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Konstantinos-Manthos Vlachos

Deformations of symplectomorphisms by mean curvature flow

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Andreas Savas-Halilaj	Assistant Professor (Supervisor)
Stavros Papadakis	Assistant Professor
Nikolaos Roidos	Assistant Professor

The acceptance of this dissertation by the Department of Mathematics of the University of Ioannina does not imply the approval of the opinions of the author (section 202 par. 2 Law 5343/1932 and section 50 par. 8 Law 1268/1982).

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Dedicated to my family

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Abstract

According to a beautiful result of Gromov [10] any symplectomorphism of $\mathbb{CP}^2$ can be deformed into a biholomorphic isometry of $\mathbb{CP}^2$. Medoš and Wang [22] applied the mean curvature flow (MCF) method to deform a symplectomorphism of $\mathbb{CP}^m$ with $m \geq 2$. Roughly speaking they proved that if $f : \mathbb{CP}^m \to \mathbb{CP}^m$ is a symplectomorphism which is close to a biholomorphic isometry, then the MCF will smoothly deform f into a biholomorphic isometry. The purpose of this Master Thesis is to analyse the work of Medoš and Wang [22] and prove the following:

Main Theorem: *There exists a number $\varepsilon(m) > 1$, which depends only on the dimension $m \in \mathbb{N}$, such that if f is a symplectomorphism of $\mathbb{CP}^m$, with the property*

$$\varepsilon^{-2}(m) < |df|^2 < \varepsilon^2(m),$$

then the (MCF) *smoothly deforms f into a biholomorphic isometry of $\mathbb{CP}^m$.*

ΠΕΡΙΛΗΨΗ

Σε αυτή τη μεταπτυχιακή εργασία θα δούμε πως μπορεί να χρησιμοποιηθεί η ροή μέσης καμπυλότητας για την απόδειξη τοπολογικών αποτελεσμάτων. Εικάζεται ότι: *κάθε συμπλεκτομορφισμός του μιγαδικού προβολικού χώρου $\mathbb{CP}^m$ δύναται να παραμορφωθεί με συνεχή τρόπο σε ολόμορφη ισομετρία του $\mathbb{CP}^m$*. Για $m = 1$ και για $m = 2$ η παραπάνω εικασία έχει αποδειχθεί από τους Smale [31] και Gromov [10], αντίστοιχα. Στο κεντρικό θεώρημα της διατριβής θα αναλύσουμε μια εργασία των Medoš και Wang [22] όπου αποδεικνύεται το εξής αποτέλεσμα:

Κεντρικό Θεώρημα: *Υπάρχει αριθμός $\varepsilon(m) > 1$, που εξαρτάται μόνο από τη διάσταση $m \in \mathbb{N}$, έτσι ώστε εάν $f : \mathbb{CP}^m \to \mathbb{CP}^m$ είναι συμπλεκτομορφισμός με*

$$\varepsilon^{-2}(m) < |df|^2 < \varepsilon^2(m),$$

τότε η ροή μέσης καμπυλότητας παραμορφώνει με λείο τρόπο την f σε μια ολόμορφη ισομετρία του μιγαδικού προβολικού χώρου $\mathbb{CP}^m$.

Contents

CHAPTER 1

Complex Differential Geometry

In this section we set up the notation and will quickly review some basic facts from Riemannian, Kählerian, and submanifold geometry. We closely follow the exposition in [2], [5], [6], [7], [17] and [46].

1.1 Connections and curvature

1.1.1 Riemannian manifolds. Let M be a smooth connected without boundary manifold of dimension m. We denote the tangent space of M at a point $x \in M$ by T_xM and the space of smooth functions of M by $C^\infty(M)$. It is well-known that any manifold admits a Riemannian metric g. When there is no possibility of confusion, we denote the metric g simply by $\langle \cdot\,,\cdot \rangle$. To the metric g we can assign a unique linear connection ∇ which is torsion free and compatible with g, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

and

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of all smooth vector fields on M. The associated connection ∇ is called the *Levi-Civita connection* and is given explicitly by the Koszul formula

$$\begin{aligned} 2\langle \nabla_Y X, Z\rangle \; = \; & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & -\langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle, \end{aligned} \tag{1.1}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

The *curvature tensor* R of a Riemannian manifold M is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(M)$ a mapping $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where ∇ is the Levi-Civita connection of M. Multiplying with g we get a 4-tensor which, for simplicity, we denote with the same symbol, i.e.

$$R(X,Y,Z,W) = -\langle R(X,Y)Z, W \rangle.$$

Let X, Y be two linearly independent tangent vectors at a point x on M. The *sectional curvature* K, for the plane spanned by X and Y, is defined by

$$K(X,Y) = \frac{R(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X, Y \rangle^2}.$$

Suppose that $\{e_1, \ldots, e_m\}$ is a local orthonormal frame defined on an open neighbourhood of M. Then,

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{m} R(X, e_i, Y, e_i)$$

defines a symmetric 2-tensor, which is called the *Ricci tensor*. We say that M is *Einstein*, if

$$\mathrm{Ric} = kg,$$

for some constant k. Taking the trace of the Ricci tensor we obtain the *scalar curvature* Sc by

$$\mathrm{Sc} = \sum_{i=1}^{m} \mathrm{Ric}(e_i, e_i).$$

Let $f \in C^\infty(M)$. The *gradient* ∇f is defined to be the vector field given by

$$\langle \nabla f, X \rangle = df(X),$$

for every $X \in \mathfrak{X}(M)$. The *Hessian* $\nabla^2 f$ is given by

$$\nabla^2 f(X,Y) = \langle \nabla_X \nabla f, Y \rangle,$$

for every $X, Y \in \mathfrak{X}(M)$, and the *Laplacian* Δf is defined by

$$\Delta f = \sum_{i=1}^{m} \nabla^2 f(e_i, e_i).$$

1.1.2. Vector bundles. Often we will need to explore how tensorial quantities vary along a manifold. The best way to formulate the concept of derivatives of tensorial quantities is through the theory of vector bundles. Roughly speaking, a vector bundle is a geometric construction that makes precise the idea of a family of vector spaces parametrised by a manifold M such that to every point $x \in M$ we attach a vector space E_x so that these vector spaces fit together to form another manifold. The precise definition of a vector bundle is the following.

Definition 1.1.1. *Let E and M be smooth manifolds and $\pi : E \to M$ a smooth surjective map. The triple (E, π, M) is a smooth real vector bundle of rank k, or simply a vector bundle, if for each $x \in M$, the following conditions are satisfied:*

(a) *For any point $x \in M$, the set $E_x = \pi^{-1}(x)$ possesses the structure of a k-dimensional real vector space. The space E_x is called the fiber of E over the point x.*

(b) *For any point $x \in M$, there exists an open neighbourhood U of x in M and a diffeomorphism*

$$\varphi : U \times \mathbb{R}^k \to \pi^{-1}(U),$$

with the property $\varphi(y, \xi) \in E_y$, for any $(y, \xi) \in U \times \mathbb{R}^k$. The map φ is called a local trivialisation of E.

(c) *For any point $x \in U$, the map $\varphi_x : \mathbb{R}^k \to E_x$ given by*

$$\varphi_x(\xi) = \varphi(x, \xi)$$

is a $\mathbb{R}$-linear isomorphism.

The space E is called the total space of the bundle, M is called its base, and π its projection. For simplicity, we usually denote a vector bundle only by E.

Definition 1.1.2. *Let E be a vector bundle over a manifold M. A n-dimensional submanifold $F \subset E$ is called subbundle of rank n over M if $(F, \pi_{|_F}, M)$ is a vector bundle of rank n over M. Here $\pi_{|_F}$ denotes the restriction of the projection map $\pi : E \to M$ on F.*

Let us now introduce the notion of a section that, roughly speaking, might be considered as a generalisation of a smooth vector field on the tangent bundle of a manifold.

Definition 1.1.3. *A section on a vector bundle (E, π, M) is a smooth map $\sigma : M \to E$ such that*

$$\pi \circ \sigma = I,$$

where I is the identity map. We often denote the value $\sigma(x)$ simply by σ_x.

The set of sections of a vector bundle is an infinite-dimensional vector space under point-wise addition and multiplication by constants, whose zero element is the zero section. This set is denoted by $\Gamma(E)$. More precisely, $\Gamma(E)$ is a module over $C^\infty(M)$. There exists a natural way of differentiation on vector bundles.

The investigation of geometric properties of vector bundles requires the notion of the differentiation. Here we shall give the basic facts about metrics and connections associated to them.

Definition 1.1.4. *Let E be a vector bundle over M. A linear connection on E is a map*

$$\nabla^E : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

denoted by

$$(X, \sigma) \mapsto \nabla^E_X \sigma,$$

satisfying the following properties:

(a) *For every $X, Y \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$, it holds*

$$\nabla^E_{X+Y}\sigma = \nabla^E_X \sigma + \nabla^E_Y \sigma.$$

(b) *For every $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$, it holds*

$$\nabla^E_{fX}\sigma = f\nabla^E_X \sigma.$$

(c) *For every $X \in \mathfrak{X}(M)$ and $\sigma_1, \sigma_2 \in \Gamma(E)$, it holds*

$$\nabla^E_X(\sigma_1 + \sigma_2) = \nabla^E_X \sigma_1 + \nabla^E_X \sigma_2.$$

(d) *For every $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$, it holds*

$$\nabla^E_X(f\sigma) = (Xf)\sigma + f\nabla^E_X \sigma.$$

The usual directional derivative in the euclidean space $\mathbb{R}^m$ is a connection. Concerning this connection, any constant vector field on the euclidean space $\mathbb{R}^m$ is parallel. Hence, we give the following general definition.

Definition 1.1.5. *A section $\sigma \in \Gamma(E)$ is said to be parallel with respect to the connection ∇^E if*

$$\nabla^E_X \sigma = 0,$$

for any vector field X on M.

Definition 1.1.6. *Suppose that M is a smooth manifold and (E, π, M) a vector bundle over M. Let ∇^M be a connection of TM and ∇^E a connection on E. For any pair of vector fields $X, Y \in \mathfrak{X}(M)$, the map*

$$\nabla^2_{X,Y} : \Gamma(E) \to \Gamma(E)$$

defined by,

$$\nabla^2_{X,Y}\sigma = \nabla^E_X \nabla^E_Y \sigma - \nabla^E_{\nabla^M_X Y}\sigma,$$

is called the second covariant derivative of σ with respect to the directions X, Y. By coupling the connections ∇^M and ∇^E, one may define similarly, the k-th derivative ∇^k of a section σ in $\Gamma(E)$.

To each connection, we associate an important operator which measures the non-commutativity of the second covariant derivative.

Definition 1.1.7. *The operator $R^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, defined by the formula*

$$R^\nabla(X, Y)\sigma = \nabla^2_{X,Y}\sigma - \nabla^2_{Y,X}\sigma,$$

for any $X, Y \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$, is called the curvature operator of the connection ∇.

Now we give the definition of a Riemannian metric on a vector bundle.

Definition 1.1.8. *A Riemannian metric on the vector bundle (E, π, M) over the manifold M is a map*

$$\mathrm{g}_E : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$$

such that its restriction to the fibers is a positive definite inner product. As usual, we occasionally denote Riemannian metrics by $\langle \cdot\,, \cdot \rangle$.

It is well-known that any vector bundle admits a Riemannian metric. The proof uses the partition of unity to glue local Riemannian metrics on each fiber. A connection ∇ is called compatible with the Riemannian metric if it satisfies the product rule

$$X\mathrm{g}_E(\omega,\sigma) = \mathrm{g}_E(\nabla_X\omega,\sigma) + \mathrm{g}_E(\omega,\nabla_X\sigma),$$

for any $X \in \mathfrak{X}(M)$ and $\omega, \sigma \in \Gamma(E)$. A vector bundle equipped with both these structures is called *Riemannian vector bundle endowed with a compatible linear connection.*

The most simple vector bundle over a given manifold M is the trivial vector bundle $M \times \mathbb{R}^k$. However, there is a plethora of non-trivial vector bundles. As a matter of fact, one can use the operations of Linear Algebra to produce new vector bundles from given ones. Let us briefly see the most important examples of vector bundles that we will need in this thesis.

Example 1.1.9 (The direct product). Let (E, π_1, M) and (F, π_2, N) be two vector bundles over the manifolds M and N.

- The direct product $E \otimes F$ is the vector bundle over the manifold $M \times N$ whose total space the manifold $E \times F$ and projection map π given by

  $$\pi(\sigma,\omega) = (\pi_1(\sigma), \pi_2(\omega)) \in M \times N.$$

  Note that

  $$(E \otimes F)_{(x,y)} = E_x \times F_y,$$

  for any $(x, y) \in M \times N$. Given $\sigma \in \Gamma(E)$ and $\omega \in \Gamma(F)$, the map $\sigma \otimes \omega$ given by

  $$(\sigma \otimes \omega)(x, y) = (\sigma_x, \omega_y)$$

  is clearly a section of $E \oplus F$.

- If ∇^E is a connection on the bundle E and ∇^F a connection on the bundle F, then the map $\nabla^{E\otimes F}$ given by

  $$\nabla^{E\otimes F}_X(\sigma \otimes \omega) = (\nabla^E_X\sigma) \otimes \omega + \sigma \otimes (\nabla^F_X\omega),$$

  where $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$ and $\omega \in \Gamma(F)$, consist a connection of the product $E \otimes F$.

- The curvature operator $R^{E\otimes F}$ of the linear connection $\nabla^{E\otimes F}$ is given by the formula

$$R^{E\otimes F}(X,Y)\sigma\otimes\omega = \big(R^E(X,Y)\sigma\big)\otimes\omega + \sigma\otimes\big(R^F(X,Y)\omega\big),$$

where R^E and R^F are the curvature operators associated with ∇^E and ∇^F, respectively.

- If g_E and g_F are Riemannian metrics that are compatible with E and F, respectively, then

$$g_{E\otimes F}(\sigma_1\otimes\omega_1, \sigma_2\otimes\omega_2) = g_E(\sigma_1,\sigma_2)\cdot g_F(\omega_1,\omega_2),$$

where $\sigma_1,\sigma_2 \in \Gamma(E)$ and $\omega_1,\omega_2 \in \Gamma(F)$, forms a Riemannian metric compatible with the linear connection $\nabla^{E\otimes F}$.

Example 1.1.10 (The Whitney sum). Let (E,π_1,M) and (F,π_2,M) be two vector bundles over the same manifold M.

- The *Whitney sum* $E\oplus F$ is the vector bundle over the manifold M whose total space is the set

$$E\oplus F = \{(\sigma,\omega)\in E\times F : \pi_1(\sigma) = \pi_2(\omega)\} \subset E\times F,$$

and with projection π given by

$$\pi(\sigma,\omega) = \pi_1(\sigma) = \pi_2(\omega).$$

Observe that for any point $x\in M$, we have that $(E\oplus F)_x = E_x\oplus F_x$. We denote sections of the bundle $E\oplus F$ by $\sigma\oplus\omega$, where $\sigma\in\Gamma(E)$ and $\omega\in\Gamma(F)$. Note that the total space of the direct sum certainly is not $E\otimes F$. The latter consists of all pairs (σ,ω) such that $\sigma\in E_x$ and $\omega\in F_y$ for any $(x,y)\in M\times M$, while $(\sigma,\omega)\in E\oplus F$ if and only if $x=y$, i.e. σ and ω are in fibers over the same point of the base.

- If E and F are endowed with linear connections ∇^E and ∇^F, respectively, then the map $\nabla^{E\oplus F}$ given by

$$\nabla^{E\oplus F}_X(\sigma\oplus\omega) = (\nabla^E_X\sigma)\oplus(\nabla^F_X\omega),$$

is the natural connection of the Whitney sum $E\oplus F$.

- The curvature operator associated with $\nabla^{E\oplus F}$ is given by

$$R^{E\oplus F}(X,Y)\sigma\oplus\omega = \big(R^E(X,Y)\sigma\big)\oplus\big(R^F(X,Y)\omega\big),$$

  where R^E and R^F are the curvature operators associated with ∇^E and ∇^F, respectively.

- If g_E is a Riemannian metric of E and g_F is a Riemannian metric of F, then the map $g_{E\oplus F}$ given by

$$g_{E\oplus F}((\sigma_1,\omega_1),(\sigma_2,\omega_2)) = g_E(\sigma_1,\sigma_2)+g_F(\omega_1,\omega_2)$$

  is a Riemannian metric of the vector bundle $E\oplus F$.

- If in addition ∇^E is a connection compatible with g_E and ∇^F is compatible with g_F, then $\nabla^{E\oplus F}$ is compatible with $g_{E\oplus F}$.

Example 1.1.11 (The dual bundle). Let (E,π,M) be a vector bundle of rank k over a manifold M endowed with a linear connection ∇^E.

- The *dual bundle* E^* is the vector bundle over M with total space

$$E^* = \cup_{x\in M}E^*_x,$$

  and with projection the map π^* given by

$$\pi^*(x,\sigma)=x.$$

- A natural connection ∇^{E^*} on E^* is given by

$$(\nabla^{E^*}_X L)\sigma := X\{L(\sigma)\} - L(\nabla^E_X\sigma),$$

  for any $X\in\mathfrak{X}(M)$, $L\in\Gamma(E^*)$ and $\sigma\in\Gamma(E)$.

- Suppose that E is endowed with a metric g_E that is compatible with ∇^E and g is a Riemannian metric on M. Define $g_{E^*}: E^*_x\times E^*_x\to\mathbb{R}$ given by

$$g_{E^*}(L_x,T_x)=\sum_{i=1}^k L_x(\sigma_i)\cdot T_x(\sigma_i),$$

  where $\{\sigma_1,\dots,\sigma_k\}$ is a local orthonormal frame of E_x with respect to g_E. One can easily check that g_E gives rise to a Riemannian metric on the dual bundle that is compatible with ∇^{E^*}.

Example 1.1.12 (The homomorphism bundle). Let (E, π_1, M) be a vector bundle of rank k and (V, π_2, M) a vector bundle of rank l over the manifold M endowed with linear connections ∇^E and ∇^V, respectively.

- The homomorphism bundle $\mathrm{Hom}(E^r; V)$, of r-copies $E^r = E \times \cdots \times E$ of E to V, is the vector bundle with total space

$$\mathrm{Hom}(E^r; V) = \cup_{x \in M} \mathrm{Hom}(E^r_x; \mathbb{R}^l).$$

  The projection map is given by

$$\pi(x, \sigma) = x.$$

- A natural connection ∇^{Hom} on the homomorphism bundle is given by

$$\begin{aligned}(\nabla^{\mathrm{Hom}}_X T)(\sigma_1, \dots, \sigma_r) = \nabla^V_X \{T(\sigma_1, \dots, \sigma_r)\} \\ -T(\nabla^E_X \sigma_1, \dots, \sigma_r) - \cdots - T(\sigma_1, \dots, \nabla^E_X \sigma_r),\end{aligned}$$

  for any $X \in \mathfrak{X}(M), T \in \Gamma(\mathrm{Hom}(E^r, V))$ and $\sigma_1, \dots, \sigma_r \in \Gamma(E)$.

- Let g_E and g_V be Riemannian metrics which are compatible with the connections ∇^E and ∇^V. Then a natural metric on Hom that is compatible with ∇^{Hom} is given by

$$g_{\mathrm{Hom}}(T_x, P_x) = \sum_{i_1, \dots, i_r = 1}^{k} g_V\big(T(\sigma_{i_1}, \dots, \sigma_{i_r}), P(\sigma_{i_1}, \dots, \sigma_{i_r})\big),$$

  where $\{\sigma_1, \dots, \sigma_k\}$ is an orthonormal basis at x with respect to g_E.

Example 1.1.13 (The pull-back bundle). Let M and N be two manifolds, let (E, π, N) be a vector bundle of rank k over N and $f : M \to N$ a smooth map. The map f induces a new vector bundle of rank k over M.

- Take as total space the set

$$f^*E = \big\{(x, \xi) : x \in M \text{ and } \xi \in E_{f(x)}\big\},$$

  and as projection the map $\pi_f : f^*E \to M$ given by

$$\pi_f(x, \xi) = x.$$

  The space f^*E contains all sections of E with base point at $f(M)$.

- Let ∇^M and ∇^E be linear connections on TM and E, respectively. Let $\{\varphi_1, \dots, \varphi_k\}$ be a frame field of E in a neighborhood of $f(x) \in N$. Then, any section $\sigma \in \Gamma\big(f^*E\big)$ can be written in the form

$$\sigma(x) = \left(x, \sum_{\alpha=1}^{k} \sigma^\alpha(x)(\varphi_\alpha \circ f)(x)\right) \cong \sum_{\alpha=1}^{k} \sigma^\alpha(x)(\varphi_\alpha \circ f)(x),$$

where σ^α, $\alpha \in \{1, \dots, k\}$, are the components of σ with respect to the given frame field. These functions are defined in a neighborhood of M and they are smooth. Define now,

$$\nabla^f_X \sigma = \sum_{\alpha=1}^{k} (X\sigma^\alpha)\varphi_\alpha \circ f + \sum_{\alpha=1}^{k} \sigma^\alpha(\nabla^E_{df(X)}\varphi_\alpha) \circ f,$$

for any $X \in \mathfrak{X}(M)$. One can easily verify that the above definition of the pull-back connection is independent of the choice of the frame field.

- The curvature operator R^f of the pull-back bundle is given by

$$R^f(X,Y)\sigma = R^E(df(X), df(Y))\sigma,$$

for any $X, Y \in T_xM$ and $\sigma \in \Gamma(f^*E)$.

- In the case $E = TN$, the following formula holds

$$\nabla^f_X df(Y) - \nabla^f_Y df(X) = df([X,Y]),$$

for any $X, Y \in \mathfrak{X}(M)$.

Example 1.1.14 (Time dependent metric on vector bundles). Let I be an open interval of $\mathbb{R}$. Suppose that $\{g_t\}_{t\in\mathbb{R}}$, is a smooth family of Riemannian metrics on a manifold M. More precisely, for any $(x,t) \in M \times I$ we have an inner product $g_{(x,t)}$ on T_xM. We can regard each g_t as a metric g acting on the spatial tangent bundle

$$\mathcal{H} = \{v \in T(M \times \mathbb{R}) : d\pi_2(v) = 0\},$$

where the map π_2 is the projection on the second component. Observe that each g_t is a metric on $\mathcal{H}$ since $\mathcal{H}_{(x,t)}$ is isomorphic to T_xM via the map π_2. We can extend g into a metric on $M \times I$, for which we have the orthogonal decomposition

$$T(M \times I) = \mathcal{H} \oplus \mathbb{R}\partial_t.$$

Since $\mathcal{H}$ is a subbundle of $T(M \times I)$, any section of $\mathcal{H}$ is a section of $T(M \times I)$. Sections of $\mathcal{H}$ are called spatial vector fields. There is a natural connection ∇ on $M \times I$. Namely, define ∇ by

$$\nabla_X Y = \nabla^{g_t}_X Y,\ \nabla_X \partial_t = 0,\ \nabla_{\partial_t}\partial_t = 0 \text{ and } \nabla_{\partial_t} X = [\partial_t, X], \tag{1.2}$$

for any $X, Y \in \Gamma(\mathcal{H})$, where ∇^{g_t} denotes the Levi-Civita connection of g_t. It is easy to see that ∇ is compatible with g, i.e.

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

for any $X \in \mathfrak{X}(M \times \mathbb{R})$, and $Y, Z \in \Gamma(\mathcal{H})$. Moreover, the connection ∇ is spatially torsion free, i.e

$$\nabla_X Y - \nabla_Y Z = [X, Y],$$

for any $X, Y \in \Gamma(\mathcal{H})$.

Example 1.1.15. We complete this subsection with an important example, where the situation we discussed above occurs. Let N be a manifold equipped with a Riemannian metric g_N. Suppose that $F : M \times I \to N$ is a family of immersions. Then, F^*g_N defines a time-dependent family of Riemannian metrics on M. If we equip $M \times I$ with the above natural connection ∇, for any $X \in \mathfrak{X}(M)$, we get

$$\nabla^{F^*TN}_{\partial_t} dF(X) - \nabla^{F^*TN}_X dF(\partial_t) = dF([\partial_t, X]) = dF(\nabla_{\partial_t} X).$$

1.2 Kählerian manifolds

Let M be a $2m$-dimensional manifold endowed with a Riemannian metric g and associated connection Levi-Civita ∇. An *almost complex* structure on M is a tensor field J of type $(1, 1)$, satisfying

$$J^2 = J \circ J = -I,$$

where I stands for the identity bundle map on TM, i.e. for any $x \in M$ the map $I_x : T_xM \to T_xM$ is the identity. The pair (M, J) is called an *almost complex manifold*. Each tangent space T_xM of an almost complex manifold has a basis of the form $\{e_1, Je_1, \ldots, e_m, Je_m\}$. Such a base is called J-*base*. It turns out that any such two bases differ by an isomorphism with positive determinant. This means that any almost complex manifold is orientable.

Definition 1.2.1. *The triple (M, g, J) is called a Kähler manifold if:*

(a) *The almost complex structure J is an isometry with respect to g, that is*

$$g(JX, JY) = g(X, Y),$$

for any $X, Y \in \mathfrak{X}(M)$.

(b) *The almost complex structure J is parallel with respect to ∇, that is*

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y = 0,$$

for any $X, Y \in \mathfrak{X}(M)$.

It turns out that on a Kähler manifold the 2-form ω, given by

$$\omega(X, Y) = g(JX, Y),$$

for any $X, Y \in \mathfrak{X}(M)$, is closed. We call ω the associated *Kähler form* on M. The *Ricci form* $\mathcal{R}$ is defined by

$$\mathcal{R}(X, Y) = \mathrm{Ric}(JX, Y).$$

Theorem 1.2.2 (Kähler identities). *Let (M, g, J) be a Kähler manifold. Then:*

(a) *The curvature operator R satisfies the identities*

$$R(X, Y)JZ = JR(X, Y)Z \text{ and } R(JX, JY)Z = R(X, Y)Z,$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

(b) *The Ricci tensor* Ric *satisfies the relation*

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}(JX, JY) = -\frac{1}{2}\sum_{k=1}^{m} R(JX, Y, e_k, Je_k),$$

where $X, Y \in \mathfrak{X}(M)$ and $\{e_1, \ldots, e_{2m}\}$ is orthonormal frame with respect to the metric g.

