A short and simple solution of the millennium problem about the Navier-Stokes equations and similarly for the Euler equations.

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Abstract In this paper is presented a very short solution to the 4th Millennium problem about the Navier-Stokes equations. The solution proves that there cannot be a blow up in finite or infinite time, and the local in time smooth solutions can be extended for all times, thus regularity. This happily is proved not only for the Navier-Stokes equations but also for the inviscid case of the Euler equations both for the periodic or non-periodic formulation and without external forcing (homogeneous case). The proof is based on an appropriate modified extension in the viscous case of the well-known Helmholtz-Kelvin-Stokes theorem of invariance of the circulation of velocity in the Euler inviscid flows. This is essentially a 1D line density of (rotatory) momentum conservation. We discover a similar 2D surface density of (rotatory) momentum conservation. These conservations are indispensable, besides to the ordinary momentum conservation, to prove that there cannot be a blow-up in finite time, of the point vorticities, thus regularity.

Keywords *Incompressible flows, regularity, blow-up, Navier-Stokes equations, Clay millennium problem*

Mathematical Subject Classification: 76A02

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1. Introduction

The Clay millennium problem about the Navier-Stokes equations is one of the 7 famous problem of mathematics that the Clay Mathematical Institute has set a high monetary award for its solution. It is considered a difficult problem as it has resisted solving it for almost a whole century. The Navier-Stokes equations are the equations that are considered to govern the flow of fluids, and had been formulated long ago in mathematical physics before it was known that matter consists from atoms. So actually they formulate the old infinite divisible material fluids. Although it is known that under its assumptions of the millennium problem the Navier-Stokes equations have a unique smooth and local in time solution, it was not known if this solution can be extended smoothly and globally for all times, which would be called the **regularity of the**

Navier-Stokes equations in 3 dimensions. The corresponding case of regularity in 2 dimensions has long ago been proved to hold but the 3-dimensions had resisted proving it. Of course the natural outcome would be that regularity holds also in 3-Dimensions. Many people felt that this difficulty hides our lack of understanding of the laws of 3-dimensional flow of the incompressible fluids.

In this paper is presented a very short solution to the Clay Millennium problem about the Navier-Stokes equations. The solution proves that there cannot be a blow up in finite or infinite time, and the local in time smooth solutions can be extended for all times, thus regularity. This happily is proved not only for the Navier-Stokes equations but also for the inviscid case of the Euler equations both for the periodic or non-periodic formulation and without external forcing (homogeneous case). But it is also proved that once the hypotheses of external forcing of the millennium problem allow for the existence of a unique smooth solution local in time, then the same result of regularity (no blow up) holds also for this inhomogeneous case. I try to keep the length of this paper as short as possible so as to encourage reading it, and make the solution as easy to be understood.as much as possible.

My first attempt to solve the millennium problem about the regularity of the Navier-Stokes equations problem was during the spring 2013 (uploaded at that time see [4] Kyritsis K. October 2013). Later attempts and solutions were published between 2017 and 2022 (see references [7], [8], [9], [11], [10] , [17]). All of them in the same direction of regularity and no Blow-up. But some of the proofs contained errors, that in the current paper have been eliminated and the solution shortened. In the current paper we prove also something more compared to my previous publications that the regularity holds also for the Euler inviscid equations, with the same hypotheses of the millennium problem putting $\nu=0$, for the viscosity coefficient.
The author has also solved the 3rd Millennium problem P vs NP in computational complexity with 3 different successive solutions each one simpler that the previous.
(see references [8], [10], [12], [13], [18])
This millennium problem seems by the title of the articles as if solved by other authors like [2] Durmagambetov Asset et al 2015 also [20] Moschandreou. T. 2021 , and [23] Ramm G. A. 2021.
Nevertheless, in my assessment they do not really solve it but eventually prove something else. In [9] Durmagambetov Asset et al 2015, the authors do not utilize the strict hypotheses of the formulation of the millennium problem, and the existence in general of blows-ups that they prove is a rather known fact. In [22] Ramm G. A. the strict hypotheses of the formulation of the millennium problem are indeed utilized but the solution essentially gives the existence of a smooth solution locally in time. Because the local in time $[0, t_1]$ smooth solution that he produces does depend on the initial data, we cannot repeated it in $[t_1 , t_2]$, $[t_2 , t_3]$ till infinite with certainty because we cannot claim that $t_1= t_2- t_1 = t_2 -t_3$ etc. Thus there is no really a proof for no blow up and regularity. On the other hand in [23] Ramm G. A. 2021 he proves that **<u>any</u>** solution of the Navier-Stokes equations, with the hypotheses of the millennium problem it will blow-up in finite time. There is obviously the counter example of potential

(irrotational) flows that it is known that they do not blow up, and plenty many other specific counter examples in various publications of various authors , that do not blow up. Thus his solution cannot be correct (although I could not find the error in his arguments). And finally in [20] Moschandreou T. the solution as he writes in the conclusions is regular but he leaves open that fact that for a set of measure zero of the 3-space there might be a blow-up in finite time. Thus it does not really proved either regularity or the existence with certainty of a blow up.

§ 2. The formulation of the millennium problem and the 4 sub-problems (A) , (B), (C), (D)

In this paragraph we highlight the basic parts of the standard formulation of the 4th Clay millennium problem.

The **Navier-Stokes** equations are given by (by R we denote the field of the real numbers, ν>0 is the density normalized viscosity coefficient)

$$\frac{\partial}{\partial t}u_i+\sum_{j=1}^{n}u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\nu\Delta u_i \qquad (x\in R^3 , t>=0 , n=3) \qquad \text{(eq.2.1)}$$

$$divu=\sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i}=0 \qquad (x\in R^3 , t>=0 , n=3) \qquad \text{(eq.2.2)}$$

with initial conditions $u(x,0)=u^0(x)$ **$x\in\mathbf{R}^3$**
and $u^0(x)$ C^∞ divergence-free vector field on **$\mathbf{R}^3$** (eq.2.3)
If ν=0 then we are taking about the **Euler** equations and **inviscid** case.

$\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator .The Euler equations are (eq2.1), (eq2.2), (eq2.3) when ν=0.
It is reminded to the reader, that in the equations of Navier-Stokes, as in (eq. 2.1) the density ρ, is constant, it is custom to normalized to 1 and omit it.

For physically meaningful solutions we want to make sure that $u^0(x)$ does not grow large as $|x|\to\infty$. This is set by defining $u^0(x)$, and f(x,t) and called in this paper **Schwartz initial conditions** , in other words
$\left|\partial_x^a u^0(x)\right|\le C_{a,K}(1+|x|)^{-K}$ on **$\mathbf{R}^3$** for any α and K (eq.2.4)

(Schwartz used such functions to define the space of Schwartz distributions)

Remark 2.1. It is important to realize that smooth Schwartz initial velocities after

(eq 4) will give that the initial vorticity ω_0 =curl(u^0) , in its supremum norm, is bounded over all 3-space.

$\left|\partial_x^a \partial_t^m f \quad (x,t)\right| \leq C_{a,m,K}(1+|x|+t)^{-K}$ on $\mathbf{R}^3 \times [\mathbf{0},+\infty)$ for any α, m, K (eq.2.5)

We accept as physical meaningful solutions only if it satisfies

p, u $\in C^{\infty}(R^3 \times [0,\infty))$ (eq.2.6)

and

$\int_{\Re^3} |u(x,t)|^2 \, dx < C$ for all t>=0 (**Bounded or finite energy**) (eq.2.7)

Remark 2.2 It is important to realize that smooth external force (densities) with the Schwartz property as in (eq.2.5) , have not only a rule for upper bounded spatial partial derivatives but also the same rule for time upper bounded partial derivatives.

