UNIVERSITY OF IOANNINA
SCHOOL OF SCIENCES
DEPARTMENT OF MATHEMATICS
"SECTOR OF ANALYSIS"

Gelfand Duality

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Master's Thesis

Ioannina, 2022

This thesis was fulfilled for the acquisition of Master's Diploma in the field of

"Mathematics (Analysis–Algebra–Geometry)"

which was handed to me from the Department of Mathematics of the University of Ioannina.

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Statutory Declaration

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Eleftherios Papadopoulos

Dedicated to my beloved parents,
Giorgos & Stavroula.

Abstract in English

Gelfand Duality establishes a duality, i.e. a contravariant equivalence, between convenient categories of topological spaces and certain algebras of continuous functions. There has been much research in the case of commutative algebras, whereas the noncommutative case is still being developed. In the classical commutative case, Gelfand Duality gives a duality between compact Hausdorff spaces and commutative C^*-Algebras and since then has been extended to many more classes of spaces. The noncommutative case includes Noncommutative Measure Theory, Noncommutative K-Theory and many other subjects incorporated in the field of Noncommutative Geometry. This thesis presents an overview of the aforementioned subjects, concentrating in the commutative case and commenting on topics of current research in the direction of Noncommutative Geometry.

Abstract in Greek

Η Δυϊκότητα Gelfand αφορά μια σχέση δυϊκότητας, δηλαδή μια αντισταθμιστική ισοδυναμία μεταξύ κατηγοριών, με συγκεκριμένες καλές ιδιότητες, τοπολογικών χώρων και αλγεβρών συνεχών συναρτήσεων. Η περίπτωση των μεταθετικών αλγεβρών έχει μελετηθεί διεξοδικά, ενώ αυτή των μη μεταθετικών είναι ακόμα υπό εξέλιξη. Στην κλασική μεταθετική περίπτωση, η Δυϊκότητα Gelfand μας παρέχει μια δυϊκότητα μεταξύ συμπαγών χώρων Hausdorff και μεταθετικών C^*-αλγεβρών και, περαιτέρω, έχει επεκταθεί σε πολλές άλλες κλάσεις τοπολογικών χώρων. Η μη μεταθετική περίπτωση περιλαμβάνει τη Μη Μεταθετική Θεωρία Μέτρου, τη Μη Μεταθετική K-θεωρία αλλά και αρκετά ακόμη θέματα, τα οποία εντάσσονται στον ευρύτερο κλάδο της Μη Μεταθετικής Γεωμετρίας. Η παρούσα διατριβή αποτελεί μια επισκόπηση των θεμάτων που προαναφέραμε, δίνοντας έμφαση στην μεταθετική περίπτωση και κάνοντας αναφορά σε θέματα που αποτελούν αντικείμενο ενεργούς έρευνας, προς την κατεύθυνση της Μη Μεταθετικής Γεωμετρίας.

Contents

Introduction

Based on a profound idea of Alain Connes [15], one can associate geometric objects to noncommutative algebras in the framework of analysis. This theory, called Noncommutative Geometry, is a rapidly growing area of mathematics which interacts with and contributes to many disciplines in mathematics and physics. Examples of such interactions and contributions include the theory of operator algebras, index theory of elliptic operators, algebraic and differential topology, number theory, quantum field theory and string theory.

To understand the basic ideas of Noncommutative Geometry one should first become familiar with the idea of a noncommutative space. The notion of a noncommutative space is based on one of the most profound ideas in mathematics, namely a duality or correspondence between Algebra and Geometry, according to which concepts or statements in Geometry correspond to, and can be equally formulated to, similar concepts and statements in Algebra. Specifically, one can formulate a correspondence

$$\Phi : \text{Geometric object} \longrightarrow \text{Algebraic object}.$$

Based on Φ, a "commutative" object, corresponds to an algebraic one, usually a commutative algebra. This correspondence is more complicated when one tries to associate a "noncommutative" algebraic object R with a "noncommutative" geometric object X, for which $\Phi(X) = R$. Here comes the notion of a "noncommutative space". We already have in hand many generalizations of commutative algebraic objects and we would like to associate to these, noncommutative geometric ones. The field of Noncommutative Geometry is a rapidly growing area of mathematics that emphasizes in the above association and extends to many more fields of both Geometry and Topology.

We need to emphasize, though, that this correspondence is, by no means, a new trend in mathematics. In fact, this duality is utilized in mathematics and its applications very often. A trivial example is the use of numbers in counting. It is, however, the case that throughout history, each new generation of mathematicians has found new ways of formulating this principle and at the same time broadening its scope. Just to mention a few highlights of this rich history, we quote Descartes (analytic geometry–1630's), Hilbert (affine varieties and commutative algebras–1900's), Gelfand–Naimark (locally compact spaces and commutative C^*-algebras–1940's), and Grothendieck (affine schemes and commutative rings–1960's). Precisely, in what concerns the approach of Gelfand and Naimark in applying the aforementioned Algebra–Geometry correspondence, is what this dissertation is all about, namely Gelfand Duality. In particular, we will prove both of the following unital and nonunital cases of Gelfand Theorems.

Theorem (Unital case of Gelfand Theorem, [16], p.236)**.** *Let A be a unital commutative C^*-algebra, with norm $\|\cdot\|$. Then, the Gelfand transform, which is a map $\gamma : A \to C(\Delta)$, is an isometric *-isomorphism.*

Theorem (Nonunital case of Gelfand Theorem, [17], p.7)**.** *Let A be a nonunital commutative C^*-algebra. Then, the Gelfand transform, which is a map $\gamma : A \to C_0(\Delta)$, is an isometric *-isomorphism.*

Utilizing the theorems above, we will establish the main goal of this dissertation, which is Gelfand Duality and consists of the following two theorems.

Theorem ([27], p.4)**.** *The category of compact Hausdorff spaces $\mathcal{CS}$ is dual to the category of unital commutative C^*-algebras $C^*Alg_{com,u}$.*

Theorem ([27], p.5)**.** *The category of locally compact Hausdorff spaces $\mathcal{LCS}$ is dual to the category of nonunital commutative C^*-algebras $C^*Alg_{com,nu}$.*

Last to mention is a noncommutative Gelfand Theorem which is analyzed in Chapter 4 and depends heavily on certain concepts in the theory of projections in a von Neumann algebra. Specifically, we will prove the following theorem.

Theorem (Noncommutative Gelfand Theorem, [5], p.6)**.** *The hermitian elements of A are exactly those q-continuous elements b of M, such that the spectral projections of b corresponding to closed subsets of the spectrum of b, which don't contain 0, are q-compact, that is, b "vanishes at ∞".*

In Chapter 1 we present some fundamentals about the theory of Banach algebras. Specifically, we define the notion of a Banach algebra, the notion of the spectrum, the notion of an ideal in a Banach algebra and we prove, using the quotient map, that the quotient algebra A/J is also a Banach algebra, provided that A is a Banach algebra and J is a proper closed ideal of A. We close up this chapter by defining the Gelfand transform and its main properties and by proving that the maximal ideal space of a commutative Banach algebra is a compact Hausdorff space.

Chapter 2 is concerned with C^*-algebras. We emphasize on certain kinds of elements in C^*-algebras, that is, hermitian (or self-adjoint), normal, positive and unitary ones. Using those elements and some of their properties, we prove the Gelfand Theorems 2.2.13 and 2.2.14. We close up this chapter with the notion of an approximate identity.

Chapter 3 is the main goal of this thesis, that is, Gelfand Duality. This chapter consists of some fundamental concepts of Category Theory, from which we distinguish the natural transformations. Two kinds of them are of our concern, namely the one of an equivalence between categories and that of a duality between categories. We prove Gelfand Duality in both cases (Theorems 3.4.1 and 3.4.2). To be specific, for a given C^*-algebra without unit, we construct a duality between the category of locally compact Hausdorff spaces and the category of nonunital commutative C^*-algebras, whereas for a given unital C^*-algebra, we have a duality between the category of compact Hausdorff spaces and the category of unital commutative C^*-algebras. We close up this chapter with some consequences of Gelfand Duality.

In Chapter 4 we analyze the fundamentals of the theory of representations in C^*-algebras, making our way to the Gelfand-Naimark Theorem (Theorem 4.1.35), which associates an arbitrary C^*-algebra, with a subalgebra of $B(H)$, i.e. the algebra of all linear and bounded operators on a Hilbert space. We define certain notions in the theory of von Neumann algebras and we prove a generalized (up to commutativity) version of Gelfand Theorem (Theorem 4.3.15), which associates the hermitian elements of an arbitrary C^*-algebra, with spectral projections of q-continuous operators, that are q-compact and do not contain 0. Finally, we comment on some topics of modern research incorporated in the field of Noncommutative Geometry.

Chapter 1

Basics of Banach Algebras

In this chapter we present some preliminaries for the theory of Banach algebras. Particularly, we present the basic concepts that lead to the construction of a Banach algebra, that is, the notions of an algebra, a normed algebra and an algebra norm. Then we define the spectrum of an element in a Banach algebra. Using this notion, we go further deep into the theory of Banach algebras and Banach algebra homomorphisms. We prove some useful theorems that concern properties of the spectrum, from which we derive results, such as the fact that the spectrum of an element in a Banach algebra is compact and nonempty. Furthermore, we define the notion of the Gelfand transform and we prove that the maximal ideal space Δ of a commutative Banach algebra A is a Hausdorff space which is compact if, and only if, A is unital. We will use the aforementioned notions in the proofs of Gelfand Theorems in Chapter 2.

1.1 Introduction to Banach Algebras

We begin this section with some definitions about various kinds of algebras.

Definition 1.1.1 ([33], p.227)**.** An **algebra** is a vector space A, over some field K, such as the field of complex numbers $\mathbb{C}$, together with a binary operation $* : (x, y) \to x * y$, called **multiplication**, that satisfies the following

(i) $(a * b) * c = a * (b * c)$ (Associativity).

(ii) $(a + b) * c = a * b + a * c$ (Right distributivity).

(iii) $a * (b + c) = a * b + a * c$ (Left distributivity).

(iv) $\lambda \cdot (a * b) = (\lambda \cdot a) * b = a * (\lambda \cdot b)$, for all $a, b, c \in A$ and all $\lambda \in K$.

Definition 1.1.2 ([33], p.227)**.** A **unital algebra** A is an algebra, which has a **multiplicative identity (called unit)**, that is, a nonzero element 1_A, such that

$$1_A * a = a * 1_A = a, \quad \text{for all } a \in A.$$

Definition 1.1.3 ([33], p.228)**.** A **commutative algebra** A is an algebra for which

$$a * b = b * a, \quad \text{for all } a, b \in A.$$

Definition 1.1.4 ([33], p.230)**.** A map $f : A \to B$ between algebras is called an **algebra homomorphism**, if it is linear and, also, if it is multiplicative, that is

$$f(a * a') = f(a) * f(a'), \quad \text{for all } a, a' \in A.$$

Definition 1.1.5 ([29], p.5)**.** If A and B are unital algebras with units 1_A and 1_B respectively, then an algebra homomorphism $f : A \to B$ is called **unital**, if

$$f(1_A) = 1_B.$$

Definition 1.1.6 ([29], p.1)**.** A subset B of an algebra A is called a **subalgebra** of A, if B is an algebra by itself, under the operations inherited from A.