(c) *The Ricci form $\mathcal{R}$ is given by the formula*

$$\mathcal{R}(X, Y) = \mathrm{Ric}(JX, Y) = \sum_{k=1}^{m} R(X, Y, e_k, Je_k),$$

where $X, Y \in \mathfrak{X}(M)$ and $\{e_1, \ldots, e_{2m}\}$ is orthonormal frame with respect to the metric g.

A 2-plane is said to be *complex* if it is invariant by the complex structure J. The restriction of the sectional curvature to a complex plane is called the *holomorphic sectional curvature* and will be denoted by Hol. That is

$$\mathrm{Hol}(X) = K(X, JX),$$

for any non-zero vector field $X \in \mathfrak{X}(M)$.

Theorem 1.2.3. *Let (M, g, ω) be a Kähler manifold with constant holomorphic sectional curvature σ. Then,*

$$\begin{aligned} R(X,Y,Z,W) \;=\; \frac{\sigma}{4}\Big(& g(X,Z)\,g(Y,W) - g(X,W)\,g(Y,Z) \\ & +\omega(X,Z)\,\omega(Y,W) - \omega(X,W)\,\omega(Y,Z) \\ & +2\,\omega(X,Y)\,\omega(Z,W)\Big), \end{aligned}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

1.3 Immersions and submersions

1.3.1 The second fundamental form. Let $f\colon M \to N$ be an immersion, i.e. f smooth whose differential df_x is injective for any x in M. If N has a Riemannian metric g_N, then the immersion f induces a Riemannian metric on M given by

$$(f^* g_N)(X, Y) = g_N(df(X), df(Y)),$$

for all $X, Y \in \mathfrak{X}(M)$. When M is already equipped with a Riemannian metric g, then the map f is called an *isometric immersion* if the induced metric $f^* g_N$ coincides with the metric g. In this case, we say that $f(M)$ is an *immersed submanifold* of N. At any $x \in M$, the ambient space $T_{f(x)}N$ splits as

$$T_{f(x)}N = df_x(T_xM) \oplus N_{f(x)}M, \tag{1.3}$$

where $N_{f(x)}M$ is the orthogonal complement of $df_x(T_xM)$ with respect to the metric g_N. The space

$$N_f M = \cup_{x \in M} N_{f(x)}M,$$

is a vector bundle over M, it is called the *normal bundle* of f with rank equal to $\dim N - \dim M$. The restriction of g_N on $N_f M$ gives a Riemannian metric on the normal bundle. Then the splitting given in equation (1.3) becomes orthogonal.

From now on let us assume that $f : M \to N$ is an isometric immersion, denote by g the Riemannian metric on M, by ∇ the Levi-Civita connection associated with g, and by g_N is the Riemannian metric on N. Any section $V \in \Gamma(f^*TN)$ can be uniquely decomposed in a unique way as

$$V = V^\top + V^\perp,$$

where $\{\cdot\}^\top$ stands for the orthogonal projection on the tangent bundle and $\{\cdot\}^\perp$ denotes the orthogonal projection on the normal bundle of the submanifold. Then, the natural connection on the normal bundle is given by

$$\nabla^\perp_X \xi = \big(\nabla^{f^*TN}_X \xi\big)^\perp,$$

and its associated curvature tensor $R^\perp$ is

$$R^\perp(X,Y)\xi = \nabla^\perp_X \nabla^\perp_Y \xi - \nabla^\perp_Y \nabla^\perp_X \xi - \nabla^\perp_{[X,Y]}\xi,$$

where $X, Y \in \mathfrak{X}(M)$ and $\xi \in N_f M$. Multiplying with the Riemannian metric on the normal bundle, we can form from $R^\perp$ a $C^\infty(M)$-valued tensor which, by abuse of notation, we denote again by $R^\perp$, i.e. we set

$$R^\perp(X,Y,\xi,\eta) = -\langle R^\perp(X,Y)\xi, \eta\rangle,$$

for any $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in N_f M$. It is a well-known fact in submanifold geometry that, for any $X, Y \in \mathfrak{X}(M)$, we have the decomposition

$$\nabla^{f^*TN}_X df(Y) = df(\nabla_X Y) + A(X,Y),$$

where A is the *second fundamental form* of f. Note that A is a symmetric tensor which takes values on the normal bundle of the submanifold. If ξ is a normal vector, then the symmetric 2-tensor A^ξ given by

$$A^\xi(X,Y) = \langle A(X,Y), \xi\rangle,$$

for any tangent vector fields X, Y, is called the *shape operator* with respect to the direction ξ. The *Weingarten operator* A_ξ associated with ξ is defined by

$$\langle A_\xi X, Y\rangle = A^\xi(X,Y) = \langle A(X,Y), \xi\rangle,$$

for any $X, Y \in \mathfrak{X}(M)$. Finally, the trace H of A with respect to the metric g, is called the *mean curvature vector field*. A submanifold with zero mean curvature is called *minimal*.

The curvature tensor R of M, the curvature tensor $\widetilde{R}$ of the manifold N, and the normal curvature $R^\perp$ are related to the second fundamental form through the *Gauss-Codazzi-Ricci equations:*

(a) **Gauss equation:**

$$R(X,Y,Z,W) = \widetilde{R}\big(df(X), df(Y), df(Z), df(W)\big) + \langle A(X,Z), A(Y,W)\rangle - \langle A(Y,Z), A(X,W)\rangle, \tag{1.4}$$

(b) **Codazzi equation:**

$$\big(\nabla^{\perp}_{X} A\big)(Y,Z) - \big(\nabla^{\perp}_{Y} A\big)(X,Z) = \big\{\widetilde{R}(df(X), df(Y))df(Z)\big\}^{\perp}, \tag{1.5}$$

(c) **Ricci equation:**

$$R^{\perp}(X,Y,\xi,\eta) = \widetilde{R}\big(df(X), df(Y), \xi, \eta\big) + \sum_{k=1}^{m} \big(A^{\xi}(X,e_k)A^{\eta}(Y,e_k) - (A^{\eta}(X,e_k)A^{\xi}(Y,e_k)\big), \tag{1.6}$$

where $X,Y,Z,W \in \mathfrak{X}(M)$, $\xi,\eta \in NM$ and $\{e_1,\ldots,e_m\}$ is a local orthonormal frame field on M with respect to g.

1.3.2 Riemannian submersions. Let M and N be two smooth manifolds with dimensions

$$m = \dim M > \dim N = n.$$

A smooth and surjective map $f : M \to N$ is called *submersion* if, for any $x \in M$, the differential of f has constant rank n. According to the Rank Theorem [20, Theorem 4.12], for each $x_0 \in M$ there exist charts (U,φ) around x_0 and (V,ψ) around $f(x_0)$ in which f has a coordinate representation $F = \psi\circ f\circ\varphi^{-1}$ of the form

$$F(x_1,\ldots,x_n;x_{n+1}\ldots,x_m) = (x_1,\ldots,x_n).$$

In particular, for any $p \in N$, each *fiber* $\mathcal{F}_p = f^{-1}(p)$ is an $(m-n)$-dimensional submanifold of M. Let us suppose in the sequel that the manifolds M and N are equipped with Riemannian metrics. Denote by $\mathcal{V} = \ker(df)$ the kernel of the differential of f and by $\mathcal{H}$ its orthogonal complement. The space $\mathcal{V}$ is called the *vertical bundle* and $\mathcal{H}$ is called the *horizontal bundle* of the submersion. The restriction of the Riemannian metric of M gives rise to Riemannian metrics on the vertical and horizontal bundle of the submersion. Now we may decompose the tangent bundle of M in the form

$$TM = \mathcal{V} \oplus \mathcal{H}.$$

Therefore any vector field $X \in \mathfrak{X}(M)$ can be uniquely decomposed in the form

$$X = X^{\mathcal{V}} + X^{\mathcal{H}},$$

where $\{\cdot\}^{\mathcal{V}}$ denotes the orthogonal projection on the vertical bundle $\mathcal{V}$ and $\{\cdot\}^{\mathcal{H}}$ the orthogonal projection on the horizontal bundle $\mathcal{H}$. The vertical bundle $\mathcal{V}$ is integrable. As a matter of fact, $\mathcal{V}_x$ is the tangent space of the fiber $\mathcal{F}_{f(x)} \subset M$ at $x \in M$. However, in general, the horizontal bundle $\mathcal{H}$ of the submersion is not integrable. There are six interesting categories of vector fields on M and N. Namely:

(A) A vector field $X \in \mathfrak{X}(M)$ is called *vertical* if $X \in \Gamma(\mathcal{V})$.

(B) A vector field $X \in \mathfrak{X}(M)$ is called *horizontal* if $X \in \Gamma(\mathcal{H})$.

(C) A vector field $X \in \mathfrak{X}(M)$ is called *projectable* if

$$df_x(X_x) = df_y(X_y),$$

for any x, y along a fiber $\mathcal{F}_p \subset M$. This means that $df(X)$ is a well-defined smooth vector field on N.

(D) A vector field $X \in \mathfrak{X}(M)$ is called *basic* if it is horizontal and projectable.

(E) The vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called f*-related* if

$$df_x(X_x) = Y_{f(x)},$$

for any $x \in M$.

(F) Using the Rank Theorem [20], one can show that for any $X \in \mathfrak{X}(N)$, there exist a unique f-related with X vector field $\widetilde{X} \in \Gamma(\mathcal{H})$, which we call the *horizontal lift* of X.

Lemma 1.3.1. *Let $V \in \Gamma(\mathcal{V})$ be a vertical vector field, $X, Y \in \mathfrak{X}(N)$ and $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathcal{H})$ be their horizontal lifts, respectively. Then, the following facts hold:*

$$[V, \widetilde{X}] \in \Gamma(\mathcal{V}) \quad \textit{and} \quad [\widetilde{X}, \widetilde{Y}]^{\mathcal{H}} = \widetilde{[X, Y]} \in \Gamma(\mathcal{H}),$$

where $\widetilde{[X, Y]}$ is the horizontal lift of $[X, Y]$.

Proof. Consider $h \in C^\infty(N)$ and set $g = X(h)$. Since V is a vertical vector field, we have that $df(V) = 0$. Moreover, for any fixed $x \in M$, we obtain

$$\begin{aligned} df_x\big([V,\widetilde{X}]\big)(h) &= \big[V,\widetilde{X}\big]_x(h\circ f) = V_x(\widetilde{X}(h\circ f)) - \widetilde{X}_x(V(h\circ f)) \\ &= V_x(X(h)\circ f) = V_x(g\circ f) = dg_{f(x)}(df(V)) \\ &= 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} df_x\big([\widetilde{X},\widetilde{Y}] - \widetilde{[X,Y]}\big)(h) &= df_x\big([\widetilde{X},\widetilde{Y}]\big)(h) - df_x\big(\widetilde{[X,Y]}\big)(h) \\ &= \widetilde{X}_x\big(\widetilde{Y}(h\circ f)\big) - \widetilde{Y}_x\big(\widetilde{X}(h\circ f)\big) - [X,Y]_{f(x)}(h) \\ &= \widetilde{X}_x(Y(h)\circ f) - \widetilde{Y}_x(X(h)\circ f) - [X,Y]_{f(x)}(h) \\ &= X_{f(x)}(Y(h)) - Y_{f(x)}(X(h)) - [X,Y]_{f(x)}(h) \\ &= 0. \end{aligned}$$

This completes the proof. □

Let us restrict ourselves to a special class of smooth maps between Riemannian manifolds. A submersion f is called *Riemannian submersion* if, for any $x \in M$, the differential

$$df_x : \mathcal{H}_x \subset T_xM \to T_{f(x)}N$$

is an isometry.

Theorem 1.3.2 (O'Neill's formulas [23]). *Let $f : (M, g_M, \nabla^M) \to (N, g_N, \nabla^N)$ be a Riemannian submersion.*

(a) *If $X, Y \in \mathfrak{X}(N)$, the following formula holds*

$$\nabla^M_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla^N_X Y} + \tfrac{1}{2}\big[\widetilde{X},\widetilde{Y}\big]^{\mathcal{V}}.$$

(b) *If $X, Y \in \mathfrak{X}(N)$ is a local orthonormal frame, then*

$$K_N(X,Y) = K_M(\widetilde{X},\widetilde{Y}) + \tfrac{3}{4}\big\|\big[\widetilde{X},\widetilde{Y}\big]^{\mathcal{V}}\big\|^2,$$

where K_M and K_N are the sectional curvatures of M and N, respectively.

Proof. Let $V \in \Gamma(\mathcal{V})$ be a vertical vector field, $Z \in \mathfrak{X}(N)$ and $\widetilde{Z} \in \mathfrak{X}(M)$ be its horizontal lift.

(a) Since by assumption f is a Riemannian submersion, we have

$$\begin{aligned} \widetilde{X} g_M(\widetilde{Y}, \widetilde{Z}) &= \widetilde{X}(g_N(df(\widetilde{Y}), df(\widetilde{Z})) \circ f) = \widetilde{X}(g_N(Y, Z) \circ f) \\ &= X_{f(x)} g_N(Y, Z). \end{aligned}$$

Moreover, by Lemma 1.3.1, we get that

$$\begin{aligned} g_M\big([\widetilde{X}, \widetilde{Y}], \widetilde{Z}\big) &= g_N\big(df\big([\widetilde{X}, \widetilde{Y}]\big), df(\widetilde{Z})\big) \circ f = g_N\big(df\big(\widetilde{[X, Y]}\big), Z\big) \circ f \\ &= g_N\big([X, Y], Z\big) \circ f. \end{aligned}$$

By Koszul's formula (1.1), we obtain that

$$g_M\big(\nabla^M_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}\big) = g_N\big(\nabla^N_X Y, Z\big) \circ f = g_M\big(\widetilde{\nabla^N_X Y}, Z\big) \circ f. \tag{1.7}$$

Again by Koszul's formula, Lemma 1.3.1 and

$$V\big(g_M\big(\widetilde{X}, \widetilde{Y}\big)\big) = V\big(g_N(X, Y) \circ f\big) = df(V)(g_N(X, Y)) = 0,$$

it follows that

$$2 g_M\big(\nabla^M_{\widetilde{X}} \widetilde{Y}, V\big) = g_M\big([\widetilde{X}, \widetilde{Y}], V\big). \tag{1.8}$$

The desired result follows immediately from (1.7) and (1.8).

(b) From part (a), we see that

$$2 g_M\big(\nabla^M_{\widetilde{X}} V, \widetilde{Y}\big) = -2 g_M\big(V, \nabla^M_{\widetilde{X}} \widetilde{Y}\big) = -g_M\big(V, [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}\big).$$

Moreover,

$$\begin{aligned} \nabla^M_{\widetilde{X}} \nabla^M_{\widetilde{Y}} \widetilde{X} &= \nabla^M_{\widetilde{X}}\big(\widetilde{\nabla^N_Y X} + \tfrac{1}{2}[\widetilde{Y}, \widetilde{X}]^{\mathcal{V}}\big) = \widetilde{\nabla^N_X \nabla^N_Y X} + \tfrac{1}{2}[\widetilde{X}, \widetilde{\nabla^N_Y X}]^{\mathcal{V}} \\ &= -\tfrac{1}{2} \nabla^M_{\widetilde{X}} [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, \end{aligned}$$

and

$$\begin{aligned} g_M\big(\nabla^M_{\widetilde{X}} \nabla^M_{\widetilde{Y}} \widetilde{X}, \widetilde{Y}\big) &= g_M\big(\widetilde{\nabla^N_X \nabla^N_Y X}, \widetilde{Y}\big) - \tfrac{1}{2} g_M\big(\nabla^M_{\widetilde{X}} [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, \widetilde{Y}\big) \\ &= g_N\big(\nabla^N_X \nabla^N_Y X, Y\big) + \tfrac{1}{2} g_M\big([\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, \nabla^M_{\widetilde{X}} \widetilde{Y}\big) \\ &= g_N\big(\nabla^N_X \nabla^N_Y X, Y\big) + \tfrac{1}{2} g_M\big([\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, \widetilde{\nabla^N_X Y} + \tfrac{1}{2}[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}\big) \\ &= g_N\big(\nabla^N_X \nabla^N_Y X, Y\big) + \tfrac{1}{4} \|[\widetilde{X}, \widetilde{Y}]\|^2. \end{aligned}$$

Furthermore,

$$g_M\big(\nabla^M_{\widetilde{Y}}\nabla^M_{\widetilde{X}}\widetilde{X},\widetilde{Y}\big) = g_M\big(\nabla^M_{\widetilde{Y}}\widetilde{\nabla^N_X X},\widetilde{Y}\big) = g_N\big(\nabla^N_Y\nabla^N_X X, Y\big),$$

and

$$\begin{aligned} g_M\big(\nabla^M_{[\widetilde{X},\widetilde{Y}]}\widetilde{X},\widetilde{Y}\big) &= g_M\big(\nabla^M_{\widetilde{[X,Y]}}\widetilde{X},\widetilde{Y}\big) + g_M\big(\nabla^M_{[\widetilde{X},\widetilde{Y}]^{\mathcal{V}}}\widetilde{X},\widetilde{X}\big) \\ &= g_N\big(\nabla^N_{X,Y}X,Y\big) - \tfrac{1}{2}\big\|[X,Y]^{\mathcal{V}}\big\|^2. \end{aligned}$$

As a matter of fact

$$R_M\big(\widetilde{X},\widetilde{Y},\widetilde{X},\widetilde{Y}\big) = R_N\big(X,Y,X,Y\big) + \tfrac{3}{4}\big\|\big[\widetilde{X},\widetilde{Y}\big]^{\mathcal{V}}\big\|^2,$$

and this completes the proof. □

1.3.3. The complex projective space. We will now present an important example of a complex manifold. Let

$$\mathbb{C}^{m+1} = \big\{z = (z_0,\ldots,z_m) : z_k = x_k + iy_k \in \mathbb{C} \text{ for all } k = 0,\ldots,m\big\}$$

be the $(m+1)$-dimensional complex euclidean space. We say that two points $z, w \in \mathbb{C}^{m+1} - \{0\}$ are equivalent, and we write $z \sim w$, if there exists a complex number λ such that $z = \lambda w$. Namely, two non-zero points of $\mathbb{C}^{m+1}$ are equivalent if and only if they lie on the same complex line. We denote by $[z]$ the equivalence class of a point $z \in \mathbb{C}^{m+1} - \{0\}$. The set of all such classes is called the *complex projective space*, and it is denoted by $\mathbb{CP}^m$.

Theorem 1.3.3. *Let $\mathbb{CP}^m$ be the complex projective space. Then the following statements hold:*

(a) *$\mathbb{CP}^m$ can be equipped with a natural smooth structure of a m-dimensional complex manifold.*

(b) *$\mathbb{CP}^1$ is diffeomorphic to the sphere $\mathbb{S}^2$; but $\mathbb{CP}^m$ is not diffeomorphic to $\mathbb{S}^{2m}$ for dimensions $m > 1$.*

(c) *$\mathbb{CP}^m$ carries a Kähler-Einstein metric with positive constant holomorphic curvature.*

Proof. The standard hermitian product $(\cdot\,,\cdot)$ of $\mathbb{C}^{m+1}$ can be written in the form

$$(z, w) = z_0\overline{w_0} + \cdots + z_m\overline{w_m},$$

where $z = (z_0, \ldots, z_m)$ and $w = (w_0, \ldots, w_m)$. Let $\mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}$ be the unit sphere. Then,

$$\mathbb{S}^{2m+1} = \{z \in \mathbb{C}^{m+1} : (z, z) = 1\}.$$

(a) Define the canonical projection map $\pi : \mathbb{S}^{2m+1} \to \mathbb{CP}^m$ given by $\pi(z) = [z]$, for any $z \in \mathbb{S}^{2m+1}$.

(1) **Topology:** We equip $\mathbb{CP}^m$ with the induced by $\pi : \mathbb{S}^{2m+1} \to \mathbb{CP}^m$ quotient topology, i.e. we say that a set $U \subseteq \mathbb{CP}^m$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{S}^{2m+1}$. Since the quotient map is open and $\mathbb{S}^{2m+1}$ is second countable, a classical result from point set topology ensures that the quotient space is also second countable; see for example [37, Corollary 7.10]. Moreover, note that $\mathbb{CP}^m$ is compact. It remains to show that the quotient topology is Hausdorff. Indeed, let $[z]$ and $[w]$ be two distinct points in $\mathbb{CP}^m$. Then,

$$\ell_1 = \pi^{-1}([z]) = \big\{e^{i\theta}z \in \mathbb{S}^{2m+1} : \theta \in [0, 2\pi]\big\}$$

and

$$\ell_2 = \pi^{-1}([w]) = \big\{e^{i\theta}w \in \mathbb{S}^{2m+1} : \theta \in [0, 2\pi]\big\}$$

are two disjoint great circles in the unit sphere $\mathbb{S}^{2m+1}$. Let

$$r = \min\big\{|e^{i\theta_1}z - e^{i\theta_2}w| : (\theta_1, \theta_2) \in [0, 2\pi] \times [0, 2\pi]\big\}.$$

In some sense r measures the distance between the circles ℓ_1 and ℓ_2. Since $[0, 2\pi] \times [0, 2\pi]$ is compact and $\ell_1 \cap \ell_2 = \varnothing$, it follows that $r > 0$. Consider the open disjoint subsets of the sphere $\mathbb{S}^{2m+1}$ given by

$$U_1 = \big\{p \in \mathbb{S}^{2m+1} : |e^{i\theta_1}p - e^{i\theta_2}z| < r/2 \text{ for all } (\theta_1, \theta_2) \in [0, 2\pi]\times[0, 2\pi]\big\}$$

and

$$U_2 = \big\{p \in \mathbb{S}^{2m+1} : |e^{i\theta_1}p - e^{i\theta_2}w| < r/2 \text{ for all } (\theta_1, \theta_2) \in [0, 2\pi]\times[0, 2\pi]\big\}.$$

Observe that for any $\theta \in [0, 2\pi]$ we have that $e^{i\theta}U_1 = U_1$ and $e^{i\theta}U_2 = U_2$. Hence, $\pi^{-1}(\pi(U_1)) = U_1$ and $\pi^{-1}(\pi(U_2)) = U_2$. Consequently, $\pi(U_1)$ and $\pi(U_2)$ are open disjoint subsets in $\mathbb{CP}^m$ and so $\mathbb{CP}^m$ is Hausdorff.

(2) **Smooth structure:** Consider the open covering of $\mathbb{CP}^m$ given by

$$U_j = \big\{[z_0, \ldots z_{j-1}, z_j, z_{j+1}, \ldots, z_m] \in \mathbb{CP}^m : z_j \neq 0\big\},$$

where $j \in \{0, \ldots, m\}$, and define the family $\{(U_j, \varphi_j)\}_{i \in \{1,\ldots,m\}}$, where the maps $\varphi_j : U_j \to \mathbb{C}^m$ are given by

$$\varphi_j\big([z_0, \ldots, z_m]\big) = \big(z_0/z_j, \ldots, z_{j-1}/z_j, z_{j+1}/z_j, \ldots, z_m/z_j\big).$$

Clearly each pair (U_j, φ_j) forms a chart on $\mathbb{CP}^m$. Moreover, the transition maps

$$\varphi_j \circ \varphi_k^{-1} : \varphi_k(U_k \cap U_j) \to \varphi_j(U_k \cap U_j),$$

are given by

$$\begin{aligned}&\varphi_j \circ \varphi_k^{-1}(z_1, \ldots, z_m)\\ &\quad= (z_1/z_j, \ldots, z_{i-1}/z_j, 1/z_j, z_{i+1}/z_j, \ldots, z_{j-1}/z_j, z_{j+1}/z_j, \ldots, z_m/z_j),\end{aligned}$$

and are biholomorphic. Thus $\mathbb{CP}^m$ can be equipped with a smooth structure of a complex manifold with complex dimension m.

(b) We will show now that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$. To achieve this recall that the differentiable structure of $\mathbb{S}^2$ is described by the charts $(\mathbb{S}^2 - \{(0,0,1)\}, \psi_1)$ and $(\mathbb{S}^2 - \{(0,0,-1)\}, \psi_2)$ given by

$$\psi_1(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3} \quad \text{and} \quad \psi_2(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 + x_3}.$$

For $\mathbb{CP}^1$ we consider the charts (U_1, θ_1) and (U_2, θ_2) given by

$$U_1 = \big\{[z_0, z_1] \in \mathbb{CP}^1 : z_0 \neq 0\big\} \;\&\; U_2 = \big\{[z_0, z_1] \in \mathbb{CP}^1 : z_1 \neq 0\big\}$$

and

$$\theta_1([z_0, z_1]) = z_1/z_0 \;\&\; \theta_2([z_0, z_1]) = \overline{z_0/z_1}.$$

It turns out that

$$\theta_2 \circ \theta_1^{-1} = \psi_2 \circ \psi_1^{-1}.$$

Thus the diffeomorphisms $\psi_1^{-1} \circ \theta_1$ and $\psi_2^{-1} \circ \theta_2$ agree on the intersection of their domains of definition, and together they define a global diffeomorphism of $\mathbb{CP}^1$ onto $\mathbb{S}^2$. For a proof that $\mathbb{CP}^m$ is not diffeomorphic to $\mathbb{S}^{2m}$ we refer to [46, Proposition 5.1.3].

(c) With the above differentiable structure, the map $\pi : \mathbb{S}^{2m+1} \to \mathbb{CP}^m$ becomes a smooth submersion. We denote by $\mathcal{V}$ and $\mathcal{H}$ the vertical and the horizontal bundles of π, respectively. Fix now an arbitrary point $p = [z]$. Then the fiber

$$\mathcal{F}_p = \pi^{-1}(p) = \{e^{i\theta} z,\ 0 \leq \theta \leq 2\pi\}$$

is great circle of $\mathbb{S}^{2m+1}$. Let us now define the 1-parameter family of smooth maps $\{f_\theta\}_{\theta \in [0,2\pi]} \in C^\infty(\mathbb{S}^{2m+1}; \mathbb{S}^{2m+1})$ given by $f_\theta = e^{i\theta} I$, where I is the identity map on $\mathbb{S}^{2m+1}$. Note that:

- The vectors z and $\xi_z = iz$ are orthogonal with respect to the euclidean inner product of $\mathbb{C}^{m+1}$. Hence, for any fixed $z \in \mathbb{S}^{2m+1}$, the curve

  $$[0, 2\pi] \ni \theta \to f_\theta(z) \in \mathbb{S}^{2m+1}$$

  is a great circle passing through z with unit tangent the vector iz. Therefore,

  $$\mathcal{V}_z = \text{span}\{\xi_z = iz\}.$$

  ξ is called the *Hopf vector field* and its integral curves are great circles.

