ORDINAL REAL NUMBERS 1. The ordinal characteristic.

Constantine E. Kyritsis*

** Associate Prof. of the University of Ioannina.* ckiritsi@uoi.gr C_kyrisis@yahoo.com,, *Dept. Accounting-Finance Psathaki Preveza 48100*

Abstract

In this paper are introduced the ordinal integers ,the ordinal rational numbers ,the ordinal real numbers ,the ordinal p-adic numbers ,the ordinal complex numbers and the ordinal quaternion numbers .It is also introduced the ordinal characteristic of linearly ordered fields. The final result of this series of papers shall be that the three different techniques of surreal numbers, of transfinite real numbers ,of ordinal real numbers give by inductive limit or union the same class of numbers known already as the class No and that would deserve the name the "infinitary totally ordered Newton-Leibniz realm of numbers ".

Key words : Hessenberg natural operations, principal ordinal,linearly ordered commutative fields,Archimedean complete fields, abstract additive valuation,formal power series fields,transfinite real numbers

Subject Classification of AMS *03,04,08,13,46*

§0 **Introduction.** This is the third paper of a series of five papers that have as goal the definition of topological complete linearly ordered fields (continuous numbers) that include the real numbers and are obtained from the ordinal numbers in a method analogous to the way that Cauchy derived the real numbers from the natural numbers. We may call them linearly ordered Newton-Leibniz numbers. The author initiated and completed this research in the island of Samos in Greece during 1990-1992 .

The present papers define the topological and algebraic structure of the ordinal real numbers and does not refer at all to their stochastic interpretation. Nevertheless as in practical applications of pre-emptive Goal Programming in Operations Research and operating systems of computers, the non-Archimedean or lexicographic order is usually called **pre-emptive prioritization order** .the ordinal real numbers could as well be called (for the sake of practical applications) **Linearly ordered pre-emptive real numbers .**

In a communication (1992) that the author had with N.L. Alling and his group of researchers on analysis on surreal numbers, suggested the term ordinal real numbers instead of surreal numbers. Some years later and before the present work appears for publication, it appeared in the bibliography conferences about*real ordinal numbers* .

In these last three papers is studied a special Hierarchy of transcendental over the real numbers, linearly ordered fields that are characterized by the property that they are fundamentally (Cauchy) complete. It shall turn out that they are isomorphic to the transfinite real numbers (see [Glayzal A. (1937)]).The author was not familiar with the 5 pages paper of [Glayzal A. (1937)] ,and his original term was “transfinite real numbers”. When one year later (1991) he discovered the paper by A. Glayzal ,he changed the term to the next closest :”Ordinal Real Numbers” .One more year

later he proved that the transfinite real numbers ,the surreal numbers and the ordinal real numbers were three different techniques leading to isomorphic field of numbers. He then suggested (1992) to researchers of surreal numbers, like N.L.Alling to use the more casual term “ordinal or transfinite real numbers “ for the surreal numbers.
In the present work it is introduced a new, better, classifying and more natural technique in order to define them. This technique I call *"free operations-fundamental completion"*. It
is actually the same ideas that lead to the process of construction of the real numbers from the natural numbers through fundamental (Cauchy) sequenses. In the modern conceptual context of the theory of categories this may demand at least three adjunctions (see[MacLane S 1971]).It is developed their elementary theory which belongs to algebra. Their definition uses the Hessenberg operations of the ordinal numbers .It may be considered as making use of an infinite dimensional K-theory which is mainly not created yet. In this first paper it is also introduced the ordinal characteristic of any linearly ordered field .It is a principal ordinal number, that is of type $\omega^{\omega^{\lambda}}$. These numbers ,as defined with the present technique of the "free operations-fundamental completion " and prior to the proof that the resulting linearly ordered fields are isomorphic to the transfinite real numbers (as in [Glayzal A. (1937)]) ,we shall call Ordinal real numbers. The relevancy with the surreal numbers and the non-standard (hyper) real numbers ,shall be studied in a later paper. In detail, the next Hierarchies are defined:

1) The **Ordinal natural numbers**, denoted by $\mathbf{N}_{\alpha}$ .2) The **Ordinal integral numbers**, denoted by $\mathbf{Z}_{\alpha}$ 3)The **Ordinal rational numbers**, denoted by $\mathbf{Q}_{\alpha}$ 4)The **Ordinal p-adic numbers**, denoted by $\mathbf{Q}_{\alpha,p}$ 5)The **Ordinal real numbers**, denoted by $\mathbf{R}_{\alpha}$ 6)The **Ordinal comlpex numbers**, denoted by $\mathbf{C}_{\alpha}$7)The **Ordinal quaternion numbers**, denoted by $\mathbf{H}_{\alpha}$, of characteristic α . The fields $\mathbf{Q}_{\alpha,p}$, $\mathbf{R}_{\alpha}$, $\mathbf{C}_{\alpha}$, $\mathbf{H}_{\alpha}$ are fundamentaly (Cauchy)complete topological fields.

The field $\mathbf{R}_{\alpha}$ is also the unique maximal field of characteristic α (that is, it is Hilbert complete) , and the unique fundamentally (Cauchy) complete field of characteristic α. It is also a real closed field , according to the theory of Artin-Schreier . These will be proved in the next paper on ordinal real numbers.