Definition 1.1.7 ([29], p.1)**.** A subalgebra B of a unital algebra A is called a **unital subalgebra** of A, if B contains 1_A (the unit of A).

Remark 1.1.8. In order for B to be a unital subalgebra of A, it is not enough that B has a unit 1_B of its own. It must contain the unit 1_A of A, in which case we demand that $1_A = 1_B$, as proven in Remark 1.1.9. Thus, an algebra B can be both unital and a subalgebra of A, without being a unital subalgebra of A.

Remark 1.1.9. An algebra A can have, at most, one multiplicative identity. If this was not the case, there would exist elements $1_A, 1'_A \in A$, with $1_A \neq 1'_A$, such that

$$1_A = 1_A * 1'_A = 1'_A$$

which is a contradiction.

Example 1.1.10. Some elementary examples of algebras are the fields of **real** and **complex** numbers, denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Both are unital and commutative. □

Example 1.1.11. If X is a compact Hausdorff space, then the set $C(X) = C(X, \mathbb{C})$, of all continuous complex-valued functions defined on X, is an algebra under the usual operations of addition, multiplication and scalar multiplication. It is both unital and commutative. □

Example 1.1.12. If X is a locally compact Hausdorff space, then, under the usual operations of addition, multiplication and scalar multiplication, the family $C_0(X) = C_0(X, \mathbb{C})$ of all continuous complex-valued functions defined on X that **vanish at infinity**, meaning that

$$\text{for all } \epsilon > 0, \text{ there exists } K \subseteq X \text{ compact, such that } |f(x)| < \epsilon, \quad \text{for all } x \notin K,$$

is a commutative algebra, which is nonunital. Indeed, if it was unital, there would exist a function $1 \in C_0(X)$, such that

$$f \cdot 1 = 1 \cdot f = f, \quad \text{for all } f \in C_0(X).$$

But this can not happen, since $1 \in C_0(X)$ is the usual constant function of value 1, which does not vanish. □

Example 1.1.13. Let A be an algebra. The family $M_n(A)$ of $n \times n$ matrices consisting of elements of A is a unital algebra, under the usual matrix operations of addition, multiplication and scalar multiplication, if, and only if, A is unital. In such a case, the unit of $M_n(A)$ is I_n, the $n \times n$ matrix with 1_A on the main diagonal and zeros elsewhere. It is noncommutative for $n > 1$. □

Example 1.1.14. If V is a normed linear space, then the set

$$B(V) = \{T : \ T \text{ is a linear and bounded map from } V \text{ onto itself}\}$$

is a unital algebra. If $\dim V > 1$, then $B(V)$ is noncommutative. Indeed, each $S, T \in B(V)$ has a matrix representation. Following Example 1.1.13, we deduce that $ST \neq TS$ since $M_n(V)$ is noncommutative, for $n > 1$. □

Definition 1.1.15 ([29], p.1)**.** If an algebra A is equipped with a norm $\|\cdot\| : A \to \mathbb{R}$, satisfying

$$\|xy\| \leq \|x\|\|y\|, \quad \text{for all } x, y \in A,$$

then A is called a **normed algebra**, and $\|\cdot\| : A \to \mathbb{R}$ is called an **algebra norm**. If A is unital, with unit 1_A, then the norm must satisfy the relation $\|1_A\| = 1$.

Definition 1.1.16 ([29], p.2)**.** A complete normed algebra is called a **Banach algebra**.

Definition 1.1.17 ([29], p.5)**.** A map $f : A \to B$ between Banach algebras is called a **Banach algebra homomorphism**, if it is both an algebra homomorphism and a bounded linear map.

Definition 1.1.18 ([33], p.275)**.** Suppose K is a field in which an involution $a \mapsto a'$ has been defined. An **involution** on an algebra A is a map $x \mapsto x^*$, from A into A, such that for all $x, y \in A$ and all $a \in K$, it holds that

(i) $(x + y)^* = x^* + y^*$

(ii) $(ax)^* = a' x^*$

(iii) $(x^*)^* = x$

(iv) $(xy)^* = y^* x^*$

Remark 1.1.19. In Definition 1.1.1 we defined the algebra A over the field K. In most cases, K is the field of complex numbers $\mathbb{C}$, so property (ii) in the Definition 1.1.18 can be replaced by $(ax)^* = \overline{a}x$, where $a \mapsto \overline{a}$ is the usual complex conjugation.

Definition 1.1.20 ([29], p.35)**.** An algebra A with an involution is called a ***-algebra** and an algebra homomorphism $f : A \to B$ between *-algebras which preserves involution, meaning that

$$f(a^*) = f(a)^*, \quad \text{for all } a \in A$$

is called a ***-homomorphism**. If the *-algebras A and B are unital, then $f : A \to B$ is said to be **unital**, if $f(1_A) = 1_B$.

Definition 1.1.21 ([29], p.35)**.** Let S be a subset of a *-algebra A and define the set

$$S^* = \{s^* : s \in S\}.$$

If $S^* = S$, then S is called **self-adjoint**.

Definition 1.1.22 ([29], p.35)**.** A nonempty self-adjoint subalgebra of a *-algebra A is called a ***-subalgebra of** A.

The next definition concerns C^*-algebras, which is the key component that leads to Gelfand Duality.

Definition 1.1.23 ([29], p.36)**.** A ***C**-algebra** is a Banach algebra A with an isometric involution that satisfies

$$\|a^*a\| = \|a\|^2, \quad \text{for all } a \in A.$$

This property of the norm is usually referred to, as the ***C**-condition** and an algebra norm that satisfies this condition is called a ***C**-norm**.

Definition 1.1.24 ([29], p.36)**.** A ***C**-subalgebra** of a C^*-algebra A is a closed *-subalgebra of A.

Example 1.1.25. The field $\mathbb{C}$ of complex numbers with the usual operations and the complex conjugation, as involution, is a unital commutative C^*-algebra with unit being the element $1 \in \mathbb{C}$. □

Example 1.1.26. If S is a set, then the algebra $l^\infty(S)$ of all bounded complex-valued functions on S is a unital commutative C^*-algebra, under the usual operations and the complex conjugation as involution, with respect to the norm $\|f\| = \sup\{f(x) : x \in X\}$.

Example 1.1.27. If X is a compact Hausdorff space, then the algebra $C(X) = C(X, \mathbb{C})$ of all continuous complex-valued functions on X, is a unital commutative C^*-algebra, with the usual operations and the complex conjugation as involution, with respect to the norm $\|f\| = \sup\{f(x) : x \in X\}$. Its unit is the constant function with value 1. □

Example 1.1.28. If X is a locally compact Hausdorff space, then the algebra $C_0(X) = C_0(X, \mathbb{C})$ of all continuous complex valued functions on X that vanish at infinity is a nonunital commutative C^*-algebra, under the usual operations and the complex conjugation as involution, with respect to the norm $\|f\| = \sup\{f(x) : x \in X\}$. The fact that it is nonunital was proved in Example 1.1.12. □

Example 1.1.29. If (X, μ) is a measure space, then the algebra $L^\infty(X, \mu)$ of (classes of) essentially bounded complex-valued measurable functions on X is a unital commutative C^*-algebra, under the usual operations and the complex conjugation as involution, with respect to the essential supremum norm $\|f\| = \sup\{f(x) : x \in X\}$.

Example 1.1.30. If (X, μ) is a measure space, then the algebra $B_\infty(X, \mu)$ of all bounded complex-valued measurable functions on X is a unital commutative C^*-algebra under the usual operations and the complex conjugation as involution, with respect to the norm $\|f\| = \sup\{f(x) : x \in X\}$.

Example 1.1.31. The algebra $B(H)$ of all bounded linear operators on a Hilbert space H is a unital noncommutative C^*-algebra with the involution $T \mapsto T^*$, that maps each operator to its adjoint. Its unit is the identity operator $1 \in B(H)$ that maps each bounded subset $V \subseteq H$ to itself. The fact that it is noncommutative was proved in Example 1.1.14. □

Example 1.1.32. We have seen in Example 1.1.13 that the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices of complex numbers is a unital algebra, since the field of complex numbers $\mathbb{C}$ is a unital algebra with unit $1 \in \mathbb{C}$ (Example 1.1.25). In order to make it into a C^*-algebra, we relate each $n \times n$ matrix in $M_n(\mathbb{C})$ with the bounded linear operator in $B(C^n)$ which it represents. The unit of $M_n(\mathbb{C})$ is I_n, the $n \times n$ matrix with ones on the main diagonal and zeros elsewhere. □

1.2 The Spectrum

In this section we define the spectrum of an element in a Banach algebra. We prove a theorem which states that the spectrum, regarded as a subset of $\mathbb{C}$, is compact and nonempty and, also, that the spectral radius formula is relevant and applicable (Theorem 1.2.14). We conclude this section with the Gelfand-Mazur Theorem (Theorem 1.2.18). All algebras are defined over the field of complex numbers $\mathbb{C}$, unless explicitly stated otherwise.

Definition 1.2.1 ([33], p.234)**.** If A is a unital Banach algebra and $x \in A$, then the **spectrum** $\sigma(x)$ of x is the set

$$\sigma(x) = \{\lambda \in \mathbb{C} : (\lambda 1_A - x) \notin \mathrm{inv}(A)\},$$

where $\mathrm{inv}(A)$ is the group of all invertible elements of A.

Remark 1.2.2. The fact that $\mathrm{inv}(A)$ is a group is evident. For if $x, y \in \mathrm{inv}(A)$, then $x^{-1}, y^{-1} \in \mathrm{inv}(A)$ and

- $xy = (y^{-1}x^{-1})^{-1} \Rightarrow xy \in \mathrm{inv}(A)$,
- $x^{-1}x = xx^{-1} = 1_A \Rightarrow 1_A \in \mathrm{inv}(A)$,
- $x1_A = 1_A x = x$.

Definition 1.2.3 ([33], p.234)**.** The complement of $\sigma(x)$ is called the **resolvent set** of x and consists of all $\lambda \in \mathbb{C}$, for which $(\lambda 1_A - x)^{-1}$ exists.

Definition 1.2.4 ([33], p.234)**.** The **spectral radius** of x is the number

$$p(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Remark 1.2.5. The spectral radius of x can be intuitively thought of, as the radius of the smallest closed disk in $\mathbb{C}$, centered at the origin, that contains $\sigma(x)$. Of course, the definition of the spectral radius of an element x does not make sense if $\sigma(x) = \emptyset$, but that is never the case, as we shall see in Theorem 1.2.14.

Remark 1.2.6. We observe that if an element $x \in A$ is invertible, then $xx^{-1} = x^{-1}x = 1_A$. This implies that x is nonzero. On the other hand, it is well known that in any field, every nonzero element is invertible. So, in any field, we have that an element is nonzero if, and only if, it is invertible.