- Let $z, w \in \mathcal{F}_p$ and $\theta \in [0, 2\pi]$ such that $w = f_\theta(z) = e^{i\theta} z$. Then,

  $$df_\theta(V) = e^{i\theta} V \in \mathcal{H}_w,$$

  for any $V \in \mathcal{H}_z$. Consequently, $df_\theta : \mathcal{H}_z \to \mathcal{H}_w$ is a linear isometry. Moreover, from the identity $\pi \circ f_\theta = \pi$, we deduce that

  $$d\pi_z(V) = d\pi_w(e^{i\theta} V).$$

- Let X be a tangent vector of $\mathbb{CP}^m$ at the point p and $w = e^{i\theta} z$ two points on the fiber $\mathcal{F}_p$. Then we have that

  $$\widetilde{X}_w = e^{i\theta} \widetilde{X}_z. \tag{1.9}$$

(1) **The Riemannian metric:** Let $z \in \mathbb{S}^{2m+1}$. Let X_p, Y_p be tangent vectors of the complex projective space $\mathbb{CP}^m$ at p. Define the metric 2-tensor g_{FS} on $\mathbb{CP}^m$ given by

$$g_{FS}(X_p, Y_p) = \langle \widetilde{X}_w, \widetilde{Y}_w \rangle, \tag{1.10}$$

for any $w \in \mathcal{F}_p$. From (1.9) it follows that the metric g_{FS} does not depend on the choice of the point $w \in \mathcal{F}_p$. The Riemannian metric g_{FS} is called the *Fubini-Study* metric of the complex projective space. We conclude that with respect to these Riemannian metrics the projection $\pi : \mathbb{S}^{2m+1} \to \mathbb{CP}^m$ becomes a Riemannian submersion.

(2) **The complex structure:** Let $X \in \mathfrak{X}(\mathbb{CP}^m)$. Recall from Lemma 1.3.1 that $[\xi, \widetilde{X}] \in \Gamma(\mathcal{V})$. Since the integral curves of ξ are geodesics we get,

$$\nabla^{\mathbb{S}^{2m+1}}_{\xi}\xi = 0, \ \left(\nabla^{\mathbb{S}^{2m+1}}_{\xi}\widetilde{X}\right)^{\mathcal{V}} = 0 = \left(\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\xi\right)^{\mathcal{V}} \text{ and } \left[\xi, \widetilde{X}\right] = 0. \tag{1.11}$$

- Define now the $(1,1)$-tensor field J on $\mathbb{CP}^m$ given by

$$JX = d\pi(\nabla^{\mathbb{S}^{2m+1}}_{\xi}\widetilde{X}) = d\pi(\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\xi). \tag{1.12}$$

From the last identity and (1.11), we see that

$$\widetilde{JX} = \nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\xi = \nabla^{\mathbb{S}^{2m+1}}_{\xi}\widetilde{X}. \tag{1.13}$$

According to (1.11), (1.12), (1.13), we deduce that

$$\begin{aligned} J^2X &= d\pi(\nabla^{\mathbb{S}^{2m+1}}_{\xi}\widetilde{JX}) = d\pi(\nabla^{\mathbb{S}^{2m+1}}_{\xi}\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\xi) = d\pi\big(R(\xi,\widetilde{X})\xi\big) \\ &= -X, \end{aligned}$$

where R stands for the curvature tensor of the sphere. Hence, J is an almost complex structure.

- Let $\{e_1, \ldots, e_{2m}\}$ be a local orthonormal frame field on $\mathbb{CP}^m$. Since π is a Riemannian submersion, we have

$$g_{FS}(Je_i, e_j) = \langle \nabla_{\xi}\widetilde{e}_i, \widetilde{e}_j\rangle = \xi\langle \widetilde{e}_i, \widetilde{e}_j\rangle - \langle \widetilde{e}_i, \nabla_{\xi}\widetilde{e}_j\rangle = -g_{FS}(e_i, Je_j),$$

for any $i, j \in \{1, \ldots, 2m\}$. This implies that

$$g_{FS}(JX, JY) = -g_{FS}(X, J^2Y) = g_{FS}(X, Y),$$

for any $X, Y \in \mathfrak{X}(\mathbb{CP}^m)$. Therefore, J is an isometry.

- Let $\nabla^{\mathbb{CP}^m}$ be the Levi-Civita connection of the Fubini-Study metric. Then, taking into account Theorem 1.3.2(a) and (1.13), we obtain

$$\begin{aligned} (\nabla^{\mathbb{CP}^m}_X J)Y &= \nabla^{\mathbb{CP}^m}_X JY - J\nabla^{\mathbb{CP}^m}_X Y \\ &= d\pi\big(\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\widetilde{JY}\big) - d\pi(\nabla^{\mathbb{S}^{2m+1}}_{\xi}\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\widetilde{Y}) \\ &= d\pi\big(\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\nabla^{\mathbb{S}^{2m+1}}_{\xi}\widetilde{Y} - \nabla^{\mathbb{S}^{2m+1}}_{\xi}\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\widetilde{Y}\big) \\ &= d\pi\big(R(\widetilde{X},\xi),\widetilde{Y})\big) \\ &= 0. \end{aligned}$$

Hence J is parallel and g_{FS} is a Kähler metric.

(3) **Curvature:** Let us now compute the sectional curvature, the holomorphic, and the Ricci curvature of the complex projective space. We have:

$$\begin{aligned}
\tfrac{1}{2}\big[\widetilde{X},\widetilde{Y}\big]^{\mathcal{V}} &= \big(\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\widetilde{Y}\big)^{\mathcal{V}} - \big(\nabla^{\mathbb{S}^{2m+1}}_{\widetilde{Y}}\widetilde{X}\big)^{\mathcal{V}} \\
&= \langle \nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\widetilde{Y},\xi\rangle\xi - \langle \nabla^{\mathbb{S}^{2m+1}}_{\widetilde{Y}}\widetilde{X},\xi\rangle\xi \\
&= \langle \nabla^{\mathbb{S}^{2m+1}}_{\widetilde{Y}}\xi,\widetilde{X}\rangle\xi - \langle \nabla^{\mathbb{S}^{2m+1}}_{\widetilde{X}}\xi,\widetilde{Y}\rangle\xi \\
&= \langle \widetilde{JY},\widetilde{X}\rangle\xi - \langle \widetilde{JX},\widetilde{Y}\rangle\xi \\
&= g_{FS}(JY,X) - g_{FS}(JX,Y) \\
&= 2g_{FS}(JY,X).
\end{aligned}$$

- According to Proposition 1.3.2(b), the sectional curvature K of $\mathbb{CP}^m$ satisfies
$$K(X,Y) = 1 + 3g_{FS}(JX,Y)^2,$$
where $\{X,Y\}$ is an arbitrary local orthonormal frame field with respect to the Fubini-Study metric g_{FS}. As a matter of fact $(\mathbb{CP}^m, g_{FS})$ is a symmetric space. If $m=1$, then $K \equiv 4$. On the other hand, if $m>1$, then the sectional curvature of $\mathbb{CP}^m$ is non-constant and satisfy
$$1 \leq K \leq 4.$$
Let us remind here that for $m>1$ the complex projective space $\mathbb{CP}^m$ is not diffeomorphic with the sphere $\mathbb{S}^{2m}$.
- The Ricci curvature Ric of the complex projective space is
$$\mathrm{Ric} = 2(m+1)g_{FS}.$$
Therefore, $\mathbb{CP}^m$ is Kähler-Einstein manifold.
- If X is a unit vector field on the complex projective space, then
$$\mathrm{Hol}(X) = K(X,JX) = 4.$$
Hence the Fubini-Study metric g_{FS} has constant holomorphic curvature equal to 4.

This completes the proof of the theorem. □

1.4 Index notation

In the following chapters of the thesis, we will perform computations regarding tensorial quantities with respect to orthonormal frames. Let $\{e_1, \ldots, e_m\}$ be a local tangent frame field on a m-dimensional Riemannian manifold M. If $S : TM \to TM$ is a (1,1)-tensor then we set

$$S_i = S(e_i),$$

for any $i \in \{1, \ldots, m\}$. If $S : TM \times TM \to C^\infty(\mathbb{R})$ is a bilinear form, then we may represent the coefficients of its matrix with respect to the basis $\{e_1, \ldots, e_m\}$ by S_{ij}, that is we set

$$S_{ij} = S(e_i, e_j).$$

On the other hand, if $S : TM \times \cdots \times TM \to C^\infty(\mathbb{R})$ is an $(r, 0)$-tensor, then its coefficients with respect to the frame $\{e_1, \ldots, e_m\}$ will be denoted by $S_{i_1 \ldots i_r}$, that is

$$S_{i_1 \ldots i_r} = S(e_{i_1}, \ldots, e_{i_r}).$$

Suppose now that $F : M \to N$ is an isometric immersion, where here M is an m-dimensional and N is an n-dimensional Riemannian manifold. Let $\{e_1, \ldots, e_m\}$ be a local tangent frame on M and let $\{\xi_{m+1}, \ldots, \xi_n\}$ be a local frame of the normal bundle NM. We will use Latin indices to denote components of tensorial quantities on the tangent bundle TM and Greek indices for components on the normal bundle of the submanifold. Then the set of vector fields

$$\{e_1, \ldots, e_m; \xi_{m+1}, \ldots, \xi_n\}$$

is called *adapted frame* along the submanifold. According to the aforementioned setup, we decide to use the following notation throughout this thesis:

$$F_i = dF(e_i), \; A_{ij} = A(e_i, e_j), \; h^\alpha_{ij} = \langle A_{ij}, \xi_\alpha \rangle \; \text{ and } \; H^\alpha = \langle H, \xi_\alpha \rangle.$$

Moreover,

$$\widetilde{R}_{ijsl} = \widetilde{R}(F_i, F_j, F_s, F_l) \; \text{ and } \; \widetilde{R}_{ij\alpha\beta} = \widetilde{R}(F_i, F_j, \xi_\alpha, \xi_\beta).$$

CHAPTER 2

Singular Value Decomposition

In this chapter we discuss a factorisation of the differential of a smooth map which generalises the eigendecomposition of a symmetric bilinear form.

2.1 Algebraic facts

Let (M, g_M) and (N, g_N) be smooth Riemannian manifolds of dimensions m and n, respectively, and $f : M \to N$ a smooth map. Consider the pull-back tensor S given by

$$S(X, Y) = g_N\big(df(X), df(Y)\big),$$

for any $X, Y \in \mathfrak{X}(M)$. Observe that S is non-negative definite and symmetric. Hence, we can diagonalise S with respect to g_M. More precisely, at a fixed point $x \in M$, there exists an orthonormal basis $\{\alpha_1, \ldots, \alpha_m\}$ of T_xM, with respect to g_M, such that

$$S(\alpha_i, \alpha_j) = \lambda_i^2 \delta_{ij},$$

for any $i, j \in \{1, \ldots, m\}$. The eigenvalues are arranged such that

$$\lambda_1^2 \leq \cdots \leq \lambda_m^2.$$

The numbers $0 \leq \lambda_1 \leq \cdots \leq \lambda_m$ are called the *singular values* of df at $x \in M$.

Let now $r = \text{rank}(df_x) \leq \min\{m, n\}$. At $f(x) \in N$ consider the orthonormal basis $\{\beta_1, \ldots, \beta_{n-r}; \beta_{n-r+1}, \ldots, \beta_n\}$, with respect to g_N, such that

$$df(\alpha_i) = \lambda_i \beta_{n-m+i},$$

for any $i \in \{m - r + 1, \ldots, m\}$. This process is known as the *singular value decomposition* of df.

2.2 Symplectomorphisms

We will define in this section the notion of a symplectomorphism and will show that such maps form an infinite group. Let start by recalling some definitions. Let $\Omega^k(M)$ be the space of differentiable k-forms on M. If X is a vector field on M, then

$$i_X : \Omega^k(M) \to \Omega^{k-1}(M)$$

is the map given by

$$(i_X\omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}),$$

for any vector fields $X_1, \dots, X_{k-1}$. The map i_X is called the *interior product*. Another common notation for the interior product is $X \lrcorner\, \omega$. According to Cartan's formula, we have

$$\mathcal{L}_X = d\, i_X + i_X d,$$

where $\mathcal{L}_X$ is the Lie derivative and d the exterior derivative.

A *symplectic form* ω on a m-dimensional manifold M is non-degenerate closed 2-form. Non-degenerate means that the mapping given by

$$TM \ni X \to i_X\omega \in T^*M,$$

is an isomorphism. The requirement that ω is non-degenerate forces M to be even dimensional and oriented. In this case the pair (M, ω) is called a symplectic manifold.

One can easily verify that any Kähler manifold is a symplectic manifold. The classical example of a symplectic manifold is the euclidean space $\mathbb{R}^{2m}$ with Cartesian coordinates $(x_1, y_1, \dots, x_m, y_m)$ and symplectic form

$$\omega_0 = \sum_{i=1}^{m} dx_i \wedge dy_i. \qquad (2.1)$$

The first important theorem in symplectic geometry is due to Darboux, which says that locally all symplectic manifolds look like $(\mathbb{R}^{2m}, \omega_0)$. More precisely, the following result hold:

Theorem 2.2.1. *Let (M, ω) be a $2m$-dimensional symplectic smooth manifold. For each point $x \in M$, there is a local chart (U, φ) where U is an open neigborhood of x, and a diffeomorphism $\varphi : U \to \mathbb{R}^{2m}$ such that $\varphi^*\omega_0 = \omega|_U$, where ω_0 is the standard symplectic form of $\mathbb{R}^{2m}$ given in* (2.1).

Therefore there are no symplectic local invariants. In particular, all symplectic invariant are of a global nature.

Definition 2.2.2. *Let (M, ω) be a symplectic manifold. A smooth map $f : (M, \omega) \to (M, \omega)$ is called a symplectomorphism if and only if $f^*\omega = \omega$.*

The set of all symplectomorphisms form a group with the law of composition of mappings. We denote the symplectomorphism group by Sympl(M, ω). It turns out that any smooth function with compact support on (M, ω) gives rise to a symplectomorphism. Let $u : M \to \mathbb{R}$ be a smooth function and X the vector field defined uniquely by the equation

$$i_X\omega = du.$$

The vector field X is called the *Hamiltonian vector field* associated with u. Suppose now that either u has compact support or, more generally, that $\mathcal{X}$ is complete. Denote by $\varphi : M \times \mathbb{R} \to M$ the flow which is generated by the vector field X, i.e. let φ be the solution of the system

$$\begin{cases} d\varphi_{(x,t)}(\partial_t) = X_{\varphi(x,t)}, \\ \varphi(x, 0) = I, \end{cases}$$

for any $x \in M$, where $I : M \to M$ is the identity map. For each $x \in M$, the curve

$$t \to \varphi(x, t)$$

is an integral curve of X passing through the point x and, for each fixed $t \in \mathbb{R}$, the map

$$x \to \varphi_t(x) = \varphi(x, t)$$

is a diffeomorphism. We claim that the 1-parameter family of diffeomorphisms $\varphi_t : M \to M$ is a symplectomorphism. Indeed, from Cartan's formula we get

$$\mathcal{L}_X\omega = d\, i_X\omega + i_X d\omega = d(du) = 0.$$

Fix now two (time-independent) vector fields $V, W \in \mathfrak{X}(M)$. Then,

$$\begin{aligned} \partial_t\{\varphi_t^*\omega(V, W)\} &= \lim_{s\to 0} \frac{\varphi_{t+s}^*\omega(V, W) - \varphi_t^*\omega(V, W)}{s} \\ &= \lim_{s\to 0} \frac{(\varphi_s^*\omega - \omega)(d\varphi_t(V), d\varphi_t(W))}{s} \\ &= (\mathcal{L}_X\omega)(d\varphi_t(V), d\varphi_t(W)) \\ &= 0. \end{aligned}$$

Thus we have shown the following result:

Theorem 2.2.3. *The symplectomorphism group of a symplectic manifold is infinite.*

Suppose now that (M, g, J, ω) is a Kähler manifold and let $f : M \to M$ be a *symplectomorphism*. Then, we can easily see that

$$df^* J df = J, \tag{2.2}$$

where df^* is the adjoint operator of df with respect to the metric g. Define the bundle map E, given by

$$E = df(df^* df)^{-1/2}. \tag{2.3}$$

Since for any $x \in M$ the differential df_x is an isomorphism, it follows that $df^* df$ is a positive definite self-adjoint automorphism of TM and the square root of $df^* df$ is well defined.

Lemma 2.2.4. *Let $f : M \to M$ be a symplectic map. Then the following facts hold:*

(a) *The map $E : M \to M$ is an isometry. Equivalently, E satisfies $EE^* = I$.*

(b) *The map $E : M \to M$ is a symplectic isometry. Equivalently, E satisfies $E^* JE = J$.*

Proof. (a) We compute

$$\begin{aligned} EE^* &= df(df^* df)^{-1/2}(df(df^* df)^{-1/2})^* \\ &= df(df^* df)^{-1/2}(df^* df)^{-1/2} df^* \\ &= df(df^* df)^{-1} df^* = df df^{-1}(df^*)^{-1} df^* \\ &= I. \end{aligned}$$

(b) We will show at first that

$$(-J(df^* df)^{-1/2} J)^2 = df^* df.$$

Indeed! Using the symplectic condition (2.2), we have that

$$\begin{aligned}(-J(df^*df)^{-1/2}J)^2 &= \big(J(df^*df)^{-1/2}J\big)\big(J(df^*df)^{-1/2}J\big)\\ &= J(df^*df)^{-1/2}J^2(df^*df)^{-1/2}J\\ &= -J(df^*df)^{-1/2}(df^*df)^{-1/2}J = -J(df^*df)^{-1}J\\ &= -df^*Jdf(df^*df)^{-1}df^*Jdf = -df^*J^2df\\ &= \ df^*df.\end{aligned}$$

Since both $(df^*df)^{1/2}$ and $-J(df^*df)^{-1/2}J$ are positive definite, it follows that

$$-J(df^*df)^{-1/2}J = (df^*df)^{1/2}. \tag{2.4}$$

Using (2.2) and (2.4), we obtain that

$$\begin{aligned}E^*JE &= E^{-1}JE = E^{-1}(df^*)^{-1}J(df)^{-1}E = (df^*E)^{-1}J(df)^{-1}E\\ &= \big(df^*df(df^*df)^{-1/2}\big)^{-1}J(df)^{-1}df(df^*df)^{-1/2}\\ &= (df^*df)^{-1/2}J(df^*df)^{-1/2}\\ &= J(df^*df)^{1/2}(df^*df)^{-1/2}\\ &= J.\end{aligned}$$

Hence E is a symplectic isometry and this completes the proof. □

Let $\{\alpha_1, \ldots, \alpha_{2m}\}$ be an orthonormal basis of T_xM that diagonalises df^*df. Then df^*df is the positive definite and it has a matrix representation of the form

$$df^*df = \begin{pmatrix} \lambda_1^2 & 0 & \ldots & 0\\ 0 & \lambda_2^2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_{2m}^2\end{pmatrix}$$

in terms of the singular values of df_x. Then

$$(df^*df)(\alpha_i) = \lambda_i^2\alpha_i \quad \text{and} \quad (df^*df)^{1/2}(\alpha_i) = \lambda_i\alpha_i.$$

From the last identity we deduce that

$$\alpha_i = \lambda_i(df^*df)^{-1/2}(\alpha_i).$$

Therefore,

$$df(\alpha_i) = \lambda_i df(df^*df)^{-1/2}(\alpha_i) = \lambda_i E(\alpha_i).$$

Consequently, df has a matrix representation of the form

$$df = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{2m} \end{pmatrix}$$

with respect to the basis $\{\alpha_1, \dots, \alpha_{2m}\}$ and $\{\beta_1 = E(\alpha_1), \dots, \beta_{2m} = E(\alpha_{2m})\}$.

Lemma 2.2.5. *The following formula holds*

$$(\lambda_i \lambda_j - 1) g(J\alpha_i, \alpha_j) = 0,$$

for any $i, j \in \{1, \dots, 2m\}$.

Proof. By the symplectic condition and Lemma 2.2.4, we have

$$\begin{aligned} g(J\alpha_i, \alpha_j) &= \omega(\alpha_i, \alpha_j) = f^*\omega(\alpha_i, \alpha_j) = g(Jdf(\alpha_i), df(\alpha_j)) \\ &= \lambda_i \lambda_j g(JE(\alpha_i), E(\alpha_j)) = \lambda_i \lambda_j g(EJ(\alpha_i), E(\alpha_j)) \\ &= \lambda_i \lambda_j g(J\alpha_i, \alpha_j). \end{aligned}$$

This completes the proof. □

Lemma 2.2.6. *Let* $f : M \to M$ *be a symplectomorphism,* x *an arbitrary point in* M *and* $\{\alpha_1, \dots, \alpha_m\}$*,* $\{\beta_1, \dots, \beta_m\}$ *orthonormal bases of the singular decomposition of* df_x*. Then, the following facts hold:*

(a) *If* λ *is a singular value of* df *at* $x \in M$*, then* $1/\lambda$ *is also a singular value. Hence, the singular values can be split into pairs whose product is* 1*.*

(b) *If* $V(\lambda)$ *denotes the eigenspace associated to the singular value* λ*, then*

$$\dim V(\lambda) = \dim V(1/\lambda).$$

Moreover, the restriction of J *on the eigenspace* $V(\lambda)$ *gives rise to an isometry between the eigenspaces* $V(\lambda)$ *and* $V(1/\lambda)$*.*

(c) *The tangent space* T_xM *splits as the direct sum*

$$T_xM = V(1)^{m_0} \oplus V(\lambda_1)^{m_1} \oplus V(1/\lambda_1)^{m_1} \oplus \cdots \oplus V(\lambda_s)^{m_s} \oplus V(1/\lambda_s)^{m_s},$$

where the singular values are in ascending order and the superscripts $m_0 \geq 0$ *and* $m_j > 0$*,* $j = \{1, \dots, s\}$*, denotes the dimension of each corresponding eigenspace.*

(d) *Let us rearrange the order of the vectors of the frame* $\{\alpha_1, \ldots, \alpha_m\}$ *to become compatible with the decomposition given in part* (c)*. Then, the complex structure* J *has the following representation*

$$J = \begin{pmatrix} 0 & -1 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 \\ 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{2.5}$$

Proof. (a) Let $\{\alpha_1, \ldots, \alpha_{2m}\}$ be the basis of the singular decomposition. Fix an index $i \in \{1, \ldots, 2m\}$. Then, since $J\alpha_i$ is a unit vector, there exists an index $j \in \{1, \ldots, 2m\}$ such that

$$g(J\alpha_i, \alpha_j) \neq 0.$$

By Lemma 2.2.5, it follows that $\lambda_j = 1/\lambda_i$.

(b) The statement is trivial if $\lambda = 1$. So let us suppose from now on that $\lambda \neq 1$. Furthermore, assume that

$$\dim V(\lambda) = k \quad \text{and} \quad \dim V(1/\lambda) = l.$$

Let $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ be a frame that spans $V(\lambda)$. Then,

$$\lambda_{i_1} = \cdots = \lambda_{i_k} = \lambda.$$

We claim now that the vectors $\{J\alpha_{i_1}, \ldots, J\alpha_{i_k}\}$ are orthonormal and belong to $V(1/\lambda)$. Indeed! From Lemma 2.2.5, it follows that if λ_i and λ_j are singular values such that $\lambda_i\lambda_j \neq 1$, then

$$g(J\alpha_i, \alpha_j) = 0.$$

In other words, the vector $J\alpha_i$ is orthogonal to each singular vector corresponding to a singular value not equal to $1/\lambda_i$. But

$$T_xM = V(1/\lambda_i) \oplus V,$$

where V is the orthogonal complement of $V(1/\lambda_i)$. Because $J\alpha_i$ is orthogonal to V, it follows that $J\alpha_i \in V(1/\lambda_i)$. Since J is an isometry, it follows that $\{J\alpha_{i_1}, \ldots, J\alpha_{i_k}\}$ is an orthonormal basis. Therefore, we conclude that $k \leq l$.

We may apply the same argument to $V(1/\lambda)$ as well. Namely, if $\{\alpha_{j_1}, \ldots, \alpha_{j_l}\}$ span $V(1/\lambda)$, then the vectors $\{J\alpha_{j_1}, \ldots, J\alpha_{j_l}\}$ are orthonormal and belong to the eigenspace $V(\lambda)$. This implies that $k \geq l$. So we conclude that $k = l$ and that $J : V(\lambda) \to V(1/\lambda)$ is an isometry. Observe now that necessarily J maps $V(1)$ onto $V(1)$.

The parts (c) and (d) of the lemma are immediate consequences of the above observations.

This completes the proof. □

2.3 Holomorphic maps

Let (M, g, J) be a Kähler manifold of real dimension $2m$. A map $f : M \to M$ is called *holomorphic* if it satisfies

$$df J = J df. \tag{2.6}$$

Let $\{\alpha_1, \ldots, \alpha_{2m}\}$ and $\{\beta_1, \ldots, \beta_{2m}\}$ be two orthonormal bases with respect to g arising from the singular decomposition of df. Suppose that λ_i is a singular value with corresponding eigendirections the vectors α_i and β_i. Then, from the condition (2.6) we see that

$$df(J\alpha_i) = Jdf(\alpha_i) = \lambda_i J\beta_i.$$

Hence, we see that if α_i and β_i are eigendirections corresponding to the singular value λ_i, then the vectors $J\alpha_i$ and $J\beta_i$ are again eigenvectors corresponding to the same singular value. As a conclusion we see that each eigenspace $V(\lambda_j)$ has even dimension $2m_j$ and contains an orthonormal J-basis.

