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Unprojection Theory, Applications to Algebraic Geometry and Anisotropy of Simplicial Spheres

Vasiliki Petrotou

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Θεωρία Αντιπροβολής, Εφαρμογές στην Αλγεβρική Γεωμετρία και Ανισοτροπία των Simplicial Σφαιρών

Βασιλική Πετρωτού

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Members of three-member Advisory Committee :

Supervisor: Stavros Papadakis (Assistant Professor, University of Ioannina)

Members: 1) Apostolos Beligiannis (Professor, University of Ioannina)

2) Apostolos Thoma (Professor, University of Ioannina)

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Members: 1) Stavros Papadakis (Supervisor, Assistant Professor, University of Ioannina)

2) Apostolos Beligiannis (Professor, University of Ioannina)

3) Epaminondas Kechagias (Professor, University of Ioannina)

4) Christos Athanasiadis (Professor, National and Kapodistrian University of Athens)

5) Nikolaos Tziolas (Professor, University of Cyprus)

6) Chara Charalambous (Professor, Aristotle University of Thessaloniki)

7) Chrysostomos Psaroudakis (Assistant Professor, Aristotle University of Thessaloniki)

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<u>Επιβλέπων</u> Σταύρος Παπαδάκης (Επίκουρος Καθηγητής, Πανεπιστήμιο Ιωαννίνων)

<u>Μέλη</u> 1) Απόστολος Μπεληγιάννης (Καθηγητής, Πανεπιστήμιο Ιωαννίνων)

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<u>Μέλη:</u>

1) Σταύρος Παπαδάκη (Επιβλέπων, Επίκουρος Καθηγητής, Πανεπιστήμιο Ιωαννίνων)

2) Απόστολος Μπεληγιάννης (Καθηγητής, Πανεπιστήμιο Ιωαννίνων)

3) Επαμεινώνδας Κεχαγιάς (Καθηγητής, Πανεπιστήμιο Ιωαννίνων)

4) Χρήστος Αθανασιάδης (Καθηγητής, Εθνικό και Καποδιστριακό Πανεπιστήμιο Αθηνών)

5) Νικόλαος Τζιόλας (Καθηγητής, Πανεπιστήμιο Κύπρου)

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Περιληψη

Η παρούσα διατριβή αποτελείται από δύο μέρη. Το πρώτο μέρος της διατριβής σχετίζεται με τη μελέτη της θεωρίας της αντιπροβολής και την κατασκευή τριών συνδιάστασης 6 οικογενειών τρισδιάστατων Fano πολυπτυγμάτων αντικανονικά εμφυτευμένων στον βαθμωτό προβολικό χώρο. Το δεύτερο μέρος αφορά τη μελέτη των ιδιοτήτων Lefschetz και της ανισοτροπίας του Stanley-Reisner δακτυλίου των μονοπλεκτικών σφαιρών.

Στο Κεφάλαιο 2, εισάγουμε κάποιες εισαγωγικές έννοιες και ήδη γνωστά αποτελέσματα από τους κλάδους της Μεταθετικής Άλγεβρας, της Αλγεβρικής Γεωμετρίας και της Συνδυαστικής Άλγεβρας, με ιδιαίτερη έμφαση στους δακτυλίους Gorenstein, στα τρισδιάστατα Fano πολυπτύγματα και στα μονοπλεκτικά συμπλέγματα.

Στο Κεφάλαιο 3, υπενθυμίζουμε κάποια ήδη υπάρχοντα αποτελέσματα που σχετίζονται με τη θεωρία της αντιπροβολής. Η θεωρία της αντιπροβολής, η οποία οφείλεται στον Miles Reid, χρησιμοποιεί ιδέες της Αμφίρητης Γεωμετρίας για να κατασκευάσει πιο περίπλοκους μεταθετικούς δακτυλίους ξεκινώντας από απλούστερα αρχικά δεδομένα. Είναι το κύριο μας εργαλείο για τις γεωμετρικές εφαρμογές.

Στο Κεφάλαιο 4, αναπτύσουμε μία νέα μορφή παράλληλης αντιπροβολής, την οποία ονομάζουμε Tom και Jerry τριάδες. Χρησιμοποιούμε αυτή τη μορφή για να αποδείξουμε, ξεκινώντας από συνδιάσταση 3, την ύπαρξη δύο συνδιάστασης 6 οικογενειών από τρισδιάστατα Fano πολυπτύγματα.

Στο Κεφάλαιο 5, αναπτύσουμε μία δεύτερη μορφή παράλληλης αντιπροβολής, την οποία καλούμε 4-διατομή. Χρησιμοποιούμε αυτή τη μορφή για να αποδείξουμε, ξεκινώντας από συνδιάσταση 2, την ύπαρξη μιας συνδιάστασης 6 οικογένειας από τρισδιάστατα Fano πολυπτύγματα.

Στο Κεφάλαιο 6, το οποίο είναι σε συνεργασία με τον Σταύρο Αναργύρου Παπαδάκη, εισάγουμε την έννοια της γενικής ανισοτροπίας μιας μονοπλεκτικής σφαίρας. Αποδεικνύουμε ότι μία μονοπλεκτική σφαίρα είναι γενικά ανισοτροπική

υπεράνω οποιουδήποτε σώματος χαρακτηριστικής 2, και ότι μία μονοδιάστατη μονοπλεκτική σφαίρα είναι γενικά ανισοτροπική υπεράνω οποιουδήποτε σώματος. Ως εφαρμογή, δίνουμε μια δεύτερη απόδειξη της g-εικασίας του McMullen για μονοπλεκτικές σφαίρες.

Abstract

The present thesis consists of two parts. The first part of the thesis is related to the study of unprojection theory and the construction of three codimension 6 families of Fano 3-folds anticanonically embedded in weighted projective space. The second part concerns the study of the Lefschetz and anisotropy properties of the Stanley-Reisner ring of simplicial spheres.

In Chapter 2, we introduce some preliminary notions and known results from Commutative Algebra, Algebraic Geometry and Combinatorial Algebra, with a particular emphasis to Gorenstein rings, Fano 3-folds and simplicial complexes.

In Chapter 3, we recall some existing results related to unprojection theory. Unprojection theory, which is due to Miles Reid, uses ideas from birational geometry to construct more complicated commutative rings starting from simpler data. It is our main tool for the geometric applications.

In Chapter 4, we develop a new parallel unprojection format, for which we give the name Tom & Jerry triples format. We use the format to prove, starting from codimension 3, the existence of two codimension 6 families of Fano 3-folds.

In Chapter 5, we develop a second parallel unprojection format, which we call the 4-intersection format. We use the format to prove, starting from codimension 2, the existence of a codimension 6 family of Fano 3-folds.

In Chapter 6, which is joint work with Stavros Argyrios Papadakis, we introduce the notion of generic anisotropy of a simplicial sphere. We prove that a simplicial sphere is generically anisotropic over any field of characteristic 2, and that a 1-dimensional simplicial sphere is generically anisotropic over any field. As an application, we give a second proof of McMullen's g-conjecture for simplicial spheres.

Contents

Chapter 1

Introduction

One of the most important problems in Algebraic Geometry is the classification of algebraic varieties. In the early 1980s, the Minimal Model Program (also known as Mori program) [24, 35, 36, 47] appeared as an effective approach for the birational classification of the higher dimensional algebraic varieties. Fano 3-folds, as a building block of this program in dimension 3, is an important topic of current research. In this direction we develop two new methods of parallel unprojection and use them to establish the existence of 3 new families of singular Fano 3-folds of codimension 6.

The second part of this thesis, which is joint work with Stavros Argyrios Papadakis, was motivated by McMullen's g-conjecture for simplicial spheres. The g-conjecture, concerns the complete characterization of the set of f-vectors of simplicial spheres, and was recently proven by Adiprasito [1, 2]. In the present work we investigate some algebraic properties of Artinian reductions of Stanley-Reisner rings of simplicial spheres. We introduce the notion of a simplicial sphere being generically anisotropic over a field and establish the generic anisotropy of any simplicial sphere over any field of characteristic 2 and the generic anisotropicy of 1-dimensional simplicial spheres over an arbitrary field. As an application, we obtain a second proof of the g-conjecture for simplicial spheres.

1.1 Some aspects of unprojection theory

Gorenstein rings form an important class of rings which appear often in Algebraic Geometry. The anticanonical ring of a Fano n-fold, the canonical ring of a regular surface of general type and the ring associated to an ample divisor on a smooth $K3$ surface are some examples of Gorenstein rings.

If $R = k[x_1, \dots, x_n]/I$ is a Gorenstein graded ring, quotient of a polynomial ring and the codimension of I is at most 3 then the structure of R is well-understood, see Subsection 2.1.5. An important open question is to find structure theorems when the codimension of I is 4 or higher.

In the 1980s, Kustin and Miller tried to find a structure theorem for Gorenstein rings of codimension 4 with a series of papers [38, 39, 40, 41, 42]. In this context, in 1983 they introduced a procedure [37] which constructs more complicated Gorenstein rings starting from simpler ones, by increasing the codimension. This procedure is called Kustin-Miller unprojection.

Around 1995, Reid rediscovered what was essentially the same procedure while working with Gorenstein rings arising from K3 surfaces and Fano 3-folds. Geometrically, unprojection, as indicated by its name, is an inverse of certain projections and can be considered as a modern and explicit version of Castelnuovo contractibility theorem.

We now summarise Reid's formulation of unprojection:

Assume that $J \subset R$ is a codimension 1 ideal with R, R/J being Gorenstein. Denote by $i\colon J \to R$ the inclusion map. Then there exists ϕ such that $\operatorname{Hom}_R(J, R)$ is generated by the set $\{i, \phi\}$ as an R-module. Using ϕ, Reid defined the new unprojection ring as in Definition 3.1.1. Some years later Papadakis and Reid [57] proved that the unprojection ring is Gorenstein (see Theorem 3.1.2). We refer the reader to Example 3.1.3 for the simplest example of Kustin-Miller unprojection.

Reid developed two families of unprojections which he called Tom and Jerry [54, 55, 59]. Each of them is a way starting from a codimension 3 Gorenstein ring with some additional properties to construct a new codimension 4 Gorenstein ring. We recall the definitions of the Tom and Jerry families in Subsection 3.1.2. Papadakis [55] computed, using multilinear and homological algebra, the equations of the Tom and Jerry families. For more details in the case of Tom we refer to Subsection 3.1.3.

Unprojection theory has found many applications in Algebraic Geometry. In particular, in the construction of new interesting algebraic surfaces and 3-folds, especially in codimension four [3, 4, 14, 15, 16, 52, 53, 67]. In the context of explicit birational geometry it allows one to explicitly write down varieties, morphisms and rational maps that arise in the Minimal Model Program [21, 22]. It has also found applications in Algebraic Combinatorics [9, 10, 11, 12].

Unprojection can be used many times over in an inductive way in order to produce Gorenstein rings of arbitrary codimension, whose properties are, nevertheless, controlled by just a few equations as new unprojection variables are adjoined. Neves and Papadakis [53] developed such a theory which is called parallel Kustin-Miller unprojection. More presicely, they discovered sufficient conditions on a positively graded Gorenstein ring R and a finite set of codimension 1 ideals which ensure the series of unprojections. Furthermore, they gave an explicit description of the end product ring which corresponds to the unprojection of the ideals. We recall the results in Subsection 3.1.1.

We develop two new formats of parallel Kustin-Miller unprojection, which we call

Tom & Jerry triples and 4-intersection respectively.

The Tom & Jerry triples unprojection format, which is discussed in Chapter 4, uses Tom & Jerry unprojections in order to set up the unprojection data. In more detail, we set conditions on the entries of a 5×5 skewsymmetric matrix M such that M can be considered simultaneously as Tom or Jerry matrix in three codimension 4 complete intersection ideals J_1, J_2, J_3. Then, the ideal of Pfaffians of M is contained in the ideals J_1, J_2, J_3. Using parallel Kustin-Miller unprojection we construct a Gorenstein ring of codimension 6.

The 4-intersection format, which is discussed in Chapter 5, defines a codimension 2 complete intersection ideal I such that I is contained in four codimension 3 ideals $J_1, \dots, J_4$. Using parallel Kustin-Miller unprojection, this format also leads to the construction of a codimension 6 Gorenstein ring.

Brown's online Graded Ring Database [4, 13] contains a large number of K3 surfaces, Fano 3-folds and Calabi-Yau 3-folds of high codimension which, conjecturally, exist and are, again conjecturally, related to varieties of small codimension. Using the Tom & Jerry triples and 4-intersection unprojection formats we establish, in Sections 4.3 and 5.2, the existence of three new families of Fano 3-folds which appear in the Graded Ring Database.

1.2 McMullen's g-conjecture

In 1971 McMullen conjectured a complete characterization of the f-vectors of the class of simplicial polytopes. Around 1979, the sufficiency of the conditions were proven by an explicit constuction due to joint work of Billera and Lee [8], while Stanley [61] proved their necessity using tools from Algebraic Geometry.

Given two integers $a, i > 0$ there exists the following unique expansion

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$, see [17, Section 4.2]. We define

$$a^{<i>} = \binom{a_i + 1}{i+1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j+1}$$

and $0^{<i>} = 0$ for all i.

Definition 1.2.1 Assume that $(g_0, \dots, g_s)$ is a sequence of nonnegative integers. We call $(g_0, \dots, g_s)$ a Macaulay vector if $g_0 = 1$ and $0 \leq g_{i+1} \leq g_i^{<i>}$, for all $i \geq 1$.

Macaulay discovered the Macaulay vectors in his study of the growth of Hilbert functions of graded rings [46], see also [17, Section 4.2]. We refer the reader to Subsection 2.3.1 for the definitions of the f-vector, h-vector and g-vector of a simplicial complex, and also for the definition of the geometric realization of a simplicial complex. By definition, a simplicial sphere is a simplicial complex such that its geometric realization is homeomorphic to the unit sphere S^n for some $n \geq 1$.

The combination of the following two famous theorems established McMullen's g-conjecture for the class of simplicial polytopes.

Theorem 1.2.2 *(Billera and Lee [8]) Assume $f = (f_0, \dots, f_n)$ is a finite sequence of integers. Denote by h and g the corresponding sequences of integers obtained from f as in Subsection 2.3.1. Assume that $h_i = h_{n+1-i}$ for all i and that $(1, g_1, \dots, g_{[(n+1)/2]})$ is a Macaulay vector. Then, there exists a simplicial polytope of dimension $n+1$ with f-vector of its boundary complex equal to f.*

Theorem 1.2.3 *(Stanley [61]) Assume Δ is the boundary of a simplicial polytope of dimension $n+1$. Then, the g-vector of Δ is a Macaulay vector.*

The following recent result of Adiprasito was known for more than 35 years as the g-conjecture for simplicial spheres [66].

Theorem 1.2.4 *(Adiprasito [1, 2]) Assume Δ is a simplicial sphere. Then, the g-vector of Δ is a Macaulay vector.*

Our approach for the second proof of the g-conjecture for simplicial spheres is based on the well-known result that to prove the g-conjecture for a simplicial sphere D it is enough to find a field k such that the Stanley-Reisner ring (also known as face ring) $k[D]$ has the Weak Lefschetz Property [23, 46, 61].

Instead of working directly with the Weak Lefschetz Property, we exploit some algebraic properties of the generic Artinian reduction of the Stanley-Reisner ring of a simplicial sphere. More precisely, we introduce in Definition 6.2.2 the notion of a simplicial sphere D being generically anisotropic over a field k_1. This means that for a certain purely transcendental field extension k of k_1, a certain Artinian reduction A of the Stanley-Reisner ring $k[D]$ has the following property: All nonzero homogeneous elements $u \in A$ of degree less or equal to $(\dim D + 1)/2$ have nonzero square.

We investigate the property of generic anisotropy of simplicial spheres. In particular, we prove that a 1-dimensional simplicial sphere is generically anisotropic over any (finite or infinite) field, see Theorem 6.9.1. Moreover, in Theorem 6.2.3 we show, using suitable differential operators, that over any (finite or infinite) field of characteristic 2, every simplicial sphere is generically anisotropic. We expect that the last statement is also true over any field of arbitrary characteristic but, so far, we have been unable to prove it. A main obstacle is that even though the differential operators we use can be defined over any field, we need certain properties of them that hold only in characteristic 2.

Using some ideas and results of Swartz [65], we prove in Theorem 6.8.1 that the generic anisotropy of the suspension $S(D)$ of a simplicial sphere D over a field k_1 implies the Weak Lefschetz Property of the Stanley-Reisner ring of D over a certain field extension k of k_1. Combining that with the generic anisotropy of all simplicial spheres over any field of characteristic 2 we obtain, in Theorem 6.8.2, a second proof of the g-conjecture for simplicial spheres.

1.3 Structure of thesis

The present thesis is organised as follows.

Chapter 2 contains background material. In Section 2.1 we recall a number of basic results and definitions of Commutative Algebra related to graded rings, graded free resolutions, Hilbert series and the Lefschetz properties. We emphasize two important classes of rings, namely Cohen-Macaulay and Gorenstein rings. In Subsection 2.1.5 we recall the structure theorems for Gorenstein ideals of codimension ≤ 3 while in Subsection 2.1.6 we briefly discuss the Lefschetz Properties of a graded algebra. In Section 2.2 we recall some notions of Algebraic Geometry. In more detail, in Subsection 2.2.1 we discuss the Proj construction of a variety starting from a graded ring, while Subsection 2.2.2 is about Fano 3-folds. In Section 2.3 we recall some basic notions of Combinatorial Algebra related to simplicial complexes and their associated Stanley-Reisner rings.

Chapter 3 contains some existing results and definitions related to Kustin-Miller unprojection and parallel Kustin-Miller unprojection. We recall the conditions defining the Kustin-Miller unprojection of a pair $J \subset R$ and the definition of the unprojection ring of the pair due to Reid [59, 57]. In Subsection 3.1.1 we recall the parallel unprojection theory due to Neves and Papadakis [53]. In Subsection 3.1.2, we recall the Tom and Jerry unprojection families. Subsection 3.1.3 contains the calculation of the unprojection ring for the Tom family due to Papadakis [55]. We close this chapter with the explicit description of the unprojection ring of a certain codimension 2 complete intersection ideal contained in a certain codimension 3 complete intersection ideal.

In Chapter 4 we introduce the new Tom and Jerry triples format of unprojection. Section 4.1 describes a number of alternative ways which guarantee that a codimension 3 ideal defined by the Pfaffians of a 5×5 skewsymmetric matrix is contained in three codimension 4 complete intersection ideals J_1, J_2, J_3. We study in detail one of the cases in Subsection 4.2.1. Our main result is Theorem 4.2.8 which establishes, using the theory of parallel unprojection, the construction of a codimension 6 Gorenstein ring. We discuss a similar result for the remaining cases in Subsection 4.2.2. Using this new format of unprojection we prove the existence of two families of Fano 3-folds of codimension 6 embedded in weighted projective space which correspond to the entries with ID: 14885 and ID: 12979 in Brown's Graded Ring Database [4, 13].

In Chapter 5 we introduce a second new format of parallel unprojection which we call the 4-intersection format. In Section 5.1 we define this notion, which consists of a codimension 2 complete intersection ideal I contained in four complete intersection codimension 3 ideals $J_1, \dots, J_4$. In Subsection 5.1.1, we introduce a specific example of 4-intersection format, and we construct, using parallel unprojection, a codimension 6 Gorenstein ring. We use the format to prove in Section 4.3 the existence of a family of Fano 3-folds of codimension 6 embedded in weighted projective space which corresponds to the entry ID: 29376 in Brown's Graded Ring Database.

In Chapter 6, which is joint work with Stavros Argyrios Papadakis, we introduce the notion of a simplicial sphere D being generically anisotropic over a field k_1, see Definition 6.2.2. We show in Theorem 6.8.1 that if the suspension $S(D)$ of D is generically anisotropic over k_1, then the Stanley-Reisner ring $k[D]$ has the Weak Lefschetz Property, where k is a certain purely transcendental field extension of k_1. We establish two results related to generic anisotropy. In Theorem 6.9.7, we prove that a simplicial sphere of dimension 1 is generically anisotropic over any (finite or infinite) field k_1. In Theorem 6.2.3 we prove that over any (finite or infinite) field of characteristic 2, every simplicial sphere is generically anisotropic. The key results for these theorems are Proposition 6.9.1, which works in all characteristics but only for simplicial spheres of dimension 1, and Theorem 6.3.14 which is valid in any dimension but only in characteristic 2. Finally, combining Theorem 6.2.3 with Theorem 6.8.1 we get a second proof of McMullen's g-conjecture for simplicial spheres in Theorem 6.8.2.

Chapter 2

Preliminary notions

In this chapter, we recall some basic notions of Commutative Algebra, Algebraic Geometry and Combinatorial Algebra that we use throughout this thesis.

2.1 A review of some basic notions of Commutative Algebra

Throughout this thesis, all rings are assumed to be commutative with unit. We denote by

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

the set of natural numbers. For more details related to the notions that follow we refer to [17, 27, 63].

2.1.1 Graded rings and modules

In this subsection, we study rings, modules and ideals which are endowed with a decomposition of their elements into homogeneous parts of nonnegative degree.

Definition 2.1.1 A ring R is called *graded* if there exists a family of subgroups $\{R_d\}_{d\geq 0}$ of R such that

1. $R = \bigoplus_d R_d$ as abelian group and
2. $R_d R_e \subset R_{d+e}$ for all $d, e \geq 0$.

We call R_i the *i-th homogeneous component* of R. An element $x \in R_i$ is called a *homogeneous element* of R of degree i.

An element f of a graded ring R can be written uniquely as a sum of homogeneous elements $f_i \in \{R_i\}_{i \geq 0}$. The elements f_i are called *the homogeneous parts* of f. The simplest example of a graded ring is the polynomial ring in n variables as indicated in the following example.

Example 2.1.2 Denote by $R = S[x_1, \dots, x_n]$ the polynomial ring in n variables over a ring S. For $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ we set $x^m = x_1^{m_1} \dots x_n^{m_n}$. For every choice of natural numbers $d_1, \dots, d_n$ there exists a unique grading on R such that $\deg x_i = d_i$ for all i and $\deg s = 0$ for all $s \in S$. We have

$$R_d = \{ \sum_{m \in \mathbb{N}^n} s_m x^m \mid s_m \in S \text{ and } d_1 m_1 + \dots + d_n m_n = d \}.$$

The choice of $\deg x_i = 1$ for all i is called the *standard grading* on R.

Definition 2.1.3 An ideal I of a graded ring R is called *homogeneous* if it is generated by homogeneous elements of R.

Example 2.1.4 Let $R = k[x_1, x_2]$ be the polynomial ring in 2 variables over a field k. We set $f = x_1^2 + x_2^3$ and consider the ideal $I = (f)$ of R. Under the standard grading on R the ideal I is not homogeneous because f is not a homogeneous element of R. However, if we endow the ring R with the grading $\deg x_1 = 3$, $\deg x_2 = 2$, then f is a homogeneous element of R of degree 6 and the ideal I is homogeneous.

Definition 2.1.5 Assume k is a field. A *graded* k-algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ is a graded ring that at the same time is a vector space over k and each component A_i is a k-vector subspace. A graded k-algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ is called *positively graded* if $A_0 = k$.

In the following, whenever we talk of a homogeneous ideal I of a polynomial ring $S = k[x_1, \dots, x_n]$ over a field k we will always assume that each variable x_i is homogeneous of positive degree. This will imply that S is a positively graded k-algebra and, when $I \neq S$, the same will be true for the quotient ring S/I.

An important example of a positively graded algebra which comes from Algebraic Geometry is the homogeneous coordinate ring of a projective variety.

Example 2.1.6 Assume k is a field. Let $S = k[x_0, \dots, x_n]$ be the standard graded polynomial ring in $n + 1$ variables. The homogeneous coordinate ring $A(X)$ of a projective variety $X \subset \mathbb{P}^n$ is

$$A(X) = S/I(X),$$

where $I(X)$ is the ideal of S generated by the set

$$\{f \in S \mid f \text{ homogeneous and } f(P) = 0 \text{ for all } P \in X\}.$$

The ring $A(X)$, as a quotient of a positively graded k-algebra with a homogeneous ideal, is also a positively graded k-algebra.

Definition 2.1.7 Let R be a graded ring. Denote by $R_+ = R_1 \oplus R_2 \oplus \dots$ the ideal consisting of all elements of degree greater than zero. The homogeneous ideal R_+ is called the *irrelevant* ideal.

The terminology for the *irrelevant* ideal arises from the connection with projective geometry. Working in $\mathbb{P}^n$, the irrelevant ideal in the standard graded polynomial ring in $n+1$ variables $k[x_0, \dots, x_n]$ contains all homogeneous polynomials of positive degree. These have no common zero in projective space. So, the common zero locus of the irrelevant ideal is the empty set.

Definition 2.1.8 Let R and S be graded rings. A ring homomorphism $f\colon R \to S$ is called *graded* or *homogeneous* if $f(R_d) \subseteq S_d$, for all d.

Definition 2.1.9 A *graded module* over a graded ring R is an R-module M with a decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

as abelian groups, such that $R_d M_i \subset M_{d+i}$ for all $d \in \mathbb{Z}_{\geq 0}$, $i \in \mathbb{Z}$.

If R is a graded ring then R is a graded module over itself. We can construct many other examples of graded modules considering graded submodules, direct sums and quotients of graded modules by graded submodules. Given a graded R-module M, we can form a new graded R-module by twisting the grading on M as follows.

Definition 2.1.10 Let n be an integer. Given a graded R-module M we define the *twist* $M(n)$ to be equal to M as an (ungraded) R-module with grading defined by

$$M(n)_k = M_{n+k}$$

for all $k \in \mathbb{Z}$.

Definition 2.1.11 Let R be a graded ring and M, N be graded R-modules. A *graded* R-module homomorphism of degree d, $f\colon M \to N$ is an R-module homomorphism with the property $f(M_i) \subset N_{i+d}$ for all $i \in \mathbb{Z}$. Two graded R-modules M, N are called *isomorphic* if there exists a bijective graded R-module homomorphism of degree 0 between them.

Definition 2.1.12 Let R be a graded ring. A finitely generated R-module M is called *graded free* if there exist integers $k_1, \dots, k_s$ such that the graded modules M and $\oplus_{i=1}^{s} R(k_i)$ are isomorphic.

2.1.2 Graded complexes and graded free resolutions

In the present subsection we assume that k is a field, $R = k[x_1, \ldots, x_n]$ is a polynomial ring over k and we have a grading on R such that each variable x_i is homogeneous of positive degree.

Definition 2.1.13 A sequence of R-modules and homomorphisms between them

$$\mathbf{F} : \ldots \to F_{i+2} \xrightarrow{d_{i+2}} F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \to \ldots$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$ is called a *chain complex* or just *complex* over R. The set of maps $d = \{d_i\}_{i\in\mathbb{Z}}$ is called the *differential* of $\mathbf{F}$. If the modules F_i are graded and each d_i is a graded homomorphism then the complex $\mathbf{F}$ is called *graded.*

Definition 2.1.14 Let $\mathbf{F}$ be a complex. The complex $\mathbf{F}$ is *exact at the position i* if $\mathrm{Ker}(d_i) = \mathrm{Im}(d_{i+1})$. A complex which is exact at every position i is called *exact.*

In 1973, a criterion for the exactness of a finite complex of finitely generated free modules over a Noetherian ring was given by Buchsbaum and Eisenbud. For details we refer to [18].

Definition 2.1.15 Let $\mathbf{F}$ be a complex. The *homology* of $\mathbf{F}$ is defined by

$$H_i(\mathbf{F}) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

The elements in $\mathrm{Ker}(d_i)$ are called *cycles* and the elements in $\mathrm{Im}(d_{i+1})$ are called *boundaries.*

Definition 2.1.16 Let $(\mathbf{F}, d)$ and $(\mathbf{G}, h)$ be two complexes of R-modules. A *homomorphism of complexes* $\phi\colon \mathbf{F} \to \mathbf{G}$ is a set of R-modules homomorphisms $\phi_i\colon \mathbf{F_i} \to \mathbf{G_i}$ such that $\phi_{i-1} \circ d_i = h_i \circ \phi_i$ for all $i \in \mathbb{Z}$. If $\mathbf{F}$ and $\mathbf{G}$ are graded, ϕ is called *homomorphism of graded complexes* if $\phi_i\colon F_i \to G_i$ is a homomorphism of fixed degree for all $i \in \mathbb{Z}$.

Let $\mathbf{F}$ be a complex such that each F_i be a finitely generated graded free R-module. Then,

$$F_i = \oplus_{j\in\mathbb{Z}} R(-j)^{c_{i,j}},$$

where $c_{i,j}$ are nonnegative integers and all except finitely many of them are equal to zero. Hence, a complex of graded free finitely generated modules is of the form

$$\mathbf{F}\colon \ \ldots \to \oplus_{j\in\mathbb{Z}} R(-j)^{c_{i,j}} \xrightarrow{d_i} \oplus_{j\in\mathbb{Z}} R(-j)^{c_{i-1,j}} \to \ldots$$

The numbers $c_{i,j}$ are called the *graded Betti numbers* of the complex $\mathbf{F}$.

Definition 2.1.17 A *free resolution* of a finitely generated R-module M is a complex of finitely generated free R-modules

$$\mathbf{F} : \ldots \to F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \ldots \to F_1 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0,$$

which is exact and such that $\operatorname{Coker} d_1$ is isomorphic to M. Sometimes, we use the following notation for a free resolution

$$\mathbf{F} : \ldots \to F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \ldots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0.$$

Every module has a free resolution which can be constructed as follows. As a first step, we take a set of generators for M. A free module is mapped onto M by sending the free generators of the free module to the given generators of M. Subsequently, we consider the kernel of this map which is denoted by M_1. Now, we repeat the same procedure starting with M_1 and so on. For more details, we refer to [58, Construction 4.2]

Definition 2.1.18 Assume M is a finitely generated graded R-module and $\mathbf{F}$ is a resolution of M. We say that $\mathbf{F}$ is a *graded free resolution* of M if $\mathbf{F}$ is graded, each F_i is finitely generated and graded free and the isomorphism $M \simeq \operatorname{Coker} d_1$ is of degree 0. The *length* of $\mathbf{F}$ is defined as $\sup\{i \in \mathbb{N} \colon F_i \neq 0\}$. We say that $\mathbf{F}$ is a *finite resolution* if it has finite length.

Given a graded free resolution $\mathbf{F}$ of M, we fix a homogeneous basis of each graded free module F_i. Then, for each i the differential d_i is given by a matrix A_i with entries homogeneous elements of R. These matrices are called *differential matrices*. We note that the differential matrices depend on the given basis.

In what follows, we define the *minimal graded free resolution* $\mathbf{F}$ of a graded finitely generated R-module M. The minimal graded free resolution of a module is closely related with its structure. More precisely, it has the following form

$$F_2 \xrightarrow{\begin{bmatrix} \textit{a minimal system} \\ \textit{of homogeneous} \\ \textit{relations on the} \\ \textit{relations in } d_1 \end{bmatrix}} F_1 \xrightarrow{\begin{bmatrix} \textit{a minimal system} \\ \textit{of homogeneous} \\ \textit{relations on the} \\ \textit{minimal generators} \\ \textit{of } M \end{bmatrix}} F_0 \xrightarrow{\begin{bmatrix} \textit{a minimal} \\ \textit{system of} \\ \textit{homogeneous} \\ \textit{generators of } M \end{bmatrix}} M \to 0$$

It is remarkable that the structure of $\mathbf{F}$ reflects many properties of M.

Definition 2.1.19 Let $\mathbf{F}$ be a graded free resolution of a graded finitely generated R-module M. We say that $\mathbf{F}$ is *minimal* if for all $i \geq 0$ it holds that

$$d_{i+1}(F_{i+1}) \subset (x_1, \ldots, x_n)F_i.$$

In other words, $\mathbf{F}$ is *minimal* if there are no invertible elements (non-zero constants) in the entries of the differential matrices.

Due to the following theorem it is possible to say "the" minimal graded free resolution of M.

Theorem 2.1.20 *Let M be a finitely generated graded R-module. Then there exists a minimal graded free resolution of M which is unique up to isomorphism.*

Proof For the proof of the theorem see [58, Theorem 7.5]. $\square$

Definition 2.1.21 Let $\mathbf{F}$ be a minimal graded free resolution of a finitely generated graded R-module M. The i-th *Betti number of M over R*, denoted by $b_i^R(M)$, is defined as

$$b_i^R(M) = \operatorname{rank}(F_i).$$

Due to Theorem 2.1.20, the Betti numbers of M are independent of the choice of the minimal graded free resolution of M. Betti numbers, as numerical invariants of the resolution can be used to obtain some useful information for the resolution especially when it is complicated to have a description of the differentials.

Definition 2.1.22 Let $\mathbf{F}$ be a minimal graded free resolution of a finitely generated graded R-module M. The *graded Betti numbers* of M are defined as

$$b_{i,p}^R(M) = \text{number of summands in } F_i \text{ of the form } R(-p).$$

We use the notation b_i instead of $b_i^R(M)$ and $b_{i,p}$ instead of $b_{i,p}^R(M)$ when it is obvious which is the module and the ring that we use.

The Betti numbers are contained in a matrix of the following form

$$\begin{array}{c|cccc} & b_0 & b_1 & \cdots & b_i \\ \hline 0 & b_{0,0} & b_{1,1} & \cdots & b_{i,i} \\ 1 & b_{0,1} & b_{1,2} & \cdots & b_{i,i+1} \\ \vdots & \vdots & \vdots & & \vdots \\ p & b_{0,p} & b_{1,1+p} & \cdots & b_{i,i+p} \end{array}$$

The entry in the i-th row and p-th column is $b_{i,i+p}$. The i-th step of the minimal graded free resolution is contained in the i-th column. At the top there is an additional row which contains the i-th Betti number b_i. The column to the left which is seperated by a vertical line from the others columns contains the labels of the rows. A zero Betti number is denoted by $\cdot$ or $-$. This matrix is called a *Betti table*.

Proposition 2.1.23 *Denote by c the minimal degree of an element in a minimal system of homogeneous generators of M. Then, $b_{i,p}^R(M) = 0$ for $p < i + c$.*

Proof For a proof see [58, Proposition 12.3.]. □

Definition 2.1.24 Let F be a minimal graded free resolution of a finitely generated graded R-module M. The *projective dimension* of M is defined as

$$\text{proj.dim}_R(M) = \max\{i \mid b_i^R(M) \neq 0\}.$$

It is immediate that $\text{proj.dim}_R(M)$ is the length of the minimal graded free resolution of M.

Theorem 2.1.25 *(Hilbert's syzygy theorem) Every finitely generated graded R-module M has a finite graded free resolution of length at most n. More generally, every finitely generated R-module M has a finite free resolution of length at most n.*

Proof For the proof, we refer to [17, Corollary 2.2.14]. □

2.1.3 Hilbert functions

In the present subsection we assume that k is a field, $R = k[x_1, \ldots, x_n]$ is a polynomial ring over k and we have a grading on R such that each variable x_i is homogeneous of positive degree.

Assume $I \subset R$ is a homogeneous ideal. The Hilbert function of R/I is an important numerical invariant of I which gives the sizes of the graded components I_j of degree j of I. From the point of view of Algebraic Geometry, it encodes important information, such as the dimension and the degree of the associated projective variety $V(I)$.