Example 1.2.7. If $z \in \mathbb{C}$, then

$$\sigma(z) = \{\lambda \in \mathbb{C} : (\lambda - z) \notin \text{inv}(\mathbb{C})\}.$$

However, the condition $\lambda - z \notin \text{inv}(\mathbb{C})$ implies that $\lambda - z = 0$. Thus $\sigma(z) = \{z\}$. $\square$

Example 1.2.8. Let X be a compact Hausdorff space and $f \in C(X)$. Then

$$\begin{aligned} \sigma(f) &= \{\lambda \in \mathbb{C} : \ (\lambda - f) \notin \text{inv}(C(X))\} \\ &= \{\lambda \in \mathbb{C} : (\lambda - f)(x) = 0, \quad \text{for all } x \in X\}. \end{aligned}$$

However, the condition $(\lambda - f)(x) = 0$ for all $x \in X$ implies that $\lambda = f(x)$ for all $x \in X$. Thus $\sigma(f) = \{f(x) : x \in X\} = f(X)$. $\square$

Example 1.2.9. The set $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$ is a unital C^*-algebra, as we saw in Example 1.1.32. For any matrix $A \in M_n(\mathbb{C})$, we have that

$$\sigma(A) = \{(\lambda I_n - A) \notin \text{inv}(M_n(\mathbb{C}))\}.$$

The above condition implies that $det(\lambda I_n - A) = 0$. Thus, $\sigma(A)$ consists of all eigenvalues of A. $\square$

Now, we will state and prove some useful tools, which will be needed in spectral theory.

Theorem 1.2.10 ([33], p.231)**.** *Suppose A is a unital Banach algebra with unit 1_A and $x \in A$ with $\|x\| < 1$. Then*

(i) $1_A - x \in \text{inv}(A)$.

(ii) $\|(1_A - x)^{-1} - 1_A - x\| \leq \dfrac{\|x\|^2}{1 - \|x\|}$.

Proof. (i) By Definition 1.1.15 we can clearly see that $\|x^n\| \leq \|x\|^n$, for all $x \in A$. Consider the elements

$$S_n = 1_A + x + x^2 + \dots x^n, \quad \text{for all } n \in \mathbb{N}. \tag{1.1}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have that

$$\begin{aligned} \|S_n - S_m\| &= \|x^{n+1} + \cdots + x^m\| \\ &\leq \|x^{n+1}\|\|1_A + x + \cdots + x^{m-n-1}\|. \end{aligned}$$

Since $\|x\| < 1$, for all $x \in A$, we have that $x^n \longrightarrow 0$, as $n \longrightarrow +\infty$. Thus

$$\|S_n - S_m\| \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

So $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. However, A is a Banach algebra and, therefore, $(S_n)_{n\in\mathbb{N}}$ is convergent, that is, there exists an element $s \in A$, such that $S_n \longrightarrow s$. Since $x^n \longrightarrow 0$ and

$$\begin{aligned} S_n(1_A - x) &= (1_A + x + x^2 + \dots x^n)(1_A - x) = (1_A - x^{n+1}) \\ &= (1_A - x)(1_A + x + x^2 + \dots x^n) = (1_A - x)S_n \end{aligned}$$

we have that

$$s(1_A - x) = \lim_{n\to+\infty} S_n(1_A - x) = \lim_{n\to+\infty} (1_A - x)S_n = (1_A - x)s$$

and

$$\lim_{n\to+\infty} (1_A - x^{n+1}) = 1_A.$$

So

$$s(1_A - x) = 1_A = (1_A - x)s$$

which means that s is the inverse of $1_A - x$ and so $(1_A - x) \in \mathrm{inv}(A)$.

(ii) By (1.1), we have that

$$\begin{aligned} \|(1_A - x)^{-1} - 1_A - x\| &= \|s - 1_A - x\| = \|x^2 + x^3 + \dots\| \\ &\le \sum_{n=2}^{+\infty} \|x\|^n = \|x\|^2 \sum_{n=0}^{+\infty} \|x\|^n = \frac{\|x\|^2}{1 - \|x\|}. \end{aligned}$$

Thus

$$\|(1_A - x)^{-1} - 1_A - x\| \le \frac{\|x\|^2}{1 - \|x\|}.$$

□

Theorem 1.2.11 ([33], p.235)**.** *Suppose A is a unital Banach algebra, $x \in \mathrm{inv}(A)$ and $h \in A$, such that $\|h\| < \frac{1}{2}\|x^{-1}\|^{-1}$. Then*

(i) $(x + h) \in \mathrm{inv}(A)$.

(ii) $\|(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1}\| \le 2\|x^{-1}\|^3\|h\|^2$.

Proof. (i) Since $x + h = x(1_A + x^{-1}h)$ and $\|x^{-1}h\| < \frac{1}{2}$, Theorem 1.2.10 implies that $(1_A + x^{-1}h) \in \mathrm{inv}(A)$. Because $\mathrm{inv}(A)$ is a group, we deduce that $(x+h) \in \mathrm{inv}(A)$.

(ii) We have that

$$\begin{aligned} \|(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1}\| &\le \|(1_A + x^{-1}h)^{-1} - 1_A + x^{-1}h\|\|x^{-1}\| \\ &\le \frac{\|x^{-1}h\|^2}{1 - \|x^{-1}h\|}\|x^{-1}\| \le 2\|x^{-1}\|^3\|h\|^2. \end{aligned}$$

The last inequality is a consequence of Theorem 1.2.10. □

Theorem 1.2.12 ([33], p.235). *If A is a unital Banach algebra, then* $\mathrm{inv}(A)$ *is open and for all $x \in \mathrm{inv}(A)$, the map $x \mapsto x^{-1}$ is a homeomorphism.*

Proof. Using Theorem 1.2.11 we have, for any $x \in \mathrm{inv}(A)$ and any element $h \in A$ with $\|h\| < \frac{1}{2}\|x^{-1}\|^{-1}$, that $(x+h) \in \mathrm{inv}(A)$. So, we get

$$\|x - (x+h)\| = \|h\| < \frac{1}{2}\|x^{-1}\|^{-1}.$$

Thus, the ball centered at $x \in \mathrm{inv}(A)$ with radius $\frac{1}{2}\|x^{-1}\|^{-1}$ is entirely contained in $\mathrm{inv}(A)$. Hence $\mathrm{inv}(A)$ is open.

Now, let $f : \mathrm{inv}(A) \to \mathrm{inv}(A)$ be defined as $f(x) = x^{-1}$. For any $y \in \mathrm{inv}(A)$ we have that $f(y^{-1}) = y$, so f is surjective. Also

$$\begin{aligned} \ker f &= \{x \in \mathrm{inv}(A) : f(x) = 0\} \\ &= \{x \in \mathrm{inv}(A) : x^{-1} = 0\} \\ &= \{0\}. \end{aligned}$$

Therefore, f is injective. Thus, we have that $f^{-1}(x) = x^{-1}$ and for any open set $U \subset \mathrm{inv}(A)$, with $x \in U$, we get

$$f(f^{-1}(x)) = f(x^{-1}) = x \in U.$$

Thus, f is continuous and so is f^{-1}. Hence, the map f is a homeomorphism. $\square$

Recall that the index of a complex number $z \in \mathbb{C}$, with respect to a closed path Γ that does not pass through z, is the integer $\mathrm{Ind}_\Gamma(z) = \dfrac{1}{2\pi i}\int_\Gamma \dfrac{d\zeta}{\zeta - z}$. Next, we recall the Cauchy Theorem, omitting its proof.

Theorem 1.2.13 (Cauchy Theorem, [35], p.259). *Suppose $U \subset \mathbb{C}$ is an open set and $f : U \to \mathbb{C}$ is an holomorphic function.*

(i) If γ is a closed path in U such that $\mathrm{Ind}_\gamma(a) = 0$*, for all $a \notin U$, then*

$$\int_\gamma f(z)dz = 0$$

and if $z \in U$, with $\mathrm{Ind}_\gamma(z) = 1$*, then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\lambda)}{\lambda - z}d\lambda.$$

(ii) If γ_0 and γ_1 are closed paths in U, such that

$$\mathrm{Ind}_{\gamma_0}(a) = \mathrm{Ind}_{\gamma_1}(a), \quad \textit{for all } a \notin U,$$

then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Theorem 1.2.14 ([33], p.235)**.** *Suppose that A is a unital Banach algebra and $x \in A$. The following hold.*

(i) The spectrum $\sigma(x)$ of x is compact and nonempty.

(ii) The spectral radius $p(x)$ of x, satisfies the spectral radius formula

$$p(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

Proof. (i) First, we will show that $\sigma(x)$ is bounded. In order to do this, we will show that $p(x) \leq \|x\|$. If $\lambda \in \sigma(x)$, with $|\lambda| \leq \|x\|$, then it is evident that

$$p(x) = \sup\{|\lambda| : \ \lambda \in \sigma(x)\} \leq \|x\|.$$

If $\lambda \in \sigma(x)$ with $|\lambda| > \|x\|$, then

$$\|\lambda^{-1}x\| \leq |\lambda^{-1}|\|x\| < |\lambda|^{-1}|\lambda| = 1.$$

So, by Theorem 1.2.10, we have that $(1_A - \lambda^{-1}x) \in \mathrm{inv}(A)$ and so does the element $\lambda 1_A - x = \lambda(1_A - \lambda^{-1}x)$, because $\mathrm{inv}(A)$ is a group. Thus $(\lambda 1_A - x) \in \mathrm{inv}(A)$ or, equivalently, $\lambda \notin \sigma(x)$, which is a contradiction. So, in any case, we have that $p(x) \leq \|x\|$. Consequently, $\sigma(x)$ is a bounded set.

In order to prove that $\sigma(x)$ is closed, we define the map $g : \mathbb{C} \to A$, by

$$g(\lambda) = \lambda 1_A - x, \quad \text{for all } \lambda \in \mathbb{C}.$$

We can see that g is continuous. Indeed, let $\lambda \in \mathbb{C}$. For any sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ with $\lambda_n \longrightarrow \lambda$, we have that

$$\lambda_n 1_A - x \longrightarrow \lambda 1_A - x.$$

So, the complement of $\sigma(x)$ is the set $\Omega = g^{-1}(\mathrm{inv}(A))$, which is open by Theorem 1.2.12. Thus $\sigma(x)$ is compact.

We will now prove that $\sigma(x) \neq \emptyset$. Define the map $f : \Omega \to \mathrm{inv}(A)$, by

$$f(\lambda) = (\lambda 1_A - x)^{-1}, \quad \text{for all } \lambda \in \Omega.$$

For the remainder of the proof, we apply Theorem 1.2.11 for $h = (\mu - \lambda)1_A$ and $\lambda 1_A - x$ in place of x, where $\mu, \lambda \in \Omega$ and μ is arbitrarily close to λ. So, from Theorem 1.2.11, we have that

$$\|f(\mu) - f(\lambda) + (\mu - \lambda)f^2(\lambda)\| \leq 2\|f(\lambda)\||\mu - \lambda|^2$$

and the continuity of the norm implies that

$$\lim_{\mu\to\lambda} \frac{f(\mu) - f(\lambda)}{\mu - \lambda} = -f^2(\lambda).$$

Thus, f is a strongly holomorphic function. Recall that a function $f : \Omega \to A$ is called **strongly holomorphic**, if the limit

$$\lim_{\mu \to \lambda} \frac{f(\mu) - f(\lambda)}{\mu - \lambda}$$

exists, for all $\lambda \in \Omega$. For any $\lambda \in \Omega$ we have that $|\lambda| > \|x\|$. Using an argument, similar to the one used in Theorem 1.2.10, we get that

$$f(\lambda) = \sum_{n=0}^{+\infty} \lambda^{-n-1} x^n = \lambda^{-1} 1_A + \lambda^{-2} x + \dots \tag{1.2}$$

This series converges uniformly on every circle centered at 0, with radius $r > \|x\|$. Indeed, for every $\epsilon > 0$, there exists a $k_\epsilon \in \mathbb{N}$, such that for all $k > k_\epsilon$ and all $x \in A$, we have that

$$\sum_{n=0}^{+\infty} \lambda^{-n-1} x^n - \sum_{n=0}^{k} \lambda^{-n-1} x^n = \sum_{n=k+1}^{+\infty} \lambda^{-n-1} x^n \longrightarrow 0, \text{ as } k \longrightarrow +\infty.$$

So

$$\left\| f(\lambda) - \sum_{n=0}^{k} \lambda^{-n-1} x^n \right\| < \epsilon.$$

Hence, we can integrate (1.2) term-by-term and get

$$x^n = \frac{1}{2\pi i} \int_{\Gamma_r} \lambda^n f(\lambda) d\lambda, \quad \text{for any } n \in \mathbb{N} \tag{1.3}$$

Now, suppose that $\sigma(x)$ is empty. Then $\Omega = \mathbb{C}$ and by Cauchy Theorem (Theorem 1.2.13), every integral in (1.3) would be equal to 0. But, when $n = 0$, we deduce from (1.3) that $1 = 0$. Thus $\sigma(x) \neq \emptyset$.