A holomorphic map $f : M \to M$ is called bi-holomorphic if it is 1-1 and its inverse is also holomorphic. It is a well-known fact in Algebraic Geometry that any biholomorphic map $f : \mathbb{CP}^{m-1} \to \mathbb{CP}^{m-1}$ can be written in the form

$$f\left(\begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}\right) = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}, \text{where } a_{ij} \in \mathbb{C}, \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in \mathbb{CP}^{m-1};$$

for more details we refer to [9, pp. 170-171].

CHAPTER 3

Mean Curvature Flow

In this chapter we introduce the notion of the mean curvature flow, following the exposition in [35]. Let N be a Riemannian manifold. We say that a family of immersions $F : M \times [0,T) \to N$ evolves by *mean curvature flow* (MCF for short) with initial data the immersion $F_0 : M \to N$ if it satisfies the initial value problem

$$\begin{cases} dF_{(x,t)}(\partial_t) = H_{F(x,t)}, \\ F(x,0) = F_0(x), \end{cases} \tag{MCF}$$

for any $(x,t) \in M \times [0,T)$, where $H_{F(x,t)}$ denotes the mean curvature vector of the immersion $F(\cdot\,,t) : M \to N$ at the point $F(x,t)$.

3.1 Existence of the flow

Writing the mean curvature flow in local coordinates one can see that we have to deal with a degenerate system of parabolic equations. Therefore, the existence of the mean curvature flow is not a simple consequence from the available classical theorems of partial differential equations. Short-time existence and uniqueness of the mean curvature flow was originally proven using results of Hamilton [13, 14] based on the Nash-Moser iteration method. However, it is possible to give a shorter proof of the short time existence of the mean curvature flow adapting a variant of the so called DeTurck's trick [8] which was first used in Ricci flow; for more details we refer to [4, 25, 38]. It is well-known that in general, long-time existence of the mean curvature flow cannot be expected. For example, the maximal time of existence of (MCF) in the euclidean space is always finite. On the other hand, if the ambient space is a Riemannian manifold there are situations where it is possible to get long-time existence and convergence of the flow.

Let us collect the most important facts about the existence and the maximal time of the existence of (MCF) in the following theorem.

Theorem 3.1.1. *Let M be a compact manifold and $F_0 \colon M \to N$ an immersion into a complete Riemannian manifold N. Then, the following facts hold:*

(a) *The mean curvature flow with initial data the immersion F_0 admits a unique up to diffeomorphisms, smooth solution on a maximal time interval $[0, T_{\max})$, where $0 < T_{\max} \leq \infty$.*

(b) *If the Riemannian metric of N is real analytic, then the mean curvature flow is real analytic in $M \times (0, T_{\max})$, i.e. the evolved submanifolds have real analytic Riemannian metrics.*

Let us mention here that part (b) of the above theorem follows from the standard regularity theory of systems of quasilinear parabolic equations; see [19].

A powerful tool to study the behaviour of solutions of the flow is the *maximum principle*. More specifically, in the analysis of singularities, a crucial step is to obtain a priori, integral, or point-wise, estimates. Let us recall here, the parabolic maximum principle for solutions of parabolic equations of second order; for the proofs see for example [2, Chapter 7].

Theorem 3.1.2. *Let $\{g_t\}_{t\in[0,T)}$ be a smooth family of Riemannian metrics on a compact manifold M and suppose that $f \in C^\infty(M \times [0,T))$ is a solution of the differential inequality*

$$f_t - \Delta_{g_t} f \geq g(V, \nabla^{g_t} f) + Q(f,t),$$

where here ∇^{g_t} is the Levi-Civita connection associated with g_t, Δ_{g_t} is the Laplacian operator associated with g_t, V is a bounded time-dependent vector field and Q is continuous in time and locally Lipschitz in space. If $\phi : [0,T) \to \mathbb{R}$ is the solution of the associated ODE

$$\begin{cases} \phi'(t) = Q(\phi(t), t), \\ \phi(0) = \min_M f(\cdot\,, 0), \end{cases}$$

then

$$f(x,t) \geq \phi(t),$$

for every $x \in M$ and t in the definition domain of ϕ.

An analogous result holds for the behaviour of maximum. More precisely, the following result holds.

Theorem 3.1.3. *Let $\{g_t\}_{t\in[0,T)}$ be a smooth family of Riemannian metrics on a compact manifold M and suppose that $f \in C^\infty(M \times [0,T))$ is a solution of the differential inequality*

$$f_t - \Delta_{g_t} f \leq g(V, \nabla^{g_t} f) + Q(f,t),$$

where here ∇^{g_t} is the Levi-Civita connection associated with g_t, Δ_{g_t} is the Laplacian operator associated with g_t, V is a bounded time-dependent vector field and Q is continuous in time and locally Lipschitz in space. If $\theta : [0,T) \to \mathbb{R}$ is the solution of the associated ODE

$$\begin{cases} \theta'(t) = Q(\theta(t), t), \\ \theta(0) = \max_M f(\cdot\,, 0), \end{cases}$$

then

$$f(x,t) \leq \theta(t),$$

for every $x \in M$ and t in the definition domain of θ.

In the next theorem, we give a characterisation of the maximal time of solutions of the mean curvature flow.

Theorem 3.1.4. *Let M be a compact manifold and $F_0 \colon M \to N$ a smooth immersion into a complete Riemannian manifold N. Then, the maximal time $T_{\max}$ of the solution of the mean curvature flow, with initial data the immersion F_0, is finite if and only if*

$$\limsup_{t\to T}\big(\max_{M\times[0,t]}|A|^2\big) = \infty.$$

Equivalently, if $F : M \times [0,T) \to N$ is solution of the mean curvature flow with initial data the immersion F_0, and the second fundamental forms of the evolved submanifolds are uniformly bounded in time, then there exist $\varepsilon > 0$ such that the flow smoothly extends in the interval $[0, T+\varepsilon)$.

The characterisation of the maximal time of the solution has been done by Huisken [15, 16] and is based on the parabolic maximum principle. The key observation is that all higher derivatives $\nabla^k A$, $k \in \mathbb{N}$, of the second fundamental tensor are uniformly bounded, once A is uniformly bounded.

3.2 Evolution equations

We will compute in this section the evolution equations of several important quantities. In order to simplify the notation, we omit upper or lower indices on connections and Laplacians which identify the corresponding bundles where they are defined. Most of these computations can be found in [3, 26–29, 35, 41, 42]. We follow the index notation introduced in Subsection 1.4.

Lemma 3.2.1. *Suppose that $F : M \times [0,T) \to N$ is a solution of the mean curvature flow. Then, the following facts are true:*

(a) *The induced metrics g evolve in time under the equation*

$$\big(\nabla_{\partial_t} g\big)(X,Y) = -2\langle H, A(X,Y)\rangle = -2A^H(X,Y).$$

(b) *There exists a local smooth time-dependent tangent orthonormal frame field and a local smooth time-dependent orthonormal frame field along the normal bundle of the evolving submanifolds.*

(c) *The induced volume form $d\mu$ on (M,g) evolves according to the equation*

$$\nabla_{\partial_t} d\mu = -|H|^2 d\mu.$$

Moreover, the volumes $\mathrm{Vol}(M, g_t)$ *of the evolved submanifolds satisfy*

$$\partial_t \mathrm{Vol}(M, g_t) = -\int |H|^2 d\mu.$$

Proof. (a) Let $v_1, \dots, v_m$ be time-independent tangent vector fields. Keeping in mind the notation introduced in Example 1.1.14, we have

$$\nabla_{\partial_t} F_i = \nabla_{v_i} F_t + dF\big([\partial_t, v_i]\big) = \nabla_{v_i} H,$$

for any $i \in \{1, \dots, m\}$. Therefore, for any $i, j \in \{1, \dots, m\}$, we deduce that

$$\begin{aligned}
(\nabla_{\partial_t} g)_{ij} &= \partial_t(g(v_i, v_j)) - g(\nabla_{\partial_t} v_i, v_j) - g(v_i, \nabla_{\partial_t} v_j) \\
&= \partial_t\langle F_i, F_j\rangle = \langle \nabla_{v_i} H, F_j\rangle + \langle \nabla_{v_j} H, F_i\rangle \\
&= -\langle H, \nabla_{v_i} F_j\rangle - \langle H, \nabla_{v_j} F_i\rangle \\
&= -2\langle H, A_{ij}\rangle.
\end{aligned}$$

(b) The associated adjoint operator $P\colon (TM, g_t) \to (TM, g_t)$ of A^H satisfies

$$A^H(X,Y) = g_t(PX,Y) = g_t(X,PY). \tag{3.1}$$

Consider the family $U_t : (TM, g_0) \to (TM, g_t)$, given as the solution of

$$\begin{cases} \nabla_{\partial_t} U_t = P \circ U_t, \\ U_0 = I. \end{cases}$$

We claim that $U_t^* g_t = g_0$. Indeed! Choose a local coordinate basis $\{\partial_1, \dots, \partial_m\}$ around a point x_0. Using the result in part (a) and that $[\partial_t, \partial_i] = 0$, we have

$$\begin{aligned} \partial_t(U_t^* g_t(\partial_i, \partial_j)) &= \partial_t(g_t(U_t\partial_i, U_t\partial_j)) \\ &= (\nabla_{\partial_t} g_t)(U_t\partial_i, U_t\partial_j) + g_t(\nabla_{\partial_t} U_t\partial_i, U_t\partial_j) + g_t(U_t\partial_i, \nabla_{\partial_t} U_t\partial_j) \\ &= -2A^H(U_t\partial_i, U_t\partial_j) + g_t((\nabla_{\partial_t} U_t)\partial_i, U_t\partial_j) + g_t(U_t\partial_i, (\nabla_{\partial_t} U_t)\partial_j) \\ &= -2A^H(U_t\partial_i, U_t\partial_j) + g_t(PU_t\partial_i, U_t\partial_j) + g_t(U_t\partial_i, PU_t\partial_j). \end{aligned}$$

From (3.1) we deduce that $U_t^* g_t = U_0 g_0 = g_0$. Hence, if $\{e_1(0), \dots, e_m(0)\}$ is orthonormal with respect to g_0, then

$$\{e_1 = U_t e_1(0), \dots, e_m = U_t e_m(0)\}$$

is orthonormal with respect to g_t. In fact,

$$\nabla_{\partial_t} e_i = P e_i = \sum_{\alpha, j} H^\alpha h_{ij}^\alpha e_j. \tag{3.2}$$

By taking the orthogonal complement of $\{e_1, \dots, e_m\}$, we get a time-dependent frame field on the normal bundles of the evolving submanifolds.

(c) Consider a time-dependent orthonormal frame field $\{e_1, \dots, e_m\}$ satisfying (3.2) and denote by $\{\omega_1, \dots, \omega_m\}$ the corresponding dual frame. Then,

$$\nabla_{\partial_t} \omega_i = -\sum_\alpha H^\alpha h_{i1}^\alpha \omega_1 - \cdots - \sum_\alpha H^\alpha h_{im}^\alpha \omega_m,$$

for any $i \in \{1, 2\}$. Hence,

$$\nabla_{\partial_t} d\mu = \nabla_{\partial_t}(\omega_1 \wedge \cdots \wedge \omega_m) = -|H|^2 \omega_1 \wedge \cdots \wedge \omega_m = -|H|^2 d\mu.$$

By integrating we get

$$\partial_t \mathrm{Vol}(M, g_t) = -\int |H|^2 d\mu,$$

and this completes the proof. □

Lemma 3.2.2. *The time-derivative of the second fundamental form is given by*

$$\big(\nabla^{\perp}_{\partial_t} A\big)^{\alpha}_{ij} = \big(\nabla^{\perp 2} H\big)^{\alpha}_{ij} - \sum_{k,\beta} H^{\beta} h^{\beta}_{jk} h^{\alpha}_{ik} - \sum_{\beta} H^{\beta} \widetilde{R}_{\beta ij\alpha},$$

where the indices are with respect to a local orthonormal frame.

Proof. Suppose that $\{e_1, \dots, e_m; \xi_{m+1}, \dots, \xi_n\}$ is a local adapted orthonormal frame field around a fixed point (x_0, t_0). Recall that

$$\nabla_{\partial_t}\partial_t = 0,\ \nabla_{e_i}\partial_t = 0 \ \text{ and } \ [\partial_t, e_i] = \nabla_{\partial_t} e_i = \sum_{j,\beta} H^{\beta} h^{\beta}_{ij} e_j. \tag{3.3}$$

In order to simplify the computations, we may assume that the frame $\{e_1, \dots, e_m\}$ is a normal frame at (x_0, t_0). Under these considerations, we have that at (x_0, t_0)

$$\begin{aligned}(\nabla_{\partial_t} A)_{ij} &= \nabla_{\partial_t}\nabla_{e_i} F_j - \nabla_{\partial_t} dF(\nabla_{e_i} e_j) - A(\nabla_{\partial_t} e_i, e_j) - A(e_i, \nabla_{\partial_t} e_j)\\ &= \nabla_{e_i}\nabla_{\partial_t} F_j + \widetilde{R}(H, F_i)F_j + \nabla_{\nabla_{\partial_t} e_i} F_j\\ &\quad - dF\big(\nabla_{\partial_t}\nabla_{e_i} e_j\big) - A(\nabla_{\partial_t} e_i, e_j) - A(e_i, \nabla_{\partial_t} e_j).\end{aligned}$$

Hence,

$$\begin{aligned}(\nabla_{\partial_t} A)_{ij} &= \nabla_{e_i}\big(\nabla_{e_j} H + dF(\nabla_{\partial_t} e_j)\big) + \widetilde{R}(H, F_i)F_j\\ &\quad + \nabla_{\nabla_{\partial_t} e_i} F_j - dF\big(\nabla_{\partial_t}\nabla_{e_i} e_j\big) - A(\nabla_{\partial_t} e_i, e_j) - A(e_i, \nabla_{\partial_t} e_j)\\ &= \nabla^2_{e_i, e_j} H + \widetilde{R}(H, F_i)F_j + \nabla_{e_i} dF(\nabla_{\partial_t} e_j)\\ &\quad + \nabla_{\nabla_{\partial_t} e_i} F_j - dF\big(\nabla_{\partial_t}\nabla_{e_i} e_j\big) - A(\nabla_{\partial_t} e_i, e_j) - A(e_i, \nabla_{\partial_t} e_j)\end{aligned}$$

and so

$$(\nabla_{\partial_t} A)_{ij} = \nabla^2_{e_i, e_j} H + \widetilde{R}(H, F_i)F_j - dF\big(R^{\nabla}(\partial_t, e_i)e_j\big),$$

where R^{∇} is the curvature operator of ∇ on $T(M \times (0, T))$. Hence, at (x_0, t_0) we have

$$\begin{aligned}(\nabla^{\perp}_{\partial_t} A)_{ij} &= \sum_{\alpha} \langle (\nabla^{\perp}_{\partial_t} A)_{ij}, \xi_\alpha\rangle \xi_\alpha = \sum_{\alpha} \langle (\nabla_{\partial_t} A)_{ij}, \xi_\alpha\rangle \xi_\alpha\\ &= \sum_{\alpha} \langle \nabla_{e_i}\nabla_{e_j} H, \xi_\alpha\rangle \xi_\alpha - \sum_{\alpha,\beta} H^{\beta} \widetilde{R}_{\beta ij\alpha} \xi_\alpha.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \nabla_{e_i}\nabla_{e_j}H,\xi_\alpha\rangle &= \langle \nabla^{\perp}_{e_i}\big(\nabla^{\perp}_{e_j}H+\sum_k\langle\nabla_{e_j}H,F_k\rangle F_k\big),\xi_\alpha\rangle\\ &= (\nabla^{2\perp}H)^\alpha_{ij}-\sum_{k,\beta}H^\beta h^\beta_{jk}h^\alpha_{ik}.\end{aligned}$$

Combining the last two equalities we obtain the result. □

Lemma 3.2.3. *The mean curvature H evolves in time under the equation*

$$(\nabla^{\perp}_{\partial_t}H)^\alpha = (\Delta^\perp H)^\alpha - \sum_{i,\beta}H^\beta\widetilde{R}_{\beta ii\alpha}+\sum_{i,j,\beta}H^\beta h^\beta_{ij}h^\alpha_{ij}.$$

Moreover,

$$\partial_t|H|^2=\Delta|H|^2-2|\nabla^\perp H|^2+2|A^H|^2-2\sum_{i,\alpha,\beta}H^\alpha H^\beta\widetilde{R}_{\alpha ii\beta},$$

where the indices are with respect to a local orthonormal frame.

Proof. Let $(x_0,t_0)\in M\times(0,T)$ and $\{e_1,\dots,e_m;\xi_{m+1},\dots,\xi_n\}$ be a local orthonormal frame field around of (x_0,t_0). From (3.3) and Lemma 3.2.2, we have

$$\begin{aligned}\big(\nabla^{\perp}_{\partial_t}H\big)^\alpha &= \sum_i\big(\nabla^{\perp}_{\partial_t}A_{ii}\big)^\alpha = \sum_i(\nabla^{\perp}_{\partial_t}A)^\alpha_{ii}+2\sum_i A^\alpha(\nabla_{\partial_t}e_i,e_i)\\ &= (\Delta^\perp H)^\alpha+\sum_{i,\beta}H^\beta\widetilde{R}_{\beta ii\alpha}-\sum_{i,j,\beta}H^\beta h^\beta_{ij}h^\alpha_{ij}+2\sum_{i,j,\beta}H^\beta h^\beta_{ij}h^\alpha_{ij},\end{aligned}$$

from where we deduce the evolution equation for H. Moreover

$$\begin{aligned}\partial_t|H|^2 &= \partial_t\langle H,H\rangle = 2\langle\nabla^{\perp}_{\partial_t}H,H\rangle = 2\sum_\alpha(\nabla^{\perp}_{\partial_t}H)^\alpha H^\alpha\\ &= 2\sum_\alpha(\Delta H)^\alpha H^\alpha-2\sum_{i,\alpha,\beta}H^\alpha H^\beta\widetilde{R}_{\alpha ii\beta}+2\sum_{i,j,\alpha,\beta}H^\alpha H^\beta h^\alpha_{ij}h^\beta_{ij}.\end{aligned}$$

On the other hand

$$\sum_\alpha\Delta(H^\alpha)^2=2\sum_\alpha(\Delta H)^\alpha H^\alpha+2\sum_\alpha|\nabla H^\alpha|^2.$$

Combining the last two identities we obtain the desired identity. □

Lemma 3.2.4 (Simons' formula)**.** *The Laplacian of the second fundamental form satisfies the following equation*

$$\begin{aligned}
\left(\Delta^{\perp} A\right)_{ij}^{\alpha} = & \left(\nabla^{\perp 2} H\right)_{ij}^{\alpha} - \sum_{k} (\widetilde{\nabla}_{e_k} \widetilde{R})_{kij\alpha} - \sum_{k} (\widetilde{\nabla}_{e_i} \widetilde{R})_{kjk\alpha} \\
& - 2 \sum_{k,\beta} h_{ik}^{\beta} \widetilde{R}_{kj\beta\alpha} - 2 \sum_{k,\beta} h_{jk}^{\beta} \widetilde{R}_{ki\beta\alpha} + 2 \sum_{k,l} h_{kl}^{\alpha} \widetilde{R}_{kijl} \\
& - \sum_{k,\beta} h_{ij}^{\beta} \widetilde{R}_{k\beta k\alpha} + \sum_{k,l} h_{il}^{\alpha} \widetilde{R}_{kjkl} + \sum_{k,l} h_{jl}^{\alpha} \widetilde{R}_{iklk} - \sum_{\beta} H^{\beta} \widetilde{R}_{\beta ij\alpha} \\
& + \sum_{k,l,\beta} h_{kl}^{\alpha} \left(h_{kj}^{\beta} h_{il}^{\beta} - h_{ij}^{\beta} h_{kl}^{\beta}\right) + \sum_{k,l,\beta} h_{jl}^{\alpha} \left(h_{kk}^{\beta} h_{il}^{\beta} - h_{ik}^{\beta} h_{kl}^{\beta}\right) \\
& + \sum_{k,l,\beta} h_{jk}^{\beta} \left(h_{kl}^{\alpha} h_{il}^{\beta} - h_{il}^{\alpha} h_{kl}^{\beta}\right),
\end{aligned}$$

where the indices are with respect to a local orthonormal frame.

Proof. Since the formula is tensorial, all computations can be made at a fixed point x_0, where we may suppose that we have an orhonormal frame such that $\nabla_{e_j} e_i = 0$. Consequently, at this point, we get

$$A_{ij} = \nabla_{e_j} dF(e_i) - dF(\nabla_{e_j} e_i) = \nabla_{e_j} dF(e_i).$$

From the Codazzi equation (1.5), we have

$$(\nabla_{e_k}^{\perp} A)_{ij} = (\nabla_{e_i}^{\perp} A)_{kj} - \sum_{\alpha} \widetilde{R}_{kij\alpha} \xi_{\alpha}.$$

Differentiating once more, we obtain

$$\begin{aligned}
(\nabla_{e_k}^{\perp} \nabla_{e_k}^{\perp} A)_{ij} &= \nabla_{e_k}^{\perp} \left((\nabla_{e_k}^{\perp} A)_{ij}\right) \\
&= \nabla_{e_k}^{\perp} \left(\nabla_{e_i}^{\perp} A_{kj} - A(\nabla_{e_i} e_k, e_j) - A(e_k, \nabla_{e_i} e_j)\right) \\
&\quad - \sum_{\alpha} e_k \left(\widetilde{R}_{kij\alpha}\right) \xi_{\alpha} - \sum_{\alpha} \widetilde{R}_{kij\alpha} \nabla_{e_k}^{\perp} \xi_{\alpha}.
\end{aligned}$$

Note that

$$\nabla_{e_k}^{\perp} \nabla_{e_i}^{\perp} A_{kj} = \sum_{\alpha} h_{kj}^{\alpha} \widetilde{R}^{\perp}(e_k, e_i) \xi_{\alpha} + \nabla_{e_i}^{\perp} \nabla_{e_k}^{\perp} A_{kj}.$$

Denote by

$$\omega_{ij}(X) = \langle \nabla_X e_i, e_j \rangle \quad \text{and} \quad \omega_{\alpha\beta}(X) = \langle \nabla_X \xi_\alpha, \xi_\beta \rangle$$

the connection forms. By Weingarten's Formula, we have

$$\widetilde{\nabla}_{e_k} \xi_\alpha = -\sum_l h^\alpha_{kl} e_l + \sum_\beta \omega_{\alpha\beta}(e_k) \xi_\beta. \tag{3.4}$$

Using equation (3.4), we compute

$$\sum_\alpha \widetilde{R}(e_k, e_i, e_j, \widetilde{\nabla}_{e_k} \xi_\alpha) \xi_\alpha = -\sum_{\alpha,l} h^\alpha_{kl} \widetilde{R}_{kijl} \xi_\alpha + \sum_\alpha \widetilde{R}(e_k, e_i, e_j, \nabla^\perp_{e_k} \xi_\alpha) \xi_\alpha.$$

Observe that

$$\sum_\alpha \widetilde{R}(e_k, e_i, e_j, \nabla^\perp_{e_k} \xi_\alpha) \xi_\alpha = -\sum_\alpha \widetilde{R}_{kij\alpha} \nabla^\perp_{e_k} \xi_\alpha.$$

Therefore,

$$\begin{aligned}
(\nabla^\perp_{e_k} \nabla^\perp_{e_k} A)_{ij} = {} & \widetilde{R}^\perp(e_k, e_i) A_{kj} + \nabla^\perp_{e_i} \nabla^\perp_{e_k} A_{kj} - A(\nabla_{e_k} \nabla_{e_i} e_k, e_j) \\
& - A(e_k, \nabla_{e_k} \nabla_{e_i} e_j) - \sum\nolimits_\alpha (\widetilde{\nabla}_{e_k} \widetilde{R})_{kij\alpha} \xi_\alpha - \sum\nolimits_{\alpha,\beta} h^\beta_{kk} \widetilde{R}_{\beta ij\alpha} \xi_\alpha \\
& - \sum\nolimits_{\alpha,\beta} h^\beta_{ki} \widetilde{R}_{k\beta j\alpha} \xi_\alpha - \sum\nolimits_{\alpha,\beta} h^\beta_{kj} \widetilde{R}_{ki\beta\alpha} \xi_\alpha \\
& + \sum\nolimits_{l,\alpha} h^\alpha_{kl} \widetilde{R}_{kijl} \xi_\alpha.
\end{aligned} \tag{3.5}$$

Using again the Codazzi equation (1.5), we get that

$$\begin{aligned}
\nabla^\perp_{e_i} \nabla^\perp_{e_k} A_{kj} = {} & \nabla^\perp_{e_i} \big((\nabla^\perp_{e_k} A)_{kj} \big) + \nabla^\perp_{e_i} \big(A(\nabla_{e_k} e_j, e_k) + A(e_j, \nabla_{e_k} e_k) \big) \\
= {} & \nabla^\perp_{e_i} \nabla^\perp_{e_j} A_{kk} - 2A(\nabla_{e_i} \nabla_{e_j} e_k, e_k) - \sum\nolimits_\alpha (\widetilde{\nabla}_{e_i} \widetilde{R})_{kjk\alpha} \xi_\alpha \\
& - \sum\nolimits_{\alpha,\beta} h^\beta_{ki} \widetilde{R}_{\beta jk\alpha} \xi_\alpha - \sum\nolimits_{\alpha,\beta} h^\beta_{ij} \widetilde{R}_{k\beta k\alpha} \xi_\alpha - \sum\nolimits_{\alpha,\beta} h^\beta_{ik} \widetilde{R}_{kj\beta\alpha} \xi_\alpha \\
& + \sum\nolimits_{l,\alpha} h^\alpha_{il} \widetilde{R}_{kjk\alpha} \xi_\alpha + A(\nabla_{e_i} \nabla_{e_k} e_j, e_k) \\
& + A(e_j, \nabla_{e_i} \nabla_{e_k} e_k).
\end{aligned} \tag{3.6}$$

