly violates the second condition for the existence of unbroken supersymmetry.

3.2.6 $E_7^2 \times SU(2)^2$

Consider now the basis:

$$\begin{aligned} u_1 &= \mathbb{1} = \{\psi^\mu|\bar{\phi}^{1,\dots,16}\}\ , \\ u_2 &= S + z_1 = \{\psi^\mu|\bar{\phi}^{1,\dots,8}\}\ , \\ u_3 &= z_1 + z_2 = \{\bar{\phi}^{1,\dots,6},\bar{\phi}^{9,10}\}\ . \end{aligned} \tag{3.70}$$

The 0-sector gauge bosons surviving the GGSO projections in this case are:

$$\begin{aligned} |V_1\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^{1,\dots,6}_{-\frac{1}{2}}\bar{\phi}^{1,\dots,6}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,\dots,6}_{-\frac{1}{2}}\bar{\phi}^{1,\dots,6}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,\dots,6}_{-\frac{1}{2}}\bar{\phi}^{*1,\dots,6}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ , \\ |V_2\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^{7,8}_{-\frac{1}{2}}\bar{\phi}^{7,8}_{-\frac{1}{2}} \\ \bar{\phi}^{*7,8}_{-\frac{1}{2}}\bar{\phi}^{7,8}_{-\frac{1}{2}} \\ \bar{\phi}^{*7,8}_{-\frac{1}{2}}\bar{\phi}^{*7,8}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ , \\ |V_3\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^{9,10}_{-\frac{1}{2}}\bar{\phi}^{9,10}_{-\frac{1}{2}} \\ \bar{\phi}^{*9,10}_{-\frac{1}{2}}\bar{\phi}^{9,10}_{-\frac{1}{2}} \\ \bar{\phi}^{*9,10}_{-\frac{1}{2}}\bar{\phi}^{*9,10}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ , \\ |V_4\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^{11,\dots,16}_{-\frac{1}{2}}\bar{\phi}^{11,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*11,\dots,16}_{-\frac{1}{2}}\bar{\phi}^{11,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*11,\dots,16}_{-\frac{1}{2}}\bar{\phi}^{*11,\dots,16}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ , \end{aligned} \tag{3.71}$$

transforming as $(\mathbf{66},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{6},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{6},\mathbf{1})$, and $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{66})$ under an $SO(12)\times SO(4)\times SO(4)\times SO(12) \simeq SO(12)_1 \times SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4 \times SO(12)_2$ gauge symmetry. The $SO(4)$ sextuplets of $|V_2\rangle$ and $|V_3\rangle$ can be decomposed to triplets transforming in the adjoint of $SU(2)$.

In addition to these, the 0-sector gives rise to four tachyons:

$$|\mathcal{T}\rangle = |0\rangle_L \times \begin{pmatrix} \bar{\phi}^{7,8}_{-\frac{1}{2}} \\ \bar{\phi}^{*7,8}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ . \tag{3.72}$$

Furthermore, additional gauge bosons are also present in the massless spectrum. The $z_1 + z_2$ sector generates the states:

$$|V_5\rangle = \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \big|(\bar{\phi}^{1,\dots,6},\bar{\phi}^{9,10})\big\rangle_R\ , \tag{3.73}$$

which lie in the $(\mathbf{32},\mathbf{1},\mathbf{2},\mathbf{1})$ representation. These, along with $|V_1\rangle$ and a triplet from $|V_3\rangle$ comprise the adjoint **133** representation of E_7, thus leading to the enhancement of $SO(12)_1 \times SU(2)_{3/4} \to E_7$. A similar enhancement can be attributed to gauge bosons arising in $\mathbb{1} + S + z_1$:

$$|V_6\rangle = \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \big|(\bar{\phi}^{9,10},\bar{\phi}^{11,\dots,16})\big\rangle_R\ , \tag{3.74}$$

transforming as $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{32})$, which together with $|V_4\rangle$ and the second triplet of $|V_3\rangle$ comprise a second **133** representation of E_7, enhancing $SO(12)_2 \times SU(2)_{4/3}$ to E_7. The full gauge symmetry in this case therefore turns out to be $E_7^2 \times SU(2)^2$.

In addition to the above, the model also exhibits massless fermionic states which can be traced to the $S + z_1$, $S + z_2$ and $\mathbb{1} + z_1 + z_2$ sectors:

$$\begin{aligned} |u_1\rangle &= |(\psi^\mu)\rangle_L \times \big|(\bar{\phi}^{1,\dots,6},\bar{\phi}^{7,8})\big\rangle_R\ , \\ |u_2\rangle &= |(\psi^\mu)\rangle_L \times \begin{pmatrix} \bar{\phi}^{1,\dots,6}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,\dots,6}_{-\frac{1}{2}} \end{pmatrix} \big|(\bar{\phi}^{7,8},\bar{\phi}^{9,10})\big\rangle_R\ , \\ |u_3\rangle &= |(\psi^\mu)\rangle_L \times \begin{pmatrix} \bar{\phi}^{11,\dots,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*11,\dots,16}_{-\frac{1}{2}} \end{pmatrix} \big|(\bar{\phi}^{7,8},\bar{\phi}^{9,10})\big\rangle_R\ , \\ |u_3\rangle &= |(\psi^\mu)\rangle_L \times \big|(\bar{\phi}^{7,8},\bar{\phi}^{11,16})\big\rangle_R\ , \end{aligned} \tag{3.75}$$

and fall into the representations $(\mathbf{56},\mathbf{2};\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$, $(\mathbf{56},\mathbf{1};\mathbf{1},\mathbf{2})$, and $(\mathbf{1},\mathbf{2};\mathbf{56},\mathbf{1})$ under the enhanced $E_7^2 \times SU(2)^2$ symmetry.

3.2.7 $U(16)$

We conclude the investigation of 10-dimensional heterotic string theories by considering dimension-4 vector bases. As exhibited in the previous section, the enlargement of the vector basis can be achieved in many inequivalent ways. In dimension-4 bases, there is one final distinct model to be encountered, which we choose to present in the vector basis:

$$\begin{aligned} u_1 &= \mathbb{1} = \{\psi^\mu|\bar{\phi}^{1,\dots,16}\}\ , \\ u_2 &= S + z_1 = \{\psi^\mu|\bar{\phi}^{1,\dots,8}\}\ , \\ u_3 &= z_1 + z_2 = \{\bar{\phi}^{1,\dots,6},\bar{\phi}^{9,10}\}\ , \\ u_4 &= z_3 = \{\bar{\phi}^{2n-1}\}\ ,\quad n = 1,\dots,8\ , \end{aligned} \tag{3.76}$$

obtained by adding the vector z_3 which implies periodicity in all odd-labelled fermions to the $E_7^2 \times SU(2)^2$ basis (3.70). The addition of z_3 further truncates the 0-sector spectrum, admitting the following gauge bosons:

$$\begin{aligned} |V_1\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^{1,3,5}_{-\frac{1}{2}}\bar{\phi}^{1,3,5}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,3,5}_{-\frac{1}{2}}\bar{\phi}^{1,3,5}_{-\frac{1}{2}} \\ \bar{\phi}^{*1,3,5}_{-\frac{1}{2}}\bar{\phi}^{*1,3,5}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ , \\ |V_2\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^{2,4,6}_{-\frac{1}{2}}\bar{\phi}^{2,4,6}_{-\frac{1}{2}} \\ \bar{\phi}^{*2,4,6}_{-\frac{1}{2}}\bar{\phi}^{2,4,6}_{-\frac{1}{2}} \\ \bar{\phi}^{*2,4,6}_{-\frac{1}{2}}\bar{\phi}^{*2,4,6}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ , \\ |V_{i-4}\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \bar{\phi}^{*i}_{-\frac{1}{2}}\bar{\phi}^{i}_{-\frac{1}{2}}\,|0\rangle_R\ ,\ i = 7,8,9,10\ , \\ |V_7\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^{11,13,15}_{-\frac{1}{2}}\bar{\phi}^{11,13,15}_{-\frac{1}{2}} \\ \bar{\phi}^{*11,13,15}_{-\frac{1}{2}}\bar{\phi}^{11,13,15}_{-\frac{1}{2}} \\ \bar{\phi}^{*11,13,15}_{-\frac{1}{2}}\bar{\phi}^{*11,13,15}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ , \\ |V_8\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \begin{pmatrix} \bar{\phi}^{12,14,16}_{-\frac{1}{2}}\bar{\phi}^{12,14,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*12,14,16}_{-\frac{1}{2}}\bar{\phi}^{12,14,16}_{-\frac{1}{2}} \\ \bar{\phi}^{*12,14,16}_{-\frac{1}{2}}\bar{\phi}^{*12,14,16}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ , \end{aligned} \tag{3.77}$$

responsible for an $SO(6)^4 \times U(1)^4$ gauge symmetry. Moreover, the 4 tachyons of the $E_7^2 \times SU(2)^2$ theory are now reduced to two:

$$|\mathcal{T}\rangle = |0\rangle_L \times \begin{pmatrix} \bar{\phi}^{8}_{-\frac{1}{2}} \\ \bar{\phi}^{*8}_{-\frac{1}{2}} \end{pmatrix} |0\rangle_R\ . \tag{3.78}$$

The gauge symmetry is enhanced to $U(16) \simeq SU(16) \times U(1)$ by the presence of the following vector bosons:

$$\begin{aligned} |V_9\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \big|(\bar{\phi}^{2n-1})\big\rangle_R\ , n = 1,\dots,8\ , \\ |V_{10}\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \big|(\bar{\phi}^{1,\dots,6},\bar{\phi}^{9,10})\big\rangle_R\ , \\ |V_{11}\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \big|(\bar{\phi}^{2,4,6},\bar{\phi}^{7},\bar{\phi}^{10}\bar{\phi}^{11,13,15})\big\rangle_R\ , \\ |V_{12}\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \big|(\bar{\phi}^{9},\bar{\phi}^{10},\bar{\phi}^{11,13,15},\bar{\phi}^{12,14,16})\big\rangle_R\ , \\ |V_{13}\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \big|(\bar{\phi}^{1,3,5},\bar{\phi}^{7},\bar{\phi}^{10},\bar{\phi}^{12,14,16})\big\rangle_R\ , \\ |V_{14}\rangle &= \psi^\mu_{-\frac{1}{2}}\,|0\rangle_L \times \big|(\bar{\phi}^{2,4,6},\bar{\phi}^{7},\bar{\phi}^{9},\bar{\phi}^{12,14,16})\big\rangle_R\ , \end{aligned} \tag{3.79}$$

arising in the z_3, z_1+z_2, z_1+z_3, $\mathbb{1}+S+z_1$, $\mathbb{1}+S+z_1+z_3$, and $\mathbb{1}+S+z_2+z_3$ sectors respectively. The fermionic content can be traced to the $S+z_1$, $S+z_1+z_3$, $S+z_2$, $S+z_2+z_3$, $\mathbb{1}+z_3$, $\mathbb{1}+z_1+z_2$, and $\mathbb{1}+z_1+z_2+z_3$ sectors, and transforms as $(\mathbf{120},\pm2)$ and $(\overline{\mathbf{120}},\pm2)$ under $SU(16)\times U(1)$.

3.2.8 Summary

Utilising the tools offered by the Free Fermionic Formulation, subject to the consistency conditions previously outlined, we have systematically analysed all possible realisations of heterotic string theory in $D = 10$ dimensions. In addition to the two supersymmetric theories, $SO(32)$ and $E_8 \times E_8$, we have also obtained six non supersymmetric theories exhibiting rank-16 gauge symmetries. All six theories turn out to be anomaly-free [107], either due to being non-chiral, due to the trivial cancellation of the $SO(8)$ and $SO(16)$ anomalies, or in the cases of $E_7^2 \times SU(2)^2$ and $U(16)$, by application of the Green–Schwarz anomaly cancellation mechanism [100]. While the absence of spacetime supersymmetry predictably leads to tachyonic behaviour in 5 of the above theories, in the case of the $SO(16)\times SO(16)$ heterotic string theory all physical tachyons are eliminated from the spectrum. All non supersymmetric rank 16 vacua outlined above have been shown to be connected to one another by intricate symmetries upon compactification in the presence of constant background fields [121, 188, 217].

In the aforementioned analysis, we have restricted ourselves to models realising a level $k = 1$ Kač–Moody algebra, which we summarise in table 3.2. We will retain this restriction in all following chapters, including four dimensional theories. For the sake of completeness, we note that there is an additional tachyonic non supersymmetric solution in $D = 10$-dimensions at the $k = 2$ level, with an E_8 gauge symmetry [215].

	Basis Vector(s)	Gauge Symmetry	SUSY	Tachyons	Fermion Representations
1	$\mathbb{1}$	$SO(32)$	NO	32	–
2	$\mathbb{1}, S$	$SO(32)$	YES	0	$\mathbf{496}$
3	$\mathbb{1}, z_1$	$E_8 \times SO(16)$	NO	16	$2 \times (\mathbf{1}, \mathbf{128})$
4	$\mathbb{1}, S + z_2$	$SO(24) \times SO(8)$	NO	8	$2 \times (\mathbf{24}, \mathbf{8})$
5	$\mathbb{1}, S, z_1$	$E_8 \times E_8$	YES	0	$(\mathbf{248}, \mathbf{1})$, $(\mathbf{1}, \mathbf{248})$
		$SO(16) \times SO(16)$	NO	0	$(\mathbf{128}, \mathbf{1})$, $(\mathbf{1}, \mathbf{128})$, $(\mathbf{16}, \mathbf{16})$
6	$\mathbb{1}, S + z_1, z_1 + z_2$	$E_7^2 \times SU(2)^2$	NO	4	$(\mathbf{56}, \mathbf{2}; \mathbf{1}, \mathbf{1})$, $(\mathbf{1}, \mathbf{1}; \mathbf{56}, \mathbf{2})$,
					$(\mathbf{56}, \mathbf{1}; \mathbf{1}, \mathbf{2})$, $(\mathbf{1}, \mathbf{2}; \mathbf{56}, \mathbf{1})$
7	$\mathbb{1}, S + z_1, z_1 + z_2, z_3$	$U(16)$	NO	2	$2 \times \mathbf{120}$, $2 \times \overline{\mathbf{120}}$

Table 3.2: *Minimal vector bases generating all possible level $k = 1$ heterotic string theories in $D = 10$ dimensions. The vectors $\mathbb{1}$, S, z_1, z_2 and z_3 follow the definitions of* (3.29), (3.34), (3.41), (3.51), *and* (3.76) *respectively.*

3.3 Finiteness and Misaligned Supersymmetry

The realisation of a tachyon-free heterotic string theory without spacetime supersymmetry was a seminal result [106], motivating the further study of the $SO(16)\times SO(16)$ theory, as well as the investigation of non supersymmetric theories in four dimensions [134, 135, 218, 219], which can be considered compactifications of $SO(16) \times SO(16)$. This subject has recently seen renewed interest [2, 3, 91, 132, 133, 141, 143–146, 220–229], in light of the absence of experimental evidence towards supersymmetry up to this point.

One of the most striking features of the five superstring theories is the complete elimination of IR and UV divergences. This result is owed to the presence of unbroken spacetime supersymmetry, which leads to exact cancellations in the contributions to the vacuum energy, ensuring it vanishes to all orders, and providing a solution to both the hierarchy and cosmological constant problems. Given the fact that nature does not exhibit unbroken supersymmetry, the existence of non supersymmetric string theories offers interesting opportunities for string model building, but brings the issue of finiteness to the forefront. In the absence of supersymmetry, we are driven to search for an alternative mechanism by which divergences can be eliminated, if the theory is to remain finite. Finiteness is, of course, trivially violated in theories where physical tachyons are present, as on-shell tachyons inadvertently lead to a divergence in the partition function. In the case of tachyon-free models such as the aforementioned $SO(16) \times SO(16)$, however, resolving the issue of finiteness is critical. If the theory is not finite, then spacetime supersymmetry would appear to be a necessary ingredient of any realistic attempt at string model building, which in conjunction with the absence of experimental evidence towards supersymmetry in modern experiments would be problematic. If, on the other hand, the theory is indeed finite, that implies the existence of a mechanism besides supersymmetry by which finiteness is achieved. Such a mechanism was first proposed in [126] and further expanded in [222, 230–237].

In order to investigate the properties of the non supersymmetric vacua, we must analyse the string partition function. In general, the partition function takes the form of a double expansion in terms of $q, \bar{q}$:

$$Z(q, \bar{q}) = \tau_2^{1-D/2} \sum_{m,n} a_{mn} q^m \bar{q}^n \ , \tag{3.80}$$

where the coefficients a_{mn} encode the difference between bosonic and fermionic degrees of freedom at the mass level (m, n), with $m = n$ corresponding to the level-matching condition identifying on-shell states. In supersymmetric vacua, the partition function trivially vanishes due to the exact cancellation of all a_{mn}. In non supersymmetric vacua, on the other hand, this does not happen. Besides supersymmetry, however, the string spectrum is also restricted by the invariance of the theory under modular transformations. UV divergences can be traced to the $\tau \to 0$ region of the fundamental domain of the modular group, which is excluded from all calculations due to modular invariance. In addition to being well-behaved in the UV regime, the theory is also protected from IR divergences as long as physical tachyons are not present. Thus, while exact cancellations cannot occur at each mass level, we expect modular invariance in conjunction with the absence of physical tachyons to prevent divergences by imposing highly non-trivial constraints on the string spectrum.

The string spectrum consists of a set of infinite towers of states generated in the various sectors of the theory. Each sector includes a ground state with a vacuum energy H_i and excited states with energies $n = H_i + \ell$, where $\ell \in \mathbb{Z}$, which generate a tower of physical states given by:

$$\sum_{n \in \mathbb{Z}+H_i} a_{nn}^{(i)} q^n \bar{q}^n \ . \tag{3.81}$$

Since the ground state energies H_i are not identical in all sectors of the theory, what we obtain are towers of states that are shifted relevant to one another, thus precluding the possibility of their spectra cancelling against each other at each mass level. In order to retain finiteness, instead, it is the entirety of the physical string spectrum that must conspire into cancelling out divergences. The net degeneracies of physical states in a given sector can be analytically continued into the smooth, continuous function $\Phi^{(i)}(n)$ [238, 239], which reproduces $a_{nn}^{(i)}$ for all appropriate $n \in \mathbb{Z}+H_i$. We now define the sector-averaged state degeneracies $\langle a_{nn} \rangle$ as a sum of the functions $\Phi^{(i)}(n)$ over all sectors of the theory:

$$\langle a_{nn} \rangle = \sum_i \Phi^{(i)}(n) \ . \tag{3.82}$$

We note here that the sector-averaged state degeneracies do not coincide with the degeneracies of any sector, nor with the total degeneracies a_{nn} of the physical states. In order for the theory to remain finite, these sector-averaged degeneracies must vanish, even though the physical degeneracies $a_{nn}^{(i)}$ do not.

In order to see how the cancellation of divergences comes forth, consider a toy model with two sectors, A, B, contributing towers of states at energy levels $n \in \mathbb{Z} + H_A$ and $n \in \mathbb{Z} + H_B$ respectively, with $H_A < H_B$. The cancellation of the sector-averaged degeneracies implies that

$$\Phi^{(A)}(n) + \Phi^{(B)}(n) = 0 \ . \tag{3.83}$$

Due to the difference in the ground state energy of the two sectors, even when $\langle a_{nn} \rangle$ vanish, there is no cancellation of the physical degeneracies a_{nn} at any mass level, as these are simply equal to $\Phi^{(A)}(n)$, when $n \in \mathbb{Z} + H_A$ and $\Phi^{(B)}(n)$, when $n \in \mathbb{Z} + H_B$, and are non-vanishing in general. By inspecting a specific mass level in both sectors, we can see that the net number of states at the mass level m of sector A are given by $\Phi^{(A)}(m+H_A)$, while sector B contributes $\Phi^{(B)}(m+H_B) = -\Phi^{(A)}(m+H_B)$. This implies that a surplus of bosons (fermions) must be followed by an even larger surplus of fermions (bosons) at the next mass level, leading to an oscillation between bosonic and fermionic excesses with growing amplitude. Figure 3.1 depicts an example of this oscillation occurring in the $SO(16) \times SO(16)$ theory.

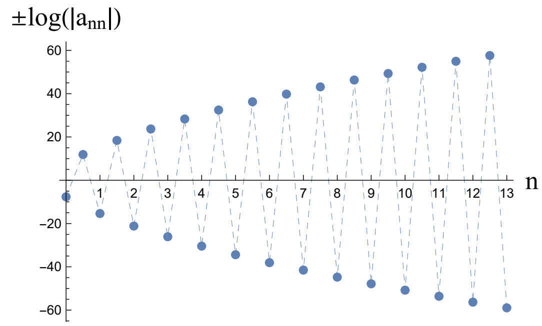

Figure 3.1: Boson-fermion oscillations in the physical spectrum of the non supersymmetric 10-dimensional $SO(16) \times SO(16)$ heterotic string theory, indicating the presence of misaligned supersymmetry. The sign of $\log(|a_{nn}|)$ is chosen according to the sign of the physical state degeneracies a_{nn} at a given mass level n, with positive (negative) values corresponding to an excess of bosons (fermions).

The physical state degeneracy in the asymptotic limit $n \to \infty$ has been shown to grow exponentially [239]:

$$a_{nn}^{(i)} \sim e^{C_{\text{tot}}\sqrt{n}} \,, \tag{3.84}$$

where C_{tot} is the inverse Hagedorn temperature. The sector-averaged degeneracies meanwhile exhibit similar asymptotic behaviour, $\langle a_{nn} \rangle \sim e^{C_{\text{eff}}\sqrt{n}}$. However, the exponential growth is cancelled out in the high-energy limit [125], thus leading to a slower growth of $\langle a_{nn} \rangle$, as $C_{\text{eff}} < C_{\text{tot}}$. It is further conjectured [126] that in some constructions $C_{\text{eff}} = 0$, in which case the growth in the sector-averaged degeneracies is merely polynomial. This conjecture has been proven to hold in a number of string theory constructions, including heterotic theories such as $SO(16) \times SO(16)$ [237].

The finiteness of the theory is thus closely tied to the exponential increase in the number of string states as the mass level increases. The restricted growth in the sector-averaged degeneracies suggests a constraint on the degree to which supersymmetry can be broken, as the cancellation of bosonic and fermionic contributions appears to not be entirely absent, but rather misaligned. Any tachyon-free closed string theory therefore possesses a mechanism guaranteeing finiteness, attributed to a residual "misaligned supersymmetry" in the string spectrum [126,230–233]. This is a generic property of all modular invariant tachyon-free string models, and has been shown to hold in any spacetime with $D \geq 2$ dimensions, regardless of the compactification scheme [126].

Misaligned supersymmetry has been explicitly demonstrated in semi-realistic non supersymmetric constructions [141], and has also been identified in theories incorporating open strings [236,237]. We note here that while misaligned supersymmetry was derived under the assumption of a tachyon-free theory, the characteristic boson-fermion oscillations can also be observed in the spectra of the five tachyonic non supersymmetric theories defined in the previous section, as depicted in figure 3.2. In fact, similar oscillations have been encountered in other tachyonic constructions [131]. This hints at the possibility that misaligned supersymmetry might still persist in the spectrum even in the presence of tachyons, though further analysis is required.

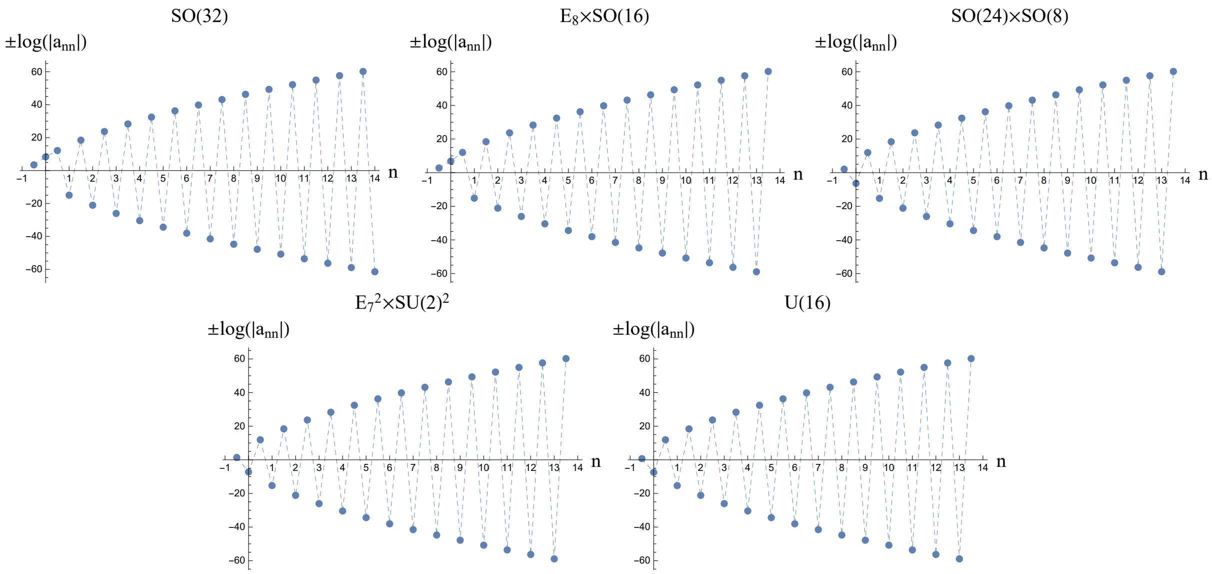

Figure 3.2: Boson-fermion oscillations in the physical spectrum of the five tachyonic non supersymmetric 10-dimensional theories.

3.4 String Compactifications

The heterotic string theories outlined thus far exhibit some appealing features, such as the incorporation of gravity in a quantum mechanical framework and the natural way in which gauge symmetries large enough to accommodate the Standard Model appear as a result of anomaly cancellation. There is, however, a significant hurdle we need to overcome if we are to make contact with low energy physics modern experiments can probe. This hurdle is the critical dimension of the string. String theories are constrained by consistency conditions to live in a $D = 10$-dimensional spacetime, while our current experimental observations imply a $(3, 1)$-dimensional spacetime at low energies. In the previous sections, we introduced the techniques of the fermionic formulation which can be used to directly construct strings in $D \leq 10$ dimensions. The lower-dimensional theories can be shown to be equivalent to a compactification of a critical, 10-dimensional theory. This is, however, not manifest in the fermionic language, as the additional degrees of freedom corresponding to the free fermions needed to cancel the conformal anomaly are only defined at the self-dual radius, obscuring the moduli-dependent dynamics of the theory.

In this section, we will develop an equivalent description of heterotic strings in which the effect of the compactified space will be made explicit. In this context, we assume that the 10-dimensional target-space factorises into a product of a 4-dimensional non-compact spacetime describing our observable universe, and a 6-dimensional compact space, whose size is assumed to be small enough to not produce any detectable effects at low energies. In this description, the low-energy predictions of the theory depend on the geometry of the compactified space.

While 10-dimensional string theories form a small set of unique solutions, upon compactification we arrive at an astounding number of possible solutions, the so-called string landscape, with wildly differing physics. This necessitates the search for viable solutions and a classification of the parameter space of possible models.

3.4.1 Toroidal Compactifications

Let us first consider the simplest example, by compactifying the bosonic string on a circle of radius R. The coordinate along the compactified dimension will exhibit periodicity:

$$X^{25}(\sigma + 2\pi, \tau) = X^{25}(\sigma, \tau) + 2\pi n R \,, \tag{3.85}$$

where n is the winding number, which takes into account the possibility of the string winding around the compactified direction. The mode expansions in the remaining 25 directions are unaffected by the compactification and follow the definitions of (2.16), (2.17), while the mode expansion along the compactified direction is modified due to the boundary condition above:

$$X_R^{25}(\tau-\sigma) = \frac{1}{2}(x^{25}-\tilde{x}^{25}) + \big(p^{25}-mR\big) + \dots \,, \qquad X_L^{25}(\tau+\sigma) = \frac{1}{2}(x^{25}+\tilde{x}^{25}) + \big(p^{25}+mR\big) + \dots \,, \tag{3.86}$$

where the ellipses correspond to the oscillator excitations. In addition to this, the momentum of the string will also be affected, with the periodicity of the position implying a quantisation in the momentum eigenvalues. The left and right-moving components of the momentum are given by:

$$p_L = \frac{1}{\sqrt{2}}\left(\frac{m}{R}+nR\right) \,, \qquad p_R = \frac{1}{\sqrt{2}}\left(\frac{m}{R}-nR\right) \,, \tag{3.87}$$

where m, and n are the momentum and winding numbers respectively. This also results in a modification of the mass formula:

$$\alpha' M^2 = \alpha' \left[\left(\frac{m}{R}\right)^2 + \frac{nR}{\alpha'}^2\right] + 2N_L + 2N_R - 4 \,, \tag{3.88}$$

which exhibits the T-duality invariance under the simultaneous exchange of $R \leftrightarrow \alpha'/R$ and $m \leftrightarrow n$. The level matching condition is now given by

$$N_R - N_L = mn \,. \tag{3.89}$$

In chapter 2, the partition functions of bosons in compactified dimensions were shown to be proportional to the momentum lattice:

$$\sum_{(\vec{p}_L,\vec{p}_R)\in\Gamma} e^{i\pi\tau\vec{p}_L^2} e^{-i\pi\bar{\tau}\vec{p}_R^2} \,. \tag{3.90}$$

The contribution to the partition function of bosons with momenta (3.87) will therefore be:

$$= Z(R) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n} \exp\left[i\pi\tau\frac{1}{2}\left(\frac{m}{R}+nR\right)^2 - i\pi\bar{\tau}\frac{1}{2}\left(\frac{m}{R}-nR\right)^2\right] \,,$$

which after a Poisson resummation can be brought to the form:

$$Z(R) = \frac{R}{\sqrt{\tau_2}}\frac{1}{|\eta|^2} \sum_{m,n\in\mathbb{Z}} \exp\left[-\frac{\pi R^2}{\tau_2}|m-n\tau|^2\right] \,. \tag{3.91}$$

This result can be straightforwardly generalised to multi-dimensional compactified spaces. In that case, the momentum will be defined by a d-dimensional lattice:

$$\Gamma_{d,d}(G,B) = \sum_{\vec{m},\vec{n}\in\mathbb{Z}^d} q^{p_L^2}\bar{q}^{p_R^2}, \tag{3.92}$$

which we explicitly depict as a function of the metric G_{ij} and the antisymmetric tensor B_{ij}. The general solution for the left-and right-moving momenta is obtained by:

$$p_L^i = G^{ij}\left(m_j (B_{jk}+G_{jk})n_k\right) \,, \qquad p_R^i = G^{ij}\left(m_j (B_{jk}-G_{jk})n_k\right) \,, \tag{3.93}$$

and their square is defined as:

$$p_{L(R)}^2 = p_{L(R)}^i G_{ij} p_{L(R)}^j \,. \tag{3.94}$$

The partition function contribution we obtain in this case is:

$$Z(G,B) = \frac{\sqrt{\det G}}{\sqrt{\tau_2}^d |\eta|^{2d}} \sum_{\vec{m},\vec{n}\in\mathbb{Z}^d} \exp\left[-\frac{\pi(G_{ij}+B_{ij})}{\tau_2}\left(m_i-n_i\tau\right)\left(m_j+n_j\bar{\tau}\right)\right]. \tag{3.95}$$

While such toroidal compactifications are appealing due to the exactly solvable nature of their underlying CFT, they are unfortunately unable to produce realistic phenomenology when the low energy limit of the theory is considered. This is due to the fact that such compactifications to 4 dimensions exhibit extended supersymmetry and are therefore incapable of generating chiral matter, a key ingredient of the SM. In order to preserve $\mathcal{N}=1$ supersymmetry and obtain chiral spectra, the compactification manifold must be Calabi–Yau [240]. Further analyses showed that the gauge symmetry in that case could be reduced to $E_6 \times E_8$, and even further towards the SM gauge symmetry by the introduction of Wilson lines [241–243].

The difficulty in approaching low energy phenomenology is further increased by the complexity of Calabi–Yau manifolds. In fact, the only Calabi–Yau metrics explicitly known are those of the simplest constructions: the 2-dimensional torus and the K3 surface. In order to overcome this issue orbifold compactifications [244,245] can be employed, in which the manifold contains singularities, which however admit exact solutions to the equations of motion. In this thesis, we consider compactifications on toroidal orbifolds, which correspond to the singular limit of a Calabi–Yau manifold.

3.4.2 Orbifold Compactifications

An orbifold is defined as the quotient of a smooth manifold by a discrete group Γ. Under the action of Γ, the points x and $g \cdot x$, $g \in \Gamma$ are identified, so in addition to the states $X^i(\sigma+2\pi) = X^i(\sigma)$ we also have to take into account the states $X^i(\sigma+2\pi) = g \cdot X^i(\sigma)$, which give rise to the "twisted" sectors.

Consider the simplest orbifold: $S^1/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ symmetry acts on the points of S^1 as: $x \to g \cdot x = -x$. The partition function is given by:

$$Z_{Orb} = (q\bar{q})^{-1/24}\left[\mathrm{Tr}_{\mathrm{untwisted}}\left(\frac{1}{2}(1+g)q^{L_0}\bar{q}^{\bar{L}_0}\right) + \mathrm{Tr}_{\mathrm{twisted}}\left(\frac{1}{2}(1+g)q^{L_0}\bar{q}^{\bar{L}_0}\right)\right], \tag{3.96}$$

where the twisted and untwisted sectors are defined by the action of the $\mathbb{Z}_2$ symmetry. This partition function can be split into four sectors: the untwisted-untwisted sector which gives the same contribution as (3.91), and the twisted sectors which have the following contributions:

$$\begin{aligned}
(q\bar{q})^{-1/24}\mathrm{Tr}_{\mathrm{untwisted}}\left(\frac{1}{2}gq^{L_0}\bar{q}^{\bar{L}_0}\right) &= \frac{1}{2}\frac{(q\bar{q})^{-1/24}}{\prod_{n=1}^{\infty}(1+q^n)(1+\bar{q}^n)} = \left|\frac{\eta}{\vartheta_2}\right|, \\
(q\bar{q})^{-1/24}\mathrm{Tr}_{\mathrm{twisted}}\left(\frac{1}{2}q^{L_0}\bar{q}^{\bar{L}_0}\right) &= \frac{1}{2}\frac{(q\bar{q})^{-1/24}}{\prod_{n=1}^{\infty}(1-q^{n-1/2})(1-\bar{q}^{n-1/2})} = \left|\frac{\eta}{\vartheta_4}\right|, \\
(q\bar{q})^{-1/24}\mathrm{Tr}_{\mathrm{twisted}}\left(\frac{1}{2}gq^{L_0}\bar{q}^{\bar{L}_0}\right) &= \frac{1}{2}\frac{(q\bar{q})^{-1/24}}{\prod_{n=1}^{\infty}(1+q^{n-1/2})(1+\bar{q}^{n-1/2})} = \left|\frac{\eta}{\vartheta_3}\right|.
\end{aligned} \tag{3.97}$$

The contribution of the twisted sectors can be seen as a result of a new kind of boundary conditions for the fermions, in which the string is wrapped around the extra dimension. The total contribution to the partition function is:

$$Z(R) = \frac{1}{2}\frac{\Gamma_{1,1}(R)}{|\eta|^2} + \left|\frac{\eta}{\vartheta_2}\right| + \left|\frac{\eta}{\vartheta_3}\right| + \left|\frac{\eta}{\vartheta_4}\right|, \tag{3.98}$$

which can be recast in the more compact form:

$$Z(R) = \frac{1}{2}\sum_{h,g=0,1}\frac{\Gamma_{1,1}\left[{}^{h}_{g}\right](R)}{|\eta|^2}, \tag{3.99}$$

where $\Gamma_{1,1}\left[{}^{h}_{g}\right](R)$ is defined as follows:

$$\Gamma_{1,1}\left[{}^{h}_{g}\right](R) = \begin{cases} \Gamma_{1,1}(R)\,, & (h,g) = (0,0) \\ \left|\frac{2\eta^3}{\vartheta\left[{}^{1-h}_{1-g}\right]}\right|, & (h,g) \neq (0,0) \end{cases}. \tag{3.100}$$

The parameter h labels the twisted and untwisted sectors of the orbifold, while summation over $g=0,1$ imposes the invariance projections associated to them.