Remark 2.3 We must stress here that imposing smoothness of the coordinate functions of velocities and external forces of the initial t=0 data and later time t data in the Cartesian coordinates plus and Schwartz condition as in (eq 2.5) is not equivalent with imposing similar such smoothness of the coordinate functions and conditions in the cylindrical or spherical coordinates. We will give in the paragraph 4, remark 4.5 an example of a strange blowup, where at any time t>0 the coordinates of the velocities are smooth and bounded in all space as functions in the polar coordinates and still the vorticity has infinite singularity at zero.

Alternatively, to rule out problems at infinity, we may look for spatially periodic solutions of (1), (2), (3). Thus we assume that $u^0(x)$, and f(x,t) satisfy

$u^0(x+e_j)= u^0(x)$, $f(x+e_j,t)= f(x,t)$, $p(x+e_j,0)=p(x,0)$, for 1<=j<=3 (eq.2.8)

(e_j is the jth unit vector in R^3)

In place of (4) and (5), we assume that $u^0(x)$, is smooth and that

$\left|\partial_x^a \partial_t^m f \quad (x,t)\right| \leq C_{a,m,K}(1+t)^{-K}$ on $\mathbf{R}^3 \times [\mathbf{0},+\infty)$ for any α, m, K (eq.2.9)

We then accept a solution of (1), (2) , (3) as physically reasonable if it satisfies

$u(x+e_j,t)= u(x, t)$, $p(x+e_j, t)=p(x,t)$, on $\mathbf{R}^3 \times [\mathbf{0},+\infty)$ for 1<=j<=3 (eq.2.10)

and p, u $\in C^{\infty}(R^3 \times [0,\infty))$ (eq.2.11)

In the next paragraphs we may also write u_0 instead of u^0 for the initial data velocity.

We denote Euclidean balls by $B(a,r) := \{x \in R^3 : ||x-a|| \leq r\}$, where ||x|| is the Euclidean norm.

The 4 sub-problems or conjectures of the millennium problem are the next:

(Conjecture A) Existence and smoothness of Navier-Stokes solution on R^3.

Take v>0 and n=3. Let $u_0(x)$ be any smooth, divergent-free vector field satisfying (4). Take f(x,t) to be identically zero. Then there exist smooth functions p(x,t) , u(x,t) on $R^3x[0,+\infty)$ that satisfy (2.1), (2.2), (2.3) , (2.6) , (2.7).

(Conjecture B) Existence and smoothness of Navier-Stokes solution on R^3/Z^3.

Take v>0 and n=3. Let $u_0(x)$ *be any smooth, divergent-free vector field satisfying (8); we take f(x,t) to be identically zero. Then there exist smooth functions p(x,t) , u(x,t) on* $R^3x[0,+\infty)$ *that satisfy (2.1), (2.2), (2.3) , (2.10) , (2.11).*

(Conjecture C) Breakdown of Navier-Stokes solution on R^3

Take v>0 and n=3. Then there exist a smooth, divergent-free vector field $u_0(x)$ *on* R^3 *and a smooth f(x,t) on* $R^3x[0,+\infty)$ *satisfying (4), (5) for which there exist no smooth solution (p(x,t) ,u(x,t)) of (2.1), (2.2), (2.3) , (2.6) , (2.7) on* $R^3x[0,+\infty)$.

(Conjecture D) Breakdown of Navier-Stokes solution on R^3/Z^3

Take v>0 and n=3. Then there exist a smooth, divergent-free vector field $u_0(x)$ *on* R^3 *and a smooth f(x,t) on* $R^3x[0,+\infty)$ *satisfying (2.8), (2.9) for which there exist no smooth solution (p(x,t) ,u(x,t)) of (2.1), (2.2), (2.3) , (2.10) , (2.11) on* $R^3x[0,+\infty)$.

In the next the $\| \|_m$ is the corresponding Sobolev spaces norm and . We denote by V^m $=\{u$ in $H^m(R^n)$ and divu=0$\}$ where $H^m(R^n)$ are the Sobolev spaces with the L^2 norm.

Remark 2.4. It is stated in the same formal formulation of the Clay millennium problem by C. L. Fefferman see [3] Fefferman C.L. 2006 (see page 2nd line 5 from below) that the conjecture (A) has been proved to holds locally. "..if the time internal $[0,\infty)$, is replaced by a small time interval [0,T), with T depending on the initial data....". In other words there is $\infty>T>0$, such that there exists a unique and smooth solution $u(x,t)\in C^\infty(R^3\times[0,T))$. See also [19] A.J. Majda-A.L. Bertozzi ,Theorem 3.4 pp 104. In this paper, as it is standard almost everywhere, the term smooth refers to the space C^∞

In these next the $\| \|_m$ is the corresponding Sobolev spaces norm and . We denote by $V^m=\{u$ in $H^m(R^n)$ and divu=0$\}$ where $H^m(R^n)$ are the Sobolev spaces with the L^2 norm.

We must mention that in A.J. Majda-A.L. Bertozzi [19] ,Theorem 3.4 pp 104, Local in Time existence of Solutions to the Euler and Navier-Stokes equations it is proved that indeed if the initial velocities belong to V^m m>=[3/2]+2 there exist unique smooth solutions locally in time [0,t]. Here, in the formulation of the millennium problem the hypotheses of smooth with Schwartz condition initial velocities satisfies this condition therefore we have the existence and uniqueness of smooth solution locally in time, both in the non-periodic and the periodic setting without external forcing (homogeneous case).

The existence and uniqueness of a smooth solutions locally in time is stated in the formulation by C.L. Fefferman [3] for the homogeneous cases and conjectures (A), (B). When a smooth Schwartz condition external force is added (inhomogeneous case) , it is natural to expect that also there should exist a local in time unique sooth solution.

But this I did not find to be stated in the A.J. Majda-A.L. Bertozzi [19], **so I will avoid assuming it.**

We state here also two, very well-known criteria of no blow-up and regularity.

In this theorem the $\| \|_m$ is the corresponding Sobolev spaces norm and . We denote by V^m={u in $H^m(R^n)$ and divu=0} where $H^m(R^n)$ are the Sobolev spaces with the L^2 norm.

Theorem 2.1 Velocities Sobolev norm sufficient condition of regularity. *Given an initial condition* $u_0 \in V^m$ $m>=[3/2]+2=3.5$ *e.g.* $m=4$ *, then for any viscosity* $\nu>=0$ *. there exists a maximal time T* (possibly infinite) of existence of a unique smooth solution* $u \in C([0,T^*];V^m) \cap C^1([0,T^*];V^{m-2})$ *to the Euler or Navier-Stokes equation. Moreover, if* $T^*<+\infty$ *then necessarily* $\lim_{t->T^*} ||u(.,t)||_m =+\infty$.

Proof: See A.J. Majda-A.L. Bertozzi [19] , Corollary 3.2 pp 112). QED

Remark 2.5 Obviously this proposition covers the periodic case too.