From Ricci equation (1.6), we have

$$\sum_\alpha h^\alpha_{kj} \widetilde{R}^\perp_{ki\alpha} = -\sum_{\alpha,\beta} h^\beta_{kj} \widetilde{R}_{ki\beta\alpha} \xi_\alpha + \sum_{l,\alpha} h^\alpha_{kj} h^\alpha_{il} A_{kl} - \sum_{l,\alpha} h^\alpha_{kj} h^\alpha_{kl} A_{il}. \tag{3.7}$$

Plugging (3.6) and (3.7) into (3.5), we deduce

$$\begin{aligned}(\nabla^\perp_{e_k}\nabla^\perp_{e_k}A)_{ij} &= \nabla^\perp_{e_i}\nabla^\perp_{e_j}A_{kk} - 2A(\nabla_{e_i}\nabla_{e_j}e_k, e_k) + A(\widetilde{R}_{ikk}, e_j) + A(\widetilde{R}_{ikj}, e_k)\\ &\quad -2\sum_{\alpha,\beta} h^\beta_{kj}\widetilde{R}_{ki\beta\alpha}\xi_\alpha + \sum_{l,\alpha} h^\alpha_{kj}h^\alpha_{il}A_{kl} - \sum_{l,\alpha} h^\alpha_{kj}h^\alpha_{kl}A_{il}\\ &\quad -\sum_{\alpha}(\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha}\xi_\alpha - \sum_{\alpha,\beta} h^\beta_{kk}\widetilde{R}_{\beta ij\alpha}\xi_\alpha - \sum_{\alpha,\beta} h^\beta_{ki}\widetilde{R}_{k\beta j\alpha}\xi_\alpha\\ &\quad +\sum_{l,\alpha} h^\alpha_{kl}\widetilde{R}_{kijl}\xi_\alpha - \sum_{\alpha}(\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha}\xi_\alpha - \sum_{\alpha,\beta} h^\beta_{ki}\widetilde{R}_{\beta jk\alpha}\xi_\alpha\\ &\quad -\sum_{\alpha,\beta} h^\beta_{ij}\widetilde{R}_{k\beta k\alpha}\xi_\alpha - \sum_{\alpha,\beta} h^\beta_{ik}\widetilde{R}_{kj\beta\alpha}\xi_\alpha + \sum_{l,\alpha} h^\alpha_{il}\widetilde{R}_{kjkl}\xi_\alpha.\end{aligned}$$

Differentiating and estimating at the point x_0, we have

$$\nabla_{e_i}\nabla_{e_j}e_k = \sum_l e_i\omega_{kl}(e_j)e_l.$$

Therefore,

$$\sum_k A(\nabla_{e_i}\nabla_{e_j}e_k, e_k) = \sum_{k>l} e_i\omega_{kl}(e_j)A_{kl} + \sum_{k<l} e_i\omega_{kl}(e_j)A_{kl} = 0.$$

Taking a trace and using the Gauss equation (1.4), we get

$$\begin{aligned}(\Delta^\perp A)_{ij} &= \sum_k \nabla^\perp_{e_i}\nabla^\perp_{e_j}H - \sum_{k,\alpha}(\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha}\xi_\alpha - \sum_{k,\alpha}(\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha}\xi_\alpha\\ &\quad -\sum_{k,\alpha,\beta} h^\beta_{ki}\widetilde{R}_{\beta jk\alpha}\xi_\alpha - \sum_{k,\alpha,\beta} h^\beta_{ij}\widetilde{R}_{k\beta k\alpha}\xi_\alpha - \sum_{k,\alpha,\beta} h^\beta_{ik}\widetilde{R}_{kj\beta\alpha}\xi_\alpha\\ &\quad -\sum_{\alpha,\beta} H^\beta\widetilde{R}_{\beta ij\alpha}\xi_\alpha - \sum_{k,\alpha,\beta} h^\beta_{ik}\widetilde{R}_{k\beta j\alpha}\xi_\alpha - 2\sum_{k,\alpha,\beta} h^\beta_{kj}\widetilde{R}_{ki\beta\alpha}\xi_\alpha\\ &\quad +\sum_{k,l,\alpha} h^\alpha_{il}\widetilde{R}_{kjkl}\xi_\alpha + \sum_{k,l,\alpha} h^\alpha_{kl}\widetilde{R}_{kijl}\xi_\alpha - \sum_{k,l,\alpha} h^\alpha_{lj}\widetilde{R}_{ikkl}\xi_\alpha\\ &\quad -\sum_{k,l,\alpha,\beta} h^\alpha_{lj}h^\beta_{ik}h^\beta_{kl}\xi_\alpha + \sum_{k,l,\alpha,\beta} h^\alpha_{lj}H^\beta h^\beta_{il}\xi_\alpha - \sum_{k,l,\alpha} h^\alpha_{lk}\widetilde{R}_{ikjl}\xi_\alpha\\ &\quad -\sum_{k,l,\alpha,\beta} h^\alpha_{lk}h^\beta_{ij}h^\beta_{kl}\xi_\alpha + \sum_{k,l,\alpha,\beta} h^\alpha_{lk}h^\beta_{kj}h^\beta_{il}\xi_\alpha + \sum_{k,l,\alpha,\beta} h^\beta_{kj}h^\beta_{il}h^\alpha_{kl}\xi_\alpha\\ &\quad -\sum_{k,l,\alpha,\beta} h^\beta_{kj}h^\beta_{kl}h^\alpha_{il}\xi_\alpha.\end{aligned}$$

From the 1st Bianchi identity it follows that

$$\sum_{k,\alpha,\beta} h^{\beta}_{ki}\widetilde{R}_{\beta jk\alpha}\xi_\alpha + \sum_{k,\alpha,\beta} h^{\beta}_{ki}\widetilde{R}_{k\beta j\alpha}\xi_\alpha = \sum_{k,\alpha,\beta} h^{\beta}_{ki}\widetilde{R}_{kj\beta\alpha}\xi_\alpha.$$

Now, we deduce that

$$\begin{aligned}
(\Delta^{\perp}A)^{\alpha}_{ij} \;=\; & (\nabla^{\perp 2}H)^{\alpha}_{ij} - \sum_k (\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha} - \sum_k (\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha} \\
& -2\sum_{k,\beta} h^{\beta}_{ik}\widetilde{R}_{kj\beta\alpha} - 2\sum_{k,\beta} h^{\beta}_{jk}\widetilde{R}_{ki\beta\alpha} + 2\sum_{k,l} h^{\alpha}_{kl}\widetilde{R}_{kijl} \\
& -\sum_{k,\beta} h^{\beta}_{ij}\widetilde{R}_{k\beta k\alpha} + \sum_{k,l} h^{\alpha}_{il}\widetilde{R}_{kjkl} + \sum_{k,l} h^{\alpha}_{jl}\widetilde{R}_{iklk} - \sum_{\beta} H^{\beta}\widetilde{R}_{\beta ij\alpha} \\
& +\sum_{k,l,\beta} h^{\alpha}_{kl}\big(h^{\beta}_{kj}h^{\beta}_{il} - h^{\beta}_{ij}h^{\beta}_{kl}\big) + \sum_{k,l,\beta} h^{\alpha}_{jl}\big(h^{\beta}_{kk}h^{\beta}_{il} - h^{\beta}_{ik}h^{\beta}_{kl}\big) \\
& +\sum_{k,l,\beta} h^{\beta}_{jk}\big(h^{\alpha}_{kl}h^{\beta}_{il} - h^{\alpha}_{il}h^{\beta}_{kl}\big).
\end{aligned}$$

This completes the proof. □

Lemma 3.2.5. *The second fundamental form evolves in time under the equation*

$$\begin{aligned}
(\nabla^{\perp}_{\partial_t}A - \Delta^{\perp}A)^{\alpha}_{ij} = & \sum_k (\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha} + \sum_k (\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha} \\
& +2\sum_{k,\beta} h^{\beta}_{ik}\widetilde{R}_{kj\beta\alpha} + 2\sum_{k,\beta} h^{\beta}_{jk}\widetilde{R}_{ki\beta\alpha} - 2\sum_{k,l} h^{\alpha}_{kl}\widetilde{R}_{kijl} \\
& +\sum_{k,\beta} h^{\beta}_{ij}\widetilde{R}_{k\beta k\alpha} - \sum_{k,l} h^{\alpha}_{il}\widetilde{R}_{kjkl} - \sum_{k,l} h^{\alpha}_{jl}\widetilde{R}_{kikl} \\
& -\sum_{k,l,\beta} h^{\alpha}_{kl}\big(h^{\beta}_{kj}h^{\beta}_{il} - h^{\beta}_{ij}h^{\beta}_{kl}\big) - \sum_{k,l,\beta} h^{\alpha}_{jl}\big(h^{\beta}_{kk}h^{\beta}_{il} - h^{\beta}_{ik}h^{\beta}_{kl}\big) \\
& -\sum_{k,l,\beta} h^{\beta}_{jk}\big(h^{\alpha}_{kl}h^{\beta}_{il} - h^{\alpha}_{il}h^{\beta}_{kl}\big) - \sum_{k,\beta} H^{\beta}h^{\beta}_{jk}h^{\alpha}_{ik},
\end{aligned}$$

where the indices are with respect to a local orthonormal frame.

Proof. The result is a direct consequense of Lemma 3.2.2 and Lemma 3.2.4. □

3.3 Evolution equations of parallel forms

Let $F : M \times [0, T) \to N$ be a solution of the mean curvature flow and suppose that Φ is a parallel k-tensor on N. Then, the pullback via F of Φ gives rise to a time-dependent k-form on M. As we will see in the next section, interesting situations occurs when N is a Kähler manifold and we consider as Φ the Kähler form of N, or when N is the Riemannian product $N_1 \times N_2$ and we consider the volume forms Ω_1 and Ω_2 of the manifolds N_1 and N_2, respectively. These evolution equations will be used extensively to examine if the mean curvature flow preserves the Lagrangian or the graphical condition of initial data.

Lemma 3.3.1. *The covariant derivative of the tensor* $S = F^*\Phi$ *is given by*

$$(\nabla_{e_s} S)_{i_1 \dots i_k} = \sum_{\alpha} \big(h^{\alpha}_{si_1} \Phi_{\alpha i_2 \dots i_k} + \cdots + h^{\alpha}_{si_k} \Phi_{i_1 \dots i_{k-1}\alpha} \big),$$

for any adapted local orthonormal frame field $\{e_1, \dots, e_m; \xi_{m+1}, \dots, \xi_n\}$ *along the submanifold.*

Proof. By a direct computation, we get that

$$\begin{aligned}(\nabla_{e_s} S)_{i_1 \dots i_k} &= e_s \Phi(F_{i_1}, F_{i_2}, \dots, F_{i_k}) \\ &= \Phi(\nabla_{e_s} F_{i_1}, F_{i_2}, \dots, F_{i_k}) + \cdots + \Phi(F_{i_1}, F_{i_2}, \dots, \nabla_{e_s} F_{i_k}) \\ &= \Phi(A_{si_1}, F_{i_2}, \dots, F_{i_k}) + \cdots + \Phi(F_{i_1}, F_{i_2}, \dots, A_{si_k}).\end{aligned}$$

Since, for any i, j we have that

$$A_{ij} = \sum_{\alpha,i,j} h^{\alpha}_{ij} \xi_{\alpha},$$

we obtain that

$$(\nabla_{e_s} S)_{i_1 \dots i_k} = \sum_{\alpha} \big(h^{\alpha}_{si_1} \Phi_{\alpha i_2 \dots i_k} + \cdots + h^{\alpha}_{si_k} \Phi_{i_1 \dots i_{k-1}\alpha} \big).$$

This completes the proof. □

By a direct computation we can derive the expression for the Laplacian of the pullback of a parallel k-tensor on N.

Lemma 3.3.2. *The Laplacian of the k-tensor $S = F^*\Phi$ is given by*

$$\begin{aligned}(\Delta S)_{i_1\dots i_k} \;=\; & \sum_\alpha (\nabla^\perp_{e_{i_1}} H)^\alpha \Phi_{\alpha i_2\dots i_k} + \dots + \sum_\alpha (\nabla^\perp_{e_{i_k}} H)^\alpha \Phi_{i_1\dots i_{k-1}\alpha} \\ & +2\sum_{s,\alpha,\beta} h^\alpha_{si_1} h^\beta_{si_2} \Phi_{\alpha\beta i_3\dots i_k} + \dots + 2\sum_{s,\alpha,\beta} h^\alpha_{si_{k-1}} h^\beta_{si_k}\Phi_{i_1\dots\alpha\beta} \\ & -\sum_{s,l,\alpha}\big(h^\alpha_{si_1}h^\alpha_{sl}\Phi_{li_2\dots i_k} + \dots + h^\alpha_{si_k}h^\alpha_{sl}\Phi_{i_1\dots i_{k-1}l}\big) \\ & -\sum_{s,\alpha}\big(\widetilde{R}_{s\alpha s i_1}\Phi_{\alpha i_2\dots i_k} + \dots + \widetilde{R}_{s\alpha s i_k}\Phi_{i_1\dots i_{k-1}\alpha}\big),\end{aligned}$$

for any adapted orthonormal frame field $\{e_1,\dots,e_m;\xi_{m+1},\dots,\xi_n\}$.

Proof. Let $\{e_1,\dots,e_m;\xi_{m+1},\dots,\xi_n\}$ be an adapted orthonormal frame which is normal at the point (x_0,t_0). We compute,

$$\begin{aligned}(\nabla_{e_s}\nabla_{e_s}S)_{i_1\dots i_k} &= e_s\big\{\Phi(A_{si_1},F_{i_2},\dots,F_{i_k}) + \dots + \Phi(F_{i_1},F_{i_2},\dots,A_{ki_k})\big\} \\ &= \Phi((\nabla_{e_s}A)_{si_1},F_{i_2},\dots,F_{i_k}) + \dots + \Phi(F_{i_1},F_{i_2},\dots,(\nabla_{e_s}A)_{si_k}) \\ &\quad +2\Phi(A_{si_1},A_{si_2},F_{i_3},\dots,F_{i_k}) + \dots + 2\Phi(F_{i_1},F_{i_2},\dots,A_{si_{k-1}},A_{si_k}).\end{aligned}$$

Making use of the Codazzi equation we obtain that

$$\begin{aligned}(\nabla_{e_s}A)_{si} &= (\nabla^\perp_{e_s}A)_{si} + \langle\nabla_{e_s}A_{si},F_l\rangle F_l = (\nabla^\perp_{e_s}A)_{is} - \langle A_{si},\nabla_{e_s}F_l\rangle F_l \\ &= (\nabla^\perp_{e_i}A)_{ss} - \widetilde{R}_{s\alpha si}\xi_\alpha - h^\alpha_{si}h^\alpha_{sl}F_l.\end{aligned}$$

Combining the last two identities we get the result. □

Lemma 3.3.3. *Suppose that $F : M\times[0,T)\to N$ is a solution of the mean curvature flow and let Φ be a parallel m-tensor on N. Then, $\varphi = *(F^*\Phi)$, where $*$ is the Hodge star operator with respect to the induced Riemannian metric g, evolves in time under the equation*

$$\begin{aligned}\partial_t\varphi - \Delta\varphi \;=\; & -2\sum_{k,\alpha,\beta} h^\alpha_{k1}h^\beta_{k2}\Phi_{\alpha\beta 3\dots m} - \dots - 2\sum_{k,\alpha,\beta} h^\alpha_{km-1}h^\beta_{km}\Phi_{1\dots\alpha\beta} \\ & +\sum_{k,l,\alpha}\big(h^\alpha_{k1}h^\alpha_{kl}\Phi_{l2\dots m} + \dots + h^\alpha_{km}h^\alpha_{kl}\Phi_{1\dots(m-1)l}\big) \\ & +\sum_{k,\alpha}\big(\widetilde{R}_{k\alpha k1}\Phi_{\alpha 2\dots m} + \dots + \widetilde{R}_{k\alpha km}\Phi_{1\dots(m-1)\alpha}\big),\end{aligned}$$

for any adapted orthonormal frame field $\{e_1,\dots,e_m;\xi_{m+1},\dots,\xi_n\}$.

Proof. Let us make our computations again, with respect to a time-dependent orthonormal frame field as in Lemma 3.2.1. We compute,

$$\begin{aligned} \partial_t u &= \partial_t (F^* \Phi)_{1\dots m} \\ &= \Phi(\nabla_{\partial_t} F_1, \dots, F_m) + \dots + \Phi(F_1, \dots, \nabla_{\partial_t} F_m). \end{aligned}$$

Taking into account the formulas (3.3), we have

$$\begin{aligned} \nabla_{\partial_t} F_i &= \nabla_{e_i} dF(\partial_t) + dF(\nabla_{\partial_t} e_i) \\ &= \nabla_{e_i} H + \sum_{k,\beta} H^\beta h^\beta_{ik} F_k, \end{aligned}$$

from where we see that

$$\nabla_{\partial_t} F_i = \nabla^\perp_{e_i} H,$$

for any $i \in \{1, \dots, m\}$. Hence, putting everything together, we deduce that

$$\partial_t u = \Phi\big(\nabla^\perp_{e_1} H, \dots, F_m\big) + \dots + \Phi\big(F_1, \dots, \nabla^\perp_{e_m} H\big).$$

Combining with Lemma 3.3.2 we obtain the result. □

Lemma 3.3.4. *Suppose that $F : M \times [0, T) \to N$ is a solution of the mean curvature flow and let Ω be a parallel m-form on N. Then,*

$$u = *(F^*\Omega),$$

where $*$ is the Hodge star operator with respect to the induced Riemannian metric g, evolves in time under the equation

$$\begin{aligned} u_t &= \Delta u + u|A|^2 \\ &\quad - \sum_{\alpha,\beta,k} \big(2h^\alpha_{1k} h^\beta_{2k} \Omega_{\alpha\beta 3\dots m} + \dots + 2h^\alpha_{(m-1)k} h^\beta_{mk} \Omega_{1\dots(m-2)\alpha\beta}\big) \\ &\quad - \sum_{\alpha,k} \big(\Omega_{\alpha 2\dots m} \widetilde{R}_{\alpha kk1} + \dots + \Omega_{1\dots(n-1)\alpha} \widetilde{R}_{\alpha kkm}\big), \end{aligned}$$

for any adapted orthonormal frame field $\{e_1, \dots, e_m; \xi_{m+1}, \dots, \xi_n\}$ along the evolved submanifolds.

3.4 Formation of singularities

Let $F : M \times [0, T) \to N$ be a solution of the (MCF) and assume that a singularity is formed in finite time. By Nash's Theorem, there exists an isometric embedding $i : N \to \mathbb{R}^N$ into a euclidean space. Consider $\overline{F} = i \circ F : M \times [0, T) \to \mathbb{R}^N$. We denote by $\overline{A}$ and $\overline{H}$ the second fundamental form and the mean curvature of the immersions $\{\overline{F}(\cdot\,, t)\}_{t\in[0,T)}$, and by A_N the second fundamental form of the Nash isometric embedding. Then

$$\overline{A}(X, Y) = A_N(dF(X), dF(Y)) + di(A(X, Y)),$$

for any $X, Y \in \mathfrak{X}(M)$. Consequently,

$$\overline{H} - di(H) = \text{trace}_g(A_N) = -V.$$

where g is the induced by F (time-dependent) metric on M. Observe that V is a bounded lower order term and $\overline{F}$ evolves under

$$d\overline{F}(\partial_t) = di(H) = \overline{H} + V. \tag{MMCF}$$

A solution of the form (MMCF), where V is a bounded lower-order term, is called solution of the *mean curvature flow with bounded additional force*.

To investigate the singularity formation along the mean curvature flow, let us introduce two important notions: the density, and the parabolic dilation.

3.4.1. Gaussian densities. Let us start by giving the definition of the density.

Definition 3.4.1. *Let $F : M \times [0, t_0) \to N \hookrightarrow \mathbb{R}^N$ be a solution of the* (MCF) *where M is compact and $t_0 < \infty$ is the maximal time of existence of the flow.*

(a) *For every point $(y, t) \in \mathbb{R}^N \times (\mathbb{R} - \{t_0\})$ the function*

$$\rho_{(y_0,t_0)}(y, t) = \frac{1}{4\pi(t_0 - t)^{\frac{m}{2}}} e^{\frac{-|y-y_0|^2}{4(t_0-t)}}$$

is called the backward heat kernel of $\mathbb{R}^m$ at $(y_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$.

(b) *The function $\Theta_{y_0} : [0, t_0) \to \mathbb{R}$ given by*

$$\Theta_{y_0}(t) = \int \rho_{(y_0,t_0)}(F, t)d\mu$$

is called the density function.

The following result is essentially due to Huisken [16] and is known as *Huisken's monotonicity formula.*

Theorem 3.4.2. *Let M be a compact m-dimensional smooth manifold and let $F : M \times [0,T) \to \mathbb{R}^N$ be a solution of* (MMCF)*. Then*

$$\frac{d}{dt}\int_{M_t}\rho_{(y_0,t_0)}(F,t)d\mu \leq C - \int_{M_t}\left|H + \frac{F^\perp}{2(t_0-t)} + \frac{V}{2}\right|^2 \rho_{(y_0,t_0)}(F,t)d\mu,$$

where C is a time-independent constant, $d\mu$ denotes the volume element of the evolved submanifold $M_t \subset \mathbb{R}^N$ and $F^\perp$ is the normal component of the position vector F.

Proof. Without loss of generality we assume that y_0 is the origin of $\mathbb{R}^N$. For simplicity let us introduce the function

$$\rho(x,t) = \rho_{(y_0,t_0)}(F(x,t),t) = \frac{1}{4\pi(t_0-t)^{\frac{m}{2}}} e^{\frac{-|F(x,t)|^2}{4(t_0-t)}}.$$

By straightforward computation, we have

$$\frac{d\rho}{dt} = \rho\left(\frac{m}{2(t_0-t)} - \frac{|F|^2}{4(t_0-t)^2} - \frac{\langle F, H+V\rangle}{2(t_0-t)}\right). \tag{3.8}$$

We will compute now the Laplacian of ρ. Let D be the Levi-Civita on $\mathbb{R}^N$ and $\{e_1,\ldots,e_m\}$ be a local tangent frame which is normal at a fixed point $x \in M$. At x we have

$$e_i(\rho) = -\frac{\rho}{2(t_0-t)}\langle F, F_i\rangle,$$

and

$$\begin{aligned} e_ie_i(\rho) &= -\frac{e_i(\rho)}{2(t_0-t)}\langle F,F_i\rangle - \frac{\rho}{2(t_0-t)}\big(\langle F_i,F_i\rangle + \langle F, D_{e_i}F_i\rangle\big) \\ &= \frac{\rho}{4(t_0-t)^2}\langle F,F_i\rangle^2 - \frac{\rho}{2(t_0-t)}\big(1+\langle F, A_{ii}\rangle\big). \end{aligned}$$

Summing over i, we obtain that

$$\Delta\rho = \rho\left(\frac{|F^\top|^2}{4(t_0-t)^2} - \frac{m}{2(t_0-t)} - \frac{\langle F,H\rangle}{2(t_0-t)}\right). \tag{3.9}$$

By (3.8) and (3.9) we deduce that

$$\frac{d\rho}{dt} + \Delta\rho = -\rho\left(\frac{|F^\perp|^2}{4(t_0-t)^2} + \frac{\langle F^\perp, H\rangle}{t_0-t} + \frac{\langle F^\perp, V\rangle}{2(t_0-t)}\right). \tag{3.10}$$

Following the same steps as in the proof of Lemma 3.2.1(c), we can show

$$\frac{d(d\mu)}{dt} = -\langle H, H+V\rangle d\mu. \tag{3.11}$$

Integrating, and using the formulas (3.10) and (3.11), we get

$$\begin{aligned}\frac{d}{dt}\int \rho\, d\mu &= \int \left(\frac{d\rho}{dt} - \langle H, H+V\rangle\rho\right) d\mu \\ &= \int \left(\frac{|V|^2}{4} - \frac{\langle F^\perp, V\rangle}{4(t_0-t)}\right)\rho\, d\mu - \int \left|\frac{F^\perp}{2(t_0-t)} + H + \frac{V}{2}\right|^2 \rho\, d\mu.\end{aligned}$$

Because $(t_0-t)^{-1}\rho$ is uniformly bounded as $t \to t_0$, we deduce that there exists a time-independent constant C such that

$$\frac{d}{dt}\int \rho\, d\mu \le C - \int \left|\frac{F^\perp}{2(t_0-t)} + H + \frac{V}{2}\right|^2 \rho\, d\mu.$$

This completes the proof. □

Corollary 3.4.3. *Let F be a solution of the* (MMCF) *as in Theorem 3.4.2. Then*

$$\lim_{t\to t_0}\int \rho_{(y_0,t_0)}(F,t)d\mu < \infty.$$

Hence, the density function has a limit as we are approaching the maximal time of existence.

From the result of Corollary 3.4.3 we are led to the following definition.

Definition 3.4.4. *Let F be a solution of the* (MMCF) *as in Theorem 3.4.2. Then the number*

$$\Theta(y_0,t_0) = \lim_{t\to t_0}\int \rho_{(y_0,t_0)}d\mu < \infty,$$

is called density at the point (y_0,t_0).

3.4.2. Parabolic rescalings. Now we will introduce a scaling method to model the formed singularities.

Definition 3.4.5. *Let $F : M \times [0, t_0) \to N \hookrightarrow \mathbb{R}^N$ be a solution of the* (MCF) *defined in a maximal time interval $[0, t_0)$ and $y_0 \in \mathbb{R}^N$. Then:*

(a) *The point y_0 is called a singular or a blow-up point of the flow, if there exist $x \in M$ such that*

$$\lim_{t\to t_0} F(x,t) = y_0 \quad \textit{and} \quad \limsup_{t\to t_0} |A(x,t)| = \infty.$$

In this case, a sequence $\{(x_i, t_i)\}_{i\in\mathbb{N}}$ is called blow-up sequence if

$$(x_i, t_i) \to (x, t_0),\ F(x_i, t_i) \to y_0 \textit{ and } |A|(x_i, t_i) = \max_{M\times[0,t_i]}|A| \to \infty.$$

(b) *The point y_0 is called a regular point of* (MCF)*, if there is $x \in M$ such that*

$$\lim_{t\to t_0} F(x,t) = y_0 \quad \textit{and} \quad \limsup_{t\to t_0} |A(x,t)| < \infty.$$

(c) *We say that a singular point y_0 is a* Type-I *singularity if there exists a blow-up sequence such that*

$$|A|^2(x_i, t_i) \le \frac{C}{t_0 - t_i}$$

for some constant C. Otherwise, we say that y_0 is a Type-II *singularity.*

So if y_0 is a singular point then for $t \to t_0$ a singularity of Type-I or Type-II will form at $y_0 \in N$ (and perhaps at other points as well).