Up to this point, we have considered the contribution of string states that either close in the compactified dimensions or are wound around them. The presence of the $\mathbb{Z}_2$ symmetry allows a third kind of boundary conditions: we can also consider transporting the string half way along the compactified dimension. This transportation is known as a $\mathbb{Z}_2$ shift, as the action of two consecutive shifts is the same as the identity.

To construct a modular invariant partition function that incorporates shifts, we must consider the action of the shift on both directions of the string worldsheet. While asymmetric orbifolds have received attention in the literature [217, 246–249], throughout this work we will restrict ourselves to symmetric shifts, which are introduced simultaneously on both the left and right-moving sectors. The $\mathbb{Z}_2$ shifts will have an effect on the left-moving and right-moving momenta. In order to avoid having to relabel the indices we introduced in equation (3.87), we add an extra contribution to the momenta, which in the case of $S^1/\mathbb{Z}_2$ will be:

$$\begin{aligned} p_L &= \frac{m}{R} + \left(n + \frac{H}{2}\right) R \,, \\ p_R &= \frac{m}{R} - \left(n + \frac{H}{2}\right) R \,. \end{aligned} \tag{3.101}$$

The corresponding contribution to the partition function is found to be:

$$Z_{\text{shift}}(R) = \frac{1}{2} \sum_{H,G=0,1} \sum_{m,n\in\mathbb{Z}} \frac{\Gamma_{1,1}^{\text{shift}}[^H_G](R)}{|\eta|^2} \,, \tag{3.102}$$

where the shifted lattice is defined as:

$$\Gamma_{1,1}^{\text{shift}}[^H_G](R) = \sum_{m,n\in\mathbb{Z}} (-1)^{Gm} q^{\frac{1}{4}|p_L|^2} \bar{q}^{\frac{1}{4}|p_R|^2} \,. \tag{3.103}$$

The full twisted and shifted lattice is then defined as follows:

$$\Gamma_{1,1}[^H_G|^h_g](R) = \begin{cases} \Gamma_{1,1}^{\text{shift}}[^H_G](R), & (h,g) = (0,0) \\ \left|\frac{2\eta^3}{\vartheta[^{1-h}_{1-g}]}\right|, & (H,G) = (0,0) \text{ or } (H,G) = (h,g) \\ 0 \,, & \text{otherwise} \end{cases} \,, \tag{3.104}$$

where the contributions of a simultaneous shift and twist vanish.

At the self-dual point, $R = \sqrt{2}$, the partition function factorises into Jacobi functions:

$$\Gamma_{1,1}[^H_G|^h_g](\sqrt{2}) = \frac{1}{2} \sum_{\epsilon,\zeta=0,1} \left|\vartheta[^\epsilon_\zeta]\vartheta[^{\epsilon+h}_{\zeta+g}]\right| (-1)^{H\zeta+G\epsilon+HG} \,. \tag{3.105}$$

This is directly related to fermionisation, which allows the redefinition of the theory in the free fermionic formulation.

The generalisation of the above to higher-dimensional compact spaces is straightforward. The two-dimensional (un)twisted toroidal orbifold can be described by:

$$\Gamma_{2,2}[^h_g](T,U) = \begin{cases} \Gamma_{2,2}(T,U), & (h,g) = (0,0) \\ \left|\frac{2\eta^3}{\vartheta[^{1-h}_{1-g}]}\right|^2, & (h,g) \neq (0,0) \end{cases} \,. \tag{3.106}$$

The orbifolds we will be employing in our attempt to construct semi-realistic 4-dimensional models will be of the form $(T^2 \times T^2 \times T^2)/\Gamma$, where Γ is a product of $\mathbb{Z}_2$ factors. These will include a $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist responsible for the $\mathcal{N} = 4 \to 1$ reduction of supersymmetry, a $\mathbb{Z}_2$ realising the Scherk–Schwarz breaking of the remaining supersymmetry via an order-2 shift along the first torus, as well as additional factors encoding model dependent information. This choice of compactification is a compromise between the complexity of Calabi-Yau manifolds and the trivial but exactly solvable nature of toroidal manifolds and is motivated by the capacity of toroidal orbifold compactifications to produce chiral matter and family repetition, two defining features of the SM any realistic model must possess.

The implementation of shifts on the toroidal orbifold can be performed in two ways: either as simultaneous shifts along the two directions of the torus, or as two independent shifts. Let us first consider the case of two independent shifts $H_{1,2}$. The momenta, expressed explicitly in terms of the Kähler and complex structure moduli can be cast in the form[5]:

$$\begin{aligned} p_L &= \frac{m_2 + \frac{H_2}{2} - U m_1 + T(n_1 + \frac{H_1}{2} + U n_2)}{\sqrt{T_2 U_2}} \,, \\ p_R &= \frac{m_2 + \frac{H_2}{2} - U m_1 + \overline{T}(n_1 + \frac{H_1}{2} + U n_2)}{\sqrt{T_2 U_2}} \,, \end{aligned} \tag{3.107}$$

and the shifted lattice is obtained by taking:

$$\Gamma^{\text{shift}}_{2,2}[^{H_1,H_2}_{G_1,G_2}](T,U) = \sum_{\substack{m_1,m_2\\ n_1,n_2 \in \mathbb{Z}}} (-1)^{G_1 m_1 + G_2 n_2} q^{\frac{1}{4}|p_L|^2} \bar{q}^{\frac{1}{4}|p_R|^2} \,. \tag{3.108}$$

The full twisted/shifted lattice is given by:

$$\Gamma_{2,2}[^{H_1,H_2}_{G_1,G_2}|^h_g](T,U) = \begin{cases} \Gamma^{\text{shift}}_{2,2}[^{H_1,H_2}_{G_1,G_2}](T,U), & h = g = 0 \\ \left| \frac{2\eta^3}{\vartheta[^{1-h}_{1-g}]} \right|^2, & \begin{array}{l} (H_1,G_1) = (H_2,G_2) = (0,0) \\ \text{or } (H_i,G_i) = (0,0) \ \& \ (H_j,G_j) = (h,g)\,,\ i \neq j = 1,2 \\ \text{or } (H_1,G_1) = (H_2,G_2) = (h,g) \end{array} \\ 0, & \text{otherwise} \end{cases} . \tag{3.109}$$

The maximal symmetry point is now defined as $T = 2U = i$, in which case the fermionic description is once again recovered:

$$\Gamma_{2,2}[^{H_1,H_2}_{G_1,G_2}|^h_g](i,i/2) = \frac{1}{4} \sum_{\substack{\epsilon_1,\epsilon_2\\ \zeta_1,\zeta_2 \in \mathbb{Z}_2}} \left| \vartheta[^{\epsilon_1}_{\zeta_1}] \vartheta[^{\epsilon_1+h}_{\zeta_1+g}] \vartheta[^{\epsilon_2}_{\zeta_2}] \vartheta[^{\epsilon_2+h}_{\zeta_2+g}] \right| (-1)^{H_1\zeta_1 + G_1\epsilon_1 + H_1 G_1 + H_2\zeta_2 + G_2\epsilon_2 + H_2 G_2}. \tag{3.110}$$

At the fermionic point, the $(2,2)$ lattice can be seen to decompose to a product of two $(1,1)$ lattices, taken at their respective fermionic points defined as $R_{1,2} = \sqrt{2}$:

$$\Gamma_{2,2}[^{H_1,H_2}_{G_1,G_2}|^h_g](i,i/2) = \Gamma^{(1)}_{1,1}[^{H_1}_{G_2}|^h_g](\sqrt{2}) \Gamma^{(2)}_{1,1}[^{H_2}_{G_2}|^h_g](\sqrt{2}). \tag{3.111}$$

We stress that this decomposition is only valid at the fermionic point. In fact, when deformations to generic points are considered, the two $\Gamma_{1,1}$ fail to properly account for all the moduli. While $\Gamma_{2,2}$ explicitly depends on four real moduli, the product of two $\Gamma_{1,1}$ lattices only involves two radii. The lattices defined in (3.108), (3.109) will be a key component in our attempts to construct semi-realistic 3-generation models, as we will see in chapter 6.

Alternatively, we may consider a simultaneous shift in the two directions, by taking $H_1 = H_2$ and $G_1 = G_2$ in (3.107) and (3.108), in which case the lattice is given by:

$$\Gamma_{2,2}[^H_G|^h_g](T,U) = \begin{cases} \Gamma^{\text{shift}}_{2,2}[^H_G](T,U), & h = g = 0 \\ \left| \frac{2\eta^3}{\vartheta[^{1-h}_{1-g}]} \right|^2, & \begin{array}{l} (H,G) = (0,0) \\ \text{or } (H,G) = (h,g) \end{array} \\ 0, & \text{otherwise} \end{cases} , \tag{3.112}$$

[5]While this is not the most general form, we will nevertheless adopt it throughout this thesis as it can be easily translated to the framework of the free fermionic formulation.

In this case, while the fermionic point now defined as $T = i$, $U = (1+i)/2$ does factorise into a product of Jacobi theta functions:

$$\Gamma_{2,2}[^H_G|^h_g](i, \frac{1+i}{2}) = \frac{1}{2} \sum_{\epsilon,\zeta \in \mathbb{Z}_2} \left| \vartheta[^\epsilon_\zeta] \vartheta[^{\epsilon+h}_{\zeta+g}] \right|^2 (-1)^{H\zeta + G\epsilon + HG} , \tag{3.113}$$

offering an equivalent fermionic picture, it cannot be expressed as a product of two $\Gamma_{1,1}$ lattices. This restriction also manifests itself in the corresponding fermionic description, where the internal space is parametrised by complex rather than real fermions. While this description tends to greatly reduce the parameter space of possible vacua, thus simplifying the search for models with specific characteristics, it comes at a phenomenological cost. Models constructed in this manner do not exhibit fully realistic phenomenology, but they do retain important features such as chirality and can be considered to be "semi-realistic". We construct such models with $SO(10)$ gauge symmetry in the following chapter, as a prelude to Pati–Salam constructions which are the focus of chapter 5.

Chapter 4

A Warm-up: Four Dimensional $SO(10)$ Models

In this chapter, we will outline the procedure we follow in the construction of four dimensional models in a step-by-step approach. We will focus on a class of models with unbroken $SO(10)$ gauge symmetry and employ the fermionic formulation in conjunction with orbifold compactifications in order to identify models with semi-realistic phenomenological properties.

4.1 Model Construction

As we saw in the previous chapter, the cancellation of the conformal anomaly requires $3(10-D)$ real left-moving and $2(26-D)$ real right-moving fermions on the worldsheet, in addition to the $D-2$ left-and right moving bosons X^μ and $\overline{X}^\mu$ in the light-cone gauge of a D dimensional heterotic string theory. For the purposes of this section, all real fermions will be grouped in pairs forming complex fermions in the following manner: $\{\chi^i,\chi^j\}\to\chi^{ij}$, $\{y^i,y^j\}\to y^{ij}$, $\{\omega^i,\omega^j\}\to\omega^{ij}$, $\{\bar{y}^i,\bar{y}^j\}\to\bar{y}^{ij}$, $\{\bar{\omega}^i,\bar{\omega}^j\}\to\bar{\omega}^{ij}$. This allows for a simpler vector basis and corresponds to a significantly smaller class of models which can then be exhaustively analysed, at the cost of introducing degeneracies in the massless spectrum, which notably exclude the possibility of models with three fermion generations. Since the purpose of this chapter is to merely showcase our dual approach and highlight only the most important features that will be relevant to more realistic constructions, we postpone further discussion of these issues to the following chapters.

The worldsheet spectrum in $D=4$ dimensions is comprised of the following: (i) 2 left and 2 right-moving bosons X^μ, $\bar{X}^\mu$ corresponding to the non compactified spacetime coordinates. (ii) 10 complex left-moving fermions which include the light-cone gauge spacetime fermions ψ^μ, the internal fermionic coordinates $x^{12,34,56}$, as well as the fermionised internal bosonic coordinates $y^{12,34,56}$, $\omega^{12,34,56}$. (iii) 22 complex right-moving fermions, with $\bar{y}^{12,34,56},\bar{\omega}^{12,34,56}$ coming from the fermionisation of the internal coordinates, in addition to the 16 complex fermions $\bar{\psi}^{1,\dots,5},\bar{\eta}^{1,2,3},\bar{\phi}^{1,\dots,8}$ realising the right-moving gauge currents.

The starting point of our vector basis corresponds to periodic boundary conditions for all fermions, as required for consistency:

$$u_1=\mathbb{1}=\{\psi^\mu,x^{12,34,56},y^{12,34,56},\omega^{12,34,56}|\bar{y}^{12,34,56},\bar{\omega}^{12,34,56},\bar{\psi}^{1,\dots,5},\bar{\eta}^{1,2,3},\bar{\phi}^{1,\dots,8}\}\ . \tag{4.1}$$

In order to generate spacetime supersymmetry and eliminate all 0 sector tachyons of the form

$$|0\rangle_L\times\begin{pmatrix}\bar{\phi}_{-\frac{1}{2}}\\ \bar{\phi}^*{}_{-\frac{1}{2}}\end{pmatrix}|0\rangle_R\ ,\quad \bar{\phi}_{-\frac{1}{2}}\in\{\bar{y}^{12,34,56},\bar{\omega}^{12,34,56},\bar{\psi}^{1,\dots,5},\bar{\eta}^{1,2,3},\bar{\phi}^{1,\dots,8}\}\ , \tag{4.2}$$

we introduce the second basis vector:

$$u_2=S=\{\psi^\mu,\chi^{12,34,56}\}\ . \tag{4.3}$$

This is the simplest vector basis that leads to consistent supersymmetric models, which exhibit an $SO(44)$ gauge symmetry with $\mathcal{N}=4$ supersymmetry, the maximal amount a four dimensional heterotic model can have. In this framework, the left-moving sector supersymmetry is realised in a non-linear manner, with the supercharge being [190]:

$$T_F = \psi^\mu \partial X_\mu + \sum_{i=1}^{6} \chi^i y^i \omega^i . \tag{4.4}$$

These models correspond to a compactification on a Narain lattice $\Gamma_{6,22}(G, B)$ [188]. The corresponding partition function, following the discussion of the previous chapter will be:

$$Z = \frac{1}{2} \frac{1}{\eta^2 \bar{\eta}^2} \sum_{a,b \in \mathbb{Z}_2} (-1)^{a+b+\mu ab} \frac{\vartheta[^a_b]^4}{\eta^4} \frac{\Gamma_{6,22}}{\eta^6 \bar{\eta}^{22}} . \tag{4.5}$$

Here, the $1/\eta^2\bar{\eta}^2$ factor encodes the contribution of the two left and right-moving bosons[1], while the ψ^μ and $x^{12,34,56}$ fermions periodic under S are responsible for the $\vartheta[^a_b]^4/\eta^4$ factor. In general, the lattice partition function depends on 6×22 moduli: the metric G_{ij}, the antisymmetric tensor B_{ij} as well as the Wilson lines that appear in the two dimensional worldsheet. For the sake of simplicity, we do not explicitly denote the dependency on the moduli of the lattice.

The parameter μ determines the overall chirality of the fermions. The free fermionic partition function can be reproduced in the orbifold formalism by fixing the lattice of (4.5) to its maximal symmetry point, termed the "fermionic" point, where the description simplifies to a product of Jacobi functions

$$\Gamma_{6,22} = \frac{1}{2} \sum_{k,\ell} \vartheta[^k_\ell]^6 \bar{\vartheta}[^k_\ell]^{22} , \tag{4.6}$$

and associating μ with the GGSO phase $c[^{\mathbb{1}}_S]$:

$$\mu = \frac{1}{2} \left(1 - c[^{\mathbb{1}}_S]\right) , \tag{4.7}$$

in which case the full partition function reads:

$$Z_{SO(44)} = \frac{1}{\eta^2 \bar{\eta}^2} \frac{1}{2} \sum_{a,b \in \mathbb{Z}_2} \frac{1}{2} \sum_{k,\ell \in \mathbb{Z}_2} (-1)^{a+b+\mu ab} \frac{\vartheta[^a_b]^4}{\eta^4} \frac{\vartheta[^k_\ell]^6}{\eta^6} \frac{\bar{\vartheta}[^k_\ell]^{22}}{\bar{\eta}^{22}} . \tag{4.8}$$

Let us now add three more vectors to our basis:

$$\begin{aligned} u_3 = T_1 &= \{y^{12}, \omega^{12} | \bar{y}^{12}, \bar{\omega}^{12}\} , \\ u_4 = T_2 &= \{y^{34}, \omega^{34} | \bar{y}^{34}, \bar{\omega}^{34}\} , \\ u_5 = T_3 &= \{y^{56}, \omega^{56} | \bar{y}^{56}, \bar{\omega}^{56}\} . \end{aligned} \tag{4.9}$$

These additions break $SO(44)$ to $SO(32) \times SO(4)^3$ and allow us to have a more clear geometrical interpretation, as the Narain lattice can now be factorised into the product of a 6 dimensional internal space and an extra contribution due to the fermionised degrees of freedom of the right-moving bosonic string: $\Gamma_{6,22} = \Gamma_{6,6} \times \Gamma_{16}$. In fact, the three T_i basis vectors split the $\Gamma_{6,6}$ lattice into 3 complex planes:

$$\Gamma_{6,6}[^{H_1,H_2,H_3}_{G_1,G_2,G_3}] = \prod_{i=1}^{3} \Gamma_{2,2}[^{H_i}_{G_i}] , \tag{4.10}$$

which correspond to the shifted lattices of (3.112) and factorise into Jacobi functions at the fermionic point according to (3.113), with $h = g = 0$.

In order to reduce supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 1$, we introduce two additional basis vectors:

$$\begin{aligned} u_6 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^1\} , \\ u_7 = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \bar{y}^{12}, \bar{y}^{56}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^2\} . \end{aligned} \tag{4.11}$$

[1] The bosonic part also contributes a factor of $1/\tau_2$ which we drop for simplicity.

The introduction of each of these vectors corresponds to a $\mathbb{Z}_2$ twist reducing supersymmetry by half. While the end goal of our analysis are non supersymmetric models, we nevertheless refrain from adding a third $\mathbb{Z}_2$ twist to explicitly break supersymmetry to $\mathcal{N}=0$, as such supersymmetry breaking generically occurs at the string scale. A spontaneous breaking of $\mathcal{N}=1\to 0$ is utilised instead, via a stringy realisation of the Scherk–Schwarz mechanism, which we outline in a later part of this chapter.

In addition to the $\mathcal{N}=4\to 1$ supersymmetry breaking, b_1 and b_2 are also responsible for breaking the $SO(32)\times SO(4)^3$ gauge symmetry to $U(1)^6\times SO(10)\times U(1)^2\times SO(18)$. The $SO(10)$ factor will act as the observable gauge symmetry of our models, while the rest of the gauge factors will constitute the "hidden sector".

We finally introduce two more vectors in order to reduce the hidden sector gauge symmetry:

$$
\begin{aligned}
u_8 &= z_1 = \{\bar\phi^{1,\dots,4}\}\ ,\\
u_9 &= z_2 = \{\bar\phi^{5,\dots,8}\}\ .
\end{aligned}
\tag{4.12}
$$

The final gauge symmetry of this class of models, gauge symmetry enhancements aside, will then be:

$$
G = SO(10)\times SO(8)_1\times SO(8)_2\times U(1)^3, \tag{4.13}
$$

where we have suppressed the standard $U(1)^6$ factor originating from the compactification. The partition function, at the fermionic point takes the form:

$$
\begin{aligned}
Z =&\frac{1}{\eta^2\bar\eta^2}\frac{1}{2^3}\sum_{\substack{a,k,\rho\\ b,\ell,\sigma}}\frac{1}{2^3}\sum_{\substack{\gamma_1,\gamma_2,\gamma_3\\ \delta_1,\delta_2,\delta_3}}\frac{1}{2^3}\sum_{\substack{H,h_1,h_2\\ G,g_1,g_2}}(-1)^{\xi[^{\mathbf{a}}_{\mathbf{b}}]}\,\frac{\vartheta[^a_b]}{\eta}\,\frac{\vartheta[^{a+h_1}_{b+g_1}]}{\eta}\,\frac{\vartheta[^{a+h_2}_{b+g_2}]}{\eta}\,\frac{\vartheta[^{a-h_1-h_2}_{b-g_1-g_2}]}{\eta}\\
&\times\frac{\vartheta[^{\gamma_1}_{\delta_1}]\vartheta[^{\gamma_1+h_1}_{\delta_1+g_1}]}{\eta^2}\,\frac{\vartheta[^{\gamma_2}_{\delta_2}]\vartheta[^{\gamma_2+h_2}_{\delta_2+g_2}]}{\eta^2}\,\frac{\vartheta[^{\gamma_3}_{\delta_3}]\vartheta[^{\gamma_3-h_1-h_2}_{\delta_3-g_1-g_2}]}{\eta^2}\\
&\times\frac{\bar\vartheta[^{\gamma_1}_{\delta_1}]\bar\vartheta[^{\gamma_1+h_1}_{\delta_1+g_1}]}{\bar\eta^2}\,\frac{\bar\vartheta[^{\gamma_2}_{\delta_2}]\bar\vartheta[^{\gamma_2+h_2}_{\delta_2+g_2}]}{\bar\eta^2}\,\frac{\bar\vartheta[^{\gamma_3}_{\delta_3}]\bar\vartheta[^{\gamma_3-h_1-h_2}_{\delta_3-g_1-g_2}]}{\bar\eta^2}\\
&\times\frac{\bar\vartheta[^k_\ell]^5}{\bar\eta^5}\,\frac{\bar\vartheta[^{k+h_1}_{\ell+g_1}]}{\bar\eta}\,\frac{\bar\vartheta[^{k+h_2}_{\ell+g_2}]}{\bar\eta}\,\frac{\bar\vartheta[^{k-h_1-h_2}_{\ell-g_1-g_2}]}{\bar\eta}\,\frac{\bar\vartheta[^\rho_\sigma]^4}{\bar\eta^4}\,\frac{\bar\vartheta[^{\rho+H}_{\sigma+G}]^4}{\bar\eta^4}\ .
\end{aligned}
\tag{4.14}
$$

The Ramond and Neveu–Schwarz sectors of the left-moving fermions are defined by $a=1$ and $a=0$ respectively, while summation over b introduces the GSO projection. For the right-movers, (k,ℓ) and (ρ,σ) label the boundary conditions of the 16 complex fermions realising the two E_8 factors of the ten-dimensional theory, while (H,G) are responsible for a twist in the gauge currents which breaks the hidden sector gauge symmetry to $SO(8)^2$. The naming scheme of the right-moving fermions is now apparent: $\bar\psi^{1,\dots,5}$ generate the observable $SO(10)$ gauge symmetry, while $\bar\eta^{1,2,3}$ produce three $U(1)$ factors. The hidden sector includes two $SO(8)$ groups associated to $\bar\phi^{1,\dots,4}$ and $\bar\phi^{5,\dots,8}$ respectively. The twisted sectors of the non-freely acting $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold are labeled by (h_1,h_2), with $(h_1,h_2)=(0,0)$ defining the untwisted sector, while $(h_1,h_2)=\{(1,0),(0,1),(1,1)\}$ make up the three twisted sectors of the model. The corresponding invariance projection is obtained by summing over g_1 and g_2.

We have also introduced the orbifold phase $\xi[^{\mathbf{a}}_{\mathbf{b}}]$ in terms of the two vectors $\mathbf{a}=(a,a+h_1,a+h_2,\dots,\rho+H)^T$ and $\mathbf{b}=(b,b+g_1,b+g_2,\dots,\sigma+G)^T$ containing all upper and lower theta function characteristics respectively. This phase plays a crucial role in our constructions. In addition to ensuring that the partition function is modular invariant, it also carries all model dependent information and its exact form can determine whether supersymmetry is broken spontaneously or explicitly. It is straightforward to show that any sector of the fermionic theory can be reproduced by fixing the elements of $\mathbf{a}$ modulo 2, while all possible projections within a sector can be matched by also fixing $\mathbf{b}$ in a similar manner, implying the existence of a $1-1$ correspondence between the GGSO coefficients and the orbifold parameters.

In order to ensure that the partition function of (4.14) is invariant under modular transformations at one and two-loops, as well as that it factorises into two one-loop amplitudes at the two-loop level, the orbifold phase must satisfy the following conditions modulo 2:

$$
\begin{aligned}
&\xi[^{\mathbf{a}}_{\mathbf{a}+\mathbf{b}+\mathbb{1}}] = \xi[^{\mathbf{a}}_{\mathbf{b}}]-1-\frac{1}{4}\mathbf{a}\cdot\mathbf{a}\ ,\\
&\xi[^{\mathbf{b}}_{\mathbf{a}}] = \xi[^{\mathbf{a}}_{\mathbf{b}}]-\frac{1}{2}\mathbf{a}\cdot\mathbf{b}\ ,\\
&\xi[^{\mathbf{a}}_{\mathbf{b}+\mathbf{b}'}] = \xi[^{\mathbf{a}}_{\mathbf{b}}]+\xi[^{\mathbf{a}}_{\mathbf{b}'}]+\mathbf{a}\ .
\end{aligned}
\tag{4.15}
$$

These are analogous to the conditions (3.6), (3.7), and (3.9) imposed on the GGSO coefficients in the free fermionic formulation.

Under these conditions, the orbifold phase takes the form:

$$\xi[^{\mathbf{a}}_{\mathbf{b}}] = a + b + HG + \Phi[^{\mathbf{a}}_{\mathbf{b}}]\ , \tag{4.16}$$

where $\Phi[^{\mathbf{a}}_{\mathbf{b}}]$ is comprised of modular invariant bilinears of the upper and lower theta function parameters. In order to complete the mapping between the two formalisms, there is one final issue we need to address. In the free fermionic formulation, the theta function characteristics are always defined modulo 2, and the sign of each term in the partition function is fully encoded in the corresponding GGSO phase $c[^{u_i}_{u_j}]$. This, however, is not the case in (4.14), as the sign of each term is sensitive to additional contributions arising due to the periodicity of the theta functions:

$$\vartheta[\,^{\ a}_{b+2}\,] = e^{i\pi a}\vartheta[\,^a_b\,]\ . \tag{4.17}$$

In order to properly take these contributions into account in the orbifold phase, we include an additional term

$$\Lambda[^{\mathbf{a}}_{\mathbf{b}}] = e^{i\pi(\mathbf{a}-[\mathbf{a}])\mathbf{b}/2}\ , \tag{4.18}$$

where $[\mathbf{a}] = \mathbf{a} \bmod 2$ and the orbifold phase finally takes the form

$$\xi[^{\mathbf{a}}_{\mathbf{b}}] = a + b + HG + \Phi[^{\mathbf{a}}_{\mathbf{b}}] + \Lambda[^{\mathbf{a}}_{\mathbf{b}}]\ . \tag{4.19}$$

The bilinears that enter the orbifold phase are correlated to the values of the GGSO coefficients. We define the vectors $\mathbf{X} = (a, k, \rho, \gamma_1, \gamma_2, \gamma_3, H, h_1, h_2)^T$, $\mathbf{Y} = (b, \ell, \sigma, \delta_1, \delta_2, \delta_3, G, g_1, g_2)^T$ and the 9×9 symmetric matrix $\mathbf{M}$ with $\mathbb{Z}_2$ elements, which encode the model dependent component of the phase:

$$\Phi[^{\mathbf{a}}_{\mathbf{b}}] = \mathbf{X}^T\mathbf{M}\mathbf{Y}\ . \tag{4.20}$$

The exact form of $\Phi[^{\mathbf{a}}_{\mathbf{b}}]$ can be computed by evaluating $\mathbf{M}$ for all choices of $\mathbf{X}$, $\mathbf{Y}$ which correspond to the basis vectors of table (4.1) and identifying each element with the corresponding GGSO phase. The symmetry conditions fully determine half of the non-diagonal elements of $\mathbf{M}$, by requiring $\mathbf{M}_{ji} = \mathbf{M}_{ij}$, while invariance under T-transformations introduces additional conditions. This procedure always leads to an equal number of free parameters as the free fermionic formulation, as for a general basis of n vectors the $n \times n$ matrix $\mathbf{M}$ is subject to $n(n-1)/2 + (n-1)$ conditions, resulting in $n(n-1)/2 + 1$ independent choices, and therefore $2^{\frac{n(n-1)}{2}+1}$ vacua, in agreement with the discussion in section 3.1.

Having established the equivalence of the fermionic and orbifold formulations, we finally reintroduce the full twisted / shifted 2,2 lattices by employing (3.113). After taking everything into account, the partition function of the orbifold will be:

$$\begin{aligned} Z = \frac{1}{\eta^2\bar\eta^2}\frac{1}{2^3}\sum_{\substack{a,k,\rho\\ b,l,\sigma}}\frac{1}{2^3}\sum_{\substack{h_1,h_2,H\\ g_1,g_2,G}}\frac{1}{2^3}\sum_{\substack{H_1,H_2,H_3\\ G_1,G_2,G_3}}(-1)^{a+b+HG+\Phi}\,\frac{\vartheta[^a_b]}{\eta}\frac{\vartheta[^{a+h_1}_{b+g_1}]}{\eta}\frac{\vartheta[^{a+h_2}_{b+g_2}]}{\eta}\frac{\vartheta[^{a-h_1-h_2}_{b-g_1-g_2}]}{\eta} \\ \times\frac{\Gamma_{2,2}[^{H_1}_{G_1}|^{h_1}_{g_1}](T^{(1)},U^{(1)})}{\eta^2\bar\eta^2}\frac{\Gamma_{2,2}[^{H_2}_{G_2}|^{h_2}_{g_2}](T^{(2)},U^{(2)})}{\eta^2\bar\eta^2}\frac{\Gamma_{2,2}[^{H_3}_{G_3}|^{h_1+h_2}_{g_1+g_2}](T^{(3)},U^{(3)})}{\eta^2\bar\eta^2} \\ \times\frac{\bar\vartheta[^k_\ell]^5}{\bar\eta^5}\frac{\bar\vartheta[^{k+h_1}_{\ell+g_1}]}{\bar\eta}\frac{\bar\vartheta[^{k+h_2}_{\ell+g_2}]}{\bar\eta}\frac{\bar\vartheta[^{k-h_1-h_2}_{\ell-g_1-g_2}]}{\bar\eta}\frac{\bar\vartheta[^\rho_\sigma]^4}{\bar\eta^4}\frac{\bar\vartheta[^{\rho+H}_{\sigma+G}]^4}{\bar\eta^4}\ , \end{aligned} \tag{4.21}$$

where we use the short-hand notation Φ to denote the $\Phi[^{\mathbf{a}}_{\mathbf{b}}]$ and $\Lambda[^{\mathbf{a}}_{\mathbf{b}}]$ components of the orbifold phase, which are now functions of the lattice parameters H_i, G_i instead of γ_i, δ_i.

Given the set of rules outlined in the previous chapter, the vector basis we employ, summarised below in table 4.1, will generate a family of models, which after choosing $c[^{S}_{T_1}] = +1$ in order to limit our focus to non supersymmetric vacua, encompasses 2^{36} models. A detailed analysis of the phenomenology of these models is beyond the scope of this thesis. We will instead provide a brief introduction of some key elements which will be relevant to the construction of Pati–Salam models in the following chapters.

$u_1 = \mathbb{1} = \{\psi^\mu, \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} \vert \bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\dots,8}\}$
$u_2 = S = \{\psi^\mu, \chi^{1,\dots,6}\}$
$u_3 = T_1 = \{y^{12}, \omega^{12} \vert \bar{y}^{12}, \bar{\omega}^{12}\}$
$u_4 = T_2 = \{y^{34}, \omega^{34} \vert \bar{y}^{34}, \bar{\omega}^{34}\}$
$u_5 = T_3 = \{y^{56}, \omega^{56} \vert \bar{y}^{56}, \bar{\omega}^{56}\}$
$u_6 = b_1 = \{\chi^{34}, \chi^{56}, y^{34}, y^{56} \vert \bar{y}^{34}, \bar{y}^{56}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^{1}\}$
$u_7 = b_2 = \{\chi^{12}, \chi^{56}, y^{12}, y^{56} \vert \bar{y}^{12}, \bar{y}^{56}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^{2}\}$
$u_8 = z_1 = \{\bar{\phi}^{1,\dots,4}\}$
$u_9 = z_2 = \{\bar{\phi}^{5,\dots,8}\}$

Table 4.1: *The full vector basis of the $\mathcal{N}=1$ supersymmetric $SO(10)\times SO(8)^2\times U(1)^3$ class of models.*

4.2 Elimination of Tachyons

In the absence of supersymmetry, or constructions are vulnerable to the appearance of tachyons in the string spectrum, which imply an instability of the vacuum and invalidate the perturbative approach. Before proceeding with their analysis, we must therefore ensure that the models are tachyon free. In order to do that, we must identify all possible sectors in which tachyons can potentially arise, and make sure that the GGSO projection eliminates all of them from the spectrum. This results in a constraint on the allowed combinations of GGSO phases.

In the $SO(10)$ models under consideration, tachyons can either arise from the Ramond vacuum of the sectors z_1 or z_2, in which case their mass is $M^2 = -1/2$, or from the sectors T_i, $T_i + z_1$ and $T_i + z_2$, $i = 1, 2, 3$, with mass $M^2 = -1/4$. In the $T_i + z_{1,2}$ sectors, these states are also obtained from the Ramond vacuum, while T_i tachyons require the action of a right-moving fermion oscillator. The first constraint imposed on the models must therefore be the elimination of tachyons from all the aforementioned sectors. While this can be fully translated to conditions on the GGSO phases themselves, we will defer the full derivation to the next chapter where Pati–Salam models are analysed. We note that an evaluation of the conditions allowing tachyon-free constructions reveals that roughly $\sim 50\%$ of the aforementioned models exhibit tachyons and are therefore not consistent $SO(10)$ vacua.

4.3 Basic Elements of the Massless Spectrum

Assuming that all tachyons have been eliminated via appropriate GGSO projections, we are now free to dive into some of the features of their massless spectrum. In the analysis of the spectrum of these models, two linear combinations of the basis vectors of table 4.1 are of high relevance. These are

$$x = \mathbb{1} + S + \sum_{i=1}^{3} T_i + \sum_{k=1}^{2} z_k = \{\bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}\} \tag{4.22}$$

and

$$b_3 = b_1 + b_2 + x = \{\chi^{12}, \chi^{34}, y^{12}, y^{34} \vert \bar{y}^{12}, \bar{y}^{34}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^{3}\} \, . \tag{4.23}$$

In order to make contact with low energy physics, we must ensure that these models reproduce some of the key features of the SM. The first property we will investigate is chirality. As presented in chapter 1, the chiral matter of the SM, in addition to right-handed neutrinos can be accommodated in the **16** and $\overline{\mathbf{16}}$ representations of $SO(10)$. Fermions in the above representations can arise in the sectors labeled $\mathcal{S}^i_{pq} = S + b_i + pT_j + qT_k$, where $p, q = 0, 1$ and $\{i, j, k\} = \{(1,2,3), (2,1,3), (3,1,2)\}$. The GGSO projection which determines whether states generated in those sectors are present in the physical spectrum follows the definition of (3.22), which can be determined by suitable projection operators:

$$P_{\mathcal{S}^i_{pq}} = \prod_{j=1}^{9} \frac{1}{2}\left(1 - c\left[\begin{smallmatrix} \mathcal{S}^i_{pq} \\ u_j \end{smallmatrix}\right]^*\right) . \tag{4.24}$$

We can take advantage of the fact that any vectors that have nonzero overlap with $\mathcal{S}^i_{pq}$ do not project out the states in question, but rather fix the internal chiralities of the worldsheet fermions, to define the generalised projectors, which only include the vectors that can eliminate a state from the spectrum. Given a state constructed purely out of the Ramond vacuum of a sector a, such vectors form a set $\Xi^{\pm}(a)$ and the result of the GGSO projection can be determined by the operator

$$\mathbb{P}^{\pm}_{a} = \prod_{\xi\in\Xi^{\pm}(a)} \frac{1}{2}\left(1 \pm c\begin{bmatrix} a \\ \xi \end{bmatrix}^*\right) , \tag{4.25}$$

with physical states satisfying the condition $\mathbb{P}^{\pm}_{a} = 1$, while states where $\mathbb{P}^{\pm}_{a} = 0$ are eliminated from the spectrum.