Remark 2.1.26 We note that if M is a finitely generated graded R-module then $\dim_k(M_i) < \infty$ for all $i \in \mathbb{Z}$ and there exists $N \in \mathbb{Z}$ such that $M_i = 0$ for all $i < N$.

Proposition 2.1.27 *Assume R is standard graded and I is an ideal of R generated by monomials. Then, for all $j \geq 0$, $(R/I)_j$ is a vector space with basis the set*

$$\{\textit{monomial } m \in R : m \notin I, \deg(m) = j\}$$

of monomials of degree j which are not elements of I.

Proof For a proof of the theorem we refer to [58, Proposition 1.8]. □

From Proposition 2.1.27 it follows that if I is an ideal generated by monomials, then $\dim_k(R/I)_j$ is equal to the number of the degree j monomials of R which are not elements of I.

Definition 2.1.28 Let M be a finitely generated graded R-module. The numerical function $\mathrm{H}_M : \mathbb{Z} \to \mathbb{N}$ defined by

$$\mathrm{H}_M(i) := \dim_k(M_i)$$

is called the *Hilbert function* of M. The formal power series

$$\mathrm{Hilb}_M(t) = \sum_{i \in \mathbb{Z}} \dim_k(M_i)t^i \in \mathbb{Z}[[t]]$$

is called the *Hilbert series* of M.

According to the following theorem, in the standard graded case one can also define the Hilbert polynomial of a finitely generated graded module.

Theorem 2.1.29 *(Hilbert) Assume that R is standard graded and M is a finitely generated graded R-module. Then there exists a unique polynomial $P_M(t) \in \mathbb{Q}[t]$ of degree $\leq n-1$ and a positive integer N such that $\mathrm{H}_M(i) = P_M(i)$ for all $i > N$.*

Proof For a proof of the theorem we refer to [27, Theorem 1.11]. □

We call the polynomial $P_M(t)$ in the above theorem the *Hilbert polynomial* of M.

The following proposition shows that the Hilbert series is additive on short exact sequences of graded R-modules.

Proposition 2.1.30 *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of finitely generated graded R-modules and homomorphisms of degree 0. *Then,*

$$\mathrm{Hilb}_{M_2}(t) = \mathrm{Hilb}_{M_1}(t) + \mathrm{Hilb}_{M_3}(t).$$

Proof For the proof see [58, Proposition 16.1]. □

For the way to compute the Hilbert series of a finitely generated graded R-module from a graded free resolution of M we refer the reader to [58, Section 16].

The following theorem is related to the rationality of the Hilbert series.

Theorem 2.1.31 *Assume that the variable x_i has degree a_i and I is a homogeneous ideal of R. Then $\mathrm{Hilb_{R/I}(t)}$ is a rational function of t, in the sense that there exists a polynomial $p(t) \in \mathbb{Z}[t]$ such that*

$$\mathrm{Hilb}_R(t) = \frac{p(t)}{\prod_{i=1}^n (1 - t^{a_i})}.$$

Proof It follows from [17, Proposition 4.4.1]. □

Example 2.1.32 If R is standard graded then

$$\mathrm{Hilb}_R(t) = \frac{1}{(1-t)^n}.$$

2.1.4 Cohen-Macaulay and Gorenstein rings

In this subsection we discuss the Cohen-Macaulay and Gorenstein rings. These are two classes of rings which play important role in Commutative Algebra, Algebraic Geometry and Algebraic Combinatorics. Cohen-Macaulay rings include the rings which are associated to some interesting classes of singular varieties and schemes. The anti-canonical ring of a Fano n-fold and the canonical ring of a regular surface of general type are examples of Gorenstein rings. Our main reference is [17].

Definition 2.1.33 Assume R is a ring and $N \neq 0$ is an R-module. An element $r \in R$ is called *N-regular* if $rn \neq 0$ for all nonzero $n \in N$.

Definition 2.1.34 ([17, Definition 1.1.1]) Let R be a ring and M be an R-module. A sequence $x_1, \dots, x_n$ of elements of R is called *regular sequence* for M or *M-regular sequence* if it satisfies the following conditions

1. $M/(x_1, \dots, x_n)M \neq 0$.
2. x_1 is M-regular.
3. For all $2 \leq i \leq n$, the element x_i is $M/(x_1, \dots, x_{i-1})M$-regular.

Remark 2.1.35 Assume that R is a local ring with maximal ideal $\mathfrak{m}$ and $M \neq 0$ is a finitely generated R-module. If the ideal $(x_1, \dots, x_n)$ of R is contained in $\mathfrak{m}$ then by Nakayama's Lemma [27, Corollary 4.8] the first condition of Definition 2.1.34 is automatically satisfied.

Example 2.1.36 The sequence $x_1, \dots, x_n$ of the variables in the polynomial ring $R = k[x_1, \dots, x_n]$ over a field k is a regular sequence on R.

Definition 2.1.37 Let R be a Noetherian ring and M be an R-module. Assume that I is an ideal of R. An M-regular sequence $x_1, \dots, x_n$ which is contained in I is called *maximal* if there is no element $x_{n+1} \in I$ such that $x_1, \dots, x_n, x_{n+1}$ is an M-regular sequence in I.

Theorem 2.1.38 *(Rees) Let M be a finitely generated R-module over a Noetherian ring R and I be an ideal such that $IM \neq M$. Then all maximal M-regular sequences contained in I have the same length n given by*

$$n = \min\{i \geq 0 : \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Proof For the proof see [17, Theorem 1.2.5]. □

Definition 2.1.39 ([17, p. 65]) A Noetherian local ring R is called *regular* if the dimension of R is equal to the minimal number of generators of its unique maximal ideal.

Definition 2.1.40 ([17, p. 68]) A Noetherian ring R is called *regular* if for every maximal ideal $\mathfrak{m}$ of R the localization $R_{\mathfrak{m}}$ is regular.

Example 2.1.41 The polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k is a regular ring.

Definition 2.1.42 ([17, p. 12]) Let R be a Noetherian ring and $I \subset R$ a proper ideal. We call *grade* of I the common length of all maximal R-sequences contained in I.

Definition 2.1.43 ([17, p. 412]) Let R be a Noetherian ring and $I \subset R$ a proper ideal. We call *height* of I in R and denote by height I the minimum of $\dim R_p$, where p takes value in the set of prime ideals of R containing I.

Definition 2.1.44 ([17, p. 413]) Let R be a Noetherian ring and $I \subset R$ an ideal. We define the *codimension* of I in R, denoted by $\operatorname{codim} I$, as follows:

$$\operatorname{codim} I = \dim R - \dim R/I.$$

The basic inequality between grade and height of an ideal is described in the following proposition.

Proposition 2.1.45 *Assume that R be a Noetherian ring and $I \subset R$ a proper ideal. Then,*

$$\operatorname{grade} I \leq \operatorname{height} I.$$

Proof For the proof see [17, Theorem 1.2.14]. □

Definition 2.1.46 ([17, Definition 1.2.7]) Assume R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and N is a finitely generated R-module. We define as *depth* of N the common length of all maximal N-sequences contained in $\mathfrak{m}$.

Definition 2.1.47 ([17, Definition 2.1.1]) A Noetherian local ring R is called *Cohen-Macaulay* if the depth of R as an R-module is equal to the dimension of R. More generally, we call a Noetherian ring R *Cohen-Macaulay* if for every maximal ideal $\mathfrak{m}$ the localization $R_{\mathfrak{m}}$ is Cohen-Macaulay.

Example 2.1.48 The polynomial ring $R = k[x_1, \dots, x_n]$ over a field k is a Cohen-Macaulay ring.

Theorem 2.1.49 *Let R be a Cohen-Macaulay ring and I an ideal of R with $I \neq R$. Then,*

$$\operatorname{grade} I = \operatorname{height} I.$$

Moreover, if R is local then

$$\operatorname{height} I = \operatorname{codim} I.$$

Proof For the proof see [17, Corollary 2.1.4]. □

Remark 2.1.50 Theorem 2.1.49 also holds for the case of a graded ring R and a homogeneous ideal I.

Remark 2.1.51 Assume k is a field and R is a finitely generated k-algebra which is an integral domain. Then, by [27, p. 226], for all proper ideals I of R we have $\operatorname{height} I = \operatorname{codim} I$. In particular, this holds when R is a polynomial ring over a field in finitely many variables.

Theorem 2.1.52 *(Krull's Principal Ideal Theorem) Assume that R is a local Noetherian ring and I is an ideal of R which is generated by n elements. Then,* $\operatorname{codim} I \leq n$.

Proof For the proof see [17, p. 414]. □

We now introduce the class of Gorenstein rings. There are many equivalent definitions for Gorenstein rings, we give one of them. For a more extensive treatment we refer to [6, 17, 27].

Definition 2.1.53 ([17, Definition 3.1.18]) A Noetherian local ring R is called *Gorenstein* if it has finite injective dimension as an R-module. More generally, a Noetherian ring R is called *Gorenstein* if for every maximal ideal $\mathfrak{m}$ of R the localization $R_{\mathfrak{m}}$ is Gorenstein.

Definition 2.1.54 An ideal I of a Gorenstein ring R is called *Gorenstein* if the quotient ring R/I is Gorenstein.

Example 2.1.55 The polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k is Gorenstein.

Theorem 2.1.56 *Let R be a polynomial ring of dimension n and I a homogeneous ideal of R. If the ring R/I is Gorenstein ring of dimension q then*

$$b_i^R(R/I) = b_{n-q-i}^R(R/I),$$

for all $0 \leq i \leq n-q$.

Proof For the proof see [58, Theorem 25.6]. □

The following theorem gives a criterion for a ring to be Gorenstein.

Theorem 2.1.57 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and I a homogeneous ideal of R. We set $q = \dim R/I$. The ring R/I is Gorenstein if and only if*

$$\text{proj.dim}_R(R/I) = n - q \quad \textit{and} \quad b_{n-q}^R(R/I) = 1.$$

Proof For the proof see [58, Theorem 25.7]. □

In the present work, we will call an ideal I of a polynomial ring $k[x_1, \ldots, x_n]$ over a field k a *complete intersection ideal* if I can be generated by $\operatorname{codim} I$ elements. We refer to [17, Section 2.3] for more details about this notion.

We conclude with a theorem which shows how the classes of regular rings, complete intersections, Cohen-Macaulay and Gorenstein rings are related.

Theorem 2.1.58 *Let R be a polynomial ring and I a homogeneous ideal of R. We set $S = R/I$. Then we have the following implications*

S regular $\Rightarrow$ S complete intersection $\Rightarrow$ S Gorenstein $\Rightarrow$ S Cohen-Macaulay.

Proof It follows from [17, Proposition 3.1.20]. □

2.1.5 Structure theorems for Gorenstein ideals of codimension ≤ 3

If $R = k[x_1, \ldots, x_n]/I$ is a Gorenstein graded ring and the codimension of the homogeneous ideal I is less or equal to 3 then there are good structure theorems. Serre proved that if $\operatorname{codim} I = 1$ or 2 then R is a complete intersection while Buchsbaum and Eisenbud [19] showed that if $\operatorname{codim} I = 3$ then I is generated by the $2n \times 2n$ Pfaffians of a skewsymmetric $(2n+1) \times (2n+1)$ matrix. In this subsection we recall these structure theorems.

Definition 2.1.59 Assume that $M = [m_{ij}]$, $1 \leq i, j \leq n$, is an $n \times n$ skewsymmetric matrix (i.e., $m_{ji} = -m_{ij}$ and $m_{ii} = 0$) with entries in a commutative ring S.

1. If n is even, then there exists a unique polynomial $\operatorname{Pf}(M)$ in m_{ij} with the following properties

   (a) $(\operatorname{Pf}(M))^2 = \det M$

   (b) $\operatorname{Pf}(\begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}) = 1_S$,

   where $I_{n/2}$ is the $n/2 \times n/2$ identity matrix. The polynomial $\operatorname{Pf}(M)$ is called the *Pfaffian* of the matrix M.

2. If n is odd by *Pfaffians* of M we mean the set $\{\operatorname{Pf}(M_1), \operatorname{Pf}(M_2), \ldots, \operatorname{Pf}(M_n)\}$, where M_i denotes the skewsymmetric submatrix of M obtained by deleting the i-th row and i-th column of M.

For more details about Pfaffians we refer to [44, Chapter XV, Section 9].

Example 2.1.60 1. For $n = 2$:

$$\operatorname{Pf}(\begin{pmatrix} 0 & m_{12} \\ -m_{12} & 0 \end{pmatrix}) = m_{12}.$$

2. For $n = 5$:

$$\operatorname{Pf}(\begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ -m_{12} & 0 & m_{23} & m_{24} & m_{25} \\ -m_{13} & -m_{23} & 0 & m_{34} & m_{35} \\ -m_{14} & -m_{24} & -m_{34} & 0 & m_{45} \\ -m_{15} & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix}) = \{\operatorname{Pf}(M_1), \operatorname{Pf}(M_2), \ldots, \operatorname{Pf}(M_5)\}$$

where

$$\begin{aligned}
\mathrm{Pf}(M_1) &= m_{23}m_{45} - m_{24}m_{35} + m_{25}m_{34}, \\
\mathrm{Pf}(M_2) &= m_{13}m_{45} - m_{14}m_{35} + m_{15}m_{34}, \\
\mathrm{Pf}(M_3) &= m_{12}m_{45} - m_{14}m_{25} + m_{15}m_{24}, \\
\mathrm{Pf}(M_4) &= m_{12}m_{35} - m_{13}m_{25} + m_{15}m_{23}, \\
\mathrm{Pf}(M_5) &= m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}.
\end{aligned}$$

Theorem 2.1.61 *Assume k is a field and that the polynomial ring $S = k[x_1, \dots, x_n]$ has a grading with $\deg x_i > 0$ for all i. Assume $I \subset S$ is a homogeneous ideal. We set $R = S/I$.*

1. *(Serre) If $\operatorname{codim} I = 1$ then R is Gorenstein if and only if the ideal I is generated by a single element of S. If $\operatorname{codim} I = 2$ then R is Gorenstein if and only if the ideal I is generated by two elements of S.*

2. *(Buchsbaum-Eisenbud [19]) If $\operatorname{codim} I = 3$ then R is Gorenstein if and only if I is generated by the $2n \times 2n$ Pfaffians of a skewsymmetric $(2n+1) \times (2n+1)$ matrix with entries in S.*

Proof The first part of the theorem follows from [27, Corollary 21.20]. The second part of the theorem follows from [17, Theorem 3.4.1]. □

2.1.6 Lefschetz properties

In this subsection we recall the notions of Weak and Strong Lefschetz Property of a graded k-algebra. Good general references are [29, 49].

Assume k is a field. In this subsection all graded k-algebras will be Noetherian and of the form $G = \oplus_{i \geq 0} G_i$ with $G_0 = k$ and $\dim_k G_i < \infty$ for all i. Recall that G is called standard graded if it is generated, as a k-algebra, by G_1. We denote by $\dim G$ the Krull dimension of G.

Definition 2.1.62 Assume F is an Artinian graded k-algebra. There exists a largest integer d such that the d-th graded part F_d is nonzero, and we call d the *socle degree* of F. An element $\omega \in F_1$ is called a *Weak Lefschetz element* if, for all $i \geq 0$, the multiplication by ω map $F_i \to F_{i+1}$ is of maximal rank, which means that it is injective or surjective (or both). We say that F has the *Weak Lefschetz Property* if there exists a Weak Lefschetz element $\omega \in F_1$.

Definition 2.1.63 Assume F is an Artinian Gorenstein graded k-algebra of socle degree d. An element $\omega \in F_1$ is called a *Strong Lefschetz element* if, for all i with $0 \leq 2i \leq d$, the multiplication by ω^{d-2i} map $F_i \to F_{d-i}$ is bijective. We say that F has the *Strong Lefschetz Property* if there exists a Strong Lefschetz element $\omega \in F_1$.

Definition 2.1.64 We say that a standard graded k-algebra G with positive Krull dimension has the *Weak Lefschetz Property* if it is Cohen-Macaulay, the field k is infinite and for Zariski general homogeneous degree 1 elements $f_1, \dots, f_{\dim G}$ of G the Artinian k-algebra $G/(f_1, \dots, f_{\dim G})$ has the Weak Lefschetz Property.

Definition 2.1.65 We say that a standard graded k-algebra G with positive Krull dimension has the *Strong Lefschetz Property* if it is Gorenstein, the field k is infinite and for Zariski general homogeneous degree 1 elements $f_1, \dots, f_{\dim G}$ of G the Artinian k-algebra $G/(f_1, \dots, f_{\dim G})$ has the Strong Lefschetz Property.

Remark 2.1.66 We refer to [29, Theorem 2.79] for the following well-known fact. Assume $F = \oplus_{i=0}^{d} F_i$ with $F_d \neq 0$ is a standard graded Gorenstein Artinian k-algebra. Then F_d is 1-dimensional, and, for all i with $0 \leq i \leq d$, the multiplication map $F_i \times F_{d-i} \to F_d \cong k$ is a perfect pairing. As a consequence, given i, j with $0 \leq i \leq j \leq d$ and a nonzero element $u \in F_i$, there exists $w \in F_{j-i}$ such that $uw \neq 0$. The reason is that by the perfect pairing property there exists $w_1 \in F_{d-i}$ such that $uw_1 \neq 0$, and since F is standard graded, w_1 is a sum of products of elements of F_{j-i} with elements of F_{d-j}.

2.2 Some notions of Algebraic Geometry

In this section we recall some notions of Algebraic Geometry that we need. More precisely, we discuss the Proj construction which assigns a projective scheme to a graded ring, a construction that will allow us to pass from algebra to geometry in Sections 4.3 and 5.2. Moreover, we discuss the notion of the Mori category and Fano 3-folds.

2.2.1 The Proj of a graded ring

We briefly recall the Proj construction, which given a graded ring R produces a projective scheme.

Definition 2.2.1 Let R be a graded ring and R_+ be the irrelevant ideal of R. We define as Proj R the set of all homogeneous prime ideals of R which do not contain the ideal R_+.

Proj R can be viewed as topological space as follows. If I is a homogeneous ideal of R, we define the subset

$$V(I) = \{P \in \text{Proj } R \mid I \subseteq P\}.$$

Due to [30, I.2, Proposition 2.1], we can define a topology on Proj R by taking the closed sets to be the subsets of the form $V(I)$. This topology is called *Zariski topology.*

There is a natural construction of a sheaf of rings on Proj R which makes Proj R a projective scheme. For details we refer to [30, II.7].

Definition 2.2.2 Assume that k is a field and $S = k[x_0, \dots, x_n]$ is the polynomial ring which has a grading with $\deg x_i = a_i > 0$ for all i. The *weighted projective space* denoted by $\mathbb{P}(a_0, \dots, a_n)$ is defined as

$$\mathbb{P}(a_0, \dots, a_n) = \text{Proj } S.$$

For more details related to weighted projective space we refer to [7, 26, 33].

2.2.2 Fano 3-folds

In this subsection we recall some notions of Algebraic Geometry related to Fano 3-folds. Good references are [4, 34, 47].

Assume X is an irreducible normal variety. A *Weil divisor* of X is a formal sum

$$D = \sum_i k_i D_i,$$

where the sum is over all irreducible codimension 1 subvarieties D_i of X, the k_i are integers and the set $\{i \ : \ k_i \neq 0\}$ is finite. For the definition of Cartier divisors we refer the reader to [30, Chapter II, Section 6]. In the following, we denote by K_X the canonical divisor of X and by $-K_X$ the anticanonical divisor of X.

Definition 2.2.3 An irreducible normal variety X has *terminal singularities* if it satisfies the following conditions:

1. For some positive integer r, rK_X is a Cartier divisor.

2. If $f : Y \to X$ is a resolution of singularities of X and $\{E_i\}$ is the family of all exceptional prime divisors of f then $K_Y = f^*K_X + \sum a_i E_i$ with all $a_i > 0$.

For more details on terminal singularities we refer the reader to [47, Section 4.1].

Definition 2.2.4 A normal variety X is called $\mathbb{Q}$-*factorial* if every Weil divisor on X has a positive integer multiple which is a Cartier divisor.

Definition 2.2.5 We say that a normal projective variety belongs to the *Mori category* if it has at worst $\mathbb{Q}$-factorial terminal singularities.

For an important property that characterizes the Mori category we refer the reader to [47, Theorem 4.1.3].

For a divisor $D = \sum_i k_i D_i$, we write $D \geq 0$ if $k_i \geq 0$ for all i. For the definition of the divisor $\mathrm{div}(f)$ of a nonzero rational function we refer the reader to [30, Chapter II, Section 6].

Definition 2.2.6 Let X be an irreducible normal projective variety over an algebraically closed field k. Assume $D = \sum_i k_i D_i$ is a Weil divisor on X. The *Riemann-Roch space* of D is defined as

$$H^0(X, D) := \{f \in k(X) \setminus \{0\} \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

In other words, it is the finite dimensional vector space of rational functions $f \in k(X)$ consisting of 0 together with all nonzero $f \in k(X)$ such that for all i

1. If $k_i = 0$, then f can have a zero along D_i, or no pole nor zero along D_i.
2. If $k_i < 0$, then f has a zero along D_i with the multiplicity of the zero at least $-k_i$.
3. If $k_i > 0$, then f can have a zero along D_i, or no pole nor zero along D_i, or a pole along D_i with the multiplicity of the pole in the set $\{1, 2, \ldots, k_i\}$.

Example 2.2.7 Consider $X = \mathbb{P}^1$ the projective line and $P = [0 : 1]$, $Q = [1 : 1]$ two points of X. Let $D = 2P - Q$. Then, $f \in H^0(X, D)$ if and only if either $f = 0$ or f is nonzero and

1. f has a zero of multiplicity at least 1 at Q and
2. f has no pole on $X \setminus \{P\}$ and
3. f has a zero at P, or f has no pole nor zero at P, or f has a pole at P of multiplicity 1 or 2.

Definition 2.2.8 The *anticanonical ring* of a normal variety X is the graded ring defined by

$$R(X, -K_X) := \bigoplus_{n \geq 0} H^0(X, -nK_X).$$

The multiplication is given by the natural maps

$$H^0(X, -nK_X) \otimes_k H^0(X, -mK_X) \to H^0(X, (-n-m)K_X).$$

Definition 2.2.9 A *Fano* 3-*fold* is a normal projective variety X of dimension 3 which belongs to the Mori category and the anticanonical divisor $-K_X$ is ample.

The maximal integer f such that $-K_X$ is divisible by some Weil divisor A, that is $-K_X = fA$, is called the *Fano index* of X. The anticanonical ring $R(X, -K_X)$ of a Fano 3-fold X is Gorenstein, finitely generated and $X \cong \text{Proj } R(X, -K_X)$.

Definition 2.2.10 Let X be a quasi-projective variety over $\mathbb{C}$ and x, y, z be coordinates of $\mathbb{C}^3$. Suppose that the group $\mathbb{Z}_r$ of r-th roots of unity acts on $\mathbb{C}^3$ via:

$$(x, y, z) \mapsto (\epsilon^a x, \epsilon^b y, \epsilon^c z),$$

where ϵ is a fixed primitive r-th root of unity and a, b, c are integers. A singularity $P \in X$ is a *quotient singularity of type* $\frac{1}{r}(a, b, c)$ if (X, P) is isomorphic to an analytic neighborhood of $(\mathbb{C}^3, 0)/\mathbb{Z}_r$. A *basket of singularities* is a collection of quotient singularities of type $\{\frac{1}{r_1}(a_1, b_1, c_1), \frac{1}{r_2}(a_2, b_2, c_2), \ldots, \frac{1}{r_s}(a_s, b_s, c_s)\}$.

In the case of Fano 3-folds of index f, Suzuki ([64, Lemma 1.2]) proves that in Definition 2.2.10 we can assume that $b = -a$, $c = f$ and r is coprime to a, b, c. Therefore, for a Fano 3-fold of index f, a basket of singularities is a collection of singularities $\frac{1}{r}(a, -a, f)$.

Definition 2.2.11 Let X be a closed subvariety of a weighted projective space with homogeneous ideal $I(X) \subset k[x_0, \ldots, x_n]$. The *affine cone* over X, denoted by C_X, is the zero set $V(I(X)) \subset \mathbb{A}^{n+1}$. We define the *vertex* of C_X to be the point $(0, 0, \ldots, 0)$.

Definition 2.2.12 Let X be a closed subvariety of a weighted projective space. X is called *quasismooth* if its affine cone C_X is smooth outside its vertex.

2.3 A review of some basic notions of Combinatorics

In this section we introduce some notions from Combinatorial Commutative Algebra. We discuss simplicial complexes to which we assign algebraic objects, the Stanley-Reisner rings. We investigate the main properties of these objects and we see how these are related to the algebraic notions introduced in Section 2.1. Good general references are [17, 31, 32, 50, 63].

2.3.1 Simplicial complexes

In this subsection we recall the notion of a simplicial complex.

Definition 2.3.1 Let $V = \{v_1, \ldots, v_m\}$ be a finite set. A *simplicial complex* Δ on V is a collection of subsets of V such that $G \in \Delta$ and $F \subseteq G$ implies that $F \in \Delta$ and such that $\{v_i\} \in \Delta$ for all $v_i \in V$. The elements of Δ are called *faces*. The *dimension* of a face F, $\dim F$ is the number $\#F - 1$, where $\#F$ denotes the cardinality of the set F. A zero dimensional face of Δ is called *vertex*. A one dimensional face of Δ is called *edge*. The *dimension* $\dim \Delta$ of the simplicial complex Δ is defined as the maximum of the dimensions of the faces of Δ.

Definition 2.3.2 Let Δ be a simplicial complex with vertex set $V = \{v_1, \ldots, v_m\}$. A face of Δ is called a *facet* if it is a maximal face under inclusion. The set of facets is denoted by $\mathcal{F}(\Delta)$. A *nonface* of Δ is a subset F of V with the property $F \notin \Delta$.

Remark 2.3.3 By convention, the empty set $\emptyset$ is the unique face of dimension -1 in any simplicial complex. A simplicial complex Δ is determined by the set $\mathcal{F}(\Delta)$.

A simplicial complex in which all of its facets have the same dimension is called *pure*.

Definition 2.3.4 Let V, W be disjoint sets and Γ, Δ be simplicial complexes on V and W respectively. The *join* $\Gamma \star \Delta$ is defined as the simplicial complex with vertex set $V \cup W$ and faces $F \cup G$ where $F \in \Gamma$ and $G \in \Delta$.

Definition 2.3.5 A *simplex* is a simplicial complex with a unique facet.

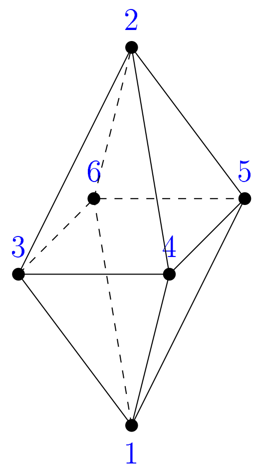
Definition 2.3.6 Let Δ be an arbitrary simplicial complex of dimension $n \geq 0$ on a vertex set V. Denote by f_i the number of i-dimensional faces of Δ. The $(n+1)$-tuple

$$f(\Delta) = (f_0, \ldots, f_n)$$

is called the *f-vector* of Δ.

We note that f_0 is the number of vertices of Δ. We set $f_{-1} = 1$.

Example 2.3.7 Consider the octahedron Δ with vertex set $V = \{1, \ldots, 6\}$.



The set of facets $\mathcal{F}(\Delta)$ is described as

$$\mathcal{F}(\Delta) = \{\{1,3,4\},\{1,4,5\},\{1,5,6\},\{1,3,6\},\{2,3,4\},\{2,4,5\},\{2,5,6\},\{2,3,6\}\}.$$

The f-vector $f(\Delta)$ is equal to

$$f(\Delta) = (6, 12, 8).$$

Definition 2.3.8 Assume Δ is a simplicial complex of dimension n. We define the *h-vector* $h(\Delta) = (h_0, \ldots, h_{n+1})$ of Δ by the equality

$$\sum_{i=0}^{n+1} h_i x^{n+1-i} = \sum_{i=0}^{n+1} f_{i-1}(x-1)^{n+1-i}.$$

Example 2.3.9 Consider the octahedron Δ (see Example 2.3.7). The h-vector $h(\Delta)$ is equal to

$$h(\Delta) = (1, 3, 3, 1).$$

Definition 2.3.10 Let Δ be a simplicial complex of dimension n. The *g-vector* $g(\Delta) = (g_0, \ldots, g_{[(n+1)/2]})$ of Δ is defined by

$$g_0 = 1, \quad g_i = h_i - h_{i-1}$$

for $1 \leq i \leq [(n+1)/2]$.

Definition 2.3.11 Let Δ be a simplicial complex with vertex set $V = \{v_1, \ldots, v_m\}$ and F a subset of V. The *star* of F is denoted by $st_\Delta F$ and described as

$$\mathrm{st}_\Delta F = \{G \in \Delta : F \cup G \in \Delta\}.$$

The *link* of F is denoted by $\mathrm{link}_\Delta F$ and described as

$$\mathrm{link}_\Delta F = \{G \in \Delta : F \cup G \in \Delta,\ F \cap G = \emptyset\}.$$

Example 2.3.12 Consider the octahedron Δ (see Example 2.3.7) with vertex set $V = \{1, \dots, 6\}$ and $F = \{4, 5\}$, $G = \{4\}$ two subsets of V. Then, the set of facets of $\mathrm{link}_\Delta F$ is

$$\{\{1\}, \{2\}\},$$

hence $\mathrm{link}_\Delta F$ consists of two nonconnected points. Moreover, the set of facets of $\mathrm{link}_\Delta G$ is

$$\{\{1, 5\}, \{1, 3\}, \{2, 3\}, \{2, 5\}\},$$

hence $\mathrm{link}_\Delta G$ is the 4-gon with vertex set {2,5,1,3}. The set of facets of $\mathrm{st}_\Delta F$ is

$$\{\{1, 4, 5\}, \{2, 4, 5\}\},$$

hence $\mathrm{st}_\Delta F$ consists of two solid triangles with common edge $\{4, 5\}$. Finally, the set of facets of $\mathrm{st}_\Delta G$ is

$$\{\{1, 5, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{2, 5, 4\}\},$$

hence $\mathrm{st}_\Delta G$ is the join of the 4-gon with vertex set {2,5,1,3} with the vertex 4.

Definition 2.3.13 Let $V = \{v_1, \dots, v_m\}$ be a finite set. Denote by e_i the i-th unit vector of $\mathbb{R}^m$. For a subset F of V, we define

$$|F| = \text{convex hull}\,\{e_i \,|\, v_i \in F\}.$$

The *geometric realization* of a simplicial complex Δ, denoted by $|\Delta|$ is defined as

$$|\Delta| = \bigcup_{F \in \Delta} |F|.$$

The set $|\Delta|$ is a subset of $\mathbb{R}^n$. Hence, it becomes a topological space with the subspace topology.

Definition 2.3.14 Let $n \geq 1$ be an integer. A *simplicial sphere* of dimension n is a simplicial complex D of dimension n such that its geometric realization is homeomorphic to the unit sphere S^n.

2.3.2 Stanley-Reisner rings

Assume k is a field. In this subsection we recall the construction, due to Stanley, that associates to a simplicial complex D the Stanley-Reisner ideal of it, which is an ideal of the polynomial ring $k[x_1, \ldots, x_m]$ generated by squarefree monomials. Here m is the number of vertices of D. Good references are [17, 31, 32, 50, 63].

Definition 2.3.15 Let $R = k[x_1, \ldots, x_m]$ be the polynomial ring over the field k and assume $(a_1, \ldots, a_m) \in \mathbb{N}^m$. The monomial $x_1^{a_1} x_2^{a_2} \ldots x_m^{a_m}$ of R is called *squarefree* if $a_i \in \{0, 1\}$ for all i. An ideal which is generated by monomials is called a *monomial ideal*. An ideal which is generated by squarefree monomials is called a *squarefree monomial ideal*.

Definition 2.3.16 Let Δ be a simplicial complex with vertex set $V = \{v_1, \ldots, v_m\}$ and k be a field. Denote by I_Δ the ideal generated by all monomials $x_{i_1} x_{i_2} \ldots x_{i_s}$ such that $\{v_{i_1}, \ldots, v_{i_s}\} \notin \Delta$. That is, I_Δ is the squarefree monomial ideal generated by monomials corresponding to nonfaces of Δ. The ideal I_Δ is called the *Stanley-Reisner ideal* (or *face ideal*) of Δ. The quotient ring

$$k[\Delta] = k[x_1, \ldots, x_m]/I_\Delta$$

is called the *Stanley-Reisner ring* (or *face ring*) of Δ over k.

Theorem 2.3.17 *Assume k is a field. Associating to a simplicial complex its Stanley-Reisner ideal over k induces a bijection between simplicial complexes with vertex set $\{1, \ldots, m\}$ and squarefree monomial ideals of the polynomial ring $R = k[x_1, \ldots, x_m]$.*

Proof For the proof see [50, Theorem 1.7]. □

Theorem 2.3.18 *Let Δ be a simplicial complex. It holds that*

$$I_\Delta = \bigcap_{F \in \Delta} J_F,$$

where J_F is the ideal generated by all x_i such that $v_i \notin F$. In particular,

$$\dim k[\Delta] = \dim \Delta + 1.$$

Proof For the proof see [17, Theorem 5.1.4]. □

Example 2.3.19 Consider the octahedron Δ with vertex set $V = \{1, \ldots, 6\}$ (see Example 2.3.7). The Stanley-Reisner ideal of Δ is

$$I_\Delta = (x_1x_2, x_3x_5, x_4x_6)$$

and

$$k[\Delta] = k[x_1, \ldots, x_6]/I_\Delta.$$

Definition 2.3.20 A simplicial complex Δ is called *Gorenstein* over a field k if $k[\Delta]$ is Gorenstein.

For the relation of this notion to the simplicial homology of the geometric realization of Δ we refer to [17, Section 5.6].

Theorem 2.3.21 *Asume k is a field and D is a simplicial sphere. Then D is Gorenstein over k.*

Proof For the proof see [17, Theorem 5.6.2]. $\square$

Chapter 3

Unprojection Theory

In this chapter, we recall some basic facts related to Kustin-Miller unprojection. For more details see [54, 55, 57, 59].

3.1 Kustin-Miller unprojection

Assume that R is a Gorenstein local ring and $J \subset R$ is a codimension 1 ideal such that the quotient ring R/J be Gorenstein. Then, $\mathrm{Hom}_R(J, R)$ is generated as an R-module by the inclusion map $i \colon J \to R$ and an extra homomorphism $\phi \colon J \to R$, as it follows from [57, Lemma 1.1].

Definition 3.1.1 We define the *Kustin-Miller unprojection ring*, $\mathrm{Unpr}(J, R)$, of the pair $J \subset R$ to be the graph of ϕ, that is the quotient

$$\mathrm{Unpr}(J, R) = \frac{R[T]}{(Tr - \phi(r) : r \in J)},$$

where T is a new variable called the unprojection variable.

Theorem 3.1.2 *([57, Theorem 1.5]) The ring* $\mathrm{Unpr}(J, R)$ *is Gorenstein.*

The simplest example of Kustin-Miller unprojection, which nevertheless has important consequences in birational geometry, is the example of a hypersurface which contains a codimension 2 complete intersection.