Definition 3.4.6. *Let $F : M \times [0, t_0) \to N \hookrightarrow \mathbb{R}^N$ be a solution of the* (MCF) *defined in a maximal time interval $[0, t_0)$.*

(a) *The image $\mathfrak{M}$ of the map $F \times I : M \times [0, t_0) \to N \times \mathbb{R}$ given by*

$$(F \times I)(x,t) = (F(x,t), t)$$

is called the space-time track of the flow. Since N is isometrically embedded into $\mathbb{R}^N$ we can regard $\mathfrak{M}$ as subspace of $\mathbb{R}^N \times \mathbb{R}$.

(b) *The map $D_\nu : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$, $\nu > 0$ given by*

$$D_\nu(y,t) = \big(\nu(y - y_0), \nu^2(t - t_0)\big)$$

is called parabolic dilation of scale ν at (y_0, t_0).

The following general theorem is well-known and shows how one can analyse forming singularities of the MCF by parabolic dilations around points where the norm of the second fundamental form attains its maximum; for details see [16].

Theorem 3.4.7. *Let $F : M \times [0, t_0) \to N \hookrightarrow \mathbb{R}^N$ be a solution of the* (MCF) *defined in a maximal time interval $[0, t_0)$ and y_0 a singular point of the flow. For $\nu > 0$, consider the immersion $F^\nu : M \times [-\nu^{-2} t_0, 0) \to \mathbb{R}^N$ given by*

$$F^\nu(x, s) = \nu\big(F(x, t_0 + \nu^{-2} s) - y_0\big). \tag{3.12}$$

Hence,

$$M^\nu_s = \nu\big(M_{t_0+\nu^{-2}s} - y_0\big) \subset \mathbb{R}^N, \quad s \in [-\nu^2 t_0, 0),$$

where M^ν_s are the scaled submanifolds. Then the following facts hold:

(a) *If $\{e_1, \dots, e_m\}$ is a local tangent orthonormal frame along $M_{t_0+\nu^{-2}s}$, then $\{e^\nu_1 = \nu^{-1} e_1, \dots, e^\nu_m = \nu^{-1} e_m\}$ is a local tangent orthonormal frame along M^ν_s. Moreover, the volume form $d\mu^\nu_s$, the second fundamental form A^ν and the mean curvature H^ν of M^ν_s are given by the formulas:*

- $d\mu^\nu_s = \nu^m d\mu$,
- $A^\nu_{(x,s)} = (A + A_N)_{(x, t_0+\nu^{-2}s)}$,
- $H^\nu_{(x,s)} = \nu^{-1}(H - V)_{(x, t_0+\nu^{-2}s)}$,
- $|A^\nu|^2(x, s) = \nu^{-2}|A + A_N|^2(x, t_0 + \nu^{-2}s)$,

where A_N stands for the second fundamental form of the Nash's isometric embedding $N \hookrightarrow \mathbb{R}^N$ and

$$V = \sum_i A_N(e^\nu_i, e^\nu_i).$$

(b) *The family $\{M^\nu_s\}_{s\in[-\nu^{-2}t_0,0)}$ evolves by a mean curvature flow with bounded additional force. More precisely,*

$$dF^\nu_{(x,s)}(\partial_s) = H^\nu_{(x,s)} + \nu^{-1} V_{(x, t_0+\nu^{-1}s)}.$$

(c) *If the point y_0 is a Type-I singularity, then for fixed $s \le 0$, the sequence $\{M^\nu_s\}_{i\in\mathbb{N}}$ converge subsequentially and smoothly to a submanifold $M^\infty_s \subset \mathbb{R}^N$ as $\nu \to \infty$. Additionally, $\{M^\infty_s\}_{s\in(-\infty,0]}$ evolves by the standard mean curvature flow.*

From Theorem 3.4.7, we immediately see that the following result holds.

Theorem 3.4.8. *Let* $F : M \times [0, t_0) \to N \hookrightarrow \mathbb{R}^N$ *be a solution of the* (MCF) *defined in a maximal time interval* $[0, t_0)$ *and* y_0 *a singular point of the flow. Then the density function is invariant under parabolic dilations of the form* (3.12)*. That is*

$$\int_{M_t} \rho_{(y_0,t_0)}(F, t) d\mu_t = \int_{M_s^\nu} \rho_{(0,0)}(F^\nu, s) d\mu_s^\nu,$$

where $d\mu_s^\nu$ *is the volume form of* M_s^ν*.*

Let us emphasise here that if in the above Theorem 3.4.7 the point y_0 is a Type-II singularity of the flow, then the sequence $\{M_s^\nu\}_{\nu>0}$ of the parabolic rescalings converge to a limiting flow but in a weak sense. That being said the limiting object is no longer smooth. To overcome this problem, the trick is to take a blow-up sequence of space-time points $\{(x_i, t_i)\}_{i\in\mathbb{N}}$ and then perform appropriate parabolic rescalings with factors $\nu_i = |A|(x_i, t_i)$. More precisely, we have the following result:

Theorem 3.4.9. *Let* $M_t \subset N \hookrightarrow \mathbb{R}^N$, $0 \le t < t_0$*, be a family evolving by* (MCF) *and suppose that* y_0 *is a singular point of Type-II. Let* $\{t_i\}_{i\in\mathbb{N}}$ *be a sequence of times in* $[0, t_0 - 1/i]$ *and points* $\{x_i\}_{i\in\mathbb{N}} \subset M_t$ *such that* $x_i \to y_0$ *and*

$$(t_0 - 1/i - t_i)|A|^2(x_i, t_i) = \max_{M\times[0,t_0-1/i]} \big((t_0 - 1/i - t)|A|^2(x, t)\big).$$

Furthermore, set

$$\nu_i = |A|(x_i, t_i), \;\; a_i = -\nu_i^2 t_i \text{ and } b_i = \nu_i^2(t_0 - 1/i - t_i)$$

and form the rescalings

$$M_s^i = \nu_i\big(M_{t_i+\nu_i^{-2}s} - x_i\big), \quad s \in [a_i, b_i]. \tag{3.13}$$

Then the following facts hold:

(a) *We have that* $t_i \to t_0$, $\nu_i \to \infty$, $a_i \to -\infty$ *and* $b_i \to \infty$*. Moreover, the second fundamental form at time* t_i *is maximised at* x_i*.*

(b) *We can choose a subsequence of* $\{(x_i, t_i)\}_{i\in\mathbb{N}}$*, which for simplicity we denote with the same symbol, such that*

- $x_i \to y_0$.
- $|A|(x_i, t_i) \to \infty$ *monotonically.*
- $|A|(x_i, t_i)(t_0 - 1/i - t_i) \to \infty$.

Then the rescalings (3.13) *will converge locally smoothly to a limiting mean curvature flow* $\{M_s^\infty\}_{s\in(-\infty,\infty)} \subset \mathbb{R}^N$.

3.4.3. White's regularity theorem. A deep theorem of Allard [1, 33] provides a criterion for whether a point on a stationary integral varifold $\mathcal{V} \subset \mathbb{R}^n$ is regular. Let $B_r(p) \subset \mathbb{R}^n$ the ball of radius r centered at ta point p and ω_m the area of the unit ball in $\mathbb{R}^n$. Roughly speaking, according to Allard's Theorem, if the density

$$\vartheta(\mathcal{V}, p) = \lim_{r\to 0} \frac{\mathrm{Vol}(\mathcal{V} \cap B_r(p))}{\omega_n r^n},$$

at a point $p \in \mathcal{V}$ is sufficiently close to 1, then $\mathcal{V}$ is regular near p. White's Regularity Theorem [44] essentially says that the Gaussian density plays the same role also in the mean curvature flow. More precisely, the following result hold.

Theorem 3.4.10. *Let* $F : M \times [0, t_0) \to \mathbb{R}^N$ *be a solution of the* (MMCF) *and* (y_0, t_0) *be a point in the space-track of the flow. If*

$$\Theta(y_0, t_0) = \lim_{t\to t_0} \int \rho_{(y_0,t_0)}(F, t)d\mu \leq 1,$$

then y_0 *is a regular point of the flow.*

CHAPTER 4

Lagrangian Mean Curvature Flow

In this chapter, we will introduce the central object of our study: the Lagrangian mean curvature flow, which will be abbreviated by LMCF. The name LMCF is given because of a beautiful result proved by Smoczyk in [36], according to which compact Lagrangian submanifolds in a Kähler-Einstein manifold remain Lagrangian under the evolution by MCF. This phenomenon is very surprising because the MCF is a concept of Riemannian submanifold geometry, rather than one of symplectic geometry. Since Chapter 5 of this thesis is concerned with the evolution of symplectomorphisms by LMCF, we will present a detailed proof of Smoczyk's theorem.

4.1 Lagrangian submanifolds

Let $F : M \to N$ be an isometric immersion of an m-dimensional Riemannian manifold into an $2m$-dimensional Kähler manifold with complex structure J and corresponding Kähler form ω. We say that F is *Lagrangian* if $F^*\omega = 0$ or, equivalently, if

$$\omega(dF(X), dF(Y)) = \langle JdF(X), dF(Y)\rangle = 0,$$

for every $X, Y \in \mathfrak{X}(M)$. Note that if $F : M \to N$ is a Lagrangian submanifold, then the complex structure of the ambient space maps the tangent bundle of F onto the normal bundle. Hence, we may associate to the map F the trilinear form $C : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$, given by

$$C(X, Y, Z) = \langle A(X, Y), JdF(Z)\rangle,$$

where A is the second fundamental form of F. The trilinear form C is called the *fundamental cubic* of the Lagrangian submanifold.

Lemma 4.1.1. *The fundamental cubic C of a Lagrangian submanifold is fully symmetric. If, in addition, the Lagrangian is minimal, then C is traceless.*

Proof. By the definition of the second fundamental form, C is symmetric in the first two indices. Moreover

$$\begin{aligned} C(X,Y,Z) &= \langle A(X,Y), JdF(Z)\rangle = \langle \nabla_Y dF(X), JdF(Z)\rangle \\ &= \langle dF(X), -\nabla_Y JdF(Z)\rangle = \langle dF(X), -J\nabla_Y dF(Z)\rangle \\ &= \langle JdF(X), \nabla_Y dF(Z)\rangle = \langle JdF(X), A(Y,Z)\rangle \\ &= C(Z,Y,X), \end{aligned}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. This completes the proof. $\square$

Lemma 4.1.2. *The fundamental cubic C satisfies the identity*

$$\begin{aligned} (\nabla_X C)(Y,Z,W) \;=\; & (\nabla_Y C)(X,Z,W) \\ & -\widetilde{R}(dF(X), dF(Y), dF(Z), JdF(W)), \end{aligned}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$, where $\widetilde{R}$ is the curvature tensor on N.

Proof. Without loss of generality, we may assume that $\{X, Y, Z\}$ is part of the normal frame at a fixed point $x \in M$. Differentiating and estimating at x, we have

$$\begin{aligned} (\nabla_X C)(Y,Z,W) &= XC(Y,Z,W) \\ &= X\langle A(Y,Z), JdF(W)\rangle \\ &= \langle \nabla^{\perp}_X A(Y,Z), JdF(W)\rangle + \langle A(Y,Z), \nabla_X JdF(W)\rangle \\ &= \langle (\nabla^{\perp}_X A)(Y,Z), JdF(W)\rangle. \end{aligned}$$

By Codazzi equation we deduce

$$\begin{aligned} (\nabla_X C)(Y,Z,W) \;-\; & (\nabla_Y C)(Y,Z,W) \\ &= \langle (\nabla^{\perp}_X A)(Y,Z) - (\nabla^{\perp}_Y A)(X,Z), JdF(W)\rangle \\ &= -\widetilde{R}(dF(X), dF(Y), dF(Z), JdF(W)). \end{aligned}$$

This completes the proof. $\square$

4.2 Lagrangian MCF

In this section, we prove the result of Smoczyk [36] about the preservation of the Lagrangian property under the mean curvature flow. Before stating and proving the result let us introduce a definition and two important tensors.

Definition 4.2.1. *A submanifold $F : M \to N$ of a Kähler manifold N is called totally real if*

$$J(dF(T_xM)) \cap dF(T_xM) = \{0\},$$

for any $x \in M$.

For example, every Lagrangian submanifold of N is totally real. Consider the bundle morphisms $K : TM \to NM$ and $\Pi : TM \to TM$ given by

$$K(X) = (JdF(X))^{\perp} \quad \text{and} \quad \Pi(X) = (JdF(X))^{\top}.$$

In a totally real submanifold, both these tensors are isomorphisms. Suppose now that $\{e_1, \dots, e_m; \xi_{m+1}, \dots, \xi_{2m}\}$ is a local adapted orthonormal frame along the submanifold. Then,

$$K(F_i) = \sum_{\alpha} \langle JF_i, \xi_\alpha \rangle \xi_\alpha = \sum_{\alpha} \omega_{i\alpha} \xi_\alpha \tag{4.1}$$

and

$$\Pi(F_i) = \sum_{j} \langle JF_i, F_j \rangle F_j = \sum_{j} \omega_{ij} F_j. \tag{4.2}$$

Moreover, for any $i \in \{1, \dots, m\}$, we have

$$1 = |JF_i|^2 = |\Pi(F_i)|^2 + |K(F_i)|^2 = \sum_j \omega_{ij}^2 + |K(F_i)|^2.$$

Setting

$$a_i = |\Pi(F_i)|^2 = \sum_j \omega_{ij}^2,$$

it follows that

$$|K(F_i)| = \frac{1}{\sqrt{1 - a_i}},$$

for any $i \in \{1, \dots, m\}$. Additionally, the set of vectors

$$\left\{ F_1, \dots, F_m; \frac{K(F_1)}{\sqrt{1 - a_1}}, \dots, \frac{K(F_m)}{\sqrt{1 - a_m}} \right\} \tag{4.3}$$

forms a local orthonormal frame on the normal bundle NM.

Theorem 4.2.2 (Smoczyk). *Let N be a complete Kähler-Einstein manifold and $F_0 : M \to N$ a Lagrangian immersion, where M is a compact manifold. Then the mean curvature flow with initial data the immersion F_0 will preserve the Lagrangian condition.*

Proof. Let $F^*\omega$ be the pull-back via F on M of the Kähler form ω of N. Without loss of generality we may work locally along the images

$$M_t = F(\cdot\,, t)(M), \quad t \in [0, T),$$

and regard S as the restriction of ω on M_t. Define the function

$$f = \tfrac{1}{2}|F^*\omega|^2.$$

The goal is to show that satisfies a differential inequality of the form

$$\partial_t f - \Delta f \leq Cf,$$

so that we may apply the maximum principle.

Step 1: We compute the evolution equation of the function f. Let

$$\{e_1, \dots, e_m; \xi_{m+1}, \dots, \xi_{2m}\}$$

be a local adapted frame along the evolved submanifolds. Following the index notation introduced in Subsection 1.4 we denote the components of ω by

$$\omega_{ij} = \omega(F_i, F_j), \quad \omega_{\alpha j} = \omega(\xi_\alpha, F_j) \quad \text{and} \quad \omega_{\alpha\beta} = \omega(\xi_\alpha, \xi_\beta).$$

Then, with respect to such a frame, the function f can be written in the form

$$f = \tfrac{1}{2}\omega_{ij}^2.$$

According to Lemma 3.3.2 and keeping in mind the fact that ω is skew-symmetric, we obtain that

$$\begin{aligned}\Delta f \;\geq\; & 2\sum_{\alpha,i,j} \omega_{ij}\omega_{\alpha j}(\nabla^\perp_{e_i} H)^\alpha - 2\sum_{\alpha,i,j,k} \omega_{ij}\big(\omega_{\alpha j}\widetilde{R}_{kik\alpha} + \omega_{i\alpha}\widetilde{R}_{kjk\alpha}\big)\\ & +2\sum_{\alpha,\beta,i,j,k} \omega_{ij}\omega_{\alpha\beta}h^\alpha_{ki}h^\beta_{kj} - 2\sum_{\alpha,i,j,k,l} \omega_{ij}\omega_{lj}h^\alpha_{ki}h^\alpha_{kl}.\end{aligned}$$

On the other hand, proceeding as in Lemma 3.3.3, we can show that

$$\partial_t f = \sum_{i,j} \omega_{ij}\partial_t(\omega_{ij}) = 2\sum_{i,j}\omega_{ij}\omega_{\alpha j}(\nabla_{e_i} H)^\alpha.$$

Consequently,

$$\begin{aligned}\partial_t f - \Delta f \;\leq\; & 2 \sum_{\alpha,i,j,k,l} \omega_{ij}\omega_{lj} h^{\alpha}_{ki} h^{\alpha}_{kl} - 2 \sum_{\alpha,\beta,i,j,k} \omega_{ij}\omega_{\alpha\beta} h^{\alpha}_{ki} h^{\beta}_{kj} \\ & +2 \sum_{\alpha,i,j,k} \omega_{ij} \big(\omega_{\alpha j} \widetilde{R}_{kik\alpha} + \omega_{i\alpha} \widetilde{R}_{kjk\alpha}\big). \end{aligned} \tag{4.4}$$

Step 2: Now we estimate the first two terms of (4.4). Let $T_{\max}$ be the maximal time of solution of the mean curvature flow. Since our initial submanifold is Lagrangian, there exists $0 < T < T_{\max}$ so that the evolving submanifolds $\{M_t\}_{t\in[0,T)}$ are totally real, i.e.

$$J(T_x M_t) \cap T_x M_t = \{0\},$$

for any $x \in M_t$. This means that for any time $t \in [0, T]$ the bundle morphisms $K : TM_t \to NM_t$ and $\varPi : TM_t \to TM_t$ given by

$$K(X) = (JX)^{\perp} \quad \text{and} \quad \varPi(X) = (JX)^{\top}$$

are isomorphisms. Hence, for fixed normal vectors ξ and η along M_t, there are tangent vectors v and w such that

$$\xi = K(v) = (Jv)^{\perp} \quad \text{and} \quad \eta = K(w) = (Jw)^{\perp}.$$

Observe that

$$\begin{aligned} \omega(\xi,\eta) &= \omega(Jv - \varPi(v), Jw - \varPi(w)) \\ &= \omega(Jv, Jw) - \omega(\varPi(v), Jw) - \omega(Jv, \varPi(w)) + \omega(\varPi(v), \varPi(w)) \\ &= \omega(v, w) - \langle Jv, w\rangle + \langle v, Jw\rangle + \omega(\varPi(v), \varPi(w)) \\ &= \omega(\varPi(v), \varPi(w)) - \omega(v, w). \end{aligned}$$

This means that the quantity $\omega|_{NM_t}$ depends on values of the form $\omega|_{TM_t}$. Therefore, there exists a constant C_1 such that

$$2 \sum_{\alpha,i,j,k,l} \omega_{ij}\omega_{lj} h^{\alpha}_{ki} h^{\alpha}_{kl} - 2 \sum_{\alpha,\beta,i,j,k} \omega_{ij}\omega_{\alpha\beta} h^{\alpha}_{ki} h^{\beta}_{kj} \leq C_1 f.$$

Step 3: Now we claim that the last two terms in the differential inequality (4.4) can be also bounded by a term of the form $C_2 f$, where C_2 is a constant. Indeed! Let $\{e_1, \ldots, e_m\}$ be a local orthonormal tangent frame on M_t. Then,

$$\begin{aligned} B_{ijk} &= \sum\nolimits_{\alpha} \big(\omega_{\alpha j} \widetilde{R}_{kik\alpha} + \omega_{i\alpha} \widetilde{R}_{kjk\alpha}\big) = -\omega(\widetilde{R}^{\perp}_{kik}, F_j) - \omega(F_i, \widetilde{R}^{\perp}_{kjk}) \\ &= -\omega(\widetilde{R}_{kik} - \widetilde{R}^{\top}_{kik}, F_j) - \omega(F_i, \widetilde{R}_{kjk} - \widetilde{R}^{\top}_{kjk}). \end{aligned} \tag{4.5}$$

For our purpose, we need to investigate the behaviour of the terms

$$-\omega(\widetilde{R}_{kik}, F_j) - \omega(F_i, \widetilde{R}_{kjk}) = \widetilde{R}(F_i, F_j, F_k, JF_k),$$

where the last identity follows using the first Bianchi identity and the Kähler identities of Theorem 1.2.2. More precisely, it suffices to investigate only the behaviour of the terms

$$C_{ij} = \sum_k \widetilde{R}(F_i, F_j, F_k, JF_k). \tag{4.6}$$

Set for simplicity $X = F_i$ and $Y = JF_j$ and denote the Einstein constant of the Riemannian metric of N by k. Using the Kähler identities of Theorem 1.2.2, we see that

$$\begin{aligned}
k\,\omega(F_j, F_i) &= \widetilde{\mathrm{Ric}}(X, Y) \\
&= \sum_k \widetilde{R}(X, F_k, Y, F_k) + \sum_k \frac{1}{1-a_k}\widetilde{R}(X, K(F_k), Y, K(F_k)) \\
&= \sum_k \widetilde{R}(X, F_k, JY, JF_k) + \sum_k \frac{1}{1-a_k}\widetilde{R}(X, K(F_k), JY, JK(F_k)) \\
&= \sum_k \widetilde{R}(X, F_k, JY, K(F_k)) + \sum_k \widetilde{R}(X, F_k, JY, \Pi(F_k)) \\
&+ \sum_k \frac{a_k}{1-a_k}\widetilde{R}(X, K(F_k), JY, JK(F_k)) + \sum_k \widetilde{R}(X, K(F_k), JY, JK(F_k)) \\
&= \sum_k \widetilde{R}(X, F_k, JY, K(F_k)) + \sum_k \widetilde{R}(X, F_k, JY, \Pi(F_k)) \\
&+ \sum_k \frac{a_k}{1-a_k}\widetilde{R}(X, K(F_k), JY, JK(F_k)) \\
&+ \sum_k \widetilde{R}(X, K(F_k), JY, J(JF_k - \Pi(F_k))) \\
&= \sum_k \widetilde{R}(X, F_k, JY, K(F_k)) - \sum_k \widetilde{R}(X, K(F_k), JY, F_k) \\
&+ \sum_k \frac{a_k}{1-a_k}\widetilde{R}(X, K(F_k), JY, JK(F_k)) + \sum_k \widetilde{R}(X, F_k, JY, \Pi(F_k)) \\
&+ \sum_k \widetilde{R}(X, K(F_k), JY, J\Pi(F_k)).
\end{aligned}$$

Using the Bianchi identity in the first two terms of the last equality we see that

$$\sum_k \widetilde{R}(X, JY, F_k, JF_k) = k\,\omega_{ij} + \sum_k \widetilde{R}(X, K(F_k), JY, J\Pi(F_k))$$
$$+ \sum_k \frac{a_k}{1-a_k} \widetilde{R}(X, K(F_k), JY, JK(F_k)) + \sum_k \widetilde{R}(X, F_k, JY, \Pi(F_k)).$$

From (4.1), (4.2) and the fact that $a_i \le f$, we deduce that

$$\sum_{i,j} \omega_{ij} C_{ij} \le C_2 f,$$

where C_2 is a constant. Consequently, there exists a constant C such that $f : M \times [0, T] \to \mathbb{R}$ satisfies the inequality

$$\partial_t f - \Delta f \le C f.$$

From the maximum principle in Theorem 3.1.3 it follows that $f \equiv 0$. This completes the proof. □

CHAPTER 5

The Main Theorem

5.1 Statement of the main result

According to a beautiful result of Gromov [10], any symplectomorphism of $\mathbb{CP}^2$ can be continuously deformed into a biholomorphic isometry. It is not known whether the same result holds in any dimension. Medoš and Wang [22] applied the Lagrangian mean curvature flow to smoothly deform a symplectomorphism f of $\mathbb{CP}^m$. They proved that if f is sufficiently close to a biholomorphic isometry, then LMCF will smoothly deform f into a biholomorphic isometry. To explicitly state Medoš and Wang's theorem, we need the following:

Definition 5.1.1. *The map f is called Λ-pinched if*

$$\Lambda^{-2}g \leq f^*g \leq \Lambda^2 g, \tag{5.1}$$

for some constant number $\Lambda \geq 1$.

Main Theorem. *Given $m \in \mathbb{N}$, there exists a constant $\Lambda(m) > 1$, such that if $f : \mathbb{CP}^m \to \mathbb{CP}^m$ is an Λ-pinched symplectomorphism with $1 < \Lambda < \Lambda(m)$, then the following facts hold:*

(a) *There exists a family of symplectomorphisms $f_t : \mathbb{CP}^m \to \mathbb{CP}^m$, $t \in [0, \infty)$, $f_0 = f$, such that the corresponding graphs Σ_t of f_t move by Lagrangian mean curvature flow in $\mathbb{CP}^m \times \mathbb{CP}^m$.*

(b) *The family $\{f_t\}_{t\in[0,\infty)}$ of symplectomorphisms converges smoothly to a biholomorphic isometry of $\mathbb{CP}^m$, as $t \to \infty$.*

As a corollary of the above Main Theorem one immediately obtains the following topological result.

Corollary 5.1.2. *For any $m \in \mathbb{N}$, there exists a constant $\Lambda(m) > 1$, such that any Λ-pinched symplectomorphism $f : \mathbb{CP}^m \to \mathbb{CP}^m$ with $1 < \Lambda < \Lambda(m)$, is symplectically isotopic to a biholomorhic isometry.*

We would like to point out that this theorem generalises a previous theorem of Smale [31] and Wang [43] for symplectomorphisms of $\mathbb{CP}^1 = \mathbb{S}^2$ in which no pinching condition is required. It would be interesting to prove the conclusion of Corollary 5.1.2 without any hypothesis. It would be also very interesting to study the behaviour of the LMCF generated by symplectomorphisms between Kähler manifolds with constant non-positive holomorphic curvature.