The projection operators which determine which of the $\mathcal{S}^i_{pq}$ sectors give rise to physical states are therefore $\mathbb{P}^{-}_{\mathcal{S}^i_{pq}}$, with

$$\Xi^{-}\left(\mathcal{S}^i_{pq}\right) = \{T_i, z_1, z_2\} . \tag{4.26}$$

While these operators determine the presence of chiral matter in each of the $\mathcal{S}^i_{pq}$ sectors, they cannot distinguish between states that transform as $\mathbf{16}$ and $\overline{\mathbf{16}}$. In order to determine the exact representation they will fall into, we utilise the representation operators:

$$X^{SO(10)}_{\mathcal{S}^i_{pq}} = \begin{cases} -c\begin{bmatrix} \mathcal{S}^i_{pq} \\ \mathcal{S}^j_{0,1-q} \end{bmatrix}^* , & j \neq i = 1,2 \\ -c\begin{bmatrix} \mathcal{S}^i_{pq} \\ \mathcal{S}^1_{1-q,0} \end{bmatrix}^* , & i = 3 \end{cases} , \tag{4.27}$$

which return $+1$ and -1 for fermions in the $\mathbf{16}$ and $\overline{\mathbf{16}}$ respectively. The net number of fermion generations can then be expressed as:

$$n_g = n_{\mathbf{16}} - n_{\overline{\mathbf{16}}}, \tag{4.28}$$

where

$$\begin{aligned} n_{\mathbf{16}} =& 4\mathrm{ch}(\psi^\mu)\sum_{i=1}^{3}\sum_{p,q=0}^{1} \mathbb{P}^{-}_{\mathcal{S}^i_{pq}} \frac{1}{2}\left(1 + X^{SO(10)}_{\mathcal{S}^i_{pq}}\right) , \\ n_{\overline{\mathbf{16}}} =& 4\mathrm{ch}(\psi^\mu)\sum_{i=1}^{3}\sum_{p,q=0}^{1} \mathbb{P}^{-}_{\mathcal{S}^i_{pq}} \frac{1}{2}\left(1 - X^{SO(10)}_{\mathcal{S}^i_{pq}}\right) , \end{aligned} \tag{4.29}$$

with $\mathrm{ch}(\psi^\mu)$ denoting the chirality of the spacetime fermions. Spinorial fermions can also arise in the $S + x$ sector, but they always come in pairs of $\mathbf{16} + \overline{\mathbf{16}}$ and thus have no effect on the net chirality. In figure 4.1, we present a plot of the net chirality exhibited by the non-tachyonic models of the class defined in table 4.1.

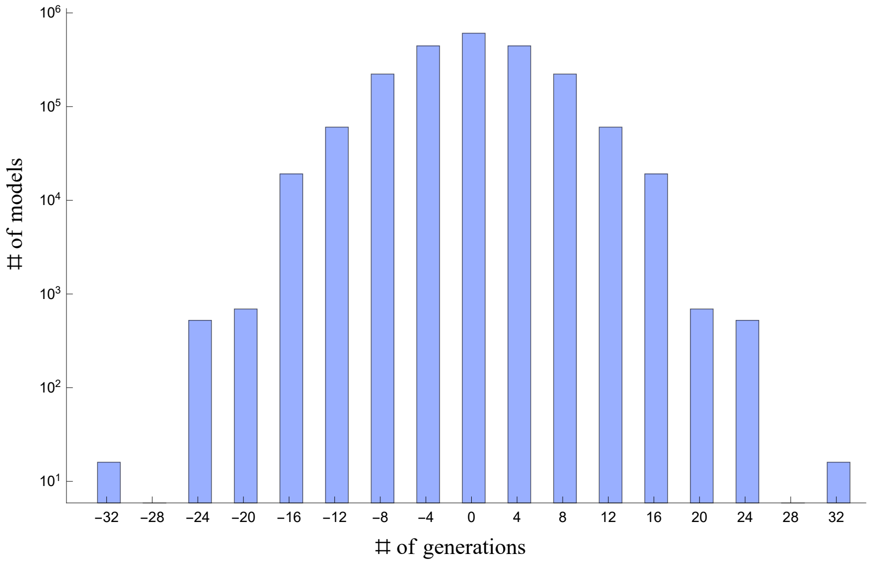

Figure 4.1: A plot of the number of tachyon-free models along with the corresponding net chirality. This roughly follows a normal distribution around zero. The lack of models with ± 28 generations appears to be in line with [250], where supersymmetric models with real fermions were considered.

The bosonic sectors $\Sigma^i_{pq} = b_i + pT_j + qT_k$ similarly give rise to spinorial states and are subject to the projection operators $\mathbb{P}^+_{\Sigma^i_{pq}}$, with

$$\Xi^+\left(\Sigma^i_{pq}\right) = \{T_i, z_1, z_2\} \ . \tag{4.30}$$

In this case, there is no well-defined representation operator implying that scalars always come in $\mathbf{16} + \overline{\mathbf{16}}$ pairs. These sectors will be of high importance in the Pati–Salam constructions, as they include the heavy Higgs $H(\mathbf{4}, \mathbf{1}, \mathbf{2})$ necessary to induce the spontaneous symmetry breaking of the Pati–Salam symmetry. As figure 4.2 shows, the existence of such states in the string spectrum is common, as a majority of the models include at least one sector giving rise to scalars in the spinorial representation.

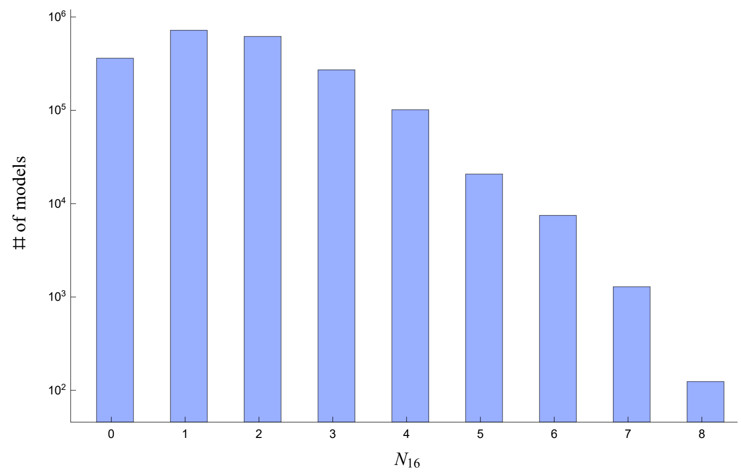

Figure 4.2: A plot of the number of tachyon-free models along with the number of sectors generating scalars in the (anti-)spinorial $\mathbf{16}/\overline{\mathbf{16}}$ representation.

Moving on, the sectors $\Upsilon^i_{pq} = \Sigma^i_{pq} + x$ can give rise to various massless bosonic states depending on which right-moving fermionic oscillator is acting on the vacuum. Among these is

$$|(\phi_L)\rangle_L \times \begin{pmatrix} \bar{\psi}^{1,\dots,5}_{-\frac{1}{2}} \\ \bar{\psi}^{*1,\dots,5}_{-\frac{1}{2}} \end{pmatrix} |(\phi_R)\rangle_R \ , \quad \phi_L, \phi_R \in \Upsilon^i_{pq} \ , \tag{4.31}$$

which accommodates the SM Higgs in the vectorial $\mathbf{10}$ representation. In order to analyse the GGSO projection on states which contain fermion oscillators $\bar{\varphi}$, which take the general form

$$|a\rangle_L \times \begin{pmatrix} \bar{\varphi}_{-\frac{1}{2}} \\ \bar{\varphi}^*_{-\frac{1}{2}} \end{pmatrix} |a\rangle_R \ , \ \bar{\varphi} \in \mathbb{1}_R - a_R \, , \tag{4.32}$$

we introduce the generalised projectors

$$\mathbb{P}^{\varphi}_{a} = \prod_{\beta \in \Xi(a)} \frac{1}{2} \left(1 + \delta_a \delta^{\varphi}_{\beta} c \begin{bmatrix} a \\ \beta \end{bmatrix}^* \right) \ , \tag{4.33}$$

with

$$\delta^{\varphi}_{\beta} = \begin{cases} -1 \ , \ \varphi \in \beta \\ +1 \ , \ \varphi \notin \beta \end{cases} \quad . \tag{4.34}$$

In this notation, $\mathbb{P}^{\varphi}_{a} = 1$ identify physical states, while $\mathbb{P}^{\varphi}_{a} = 0$ project the state under question out of the massless spectrum.

Returning to the Υ^i_{pq} sectors, the state (4.31) including the SM Higgs is identified by the operator

$$\mathbb{P}^{\overline{\psi}^{4,5}}_{\Upsilon^i_{pq}} = 1 \, , \ \Xi\left(\Upsilon^i_{pq}\right) = \{T_i, z_1, z_2\} \ . \tag{4.35}$$

Figure 4.3 clearly shows that as was the case with the spinorials, a majority of the models include at least one sector responsible for physical scalars in the vectorial representation.

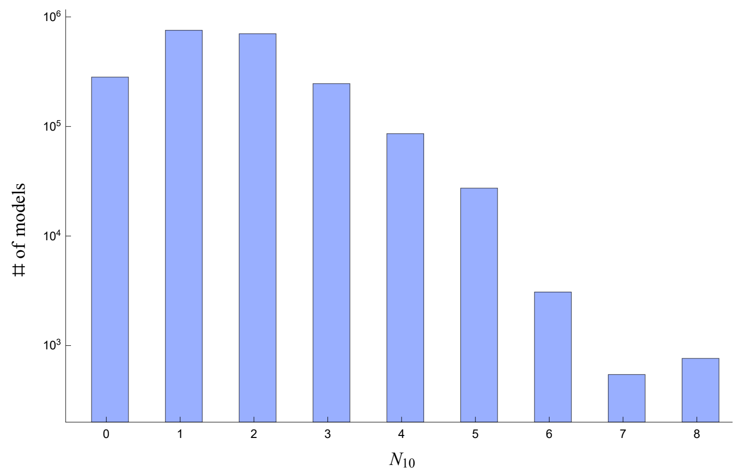

Figure 4.3: A plot of the number of tachyon-free models along with the number of sectors generating scalars in the vectorial **10** representation.

By proceeding in a similar manner we can analyse the entire massless spectrum of the theory, extracting algebraic expressions in terms of the GGSO coefficients which encode phenomenological features in a model independent way. We will explicitly do this in the following chapters.

While the fermionic formulation offers powerful tools for the analysis of the string spectrum, its description of the dynamics of the theory is restricted to the "fermionic point". All moduli are fixed to specific values and the behaviour of the theory in generic points of the moduli space is obscured. In addition to this, the fermionic description offers no insight on the mechanism which induces supersymmetry breaking, revealing only whether unbroken supersymmetry is present or not. These limitations are what motivates the redefinition of the theory in the orbifold language. In the following sections, we briefly discuss some of the main phenomenological features this allows us to investigate.

4.4 Spontaneous Supersymmetry Breaking

In this section, we will provide a short argument for the consistency of supersymmetry breaking implemented as a $\mathbb{Z}_2$ momentum / winding shift along the first torus defined by the lattice (3.112) with the Scherk–Schwarz mechanism.

Consider a model of the class defined by the vector basis of table 4.1, described in the orbifold notation by (4.21). In addition to the conditions (4.15) imposed by modular invariance and factorisation, the modular invariant phase Φ is subject to strict constraints by the requirement of spontaneous supersymmetry breaking. More specifically, if we consider the modified Riemann identity

$$\frac{1}{2}\sum_{a,b=0,1}(-1)^{a(1+Y)+b(1+X)}\vartheta[^a_b]\vartheta[^{a+h_1}_{b+g_1}]\vartheta[^{a+h_2}_{b+g_2}]\vartheta[^{a-h_1-h_2}_{b-g_1-g_2}]=\vartheta[^{1+X}_{1+Y}]\vartheta[^{1+X+h_1}_{1+Y+g_1}]\vartheta[^{1+X+h_2}_{1+Y+g_2}]\vartheta[^{1+X-h_1-h_2}_{1+Y-g_1-g_2}]\,, \tag{4.36}$$

which appears as a multiplicative factor in the one-loop partition function, we can see that the right-hand side is non-vanishing if any $X\in\{k,\rho,H,H_i\}$, $Y\in\{l,\sigma,G,G_i\}$ are coupled to b and a in the orbifold phase. Since this factor is precisely what causes the partition function to vanish in supersymmetric theories, the presence of such couplings indicate explicit supersymmetry breaking and must therefore be eliminated. The only allowed couplings to a,b come from the modular invariant term $aG_1+bH_1+H_1G_1$, which must be present in order to implement the Scherk–Schwarz breaking[2], in addition to potential

[2]We assume for simplicity that the Scherk–Schwarz mechanism is implemented solely along the direction of the first torus, avoiding more complicated implementations involving the other two tori.

model-dependent terms of the form $ag_i + bh_i + h_i g_i$, which trivially reshuffle the Jacobi functions on the right-hand side.

In order to verify that supersymmetry is spontaneously broken under the aforementioned assumptions, we focus on the sector of the theory which is responsible for generating the gravitino. In the fermionic formulation, this is sector S. In the orbifold notation, this corresponds to the following choice for the parameters in (4.21):

$$a = 1\ , \quad k = \rho = H = h_1 = h_2 = H_1 = H_2 = H_3 = 0\ . \tag{4.37}$$

Upon closer inspection, the choice $a = 1$ restricts non-vanishing contributions to the sub sector $b = g_1 = g_2 = 0$, as otherwise the partition function vanishes due to the appearance of $\vartheta[^1_1]$ factors. Moreover, the vanishing of all other upper arguments implies that no terms containing G_1 besides aG_1 remain in the phase. Since $h_1 = g_1 = 0$, the shifted lattice is involved, contributing

$$\sum_{\substack{m_1,m_2\\ n_1,n_2}\in\mathbb{Z}} (-1)^{G_1(m_1+n_2)} q^{\frac{1}{4}|p_L|^2} \bar{q}^{\frac{1}{4}|p_R|^2}\ , \tag{4.38}$$

with lattice momenta following the definition of (3.107), with identical shifts in the two directions of the torus.

Given the presence of the additional G_1 factor in the modular invariant phase Φ, when summation over G_1 is performed, m_1+n_2 is forced to be odd. The level matching condition places further constraints, imposing the condition $n_1 = n_2 = 0$ and therefore restricting m_1 to odd values. The lowest mass state, which can be associated with the gravitino mass, is then obtained by taking $m_2 = 0$ and $m_1 = 1$, in which case the mass takes the form

$$m_{3/2} = \frac{|U|}{\sqrt{T_2 U_2}}\ . \tag{4.39}$$

In the simplest case of a square torus, where $T = iR_1R_2$ and $U = iR_2/R_1$, the mass then simplifies to

$$m_{3/2} = \frac{1}{R_1}\ , \tag{4.40}$$

recovering the Scherk–Schwarz prediction.

We emphasise that the exact form of (4.36) and the constraints it places on the couplings of a and b are common to all constructions making use of the vectors S, b_1, and b_2, while the vanishing of all upper arguments besides a in (4.37) follows from the definition of the S-sector, thus implying that (4.39) is also valid in more general constructions, including those with Pati–Salam gauge symmetry we investigate in the following chapters.

4.5 Effective Potential and the Cosmological Constant Problem

In addition to tachyonic instabilities, in order to construct realistic models without spacetime supersymmetry we must ensure that the predicted value of the cosmological constant is sufficiently low, in line with cosmological observations. The cosmological constant can be calculated at the one-loop level by integrating the partition function over the moduli space of the worldsheet torus. The vacuum-to-vacuum amplitude then appears as a function of the moduli:

$$V_{\text{1-loop}} = -\frac{1}{2(2\pi)^4} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{Z(\tau, \bar{\tau}; t_I)}{\tau_2}\ , \tag{4.41}$$

where $t_I = \{T_1^{(I)} + iT_2^{(I)}, U_1^{(I)} + iU_2^{(I)}, I = 1, 2, 3\}$. While the partition function depends explicitly on the 12 moduli $T^{(1,2,3)}$, $U^{(1,2,3)}$, a full analysis of the dynamics when taking them all into account is beyond our technical capabilities. In order to probe the modular dependency of the theory, we resort to a series of numerical simplifications.

Firstly, while spontaneous supersymmetry breaking can be achieved by appropriate $\mathbb{Z}_2$ shifts along any of the three tori, or even in combinations of them, we only consider the case where the shift responsible for breaking supersymmetry is solely along the direction of the first torus. We then fix the remaining two tori to their fermionic points, by taking $T^{(2)} = T^{(3)} = i$ and $U^{(2)} = U^{(3)} = (1+i)/2$ and replace them with their equivalent description in terms of theta functions.

Further simplification can be achieved, without loss of generality, by taking advantage of the Riemann identity (4.36) to perform the summation over the spin structures a, b. In addition to this, we can restrict our attention to the sub-sector $(H_1, G_1) \neq (0,0)$, as in that case supersymmetry is restored. Supersymmetry is also restored in the sub-sectors satisfying $(H_1, G_1) \neq (h_1, g_1)$. Since simultaneous shifts and twists also lead to vanishing contributions due to the lattice definition (3.112), numerical calculations can safely be performed by taking $h_1 = g_1 = 0$, in which case only the shifted lattice contributes. The partition function then takes the simplified form:

$$Z = \frac{1}{2} \sum_{H_1, G_1 \in \mathbb{Z}_2} \Psi[^{H_1}_{G_1}] \Gamma^{\text{shift}}_{2,2}[^{H_1}_{G_1}](T,U) \; , \tag{4.42}$$

where

$$\begin{aligned} \Psi[^{H_1}_{G_1}] = & \frac{1}{\eta^{12}\bar\eta^{24}} \frac{1}{2^6} \sum_{\substack{k,\rho \\ l,\sigma}\in\mathbb{Z}_2} \sum_{\substack{\gamma_2,\gamma_3 \\ \delta_2,\delta_3}\in\mathbb{Z}_2} \sum_{\substack{h_2,H \\ g_2,G}\in\mathbb{Z}_2} (-1)^{HG+\hat\Phi} \vartheta[^{1+H_1}_{1+G_1}]^2 \vartheta[^{1+H_1+h_2}_{1+G_1+g_2}]^2 \\ & \times \vartheta[^{\gamma_2}_{\delta_2}] \vartheta[^{\gamma_2+h_2}_{\delta_2+g_2}] \vartheta[^{\gamma_3}_{\delta_3}] \vartheta[^{\gamma_3-h_2}_{\delta_3-g_2}] \bar\vartheta[^{\gamma_2}_{\delta_2}] \bar\vartheta[^{\gamma_2+h_2}_{\delta_2+g_2}] \bar\vartheta[^{\gamma_3}_{\delta_3}] \bar\vartheta[^{\gamma_3-h_2}_{\delta_3-g_2}] \\ & \times \bar\vartheta[^{k}_{\ell}]^6 \bar\vartheta[^{k+h_2}_{\ell+g_2}] \bar\vartheta[^{k-h_2}_{\ell-g_2}] \bar\vartheta[^{\rho}_{\sigma}]^4 \bar\vartheta[^{\rho+H}_{\sigma+G}]^4 \end{aligned} \tag{4.43}$$

with the phase now relabeled to $\hat\Phi$ to reflect the above simplifications have been taken into account.

The lattice-independent factor can be Fourier expanded in terms of $q, \bar q$:

$$\Psi[^{H_1}_{G_1}] = \sum_{\Delta \geq -\frac{1}{2}} \sum_{\bar\Delta \geq -1} C[^{H_1}_{G_1}](\Delta, \bar\Delta) q^{\Delta} \bar q^{\bar\Delta} \; . \tag{4.44}$$

We finally fix the T_1, U_1 and U_2 moduli of the Scherk–Schwarz torus to their fermionic point values and consider deformations of T_2. While the exact shape and characteristics of the effective potential are highly model dependent, we can deduce some of its properties which will be encountered in all cases. First of all, assuming supersymmetry is spontaneously broken by a stringy realisation of the Scherk–Schwarz mechanism, in which the masses generated are inversely proportional to the volume of the internal space, the potential must asymptotically vanish as the limit $T_2 \to \infty$ where supersymmetry is restored in a higher-dimensional theory is approached. Additionally, since the momentum lattice respects the T-duality symmetry as $T_2 \leftrightarrow 1/T_2$, the same asymptotic behaviour will be present in the limit $T_2 \to 0$. This further implies that the self-dual fermionic point $T_2 = 1$ will correspond to a local extremum. In figure 4.4 we illustrate the typical form of the potential, taking all of the above into account.

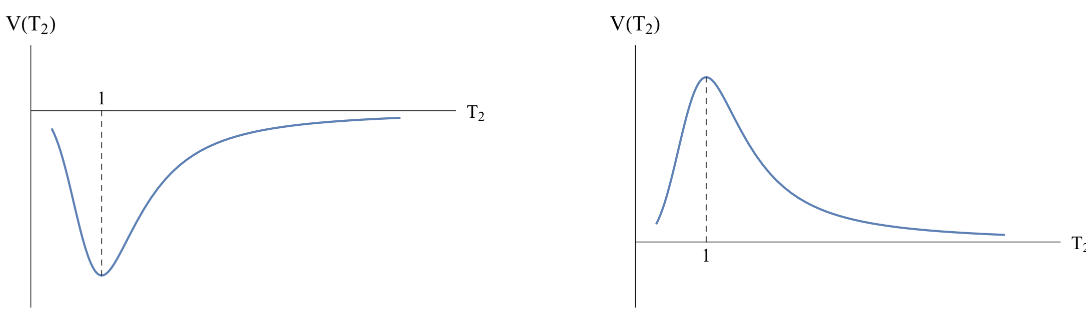

Figure 4.4: The typical form of the one-loop potential as a function of the T_2 modulus.

In the vast majority of models, this potential takes the form of a puddle, with a global minimum at $T_2 = 1$. In addition to predicting large, negative values for the cosmological constant, this implies that supersymmetry is broken at the string scale, while deformations of the moduli around the minimum typically give rise to tachyons.

Even when the potential is positive definite, however, the situation is still not ideal. The asymptotic behaviour of the effective potential can be extracted by using the unfolding method [245, 251, 252] by decomposing the lattice into modular orbits. A more detailed showcase of this procedure can be found in [91]. For the purposes of this work, we will merely reiterate the result, in which the asymptotic form is given by:

$$-2(2\pi)^4 V_{1-\mathrm{loop}} \simeq \frac{C[^0_1](0,0)}{\pi^3 T_2^2} \sum_{m_1,m_2} \frac{U_2^3}{\left|m_1+\frac{1}{2}+Um_2\right|^6} + \dots \ , \tag{4.45}$$

where the ellipses denote exponentially suppressed terms. If we take into account the current experimental constraints on the size of the extra dimensions and assume that the compactification scale is of the order of a few TeV [253], then the value of the one-loop potential exceeds the observed value of the cosmological constant by many orders of magnitude. In order to obtain small values for the cosmological constant, within the bounds set by observations, it is therefore necessary to ensure the exponential suppression of the effective potential by requiring that the power-law term vanishes:

$$C[^0_1](0,0) = 0 \ . \tag{4.46}$$

In order to obtain a physical interpretation of the above condition, we need to investigate the components of the full partition function in which it appears by incorporating the lattice contribution. The lattice generates moduli dependent contributions, with one exception: the term $\Gamma^{\mathrm{shift}}_{2,2}[^0_1]$ always takes the form

$$\Gamma^{\mathrm{shift}}_{2,2}[^0_1] = 1 + f(T_2) \ , \tag{4.47}$$

where $f(T_2)$ includes all moduli-dependent terms. The moduli independent term equal to unity gives rise to physical massless states when taken together with the $(\Delta,\bar{\Delta})=(0,0)$ contribution of Ψ. These will be the only massless states of the theory at generic points, as all other (H_1,G_1) sectors exhibit moduli-dependent masses. We note that in the $(H_1,G_1)=(0,0)$ case the partition function vanishes identically, in which case we can recast the condition in the form.

$$\sum_{G_1} C[^{\,0}_{G_1}](0,0) = 0 \ . \tag{4.48}$$

This condition can be reinterpreted as a constraint on the physical massless states of the theory, as summation over G_1 imposes the appropriate invariance projection. The requirement of an exponentially suppressed cosmological constant therefore imposes a strict cancellation between massless bosonic and fermionic degrees of freedom, in which case we arrive at the conventional condition of

$$n_B = n_F \ , \tag{4.49}$$

which defines a class of models that exhibit boson-fermion degeneracy in their massless spectra. These models, known in the literature as super no-scale models [137–140] yield exponentially small values for the vacuum amplitude for sufficiently large values of the compactification radius.

Chapter 5

Classification of Four Dimensional non supersymmetric Pati–Salam Models

Having outlined all the necessary mathematical tools, we are now ready to introduce the class of non supersymmetric Pati–Salam heterotic string models comprising the main focus of this thesis. Using the tools offered by the free fermionic formulation we analyse the massless string spectrum, as well as possible tachyonic modes that may appear, and formulate a set of model-independent conditions that guarantee the consistency of the theory with low energy phenomenology. We then switch to the orbifold notation which allows us to investigate the theory at generic points of the moduli space, away from the fermionic point. In this framework, additional conditions are needed in order to ensure that supersymmetry is broken spontaneously, in a manner consistent with the stringy realisation of the Scherk–Schwarz mechanism as outlined in section 4.4. We also require that the models be Bose–Fermi degenerate in their massless spectrum, in order to facilitate the construction of vacua with an exponentially suppressed cosmological constant, in line with observational constraints. Utilising the basic properties of the one-loop effective potential for models satisfying all the above criteria, we then present a classification of the full parameter space of compatible models and present some explicit examples with appealing phenomenology.

5.1 Model Setup

The construction of the class of models under consideration follows the procedure of the $SO(10)$ models outlined in the previous chapter, with the addition of the basis vector $u_{10} = \alpha = \{\bar{\psi}^{4,5}, \bar{\phi}^{1,2}\}$. The full vector basis is:

$$\begin{aligned}
u_1 &= \mathbb{1} = \{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} | \bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\phi}^{1,\dots,8}\}\ ,\\
u_2 &= S = \{\psi^\mu, \chi^{1,\dots,6}\}\ ,\\
u_3 &= T_1 = \{y^{12}, \omega^{12} | \bar{y}^{12}, \bar{\omega}^{12}\}\ ,\\
u_4 &= T_2 = \{y^{34}, \omega^{34} | \bar{y}^{34}, \bar{\omega}^{34}\}\ ,\\
u_5 &= T_3 = \{y^{56}, \omega^{56} | \bar{y}^{56}, \bar{\omega}^{56}\}\ ,\\
u_6 &= b_1 = \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^1\}\ ,\\
u_7 &= b_2 = \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \bar{y}^{12}, \bar{y}^{56}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^2\}\ ,\\
u_8 &= z_1 = \{\bar{\phi}^{1,\dots,4}\}\ ,\\
u_9 &= z_2 = \{\bar{\phi}^{5,\dots,8}\}\ ,\\
u_{10} &= \alpha = \{\bar{\psi}^{4,5}, \bar{\phi}^{1,2}\}\ .
\end{aligned} \tag{5.1}$$

The addition of α to the vector basis breaks the observable $SO(10)$ gauge symmetry to $SO(6) \times SO(4) \simeq SU(4) \times SU(2)_L \times SU(2)_R$, while also reducing the hidden sector: $SO(8) \times SO(8) \to SO(4) \times SO(4) \times$

$SO(8) \simeq SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4 \times SO(8)$. The full gauge symmetry these models possess is

$$G = SU(4) \times SU(2)_L \times SU(2)_R \times U(1)^3 \times SU(2)^4 \times SO(8) \;, \tag{5.2}$$

with special choices of the GGSO phases potentially giving rise to various enhancements, which we will cover in a later part of this chapter. A standard $U(1)^6$ factor arising due to the compactification has been suppressed. We note that the models exhibit a cyclical symmetry among the three $SO(4)$ factors. In the following we adopt the standard convention where the $SU(2)_L \times SU(2)_R$ factor of the Pati–Salam gauge symmetry is attributed to the fermions $\bar\psi^{4,5}$, while $\bar\phi^{1,2}$ and $\bar\phi^{3,4}$ are responsible for the hidden sector $SU(2)$ gauge groups.

The string spectrum is organised in 2^{10} sectors given by the linear combinations of basis vectors

$$\sum_{i=1}^{10} m_i \beta_i \;, \quad m_1, \ldots, m_{10} = \{0,1\} \;, \tag{5.3}$$

subject to GGSO projections, encoded in the generalised projectors (4.25), (4.33). This class of models comprises $2^{46} \approx 7 \times 10^{13}$ a priori distinct vacua, each defined by specifying the sign of the independent GGSO phases $c\left[\begin{smallmatrix} u_i \\ u_j \end{smallmatrix}\right]$, $i = j = 1$, $i < j = 1, \ldots, 10$. In order to focus our efforts on non supersymmetric vacua, we project out the S-sector gravitino by explicitly fixing the GGSO coefficient

$$c\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = c\begin{bmatrix} S \\ T_1 \end{bmatrix} = +1 \,, \tag{5.4}$$

associated with the first torus. We stress that while this ensures the construction of non supersymmetric vacua, it is not sufficient by itself to ensure that supersymmetry breaking is in line with the stringy Scherk–Schwarz mechanism. In order to ensure the spontaneous breaking of supersymmetry, we employ additional conditions as discussed in section 4.4.

The one-loop partition function of the models in the orbifold picture is given by:

$$\begin{aligned} Z = & \frac{1}{\eta^2\bar\eta^2}\, \frac{1}{2^4} \sum_{\substack{h_1,h_2,H,H' \\ g_1,g_2,G,G'}} \frac{1}{2^3} \sum_{\substack{a,k,\rho \\ b,\ell,\sigma}} \frac{1}{2^3} \sum_{\substack{H_1,H_2,H_3 \\ G_1,G_2,G_3}} (-1)^{a+b+HG+H'G'+\Phi} \\ & \times \frac{\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}{\eta} \frac{\vartheta\left[\begin{smallmatrix} a+h_1 \\ b+g_1 \end{smallmatrix}\right]}{\eta} \frac{\vartheta\left[\begin{smallmatrix} a+h_2 \\ b+g_2 \end{smallmatrix}\right]}{\eta} \frac{\vartheta\left[\begin{smallmatrix} a-h_1-h_2 \\ b-g_1-g_2 \end{smallmatrix}\right]}{\eta} \\ & \times \frac{\bar\vartheta\left[\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right]^3}{\bar\eta^3} \frac{\bar\vartheta\left[\begin{smallmatrix} k+H' \\ \ell+G' \end{smallmatrix}\right]}{\bar\eta} \frac{\bar\vartheta\left[\begin{smallmatrix} k-H' \\ \ell-G' \end{smallmatrix}\right]}{\bar\eta} \frac{\bar\vartheta\left[\begin{smallmatrix} k+h_1 \\ \ell+g_1 \end{smallmatrix}\right]}{\bar\eta} \frac{\bar\vartheta\left[\begin{smallmatrix} k+h_2 \\ \ell+g_2 \end{smallmatrix}\right]}{\bar\eta} \frac{\bar\vartheta\left[\begin{smallmatrix} k-h_1-h_2 \\ \ell-g_1-g_2 \end{smallmatrix}\right]}{\bar\eta} \\ & \times \frac{\bar\vartheta\left[\begin{smallmatrix} \rho+H' \\ \sigma+G' \end{smallmatrix}\right]}{\bar\eta} \frac{\bar\vartheta\left[\begin{smallmatrix} \rho-H' \\ \sigma-G' \end{smallmatrix}\right]}{\bar\eta} \frac{\bar\vartheta\left[\begin{smallmatrix} \rho \\ \sigma \end{smallmatrix}\right]^2}{\bar\eta^2} \frac{\bar\vartheta\left[\begin{smallmatrix} \rho+H \\ \sigma+G \end{smallmatrix}\right]^4}{\bar\eta^4} \\ & \times \frac{\Gamma^{(1)}_{2,2}\left[\begin{smallmatrix} H_1 \\ G_1 \end{smallmatrix}\middle|\begin{smallmatrix} h_1 \\ g_1 \end{smallmatrix}\right](T^{(1)},U^{(1)})}{\eta^2\bar\eta^2} \frac{\Gamma^{(2)}_{2,2}\left[\begin{smallmatrix} H_2 \\ G_2 \end{smallmatrix}\middle|\begin{smallmatrix} h_2 \\ g_2 \end{smallmatrix}\right](T^{(2)},U^{(2)})}{\eta^2\bar\eta^2} \frac{\Gamma^{(3)}_{2,2}\left[\begin{smallmatrix} H_3 \\ G_3 \end{smallmatrix}\middle|\begin{smallmatrix} h_1+h_2 \\ g_1+g_2 \end{smallmatrix}\right](T^{(3)},U^{(3)})}{\eta^2\bar\eta^2} \;. \end{aligned} \tag{5.5}$$

This notation follows the conventions of the $SO(10)$ constructions outlined in the previous chapter, and the parametrisation is consistent with that of (4.21), with the additional $SO(10)$ breaking twist introduced via the vector α parametrised by (H', G'). Supersymmetry can be broken spontaneously à la Scherk–Schwarz, generating a gravitino mass $m_{3/2} = 1/R$ as long as the orbifold phase Φ satisfies the constraints described in section 4.4.

In order to probe the modular dependency of the theory, we proceed with the numerical simplifications outlined in section 4.5. The partition function can then be cast in the simplified form

$$Z = \frac{1}{2} \sum_{H_1,G_1} \Psi\left[\begin{smallmatrix} H_1 \\ G_1 \end{smallmatrix}\right] \Gamma^{\text{shift}}_{2,2}\left[\begin{smallmatrix} H_1 \\ G_1 \end{smallmatrix}\right] \,, \tag{5.6}$$

with the lattice-independent factor

$$\begin{aligned}\Psi[^{H_1}_{G_1}] =& \frac{1}{\eta^{12}\bar{\eta}^{24}}\frac{1}{2^7}\sum_{\substack{k,\rho,\gamma_2,\gamma_3\\ l,\sigma,\delta_2,\delta_3}}\ \sum_{\substack{h_2,H,H'\\ g_2,G,G'}}(-1)^{HG+H'G'+\hat{\Phi}}\vartheta[^{1+H_1}_{1+G_1}]^2\vartheta[^{1+H_1+h_2}_{1+G_1+g_2}]^2\\ &\times\vartheta[^{\gamma_2}_{\delta_2}]\vartheta[^{\gamma_2+h_2}_{\delta_2+g_2}]\vartheta[^{\gamma_3}_{\delta_3}]\vartheta[^{\gamma_3-h_2}_{\delta_3-g_2}]\bar{\vartheta}[^{\gamma_2}_{\delta_2}]\bar{\vartheta}[^{\gamma_2+h_2}_{\delta_2+g_2}]\bar{\vartheta}[^{\gamma_3}_{\delta_3}]\bar{\vartheta}[^{\gamma_3-h_2}_{\delta_3-g_2}]\\ &\times\bar{\vartheta}[^k_\ell]^4\bar{\vartheta}[^{k+H'}_{\ell+G'}]\bar{\vartheta}[^{k-H'}_{\ell-G'}]\bar{\vartheta}[^{k+h_2}_{\ell+g_2}]\bar{\vartheta}[^{k-h_2}_{\ell-g_2}]\bar{\vartheta}[^{\rho+H'}_{\sigma+G'}]\bar{\vartheta}[^{\rho-H'}_{\sigma-G'}]\bar{\vartheta}[^\rho_\sigma]^2\bar{\vartheta}[^{\rho+H}_{\sigma+G}]^4\,.\end{aligned} \tag{5.7}$$

Since the lattices we employ in the parametrisation of the compactified space are identical to those used in the $SO(10)$ construction of the previous chapter, the asymptotic behaviour of the one-loop effective potential is similar to 4.45, necessitating Bose–Fermi degeneracy in the massless spectrum in order to exponentially suppress the cosmological constant. We note that we require the exponential suppression of the cosmological constant in the $T_2 \gg 1$ limit, which implies that the super no-scale condition $n_B = n_F$ must be imposed at the generic point. In fact, it is easy to show that none of the models in the framework defined by (5.1) which exhibits a degeneracy between bosons and fermions at generic points of the moduli space will retain this feature at the fermionic point, as the additional massless contributions arising at the maximal symmetry point can never cancel out against each other due to the form of (5.7) [2].