As it is known there are three more techniques and Hierarchies of transcendental over the real numbers, linearly ordered fields. Namely (in the historical order): The transfinite real numbers (see [Glayzal A. 1937]), and the surreal numbers (see [Conway J.H (1976)]).

In this series of papers, it is proved (among other results) that all the previous three different techniques and Hierarchies give by inductive limit, or by union, the same class of numbers (already known as the class **No**).

§ **1. The ordinal characteristic of linearly ordered fields**.

Corrective Remark (2022) : The ordinal characteristic is essentially a measurement of the size of a linearly ordered commutative field with a semiring of Ordinal natural numbers (Hessenberg natural commutative operations in the ordinal numbers, as developed in the two previous papers-sections). We embed systems of ordinal natural numbers in a linearly ordered field, so that not “gaps” exist. There is always a minimal such system the natural numbers themselves. The definition of the ordinal characteristic of such ordinal natural numbers is always the supremum of the ordinals which are contained in it, and it is a principal ordinal numbers as we have described in the previous paper-section. Then we embed with monomorphisms and with 1-1 functions , such semi-rings of ordinal natural numbers in a linearly ordered commutative field so that the 0 and 1 of the ordinal real numbers goes to the 0, 1 of the linearly ordered field and there are no “gaps”, in other words the image is the minimal such possible set in the linearly ordered field. All such possible monomorphic with no gaps in a linearly ordered field, which is a set, give a set of corresponding ordinal characteristics of such semi-rings of ordinal natural numbers which is upper bounded, because of the cardinal and corresponding ordinal of the set and linearly ordered field. Thus as such ordinal are a subset of a well ordered set of ordinals it holds the supremum property, and there is such a supremum ordinal. Since also such a maximal embedding is also a semi-ring of ordinal natural numbers, this supremum is also a principal ordinal number which exist and its unique, it measures the size of the linearly ordered field and we call the it its ordinal characteristic. Having defined the ordinal characteristic as above the next definitions follow with minor corrections.

Definition 0. We remind the reader that a linearly (totally) *ordered, double abelian semigroup (semiring)* M is a set with two operations denoted by +,., such that with each one of them it is an abelian semigroup. Furthermore the distribution law holds for multiplication over addition. A linear ordering is supposed defined in M that satisfies the following compatibility conditions with the two operations 1) if x>y, x'>y' x,x',y,y' ∈M then x+x'>y+y' and xx'+yy'>xy'+yx' (The symbol < is used for ≤ and not equal) if M is also a monoid relative to the two operations, and zero is absorbent unit for M, M is called *ordered double abelian monoid*. (semiring)

(e.g.The set of natural numbers ,denoted by N).

In the next we shall consider linearly (totally) ordered fields.(For a definition see [Lang S.] ch xi §1 pp 391).

Also in the next we shall use ordinal numbers. (For a reference to standard symbolism and definitions see [Kuratowski K.-Mostowski A. 1968] ch vii, [Cohn P.M. 1965] pp 1-36)

In the following paragraphs we will not avoid the use of larger totalities than the sets of the Zermelo-Frankel set theory, namely classes.

We may suppose that we work in the Zermelo-Frankel set theory, augmented with axioms for classes also, as is presented for instance in bibliography [Cohn P.M. 1965] p.1-36 with axioms A1-A11. Wee denote by Ω_1 the class of the ordinal numbers. (The last capital letter of the Greek alphabet with subscript 1). The axioms A1-A11 allow for larger entities than sets, to define algebraic fields or integral domains or semi-groups. Hence we will also study classes that have two algebraic operations (Their Cartesian square treated as classes of sets of the form {{x, y}, {x}}, that is of ordered pairs) that satisfy the axioms of an algebraic field and have a subclass called the class of positive elements, with properties 1. 2., that they define a compatible ordering in the field (again as a class of ordered pairs) such classes that are ordered fields we will call again ordered fields and if we want to discriminate them from set-fields, especially when they are classes that are not sets, we will write for them that they are c-fields similarly we write c-integral domains or c-semigroups. We must not confuse the term "c-field" with the term "class-field" of the ordinary set-fields of "class-fields theory" (see [Van der Waerden B.L 1970], [Artin E.-Tate J. 1967]). A subset (or subclass) denoted by $X \subseteq F$ of a linearly ordered field F, is said to be <u>cofinal</u> with F, if for every $a \in F$ there is a $b \in X$ with $a \leq b$.

Definition 1. We say that an ordinal number α' is contained in a linearly ordered field (or integral domain or double abelian monoid) denoted by F, if there is an ordinal α, α'<α and $\alpha = \omega^{\omega^x}$, (omega in the power of omega in the power of x) where x is an ordinal ,and a subset A of $F^+ \cup \{O\}$ and a function h: $W(\alpha) \to A$ which is an order isomorphism (similarity) of W(α) and A and such that h(0)=0 and if β is an ordinal number with β<α then h(s(β))=h(β)+1 in the field operations and furthermore the set A is closed to sum and product in the field (integral domain or double monoid) operations and isomorphic by h to the W(α) relative to the Hessenberg natural operations , furthermore the closure in the order topology of the field of the set A (range of h) is the **minimal** such set with the previous properties (so we ensure that there are no gaps at the limits that are included).