We will divide the proof of the Main Theorem into the following 6 steps:

Step 1: Consider the graph of f in the Riemannian product $\mathbb{CP}^m \times \mathbb{CP}^m$. It turns out that the graph of f gives rise to a Lagrangian submanifold of the product.

Step 2: Consider the mean curvature flow of the graph generated by the initial symplectomorphism f. According to the result of Smoczyk [36], the flow will preserve the Lagrangian property.

Step 3: The graphical property is preserved under the LMCF. This fact follows from the parabolic maximum principle. Then the LMCF gives rise to a smooth family of symplectomorphisms which are isotopic to the given initial one.

Step 4: The LMCF exist for all times. To prove this result one needs to prove that the second fundamental forms of the evolved submanifolds stay uniformly bounded. Unfortunately, a-priori, such curvature estimates are not yet available. To overcome the problem, White's Regularity Theorem [44] is employed to show that there are no finite time singularities.

Step 5: From the fact that the LMCF exists for all times, the maximum principle implies that the pinching condition is improved; in particular the singular values of the evolved symplectomorphisms are approaching the value 1. Also this fact can be used now to show that the norms of the second fundamental forms are uniformly bounded.

Step 6: Smooth convergence is now achieved using a very deep general result of Simon [32] which ensures that the LMCF smoothly converges into a unique minimal limiting map. In this step the analytic structure of $\mathbb{CP}^m$ is required. Moreover, from the parabolic maximum principle we can show that the limiting map is actually a biholomorphic isometry.

5.2 Symplectomorphisms and Lagrangians

Let (M, g_M) be a Riemannian manifold and $M \times M$ be the product manifold. Denote by $\pi_1 : M \times M \to M$ and $\pi_2 : M \times M \to M$ the natural projections given by

$$\pi_1(x, y) = x \quad \text{and} \quad \pi_2(x, y) = y.$$

Clearly π_1 and π_2 are submersions. We can use the differentials $d\pi_1$ and $d\pi_2$ to define a canonical isomorphism

$$\Pi_{(x,y)} : T_{(x,y)}(M \times M) \to T_x M \times T_y M$$

given by

$$\Pi_{(x,y)}(V) = \big(d\pi_1(V), d\pi_2(V)\big),$$

for any $V \in T_{(x,y)}(M \times M)$. The 2-tensor $g_{M\times M}$ given by

$$g_{M\times M} = \pi_1^* g_M + \pi_2^* g_M$$

gives rise to a Riemannian metric on $M \times M$. With respect to $g_{M\times M}$ and g_M both projections becomes Riemannian submersions. The Levi-Civita connection $\nabla^{g_{M\times M}}$ associated to the Riemannian metric $g_{M\times M}$ on $M \times M$ is related to the Levi-Civita connection ∇^M on M by

$$\nabla^{g_{M\times M}} = \big(\pi_1^* \nabla^M, \pi_2^* \nabla^M\big).$$

Moreover, the corresponding curvature operator $R_{M\times M}$ on $M \times M$ with respect to the metric $g_{M\times M}$ is related to the curvature operators R_M on M

$$R_{M\times M} = \big(\pi_1^* R_M, \pi_2^* R_M\big). \tag{5.2}$$

Suppose that M has a complex structure J_M with associated Kähler form ω_M. One can easily verify that

$$J_{M\times M} = \big(\pi_1^* J_M, -\pi_2^* J_M\big) \tag{5.3}$$

forms a natural complex structure on the product whose associated Kähler form is

$$\omega_{M\times M} = \pi_1^* \omega_M - \pi_2^* \omega_M.$$

Consequently, the product of two Kähler manifolds is again a Kähler manifold. In particular, if M is Kähler-Einstein then also the Riemannian product $M \times M$ is Kähler-Einstein.

Suppose now that $f : M \to M$ is a smooth map. The *graph* of f is defined to be the submanifold

$$\Sigma = \{(x, f(x)) \in M \times M : \, x \in M\}$$

of the Riemannian product $M \times M$. The graph can be globally parametrized via the embedding $F : M \to M \times M$ given by

$$F = (I, f),$$

where $I : M \to M$ is the identity map. Since F is an embedding, it induces another Riemannian metric

$$g = F^* g_{M\times M} = g_M + f^* g_M.$$

on M. The following elementary observation will be very crucial.

Lemma 5.2.1. *Let $f : (M, g_M, J_M) \to (M, g_M, J_M)$ be a diffeomorphism of a Kähler manifold. Then f is a symplectomorphism if and only if its graph is a Lagrangian submanifold of $(M \times M, g_{M\times M}, J_{M\times M})$.*

Proof. Let $X, Y \in \mathfrak{X}(M)$. Then, by a direct computation we see that

$$\begin{aligned} F^*\omega_{M\times M}(X, Y) &= g_{M\times M}(J(dF(X)), dF(Y)) \\ &= g_M(J_M X, Y) - g_M(J_M df(X), df(Y)) \\ &= \omega_M(X, Y) - f^*\omega_M(X, Y). \end{aligned}$$

Thus $F^*\omega_{M\times M} = 0$ if and only if $f^*\omega_M = \omega_M$ and this completes the proof. □

5.3 Evolution of symplectomorphisms

Let M be a compact Kähler-Einstein manifold, assume that $f : M \to M$ is a symplectomorphism and let Σ be its graph in the product manifold $M \times M$. According to Lemma 5.2.1, the submanifold $\Sigma \subset M \times M$ is Lagrangian, and the mean curvature flow will preserve this property. Denote by $\{\Sigma_t\}_{t\in[0,T)}$ the evolved by the LMCF submanifolds, where T is the maximal time of the flow, that is $\Sigma_t = F(M \times [0, T))$, where $F : M \times [0, T) \to M \times M$ is the solution of the LMCF. Since M is compact, the evolving submanifolds will stay graphical at least on some time maximal interval $[0, T_g)$, with $0 < T_g \leq T$.

Let now Ω_M be the volume form on M. We can extend Ω_M to a parallel form on the product manifold $M \times M$ by pulling it back via the projection map π_1. That is, consider the parallel form

$$\Omega = \pi_1^* \Omega_M$$

and define the smooth functions

$$u = *(F^*\Omega) = *\{(\pi_1 \circ F)^*\Omega\} = *(I^*\Omega),$$

where $*$ stands for the Hodge star operator with respect to the induced metric g. Note that u is the Jacobian of the projection map from Σ_t to the first factor of $M \times M$. Therefore, the evolving submanifolds stay graphical as long as u is positive. In this case, there exists an 1-parameter family of diffeomorphisms $\phi_t : M \to M$ and maps $f_t : M \to M$ such that

$$F(\phi_t(x), t) = (x, f_t(x)), \quad x \in M.$$

In order to decide whether the graphical property is preserved under the flow we need to compute the evolution equation of the function u. For technical reasons, it is better to estimate quantities using the special frames of the singular value decomposition of df_x given in Chapter 2. Consider the isometry E given by

$$E = df_x(df_x^* df_x)^{-1/2}.$$

Let $\{\alpha_1, \alpha_2 = J_M\alpha_1, \ldots, \alpha_{2m-1}, J_M\alpha_{2m-1}\}$ and $\{E(\alpha_1), \ldots, E(\alpha_{2m})\}$ be the special orthonormal basis of T_xM and $T_{f(x)}M$ with respect to g_M, given in Lemma 2.2.6. Then, the vectors

$$e_i = \frac{1}{\sqrt{1+\lambda_i^2}}\big(\alpha_i, df_x(\alpha_i)\big) = \frac{1}{\sqrt{1+\lambda_i^2}}\big(\alpha_i, \lambda_i E(\alpha_i)\big), \ 1 \le i \le 2m, \quad (5.4)$$

form an orthonormal tangent basis with respect to g and

$$\begin{aligned} e_{2m+i} &= J_{M\times M} e_i = \frac{1}{\sqrt{1+\lambda_i^2}}\big(J_M\alpha_i, -J_M\lambda_i E(\alpha_i)\big) \\ &= \frac{1}{\sqrt{1+\lambda_i^2}}\big(J_M\alpha_i, -\lambda_i E(J_M\alpha_i)\big), \quad 1 \le i \le 2m, \end{aligned} \quad (5.5)$$

form an orthonormal basis of the normal bundle. In terms of this basis we have

$$u = \Omega(d\pi_1(e_1), \ldots, d\pi_1(e_{2m})) = \prod_{i=1}^{2m} \frac{1}{\sqrt{1+\lambda_i^2}}.$$

Because of the Lagrangian property, the second fundamental form A of Σ_t is characterised by coefficients

$$h_{ijk} = C(e_i, e_j, e_k) = g_{M\times M}\big(\nabla^{M\times M}_{e_i} e_j, J_{M\times M} e_k\big).$$

Recall from Lemma 4.1.1 that h_{ijk} is fully symmetric. Therefore, the information concerning the components of the second fundamental form is encoded in the vector h whose elements are formed by the "different" terms

$$h_{iii};\ h_{iij} \text{ with } i < j \text{ and } h_{ijk} \text{ with } i < j < k.$$

That is

$$h = (h_{111}, h_{222}, \dots; h_{112}, h_{113}, \dots; h_{123}, h_{124}, \dots). \tag{5.6}$$

Observe that

$$|h|^2 = \sum_i h^2_{iii} + \sum_{i<j} h^2_{iij} + \sum_{i,j,k} h^2_{ijk} \quad \text{and} \quad |A|^2 = \sum_{i,j,k} h^2_{ijk}.$$

Let us also introduce the singular value vector

$$\ell = (\lambda_1, \dots, \lambda_{2m}). \tag{5.7}$$

Following the index notation in Section 1.4, we have

$$R_{ijkl} = R(\alpha_i, \alpha_j, \alpha_k, \alpha_l) \quad \text{and} \quad \overline{R}_{ijkl} = (E^*R)(\alpha_i, \alpha_j, \alpha_k, \alpha_l).$$

Additionally, for any index $i \in \{1, \dots, 2m\}$, we set

$$i' = i + (-1)^{i+1}.$$

For instance, $1' = 2$ and $2' = 1$.

Lemma 5.3.1. *The function u satisfies the evolution equation*

$$u_t = \Delta u + u\left(Q(\ell, h) + \sum_{i,k} \frac{\lambda_i^2(R_{ikik} - \lambda_k^2 \overline{R}_{ikik})}{(1+\lambda_k^2)(1+\lambda_i^2)}\right),$$

where

$$Q(\ell, h) = \sum_{i,j,k} h^2_{ijk} - 2\sum_k \sum_{i<j} (-1)^{i+j} \lambda_i \lambda_j (h_{i'ik} h_{j'jk} - h_{i'jk} h_{j'ik}). \tag{5.8}$$

and h and ℓ are defined in (5.6) *and* (5.7)*, respectively. If $M \equiv \mathbb{CP}^m$ then*

$$u_t = \Delta u + u\left(Q(\ell, h) + \sum_{k=\text{odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2}\right), \tag{5.9}$$

where the indices are with respect to the special bases given in (5.4) *and* (5.5).

Proof. For simplicity let us set $J = J_{M\times M}$. By Lemma 3.3.4, we have

$$\begin{aligned} u_t \;=\; & \Delta u + u\sum_{i,j,k} h_{ijk}^2 \\ & -2\sum_{p,q,k}\sum_{i<j}\Omega(e_1,\dots,\underbrace{Je_p}_{i^{\text{th}}-\text{position}},\dots,\underbrace{Je_q}_{j^{\text{th}}-\text{position}},\dots,e_{2m})h_{pik}h_{qjk} \\ & -\sum_{p,k,i}\Omega(e_1,\dots,\underbrace{Je_p}_{i^{\text{th}}-\text{position}},\dots,e_{2m})R_{M\times M}(Je_p,e_k,e_k,e_i). \end{aligned}$$

Let

$$\mathcal{A} = u\sum_{i,j,k} h_{ijk}^2 - 2\sum_{p,q,k}\sum_{i<j}\Omega(e_1,\dots,\underbrace{Je_p}_{i^{\text{th}}-\text{position}},\dots,\underbrace{Je_q}_{j^{\text{th}}-\text{position}},\dots,e_{2m})h_{pik}h_{qjk}$$

and

$$\mathcal{B} = -\sum_{i,k,p}\Omega(e_1,\dots,\underbrace{Je_p}_{i^{\text{th}}-\text{position}},\dots,e_{2m})R_{M\times M}(Je_p,e_k,e_k,e_i).$$

From (5.5) we have that

$$\pi_1(Je_p) = \frac{1}{\sqrt{1+\lambda_p^2}}J\alpha_p.$$

Hence,

$$\begin{aligned} \mathcal{A} = & u\sum_{i,j,k} h_{ijk}^2 \\ & -2u\sum_{p,q,k}\sum_{i<j}\frac{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}}{\sqrt{(1+\lambda_p^2)(1+\lambda_q^2)}}\Omega(\alpha_1,\dots,\underbrace{J\alpha_p}_{i^{\text{th}}-\text{position}},\dots,\underbrace{J\alpha_q}_{j^{\text{th}}-\text{position}},\dots,\alpha_{2m})h_{pik}h_{qjk}. \end{aligned}$$

By the convention we use for the indices we deduce that $J\alpha_p = (-1)^{p+1}\alpha_{p'}$. Hence,

$$\Omega(\alpha_1,\dots,\underbrace{J\alpha_p}_{i^{\text{th}}-\text{position}},\dots,\underbrace{J\alpha_q}_{j^{\text{th}}-\text{position}},\dots,\alpha_{2m}) = (-1)^{i+j}(\delta_{pi'}\delta_{qj'} - \delta_{pj'}\delta_{qi'}),$$

since only non-zero terms are those for which $p = i'$ and $q = j'$ or $p = j'$ and $q = i'$. Observe that

$$\lambda_{i'} = \frac{1}{\lambda_i} \quad\text{and}\quad \frac{\sqrt{(1+\lambda_i^2)}}{\sqrt{(1+\lambda_{i'}^2)}} = \lambda_i.$$

Putting everything together, it follows that

$$\mathcal{A} = u\, Q(\ell, h).$$

Similarly as above, using (5.2) and the skew-symmetry of the curvature tensor $R_{M\times M}$ we derive

$$\mathcal{B} = u \sum_{i,k} (-1)^i \lambda_i R_{M\times M}(Je_{i'}, e_k, e_i, e_k).$$

Moreover,

$$R_{M\times M}(Je_{i'}, e_k, e_i, e_k) = \frac{(-1)^i \lambda_i}{(1+\lambda_i^2)(1+\lambda_k^2)}(R_{ikik} - \lambda_k^2 \overline{R}_{ikik}),$$

from where we deduce that

$$\mathcal{B} = u \sum_{i,k} \frac{\lambda_i^2}{(1+\lambda_i^2)(1+\lambda_k^2)}(R_{ikik} - \lambda_k^2 \overline{R}_{ikik}).$$

Suppose now that M is the complex projective space $\mathbb{CP}^m$. By the formula of Theorem 1.2.3 and direct straightforward computations, it follows that

$$\mathcal{B} = u \sum_{k=\text{odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2}.$$

This completes the proof. ☐

It will be very important in our analysis to understand the nature of the quadratic term

$$Q = Q(\ell, h)$$

given in equation (5.8). Observe at first that we can write this term in the form

$$\begin{aligned} Q &= \sum_{i,j,k} h_{ijk}^2 \qquad (5.10)\\ &-2 \sum_k \sum_{i=\text{odd}} (h_{iik} h_{i'i'k} - h_{ii'k}^2)\\ &-2 \sum_k \sum_{i=\text{odd}<j=\text{odd}} \big\{ -(\lambda_i\lambda_j + \lambda_{i'}\lambda_{j'}) h_{i'jk} h_{j'ik} + (\lambda_{i'}\lambda_j + \lambda_i\lambda_{j'}) h_{ijk} h_{j'i'k} \big\}\\ &-2 \sum_k \sum_{i=\text{odd}<j=\text{odd}} (\lambda_i - \lambda_{i'})(\lambda_j - \lambda_{j'}) h_{i'ik} h_{j'jk}. \end{aligned}$$

Lemma 5.3.2. *For $\ell = \ell_0 = (1, \dots, 1)$, we have that*

$$\mathcal{Q} = Q(\ell_0, h) \geq (3 - \sqrt{5})|h|^2.$$

Proof. Denote by A, B and C the three first summands of $\mathcal{Q}$. Since we assume that $\ell_0 = (1, \dots, 1)$ we get that

$$\begin{aligned} A \;=\; & \sum_i h_{iii}^2 + 3 \sum_{i=\text{odd}} \left(h_{ii'i'}^2 + h_{i'ii}^2\right) \\ & +3 \sum_{i=\text{odd}<j=\text{odd}} \left(h_{ijj}^2 + h_{ij'j'}^2 + h_{i'jj}^2 + h_{i'j'j'}^2 + h_{jii}^2 + h_{j'ii}^2 + h_{ji'i'}^2 + h_{j'i'i'}^2\right) \\ & +6 \sum_{i=\text{odd}<j=\text{odd}} \left(h_{ii'j}^2 + h_{ii'j'}^2 + h_{ijj'}^2 + h_{i'jj'}^2\right) \\ & +6 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} \left(h_{ijk}^2 + h_{ijk'}^2 + h_{ij'k}^2 + h_{ij'k'}^2 + h_{i'jk}^2 + h_{i'jk'}^2 + h_{i'j'k}^2 + h_{i'j'k'}^2\right) \end{aligned}$$

and

$$\begin{aligned} B \;=\; & -2 \sum_{i=\text{odd}} h_{iii} h_{i'i'i} + 2 \sum_{i=\text{odd}} h_{ii'i}^2 - 2 \sum_{i=\text{odd}} h_{iii'} h_{i'i'i'} + 2 \sum_{i=\text{odd}} h_{ii'i'}^2 \\ & -2 \sum_{i=\text{odd}\,<j=\text{odd}} \left(h_{iij} h_{i'i'j} - h_{ii'j}^2 + h_{iij'} h_{i'i'j'} - h_{ii'j'}^2\right) \\ & -2 \sum_{i=\text{odd}<j=\text{odd}} \left(h_{jji} h_{j'j'i} - h_{jj'i}^2 + h_{jji'} h_{j'j'i'} - h_{jj'i'}^2\right). \end{aligned}$$

Moreover,

$$\begin{aligned} C \;=\; & 4 \sum_{i=\text{odd}<j=\text{odd}} \left(h_{i'ji} h_{j'ii} - h_{iji} h_{j'i'i} + h_{i'ji'} h_{j'ii'} - h_{iji'} h_{j'i'i'}\right) \\ & +4 \sum_{i=\text{odd}<j=\text{odd}} \left(h_{i'jj} h_{j'ij} - h_{ijj} h_{j'i'j} + h_{i'jj'} h_{j'ij'} - h_{ijj'} h_{j'i'j'}\right) \\ & +4 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} \left(h_{i'jk} h_{j'ik} - h_{ijk} h_{j'i'k} + h_{i'jk'} h_{j'ik'} - h_{ijk'} h_{j'i'k'}\right) \\ & +4 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} \left(h_{j'ki} h_{k'ji} - h_{jki} h_{k'j'i} + h_{j'ki'} h_{k'ji'} - h_{jki'} h_{k'j'i'}\right) \\ & +4 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} \left(h_{i'kj} h_{k'ij} - h_{ikj} h_{k'i'j} + h_{i'kj'} h_{k'ij'} - h_{ikj'} h_{k'i'j'}\right). \end{aligned}$$

Therefore, we may write the crucial terms $\mathcal{Q}$ in the form

$$\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3$$

where $\mathcal{Q}_1$ contain terms which depends only on i, that is

$$\mathcal{Q}_1 = \sum_{i=\text{odd}} \left\{ h_{iii}^2 + h_{i'i'i'}^2 + 5(h_{ii'i'}^2 + h_{i'ii}^2) - 2h_{iii}h_{i'i'i} - 2h_{iii'}h_{i'i'i'} \right\},$$

the term $\mathcal{Q}_2$ contain quantities depending on $i$and j with $i < j$, namely

$$\begin{aligned}
\mathcal{Q}_2 &= 3 \sum_{i=\text{odd}<j=\text{odd}} (h_{ijj}^2 + h_{ij'j'}^2 + h_{i'jj}^2 + h_{i'j'j'}^2 + h_{jii}^2 + h_{j'ii}^2 + h_{ji'i'}^2 + h_{j'i'i'}^2) \\
&+ 8 \sum_{i=\text{odd}<j=\text{odd}} (h_{ii'j}^2 + h_{ii'j'}^2 + h_{ijj'}^2 + h_{i'jj'}^2) - 2 \sum_{i=\text{odd}<j=\text{odd}} (h_{iij}h_{i'i'j} + h_{iij'}h_{i'i'j'}) \\
&- 2 \sum_{i=\text{odd}<j=\text{odd}} (h_{jji}h_{j'j'i} + h_{jji'}h_{j'j'i'}) + 4 \sum_{i=\text{odd}<j=\text{odd}} (h_{i'ji}h_{j'ii} - h_{iji}h_{j'i'i}) \\
&+ 4 \sum_{i=\text{odd}<j=\text{odd}} (h_{i'ji'}h_{j'ii'} - h_{ijj'}h_{j'i'i'}) + 4 \sum_{i=\text{odd}<j=\text{odd}} (h_{i'jj}h_{j'ij} - h_{ijj}h_{j'i'j}) \\
&+ 4 \sum_{i=\text{odd}<j=\text{odd}} (h_{i'jj'}h_{j'ij'} - h_{ijj'}h_{j'i'j'}),
\end{aligned}$$

and finally $\mathcal{Q}_3$ contain terms with three different indices (i, j, k), that is

$$\begin{aligned}
\mathcal{Q}_3 &= 6 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} (h_{ijk}^2 + h_{ijk'}^2 + h_{ij'k}^2 + h_{ij'k'}^2 + h_{i'jk}^2 + h_{i'jk'}^2 + h_{i'j'k}^2 + h_{i'j'k'}^2) \\
&+4 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} (h_{j'ki}h_{k'ji} - h_{jki}h_{k'j'i} + h_{j'ki'}h_{k'ji'} - h_{jki'}h_{k'j'i'}) \\
&+4 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} (h_{i'kj}h_{k'ij} - h_{ikj}h_{k'i'j} + h_{i'kj'}h_{k'ij'} - h_{ikj'}h_{k'i'j'}) \\
&+4 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} (h_{i'jk}h_{j'ik} - h_{ijk}h_{j'i'k} + h_{i'jk'}h_{j'ik'} - h_{ijk'}h_{j'i'k'}).
\end{aligned}$$

Observe that:

(1) $\mathcal{Q}_1$ is the sum of two identical quadratic forms with 2-variables, each of which having $3 - \sqrt{5}$ as the smallest eigenvalue. Hence,

$$\mathcal{Q}_1 \geq (3 - \sqrt{5}) \sum_{i=\text{odd}} (h_{iii}^2 + h_{i'i'i'}^2 + h_{ii'i'}^2 + h_{i'ii}^2).$$

(2) $\mathcal{Q}_2$ is the sum of the quadratic forms of 3-variables, each having the number 2 as smallest eigenvalue. Consequently,

$$\mathcal{Q}_2 \geq 2 \sum_{i=\text{odd}<j=\text{odd}} \left(h^2_{ijj} + h^2_{ij'j'} + h^2_{i'jj} + h^2_{i'j'j'} + h^2_{jii} + h^2_{j'ii} + h^2_{j'i'i'} \right.$$
$$\left. + h^2_{j'i'i'} + h^2_{ii'j} + h^2_{i'j'} + h^2_{ijj'} + h^2_{i'jj'}\right).$$

(3) $\mathcal{Q}_3$ can written as the sum of two identical quadratic forms of 4-variables, each having smallest eigenvalue 4. Hence,

$$\mathcal{Q}_3 \geq 4 \sum_{i=\text{odd}<j=\text{odd}<k=\text{odd}} (h^2_{ijk} + h^2_{ijk'} + h^2_{ij'k} + h^2_{ij'k'} + h^2_{i'jk} + h^2_{i'jk'} + h^2_{i'j'k} + h^2_{i'j'k'}).$$

Therefore,

$$\mathcal{Q} \geq (3 - \sqrt{5})|h|^2$$

and this completes the proof. □

Lemma 5.3.3. *The following statements hold:*

(a) *In each dimension $m \in \mathbb{N}$, there exists a number $\Lambda_0(m)$ such that the quadratic term $Q(\ell, h)$ is non-negative whenever*

$$\Lambda_0^{-2}(m) \leq \lambda_i^2 \leq \Lambda_0^2(m),$$

for any index $i \in \{1, \dots, 2m\}$.

(b) *For any $1 \leq \Lambda_1 < \Lambda_0(m)$, there exists a positive number δ such that*

$$Q(\ell, h) \geq \delta \sum_{i,j,k} h^2_{ijk},$$

whenever

$$\Lambda_1^{-2} \leq \lambda_i^2 \leq \Lambda_1^2,$$

for any index $i \in \{1, \dots, 2m\}$.

(c) *If $M \equiv \mathbb{CP}^m$ and the singular values satisfy the pinching condition of part* (b)*, then u evolves in time under the equation*

$$u_t - \Delta u \geq \delta u|A|^2 + u \sum_{k=\text{odd}} \frac{(1 - \lambda_k^2)^2}{(1 + \lambda_k^2)^2}, \tag{5.11}$$

where δ is the constant given in part (b)*.*

Proof. Since

$$\frac{1}{6}\sum_{i,j,k} h_{ijk}^2 \leq |h|^2 \leq \sum_{i,j,k} h_{ijk}^2,$$

from Lemma 5.3.2, it follows that

$$Q((1,\ldots,1),h) \geq \frac{3-\sqrt{5}}{6}\sum_{i,j,k} h_{ijk}^2.$$

Since being a positive definite bilinear form is an open condition, it follows that there is an open subset U around $(1,\ldots,1)$ such that $\ell = (\lambda_1,\ldots,\lambda_m) \in U$ implies that $Q = Q(\ell,h)$ is positive definite. Let δ_ℓ be the smallest eigenvalue of Q at ℓ. Note that δ_ℓ depends continuously on ℓ. For fixed $\Lambda \geq 1$, set

$$\delta_\Lambda = \min\{\delta_\ell : \ell = (\lambda_1,\ldots,\lambda_m) \text{ and } \Lambda^{-2} \leq \lambda_1^2 \leq \cdots \leq \lambda_{2m}^2 \leq \Lambda^2\}.$$

The constant Λ_0 defined by

$$\Lambda_0 = \sup\{\Lambda : \Lambda \geq 1 \text{ and } \delta_\Lambda > 0\}$$

has the desired property. Now the claims of the lemma are clear. This completes the proof. □

5.4 Preservation of the initial conditions

We will show in this section that the LMCF will preserve the initial conditions under the assumptions of the main theorem. We start our investigation with some preliminary algebraic observations.