5.2 Tachyonic Modes

Since non supersymmetric constructions are the focus of this chapter, we must first ensure that the corresponding vacua are free of physical tachyons. In the absence of a mechanism automatically projecting out these states, we must ensure that the GGSO projections resolve in such a way as to eliminate all of them.

By evaluating the mass-shell condition (3.24), tachyons can be traced back to 19 sectors. States with mass $M^2 = -1/2$ may arise out of the vacuum of the $\mathcal{T}_{-1/2} \in \{z_1, z_2, \alpha, z_1+\alpha\}$ sectors. The corresponding projection operators that must be evaluated are thus $\mathbb{P}^+_{\mathcal{T}_{-1/2}}$, where

$$\begin{aligned}\Xi^+(z_1) &= \{S, T_1, T_2, T_3, b_1, b_2, z_2\}\ ,\\ \Xi^+(z_2) &= \{S, T_1, T_2, T_3, b_1, b_2, z_1, \alpha\}\ ,\\ \Xi^+(\alpha) &= \{S, T_1, T_2, T_3, b_1+b_2, b_1+z_1+\alpha, z_2\}\ ,\\ \Xi^+(z_1+\alpha) &= \{S, T_1, T_2, T_3, b_1+b_2, b_1+\alpha, z_2\}\ .\end{aligned} \tag{5.8}$$

The ground states of the sectors $\mathcal{T}_{-1/4} \in \{T_i+z_1, T_i+z_2\,,\ T_i+pz_1+\alpha\}$, where $i = 1,2,3$ and $p = 0,1$ are also tachyonic, with $M^2 = -1/4$. The associated projector operators are $\mathbb{P}^+_{\mathcal{T}_{-1/4}}$ with

$$\begin{aligned}\Xi^+(T_i+z_1) &= \{S, T_j, T_k, b_i, z_2\}\ ,\\ \Xi^+(T_i+z_2) &= \{S, T_j, T_k, b_i, z_1, \alpha\}\ ,\\ \Xi^+(T_i+\alpha) &= \{S, T_j, T_k, b_i+x, b_i+z_1+\alpha, z_2\}\ ,\\ \Xi^+(T_i+z_1+\alpha) &= \{S, T_j, T_k, b_i+x, b_i+\alpha, z_2\}\ ,\end{aligned} \tag{5.9}$$

where $i = 1,2,3, i \neq j \neq k$.

Finally, tachyons with mass $M^2 = -1/4$ can also be generated in the sectors T_1, T_2, T_3 by acting on the ground state with a right-moving fermion oscillator. In that case, the associated projectors are $\mathbb{P}^{\bar{\varphi}}_{T_i}$ with $\bar{\varphi} \in \mathbb{1}_R - T_{iR}$ and

$$\Xi(T_i) = \{S, T_j, T_k, b_i, z_1, z_2, \alpha\}\,,\ i \neq j \neq k\,. \tag{5.10}$$

The first condition we impose on our models, in order to ensure that they are free of physical tachyons, is that all projectors defined above vanish:

$$\begin{aligned}&\mathbb{P}^+_a = 0\ ,\ a \in \{\mathcal{T}_{-1/2}, \mathcal{T}_{-1/4}\}\,,\\ &\mathbb{P}^{\bar{\varphi}}_{T_i} = 0\ ,\ \bar{\varphi} \in \mathbb{1}_R - T_{iR}\ ,\ i = 1,2,3\,.\end{aligned} \tag{5.11}$$

While tachyons carrying oscillators of the internal coordinates $\{\bar{y}^{12,34,56}, \bar{\omega}^{12,34,56}\}$ are not present in the spectrum at generic points of the moduli space, we insist on their elimination, in order to ensure that our models are consistent at the fermionic point as well.

5.3 Analysis of the Massless Spectrum

We now proceed with the analysis of all sectors of the theory which give rise to massless states. The models introduced in (5.1) exhibit a total of 229 such sectors, with 122 giving rise to bosons and 107 generating fermions. This mismatch in the number of bosonic and fermionic sectors can be attributed to the absence of spacetime supersymmetry, as will become clear in a later part of this section.

In order to proceed with the analysis in an organised manner, we divide the spectrum as follows. Firstly, we consider all sectors that can give rise to gauge bosons, introducing the necessary conditions which ensure that the observable Pati–Salam gauge symmetry is not enhanced, followed by the fermionic sectors which in supersymmetric theories would be responsible for the gauginos. We then focus on the subset of sectors that are responsible for the particle content which is compatible with a minimal low energy implementation of the Pati–Salam theory, deriving additional constraints to ensure the consistency of the models with such constructions. The next step of the analysis concerns sectors which may give rise to fractionally charged exotic states. We discuss the implications of the existence of such states in the spectrum and introduce constraints that ensure they decouple from the low energy theory. The next subset of sectors corresponds to those giving rise to Pati–Salam singlets which transform non-trivially under the hidden sector gauge groups. We conclude the analysis by outlining the 15 unpaired bosonic sectors which are unique to non supersymmetric models. We note here that the above organisation of the spectrum is purely for bookkeeping purposes and does not reflect any underlying structure in the GGSO projections, or the massless spectrum in general.

5.3.1 The Gauge Sector

The starting point of our analysis is the gauge sector of the theory, comprising all sectors where gauge bosons may arise. We begin by focusing on the 0 sector, whose spectrum is model-independent, as all relevant GGSO projections are fixed. This sector is responsible for providing the spectrum of all models with a massless graviton and dilaton. In addition to these, the vector bosons responsible for the gauge symmetry of (5.2) are also present. Finally, the 0 sector generates various scalars, outlined in Table 5.1.

$SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^3\times SU(2)^4\times SO(8)$ representation(s)
$(\mathbf{6},\mathbf{1},\mathbf{1},\pm1,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\ (\mathbf{6},\mathbf{1},\mathbf{1},0,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$
$(\mathbf{6},\mathbf{1},\mathbf{1},0,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\ (\mathbf{1},\mathbf{1},\mathbf{1},\pm1,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$
$(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\ (\mathbf{1},\mathbf{1},\mathbf{1},0,\pm1,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$
$12\times(\mathbf{1},\mathbf{1},\mathbf{1},0,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$

Table 5.1: *Model independent spectrum of massless scalars generated in the* 0 *sector.*

We now proceed with the analysis of the model dependent sectors of the theory, starting with those that can generate additional vector bosons, resulting in the enhancement of the gauge symmetry. These are the 10 sectors

$$\mathcal{G}\in\{z_1,z_2,\alpha,z_1+z_2,z_1+\alpha,z_2+\alpha,z_1+z_2+\alpha,x,\alpha+x,z_1+\alpha+x\}\ .$$

Besides gauge bosons, these sectors may also generate scalars. However, we do not consider these to be of interest for the purposes of our analysis.

Let us first discuss the sectors which are responsible for enhancements of the observable gauge group. The only sector which can enhance the observable gauge group without also affecting the hidden gauge symmetry is $x=\{\bar\psi^{1,\dots,5},\bar\eta^{1,2,3}\}$. The gauge bosons of this sector are of the form:

$$\psi^\mu_{-\frac12}\left|0\right\rangle_L\times\left|(\bar\psi^{1,\dots,5},\bar\eta^{1,2,3})\right\rangle_R\ , \tag{5.12}$$

and their presence in the spectrum, signaled by the projector

$$\mathbb{P}^+_x\mathbb{P}^-_x=1\ , \tag{5.13}$$

with

$$\begin{aligned} \Xi^+(x) &= \{T_1, T_2, T_3, z_1, z_2\} \ , \\ \Xi^-(x) &= \{S\} \ , \end{aligned} \tag{5.14}$$

leads to the enhancement

$$SU(4) \times SU(2)_{L/R} \times U(1)' \to SU(6) \ , \tag{5.15}$$

where $U(1)'$ denotes a linear combination of the three $U(1)$ factors. This symmetry enhancement can be utilised to investigate $SU(6) \times SU(2)$ models in the framework of heterotic string theory. While such models have interesting phenomenological implications [254–259], they lie beyond the scope of this thesis.

The enhancement of parts of both the observable and hidden sector gauge groups can be realised by gauge bosons in the sectors $\{z_1, z_2, \alpha, z_1+\alpha, z_2+\alpha, z_1+z_2+\alpha, \alpha+x, z_1+\alpha+x\}$. Gauge bosons in sectors z_1, z_2, α and $z_1+\alpha$ are constructed by acting on the right-moving vacuum with a fermion oscillator. In the cases of z_1 and z_2, in order to obtain an enhancement of both the observable and hidden symmetries the right-moving oscillator must be $\bar{\psi}^{1,2,3}$ or $\bar{\psi}^{4,5}$, while in α and $z_1+\alpha$ mixed enhancements occur for all oscillator choices. The exact GGSO combinations which result in these enhancements are equivalent to those derived for supersymmetric models in [117], subject to slight modifications due to the complexification of the internal fermions, and will not be repeated here.

Since we are only interested in models with an observable Pati–Salam gauge group, we must ensure that all gauge bosons generating the enhancements presented in (5.15) and table 5.2 are eliminated from the spectrum. In order to do so, we impose the conditions:

$$\begin{aligned} &\mathbb{P}_a^+ \mathbb{P}_a^- = 0,\ a \in \{x, \alpha+x, z_1+\alpha+x, z_2+\alpha, z_1+z_2+\alpha\} \ , \\ &\mathbb{P}_a^{\bar{\phi}} = 0,\ \bar{\phi} \in \mathbb{1}_R - a_R,\ a \in \{\alpha, z_1+\alpha\} \ , \\ &\mathbb{P}_a^{\bar{\phi}} = 0,\ \bar{\phi} \in \mathbb{1}_R - a_R,\ \bar{\phi} \notin \{\bar{\eta}^{1,2,3}, \bar{\phi}^{1,\dots,8}\},\ a \in \{z_1, z_2\} \ , \end{aligned} \tag{5.16}$$

where:

$$\begin{aligned} &\Xi^+(x) = \{T_1, T_2, T_3, z_1, z_2\} \ , \\ &\Xi^+(\alpha+x) = \{T_1, T_2, T_3, z_2, z_1+\alpha\} \ , \\ &\Xi^+(z_1+\alpha+x) = \{T_1, T_2, T_3, z_2, \alpha\} \ , \\ &\Xi^+(z_2+\alpha) = \{T_1, T_2, T_3, b_1+b_2, b_1+z_1+\alpha\} \ , \\ &\Xi^+(z_1+z_2+\alpha) = \{T_1, T_2, T_3, b_1+b_2, b_1+\alpha\} \ , \\ &\Xi^-(a) = \{S\}\,, a \in \{x, \alpha+x, z_1+\alpha+x, z_2+\alpha, z_1+z_2+\alpha\} \ , \\ &\Xi(\alpha) = \{S, T_1, T_2, T_3, b_1+b_2, b_1+z_1+\alpha, z_2\} \ , \\ &\Xi(z_1+\alpha) = \{S, T_1, T_2, T_3, b_1+b_2, b_1+\alpha, z_2\} \ , \\ &\Xi(z_1) = \{S, T_1, T_2, T_3, b_1, b_2, z_2\} \ , \\ &\Xi(z_2) = \{S, T_1, T_2, T_3, b_1, b_2, z_1, \alpha\} \ . \end{aligned} \tag{5.17}$$

Sector	Gauge Symmetry Enhancement
z_1	$SU(4)\times SU(2)_L\times SU(2)_R\times SU(2)^4\to SO(10)\times SO(8)$
z_2	$SU(4)\times SO(8)\to SO(14)$ $SU(2)_L\times SU(2)_R\times SO(8)\to SO(12)$
α	$SU(4)\times SU(2)_L\times SU(2)_R\times SU(2)^4\to SO(10)\times SO(8)$ $SU(2)_{L/R}\times SU(2)_{1/2}\times SO(8)\to SO(12)$ $SU(2)_{L/R}\times SU(2)_{1/2}\times U(1)\to SO(6)$ $SU(2)_{L/R}\times SU(2)_{1/2}\to SO(5)$
$z_1+\alpha$	$SU(4)\times SU(2)_L\times SU(2)_R\times SU(2)^4\to SO(10)\times SO(8)$ $SU(2)_{L/R}\times SU(2)_{3/4}\times SO(8)\to SO(12)$ $SU(2)_{L/R}\times SU(2)_{3/4}\times U(1)\to SO(6)$ $SU(2)_{L/R}\times SU(2)_{3/4}\to SO(5)$
$z_2+\alpha$	$SU(2)_{L/R}\times SU(2)_{1/2}\times SO(8)\to SO(12)$
$z_1+z_2+\alpha$	$SU(2)_{L/R}\times SU(2)_{3/4}\times SO(8)\to SO(12)$
$\alpha+x$	$SU(2)_{L/R}\times SU(2)_{1/2}\times U(1)'\to SU(6)$
$z_1+\alpha+x$	$SU(2)_{L/R}\times SU(2)_{3/4}\times U(1)'\to SU(6)$

Table 5.2: *All possible simultaneous enhancements of the observable and hidden gauge symmetries.*

Finally, there is the possibility of enhancements affecting only the hidden part of the gauge symmetry. Since such enhancements leave the observable Pati–Salam gauge symmetry unaffected, the corresponding models in which they occur are relevant to our analysis. The enhancement of the hidden gauge groups can be attributed to massless gauge bosons arising in the sectors z_1, z_2 or z_1+z_2. Vector bosons constructed by taking oscillators of $\bar{\eta}^{1,2,3}$ or $\bar{\phi}^{1,\dots,8}$ in the z_1 and z_2 sectors enhance the hidden gauge symmetry, leaving the Pati–Salam group unaffected. More specifically, the z_1 sector gauge bosons of the form

$$\psi^\mu_{-\frac{1}{2}}\left|0\right\rangle_L\times\begin{pmatrix}\bar{\eta}^{1,2,3}_{-\frac{1}{2}}\\ \bar{\eta}^{*1,2,3}_{-\frac{1}{2}}\end{pmatrix}\left|\left(\bar{\phi}^{1,\dots,4}\right)\right\rangle_R \tag{5.18}$$

lead to the enhancement $SU(2)_{1/2}\times SU(2)_{3/4}\times U(1)\to SO(6)$, while

$$\psi^\mu_{-\frac{1}{2}}\left|0\right\rangle_L\times\begin{pmatrix}\bar{\phi}^{5,\dots,8}_{-\frac{1}{2}}\\ \bar{\phi}^{*5,\dots,8}_{-\frac{1}{2}}\end{pmatrix}\left|\left(\bar{\phi}^{1,\dots,4}\right)\right\rangle_R \tag{5.19}$$

enhance the $SU(2)_{1/2}\times SU(2)_{3/4}\times SO(8)$ factor to $SO(12)$. Furthermore, in sector z_2, the states

$$\psi^\mu_{-\frac{1}{2}}\left|0\right\rangle_L\times\begin{pmatrix}\bar{\eta}^{1,2,3}_{-\frac{1}{2}}\\ \bar{\eta}^{*1,2,3}_{-\frac{1}{2}}\end{pmatrix}\left|\left(\bar{\phi}^{5,\dots,8}\right)\right\rangle_R \tag{5.20}$$

are responsible for the $U(1)\times SO(8)\to SO(10)$ enhancement, and

$$\psi^\mu_{-\frac{1}{2}}\left|0\right\rangle_L\times\begin{pmatrix}\bar{\phi}^{1,2}_{-\frac{1}{2}}\\ \bar{\phi}^{*1,2}_{-\frac{1}{2}}\end{pmatrix}\left|\left(\bar{\phi}^{5,\dots,8}\right)\right\rangle_R\ ,\quad \psi^\mu_{-\frac{1}{2}}\left|0\right\rangle_L\times\begin{pmatrix}\bar{\phi}^{3,4}_{-\frac{1}{2}}\\ \bar{\phi}^{*3,4}_{-\frac{1}{2}}\end{pmatrix}\left|\left(\bar{\phi}^{5,\dots,8}\right)\right\rangle_R \tag{5.21}$$

generate $SU(2)_1 \times SU(2)_2 \times SO(8) \to SO(12)$ and $SU(2)_3 \times SU(2)_4 \times SO(8) \to SO(12)$ enhancements respectively. The above states are identified by the projection operators

$$\begin{aligned} &\mathbb{P}^{\bar\phi}_{z_1} = 1 \ , \bar\phi \in \{\bar\eta^{1,2,3}, \bar\phi^{5,\dots,8}\} \ , \\ &\mathbb{P}^{\bar\phi}_{z_2} = 1 \ , \bar\phi \in \{\bar\eta^{1,2,3}, \bar\phi^{1,\dots,4}\} \ . \end{aligned} \tag{5.22}$$

In addition to these, the $z_1 + z_2$ sector gauge bosons result in the enhancement of the hidden $SU(2)_{1/2} \times SU(2)_{3/4} \times SO(8)$ gauge symmetry to $SO(12)$. The corresponding projection operator indicating this enhancement is:

$$\mathbb{P}^{+}_{z_1+z_2}\mathbb{P}^{-}_{z_1+z_2} = 1 \ , \tag{5.23}$$

where

$$\begin{aligned} &\Xi^{+}(z_1+z_2) = \{T_1, T_2, T_3, b_1, b_2\} \ , \\ &\Xi^{-}(z_1+z_2) = \{S\} \ . \end{aligned} \tag{5.24}$$

We note that the appearance of gauge bosons in a given sector does not preclude additional gauge bosons from emerging in other sectors as well, in which case more complicated gauge symmetry enhancements may arise. After ensuring that the observable gauge symmetry is unaffected, and imposing further phenomenological conditions which we outline in the following sections, the hidden sector gauge symmetry is restricted to four combinations: $U(1)^2 \times SO(6) \times SO(12)$, $U(1)^3 \times SU(2)^2 \times SO(12)$, $U(1)^2 \times SU(2)^4 \times SO(10)$, and $U(1)^2 \times SU(2)^2 \times SO(6) \times SO(8)$, in addition to the unenhanced $U(1)^3 \times SU(2)^4 \times SO(8)$.

In a supersymmetric theory, the 11 fermionic sectors $\{\mathcal{G}, S+\mathcal{G}\}$ are responsible for the gauginos of the theory and their spectrum is fully determined by the same projection operators as the ones introduced above for the gauge bosons. Since we have ensured the absence of supersymmetry by making the choice $c\left[{S \atop T_1}\right] = +1$, this is no longer the case. The GGSO projections for states in these sectors are decoupled from the corresponding bosons and their spectra are highly model dependent. We will present the most critical states arising in these sectors in the following sections.

5.3.2 Minimal Pati–Salam Content

We now shift our focus towards the twisted sectors of the theory, starting from those providing the necessary states for the construction of a minimal Pati–Salam low energy model. These must include at least 3 chiral fermion generations, while also exhibiting copies of the heavy and light Higgs bosons required to implement the spontaneous symmetry breaking chain

$$SU(4) \times SU(2)_L \times SU(2)_R \to SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_{em} \ . \tag{5.25}$$

Chiral fermionic matter can be generated in the 12 sectors labeled $\mathcal{S}^i_{pq} = S + b_i + pT_j + qT_k$, where $p, q = 0, 1$, $(i, j, k) = \{(1, 2, 3), (2, 1, 3), (3, 1, 2)\}$. Each of these sectors may give rise to chiral or anti-chiral matter in the PS representations $F_L(\mathbf{4}, \mathbf{2}, \mathbf{1}), \overline{F}_R(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}), \overline{F}_L(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}), F_R(\mathbf{4}, \mathbf{1}, \mathbf{2})$. In this framework, each (anti-)family requires the combination of appropriate states arising in two different sectors. Due to the asymmetric nature of the GGSO projections of α, the existence of complete generations is not guaranteed and must be imposed as a constraint.

In order to determine which $\mathcal{S}^i_{pq}$ sectors give rise to chiral fermions that survive the GGSO projection, we employ the projection operators

$$\mathbb{P}^{-}_{\mathcal{S}^i_{pq}} \ , \ \Xi^{-}\left(\mathcal{S}^i_{pq}\right) = \{T_i, z_1, z_2\} \ , \tag{5.26}$$

while the exact representation of the resulting states can be determined by the operators

$$X^{SU(4)}_{\mathcal{S}^i_{pq}} = \begin{cases} -c\begin{bmatrix} \mathcal{S}^i_{pq} \\ \mathcal{S}^j_{0,1-q}+\alpha \end{bmatrix}^* , & j \neq i = 1,2 \\ -c\begin{bmatrix} \mathcal{S}^i_{pq} \\ \mathcal{S}^1_{1-q,0}+\alpha \end{bmatrix}^* , & i = 3 \end{cases} \ , \tag{5.27}$$

which distinguish between fermions transforming as $\mathbf{4}$ and $\bar{\mathbf{4}}$, in conjunction with

$$X^{SO(4)}_{\mathcal{S}^i_{pq}} = -c\begin{bmatrix} \mathcal{S}^i_{pq} \\ \alpha \end{bmatrix}^* , i = 1, 2, 3 \,, \tag{5.28}$$

which identify whether a given sector generates doublets of $SU(2)_L$ or $SU(2)_R$. These operators allow the derivation of expressions that count the multiplicities of F_L, $\overline{F}_R$, $\overline{F}_L$ and F_R, denoted n_L, $\bar{n}_R$, $\bar{n}_L$ and n_R respectively. The latter take the form[1]:

$$\begin{aligned}
n_L &= 4\sum_{i=1}^{3}\sum_{p,q=0}^{1} \mathbb{P}^-_{\mathcal{S}^i_{pq}} \frac{1}{2}\left(1 + X^{SU(4)}_{\mathcal{S}^i_{pq}}\right)\frac{1}{2}\left(1 + X^{SO(4)}_{\mathcal{S}^i_{pq}}\right) \,, \\
\bar{n}_R &= 4\sum_{i=1}^{3}\sum_{p,q=0}^{1} \mathbb{P}^-_{\mathcal{S}^i_{pq}} \frac{1}{2}\left(1 - X^{SU(4)}_{\mathcal{S}^i_{pq}}\right)\frac{1}{2}\left(1 - X^{SO(4)}_{\mathcal{S}^i_{pq}}\right) \,, \\
\bar{n}_L &= 4\sum_{i=1}^{3}\sum_{p,q=0}^{1} \mathbb{P}^-_{\mathcal{S}^i_{pq}} \frac{1}{2}\left(1 - X^{SU(4)}_{\mathcal{S}^i_{pq}}\right)\frac{1}{2}\left(1 + X^{SO(4)}_{\mathcal{S}^i_{pq}}\right) \,, \\
n_R &= 4\sum_{i=1}^{3}\sum_{p,q=0}^{1} \mathbb{P}^-_{\mathcal{S}^i_{pq}} \frac{1}{2}\left(1 + X^{SU(4)}_{\mathcal{S}^i_{pq}}\right)\frac{1}{2}\left(1 - X^{SO(4)}_{\mathcal{S}^i_{pq}}\right) \,,
\end{aligned} \tag{5.29}$$

with the number of generations given by

$$n_g \equiv n_L - \bar{n}_L = \bar{n}_R - n_R \,. \tag{5.30}$$

The condition $n_L - \bar{n}_L = \bar{n}_R - n_R$ must be imposed as a constraint in order to ensure that our models exhibit complete generations. In addition to the $\mathcal{S}^i_{pq}$ sectors, chiral fermions may also be generated in the $S + x$ sector. In this case, however, there is no well-defined $X^{SU(4)}_{S+x}$ operator that distinguishes between the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations, indicating that chiral fermions in this sector are forced to come in pairs of $\big(\mathbf{4}+\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}\big)$ or $\big(\mathbf{4}+\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}\big)$. These have no effect on the net chirality of (5.30) and are therefore excluded from our analysis.

At this point we must address a severe limitation of the current class of models regarding the number of fermion generations. As is apparent from (5.29), the number of generations will always be a multiple of four and the string spectrum can therefore not be considered to be “realistic”. This multiplicity can be traced back to our choice of parametrising the internal coordinates in terms of the complex fermions $\{y^{12}, y^{34}, y^{56}, \omega^{12}, \omega^{34}, \omega^{56}\}$ and their right-moving counterparts $\{\bar{y}^{12}, \bar{y}^{34}, \bar{y}^{56}, \bar{\omega}^{12}, \bar{\omega}^{34}, \bar{\omega}^{56}\}$. In fact, it has been shown that models with three generations can be constructed in theories where all internal coordinates are parametrised by real fermions [3, 114–120, 259–264], a fact we will explicitly demonstrate in the following chapter.

We note that the complexification of the fermions was not mandated by any consistency conditions, nor was it required by phenomenology, but was rather chosen due to technical reasons. First of all, the mapping between the free fermionic and orbifold formulations is significantly simpler when all fermions are taken to be complex. Moreover, spontaneous supersymmetry breaking in a similar framework [91, 143] has been shown to produce models with appealing features and, as we saw in section 4.5, the conditions which allow for small values of the cosmological constant have a clear, physical interpretation in terms of a degeneracy in the massless string spectrum. Furthermore, when the theory is formulated in terms of real fermions, the parameter space of possible models is enlarged to such a degree that a systematic scan of all models is rendered unfeasible.

Moving on, the 13 corresponding bosonic partners of the aforementioned sectors, namely $\mathit{\Sigma}^i_{pq} = b_i + pT_j + qT_k$, where $p, q = 0, 1$, $(i, j, k) = \{(1, 2, 3), (2, 1, 3), (3, 1, 2)\}$ and x, can potentially give rise to scalars in the spinorial representations introduced above. As we mentioned in chapter 1, the spontaneous breaking of the Pati–Salam gauge symmetry in this framework is realised by giving a vev to the neutral component of the heavy Higgs, $H(\mathbf{4}, \mathbf{1}, \mathbf{2})$, and thus the outcome of the GGSO projection on these sectors is critical.

[1] Here we have chosen the chirality of the spacetime fermions, $\text{ch}(\psi^\mu)$, to be positive without loss of generality.

The projection operators determining whether states survive the GGSO projection are $\mathbb{P}^{+}_{\Sigma^i_{pq}}\mathbb{P}^{-}_{\Sigma^i_{pq}}$, where

$$\Xi^+\left(\Sigma^i_{pq}\right)=\{T_i,z_1,z_2\}\ ,\ \Xi^-\left(\Sigma^i_{pq}\right)=\{\alpha\}\ , \tag{5.31}$$

with the α projection essentially acting as $X^{SO(4)}_{\Sigma^i_{pq}}$, identifying the $SU(2)_R$ doublets. The x sector, like $S+x$, also turns out to be irrelevant to our analysis, though in this case the reason is more subtle. Consider a massless scalar state in the x sector. Since the left-moving vacuum is tachyonic, such a state must be of the form

$$\phi_{-\frac{1}{2}}\left|0\right\rangle_L\times\left|(\bar\psi^{1,\dots,5},\bar\eta^{1,2,3})\right\rangle_R\ , \tag{5.32}$$

where $\phi_{-\frac{1}{2}}$ is a left-moving fermion oscillator. However, the choice $\phi_{-\frac{1}{2}}=\{\chi^{12,34,56}\}$ is problematic, since the GGSO projection cannot distinguish physical scalars of this form from the gauge bosons obtained when $\phi_{-\frac{1}{2}}=\psi^\mu_{-\frac{1}{2}}$ which enhance the observable Pati–Salam gauge symmetry. Such scalars are thus irrelevant to our analysis as they are never encountered in models with a Pati–Salam gauge symmetry. If we choose, instead, to act with a $y^{12,34,56}$ or $\omega^{12,34,56}$ oscillator, then the resulting states carry non-trivial charge under the compactification lattice and, as such, disappear from the massless spectrum when we deform the theory away from the fermionic point.

Taking the above into consideration, the number of copies of the heavy Higgs multiplets responsible for breaking the PS symmetry can be determined by

$$n_H=4\sum_{i=1}^{3}\sum_{p,q=0}^{1}\mathbb{P}^{+}_{\Sigma^i_{pq}}\mathbb{P}^{-}_{\Sigma^i_{pq}}\ . \tag{5.33}$$

In addition to the above, the minimal implementation of the PS models also requires the presence of the SM Higgs boson $h(\mathbf{1},\mathbf{2},\mathbf{2})$, in order to realise the spontaneous breaking of the electroweak symmetry. This scalar bi-doublet can be generated in the sectors $\Upsilon^i_{pq}=\Sigma^i_{pq}+x=b_i+pT_j+qT_k+x, i\neq j\neq k$, and T_a+T_b, $a\neq b=\{1,2,3\}$, by acting on the vacuum with the right-moving fermionic oscillator $\bar\psi^{4,5}_{-\frac{1}{2}}$. We will, however, not take the sectors T_a+T_b into consideration, as they only generate massless states at the fermionic point. The number of SM Higgs scalars can therefore be expressed as

$$n_h=4\sum_{i=1}^{3}\sum_{p,q=0}^{1}\mathbb{P}^{\bar\psi^{4,5}}_{\Upsilon^i_{pq}}\ , \tag{5.34}$$

where $\Xi\left(\Upsilon^i_{pq}\right)=\{T_i,z_1,z_2,\alpha\}$.

The minimal Pati–Salam realisation in our framework includes 4 fermion generations as well as 4 copies of the heavy and light Higgs scalars. The massless string spectrum of the aforementioned sectors, however, may potentially include many additional states. In fact, the Υ^i_{pq} sectors can generate various states depending on which fermion oscillator acts on the vacuum. When $\bar\psi^{1,2,3}$ acts on the vacuum, the results are scalars transforming as $(\mathbf{6},\mathbf{1},\mathbf{1})$, which give rise to SM triplets, while $\bar\phi^{1,2}$ and $\bar\phi^{3,4}$ generate Pati–Salam singlets transforming as bi-doublets of $SU(2)_1\times SU(2)_2$ and $SU(2)_3\times SU(2)_4$ respectively. The action of a $\bar\phi^{5,\dots,8}$ oscillator is responsible for vectorials of the hidden $SO(8)$, which transform as singlets under the Pati–Salam gauge symmetry. Finally, the $\bar\eta^{1,2,3}$ oscillators generate scalars transforming as singlets of all non-abelian gauge groups. In addition to these scalars, the sectors

$$\mathcal{Y}^i_{pq}=\mathcal{S}^i_{pq}+x=S+b_i+pT_j+qT_k+x,\ i\neq j\neq k\ ,$$

can give rise to fermions in the aforementioned representations. All states presented above can be identified by their corresponding projection operators $\mathbb{P}^{\bar\varphi}_{\Upsilon^i_{pq}}$ and $\mathbb{P}^{\bar\varphi}_{\mathcal{Y}^i_{pq}}$.

5.3.3 Fractionally Charged Exotics

Up to this point we have discussed states transforming non-trivially under either the observable Pati–Salam gauge symmetry or the hidden sector groups, but not under both simultaneously. All these states exhibit standard electric charge assignments, given by the generator

$$Q_{em}=\frac{1}{\sqrt{6}}T_{15}+\frac{1}{2}I_{3_L}+\frac{1}{2}I_{3_R}, \tag{5.35}$$

where T_{15} and $I_{3_{L,R}}$ correspond to the diagonal generators of $SU(4)$ and $SU(2)_{L,R}$.

In addition to these, however, a generic feature of string compactifications is the appearance of exotic fractionally charged states [265–269]. The appearance of these states can be attributed to the breaking of the $SO(10)$ gauge symmetry and their presence can thus be traced to sectors which include the basis vector α. In the Pati–Salam framework we are investigating, these states either transform as colour (anti-) quadruplets in the representation $(\mathbf{4},\mathbf{1},\mathbf{1})$ or $(\bar{\mathbf{4}},\mathbf{1},\mathbf{1})$, or as colour singlets in the $(\mathbf{1},\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{1},\mathbf{2})$. The (anti-)quadruplets will give rise to SM triplets and singlets with fractional charge $Q = \pm 1/6$, while $(\mathbf{1},\mathbf{2},\mathbf{1})$ leads to additional leptons with $Q = \pm 1/2$. Finally, $(\mathbf{1},\mathbf{1},\mathbf{2})$ generates SM singlets with $Q = \pm 1/2$. In addition to the Pati–Salam transformations outlined above, these states also transform as doublets under one of the hidden sector $SU(2)$ groups.

Let us first consider the fermions carrying colour charge. These states can be traced to the sectors:

$$\mathcal{E}^i_{pqr} = \mathcal{S}^i_{pq} + rz_1 + \alpha = S + b_i + pT_j + qT_k + rz_1 + \alpha,$$

where $i \neq j \neq k$ and $p, q, r = 0, 1$. Following the procedure outlined above, the projection operators signaling their presence in the spectrum are:

$$\mathbb{P}^-_{\mathcal{E}^i_{pqr}} = 1,\ \Xi^-(\mathcal{E}^i_{pqr}) = \{T_i, (1-r)z_1 + \alpha, z_2\}\ , \tag{5.36}$$

while the representation operators determining their transformation properties under $SU(4)$ are given by:

$$X^{SU(4)}_{\mathcal{E}^i_{pqr}} = \begin{cases} -c\begin{bmatrix} \mathcal{E}^i_{pqr} \\ \mathcal{S}^j_{0,1-q} \end{bmatrix}^*, & i \neq j = 1,2 \\ -c\begin{bmatrix} \mathcal{E}^i_{pqr} \\ \mathcal{S}^1_{1-q,0} \end{bmatrix}^*, & i = 3 \end{cases} . \tag{5.37}$$

The number of exotic quadruplets and anti-quadruplets is then determined by:

$$\begin{aligned} n_{\mathbf{4}} &= 4\sum_{i=1}^{3}\sum_{p,q,r=0}^{1} \mathbb{P}^-_{\mathcal{E}^i_{pqr}} \frac{1}{2}\left(1 + X^{SU(4)}_{\mathcal{E}^i_{pqr}}\right)\ , \\ n_{\bar{\mathbf{4}}} &= 4\sum_{i=1}^{3}\sum_{p,q,r=0}^{1} \mathbb{P}^-_{\mathcal{E}^i_{pqr}} \frac{1}{2}\left(1 - X^{SU(4)}_{\mathcal{E}^i_{pqr}}\right)\ . \end{aligned} \tag{5.38}$$

In addition to $\mathcal{E}^i_{pqr}$, the untwisted sectors $S + \alpha + x$, and $S + z_1 + \alpha + x$ can also generate exotic fermion quadruplets. In that case, however, they are always generated in pairs of $(\mathbf{4} + \bar{\mathbf{4}}, \mathbf{1}, \mathbf{1})$. The existence of the aforementioned colour-charged states in the massless spectrum poses a problem for our models, since such violation of charge quantisation has never been observed experimentally [270–274]. In order to obtain models with compatible low energy phenomenology, we therefore need to add another condition to our models, ensuring the elimination of all exotic states from the low energy spectrum. Viable models satisfying such a constraint have been shown to exist in theories where the internal coordinates are described using real fermions [116, 259, 275], but when complex fermions are introduced instead, as is the case in our analysis, this possibility is eliminated.