Remark 2. If an ordinal number α' is contained in the field F, then also the sequent of α', S(α') is contained in F. This holds since the sequent of α' is again in W(α) where α as in the definition above.

Remark 3. If the ordinal number α is contained in the field F, then obviously every ordinal number less than α, is also contained in the field F. *In the next, we will suppose (for simplification of symbolism) that if the ordinal α is contained in F, the set α is a subset of F, and also α is the element h(α) of the field F.* We fix a mapping h for each ordinal that is contained in F .So we can talk about the set of ordinals contained in F as if it is a subset of F .The set of ordinal numbers that are contained in a linearly ordered set-field, is obviously a non-empty set. (Because as F is linearly ordered, charF = ∞ hence for every natural number n, we have that it is a (finite) ordinal contained in the field F).

But even more by the remarks 2, 3, we have that the set of ordinals contained in a linearly ordered field ,which of course by the non-Neuman definition of ordinals is itself an ordinal , is either of the form W(x) or $W(x) \cup \{x\} = W(S(x))$ for some ordinal number x (in other words either it shall be a limit ordinal or it shall have a immediately previous ordinal). The last case is directly excluded (by remark 2) hence it is of the form W(x) = x, that is this set is itself a limit ordinal number. In case the linearly ordered field F, is a c-field then all the ordinals contained in F is again a set which is limit ordinal number, or the class Ω_1 of all ordinal numbers.

Definition 4. Let a linearly ordered set-field (or integral domain or double abelian monoid) demoted by F. Let α be the set of ordinals contained in F (which is itself a limit ordinal number). We say that the field (or integral domain or double abelian monoid) F is of characteristic α and we shall write charF = α.

If F is a c-field we include the case of characteristic Ω_1 and we write charF = Ω_1 if all ordinals contained in F is the class Ω_1 and also it is a cofinal subclass with F.

Remark . In the case of a set-field F with α = charF, we do not need to suppose that the subset of elements of F corresponding to the ordinal in α by the definition 1 (it always exists ,by making use of the definition by transfinite induction and its version that uses only a set of functions sufficient for an inductive rule), see appendix A), is cofinal with F, as this is a consequence of the definition. For, if there is an element $X_0 \in F$ with $\beta < X_0$ for every ordinal number β with $\beta \leq a$, then the set $\alpha \cup \{X_0\}$ can be extended , with the field operations ,to its closure in the natural Hessenberg operations (a semiring) (see [Kyritsis C. Alt]) and it becomes similar to an initial segment of a principal ordinal number Thus α+1 is an ordinal contained in F, contradiction with the definition of a .

By the previous definitions we realize that every linearly ordered set-field has characteristic which is a limit ordinal number.

The fact that the linearly ordered field F has characteristic ω (the least infinite ordinal) is equivalent with the statement that the field F is Archimedean.

In the followings when we will work on a linearly ordered field denoted by F of ordinal characteristic α, α=charF (or Ω_1= charF) we will supposed that is fixed an embedding of the ordinal numbers of the initial segment w(α) in the set F (or of Ω_1 in F).

If the characteristic is ω, the embedding is obviously unique as it can be proved by finite induction.

Remark.5 Let a linearly ordered field denoted by F .Obviously there is an extension which is a real field .Let us denote by R(F) the real closure of F .(For results of the theory of Artin-Schreir on real and real closed fields see e.g.[Lang S. 1984] ch xi .or [Artin E.-Shreier O. 1927]) Since R(F) can be obtained by adjunction of the square roots of the positive elements of F and Zorn's Lemma on algebraic extensions see[Lang S. 1984] ch i proposition 2.10 theorem 2.11 pp 397), it is direct that the characteristic of the real closure R(F) is the same with that of F.

For the definitions of the terms <u>infinite</u>, <u>finite</u>, <u>infinitesimal</u> elements in an extension of such fields, see e.g.[Lang S] ch xi paragraph 1 pp 391, the definitions can be given relative to extensions of any linearly ordered field to an other linearly ordered field ,and not only extensions of the real numbers.

§2 The ordinal natural numbers N . The ordinal- integers Z .

Let w(α) a principal initial segment of ordinal numbers. Let us denote by + and . the Hessenberg's natural sum and product in w(α). They satisfy properties 0.1.2.3.4.5.6. after lemma 1 in §1 in [Kyritsis C.1991 Alter]

Definition 6. *The set w(α)=α where* $\alpha = \omega^{\omega^x}$ *for some ordinal x, is an abelian double monoid relative to sum and product and furthermore it satisfies the cancellation low (see [* Kyritsis C. 1991 Alter*] lemma 1).This set I call the (double abelian) monoid of* <u>*ordinal natural numbers*</u> *of characteristic a and I denote it by* N_α. *Thus* $N_\alpha = \alpha$.

Remark 7. It is obvious that the (double abelian, well ordered) monoid N_α, is <u>the</u> minimal such monoid of characteristic α and the embedding of the ordinal numbers of W(α) in it is unique . Furthermore it can be proved by transfinite induction that it is a unique factorization monoid (called simply factorial monoid also).