Theorem 2.2 Supremum of vorticity sufficient condition of regularity

Let the initial velocity $u_0 \in V^m$ $m>=[3/2]+2$ *, e.g.* $m=4$*, so that there exists a classical solution* $u \in C^1([0,T] ; C^2 \cap V^m)$ *to the 3D Euler or Navier-Stokes equations. Then :*

(i) If for any T>0 there is $M_1>0$ *such that the vorticity* ω*=curl(u) satisfies*

$$\int_0^T |\omega(.,\tau)|_{L^\infty}\, d\tau \le M_1$$

Then the solution u exists globally in time, $u \in C^1([0,+\infty] ; C^2 \cap V^m)$

(ii) If the maximal time T^* *of the existence of the solution* $u \in C^1([0,T] ; C^2 \cap V^m)$ *is finite,*

then necessarily the vorticity accumulates so rapidly that

$$lim_{t\to T^*} \int_0^T |\omega(.,\tau)|_{L^\infty}\, d\tau = +\infty \qquad \text{(eq. 11.1) (eq. 2.12)}$$

Proof: See A.J. Majda-A.L. Bertozzi [19] ,Theorem 3.6 pp 115, L^∞ vorticity control of regularity. QED.

Remark 2.6 Obviously this proposition covers the periodic case too.

§ 3. What is that we do not understand with the Navier-Stokes equations? The need for more consciousness for interpretations. Why we chose the geometric calculus approach for the solution?

It has been written in the initial formulation of the problem, that our difficulty of solving this millennium problem shows that there several things that we do not understand very well in the Navier-Stokes equations. In this paragraph we will investigate this issue. We will explain also why the rather elementary geometric calculus approach is better so as to solve the millennium problem, compared to more advanced functional analysis.

1) One primary point, known but often forgotten is the next. The Euler and the Navier-Stokes equations are the equations that are considered to govern the flow of fluids, and had been formulated long ago in mathematical physics before it was known that matter consists from atoms. So actually they formulated the old **infinite divisible material fluids**. After L. Boltzmann and the discovery of material atoms, the more true model is that of statistical mechanics. We may consider that the two different types of matter, a) infinite divisible b) made from finite atoms, behave the same as far as flows in fluid dynamics, and certainly there are many common properties but ultimately are mathematically and logically different. One example of the difference is that in the atomic structured material fluid model, the angular velocity of the spin e.g. of electrons, protons, neutrons which is about 1 terahertz (infrared range) can vary increase or decrease, independently from the vorticity, which only the part of the angular velocity which is "geared to the environmental" rotation of the fluid. In the classical Weierstrass calculus of infinite divisible material fluids (Euler and Navier-Stokes equations) this distinction does not exist and all the angular velocity of a point is due to the vorticity. In [21] Muriel, A 2000 a corresponding to the millennium problem in statistical mechanics has been solved in the direction of regularity. Similarly, in [6] Kyritsis, K. November 2017 a solution of the current millennium problem has been proved in the direction of regularity, but only if adding an additional hypothesis to the initial formulation, that of existence of finite atomic particles that are conserved during the flow. Strictly speaking a mathematical model of the material fluids and their flow which will have a high degree of exactness should take in to account that matter consists of atoms, (the electron range of magnitudes is of the order 10^{-15} meters) and this it should avoid utilizing concepts of continuity and smoothness that use $\varepsilon>0$ $\delta>0$ in their definition smaller than 10^{-15} meters. To address this difficulty of our current (Weierstrass) calculus the author developed the Democritus digital and finite decimal differential calculus (see [16] Kyritsis K. 2019b , [15] Kyritsis K. 2017 B , [14] Kyritsis K. 2022) In this finite calculus, we define concepts, of seemingly infinitesimal numbers (they are finite), seemingly infinite numbers (they are finite) and feasible finite numbers, so as to develop a differential and integral calculus up to decimal numbers with only a fixed finite number decimal (decimal density of level of precision). Different levels of precision give different definitions of continuity and smoothness. These multi-precision levels Democritus calculi is what an applied mathematician is doing when applying the Newton-Leibniz and Weierstrass calculus with the infinite (and infinitesimals). The Democritus calculus strictly speaking is not logically equivalent to the Newton-Leibniz calculus or to the Weierstrass calculus. E.g. classical Weisstrass calculus continuity corresponds in the Democritus calculus of being continuous not only to a single precision level but to all possible personal levels. Because in the Democritus calculus continuity and smoothness is only up to a precision

level, the turbulence can be defined in a way that in Weierstrass calculus cannot be defined. In a turbulent flow, the flow in the Democritus calculus may be smooth relative to a precision level but non-smooth relative to a coarser precision level (or the opposite) in the Weierstrass calculus this is impossible. Furthermore, now when a computer scientist is experimenting with computers to discover if in a flow there will be a blow up or not in finite time, within the Democritus calculus and its Navier-Stokes equations he will have an absolute proof and criterion. If the vorticity will become seemingly infinite (still finite) in a feasible finite time interval there is a blow up. If it becomes only feasible finite in any feasible finite time interval, there is no blow up. Of course blow-up in the Democritus calculus is not equivalent with a blow up in the Weisstrass calculus. Finally, with the Democritus calculus the applied mathematician acquires the subjective quality of congruence. In other words, what he thinks, sais and writes is what he acts and applies. With the infinite in the ontology of calculus this is not possible and it is unavoidable the incongruence, because infinite cannot be acted in the applications in a material reality where all are finite.

2) It is known that when the calculus (which is used in modeling the fluids) was discovered by Newton and Leibniz, the original mathematical ontology was utilizing infinitesimals, smaller than any positive real numbers but not zero. Then later with Weierstrass calculus this ontology was abandoned, we restricted ourselves to the real numbers only, and we utilized limits and convergence. So when we take the law of force (momentum conservation) of Newton $F=m*\gamma$ on a solid finite particle and then take the limit by shrinking it to a point to derive the Euler and Navier-Stokes equations, we must not forget, that originally the limit was not to a point but to an infinitesimal solid body particle. **This is not the same!** In [30x3] Kyritsis K. 2022, I have restored with strict mathematics the original ontology of infinitesimals of Newton-Leibniz , utilizing algebra of intervals (or inverses of ordinal numbers as J. H Conway has also done with the surreal numbers see [1] J H. Conway and [5] K Kyritsis ordinal real numbers 1,2,3). Then we have a two-density calculus with two different linearly ordered fields, a) the real numbers b) a larger such field of Newton-Leibniz fluxions, with infinitesimal, finite and infinite numbers. The topologies of convergence of a solid finite particle by shrinking it to a point ot to an infinitesimal solid particle are different! And this affects the issue of vorticity and angular velocity of infinitesimal particle. When I was a University student, and I was learning about the equations of Navier-Stokes, I was satisfied to see that the simple law of force (momentum conservation) of Newton $F=m*\gamma$ was converted to the Navier-Stokes equations, but I was shocked to realize, that the rest of the independent information about the motion of the solid finite particle, namely its rotational momentum, was not shanked to an angular velocity ω of the infinitesimal solid particle. So we see now that this is not reasonable in the Weisstrass calculus, which shrinks to a point, while it is possible in the older Newton-Leibniz calculus which shrinks to an infinitesimal solid body, and would lead to a different model of flows of fluids, with independent initial data of angular velocities, besides linear velocities and besides the derived from them vorticity.

3) In the current solution of the millennium problem, we may observe a 20%-80% Pareto rule. In other words, more than 80% of the equations utilized as governing equations of the flow, are those derived from fundamental theorem of the calculus, (in the form of Stokes theorem, divergence theorem, green theorem, Helmholtz-kelvin theorem, fundamental theorem of calculus etc.) and less that 30% the PDE of the Navier-Stokes equations. So I might say that the main equations governing the phenomenon of flow is the machinery of exterior differential algebra (wedge product) differentiation (differential forms) etc. rather than simply PDE equations. For reasons of simplicity and because we are restricted here to only 3 spatial dimensions, we do not utilize the symbolism of the wedge products and differential forms, but only the Stokes theorem, divergence theorem etc.