Since the appearance of exotic states in the massless spectrum of our models is inevitable, we must ensure that they acquire heavy masses, rendering them irrelevant at low energies. As a minimal phenomenological requirement in order to have models whose low energy predictions are compatible with experimental data, we therefore demand that exotic fermion quadruplets always be vector-like, satisfying the condition $n_4 = n_{\bar{4}}$. Since the states arising in $S+\alpha+x$, and $S+z_1+\alpha+x$ automatically satisfy this, we will not be taking them into account in the counting of n_4 and $n_{\bar{4}}$. The condition $n_4 = n_{\bar{4}}$ is sufficient to ensure all exotic states decouple from the low energy spectrum, as all exotic fermion representations are real.

We now move on to exotic fermions transforming as bi-doublets of $SU(2)_{L/R} \times SU(2)_{1/2/3/4}$. We can encounter such fermions in the sectors

$$\mathcal{E'}^i_{pqr} = \mathcal{Y}^i_{pq} + rz_1 + \alpha = S + b_i + pT_j + qT_k + x + rz_1 + \alpha\ ,$$

when the corresponding projection operators:

$$\mathbb{P}^-_{\mathcal{E'}^i_{pqr}}\ ,\ \ \Xi^-(\mathcal{E'}^i_{pqr}) = \{T_i, z_2, (1-p)T_j + (1-q)T_k + b_i + (1-r)z_1 + \alpha\} \tag{5.39}$$

take non-vanishing values. Moreover, the Ramond vacuum of the untwisted sectors $a = \{S+z_2+\alpha, S+z_1+z_2+\alpha\}$ also generates exotic bi-doublets. The projection operators in this case are $\mathbb{P}_a^-$, where:

$$\begin{aligned}\Xi^-(S+z_2+\alpha) &= \{T_1, T_2, T_3, x+z_1+\alpha\}\ ,\\ \Xi^-(S+z_1+z_2+\alpha) &= \{T_1, T_2, T_3, x+\alpha\}\ .\end{aligned} \tag{5.40}$$

Furthermore, exotic fermions can also arise in the sectors $\beta = \{S+\alpha, S+z_1+\alpha\}$ upon the action of a right-moving fermion oscillator. Their exact transformations depend on the choice of oscillator, with:

$$\left|(\psi^\mu, \chi^{12,34,56})\right\rangle_L \times \begin{pmatrix} \bar{\phi}_{-\frac{1}{2}} \\ \bar{\phi}^*_{-\frac{1}{2}} \end{pmatrix} \left|(\bar{\psi}^{4,5}, \bar{\phi}^{1,2})\right\rangle_R\ ,\ \bar{\phi}_{-1/2} \in \{\bar{\eta}^{1,2,3}, \bar{\phi}^{3,\dots,8}\} \tag{5.41}$$

and

$$\left|(\psi^\mu, \chi^{12,34,56})\right\rangle_L \times \begin{pmatrix} \bar{\phi}_{-\frac{1}{2}} \\ \bar{\phi}^*_{-\frac{1}{2}} \end{pmatrix} \left|(\bar{\psi}^{4,5}, \bar{\phi}^{3,4})\right\rangle_R\ ,\ \bar{\phi}_{-1/2} \in \{\bar{\eta}^{1,2,3}, \bar{\phi}^{1,2}, \bar{\phi}^{5,\dots,8}\} \tag{5.42}$$

transforming as $SU(2)_{L/R} \times SU(2)_{1/2/3/4}$ bi-doublets, while

$$\left|(\psi^\mu, \chi^{12,34,56})\right\rangle_L \times \begin{pmatrix} \bar{\psi}^{1,2,3}_{-\frac{1}{2}} \\ \bar{\psi}^{*1,2,3}_{-\frac{1}{2}} \end{pmatrix} \left|(\bar{\psi}^{4,5}, \bar{\phi}^{1,2})\right\rangle_R \tag{5.43}$$

and

$$\left|(\psi^\mu, \chi^{12,34,56})\right\rangle_L \times \begin{pmatrix} \bar{\psi}^{1,2,3}_{-\frac{1}{2}} \\ \bar{\psi}^{*1,2,3}_{-\frac{1}{2}} \end{pmatrix} \left|(\bar{\psi}^{4,5}, \bar{\phi}^{3,4})\right\rangle_R \tag{5.44}$$

transform as $SU(4)$ vectorials. The appearance of the above states in the spectrum is determined by the projection operators $\mathbb{P}_\beta^{\bar{\phi}}$, where:

$$\begin{aligned}\Xi(S+\alpha) &= \{T_1, T_2, T_3, z_2, x+z_1+\alpha\}\ ,\\ \Xi(S+z_1+\alpha) &= \{T_1, T_2, T_3, z_2, x+\alpha\}\ .\end{aligned} \tag{5.45}$$

We do not consider states containing $\bar{y}^{12,34,56}$ or $\bar{\omega}^{12,34,56}$ oscillators to be relevant to our analysis, as these are not present in the massless spectrum at generic points of the moduli space.

Fractionally charged scalars can arise in the 48 bosonic counterparts of the sectors $\mathcal{E}^i_{pqr}$ and $\mathcal{E'}^i_{pqr}$. Exotic bosons transforming as colour quadruplets can be traced to the sectors

$$E^i_{pqr} = \Sigma^i_{pq} + rz_1 + \alpha = b_i + pT_j + qT_k + rz_1 + \alpha\ ,$$

while bi-doublets occur in

$$E'^i_{pqr} = \Upsilon^i_{pq} + rz_1 + \alpha = b_i + pT_j + qT_k + x + rz_1 + \alpha\ .$$

All bosonic states in these sectors come in vector-like pairs, and can therefore acquire heavy masses.

5.3.4 Hidden Sector States

After concluding the analysis of all the sectors that generate states charged under the observable gauge symmetries, we now move to the sectors which generate "hidden" states. These include states with standard charge assignment that are singlets under the observable gauge symmetry but transform non-trivially under the hidden sector gauge groups. We note that we have already encountered such states in the sectors Υ^i_{pq} and $\mathcal{Y}^i_{pq}$. Since these states are not part of the minimal Pati–Salam setup, they are not a focus of our analysis. Nevertheless, we outline them here for the sake of completion.

The twelve sectors

$$\mathcal{H}^i_{pq} = \mathcal{Y}^i_{pq} + z_1 = S + b_i + pT_j + qT_k + x + z_1$$

can generate Pati–Salam singlets transforming as bi-doublets of the hidden sector $SU(2)_{1/2} \times SU(2)_{3,4}$, while fermions from the sectors

$$\mathcal{H}'^i_{pq} = \mathcal{Y}^i_{pq} + z_2 = S + b_i + pT_j + qT_k + x + z_2$$

transform as $SO(8)$ octuplets. The corresponding bosonic states can be obtained from the sectors

$$H^i_{pq} = \Upsilon^i_{pq} + z_1 = b_i + pT_j + qT_k + x + z_1$$

and

$$H'^i_{pq} = \Upsilon^i_{pq} + z_2 = b_i + pT_j + qT_k + x + z_2$$

respectively.

5.3.5 Lone sectors

We conclude the analysis of the massless spectrum with the group of 15 sectors:

$$L \in \{T_i + T_j + pz_1 + q\alpha,\, T_i + T_j + z_2\}\ ,\ \ j \neq i = 1,2,3\ ,\ p,q = 0,1\ .$$

The level-matched states arising in the fermionic sectors $S + L$ that would give rise to the superpartners of the above are always massive, as can easily be deduced from (3.24). The GGSO projection in supersymmetric models therefore projects out the sectors L in their entirety. Massless states arising in these sectors are thus unique to non supersymmetric models, where the GGSO projections of bosons and fermions decouple. Moreover, the massless states arising in these sectors acquire moduli-dependent masses when deformations of the moduli are considered. Despite possibly contributing to the massless spectrum at the fermionic point, the sectors L are consequently excluded from our analysis.

5.4 Full Analysis of the Parameter Space and Classification of Viable Models

We now proceed with the analysis of the parameter space of all possible vacua. As shown in the beginning of this chapter, (5.1) defines a class of $2^{45} \sim 3.5 \times 10^{13}$ a priori distinct non supersymmetric models. Following the discussion of the previous sections, however, it is obvious that the phenomenology of a large number of these models will not be consistent with low energy experiments. In this section, we focus on identifying models which are consistent with a set of conditions that ensure they exhibit “semi-realistic” behaviour. These conditions are:

(i) Absence of on-shell tachyons, imposed by (5.11).

(ii) Absence of enhancements of the observable Pati–Salam gauge symmetry, by demanding (5.16).

(iii) Existence of complete chiral generations under $SU(4)\times SU(2)_L\times SU(2)_R$, requiring $n_g = n_L - \bar{n}_L = \bar{n}_R - n_R \neq 0$ as defined in (5.29), (5.30).

(iv) Existence of heavy and light Higgs bosons responsible for the spontaneous breaking of the Pati–Salam and SM gauge symmetries, implying $n_H > 0$ and $n_h > 0$ in (5.33) and (5.34).

(v) Vector-like fractionally charged exotics, satisfying $n_4 = n_{\bar{4}}$ in (5.38).

(vi) Spontaneous supersymmetry breaking, consistent with the Scherk–Schwarz mechanism, as covered in section 4.4.

(vii) Exponential suppression of the cosmological constant in the region $T_2 \gg 1$, by imposing $n_B = n_F$, satisfying the condition (4.48).

The set of conditions (i)-(vii) can be directly expressed as a set of algebraic equations involving the GGSO coefficients. These can then be included into a computer program which scans over all relevant GGSO coefficients, applying the classification techniques introduced in [114,117,276]. However, it turns out that not all GGSO phases are relevant to our analysis. In fact, the phases $c[{}^{\mathbb{1}}_{\mathbb{1}}]$, $c[{}^{\mathbb{1}}_{S}]$ and $c[{}^{S}_{b_{1,2}}]$ correspond to conventions and can be safely fixed to $+1$ without loss of generality. Moreover, $c[{}^{b_1}_{b_2}]$ is only responsible for an overall chirality flip, while the six additional coefficients $c[{}^{T_1}_{b_2}]$, $c[{}^{T_2}_{b_1}]$, $c[{}^{\mathbb{1}}_{b_{1,2}}]$ and $c[{}^{\mathbb{1}}_{z_{1,2}}]$ do not enter any of the algebraic expressions related to the above criteria. We are therefore free to also fix these phases to $+1$, without affecting any of the phenomenological characteristics under investigation, essentially reducing the parameter space of vacua to $2^{34} \sim 1.7 \times 10^{10}$. We note that each of these vacua actually corresponds to a family of 2^{11} models with identical behaviour under the criteria (i)-(vii), obtained by iterating over all combinations of the 11 coefficients we have fixed.

At the technical level, the scan can be significantly accelerated by applying a two-stage process [2,3,119,264], in which we take advantage of the fact that some critical properties of the models are already established at the intermediate $SO(10)$-level, obtained by ignoring the $SO(10)$-breaking vector α and the relevant projections. More specifically, since Pati–Salam generations are obtained by a combination of states arising in two sectors, all Pati–Salam models with complete generations will descend from $SO(10)$ models with an even number of sectors giving rise to spinorial fermions, whose net chirality as defined in (4.28), (4.29) is non-vanishing and equal to $n_g = 0$ mod 8. Further constraints can be identified by requirement (iv). In order for the necessary Higgs bosons to be present in the spectrum, at least one sector generating scalars in the **10** and $\mathbf{16}/\overline{\mathbf{16}}$ representations of $SO(10)$ must lead to physical states, which translates to the requirements

$$\begin{aligned} &\sum_{\substack{i=1,2,3\\p,q=0,1}} \mathbb{P}^{+}_{\Sigma^i_{pq}} \neq 0 \, , \\ &\sum_{\substack{i=1,2,3\\p,q=0,1}} \mathbb{P}^{\overline{\psi}^{4,5}}_{\Upsilon^i_{pq}} \neq 0 \, , \end{aligned} \tag{5.46}$$

employing the projection operators defined in (4.30), (4.35). We note, that while these conditions are necessary for the existence of heavy and light Higgs bosons, they are not sufficient to guarantee their presence in the massless spectrum, as these states will also be subject to projections associated with $u_{10} = \alpha$, which can only be determined at the full Pati–Salam level. Moreover, while the full conditions that ensure the absence of physical tachyons and symmetry enhancing gauge bosons of conditions (i) and (ii) must be evaluated at the Pati–Salam level, a subset of models can be eliminated at the $SO(10)$ level by noticing that when tachyonic states arise in sectors β such that $\beta \cap \alpha \neq \emptyset$, the α-related GGSO projection will never be able to eliminate them.

We therefore perform a first scan at the $SO(10)$ level, identifying models which satisfy the conditions outlined above. These models then form "fertile cores" which have a higher likelihood of generating semi-realistic Pati–Salam models when the α-related GGSO phases are reintroduced and iterated over. The implementation of the $SO(10)$-level conditions involves 22 GGSO phases, as in addition to the nine $c[{}^{u_i}_{\alpha}]$ phases we omit, $c[{}^{\mathbb{1}}_{T_{1,2,3}}]$ are also irrelevant at this stage of the analysis.

After imposing the relevant conditions, and performing the computer-aided scan, $\sim 5 \times 10^5$ models satisfying the $SO(10)$-level criteria are found. Upon reintroducing all Pati–Salam relevant phases, these will give rise to a class of $\sim 2 \times 10^9$ models, significantly reducing the parameter space that needs to be evaluated. Each of these models is then subjected to the full Pati–Salam constraints (i)-(vii). In total, approximately one in a hundred Pati–Salam models satisfy conditions (i)-(v), resulting in $\sim 2.4 \times 10^8$ models. When introducing conditions (vi) and (vii), this number is significantly reduced to $\sim 5.6 \times 10^5$ models, implying that consistency with the Scherk–Schwarz mechanism and the super no-scale condition are difficult to achieve. In total, approximately one in 30000 models satisfy all constraints.

Each of the models satisfying all the above constraints is further analysed in terms of its one-loop partition function and the corresponding effective potential. After implementing the numerical simplifications outlined in section 4.5, we calculate the effective potential as a function of the T_2 modulus of the first torus by applying numerical methods. It turns out that the partition function and effective potential offer an efficient way to classify the above models, as they can be grouped into 26 subclasses, each corresponding to a family of models with identical partition functions. It is important to note, however, that models with identical partition functions do not necessarily exhibit the same characteristics, as their spectra can vary. Each subclass then corresponds to a specific effective potential, shared by all the models in the class. The overall shape and characteristics of the effective potentials can then be used to organise all models in four broad classes, labelled A through D.

The first class of models, labelled class A, includes models which exhibit a positive semi-definite potential, with a global maximum at the fermionic point $T_2 = 1$. As a consequence of the super no-scale property, these models exhibit an exponential suppression of the cosmological constant at large volume, while preserving the positivity of the potential. In addition to this, the shape of the potential dynamically leads to a supersymmetry breaking scale significantly lower than the string scale. Class A is further divided into two subclasses, $A1$ and $A2$ based on the exact form of the partition function, whose breakdown in terms of the net chirality, as well as the number of heavy and light Higgs bosons is presented in table 5.3.

Class	$\lvert n_g \rvert$	n_H	n_h	# of models
$A1$	4	4	8	1536
$A2$	8	4	8	2048

Table 5.3: *Class A model synopsis.*

The second class of models, class B, also contains two subclasses: $B1$ and $B2$. While the effective potential of models in class B is also positive semi-definite, with an exponential suppression at large volumes, what sets them apart from class A models is the behaviour near the fermionic point. In this case, instead of a global maximum, the fermionic point corresponds to a local minimum, with the potential exhibiting two maxima at nearby T_2 values. While a naively one might interpret this as a metastable vacuum, we note that further analysis is required, as tachyonic excitations typically appear when considering marginal deformations of all moduli near the self-dual fermionic point, in which case the region is instead unstable already at tree level. Table 5.4 contains a synopsis of the models of class B.

Class	$\lvert n_g \rvert$	n_H	n_h	# of models
$B1$	4	4	4	3584
	4	8	8	1792
	8	4	8	4096
$B2$	4	4	4	1792

Table 5.4: *Class B model synopsis.*

Figure 5.1 illustrates some of the results of our analysis, including the number of fermion generations each model exhibits. The light-shaded bars correspond to models satisfying conditions (i)-(v), while medium shading covers models satisfying all conditions (i)-(vi). Finally, the dark-shaded bars correspond to the 1.5×10^4 models of classes A and B which in addition to satisfying all aforementioned conditions also exhibit a positive semi-definite one-loop effective potential.

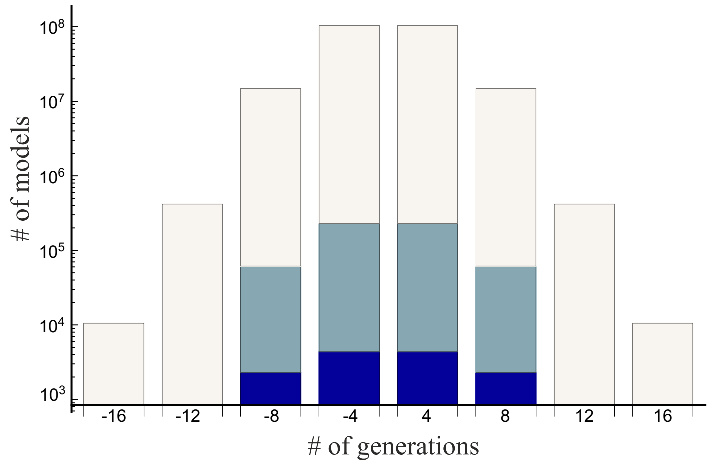

Figure 5.1: Number of PS models exhibiting each number of chiral matter generations. Light shaded bars correspond to configurations satisfying criteria (i)-(v), whereas medium shading corresponds to models satisfying all conditions (i)-(vii). Dark shaded bars represent the subset of models satisfying all previous criteria while simultaneously exhibiting a positive semi-definite one-loop effective potential.

The third class of models contains 10 subclasses, labelled $C1$ through $C10$. In this case the potential exhibits an oscillation between positive and negative values. The fermionic point corresponds to a global minimum at negative values, while at large volumes the potential asymptotically vanishes from above. The dynamics imply a stabilisation of the vacuum at the fermionic point, which is problematic on many accounts. First of all, the cosmological constant takes large negative values, causing a backreaction to the tree-level geometry. In addition to this, supersymmetry appears to be broken at the string scale, spoiling the resolution of the hierarchy problem, while tachyonic excitations typically lead to a breakdown of the perturbative analysis when deformations around the fermionic point are considered. We present a further classification of the models within class C based on their chiral fermion and Higgs spectrum in table 5.5.

Class	$\lvert n_g \rvert$	n_H	n_h	# of models
$C1$	4	4	4	9984
	4	4	8	22784
	4	4	12	3584
	4	8	4	8192
	4	8	8	3072
	8	4	4	8192
	8	4	8	6144
	8	8	4	4096
	8	8	8	2048
$C2$	4	4	4	20480
	4	4	8	12544
	4	4	12	1792
	4	8	4	15104
	4	8	8	1792
	8	4	4	18432
	8	8	4	6144
$C3$	4	4	4	3584
	4	4	12	1792
	4	8	12	896

Class	$\lvert n_g \rvert$	n_H	n_h	# of models
$C4$	4	4	8	1792
	8	4	8	2048
$C5$	4	4	4	43264
	4	4	8	11776
	4	4	12	6144
	4	8	4	1792
	8	8	4	4096
	8	8	8	1024
$C6$	4	8	4	1792
	4	8	8	1792
$C7$	4	4	4	1792
	4	4	8	1792
	4	8	4	1792
	8	4	4	4096
$C8$	4	4	4	12288
	4	4	8	9600
	4	4	12	6144
$C9$	4	4	4	6144
$C10$	4	4	4	1536

Table 5.5: *Class C model synopsis.*

Finally, we have class D, comprising 12 subclasses, $D1$ through $D12$. The effective potential in this class is negative semi-definite, with a global minimum at the fermionic point and an exponential suppression asymptotically leading to vanishing from negative values. The potential is once again stabilised at the fermionic point, and the cosmological constant, supersymmetry breaking and tachyon instability problems of class C are also present here. Table 5.6 summarises the subclasses of class D.

Class	$\|n_g\|$	n_H	n_h	# of models
$D1$	8	8	8	3072
$D2$	4	4	8	8192
	8	8	8	3072
	8	16	8	1024
$D3$	4	4	4	61184
	4	4	8	34560
	4	4	12	3072
	4	4	16	1536
	4	8	4	5376
	8	4	4	2048
	8	8	4	2048
	8	8	8	14336
	8	16	8	512
$D4$	4	4	4	7168
	4	4	8	6656
	8	4	4	2048
	8	8	4	2048
$D5$	4	4	8	1152
$D6$	4	4	8	1152
	8	8	8	768

Class	$\|n_g\|$	n_H	n_h	# of models
$D7$	4	4	4	1792
$D8$	4	4	4	19712
	4	4	8	14336
	4	4	12	1792
	4	8	4	8704
	4	8	8	1792
	8	4	4	10240
	8	4	8	2048
	8	8	4	10240
	8	8	8	2048
$D9$	4	4	4	19520
	4	4	8	8960
	4	4	12	1344
	4	8	4	6528
	4	8	8	1344
$D10$	4	4	4	3584
	4	4	12	1792
	4	8	12	896
$D11$	4	4	8	384
$D12$	4	4	8	224
	4	12	8	96

Table 5.6: *Class D model synopsis.*

In figure 5.2 we collect the effective potentials of all subclasses into a single plot. This showcases the combined effect of the T-duality symmetry, the asymptotic recovery of supersymmetry in the $T_2 \to \infty$ (and its dual $T_2 \to 0$) limit and the super no-scale structure of the potential, which act as boundary conditions bounding the curves between the maximum value 43.7351 (class $A1$) and the minimum at -172.913 (class $D12$) as the effective potential varies discontinuously. Figure 5.3 illustrates the one-loop effective potential, along with the number of models constituting each subclass.

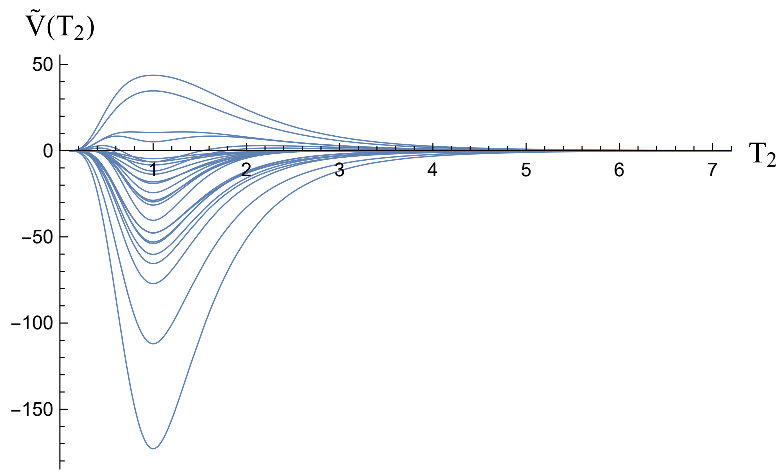

Figure 5.2: The rescaled effective potentials corresponding to the 26 distinct subclasses of models.

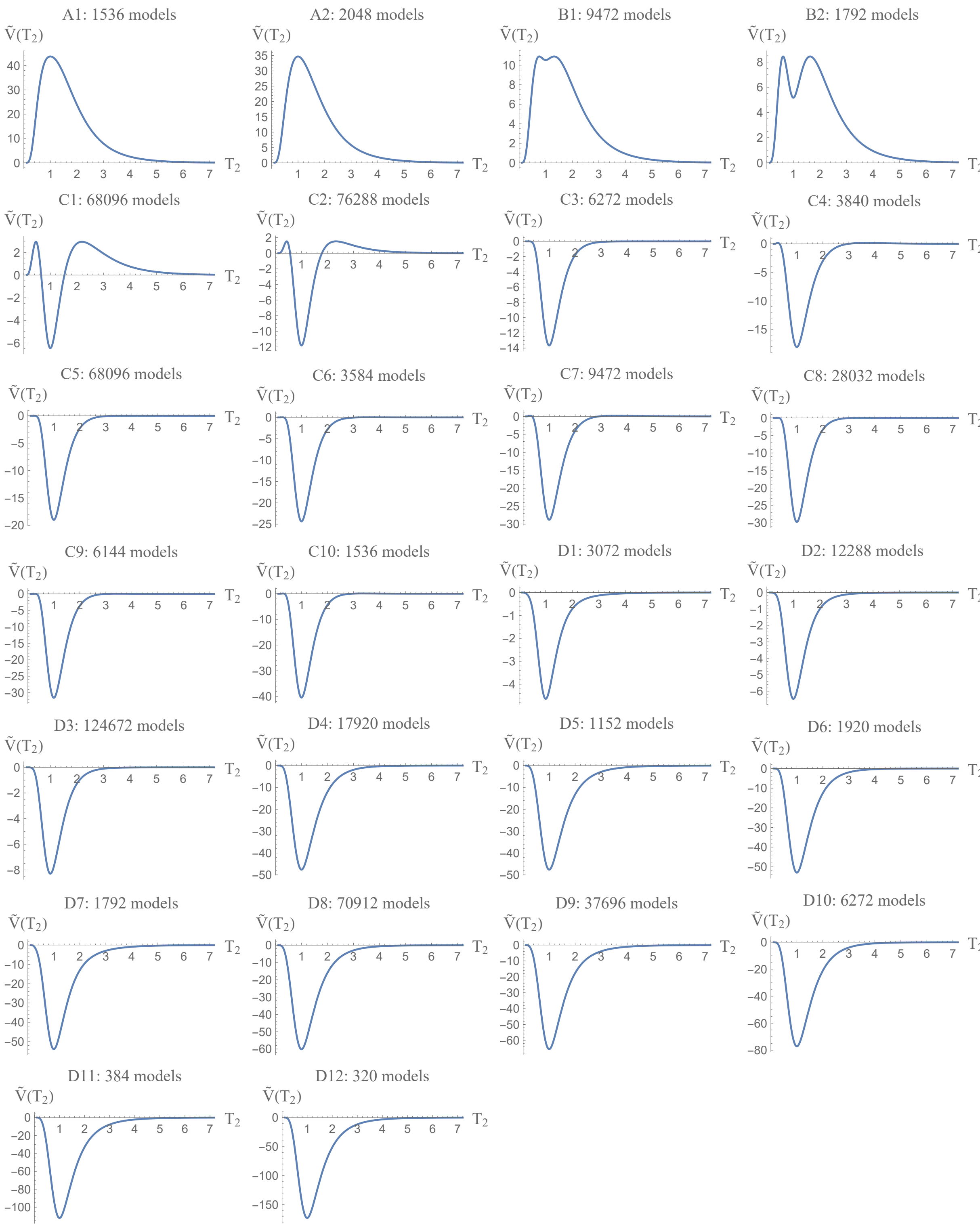

Figure 5.3: The rescaled effective potential $\tilde{V}(T_2) = 2(2\pi)^4 V(T_2)$ for each of the 26 distinct subclasses of models. The number of models in each class is also displayed.

The classification of Pati–Salam models presented above offers an interesting opportunity to study the effect the reduction of the gauge symmetry has on the moduli-dependent dynamics of the theory. Given the fact that we use $SO(10)$ models defined by the vector basis of table 4.1 as an intermediate step in the construction of our Pati–Salam models, and that the corresponding class has been studied in the literature [91], direct comparisons between the two can be made.

The main results of this comparison can be extracted without the need for a systematic analysis. Starting from any Pati–Salam model within our classes, we can obtain an $SO(10)$ "parent" model by removing the vector $u_{10} = \alpha$ and ignoring the related GGSO phases. In the orbifold picture of (5.5) this corresponds to eliminating the sectors and projection labeled by $(H', G') \neq (0, 0)$. We note that in doing this, Pati–Salam models with spontaneous supersymmetry breaking will always correspond to $SO(10)$ models in which supersymmetry is also spontaneously broken. The opposite, however, is not true. Any $SO(10)$ model can generate a family of 2^9 Pati–Salam models by introducing the aforementioned projections, in precisely the same manner as the second stage of our scan. The additional $\mathbb{Z}_2$ orbifold introduced, however, can lead to explicit supersymmetry breaking, unless the GGSO phases are appropriately chosen.

The super no-scale structure of the models and the behaviour of the effective potential appear to be completely unrelated and are generally not preserved when interpolating between the two classes. A super no-scale $SO(10)$ model with a positive potential such as the ones presented in [91] can generate Pati–Salam models which may or may not be super no-scale themselves, and which may exhibit both positive and negative potentials, with no strong preference for either. In fact, super no-scale Pati–Salam models with a positive potential may originate from $SO(10)$ models which are neither super no-scale, nor positive. This is not surprising, as these properties involve the entire spectrum of the theory, including massive and non level-matched states at both the fermionic and generic points of moduli space. The string spectrum undergoes a significant rearrangement when a $\mathbb{Z}_2$ orbifold is added or removed and there is no mechanism guaranteeing that the theory will continue to exhibit similar structure under such a transition.

The main conclusion of this comparison is that the shape of the effective potential by itself does not constitute a criterion by which a model should be excluded. Thus, while a majority of the Pati–Salam models of figure 5.3 appear to be unsatisfying at first glance due to their negative potentials, further reductions of the gauge symmetry and the corresponding rearrangement of the spectrum may cause significant changes. The models residing in classes C and D may therefore still be useful as parent models themselves, in the construction of models where the observable Pati–Salam gauge symmetry is further reduced, such as SM-like vacua.

5.5 Explicit Constructions

In this section we present three explicit examples of models satisfying all phenomenological conditions outlined earlier in this chapter, in order to illustrate some of their properties in more detail. We provide a comprehensive definition of each model by defining both the GGSO matrix defining the free fermionic description, as well as the modular invariant phase which realises the equivalent orbifold picture. We then present the full massless fermionic and scalar spectrum, omitting the gauge and gravitational states, as well as the partition function at the fermionic point as an expansion in terms of q_r, q_i. We reiterate that these models cannot be considered fully realistic, as they always involve a number of generations that is a multiple of four, as a result of the complexification of the worldsheet fermions we imposed for technical reasons. Despite this, all models presented below are tachyon-free, chiral, and possess the necessary Higgs scalars that are responsible for gauge symmetry breaking. Additionally, they exhibit spontaneous supersymmetry breaking compatible with the Scherk–Schwarz mechanism and satisfy the super no-scale condition. We choose to present two models which exhibit positive semi-definite potentials and a third which offers an intriguing way to avoid exotic states in the low energy spectrum, albeit whose potential is negative semi-definite potential.

5.5.1 Model A: A Case With Positive Semi-Definite Potential

As a first example, we present a model from class $A2$, labelled Model A, defined by the GGSO matrix:

$$c\left[\begin{smallmatrix}u_i\\u_j\end{smallmatrix}\right]=\begin{pmatrix}+1&+1&-1&+1&+1&+1&+1&+1&+1&+1\\+1&+1&+1&+1&+1&+1&+1&-1&+1&-1\\-1&+1&+1&-1&-1&+1&+1&+1&-1&+1\\+1&+1&-1&-1&+1&+1&+1&-1&-1&-1\\+1&+1&-1&+1&-1&+1&-1&-1&-1&+1\\+1&-1&+1&+1&+1&+1&-1&-1&+1&+1\\+1&-1&+1&+1&-1&-1&+1&+1&+1&+1\\+1&-1&+1&-1&-1&-1&+1&+1&+1&-1\\+1&+1&-1&-1&-1&+1&+1&+1&+1&+1\\+1&-1&+1&-1&+1&-1&-1&+1&+1&+1\end{pmatrix}. \tag{5.47}$$

This model possesses 8 fermion generations, originating in the sectors $S+b_1+T_2$, $S+b_1+T_3$, $S+b_2+T_1$, and $S+b_3$. In addition to this, the Pati–Salam gauge symmetry is spontaneously broken by non vanishing vevs of the neutral component of scalars generated in sector b_3+T_1, while 8 copies of the SM Higgs are accommodated in the bi-doublets arising in sectors b_2+T_3+x and $b_3+T_1+T_2+x$. Moreover, the exotic colour quadruplets from sectors $S+b_1+T_3+\alpha$ and $S+b_1+T_2+z_1+\alpha$ form a vector-like pair and can be decoupled from the low energy spectrum by acquiring large masses. The full massless spectrum is presented in Tables 5.7 and 5.8.

Sector	$SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^3\times SU(2)^4\times SO(8)$ representation(s)
$S+z_2$	$(\mathbf{6},\mathbf{1},\mathbf{1},0,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_c})$, $(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_c})$
	$(\mathbf{1},\mathbf{1},\mathbf{1},0,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_c})$, $(\mathbf{1},\mathbf{1},\mathbf{1},0,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_c})$
$S+b_1+T_2$	$4\times(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+T_3$	$4\times(\mathbf{4},\mathbf{2},\mathbf{1},\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+T_2+T_3+x$	$8\times(\mathbf{1},\mathbf{1},\mathbf{1},0,\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $8\times(\mathbf{1},\mathbf{1},\mathbf{1},0,-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+T_2+T_3+z_2+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},0,\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_s})$
$S+b_1+T_3+\alpha$	$4\times(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},-\frac{1}{2},0,0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+T_2+z_1+\alpha$	$4\times(\mathbf{4},\mathbf{1},\mathbf{1},-\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_2+T_1$	$4\times(\mathbf{4},\mathbf{2},\mathbf{1},0,\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+T_1+T_3+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},-\frac{1}{2},0,\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+z_1+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},0,-\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_2+\alpha+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+T_1+T_3+z_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_3$	$4\times(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},0,0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_3+T_2+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_3+T_1+z_1+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},0,\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_3+T_2+\alpha+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},\frac{1}{2},\frac{1}{2},0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_3+T_1+z_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$

Table 5.7: *Spectrum of massless fermionic matter and their representations at the fermionic point of Model A.*

Sector	$SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^3\times SU(2)^4\times SO(8)$ representation(s)
0	$(\mathbf{6},\mathbf{1},\mathbf{1},\pm1,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{6},\mathbf{1},\mathbf{1},0,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$ $(\mathbf{6},\mathbf{1},\mathbf{1},0,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$ $(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{1},\mathbf{1},\mathbf{1},0,\pm1,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$ $12\times(\mathbf{1},\mathbf{1},\mathbf{1},0,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
α	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},0,0,0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$z_1+\alpha$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},0,0,0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_1+T_2+T_3+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},0,\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_v})$
$b_1+T_2+T_3+\alpha$	$4\times(\mathbf{4},\mathbf{1},\mathbf{1},\frac{1}{2},0,0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_1+T_2+T_3+z_1+\alpha$	$4\times(\mathbf{4},\mathbf{1},\mathbf{1},-\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
b_2	$4\times(\mathbf{4},\mathbf{2},\mathbf{1},0,\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
b_2+T_3+x	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},-\frac{1}{2},0,\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_2+T_1+z_1+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},0,-\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$b_2+T_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0-\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_2+T_3+z_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
b_3+T_1	$4\times(\mathbf{4},\mathbf{1},\mathbf{2},0,0,\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+T_1+T_2+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},\frac{1}{2},\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
b_3+z_1+x	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},0,\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_3+z_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$b_3+T_1+T_2+\alpha+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},\frac{1}{2},\frac{1}{2},0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$

Table 5.8: *Spectrum of massless scalar matter and their representations at the fermionic point of Model A.*

At the fermionic point, the expansion of the partition function takes the form:

$$\begin{aligned} Z =&\frac{2q_i}{q_r}-\frac{16q_i}{\sqrt{q_r}}+\left(-24+128q_i+56q_i^2\right)+\left(1504+\frac{7424}{q_i}-1280q_i-416q_i^2\right)\sqrt{q_r} \\ &+\left(16384+\frac{17408}{q_i^2}+\frac{96384}{q_i}+3424q_i+1536q_i^2+792q_i^3\right)q_r+\mathcal{O}(q_r^{3/2})\ , \end{aligned} \tag{5.48}$$

where $q_i=e^{2\pi i\tau_1}$ and $q_r=e^{-2\pi\tau_2}$. The partition function at generic points can be calculated from (5.5) using the orbifold phase:

$$\begin{aligned} \Phi =&ab+a(G_1+g_1+g_2)+b(H_1+h_1+h_2)+k(G+G'+g_1+g_2)+\ell(H+H'+h_1+h_2) \\ &+\rho(G+G_2+g_2)+\sigma(H+H_2+h_2)+H(G_1+g_1+g_2)+G(H_1+h_1+h_2) \\ &+H'(G_1+G_3+g_2)+G'(H_1+H_3+h_2)+H'G' \\ &+H_1G_1+H_1G_2+G_1H_2+H_2G_2+H_2(G_3+g_1)+G_2(H_3+h_1)+H_3g_1+G_3h_1 \\ &+h_1g_2+g_1h_2+h_2g_2\ . \end{aligned} \tag{5.49}$$

A numerical evaluation of its one-loop effective potential as a function of the T_2 modulus of the Scherk–Schwarz torus, with all other moduli fixed at the fermionic point is presented in figure 5.4. It exhibits a positive semi-definite structure, with an exponential suppression in the region $T_2\gg1$, as expected for a model in class A.