The additive cancellation low in α has as a consequence that α is monomorphicaly embedded in its Grothendieck group denoted by k(α) (see [Lang S. 1984] Ch.1 §9 p. 44). Furthermore the Grochendieck group $k(N_\alpha)$ can be ordered by defining the set of positive elements $k(\alpha)^+ = \{v/v = (x,y)$ with $x,y \in w(\alpha)$ and $x > y\}$. We remind the reader that if we denote by $F_{ab}(\alpha)$ the free abelian group generated by α, and by ((x+y)-x-y) the normal subgroup of $F_{ab}(\alpha)$ generated by elements of the form (x+y)-x-y, then $k(\alpha) \cong \dfrac{F_{ab}(\alpha)}{((x+y)-x-y)}$

By (x,y) we denote the equivalence class that is defined in $F_{ab}(\alpha)$ in the process of taking the quotient group $F_{ab}(\alpha)/((x+y)-x-y)$ by the representative x+(-y).

The first part of property 6. (lemma 1 in [Kyritsis C.1991 Alter]) guarantees that this ordering in k(α) restricted on α coincides with the usual ordering of ordinal numbers.

Definition 8. *The ordered Grothendieck group k(α) of an initial segment of ordinals relative to natural sum, we call transfinite cyclic group of exponent α and we denote it by Γ_α. (by [Kuratowski K. Mostowski A. 1968] ch vii §7 pp 252-253 exercises 1.2.3.the ordinal α has to be of the type ω^x. If the ordinal α is principal then I denote it also by Z_α).*

Every element of the group Z_α is represented as a difference x-y with $x,y \in w(\alpha)$. Then we define multiplication in Z_α by the rule

(*) (x-y).(x'-y')=(x.x'+y.y')-(xy'+x'y)

where sum and product are the natural sum and product in w(α). This makes Z_α a commutative ring with unit (the element 1).

If (x-y)(x'-y') = 0 and both (x-y), (x'-y') are not zero, we get by property 6 in lemma 1 in [Kyritsis C. 1991Alter] that xx'+yy' ≠ xy'+yx' or (x-y)(x'-y') ≠ 0, contradiction. Then one of

(x-y), (x'-y') is zero that is the ring Z_α has no divisors of zero and it is an integral domain. Remembering that $Z_\alpha^+ = \{v | v \in Z\alpha$ and $v = (x,+y)$ with $x,y \in w(\alpha)$ $x > y\}$, by property 6 lemma 1 in [Kyritsis C. 1991 Alter], we get that the sum and product of elements of Z_α^+ are again elements of Z_α^+. From all these we get:

Lemma 9. *The ring Z_α is a linearly ordered integral domain of characteristic the principal ordinal α (see § 1 Def.1).The set Z_α^+ is a linearly ordered double abelian monoid and $Z_\alpha^+ \neq N_\alpha$*

Definition 10 . *The integral domain Z_α I call ordinal integers of characteric α .*

The integral domain Z_α of characteristic α has minimality relative to its property of being an integral domain of characteristic α, in the following sense: Every integral domain of characteristic α contains a monomorphic image of Z_α.

Theorem 11 (Minimality).

Every integral domain Z_α is minimal integral domain of characteristic α. That is every integral domain of characteristic α, contains a monomorphic image of Z_α.

Proof. Put R_α an integral domain of characteristic α, where α is a principal ordinal number ($\alpha = \omega^{\omega^\lambda}$).

Then the initial segment w(α) is contained in R_α (more precisely an order preserving image of w(α)). The principal initial segment is closed to the integral domain operations and by theorem 13,14 of [Kyritsis C. 1991 Alter], they coincide with the natural sum and product of Hessenberg. Then, applying the construction of this paragraph for the integral-domain Z_α, we remain inside the integral-domain R_α, that is $Z_\alpha \subseteq R_\alpha$. This proves the minimality.

Remark 12. The ordinal integers are semigroup-rings of quotient monoids of semigroups that are used to define as semigroup-rings the hierarchy of integral domains of the transfinite integers (see [Gleyzal A. 1937] pp 586).I use the term hierarchy not only as a well ordered sequence but also as a net (thus partially ordered). The transfinite real numbers are thus an hierarchy.