4) These versions of the fundamental theorem of the calculus (Stokes theorem etc) lead to an extension of the law of momentum conservation of 3D fluid parts to a law of 1D line density (rotatory) momentum conservation (Theorem 4.1) and law of 2D surface density (rotatory) momentum conservation (Theorem 4.2). These laws are very valuable for infinite divisible fluids so valuable as the existence of finite atoms in the atomics structured fluids. Without these extra laws of momentum density conservation, we would have a hope to solve the millennium problem. As T. Tao had remarked, only an integral of 3D energy conservation and an integral of 3D momentum conservation is not adequate to derive that momentum point densities ρ•u, or energy point densities (1/2)ρ•u^2 will not blow up.

5) Besides the forgotten conservation law of finite particles, which unfortunately we cannot utilize in the case of infinite divisible fluids to solve the millennium problem, there are **two more forgotten laws of conservation or invariants**. The first of them is the obvious that during the flow, the physical measuring units dimensions (dimensional analysis) of the involved physical quantities (mass density, velocity, vorticity, momentum, energy, force point density, pressure, etc.) are conserved. It is not very wise to eliminate the physical magnitudes interpretation and their dimensional analysis when trying to solve the millennium problem, because the dimensional analysis is a very simple and powerful interlink of the involved quantities and leads with the physical interpretation, to a transcendental shortcut to symbolic calculations. **By eliminating the dimensional analysis we lose part of the map to reach our goal.**

6) The 2nd forgotten conservation law or invariant, is related to the viscosity (friction). Because we do know that at each point (pointwise), the viscosity is only subtracting kinetic energy, with an irreversible way, and converting it to thermal energy, (negative energy point density), and this is preserved in the flow, (it can never convert thermal energy to macroscopic kinetic energy), we know that its sign does not change too it is a flow invariant , so the integrated 1D or 2D work density is always of the same sign (negative) and as sign an invariant of the flow. The **conservation or invariance of the sign of work density by the viscosity (friction)** is summarized in the lemma 3.1 below.

7) Finally we must not understate the elementary fact that the force densities F_p due to the pressures p, $F_p = -\nabla p$ are conservative , irrotational vector field, and they do not contribute to the increase or decrease of the rotational momentum and vorticity of the fluid during the flow. Because of this we get that the conserved 1D and 2D densities of momentum in Theorems 4.1 and 4.2 are only of the rotatory type.

8) Anyone who has spent time to try to prove existence of Blow up or regularity in the various physical quantities of the fluid like velocity, vorticity, acceleration, force density, momentum, angular momentum, energy etc. he will observe that in the arguments the regularity and uniform in time boundedness propagates easily from derivatives to lower order of differentiation, while the blowup propagates easily from the magnitudes to their derivatives. The converses are hard in proving. This is due to the usual properties of the calculus derivatives and integrals. The hard part of the proofs, must utilize forms of the fundamental theorem of the calculus like stokes theorem, divergence theorem etc.

9) Based on the above 8 remarks about what is not very well understood with Navier-Stokes equations I decided that **elementary geometric calculus should be the appropriate to solve the millennium problem**, and this I did indeed.

Lemma 3.1 The viscosity sign forgotten invariant.

If we integrate the force density of the viscosity, over a line (1D work density) or surface (2D work density) or a volume (work) its sign will remain the same during the flow.

Proof: Because we do know that pointwise, the viscosity is only subtracting kinetic energy, with an irreversible way, and converting it to thermal energy, (negative energy point density), and this is preserved in the flow, (it can never convert thermal energy to macroscopic kinetic energy), we deduce that its sign does not change too it is a flow invariant , so the integrated 1D or 2D work density is always of the same sign (negative) and as sign an invariant of the flow. QED.

§ 4. The ***Helmholtz-Kelvin-Stokes theorem*** in the case of viscous flows. New monotone semi-invariants of viscous flows with the interpretation of average rotational momentum axial 1-D line densities.

Here we apply the idea that the most valuable equations that govern he flow of the fluid are not literally the Navier-Stokes equations but the invariants or semi-invariant properties of the flow, derived from the abstract multi-dimensional fundamental theorems of calculus, in the forms of divergence theorems, Stokes theorems, Greens theorems etc. Actually this is the mechanism of wedge-products and abstract algebra of differential forms which is beyond classical partial differential equations. We do not utilize though definitions and symbolism of wedge-products and differential forms in his paper so as to keep it elementary and easy to read. The main discovery of this paragraph is the **Helmholtz-Kelvin-Stokes theorems 4.3 in the case of viscous flows and the resulting general no-blow-up theorem 4.4 for the viscous flows without external forcing.** A blow-up when it occurs, it will occur at least as blow-up of the vorticity, or of ρ•ω. If we discover average value invariants of the flow with physical units dimensions ρ•ω, that in the limit can give also the point value of the ρ•ω, **and that are invariants independent from the size of averaging,** it is reasonable that we can deduce conclusions, if the point densities can blow-up or not.

Theorem 4.1 The ***Helmholtz-Kelvin-Stokes theorem*** in the case of inviscid Euler equations flows without external force or homogeneous case. (A 1D line density of rotatory momentum, conservation law).

Let initial data in R^3 *so that they guarantee the existence of a unique smooth solution to the Euler equation in a local time interval [0,T]. Then at any time t ∈ [0,T] the circulation Γ(c) of the velocities on a closed smooth loop is equal to the flux of the vorticity on smooth surface S with boundary the loop c, and is constant and preserved as both loop and surface flow with the fluid. In symbols (ρ=1 is the density of the incompressible fluid)*

$$\Gamma_{c(t)} = \rho \oint_{c=\partial S} u dl = \rho \iint_S \omega \bullet ds \qquad \text{(eq. 4.1)}$$

Proof:

See [19] Majda, A.J-Bertozzi, A. L. 2002, Proposition 1.11 and Corollary 1.3 , in page 23. The proof is carried actually by integrating the Euler equations on a loop c and utilizing that the integral of the pressure forces (densities) defined as –∇p are zero as it is a conservative (irrotational) field of force (densities). Then by applying also the Stokes theorem that makes the circulation of the velocity on a loop equal to the flux of the vorticity on a smooth surface with boundary the loop (see e.g. Wikipedia Stokes theorem https://en.wikipedia.org/wiki/Stokes%27_theorem) the claim is obtained. QED.

We may notice that this circulation and surface vorticity flux has physical measuring units [ρ]*[ω]*[s]^2=[m]*[s]^(-3)*[t]^(-1)[s]^2=[m]*[s]^(-1)*[t]^(-1) =[moment_of_inertia]*[ω]*[s]^(-3) thus angular momentum point density. While the ρ*ω has physical measuring units dimensions [ρ]*[ω]=[m]*[s]^(-3]*[t]^(-1) =

[moment_of_inertia]*[ω]*[s]^(-2) thus 2nd spatial derivative of rotational momentum of point density .

A blow-up when it occurs, it will occur at least as blow-up of the vorticity, or of ρ•ω. If we discover bounded average value invariants of the flow with physical units dimensions ρ•ω, that in the limit can give also the point value of the ρ•ω, and that are invariants and bounded independent from the size of averaging, it is reasonable that we can deduce conclusions, if the point densities can blow-up or not.

Here we convert the surface vorticity flux invariant of Helmholtz-Kelvin-Stokes to one with 3D integration which will be more convenient in the arguments as the volumes are preserved by incompressible flows and most important, the integration is 3-dimensional which can be utilized to define average values of the vorticity (flux) on 3D finite particles..