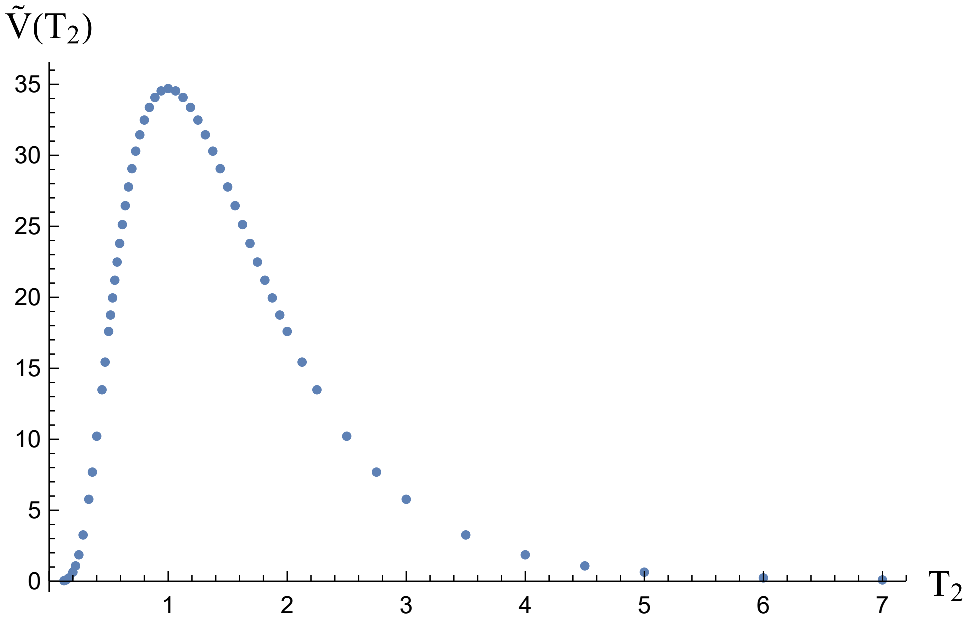

Figure 5.4: Numerical evaluation of the one-loop potential of Model A as a function of the T_2 modulus.

5.5.2 Model B: A Local Minimum at the Fermionic Point

As a second example, we present a model from class $B2$, labelled model B, obtained by the GGSO matrix

$$c\left[\begin{smallmatrix}u_i\\u_j\end{smallmatrix}\right] = \begin{pmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & +1 \\ +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & -1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 & +1 \\ +1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \end{pmatrix} \tag{5.50}$$

This model exhibits 4 fermion generations arising in the sectors $S + b_1 + T_2$ and $S + b_3$. The heavy and light Higgs bosons are generated in the sectors b_3+T_1 and $b_1+T_2+T_3+x$ respectively, while exotic colour (anti-)quadruplets arise in the sectors, $S + z_1 + \alpha + x$, in vector-like pairs. The full massless spectrum of model B is presented in Tables 5.9 and 5.10.

Sector	$SU(4) \times SU(2)_L \times SU(2)_R \times U(1)^3 \times SU(2)^4 \times SO(8)$ representation(s)
$S + \alpha$	$4 \times (\mathbf{1}, \mathbf{2}, \mathbf{1}, 0, 0, 0, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$S + z_2 + \alpha$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, 0, 0, 0, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{8_s})$
$S + z_1 + \alpha + x$	$(\mathbf{4}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$, $(\mathbf{4}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$, $(\mathbf{4}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$, $(\mathbf{4}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$, $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$, $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$, $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$, $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
$S + b_1 + T_2$	$4 \times (\mathbf{4}, \mathbf{2}, \mathbf{1}, -\frac{1}{2}, 0, 0, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$S + b_1 + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0, \frac{1}{2}, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8_v})$
$S + b_1 + T_2 + x$	$8 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0, \frac{1}{2}, \frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, $8 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0, -\frac{1}{2}, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$S + b_1 + T_2 + T_3 + z_1 + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0, \frac{1}{2}, -\frac{1}{2}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$S + b_1 + z_2 + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0, -\frac{1}{2}, \frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8_c})$
$S + b_1 + T_2 + z_1 + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0, -\frac{1}{2}, -\frac{1}{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
$S + b_1 + T_2 + \alpha + x$	$4 \times (\mathbf{1}, \mathbf{2}, \mathbf{1}, 0, \frac{1}{2}, -\frac{1}{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$S + b_1 + T_2 + T_3 + \alpha + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{2}, 0, -\frac{1}{2}, \frac{1}{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$S + b_2 + T_3 + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8_v})$
$S + b_3$	$4 \times (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, 0, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$S + b_3 + z_1 + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, -\frac{1}{2}, 0, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$S + b_3 + T_1 + z_1 + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, \frac{1}{2}, 0, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$S + b_3 + z_1 + \alpha + x$	$4 \times (\mathbf{1}, \mathbf{2}, \mathbf{1}, \frac{1}{2}, -\frac{1}{2}, 0, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$S + b_3 + T_1 + z_1 + \alpha + x$	$4 \times (\mathbf{1}, \mathbf{1}, \mathbf{2}, -\frac{1}{2}, \frac{1}{2}, 0, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$

Table 5.9: *Spectrum of massless fermionic matter and quantum numbers under the gauge bundle for Model B.*

Sector	$SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^3\times SU(2)^4\times SO(8)$ representation(s)
0	$(\mathbf{6},\mathbf{1},\mathbf{1},\pm1,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{6},\mathbf{1},\mathbf{1},0,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{6},\mathbf{1},\mathbf{1},0,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},0,\pm1,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $12\times(\mathbf{1},\mathbf{1},\mathbf{1},0,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
z_1	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},0,0,0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$
z_2	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},0,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_s})$
α	$(\mathbf{1},\mathbf{2},\mathbf{1},0,0,\pm1,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{2},\mathbf{1},0,\pm1,0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$z_1+\alpha$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},0,0,0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
b_1+T_2+x	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_1+T_2+T_3+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},0,\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_1+T_2+\alpha$	$4\times(\mathbf{4},\mathbf{1},\mathbf{1},-\frac{1}{2},0,0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_1+T_2+z_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},0,+\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$b_1+T_2+T_3+z_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_2+T_1+T_3+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_v})$
b_3+T_1	$4\times(\mathbf{4},\mathbf{1},\mathbf{2},0,0,+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
b_3+z_1+x	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_3+T_1+z_1+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},0,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_3+z_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$b_3+T_1+z_1+\alpha+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$

Table 5.10: *Spectrum of massless scalar matter (at the fermionic point) and quantum numbers under the gauge bundle for Model B.*

The fermionic point partition function is given by:

$$
\begin{aligned}
Z =& \frac{2q_i}{q_r} - \frac{16q_i}{\sqrt{q_r}} + \left(40 + \frac{48}{q_i} + 144q_i + 56q_i^2\right) + \left(1248 + \frac{6528}{q_i} - 1152q_i - 416q_i^2\right)\sqrt{q_r} \\
&+ \left(8192 + \frac{18816}{q_i^2} + \frac{105600}{q_i} + 4960q_i + 2688q_i^2 + 792q_i^3\right) q_r + \mathcal{O}(q_r^{3/2}) \ ,
\end{aligned} \tag{5.51}
$$

while the phase identifying the model in the equivalent orbifold description of (5.5), is:

$$
\begin{aligned}
\Phi =& ab + a(G_1 + g_1 + g_2) + b(H_1 + h_1 + h_2) + k(G' + G_1 + G_3 + g_2) + \ell(H' + H_1 + H_3 + h_2) + k\ell \\
&+ \rho(G_1 + G_3) + \sigma(H_1 + H_3) + H(G' + G_1 + G_2 + g_1 + g_2) + G(H' + H_1 + H_2 + h_1 + h_2) \\
&+ H'G_2 + G'H_2 + H'G' + H_1G_1 + H_1(G_2 + g_1 + g_2) + G_1(H_2 + h_1 + h_2) + H_3g_2 + G_3h_2 + h_1g_1 .
\end{aligned} \tag{5.52}
$$

Being a model of class B, this model exhibits a positive semi-definite potential as we vary T_2, with a local minimum at the fermionic point, as illustrated in figure 5.5.

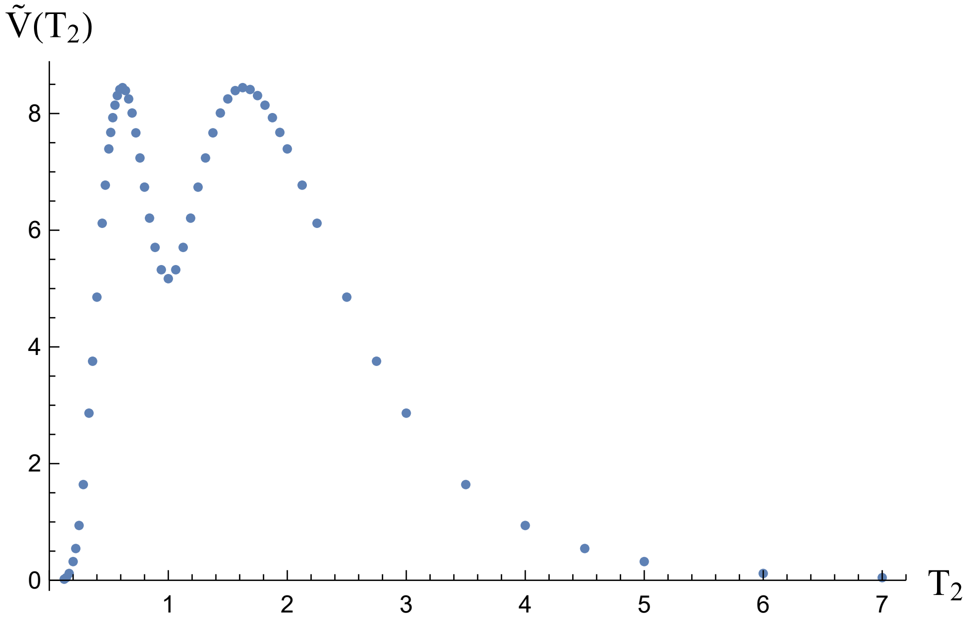

Figure 5.5: Numerical evaluation of the one-loop potential of Model B.

5.5.3 Model C: Confinement of all Exotic States

As a third and final example, we present model C, defined by the following GGSO matrix:

$$
c\left[\begin{smallmatrix} u_i \\ u_j \end{smallmatrix}\right] = \begin{pmatrix}
+1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 \\
+1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 \\
-1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 \\
-1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\
+1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\
+1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\
-1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 & -1 & -1
\end{pmatrix} . \tag{5.53}
$$

Model C resides in class $D3$ and exhibits a negative semi-definite potential. While this is problematic from a phenomenological point of view, as discussed in the previous section, model C is exhibits a very interesting property regarding its spectrum of exotic states. As previously discussed, the complexification of the worldsheet fermions is intricately tied to the appearance of fractionally charged exotic fermions in the massless spectrum of non supersymmetric models. While these exotic states, transforming as $(\mathbf{4}+\bar{\mathbf{4}},\mathbf{1},\mathbf{1})$ or $(\mathbf{1},\mathbf{2},\mathbf{1})/(\mathbf{1},\mathbf{1},\mathbf{2})$ under the observable Pati–Salam gauge group, can become super-heavy by forming vector-like pairs, following the discussion of section 5.3.3 and illustrated in models A and B, an interesting opportunity arises when considering models with gauge symmetry enhancements.

More specifically, there exist models in which all the non-abelian gauge groups of the hidden sector are enhanced forming groups whose gauge couplings run faster than $SU(3)$. In this case, all fractionally charged exotics transforming non-trivially under this gauge symmetry may exhibit confinement at energies higher than the TeV scale, providing an alternative resolution to the problem of fractionally charged exotic states.

Model C is such an example, which in addition to satisfying all of our phenomenological constraints, also exhibits simultaneous gauge symmetry enhancements due to gauge bosons arising in sectors z_1 and z_1+z_2. The full gauge symmetry of model C is:

$$G = SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^2\times SO(6)\times SO(12)\,, \tag{5.54}$$

and its partition function at the fermionic point reads:

$$\begin{aligned} Z =& \frac{2q_i}{q_r}-\frac{16q_i}{\sqrt{q_r}}+\left(8-32q_i+56q_i^2\right)+\left(4064+\frac{4096}{q_i}+512q_i-416q_i^2\right)\sqrt{q_r} \\ &+\left(\frac{16384}{q_i^2}+\frac{122880}{q_i}-8480q_i-768q_i^2+792q_i^3\right)q_r+\mathcal{O}(q_r^{3/2})\,, \end{aligned} \tag{5.55}$$

The corresponding orbifold phase is given by:

$$\begin{aligned} \Phi =& ab+a(G_1+g_1+g_2)+b(H_1+h_1+h_2)+k(G+G')+\ell(H+H') \\ &+H(G'+g_1+g_2)+G(H'+h_1+h_2)+HG+H'(G_1+g_1)+G'(H_1+h_1)+H'G' \\ &+H_1G_1+H_1G_2+G_2H_1+h_1g_1+h_2g_2\,. \end{aligned} \tag{5.56}$$

The full massless spectrum of Model C is presented in Tables 5.11 and 5.12.

Sector(s)	$SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^2\times SO(6)\times SO(12)$ representation(s)
$S+\alpha(+z_1+z_2)$	$(\mathbf{1},\mathbf{2},\mathbf{1},0,0,\mathbf{1},\mathbf{32})$
$S+\alpha+z_1(+z_1+z_2)$	$(\mathbf{1},\mathbf{1},\mathbf{2},0,0,\mathbf{1},\mathbf{32})$
$S+x+\alpha(+z_1)$	$(\mathbf{4},\mathbf{1},\mathbf{1},\pm\frac{1}{2},\pm\frac{1}{2},\bar{\mathbf{4}},\mathbf{1})\,,\,(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},\pm\frac{1}{2},\pm\frac{1}{2},\mathbf{4},\mathbf{1})$
$S+b_1+T_2$	$4\times(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1})$
$S+b_1+T_2+T_3$	$4\times(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1})$
$S+b_1+x+\alpha(+z_1)$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},0,-\frac{1}{2},\bar{\mathbf{4}},\mathbf{1})$
$S+b_1+T_3+x+\alpha(+z_1)$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},0,-\frac{1}{2},\bar{\mathbf{4}},\mathbf{1})$
$S+b_2+T_1$	$4\times(\mathbf{4},\mathbf{2},\mathbf{1},0,-\frac{1}{2},\mathbf{1},\mathbf{1})$
$S+b_2+T_1+T_3$	$4\times(\mathbf{4},\mathbf{2},\mathbf{1},0,-\frac{1}{2},\mathbf{1},\mathbf{1})$
$S+b_2+x+\alpha(+z_1)$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0,\mathbf{4},\mathbf{1})$
$S+b_2+T_3+x+\alpha(+z_1)$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0,\mathbf{4},\mathbf{1})$
$S+b_3+T_1+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})$
$S+b_3+T_2+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})$
$S+b_3+x(+z_1)$	$4\times(\mathbf{6},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})\,,\,4\times(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},\mathbf{6},\mathbf{1})\,,\,4\times(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})\,,$ $4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1})$
$S+b_3+T_1+T_2+x(+z_1)$	$4\times(\mathbf{6},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1})\,,\,4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},\mathbf{6},\mathbf{1})\,,\,4\times(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})\,,$ $4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1})$
$S+b_3+T_1+\alpha(+z_1)$	$4\times(\mathbf{4},\mathbf{1},\mathbf{1},0,0,\bar{\mathbf{4}},\mathbf{1})$
$S+b_3+T_2+\alpha(+z_1)$	$4\times(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},0,0,\mathbf{4},\mathbf{1})$
$S+b_3+T_1+x+z_1(+z_1+z_2)$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{12})$
$S+b_3+T_2+x+z_1(+z_1+z_2)$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{12})$

Table 5.11: *Spectrum of massless fermionic matter and quantum numbers under the gauge group of Model C. Multiplets of the hidden $SO(6)$ and $SO(12)$ gauge groups are obtained from sector pairs of the form $\xi(+z_1)=\{\xi,\xi+z_1\}$ and $\xi(+z_1+z_2)=\{\xi,\xi+z_1+z_2\}$ respectively.*

Sector(s)	$SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^2\times SO(6)\times SO(12)$ representation(s)
$0,\ z_1, z_2$	$(\mathbf{6},\mathbf{1},\mathbf{1},0,0,\mathbf{6},\mathbf{1})\,,\ (\mathbf{6},\mathbf{1},\mathbf{1},\pm1,0,\mathbf{1},\mathbf{1})\,,\ (\mathbf{6},\mathbf{1},\mathbf{1},0,\pm1,\mathbf{1},\mathbf{1})\,,$ $(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,0,\mathbf{6},\mathbf{1})\,,\ (\mathbf{1},\mathbf{1},\mathbf{1},0,\pm1,\mathbf{6},\mathbf{1})\,,\ (\mathbf{1},\mathbf{1},\mathbf{1},\pm1,\pm1,\mathbf{1},\mathbf{1})\,,$ $12\times(\mathbf{1},\mathbf{1},\mathbf{1},0,0,\mathbf{1},\mathbf{1})\,,\ (\mathbf{1},\mathbf{2},\mathbf{2},0,0,\mathbf{1},\mathbf{12})$
b_1	$4\times(\mathbf{4},\mathbf{1},\mathbf{2},\frac{1}{2},0,\mathbf{1},\mathbf{1})$
b_1+T_3	$4\times(\mathbf{4},\mathbf{1},\mathbf{2},\frac{1}{2},0,\mathbf{1},\mathbf{1})$
$b_1+T_2+x+\alpha(+z_1)$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},0,\frac{1}{2},\mathbf{4},\mathbf{1})$
$b_1+T_2+T_3+\alpha(+z_1)$	$4\times(\mathbf{1},\mathbf{1},\mathbf{2},0,\frac{1}{2},\mathbf{4},\mathbf{1})$
b_2	$4\times(\mathbf{4},\mathbf{2},\mathbf{1},0,-\frac{1}{2},\mathbf{1},\mathbf{1})$
b_2+T_3	$4\times(\mathbf{4},\mathbf{2},\mathbf{1},0,-\frac{1}{2},\mathbf{1},\mathbf{1})$
$b_2+T_1+x+\alpha(+z_1)$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0,\mathbf{4},\mathbf{1})$
$b_2+T_1+T_3+x+\alpha(+z_1)$	$4\times(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0,\mathbf{4},\mathbf{1})$
b_3+x	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})$
$b_3+T_1+x(+z_1)$	$4\times(\mathbf{6},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})\,,\ 4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},\mathbf{6},\mathbf{1})\,,\ 16\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1})$
$b_3+T_2+x(+z_1)$	$4\times(\mathbf{6},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})\,,\ 4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},\mathbf{6},\mathbf{1})\,,\ 16\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1})$
$b_3+T_1+T_2+x$	$4\times(\mathbf{1},\mathbf{2},\mathbf{2},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{1})$
$b_3+\alpha(+z_1)$	$4\times(\mathbf{4},\mathbf{1},\mathbf{1},0,0,\bar{\mathbf{4}},\mathbf{1})$
$b_3+T_1+T_2+\alpha(+z_1)$	$4\times(\mathbf{4},\mathbf{1},\mathbf{1},0,0,\bar{\mathbf{4}},\mathbf{1})$
$b_3+x+z_1(+z_1+z_2)$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{12})$
$b_3+T_1+T_2+x+z_1(+z_1+z_2)$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\frac{1}{2},\frac{1}{2},\mathbf{1},\mathbf{12})$

Table 5.12: *Spectrum of massless scalar matter and quantum numbers under the gauge bundle for Model C. Multiplets of the hidden $SO(6)$ and $SO(12)$ gauge groups are obtained from sector pairs of the form $\xi(+z_1)=\{\xi,\xi+z_1\}$ and $\xi(+z_1+z_2)=\{\xi,\xi+z_1+z_2\}$ respectively.*

By inspecting the spectrum, we observe that all fractionally charged exotic states transform either as vectorials or anti-vectorials of the hidden sector $SO(6) \sim SU(4)$ group, implying that they may be confined. This is the first realisation of such a scenario in the context of Pati–Salam models [2], while confinement of exotic states along similar lines has also been encountered in the case of supersymmetric flipped $SU(5)$ models [110, 277].

Chapter 6

Four Dimensional super no-scale Pati–Salam Models with Three Fermion Generations

In the previous chapter, we focused on a class of non supersymmetric super no-scale models exhibiting an observable Pati–Salam gauge symmetry. While models with appealing phenomenological features, such as the absence of tachyons, chirality and the presence of heavy and light Higgs bosons necessary for the realisation of spontaneous symmetry breaking were identified, they nevertheless all shared a critical flaw: the number of fermion generations was always a multiple of four, failing to reproduce the experimentally observed three fermion generations.

In this chapter, we proceed to investigate a new class of Pati–Salam models, in which the worldsheet fermions corresponding to the compactified dimensions are not complexified, in order to obtain models with three fermion generations. The class of models under investigation is defined in the free fermionic formulation by the set of 13 basis vectors:

$$\begin{aligned}
\beta_1 &= \mathbb{1} = \{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} | \bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\phi}^{1,\dots,8}\}\ ,\\
\beta_2 &= S = \{\psi^\mu, \chi^{1,\dots,6}\}\ ,\\
\beta_3 &= e_1 = \{y^1, \omega^1 | \bar{y}^1, \bar{\omega}^1\}\ ,\\
\beta_4 &= e_2 = \{y^2, \omega^2 | \bar{y}^2, \bar{\omega}^2\}\ ,\\
\beta_5 &= e_3 = \{y^3, \omega^3 | \bar{y}^3, \bar{\omega}^3\}\ ,\\
\beta_6 &= e_4 = \{y^4, \omega^4 | \bar{y}^4, \bar{\omega}^4\}\ ,\\
\beta_7 &= e_5 = \{y^5, \omega^5 | \bar{y}^5, \bar{\omega}^5\}\ ,\\
\beta_8 &= e_6 = \{y^6, \omega^6 | \bar{y}^6, \bar{\omega}^6\}\ ,\\
\beta_9 &= b_1 = \{\chi^{34}, \chi^{56}, y^3, y^4, y^5, y^6 | \bar{y}^3, \bar{y}^4, \bar{y}^5, \bar{y}^6, \bar{\psi}^{1,\dots,5}, \bar{\eta}^1\}\ ,\\
\beta_{10} &= b_2 = \{\chi^{12}, \chi^{56}, y^1, y^2, y^5, y^6 | \bar{y}^1, \bar{y}^2, \bar{y}^5, \bar{y}^6, \bar{\psi}^{1,\dots,5}, \bar{\eta}^2\}\ ,\\
\beta_{11} &= z_1 = \{\bar{\phi}^{1,\dots,4}\}\ ,\\
\beta_{12} &= z_2 = \{\bar{\phi}^{5,\dots,8}\}\ ,\\
\beta_{13} &= \alpha = \{\bar{\psi}^{4,5}, \bar{\phi}^{1,2}\}\ .
\end{aligned} \tag{6.1}$$

This basis closely resembles that of (5.1), with the difference being the replacement of the vectors T_i, $i = 1, 2, 3$, encoding the complexified y, ω fermions with six new vectors e_j, $j = 1, \dots, 6$ reflecting the change to real y, ω. This basis gives rise to 2^{79} a priori distinct vacua, defined by the set of phases $c\left[{u_i \atop u_j}\right] = \pm 1$, $i, j = 1, 2, \dots, 13$, subject to the conditions outlined in section 3.1. In this chapter, we will proceed by analysing the critical components of the string spectrum. Moreover, we will demand that these non supersymmetric models are consistent with the stringy implementation of the Scherk–Schwarz mechanism, and exhibit an exponential suppression of the cosmological constant in the large volume limit.

6.1 Spectrum Analysis

The massless and tachyonic spectrum in this case is similar to the previous constructions and we will not outline it in its entirety. Instead, we will focus on the critical components of the string spectrum, including sectors generating tachyonic states, vector bosons, chiral fermions, the vectorial and spinorial scalar representations needed to accommodate the PS and SM Higgs bosons, as well as the fractionally charged exotics.

6.1.1 Tachyonic States

We begin by outlining all sectors which can generate on-shell tachyons. In the class of models under consideration, the mass of level-matched tachyons can take the values $M^2 = -1/2$, $M^2 = -3/8$, $M^2 = -1/4$ or $M^2 = -1/8$. In total, there are 209 sectors which can potentially generate such states.

The projection operators signalling the existence of $M^2 = -1/2$ tachyons, generated in the $\mathcal{T}_{-1/2} \in \{z_1, z_2, \alpha, z_1 + \alpha\}$ sectors are $\mathbb{P}^+_{\mathcal{T}_{-1/2}}$, where

$$\begin{aligned} \Xi^+(z_1) &= \{S, e_i, b_1, b_2, z_2\} \ , \\ \Xi^+(z_2) &= \{S, e_i, b_1, b_2, z_1, \alpha\} \ , \\ \Xi^+(\alpha) &= \{S, e_i, b_1 + b_2, b_1 + z_1 + x, z_2\} \\ \Xi^+(z_1 + \alpha) &= \{S, e_i, b_1 + b_2, b_1 + \alpha, z_2\} \ , \end{aligned} \tag{6.2}$$

and $i = 1, \ldots, 6$.

Level-matched tachyons with $M^2 = -3/8$ can be encountered in the sectors

$$\mathcal{T}_{-3/8} \in \{e_i, e_i + z_1, e_i + z_2, e_i + \alpha, e_i + z_1 + \alpha\} \ , \ \ i = 1, \ldots, 6 \ .$$

The corresponding projection operators $\mathbb{P}^+_{\mathcal{T}_{-3/8}}$ and $\mathbb{P}^\varphi_{e_i}$ are defined using:

$$\begin{aligned} \Xi^+(e_i + z_1) &= \{S, e_j, b_I, z_2\} \ , \\ \Xi^+(e_i + z_2) &= \{S, e_j, b_I, z_1, \alpha\} \ , \\ \Xi^+(e_i + \alpha) &= \{S, e_j, b_I + x, b_I + z_1 + x, z_2\} \ , \\ \Xi^+(e_i + z_1 + \alpha) &= \{S, e_j, b_I + x, b_I + \alpha, z_2\} \ , \\ \Xi(e_i) &= \{S, e_j, b_I, z_1, z_2, \alpha\} \ , \end{aligned} \tag{6.3}$$

where $i, j = 1, \ldots, 6$, $i \neq j$ and $I = \min\{n \in \mathbb{Z} | n \geq i/2\}$.

The sectors $\mathcal{T}_{-1/4} \in \{e_i + e_j, e_i + e_j + z_1, e_i + e_j + z_2, e_i + e_j + \alpha, e_i + e_j + z_1 + \alpha\}$ with $i, j = 1, \ldots, 6$, $i \neq j$ can give rise to tachyons with $M^2 = -1/4$. The corresponding definitions for $\mathbb{P}^+_{\mathcal{T}_{-1/4}}$ and $\mathbb{P}^\varphi_{e_i+e_j}$ are:

$$\begin{aligned} \Xi^+(e_i + e_j + z_1) &= \begin{cases} \{S, e_k, b_I, z_2\} \ , \ (i,j) = (2n-1, 2n), \ n \in \mathbb{Z}^* \\ \{S, e_k, z_2\} \ , \ \text{otherwise} \end{cases} \ , \\ \Xi^+(e_i + e_j + z_2) &= \begin{cases} \{S, e_k, b_I, z_1, \alpha\} \ , \ (i,j) = (2n-1, 2n), \ n \in \mathbb{Z}^* \\ \{S, e_k, z_1, \alpha\} \ , \ \text{otherwise} \end{cases} \ , \\ \Xi^+(e_i + e_j + \alpha) &= \begin{cases} \{S, e_k, b_I + x, z_1 + \alpha + x, z_2\} \ , \ (i,j) = (2n-1, 2n), \ n \in \mathbb{Z}^* \\ \{S, e_k, z_1 + \alpha + x, z_2\} \ , \ \text{otherwise} \end{cases} \ , \\ \Xi^+(e_i + e_j + z_1 + \alpha) &= \begin{cases} \{S, e_k, b_I + x, b_I + \alpha, z_2\} \ , \ (i,j) = (2n-1, 2n), \ n \in \mathbb{Z}^* \\ \{S, e_k, x + \alpha, z_2\} \ , \ \text{otherwise} \end{cases} \ , \\ \Xi(e_i + e_j) &= \begin{cases} \{S, e_k, b_I, z_1, z_2, \alpha\} \ , \ (i,j) = (2n-1, 2n), \ n \in \mathbb{Z}^* \\ \{S, e_k, z_1, z_2, \alpha\} \ , \ \text{otherwise} \end{cases} \ , \end{aligned} \tag{6.4}$$

where $i, j, k = 1, \ldots, 6$, $i \neq j \neq k$ and $I = \min\{n \in \mathbb{Z} | n \geq i/2\}$.

Finally, tachyons with $M^2 = -1/8$ can be produced in the sectors

$$\mathcal{T}_{-1/8} \in \{e_i + e_j + e_k, e_i + e_j + e_k + z_1, e_i + e_j + e_k + z_2, e_i + e_j + e_k + \alpha, e_i + e_j + e_k + z_1 + \alpha\} \ ,$$

with $i, j, k = 1, \ldots, 6$, $i \neq j \neq k$. The corresponding projection operators $\mathbb{P}^+_{\mathcal{T}_{-1/8}}$ and $\mathbb{P}^{\varphi}_{e_i+e_j+e_k}$ are given by:

$$\begin{aligned}
\Xi^+(e_i + e_j + e_k + z_1) &= \{S, e_l, z_2\} \ , \\
\Xi^+(e_i + e_j + e_k + z_2) &= \{S, e_l, z_1, \alpha\} \ , \\
\Xi^+(e_i + e_j + e_k + \alpha) &= \{S, e_l, z_1 + \alpha + x, z_2\} \ , \\
\Xi^+(e_i + e_j + e_k + z_1 + \alpha) &= \{S, e_l, x + \alpha, z_2\} \ , \\
\Xi(e_i + e_j + e_k) &= \{S, e_l, z_1, z_2, \alpha\} \ ,
\end{aligned} \tag{6.5}$$

with $i, j, k, l = 1, \ldots, 6$, $i \neq j \neq k \neq l$.

In order to ensure our models are free of physical tachyons we impose the conditions:

$$\begin{aligned}
&\mathbb{P}^+_a = 0 \ , \ a \in \left\{\mathcal{T}_{-1/2}, \mathcal{T}_{-3/8}, \mathcal{T}_{-1/4}, \mathcal{T}_{-1/8}\right\} \ , \ a \notin \{e_i + pe_j + qe_k\} \ , \ i \neq j \neq k = 1, \ldots, 6 \ , \ p, q = 0, 1 \, , \\
&\mathbb{P}^{\bar\varphi}_a = 0 \ , \ a \in \{e_i + pe_j + qe_k\} \ , \ \bar\varphi \in \mathbb{1}_R - a_R \ ,
\end{aligned} \tag{6.6}$$

which are analogous to (5.11).

6.1.2 Massless States

The massless spectrum of the theory can be organised in the same manner as in the previous chapter, with the combinations $b_i + pT_j + qT_k$ now replaced by $b_i + pe_j + qe_k + re_l + se_m$, $i = 1, 2, 3$, $j \neq k \neq l \neq m = 1, \ldots, 6$, $j, k, l, m \neq \{2i - 1, 2i\}$, $p, q, r, s = 0, 1$. In this case, it is easy to see that the multiplicity of four that was always present in the previous class of models is now lifted. This allows the identification of models with three generations, arising in the sectors $\mathcal{S}^i_{pqrs} = S + b_i + pe_j + qe_k + re_l + se_m$, where $p, q, r, s = 0, 1$, $\{i, j, k, l, m\} = \{(1, 3, 4, 5, 6), (2, 1, 2, 5, 6), (3, 1, 2, 3, 4)\}$, with the modified form of (5.29) being:

$$\begin{aligned}
n_L &= \text{ch}(\psi^\mu) \sum_{i=1}^{3} \sum_{p,q,r,s=0}^{1} \mathbb{P}^-_{\mathcal{S}^i_{pqrs}} \frac{1}{2}\left(1 + X^{SU(4)}_{\mathcal{S}^i_{pqrs}}\right) \frac{1}{2}\left(1 + X^{SO(4)}_{\mathcal{S}^i_{pqrs}}\right) \ , \\
\bar n_R &= \text{ch}(\psi^\mu) \sum_{i=1}^{3} \sum_{p,q,r,s=0}^{1} \mathbb{P}^-_{\mathcal{S}^i_{pqrs}} \frac{1}{2}\left(1 - X^{SU(4)}_{\mathcal{S}^i_{pqrs}}\right) \frac{1}{2}\left(1 - X^{SO(4)}_{\mathcal{S}^i_{pqrs}}\right) \ , \\
\bar n_L &= \text{ch}(\psi^\mu) \sum_{i=1}^{3} \sum_{p,q,r,s=0}^{1} \mathbb{P}^-_{\mathcal{S}^i_{pqrs}} \frac{1}{2}\left(1 - X^{SU(4)}_{\mathcal{S}^i_{pqrs}}\right) \frac{1}{2}\left(1 + X^{SO(4)}_{\mathcal{S}^i_{pqrs}}\right) \ , \\
n_R &= \text{ch}(\psi^\mu) \sum_{i=1}^{3} \sum_{p,q,r,s=0}^{1} \mathbb{P}^-_{\mathcal{S}^i_{pqrs}} \frac{1}{2}\left(1 + X^{SU(4)}_{\mathcal{S}^i_{pqrs}}\right) \frac{1}{2}\left(1 - X^{SO(4)}_{\mathcal{S}^i_{pqrs}}\right) \ ,
\end{aligned} \tag{6.7}$$

resulting in the standard definition for the number of generations:

$$n_g = n_L - \bar n_L = \bar n_R - n_R \ , \tag{6.8}$$

which must, once again, be imposed as a constraint.