Example 3.1.3 (Reid's $Ax - By$ argument)
Assume $A, B \in k[x, y, z, w]$ such that $Ax - By$ nonzero. We set

$$R = \frac{k[x, y, z, w]}{(Ax - By)}$$

and

$$J = (x, y) \subset R.$$

We define $\phi \colon J \to R$ to be the unique R-module homomorphism such that $\phi(x) = B$ and $\phi(y) = A$. Then, $\mathrm{Hom}_R(J, R)$ is generated as R-module by i and ϕ.
Moreover,

$$\mathrm{Unpr}(J, R) = \frac{R[T]}{(Tx - B, Ty - A)}.$$

For more details we refer to [59, Section 2].

For applications of the Kustin-Miller unprojection to graded rings and birational geometry we refer to [57, 59].

3.1.1 Parallel Kustin-Miller unprojection

Sometimes, especially for applications, it is necessary to perform not only one but several Kustin-Miller unprojections. Neves and Papadakis [53] developed such a theory which they named parallel Kustin-Miller unprojection. In this subsection, we recall their formulation.

Assume k is a field and $\mathcal{L}$ is a nonempty finite indexing set. Assume that R is a Gorenstein positively graded k-algebra and $\{J_\alpha, \alpha \in \mathcal{L}\}$ is a set of codimension 1 homogeneous ideals of R such that, for all $\alpha \in \mathcal{L}$, the quotient ring R/J_α is Gorenstein.

For each $\alpha \in \mathcal{L}$ we fix a graded R-module homomorphisms $\phi_\alpha \colon J_\alpha \to R$ such that $\mathrm{Hom}_R(J_\alpha, R)$ is generated as an R-module by $\{i_\alpha, \phi_\alpha\}$, where $i_\alpha \colon J_\alpha \to R$ is the inclusion map. We make the assumption that for distinct α, $\beta \in \mathcal{L}$ there exists a homogeneous element $r_{\alpha\beta} \in R$ with $\deg\ r_{\alpha\beta} = \deg\ \phi_\alpha$ such that

$$(\phi_\alpha + r_{\alpha\beta} i_\alpha)(J_\alpha) \subset J_\beta \tag{3.1}$$

and that for all distinct $\alpha, \beta \in \mathcal{L}$,

$$\mathrm{codim}_R(J_\alpha + J_\beta) \geq 2. \tag{3.2}$$

Denote by $\phi_{\alpha\beta} = \phi_\alpha + r_{\alpha\beta} i_\alpha$. By [53, Proposition 2.1] for distinct α, $\beta \in \mathcal{L}$ there exists a unique homogeneous element $A_{\beta\alpha} \in R$ of degree $\deg\ \phi_\alpha + \deg\ \phi_\beta$ such that

$$\phi_{\beta\alpha}(\phi_{\alpha\beta}(s)) = A_{\beta\alpha} s \quad \text{for all } s \in J_\alpha.$$

Assume $\mathcal{M} \subset \mathcal{L}$ is a nonempty subset, and denote by $\{T_u \mid u \in \mathcal{M}\}$ a set of new variables with degree of T_u equal to $\deg\ \phi_u$ for all $u \in \mathcal{M}$. Denote by $R_\mathcal{M}$ the graded ring given as quotient of polynomial ring $R[T_u \mid u \in \mathcal{M}]$ by the ideal generated by the set

$$\{T_u s - \phi_u(s) \mid u \in \mathcal{M}, s \in J_u\} \cup \{(T_v + r_{vu})(T_u + r_{uv}) - A_{vu} \mid u, v \in \mathcal{M}, u \neq v\}.$$

Theorem 3.1.4 *([53, Theorem 2.3]) The ring $R_{\mathcal{M}}$ is Gorenstein with dimension equal to the dimension of R.*

3.1.2 Tom and Jerry unprojections

We fix a codimension 4 complete intersection ideal J. The question is to find a 5×5 skewsymmetric matrix M such that if we denote by I the ideal given by the Pfaffians of M we have $I \subseteq J$ and I has codimension 3. Tom and Jerry are two different answers to this question.

Tom and Jerry are two families of unprojections which were defined and named by Reid. These families occur in many constructions of Gorenstein codimension 4 ideals with 9×16 resolution (i.e., 9 equations and 16 first syzygies) and, by [55, Section 5], can be considered as a type of deformation of the homogeneous coordinate rings of the Segre embeddings $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ respectively.

Definition 3.1.5 Assume $M = (m_{kl})$ is a 5×5 skewsymmetric matrix and J is a codimension 4 ideal.

1. Assume $1 \leq i \leq 5$. The matrix M is called Tom_i in J if $m_{kl} \in J$ whenever $k \neq i$ and $l \neq i$.

2. Assume $1 \leq i < j \leq 5$. The matrix M is called Jerry_{ij} in J if $m_{kl} \in J$ whenever $k \in \{i, j\}$ or $l \in \{i, j\}$.

Remark 3.1.6 In other words, M is Tom_i in J if all entries of the submatrix of M obtained by deleting the i-th row and the i-th column of M are elements of J. Moreover, M is Jerry_{ij} in J if each entry of the i-th row of M is in J, and the same is true for each each entry of the j-th row, the i-th column and the j-th column.

Remark 3.1.7 For an example which is Tom_1 in $J = (z_1, z_2, z_3, z_4)$ see Subsection 3.1.3.

Remark 3.1.8 Assume $1 \leq i, j \leq 5$ and M is a Tom_i matrix in J. Then there exists a suitable permutation matrix A such that AMA^t is Tom_j in J, where A^t is the transpose of A. For example, consider a matrix M which is Tom_2 matrix in J. Denote by A the following invertible 5×5 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then the matrix AMA^t is Tom_1 in J. A similar statement holds for the case of a Jerry_{ij} matrix in J.

3.1.3 The fundamental calculation for Tom

Papadakis [54] gave an explicit presentation of the unprojection ring for the Tom and Jerry families. In what follows, we give a quick review of the main steps of the Tom case.

We work over the polynomial ring $R = k[x_k, z_k, m_{ij}^k]$, where the indices are as follows: $1 \leq k \leq 4$, $2 \leq i < j \leq 5$. We denote by

$$N = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ -x_1 & 0 & m_{23} & m_{24} & m_{25} \\ -x_2 & -m_{23} & 0 & m_{34} & m_{35} \\ -x_3 & -m_{24} & -m_{34} & 0 & m_{45} \\ -x_4 & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix},$$

where

$$m_{ij} = \sum_{k=1}^{4} m_{ij}^k z_k.$$

Denote by P_i the Pfaffian of the submatrix of N obtained by deleting the $(i+1)$-th row and column of N. We set $J = (z_1, z_2, z_3, z_4)$ and $I = (P_0, \dots, P_4)$. We have that $I \subset J$ and N is a matrix which is Tom_1 in the codimension 4 complete intersection ideal J.

Since $P_1, \dots, P_4$ are linear in z_1, z_2, z_3, z_4, there exists a unique 4×4 matrix Q such that

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = Q \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}.$$

We denote by Q_i the submatrix of Q which obtained by deleting the i-th row of Q. For $i = 1, \dots, 4$, let H_i be the 1×4 matrix whose i-th entry is equal to $(-1)^{i+1}$ times the determinant of the submatrix of Q_i. For all i, j, it holds that

$$x_i H_j = x_j H_i.$$

Using the last equality, we can define four polynomials g_1, g_2, g_3, g_4 as follows. We fix $1 \leq j \leq 4$ and we set

$$(g_1, g_2, g_3, g_4) = H_j / x_j.$$

We note that this definition is independent of the choice of j.

Denote by ϕ the map which is defined by

$$\phi\colon J/I \to R/I$$

with

$$z_i + I \mapsto g_i + I,$$

for all $1 \leq i \leq 4$.

By [54], $\mathrm{Hom}_{R/I}(J/I, R/I)$ is generated as R/I-module by the inclusion map i and ϕ. Moreover, the ideal

$$(P_0, \ldots, P_4, Tz_1 - g_1, Tz_2 - g_2, Tz_3 - g_3, Tz_4 - g_4)$$

of the polynomial ring $R[T]$ is Gorenstein of codimension 4.

3.1.4 Unprojection of a codimension 2 ideal inside a codimension 3

In this subsection we specify a codimension 2 complete intersection ideal I and a codimension 3 complete intersection J such that $I \subset J$. Following [54, Subsection 2.5.1], we give the explicit description of the unprojection ring $\mathrm{Unpr}(J/I, R/I)$ of the pair $J/I \subset R/I$.

Let $R = k[a_i, b_i, x_j]$, where $1 \leq i \leq 3$ and $j \in \{1, 3, 5\}$, be the standard graded polynomial ring in 9 variables over a field k. We set

$$f_1 = a_1x_1 + a_2x_3 + a_3x_5, \qquad f_2 = b_1x_1 + b_2x_3 + b_3x_5,$$

and consider the ideals

$$I = (f_1, f_2), \qquad J = (x_1, x_3, x_5)$$

of R. We denote by A the 2×3 matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

and, for $1 \leq i \leq 3$, by A_i the 2×2 submatrix of A obtained by removing the i-th column of A.

Proposition 3.1.9 *The ideal I is a homogeneous codimension 2 Gorenstein ideal of R and the ideal J is a homogeneous codimension 3 Gorenstein ideal. Moreover, I is a subset of J.*

Proof We first prove that $\operatorname{codim} I = 2$. The ideal I is generated by two homogeneous polynomials of R of degree 2. Hence, by Theorem 2.1.52 $\operatorname{codim} I \leq 2$. To prove the claim it is enough to show that $\operatorname{codim} I \geq 2$. We set $f_3 = -b_1 f_1 + a_1 f_2$. Let $>$ be the lexicographic order on R with

$$a_1 > \cdots > a_3 > b_1 > \cdots > b_3 > x_1 > \cdots > x_3.$$

We denote by Q the initial ideal of I with respect $>$. It is well-known that it holds that $\operatorname{codim} I = \operatorname{codim} Q$.

We set

$$L = (a_1 x_1, b_1 x_1, a_1 b_2 x_3).$$

Since the initial term of f_1 is $a_1 x_1$, the initial term of f_2 is $b_1 x_1$ and the initial term of f_3 is $a_1 b_2 x_3$ we have $L \subset Q$, hence $\operatorname{codim} L \leq \operatorname{codim} Q$.

We consider the affine variety $X = V(L) \subset \mathbb{A}^9$. It holds that

$$X = V(x_1, x_3) \cup V(b_2, x_1) \cup V(a_1, b_1) \cup V(a_1, x_1),$$

hence $\dim X = 9 - 2 = 7$. Using that

$$\dim\ R/L = \dim\ X,$$

it follows that $\operatorname{codim} L = 2$. Hence $\operatorname{codim} I \geq 2$.

We now prove that $\operatorname{codim} J = 3$. According to the Third Isomorphism Theorem of rings

$$R/J \cong k[a_1, a_2, a_3, b_1, b_2, b_3].$$

So, $\dim\ R/J = 6$. Hence,

$$\operatorname{codim}\ J = \dim\ R - \dim\ R/J = 3.$$

By Theorem 2.1.61, the ideals I and J are Gorenstein. By the equality of matrices

$$\begin{pmatrix} f_1 & f_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix}$$

it follows that $I \subset J$. $\square$

We set, for $1 \leq i \leq 3$, h_i to be the determinant of the matrix A_i. Denote by

$$\phi\colon J/I \to R/I$$

the map such that

$$\phi(x_1 + I) = h_1 + I, \quad \phi(x_3 + I) = -h_2 + I, \quad \phi(x_5 + I) = h_3 + I.$$

By [54, Theorem 2.5.6], $\operatorname{Hom}_{R/I}(J/I, R/I)$ is generated as R/I-module by the inclusion map i and ϕ. As a corollary,

$$\operatorname{Unpr}(J/I, R/I) = \frac{R[T]}{I + (Tx_1 - h_1, Tx_3 - (-h_2), Tx_5 - h_3)}.$$

Chapter 4

Tom & Jerry triples unprojection format with an application to Fano 3-folds

In this chapter we introduce a new format of parallel unprojection which we call Tom and Jerry triples. The initial data of the Tom and Jerry triples format provides one answer to the following question:

Question 4.1 Assume we are given three codimension 4 complete intersection ideals $J_1, \dots, J_3$. How can one construct a 5×5 skewsymmetric matrix M such that $I \subset J_t$, for all $1 \leq t \leq 3$, where I denotes the ideal generated by the 5 Pfaffians of M?

The motivation for Question 4.1 is that under favourable conditions for J_t and I one can hope to use parallel unprojection to construct codimension 6 Gorenstein rings which will correspond to interesting geometric objects.

Our approach for answering Question 4.1 is to insist that M is Tom_a (or Jerry_{ab}) with respect to the ideal J_1, Tom_c (or Jerry_{cd}) with respect to the ideal J_2 and Tom_e (or Jerry_{ef}) with respect to the ideal J_3, for a suitable choice of integers $a, \dots, f$. (We recall that the notions Tom_a and Jerry_{ab} were defined in Definition 3.1.5).

We will use the following notation: A 5×5 skewsymmetric matrix M will be called Tom_a+Tom_c+Tom_e for the triple of ideals $J_1, \dots, J_3$ if M is Tom_a with respect to J_1, Tom_c with respect to J_2 and Tom_e with respect to J_3. Similarly, it will be called Tom_a+Tom_c+Jerry_{ef} for the triple of ideals $J_1, \dots, J_3$ if M is Tom_a with respect to J_1, Tom_c with respect to J_2 and Jerry_{ef} with respect to J_3, etc. We will often avoid mentioning the triple of ideals $J_1, \dots, J_3$ when no confusion is likely to arise.

In Section 4.1 we study the problem of what different Tom and Jerry triples we have up to the obvious symmetry obtained by permutation of the indices.

Assume now that we have fixed explicit ideals $J_1, \dots, J_3$ and we have made the choice to use for example the Tom_1+Tom_2+Tom_3 configuration. Then, Definition 4.1.1

gives explicit conditions for the matrix M, and it makes sense to define the entries of M as the most general linear combination of the generators of the ideals J_t that satisfy the conditions. An explicit example is the matrix $\mathrm{Tom}(1,2,3)$ defined in Subsection 4.2.1.

A second question, very important for applications to Algebraic Geometry, is the following:

Question 4.2 How can one choose three specific complete intersection ideals $J_1, \dots, J_3$ and a choice of a configuration (for example, $\mathrm{Tom}_1+\mathrm{Tom}_2+ \mathrm{Tom}_3$) such that the parallel unprojection works with respect to this initial data, and produces a Gorenstein codimension 6 ring with good properties such as being an integral domain and the associated projective varieties have good properties such as being in the Mori category?

We believe that Question 4.2 has no obvious answers, since one needs to balance a number of contradicting factors. For example, if the three ideals J_t intersect a lot then the theory of parallel unprojection may not be applicable, while if the ideals J_t intersect very little this imposes many restrictions on M and we may lose essential properties such as I being a prime ideal.

In Subsection 4.2.1 we provide an answer to Question 4.2. More precisely, we make a specific choice of ideals $J_1, \dots, J_3$ and the choice of $\mathrm{Tom}_1+\mathrm{Tom}_2+ \mathrm{Tom}_3$ configuration and we use parallel unprojection to produce a Gorenstein codimension six family of rings. In Section 4.3, we use this family of rings in two different ways to construct two families of Fano 3-folds of codimension 6 embedded in weighted projective space which correspond to the entries with ID: 14885, ID: 12979 in Brown's Graded Ring Database [13].

4.1 Tom and Jerry triples

In this section we introduce the Tom and Jerry triples format. One of this triples is studied in detail in Subsection 4.2.1 and leads to an application to Fano 3-folds in Section 4.3. We discuss the remaining cases in Subsection 4.2.2.

Consider the 5×5 skewsymmetric matrix

$$M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ -m_{12} & 0 & m_{23} & m_{24} & m_{25} \\ -m_{13} & -m_{23} & 0 & m_{34} & m_{35} \\ -m_{14} & -m_{24} & -m_{34} & 0 & m_{45} \\ -m_{15} & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix}$$

and three complete intersection ideals J_1, J_2, J_3 of codimension 4. In each of the following cases we set conditions in the entries of M such that the ideal I of Pfaffians is contained in each of the ideals $J_1, \dots, J_3$. We denote by S_5 the symmetric group of permutations of the set $\{1, \dots, 5\}$.

4.1.1 Tom & Tom & Tom case

Consider

$$\mathrm{Tom}_i+\mathrm{Tom}_j+\mathrm{Tom}_k,$$

for $1 \leq i < j < k \leq 5$. Given $\sigma \in S_5$ we obtain

$$\mathrm{Tom}_{\sigma(i)}+\mathrm{Tom}_{\sigma(j)}+\mathrm{Tom}_{\sigma(k)},$$

for $1 \leq \sigma(i) < \sigma(j) < \sigma(k) \leq 5$. In other words, S_5 acts on the set $\{i, j, k\}$. It is not difficult to see that there is a unique orbit of this action with representative $\mathrm{Tom}_1+\mathrm{Tom}_2+\mathrm{Tom}_3$.

Definition 4.1.1 We say that M is a $\mathrm{Tom}_1+\mathrm{Tom}_2+\mathrm{Tom}_3$ matrix if the entries of M satisfy the following conditions:

$$m_{12} \in J_3, \quad m_{13} \in J_2, \quad m_{14}, m_{15} \in J_2 \cap J_3, \quad m_{23} \in J_1,$$

$$m_{24}, m_{25} \in J_1 \cap J_3, \quad m_{34}, m_{35} \in J_1 \cap J_2, \quad m_{45} \in J_1 \cap J_2 \cap J_3.$$

Then, M is Tom_1 in J_1, Tom_2 in J_2 and Tom_3 in J_3.

4.1.2 Jerry & Jerry & Jerry case

Working as before, consider

$$\mathrm{Jerry}_{ij}+\mathrm{Jerry}_{kl}+\mathrm{Jerry}_{mn},$$

for $1 \leq i < j \leq 5$, $1 \leq k < l \leq 5$, $1 \leq m < n \leq 5$ and $(i, j), (k, l), (m, n)$ pairwise different. Given $\sigma \in S_5$ we obtain

$$\mathrm{Jerry}_{\sigma(i)\sigma(j)}+\mathrm{Jerry}_{\sigma(k)\sigma(l)}+\mathrm{Jerry}_{\sigma(m)\sigma(n)}.$$

In this case, the following representatives of the orbits of the action occur:

1. $\mathrm{Jerry}_{12}+\mathrm{Jerry}_{13}+\mathrm{Jerry}_{14}$
2. $\mathrm{Jerry}_{12}+\mathrm{Jerry}_{13}+\mathrm{Jerry}_{23}$
3. $\mathrm{Jerry}_{12}+\mathrm{Jerry}_{14}+\mathrm{Jerry}_{23}$
4. $\mathrm{Jerry}_{14}+\mathrm{Jerry}_{15}+\mathrm{Jerry}_{23}$.

Definition 4.1.2 We say that M is a $\mathrm{Jerry}_{12}+\mathrm{Jerry}_{13}+\mathrm{Jerry}_{14}$ matrix if M is Jerry_{12} in J_1, Jerry_{13} in J_2 and Jerry_{14} in J_3. Similar definitions also hold for the remaining representatives of the orbits of the action above and we will not write them explicitly.

To avoid repetition of the same arguments and definitions in the cases that follow we write down explicitly only the representatives of the orbits of the action.

4.1.3 Tom & Tom & Jerry case

Consider

$$\text{Tom}_i+\text{Tom}_j+\text{Jerry}_{kl},$$

for $1 \leq i < j \leq 5$, $1 \leq k < l \leq 5$.

In this case, the following representatives arise:

(5) $\text{Tom}_1+\text{Tom}_2+\text{Jerry}_{12}$

(6) $\text{Tom}_1+\text{Tom}_2+\text{Jerry}_{13}$

(7) $\text{Tom}_1+\text{Tom}_2+\text{Jerry}_{34}$.

4.1.4 Tom & Jerry & Jerry case

Consider

$$\text{Tom}_i+\text{Jerry}_{jk}+\text{Jerry}_{lm},$$

for $1 \leq i \leq 5$, $1 \leq j < k \leq 5$, $1 \leq l < m \leq 5$ and $(j,k) \neq (l,m)$.

So, for this case we have the following list of representatives:

(8) $\text{Tom}_1+\text{Jerry}_{12}+\text{Jerry}_{13}$

(9) $\text{Tom}_1+\text{Jerry}_{12}+\text{Jerry}_{23}$

(10) $\text{Tom}_1+\text{Jerry}_{12}+\text{Jerry}_{34}$

(11) $\text{Tom}_1+\text{Jerry}_{23}+\text{Jerry}_{24}$

(12) $\text{Tom}_1+\text{Jerry}_{23}+\text{Jerry}_{45}$.

4.2 The main results

This section consists of two subsections. In Subsection 4.2.1, we establish a result which concerns the construction of a codimension 6 Gorenstein ring using one of the formats which were developed in Subsection 4.1.1. In Subsection 4.2.2, we present a theorem related to the other formats which were described in Section 4.1.

4.2.1 Tom & Tom & Tom format

In the present subsection, we specify three codimension 4 complete intersection ideals J_1, J_2, J_3 and a codimension 3 ideal I generated by the Pfaffians of a specific $\mathrm{Tom}_1+\mathrm{Tom}_2+\mathrm{Tom}_3$ matrix. We prove that this data satisfies the conditions for parallel Kustin-Miller unprojection established by Neves and Papadakis and recalled in Theorem 3.1.4. Moreover, using Theorem 3.1.4 we give a description of the final ring as a quotient of a polynomial ring by a codimension 6 ideal. This format will be used in Section 4.3 to prove the existence of two families of codimension 6 Fano 3-folds. As Part 2 of Theorem 4.2.10 demonstrates, there are also other alternative choices of ideals J_1, J_2, J_3 which are leading to the construction of Gorenstein rings of codimension six which could be useful for the construction of some interesting geometric objects.

We work over the standard graded polynomial ring $R = k[z_i, c_j]$, where $1 \le i \le 7$ and $1 \le j \le 25$. Denote by $\mathrm{Tom}(1,2,3)$ the following 5×5 skewsymmetric matrix

$$\begin{pmatrix} 0 & c_1z_1+c_2z_2+c_3z_3+c_4z_6 & c_5z_1+c_6z_2+c_7z_4+c_8z_5 & c_9z_1+c_{10}z_2 & c_{11}z_1+c_{12}z_2 \\ & 0 & c_{13}z_2+c_{14}z_3+c_{15}z_5+c_{16}z_7 & c_{17}z_2+c_{18}z_3 & c_{19}z_2+c_{20}z_3 \\ & & 0 & c_{21}z_2+c_{22}z_5 & c_{23}z_2+c_{24}z_5 \\ & -Sym & & 0 & c_{25}z_2 \\ & & & & 0 \end{pmatrix}$$

which is a $\mathrm{Tom}_1+\mathrm{Tom}_2+\mathrm{Tom}_3$ matrix in the ideals

$$J_1 = (z_2, z_3, z_5, z_7),\ J_2 = (z_1, z_2, z_4, z_5),\ J_3 = (z_1, z_2, z_3, z_6). \tag{4.1}$$

Let I be the ideal generated by the Pfaffians of $\mathrm{Tom}(1,2,3)$.

Proposition 4.2.1 *(i) For all t with $1 \le t \le 3$, the ideal J_t/I is a codimension 1 homogeneous ideal of the quotient ring R/I such that the ring R/J_t is Gorenstein.*

(ii) For all t, s with $1 \le t < s \le 3$, it holds that $\operatorname{codim}_{R/I}(J_t/I + J_s/I) = 3$.

Proof We first prove (i). According to the Third Isomorphism Theorem of rings

$$R/J_1 \cong k[z_1, z_4, z_6, c_1, \dots, c_{25}],\ \ R/J_2 \cong k[z_3, z_6, z_7, c_1, \dots, c_{25}], \tag{4.2}$$

$$R/J_3 \cong k[z_4, z_5, z_7, c_1, \dots, c_{25}].$$

So, we conclude that for all t with $1 \le t \le 3$,

$$\dim\ R/J_t = 28.$$

We claim that

$$\dim\ R/I = 29.$$

Denote by $\tilde{I} = (c_1, c_2, c_3, c_5, c_6, c_7, c_9, c_{12}, c_{13}, c_{15}, c_{16}, c_{18}, c_{19}, c_{21}, c_{23})$, the ideal generated by some variables of R. We set $J^{new} = I + \tilde{I}$. The ideal J^{new} is a homogeneous ideal of R. Hence, from Krull's principal ideal theorem it follows that

$$\dim\ R/J^{new} \geq \dim\ R/I - 15.$$

We call $\hat{I}$ the ideal obtained from the ideal I by setting the variables

$$c_1, c_2, c_3, c_5, c_6, c_7, c_9, c_{12}, c_{13}, c_{15}, c_{16}, c_{18}, c_{19}, c_{21}, c_{23}$$

equal to zero. Using the Third Isomorphism Theorem of rings as before we have that

$$R/J^{new} \cong k[z_1, \ldots, z_7, c_4, c_8, c_{10}, c_{11}, c_{14}, c_{17}, c_{20}, c_{22}, c_{24}, c_{25}]/\hat{I}.$$

For the computation of the Krull dimension of

$$k[z_1, \ldots, z_7, c_4, c_8, c_{10}, c_{11}, c_{14}, c_{17}, c_{20}, c_{22}, c_{24}, c_{25}]/\hat{I}$$

we used the computer algebra program Macaulay2 [28].

It occurs that

$$\dim\ k[z_1, \ldots, z_7, c_4, c_8, c_{10}, c_{11}, c_{14}, c_{17}, c_{20}, c_{22}, c_{24}, c_{25}]/\hat{I} = 14$$

and therefore

$$\dim\ R/J^{new} = 14.$$

As a consequence, $\dim\ R/I \leq 29$.

It is well-known that, see for example [17, Theorem 3.4.1(a)] the ideal generated by the Pfaffians of a skewsymmetric matrix has codimension ≤ 3. Hence, $\operatorname{codim}\ I \leq 3$. Equivalently,

$$\dim\ R/I \geq 29,$$

which completes the proof of the claim. As a consequence,

$$\operatorname{codim}\ I = \dim\ R - \dim\ R/I = 3.$$

Hence, by the second part of the Theorem 2.1.61, R/I is a Gorenstein ring. Using again the definition of codimension for all t with $1 \leq t \leq 3$, we get codim $J_t/I = 1$.

Due to the isomorphisms (4.2) for all t with $1 \leq t \leq 3$, the ring R/J_t is Gorenstein.

We now prove (ii). Third Isomorphism Theorem of rings implies that

$$R/(J_1 + J_2) \cong k[z_6, c_1, \dots, c_{25}], \quad R/(J_1 + J_3) \cong k[z_4, c_1, \dots, c_{25}],$$
$$R/(J_2 + J_3) \cong k[z_7, c_1, \dots, c_{25}].$$

From the later isomorphisms it holds that for t, s with $1 \leq t < s \leq 3$,

$$\dim\ R/(J_t + J_s) = 26.$$

Recall that dim $R/I = 29$. Taking into account the definition of codimension we conclude that for all t, s with $1 \leq t < s \leq 3$,

$$\operatorname{codim}\ (J_t/I + J_s/I) = 3.$$

□

For all t, with $1 \leq t \leq 3$, we denote by $i_t\colon J_t/I \to R/I$ the inclusion map. Our aim is to define $\phi_t\colon J_t/I \to R/I$ for all t, with $1 \leq t \leq 3$, and prove that these maps satisfy the assumptions of the Theorem 3.1.4. As a first step for the definition of ϕ_t, we relate Tom_t matrix in J_t (for the definition see Subsection 3.1.2) to the matrix N which was defined in Subsection 3.1.3.

Assume D is a Tom_1 matrix in J_1. It is clear that D is a specialization of the matrix N. For an example, see Equation (4.3) below.

Assume D is a Tom_2 matrix in J_2. Let A be the invertible 5×5 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix ADA^t, where A^t is the transpose of A, is a specialization of the matrix N.

Assume D is a Tom_3 matrix in J_3. Denote by B the following invertible 5×5 matrix

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix BDB^t is a specialization of the matrix N.

We remark that the ideal generated by the Pfaffians of the matrix ADA^t is equal to the ideal generated by the Pfaffians of the matrix D. The same is true for the ideal generated by the Pfaffians of the matrix BDB^t.

As a second step we apply the above considerations to obtain, for $t = 1, 2, 3$, the matrix Tom(1, 2, 3) in J_t as a specialization of N. In what follows until the end of the proof when we write "$a = b$" we mean that a is replaced by b. We set

$$D_1 = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ & 0 & c_1z_2 + c_2z_3 + c_3z_5 + c_4z_7 & c_5z_2 + c_6z_3 + c_7z_5 + c_8z_7 & u_1 \\ & & 0 & c_{13}z_2 + c_{14}z_3 + c_{15}z_5 + c_{16}z_7 & u_2 \\ & -Sym & & 0 & u_3 \\ & & & & 0 \end{pmatrix} \tag{4.3}$$

where

$$u_1 = c_9z_2 + c_{10}z_3 + c_{11}z_5 + c_{12}z_7, \quad u_2 = c_{17}z_2 + c_{18}z_3 + c_{19}z_5 + c_{20}z_7,$$

$$u_3 = c_{21}z_2 + c_{22}z_3 + c_{23}z_5 + c_{24}z_7.$$

The matrix D_1 is a Tom_1 matrix in J_1. D_1 is obtained from N by the following substitutions

$$z_1 = z_2,\ z_2 = z_3,\ z_3 = z_5,\ z_4 = z_7 \tag{4.4}$$

and the obvious substitutions of m_{ij}^k in terms of c_l. We set

$$c_7 = c_8 = c_{11} = c_{12} = c_{14} = c_{16} = c_{18} = c_{20} = c_{22} = c_{23} = c_{24} = 0 \tag{4.5}$$

in D_1. We call D_2 the matrix which occurs. It is given explicitly by,

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ & 0 & c_1z_2 + c_2z_3 + c_3z_5 + c_4z_7 & c_5z_2 + c_6z_3 & c_9z_2 + c_{10}z_3 \\ & & 0 & c_{13}z_2 + c_{15}z_5 & c_{17}z_2 + c_{19}z_5 \\ & -Sym & & 0 & c_{21}z_2 \\ & & & & 0 \end{pmatrix}$$

Finally, setting

$$x_1 = c_1z_1 + c_2z_2 + c_3z_3 + c_4z_6,\ \ x_2 = c_5z_1 + c_6z_2 + c_7z_4 + c_8z_5, \tag{4.6}$$

$$x_3 = c_9z_1 + c_{10}z_2,\ \ x_4 = c_{11}z_1 + c_{12}z_2,\ \ c_1 = c_{13},\ \ c_2 = c_{14},$$

$$c_3 = c_{15},\ \ c_4 = c_{16},\ \ c_5 = c_{17},\ \ c_6 = c_{18},\ \ c_9 = c_{19},\ \ c_{10} = c_{20},$$

$$c_{13} = c_{21},\ \ c_{15} = c_{22},\ \ c_{17} = c_{23},\ \ c_{19} = c_{24},\ \ c_{21} = c_{25}$$

in D_2 we obtain the $\mathrm{Tom}(1,2,3)$ matrix. A similar analysis applies to consider $\mathrm{Tom}(1,2,3)$ matrix as a Tom_2 in J_2 and Tom_3 in J_3.

At this point we use Papadakis Fundamental Calculation for N (see Subsection 3.1.3) in order to define the maps ϕ_1, ϕ_2 and ϕ_3.

Assume that $1 \leq t \leq 4$. We consider the polynomial g_t which was defined in Subsection 3.1.3. We denote by g'_t the polynomial obtained from g_t after the substitutions which are noted in Equation (4.4) and the obvious substitutions of m^k_{ij} in terms of c_t. We denote by $\tilde{g}_t$ the polynomial obtained by g'_t after the substitutions which are described in Equation (4.5). Finally, we denote by h_t the polynomial which occurs from the polynomial $\tilde{g}_t$ after the substitutions which are noted in Equation (4.6).

Proposition 4.2.2 *There exists a unique graded homomorphism of R/I-modules $\phi_1 \colon J_1/I \to R/I$ such that*

$$\phi_1(z_2 + I) = h_1 + I, \quad \phi_1(z_3 + I) = h_2 + I,$$

$$\phi_1(z_5 + I) = h_3 + I, \quad \phi_1(z_7 + I) = h_4 + I.$$

Proof It follows from [55, Theorem 5.6]. □

For the definitions of ϕ_2 and ϕ_3 we work similarly. We omit the details. For all t with $1 \leq t \leq 3$, the degree of ϕ_t is equal to 6. Following [53, Definition 2.2] the degree of the new unprojection variable is equal to the degree of the corresponding ϕ_t.

Proposition 4.2.3 *For all t with $1 \leq t \leq 3$, the R/I-module $\mathrm{Hom}_{R/I}(J_t/I, R/I)$ is generated by the two elements i_t and ϕ_t.*

Proof It follows from [55, Theorem 5.6]. □

For all t, s with $1 \leq t, s \leq 3$ and $t \neq s$, we define $r_{ts} = 0$.

Proposition 4.2.4 *For all t, s with $1 \leq t, s \leq 3$ and $t \neq s$, it holds that*

$$\phi_t(J_t/I) \subset J_s/I.$$

Proof It is a direct computation using the definitions of the maps ϕ_t. □

Proposition 4.2.5 *For all t, s with $1 \leq t, s \leq 3$ and $t \neq s$, there exists a homogeneous element A_{st} such that*

$$\phi_s(\phi_t(p)) = A_{st}p$$

for all $p \in J_t/I$.

Proof It follows from [53, Proposition 2.1]. □

Remark 4.2.6 It is immediate by the above considerations that the elements A_{st} are polynomial expressions in the variables c_i and z_j. We explicitly computed the elements A_{st} using the computer algebra program Macaulay2 [28].

Definition 4.2.7 Let T, S, W be three new variables of degree 6. Following Subsection 3.1.1, we define as I_{un} the ideal

$$(I)+(Tz_2-\phi_1(z_2),\ Tz_3-\phi_1(z_3),\ Tz_5-\phi_1(z_5),\ Tz_7-\phi_1(z_7),\ Sz_1-\phi_2(z_1),\ Sz_2-\phi_2(z_2),$$

$$Sz_4-\phi_2(z_4), Sz_5-\phi_2(z_5), Wz_1-\phi_3(z_1),\ Wz_2-\phi_3(z_2),\ Wz_3-\phi_3(z_3),\ Wz_6-\phi_3(z_6),$$

$$TS-A_{12},\ TW-A_{13},\ SW-A_{23})$$

of the polynomial ring $R[T, S, W]$. We set $R_{un} = R[T, S, W]/I_{un}$.

Concerning the previous definition we note that the new variables T, S, W as unprojection variables are of degree 6. Moreover, according to [53, Proposition 2.1] the degree of each A_{st} is equal to 12.