We will prove at first a lemma about the 3D volume integral of Theorem 4.2, and convergence of average values of vorticity, based on this 3D integral, to point values to vorticity.

We define an average value for the volume 3D integral of vorticity flux.

Definition 4.1

We define as average value on ball in of the vorticity ω , denoted by $\bar{\omega}_B$ *,the unique constant value of the vorticity on the interior of the ball that would give the same 3D flux of vorticity on the ball,* $\rho\int_0^{\pi}\iint_S \bar{\omega}\bullet ds = \rho\int_0^{\pi}\iint_S \omega\bullet ds$. *This average value* $\bar{\omega}$ *of the vorticity is of course the*

$$\| \bar{\omega}_B || = | \frac{\rho\int_0^{\pi}\iint_S \omega\bullet ds d\theta}{|B|} | \qquad (eq.4.2)$$

and its direction is that of the vertical axis of the ball B

*Where |B|=(4/3)*π**r^2 *is the volume of the ball B, of radius r, and* $|\,\overline{|\omega_B}||$ *is the Euclidean norm of the vector. A more detailed symbolism of the average vorticity is the* $\bar{\omega}(x_t\,, t)_{B(r,t)}$

The numerator of this average value of vorticity has also the interpretation of ***rotational momentum average axial density*** *on the ball B and relative to the axis a. A reason for this is that the physical dimensions of measuring units of this magnitude is that of rotational momentum line density. This is because the rotational momentum point density has physical dimensions [moment_of_inertia]*[ω]*[s]^(-3)=[m][s]^(-1)[t]^(-1), where [m] for mass, [s] for distance, [t] for time, and this magnitude has physical units dimensions, ([ρ][ω][s]^3)=([m][s]^(-1)[t]^(-1))[s]^(1), thus rotational momentum point density integrated on 1-d line axial density. And the full quotient therefore has physical units dimensions [m][s]^(-3)[t]^(-1))=[ρ][ω].*

A blow-up when it occurs, it will occur at least as blow-up of the vorticity, or of ρ•ω. If we discover average value invariants of the flow with physical units dimensions ρ•ω, that in the limit can give also the point value of the ρ•ω, and that are invariants and bounded independent from the size of averaging**,** it is reasonable that we can deduce conclusions, if the point densities can blow-up or not.

Lemma 4.1 *Let a ball B of radius r and center x, and the average vorticity* $\bar{\omega}_B$ *in it as in the Definition 4.1 so that its axis a that defines the average vorticity is also the axis of the point vorticity* ω_x *at the center x of the ball. By taking the limit of shrinking the ball to its center x , (r->0), the average vorticity* $\bar{\omega}_B$ *converges to the point vorticity* ω_x *. In symbols* $\lim_{r\to 0} \bar{\omega}_B = \omega_x$*. If the axis a of the ball to estimate the average vorticity is not the axis of the point vorticity, then the limit of the average vorticity will be equal to the projection component* $\omega_a(x,t)$ *of the point vorticity* $\omega(x,t)$ *on the axis a.*

Proof: We simply apply an appropriate 3-dimensional version, with iterated integrals of the 1-dimensional fundamental theorem of the calculus. QED.

Remark 4.1. Such a limit of 3D body to a point is the same as the limit that from the Newton equation of force F=mγ, We deduce the Navier-Stokes equations.

Since the flow of a fluid under the Euler or Navier-Stokes equations, with or without smooth Schwartz external force is a smooth and continuous mapping F , then such a limit will be conserved to still be a valid limit during the flow. In other words $F(\lim_{B\to 0} \bar{\omega}_B) = \lim_{F(B)\to 0} F(\bar{\omega}_B)$

and B->0 , implies $F_t(B)$->0. We define of course in an obvious appropriate way the average vorticity $F_t(\bar{\omega}_B)$ as in definition 4.1, for the flow-image of a ball B after time t. Simply the disc surfaces will no longer be flat, and the loop no longer perfect circle. But the integrals in the definition will be the same. Constancy of the average vorticity on such surfaces will only be, up to its Euclidean norm and vertical angle to the surface. We must notice though that although a relation $F(\lim_{B\to 0} \bar{\omega}_B) = \lim_{B\to 0} F(\bar{\omega}_B)$ would hold , the value of this limit **will not be** the vorticity $\omega_{F(x)}$ at the flowed point! Unfortunately, the Lemma 4.2 holds not on arbitrary 3D shapes and arbitrary integration parametrization on it, but only when we start with standard 3D shapes like a sphere, a cylinder a cube etc. and the normal parametrization on them. The reason is that we need to take in to account in a normal way the average vorticity around a point in an unbiased way, that an arbitrary shape will not give.

Another important conservation point is that the relation of the vorticity ω_x being tangent to an axis a (or general curve) is conserved during inviscid Euler flows. It is the conservation of vorticity lines (See [19] Majda, A. J. –Bertozzi, A. L. 2002, Proposition 1.9 in page 21). Therefore for inviscid (and incompressible) flows the axis of the initial point vorticity ω(0) , which is also the axis to estimate the average vorticity on the ball B, will still be after the flow and at time t, tangent to the point vorticity ω(t).

But for general viscous flows this will not be so. Notice that such limits of average values would not work for the circulation of the velocity on a loop, as in the application of the iterated 1-dimensional fundamental theorem of the calculus would require boundaries of the integration.

Lemma 4.2 *Let the Euler or Navier-Stokes equations of incompressible fluids in the non-periodic or periodic setting, with smooth initial data and we assume that the initial data in the periodic or non-periodic case, are so that the supremum of the* ***vorticity*** *is finite denoted by* F_ω *on all 3-space at time t=0. Let the average vorticity, or average rotational momentum density, defined as in Definition 4.1 but with integration parametrization one any smooth 3D shape B of any size, that of course both as a diffeomorphic image of a spherical ball with its spherical coordinates integration parametrization. Then the average vorticity or average rotational momentum density is also upper bounded by the* F_ω. *In symbols*

$$\| \bar{\omega}_B || = \left|\frac{\int_0^\pi \iint_S \ \omega \bullet ds d\theta}{|B|}\right| \le \ F_\omega \qquad \text{(eq. 4.3)}$$

Proof: Since $||\omega|| <= F_\omega = ||(\omega/||\omega||)||F_\omega$ in the flux-integration we have for the inner product of ω and the unit area vector n, $(\omega, n) <= ((\omega/||\omega||)F_\omega, n) <= F_\omega$. Thus in the integration we may factor out the F_ω

$$\left|\frac{\int_0^\pi \iint_S \ \omega \bullet ds d\theta}{|B|}\right| \le \left|\frac{\int_0^\pi \iint_S \ F_\omega ds d\theta}{|B|}\right| = F_\omega\left|\frac{\int_0^\pi \iint_S \ ds d\theta}{|B|}\right| = F_\omega \left|\frac{|B|}{|B|}\right. = F_\omega \ . \qquad \text{QED.}$$

Theorem 4.2 A 3-dimensionl integral version of the Helmholtz-Kelvin-Stokes theorem. (A 2D surface density of rotatory momentum, conservation law).