The projection operators indicating the presence of physical states in a sector $\mathcal{S}^i_{pqrs}$ now take the form

$$\mathbb{P}^-_{\mathcal{S}^i_{pqrs}} = 1, \ \Xi^-(\mathcal{S}^i_{pqrs}) = \{e_{2i-1}, e_{2i}, z_1, z_2\} \ , \tag{6.9}$$

while the representation operators determining their transformation properties under $SU(4) \times SO(4)$ are given by:

$$X^{SU(4)}_{\mathcal{S}^i_{pqrs}} = \begin{cases} -c\begin{bmatrix} \mathcal{S}^i_{pqrs} \\ \mathcal{S}^j_{0,0,1-r,1-s} + \alpha \end{bmatrix}^*, \ i, j = 1, 2, \ i \neq j \\ -c\begin{bmatrix} \mathcal{S}^i_{pqrs} \\ \mathcal{S}^1_{1-r,1-s,0,0} + \alpha \end{bmatrix}^*, \ i = 3 \end{cases} \quad , \quad X^{SO(4)}_{\mathcal{S}^i_{pqrs}} = -c\begin{bmatrix} \mathcal{S}^i_{pqrs} \\ \alpha \end{bmatrix}^* . \tag{6.10}$$

The corresponding 48 sectors giving rise to scalars in the chiral representations are: $\Sigma^i_{pqrs} = b_i + pe_j + qe_k + re_l + se_m$, where $p,q,r,s = 0,1$, $\{i,j,k,l,m\} = \{(1,3,4,5,6),(2,1,2,5,6),(3,1,2,3,4)\}$ and the projection operators determining whether states survive the GGSO projection are:

$$\mathbb{P}^+_{\Sigma^i_{pqrs}}\mathbb{P}^-_{\Sigma^i_{pqrs}} = 1,\ \Xi^+(\Sigma^i_{pqrs}) = \{e_{2i-1}, e_{2i}, z_1, z_2\},\ \Xi^-(\Sigma^i_{pqrs}) = \{\alpha\} \tag{6.11}$$

and the number of PS breaking Higgs multiplets can be determined by:

$$n_H = \sum_{i=1}^{3}\sum_{p,q,r,s=0}^{1}\mathbb{P}^+_{\Sigma^i_{pqrs}}\mathbb{P}^-_{\Sigma^i_{pqrs}}\ . \tag{6.12}$$

The number of SM Higgs multiplets arising in the sectors $\Upsilon^i_{pqrs} = \Sigma^i_{pqrs} + x = b_i + pe_j + qe_k + re_l + se_m + x$ is similarly determinded by:

$$n_h = \sum_{i=1}^{3}\sum_{p,q,r,s=0}^{1}\mathbb{P}^{\bar{\psi}^{4,5}}_{\Upsilon^i_{pqrs}},\ \Xi(\Upsilon^i_{pqrs}) = \{e_{2i-1}, e_{2i}, z_1, z_2, \alpha\}\ . \tag{6.13}$$

Finally, fractionally charged exotic fermions can arise in the sectors

$$\mathcal{E}^i_{pqrst} = \mathcal{S}^i_{pqrs} + tz_1 + \alpha = S + b_i + pe_j + qe_k + re_l + se_m + tz_1 + \alpha,$$

which give rise to colour (anti-)quadruplets, as well as $S+\alpha+x$ and $S+z_1+\alpha+x$ which give rise to pairs of $(\mathbf{4}+\bar{\mathbf{4}},\mathbf{1},\mathbf{1})$. The number of exotic (anti-)quadruplets arising in these sectors can then be determined by:

$$\begin{aligned} n_{\mathbf{4}} &= \sum_{i=1}^{3}\sum_{p,q,r,s,t=0}^{1}\mathbb{P}^-_{\mathcal{E}^i_{pqrst}}\frac{1}{2}\left(1 + X^{SU(4)}_{\mathcal{E}^i_{pqrst}}\right)\ , \\ n_{\bar{\mathbf{4}}} &= \sum_{i=1}^{3}\sum_{p,q,r,s,t=0}^{1}\mathbb{P}^-_{\mathcal{E}^i_{pqrst}}\frac{1}{2}\left(1 - X^{SU(4)}_{\mathcal{E}^i_{pqrst}}\right)\ , \end{aligned} \tag{6.14}$$

where

$$X^{SU(4)}_{\mathcal{E}^i_{pqrst}} = \begin{cases} -c\begin{bmatrix}\mathcal{E}^i_{pqrst}\\ \mathcal{S}^j_{0,0,1-r,1-s}\end{bmatrix},\ i \neq j = 1,2 \\ -c\begin{bmatrix}\mathcal{E}^i_{pqrst}\\ \mathcal{S}^1_{1-r,1-s,0,0}\end{bmatrix},\ i = 3\end{cases}\ . \tag{6.15}$$

Moreover,

$${\mathcal{E}'}^i_{pqrst} = \mathcal{Y}^i_{pqrs} + tz_1 + \alpha = S + b_i + pe_j + qe_k + re_l + se_m + x + tz_1 + \alpha\ ,$$

in addition to $S+z_2+\alpha$, $S+z_1+z_2+\alpha$, $S+\alpha$, and $S+z_1+\alpha$ may produce fermion bi-doublets. The definitions of the projection operators which can identify their presence in the spectrum and pinpoint their exact representation follow (5.36)-(5.45). While the use of real fermions in the parametrisation of the internal coordinates allows for the existence of models in which none of these exotic states survive the GGSO projection [116, 259, 275], we only enforce the relaxed condition allowing for their presence in vector-like pairs, in order to increase the odds of identifying compatible models.

Extra gauge bosons still arise in the sectors: $G = \{z_1, z_2, \alpha, z_1+z_2, z_1+\alpha, z_2+\alpha, z_1+z_2+\alpha, x, \alpha+x, z_1+\alpha+x\}$, where $x = 1 + S + \sum_{i=1}^{6} e_i + \sum_{k=1}^{2} z_k$, and the relevant conditions ensuring that no enhancement of the observable gauge symmetry occurs are given by (5.16), with:

$$
\begin{aligned}
&\Xi^{+}(x)=\{e_1,e_2,e_3,e_4,e_5,e_6,z_1,z_2\}\ ,\\
&\Xi^{+}(\alpha+x)=\{e_1,e_2,e_3,e_4,e_5,e_6,z_2,z_1+\alpha\}\ ,\\
&\Xi^{+}(z_1+\alpha+x)=\{e_1,e_2,e_3,e_4,e_5,e_6,z_2,\alpha\}\ ,\\
&\Xi^{+}(z_2+\alpha)=\{e_1,e_2,e_3,e_4,e_5,e_6,b_1+b_2,b_1+z_1+\alpha\}\ ,\\
&\Xi^{+}(z_1+z_2+\alpha)=\{e_1,e_2,e_3,e_4,e_5,e_6,b_1+b_2,b_1+\alpha\}\ ,\\
&\Xi^{-}(a)=\{S\}\,,a\in\{x,\alpha+x,z_1+\alpha+x,z_2+\alpha,z_1+z_2+\alpha\}\ ,\\
&\Xi(\alpha)=\{S,e_1,e_2,e_3,e_4,e_5,e_6,b_1+b_2,b_1+z_1+\alpha,z_2\}\ ,\\
&\Xi(z_1+\alpha)=\{S,e_1,e_2,e_3,e_4,e_5,e_6,b_1+b_2,b_1+\alpha,z_2\}\ ,\\
&\Xi(z_1)=\{S,e_1,e_2,e_3,e_4,e_5,e_6,b_1,b_2,z_2\}\ ,\\
&\Xi(z_2)=\{S,e_1,e_2,e_3,e_4,e_5,e_6,b_1,b_2,z_1,\alpha\}\ .
\end{aligned}
\tag{6.16}
$$

The rest of the massless spectrum is comprised of states charged under the hidden gauge symmetry, in addition to states arising in sectors unique to non supersymmetric constructions, which still disappear at generic points of the moduli space as discussed in 5.3.5 and, as such, are irrelevant to our current analysis.

6.2 Generic Point Behaviour

In order to obtain three generation models we are forced to modify the vector basis of (5.1) by replacing each of the $T_i=\{y^{ab},\omega^{ab}|\bar{y}^{ab},\bar{\omega}^{ab}\}$ with a pair of vectors $e_i=\{y^{a},\omega^{a}|\bar{y}^{a},\bar{\omega}^{a}\}$, $e_j=\{y^{b},\omega^{b}|\bar{y}^{b},\bar{\omega}^{b}\}$, in which the complexification of the internal fermions is undone. The equivalent orbifold description is given by the expression:

$$
\begin{aligned}
Z=&\frac{1}{\eta^2\bar{\eta}^2}\frac{1}{2^4}\sum_{\substack{h_1,h_2,H,H'\\ g_1,g_2,G,G'}\in\mathbb{Z}_2}\frac{1}{2^3}\sum_{\substack{a,k,\rho\\ b,\ell,\sigma}\in\mathbb{Z}_2}\frac{1}{2^3}\sum_{\substack{H_1,H_2,H_3\\ G_1,G_2,G_3}\in\mathbb{Z}_2}\frac{1}{2^3}\sum_{\substack{H_4,H_5,H_6\\ G_4,G_5,G_6}\in\mathbb{Z}_2}(-1)^{a+b+HG+H'G'+\Phi}\\
&\times\frac{\vartheta[^a_b]}{\eta}\frac{\vartheta[^{a+h_1}_{b+g_1}]}{\eta}\frac{\vartheta[^{a+h_2}_{b+g_2}]}{\eta}\frac{\vartheta[^{a-h_1-h_2}_{b-g_1-g_2}]}{\eta}\\
&\times\frac{\bar{\vartheta}[^k_\ell]^3}{\bar{\eta}^3}\frac{\bar{\vartheta}[^{k+H'}_{\ell+G'}]}{\bar{\eta}}\frac{\bar{\vartheta}[^{k-H'}_{\ell-G'}]}{\bar{\eta}}\frac{\bar{\vartheta}[^{k+h_1}_{\ell+g_1}]}{\bar{\eta}}\frac{\bar{\vartheta}[^{k+h_2}_{\ell+g_2}]}{\bar{\eta}}\frac{\bar{\vartheta}[^{k-h_1-h_2}_{\ell-g_1-g_2}]}{\bar{\eta}}\\
&\times\frac{\bar{\vartheta}[^{\rho+H'}_{\sigma+G'}]}{\bar{\eta}}\frac{\bar{\vartheta}[^{\rho-H'}_{\sigma-G'}]}{\bar{\eta}}\frac{\bar{\vartheta}[^\rho_\sigma]^2}{\bar{\eta}^2}\frac{\bar{\vartheta}[^{\rho+H}_{\sigma+G}]^4}{\bar{\eta}^4}\\
&\times\frac{\Gamma^{(1)}_{2,2}[^{H_1,H_2}_{G_1,G_2}|^{h_1}_{g_1}](T^{(1)},U^{(1)})}{\eta^2\bar{\eta}^2}\frac{\Gamma^{(2)}_{2,2}[^{H_3,H_4}_{G_3,G_4}|^{h_2}_{g_2}](T^{(2)},U^{(2)})}{\eta^2\bar{\eta}^2}\frac{\Gamma^{(3)}_{2,2}[^{H_5,H_6}_{G_5,G_6}|^{h_1+h_2}_{g_1+g_2}](T^{(3)},U^{(3)})}{\eta^2\bar{\eta}^2}\ .
\end{aligned}
\tag{6.17}
$$

In order to probe the moduli dependent properties of the theory, we proceed with the simplification of the partition function in a similar manner as in the previous chapters. We substitute two of the lattices with their equivalent fermionic point description in terms of Jacobi and Dedekind functions, as defined in (3.110), then take advantage of the Riemann–Jacobi identity in order to perform the summation over a,b, after which the partition function takes the form:

$$
Z=\frac{1}{4}\sum_{\substack{H_1,H_2\\ G_1,G_2}\in\mathbb{Z}_2}\Psi[^{H_1,H_2}_{G_1,G_2}]\Gamma^{\rm shift}_{2,2}[^{H_1,H_2}_{G_1,G_2}]\ ,
\tag{6.18}
$$

where

$$
\begin{aligned}
\Psi[^{H_1,H_2}_{G_1,G_2}] =& \frac{1}{\eta^{12}\bar\eta^{24}}\frac{1}{2^9}\sum_{\substack{h_2,H,H'\\ g_2,G,G'}\in\mathbb{Z}_2}\ \sum_{\substack{k,\rho\\ \ell,\sigma}\in\mathbb{Z}_2}\ \sum_{\substack{\gamma_3,\gamma_4,\gamma_5,\gamma_6\\ \delta_3,\delta_4,\delta_5,\delta_6}\in\mathbb{Z}_2} (-1)^{HG+H'G'+\hat\Phi}\,\vartheta[^{1+H_1}_{1+G_1}]^2\vartheta[^{1+H_1+h_2}_{1+G_1+g_2}]^2\\
&\times\left|\vartheta[^{\gamma_3}_{\delta_3}]\vartheta[^{\gamma_3+h_2}_{\delta_3+g_2}]\vartheta[^{\gamma_4}_{\delta_4}]\vartheta[^{\gamma_4+h_2}_{\delta_4+g_2}]\vartheta[^{\gamma_5}_{\delta_5}]\vartheta[^{\gamma_5-h_2}_{\delta_5-g_2}]\vartheta[^{\gamma_6}_{\delta_6}]\vartheta[^{\gamma_6-h_2}_{\delta_6-g_2}]\right|\\
&\times\bar\vartheta[^k_\ell]^4\,\bar\vartheta[^{k+H'}_{\ell+G'}]\,\bar\vartheta[^{k-H'}_{\ell-G'}]\bar\vartheta[^{k+h_2}_{\ell+g_2}]\bar\vartheta[^{k-h_2}_{\ell-g_2}]\bar\vartheta[^{\rho+H'}_{\sigma+G'}]\,\bar\vartheta[^{\rho-H'}_{\sigma-G'}]\,\bar\vartheta[^\rho_\sigma]^2\,\bar\vartheta[^{\rho+H}_{\sigma+G}]^4\\
=&\sum_{\Delta,\bar\Delta} C[^{H_1,H_2}_{G_1,G_2}](\Delta,\bar\Delta)q^\Delta\bar q^{\bar\Delta}\ .
\end{aligned}
\tag{6.19}
$$

While it is natural to assume that the potential still follows the same asymptotic behaviour:

$$
V_{\text{1-loop}}(T_2\gg 1)\sim -\frac{n_B-n_F}{T_2^2}+\dots\ , \tag{6.20}
$$

upon closer inspection the situation appears to be more complicated. We can now identify three main contributions to the effective potential:

$$
-2(2\pi)^4V_{1-\text{loop}}(T_2)=I_1+I_2+I_3\ , \tag{6.21}
$$

where

$$
\begin{aligned}
I_1 &= \int\frac{d\tau d\bar\tau}{\tau_2^3}\frac14\sum_{H_1,G_1}\Psi[^{H_1,0}_{G_1,0}]\Gamma^{\text{shift}}_{2,2}[^{H_1,0}_{G_1,0}]\ ,\\
I_2 &= \int\frac{d\tau d\bar\tau}{\tau_2^3}\frac14\sum_{H_1,G_1}\Psi[^{H_1,H_1}_{G_1,G_1}]\Gamma^{\text{shift}}_{2,2}[^{H_1,H_1}_{G_1,G_1}]\ ,\\
I_3 &= \int\frac{d\tau d\bar\tau}{\tau_2^3}\frac14\sum_{\substack{H_1,H_2\\ G_1,G_2}}\Psi[^{H_1,H_2}_{G_1,G_2}]\Gamma^{\text{shift}}_{2,2}[^{H_1,H_2}_{G_1,G_2}]\ ,\quad (H_2,G_2)\neq\{(0,0),(H_1,G_1)\}\ .
\end{aligned}
\tag{6.22}
$$

The asymptotic behaviour in the large volume regime can be evaluated using the unfolding method. The I_2 contribution corresponds to all sectors satisfying $H_1=H_2$ and $G_1=G_2$. This essentially reproduces the contribution to the effective potential of (4.45), which now yields:

$$
I_2=\frac{2C[^{0,0}_{1,0}](0,0)}{\pi^3T_2^2}\sum_{m_1,m_2}\frac{U_2^3}{\left|m_1+\frac12+Um_2\right|^6}+\dots \tag{6.23}
$$

Using similar methods, we can deduce the expressions for I_1 and I_3. These are:

$$
I_1=\frac{2C[^{0,0}_{1,1}](0,0)}{\pi^3T_2^2}\sum_{m_1,m_2}\frac{U_2^3}{\left|m_1+\frac12+Um_2\right|^6}+\dots \tag{6.24}
$$

and

$$
I_3=\frac{4C[^{0,1}_{1,0}](0,0)}{\pi^3T_2^2}\sum_{m_1,m_2}\frac{(-1)^{m_2}U_2^3}{\left|m_1+\frac12+Um_2\right|^6}+\dots\ , \tag{6.25}
$$

where in the case of I_3 we have taken advantage of the modular transformations to reduce it to the evaluation of

$$
I_3=\frac12\sum_{\gamma\in\{1,S,ST\}}\gamma\left(\frac12\sum_{G_2}\Psi[^{0,\,1}_{1,G_2}]\Gamma^{\text{shift}}_{2,2}[^{0,\,1}_{1,G_2}]\right)\ . \tag{6.26}
$$

These three contributions once again result in a power-law evolution of the potential, necessitating additional conditions in order to ensure an exponential suppression. Altogether, the potential now takes the asymptotic form:

$$
\begin{aligned}
\lim_{T_2\to\infty}\Big(-(2\pi)^4V_{\text{1-loop}}(T_2)\Big) =& \frac{C[^{0,0}_{1,0}](0,0)+C[^{0,0}_{1,1}](0,0)}{\pi^3T_2^2}\sum_{m_1,m_2}\frac{U_2^3}{\left|m_1+\frac12+Um_2\right|^6}\\
&+\frac{2C[^{0,1}_{1,0}](0,0)}{\pi^3T_2^2}\sum_{m_1,m_2}\frac{(-1)^{m_2}U_2^3}{\left|m_1+\frac12+Um_2\right|^6}+\mathcal{O}(e^{-\lambda\sqrt{T_2}})\ ,
\end{aligned}
\tag{6.27}
$$

where λ is a positive constant of order one. Eliminating the power-law contributions therefore requires the simultaneous cancellation of the two terms, which results in two independent conditions:

$$\begin{aligned} &C[^{0,0}_{1,0}](0,0) + C[^{0,0}_{1,1}](0,0) = 0 \ , \\ &C[^{0,1}_{1,0}](0,0) = 0 \ . \end{aligned} \tag{6.28}$$

Once again, taking advantage of the fact that the partition function vanishes identically when $H_1 = G_1 = 0$, we can rewrite the first condition as

$$\sum_{G_1, G_2 \in \mathbb{Z}_2} C[^{\ 0\ ,\ 0}_{G_1, G_2}](0,0) = 0 \ . \tag{6.29}$$

Using a similar argument as in the previous chapter, we can prove that this condition is equivalent to demanding Bose–Fermi degeneracy at the massless level, recovering the established $n_B = n_F$ condition of super no-scale models as we expected. However, while in previous constructions this condition was sufficient to ensure the exponential decay of the one-loop potential, this is now no longer the case. In fact, models which do exhibit Bose–Fermi degeneracy might still exhibit a power-law behaviour and thus unacceptably large values of the cosmological constant in the large volume regime, if they fail to also satisfy the condition $C[^{0,1}_{1,0}](0,0) = 0$. As an example we present a 3-generation model which satisfies all phenomenological conditions introduced in section 5.4, defined in the free fermionic formulation by the GGSO matrix:

$$c[^{u_i}_{u_j}] = \begin{pmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\ +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 \\ +1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 \\ +1 & -1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 \\ +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & -1 & +1 \\ +1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 & -1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 & -1 \\ +1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 \end{pmatrix} . \tag{6.30}$$

The massless spectrum of this model exhibits the desired boson-fermion degeneracy. Additionally, it features three generations of fermions which appear in the sectors: $S + b_1 + e_6$, $S + b_1 + e_3 + e_4 + e_6$, $S + b_2 + e_1 + e_6$, $S + b_2 + e_2 + e_5 + e_6$, $S + b_3 + e_1$, $S + b_3 + e_1 + e_3 + e_4$, as well as Pati–Salam and SM Higgs bosons from sectors $b_2 + e_6$ and $b_1 + e_6 + x$, $b_1 + e_3 + e_4 + e_6 +$ x respectively. The gauge symmetry is not enhanced and while it does include exotic fermions, they are organised into vector-like pairs. Moreover, supersymmetry is spontaneously broken by the Scherk–Schwarz mechanism and its one-loop effective potential is positive semi-definite.

While this model does satisfy all the phenomenological conditions we introduced to analyse models with complex worldsheet fermions, it fails to satisfy the second condition of (6.28) and a close inspection of its effective potential, presented in figure 6.2 reveals that it is not in fact exponentially suppressed.

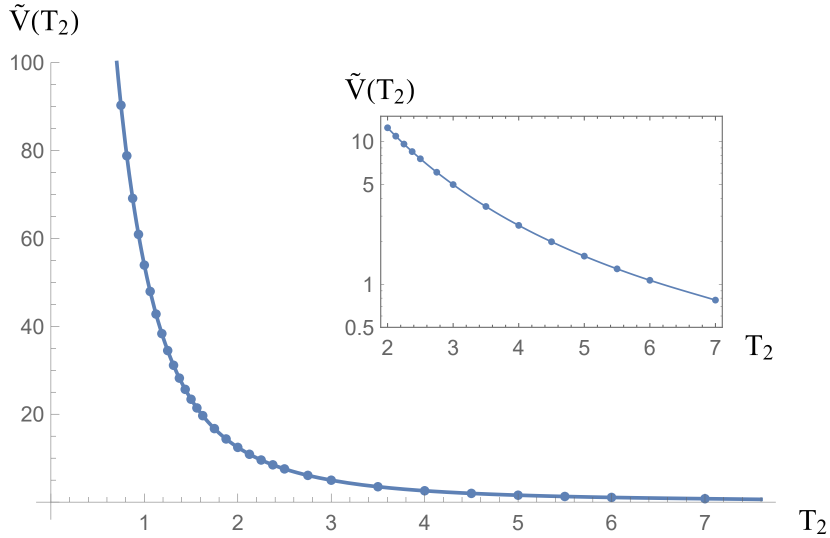

Figure 6.1: Numerical evaluation of the rescaled one-loop potential of the model defined by (6.30). In detail, we present the same potential in semi-logarithmic axes, highlighting the deviation from the exponential suppression which implies a power-law behaviour in the region $T_2 \gg 1$.

In order to obtain an effective potential with the desired exponential suppression, we must also impose the second constraint, which can be recast in the form:

$$\sum_{G_1,G_2\in\mathbb{Z}_2} C\left[\begin{smallmatrix}0 & ,\, 1\\ G_1 & ,\, G_2\end{smallmatrix}\right](0,0) = 0\ . \tag{6.31}$$

This condition can be shown to also correspond to a degeneracy of bosonic and fermionic states, but in this case this must be applied not at the massless level of the theory, but rather at the ground state of the tower of states which exhibit a winding shift due to $H_2 = 1$.

The two conditions that are necessary for the models to exhibit super no-scale structure can then be expressed as:

$$\Sigma(H_2) = \sum_{G_1,G_2\in\mathbb{Z}_2} C\left[\begin{smallmatrix}0 & ,\, H_2\\ G_1 & ,\, G_2\end{smallmatrix}\right](0,0) = 0\ , \quad H_2 = (0,1)\ . \tag{6.32}$$

6.3 The Search for Consistent Realisations

The family of models defined by the vector basis of (6.1) comprises 2^{79} a priori distinct models. However, some of the independent GGSO phases do not appear in our constraints and can therefore be considered to be irrelevant to our analysis. More specifically, the phases $c\left[\begin{smallmatrix}\mathbb{1}\\ \mathbb{1}\end{smallmatrix}\right]$, $c\left[\begin{smallmatrix}\mathbb{1}\\ S\end{smallmatrix}\right]$, $c\left[\begin{smallmatrix}S\\ b_k\end{smallmatrix}\right]$ and $c\left[\begin{smallmatrix}b_1\\ b_2\end{smallmatrix}\right]$ amount to conventions and can be freely taken to be $+1$, as before. Additionally, the phases $c\left[\begin{smallmatrix}\mathbb{1}\\ \beta\end{smallmatrix}\right]$, $\beta = \{e_i, b_k, z_a\}$, $i = 1,\ldots,6$, $k = 1,2$, $a = 1,2$, as well as $c\left[\begin{smallmatrix}e_1\\ b_2\end{smallmatrix}\right]$, $c\left[\begin{smallmatrix}e_2\\ b_2\end{smallmatrix}\right]$, $c\left[\begin{smallmatrix}e_3\\ b_1\end{smallmatrix}\right]$ and $c\left[\begin{smallmatrix}e_4\\ b_1\end{smallmatrix}\right]$ are absent from all the relevant projection operators. Consequently, we choose to fix them to $+1$ as well. Finally, we set $c\left[\begin{smallmatrix}S\\ e_1\end{smallmatrix}\right] = +1$ in order to focus our search to non supersymmetric models.

Having fixed the GGSO phases that do not affect the aforementioned characteristics of our models, the parameter space of interest consists of 2^{59} vacua. While a comprehensive scan of all models as performed in [2] and outlined in the previous chapter is beyond current technical capabilities, we will still employ the two-stage scan procedure employed in [2, 119, 264], utilising the properties of the $SO(10)$ "parent" models in order to increase the likelihood of obtaining phenomenologically appealing Pati-Salam models.

These PS models are subjected to the phenomenological criteria outlined in section 5.4, with two amendments. Firstly, since the number of generations is now an integer instead of a multiple of four, we require that at least 3 fermion generations are present in the massless spectrum. The models are still expected to follow a normal distribution with respect to the net chirality, with most chiral vacua exhibiting 1 or 2 fermion generations. This suggests that a much larger subset of models will be eliminated by this condition compared to its application on the models analysed in the previous chapter.

Secondly, the condition imposed to ensure the exponential suppression of the cosmological constant is now expanded. In addition to the Bose–Fermi degeneracy in the massless spectrum, we also require boson-fermion degeneracy in the states with winding shift due to $H_2 = 1$, which are asymptotically massless in the $T_2 \to \infty$ limit. In the previous chapter, we saw that the highly non-trivial massless degeneracy necessary for the super no-scale condition was typically violated, with only a small minority of models having the desired behaviour for large values of T_2. The additional constraint in this case is expected to shrink the parameter space of viable models even more, rendering the identification of such models even more difficult.

The updated set of constraints imposed upon the models of class (6.1) are:

(i) Absence of on-shell tachyons, imposed by (6.6).

(ii) Absence of observable gauge symmetry enhancement, by demanding (5.16), with the updated projectors given by (6.16).

(iii) Existence of at least three complete chiral generations under $SU(4) \times SU(2)_L \times SU(2)_R$, requiring $n_g = n_L - \bar{n}_L = \bar{n}_R - n_R \geq 3$ as defined in (6.7), (6.8).

(iv) Existence of heavy and light Higgs bosons responsible for the spontaneous breaking of the Pati–Salam and SM gauge symmetries, implying $n_H > 0$ and $n_h > 0$ in (6.12) and (6.13).

(v) Vector-like fractionally charged exotics, satisfying $n_4 = n_{\bar{4}}$ in (6.14).

(vi) Spontaneous supersymmetry breaking, consistent with the Scherk–Schwarz mechanism, as covered in section 4.4.

(vii) Exponential suppression of the cosmological constant in the region $T_2 \gg 1$, by imposing the condition (6.32).

Our starting point is a random sample of 2×10^9 $SO(10)$ models, corresponding to $\sim 4 \times 10^{12}$ possible models once the relevant Pati-Salam GGSO phases are introduced, which roughly amount to 1/10000 of all possible vacua. Utilising a DELL PowerEdge R630 workstation with 32 GB of memory, the parameter space was scanned over a period of 10 days. Our analysis resulted in the identification of 174 models which satisfy the above criteria. We note that after applying conditions (i) through (vi), roughly $1 : 1000$ of the remaining models exhibit Bose–Fermi degeneracy in the massless spectrum, and $\sim 1 : 5$ satisfy the second condition of (6.28), but only $\sim 1 : 50000$ models satisfy both conditions.

After analysing the resulting models in terms of their partition function and effective potential, we uncover 17 classes, labeled (I)-(XVII) which we present in figure 6.2. The first class of models is the only one which exhibits a positive potential. Classes (II) through (IX) oscillate between positive and negative values, asymptotically vanishing from above. The potential of the remaining classes is negative semi-definite. With the exception of class (II), which appears to be tachyon-free for all values of T_2, all classes exhibit tachyons in the region $T_2 < 1$.

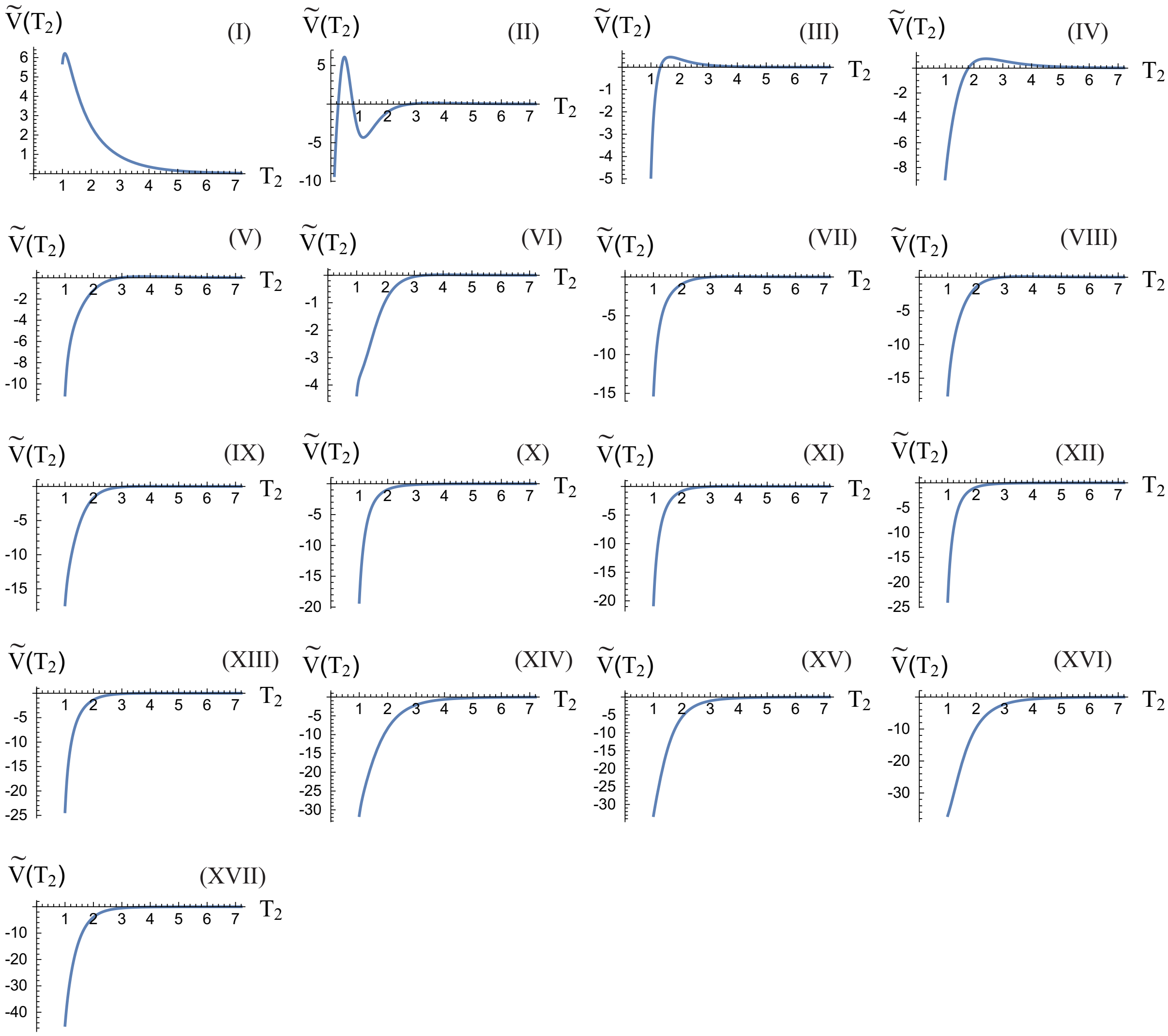

Figure 6.2: The rescaled effective potential $\tilde{V}(T_2) = 2(2\pi)^4 V(T_2)$ for each distinct subclass of models. All subclasses except for the second one exhibit tachyons in the region $T_2 < 1$.

6.4 A Three-Generation Super No-Scale Model

We conclude this chapter by presenting a specific model of class (I) which satisfies all phenomenological constraints. It is defined in the free fermionic formulation by the GGSO matrix:

$$c\left[{}^{\beta_i}_{\beta_j}\right] = \begin{pmatrix}
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\
+1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & +1 \\
+1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 \\
+1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & -1 \\
+1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & +1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 \\
+1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 \\
+1 & -1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & -1 \\
+1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & +1 \\
+1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 \\
+1 & +1 & -1 & +1 & +1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \\
-1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 & +1 & -1 & -1
\end{pmatrix} . \qquad (6.33)$$

This model exhibits 3 fermionic generations from the sectors $S+b_1+e_6$, $S+b_1+e_3+e_4+e_6$, $S+b_2+e_1+e_6$, $S+b_2+e_2+e_5+e_6$, $S+b_3+e_1$, and $S+b_3+e_1+e_3+e_4$, as well as the necessary scalars needed for the spontaneous breaking of the Pati–Salam and Standard Model gauge symmetries which can be encountered in sectors b_2+e_6, b_1+e_6+x, $b_1+e_3+e_4+e_6+x$, and $b_2+e_1+e_6+x$. It is tachyon free and supersymmetry breaking can be traced to the Scherk–Schwarz mechanism implemented as a shift along the first compactified direction. Furthermore, the model satisfies both super no-scale conditions defined in (6.32) and its potential, presented in figure 6.3, is exponentially suppressed in the large T_2 limit. Figure 6.4 highlights the exponential suppression of the one-loop effective potential in contrast to the model introduced in (6.30), which fails to satisfy both super no-scale conditions.

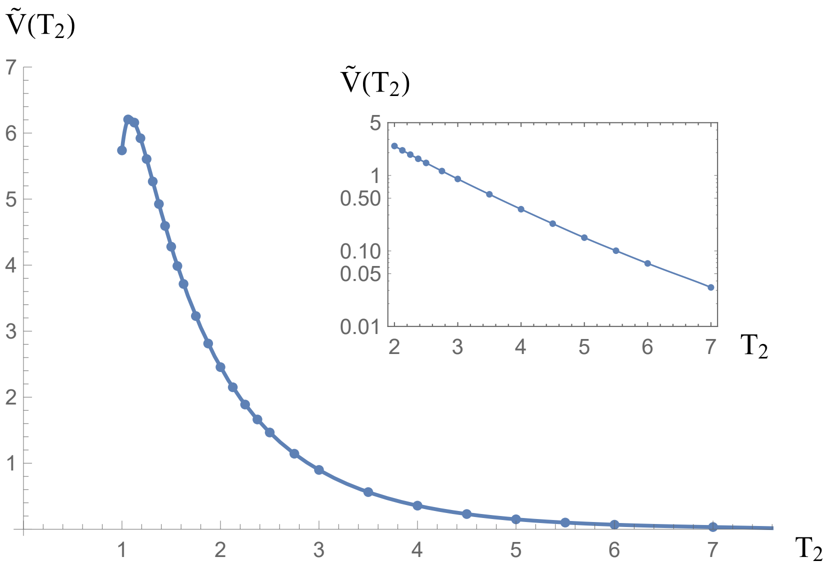

Figure 6.3: The rescaled effective potential $\tilde{V}(T_2) = 2(2\pi)^4 V(T_2)$ for the model defined by (6.33), in the non-tachyonic region $T_2 \geq 1$. In detail, we present the same potential in semi-logarithmic axes, focusing on the region $T_2 \gg 1$ which clearly exhibits an exponential decay.