Theorem 4.2.8 *The ring R_{un} is Gorenstein.*

Proof By Propositions 4.2.1, 4.2.3 and 4.2.4, the assumptions of Theorem 3.1.4 are satisfied. Hence, the ring R_{un} is Gorenstein. □

Proposition 4.2.9 *The homogeneous ideal I_{un} is a codimension 6 ideal with a minimal generating set of 20 elements.*

Proof According to the grading of the variables and the discussion before the Proposition 4.2.3 it is not difficult to see that I_{un} is a homogeneous ideal. Recall that in Kustin-Miller unprojection codimension is increasing by 1. Hence, the homogeneous ideal I_{un}, as a result of a series of three unprojections of Kustin-Miller type starting by the codimension 3 ideal I, is a codimension 6 ideal. We denote by

$$\mathcal{A} = \{2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 15, 18, 19, 21, 24, 25\}$$

and

$$\mathcal{B} = \{1, 8, 11, 16, 17, 20, 22, 23\}$$

two sets of indices. In order to prove that I_{un} is minimally generated by 20 elements we use the idea of specialization. More precisely, for $i \in \mathcal{A}$ and $j \in \mathcal{B}$ we set

$$c_i = 0 \text{ and } c_j = 1$$

in the ideal I_{un}. We call $\widetilde{I_{un}}$ the ideal which occurs after these substitutions. $\widetilde{I_{un}}$ is a homogeneous ideal with 14 monomials and 6 binomials as generators. It is not difficult to see that $\widetilde{I_{un}}$ is minimally generated by these elements. Hence, we conclude that I_{un} is generated by at least 20 elements. By Definition 4.2.7, I_{un} is generated by 20 homogeneous elements. The result follows. $\square$

4.2.2 The other formats

In this subsection we formulate a theorem related to the Cases $(1)-(12)$ of Section 4.1.

Theorem 4.2.10 *1. Consider the ideals J_1, J_2, J_3 defined in (4.1). Let I be the ideal generated by the Pfaffians of a sufficiently general 5×5 skewsymmetric matrix M which belongs to one of the Cases $(2)-(12)$ defined in Section 4.1. For the ideals J_1, J_2, J_3 the conditions of parallel Kustin-Miller unprojection are satisfied. Using the notation of Subsection 4.2.1, the final ring R_{un} is a codimension 6 Gorenstein ring.*

2. Consider the ideals $J_1 = (z_1, z_2, z_3, z_4), J_2 = (z_1, z_2, z_5, z_6), J_3 = (z_3, z_4, z_5, z_6)$ of the polynomial ring $R = k[z_i, c_j]$, where $1 \le i \le 6$ and $1 \le j \le 26$. Let I be the ideal generated by the Pfaffians of a sufficiently general 5×5 skewsymmetric matrix M which is a $Jerry_{12}$+$Jerry_{13}$+$Jerry_{14}$ matrix in J_1, J_2, J_3 (hence we are in Case (1) defined in Section 4.1). The ideals J_1, J_2, J_3 satisfy the conditions of parallel Kustin-Miller unprojection. For this choice of J_1, J_2, J_3, the final ring R_{un} is a codimension 6 Gorenstein ring.

Proof We verified the above claims using the computer algebra program Macaulay2. $\square$

Remark 4.2.11 We note that in Part (1) of the above theorem we didn't include Case (1) of Section 4.1, because it leads to a codimension 6 Cohen-Macaulay ring which is not Gorenstein.

4.3 Applications

In this section, we prove, using Theorem 4.2.8, the existence of 2 families of Fano 3-folds of codimension 6 in weighted projective space. We note that in what follows we make essential use of computer algebra systems Macaulay2 [28] and Singular [25].

The first construction is summarised in the following theorem. It corresponds to the entry 14885 of Brown's Graded Ring Database [13]. More details for the construction are given in Subsection 4.3.1.

Theorem 4.3.1 *There exists a family of quasismooth, projectively normal and projectively Gorenstein Fano* 3*-folds* $X \subset \mathbb{P}(1^3, 2^7)$*, nonsingular away from eight quotient singularities* $\frac{1}{2}(1,1,1)$*, with Hilbert series*

$$P_X(t) = \frac{1-20t^4+64t^6-90t^8+64t^{10}-20t^{12}+t^{16}}{(1-t)^3(1-t^2)^7}.$$

The second construction is summarised in the following theorem. It corresponds to the entry 12979 of Brown's Graded Ring Database. More details for the construction are given in Subsection 4.3.2.

Theorem 4.3.2 *There exists a family of quasismooth, projectively normal and projectively Gorenstein Fano* 3*-folds* $X \subset \mathbb{P}(1^3, 2^5, 3^2)$*, nonsingular away from four quotient singularities* $\frac{1}{2}(1,1,1)$*, and two quotient singularities* $\frac{1}{3}(1,1,2)$*, with Hilbert series*

$$P_X(t) = \frac{1-11t^4-8t^5+23t^6+32t^7-13t^8-48t^9-13t^{10}+32t^{11}+23t^{12}-8t^{13}-11t^{14}+t^{18}}{(1-t)^3(1-t^2)^5(1-t^3)^2}.$$

4.3.1 Construction of Graded Ring Database entry with Identifier Number 14885

In the present subsection, we give the details of the construction for the family of codimension 6 Fano 3-folds described in Theorem 4.3.1.

We note that a difficult part of the arguments for this construction is the computation of singular locus of the general member of the family. As we will see below, for this part we used the computer algebra system Singular [25].

Denote by $k = \mathbb{C}$ the field of complex numbers. Consider the polynomial ring $R = k[z_i, c_j]$, where $1 \le i \le 7$ and $1 \le j \le 25$. Let R_{un} be the ring in Definition 4.2.7 and $\hat{R} = k[z_1, \dots, z_7]$ be the polynomial ring in z_i. We substitute the variables $(c_1, \dots, c_{25})$ which appear in the definitions of the rings R and R_{un} with a general element of k^{25} (in the sense of being outside a proper Zariski closed subset of k^{25}). Let $\hat{I}$ be the ideal of $\hat{R}$ which is obtained by the ideal I and $\hat{I}_{un}$ the ideal of $\hat{R}[T,S,W]$ which is obtained by the ideal I_{un} after this substitution. We set $\hat{R}_{un} = \hat{R}[T,S,W]/\hat{I}_{un}$. In what follows z_i, for all i with $1 \le i \le 7$, and T,S,W are variables of degree 2. According to this grading the ideals $\hat{I}$ and $\hat{I}_{un}$ are homogeneous. Moreover, from the discussion before the Propositions 4.2.2 and 4.2.3 it follows that the degree of T is equal to 2. Similarly for the variables S,W. Due to the Theorem 4.2.8, Proj $\hat{R}_{un} \subset \mathbb{P}(2^{10})$ is a projectively Gorenstein 3-fold.

Let $A = k[w_1,w_2,w_3,z_1,z_2,z_3,z_5,T,S,W]$ be the polynomial ring over k with w_1,w_2,w_3 variables of degree 1 and the other variables of degree 2. Consider the graded k-algebra homomorphism

$$\psi\colon \hat{R}[T,S,W] \to A$$

with

$$\psi(z_1) = z_1, \quad \psi(z_2) = z_2, \quad \psi(z_3) = z_3, \quad \psi(z_4) = f_1,$$

$$\psi(z_5) = z_5, \quad \psi(z_6) = f_2, \quad \psi(z_7) = f_3, \quad \psi(T) = T,$$

$$\psi(S) = S, \quad \psi(W) = W$$

where

$f_1 = l_1z_1 + l_2z_2 + l_3z_3 + l_4z_5 + l_5T + l_6S + l_7W + l_8w_1^2 + l_9w_1w_2 + l_{10}w_1w_3 + l_{11}w_2^2 + l_{12}w_2w_3 + l_{13}w_3^2$,

$f_2 = l_{14}z_1 + l_{15}z_2 + l_{16}z_3 + l_{17}z_5 + l_{18}T + l_{19}S + l_{20}W + l_{21}w_1^2 + l_{22}w_1w_2 + l_{23}w_1w_3 + l_{24}w_2^2 + l_{25}w_2w_3 + l_{26}w_3^2$,

$f_3 = l_{27}z_1 + l_{28}z_2 + l_{29}z_3 + l_{30}z_5 + l_{31}T + l_{32}S + l_{33}W + l_{34}w_1^2 + l_{35}w_1w_2 + l_{36}w_1w_3 + l_{37}w_2^2 + l_{38}w_2w_3 + l_{39}w_3^2$

and $(l_1,\dots,l_{39}) \in k^{39}$ are general. In other words, f_1,f_2,f_3 are general degree 2 homogeneous elements of A.

Denote by Q the ideal of the ring A generated by the subset $\psi(\hat{I}_{un})$.

Let $X = V(Q) \subset \mathbb{P}(1^3,2^7)$. It is immediate that X is a codimension 6 projectively Gorenstein 3-fold.

Proposition 4.3.3 *The ring A/Q is an integral domain.*

Proof It is enough to show that the ideal Q is prime. The computer algebra program Macaulay2 [28] gave us that for a specific choice of rational values for the parameters c_i, l_j, for $1 \le i \le 25$ and $1 \le j \le 39$ the ideal which was obtained by Q is a homogeneous, codimension 6, prime ideal with the right Betti table. □

In what follows, we show that the only singularities of $X \subset \mathbb{P}(1^3,2^7)$ are 8 quotient singularities of type $\frac{1}{2}(1,1,1)$. According to the discussion after Definition 2.2.10, X belongs to the Mori category.

Proposition 4.3.4 *Consider $X = V(Q) \subset \mathbb{P}(1^3, 2^7)$. Denote by $X_{cone} \subset \mathbb{A}^{10}$ the affine cone over X. The scheme X_{cone} is smooth outside the vertex of the cone.*

Proof Our approach is similar to the approach in [60, p. 18]. We work over the finite field $\mathbb{Z}/(1021)$. Differentiating the 20 equations of Q with respect to the ten variables gives the 10×20 Jacobian matrix M^{Jac}. Let J be the ideal of 6×6 minors of M^{Jac}. The ideal $Q + J$ defines the singular locus of X_{cone}. Our claim is that the only singularity of the scheme X_{cone} is the vertex of the cone. Consider the ideal $Q + J$. Using the computer algebra program Singular we proved that $\dim(A/(Q + J)) = 0$. The ideal $Q + J$ is homogeneous. Hence, the claim is proven. $\square$

Proposition 4.3.5 *Consider the singular locus $Z = V(w_1, w_2, w_3)$ of the weighted projective space $\mathbb{P}(1^3, 2^7)$. The intersection of X with Z consists of exactly eight reduced points which are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ for X.*

Proof We checked using the computer algebra program Macaulay2 that the intersection of X with Z consists of eight reduced points. We observe that the first three rows of the matrix which occurs from the jacobian matrix M^{Jac} of Q by setting the variables w_1, w_2, w_3 be equal to zero, are zero. Hence, due to the Proposition 4.3.4, there exists a non-zero 6×6 minor in six out of seven variables of degree 2. In that way, we conclude that the eight points are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ for X. $\square$

Lemma 4.3.6 *Let $\omega_{\hat{R}/\hat{I}}$ be the canonical module of $\hat{R}/\hat{I}$. It holds that the canonical module $\omega_{\hat{R}/\hat{I}}$ is isomorphic to $\hat{R}/\hat{I}(-4)$.*

Proof From the minimal graded free resolution of $\hat{R}/\hat{I}$ as $\hat{R}$-module

$$0 \to \hat{R}(-10) \to \hat{R}(-6)^5 \to \hat{R}(-4)^5 \to \hat{R}$$

and the fact that the sum of the degrees of the variables is equal to 14 we conclude that

$$\omega_{\hat{R}/\hat{I}} = \hat{R}/\hat{I}(10 - 14) = \hat{R}/\hat{I}(-4).$$

$\square$

Proposition 4.3.7 *The minimal graded resolution of A/Q as A-module is equal to*

$$0 \to A(-16) \to A(-12)^{20} \to A(-10)^{64} \to A(-8)^{90} \to A(-6)^{64} \to A(-4)^{20} \to A \tag{4.7}$$

Moreover, the canonical module of A/Q is isomorphic to $(A/Q)(-1)$ and the Hilbert series of A/Q as graded A-module is equal to

$$\frac{1 - 20t^4 + 64t^6 - 90t^8 + 64t^{10} - 20t^{12} + t^{16}}{(1-t)^3(1-t^2)^7}.$$

Proof To compute the minimal graded free resolution of A/Q we followed the method described in the proof of [52, Proposition 3.4]. From the minimal graded free resolution (4.7) of A/Q and the fact that the sum of the degrees of the variables is equal to 17 we conclude that

$$\omega_{A/Q} = A/Q(16 - 17) = A/Q(-1).$$

The last conclusion of Proposition 4.3.7 follows easily from the resolution (4.7). □

Taking into account the Propositions 4.3.4, 4.3.5 and 4.3.7, we conclude that X is a Fano 3-fold.

4.3.2 Construction of Graded Ring Database entry with Identifier Number 12979

In this subsection, we sketch the construction for the family of the codimension 6 Fano 3-folds described in Theorem 4.3.2.

Denote by $k = \mathbb{C}$ the field of complex numbers. Working as before, consider the polynomial ring $R = k[z_i, c_j]$, where $1 \leq i \leq 7$ and $1 \leq j \leq 25$. Let R_{un} be the ring in Definition 4.2.7 and $\hat{R} = k[z_1, z_2, z_3, z_4, z_5, z_6, z_7, c_1, c_5, c_9, c_{11}]$ be the polynomial ring. We substitute the variables $(c_2, c_3, c_4, c_6, c_7, c_8, c_{10}, c_{12}, c_{13}, \dots, c_{25})$ which appear in the definitions of the rings R and R_{un} with a general element of k^{21} (in the sense of being outside a proper Zariski closed subset of k^{21}). Let $\hat{I}$ be the ideal of $\hat{R}$ which is obtained by the ideal I and $\hat{I}_{un}$ the ideal of $\hat{R}[T, S, W]$ which is obtained by the ideal I_{un} after this substitution. We set $\hat{R}_{un} = \hat{R}[T, S, W]/\hat{I}_{un}$. In what follows we assume that the variables $z_1, c_1, c_5, c_9, c_{11}$ are of degree 1, the variables $z_2, \dots, z_7, T$ are of degree 2 and the variables S, W are of degree 3. Under this grading, the ideals $\hat{I}$ and $\hat{I}_{un}$ are homogeneous. Due to the Theorem 4.2.8, Proj $\hat{R}_{un} \subset \mathbb{P}(1^5, 2^7, 3^2)$ is a projectively Gorenstein 7-fold.

Let $A = k[z_1, c_5, c_9, z_2, z_3, z_5, z_6, T, S, W]$ be the polynomial ring with z_1, c_5, c_9 variables of degree 1, z_2, z_3, z_5, z_6, T variables of degree 2 and S, W are variables of degree 3. We consider the graded k-algebra homomorphism

$$\psi \colon \hat{R}[T, S, W] \to A$$

with

$$\psi(z_1) = z_1, \quad \psi(c_1) = g_1, \quad \psi(c_5) = c_5, \quad \psi(c_9) = c_9,$$

$$\psi(c_{11}) = g_2, \quad \psi(z_2) = z_2, \quad \psi(z_3) = z_3, \quad \psi(z_4) = f_1,$$

$$\psi(z_5) = z_5, \quad \psi(z_6) = z_6, \quad \psi(z_7) = f_2, \quad \psi(T) = T,$$

$$\psi(S) = S, \quad \psi(W) = W$$

where
$g_1 = l_1 z_1 + l_2 c_5 + l_3 c_9$,
$g_2 = l_4 z_1 + l_5 c_5 + l_6 c_9$,
$f_1 = l_7 z_2 + l_8 z_3 + l_9 z_5 + l_{10} z_6 + l_{11} T + l_{12} z_1^2 + l_{13} z_1 c_5 + l_{14} z_1 c_9 + l_{15} c_5^2 + l_{16} c_5 c_9 + l_{17} c_9^2$,
$f_2 = l_{18} z_2 + l_{19} z_3 + l_{20} z_5 + l_{21} z_6 + l_{22} T + l_{23} z_1^2 + l_{24} z_1 c_5 + l_{25} z_1 c_9 + l_{26} c_5^2 + l_{27} c_5 c_9 + l_{28} c_9^2$
and $(l_1, \dots, l_{28}) \in k^{28}$ are general. In other words, g_1, g_2 are general degree 1 homogeneous elements of A, while g_1, g_2 are general degree 2 homogeneous elements of A.

Denote by Q the ideal of the ring A generated by the subset $\psi(\hat{I}_{un})$.

Let $X = V(Q) \subset \mathbb{P}(1^3, 2^5, 3^2)$. It is immediate that X is a codimension 6 projectively Gorenstein 3-fold.

Repeating the arguments which were used for the construction which was described in Subsection 4.3.1 we proved that $X \subset \mathbb{P}(1^3, 2^5, 3^2)$ is a Gorenstein Fano 3-fold nonsingular away from four quotient singularities $\frac{1}{2}(1,1,1)$ and two quotient singularities $\frac{1}{3}(1,1,2)$.

Chapter 5

The 4-intersection unprojection format with an application to Fano 3-folds

In this chapter we introduce a new format of parallel unprojection which we name the 4-intersection format. The 4-intersection format is specified by a codimension 2 complete intersection ideal I with the property that it is contained in four codimension 3 ideals $J_1, \dots, J_4$. It leads to the construction of codimension 6 Gorenstein rings. As an application, in Section 5.2 we construct a family of Fano 3-folds of codimension 6 embedded in weighted projective space which corresponds to the entry ID: 29376 in Brown's Graded Ring Database [13].

5.1 The 4-intersection unprojection format

We now define the notion of 4-intersection unprojection format.

Definition 5.1.1 Assume that $J_1, \dots, J_4$ are four codimension 3 complete intersection ideals and I is a codimension 2 complete intersection ideal. We say that I is a 4-*intersection* ideal in $J_1, \dots, J_4$ if $I \subset J_t$ for all $1 \leq t \leq 4$.

An important question is how to explicitly construct I and J_t such that I is a 4-intersection ideal in $J_1, \dots, J_4$. In Subsection 5.1.1 we present such a construction.

5.1.1 An example of 4-intersection unprojection format

In the present subsection we specify the following: a codimension 2 complete intersection ideal I and four codimension 3 complete intersection ideals $J_1, \dots, J_4$ such that I is a 4-intersection ideal in $J_1, \dots, J_4$. Using this configuration as initial data, we construct, by parallel Kustin-Miller unprojection [53], a codimension 6 Gorenstein ring.

Assume that k is a field. We consider the standard graded polynomial ring $R = k[c_i, x_i]$, where $1 \leq i \leq 6$. We set

$$f = c_1x_1x_2 + c_2x_3x_4 + c_3x_5x_6, \qquad g = c_4x_1x_2 + c_5x_3x_4 + c_6x_5x_6,$$

$I = (f, g)$ and

$$J_1 = (x_1, x_3, x_5),\ J_2 = (x_1, x_4, x_6),\ J_3 = (x_2, x_3, x_6),\ J_4 = (x_2, x_4, x_5).$$

It is clear that f, g are homogeneous elements of degree 3 and I is a 4-intersection ideal in the ideals $J_1, \dots, J_4$.

In the applications we need to specialize the variables c_i to elements of k. We now give a precise way to do that. Consider the Zariski open subset

$$\mathcal{U} = \{(u_1, \dots, u_6) \in \mathbb{A}^6 : u_i \neq 0 \text{ for all } 1 \leq i \leq 6\}.$$

We assume that $(d_1, \dots, d_6) \in \mathcal{U}$. We denote by $\hat{R} = k[x_1, \dots, x_6]$ the polynomial ring in the variables x_i. Let

$$\hat{\phi}\colon R \to \hat{R}$$

be the unique k-algebra homomorphism such that

$$\hat{\phi}(x_i) = x_i, \quad \hat{\phi}(c_i) = d_i$$

for all $1 \leq i \leq 6$. We denote by $\hat{I}$ the ideal of the ring $\hat{R}$ generated by the subset $\hat{\phi}(I)$.

Proposition 5.1.2 *The ideals I and $\hat{I}$ are homogeneous codimension 2 Gorenstein ideals.*

Proof Since I is generated by two elements, we have, by Theorem 2.1.52, that $\operatorname{codim} I \leq 2$. Now we show that $\operatorname{codim} I \geq 2$. We set

$$r_1 = -c_4f + c_1g, \qquad r_2 = g, \qquad r_3 = f.$$

Let $>$ be the lexicographic order on R with $c_1 > \cdots > c_6 > x_1 > \cdots > x_6$. Consider the ideal

$$L = (\mathrm{in}_>(r_1), \mathrm{in}_>(r_2), \mathrm{in}_>(r_3)),$$

where $\mathrm{in}_>(r_1) = x_3x_4c_1c_5, \mathrm{in}_>(r_2) = x_1x_2c_4$ and $\mathrm{in}_>(r_3) = x_1x_2c_1$. We now prove that $\operatorname{codim} L = 2$. It is enough to show that $\dim\ R/L = 10$. Consider the affine variety $X = V(L) \subset \mathbb{A}^{12}$. It holds that

$$X = V(c_4, c_1) \cup V(c_5, x_1) \cup V(x_4, x_1) \cup V(x_3, x_1) \cup V(c_1, x_1) \cup V(c_5, x_2) \cup Z$$

where

$$Z = V(x_4, x_2) \cup V(x_3, x_2) \cup V(c_1, x_2).$$

Using that

$$\dim\ R/L = \dim\ X$$

the claim is proven. Hence, $\operatorname{codim} I \geq 2$.

In what follows we show that the ideal $\hat{I}$ is also a codimension 2 Gorenstein ideal. We set

$$\tilde{r_1} = \hat{\phi}(r_1), \qquad \tilde{r_2} = \hat{\phi}(r_2).$$

Let $>$ be the lexicographic order on $\hat{R}$ with $x_1 > \cdots > x_6$. Consider the ideal

$$Q = (\text{in}_>(\tilde{r_1}), \text{in}_>(\tilde{r_2})),$$

where $\text{in}_>(\tilde{r_1}) = x_3x_4d_1d_5$, $\text{in}_>(\tilde{r_2}) = x_1x_2d_4$. It is immediate that $Q = (x_3x_4, x_1x_2)$. It is enough to show that $\dim\ R/Q = 4$. Consider the affine variety $X = V(Q) \subset \mathbb{A}^6$. It holds that

$$X = V(x_2, x_4) \cup V(x_2, x_3) \cup V(x_1, x_3) \cup V(x_1, x_4).$$

Using that

$$\dim\ R/Q = \dim\ X$$

the claim is proven. Hence, $\operatorname{codim} \hat{I} \geq 2$. By Theorem 2.1.61, the ideals I and $\hat{I}$ are Gorenstein. $\square$

Proposition 5.1.3 *(i) For all t with $1 \leq t \leq 4$, the ideal J_t/I is a codimension 1 homogeneous ideal of the quotient ring R/I such that the ring R/J_t is Gorenstein.*

(ii) For all t, s with $1 \leq t < s \leq 4$, it holds that $\operatorname{codim}_{R/I}(J_t/I + J_s/I) = 3$.

Proof We first prove (i). According to the Third Isomorphism Theorem of rings

$$R/J_1 \cong k[c_1, \ldots, c_6, x_2, x_4, x_6], \ \ R/J_2 \cong k[c_1, \ldots, c_6, x_2, x_3, x_5], \tag{5.1}$$

$$R/J_3 \cong k[c_1, \ldots, c_6, x_1, x_4, x_5], \ \ R/J_4 \cong k[c_1, \ldots, c_6, x_1, x_3, x_6].$$

So, we conclude that for all t with $1 \leq t \leq 4$,

$$\dim\ R/J_t = 9.$$

By Proposition 5.1.2, it follows that

$$\dim\ R/I = \dim\ R - \operatorname{codim}\ I = 10.$$

Hence, using the last two equalities we have that for all t with $1 \leq t \leq 4$

$$\operatorname{codim} J_t/I = 1.$$

Due to the isomorphisms (5.1) for all t with $1 \leq t \leq 4$, the ring R/J_t is Gorenstein.

Concerning the Claim (ii), the Third Isomorphism Theorem of rings implies that

$$\begin{aligned} R/(J_1+J_2) &\cong k[c_1,\dots,c_6,x_2], \quad R/(J_1+J_3) \cong k[c_1,\dots,c_6,x_4],\\ R/(J_1+J_4) &\cong k[c_1,\dots,c_6,x_6], \quad R/(J_2+J_3) \cong k[c_1,\dots,c_6,x_5],\\ R/(J_2+J_4) &\cong k[c_1,\dots,c_6,x_3], \quad R/(J_3+J_4) \cong k[c_1,\dots,c_6,x_1]. \end{aligned}$$

From the later isomorphisms it holds that for t, s with $1 \leq t < s \leq 4$,

$$\dim \ R/(J_t + J_s) = 7.$$

Recall that $\dim \ R/I = 10$. Taking into account the definition of codimension we conclude that for all t, s with $1 \leq t < s \leq 4$,

$$\operatorname{codim} \ (J_t/I + J_s/I) = 3.$$

□

For all t, with $1 \leq t \leq 4$, we denote by $i_t \colon J_t/I \to R/I$ the inclusion map. In what follows, we define $\phi_t \colon J_t/I \to R/I$ for all t, with $1 \leq t \leq 4$, and prove that these maps satisfy the assumptions of the [53, Theorem 2.3].

Recall the polynomials h_1, h_2, h_3 which were defined in Section 3.1.4. We denote by $\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3}$ the polynomials which occur from h_1, h_2, h_3 if we substitute

$$a_1 = c_1x_2,\ a_2 = c_2x_4,\ a_3 = c_3x_6,\ b_1 = c_4x_2,\ b_2 = c_5x_4,\ b_3 = c_6x_6.$$

Proposition 5.1.4 *There exists a unique graded homomorphism of R/I-modules $\phi_1 \colon J_1/I \to R/I$ such that*

$$\phi_1(x_1 + I) = \widetilde{h_1} + I, \quad \phi_1(x_3 + I) = \widetilde{h_2} + I, \quad \phi_1(x_5 + I) = \widetilde{h_3} + I.$$

Proof It follows from [53, Theorem 4.3]. □

For the definition of ϕ_2 we replace x_3 by x_4 and x_5 by x_6. In this case, $\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3}$ are the polynomials which occur from h_1, h_2, h_3 if we substitute

$$a_1 = c_1x_2,\ a_2 = c_2x_3,\ a_3 = c_3x_5,\ b_1 = c_4x_2,\ b_2 = c_5x_3,\ b_3 = c_6x_5.$$

For the definitions of ϕ_3 and ϕ_4 we work similarly. For all t, with $1 \leq t \leq 4$, the degree of ϕ_t is equal to 3. By [53, Definition 2.2] the new unprojection variable has degree equal to the degree of the corresponding ϕ_t.

Proposition 5.1.5 *For all t, with $1 \leq t \leq 4$, the R/I-module* $\operatorname{Hom}_{R/I}(J_t/I, R/I)$ *is generated by the two elements i_t and ϕ_t.*

Proof It follows from [55, Theorem 4.3]. □

For all t, s, with $1 \leq t, s \leq 4$ and $t \neq s$, we define $r_{ts} = 0$.

Proposition 5.1.6 *For all t, s, with $1 \leq t, s \leq 4$ and $t \neq s$, it holds that*

$$\phi_t(J_t/I) \subset J_s/I.$$

Proof It is a direct computation using the definition of the maps ϕ_t. □

Proposition 5.1.7 *For all t, s, with $1 \leq t, s \leq 4$ and $t \neq s$, there exists a homogeneous element A_{st} such that*

$$\phi_s(\phi_t(p)) = A_{st} p$$

for all $p \in J_t/I$.

Proof It follows from [53, Proposition 2.1]. □

Remark 5.1.8 We note that the elements A_{st} are polynomial expressions in the variables c_i and x_j. We computed them using the computer algebra program Macaulay2 [28].

Following [53, Section 2], we write down explicitly the final ring as a quotient of a polynomial ring by a codimension 6 ideal.

Definition 5.1.9 Let T_1, T_2, T_3, T_4 be four new variables of degree 3. We define as I_{un} the ideal

$$(I) + (T_1x_1 - \phi_1(x_1),\, T_1x_3 - \phi_1(x_3),\, T_1x_5 - \phi_1(x_5),\, T_2x_1 - \phi_2(x_1),$$

$$T_2x_4 - \phi_2(x_4),\, T_2x_6 - \phi_2(x_6),\, T_3x_2 - \phi_3(x_2),\, T_3x_3 - \phi_3(x_3),$$

$$T_3x_6 - \phi_3(x_6),\, T_4x_2 - \phi_4(x_2),\, T_4x_4 - \phi_4(x_4),\, T_4x_5 - \phi_4(x_5),\, T_2T_1 - A_{21},$$

$$T_3T_1 - A_{31},\, T_4T_1 - A_{41},\, T_3T_2 - A_{32},\, T_4T_2 - A_{42},\, T_4T_3 - A_{43})$$

of the polynomial ring $R[T_1, T_2, T_3, T_4]$. We set $R_{un} = R[T_1, T_2, T_3, T_4]/I_{un}$.

Remark 5.1.10 The reason we put, for all $1 \leq i \leq 4$, $\deg T_i = 3$ is that each homomorphism ϕ_i is graded of degree 3. We also note that according to [53, Proposition 2.1] the degree of each A_{st} is equal to 6.

Theorem 5.1.11 *The ring R_{un} is Gorenstein.*

Proof By Propositions 5.1.3, 5.1.4 and 5.1.6, the assumptions of [53, Theorem 2.3] are satisfied. Hence, the ring R_{un} is Gorenstein. □

Proposition 5.1.12 *The homogeneous ideal I_{un} is a codimension 6 ideal with a minimal generating set of 20 elements.*

Proof According to the grading of the variables and the discussion before the Proposition 5.1.5 it is not difficult to see that I_{un} is a homogeneous ideal. Recall that in Kustin-Miller unprojection the codimension increases by 1. Hence, the homogeneous ideal I_{un}, as a result of a series of four unprojections of Kustin-Miller type starting by the codimension 2 ideal I, is a codimension 6 ideal. In order to prove that I_{un} is minimally generated by 20 elements we use the idea of specialization. More precisely we set

$$c_1 = c_3 = c_5 = c_6 = 0$$

and

$$c_2 = c_4 = 1$$

in the ideal I_{un}. We call $\widetilde{I_{un}}$ the ideal which occurs after these substitutions. The ideal $\widetilde{I_{un}}$ is a homogeneous ideal with 16 monomials and 4 binomials as generators. It is not difficult to see that $\widetilde{I_{un}}$ is minimally generated by these elements. Hence, we conclude that I_{un} is generated by at least 20 elements. By Definition 5.1.9, I_{un} is generated by 20 homogeneous elements. The result follows. □

5.2 Applications

In the present section we prove, using Theorem 5.1.11, the existence of a family of Fano 3-folds of codimension 6 in weighted projective space. We note that in what follows we make essential use of the computer algebra systems Macaulay2 [28] and Singular [25].

The construction is summarised in the following theorem. It corresponds to the entry 29376 of Brown's Graded Ring Database [13].

Theorem 5.2.1 *There exists a family of quasismooth, projectively normal and projectively Gorenstein Fano 3-folds $X \subset \mathbb{P}(1^8, 2, 3)$, nonsingular away from eight quotient singularities $\frac{1}{3}(1,1,2)$, with Hilbert series*

$$P_X(t) = \frac{1-6t^2+15t^4-20t^6+15t^8-6t^{10}+t^{12}}{(1-t)^8(1-t^2)(1-t^3)}.$$

We now give the explicit construction of the family of the Fano 3-folds which is described in Theorem 5.2.1.

We denote by $k = \mathbb{C}$ the field of complex numbers. We consider the polynomial ring $R = k[x_i, c_i]$, where $1 \leq i \leq 6$. Let R_{un} be the ring in Definition 5.1.9 and $\hat{R} = k[x_1, \ldots, x_6]$ be the polynomial ring in the variables x_i. We substitute the variables $(c_1, \ldots, c_6)$ which appear in the definitions of the rings R and R_{un} with a general element of k^6 (in the sense of being outside a proper Zariski closed subset of k^6). Let $\hat{I}$ be the ideal of $\hat{R}$ which is obtained by the ideal I and $\hat{I}_{un}$ the ideal of $\hat{R}[T_1, T_2, T_3, T_4]$ which is obtained by the ideal I_{un} after this substitution. We set $\hat{R}_{un} = \hat{R}[T_1, T_2, T_3, T_4]/\hat{I}_{un}$. In what follows x_1, x_3, x_5 are variables of degree 1 and x_2, x_4, x_6 are variables of degree 2. From the discussion before the Propositions 5.1.4 and 5.1.5 it follows that the degrees of T_2, T_3, T_4 are equal to 1 and the degree of T_1 is equal to 3. According to this grading the ideals $\hat{I}$ and $\hat{I}_{un}$ are homogeneous. Due to Theorem 5.1.11, Proj $\hat{R}_{un} \subset \mathbb{P}(1^6, 2^3, 3)$ is a projectively Gorenstein 3-fold.

Let $A = k[w_1, w_2, T_2, T_3, T_4, x_1, x_3, x_5, x_6, T_1]$ be the polynomial ring over k with w_1, w_2 variables of degree 1 and the other variables of degree noted as above. Consider the unique k-algebra homomorphism

$$\psi\colon \hat{R}[T_1, T_2, T_3, T_4] \to A$$

such that

$$\psi(x_1) = x_1, \quad \psi(x_2) = f_1, \quad \psi(x_3) = x_3, \quad \psi(x_4) = f_2,$$
$$\psi(x_5) = x_5, \quad \psi(x_6) = x_6, \quad \psi(T_1) = T_1, \quad \psi(T_2) = T_2,$$
$$\psi(T_3) = T_3, \quad \psi(T_4) = T_4$$

where

$f_1 = l_1x_1^2 + l_2x_1x_3 + l_3x_3^2 + l_4x_1x_5 + l_5x_3x_5 + l_6x_5^2 + l_7x_1T_2 + l_8x_3T_2 + l_9x_5T_2 + l_{10}T_2^2 + l_{11}x_1T_3 + l_{12}x_3T_3 + l_{13}x_5T_3 + l_{14}T_2T_3 + l_{15}T_3^2 + l_{16}x_1T_4 + +l_{17}T_3T_4 + l_{18}x_5T_4 + l_{19}T_2T_4 + l_{20}T_3T_4 + l_{21}T_4^2 + l_{22}x_1w_1 + l_{23}x_3w_1 + l_{24}x_5w_1 + l_{25}T_2w_1 + l_{26}T_3w_1 + l_{27}T_4w_1 + l_{28}w_1^2 + l_{29}x_1w_2 + l_{30}x_3w_2 + l_{31}x_5w_2 + l_{32}T_2w_2 + l_{33}T_3w_2 + l_{34}T_4w_2 + l_{35}w_1w_2 + l_{36}w_2^2 + l_{37}x_6$,

$f_2 = l_{38}x_1^2 + l_{39}x_1x_3 + l_{40}x_3^2 + l_{41}x_1x_5 + l_{42}x_3x_5 + l_{43}x_5^2 + l_{44}x_1T_2 + l_{45}x_3T_2 + l_{46}x_5T_2 + l_{47}T_2^2 + l_{48}x_1T_3 + l_{49}x_3T_3 + l_{50}x_5T_3 + l_{51}T_2T_3 + l_{52}T_3^2 + l_{53}x_1T_4 + l_{54}T_3T_4 + l_{55}x_5T_4 + l_{56}T_2T_4 + l_{57}T_3T_4 + l_{58}T_4^2 + l_{59}x_1w_1 + l_{60}x_3w_1 + l_{61}x_5w_1 + l_{62}T_2w_1 + l_{63}T_3w_1 + l_{64}T_4w_1 + l_{65}w_1^2 + l_{66}x_1w_2 + l_{67}x_3w_2 + l_{68}x_5w_2 + l_{69}T_2w_2 + l_{70}T_3w_2 + l_{71}T_4w_2 + l_{72}w_1w_2 + l_{73}w_2^2 + l_{74}x_6$, and $(l_1, \ldots, l_{74}) \in k^{74}$ are general. In other words, f_1, f_2 are two general degree 2 homogeneous elements of A.