The transfinite integers over the order-type λ symbolised by Z(λ), is the semigroup-ring (also module Z-algebra and integral domain) of the linearly ordered monoid $\sum_\lambda N$, where $\sum_\lambda N$ is the coproduct, or direct sum denoted also by $\coprod_\lambda N$, of a family of isomorphic copies of N with set of indices the order-type λ. Thus Z(λ) =Z[$\coprod_\lambda N$]. Thus any ring of polynomials of a linearly ordered set of variables with integer coefficients is an integral domain of transfinite integers and conversely. It can be proved with the axiom of choice and transfinite induction , as in the case of finite set of variables, that Z(λ) is a unique factorization domain . On the other hand the Cantor normal form in the Hessenberg operations of the ordinal numbers (see lemma 6 in [Kyritsis C. 1991 Alter]) gives that any element x of Z_α is of the form $x = \omega^{x_1} y_1 + \ldots + \omega^{x_n} y_n, y_i \in Z, i = 1, \ldots, n, n \in N$ x_i are ordinals with $x_1 > \ldots > x_n$. The ordinal powers of ω in Z_α is an abelian well ordered monoid (see e.g. [Neumann B.H. 1949] §2 pp 204-205) of ordinal characteristic $\beta = \omega^x$, if $\alpha = \omega^{\omega^x}$. Let us denote it by M_β. Actually $M_\beta = \beta$. Let us denote by $\log_\omega(\alpha)$, or simply by λ_α the order type of the Archimedean equivalent classes of M_β. Then we get by the Cantor normal form that $Z_\alpha = Z[M_\beta]$ (The semigroup ring of M_β). The monoid M_β can be obtained as quotient monoid of the free abelian multiplicative monoid of λ_α variables, which is the monoid $\coprod_{\lambda_\alpha} N$.

But Z [$\coprod_{\lambda_\alpha} N$]=Z($\lambda_\alpha$), which was the assertion to be proved.

Remark 13 The equation $Z_{\omega^{\omega^x}} = Z[M_{\omega^x}]$ gives an alternative, simpler definition of the ordinal integers without the use of the Hessenberg multiplication, since the ordinal powers of ω coincide n the abelian Hessenberg operations and the usual ordinal operations (see [Kyritsis C.1991 Alter] Remark 7.5)) and without the use of the Grothentick group .The monoid M_x is defined as the initial segment $W(\omega^x)$ (or simply as the ordinal ω^x) in the Hessenberg addition .

§3 The definition of the fields Q_α, R_α, C_α, H_α.

In this paragraph, I shall introduce the hierarchies of fields of ordinal rational ,real, complex ,quaternion numbers. These hierarchies give the unification of the other three techniques and hierarchies, namely of the transfinite real numbers, of the surreal numbers. Furthermore we introduced the hierarchies of transfinite complex and transfinite quaternion numbers.

Definition 14. *The localization (field of quotients) of the integral domain Z_α, I will denote by Q_α and I will call* <u>*ordinal rational numbers*</u> *(of characteristic α) (see [Lang S. 1984] ChII §3).*

Remark. Since we have that cancellation low holds, we do not have to use the Malcev-Neuman theorem (see [Cohn P.M. 1965] Ch VII §3. Theorem 3.8). We define as set of positive element of Q_α the set $Q_\alpha^{+'} = \left\{ x \middle| x = \frac{m}{l} \text{ with } m,l \in Z_\alpha \text{ and } m.l \in Z_\alpha^+ \right\}$. It is elementary in algebra that if the integral domain is linearly ordered then also its field of quotients (localization) with the previous definition for its set of positive elements, is a linearly ordered field with the restriction of its ordering on the integral domain to coincide with the ordering of the integral domain. Obviously the ordinals of the initial segment of w(α) are contained in Z_α and also in Q_α. By a direct argument, holds also that the characteristic of Q_α is a: <u>Char $Q_\alpha = \alpha$</u>.

Remark From the construction of Q_α we infer easily that $\aleph(Q_\alpha) = \aleph(\alpha)$ and if $\alpha < \beta$ where α, β are two principal ordinals then $Q_\alpha \subseteq Q_\beta$. The converse obviously holds.

Lemma 15. Every element x of the field Q_α is of the form $x = \frac{\omega^{\alpha_1}.a_1 + \ldots + \omega^{\alpha_n}.a_n}{\omega^{\beta_1}.b_1 + \ldots + \omega^{\beta_m}.a_m}$ where $\alpha_i, \beta_j \in w(\alpha)$ and $\alpha_1 > \alpha_2 > \ldots > \alpha_n \geq 0$, $\beta_1 > \beta_2 > \ldots > \beta_m \geq 0$ and a_i, b_j for i = 1,...,n, j = 1,...,m are finite integers.

Proof. Direct from the definition of localization and lemma 6 in [Kyritsis C. 1991Alter].

Theorem 17. <u>(Minimality)</u>

The field Q_α is a <u>*minimal*</u> *field of characteristic α, in the sense that every field of characteristic α, contains the field Q_α (more precisely an order preserving monomorphic image of Q_α).*

Remark. This property is already obvious for the field of rational numbers, that in the statement of Theorem 17 is denoted by Q_ω.

Proof. Let a field of characteristic α, that we denote by F_α. Then the principal initial segment w(α) of ordinals is contained in F_α and the field-inherited operations coincide with the natural sum and product of Hessenberg (see theorem 14 in [Kyritsis C. 1991 Alter]). Then constructing first the integral domain Z_α and afterwards its localization Q_α we always remain in the field F_α.

Thus $Q_\alpha \subseteq F_\alpha$ (or more precisely $h(Q_\alpha) \subseteq F_\alpha$ where h is a order-preserving monomorphism of Q_α in to F_α) Q.E.D.