Let initial data in R^3 *so that they guarantee the existence of a unique smooth solution to the Euler equation in a local time interval [0,T]. Then at any time t* ∈ *[0,T] let a sphere B of radius r (as in figure 4.)* ***considered as a finite particle****, then the azimuthal* θ*-angle,* θ*-integral on a meridian in spherical coordinates of the circulations* $\Gamma(c)$ *of the velocities on all closed longitude smooth loops parallel to the equatorial loop is equal to the same* θ*-integral of the surface flux of the vorticity on smooth flat disc surfaces S with boundary the loops c (as in figures 4.2, 4.3) , and both integrals are constant and preserved as both surface and volume integrals during the flow with the fluid. In symbols (*$\rho=1$ *is the density of the incompressible fluid)*

$$\rho \int_0^\pi \oint_{c=\partial S} udl = \rho \int_0^\pi \iint_S \ \omega \bullet ds \qquad \textit{(eq. 4.4)}$$

After (eq. 4.2) $\| \overline{\omega}_B|| = |\frac{\int_0^\pi \iint_S \ \omega \bullet dsd\theta}{|B|}|$ *it holds also*

for t ∈ [0,T] $||\overline{\omega}_{B(0)}|| = ||\overline{\omega}_{B(t)}||$ *(eq. 4.5)*

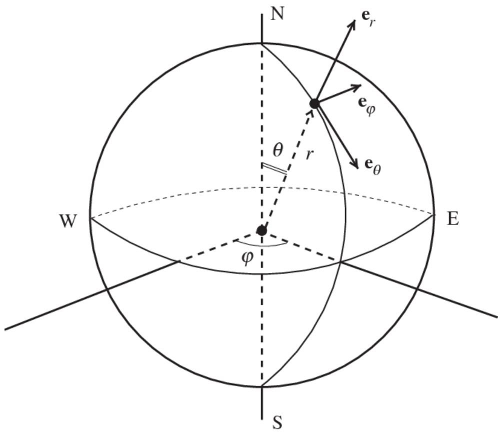

Figure 4.1

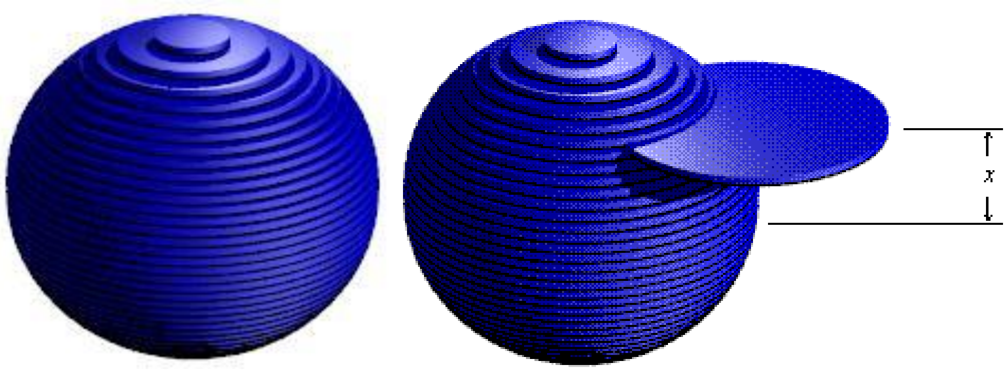

Figure 4.2-4.3

Proof: We simply take the θ-azimuthal angle θ-integral of both sides of the equation 4.1 in the theorem 4.1. Both sides are preserved during the flow and so is their θ-integrals too. We notice that the measuring physical units dimensions of the conserved quantity $\rho \int_0^\pi \oint_{c=\partial S} udl$ is [mass]*[length]^(-3)*[velocity]*[length]^(2)= [mass]*[length]^(-2)*[velocity] thus integration in 2-dimension surface of momentum 3D-point-density, or equivalently momentum 1D density QED

Theorem 4.3. The *Helmholtz-Kelvin-Stokes theorem* in the case of viscous Navier-Stokes equations flows without external force (homogeneous case) .

Let initial data in R^3 *so that they guarantee the existence of a unique smooth solution to the Navier-Stokes equation with viscosity coefficient* $v>0$ *, in a local time interval [0,T]. Then at any time t ∈ [0,T] the circulation Γ(c) of the velocities on a closed smooth loop is equal to the flux of the vorticity on smooth surface S with boundary the loop c, and is decreasing as both loop and surface flow with the fluid. In symbols (ρ=1 is the density of the incompressible fluid)*

$$\rho \oint_{c=\partial S} u dl = \rho \iint_S \omega \bullet ds \qquad \textit{(eq. 4.1)}$$

and for t ∈ [0,T] $\oint_{c=\partial S} u(0) dl > \oint_{c=\partial S} u(t) dl$ *(eq. 4.6)*

and similarly for the 3D volume integration as in Theorem 4.2

for t ∈ [0,T] $\rho \int_0^{\pi} \iint_S \omega(0) \bullet ds d\theta > \rho \int_0^{\pi} \iint_S \omega(t) \bullet ds d\theta$ *(eq. 4.7)*

After (eq. 4.2) $\| \bar{\omega}_B || = |\frac{\int_0^{\pi} \iint_S \omega \bullet ds d\theta}{|B|}|$ *it holds also for initial finite spherical particles for t ∈ [0,T]* $||\overline{\omega}_{B(0)}|| > ||\overline{\omega}_{B(t)}|| \ |$ *(eq. 4.8)*

Proof: Again The (eq. 4.1) is nothing else of course but the Stokes theorem as in (eq 4.1)

We shall utilize here the next equation (See [19] Majda, A.J-Bertozzi, A. L. 2002, (eq 1.61) , in page 23.) in the case of viscous incompressible flows under the Navier-Stokes equations

$$\frac{d}{dt} \Gamma_{c(t)} = \frac{d}{dt} \oint_{c(t)} u dl = v \oint_{c(t)} \Delta u dl = - v \oint_{c(t)} curl\ \omega dl \qquad \textit{(eq. 4.9)}$$

This equation is derived after applying as in Theorem 4.1 the loop integral of the circulation at the Navier-Stokes equations instead at the Euler equations taking the material-flow derivative outside the integral, and eliminating the conservative, irrotational part of the pressure forces as gradient of the pressure. Here the viscosity is not zero thus the left hand of the equations is not zero as in the case of Euler equations, where it is conserved. **The right hand side is nothing else than the loop work density of the point density of the force of viscosity at any time t. And as the viscosity always subtracts energy, this right hand side work density is always negative during the flow.** We notice after the **Lemma 3.1** that the viscosity force point density keeps constant sign on the trajectory path as orbital component during the flow and relative to the velocity on the trajectory. It is always as orbital component opposite to the motion and represents the always irreversible energy absorption and linear momentum and angular momentum decrease. Similarly, for any rotation of the fluid e.g. with axis the trajectory path. The viscosity force point density as component on the loop is always opposite to the rotation, it never converts thermal energy to add to linear or angular momentum. This opposite to motion monotonicity of the viscosity force density applies to the Navier-Stokes equations but also as opposite to rotation monotonicity in the vorticity equation $\frac{D\omega}{Dt} = \omega * \nabla u + \nu\Delta\omega$ (see [19] Majda, A.J-Bertozzi, A. L. 2002, (eq 1.33) and (eq 1.50) in pages 13 and 20) . So if we choose a direction of the loop so that the circulation integral on the right hand side is positive then this will have the same sign during the flow (although different absolute value), and will make the left hand side of the (eq. 4.9) always negative during the flow. But this means from the left-hand side of the equation that the circulation of the velocity on the loop is always decreasing during the flow.

$$\frac{d}{dt}\oint_{c(t)} udl < 0 \qquad \text{for any t in [0,T]} \qquad \text{(eq. 4.10)}$$

Thus (eq. 4.6) is proved, and (eq. 4.7) is direct consequence.