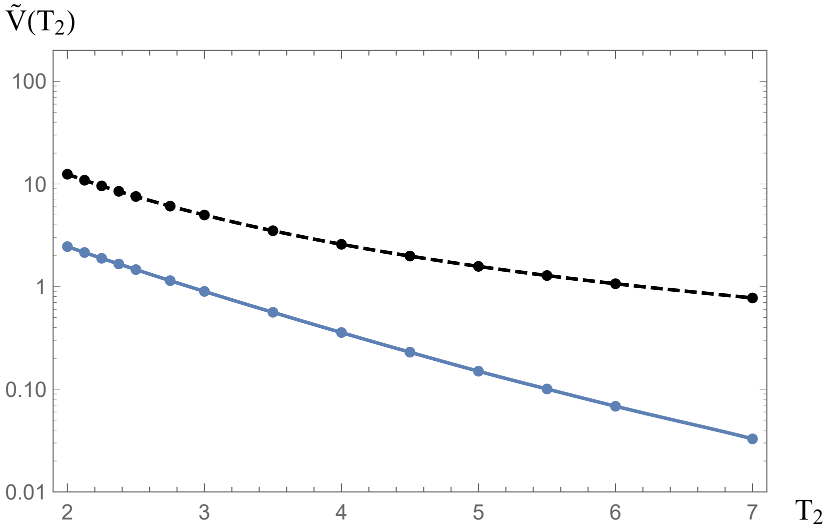

Figure 6.4: A direct comparison of the rescaled effective potentials of the two models presented in this chapter. The black, dashed line corresponds to the model defined in (6.30), whose spectrum satisfies all relevant constraints, in addition to one of the two super no-scale conditions defined in (6.32) and does not exhibit exponential suppression of the cosmological constant. In blue, we have the model of (6.33) which satisfies all conditions and exhibits the desired exponential suppression.

At the fermionic point, its partition function, expanded in terms of $q_r = e^{2\pi i\tau_1}$, $q_i = e^{-2\pi\tau_2}$, is given by:

$$\begin{aligned} Z = {} & \frac{2q_i}{q_r} - \frac{4q_i}{q_r^{3/4}} + \frac{8q_i}{\sqrt{q_r}} - \frac{16q_i}{\sqrt[4]{q_r}} + \left(32 + 32q_i^2\right) + \left(60 + \frac{384}{q_i} - 80q_i - 36q_i^2\right)\sqrt[4]{q_r} \\ & + \left(1424 + \frac{3008}{q_i} + 448q_i + 80q_i^2\right)\sqrt{q_r} + \left(-560 + \frac{14272}{q_i} + 256q_i - 304q_i^2\right)q_r^{3/4} + \dots \end{aligned} \tag{6.34}$$

The partition function at generic points of moduli space can be obtained from (6.17), using the orbifold phase:

$$\begin{aligned} \Phi = {} & ab + a(G_1 + g_2) + b(H_1 + h_2) + k\ell + k(G' + G_1 + g_2) + \ell(H' + H_1 + h_2) \\ & + \rho\sigma + \rho(G + G_1 + G_3) + \sigma(H + H_1 + H_3) \\ & + HG + H(G' + G_1 + G_3 + G_6 + g_1 + g_2) + G(H' + H_1 + H_3 + H_6 + h_1 + h_2) \\ & + H'G' + H'(G_2 + G_3 + G_5 + G_6 + g_2) + G'(H_2 + H_3 + H_5 + H_6 + h_2) \\ & + H_1G_1 + H_1(G_2 + G_3 + G_5 + g_1 + g_2) + G_1(H_2 + H_3 + H_5 + h_1 + h_2) \\ & + H_2(G_5 + g_2) + G_2(H_5 + h_2) + H_3G_3 + H_3(G_4 + G_5 + G_6) + G_3(H_4 + H_5 + H_6) \\ & + H_5 g_2 + G_5 h_2\,. \end{aligned} \tag{6.35}$$

Its massless spectrum at the fermionic point is outlined in tables 6.1 – 6.6.

Sector	$SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^3\times SU(2)^4\times SO(8)$ representation(s)
$S+x$	$(\mathbf{4},\mathbf{2},\mathbf{1},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{4},\mathbf{2},\mathbf{1},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$ $(\mathbf{4},\mathbf{2},\mathbf{1},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{4},\mathbf{2},\mathbf{1},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$ $(\bar{\mathbf{4}},\mathbf{2},\mathbf{1},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\bar{\mathbf{4}},\mathbf{2},\mathbf{1},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$ $(\bar{\mathbf{4}},\mathbf{2},\mathbf{1},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\bar{\mathbf{4}},\mathbf{2},\mathbf{1},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+z_2$	$2\times(\mathbf{1},\mathbf{1},\mathbf{1},0,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_c})$
$S+z_1+\alpha$	$2\times(\mathbf{1},\mathbf{1},\mathbf{2},0,0,0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_1+e_3+e_4+e_6$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},+\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+e_6$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},+\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+e_3+e_4+e_5+e_6+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,+\frac{1}{2},-\frac{1}{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+e_5+e_6+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,+\frac{1}{2},-\frac{1}{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+e_3+e_5+e_6+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_1+e_4+e_5+e_6+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_1+e_3+e_5+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_v})$
$S+b_1+e_4+e_5+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_v})$
$S+b_1+e_3+e_5+e_6+\alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},-\frac{1}{2},0,0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+e_4+e_5+e_6+\alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},-\frac{1}{2},0,0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+e_3+e_4+e_5+e_6+z_1+\alpha$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},+\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_1+e_5+e_6+z_1+\alpha$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},+\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_1+e_3+e_4+e_6+z_1+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},0,+\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_1+e_6+z_1+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},0,+\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_1+e_3+e_6+\alpha+x$	$(\mathbf{1},\mathbf{1},\mathbf{2},0,+\frac{1}{2},-\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_1+e_4+e_6+\alpha+x$	$(\mathbf{1},\mathbf{1},\mathbf{2},0,+\frac{1}{2},-\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$

Table 6.1: *Spectrum of massless fermionic matter arising in the untwisted sectors and twisted sectors which involve b_1 and their quantum numbers under the gauge bundle.*

Sector	$SU(4)\times SU(2)_L\times SU(2)_R\times U(1)^3\times SU(2)^4\times SO(8)$ representation(s)
$S+b_2+e_2+e_5+e_6$	$(\mathbf{4},\mathbf{2},\mathbf{1},0,-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+e_6$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},0,-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+e_2+e_5+e_6+x$	$(\mathbf{6},\mathbf{1},\mathbf{1},-\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},\pm1,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $2\times(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $2\times(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_6+x$	$(\mathbf{1},\mathbf{2},\mathbf{2},-\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},0,-\frac{1}{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_6+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},0,+\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},0,+\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_2+e_2+e_5+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+e_2+e_5+e_6+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_2+e_2+e_5+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})$
$S+b_2+e_2+e_5+e_6+z_2+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_c})$
$S+b_2+e_6+z_2+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_s})$
$S+b_2+e_1+e_6+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_v})$
$S+b_2+e_1+\alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},0,-\frac{1}{2},0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+e_2+e_5+e_6+\alpha$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},0,-\frac{1}{2},0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_6+z_1+\alpha$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},0,-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+z_1+\alpha$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},0,-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_2+e_2+e_5+e_6+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0,+\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},+\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+e_6+z_1+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_2+z_1+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},-\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S+b_2+e_1+e_6+\alpha+x$	$(\mathbf{1},\mathbf{1},\mathbf{2},+\frac{1}{2},0,+\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+e_2+e_5+\alpha+x$	$(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S+b_2+e_2+e_5+e_6+z_1+\alpha+x$	$(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S+b_2+e_1+e_2+e_5+z_1+\alpha+x$	$(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},0,+\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$

Table 6.2: *Spectrum of massless fermionic matter arising in twisted sectors which involve b_2 and their quantum numbers under the gauge bundle.*

Sector	$SU(4) \times SU(2)_L \times SU(2)_R \times U(1)^3 \times SU(2)^4 \times SO(8)$ representation(s)
$S + b_3 + e_1$	$(\mathbf{4},\mathbf{2},\mathbf{1},0,0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S + b_3 + e_1 + e_3 + e_4$	$(\mathbf{4},\mathbf{2},\mathbf{1},0,0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S + b_3 + x$	$(\mathbf{6},\mathbf{1},\mathbf{1},-\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},-\frac{1}{2},\pm 1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $2 \times (\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $2 \times (\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S + b_3 + e_3 + e_4 + x$	$(\mathbf{6},\mathbf{1},\mathbf{1},-\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},-\frac{1}{2},\pm 1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $2 \times (\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $2 \times (\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S + b_3 + e_3 + x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S + b_3 + e_4 + x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$S + b_3 + e_2 + e_3 + z_1 + x$	$(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S + b_3 + e_2 + e_4 + z_1 + x$	$(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$S + b_3 + e_2 + \alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},0,0,+\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S + b_3 + e_2 + e_3 + e_4 + \alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},0,0,+\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$S + b_3 + e_1 + e_2 + \alpha + x$	$(\mathbf{1},\mathbf{2},\mathbf{1},+\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S + b_3 + e_1 + e_2 + e_3 + e_4 + \alpha + x$	$(\mathbf{1},\mathbf{2},\mathbf{1},+\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S + b_3 + e_1 + e_3 + z_1 + \alpha + x$	$(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$S + b_3 + e_1 + e_4 + z_1 + \alpha + x$	$(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$

Table 6.3: *Spectrum of massless fermionic matter arising in twisted sectors which involve b_3 and their quantum numbers under the gauge bundle.*

Sector	$SU(4) \times SU(2)_L \times SU(2)_R \times U(1)^3 \times SU(2)^4 \times SO(8)$ representation(s)
0	$(\mathbf{6},\mathbf{1},\mathbf{1},\pm1,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{6},\mathbf{1},\mathbf{1},0,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{6},\mathbf{1},\mathbf{1},0,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,\pm1,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},\pm1,0,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{1},0,\pm1,\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$, $12 \times (\mathbf{1},\mathbf{1},\mathbf{1},0,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$e_1+e_2+e_5+e_6+\alpha$	$2 \times (\mathbf{1},\mathbf{2},\mathbf{1},0,0,0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$e_1+e_2+e_5+e_6+z_1+\alpha$	$2 \times (\mathbf{1},\mathbf{2},\mathbf{1},0,0,0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$b_1+e_3+e_4+e_6+x$	$(\mathbf{1},\mathbf{2},\mathbf{2},0,-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
b_1+e_6+x	$(\mathbf{1},\mathbf{2},\mathbf{2},0,-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_1+e_3+e_6+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_1+e_4+e_6+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_1+e_3+e_5+z_2+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,+\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_s})$
$b_1+e_4+e_5+z_2+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},0,+\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8_s})$
$b_1+e_3+e_4+e_6+z_1+\alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},-\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_1+e_6+z_1+\alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},-\frac{1}{2},0,0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_1+e_3+e_5+e_6+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},0,+\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_1+e_4+e_5+e_6+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},0,+\frac{1}{2},+\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_1+e_3+e_4+e_5+e_6+z_1+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},0,-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$b_1+e_5+e_6+z_1+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},0,-\frac{1}{2},-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$

Table 6.4: *Spectrum of massless scalar matter arising in the untwisted sectors and twisted sectors which involve b_1 and their quantum numbers under the gauge bundle.*

Sector	$SU(4) \times SU(2)_L \times SU(2)_R \times U(1)^3 \times SU(2)^4 \times SO(8)$ representation(s)
$b_2 + e_1 + e_2 + e_5 + e_6$	$(\mathbf{4}, \mathbf{2}, \mathbf{1}, 0, -\frac{1}{2}, 0, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_6$	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, 0, +\frac{1}{2}, 0, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_2 + e_5 + e_6 + x$	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, $(\mathbf{1}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, \pm 1, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, $2 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, $2 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_1 + e_6 + x$	$(\mathbf{1}, \mathbf{2}, \mathbf{2}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_1 + e_6 + z_1 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$b_2 + z_1 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
$b_2 + e_1 + e_2 + e_5 + z_1 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$b_2 + e_2 + e_5 + e_6 + z_1 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
$b_2 + e_1 + e_2 + e_5 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})$
$b_2 + e_1 + e_2 + e_5 + e_6 + z_2 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8_c})$
$b_2 + e_1 + e_6 + z_2 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, +\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8_s})$
$b_2 + e_6 + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8_v})$
$b_2 + e_2 + e_5 + e_6 + \alpha$	$(\mathbf{4}, \mathbf{1}, \mathbf{1}, 0, +\frac{1}{2}, 0, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + \alpha$	$(\mathbf{4}, \mathbf{1}, \mathbf{1}, 0, -\frac{1}{2}, 0, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_1 + e_6 + z_1 + \alpha$	$(\mathbf{4}, \mathbf{1}, \mathbf{1}, 0, +\frac{1}{2}, 0, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$b_2 + z_1 + \alpha$	$(\mathbf{4}, \mathbf{1}, \mathbf{1}, 0, -\frac{1}{2}, 0, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
$b_2 + e_1 + e_2 + e_5 + e_6 + \alpha + x$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_1 + \alpha + x$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, +\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_6 + z_1 + \alpha + x$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$b_2 + e_1 + z_1 + \alpha + x$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, -\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$
$b_2 + e_6 + \alpha + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, +\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_2 + e_5 + \alpha + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$
$b_2 + e_1 + e_2 + e_5 + e_6 + z_1 + \alpha + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, -\frac{1}{2}, 0, -\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$
$b_2 + e_2 + e_5 + z_1 + \alpha + x$	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, -\frac{1}{2}, 0, +\frac{1}{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$

Table 6.5: *Spectrum of massless scalar matter arising in twisted sectors which involve b_2 and their quantum numbers under the gauge bundle.*

Sector	$SU(4) \times SU(2)_L \times SU(2)_R \times U(1)^3 \times SU(2)^4 \times SO(8)$ representation(s)
b_3	$(\mathbf{4},\mathbf{2},\mathbf{1},0,0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_3+e_4$	$(\mathbf{4},\mathbf{2},\mathbf{1},0,0,-\frac{1}{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
b_3+e_1+x	$(\mathbf{6},\mathbf{1},\mathbf{1},-\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},-\frac{1}{2},\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$ $2\times(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,2\times(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_1+e_3+e_4+x$	$(\mathbf{6},\mathbf{1},\mathbf{1},-\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},-\frac{1}{2},\pm1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,$ $2\times(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,2\times(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_1+e_3+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_1+e_4+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},+\frac{1}{2},+\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\,(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_1+e_2+e_3+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$b_3+e_1+e_2+e_4+z_1+x$	$(\mathbf{1},\mathbf{1},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$
$b_3+e_1+e_2+\alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},0,0,+\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_1+e_2+e_3+e_4+\alpha$	$(\mathbf{4},\mathbf{1},\mathbf{1},0,0,+\frac{1}{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_2+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},+\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_2+e_3+e_4+\alpha+x$	$(\mathbf{1},\mathbf{2},\mathbf{1},+\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$
$b_3+e_3+z_1+\alpha+x$	$(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$
$b_3+e_4+z_1+\alpha+x$	$(\mathbf{1},\mathbf{1},\mathbf{2},-\frac{1}{2},-\frac{1}{2},0,\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$

Table 6.6: *Spectrum of massless scalar matter arising in twisted sectors which involve b_3 and their quantum numbers under the gauge bundle.*

Chapter 7

Summary And Outlook

In this thesis, we investigate four-dimensional non supersymmetric heterotic string models in which supersymmetry is broken via the stringy realisation of the Scherk–Schwarz mechanism and the cosmological constant is exponentially suppressed for large radii. We dedicate the first chapter to a brief presentation of the SM, along with some of the open problems it faces and some proposed frameworks in which they can be resolved. This introductory chapter includes a discussion of the $SO(10)$ and Pati–Salam models, which comprise the main focus of our work. We also introduce the Scherk–Schwarz mechanism, which when suitably generalised in the framework of string theory can be employed to spontaneously break supersymmetry. The chapter concludes with a short historical overview of string theory outlining its key advantages, as well as the challenges it is facing, and provides a motivation for the examination of non supersymmetric strings in particular.

Chapters 2-4 are focused on the introduction and development of the various tools that are necessary for our analysis. More specifically, in chapter 2 we present a concise introduction to some of the crucial aspects of string theory that are relevant to our work. We discuss the main features and quantisation of the bosonic string, as well as the 10-dimensional superstring. We then show how their hybridisation brings forth the heterotic string which provides the cornerstone of our analysis. In the same chapter, we include a discussion on string interactions, and outline the basic components of the string partition function, the calculation of which is critical to our analysis.

In chapter 3, we introduce the two main formalisms whose combined application is the basis of our analysis. The free fermionic formulation is outlined and we showcase some of the tools that it offers by explicitly analysing and classifying all heterotic string theories in ten dimensions. We also remark on the finiteness of string theories which do not exhibit spacetime supersymmetry, which is ensured by misaligned supersymmetry. We conclude the chapter by providing a technical overview of the specific toroidal lattices we employ in the orbifold compactifications describing our four-dimensional models, including their connection to the fermionic formulation for specific values of the moduli.

In chapter 4, we focus on a class of four-dimensional toy models exhibiting an unbroken $SO(10)$ gauge symmetry. While these cannot provide a realistic picture of low energy physics, we take advantage of their relative simplicity in order to explicitly present our model-building methods and highlight the equivalency of the free fermionic and orbifold descriptions. The key ingredients of the massless spectrum of these models which continue to be relevant in Pati–Salam constructions are briefly examined. We then provide the main arguments from which the conditions that ensure both the spontaneous breaking of supersymmetry á la Scherk–Schwarz, as well as the exponential suppression of the cosmological constant can be derived.

Having both set up the theoretical framework, and presented all the technical tools that we employ in our analysis, in chapter 5 we finally proceed to the investigation of four-dimensional non supersymmetric models with an observable Pati–Salam gauge symmetry. In order to restrict our analysis to a subset of vacua whose exhaustive scan is within our technical capabilities, we consider the special case in which all worldsheet fermions parametrising the compactified dimensions are complex. This choice inadvertently introduces a four-fold degeneracy in the string spectrum, preventing the construction of models with three fermion generations. While these models cannot be considered fully realistic, they still reproduce a series of phenomenological characteristics that are of significant interest. In order to probe the moduli dependence of the models, we fix the moduli of two tori to their fermionic point values, and consider deformations in the T_2 direction of the Scherk–Schwarz lattice.

We begin the analysis of the aforementioned models by deriving conditions which ensure the elimination of all physical tachyons from the string spectrum. We then outline all the massless sectors of the theory, deriving model-independent conditions that allow us to scan for models whose massless spectrum includes chiral fermions and the Higgs bosons required for the spontaneous breaking of the Pati–Salam and SM gauge symmetries, while avoiding enhancements of the observable gauge symmetry and allowing the possibility of giving large masses to all exotic fermions. We perform a comprehensive scan over the full parameter space of vacua and identify those which in addition to the above conditions are also compatible with the Scherk–Schwarz breaking of supersymmetry and exhibit an exponentially suppressed cosmological constant in the large volume regime.

We then proceed with the systematic classification of the $\sim 5 \times 10^5$ models that satisfy all our conditions based on their one-loop effective potential. This classification results in 26 distinct families of models comprising four broad classes, which include the first realisations of super no-scale models with a positive effective potential and a Pati–Salam gauge symmetry. More specifically, two classes comprising $\sim 2.6\%$ of all models possess a positive effective potential, with the fermionic point being either a global maximum (class A), or a local minimum (class B). The third class of models exhibits an oscillating potential in which the fermionic point corresponds to a global minimum at negative values, while the potential of the final class is negative.

The classification of the above models allows for a direct comparison with parent models based on $SO(10)$ gauge symmetry obtained by removing the $SO(10)$-breaking projections. This comparison results in no correlation in the super no-scale property nor the main features of the effective potential between the two classes, implying that both properties are highly sensitive to the rearrangement of the string spectrum that takes place when the gauge symmetry is reduced. The main takeaway from this comparison is that upon further reduction of the observable gauge symmetry or reductions of the hidden sector gauge groups, models exhibiting a negative potential at the Pati–Salam level might eventually produce positive potential models and as a result should not be excluded from further studies purely based on their effective potential.

We conclude this chapter by presenting specific examples of models with interesting features, providing a detailed definition of each one in terms of the GGSO phases of the free fermionic formulation and the orbifold phase. We present their massless spectra, as well as the expansion of the partition function at the fermionic point and provide a numeric evaluation of the one-loop effective potential. Two of these examples correspond to models which in addition to satisfying all conditions we imposed, exhibit a positive potential. The potential of the first model features a global maximum at the fermionic point, while in the second one the self-dual point is a local minimum. The third example, exhibiting a negative potential, also satisfies all our conditions. It involves enhancements of the hidden sector gauge symmetries whose gauge couplings could grow rapidly, providing a novel solution to the problem of exotic states in the form of confinement.

Lastly, in chapter 6, we move to a more general class of vacua, in which the degeneracies in the massless spectrum can be lifted and three-generation models can be constructed. We repeat the analysis of the tachyonic and massless sectors of the theory, and present the updated form of the constraints we impose on them. In order to lift the degeneracies and obtain models with three fermion generations, the toroidal lattices parametrising the compactification must be modified in a manner which drastically affects the large volume behaviour of the theory. By probing the asymptotics in the large volume regime, we discover that the degeneracy between massless bosons and fermions, commonly used in the literature to ensure the exponential suppression of the cosmological constant, is necessary but not sufficient to do so in such constructions and must be supplemented by an additional condition. We present an explicit example of a three-generation model with spontaneous supersymmetry breaking whose potential exhibits a power-law decay for large radii, despite the fact that its massless spectrum is Bose–Fermi degenerate at generic points.

We identify the supplementary condition needed for the exponential suppression, which we interpret as an additional degeneracy among bosons and fermions that are massive, but become light at large radii and asymptotically massless in the infinite volume supersymmetric limit. We stress that this condition is not an accidental property of the specific Pati–Salam class we analysed, but arises due to the exact form of the compactification lattices which is required in order to obtain three generation models. As a result, we expect this condition to also be relevant in other constructions as well.

We perform a random scan over a large part of the parameter space comprising 1 : 10000 out of all possible configurations and proceed with the identification and classification of models satisfying our phenomenological conditions along the lines outlined above. Though the potential no longer exhibits a T-duality symmetry under the inversion of the T_2 modulus due to the exact form of the compactification lattices and thus produces asymmetrical plots, three main categories of models are still present, them being: (i) models with positive semi-definite potential, (ii) models whose potential oscillates between positive and negative values, and (iii) models with a negative semi-definite potential. In total, we uncover 17 distinct classes of models. The appearance of tachyons in the region $T_2 < 1$ in all but one classes of vacua is notable, though further analysis is required to identify whether this is a peculiarity of this specific class, a statistical fluke due to our inability to perform an exhaustive scan of the parameter space, or a feature owed to either the form of the compactification lattices or the strictness of the super no-scale conditions.

Finally, we present in detail a specific model with spontaneously broken supersymmetry whose massless spectrum includes three fermion generations, the heavy and light Higgs bosons necessary to break the Pati–Salam and SM gauge symmetries, and vector-like pairs of exotic fermions that can acquire large masses. This model notably satisfies both degeneracy conditions and its potential is positive semi-definite and exponentially suppressed in the large volume regime, proving that three-generation super no-scale models are indeed possible.

We conclude this thesis by discussing some potential options for further study. Perhaps the most obvious path for future work is to apply the methods we have developed in the investigation of theories with alternate gauge symmetries, such as flipped-$SU(5)$, left-right symmetric or SM-like vacua. This is well-motivated given the fact that models exhibiting super no-scale structure at generic points of the moduli space have not been explicitly demonstrated in such set-ups. The study of three-generation constructions is especially warranted in light of the additional super no-scale condition we uncovered.

Alternatively, one might proceed with further investigation of the Pati–Salam vacua presented in this work, focusing on additional characteristics which were not covered in this thesis. In particular, it would be interesting to examine the running of the gauge couplings and ascertain whether the decompactification problem can be resolved within the current framework along the lines proposed by [143]. Moreover, investigating the moduli dependence of the vacua in a more general framework, in which deformations of all moduli are taken into account is also an interesting option. While this is a technically challenging endeavour, identifying vacua which exhibit super no-scale structure at the generic point of all moduli would be a seminal result. Further analysis of these models is also motivated by the fact that supersymmetric Pati–Salam models in which all exotic fermions are eliminated from the massless spectrum are known to exist [116, 117], but our analysis of their non supersymmetric counterparts uncovered no such models consistent with all our constraints.

Given the large rate at which the vacuum configurations grow, in conjunction with the strict restrictions imposed by the need for realistic phenomenology, developing improved techniques to aid in the efficient search of string vacua would also be of great benefit to further analyses. Machine learning algorithms in particular have already started to see use in this context [278–282] and their further exploration would be interesting.

Appendix A

Modular Functions and Identities

The Jacobi theta functions with characteristics a, b encode the contribution of fermions to the one-loop vacuum amplitude and are defined as follows:

$$\vartheta[^a_b](z|\tau) = \sum_{n\in\mathcal{Z}} \exp[i\pi\tau(n+a/2)^2 + 2\pi i(n+a/2)(z+b/2)] \ . \tag{A.1}$$

These functions exhibit periodicity in the arguments a, b:

$$\begin{aligned} \vartheta\left[^{a+2}_{\ b}\right](\tau) &= \vartheta[^a_b](\tau) \ , \\ \vartheta[^{\ a}_{b+2}](\tau) &= e^{i\pi a}\vartheta[^a_b](\tau) \ . \end{aligned} \tag{A.2}$$

The theta functions of interest have $z = 0$ and the characteristics a, b can take the values $0, 1$:

$$\begin{aligned} \vartheta[^1_1](0|\tau) &= \vartheta_1(\tau) = e^{i\pi/2}\sum_{n\in\mathbb{Z}}(-1)^n e^{i\pi\tau(n+1/2)^2} \ , \\ \vartheta[^1_0](0|\tau) &= \vartheta_2(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau(n+1/2)^2} \ , \\ \vartheta[^0_0](0|\tau) &= \vartheta_3(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2} \ , \\ \vartheta[^0_1](0|\tau) &= \vartheta_4(\tau) = \sum_{n\in\mathbb{Z}}(-1)^n e^{i\pi\tau n^2} \ . \end{aligned} \tag{A.3}$$

Since our interest in the theta functions stems from their potential to describe fermionic contributions to the string partition function, it is important to see how they behave under modular transformations. The T-transformation, $\tau \to \tau + 1$ of the theta functions yields:

$$\vartheta[^a_b](\tau+1) = e^{-i\pi a(a-2)/4}\vartheta[^{\quad a}_{a+b-1}](\tau) \tag{A.4}$$

Under the S-transformation, $\tau \to -1/\tau$, we have:

$$\vartheta[^a_b](-1/\tau) = \sqrt{-i\tau}e^{i\pi ab/2}\vartheta[^{-b}_{\ a}](\tau) \tag{A.5}$$

The modular transformations of the relevant functions ϑ_1, ϑ_2, ϑ_3 and ϑ_4 can easily be obtained by substituting the corresponding values of a and b in equations (A.4) and (A.5):

$$\begin{aligned} \vartheta_1(\tau+1) &= e^{i\pi/4}\vartheta_1(\tau) \ , \quad & \vartheta_1(-1/\tau) &= i\sqrt{-i\tau}\vartheta_1(\tau) \ , \\ \vartheta_2(\tau+1) &= e^{i\pi/4}\vartheta_2(\tau) \ , \quad & \vartheta_2(-1/\tau) &= \sqrt{-i\tau}\vartheta_4(\tau) \ , \\ \vartheta_3(\tau+1) &= \vartheta_4(\tau) \ , & \vartheta_3(-1/\tau) &= \sqrt{-i\tau}\vartheta_3(\tau) \ , \\ \vartheta_4(\tau+1) &= \vartheta_3(\tau) \ , & \vartheta_4(-1/\tau) &= \sqrt{-i\tau}\vartheta_2(\tau) \ . \end{aligned} \tag{A.6}$$

The theta functions satisfy interesting identities. One of the most important ones is Jacobi's "abstruse identity":

$$\vartheta_3^4(\tau) - \vartheta_2^4(\tau) - \vartheta_4^4(\tau) = 0 \ . \tag{A.7}$$

This identity appears in the GSO projection and is what guarrantees that the partition function of the supersymmetric string theories vanishes.

Another very useful identity is the triple product identity, which connects the Jacobi functions to the Dedekind eta function:

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\eta^3(\tau)\ . \tag{A.8}$$

The Dedekind eta function encodes the contribution of the worldsheet bosons and is defined as:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\ . \tag{A.9}$$

We can also cast it in the form of a sum:

$$\eta(\tau) = e^{\pi i\tau/12}\sum_{n=-\infty}^{\infty}(-1)^n e^{\pi i\tau n(3n+1)}\ . \tag{A.10}$$

Under modular transformations, it has the following properties:

$$\begin{aligned}\eta(\tau+1) &= e^{i\pi/12}\eta(\tau)\ ,\\ \eta(-1/\tau) &= \sqrt{-i\tau}\,\eta(\tau)\ .\end{aligned} \tag{A.11}$$

In order to calculate the partition functions of models in the free fermionic formulation,it is convenient to express the eta and theta functions as expansions in terms of $q = e^{2\pi i\tau}$:

$$\begin{aligned}\eta(q) &= q^{1/12} + \sum_{n=1}^{\infty}(-1)^n q^{3n^2+1/12}\left(q^n + q^{-n}\right)\ ,\\ \vartheta[{}^a_b](q) &= q^{a^2/4}e^{i\pi ab/2} + \sum_{n=1}^{\infty}\left[q^{(n+a/2)^2}e^{i\pi(n+a/2)b} + q^{(n-a/2)^2}e^{i\pi(-n+a/2)b}\right]\ .\end{aligned} \tag{A.12}$$

Instead of q, it is sometimes useful to expand these functions in terms of $q_r = e^{-2\pi\tau_2}$ and $q_i = e^{2\pi i\tau_1}$, where τ_1 and τ_2 are the real and imaginary parts of τ. Then $q = q_r\cdot q_i$ and the expansions are:

$$\begin{aligned}\eta(q_r,q_i) &= (q_r\cdot q_i)^{1/12} + \sum_{n=1}^{\infty}(-1)^n(q_r\cdot q_i)^{3n^2+1/12}\left[(q_r\cdot q_i)^n + (q_r\cdot q_i)^{-n}\right]\ ,\\ \vartheta[{}^a_b](q_r,q_i) &= (q_r\cdot q_i)^{a^2/4}e^{i\pi ab/2} + \sum_{n=1}^{\infty}\left[(q_r\cdot q_i)^{(n+a/2)^2}e^{i\pi(n+a/2)b} + (q_r\cdot q_i)^{(n-a/2)^2}e^{i\pi(-n+a/2)b}\right]\ .\end{aligned} \tag{A.13}$$

The complex conjugate functions $\bar{\eta}$ and $\bar{\vartheta}$ can be obtained by taking the complex conjugate of equations ($A.12$) or ($A.13$). In that case, q simply becomes $\bar{q}$ and the product $= q_r\cdot q_i$ is replaced by q_i/q_r.

Appendix B

Free Fermionic Vector Bases and 10-dimensional Heterotic String Theories

In this appendix, we summarise the 10-dimensional heterotic string theories constructed by the dimension-2 and dimension-3 vector bases utilising the vectors $\mathbb{1}$, S, z_1, z_2 and z_3 following the definitions of (3.29), (3.34), (3.41), (3.51), and (3.76) respectively.

	Basis Vectors	Gauge Symmetry	SUSY	Tachyons	Fermions
1	$\mathbb{1}, S$	$SO(32)$	YES	0	496
2	$\mathbb{1}, z_1$	$E_8 \times SO(16)$	NO	16	256
3	$\mathbb{1}, z_2$	$SO(32)$	NO	32	0
4	$\mathbb{1}, S + z_1$	$E_8 \times SO(16)$	NO	16	256
5	$\mathbb{1}, S + z_2$	$SO(24) \times SO(8)$	NO	8	384
7	$\mathbb{1}, z_1 + z_2$	$SO(32)$	NO	32	0
8	$\mathbb{1}, S + z_1 + z_2$	$SO(24) \times SO(8)$	NO	8	384

Table B.1: *All possible dimension 2 vector bases and the corresponding models they describe.*

	Basis Vectors	Gauge Symmetry	SUSY	Tachyons	Fermions
1	$\mathbb{1}, S, z_1$	$E_8 \times E_8$	YES	0	496
		$SO(16) \times SO(16)$	NO	0	512
2	$\mathbb{1}, S, z_2$	$SO(24) \times SO(8)$	NO	8	384
3	$\mathbb{1}, S, z_1 + z_2$	$E_8 \times E_8$	YES	0	496
		$SO(16) \times SO(16)$	NO	0	512
4	$\mathbb{1}, S, S + z_1$	$E_8 \times E_8$	YES	0	496
		$SO(16) \times SO(16)$	NO	0	512
5	$\mathbb{1}, S, S + z_2$	$SO(24) \times SO(8)$	NO	8	384
		$SO(32)$	YES	0	496
6	$\mathbb{1}, S, S + z_1 + z_2$	$E_8 \times SO(16)$	NO	16	256
7	$\mathbb{1}, z_1, z_2$	$E_8 \times SO(16)$	NO	16	256
8	$\mathbb{1}, S + z_1, z_1$	$E_8 \times E_8$	YES	0	496
		$SO(16) \times SO(16)$	NO	0	512
9	$\mathbb{1}, S + z_1, z_2$	$E_8 \times SO(16)$	NO	16	256
10	$\mathbb{1}, S + z_1, z_1 + z_2$	$E_7^2 \times SU(2)^2$	NO	4	448
11	$\mathbb{1}, S + z_1, S + z_2$	$E_7^2 \times SU(2)^2$	NO	4	448
12	$\mathbb{1}, S + z_1, S + z_1 + z_2$	$E_8 \times SO(16)$	NO	16	256
13	$\mathbb{1}, S + z_2, z_1$	$E_7^2 \times SU(2)^2$	NO	4	448
14	$\mathbb{1}, S + z_2, z_2$	$SO(24) \times SO(8)$	NO	8	384
		$SO(32)$	YES	0	496
15	$\mathbb{1}, S + z_2, z_1 + z_2$	$E_7^2 \times SU(2)^2$	NO	4	448
16	$\mathbb{1}, S + z_2, S + z_1 + z_2$	$E_7^2 \times SU(2)^2$	NO	4	448
17	$\mathbb{1}, z_1 + z_2, z_1$	$E_8 \times SO(16)$	NO	16	256
18	$\mathbb{1}, z_1 + z_2, z_2$	$E_8 \times SO(16)$	NO	16	256
19	$\mathbb{1}, z_1 + z_2, S + z_1 + z_2$	$SO(24) \times SO(8)$	NO	8	384
		$SO(32)$	YES	0	496
20	$\mathbb{1}, S + z_1 + z_2, z_1$	$E_7^2 \times SU(2)^2$	NO	4	448
21	$\mathbb{1}, S + z_1 + z_2, z_2$	$E_8 \times SO(16)$	NO	16	256

Table B.2: *All possible dimension 3 vector bases and the corresponding models they describe.*

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