(ii) If $\lambda \notin \sigma(x)$, which means that $\lambda \in \Omega$ and $|\lambda| > p(x)$, then by Cauchy Theorem (Theorem 1.2.13) we can replace the condition $r > \|x\|$, with $r > p(x)$ and the integrals in (1.3) would remain the same. This is true, since we integrate in closed paths. We define

$$M(r) = \max_\theta \|f(re^{i\theta})\|, \quad \text{for all } r > p(x) \text{ and all } \theta \in [0, 2\pi],$$

and get

$$\begin{aligned} \|x^n\| = \frac{1}{2\pi i} \left\| \int_{\Gamma_r} \lambda^n f(\lambda) d\lambda \right\| &\leq \frac{1}{2\pi i} \int_{\Gamma_r} \left\|(re^{i\theta}) f(re^{i\theta}) i r e^{i\theta}\right\| d\theta \\ &\leq r^{n+1} \frac{1}{2\pi i} \int_{\Gamma_r} \left\|e^{i\theta(n+1)}\right\| \left\|f(re^{i\theta})\right\| d\theta \\ &\leq r^{n+1} M(r). \end{aligned}$$

So, we have that $\|x^n\|^{1/n} \leq r(M(r))^{1/n}$. By taking limits, this inequality yields that $\limsup_{n\to\infty} \|x^n\|^{1/n} \leq r$. The last inequality implies that

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leq p(x) \tag{1.4}$$

On the other hand, if $\lambda \in \sigma(x)$, then

$$\lambda^n 1_A - x^n = (\lambda 1_A - x)(\lambda^{n-1} 1_A + \cdots + x^{n-1}),$$

which implies that the element $\lambda^n 1_A - x^n$ is not invertible, since $\lambda 1_A - x$ is not invertible. Thus $\lambda^n \in \sigma(x)$ and by (1.1) we deduce that

$$|\lambda^n| \leq \|x^n\|, \quad \text{for all } n \in \mathbb{N}.$$

Hence

$$p(x) \leq \inf_{n\geq 1} \|x^n\|^{1/n}. \tag{1.5}$$

Combining the relations (1.4) and (1.5), we get

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leq \inf_{n\geq 1} \|x^n\|^{1/n}.$$

Since the converse inequality always holds, we deduce that

$$p(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}. \qquad \square$$

Definition 1.2.15 ([16], p.19)**.** A map $\phi : A \to B$ between Banach algebras is called an **isometry**, if

$$\|\phi(x)\| = \|x\|, \text{ for every } x \in A.$$

Definition 1.2.16 ([16], p.93)**.** A map $\phi : A \to B$ is called an **isomorphism**, if it is an injective Banach algebra homomorphism from A onto B.

Definition 1.2.17 ([19], p.13)**.** A map $\phi : A \to B$ is called an **isometric isomorphism**, if it is both an isometry and an isomorphism.

Theorem 1.2.18 (Gelfand–Mazur, [33], p.237)**.** *If A is a unital Banach algebra in which every nonzero element is invertible, then A is isometrically isomorphic to the complex field $\mathbb{C}$.*

Proof. If $x \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, with $\lambda_1 \neq \lambda_2$, then at most one of $(\lambda_1 1_A - x)$ and $(\lambda_2 1A - x)$ is 0, because if both were equal to 0, this would imply that $\lambda_1 = \lambda_2$. Let's say that $(\lambda_2 1_A - x)$ is 0. Then, $(\lambda_1 1_A - x)$ is non-zero, thus it is invertible. This, however, is a contradiction, since λ_1 belongs in the spectrum.

Thus, $\sigma(x)$ contains only one point, say $\lambda(x)$. Hence, $\lambda(x)1_A - x$ is not invertible and, by the hypothesis, we deduce that $x = \lambda(x)1_A$. So, the map $x \mapsto \lambda(x)$ is an isomorphism of A onto $\mathbb{C}$. This map is also an isometry, since

$$|\lambda(x)| = \|\lambda(x)1_A\| = \|x\|. \qquad \square$$

Lemma 1.2.19 ([33], p.238)**.** *Suppose A is a Banach algebra, $(x_n)_{n\in\mathbb{N}}$ is a sequence in* $\mathrm{inv}(A)$ *and $x \in \overline{\mathrm{inv}(A)}$ (the closure of* $\mathrm{inv}(A)$*), such that $x_n \longrightarrow x$, as $n \longrightarrow \infty$. Then*

$$\|x_n^{-1}\| \longrightarrow \infty, \ \ \textit{as } n \longrightarrow \infty.$$

Proof. Suppose that the conclusion is false. Then there would exist $M < +\infty$, such that $\|x_n^{-1}\| < M$. We fix an $n \in \mathbb{N}$, such that

$$\|x_n^{-1}\| < M \quad \text{and} \quad \|x_n - x\| < \frac{1}{M}.$$

Hence

$$\|1_A - x_n^{-1}x\| = \|x_n^{-1}(x_n - x)\| < 1.$$

Thus, by Theorem 1.2.10, we have that $(1_A - (1_A - x_n^{-1}x)) \in \mathrm{inv}(A)$, which implies that $x_n^{-1}x \in \mathrm{inv}(A)$. Since $x = x_n(x_n^{-1}x)$ and $\mathrm{inv}(A)$ is a group, it follows that $x \in \mathrm{inv}(A)$. This means that $\mathrm{inv}(A)$ is closed, which is a contradiction. □

An application of Lemma 1.2.19 is Theorem 1.2.20, whose conclusion is the same as that of Gelfand–Mazur Theorem (Theorem 1.2.18), but with a different hypothesis.

Theorem 1.2.20 ([33], p.239)**.** *If A is a Banach algebra and if there exists an $M < \infty$, such that*

$$\|x\|\|y\| \leq M\|xy\|, \quad \textit{for all } x, y \in A,$$

then A is isometrically isomorphic to $\mathbb{C}$.

Proof. Let $y \in \overline{\mathrm{inv}(A)}$. Then, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{inv}(A)$, such that $y_n \longrightarrow y$. So, by Lemma 1.2.19, we have that $\|y_n^{-1}\| \longrightarrow \infty$. Using the hypothesis, we have that

$$\|y_n\|\|y_n^{-1}\| \leq M\|1_A\| = M. \tag{1.6}$$

So, for (1.6) to hold, we must have $\|y_n\| \longrightarrow 0$. Therefore, $y = 0$. By this construction, if we let $x \in A$ and λ be a point in the closure of $\sigma(x)$, then the element $\lambda 1_A - x$ is a point in the closure of $\mathrm{inv}(A)$ for which $\lambda 1_A - x = 0$. This implies that $x = \lambda 1_A$. Thus

$$A = \{\lambda 1_A : \ \lambda \in \mathbb{C}\}$$

and the proof is complete. □

1.3 Ideals

This section is about providing some algebraic background that concerns various types of ideals and a useful theorem about them. Particularly, we prove that every proper ideal of a unital commutative algebra A is contained in a maximal ideal of A and that every maximal ideal of a commutative Banach algebra is a closed set. We make use of Hausdorff maximal principle (Lemma 1.3.8) and Zorn Lemma (Lemma 1.3.9) for the proof of the aforementioned theorem (Theorem 1.3.10).

Definition 1.3.1 ([29], p.4)**.** A subset J of an algebra A is called a **left ideal** (resp. **right ideal**), if all of the following conditions hold

(i) J is a vector subspace of A.

(ii) $xy \in J$ (resp. $yx \in J$), for all $x \in A$, $y \in J$.

A subset that is both a left ideal and a right ideal is called a **two-sided ideal**. If $J \neq A$, then J is called a **proper ideal**. **Maximal ideals** are proper ideals that are not contained in any larger proper ideal, meaning that there does not exist $I \subset A$ with I being proper, such that

$$J \subset I \subset A.$$

Proposition 1.3.2 ([33], p.264)**.** *(i) If A is a unital commutative algebra, then no proper ideal of A contains any invertible element of A.*

(ii) If J is an ideal in a commutative Banach algebra A, then its closure $\overline{J}$ is also an ideal.

Proof. (i) Suppose that J is a proper ideal in A and that there exists an element $x \in \mathrm{inv}(A)$, such that $x \in J$. Then, by the definition of the ideal, we have that $x^{-1}x \in J$, which implies that $1_A \in J$. Thus, for any $y \in A$, we get $y1_A \in J$ or, equivalently, $J = A$, which is a contradiction, since J is proper.

(ii) Suppose that J is an ideal in a commutative Banach algebra A, where A is taken over some field K. Then, for any $x, y \in \overline{J}$ and any $\lambda, \mu \in K$, there exist sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in J, such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$. By the continuity of addition and scalar multiplication, we have that

$$\lambda x_n + \mu y_n \longrightarrow \lambda x + \mu y,$$

which implies that $\lambda x + \mu y \in \overline{J}$. So, $\overline{J}$ is a vector subspace of A. Now, for any $z \in \overline{J}$, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in J, such that $z_n \longrightarrow z$. Since multiplication in A is continuous, we have, for any $x \in A$, that $xz_n \longrightarrow xz$, which implies that $xz \in \overline{J}$. Thus $\overline{J}$ is an ideal in A. □

The following definitions and lemmas are derived from [20].

Definition 1.3.3. A **partial order** is a binary relation $\preceq$ over a set P, such that for all $a, b, c \in P$, the following conditions are satisfied

(i) $a \preceq a$ (Reflexivity).

(ii) If $a \preceq b$ and $b \preceq a$, then $a = b$ (Antisymmetry).

(iii) If $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (Transitivity).

A set with a partial order is called a **partially ordered set**.

Definition 1.3.4. If $(P, \preceq)$ is a partially ordered set, then an **upper bound** of a subset X of P is an element u of P, such that $x \preceq u$, for all $x \in X$.

Definition 1.3.5. A **maximal element** of a subset X of a partially ordered subset $(P, \preceq)$ is an element m of X, such that, for all $a \in X$, $m \preceq a$, implies that $a \preceq m$.

Definition 1.3.6. A **chain** of a partially ordered set $(P, \preceq)$ is a subset of P for which any two elements are related, with respect to the partial order $\preceq$ of P. It is called **maximal**, if no other element of P can be added, without losing the property of being totally ordered.