To prove the equation 4.8 we notice that due to incompressibility, the flow is volume preserving, thus |B(x(t))|=|B(x(0)| , and by dividing both sides of the equation 4.7 , and after the definition

$$\| \bar{\omega}_B \| = \left| \frac{\int_0^{\pi} \iint_S \omega \bullet ds d\theta}{|B|} \right| \qquad \textit{it holds also}$$

$$\text{for } t \in [0,T] \quad \|\overline{\omega}_{B(0)}\| > \|\overline{\omega}_{B(t)}\| \qquad \textit{(eq. 4.8)} \qquad \text{QED.}$$

Remark 4.2. We can extend the results of the theorems 4.1, 4.3 with Euler or Navier-Stokes equations to similar ones in the inhomogeneous case with external forces $F_{ext.}$, provided of course we have the existence and uniqueness of a smooth solution local in time. We would start from an equation

$$\frac{d}{dt}\Gamma_{c(t)} = \rho\frac{d}{dt}\oint_{c(t)} udl = -\nu\rho\oint_{c(t)} curl\ \omega dl \ +\rho\oint_{c(t)} F_{ext}dl \ <= \ \rho\oint_{c(t)} F_{ext}dl$$

Similarly

$$\frac{d}{dt}\rho\iint_S \ \omega \bullet ds = -\nu\rho\oint_{c(t)} curl\ \omega dl \ + \rho\iint_S \ curl F_{ext} \bullet ds$$

$$<= \ \rho\iint_S \ curl F_{ext} \bullet ds \quad <= |S|\ M_0$$

since as in the proof of Theorem 4.3 the friction circulation term is always negative and due to the Schwartz conditions on the external force in space and time the constant M_0 is independent from space and time and the size of the surface of the loop in the integration. |S| is the area of the flux integration.

Then from smoothness and elementary 1-dimensional calculus we would get an inequality like

$$\rho | \iint_S \omega(0) \bullet ds - \iint_S \omega(0) \bullet ds | <= |S| \, M_1(F_{ext}, t)$$

where again due to the Schwartz conditions on the external force in space and time the constant M_1 is independent from space and the trajectories paths and depends only on the time and the external force.

Similarly by dividing the first equation by |B| which does not change by time and integrating for 3D ball , we can result similarly to an inequality like

$| \frac{\int_0^{\pi} \iint_S \omega(t) \bullet ds d\theta}{|B(t)|} - \frac{\int_0^{\pi} \iint_S \omega(0) \bullet ds d\theta}{|B(0)|} | \leq \frac{|B|}{|B|} M_1(F_{ext}, t) = M_1(F_{ext}, t)$ where again the constant M_2 is independent from space and the size of the ball and depends only on the time t.

Theorem 4.4 The no blow-up theorem in finite or infinite time in the Euler, Navier-Stokes, periodic or non-periodic and homogeneous cases.

Let the Euler or Navier-Stokes equations of incompressible fluids in the non-periodic or periodic setting (homogeneous case with no external forces), with

a) smooth initial data and whatever else hypothesis is necessary so as, also to guarantee the ***existence and uniqueness of smooth solutions*** *to the equations locally in time [0, T).*

b) Furthermore we assume that the initial data in the periodic or non-periodic case, are such that the supremum of the vorticity, denoted by F_ω *, is finite* at t=0. (*In the periodic case, smoothness of the initial velocities is adequate to derive it, while in the non-periodic setting smooth Schwartz initial velocities is adequate to derive it)*

Then it holds that there cannot exist any finite or infinite time blow-up at the point vorticities during the flow.

Proof: The proof will by contradiction. The main idea of the proof is to utilize that in the case of a blow-up the vorticity will converge to infinite, so it will become larger than an arbitrary lower bound $M+F_\omega$, $M>0$, $F_\omega >0$ and by approximating it with average flux vorticity of a 3D spherical particle, and tracing it back at the initial conditions where all is bounded by F_ω ,utilizing the semi-invariance of the average vorticity that we have proved, we will get that $F_\omega > M+F_\omega$.

So let us assume that there is a blow up, in a finite time or infinite time T^*, with the hypotheses of the theorem 4.x. Then from the Theorem 2.2 and (eq. 2.12) which is the well-known result of the control of regularity or blow up by the vorticity we get that ,

$$\lim_{t\to T^*} \int_0^T |\omega(.,\tau)|_{L^\infty}\, d\tau=+\infty \qquad \text{(eq. 2.12)}$$

We conclude that there will exist an infinite sequence of points $\{x_{tn}$, n natural number, $0<t_n<T^*$, $\lim_{n\to\infty} t_n = T^* \}$ so that the point vorticity $\omega(x_{tn})$ blows-up, or equivalently $\lim_{n\to\infty} \omega(x_{tn}) = +\infty$. We do not need to assume them on the same trajectory. Therefore, for every positive arbitrary large real number M_0 , there is a n_0 such that for all natural numbers $n> n_0$, it holds that $\omega(x_{tn})>M_0$. We choose $M_0=M_{00} +F_\omega$, for an arbitrary large positive number M_{00} . So

$$\omega(x_{tn})> M_{00} +F_\omega \qquad \text{(eq. 4.11)}$$

Now we approximate this point vorticity with an average flux vorticity on a 3D particle after Definition 4.1 , theorem 4.2 and Lemma 4.1.

Let a spherical ball particle $B(r, x_{tn},)$ as in theorem 4.2. with center x_{tn} and radius $r>0$. After Definition 4.1 ,theorem 4.2 and Lemma 4.1. we have that

$$\lim_{r\to 0} \bar{\omega}_B = \omega_{x(tn)}\,,\ \text{with } \|\bar{\omega}_B\| = \left| \frac{\int_0^{2\pi} \iint_S \omega\bullet ds d\theta}{|B(r,x(tn))|} \right| \qquad \text{(eq. 4.2)}$$

Therefore for arbitrary small positive number $\varepsilon>0$, there is radius R, with

$$\bar{\omega}_{B(R)} > \omega_{x(tn)} - \varepsilon \qquad \text{or} \qquad \left| \frac{\int_0^{2\pi} \iint_S \omega\bullet ds d\theta}{|B(r,x(tn))|} > \omega_{x(tn)} - \varepsilon \right. \qquad \text{(eq. 4.12)}$$

Thus after (eq. 4.11) $$\left| \frac{\int_0^{2\pi} \iint_S \omega\bullet ds d\theta}{|B(R,x(tn))|} \right| > M_{00} + F_\omega - \varepsilon \qquad \text{(eq. 4.13)}$$

Now we trace back on the trajectory of the x_{tn} the parts of the (eq. 4.13)

At initial time t=0. We use the advantage that as the incompressible flow is volume preserving, the $|B(R, x_0,)|= \left|B\big(R, x(tn)\big)\right|$. We also utilize theorems 4.2, 4.3, and (eq. 4.5), (eq. 4.8) , which prove that at the initial conditions t=0 , this average vorticity is the same or higher than that at t_n .

$$\left|\frac{\int_0^{2\pi} \iint_S \ \omega \bullet dsd\theta}{\left|B\big(R, x(0)\big)\right|}\right| \geq \left|\frac{\int_0^{2\pi} \iint_S \ \omega \bullet dsd\theta}{\left|B\big(R, x(tn)\big)\right|}\right|$$

We conclude that

$$\left|\frac{\int_0^{2\pi} \iint_S \ \omega \bullet dsd\theta}{\left|B\big(R, x(0)\big)\right|}\right| > M_{00} + F_\omega - \varepsilon \qquad \text{(eq. 4.14)}$$

From the (eq. 4.14) and (eq. 4.3) of Lemma 4.2 we conclude that

$$F_\omega \quad > M_{00} + F_\omega - \varepsilon \qquad \text{(eq. 4.15)}$$

But M_{00} was chosen in an independent way from ε>0 to be arbitrary large, while ε>0 can be chosen to be arbitrary small. Therefore, a contradiction. Thus there cannot be any blow-up either in finite or infinite time T*. QED.