Definition 18. *The (strong) Cauchy completion of the topological field Q_α we denote by R_α and I call* <u>*ordinal real numbers*</u> *of characteristic α.*

Remark. The process of extensions ,beginning with a principal initial ordinal $\alpha = N_\alpha$ which is <u>the</u> minimal double, abelian monoid of characteristic α, and ending with the field R_α which is the maximal field of characteristic α ,we call K-fundamental densification .

Lemma 19. *The characteristic of the (strong) Cauchy completion of a linearly ordered field F ,is the same with that of the field F.*

Proof. If the characteristic ofthe field is α, let us denote it by F_α, and its completion by $\hat{F}_\alpha$. Obviously the characteristic of $\hat{F}_\alpha$ is not less than α.

Suppose that there is an ordinal β with α < β which is contained in $\hat{F}_\alpha$ (see Definition 1). Then there is a Cauchy net $\{x_i|i \in I\}$ of elements of F_α that converges to $\beta\text{:}\lim_{i\in I} x_i = \beta$. Let $\varepsilon \in F_\alpha$ $0<\varepsilon<1$, then there is $i_0 \in I$ such that for every $i \in I$ $i \geq i_0$

$x_i \in (b-\varepsilon, b+\varepsilon)$. But this gives an element of F_α greater than α, hence than every element of F_α, which is a contradiction. Thus Char $R_\alpha = \alpha$. Q.E.D.

Corollary 20. *The characteristic of R_α is α .*

From the definition of R_α we infer that $\aleph(R_\alpha) \leq 2^{\aleph(\alpha)}$ and that $\alpha<\beta \Leftrightarrow R_\alpha \subsetneq R_\beta$ for two principal ordinals denoted by α, β.

Remark.21 We denote by R(λ) the transfinite real numbers of order-base λ . It holds by definition that $R(\lambda)=R((LR^\lambda))$, where LR^λ is the lexicographic product of a family of isomorphic copies of the real numbers R ,with set of indices the order-type λ.

Remark . It is said that a field F has formal power series representation, if there is a formal power series ring R((G)) and a ideal I of it such that F has a monomorphic image in R((G))/I .From the universal embedding property of the hierarchy of transfinite real numbers we get that every linearly ordered field has formal power series representation .Thus:

Corollary 22. *The fields of ordinal real numbers R, have formal power series representation ,with real coefficients.*

Definition 23 *The field $C_\alpha = R_\alpha[i]$ I call* <u>*ordinal complex numbers*</u> *of characteristic α.*

Definition 24. *The field $C(\lambda)= R(\lambda)[i]$ we call* <u>transfinite complex numbers</u> *of base-order λ. Actually it is the field $C(\lambda)=C((LR^\lambda))$.*

Definition 25. *The quaternion extension field of the field R_α (or of C_α) by the units i, j, k with $i^2 = j^2 = k^2 = ijk = -1$, I call the* <u>*ordinal quaternion numbers of characteristic α*</u> *and I denote them by H_α. They are non-commutative fields (following the terminology e.g. of A.Weil in [Weil A. 1967]) that are transcendental extension of the non-commutative field H of quaternion numbers.*

Definition 26 . *The formal power series fields $H(\lambda)=H((LR^\lambda))$ we call* <u>*transfinite quaternion*</u> *numbers of base-order λ.*

For a proof that $H((LR^\lambda))$ is a (non commutative) field see [Neumann B.H.1949] part I.

§4 The ordinal p-adic numbers $Q_{\alpha,p}$.

As it is known if F is a linearly ordered field ,and K a linearly ordered subfield of the real numbers and $F|K$ is an extension respecting the ordering, then this extension defines the order-valuation (see [N.L.Alling 1987] ch 6 § 6.00 pp 207) .Actually every extension of any two linearly ordered fields F, K, $K\subseteq F$, respecting the ordering, defines a place, thus a valuation v. (I use the place and valuation as are defined e.g. by O.Zariski in [Zariski O.-. Samuel P.1958] vol ii ch vi §2, §8.and not as are defined by A.Weil in [Weil A. 1967] ch iii or by v.der Waerden in [Van der Waerden B.L. 1970] vol ii ch 18 .The definition of Zariski is equivalent with the definition of v.der Waerden only for the non Archimedean valuations of the latter).

The place-ring is the F_v ={x/x ∈ F and there are a, b ∈K with a<x<b }. The maximal ideal of the place (or valuation v) is the ideal of infinitesimals of K relative to F.

This valuation we call <u>extension - valuation</u> (and the corresponding place <u>extension - place</u>) It has as special case the order valuation .The rank of the extension- place (see [Zariski O.-. Samuel P.1958] vol.II §3 pp 9) we call the <u>rank of the extension</u> .If char(F)>char(K) then the extension is transcendental ,and has transcendental degree and basis ;the latter is to be found in the ideal of infinitesimals or in the set of infinite elements .

Definition 27 .

Let F a field of ordinal characteristic. Let R a subring of F that has F as its field of quotients. Let p a prime ideal of R, such that the triple (pR_p, R_p, F) where R_p is the localization of R at p, defines a place of F. Such a place (or valuation denoted by v_p) I call <u>p-adic</u> of the field F. In the valuation topology of the valuation v_p, that has a local base of zero the ideals of R) the field F is a topological field and the (strong) Cauchy completion I denote by F_p, it is a (topological field) and I call <u>p-adic extension field of F</u>.