Definition 1.3.7. A partially ordered set that is also a chain is called a **totally ordered set**.

We state two useful lemmas before introducing a key theorem of our study. We omit the proofs of these lemmas, since these are outside the scope of this thesis.

Lemma 1.3.8 (Hausdorff maximal principle)**.** *Let $(P, \preceq)$ be a nonempty partially ordered set. Then, there exists a maximal chain in P.*

Lemma 1.3.9 (Zorn lemma)**.** *Let $(P, \preceq)$ be a partially ordered set. If every nonempty chain has an upper bound in P, then P contains at least one maximal element.*

Theorem 1.3.10 ([33], p.264)**.** *(i) If A is a unital commutative algebra, then every proper ideal of A is contained in a maximal ideal of A.*

(ii) If A is a unital commutative Banach algebra, then every maximal ideal of A is a closed set.

Proof. (i) Let J be a proper ideal of A and P be the collection of all proper ideals of A that contain J, that is

$$P = \{I \subset A : I \text{ is a proper ideal and } J \subset I\}.$$

We observe that P is nonempty, since it contains the proper ideal J and J is an ideal of itself. We partially order P by the usual subset relation. If we let $K \subset \mathbb{N}$ be any countable subset of $\mathbb{N}$, we can, then, define

$$L = \{I_i \in P : i \in K\}$$

to be a maximal totally ordered subcollection of P. The existence of L is guaranteed by the Hausdorff maximal principle (Lemma 1.3.8). Now, let $M = \cup_{i \in K} I_i$. It is evident that the set M, being the union of elements in the totally ordered set L, is a proper ideal that contains J. Since L is maximal, M is a maximal ideal of A, i.e. it is greater than any other element of L.

(ii) Suppose that M is a maximal ideal of A. Since M contains no invertible elements of A (Proposition 1.3.2) and since $\operatorname{inv}(A)$ is open, we deduce that $\overline{M}$ contains no invertible elements of A as well. Indeed, let Π_x be an open neighborhood of $x \in \operatorname{inv}(A)$. We can let Π_x be a subset of $\operatorname{inv}(A)$, since $\operatorname{inv}(A)$ is open. If $\overline{M}$ contained the invertible element $x \in A$, we would have that

$$\emptyset \neq \Pi_x \cap M \subset \operatorname{inv}(A) \cap M = \emptyset.$$

Thus $1_A \notin \overline{M}$ or, equivalently, $\overline{M} \neq A$. This implies that $\overline{M}$ is proper. Since M is maximal in A, we have that $\overline{M} \subset M$. The converse relation always holds, so we deduce that $\overline{M} = M$, that is, M is closed. $\square$

1.4 Quotient Spaces

Definition 1.4.1 ([33], p.29)**.** Let N be a subspace of a vector space X over some field K. For every $x \in X$, let $\pi(x)$ be the coset of N that contains x, that is

$$\pi(x) = x + N.$$

These cosets form a vector space X/N, called the **quotient space of X modulo N**. In this space, addition and scalar multiplication are defined by

$$\pi(x) + \pi(y) = \pi(x + y)$$

and

$$\alpha\pi(x) = \pi(\alpha x), \quad \text{for all } x, y \in X,\ \alpha \in K.$$

Remark 1.4.2. Since N is itself a vector space, the operations of addition and scalar multiplication are well defined.

The next proposition concerns some fundamental properties of the quotient map and, thus, its proof will be omitted.

Proposition 1.4.3 ([16], p.370)**.** *The quotient map $\pi : X \to X/N$ is linear and* $\ker\pi = N$*, where* $\ker\pi$ *is the **kernel** (or **null space**) of π, defined to be the set*

$$\ker\pi = \{x \in X : \pi(x) = 0_{X/N} = N\}.$$

Definition 1.4.4 ([33], p.30)**.** Let X be a normed vector space and N be a vector subspace of X. We define the **quotient norm** on the vector space X/N by

$$\|\pi(x)\| = \|x + N\| = \inf\{\|x - z\| : z \in N\}.$$

Remark 1.4.5. The quotient norm can be interpreted as the distance of $x \in X$ from the vector subspace N of X.

Remark 1.4.6. The origin of X/N is $\pi(0) = N$.

Proposition 1.4.7 ([16], p.70)**.** *(i) The quotient norm $\|\cdot\| : X/N \to \mathbb{R}$ satisfies $\|\pi(x)\| \leq \|x\|$, for every $x \in X$.*

(ii) The quotient map $\pi : X \to X/N$ is continuous, with respect to the quotient norm.

Proof. (i) Since $0 \in \ker\pi$, we immediately obtain

$$\|\pi(x)\| = \inf\{\|x - z\| :\ z \in N\} \leq inf\{\|x - 0\| :\ 0 \in \ker\pi\} = \|x\|.$$

Thus

$$\|\pi(x)\| \leq \|x\|.$$

(ii) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that converges to an element $x \in X$, that is

$$x_n \xrightarrow{\|\cdot\|} x, \text{ as } n \longrightarrow \infty$$

Then, for any $\epsilon > 0$, we have that

$$\|\pi(x_n - x)\| \leq \|x_n - x\| < \epsilon.$$

Thus, $\pi(x_n) \xrightarrow{\|\cdot\|} \pi(x)$ and π is continuous. □

The next theorem plays an important role for the remainder of this thesis. It will be used to prove the completion of the space X/N, through the use of the quotient map, provided that X is complete. We will see, later on, that this statement holds for Banach algebras as well.

Theorem 1.4.8 ([33], p.29)**.** *Let N be a closed subspace of the Banach space X. Then X/N is also a Banach space.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X/N. For each $n \in \mathbb{N}$, we choose an element $x_n \in X$, such that $\pi(x_n) = y_n$. We can do that because the quotient map is onto X/N. In this way, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ in X, which is Cauchy. Indeed, since (y_n) is a Cauchy sequence, we have for all $m, n \in \mathbb{N}$ with $m \geq n$ and any $\epsilon > 0$, that $\|y_m - y_n\| < \epsilon$. This means that $\|\pi(x_m) - \pi(x_n)\| < \epsilon$, which implies that $\|\pi(x_m - x_n)\| < \epsilon$. Thus, there exists a $\delta > 0$, such that $\|x_m - x_n\| < \delta$. So, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and, since X is complete, there exists an element $x \in X$ such that

$$x_n \longrightarrow x, \text{ as } n \longrightarrow \infty.$$

From the continuity of π, we obtain

$$\pi(x_n) = y_n \longrightarrow \pi(x).$$

Thus, the sequence $(y_n)_{n\in\mathbb{N}}$ in X/N converges to $\pi(x)$. Hence, X/N is a Banach space. □

1.5 Quotient Algebras and Homomorphisms

As we saw in Theorem 1.4.8, for a Banach space X and a closed subspace N of X, the quotient space X/N is also a Banach space. A similar argument holds if X is a Banach algebra and N is a closed proper ideal of X. In this section, we will prove the aforementioned statement as well as some basic properties of Δ, the set of all complex homomorphisms on X. Also, we will describe the connection between Δ and the maximal ideals of X.

Proposition 1.5.1 ([33], p.264)**.** *If A and B are commutative Banach algebras and $\phi : A \to B$ is a Banach algebra homomorphism, then* kerϕ *is an ideal of A, which is closed if ϕ is continuous.*

Proof. Let K be the field over which A was defined. For any $x, y \in \ker\phi$ and any $\lambda, \mu \in K$ we have that

$$\begin{aligned}\phi(x) = 0 = \phi(y) &\Rightarrow \{\lambda\phi(x) = 0 = \phi(\lambda x) \text{ and } \mu\phi(y) = 0 = \phi(\mu y)\}\\ &\Rightarrow \phi(\lambda x + \mu y) = \phi(\lambda x) + \phi(\mu y) = 0\\ &\Rightarrow (\lambda x + \mu y) \in \ker\phi\\ &\Rightarrow \ker\phi \text{ is a subspace of } A.\end{aligned}$$

For any $x \in \ker\phi$ and any $y \in A$, we have

$$\begin{aligned}\phi(x) = 0 &\Rightarrow y\phi(x) = 0\\ &\Rightarrow \phi(yx) = 0\\ &\Rightarrow yx \in \ker\phi\\ &\Rightarrow \ker\phi \text{ is an ideal of } A.\end{aligned}$$

Now, suppose that ϕ is continuous. Then for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\ker\phi$ that converges to an element $x \in A$, we have that $\phi(x_n) \longrightarrow \phi(x)$, as $n \longrightarrow \infty$, so $\phi(x) = 0$. Thus $x \in \ker\phi$, which means that $\ker\phi$ is closed. □

Before stating and proving the next theorem, we will show that a multiplication can be defined in A/J. Indeed, for any $x, x', y, y' \in A$, with $\pi(x) = \pi(x')$ and $\pi(y) = \pi(y')$, we have that

$$(x' - x) \in J \text{ and } (y' - y) \in J.$$

Thus, by the identity

$$x'y' - xy = (x' - x)y' + x(y' - y)$$

we have that $(x'y' - xy) \in J$, because J is an ideal. Hence $\pi(x'y') = \pi(xy)$ and, therefore, a multiplication can be defined in A/J by the relation

$$\pi(x)\pi(y) = \pi(xy), \quad \text{for all } x, y \in A.$$

Theorem 1.5.2 ([33], p.264)**.** *If J is a proper closed ideal of the unital Banach algebra A and $\pi : A \to A/J$ is the quotient map, then A/J is a Banach algebra.*

Proof. As we have seen in Theorem 1.4.8, if A is a Banach space, so is A/J. So, we just need to prove, for every $x, y \in A$, that

$$\|\pi(x)\pi(y)\| \leq \|\pi(x)\|\|\pi(y)\| \quad \text{and} \quad \|\pi(1_A)\| = 1,$$

where 1_A is the unit of A.

Let $x_1, x_2 \in A$ and $\epsilon > 0$ be arbitrary. Then, we have that there exist $y_1, y_2 \in J$, such that

$$\|x_i - y_i\| \leq \|\pi(x_i)\| + \epsilon, \quad \text{for } i = 1, 2 \tag{1.7}$$

and, by Proposition 1.4.3,

$$\|\pi(x)\| \leq \|x\|. \tag{1.8}$$

Since

$$(x_1 - y_1)(x_2 - y_2) = x_1x_2 + (y_1y_2 - x_1y_2 - x_2y_1) \in x_1x_2 + J,$$

we have that

$$\begin{aligned}\|\pi(x_1x_2)\| &\leq \|(x_1 - y_1)(x_2 - y_2)\| \\ &\leq \|x_1 - y_1\|\|x_2 - y_2\|.\end{aligned}$$

This last inequality implies that

$$\|\pi(x_1x_2)\| \leq \|\pi(x_1)\|\|\pi(x_2)\|. \tag{1.9}$$

Also, if 1_A is the unit of A, then

$$\pi(x) = \pi(x)\pi(1_A) = \pi(1_A)\pi(x),$$

which means that $\pi(1_A)$ is the unit of A/J. It is a fact that $\pi(1_A) \neq J$ since, otherwise, we would have

$$\pi(1_A) = 1_A + J = 0 + J.$$

The last equality implies that $1_A \in J$ or, equivalently, that $J = A$ which is a contradiction because J is proper. So from (1.9) we obtain

$$\|\pi(1_A)\|\|\pi(1_A)\| \geq \|\pi(1_A)\pi(1_A)\| = \|\pi(1_A)\|.$$

Thus, $\|\pi(1_A)\| \geq 1$. Now, the relation (1.8) implies that $\|\pi(1_A)\| \leq \|1_A\| = 1$. Hence $\|\pi(1_A)\| = 1$ and A/J is a Banach algebra. $\square$

Theorem 1.5.3 ([33], p.265)**.** *Suppose that A is a unital commutative Banach algebra and that Δ is the set of all Banach algebra homomorphisms $f : A \to \mathbb{C}$. Then*

(i) Every maximal ideal of A is the kernel of some $h \in \Delta$.