Denote by Q the ideal of the ring A generated by the subset $\psi(\hat{I}_{un})$.

Let $X = V(Q) \subset \mathbb{P}(1^8, 2, 3)$. It is immediate that X is a codimension 6 projectively Gorenstein 3-fold.

Proposition 5.2.2 *The ring A/Q is an integral domain.*

Proof It is enough to show that the ideal Q is prime. For a specific choice of rational values for the parameters c_i, l_j, for $1 \leq i \leq 6$ and $1 \leq j \leq 74$ we checked using the computer algebra program Macaulay2 that the ideal which was obtained by Q is a homogeneous, codimension 6, prime ideal with the right Betti table. $\square$

In what follows, we show that the only singularities of $X \subset \mathbb{P}(1^8, 2, 3)$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$. According to the discussion after Definition 2.2.10, X belongs to the Mori category.

Proposition 5.2.3 *Consider $X = V(Q) \subset \mathbb{P}(1^8, 2, 3)$. Denote by $X_{cone} \subset \mathbb{A}^{10}$ the affine cone over X. The scheme X_{cone} is smooth outside the vertex of the cone.*

Proof For the proof we follow the steps which are described in the proof of Proposition 4.3.5. $\square$

Proposition 5.2.4 *Consider the singular locus $Z = V(x_1, x_3, x_5, T_2, T_3, T_4, w_1, w_2)$ of the weighted projective space $\mathbb{P}(1^8, 2, 3)$. The intersection of X with Z is a unique reduced point which is quotient singularity of type $\frac{1}{3}(1,1,2)$ for X.*

Proof We checked with the computer algebra program Macaulay2 that the intersection of X with Z is one reduced point. We denote this point by P. Point P corresponds to the ideal (x_i, T_j, w_k) for $i \in \{1,3,5,6\}$, $2 \leq j \leq 4$, $1 \leq k \leq 2$. By Proposition 5.2.3 X is smooth outside P. Around P we have that $T_1 = 1$. Looking the equations of Q we can eliminate the variables $x_1, x_3, x_5, T_2, T_3, T_4$ since these variables appear in the set of equations multiplied by T_1. This means that P is a quotient singularity of type $\frac{1}{3}(1,1,2)$. $\square$

Lemma 5.2.5 *Let $\omega_{\hat{R}/\hat{I}}$ be the canonical module of $\hat{R}/\hat{I}$. It holds that the canonical module $\omega_{\hat{R}/\hat{I}}$ is isomorphic to $\hat{R}/\hat{I}(-3)$.*

Proof From the minimal graded free resolution of $\hat{R}/\hat{I}$ as $\hat{R}$-module

$$0 \to \hat{R}(-6) \to \hat{R}(-3)^2 \to \hat{R}$$

and the fact that the sum of the degrees of the variables is equal to 9 we conclude that

$$\omega_{\hat{R}/\hat{I}} = \hat{R}/\hat{I}(6-9) = \hat{R}/\hat{I}(-3).$$

$\square$

Proposition 5.2.6 *The minimal graded resolution of A/Q as A-module is equal to*

$$0 \to C_6 \to C_5 \to C_4 \to C_3 \to C_2 \to C_1 \to C_0 \to 0 \tag{5.2}$$

where

$$\begin{aligned}
&C_6 = A(-12), \qquad C_5 = A(-8)^6 \oplus A(-9)^8 \oplus A(-10)^6,\\
&C_4 = A(-6)^8 \oplus A(-7)^{24} \oplus A(-8)^{24} \oplus A(-9)^8,\\
&C_3 = A(-4)^3 \oplus A(-5)^{24} \oplus A(-6)^{36} \oplus A(-7)^{24} \oplus A(-8)^3,\\
&C_2 = A(-3)^8 \oplus A(-4)^{24} \oplus A(-5)^{24} \oplus A(-6)^8,\\
&C_1 = A(-2)^6 \oplus A(-3)^8 \oplus A(-4)^6, \qquad C_0 = A.
\end{aligned}$$

Moreover, the canonical module of A/Q is isomorphic to $(A/Q)(-1)$ and the Hilbert series of A/Q as graded A-module is equal to

$$\frac{1 - 6t^2 + 15t^4 - 20t^6 + 15t^8 - 6t^{10} + t^{12}}{(1-t)^8(1-t^2)(1-t^3)}.$$

Proof To compute the minimal graded free resolution of A/Q we followed the method described in the proof of [52, Proposition 3.4]. From the minimal graded free resolution (5.2) of A/Q and the fact that the sum of the degrees of the variables is equal to 13 we conclude that

$$\omega_{A/Q} = A/Q(12-13) = A/Q(-1).$$

The last conclusion of Proposition 5.2.6 follows easily from the resolution (5.2). □

Taking into account Propositions 5.2.3, 5.2.4 and 5.2.6, we conclude that X is a Fano 3-fold.

Chapter 6

Anisotropy of Simplicial Spheres

This chapter is a joint work with Stavros Argyrios Papadakis, and was motivated by McMullen's g-conjecture for simplicial spheres [8, 61, 66]. In 2018, a proof of this important conjecture was announced by Adiprasito [1, 2].

Section 6.1 contains the construction of the generic Artinian reduction of an algebra. This useful construction appears many times throughout the chapter. In Section 6.2, we introduce the notion of the generic anisotropy of a simplicial sphere and we formulate one of our two main results of the chapter. This is Theorem 6.2.3, which states that over a field of characteristic 2 every simplicial sphere is generically anisotropic. The proof of the theorem is given in Subsection 6.4.3. The question of generic anisotropicity of simplicial spheres of dimension ≥ 2 over a field of characteristic not equal to 2 remains open.

Section 6.3 contains Theorem 6.3.14 which is a key result for the proof of Theorem 6.2.3. In order to use Theorem 6.3.14 for the proof of generic anisotropy in characteristic 2, we introduce, in Section 6.4, certain $(\dim D+1)$-th order differential operators ∂_σ and $\partial_{p,\sigma}$, associated to faces $\sigma, \sigma \cup \{p\}$ of a simplicial sphere D.

In Sections 6.5, 6.6 and 6.7 we study the differential operators in some detail, and prove Theorem 6.7.6, which states identities related to the differentiation of the product of the maximal minors of certain matrices. The theorem is used to prove the key Propositions 6.4.1 and 6.4.7. The propositions imply Corollaries 6.4.6 and 6.4.13, and the corollaries imply Theorem 6.2.3.

In Section 6.8 we prove Theorem 6.8.1 which connects the notion of generic anisotropy with the Lefschetz properties. Combining Theorem 6.2.3 with Theorem 6.8.1 we get a second proof of McMullen's g-conjecture for simplicial spheres in Theorem 6.8.2.

In Section 6.9 we prove that the simplicial spheres of dimension 1 are generically anisotropic over any field, which is the second of the main results of the chapter. A key tool is Proposition 6.9.1, which works in all characteristics.

Section 6.10 is dedicated to a specific form of Gauss elimination that we need, while in Section 6.11 we discuss a well-known technique for proving that a polynomial is nonzero. Section 6.12 contains some results related to the behaviour of the Lefschetz properties under field extensions. Finally, in Section 6.13 we state a general conjecture about our differential operators.

6.1 The generic Artinian reduction of an algebra

In this section we give a useful construction that will appear a number of times in the present chapter.

Assume $m \geq 1$ and k_1 is a field. We consider the polynomial ring $k_1[x_1, \dots, x_m]$, where the degree of the variable x_i is equal to 1, for all $1 \leq i \leq m$. Assume I is a homogeneous ideal of $k_1[x_1, \dots, x_m]$. We denote by d the Krull dimension of the quotient ring $k_1[x_1, \dots, x_m]/I$. We assume $d \geq 1$, and denote by k the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq d,\ 1 \leq j \leq m].$$

For $1 \leq i \leq d$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j.$$

Definition 6.1.1 We define the generic *Artinian reduction* of $k_1[x_1, \dots, x_m]/I$ to be the Artinian k-algebra

$$k[x_1, \dots, x_m]/((I) + (f_1, \dots, f_d)),$$

where (I) denotes the ideal of $k[x_1, \dots, x_m]$ generated by I.

6.2 Statement of the main theorem

In this section we introduce the notion of generic anisotropy and we formulate one of our main results which is related to generic anisotropy of a simplicial sphere over a field of characteristic 2.

Assume $n \geq 1$ is an integer and D is a simplicial sphere of dimension n with vertex set $\{1, \dots, m\}$. Assume k_1 is any field and denote by k the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n+1,\ 1 \leq j \leq m].$$

We define the polynomial ring $R = k[x_1, \dots, x_m]$, where we put degree 1 for all variables x_i. Denote by $I_D \subset R$ the Stanley-Reisner ideal of D. We set $k[D] = R/I_D$.

For $i = 1, \dots, n+1$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j,$$

and we define $A = k[D]/(f_1, \dots, f_{n+1})$. Hence, A is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 6.1.1. We denote by $\pi : R \to A$ the natural projection k-algebra homomorphism.

Remark 6.2.1 By [17, Section 5], the k-algebra $k[D]$ is standard graded and Gorenstein with Krull dimension equal to $n+1$. Since $a_{i,j}$ are independent variables that do not appear in the minimal monomial generating set for I_D, the sequence $f_1, \dots, f_{n+1}$ is a regular sequence for $k[D]$, see [17, Proposition 1.5.12]. Hence, A is a Gorenstein Artinian standard graded k-algebra. It has socle degree equal to $n+1$ by [17, Lemma 5.6.4]. Consequently, $A_i = 0$ for all $i \geq n+2$ and $\dim_k A_{n+1} = 1$. In particular, $\dim_k A_1 \geq 1$, which implies that $m \geq n+2$.

Definition 6.2.2 We call D *generically anisotropic* over k_1, if for all integers j with $1 \leq 2j \leq n+1$ and all nonzero elements $u \in A_j$ we have $u^2 \neq 0$.

The main result of the present Chapter is the following theorem, whose proof will be given in Subsection 6.4.3.

Theorem 6.2.3 *Assume that the field k_1 has characteristic 2, $n \geq 1$ is an integer, and D is a simplicial sphere of dimension n. Then D is generically anisotropic over k_1.*

6.3 The Artinian reduction of the Stanley-Reisner ring

In this section is contained one of the key results for the proof of the generic anisotropy in characteristic 2 and all dimensions. This is the Theorem 6.3.14, which is valid in any dimension but only in characteristic 2. An interesting open question is to establish a version of Theorem 6.3.14 valid in all characteristics.

We keep using the notations and assumptions defined in Section 6.2. In particular, we allow the field k_1 to be of arbitrary characteristic.

If $\sigma = (b_1, \dots, b_q)$ is a sequence of integers, with $1 \leq b_i \leq m$ for all i, we set

$$x_\sigma = \prod_{i=1}^{q} x_{b_i} \in R.$$

Whenever $q = n+1$, we also use the notation

$$[\sigma] = [b_1, \dots, b_{n+1}] \in k,$$

where, by definition, $[b_1, \dots, b_{n+1}]$ is the determinant of the $(n+1) \times (n+1)$ matrix with (i,j)-entry equal to a_{i,b_j}.

We denote by $F(D)$ the set of facets of D. We define an *ordered facet* of D to be a sequence $(b_1, b_2, \dots, b_{n+1})$ of positive integers such that the set $\{b_1, b_2, \dots, b_{n+1}\}$ is a facet of D. For $0 \le i \le n$, we define a *codimension* i face σ of D to be a face of dimension $n-i$. This is equivalent to $\#\sigma = n+1-i$.

Assume $g = \prod_{i=1}^m x_i^{a_i} \in R$ is a monomial. We define the *complexity* $c(g)$ of g by

$$c(g) = \sum_{i=1}^m a_i - \#\{i : a_i > 0\}.$$

It is clear that $c(g) \ge 0$ and that $c(g) = 0$ if and only if g is square-free.

The following proposition is well-known, but we provide a proof for completeness.

Proposition 6.3.1 *Assume $1 \le r \le n+1$. We have that the r-th graded piece A_r of A is spanned, as a k-vector space, by the image under π of the set of square-free monomials of R of degree r.*

Proof By finite induction, it is enough to show that if $g \in R$ is a nonzero monomial of degree r and complexity ≥ 1, then there exists $q \in R$ homogeneous of degree r, such that $\pi(q) = \pi(g)$ and q is a linear combination of monomials of complexity $c(g) - 1$.

Assume $g = \prod_{i=1}^m x_i^{a_i}$. Since $c(g) \ge 1$, by rearranging indices we can assume that $a_1 \ge 2$. Since $r \le n+1$, by rearranging indices we can assume that $a_i = 0$ for all $i \ge n+2$.

By Proposition 6.10.1, we have

$$\sum_{t=1}^m [2, 3, \dots, n+1, t] \pi(x_t) = 0.$$

Hence,

$$[2, 3, \dots, n+1, 1] \pi(x_1) = - \sum_{t=n+2}^m [2, 3, \dots, n+1, t] \pi(x_t).$$

As a consequence, multiplying by $\pi(g/x_1)$ we get

$$\pi(g) = -(\sum_{t=n+2}^m [2, 3, \dots, n+1, t] \pi(x_t g/x_1))/[2, 3, \dots, n+1, 1].$$

Since, for all $t \ge n+2$, we have $c(x_t g/x_1) = c(g) - 1$, the result follows. □

Remark 6.3.2 For a strengthening of Proposition 6.3.1 see Proposition 6.4.9.

Remark 6.3.3 We will use the following two facts, see [20, p. 111, Remark before Corollary 7.19]. Each codimension 1 face of D is contained in exactly two facets of D. Moreover, if σ_1 and σ_2 are two facets of D, then there exists a finite sequence

$$\tau_0, \tau_1, \dots, \tau_q$$

of facets of D such that $\tau_0 = \sigma_1$, $\tau_q = \sigma_2$, and, for all $0 \leq i \leq q-1$, the intersection $\tau_i \cap \tau_{i+1}$ is a codimension 1 face of D.

Proposition 6.3.4 *Assume*

$$\sigma_1 = (b_1, \dots, b_n, d_1), \quad \sigma_2 = (b_1, \dots, b_n, d_2),$$

are two ordered facets of D having codimension 1 intersection. We then have the following equality in the ring A

$$[\sigma_1]\pi(x_{\sigma_1}) = -[\sigma_2]\pi(x_{\sigma_2}).$$

Proof We set $\tau = \sigma_1 \cap \sigma_2$. Hence, $\tau = \{b_1, \dots, b_n\}$. By Proposition 6.10.1, we have that

$$\sum_{j=1}^{m} [b_1, b_2, \dots, b_n, j]\pi(x_j) = 0.$$

Hence,

$$\sum_{j=1}^{m} [b_1, b_2, \dots, b_n, j]\pi(x_j x_\tau) = 0.$$

If $j \in \tau$, we have $[b_1, b_2, \dots, b_n, j] = 0$. By Remark 6.3.3, σ_1 and σ_2 are the only facets of D which contain the codimension 1 face τ. Hence, the only terms of the last sum that are nonzero are for $j = d_1$ and $j = d_2$. The result follows. $\square$

Corollary 6.3.5 *Assume σ_1 and σ_2 are two ordered facets of D. Then there exists $\epsilon \in \{-1, 1\}$, such that*

$$[\sigma_1]\pi(x_{\sigma_1}) = \epsilon[\sigma_2]\pi(x_{\sigma_2}).$$

Proof By Remark 6.3.3, there exists a finite sequence

$$\tau_0, \tau_1, \ldots, \tau_q$$

of facets of D such that $\tau_0 = \sigma_1$, $\tau_q = \sigma_2$, and, for all $0 \leq i \leq q-1$, the intersection $\tau_i \cap \tau_{i+1}$ is a codimension 1 face of D. Using Proposition 6.3.4, we have that, for all $0 \leq i \leq q-1$, there exists $\epsilon_i \in \{-1, 1\}$, such that we have the following equality in the ring A

$$[\tau_i]\pi(x_{\tau_i}) = \epsilon_i[\tau_{i+1}]\pi(x_{\tau_{i+1}}).$$

The result follows. □

We fix an ordered facet $e = (e_1, \ldots, e_{n+1})$ of D. By Remark 6.2.1, $\dim_k A_{n+1} = 1$. Using Proposition 6.3.1, A_{n+1} is spanned, as a k-vector space, by the square-free monomials that correspond to the facets of D. Corollary 6.3.5 implies that any of them spans A_{n+1}. As a consequence, $\pi(x_e) \neq 0$ and $\pi(x_e)$ is a k-basis of A_{n+1}. Hence, there exists a unique set-theoretic map $\Psi_e : A_{n+1} \to k$ with the property that

$$u = \Psi_e(u)[e]\pi(x_e) \tag{6.1}$$

for all $u \in A_{n+1}$. It is clear that Ψ_e is an isomorphism of k-vector spaces. In addition, if n is odd, we set $p_1 = (n+1)/2$ and define the symmetric bilinear form

$$\rho_e : A_{p_1} \times A_{p_1} \to k \tag{6.2}$$

by

$$\rho_e(u, w) = \Psi_e(uw)$$

for all $u, w \in A_{p_1}$.

Remark 6.3.6 If we change the ordered facet e of D to another ordered facet σ, Corollary 6.3.5 implies that either $\Psi_\sigma = \Psi_e$ or $\Psi_\sigma = -\Psi_e$. Hence, if the field k_1 has characteristic 2 the map Ψ_e is canonical, in the sense that it is independent of the choice of the facet e of D, and we will denote it by Ψ.

Remark 6.3.7 Assume n is odd. Recall that a symmetric bilinear form

$$\delta : A_{p_1} \times A_{p_1} \to k$$

is called *anisotropic* if $\delta(u, u) \neq 0$ for all nonzero elements $u \in A_{p_1}$. Using Remark 2.1.66, it follows that ρ_e is anisotropic if and only if for all integers j with $1 \leq 2j \leq n+1$ and all nonzero elements $u \in A_j$ we have $u^2 \neq 0$. This (partially) explains the use of the term generic anisotropy in Definition 6.2.2.

Remark 6.3.8 The proof of Proposition 6.3.1 gives that for all $u \in A_{n+1}$ the element $\Psi_e(u)$ of k is a rational function in the set of all bracket polynomials

$$\{ [i_1, \dots, i_{n+1}] \ : \ 1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq m \}.$$

In addition, combined with the proof of Corollary 6.3.5, it provides an algorithm for computing $\Psi_e(u)$.

Proposition 6.3.9 *Assume k_1 is a field of characteristic 2 and $\sigma = (b_1, \dots, b_{n+1})$ is a facet of D. We have*

$$(\Psi \circ \pi)(x_\sigma) = 1/[b_1, \dots, b_{n+1}].$$

Proof By Corollary 6.3.5, we have

$$[\sigma]\pi(x_\sigma) = [e]\pi(x_e).$$

The result follows from the definition of Ψ. □

The following proposition allows the computation of $\Psi_e(u)$ in more cases.

Proposition 6.3.10 *Assume $\sigma = (b_1, \dots, b_{n-1}, c)$ is a codimension 1 ordered face of D. Denote by $\tau_1 = (b_1, \dots, b_{n-1}, c, d_1)$ and $\tau_2 = (b_1, \dots, b_{n-1}, c, d_2)$ the two ordered facets of D that contain σ. We then have the following two equalities*

$$\begin{aligned}
[b_1, \dots, b_{n-1}, c, d_1][b_1, \dots, b_{n-1}, c, d_2]\pi(x_c^2 \prod_{i=1}^{n-1} x_{b_i}) &= -[b_1, \dots, b_{n-1}, d_1, d_2][\tau_1]\pi(x_{\tau_1}) \\
&= [b_1, \dots, b_{n-1}, d_1, d_2][\tau_2]\pi(x_{\tau_2}).
\end{aligned}$$

Proof We set $S = \{1, \dots, m\} \setminus \{c\}$. By Proposition 6.10.1, we have that

$$\sum_{j=1}^{m} [b_1, b_2, \dots, b_{n-1}, d_1, j]\pi(x_j) = 0.$$

Hence,

$$[b_1, b_2, \dots, b_{n-1}, d_1, c]\pi(x_c) = -\sum_{j \in S} [b_1, b_2, \dots, b_{n-1}, c, j]\pi(x_j).$$

Consequently,

$$[b_1, b_2, \dots, b_{n-1}, d_1, c]\pi(x_c^2 \prod_{i=1}^{n-1} x_{b_i}) = -\sum_{j \in S} [b_1, b_2, \dots, b_{n-1}, d_1, j]\pi(x_j x_c \prod_{i=1}^{n-1} x_{b_i}).$$

Arguing for the last sum as in the proof of Proposition 6.3.4, we get

$$[b_1, b_2, \dots, b_{n-1}, d_1, c]\pi(x_c^2 \prod_{i=1}^{n-1} x_{b_i}) = -[b_1, b_2, \dots, b_{n-1}, d_1, d_2]\pi(x_{d_2} x_c \prod_{i=1}^{n-1} x_{b_i}).$$

Using that, by Proposition 6.3.4, $[\tau_1]\pi(x_{\tau_1}) = -[\tau_2]\pi(x_{\tau_2})$, the result follows. □

The following corollary is an immediate consequence of Proposition 6.3.10.

Corollary 6.3.11 *Assume k_1 is a field of characteristic 2 and $\sigma = (b_1, \dots, b_{n-1}, c)$ is a codimension 1 face of D. Denote by $(b_1, \dots, b_{n-1}, c, d_1)$ and $(b_1, \dots, b_{n-1}, c, d_2)$ the two facets of D that contain σ. We have*

$$(\Psi \circ \pi)(x_c^2 \prod_{i=1}^{n-1} x_{b_i}) = \frac{[b_1, \dots, b_{n-1}, d_1, d_2]}{[b_1, \dots, b_{n-1}, c, d_1][b_1, \dots, b_{n-1}, c, d_2]}.$$

FURTHER ASSUMPTION. For the rest of this section we make the additional assumption that the field k_1 has characteristic 2.

We set $Z = m + 2n$ and denote by M the $(n+1) \times Z$ matrix whose (i, j)-entry is equal to the variable $a_{i,j}$, for $1 \leq i \leq n+1$ and $1 \leq j \leq Z$. Given a subset $\mathcal{A}$ of the set $\{1, 2, \dots, Z\}$ of cardinality $n+1$, we denote by $M(\mathcal{A})$ the determinant of the $(n+1) \times (n+1)$ submatrix of M obtained by keeping the columns of M specified by the set $\mathcal{A}$.

We denote by k_2 the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n+1,\ 1 \leq j \leq Z].$$

It follows that k is a subfield of k_2.

Proposition 6.3.12 *(Recall that the field k_1 has characteristic equal to 2.) Assume n is odd. We set $l = (n+1)/2$. We assume that D is the boundary complex of the $(n+1)$-dimensional simplex with vertex set $\tau = \{c_1, \dots, c_l, g_1, \dots, g_{l+1}\}$. We then have the following equality in the field k_2*

$$(\Psi \circ \pi)(\prod_{i=1}^{l} x_{c_i}^2) = \frac{\prod_{i=1}^{l} M(\tau \setminus \{c_i\})}{\prod_{i=1}^{l+1} M(\tau \setminus \{g_i\})}.$$

Proof We set $c = \{c_1, \dots, c_l\}, g = \{g_1, \dots, g_{l+1}\}$. Assume $1 \leq i \leq l$. By Proposition 6.10.1, we have that

$$\sum_{t=1}^{l} [c \setminus \{c_i\}, g \setminus \{g_i\}, c_t]\pi(x_{c_t}) + \sum_{t=1}^{l+1} [c \setminus \{c_i\}, g \setminus \{g_i\}, g_t]\pi(x_{g_t}) = 0.$$

Hence,

$$[c \setminus \{c_i\}, g \setminus \{g_i\}, c_i]\pi(x_{c_i}) = [c \setminus \{c_i\}, g \setminus \{g_i\}, g_i]\pi(x_{g_i}),$$

since the field has characteristic 2 and the other terms in the two sums are zero.

Multiplying the above equations for $1 \leq i \leq l$, we get

$$(\prod_{i=1}^{l}[c \setminus \{c_i\}, g \setminus \{g_i\}, c_i])\ u_1 = (\prod_{i=1}^{l}[c \setminus \{c_i\}, g \setminus \{g_i\}, g_i])\ u_2, \tag{6.3}$$

where

$$u_1 = \prod_{i=1}^{l} \pi(x_{c_i}), \qquad u_2 = \prod_{i=1}^{l} \pi(x_{g_i}).$$

The result follows by multiplying both sides of Equality (6.3) by u_1 and using that, by Corollary 6.3.9,

$$\Psi(u_1 u_2) = 1/[c, g \setminus \{g_{l+1}\}].$$

□

Proposition 6.3.13 *(Recall that the field k_1 has characteristic equal to 2.) Assume n is even. We set $l = n/2$. Assume that D is the boundary complex of the simplex of dimension $n+1$ with vertex set $\tau = \{c_1, \ldots, c_l, b, g_1, \ldots, g_{l+1}\}$. We then have the following equality in the field k_2*

$$(\Psi \circ \pi)(x_b \prod_{i=1}^{l} x_{c_i}^2) = \frac{\prod_{i=1}^{l} M(\tau \setminus \{c_i\})}{\prod_{i=1}^{l+1} M(\tau \setminus \{g_i\})}.$$

Proof We set $c = \{c_1, \ldots, c_l\}, g = \{g_1, \ldots, g_{l+1}\}$. Assume $1 \leq i \leq l$. By Proposition 6.10.1, we have that

$$\sum_{t=1}^{l}[b, c \setminus \{c_i\}, g \setminus \{g_i\}, c_t]\pi(x_{c_t}) + \sum_{t=1}^{l+1}[b, c \setminus \{c_i\}, g \setminus \{g_i\}, g_t]\pi(x_{g_t}) = 0.$$

Hence,

$$[b, c \setminus \{c_i\}, g \setminus \{g_i\}, c_i]\pi(x_{c_i}) = [b, c \setminus \{c_i\}, g \setminus \{g_i\}, g_i]\pi(x_{g_i}),$$

since the field has characteristic 2 and the other terms in the two sums are zero.

Multiplying the above equalities for $1 \leq i \leq l$, we get

$$(\prod_{i=1}^{l}[b, c \setminus \{c_i\}, g \setminus \{g_i\}, c_i])\ u_1 = (\prod_{i=1}^{l}[b, c \setminus \{c_i\}, g \setminus \{g_i\}, g_i])\ u_2, \tag{6.4}$$

where

$$u_1 = \prod_{i=1}^{l} \pi(x_{c_i}), \qquad u_2 = \prod_{i=1}^{l} \pi(x_{g_i}).$$

The result follows by multiplying both sides of Equality (6.4) by $\pi(x_b)u_1$ and using that, by Corollary 6.3.9,

$$\Psi(\pi(x_b)u_1u_2) = 1/[b, c, g \setminus \{g_{l+1}\}].$$

□

We fix an integer r with $m+1 \leq r \leq Z$. Assume l is an integer with $2 \leq 2l \leq n+1$. We set $s = n + 1 - 2l$. Assume

$$\tau_1 = \{c_1, \dots, c_l\}, \quad \tau_2 = \{b_1, \dots, b_s\}$$

are two subsets of the vertex set $\{1, \dots, m\}$ of D, such that $\tau_1 \cup \tau_2$ has cardinality $l + s$ and is a face of D. We set $\tau = \tau_1 \cup \tau_2$.
Assume $\sigma \in F(D)$ is a facet of D. We define the rational function $H(\tau_1, \tau_2, \sigma)$ as follows:

1. If τ is not a subset of σ we set $H(\tau_1, \tau_2, \sigma) = 0$.
2. If τ is a subset of σ, we denote the elements of $\sigma \setminus \tau$ by $g_1, \dots, g_l$ and we set

$$H(\tau_1, \tau_2, \sigma) = \frac{\prod_{i=1}^{l} M((\sigma \cup \{r\}) \setminus \{c_i\})}{M(\sigma) \prod_{i=1}^{l} M((\sigma \cup \{r\}) \setminus \{g_i\})}.$$

Clearly,

$$H(\tau_1, \tau_2, \sigma) = \frac{\prod_{j \in \tau_1} M((\sigma \cup \{r\}) \setminus \{j\})}{M(\sigma) \prod_{j \in (\sigma \setminus (\tau_1 \cup \tau_2))} M((\sigma \cup \{r\}) \setminus \{j\})}.$$

The proof of the following theorem will be given in Subsection 6.3.1.

Theorem 6.3.14 *(Recall that the field k_1 has characteristic equal to 2.) We have the following equality in the field k_2*

$$(\Psi \circ \pi)((\prod_{i=1}^{l} x_{c_i}^2)(\prod_{i=1}^{s} x_{b_i})) = \sum_{\sigma \in F(D)} H(\tau_1, \tau_2, \sigma). \tag{6.5}$$

Remark 6.3.15 It is interesting to notice the similarities in the statement and proof of Theorem 6.3.14 with the results obtained by Lee in [45, Section 6].

Remark 6.3.16 Using the definition of the function H, it is clear that the nonzero terms of the sum in Equation (6.5) are exactly those where σ contains τ. Hence, the sum can also be considered as a sum over the facets of the link of the face τ in D.

Remark 6.3.17 Even though the left hand side in Equation (6.5) is completely independent of r, each nonzero term $H(\tau_1, \tau_2, \sigma)$ on the right hand side does depend on r. Hence, provided no denominator vanishes, we are allowed to specialise the variables $a_{i,r}$, for $1 \leq i \leq n+1$. This observation will be used in Corollaries 6.5.3 and 6.6.3.

Example 6.3.18 Assume k_1 is a field of characteristic 2, $m \geq 3$ and D is the m-gon with consecutive vertices $1, 2, \ldots, m$. By Corollary 6.3.11, we have

$$(\Psi \circ \pi)(x_2^2) = \frac{[1,3]}{[1,2][2,3]},$$

while, by Theorem 6.3.14, we have

$$(\Psi \circ \pi)(x_2^2) = H(\{2\}, \varnothing, \{1,2\}) + H(\{2\}, \varnothing, \{2,3\}) = \frac{[1,r]}{[1,2][2,r]} + \frac{[3,r]}{[2,3][2,r]}.$$

Example 6.3.19 Assume k_1 is a field of characteristic 2, and D is the simplicial complex with vertex set $\{1, 2, \ldots, 7\}$ and Stanley-Reisner ideal equal to

$$I_D = (x_1 x_2, x_3 x_4 x_5, x_6 x_7).$$

Then D is a simplicial sphere of dimension 3.

We set $\tau_1 = \{1,3\}, \tau_2 = \emptyset$. Clearly we have that τ_1 is a face of D. Moreover, since $I_D : (x_1 x_3) = (x_2, x_4 x_5, x_6 x_7)$, the link of τ_1 in D is the 4-gon with consecutive vertices $4, 6, 5, 7$. By Theorem 6.3.14

$$(\Psi \circ \pi)(x_1^2 x_3^2) = H_{4,6} + H_{6,5} + H_{5,7} + H_{7,4},$$

where

$$H_{a,b} = H(\tau_1, \tau_2, \tau_1 \cup \{a,b\}) = \frac{[1,a,b,r][3,a,b,r]}{[1,3,a,b][1,3,a,r][1,3,b,r]}.$$

Remark 6.3.20 We expect that with the correct sign adjustments there should be a version of Theorem 6.3.14 valid over a field k_1 of arbitrary characteristic. We do not pursue this direction further in the present work.

6.3.1 Proof of Theorem 6.3.14

We now give the proof of Theorem 6.3.14 by induction on $l \geq 1$.

Assume $l = 1$. We have $s = n - 1$ and

$$\tau_1 = \{c_1\}, \quad \tau_2 = \{b_1, \ldots, b_{n-1}\}.$$

Recall that $\tau = \tau_1 \cup \tau_2$. Hence, τ is a codimension 1 face of D. Using Remark 6.3.3, τ it is contained in exactly two facets of D. We denote them by

$$\sigma_1 = \{b_1, \dots, b_{n-1}, c_1, d_1\}, \quad \sigma_2 = \{b_1, \dots, b_{n-1}, c_1, d_2\}.$$

We use the notation

$$[\tau_2, i, j] = [b_1, \dots, b_{n-1}, i, j].$$

By Corollary 6.3.11,

$$(\Psi \circ \pi)(x_{c_1}^2 \prod_{i=1}^{n-1} x_{b_i}) = \frac{[\tau_2, d_1, d_2]}{[\tau_2, c_1, d_1][\tau_2, c_1, d_2]}.$$

We have $H(\tau_1, \tau_2, \sigma) = 0$ if $\sigma \in F(D) \setminus \{\sigma_1, \sigma_2\}$. Using the Plücker relation ([43, Theorem 5.2.3])

$$[\tau_2, d_1, d_2][\tau_2, c_1, r] = [\tau_2, d_1, r][\tau_2, c_1, d_2] + [\tau_2, d_1, c_1][\tau_2, d_2, r]$$

and taking into account that the field k_1 has charactersitic 2, we have

$$\begin{aligned}
(\Psi \circ \pi)(x_{c_1}^2 \prod_{i=1}^{n-1} x_{b_i}) &= \frac{[\tau_2, d_1, d_2]}{[\tau_2, c_1, d_1][\tau_2, c_1, d_2]} \\
&= \frac{[\tau_2, d_1, d_2][\tau_2, c_1, r]}{[\tau_2, c_1, d_1][\tau_2, c_1, d_2][\tau_2, c_1, r]} \\
&= \frac{[\tau_2, d_1, r][\tau_2, c_1, d_2] + [\tau_2, d_1, c_1][\tau_2, d_2, r]}{[\tau_2, c_1, d_1][\tau_2, c_1, d_2][\tau_2, c_1, r]} \\
&= \frac{[\tau_2, d_1, r]}{[\tau_2, c_1, d_1][\tau_2, c_1, r]} + \frac{[\tau_2, d_2, r]}{[\tau_2, c_1, d_2][\tau_2, c_1, r]} \\
&= H(\tau_1, \tau_2, \sigma_1) + H(\tau_1, \tau_2, \sigma_2).
\end{aligned}$$

We assume now that $l \geq 1$ with $2(l+1) \leq n+1$ and that Theorem 6.3.14 is true for l. We will prove that Theorem 6.3.14 is true for the value $l+1$. We set $s = n + 1 - 2(l+1)$. Assume

$$\tau_1 = \{c_1, \dots, c_{l+1}\}, \quad \tau_2 = \{b_1, \dots, b_s\}$$

such that $\tau_1 \cup \tau_2$ has cardinality $l + s + 1$ and is a face of D.