Definition 28. *For F=Q_a and R=Z_a in the previous definition the field $Q_{\alpha,p}$ I call <u>ordinal p-adic numbers</u> of characteristic α.*

Final remark .Using inductive limit ,or union of the elements of the hierarchies of the previous ordinal and transfinite number systems, we get corresponding classes of numbers .The classes of ordinal natural, integer, rational, real, complex, quaternion numbers denoted respectively by Ω_1, (or On), Ω_1Z, Ω_1Q, Ω_1R, Ω_1C, Ω_1H.

And the classes of transfinite integer, rational, real, complex, quaternion numbers denoted respectively by:

CZ, CQ, CR, CC, CH.

Acknowledgments. I would like to thank professors W.A.J.Laxemburg and A.Kechris (Mathematics Department of the CALTECH) for the interest they showed and that they gave to me the opportunity to lecture about the ordinal real numbers in CALTECH. Also the professors H.Enderton and G.Moschovakis (Mathematics Department of the UCLA) for their interest and encouragement to continue this project.

Bibliography

[N. L. Alling 1987] Foundations of analysis over surreal number fields North-Holland Math.Studies V. 141 1987 .

[Artin E. Schreier O.1927] Algebraishe konstruktion reellerkorper, Abh. Math. Sem.Univ. Hamburg 5 (1927) pp 85-99 .

[Artin E. - Tate J.1967] Class Field Theory Benjamin 1967.

[Bourbaki N.1952] Elemente de Mathematique algebre, chapitre II Hermann Paris 1948, chapitre VI Hermann Paris 1952.

[Clliford A. H.1954] Note on Hahn's theorem on ordered abelian groups. Proc. of the Amer. Math. Soc. 5 (1954) pp 860-863.

[Cohen L.W. - Goffman C. 1949] Theory of transfinite Convergence.Transact. of the Amer. Math. Soc. 66 (1949) pp 65-74.

[Cohn P.M.1965] Universal Algebva Harper - Row 1965 .

[Conway J.H 1976]. On numbers and games Academic press 1976 .

[Cuesta Dutardi N.1954] Algebra Ordinal Rev. Acad. Cientis Madrid 4 (1954) pp 103-145 .

[Dugundji J.1966] Topology, Allyn and Bacon inc. 1966.

[Ehreshmann Ch 1956]. Categories et structure Dunod 1956

[Ehrlich P.1988] An alternative construction of Conway's Ordered Field No, Algebra

Universalis 25 (1988) pp 7-16
.

[Ehrlich P]. The Dedekind completion of No, submitted to Algebra Universalis.

[Endler O.1972] Valuation Theory, Springer 1972.

[Erdos P.-Gillman L.- Henrkiksen M.1955] An isomorphism theorem for real closed fields. Ann. of Math.(2) 61 (1955)pp 542-554.

[Frankel A.A. 1953] Abstract set Theory. North - Holland 1953.

[Fuchs L 1963]. Partially ordered algebraic systems. Pergamon Oxford 1963.

[Gillman L.-Jerison M.1960] Rings of continuous functions. Van Nostrand Princeton 1960.

[Gleyzal A. 1937] Transfinite real numbers. Proc. of the Nat. Acad.of scien. 23 (1937) pp 581-587.

[Gravett K.A.H. 1956] Ordered abelian groups. Quart. J. Math. Oxford 7 (1956) pp 57-63.

[Hahn H.1907] Uber die nichtarhimedishen Grossensysteme.S. Ber. Akad. Wiss. Wein. Math. Natur.Wkl Abt. IIa 116 (1907) pp 601-655.

[Hausner M.-Wendel J.G. 1952] Ordered Vector Spases Proc. of the Amer. Math. Soc.3 (1952) pp 977-982.

[Hessenberg G. 1906] Grundbegriffe der Mengenlehre (Abh. der Friesschen Schule, N.S. [1] Heft 4) Gottingen 220 1906).

[Hilbert D.1977] Grundlagen der Geometry Teubner Studienbucher 1977 .

[Hilbert D. -Ackermann W.1950] Principles of Mathematical Logic. Chelsea Pub. Comp. N.Y. 1950.

[Kaplansky I. 1942] Maximal fields with valuations Duke Math. J. 9 (1942) pp 303-321.

[Krull W. 1931] Allgemeine Bewertungs theorie. J.reine angew. Math. 176 (1931) pp 160-196.

[Kuratowski K 1966]. Topology v.I v.II Academic Press 1966.

[Kuratowski K. -Mostowski A.1968] Set Theory North - Holland 1968.

[Kyritsis C.E.1991] Alternative algebraic definitions of the Hessenberg operations in the ordinal numbers.

[Lang S 1984]. lgebra . Addison-Wesley P.C. 1984 .

[Laugwitz Detler 1983] Ω Calculus as a Generalization of Field Extension. An alternative approach to non-Standard analysis "Recent developments in non-standard analysis" Lecture Notes in Math 983 Springer 1983.

[MacLane S. 1939] The Universality of Formal Power Series fields.Bull. of the Amer. Math. Soc. 45 (1939) pp 880-890.