(ii) If $h \in \Delta$, then ker$h$ *is a maximal ideal of A.*

(iii) An element $x \in A$ is invertible if, and only if, $h(x) \neq 0$, for all $h \in \Delta$.

(iv) An element $x \in A$ is invertible if, and only if, x lies in no proper ideal of A.

(v) $\lambda \in \sigma(x)$ if, and only if, $h(x) = \lambda$, for some $h \in \Delta$.

Proof. (i) Let M be a maximal ideal of A. Then, from Theorems 1.3.10 and 1.5.2, M is closed and A/M is a Banach algebra, since every maximal ideal is proper.

Now, we choose $x \in A$ with $x \notin M$ and define the set

$$J = \{\alpha x + y : \alpha \in A, y \in M\}.$$

It is evident that J is an ideal of A, for if $z \in A$ is an arbitrary element of A, then

$$z(\alpha x + y) = (z\alpha)x + y \in J, \quad \text{for all } \alpha \in A \text{ and all } y \in M.$$

Also, the ideal J is larger than M, since for $\alpha = 1_A$ and $y = 0$, we have that $x \in J$, for every $x \in A$. Thus, $J = A$ and $\alpha x + y = 1_A$, for some $\alpha \in A$ and $y \in M$. If $\pi : A \to A/M$ is the quotient map, then

$$\pi(\alpha x + y) = \pi(1_A),$$

which implies that $\pi(\alpha)\pi(x) = \pi(1_A)$. Hence, every nonzero element $\pi(x)$ of A/M is invertible in A/M. By applying the Gelfand-Mazur Theorem (Theorem 1.2.18), to the Banach algebra A/M, we get an isomorphism $j : A/M \to \mathbb{C}$.

Next, we define the map $h : A \to \mathbb{C}$, via the composition $h = j \circ \pi$. Then, $h \in \Delta$ since for all $x, y \in A$ we have that

$$\begin{aligned} h(xy) = (j \circ \pi)(x \cdot y) &= j(\pi(x) \cdot \pi(y)) \\ &= ((j \circ \pi)(x)) \cdot ((j \circ \pi)(y)) \\ &= h(x) \cdot h(y). \end{aligned}$$

Thus,

$$\begin{aligned} \ker h &= \{x \in A : h(x) = 0 = (j \circ \pi)(x)\} \\ &= \{x \in A : j \circ (x + M) = 0\} \\ &= \{x \in A : (x + M) \in \ker j = 0 + M\} \\ &= \{x \in A : \ x \in M\} \\ &= M. \end{aligned}$$

(ii) If $h \in \Delta$, then $h^{-1}(0) = \{x \in A : h(x) = 0\} = \ker h$ is an ideal of A which is maximal, because it has codimension 1. This is true, since from Theorem 1.3.10, every proper ideal is contained in a maximal one. So, if $\ker h$ was not maximal, but only proper, there would exist a maximal ideal I of A, with codimension < 1. Apparently, its codimension would be 0, which implies that I would not be proper, a contradiction.

(iii) If $x \in \mathrm{inv}(A)$ and $h \in \Delta$, then

$$h(x)h(x^{-1}) = h(xx^{-1}) = h(1_A).$$

Thus, $h(x) \neq 0$. For the converse, we assume that x is not invertible. Then, the set $B = \{\alpha x : \alpha \in A\}$ does not contain 1_A. Hence $B = \{\alpha x : \alpha \in A\} \neq A$ and, thus, B is a proper ideal of A which is contained in a maximal one, by Theorem 1.3.10. Therefore, by (i), there exists an $h \in \Delta$, such that

$$h(\alpha x) = \alpha h(x) = 0,$$

which implies that $h(x) = 0$, for every $x \in A$. This is a contradiction.

(iv) If $x \in \mathrm{inv}(A)$, then it is evident that x does not belong in any proper ideal M of A, since, otherwise, this would imply that 1_A belongs in this ideal, which leads to M not being proper. The converse can be easily shown utilizing (iii), by considering the specific proper ideal $\{\alpha x : \alpha \in A\}$ for $a = 1_A \in A$.

(v) If $\lambda \in \sigma(x)$, then $(\lambda 1_A - x) \notin \mathrm{inv}(A)$. So, we can apply (iii), taking $(\lambda 1_A - x)$ in place of x, in order to deduce that $h(\lambda 1_A - x) = 0$, which implies that $\lambda = h(x)$, for some $h \in \Delta$. For the converse, suppose there exists an $h \in \Delta$, such that $h(x) = \lambda$. Then $h(\lambda 1_A - x) = 0$. Again, by (iii), we have that $(\lambda 1_A - x) \notin \mathrm{inv}(A)$, from which we deduce that $\lambda \in \sigma(x)$. $\square$

1.6 The Gelfand Transform

For the last section of this chapter, we will define the Gelfand transform of an element $x \in X$ and the maximal ideal space, also known as the structure space Δ, which is the space Δ equipped with the, so called, Gelfand topology. For the rest of this chapter, all algebras are unital and over the field $\mathbb{C}$ unless explicitly stated otherwise.

Definition 1.6.1 ([33], p.268)**.** Let Δ be the set of all complex homomorphisms of a commutative Banach algebra A. For each $x \in A$, we define a map $\hat{x} : \Delta \to \mathbb{C}$, with $\hat{x}(h) = h(x)$. This map is called the **Gelfand transform of** x.

Definition 1.6.2 ([33], p.268)**.** For every $x \in A$, let $\hat{A}$ be the set of all Gelfand transforms $\hat{x} : \Delta \to \mathbb{C}$ and Δ^* be the continuous dual of Δ, namely

$$\Delta^* = \{f : \ \Delta \to \mathbb{C} : f \text{ is linear and continuous } \}.$$

The **Gelfand topology of** Δ is the $\mathcal{W}^*$ topology, induced by Δ^*, that is, the weakest topology that makes every $\hat{x} : \Delta \to \mathbb{C}$ continuous. We can define a **norm** on $\hat{A}$ by the formula

$$\|\hat{x}\|_\infty = \max_{h \in \Delta}\{|\hat{x}(h)|\}.$$

Remark 1.6.3. By "weakest" in the definition of Gelfand topology, we mean that $\mathcal{W}^* \subset \mathcal{T}$, for any other topology $\mathcal{T}$ on Δ which makes every $\hat{x}$ continuous. The elements of the $\mathcal{W}^*$ topology are of the form

$$V = W(x_1, x_2, \dots, x_n, \epsilon) = \{h \in \Delta^* : |h(x_i)| < \epsilon, \quad \text{for all } i = 1, 2, \dots, n\}.$$

Remark 1.6.4. We observe that $\hat{A} \subset C(\Delta)$. This implies that the set $\hat{A}$ is a commutative Banach algebra, under the usual operations and the complex conjugation, as involution.

Remark 1.6.5. The map $\|\cdot\|_\infty : \hat{A} \to \mathbb{R}$, defined by the formula

$$\|\hat{x}\|_\infty = \max_{h \in \Delta_A}\{|\hat{x}(h)|\},$$

is a norm. Indeed

$$\begin{aligned} \|\hat{x}(h)\|_\infty = 0 &\Leftrightarrow \max_{h \in \Delta_A}\{|\hat{x}(h)|\} = 0 \\ &\Leftrightarrow \hat{x}(h) = 0, \quad \text{for all } h \in \Delta_A \\ &\Leftrightarrow \hat{x} \equiv 0, \quad \text{for all } \hat{x} \in \hat{A}. \end{aligned}$$

Additionally

$$\begin{aligned}\|\hat{x}+\hat{y}\|_\infty &= \max_{h\in\Delta_A}\{|(\hat{x}+\hat{y})(h)|\}\\ &\leq \max_{h\in\Delta_A}\{|\hat{x}(h)+\hat{y}(h)|\}\\ &= \|\hat{x}\|_\infty+\|\hat{y}\|_\infty, \quad \text{for all } \hat{x},\hat{y}\in\hat{A},\ h\in\Delta_A.\end{aligned}$$

Also

$$\|\alpha\hat{x}\|_\infty = \|\widehat{\alpha x}\|_\infty = \max_{h\in\Delta_A}\{|\widehat{\alpha x}(h)|\} = |\alpha|\|\hat{x}\|_\infty, \quad \text{for all } \alpha\in\mathbb{C},\ \hat{x}\in\hat{A}.$$

We made use of the fact that the Gelfand transform, regarded as a map from A into $\hat{A}$, is a homomorphism, which is proved in Theorem 1.6.15.

Definition 1.6.6. The **radical of** A, denoted by **radA**, is the intersection of all maximal ideals of A. Since every ideal of A contains the zero element of A, we obtain that $\text{rad}A \neq \emptyset$. If $\text{rad}A = \{0\}$, then A is called **semisimple**.

Remark 1.6.7. The definition of a semisimple algebra must not be confused with the definition of a simple algebra, which is the algebra that contains only the trivial ideals, that is $\{0\}$ and the whole algebra.

Remark 1.6.8. Since there exists a one-to-one correspondence between the elements of Δ and the maximal ideals of A, Δ equipped with the Gelfand topology is usually called **the maximal ideal space of** A. The elements of Δ are usually referred to as the **characters of** A, that is, nonzero homomorphisms from A onto $\mathbb{C}$.

We recall some definitions from other branches of mathematics.

Definition 1.6.9 ([20], p.210)**.** A **directed set** is a nonempty set Λ, together with a binary relation $\leq$, that satisfies all of the following properties

(i) $x \leq x$, for all $x \in \Lambda$ (Reflexivity).

(ii) $x \leq y$ and $y \leq z$ implies that $x \leq z$, for all $x, y, z \in \Lambda$ (Transitivity).

(iii) For all $x, y \in \Lambda$ there exists $z \in \Lambda$, such that $x \leq z$ and $y \leq z$ (Upper boundedness).

Definition 1.6.10 ([20], p.210)**.** If Λ is a directed set and X is a nonempty set, then a function $f : \Lambda \to X$ with $f(\lambda) = x_\lambda$, is called a **net** in X, denoted by $(x_\lambda)_{\lambda\in\Lambda}$.

Definition 1.6.11 ([20], p.210)**.** A net $(x_\lambda)_{\lambda\in\Lambda}$ on a nonempty set X is called **increasing**, if $x_\nu \leq x_\mu$, whenever $\nu \leq \mu$.

Now, we prove an important theorem from functional analysis.