Lemma 5.4.1. *Let $\Lambda > 1$ be a constant so that $\Lambda^{-2} \leq \lambda_1^2 \leq \cdots \leq \lambda_{2m}^2 \leq \Lambda^2$. Then*

$$\frac{1}{2^m} - \varepsilon \leq u = \prod_i \frac{1}{\sqrt{(1+\lambda_i^2)}} \leq \frac{1}{2^m},$$

where

$$\varepsilon = \frac{1}{2^m} - \frac{1}{\left(\Lambda + \frac{1}{\Lambda}\right)^m} > 0.$$

Note that $u = 2^{-m}$ if and only if all the singular values are equal to 1.

Proof. We may write u equivalently in the form

$$u = \prod_i \frac{1}{\sqrt{(1+\lambda_i^2)}} = \prod_{i=\text{odd}} \frac{1}{(\lambda_{i'} + \lambda_i)}.$$

Since $\lambda_i \lambda_{i'} = 1$, we have that $\lambda_i + \lambda_{i'} \geq 2$. Therefore, the above expression has always an upper bound. As a matter of fact,

$$u = \prod_i \frac{1}{\sqrt{(1+\lambda_i^2)}} \leq \frac{1}{2^m}, \tag{5.12}$$

and the equality holds if and only if $\lambda_1 = \cdots = \lambda_{2m} = 1$. On the other hand, the function $h : (1, \infty) \to \mathbb{R}$ given by

$$h(x) = x + \frac{1}{x}$$

is increasing. Therefore, if $\Lambda^{-1} \leq \lambda_1 \leq \cdots \leq \lambda_{2m} \leq \Lambda$, then

$$\lambda_i + \lambda_{i'} = \frac{1}{\lambda_i'} + \lambda_i' = h(\lambda_i') \leq h(\Lambda) = \Lambda + \frac{1}{\Lambda}, \tag{5.13}$$

from where we deduce that

$$\frac{1}{2^m} - \varepsilon \leq \prod_i \frac{1}{\sqrt{(1+\lambda_i^2)}} \leq \frac{1}{2^m},$$

where

$$\varepsilon = \frac{1}{2^m} - \frac{1}{\left(\Lambda + \frac{1}{\Lambda}\right)^m}.$$

This completes the proof. □

We will see now that the converse of the above lemma is also true. Namely, if u has a lower positive bound, then each singular value is bounded from above.

Lemma 5.4.2. *Assume that there exists a constant* $\varepsilon \in (0, 2^{-m})$ *such that*

$$\frac{1}{2^m} - \varepsilon \leq \prod_i \frac{1}{\sqrt{(1+\lambda_i^2)}}.$$

Then,

$$\Lambda^{-2} \leq \lambda_1^2 \leq \cdots \leq \lambda_{2m}^2 \leq \Lambda^2,$$

where

$$1 < \Lambda = \frac{\frac{1}{2^m}}{\frac{1}{2^m} - \varepsilon} + \sqrt{\left(\frac{\frac{1}{2^m}}{\frac{1}{2^m} - \varepsilon}\right)^2 - 1}.$$

Proof. By assumption,

$$\frac{1}{2^m} - \varepsilon \leq \prod_i \frac{1}{\sqrt{(1+\lambda_i^2)}} = \prod_{i=\text{odd}} \frac{1}{(\lambda_{i'} + \lambda_i)}.$$

Hence,

$$\prod_{i=\text{odd}} (\lambda_{i'} + \lambda_i) \leq \frac{2^m}{1 - 2^m \varepsilon},$$

from where it follows that

$$\lambda_{i'} + \lambda_i \leq \prod_{j \neq i, j=\text{odd}} \frac{2^m}{(1 - 2^m\varepsilon)(\lambda_{j'} + \lambda_j)}.$$

Since $\lambda_j + \lambda_{j'} \geq 2$, the above inequality implies

$$\lambda_i + \lambda_{i'} \leq \frac{2^m}{(1 - 2^m\varepsilon)2^{m-1}} = 2\frac{\frac{1}{2^m}}{\frac{1}{2^m} - \varepsilon}.$$

Using the fact that $\lambda_i \lambda_{i'} = 1$, we obtain

$$\Lambda^{-2} \leq \lambda_1^2 \leq \cdots \leq \lambda_{2m}^2 \leq \Lambda^2,$$

where

$$\Lambda = \frac{\frac{1}{2^m}}{\frac{1}{2^m} - \varepsilon} + \sqrt{\left(\frac{\frac{1}{2^m}}{\frac{1}{2^m} - \varepsilon}\right)^2 - 1}.$$

This completes the proof. □

Lemma 5.4.3. *Let $f : \mathbb{CP}^m \to \mathbb{CP}^m$ be a Λ_0-pinched symplectomorphism, where $\Lambda_0 > 1$ is the constant characterised in Lemma 5.3.3. Then:*

(a) *The mean curvature flow with initial data the graph Σ of f stays graphical as long as it exists.*

(b) *The function $\ln u$ satisfies*

$$\ln u - \ln 2^{-m} \geq c_0 e^{-c_1 t},$$

where

$$c_0 = \min_{x \in \mathbb{CP}^m} \big(\ln u(x,0) - \ln 2^{-m} \big) \text{ and } c_1 = \frac{8}{(\Lambda_0 + \frac{1}{\Lambda_0})^2}.$$

Proof. From Lemma 5.4.1 we have that

$$u(x,0) \geq 2^{-m} - \varepsilon > 0,$$

for any $x \in \mathbb{CP}^m$, where

$$\varepsilon = \frac{1}{2^m}\left(1 - \frac{2}{\Lambda_0 + \frac{1}{\Lambda_0}}\right).$$

From Lemma 5.3.1, the function $\ln u$ satisfies the differential inequality

$$(\partial_t - \Delta)\ln u = \frac{u_t - \Delta u}{u} + \frac{|\nabla u|^2}{u^2} \geq \sum_{k=\text{odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2}. \tag{5.14}$$

Fix $k \in \{1, \dots, 2m\}$ and set

$$x = (\lambda_k + \lambda_{k'})^2.$$

Then, keeping in mind that $\lambda_k \lambda_{k'} = 1$, we see that

$$0 \leq \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} = \frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})^2} = \frac{x-4}{x}.$$

Moreover, by (5.13) we get that

$$(\lambda_k + \lambda_{k'})^2 = x \leq \left(\Lambda_0 + \frac{1}{\Lambda_0}\right)^2. \tag{5.15}$$

We claim now that

$$\frac{x-4}{x} \geq \frac{8}{\left(\Lambda_0 + \frac{1}{\Lambda_0}\right)^2}\left(\frac{\ln x}{2} - \ln 2\right). \tag{5.16}$$

To prove the claim, let us define the functions

$$f(x) = \frac{x-4}{x} \quad \text{and} \quad g(x) = \frac{8}{\left(\Lambda_0 + \frac{1}{\Lambda_0}\right)^2}\left(\frac{\ln x}{2} - \ln 2\right).$$

Then, from (5.15) we see that

$$f'(x) = \frac{4}{x^2}, \quad g'(x) = \frac{4}{\left(\Lambda_0 + \frac{1}{\Lambda_0}\right)^2}\frac{1}{x} \quad \text{and} \quad \frac{f'(x)}{g'(x)} = \frac{\left(\Lambda_0 + \frac{1}{\Lambda_0}\right)^2}{x} \geq 1.$$

Because,

$$f(4) = g(4) = 0 \quad \text{and} \quad f'(x) \geq g'(x),$$

for every

$$4 \leq x \leq \big(\Lambda_0 + \frac{1}{\Lambda_0}\big)^2,$$

we see that $f \geq g$, which proves our claim. Therefore, from (5.16), we deduce that

$$\sum_{k=\text{odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} = \sum_{k=\text{odd}} \frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})^2} \geq \frac{8}{\big(\Lambda_0 + \frac{1}{\Lambda_0}\big)} \sum_{k=\text{odd}} \big(\ln(\lambda_k + \lambda_{k'}) - \ln 2\big)$$
$$= -\frac{8}{\big(\Lambda_0 + \frac{1}{\Lambda_0}\big)^2}\big(\ln u - \ln 2^{-m}\big).$$

Therefore (5.14) becomes

$$(\partial_t - \Delta)\big(\ln u - \ln 2^{-m}\big) \geq -\frac{8}{\big(\Lambda_0 + \frac{1}{\Lambda_0}\big)^2}\Big(\ln u - \ln 2^{-m}\Big).$$

From the parabolic maximum principle, we see that

$$\ln u - \ln 2^{-m} \geq c_0 e^{-c_1 t},$$

where

$$c_0 = \min_{x \in \mathbb{CP}^m}\big(\ln u(x,0) - \ln 2^{-m}\big) \text{ and } c_1 = \frac{8}{(\Lambda_0 + \frac{1}{\Lambda_0})^2}.$$

Consequently, $\ln u$ cannot approach $-\infty$ or, equivalently, the function u cannot tend to 0. Therefore, the LMCF will preserve the graphical property as long as it exists. Additionally, we see that

$$u(x,t) > 2^{-m} - \varepsilon \text{ for all } (x,t) \in M \times (0,T).$$

Hence, from Lemma 5.4.2 it follows that the Λ_0-pinching condition is preserved under the flow. In particular, if the flow exists for all times, then

$$\lim_{t \to \infty} u \geq 2^{-m}$$

which implies that the singular values of the evolved symplectomorphisms would tend to 1. This completes the proof. □

5.5 Long-time existence of MCF

The goal of this section is to prove the long-time existence of the LMCF under the conditions of the Main Theorem. At first let us isometrically embed the product $\mathbb{CP}^m \times \mathbb{CP}^m$ into a euclidean space $\mathbb{R}^N$. Then, following the strategy developed in Section 3.4, the graphical LMCF gives rise to a MCF $F : M = \mathbb{CP}^m \to \mathbb{R}^N$

$$dF(\partial_t) = H + V$$

with a bounded additional force V. Suppose now to the contrary that the flow reach at $F(x_0, t_0) = (y_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$ a finite time singularity. We will arrive to a contradiction. Let $\rho_{(y_0,t_0)}$ be the backward heat kernel

$$\rho_{(y_0,t_0)}(y,t) = \frac{1}{4\pi(t_0 - t)^m} e^{\frac{-|y-y_0|^2}{4(t_0-t)}}$$

in $\mathbb{R}^N \times \mathbb{R}$. As usual we abbreviate by ρ the function given by

$$\rho(x,t) = \rho_{(y_0,t_0)}(F(x,t),t),$$

for any (x,t) in space-time.

Lemma 5.5.1. *Under the assumptions of the Main Theorem, the limit*

$$\lim_{t\to t_0} \int (1-u)\rho \, d\mu$$

exists and, moreover,

$$\frac{d}{dt}\int (1-u)\rho \, d\mu \le C - \delta \int u\rho \, |A|^2 d\mu,$$

for some positive constant $C > 0$.

Proof. By equations (3.10) and (3.11), we have

$$\frac{d\rho}{dt} = -\Delta\rho - \rho\left(\frac{|F^\perp|^2}{4(t_0-t)^2} + \frac{\langle F^\perp, H\rangle}{t_0 - t} + \frac{\langle F^\perp, V\rangle}{2(t_0-t)}\right)$$

and

$$\frac{d(d\mu)}{dt} = -\langle H, H+V\rangle d\mu.$$

Taking into account (5.9) and Green's identity, we deduce that

$$
\begin{aligned}
&\frac{d}{dt}\int (1-u)\rho\, d\mu \\
&\leq \int \big(\Delta(1-u) - \delta u|A|^2\big)\rho\, d\mu - \int (1-u)\langle H, H+V\rangle \rho\, d\mu \\
&\quad - \int (1-u)\left[\Delta\rho + \rho\Big(\frac{|F^{\perp}|^2}{4(t_0-t)^2} + \frac{\langle F^{\perp}, H\rangle}{t_0-t} + \frac{\langle F^{\perp}, V\rangle}{2(t_0-t)}\Big)\right] d\mu \\
&= \int \big(\rho\Delta(1-u) - (1-u)\Delta\rho\big) d\mu - \int \delta u|A|^2\rho\, d\mu \\
&\quad - \int (1-u)\rho\left[\left(\frac{\left|F^{\perp}\right|^2}{4\,(t_0-t)^2} + \frac{\langle F^{\perp}, H\rangle}{t_0-t} + \frac{\langle F^{\perp}, V\rangle}{2\,(t_0-t)}\right) + |H|^2 + \langle H, V\rangle\right] d\mu \\
&= -\int \delta u|A|^2\rho\, d\mu - \int (1-u)\rho\left|\frac{F^{\perp}}{2\,(t_0-t)} + H + \frac{V}{2}\right|^2 d\mu \\
&\quad + \int (1-u)\rho\left|\frac{V}{2}\right|^2 d\mu.
\end{aligned}
$$

By Huisken's monotonicity formula 3.4.2, the limit $\lim_{t\to t_0}\int \rho\, d\mu$ exists. Thus

$$
\int (1-u)\rho\, d\mu \leq \int \rho\, d\mu < \infty.
$$

Since V is bounded, it follows that

$$
\frac{d}{dt}\int (1-u)\rho\, d\mu \leq C - \delta \int u|A|^2\rho\, d\mu,
$$

for some constant C. Observe that the function

$$
h(t) = \int (1-u)\rho\, d\mu - Ct
$$

is non-increasing in the interval $[0, t_0)$, which implies that $\lim_{t\to t_0} h(t)$ exists. Thus,

$$
\lim_{t\to t_0}\int (1-u)\rho\, d\mu
$$

exists and this completes the proof. □

Lemma 5.5.2. *Consider the parabolic rescalings described in* (3.12). *Then, for any* $\tau > 0$ *and* $\nu > 0$*, it holds that*

$$\lim_{\nu\to\infty}\int_{-1-\tau}^{-1}\left(\int_{M_s^\nu}|A^\nu|^2\rho_{(0,0)}d\mu_s^\nu\right)ds = 0.$$

Proof. Since both u and $\rho_{(y_0,t_0)}d\mu$ are invariant such dilations, we have

$$\int_{M_t}(1-u)\rho_{(y_0,t_0)}d\mu_t = \int_{M_s^\nu}(1-u)\rho_{(0,0)}d\mu_s^\nu. \tag{5.17}$$

From the fact $t = t_0 + \nu^{-2}s$ and the equation (5.17), we have that

$$\frac{d}{ds}\int_{M_s^\nu}(1-u^\nu)\rho_{(0,0)}d\mu_s^\nu = \nu^{-2}\frac{d}{dt}\int_{M_t}(1-u)\rho_{(y_0,t_0)}d\mu_t.$$

Then, by Lemma 5.5.1, we have

$$\frac{d}{ds}\int_{M_s^\nu}(1-u)\rho_{(0,0)}d\mu_s^\nu \le C\nu^{-2} - \delta\nu^{-2}\int_{M_t}u|A|^2\rho_{(y_0,t_0)}d\mu_t$$

for some constant C. From the conclusions of Theorem 3.4.7, we have that

$$\nu^{-2}\int_{M_t}u|A|^2\rho_{(y_0,t_0)}d\mu_t = \int_{M_s^\nu}u|A^\nu - A_N|^2\rho_{(0,0)}d\mu_s^\nu$$

since the norm of the second fundamental form scales like the inverse of the distance. Thus

$$\frac{d}{ds}\int_{M_s^\nu}(1-u)\rho_{(0,0)}d\mu_s^\nu \le C\nu^{-2} - \delta\int_{M_s^\nu}u|A^\nu - A_N|^2\rho_{(0,0)}d\mu_s^\nu.$$

Fix $\tau > 0$ and let us integrate the above inequality from $-1-\tau$ to -1 with respect to s. Then we get

$$\begin{aligned}\delta\int_{-1-\tau}^{-1}&\left(\int_{M_s^\nu}u|A^\nu - A_N|^2\rho_{(0,0)}d\mu_s^\nu\right)ds\\ &\le -\int(1-u)\rho_{(0,0)}d\mu_{-1}^\nu + \int(1-u)\rho_{(0,0)}d\mu_{-1-\tau}^\nu + C\nu^{-2}.\end{aligned}$$

Letting $\nu\to\infty$, using the fact that u is bounded, that A_N is of bounded norm and that

$$\lim_{t\to t_0}\int(1-u)\rho_{(y_0,t_0)}d\mu$$

exists, we obtain the desired result. □

Take a blow-up sequence (x_i, t_i) converging to the singular point and then set $\nu_i = |A|(x_i, t_i)$. Then from Theorem 3.4.7 the second fundamental forms of $\{M_s^{\nu_i}\}_{i\in\mathbb{N}}$ stay uniformly bounded. By Arzela-Ascoli theorem, $M_s^{\nu_i} \to M_s^{\infty}$ for all $s \in (-\infty, 0)$. Hence

$$\int_{-1-\tau}^{-1} \left(\int_{M_s^{\nu_i}} |A^{\nu}|^2 \rho_{(0,0)} d\mu_s^{\nu_i} \right) ds \leq c_i,$$

where $c_i \to 0$ as $i \to \infty$. We first choose $\tau_i \to 0$ such that $c_i/\tau_i \to 0$ and then choose $s_i \in [-1-\tau_i, -1]$ such that

$$\int_{M_{s_i}^{\nu_i}} |A^{\nu}|^2 \rho_{(0,0)} d\mu_{s_i}^{\nu_i} \leq c_i/\tau_i \to 0.$$

Suppose that $M_{s_i}^{\nu_i}$ is the image of $F_{s_i}^{\nu_i} = F^{\nu_i}(\cdot, s_i) : M \to \mathbb{R}^N$. Then

$$\rho_{(0,0)}(F_{s_i}^{\nu_i}, s_i) = \frac{1}{4\pi(-s_i)^m} e^{\frac{-|F_{s_i}^{\nu_i}|^2}{4(-s_i)}}.$$

Let $B_r(0) \subset \mathbb{R}^N$ be the radius r ball centered at the origin of $\mathbb{R}^N$. Since each s_i is bounded and $|F_{s_i}^{\nu_i}| \leq r$ on $\Sigma_{s_j}^{\nu_j} \cap B_r(0)$, we have

$$\begin{aligned} \int_{M_{s_i}^{\nu_i}} |A^{\nu_i}|^2 \rho_{(0,0)} d\mu_{s_i}^{\nu_i} &\geq \int_{M_{s_i}^{\nu_i} \cap B_r(0)} |A^{\nu_i}|^2 \rho_{(0,0)} d\mu_{s_i}^{\nu_i} \\ &\geq c\, e^{-\frac{r^2}{2}} \int_{M_{s_i}^{\nu_i} \cap B_r(0)} |A^{\nu_i}|^2 d\mu_{s_i}^{\nu_i}, \end{aligned}$$

where $c > 0$. Hence, on any compact set K of $\mathbb{R}^N$, we have that

$$\lim_{i\to\infty} \int_{M_{s_i}^{\nu_i} \cap B_r(0)} |A^{\nu_i}|^2 d\mu_{s_j}^{\nu_j} = 0.$$

Since the convergence $M_s^{\nu_i} \to M_s^{\infty}$ is smooth we deduce that the submanifold M_{-1}^{∞} and consequently each M_s^{∞} is flat in $\mathbb{R}^N$. But then

$$\lim_{t_i\to t_0} \int_{M_t} \rho_{(y_0,t_0)} d\mu = \lim_{i\to\infty} \int_{M_{s_i}^{\nu_i}} \rho_{(0,0)} d\mu_{s_i}^{\nu_i} = \int_{M_{-1}^{\infty}} \rho_{(0,0)} d\mu_{-1}^{\infty} = 1.$$

White's Theorem [44] asserts (y_0, t_0) is a regular point which contradicts the assumption that we made in the beginning of the section. Consequently, there is no space-time singularity of the mean curvature flow.

5.6 Convergence to a biholomorphic isometry

To complete the proof of the Main Theorem it remains to show that the family of the evolving symplectomorphisms $\{f_t\}_{t\in[0,\infty)}$ converges to a biholomorphic isometry. Let $\epsilon > 0$ and define the functions

$$\eta_\epsilon = u - 2^{-m} + \epsilon.$$

Recall from Lemma 5.4.3 that the function

$$\varrho_\epsilon(t) = \min_{x\in\mathbb{CP}^m} \eta_\epsilon(x,t)$$

is non-decreasing and that

$$\lim_{t\to\infty} \varrho_\epsilon(t) \to \epsilon.$$

Let $T_\epsilon \geq 0$ be a large enough time such that $\eta_\epsilon(x,t) > 0$, for all the points $(x,t) \in \mathbb{CP}^m \times (T_\epsilon, \infty)$. Then,

$$\partial_t \eta_\epsilon \geq \Delta\eta_\epsilon + \delta u|A|^2 = \Delta\eta_\epsilon + \frac{\delta\eta_\epsilon u|A|^2}{\eta_\epsilon}.$$

Since $\mathbb{CP}^m \times \mathbb{CP}^m$ is a symmetric space its curvature tensor is parallel. Hence $|A|^2$ satisfies the inequality

$$\partial_t|A|^2 \leq \Delta|A|^2 - 2|\nabla A|^2 + K_1|A|^4 + K_2|A|^2,$$

where K_1 and K_2 are positive constants that depend only on the dimension m. Taking into account the last two inequalities, we get

$$\begin{aligned}
(\eta_\epsilon^{-1}|A|^2)_t \;\leq\; & -\eta_\epsilon^{-2}|A|^2\big(\Delta\eta_\epsilon + \delta u|A|^2\big) \\
& +\eta_\epsilon^{-1}\big(\Delta|A|^2 - 2|\nabla A|^2 + K_1|A|^4 + K_2|A|^2\big) \\
= \; & \Delta\left(\eta_\epsilon^{-1}|A|^2\right) - 2\langle\nabla\eta_\epsilon^{-1}, \nabla|A|^2\rangle - 2\eta_\epsilon|\nabla\eta_\epsilon^{-1}|^2|A|^2 \\
& -2\eta_\epsilon^{-1}|\nabla A|^2 + \eta_\epsilon^{-2}\left(\eta_\epsilon K_1 - \delta u\right)|A|^4 + \eta_\epsilon^{-1}K_2|A|^2.
\end{aligned}$$

Note that

$$-2\langle\nabla\eta_\epsilon^{-1}, \nabla|A|^2\rangle - 2\eta_\epsilon|\nabla\eta_\epsilon^{-1}|^2|A|^2 = -2\eta_\epsilon\langle\nabla\eta_\epsilon^{-1}, \nabla(\eta_\epsilon^{-1}|A|^2)\rangle.$$

Since the minimum of u is increasing and $\eta_\epsilon \leq \epsilon$, the function $\varphi = \eta_\epsilon^{-1}|A|^2$ satisfies

$$\begin{aligned}
\varphi_t &\leq \Delta\varphi - 2\eta_\epsilon\langle\nabla\eta_\epsilon^{-1}, \nabla\varphi\rangle + \left(\eta_\epsilon K_1 - \delta u\right)\varphi^2 + K_2\varphi \\
&\leq \Delta\varphi - 2\eta_\epsilon\langle\nabla\eta_\epsilon^{-1}, \nabla\varphi\rangle + \left(\epsilon K_1 - \delta c_0\right)\varphi^2 + K_2\varphi,
\end{aligned}$$

where c_0 denotes the $\min_{x\in\mathbb{CP}^m} u(x,0)$. We can chose ϵ small enough so that $\epsilon K_1 - \delta C_0 < 0$. Then by the parabolic principle $\varphi \leq y(t)$ for all $t \geq T_\epsilon$, where y is the solution of the ODE

$$\begin{cases} y' = -\left(\delta C_0 - \epsilon K_1\right) y^2 + K_2 y \\ y\left(T_\epsilon\right) = \max_{x\in\mathbb{CP}^m} \varphi(x, T_\varepsilon). \end{cases}$$

It is easy to see that $y(t)$ is given by

$$y(t) = \begin{cases} \frac{K_2}{\delta C_0 - \epsilon K_1}, & \text{if } \; y\left(T_\epsilon\right) = \frac{K_2}{\delta C_0 - \epsilon K_1}, \\ \frac{K_2 K e^{K_2 t}}{(\delta C_0 - \epsilon K_2) K e^{K_2 t} - 1}, & \text{otherwise,} \end{cases}$$

where the constant K is positive if

$$y\left(T_\epsilon\right) > \frac{K_2}{\delta C_0 - \epsilon K_1},$$

and negative if

$$y\left(T_\epsilon\right) < \frac{K_2}{\delta C_0 - \epsilon K_1}.$$

It follows that

$$|A|^2(x,t) \leq \eta_\epsilon y(t) \leq \epsilon y(t),$$

for all $(x,t) \in \mathbb{CP}^m \times (T_\epsilon, \infty)$. Sending $t \to \infty$ and then $\epsilon \to 0$ we conclude that

$$\max_{x\in\mathbb{CP}^m} |A|^2 \to 0.$$

Because the Fubini-Study metric, the induced metrics and the volume functional have analytic dependence on F, a deep theorem of Simon [32] implies that the flow converges smoothly to a unique limit f_∞. Since each singular value tends uniformly to 1 as time goes to infinity, it follows that the map f_∞ is an isometry. Being symplectic is a closed property, thus f_∞ is symplectic, which implies that the map f_∞ is a biholomorpic isometry. This completes the proof. □

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