[MacLane S 1971]. Categories for the working mathematician Springer 1971

[Monna A.F.1970] Analyse non-Archimedienne Springer 1970.

[Munkress J.R. 1975] Topology. Prenctice Hall 1975.

[Nachbin L. 1976] Topology and Order. Robert E.Krieger P.C. N.Y. 1976.

[Neumann B.H 1949]. On ordered division rings. Transact. of the Amer. Math. Soc. 66 (1949) pp 202-252. .

[Robinson A. 1974] Non-Standard analysis. North - Holland 1974 (1966).

[Rudin W. 1960] Fourier analysis on groups. Interscience Pub 1960 .

[Shilling O.F.G. 1950] The theory of valuastions. Amer. Math. Soc. 1950.

[Schubert H.1972] Categories Springer 1972.

[Sirkoski R. 1948] On an ordered algebraic field. Warsow, Towarzytwo Nankowe Warzawskie 41 (1948) pp 69-96.

[Stone A.L. 1969] Non-Standard analysis in topological algebra in Applications of Model Theory to Algebra, Analysis and Probability N.Y. (1969) PP 285-300.

[Stroyan, K.D. and Luxenburg W.A.J. 1976] Introduction to the theory of Infinitecimals N.Y.1976.

[Lynn A.Steen- Seebach J.A. Jr. 1970] Counterexamples in Toplogy Springer 1970.

[Van der Waerden B.L.1970] Algebra V1 V2 Frederick Unger Pub. Co. N.Y. 1970.

[Weil A.1967] Basic Number Theory, Springer Verlag Berlin, Heidelberg N.Y. 1967.

[Zakon E.1955] Fractions of ordinal numbers Israel Institute of Tecnology Scient. Public. 6, 94-103 1955.

[Zariski O. -Samuel P.1958] Commutative Algebra V.I.II Springer 1958.

List of special symbols

ω : Small Greek letter omega, the first infinit number.

α, β : Small Greek letter alfa, an ordinal.

$\alpha = \omega^{\omega^{x}}$ Ordinal alpha α equal to omega in the power of omega in the power of x

Ω^0 : Capital Greek letter omega with the superscript zero.

F^a : Capital letter with superscript a. The of algebraic elements of a field F.

char F : The characteristic of a field denoted by F.

$\cong$: Equiralence relation of Commensurateness.

$\sim$: Equiralence relation of comparability.

tr.d.(x) : The transcendance degree initial of words tr.(anscendance) and d.(egree).

N(x) : Aleph of x, the cardinality of the set X. N: the fisrt capital letter of the Hebrew alfabet.

cf(X)=cf(Y) : The sets x and Y are cofinal.

W(α) : Initial segment of ordinal naumbers defined by the ordinal number a.

$\oplus, \circ$: Natural sum and product of G. Hessenberg plus and point in parenthesis.

Nα,Zα,Qα,Rα,: Double-lined capital letters with subscript small Greek letters

Cα,Hα namely transfinite positive integers, intergals, rationals reats, complex and quatenion numbers.

$Z\alpha_1{}^*\omega$: The dual lually compact abelian groups of the transfinite integers Za. The capital letter Z double-lined wiuth subscripts two Greek let-α (alpha) and ω (omega) and superscript a star

T_α : Transfinite circle groups: Capital letter T with subscript a small Greek letter.

${}^*X$, ${}^*R$ et.c : A non-standard enlergement structure capital letter X with left superscript a star.

ξNo : A sureal number field of characteristic ξ. A small Greek letter followed by the symbol No.

C,RC*R,No : The c-structures (classes) previous symbols following the capital
CN,CZ,CQ,. latin letter C
CC,CH

$\hat{X}$: Strong Canchy competition of a topological space capital letter with cap.

Σ : Capital Greek letter sigma symbol for summation.

$\mathring{D}_\alpha$: The open full-linary tree of leight a. Capital latin D with subscript a small Greek letter and in upper place a small zero.

The ordinal real numbers 1. The ordinal characteristic.

APPENTIX A.

A MORE EFFECTIVE FORM OF DEFINITION BY TRANSFINITE INDUCTION.

1.Given a set Z and an ordinal α, let Φ be a set of ξ-sequences with the properties:

a) If f belongs to Φ then f/W(ξ) belongs to Φ for every ξ <= domain of f.

b) For every $\xi<\alpha$ there is at least one f belonging to Φ with ξ=w(ξ)=domain(f) and values belonging to Z.

c)If f_ξ is an α-sequence of ξ-sequences of Φ such that whenever $\gamma<\xi_1$

, $\xi_2<\alpha$, $f_{\xi 1}$ /w(γ) = $f_{\xi 2}$ /w(γ) ;then the α-sequence c_α (ξ)=f_ξ (ξ), belongs to Φ also.

For each function h in Z^Φ, there is one and only one transfinite sequence f defined on $\xi<=\alpha$,

f in Φ and such that f(ξ)=h[f/w(ξ)] for every $\xi<=\alpha$.

The function h is called **a recursive rule for Φ.** The set Φ with the properties a). b), c), is called ,sufficient for recursive rules.

Proof: Not much different than the ordinary form of definition by transfinite induction.