Theorem 1.6.12 (Banach–Alaoglu, [16], p.130)**.** *Let X be a normed space. Then, the closed unit ball B_{X^*} of X^* is $\mathcal{W}^*$ compact, that is, $(B_{X^*}, \mathcal{W}^*)$ is compact.*

Proof. By the definition of the closed unit ball, we get

$$B_{X^*} = \{f \in X^* : \ \|f\| \leq 1\} = \{f \in X^* : \ |f(x)| \leq \|x\|, \quad \text{for all } x \in X\}.$$

Next, we define the map $\phi : B_{X^*} \to \mathbb{R}^X$, via the formula

$$\phi(f) = (f(x))_{x \in X}.$$

Recall that the set $\mathbb{R}^X$ is the set of all functions from X into $\mathbb{R}$. If we let $K = \Pi_{x\in X}[-\|x\|, \|x\|]$ then, from Tychonoff Theorem, we see that K is compact. The Tychonoff Theorem states that if $(X_i, \mathcal{T}_i)$ are compact topological spaces, then $(\Pi X_i, \times\mathcal{T}_i)$ is compact, where $\times\mathcal{T}_i$ is the product topology of ΠX_i.

We, also, observe, for any net $(f_\lambda)_{\lambda\in\Lambda}$ in B_{X^*} and any $f \in B_{X^*}$ with $f_\lambda \xrightarrow{\mathcal{W}^*} f$ that

$$f_\lambda \xrightarrow{\mathcal{W}^*} f \text{ if, and only if, } f_\lambda(x) \to f(x), \quad \text{for all } x \in X$$

or, equivalently,

$$f_\lambda \xrightarrow{\mathcal{W}^*} f \text{ if, and only if, } \phi(f_\lambda) \to \phi(f).$$

Hence, the map $\phi : (B_{X^*}, \mathcal{W}^*) \to (\phi(B_{X^*}, \times\mathcal{T})$ is a homeomorphism. So, in order to deduce that (B_{X^*}, w^*) is compact, we just need to prove that $\phi(B_{X^*})$ is a closed subset of K, because if it is closed, then it will also be compact, since K is compact.

Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net in $\phi(B_{X^*})$ and $f \in K$, such that $f_\lambda \to f$, in the product topology $\times\mathcal{T}$. Then, for any $x, y \in X$ and any $\alpha, \beta \in \mathbb{R}$, we have that

$$f_\lambda(\alpha x + \beta y) = \alpha f_\lambda(x) + \beta f_\lambda(y) \longrightarrow \alpha f(x) + \beta f(y)$$

and

$$f_\lambda(\alpha x + \beta y) \longrightarrow f(\alpha x + \beta y).$$

Hence $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, since $\mathbb{R}^X$ is Hausdorff and the limit is unique. Thus, f is linear. Since $(f_\lambda)_{\lambda\in\Lambda} \in \phi(B_{X^*}) \subset K$, we have that $|f_\lambda(x)| \leq \|x\|$, for all $x \in X$ and $|f(x)| \leq \|x\|$, for all $x \in X$, since $f \in K$. Hence, f is continuous. So, $\phi(B_{X^*})$ is a closed subset of K, hence compact. Thus $(B_{X^*}, \mathcal{W}^*)$ is compact, since ϕ is a homeomorphism. □

Lemma 1.6.13 ([33], p.231)**.** *Let $\epsilon > 0$. For any Banach algebra A and any $x \in A$, with $\|x\| < \epsilon$, we have that $|\phi(x)| < \epsilon$, for every homomorphism $\phi : A \to \mathbb{C}$.*

Proof. For any $\lambda \in \mathbb{C}$, with $|\lambda| > \epsilon$, we have

$$\|\lambda^{-1}x\| \leq |\lambda|^{-1}\|x\| < 1.$$

Hence, by Theorem 1.2.10 we have

$$(1_A - \lambda^{-1}x) \in \text{inv}(A)$$

and by Theorem 1.5.3 we get

$$1 - \lambda^{-1}\phi(x) = \phi(1_A - \lambda^{-1}x) \neq 0.$$

Hence $\phi(x) \neq \lambda$, which implies that $|\phi(x)| < \epsilon$. □

Lemma 1.6.14 ([16], p.219). *Let A be a unital commutative Banach algebra and $h \in \Delta$. Then*

(i) $h(1_A) = 1$.

(ii) $\|h\| = 1$.

(iii) $|h(x)| \leq \|x\|$, *for all* $x \in A$.

Proof. (i) Since $h \in \Delta$, we have that $h(x) \neq 0$, for some $x \in A$. Thus $h(x) = h(x1_A) = h(x)h(1_A)$, which means that $h(1_A) = 1$.

(ii) Let $a \in A$ be an arbitrary element of A and $\lambda = h(a) \in \mathbb{C}$. If $|\lambda| > \|a\|$, then $\left\|\frac{a}{\lambda}\right\| < 1$ and so the element $1_A - \frac{a}{\lambda}$ is invertible, according to Theorem 1.2.10. Now, let $b = \left(a - \frac{a}{\lambda}\right)^{-1}$. Then $1_A = b\left(a - \frac{a}{\lambda}\right)$. Using (i), we have

$$1 = h(1_A) = h\left[b\left(1_A - \frac{a}{\lambda}\right)\right] = h(b) - \frac{ba}{\lambda} = h(b) - \frac{h(b)h(a)}{h(a)} = 0,$$

which is a contradiction. Hence $|h(a)| = |\lambda| \leq \|a\|$. Since $h(1_A) = 1$, we observe that this inequality can never hold, unless $\|h\| = 1$.

(iii) By (i) and (ii) we get, for any $x \in A$, that

$$|h(x)| \leq \|h\|\|x\| = \|x\|.$$

□

Theorem 1.6.15 ([33], p.268). *Suppose that Δ is the maximal ideal space of a unital commutative Banach algebra A. Then*

(i) Δ *is a compact Hausdorff space.*

(ii) The Gelfand transform, regarded as a map from A into $\hat{A}$ is a homomorphism, whose kernel is $\mathrm{rad}A$. *Therefore, the Gelfand transform is an isomorphism if, and only if, A is semisimple.*

(iii) For each $x \in A$, the range $\Delta(x)$ of $\hat{x}$ is the spectrum $\sigma(x)$ of x, that is

$$\sigma(x) = \Delta(x) = \{h(x) : \ h \in \Delta\}.$$

Hence

$$\|\hat{x}\|_\infty = p(x) \leq \|x\|$$

and

$$x \in \mathrm{rad}A, \ \textit{if, and only if, } p(x) = 0.$$

Proof. (i) Let A^* be the continuous dual of A, regarded as a Banach space, that is

$$A^* = \{f : A \to \mathbb{C} : f \text{ is linear and continuous }\}.$$

A^*, being a normed linear space, is a Hausdorff space. Thus, the closed unit ball B_{A^*} of A^* is a Hausdorff space and is $\mathcal{W}^*$ compact, by Banach–Alaoglu Theorem (Theorem 1.6.12). So, we just need to show that $\Delta \subset B_{A^*}$ and that Δ is $\mathcal{W}^*$ closed in B_{A^*}. Using Lemma 1.6.14, it is evident that $\Delta \subset B_{A^*}$.

Now, let $(h_i)_{i\in I}$ be a net in Δ and an $h \in B_{A^*}$, such that $h_i \xrightarrow{\mathcal{W}^*} h$. By the definition of the $\mathcal{W}^*$ topology, we have that

$$h_i \xrightarrow{\mathcal{W}^*} h \text{ if, and only if, } h_i(x) \longrightarrow h(x), \quad \text{for all } x \in A.$$

From the fact that $\mathbb{C}$ is a Hausdorff space, we deduce that the limit above is unique. Since left and right multiplication on $\mathbb{C}$ are continuous we have, for any $a, b \in A$, that $h_i(a) \longrightarrow h(a)$, which implies that

$$h_i(b)h_i(a) \longrightarrow h_i(b)h(a)$$

and $h_i(b) \longrightarrow h(b)$, which implies that

$$h_i(b)h(a) \longrightarrow h(b)h(a).$$

Thus

$$h_i(b)h_i(a) \longrightarrow h(b)h(a)$$

and

$$h_i(a)h_i(b) = h_i(ab) \longrightarrow h(ab).$$

Hence, $h(ab) = h(a)h(b)$. For any $\lambda, \mu \in \mathbb{C}$, we have that

$$h_i(\lambda a + \mu b) \longrightarrow h(\lambda a + \mu b)$$

and

$$h_i(\lambda a + \mu b) = h_i(\lambda a) + h_i(\mu b) = \lambda h_i(a) + \mu h_i(b) \longrightarrow \lambda h(a) + \mu h(b).$$

Thus, h is a complex homomorphism.

Since for any Banach algebra homomorphism $h : A \to B$ between unitary algebras, it holds that $h(1_A) = 1_B$, we have that

$$1 = h_i(1_A) \longrightarrow h(1_A),$$

which implies that

$$h(1_A) = 1. \tag{1.10}$$

Thus, h is a Banach algebra homomorphism which is nonzero, because of (1.10). Hence, $h \in \Delta$, which implies that Δ is $\mathcal{W}^*$ closed in B_{A^*} and since B_{A^*} is compact, we deduce that Δ is a compact Hausdorff space. Notice that every subspace of a topological Hausdorff space, is Hausdorff itself.

(ii) Let $x, y \in A$, $\alpha \in \mathbb{C}$ and $h \in \Delta$. Then

$$\widehat{(\alpha x)}(h) = h(\alpha x) = \alpha h(x) = \alpha \hat{x}(h),$$

$$\widehat{(x+y)}(h) = h(x+y) = h(x) + h(y) = \hat{x}(h) + \hat{y}(h)$$

and

$$\widehat{(xy)}(h) = h(xy) = h(x)h(y) = \hat{x}(h)\hat{y}(h).$$

Thus, the map $x \mapsto \hat{x}$ is a homomorphism, since $\widehat{1_A}(h) = h(1_A) = 1$. The kernel of the map $x \mapsto \hat{x}$ consists of all $x \in A$, for which $h(x) = 0$. By Theorem 1.5.3, the condition $h(x) = 0$ implies that x lies in every maximal ideal of A. So, the kernel of the map $x \mapsto \hat{x}$ is the intersection of all maximal ideals of A, which is equal to rad$A$ by definition. Thus, if $x \mapsto \hat{x}$ is an isomorphism, then rad$A = 0$, which means that A is semisimple. For the converse, if A is semisimple, then rad$A = 0$, which implies that the map $x \mapsto \hat{x}$ is an injective homomorphism. By the definition of the Gelfand transform, we deduce that the map $x \mapsto \hat{x}$ is surjective and, thus, an isomorphism.

(iii) Consider a $\lambda \in \mathbb{C}$, such that $\lambda = h(x)$. By Theorem 1.5.3, this implies that $\lambda \in \sigma(x)$. Thus

$$\Delta(x) = \{h(x) : \ h \in \Delta\} \subset \sigma(x).$$

Now, for any $\lambda \in \sigma(x)$, we have, again from Theorem 1.5.3, that $\lambda = h(x)$, for all $h \in \Delta$, which implies that $\lambda \in \Delta(x)$. Hence $\Delta(x) = \sigma(x)$. By the definition of the norm $\|\cdot\|_\infty : \ \hat{A} \to \mathbb{R}$, we have that

$$\begin{aligned} \|\hat{x}\|_\infty &= \max_{h \in \Delta}\{|\hat{x}(h)|\} \\ &= max\{|h(x)| : \ h \in \Delta_A\} \\ &= \sup\{|\lambda| : \ \lambda \in \sigma(x)\} \\ &= p(x) \leq \|x\|. \end{aligned}$$

The last inequality holds from Theorem 1.2.14.