We fix integers $p_1, \dots, p_l$, such that $m + 1 \leq p_i \leq Z$, for all $1 \leq i \leq l$, and the set $\{r, p_1, p_2, \dots, p_l\}$ has cardinality equal to $l+1$. We set $\mathcal{B} = \{1, \dots, m\} \setminus (\tau_1 \cup \tau_2)$ and

$$u = (\prod_{i=1}^{l+1} x_{c_i}^2)(\prod_{i=1}^{s} x_{b_i}).$$

For $1 \leq i \leq Z$, we set

$$[[i]] = [b_1, \ldots, b_s,\ c_1, \ldots, c_l,\ r,\ p_1, \ldots, p_l,\ i].$$

Using Proposition 6.10.1,

$$\sum_{i=1}^{m} [[i]]\ \pi(x_i) = 0.$$

Hence

$$\pi(x_{c_{l+1}}) = \sum_{i} \frac{[[i]]}{[[c_{l+1}]]}\ \pi(x_i),$$

with the sum for $1 \leq i \leq m$ and $i \neq c_{l+1}$. Since $[[i]] = 0$ when $i \in \{c_1, \ldots, c_l, b_1, \ldots, b_s\}$, we have that

$$\pi(x_{c_{l+1}}) = \sum_{i \in \mathcal{B}} \frac{[[i]]}{[[c_{l+1}]]}\ \pi(x_i).$$

Multiplying this equality by

$$\pi(x_{c_{l+1}}) \prod_{i=1}^{l} \pi(x_{c_i}^2) \prod_{i=1}^{s} \pi(x_{b_i})$$

we get

$$\pi(u) = \sum_{i \in \mathcal{B}} \frac{[[i]]}{[[c_{l+1}]]}\ \pi(E_i),$$

where

$$E_i = x_i x_{c_{l+1}} (\prod_{i=1}^{l} x_{c_i}^2)(\prod_{i=1}^{s} x_{b_i}).$$

Hence,

$$(\Psi \circ \pi)(u) = \sum_{i \in \mathcal{B}} \frac{[[i]]}{[[c_{l+1}]]}\ (\Psi \circ \pi)(E_i).$$

Since, for all $i \in \mathcal{B}$, the expression for E_i has l squares, we can use the inductive hypothesis for $(\Psi \circ \pi)(E_i)$ to get

$$(\Psi \circ \pi)(E_i) = \sum_{\sigma \in F(D)} H(\tau_1 \setminus \{c_{l+1}\}, \tau_2 \cup \{i, c_{l+1}\}, \sigma).$$

As a consequence,

$$(\Psi \circ \pi)(u) = \sum_{i \in \mathcal{B}} \sum_{\sigma \in F(D)} V_{i,\sigma} = \sum_{\sigma \in F(D)} \sum_{i \in \mathcal{B}} V_{i,\sigma},$$

where

$$V_{i,\sigma} = \frac{[[i]]}{[[c_{l+1}]]}\ H(\tau_1 \setminus \{c_{l+1}\}, \tau_2 \cup \{i, c_{l+1}\}, \sigma).$$

Therefore, to finish the proof it is enough to show that for all $\sigma \in F(D)$ it holds

$$\sum_{i \in \mathcal{B}} V_{i,\sigma} = H(\tau_1, \tau_2, \sigma). \tag{6.6}$$

For $i \in \mathcal{B}$ we set

$$\eta_i = (\tau_1 \setminus \{c_{l+1}\}) \cup (\tau_2 \cup \{i, c_{l+1}\}),$$

therefore $\eta_i = \tau \cup \{i\}$.

We first assume that $\sigma \in F(D)$ does not contain τ as a subset. Hence $H(\tau_1, \tau_2, \sigma)$ is equal to zero. Assume $i \in \mathcal{B}$. Since $\tau \subset \eta_i$, it follows that η_i is not a subset of σ. This implies that $H(\tau_1 \setminus \{c_{l+1}\}, \tau_2 \cup \{i, c_{l+1}\}) = 0$, therefore $V_{i,\sigma} = 0$. As a consequence, Equality (6.6) is true.

Assume now that $\sigma \in F(D)$ contains τ as a subset. We set $\mathcal{C} = \sigma \setminus \tau$ and denote the elements of $\mathcal{C}$ by $g_1, \ldots, g_{l+1}$. We set $\sigma^r = \sigma \cup \{r\}$. If $i \in \mathcal{B} \setminus \mathcal{C}$, it follows that η_i is not a subset of σ, therefore $V_{i,\sigma} = 0$. As a consequence,

$$\sum_{i \in \mathcal{B}} V_{i,\sigma} = \sum_{i \in \mathcal{C}} V_{i,\sigma} = \sum_{i=1}^{l+1} V_{g_i,\sigma}.$$

We have

$$\begin{aligned}
V_{g_i,\sigma} &= \frac{[[g_i]]}{[[c_{l+1}]]}\ H(\tau_1 \setminus \{c_{l+1}\}, \tau_2 \cup \{g_i, c_{l+1}\}, \sigma) \\
&= \frac{[[g_i]]}{[[c_{l+1}]]}\ \frac{\prod_{t=1}^{l} M(\sigma^r \setminus \{c_t\})}{M(\sigma) \prod_{t=1}^{i-1} M(\sigma^r \setminus \{g_t\}) \prod_{t=i+1}^{l+1} M(\sigma^r \setminus \{g_t\})} \\
&= \frac{[[g_i]]}{[[c_{l+1}]]}\ \frac{M(\sigma^r \setminus \{g_i\}) \prod_{t=1}^{l} M(\sigma^r \setminus \{c_t\})}{M(\sigma) \prod_{t=1}^{l+1} M(\sigma^r \setminus \{g_t\})} \\
&= \Gamma\ [[g_i]]\ M(\sigma^r \setminus \{g_i\}),
\end{aligned}$$

where

$$\Gamma = \frac{\prod_{t=1}^{l} M(\sigma^r \setminus \{c_t\})}{[[c_{l+1}]] M(\sigma) \prod_{t=1}^{l+1} M(\sigma^r \setminus \{g_t\})}.$$

Hence,

$$\sum_{i=1}^{l+1} V_{g_i,\sigma} = \Gamma \sum_{i=1}^{l+1} [[g_i]] M(\sigma^r \setminus \{g_i\}).$$

By the Plücker relation ([43, Theorem 5.2.3]),

$$\sum_{i=1}^{l+1}[[g_i]]M(\sigma^r \setminus \{g_i\}) = [[c_{l+1}]]M(\sigma^r \setminus \{c_{l+1}\}).$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{l+1} V_{g_i,\sigma} &= \Gamma\ [[c_{l+1}]]\ M(\sigma^r \setminus \{c_{l+1}\}) \\ &= \frac{\prod_{t=1}^{l+1} M(\sigma^r \setminus \{c_t\})}{M(\sigma)\prod_{t=1}^{l+1} M(\sigma^r \setminus \{g_t\})} \\ &= H(\tau_1, \tau_2, \sigma).\end{aligned}$$

As a consequence, Equality (6.6) is true, which finishes the proof of Theorem 6.3.14.

6.4 Using the differential operators to establish anisotropy

We keep using the notations introduced in Sections 6.2 and 6.3. Moreover, we assume that the field k_1 has characteristic 2.

6.4.1 Case n is odd

Assume $n \geq 1$ is odd. We set $l = (n+1)/2$. We assume that $\sigma \in D$ is a face of dimension $l-1$. We denote, in increasing order, the elements of σ by $\sigma(1), \sigma(2), \ldots, \sigma(l)$. We define $\partial_\sigma : k_2 \to k_2$ to be the $(n+1)$-th order differential operator which is differentiation with respect to the variables in the set

$$\{a_{i,\sigma(j)} \quad : \quad 1 \leq i \leq n+1,\ j = [(i+1)/2]\},$$

where $[x]$ denotes the integral part of the real number x.

Proposition 6.4.1 *Assume τ is a face of D of dimension $l-1$. We then have*

$$(\partial_\sigma \circ \Psi \circ \pi)(x_\tau^2) = \big((\Psi \circ \pi)(x_\sigma x_\tau)\big)^2. \tag{6.7}$$

Proof We define the sets

$$\mathcal{K}_1 = \{\eta \in F(D) : \tau \subset \eta\}, \qquad \mathcal{K}_2 = \{\eta \in F(D) : \tau \cup \sigma \subset \eta\}.$$

We set $\gamma_1 = \tau \cap \sigma$, $\gamma_2 = (\tau \cup \sigma) \setminus \gamma_1$. Using Theorem 6.3.14, we get

$$(\Psi \circ \pi)(x_\tau^2) = \sum_{\eta \in \mathcal{K}_1} H(\tau, \varnothing, \eta) \qquad \text{and} \qquad (\Psi \circ \pi)(x_\tau x_\sigma) = \sum_{\eta \in \mathcal{K}_2} H(\gamma_1, \gamma_2, \eta).$$

Clearly $\mathcal{K}_2 \subset \mathcal{K}_1$. If $\eta \in \mathcal{K}_1 \setminus \mathcal{K}_2$, we have that $\sigma \setminus (\eta \cap \sigma) \neq \varnothing$, which implies that $\partial_\sigma(H(\tau, \varnothing, \eta)) = 0$. Hence

$$(\partial_\sigma \circ \Psi \circ \pi)(x_\tau^2) = \sum_{\eta \in \mathcal{K}_2} \partial_\sigma(H(\tau, \varnothing, \eta)).$$

Since the field k_1 has characteristic 2, we get

$$((\Psi \circ \pi)(x_\tau x_\sigma))^2 = \sum_{\eta \in \mathcal{K}_2} (H(\gamma_1, \gamma_2, \eta))^2.$$

Assume that $\eta \in \mathcal{K}_2$. Using Corollary 6.5.5, we have

$$\begin{aligned}
\partial_\sigma(H(\tau, \varnothing, \eta)) &= \partial_\sigma\Big(\frac{\prod_{i \in \tau} M((\eta \cup \{r\}) \setminus \{i\})}{M(\eta) \cdot \prod_{i \in \eta \setminus \tau} M((\eta \cup \{r\}) \setminus \{i\})}\Big) \\
&= \frac{\prod_{i \in \tau \cap \sigma} (M((\eta \cup \{r\}) \setminus \{i\}))^2}{(M(\eta))^2 \cdot \prod_{i \in \eta \setminus (\tau \cup \sigma)} (M((\eta \cup \{r\}) \setminus \{i\}))^2} \\
&= (H(\gamma_1, \gamma_2, \eta))^2,
\end{aligned}$$

which finishes the proof. $\square$

Remark 6.4.2 Conjecture 6.13.1 contains a conjectural statement generalising Proposition 6.4.1.

Corollary 6.4.3 *Assume u is a homogeneous element of R of degree l. We then have*

$$(\partial_\sigma \circ \Psi \circ \pi)(u^2) = \big((\Psi \circ \pi)(x_\sigma u)\big)^2. \tag{6.8}$$

Proof Using Proposition 6.3.1, there exist $s > 0$, faces $\tau_1, \ldots, \tau_s$ of D of dimension $l - 1$ and elements $\lambda_1, \ldots, \lambda_s$ in k such that

$$\pi(u) = \pi(\sum_{i=1}^{s} \lambda_i x_{\tau_i}).$$

Taking into account that the field k_1 has characteristic 2 and combining Proposition 6.4.1 with Remark 6.7.1, we have

$$\begin{aligned}(\partial_\sigma \circ \Psi \circ \pi)(u^2) &= (\partial_\sigma \circ \Psi \circ \pi)(\sum_{i=1}^{s} \lambda_i^2 x_{\tau_i}^2) = \sum_{i=1}^{s} \lambda_i^2 \big((\partial_\sigma \circ \Psi \circ \pi)(x_{\tau_i}^2)\big) \\ &= \sum_{i=1}^{s} \lambda_i^2 \big((\Psi \circ \pi)(x_\sigma x_{\tau_i})\big)^2 = \big(\sum_{i=1}^{s} \lambda_i((\Psi \circ \pi)(x_\sigma x_{\tau_i}))\big)^2 \\ &= \big((\Psi \circ \pi)(\sum_{i=1}^{s} \lambda_i x_{\tau_i} x_\sigma)\big)^2 = \big((\Psi \circ \pi)(x_\sigma u)\big)^2.\end{aligned}$$

This finishes the proof of Corollary 6.4.3. □

Remark 6.4.4 If we abuse the notation by avoiding writing down the maps Ψ and π, Equations (6.7) and (6.8) take the simpler form

$$\partial_\sigma(x_\tau^2) = (x_\sigma x_\tau)^2 \qquad \text{and} \qquad \partial_\sigma(u^2) = (x_\sigma u)^2$$

respectively.

Example 6.4.5 We use the assumptions of Example 6.3.18 and the notational convention described in Remark 6.4.4. We have

$$x_2^2 = \frac{[1,3]}{[1,2][2,3]}, \qquad \partial_{\{1\}}(x_2^2) = \frac{1}{[1,2]^2} = (x_1 x_2)^2, \qquad \partial_{\{2\}}(x_2^2) = \frac{[1,3]^2}{[1,2]^2[2,3]^2} = (x_2^2)^2.$$

Assume, in addition, that $m \geq 4$. Then

$$\partial_{\{4\}}(x_2^2) = 0 = (x_4 x_2)^2.$$

Corollary 6.4.6 *Assume u is a homogeneous element of R of degree less or equal than l such that $\pi(u) \neq 0$. We then have that $(\pi(u))^2 \neq 0$.*

Proof Using Remark 6.2.1, A is Artinian, Gorenstein and standard graded with socle degree equal to $n+1$. It follows, by Remark 2.1.66, that there exists a homogeneous element $h \in R$ of degree $l - \deg(u)$ such that $\pi(uh) \neq 0$. Combining Proposition 6.3.1 with Remark 2.1.66, there exists a face σ of D of dimension $l-1$ such that $\pi(x_\sigma uh) \neq 0$.

This implies that $(\Psi \circ \pi)(x_\sigma uh) \neq 0$, hence, by Corollary 6.4.3, $(\Psi \circ \pi)((uh)^2) \neq 0$. Since π is a k-algebra homomorphism, we get $(\pi(u))^2 \neq 0$. □

6.4.2 Case n is even

Assume $n \geq 2$ is even. We set $l = n/2$. We assume $\sigma \in D$ is a face of dimension $l-1$ and that p is vertex of D such that $\sigma \cup \{p\}$ is a face of D of dimension l. We denote, in increasing order, the elements of σ by $\sigma(1), \sigma(2), \ldots, \sigma(l)$. We define $\partial_{p,\sigma} : k_2 \to k_2$ to be the $(n+1)$-th order differential operator which is differentiation with respect to the variables in the set

$$\{a_{1,p}\} \ \cup \ \{a_{i,\sigma(j)} \ \ : \ \ 2 \leq i \leq n+1, \ j = [i/2]\},$$

where $[x]$ denotes the integral part of the real number x.

Proposition 6.4.7 *Assume τ is a face of D of dimension $l-1$ which does not contain p. We then have*

$$(\partial_{p,\sigma} \circ \Psi \circ \pi)(x_\tau^2 x_p) = \big((\Psi \circ \pi)(x_\sigma x_\tau x_p)\big)^2. \tag{6.9}$$

Proof We set $\tau_1 = \tau \cup \{p\}$. If τ_1 is not a face of D, we have $\pi(x_\tau x_p) = 0$ and the proposition is true.

Hence, we can assume that τ_1 is a face of D. We define the sets

$$\mathcal{K}_1 = \{\eta \in F(D) : \tau_1 \subset \eta\}, \qquad \mathcal{K}_2 = \{\eta \in F(D) : \tau_1 \cup \sigma \subset \eta\}.$$

We set $\gamma_1 = \tau_1 \cap \sigma$, $\gamma_2 = (\tau_1 \cup \sigma) \setminus \gamma_1$. Since p is not an element of σ, we have $\gamma_1 = \tau \cap \sigma$. Using Theorem 6.3.14, we get

$$(\Psi \circ \pi)(x_\tau^2 x_p) = \sum_{\eta \in \mathcal{K}_1} H(\tau, \{p\}, \eta) \qquad \text{and} \qquad (\Psi \circ \pi)(x_\tau x_\sigma x_p) = \sum_{\eta \in \mathcal{K}_2} H(\gamma_1, \gamma_2, \eta).$$

Clearly $\mathcal{K}_2 \subset \mathcal{K}_1$. If $\eta \in \mathcal{K}_1 \setminus \mathcal{K}_2$, we have that $\sigma \setminus (\eta \cap \sigma) \neq \varnothing$, which implies that $\partial_{p,\sigma}(H(\tau, \{p\}, \eta)) = 0$. Hence

$$(\partial_{p,\sigma} \circ \Psi \circ \pi)(x_\tau^2 x_p) = \sum_{\eta \in \mathcal{K}_2} \partial_{p,\sigma}(H(\tau, \{p\}, \eta)).$$

Since the field k_1 has characteristic 2

$$((\Psi \circ \pi)(x_\tau x_\sigma x_p))^2 = \sum_{\eta \in \mathcal{K}_2} (H(\gamma_1, \gamma_2, \eta))^2.$$

Assume that $\eta \in \mathcal{K}_2$. Using Corollary 6.6.5, we have

$$\begin{aligned}
\partial_{p,\sigma}(H(\tau, \{p\}, \eta)) &= \partial_{p,\sigma}\Big(\frac{\prod_{i \in \tau} M((\eta \cup \{r\}) \setminus \{i\})}{M(\eta) \cdot \prod_{i \in \eta \setminus (\tau \cup \{p\})} M((\eta \cup \{r\}) \setminus \{i\})}\Big) \\
&= \frac{\prod_{i \in \tau \cap \sigma} (M((\eta \cup \{r\}) \setminus \{i\}))^2}{(M(\eta))^2 \cdot \prod_{i \in \eta \setminus (\tau \cup \{p\} \cup \sigma)} (M((\eta \cup \{r\}) \setminus \{i\}))^2} \\
&= (H(\gamma_1, \gamma_2, \eta))^2,
\end{aligned}$$

which finishes the proof. □

Remark 6.4.8 Conjecture 6.13.1 contains a conjectural statement generalising Proposition 6.4.7.

We will need the following strengthening of Proposition 6.3.1.

Proposition 6.4.9 *Assume $1 \le d \le n+1$ and $u \in R_d$. Then there exist $s > 0$, faces $\tau_1, \ldots, \tau_s$ of D dimension $d-1$ and elements $\lambda_1, \ldots, \lambda_s$ in k such that*

$$\pi(u) = \pi(\sum_{i=1}^{s} \lambda_i x_{\tau_i})$$

and, moreover, p is not an element of τ_i for all $1 \le i \le s$.

Proof Using Proposition 6.3.1, it is enough to assume that $u = x_\eta$, where η is a face of D of dimension $d-1$. If p is not an element of η, the result is obvious by setting $s = 1$, $\tau_1 = \eta$, $\lambda_1 = 1$.

Assume now that $p \in \eta$. Without loss of generality, we can assume that $p = 1$ and $\eta = \{1, 2, \ldots, d\}$. By Proposition 6.10.1, we have

$$\sum_{t=1}^{m} [2, 3, \ldots, n+1, t]\pi(x_t) = 0.$$

Hence,

$$\pi(x_1) = -\sum_{t=n+2}^{m} \frac{[2, 3, \ldots, n+1, t]}{[2, 3, \ldots, n+1, 1]}\pi(x_t),$$

which implies that

$$\pi(x_\eta) = -\sum_{t=n+2}^{m} \frac{[2, 3, \ldots, n+1, t]}{[2, 3, \ldots, n+1, 1]}\pi(x_t \prod_{i=2}^{d} x_i).$$

The result follows. □

Corollary 6.4.10 *Assume u is a homogeneous element of R of degree l. We then have*

$$(\partial_{p,\sigma} \circ \Psi \circ \pi)(u^2 x_p) = \big((\Psi \circ \pi)(x_\sigma u x_p)\big)^2. \tag{6.10}$$

Proof Using Proposition 6.4.9 , there exist $s > 0$, faces $\tau_1, \dots, \tau_s$ of D of dimension $l-1$ and elements $\lambda_1, \dots, \lambda_s$ in k such that

$$\pi(u) = \pi(\sum_{i=1}^{s} \lambda_i x_{\tau_i})$$

and, moreover, p is not an element of τ_i for all $1 \le i \le s$.

Taking into account that the field k_1 has characteristic 2 and combining Proposition 6.4.7 with Remark 6.7.1, we have

$$\begin{aligned}(\partial_{p,\sigma} \circ \Psi \circ \pi)(u^2 x_p) &= (\partial_{p,\sigma} \circ \Psi \circ \pi)(\sum_{i=1}^{s} \lambda_i^2 x_{\tau_i}^2 x_p) = \sum_{i=1}^{s} \lambda_i^2 \big((\partial_{p,\sigma} \circ \Psi \circ \pi)(x_{\tau_i}^2 x_p)\big) \\ &= \sum_{i=1}^{s} \lambda_i^2 \big((\Psi \circ \pi)(x_\sigma x_{\tau_i} x_p)\big)^2 = \big(\sum_{i=1}^{s} \lambda_i((\Psi \circ \pi)(x_\sigma x_{\tau_i} x_p))\big)^2 \\ &= \big((\Psi \circ \pi)(\sum_{i=1}^{s} \lambda_i x_{\tau_i} x_\sigma x_p)\big)^2 = \big((\Psi \circ \pi)(x_\sigma u x_p)\big)^2.\end{aligned}$$

□

Remark 6.4.11 If we abuse the notation by avoiding writing down the maps Ψ and π, Equations (6.9) and (6.10) take the simpler form

$$\partial_{p,\sigma}(x_\tau^2 x_p) = (x_\sigma x_\tau x_p)^2 \qquad \text{and} \qquad \partial_{p,\sigma}(u^2 x_p) = (x_\sigma u x_p)^2$$

respectively.

Example 6.4.12 Assume D is the boundary complex of the 3-simplex with vertex set $\{1, 2, 3, 4\}$. We set $p = 1, \tau = \{2\}$. Using Corollary 6.3.11 and the notational convention described in Remark 6.4.11, we have

$$x_\tau^2 x_p = \frac{[1,3,4]}{[1,2,3][1,2,4]}, \qquad \partial_{p,\{2\}}(x_\tau^2 x_p) = \frac{[1,3,4]^2}{[1,2,3]^2[1,2,4]^2} = (x_2 x_\tau x_p)^2$$

and

$$\partial_{p,\{3\}}(x_\tau^2 x_p) = \frac{1}{[1,2,3]^2} = (x_3 x_\tau x_p)^2, \qquad \partial_{p,\{4\}}(x_\tau^2 x_p) = \frac{1}{[1,2,4]^2} = (x_4 x_\tau x_p)^2.$$

Corollary 6.4.13 *Assume u is a homogeneous element of R of degree less or equal than l such that $\pi(u) \neq 0$. We then have that $(\pi(u))^2 \neq 0$.*

Proof Using Remark 6.2.1, A is Artinian, Gorenstein and standard graded with socle degree equal to $n+1$. It follows, by Remark 2.1.66, that there exists a homogeneous element $h \in R$ of degree $l - \deg(u)$ such that $\pi(uh) \neq 0$. Combining Proposition 6.3.1 with Remark 2.1.66, there exists a face σ_1 of D of dimension l such that $\pi(x_{\sigma_1} uh) \neq 0$.

We fix an element p of σ_1, and set $\sigma = \sigma_1 \setminus \{p\}$. Therefore, $\pi(x_{\sigma_1} uh) \neq 0$ implies that $(\Psi \circ \pi)(x_\sigma uhx_p) \neq 0$. Using Corollary 6.4.10, it follows that $(\Psi \circ \pi)((uh)^2 x_p)) \neq 0$. Since π is a k-algebra homomorphism, we get $(\pi(u))^2 \neq 0$. □

6.4.3 Proof of Theorem 6.2.3

We now prove Theorem 6.2.3. If n is odd, it follows from Corollary 6.4.6, while if n is even, it follows from Corollary 6.4.13.

6.5 The differential operator for n odd

The aim of the present section is to establish, in conjuction with the following two Sections 6.6 and 6.7, the results about the differential operators that were used in Section 6.4.

In the present section we work over a field k_1 of characteristic 2.

Assume $n \geq 1$ is odd and m is an integer with $m \geq n+1$. We set $Z = m + 2n$ and denote by M the $(n+1) \times Z$ matrix whose (i,j)-entry is equal to the variable $a_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq Z$. Given a subset $\mathcal{A}$ of the set $\{1, 2, \ldots, Z\}$ of cardinality $n+1$, we denote by $M(\mathcal{A})$ the determinant of the $(n+1) \times (n+1)$ submatrix of M obtained by keeping the columns of M specified by the set $\mathcal{A}$.

We denote by k_2 the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n+1,\ 1 \leq j \leq Z].$$

We set $l = (n+1)/2$. Assume

$$\tau_1 = \{c_1, \ldots, c_l\}, \quad \tau_2 = \{g_1, \ldots, g_{l+1}\}$$

are two subsets of the set $\{1, 2, \ldots, Z\}$ such that $\tau_1 \cup \tau_2$ has cardinality $2l+1$.

We set $\tau = \tau_1 \cup \tau_2$ and

$$G(\tau_1, \tau_2) = \frac{\prod_{i=1}^{l} M(\tau \setminus \{c_i\})}{\prod_{i=1}^{l+1} M(\tau \setminus \{g_i\})}.$$

For the rest of this section we make the assumption that τ is a subset of the set $\{1, 2, \ldots m\}$. We fix r with $m+1 \leq r \leq Z$ and set, for $1 \leq i \leq l+1$,

$$G_i(\tau_1, \tau_2, \{r\}) = G(\tau_1, (\tau_2 \cup \{r\}) \setminus \{g_i\}).$$

We denote by $G_i^{sp}(\tau_1,\tau_2)$ the result of substituting in $G_i(\tau_1,\tau_2,\{r\})$ the value 1 for the variable $a_{1,r}$ and the value 0 for the variables $a_{j,r}$, for $2 \le j \le n+1$. We remark that $G_i^{sp}(\tau_1,\tau_2)$ is well-defined, since the denominator of $G_i(\tau_1,\tau_2,\{r\})$ does not vanish when we perform the substitution.

Moreover, we denote by $T_{n+1} : k_2 \to k_2$ the $(n+1)$-th order differential operator which is differentiation with respect to the set of variables

$$\{a_{1,c_1}, a_{2,c_1},\ a_{3,c_2}, a_{4,c_2},\ a_{5,c_3}, a_{6,c_3}, \dots, a_{n,c_l}, a_{n+1,c_l}\}.$$

Remark 6.5.1 This set of variables can also be described as the set

$$\{a_{i,c_j} : 1 \le i \le n+1, j = [(i+1)/2]\},$$

where $[x]$ denotes the integral part of the real number x. For example, if $n = 3$, then

$$T_{n+1} = \frac{\partial^4}{\partial a_{1,c_1}\, \partial a_{2,c_1}\, \partial a_{3,c_2}\, \partial a_{4,c_2}}.$$

We remind the reader that the field k_1 has characteristic 2.

Proposition 6.5.2 *We have the following equality in the field k_2*

$$G(\tau_1,\tau_2) = \sum_{i=1}^{l+1} G_i(\tau_1,\tau_2,\{r\}).$$

Proof Denote by D the boundary complex of the simplex of dimension $n+1$ with vertex set τ. By Proposition 6.3.12, we have

$$(\Psi \circ \pi)(\prod_{i=1}^{l} x_{c_i}^2) = G(\tau_1,\tau_2).$$

Since $G_i(\tau_1,\tau_2,\{r\}) = H(\tau_1,\varnothing,\tau \setminus \{g_i\})$, by Theorem 6.3.14 we have

$$(\Psi \circ \pi)(\prod_{i=1}^{l} x_{c_i}^2) = \sum_{i=1}^{l+1} G_i(\tau_1,\tau_2,\{r\}).$$

The result follows. □

For an example related to the above Proposition 6.5.2 see Example 6.3.18.

The following corollary follows immediately from Proposition 6.5.2, by taking into account that, for all $1 \le j \le n+1$, the variable $a_{j,r}$ does not appear in $G(\tau_1,\tau_2)$.

Corollary 6.5.3 *We have the following equality in the field k_2*

$$G(\tau_1, \tau_2) = \sum_{i=1}^{l+1} G_i^{sp}(\tau_1, \tau_2).$$

The following proposition is an immediate corollary of Part 1 of Theorem 6.7.6. For simplicity of notation, for $i \in \tau$ we set $M_i = M(\tau \setminus \{i\})$.

Proposition 6.5.4 *We have the following equality in the field k_2*

$$T_{n+1}\ (\prod_{i\in\tau} M_i) = \prod_{i\in\tau_1} (M_i)^2.$$

Corollary 6.5.5 *Assume S is a subset of τ. We then have the following equality in the field k_2*

$$T_{n+1}(\frac{\prod_{i\in S} M_i}{\prod_{i\in\tau\setminus S} M_i}) = \frac{\prod_{i\in S\cap\,\tau_1}(M_i)^2}{\prod_{i\in\tau_2\setminus S}(M_i)^2}.$$

Proof Using Proposition 6.5.4 and Remark 6.7.1, we have

$$\begin{aligned} T_{n+1}(\frac{\prod_{i\in S} M_i}{\prod_{i\in\tau\setminus S} M_i}) &= T_{n+1}(\frac{\prod_{i\in\tau} M_i}{(\prod_{i\in\tau\setminus S} M_i)^2}) = \frac{T_{n+1}(\prod_{i\in\tau} M_i)}{(\prod_{i\in\tau\setminus S} M_i)^2} \\ &= \frac{\prod_{i\in\tau_1}(M_i)^2}{\prod_{i\in\tau\setminus S}(M_i)^2} = \frac{E\cdot\prod_{i\in S\cap\,\tau_1}(M_i)^2}{E\cdot\prod_{i\in\tau_2\setminus S}(M_i)^2}, \end{aligned}$$

where $E = \prod_{i\in\tau_1\setminus S}(M_i)^2$. The result follows. □

6.6 The differential operator for n even

The aim of the present section is to establish, in conjuction with the previous Section 6.5 and the following Section 6.7, the results about the differential operators that were used in Section 6.4.

In the present section we work over a field k_1 of characteristic 2.

Assume $n \geq 1$ is even and m is an integer with $m \geq n+1$. We set $Z = m + 2n$ and denote by M the $(n+1)\times Z$ matrix whose (i,j)-entry is equal to the variable $a_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq Z$. Given a subset $\mathcal{A}$ of the set $\{1, 2, \ldots, Z\}$ of cardinality $n+1$, we denote by $M(\mathcal{A})$ the determinant of the $(n+1)\times(n+1)$ submatrix of M obtained by keeping the columns of M specified by the set $\mathcal{A}$.

We denote by k_2 the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n+1,\ 1 \leq j \leq Z].$$

We set $l = n/2$. Assume

$$\tau_1 = \{c_1, \ldots, c_l\}, \quad \tau_2 = \{b\}, \quad \tau_3 = \{g_1, \ldots, g_{l+1}\}$$

are three subsets of the set $\{1, 2, \ldots, Z\}$ such that $\cup_{i=1}^{3} \tau_i$ has cardinality $2l + 2$.

We set $\tau = \cup_{i=1}^{3} \tau_i$ and

$$G(\tau_1, \tau_2, \tau_3) = \frac{\prod_{i=1}^{l} M(\tau \setminus \{c_i\})}{\prod_{i=1}^{l+1} M(\tau \setminus \{g_i\})}.$$

For the rest of this section we make the assumption that τ is a subset of the set $\{1, 2, \ldots m\}$. We fix r with $m + 1 \leq r \leq Z$ and set, for $1 \leq i \leq l + 1$,

$$G_i(\tau_1, \tau_2, \tau_3, \{r\}) = G(\tau_1, \tau_2, (\tau_3 \cup \{r\}) \setminus \{g_i\}).$$

Denote by $G_i^{sp}(\tau_1, \tau_2, \tau_3)$ the result of substituting in $G_i(\tau_1, \tau_2, \tau_3, \{r\})$ the value 1 for the variable $a_{1,r}$ and the value 0 for the variables $a_{j,r}$, for $2 \leq j \leq n + 1$. We remark that $G_i^{sp}(\tau_1, \tau_2, \tau_3)$ is well-defined, since the denominator of $G_i(\tau_1, \tau_2, \tau_3, \{r\})$ does not vanish when we perform the substitution.

Moreover, we denote by $T_{n+1} : k_2 \to k_2$ the $(n + 1)$-th order differential operator which is differentiation with respect to the set of variables

$$\{a_{1,b},\ a_{2,c_1}, a_{3,c_1},\ a_{4,c_2}, a_{5,c_2}, \ldots, a_{n,c_l}, a_{n+1,c_l}\}.$$

Remark 6.6.1 This set of variables can also be described as the set

$$\{a_{1,b}\} \cup \{a_{i,c_j} : 2 \leq i \leq n + 1, j = [i/2]\},$$

where $[x]$ denotes the integral part of the real number x. For example, if $n = 2$, then

$$T_{n+1} = \frac{\partial^3}{\partial a_{1,b}\, \partial a_{2,c_1}\, \partial a_{3,c_1}}.$$

We remind the reader that the field k_1 has characteristic 2.

Proposition 6.6.2 *We have the following equality in the field k_2*

$$G(\tau_1, \tau_2, \tau_3) = \sum_{i=1}^{l+1} G_i(\tau_1, \tau_2, \tau_3, \{r\}).$$

Proof Denote by D the boundary complex of the simplex of dimension $n+1$ with vertex set τ. By Proposition 6.3.13, we have

$$(\Psi \circ \pi)(x_b \prod_{i=1}^{l} x_{c_i}^2) = G(\tau_1, \tau_2, \tau_3).$$

Since $G_i(\tau_1, \tau_2, \tau_3, \{r\}) = H(\tau_1, \tau_2, \tau \setminus \{g_i\})$, by Theorem 6.3.14 we have

$$(\Psi \circ \pi)(x_b \prod_{i=1}^{l} x_{c_i}^2) = \sum_{i=1}^{l+1} G_i(\tau_1, \tau_2, \tau_3, \{r\}).$$

The result follows. □

The following corollary follows immediately from Proposition 6.6.2, by taking into account that, for all $1 \leq j \leq n+1$, the variable $a_{j,r}$ does not appear in $G(\tau_1, \tau_2, \tau_3)$.

Corollary 6.6.3 *We have the following equality in the field k_2*

$$G(\tau_1, \tau_2, \tau_3) = \sum_{i=1}^{l+1} G_i^{sp}(\tau_1, \tau_2, \tau_3).$$

The following proposition is an immediate corollary of Part 2 of Theorem 6.7.6. For simplicity of notation, for $i \in \tau_1 \cup \tau_3$ we set $M_i = M(\tau \setminus \{i\})$.