Remark 4.3. Infinite initial energy. We must remark that we did not utilize anywhere that the initial energy was finite, only that the vorticity initially has finite supremum. Thus this result of no-blow-up can be with infinite initial energy too. But when applying it to the millennium problem we do have there also that the initial energy is finite.

Remark 4.4. Inhomogeneous case. It is interesting to try to extend this result of no blowup, for the inhomogeneous case too of the Euler and Navier-Stokes equations and investigate where it would fail, if at all, provided of course we have the existence and uniqueness of a smooth solution local in time. We would utilize the last inequality of remark 4.2

$\left|\frac{\int_0^{\pi} \iint_S \ \omega(t) \bullet dsd\theta}{|B(t)|} - \frac{\int_0^{\pi} \iint_S \ \omega(0) \bullet dsd\theta}{|B(0)|}\right| \leq M_1(F_{ext}, t)$ and we would anticipate for the choice of the constant M_0 in (eq 4.11) , $M_0=M_{00}+M_1+F_\omega$. We would reason similarly as in the proof of the Theorem 4.4 and we end to a same contradiction

$$F_\omega \quad > M_{00} + F_\omega - \varepsilon$$

But since at least in the book [19] Majda, A.J-Bertozzi, A. L. 2002, that I took as reference on the subject, it does not claim the existence and uniqueness of a smooth solution locally in time, in the case of external forces, as we wrote in remark 2.4, I will avoid using it, and I remain only in the homogeneous case. Therefore, for the moment I will not spend space in this paper on the inhomogeneous case.

Remark 4.5. A strange blow up for any time t>0 of initially smooth data. We might be curious to ask the question if it is possible, starting with zero initial velocities and pressures, to create an artificial blow-up only with external forcing. A good candidate is the perfect circular vortex, where all the trajectory paths are perfect circles, which is known that it is an instance of the solution of the Euler and Navier-Stokes equations. We can formulate the circular vortex in 3D with cylindrical or spherical coordinates. But for simplicity we will formulate it in 2 dimensions, in spite the fact that we do know that in 2D dimensions there cannot be a blow up under the hypotheses of the millennium problem. So with an external forcing also as perfect circular vortex that in polar coordinates are as follows

$F_{r=0}$, $F_\theta=2\rho/(1+\exp(r))$ (eq 4.16)

we raise the absolute initial rest within finite time t the flow to a circular vortex which has velocities in polar coordinates

$u_{r=0}$, $u_\theta=2t/(1+\exp(r))$ (eq 4.17)

Now it is elementary to show that

1) this flow follows the Euler and Navier-Stokes equations

2) Because curlω=0 , the viscosity has no effect it is as if an inviscid flow.

Where ω is the vorticity which is calculated in polar coordinates at the verical z-axis by the formula

$$\omega_z = \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} - \frac{\partial u_r}{r\partial r} \qquad \text{(eq. 4.18)}$$

4) Although the velocity has smooth polar coordinates, the vorticity is in steady blow-up (singularity) at r=0 for any t>0. That is although at t=0 the initial data are smooth, for any t>0, there is a blow-up.

5) The 4) is so because the external forcing although it has smooth polar coordinates, in the Cartesian coordinates, it has curl(F)=+∞, at r=0, thus it does not satisfy the smooth Schwartz condition external forcing of the millennium problem.

§5 THE SOLUTION OF THE MILLENNIOUM PROBLEM FOR THE NAVIER-STOKES EQUATIONS BUT ALSO FOR THE EULER EQUATIONS.

We are now in a position to prove the Conjectures (A) and (B) , non-periodic and periodic setting , homogeneous case of the Millennium problem.

(Millennium Homogeneous Case A) Existence and smoothness of Navier-Stokes solution on R^3.

Take ν>0 and n=3. Let $u_0(x)$ be any smooth, divergent-free vector field satisfying (2.4). Take f(x,t) to be identically zero. Then there exist smooth functions p(x,t) , u(x,t) on $R^3x[0,+\infty)$ that satisfy (2.1), (2.2), (2.3) , (2.6) , (2.7).

Proof: All the hypotheses of the no-blow-up theorem 4.4 are satisfied. After remark 2.4, with the current case of the millennium problem there exist indeed a unique smooth solution locally in time [0,t] (after A.J. Majda-A.L. Bertozzi [19] ,Theorem 3.4 pp 104, Local in Time existence of Solutions to the Euler and Navier-Stokes equations) . And also the Schwartz condition of the initial data, guarantees that the supremum of the vorticity, is finite at t=0. Therefore we conclude by Theorem 4.4 that there cannot be any finite or infinite time blow-up. Thus from **Theorem 2.2 Supremum of vorticity sufficient condition of regularity** we conclude that this local in time [0,t] solution , can be extended in [0,+∞). QED

(Millennium Homogeneous Case B) Existence and smoothness of Navier-Stokes solution on R^3/Z^3.

Take ν>0 and n=3. Let $u_0(x)$ be any smooth, divergent-free vector field satisfying (8); we take f(x,t) to be identically zero. Then there exist smooth functions p(x,t) , u(x,t) on $R^3x[0,+\infty)$ that satisfy (2.1), (2.2), (2.3) , (2.10) , (2.11).

Proof: All the hypotheses of the no-blow-up theorem 4.4 are satisfied. After remark 2.4, with the current case of the millennium problem there exist indeed a unique smooth solution locally in time [0,t] (after A.J. Majda-A.L. Bertozzi [19] ,Theorem 3.4 pp 104, Local in Time existence of Solutions to the Euler and Navier-Stokes equations) . And also the compactness of the 3D torus of the initial data, guarantees that the supremum of the vorticity, is finite at t=0. Therefore we conclude by Theorem 4.4 that there cannot be any finite or infinite time blow-up. Thus from **Theorem 2.2 Supremum of vorticity sufficient condition of regularity** and **remark 2.6 (that the previous theorem covers the periodic setting too)** we conclude that this local in time [0,t] solution , can be extended in [0,+∞). QED

Remark 5.1. Now in the previous two Millennium cases we could as well take ν=0 , and we would have the same proofs and conclusions because the Theorem 4.4 of the no-blow-up covers too the **case of inviscid Euler equations flows.**

9. Epilogue

In this paper I solved the millennium problem about the Navier-Stokes equations in the homogeneous case without external forcing, and proved that there cannot be a blowup in finite or infinite time (regularity) both in the periodic and non-periodic setting without external forcing (homogeneous case). But it is also proved that once the hypotheses of external forcing of the millennium problem allow for the existence of a unique smooth solution local in time, then the same result of regularity (no blow up) holds also for this inhomogeneous case with external forcing. Furthermore I proved also the by far more difficult same result for the Euler inviscid flows. I did so by utilizing (e.g. in in the inviscid case) that not only the momentum is conserved but also rotatory versions of the momentum 1D line and 2D surface densities are conserved. Then I extended the conservation in the case of viscous Navier-Stokes flows to monotone semi invariants, in other words that these densities are monotonously decreasing due to friction. This allowed me to prove with elementary geometric calculus that there cannot be any blow up (regularity). The solution of this millennium problem gave the opportunity to discover 2 new monotone semi invariants (1D and 2D densities of (rotatory type) momentum) for the viscous Navier-Stokes equations.

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