Proposition 6.6.4 *We have the following equality in the field k_2*

$$T_{n+1} \left(\prod_{i \in \tau_1 \cup \tau_3} M_i \right) = \prod_{i \in \tau_1} (M_i)^2.$$

Corollary 6.6.5 *Assume S is a subset of $\tau_1 \cup \tau_3$. We then have the following equality in the field k_2*

$$T_{n+1}\left(\frac{\prod_{i \in S} M_i}{\prod_{i \in (\tau_1 \cup \tau_3) \setminus S} M_i}\right) = \frac{\prod_{i \in S \cap \tau_1} (M_i)^2}{\prod_{i \in \tau_3 \setminus S} (M_i)^2}.$$

Proof We set $w = \tau_1 \cup \tau_3$. Using Proposition 6.6.4 and Remark 6.7.1, we have

$$\begin{aligned} T_{n+1}\left(\frac{\prod_{i \in S} M_i}{\prod_{i \in w \setminus S} M_i}\right) &= T_{n+1}\left(\frac{\prod_{i \in w} M_i}{(\prod_{i \in w \setminus S} M_i)^2}\right) = \frac{T_{n+1}(\prod_{i \in w} M_i)}{(\prod_{i \in w \setminus S} M_i)^2} \\ &= \frac{\prod_{i \in \tau_1} (M_i)^2}{\prod_{i \in w \setminus S} (M_i)^2} = \frac{E \cdot \prod_{i \in S \cap \tau_1} (M_i)^2}{E \cdot \prod_{i \in \tau_3 \setminus S} (M_i)^2}, \end{aligned}$$

where $E = \prod_{i \in \tau_1 \setminus S} (M_i)^2$. The result follows. □

6.7 Some useful characteristic 2 identities

The aim of the present section is to establish, in conjuction with the previous two Sections 6.5 and 6.6, the results about the differential operators that were used in Section 6.4.

In the present section we work over a field k_1 of characteristic 2.

Assume $h \geq 2$ is an integer. We denote by k the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq h+2,\ 1 \leq j \leq h+1].$$

We denote by M^{big} the $(h+2) \times (h+1)$ matrix whose (i,j)-entry is equal to the variable $a_{i,j}$, for $1 \leq i \leq h+2$ and $1 \leq j \leq h+1$.

Assume $h \geq 2$ is even. We denote by $N^{(h)}$ the $h \times (h+1)$ submatrix of M^{big}, obtained by keeping the rows indexed by $1, 2, \ldots, h$. We denote by $P^{(h)}$ the $h \times (h+1)$ submatrix of M^{big}, obtained by keeping the rows indexed by $3, 4, \ldots, h+2$. We define the folowing two sets of variables

$$\mathcal{A}_{N,h} = \{a_{i,j} : 1 \leq i \leq h, j = [(i+1)/2]\}, \quad \mathcal{A}_{P,h} = \{a_{i,j} : 3 \leq i \leq h+2, j = [(i+1)/2]\},$$

where $[x]$ denotes the integral part of the real number x. For $S \in \{N, P\}$, we denote by $T_{S,h}$ the h-th order differential operator which is partial differentiation with respect to the variables in the set $\mathcal{A}_{S,h}$.

Assume $h \geq 3$ is odd. We denote by $Q^{(h)}$ the $h \times (h+1)$ submatrix of M^{big}, obtained by keeping the rows indexed by $2, 3, \ldots, h+1$. We define the folowing set of variables

$$\mathcal{A}_{Q,h} = \{a_{2,1}\} \cup \{a_{i,j} : 3 \leq i \leq h+1, j = [(i+1)/2]\}.$$

We denote by $T_{Q,h}$ the h-th order differential operator which is partial differentiation with respect to the variables in the set $\mathcal{A}_{Q,h}$.

In the present section we will use the following notational convention. Assume $l \geq 1$, S is an $l \times (l+1)$ matrix and $1 \leq i \leq l+1$. We will denote by S_i the determinant of the $l \times l$ submatrix of S obtained by deleting the i-th column of S.

Remark 6.7.1 We will use that, since the field k_1 has characteristic 2, we have

$$T_{S,h}(f^2 g) = f^2 T_{S,h}(g)$$

for all $f, g \in k$, $S \in \{N, P, Q\}$ and $h \geq 2$ as above (that is, h even if $S = N$ or $S = P$ and h odd if $S = Q$). Indeed, by the Leibnitz Rule,

$$\frac{\partial}{\partial a_{i,j}}(f^2 g) = g\frac{\partial}{\partial a_{i,j}}(f^2) + f^2\frac{\partial}{\partial a_{i,j}}(g) = 2gf\frac{\partial}{\partial a_{i,j}}(f) + f^2\frac{\partial}{\partial a_{i,j}}(g) = f^2\frac{\partial}{\partial a_{i,j}}(g),$$

and $T_{S,h}$ is a composition of such operators. Consequently, if $f, g \in k$ with $g \neq 0$, then

$$T_{S,h}(\frac{f}{g}) = T_{S,h}(\frac{fg}{g^2}) = \frac{T_{S,h}(fg)}{g^2}.$$

$\square$

Proposition 6.7.2 *Assume that $h \geq 2$ is even and that*

$$T_{N,h}(\prod_{i=1}^{h+1} N_i^{(h)}) = \prod_{i=1}^{h/2} (N_i^{(h)})^2.$$

We then have

$$T_{P,h}(\prod_{i=1}^{h+1} P_i^{(h)}) = \prod_{i=2}^{(h+2)/2} (P_i^{(h)})^2.$$

Proof We denote by N^{mod} the matrix obtained from $N^{(h)}$ by putting the last column of $N^{(h)}$ first. Since, the field k_1 has characteristic 2, we get $N_1^{mod} = N_{h+1}^{(h)}$ and that

$$N_i^{mod} = N_{i-1}^{(h)},$$

for all $2 \leq i \leq h+1$. Hence, using the assumption we have

$$T_{N,h}(\prod_{i=1}^{h+1} N_i^{mod}) = T_{N,h}(\prod_{i=1}^{h+1} N_i^{(h)}) = \prod_{i=1}^{h/2} (N_i^{(h)})^2 = \prod_{i=1}^{h/2} (N_{i+1}^{mod})^2 = \prod_{i=2}^{(h+2)/2} (N_i^{mod})^2. \quad (6.11)$$

We have that both N^{mod} and $P^{(h)}$ are $h \times (h+1)$ matrices. The entries of each matrix are independent indeterminates. For $1 \leq i \leq h$ and $1 \leq j \leq h+1$, we denote by $n_{i,j}$ the (i,j)-entry of N^{mod} and by $p_{i,j}$ the (i,j)-entry of $P^{(h)}$. By definition, $T_{N,h}$ is differentiation with respect to the variables in the set

$$\{n_{i,j} : 1 \leq i \leq h, j = 1 + [(i+1)/2]\},$$

while $T_{P,h}$ is differentiation with respect to the variables in the set

$$\{p_{i,j} : 1 \leq i \leq h, j = 1 + [(i+1)/2]\}.$$

There exists a unique isomorphism of k_1-algebras $\phi : k_1[n_{i,j}] \to k_1[p_{i,j}]$ such that $\phi(n_{i,j}) = p_{i,j}$ for all $1 \leq i \leq h$ and $1 \leq j \leq h+1$. As a consequence, the result follows from Equation (6.11). $\square$

Proposition 6.7.3 *Assume $h = 2$. We have*

$$T_{N,2}(N_1^{(2)} N_2^{(2)} N_3^{(2)}) = (N_1^{(2)})^2 \quad \text{and} \quad T_{P,2}(P_1^{(2)} P_2^{(2)} P_3^{(2)}) = (P_2^{(2)})^2.$$

Proof Using Proposition 6.7.2, it is enough to prove only the first equality. We have

$$T_{N,2} = \frac{\partial^2}{\partial a_{1,1}\,\partial a_{2,1}}$$

and

$$N_1^{(2)} = \det\begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix}, \quad N_2^{(2)} = \det\begin{pmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{pmatrix}, \quad N_3^{(2)} = \det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2}. \end{pmatrix}.$$

The result follows by an easy direct computation, taking into account that the field k_1 has characteristic 2. □

Remark 6.7.4 It is easy to see that the assumption that the field k_1 has characteristic two is crucial in order to have the equalities in the statement of Proposition 6.7.3.

Proposition 6.7.5 *1) Assume $h \geq 4$ is even and that*

$$T_{P,h-2}(\prod_{i=1}^{h-1} P_i^{(h-2)}) = \prod_{i=2}^{h/2} (P_i^{(h-2)})^2.$$

We then have

$$T_{Q,h-1}(\prod_{i=2}^{h} Q_i^{(h-1)}) = \prod_{i=2}^{h/2} (Q_i^{(h-1)})^2.$$

2) Assume $h \geq 4$ is even and that

$$T_{Q,h-1}(\prod_{i=2}^{h} Q_i^{(h-1)}) = \prod_{i=2}^{h/2} (Q_i^{(h-1)})^2.$$

We then have

$$T_{N,h}(\prod_{i=1}^{h+1} N_i^{(h)}) = \prod_{i=1}^{h/2} (N_i^{(h)})^2.$$

Proof We fix an even integer $h \geq 4$. For simplicity of notation, we set

$$N = N^{(h)}, \quad T_N = T_{N,h}, \quad Q = Q^{(h-1)}, \quad T_Q = T_{Q,h-1}, \quad P = P^{(h-2)}, \quad T_P = T_{P,h-2}.$$

We first prove Part 1). We set

$$W = \frac{\prod_{i=(h+2)/2}^{h-1} Q_i}{Q_h \prod_{i=2}^{h/2} Q_i}.$$

Using Remark 6.7.1, it is enough to prove that

$$T_Q(W) = 1/(Q_h)^2. \tag{6.12}$$

Denote by r_1 the transpose of the $1 \times (h-1)$ matrix $(1, 0, 0, \ldots, 0)$. For an element $s \in \{2, 3, \ldots, h/2\} \cup \{h\}$ we denote by $U^{<s>}$ the $(h-1) \times h$ matrix obtained by replacing the s-th column of Q with r_1.

We set, for $s \in \{2, 3, \ldots, h/2\} \cup \{h\}$,

$$W^{<s>} = \frac{\prod_{j=(h+2)/2}^{h-1} U_j^{<s>}}{U_h^{<s>} \prod_{j=2}^{h/2} U_j^{<s>}}.$$

Since $U_j^{<s>} \neq 0$, for all $1 \leq j \leq h$, we have that $W^{<s>}$ is well-defined. By Corollary 6.6.3, we have

$$W = W^{<h>} + \sum_{s=2}^{h/2} W^{<s>}.$$

For simplicity, we set $B = U^{<h>}$. Assume $s \in \{2, 3, \ldots, h/2\}$. We have that $T_Q(W^{<s>})$ is zero, since the variable $a_{2s,s}$ is an element of $\mathcal{A}_{Q,h-1}$ but does not appear in $W^{<s>}$. As a consequence, we have

$$T_Q(W) = T_Q(W^{<h>}).$$

Hence, using that $Q_h = B_h$, to prove Equation (6.12) it is enough to prove that

$$T_Q(W^{<h>}) = 1/(B_h)^2.$$

Taking into account Remark 6.7.1, it follows that to prove Equation (6.12) it is enough to prove that

$$T_Q(\prod_{i=2}^{h} B_i) = \prod_{i=2}^{h/2} (B_i)^2. \tag{6.13}$$

Using the definition of r_1, we get that $B_i = P_i$ for all $1 \leq i \leq h-1$. We set

$$K = B_h - a_{2,1} P_1.$$

By the well-known formula for the development of the determinant B_h using the row containing the element $a_{2,1}$ it follows that the variable $a_{2,1}$ does not appear in K. Consequently, the differential operator $\frac{\partial}{\partial a_{2,1}}$ annihilates K. Since the same operator annihilates P_j for all $1 \leq j \leq h$, and $T_Q = T_P \circ \frac{\partial}{\partial a_{2,1}}$, we get

$$T_Q(\prod_{i=2}^{h} B_i) = T_Q(B_h \prod_{i=2}^{h-1} B_i) = T_Q(a_{2,1} P_1 \prod_{i=2}^{h-1} P_i) = T_P(P_1 \prod_{i=2}^{h-1} P_i) = \prod_{i=2}^{h/2} (P_i)^2,$$

with the last equality by the assumption for Part 1). Since, for $1 \le i \le h-1$, we have $B_i = P_i$, Equality (6.13) follows, which finishes the proof of Part 1).

EXAMPLE (to help understand the above proof of Part 1): If $h = 4$, we have

$$Q = \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}, \qquad B = U^{<h>} = \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & 1 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix}$$

and

$$T_Q = \frac{\partial^3}{\partial a_{2,1}\, \partial a_{3,2}\, \partial a_{4,2}}, \qquad T_P = \frac{\partial^2}{\partial a_{3,2}\, \partial a_{4,2}}.$$

We now prove Part 2) using similar arguments to the ones used in the proof of Part 1). We set

$$W = \frac{\prod_{i=(h+2)/2}^{h} N_i}{N_{h+1} \prod_{i=1}^{h/2} N_i}.$$

Using Remark 6.7.1, it is enough to prove that

$$T_N(W) = 1/(N_{h+1})^2. \tag{6.14}$$

Denote by r_2 the transpose of the $1 \times h$ matrix $(1, 0, 0, \dots, 0)$. For

$$s \in \{1, 2, \dots, h/2\} \cup \{h+1\}$$

we denote by $X^{<s>}$ the $h \times (h+1)$ matrix obtained by replacing the s-th column of N with r_2.

We set, for $s \in \{1, 2, \dots, h/2\} \cup \{h+1\}$,

$$W^{<s>} = \frac{\prod_{j=(h+2)/2}^{h} X_j^{<s>}}{X_{h+1}^{<s>} \prod_{j=1}^{h/2} X_j^{<s>}}.$$

Since $X_j^{<s>} \ne 0$, for all $1 \le j \le h+1$, we have that $W^{<s>}$ is well-defined. By Corollary 6.5.3, we have

$$W = W^{<h+1>} + \sum_{s=1}^{h/2} W^{<s>}.$$

For simplicity, we set $C = X^{<h+1>}$. Assume $s \in \{1, 2, \ldots, h/2\}$. We have $T_N(W^{<s>}) = 0$, since the variable $a_{2s,s}$ is an element of $\mathcal{A}_{N,h}$ but does not appear in $W^{<s>}$. As a consequence, we have

$$T_N(W) = T_N(W^{<h+1>}).$$

Hence, using that $N_{h+1} = C_{h+1}$, to prove Equation (6.14) it is enough to prove that

$$T_N(W^{<h+1>}) = 1/(C_{h+1})^2.$$

Taking into account Remark 6.7.1, it follows that to prove Equation (6.14) it is enough to prove that

$$T_N(\prod_{i=1}^{h+1} C_i) = \prod_{i=1}^{h/2} (C_i)^2. \tag{6.15}$$

Using the definition of r_2, we get that $C_i = Q_i$ for all $1 \leq i \leq h$. We set

$$K = C_{h+1} - a_{1,1}Q_1.$$

By the well-known formula for the development of the determinant C_{h+1} using the row containing the element $a_{1,1}$ it follows that the variable $a_{1,1}$ does not appear in K. Consequently, the differential operator $\frac{\partial}{\partial a_{1,1}}$ annihilates K. Since the same operator annihilates Q_j for all $1 \leq j \leq h$, and $T_N = T_Q \circ \frac{\partial}{\partial a_{1,1}}$, we get

$$\begin{aligned} T_N(\prod_{i=1}^{h+1} C_i) &= T_N(C_{h+1} \prod_{i=1}^{h} C_i) = T_N(a_{1,1}Q_1 \prod_{i=1}^{h} Q_i) = T_N(a_{1,1}(Q_1)^2 \prod_{i=2}^{h} Q_i) \\ &= T_Q((Q_1)^2 \prod_{i=2}^{h} Q_i) = (Q_1)^2\, T_Q(\prod_{i=2}^{h} Q_i) = (Q_1)^2 \prod_{i=2}^{h/2} (Q_i)^2 \end{aligned}$$

with the last two equalities by Remark 6.7.1 and the assumption for Part 2). Since, for $1 \leq i \leq h$, we have $C_i = Q_i$, Equality (6.15) follows, which finishes the proof of Part 2).

EXAMPLE (to help understand the above proof of Part 2): If $h = 4$, we have

$$N = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \end{pmatrix}, \quad C = X^{<h+1>} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & 1 \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

and

$$T_N = \frac{\partial^4}{\partial a_{1,1}\, \partial a_{2,1}\, \partial a_{3,2}\, \partial a_{4,2}}, \qquad T_Q = \frac{\partial^3}{\partial a_{2,1}\, \partial a_{3,2}\, \partial a_{4,2}}.$$

□

Theorem 6.7.6 *1) Assume $h \geq 2$ is even. We have*

$$T_{N,h}(\prod_{i=1}^{h+1} N_i^{(h)}) = \prod_{i=1}^{h/2} (N_i^{(h)})^2.$$

2) Assume $h \geq 3$ is odd. We have

$$T_{Q,h}(\prod_{i=2}^{h+1} Q_i^{(h)}) = \prod_{i=2}^{(h+1)/2} (Q_i^{(h)})^2.$$

Proof It is obvious that Part 2) is equivalent to the statement that for all even integers $h \geq 4$ we have

$$T_{Q,h-1}(\prod_{i=2}^{h} Q_i^{(h-1)}) = \prod_{i=2}^{h/2} (Q_i^{(h-1)})^2.$$

Using induction on the even integer $h \geq 2$, the proof of the present theorem follows by combining Proposition 6.7.3, which provides the starting case $h = 2$, and Propositions 6.7.2 and 6.7.5, which provide the inductive step. □

Remark 6.7.7 Conjecture 6.13.1 contains a conjectural statement generalising Theorem 6.7.6.

6.8 Anisotropy implies the Lefschetz properties

In the present section we investigate the relations between generic anisotropy and the Lefschetz properties. As an application, in Theorem 6.8.2 we give a second proof of McMullen's g-conjecture for simplicial spheres.

Assume k_1 is a field of arbitrary characteristic, $n \geq 1$ is an integer, and D is a simplicial sphere of dimension n and vertex set $\{1, 2, \ldots, m\}$.

We denote by $S(D)$ the *suspension* of D. More precisely, it is the simplicial complex with vertex set $\{1, 2, \dots, m+2\}$ and set of facets equal to

$$\{\sigma \cup \{x_{m+1}\} : \sigma \in F(D)\} \quad \cup \quad \{\sigma \cup \{x_{m+2}\} : \sigma \in F(D)\},$$

where $F(D)$ denotes the set of facets of D. It is well-known that $S(D)$ is a simplicial sphere of dimension $n+1$. Moreover, we denote by k the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n+2,\ 1 \leq j \leq m+2].$$

The proof of the following theorem will be given in Subsection 6.8.1.

Theorem 6.8.1 *Assume that $S(D)$ is generically anisotropic over the field k_1. Then the graded k-algebra $k[D]$ has the Weak Lefschetz Property.*

All three statements in the following theorem are results originally due to Adiprasito [1, 2]. The proof of the theorem will be given in Subsection 6.8.2.

Theorem 6.8.2 *(Adiprasito) Assume D is a simplicial sphere of dimension n, with $n \geq 1$. Then*

i) McMullen's g-conjecture is true for D.

ii) Assume k_1 is an infinite field of characteristic 2. Then the Stanley-Reisner ring $k_1[D]$ has the Weak Lefschetz Property.

iii) Assume k_1 is an infinite field of characteristic 2. Then the Stanley-Reisner ring $k_1[D]$ has the Strong Lefschetz Property.

Remark 6.8.3 It is well-known that iii) implies ii). We state both ii) and iii), since in our approach we first prove ii) and then use it to establish iii). Notice also that the paper [1] contains the stronger result that for any infinite field k_1 of arbitrary characteristic the Stanley-Reisner ring $k_1[D]$ has the Strong Lefschetz Property.

6.8.1 Proof of Theorem 6.8.1

The aim of the present subsection is to prove Proposition 6.8.8, since it immediately implies Theorem 6.8.1. We use some key ideas and results of Swartz, which were developed in [65, Section 4].

We keep using the notations defined in Section 6.8. We set $R_{sm} = k[x_1, \dots, x_m]$ and $R = R_{sm}[x_{m+1}, x_{m+2}]$. We denote by $I_D \subset R_{sm}$ the Stanley-Reisner ideal of D over the field k and by $I_{S(D)} \subset R$ the Stanley-Reisner ideal of $S(D)$ over the same field k. It is clear that

$$I_{S(D)} = (I_D) + (x_{m+1}x_{m+2}).$$

We denote by $k[D] = R_{sm}/I_D$ and $k[S(D)] = R/I_{S(D)}$ the corresponding Stanley-Reisner rings over k.

For $1 \leq i \leq n+2$, we set

$$f_i = \sum_{j=1}^{m+2} a_{i,j} x_j \in R.$$

We use the notation $A = k[S(D)]/(f_1, \dots, f_{n+2})$, and denote by $\pi_A : R \to A$ the natural projection k-algebra homomorphism. Therefore, A is the generic Artinian reduction of $k_1[S(D)]$ in the sense of Definition 6.1.1.

We set $J = I_{S(D)} : (x_{m+1}) \subset R$. In other words,

$$J = \{u \in R \ : \ u x_{m+1} \in I_{S(D)}\}.$$

It is clear that $J = (I_D) + (x_{m+2})$. We use the notation

$$B = \frac{R}{J + (f_1, f_2, \dots, f_{n+2})} \ ,$$

and we denote by $\pi_B : R \to B$ the natural projection k-algebra homomorphism.

For $2 \leq i \leq n+2$ and $1 \leq j \leq m$, we set

$$c_{i,j} = \det \begin{pmatrix} a_{1,j} & a_{1,m+1} \\ a_{i,j} & a_{i,m+1} \end{pmatrix} \in k.$$

In addition, for $2 \leq i \leq n+2$, we set

$$g_i = \sum_{j=1}^{m} c_{i,j} x_j \in R_{sm}.$$

Since, for all $2 \leq i \leq n+2$, it holds

$$g_i = a_{i,m+1} f_1 - a_{1,m+1} f_i + x_{m+2}(a_{1,m+1} a_{i,m+2} - a_{i,m+1} a_{1,m+2}),$$

we get the following equality of ideals of R

$$(f_1, f_2, \dots, f_{n+2}) + (x_{m+2}) = (f_1) + (g_2, g_3, \dots, g_{n+2}) + (x_{m+2}). \tag{6.16}$$

We use the notation $C = k[D]/(g_2, g_3, \dots, g_{n+2})$, and denote by $\pi_C : R_{sm} \to C$ the natural projection k-algebra homomorphism. We set

$$\omega = -\sum_{i=1}^{m} \frac{a_{1,i}}{a_{1,m+1}} x_i \in R_{sm}.$$

It is clear that $\pi_B(\omega) = \pi_B(x_{m+1})$. We consider the unique k-algebra homomorphism $\phi_{mod} : R \to C$, such that $\phi_{mod}(x_i) = \pi_C(x_i)$ for all $1 \leq i \leq m$, $\phi_{mod}(x_{m+1}) = \pi_C(\omega)$ and $\phi_{mod}(x_{m+2}) = 0$. From the definition of ω it follows that $f_1 \in \ker \phi_{mod}$. Hence, Equation (6.16) implies that the ideal $J + (f_1, \ldots, f_{n+2})$ is contained in the kernel of ϕ_{mod}. Consequently, there exists an induced k-algebra homomorphism $\phi : B \to C$ such that $\phi \circ \pi_B = \phi_{mod}$.

Proposition 6.8.4 *The map ϕ is an isomorphism of graded k-algebras.*

Proof It is clear from the definition that ϕ preserves degrees. We consider the unique k-algebra homomorphism $R_{sm} \to B$, that sends x_i to $\pi_B(x_i)$, for all $1 \leq i \leq m$. Using Equation (6.16), it follows that the ideal $I_D + (g_2, g_3, \ldots, g_{n+2})$ of R_{sm} is inside its kernel, hence there exists an induced k-algebra homomorphism $\psi : C \to B$. It follows from the definitions that ψ is the inverse map of ϕ. $\square$

Proposition 6.8.5 *i) The k-algebra A is graded, Artinian and Gorenstein with socle degree equal to $n + 2$.*

ii) The k-algebras B and C are graded, Artinian and Gorenstein with socle degree equal to $n + 1$.

Proof We first remark that, by Proposition 6.8.4, the graded k-algebras B and C are isomorphic.

By Remark 6.2.1, the k-algebra $k[S(D)]$ is graded and Gorenstein with Krull dimension equal to $n + 2$. Moreover, by the same remark A is Artinian and Gorenstein with socle degree equal to $n + 2$.

Since $I_{S(D)} \subset J$, there exists a unique surjective homomorphism of k-algebras $\pi_{new} : A \to B$, such that $\pi_{new} \circ \pi_A = \pi_B$. Since π_{new} is surjective and A is Artinian, it follows that B is Artinian. Since C is isomorphic to B we get that C is also Artinian. By Remark 6.2.1, the k-algebra $k[D]$ is graded and Gorenstein with Krull dimension equal to $n+1$. It follows that the sequence $g_2, \ldots, g_{n+2}$ is a regular sequence for $k[D]$. This implies that the k-algebra C is Gorenstein and, using again Remark 6.2.1, that the socle degree of C is equal to $n + 1$. $\square$

We consider the homomorphism of R-modules $R \to A$, that sends u to $\pi_A(x_{m+1}u)$, for all $u \in R$. It is clear that the ideal $J + (f_1, \ldots, f_{n+2})$ of R is inside its kernel. Hence, we get an induced homomorphism of R-modules $m_{x_{m+1}} : B \to A$, such that

$$m_{x_{m+1}}(\pi_B(u)) = \pi_A(x_{m+1}u)$$

for all $u \in R$. The following proposition is a special case of [65, Proposition 4.24].

Proposition 6.8.6 *(Swartz) The homomorphism $m_{x_{m+1}}$ is injective.*

Proof Recall the map ψ defined in the proof of Proposition 6.8.4. We set

$$\delta = m_{x_{m+1}} \circ \psi \ : \ C \to A.$$

Since ψ is an isomorphism, it is enough to prove that δ is injective.

Since, for all $j \geq 0$, we have $\delta(C_j) \subset A_{j+1}$, to prove that δ is injective it is enough to assume that $0 \leq j \leq n+1$ and $u \in (R_{sm})_j$ is a homogeneous element of degree j such that $\pi_C(u) \neq 0$, and prove that $\delta(\pi_C(u)) \neq 0$. In order to get a contradiction, we assume that

$$\delta(\pi_C(u)) = 0. \tag{6.17}$$

By Proposition 6.8.5, C is a graded Artinian Gorenstein k-algebra with socle degree $n+1$. Therefore, by Remark 2.1.66, there exists $w \in (R_{sm})_{n+1-j}$ such that $\pi_C(uw)$ is nonzero. Using Equation (6.17)

$$\delta(\pi_C(uw)) = \pi_A(x_{m+1}uw) = \pi_A(x_{m+1}u)\pi_A(w) = \delta(\pi_C(u))\pi_A(w) = 0. \tag{6.18}$$

We fix a facet $\{a_1, \dots, a_{n+1}\}$ of D and consider the facet $\{a_1, \dots, a_{n+1}, m+1\}$ of $S(D)$. We set

$$z_C = \prod_{r=1}^{n+1} x_{a_r} \in R_{sm}, \qquad z_A = x_{m+1} \prod_{r=1}^{n+1} x_{a_r} \in R.$$

Using the discussion after the proof of Corollary 6.3.5, $\pi_A(z_A)$ is a nonzero element of A_{n+2}. By the same discussion, $\pi_C(z_C)$ is nonzero, hence is a basis of the 1-dimensional k-vector space C_{n+1}. Therefore, there exists a nonzero element $\lambda \in k$ such that

$$\pi_C(uw) = \lambda \pi_C(z_C).$$

Consequently,

$$\delta(\pi_C(uw)) = \delta(\lambda \pi_C(z_C)) = \lambda \delta(\pi_C(z_C)) = \lambda \pi_A(x_{m+1} z_C) = \lambda \pi_A(z_A) \neq 0,$$

which contradicts Equation (6.18). $\square$

The following corollary follows immediately from Proposition 6.8.6.

Corollary 6.8.7 *Assume $u \in R$. Then the following are equivalent*

i) We have $\pi_B(u) = 0$.

ii) We have $\pi_A(x_{m+1}u) = 0$.

Proposition 6.8.8 *Assume $S(D)$ is generically anisotropic over the field k_1. Then the element $\pi_C(\omega)$ is a Weak Lefschetz element for the Artinian k-algebra C. As a consequence, the graded k-algebra $k[D]$ has the Weak Lefschetz Property.*

Proof We denote by p the integral value of the rational number $n/2$.

Using that $\pi_B(\omega) = \pi_B(x_{m+1})$ and Proposition 6.8.4, it is enough to prove that the element $\pi_B(x_{m+1})$ is a Weak Lefschetz element for the Artinian k-algebra B. By Proposition 6.8.5, B is a graded Artinian Gorenstein k-algebra with socle degree $n+1$. Using [51, Remark 2.4], it is enough to prove that the multiplication by $\pi_B(x_{m+1})$ map from B_p to B_{p+1} is injective.

Assume $u \in R_p$ has the property

$$\pi_B(x_{m+1}u) = 0.$$

Using Corollary 6.8.7, we have $\pi_A(x_{m+1}^2 u) = 0$, hence $\pi_A(x_{m+1}^2 u^2) = 0$. Using that the socle degree of A is $n+2$ and the assumption that $S(D)$ is generically anisotropic over the field k_1, we get $\pi_A(x_{m+1}u) = 0$. Corollary 6.8.7 implies that $\pi_B(u) = 0$. □

Proposition 6.8.9 *Assume the dimension of D is even and $S(D)$ is generically anisotropic over the field k_1. Then the element $\pi_C(\omega)$ is a Strong Lefschetz element for the Artinian k-algebra C. As a consequence, the graded k-algebra $k[D]$ has the Strong Lefschetz Property.*

Proof We set $z = \pi_B(\omega)$. Using Proposition 6.8.4, it is enough to prove that the element z is a Strong Lefschetz element for the Artinian k-algebra B. By Proposition 6.8.5, B is a graded Artinian Gorenstein k-algebra with socle degree $n+1$. Hence, to finish the proof it is enough to prove that, for all i with $0 \leq 2i \leq n+1$, the multiplication by z^{n+1-2i} map $B_i \to B_{n+1-i}$ is injective.

Assume $0 \leq 2i \leq n+1$ and $u \in R_i$ has the property

$$z^{n+1-2i}\pi_B(u) = 0.$$

Using that $z = \pi_B(x_{m+1})$ and Corollary 6.8.7, we get $\pi_A(x_{m+1}^{n+2-2i}u) = 0$, which implies that $\pi_A(x_{m+1}^{n+2-2i}u^2) = 0$.

Since n is even, the socle degree of A is $n+2$ and we assumed that $S(D)$ is generically anisotropic over the field k_1, we get $\pi_A(x_{m+1}^{(n+2)/2-i}u) = 0$. Corollary 6.8.7 implies that $\pi_B(x_{m+1}^{(n+2)/2-i-1}u) = 0$, therefore

$$z^{(n+2)/2-i-1}\,\pi_B(u) = 0. \tag{6.19}$$

By the proof of Proposition 6.8.8, z is a Weak Lefschetz element for B. Hence, the multiplication by z map $B_{n/2} \to B_{n/2+1}$ is injective. Using Proposition 6.12.7, we have that, for all t with $0 \leq t \leq n/2$, the multiplication by z map $B_t \to B_{t+1}$ is injective. Consequently, Equation (6.19) implies that $\pi_B(u) = 0$. □

6.8.2 Proof of Theorem 6.8.2

We start the proof of Theorem 6.8.2. We denote by k_{mod} the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n+1,\ 1 \leq j \leq m].$$

For $1 \leq i \leq n+1$, we set

$$f_{mod,i} = \sum_{j=1}^{m} a_{i,j} x_j.$$

We use the notation

$$A_{mod} = k_{mod}[D]/(f_{mod,1}, \ldots, f_{mod,n+1}).$$

Hence, A_{mod} is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 6.1.1.

We first prove Part i). We denote by k_1 the field $\mathbb{Z}/(2)$ with two elements. By Theorem 6.2.3, $S(D)$ is generically anisotropic over the field k_1. Hence, by Theorem 6.8.1, $k[D]$ has the Weak Lefschetz Property. It is well-known ([61]) that this implies that McMullen's g-conjecture is true for D.

We now prove Part ii). Assume k_1 is an infinite field of characteristic 2. By Theorem 6.2.3, $S(D)$ is generically anisotropic over the field k_1. Hence, by Theorem 6.8.1, $k[D]$ has the Weak Lefschetz Property. Using Proposition 6.12.3, $k_1[D]$ also has the Weak Lefschetz Property.

We now prove Part iii). Assume k_1 is an infinite field of characteristic 2. By Theorem 6.2.3, $S(D)$ is generically anisotropic over the field k_1. If the dimension n of D is even, Proposition 6.8.9 implies that $k[D]$ has the Strong Lefschetz Property. Using Proposition 6.12.5, $k_1[D]$ also has the Strong Lefschetz Property.

Assume now that n is odd. By Part ii), $k_1[D]$ has the Weak Lefschetz Property. Using Proposition 6.12.4, the Artinian k_{mod}-algebra A_{mod} has the Weak Lefschetz Property. By Theorem 6.2.3, D is generically anisotropic over the field k_1. Hence, for all i with $0 \leq i \leq (n+1)/2$ and all $0 \neq u \in (A_{mod})_i$, we have $u^2 \neq 0$. Proposition 6.12.8 now implies that A_{mod} has the Strong Lefschetz Property. Since k_1 is infinite, Proposition 6.12.6 implies that $k_1[D]$ has the Strong Lefschetz Property. This finishes the proof of Theorem 6.8.2.

Corollary 6.8.10 *Assume D is a simplicial sphere of dimension $n \geq 1$, and k_1 is a (finite or infinite) field of characteristic 2. Then the k_{mod}-algebra A_{mod} has the Strong Lefschetz Property.*

Proof The field k_{mod} is infinite and has characteristic 2. Hence, Theorem 6.8.2 implies that the k_{mod}-algebra $k_{mod}[D]$ has the Strong Lefschetz Property. Using Proposition 6.12.6, the result follows. □

6.9 Anisotropy in dimension 1

In this section k_1 denotes a field of arbitrary characteristic.

We assume that $m \geq 3$ and D is the boundary of the m-gon with vertex set $\{1, \dots, m\}$. We also assume the following: the vertex 1 is connected to the vertices m and 2, the vertex i is connected to the vertices $i-1$ and $i+1$ when $2 \leq i \leq m-1$, and the vertex m is connected to the vertices $m-1$ and 1.

We denote by S_{sp} the polynomial ring

$$S_{sp} = k_1[a_{i,j} : 1 \leq i \leq 2,\ 1 \leq j \leq m]$$

and by k the field of fractions of S_{sp}. We define the polynomial ring $R = k[x_1, \dots, x_m]$. We denote by $I_D \subset R$ the Stanley-Reisner ideal of D, and we set $k[D] = R/I_D$. For $1 \leq i \leq 2$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j,$$

and we define $A = k[D]/(f_1, f_2)$. Therefore, A is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 6.1.1.

If $m \geq 4$ we have

$$I_D = (x_1 x_j : 3 \leq j \leq m-1) + (x_i x_j : 2 \leq i \leq m-2, i+2 \leq j \leq m),$$

while if $m = 3$, we have $I_D = (x_1 x_2 x_3)$.

We fix the ordered facet $(1, 2)$ of D. Following Equations (6.1) and (6.2), we set

$$\Psi = \Psi_{(1,2)} : A_2 \to k \qquad \text{and} \qquad \rho = \rho_{(1,2)} : A_1 \times A_1 \to k.$$

Proposition 6.9.1 *For $1 \leq i \leq m-1$, we have*

$$(\Psi \circ \pi)(x_i x_{i+1}) = \frac{1}{[i, i+1]}.$$

Moreover, we have

$$(\Psi \circ \pi)(x_1 x_m) = \frac{1}{[m, 1]}, \quad (\Psi \circ \pi)(x_1^2) = -\frac{[m, 2]}{[m, 1][1, 2]}, \quad (\Psi \circ \pi)(x_m^2) = -\frac{[m-1, 1]}{[m-1, m][m, 1]}$$

and

$$(\Psi \circ \pi)(x_i^2) = -\frac{[i-1, i+1]}{[i-1, i][i, i+1]}$$

for $2 \leq i \leq m-1$.

Proof Combining Proposition 6.3.4 with Proposition 6.3.10, the result is immediate. □

Proposition 6.9.2 *We have* $\dim_k A_1 = m-2$. *If* $\mathcal{S}$ *is any subset of* $\{1,\dots,m\}$ *of cardinality* $m-2$, *then the set* $\{\pi(x_i) : i \in \mathcal{S}\}$ *is a* k*-basis of* A_1.

Proof We denote by M the $2 \times m$ matrix with (i,j)-entry equal to $a_{i,j}$. The determinant of every 2×2 submatrix of M is a nonzero element of the field k. Since $A = k[D]/(f_1, f_2)$ and I_D is a homogeneous ideal with generators of degrees ≥ 2, the result follows. □

For $1 \leq i \leq m-2$, we set $e_i = \pi(x_{i+1})$. By Proposition 6.9.2, the finite sequence

$$e_1, e_2, \dots, e_{m-2}$$

is an ordered basis of A_1. We denote by N_m the $(m-2) \times (m-2)$ symmetric matrix, with (i,j)-entry equal to $\rho(e_i, e_j)$. We call N_m the matrix of ρ with respect to the ordered basis.

Remark 6.9.3 Assume $a, b, c, d \in \{1, \dots, m\}$. Then, we have the well-known Plücker identity

$$[a,b][c,d] - [a,c][b,d] + [a,d][b,c] = 0,$$

see [43, Theorem 5.2.3].

Proposition 6.9.4 *We have*

$$\det(N_m) = (-1)^m \frac{[1,m]}{\prod_{i=1}^{m-1}[i,i+1]}.$$

Proof We use induction on $m \geq 3$. For $m = 3$, it follows from Proposition 6.9.1.

Assume $m = 4$. Then we have to compute the determinant of the matrix

$$N_4 = \begin{pmatrix} -\frac{[1,3]}{[1,2][2,3]} & \frac{1}{[2,3]} \\ \frac{1}{[2,3]} & -\frac{[2,4]}{[2,3][3,4]} \end{pmatrix}.$$

It is equal to

$$\frac{[1,3]}{[1,2][2,3]} \frac{[2,4]}{[2,3][3,4]} - \Big(\frac{1}{[2,3]}\Big)^2 = \frac{[1,3][2,4] - [1,2][3,4]}{[1,2]|[2,3]^2[3,4]}.$$

Using the Plücker identity $[1,2][3,4] - [1,3][2,4] + [1,4][2,3] = 0$ (see Remark 6.9.3) the result follows.

Assume now $m \geq 5$ and that the result holds for all previous values up to $m-1$. Using Proposition 6.9.1, we have that N_m has the block format

$$N_m = \begin{pmatrix} N_{m-1} & v^t \\ v & -\frac{[m-2,m]}{[m-2,m-1][m-1,m]} \end{pmatrix},$$

where v is the $(m-3) \times 1$ matrix

$$v = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{1}{[m-2,m-1]} \end{pmatrix}.$$

Morever, a similar block format statement holds for the matrix N_{m-1}.

Developing the determinant of N_m using the last column, and using the inductive hypothesis together with the Plücker identity (see Remark 6.9.3)

$$[1,m-2][m-1,m] - [1,m-1][m-2,m] + [1,m][m-2,m-1] = 0,$$

we get

$$\det(N_m) = -\frac{[m-2,m]}{[m-2,m-1][m-1,m]}\det(N_{m-1}) - (\frac{1}{[m-2,m-1]})^2 \det(N_{m-2})$$

$$= (-1)^{m-1}\frac{-[m-2,m][1,m-1]}{[m-2,m-1][m-1,m]\prod_{i=1}^{m-2}[i,i+1]} - (-1)^{m-2}\frac{[1,m-2]}{[m-2,m-1]^2\prod_{i=1}^{m-3}[i,i+1]}$$

$$= (-1)^{m-2}(\frac{[1,m-1][m-2,m]}{[m-2,m-1]\prod_{i=1}^{m-1}[i,i+1]} - \frac{[1,m-2]}{[m-2,m-1]\prod_{i=1}^{m-2}[i,i+1]})$$

$$= (-1)^{m-2}(\frac{[1,m-1][m-2,m] - [1,m-2][m-1,m]}{[m-2,m-1]\prod_{i=1}^{m-1}[i,i+1]})$$

$$= (-1)^{m}(\frac{[1,m]}{\prod_{i=1}^{m-1}[i,i+1]}).$$

□

Remark 6.9.5 Assume $1 \leq c < d \leq m$. It is well-known that $[c,d]$ is an irreducible element of S_{sp}. Hence, there exists an induced valuation map

$$\mathrm{val}_{[c,d]} : k \setminus \{0\} \to \mathbb{Z}.$$

Recall that if $f, g \in S_{sp} \setminus \{0\}$, then $\mathrm{val}_{[c,d]}(f)$ is the largest integer s such that $[c,d]^s$ divides f in S_{sp}, and

$$\mathrm{val}_{[c,d]}(f/g) = \mathrm{val}_{[c,d]}(f) - \mathrm{val}_{[c,d]}(g).$$

Remark 6.9.6 Assume that h is any ordered basis of A_1. We denote by H the matrix of ρ with respect to h. By the basic theory of bilinear forms, there exists an invertible matrix P with entries in k such that

$$H = P^t N_m P.$$

As a consequence, using Proposition 6.9.4,

$$\det(H) = (-1)^m (\det P)^2 \frac{[1,m]}{\prod_{i=1}^{m-1} [i,i+1]}.$$

Taking into account Remark 6.9.5, we conclude that we can recover the simplicial complex D from (the determinant of) ρ, since the set of facets of D is exactly the set of ordered pairs (c,d) such that $1 \leq c < d \leq m$ and $\mathrm{val}_{[c,d]}(\det(H))$ is an odd integer. An interesting question is whether this holds for all simplicial spheres of odd dimension. In other words, assume E is a simplicial sphere of odd dimension ≥ 3 and e is an ordered facet of E. Is it possible to recover E from (the determinant of) the symmetric bilinear form ρ_e?

The proof of the following theorem will be given in Subsection 6.9.1.

Theorem 6.9.7 *The simplicial sphere D is generically anisotropic over k_1.*

6.9.1 Proof of Theorem 6.9.7

We keep using the notations of Section 6.9. Using Remark 6.3.7, to prove Theorem 6.9.7 it is enough to prove that the symmetric bilinear form $\rho : A_1 \times A_1 \to k$ is anisotropic.

We define a second basis of A_1, by using the Gram-Schmidt orthogonalization. We set $\tilde{e}_1 = e_1$, and we inductively define

$$\tilde{e}_i = e_i + \frac{[1,i]}{[1,i+1]} \tilde{e}_{i-1},$$

for $2 \leq i \leq m-2$.

Proposition 6.9.8 *For all $1 \leq i \leq m-2$, we have*

$$\tilde{e}_i = \sum_{t=2}^{i+1} \frac{[1,t]}{[1,i+1]} \pi(x_t). \tag{6.20}$$

Proof We prove Equation (6.20) using induction on i. For $i = 1$ it is true by the definition of $\tilde{e}_1$. Assume $1 \le i \le m-3$ and that Equation (6.20) is true for the value i. We have

$$\begin{aligned}\tilde{e}_{i+1} = e_{i+1} + \frac{[1,i+1]}{[1,i+2]}\tilde{e}_i &= \pi(x_{i+2}) + \frac{[1,i+1]}{[1,i+2]}\Big(\sum_{t=2}^{i+1}\frac{[1,t]}{[1,i+1]}\pi(x_t)\Big) \\ &= \pi(x_{i+2}) + \sum_{t=2}^{i+1}\frac{[1,t]}{[1,i+2]}\pi(x_t) = \sum_{t=2}^{i+2}\frac{[1,t]}{[1,i+2]}\pi(x_t).\end{aligned}$$

□

Proposition 6.9.9 *For all $1 \le i \le m-2$, we have*

$$\rho(\tilde{e}_i,\tilde{e}_i) = -\frac{[1,i+2]}{[1,i+1][i+1,i+2]}.$$

Moreover, if $1 \le j \le m-2$ and $j \neq i$, we have

$$\rho(\tilde{e}_i,\tilde{e}_j) = 0.$$

Proof Assume $1 \le i \le m-2$. We set $u = \sum_{t=2}^{i+1}[1,t]\pi(x_t)$. By Proposition 6.10.1,

$$\sum_{t=2}^{m}[1,t]\pi(x_t) = 0.$$

Hence, if $1 \le r \le i$, taking into account that $\pi(x_r x_t) = 0$ when $r+2 \le t \le m$, we get

$$u\;\pi(x_r) = 0. \tag{6.21}$$

Assume $1 \le j < i$. Using Proposition 6.9.8, Equation (6.21) implies that $\rho(\tilde{e}_i,\tilde{e}_j)$ is equal to zero. Moreover, Equation (6.21) also implies that

$$\begin{aligned}\Psi(u^2) &= \Psi\big(u\;\big(\sum_{t=2}^{i+1}[1,t]\pi(x_t)\big)\big) = \Psi\big([1,i+1]u\pi(x_{i+1})\big) \\ &= \Psi\big([1,i+1][1,i]\pi(x_i x_{i+1}) + [1,i+1]^2\pi(x_{i+1}^2)\big) \\ &= [1,i+1]\Big(\frac{[1,i]}{[i,i+1]} - \frac{[1,i+1][i,i+2]}{[i,i+1][i+1,i+2]}\Big) \\ &= [1,i+1]\frac{[1,i][i+1,i+2]-[1,i+1][i,i+2]}{[i,i+1][i+1,i+2]} \\ &= -[1,i+1]\frac{[1,i+2][i,i+1]}{[i,i+1][i+1,i+2]} = -[1,i+1]\frac{[1,i+2]}{[i+1,i+2]},\end{aligned}$$

where we used Proposition 6.9.1 and Remark 6.9.3. Since $\tilde{e}_i = u/[1, i+1]$, this proves the formula for $\rho(\tilde{e}_i, \tilde{e}_i)$. $\square$

We set

$$L = \prod_{s=2}^{m-1} [1, s][s, s+1]$$

and, for $1 \leq t \leq m-2$, we define $L_t = L/([1, t+1][t+1, t+2]) \in S_{sp}$.

Using Proposition 6.9.9, it is clear that to prove Theorem 6.9.7 it is enough to prove that, if $d_t \in k$ satisfy

$$\sum_{t=1}^{m-2} d_t^2 \frac{[1, t+2]}{[1, t+1][t+1, t+2]} = 0,$$

we then have $d_t = 0$ for all $1 \leq t \leq m-2$. By clearing denominators, it is enough to prove the following proposition.

Proposition 6.9.10 *Assume $d_1, \ldots, d_{m-2} \in S_{sp}$ satisfy*

$$\sum_{t=1}^{m-2} d_t^2 [1, t+2] L_t = 0. \tag{6.22}$$

Then, we have $d_t = 0$, for all $1 \leq t \leq m-2$.

Proof We give to the polynomial ring S_{sp} the lexicographic ordering $>$ with

$$a_{1,1} > a_{1,2} > \cdots > a_{1,m} > a_{2,1} > a_{2,2} > \cdots > a_{2,m}.$$

Using Corollary 6.11.3, it is enough to prove that if the integers i, j have the properties $1 \leq i < j \leq m-2$, $d_i \neq 0$ and $d_j \neq 0$, we then have

$$\mathrm{in}_>(d_i^2 [1, i+2] L_i) \neq \mathrm{in}_>(d_j^2 [1, j+2] L_j). \tag{6.23}$$

Using the definitions of L_i and L_j and Remark 6.11.1, we have

$$\mathrm{in}_>(d_i^2 [1, i+2] L_i) = (\mathrm{in}_>(d_i))^2 \cdot (a_{1,1})^{m-2} \cdot \prod_{s=1}^{i} a_{1,s} \cdot \prod_{s=i+2}^{m-1} a_{1,s} \cdot Q_i,$$

and

$$\mathrm{in}_>(d_j^2 [1, j+2] L_j) = (\mathrm{in}_>(d_j))^2 \cdot (a_{1,1})^{m-2} \cdot \prod_{s=1}^{j} a_{1,s} \cdot \prod_{s=j+2}^{m-1} a_{1,s} \cdot Q_j,$$

where Q_i and Q_j are monomials in the variables $a_{2,1}, \ldots, a_{2,m}$. Therefore, the variable $a_{1,j+1}$ appears in the monomial $\mathrm{in}_>(d_i^2 [1, i+2] L_i)$ with an odd power, and in the monomial $\mathrm{in}_>(d_j^2 [1, j+2] L_j)$ with an even power. Hence, Inequality (6.23) is true. $\square$

Example 6.9.11 Assume $m = 6$. Equation (6.22) becomes

$$d_1^2[1,3]L_1 + d_2^2[1,4]L_2 + d_3^2[1,5]L_3 + d_4^2[1,6]L_4 = 0,$$

where

$$L_1 = \frac{L}{[1,2][2,3]}, \quad L_2 = \frac{L}{[1,3][3,4]}, \quad L_3 = \frac{L}{[1,4][4,5]}, \quad L_4 = \frac{L}{[1,5][5,6]}$$

and

$$L = [1,2][1,3][1,4][1,5][2,3][3,4][4,5][5,6].$$

6.10 A general proposition related to elimination

In this section we describe a specific form of Gauss elimination that is used in the present chapter.

Assume R is a commutative ring with unit, and n, m, Z are positive integers with $n < m \leq Z$. Assume that, for $1 \leq j \leq m$, x_j is an elements of R and that for $1 \leq i \leq n$ and $1 \leq j \leq Z$, $a_{i,j}$ is an element of R. We denote by M the $n \times Z$ matrix with (i,j)-entry equal to $a_{i,j}$.

Assume $b_1, \dots, b_n$ are n integers, with $1 \leq b_i \leq Z$, for all i. We denote by $[b_1, \dots, b_n]$ the determinant of the $n \times n$ matrix, whose i-th column is equal to the b_i-th column of M. For $1 \leq i \leq n$, we set

$$f_i = \sum_{t=1}^{m} a_{i,t} x_t,$$

and we denote by $I = (f_1, \dots, f_n)$ the ideal of R generated by the f_i.

Proposition 6.10.1 *Assume $c_1, \dots, c_{n-1}$ are integers, with $1 \leq c_i \leq Z$ for all i. We have*

$$\sum_{t=1}^{m} [c_1, c_2, \dots, c_{n-1}, t] x_t \in I.$$

Proof Denote by N the $n \times (n-1)$ matrix, whose i-th column is the c_i-th column of M. For $1 \leq j \leq n$, we denote by N_j the determinant of the submatrix of N obtained by deleting the j-th row of N. We claim that

$$\sum_{t=1}^{m} [c_1, c_2, \dots, c_{n-1}, t] x_t = \sum_{j=1}^{n} (-1)^{j+n} N_j f_j.$$

Indeed, on the left hand side, the coefficient of x_t is $[c_1, c_2, \dots, c_{n-1}, t]$, while on the right hand side the coefficient is equal to $\sum_{j=1}^{n} (-1)^{j+n} N_j a_{j,t}$. The two quantities are equal, by developing the determinant $[c_1, c_2, \dots, c_{n-1}, t]$ using the last column. $\square$

6.11 A general technique for proving a polynomial is nonzero

Here we discuss a well-known general method which is useful for proving that certain sums of products of bracket polynomials are nonzero. We use it in the proof of Proposition 6.9.10.

Assume $m \geq 1$, k is a field and $R = k[x_i : 1 \leq i \leq m]$. We denote by $\mathcal{A}_R$ the set of all monomials of R. In other words,

$$\mathcal{A}_R = \{x_1^{a_1} \cdots x_n^{a_m} : a_i \geq 0 \text{ for all } i\}.$$

Following [27, Section 15.2], a *monomial order* on R is a total order $>$ on $\mathcal{A}_R$ such that if $u_1, u_2, w \in \mathcal{A}_R$ with $u_1 > u_2$ and $w \neq 1$, we then have $wu_1 > wu_2 > u_2$. In addition, by the same reference, the *lexicographic order* on R with $x_1 > x_2 > \cdots > x_m$ is the total order $>$ on $\mathcal{A}_R$ defined by $x_1^{a_1} \cdots x_m^{a_m} > x_1^{b_1} \cdots x_m^{b_m}$ if and only if $a_i > b_i$ for the first index i such that $a_i \neq b_i$. It is a monomial order on R.

Assume now $>$ is a monomomial order on R. It induces the *initial monomial map*, $\mathrm{in}_> : R \setminus \{0\} \to \mathcal{A}_R$, defined as follows. Assume $f \in R \setminus \{0\}$. Then, there exist (unique) $s > 0$, $g_1, \dots, g_s \in \mathcal{A}_R$ and $\lambda_1, \dots, \lambda_s \in k \setminus \{0\}$ such that

$$f = \sum_{i=1}^{s} \lambda_i g_i \qquad \text{and} \qquad g_1 > g_2 > g_3 > \cdots > g_s.$$

By definition, $\mathrm{in}_>(f) = g_1$.

Remark 6.11.1 By the definition of a monomial ordering, we have

$$\mathrm{in}_>(f_1 f_2) = (\mathrm{in}_>(f_1))(\mathrm{in}_>(f_2))$$

for all $f_1, f_2 \in R \setminus \{0\}$.

Moreover, by the definition of a monomial ordering we have the following proposition.

Proposition 6.11.2 *Assume $f_1, f_2, \dots, f_t \in R \setminus \{0\}$. Assume there exists a with $1 \leq a \leq t$ such that*

$$\mathrm{in}_>(f_a) \;>\; \mathrm{in}_>(f_b)$$

for all b with $1 \leq b \leq t$ and $b \neq a$. Then $\sum_{i=1}^t f_i \neq 0$ and $\mathrm{in}_>(\sum_{i=1}^t f_i) = \mathrm{in}_>(f_a)$.

Corollary 6.11.3 *Assume $f_1, f_2, \dots, f_t \in R \setminus \{0\}$ satisfy*

$$\mathrm{in}_>(f_i) \;\neq\; \mathrm{in}_>(f_j)$$

for all $1 \leq i, j \leq t$ with $i \neq j$. Then $\sum_{i=1}^t f_i \neq 0$.

Proof Since $\mathrm{in}_>(f_i) \neq \mathrm{in}_>(f_j)$ for all $1 \leq i, j \leq t$ with $i \neq j$, there exists a unique integer a such that $1 \leq a \leq t$ and $\mathrm{in}_>(f_a) > \mathrm{in}_>(f_b)$ for all b with $1 \leq b \leq t$ and $b \neq a$. The result follows by Proposition 6.11.2. $\square$

6.12 Lefschetz properties and base change

The statements in the present section, with the likely exception of Proposition 6.12.8, are well-known. We include them for completeness.

Proposition 6.12.1 *Assume E is an infinite field, $f \in E[x_1, \ldots, x_m]$ is a nonzero polynomial and, for $1 \leq i \leq m$, Z_i is an infinite subset of E. Then, there exists a point p in the set $Z_1 \times Z_2 \times \cdots \times Z_m$ such that $f(p) \neq 0$.*

Proof We use induction on m. If $m = 1$, it is well-known that the polynomial f has a finite number of roots in the field E, and the result follows.

Assume $m \geq 2$ and that the result is true for $m - 1$. There exist $s > 0$ and, for $0 \leq i \leq s$, a polynomial $g_i \in E[x_1, \ldots, x_{m-1}]$, such that

$$f = \sum_{i=0}^{s} g_i x_m^i.$$

Since f is nonzero, there exists c, with $0 \leq c \leq s$, such that g_c is nonzero. Hence, by the inductive hypothesis, there exists an element $(a_1, \ldots, a_{m-1}) \in Z_1 \times Z_2 \times \cdots \times Z_{m-1}$ such that $g_c(a_1, \ldots, a_{m-1}) \neq 0$. Consequently, the polynomial $h \in E[x_m]$, with

$$h = \sum_{i=0}^{s} g_i(a_1, \ldots, a_{m-1}) x_m^i,$$

is nonzero. By the case $m = 1$, there exists $a_m \in Z_m$ such that $h(a_m) \neq 0$. This implies that $f(a_1, \ldots, a_m) \neq 0$. $\square$

Corollary 6.12.2 *Assume that E is an infinite field, $m \geq 1$ is a positive integer and $f \in E[x_1, \ldots, x_m]$ is a nonzero polynomial. Assume k_1 is an infinite subfield of E. Then*

i) There exists a point $p \in k_1^m$ such that $f(p) \neq 0$.

ii) Endow the set E^m with the Zariski topology. Then the subset k_1^m of E^m is Zariski dense.

Proof Part i) follows from Proposition 6.12.1, by setting $Z_i = k_1$ for all $1 \leq i \leq m$.

Part ii) follows immediately from Part i). $\square$

Assume that $k_1 \subset E$ is a field extension. We consider the polynomial ring $k_1[x_1, \ldots, x_m]$, where the degree of the variable x_i is equal to 1, for all $1 \leq i \leq m$. Assume $I \subset k_1[x_1, \ldots, x_m]$ is a homogeneous ideal such that the quotient algebra $G = k_1[x_1, \ldots, x_m]/I$ is Cohen-Macaulay. We denote by d the Krull dimension of G.

We set $G_E = E[x_1, \ldots, x_m]/(I)$, where (I) is the ideal of $E[x_1, \ldots, x_m]$ generated by I. By [17, Theorem 2.1.10], G_E is also Cohen-Macaulay. Since, for all $i \geq 0$, $(G_E)_i = G_i \otimes_{k_1} E$, the Hilbert function of G as a graded k_1-algebra is equal to the Hilbert function of G_E as a graded E-algebra. Consequently, the Krull dimension of G_E is d.

Proposition 6.12.3 *Assume that the field k_1 is infinite. Then the following are equivalent:*

i) The graded k_1-algebra G has the Weak Lefschetz Property.

ii) The graded E-algebra G_E has the Weak Lefschetz Property.

Proof We first assume that G has the Weak Lefschetz Property. Then, there exist elements $g_1, \ldots, g_d, \omega \in G_1$ such that $g_1, \ldots, g_d$ is a regular sequence for G and ω is a Weak Lefschetz element for $G/(g_1, \ldots, g_d)$. Clearly, $g_1, \ldots, g_d$ is a regular sequence also for G_E and ω is a Weak Lefschetz element also for $G_E/(g_1, \ldots, g_d)$. Hence, the k-algebra G_E has the Weak Lefschetz Property.

For the opposite direction, we assume that G_E has the Weak Lefschetz Property. By taking the coefficients of f_i and ω, we can identify the set

$$\mathcal{S} = \{(g_1, \ldots, g_d, \omega) : g_i \in (G_E)_1, \omega \in (G_E)_1\}$$

with the affine space $(G_E)_1^{d+1}$. We denote by U the subset of $\mathcal{S}$ consisting of the element $(g_1, \ldots, g_d, \omega)$ such that $g_1, \ldots, g_d$ is a regular sequence for G_E and ω is a Weak Lefschetz element for $G_E/(g_1, \ldots, g_d)$.

By the assumption that G_E has the Weak Lefschetz Property, the set U is nonempty. Hence, by [5, Lemma 4.1], U is a nonempty Zariski open subset of $\mathcal{S}$. Using that the field k_1 is infinite, Corollary 6.12.2 implies that G_1^{d+1} is Zariski dense in $(G_E)_1^{d+1}$, hence $G_1^{d+1} \cap U \neq \varnothing$. Let $(g_1, \ldots, g_d, \omega) \in G_1^{d+1} \cap U$. Then $g_1, \ldots, g_d$ is a regular sequence for G and ω is a Weak Lefschetz element for $G/(g_1, \ldots, g_d)$. Hence, the k_1-algebra G has the Weak Lefschetz Property. $\square$

We denote by k the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq d,\ 1 \leq j \leq m].$$

We set $G_k = k[x_1, \ldots, x_m]/(I)$, where (I) is the ideal of $k[x_1, \ldots, x_m]$ generated by I. For $1 \le i \le d$, we set $f_i = \sum_{j=1}^{m} a_{i,j}x_j$. Hence, the Artinian k-algebra $G_k/(f_1, \ldots, f_d)$ is the generic Artinian reduction of the k_1-algebra G in the sense of Definition 6.1.1.

Proposition 6.12.4 *Assume $d \ge 1$. Then the following are equivalent:*

i) The Artinian k-algebra $G_k/(f_1, \ldots, f_d)$ has the Weak Lefschetz Property.

ii) If E is an infinite field containing k_1 as a subfield, then the E-algebra G_E has the Weak Lefschetz Property.

iii) There exists an infinite field F which contains k_1 as a subfield such that the F-algebra G_F has the Weak Lefschetz Property.

Proof We first prove that i) implies ii). Since the k-algebra $G_k/(f_1, \ldots, f_d)$ has the Weak Lefschetz Property, it follows that the k-algebra G_k has the Weak Lefschetz Property. Assume E is an infinite field containing k_1 as a subfield. We denote by E_1 the field of fractions of the polynomial ring

$$E[a_{i,j} : 1 \le i \le d,\ 1 \le j \le m].$$

Since k is a subfield of E_1, Proposition 6.12.3 implies that the E_1-algebra G_{E_1} has the Weak Lefschetz Property. Since E is an infinite subfield of E_1, the same proposition gives that the E-algebra G_E has the Weak Lefschetz Property.

We now prove that ii) implies iii). It is clear.

We now prove that iii) implies i). We denote by E the field of fractions of the polynomial ring in one variable $k[T]$ over k. We denote by F_1 the field of fractions of the polynomial ring

$$F[T, a_{i,j} : 1 \le i \le d,\ 1 \le j \le m].$$

Since we have that F is a subfield of F_1, both fields are infinite, and, by the assumption, the F-algebra G_F has the Weak Lefschetz Property, it follows, by Proposition 6.12.3, that the F_1-algebra G_{F_1} has the Weak Lefschetz Property. Since E is an infinite subfield of F_1, the same proposition implies that the E-algebra G_E has the Weak Lefschetz Property.

We denote by I^e the ideal of $E[x_1, \ldots, x_m]$ generated by I. We denote by V the m-dimensional E-vector subspace of $E[x_1, \ldots, x_m]$ consisting of homogeneous degree one polynomials. For $1 \le i \le d,\ 1 \le j \le m$, we define the infinite subset

$$Z_{i,j} = \{a_{i,j} + T^r \ : \ r \ge 1\}$$

of E. We denote by Z the Cartesian product, for $1 \le i \le d,\ 1 \le j \le m$, of the sets $Z_{i,j}$. By Corollary 6.12.2, Z is Zariski dense in the affine space E^{dm}.

Since G_E has the Weak Lefschetz Property, it follows that the set U consisting of all $(g_1, \ldots, g_d) \in V^d$ such that $g_1, \ldots, g_d$ is a regular sequence for G_E and $G_E/(g_1, \ldots, g_d)$

has the Weak Lefschetz Property, is a nonempty Zariski open subset of the affine space V^d.

We identify V^d with E^{dm}, by considering the coefficients of the homogeneous degree one polynomials. Since Z is Zariski dense in E^{dm}, the intersection of Z with U is nonempty. Hence, for $1 \leq i \leq d$, $1 \leq j \leq m$, there exists a positive integer $r_{i,j}$ such that, if we set

$$g_i = \sum_{j=1}^{m} (a_{i,j} + T^{r_{i,j}}) x_j,$$

we have that $g_1, \ldots, g_d$ is a regular sequence for G_E and $G_E/(g_1, \ldots, g_d)$ has the Weak Lefschetz Property.

There exists a unique k_1-linear automorphism of the polynomial ring $k_1[a_{i,j}, T]$ that sends T to T and $a_{i,j}$ to $a_{i,j} + T^{r_{i,j}}$, for all i, j. The automorphism extends first to a field automorphism of E and then to a degree preserving automorphism ϕ of the polynomial ring $E[x_1, \ldots, x_m]$ that sends x_i to x_i, for all $1 \leq i \leq m$, and is the identity when restricted to k_1. Hence, $\phi(I^e) = I^e$ and $\phi(f_i) = g_i$, for all $1 \leq i \leq d$, which imply that

$$\phi(I^e + (f_1, \ldots, f_d)) = I^e + (g_1, \ldots, g_d).$$

Consequently, $f_1, \ldots, f_d$ is a regular sequence for G_E and $G_E/(f_1, \ldots, f_d)$ has the Weak Lefschetz Property, since the same properties hold for $g_1, \ldots, g_d$ and $G_E/(g_1, \ldots, g_d)$.

Finally, since k is an infinite subfield of E, Proposition 6.12.3 implies that the k-algebra $G_k/(f_1, \ldots, f_d)$ has the Weak Lefschetz Property. □

We now discuss the corresponding statements of the last two propositions for the Strong Lefschetz Property.

Proposition 6.12.5 *Assume that the field k_1 is infinite and G is Gorenstein. Then the following are equivalent:*

i) The graded k_1-algebra G has the Strong Lefschetz Property.

ii) The graded k-algebra G_E has the Strong Lefschetz Property.

Proof With the obvious modifications, the arguments in the proof of Proposition 6.12.3 also work here. □

Proposition 6.12.6 *Assume that G is Gorenstein and $d \geq 1$. Then the following are equivalent:*

i) The Artinian k-algebra $G_k/(f_1, \ldots, f_d)$ has the Strong Lefschetz Property.

ii) If E is an infinite field containing k_1 as a subfield, then the E-algebra G_E has the Strong Lefschetz Property.

iii) There exists an infinite field F which contains k_1 as a subfield such that the F-algebra G_F has the Strong Lefschetz Property.

Proof With the obvious modifications, the arguments in the proof of Proposition 6.12.4 also work here. ☐

We also need the following two propositions. The first is a special case of Part (b) of [48, Proposition 2.1].

Proposition 6.12.7 *Assume k_1 is a field and A is a standard graded Artinian Gorenstein k_1-algebra of socle degree d. Assume s in an integer with $1 \leq s < d$. Assume $\omega \in A_1$ has the property that the multiplication by ω map $A_s \to A_{s+1}$ is injective. Then, for all t with $0 \leq t \leq s$, we have that the multiplication by ω map $A_t \to A_{t+1}$ is injective.*

Proof Assume $0 \leq t \leq s$ and $0 \neq u \in A_t$. By Remark 2.1.66, there exists $z \in A_{s-t}$ such that $uz \neq 0$. Hence $\omega(uz) \neq 0$, which implies that $\omega u \neq 0$. ☐

Proposition 6.12.8 *Assume k_1 is a field and A is a standard graded Artinian Gorenstein k_1-algebra of even socle degree d. We assume that A has the Weak Lefschetz Property and that, for all i with $0 \leq i \leq d/2$ and all $0 \neq u \in A_i$, we have $u^2 \neq 0$. Then A has the Strong Lefschetz Property.*

Proof We fix $\omega \in A_1$ such that, for all $t \geq 0$, the multiplication by ω map $A_t \to A_{t+1}$ has maximal rank. Since A is Gorenstein of even socle degree d, it follows that the multiplication by ω map form $A_{d/2-1} \to A_{d/2}$ is injective. By the definition of the Strong Lefschetz Property, and using that A is Gorenstein, to prove the proposition it is enough to show that for all i, with $0 \leq i < d/2$, the multiplication by ω^{d-2i} map from A_i to A_{d-i} is injective.

Assume $0 \leq i < d/2$ and that $z \in A_i$ has the property

$$\omega^{d-2i} z = 0.$$

As a consequence, $\omega^{d-2i} z^2 = 0$. Using the assumption, it follows that $\omega^{d/2-i} z = 0$. Proposition 6.12.7 implies that $z = 0$. ☐

6.13 A conjecture about differentiation

Assume k_1 is a field of characteristic 2. Assume $n \geq 1$ is an integer and D is a simplicial sphere of dimension n with vertex set $\{1, 2, \dots, m\}$. We denote by k the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n+1,\ 1 \leq j \leq m].$$

We define the polynomial ring $R = k[x_1, \dots, x_m]$, where we put degree 1 for all variables x_i. We denote by $I_D \subset R$ the Stanley-Reisner ideal of D. Moreover, we set $k[D] = R/I_D$. For $i = 1, \dots, n+1$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j,$$

and we define $A = k[D]/(f_1, \dots, f_{n+1})$. Hence, A is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 6.1.1. We denote by $\pi : R \to A$ the natural projection k-algebra homomorphism, and by $\Psi : A_{n+1} \to k$ the vector space isomorphism defined in Remark 6.3.6.

For a finite sequence $\delta = (\delta_1, \dots, \delta_{n+1})$ such that $1 \le \delta_i \le m$ for all $1 \le i \le n+1$ we set

$$x_\delta = \prod_{i=1}^{n+1} x_{\delta_i} \in R.$$

Assume $\sigma = (\sigma_1, \dots, \sigma_{n+1})$ and $\tau = (\tau_1, \dots, \tau_{n+1})$ are two finite sequences such that $1 \le \sigma_i, \tau_i \le m$, for all $1 \le i \le n+1$. We denote by $\partial_\sigma^{mod} : k \to k$ the $(n+1)$-th order differential operator which is differentiation with respect to the variables in the set

$$\{a_{i,\sigma_i} \ : \ 1 \le i \le n+1\}.$$

The following conjecture, if true, will generalise Theorem 6.7.6 and Propositions 6.4.1 and 6.4.7.

Conjecture 6.13.1 1) Assume the monomial $x_\sigma x_\tau$ is not the square of a monomial in R. We then have

$$(\partial_\sigma^{mod} \circ \Psi \circ \pi)(x_\tau) = 0.$$

2) Assume $x_\sigma x_\tau$ is the square of a monomial in R. Assume $\delta = (\delta_1, \dots, \delta_{n+1})$ is a finite sequence such that $1 \le \delta_i \le m$, for all $1 \le i \le n+1$, and $x_\sigma x_\tau = (x_\delta)^2$. We then have

$$(\partial_\sigma^{mod} \circ \Psi \circ \pi)(x_\tau) = \big((\Psi \circ \pi)(x_\delta)\big)^2.$$

Remark 6.13.2 We note that Conjecture 6.13.1 implies the following interesting equality

$$(\partial_\sigma^{mod} \circ \Psi \circ \pi)(x_\tau) = (\partial_\tau^{mod} \circ \Psi \circ \pi)(x_\sigma).$$

Remark 6.13.3 Assume $i \ge 1$ and $t \ge 2$ are two integers such that $ti \le n+1$. Assume $0 \ne u \in A_i$. Theorem 6.2.3 implies that if t is a power of 2 then $u^t \ne 0$. Is this also true for all values of t?

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Index