Now, if $x \in$ rad$A$, then from Theorem 1.5.3, we deduce that $h(x) = 0$, for all $h \in \Delta$, which means that $\sigma(x) = \{0\}$. Indeed, if $\lambda \in \sigma(x)$, with $\lambda \neq 0$, then $h(x) = \lambda$, for all $h \in \Delta$ by Theorem 1.5.3. This means that x does not lie in any proper ideal of A, which is a contradiction, since $x \in$ rad$A$. Thus $\sigma(x) = 0$, for all $x \in A$ and $p(x) = 0$.

Now, for an element $x \in A$ with $p(x) = 0$, we have that $0 \in \sigma(x)$. This implies that $x \notin \text{inv}(A)$ or, equivalently, that x lies in every maximal ideal of A. Thus, x lies in the intersection of those maximal ideals. This intersection is rad$A$ by definition, hence $x \in$ rad$A$. □

Remark 1.6.16. In the proof of (i) of Theorem 1.6.15, the hypothesis that A is unital was only used to prove that the complex homomorphism h preserves the identity 1_A of A. So, if we withdraw this hypothesis, we can conclude that Δ_A is a locally compact Hausdorff space since, in that case, every point of Δ_A, i.e. every complex homomorphism $h : A \to \mathbb{C}$, would have a neighborhood basis consisting of compact sets. Thus, Δ_A is compact if, and only if, A is unital.

Theorem 1.6.17 ([33], p.269). *If $\phi : B \to A$ is a homomorphism of a commutative Banach algebra B, into a semisimple commutative Banach algebra A, then ϕ is continuous.*

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in B and $x \in B$, such that $x_n \longrightarrow x$. We will show that $\phi(x_n) \longrightarrow \phi(x)$.

For this, let Δ_B, Δ_A, be the maximal ideal spaces of B and A respectively. We fix an $h \in \Delta_A$ and let $\psi = h \circ \phi$. For any $x, y \in B$ and any $\lambda, \mu \in \mathbb{C}$, we have that

$$\psi(xy) = h \circ \phi(xy) = (h \circ \phi)(x)(h \circ \phi)(y) = \psi(x)\psi(y),$$

$$\psi(\lambda x + \mu y) = h \circ \phi(\lambda x + \mu y) = \lambda(h \circ \phi)(x) + \mu(h \circ \phi)(y) = \lambda\psi(x) + \mu\psi(y)$$

and

$$\psi(x) = h \circ \phi(x) \neq 0.$$

Hence $\psi \in \Delta_B$.

Now, we pick an $\epsilon > 0$ and apply Lemma 1.6.13 to the element $(x_n - x)$, for which $\|x_n - x\| < \epsilon$. Thus, we get $|\psi(x_n - x)| < \epsilon$, which means that ψ is continuous. Following Lemma 1.6.13 and the continuity of ψ, we deduce that all complex homomorphisms of Banach algebras are continuous. Hence, h is continuous, since $h \in \Delta_A$.

Now, let $\lambda \in \mathbb{C}$, with $\lambda \neq 0$ and $\lambda \in \sigma(\lim \phi(x_n) - \phi(x))$. By Theorem 1.5.3, we have that

$$\lambda = h(\lim \phi(x_n) - \phi(x)) = \lim \psi(x_n) - \psi(x) = 0,$$

which is a contradiction, since $\lambda \neq 0$. Hence

$$\sigma(\lim \phi(x_n) - \phi(x)) = 0,$$

which implies that

$$p(\lim \phi(x_n) - \phi(x)) = 0.$$

By (iii) of Theorem 1.6.15, we get

$$(\lim \phi(x_n) - \phi(x)) \in \mathrm{rad}A = \{0\},$$

by hypothesis. Thus $\phi(x_n) \longrightarrow \phi(x)$ and ϕ is continuous. $\square$

Corollary 1.6.18. *Every isomorphism between two semisimple commutative Banach algebras is a homeomorphism.*

Lemma 1.6.19 ([33], p.270). *If A is a commutative Banach algebra and for all $x \in A$ with $x \neq 0$ it holds that*

$$r = \inf \frac{\|x^2\|}{\|x\|^2} \text{ and } s = \inf \frac{\|\hat{x}\|_\infty}{\|x\|},$$

then $s^2 \leq r \leq s$.

Proof. From Theorem 1.6.15, we have that $\|\hat{x}\|_\infty = p(x) \leq \|x\|$. Then, for any $x \in A$, we have

$$\|x^2\| \geq \|\hat{x}^2\|_\infty = \|\hat{x}\|_\infty^2 \geq s^2\|x\|^2.$$

Thus, $\{s^2\}$ is a lower bound of the set

$$\left\{ \frac{\|x^2\|}{\|x\|^2}, \quad \text{for all } x \in A \text{ with } x \neq 0 \right\}.$$

Hence $s^2 \leq r$. Since $\|x^2\| \geq r\|x\|^2$, for all $x \in A$ with $x \neq 0$, induction on $\mathbb{N}$ implies, for all $n \in \mathbb{N}$, that

$$\begin{aligned} \|x^n\| \geq r^{n-1}\|x\|^n &\Rightarrow \|x^n\|^{1/n} \geq r^{(n-1)/n}\|x\| \\ &\Rightarrow \lim \|x^n\|^{1/n} \geq r\|x\| \\ &\Rightarrow \|\hat{x}\|_\infty = p(x) \geq r\|x\|. \end{aligned}$$

Thus, r is a lower bound of the set

$$\left\{ \frac{\|\hat{x}\|_\infty}{\|x\|}, \quad \text{for all } x \in A \text{ with } x \neq 0 \right\}.$$

Hence, $r \leq s$ and $s^2 \leq r \leq s$. □

Next, we derive an important result for this thesis.

Theorem 1.6.20 ([33], p.270)**.** *Suppose that A is a commutative Banach algebra. Then, the Gelfand transform is an isometry if, and only if, $\|x^2\| = \|x\|^2$, for all $x \in A$.*

Proof. From Lemma 1.6.19, we have that the Gelfand transform is an isometry if, and only if, $s = 1$. This is true, because if the Gelfand transform is an isometry, then $\|\hat{x}\|_\infty = \|x\|$, which implies that $s = 1$. Furthermore, if $s = 1$, then for any $x \in A$, with $x \neq 0$, we have that

$$\frac{\|\hat{x}\|_\infty}{\|x\|} \geq 1.$$

Since the converse inequality was proved in Theorem 1.6.15, we finally have that

$$\|\hat{x}\|_\infty = \|x\|,$$

that is, the Gelfand transform is an isometry.

By Lemma 1.6.19, $s = 1$ implies that $r = 1$ and vice-versa. Hence, we have the result. □

Corollary 1.6.21. *Let A be a commutative C^*-algebra. Then, the Gelfand transform is an isometry.*

Proof. Since $\|x^*x\| = \|x\|^2$, for all $x \in A$, by the definition of the C^*-identity, we immediately conclude that $\|x^2\| = \|x\|^2$, which implies that the Gelfand transform is an isometry, from Theorem 1.6.20. □

We complete this chapter with examples of how a maximal ideal space can be computed and we close up with some remarks.

Example 1.6.22. Let X be a compact Hausdorff space and $A = C(X)$, induced with the supremum norm. For each $x \in X$, the map $h_x : C(X) \to \mathbb{C}$, with $h_x = f(x)$, is a complex homomorphism. By Urysohn Lemma, $C(X)$ separates the points of X, since $C(X)$ is normal and Hausdorff. So, the condition $x \neq y$ implies that $h_x \neq h_y$, for all $x, y \in X$. Thus, the map $x \mapsto h_x$ embeds X into Δ_A.

Now, we claim that each $h \in \Delta_A$ is, in fact, an h_x, for some $x \in X$. Indeed, if the conclusion is false, then there exists a maximal ideal M of $C(X)$, that contains a function f, with $f(p) \neq 0$ for all $p \in X$. The compactness of X implies that M contains finitely many functions $f_1, f_2, \ldots, f_n$, such, that at least one of them is nonzero. We let

$$g = \hat{f}_1 f_1 + \cdots + \hat{f}_n f_n.$$

Since M is an ideal, we have that $g \in M$ and $g(x) > 0$ for all $x \in X$. Hence, g is invertible and, by Theorem 1.5.3, does not lie in any proper ideal of A, which is a contradiction. Hence $x \leftrightarrow h_x$ is a one-to-one correspondence between X and Δ_A. This identification is also correct in terms of the two topologies that are involved. The Gelfand Topology of X is the weak topology induced by $C(X)$ and is, therefore, weaker that the original topology of X. But the Gelfand topology is a Hausdorff topology (Theorem 1.6.15), while the original topology is compact since X is assumed to be compact. Hence, the two topologies coincide and the identification $x \leftrightarrow h_x$ is a homeomorphism. Thus, X is the maximal ideal space of $C(X)$ and the Gelfand transform is the identity map of $C(X)$. Following a similar argument as that above, we deduce that

"If X is a locally compact Hausdorff space,
then X is the maximal ideal space of $C_0(X)$."

$\square$

For our last example, we will need some measure theoretic background. Further measure theoretic definitions will be used and the interested reader can look for more details in [22], [23] and [35].

Theorem 1.6.23 (The Riesz Representation Theorem, [35], p.40)**.** *Let X be a locally compact Hausdorff space and let Λ be a positive linear functional on $C_c(X)$ which is defined to be the collection of all continuous complex-valued functions on X with compact support. Then, there exists a σ-algebra $\mathcal{M}$ in X which contains all Borel sets in X and a unique positive measure μ on $\mathcal{M}$, such that*

$$\Lambda f = \int_X f d\mu$$

for every $f \in C_c(X)$. This positive measure satisfies the following additional properties.

(i) $\mu(K) < \infty$, for every compact set $K \subset X$.

(ii) For every $E \in \mathcal{M}$, we have

$$\mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open }\}.$$

(iii) The relation

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact }\}$$

holds, for every open set E and for all $E \in \mathcal{M}$ that satisfy $\mu(E) < \infty$.

(iv) If $E \in \mathcal{M}$, $A \subset E$ and $\mu(E) = 0$, then $A \in \mathcal{M}$.

For the next theorem, we suppose that μ is a measure on a locally compact Hausdorff space X that satisfies the properties of Theorem 1.6.23.

Theorem 1.6.24 (Lusin Theorem, [35], p.53)**.** *Let $\epsilon > 0$ be arbitrary. Also, suppose that f is a complex measurable function on X, $\mu(A) < \infty$ and $f(x) = 0$ if $x \notin A$. Then, there exists a function $g \in C_c(X)$, such that*

$$\mu(\{x : f(x) \neq g(x)\}) < \epsilon$$

and

$$\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|.$$

Example 1.6.25. Our last example is $L^\infty(\mu)$. Here μ is a Lebesgue measure on the unit interval $[0, 1]$ and $L^\infty(\mu)$ is the usual Banach space of equivalent classes (modulo sets of measure 0) of complex bounded measurable functions on $[0, 1]$, normed by the essential supremum norm $\|f\| = \sup\limits_{x\in[0,1]} \{f(x)\}$, which is defined as the usual supremum norm but without regarding the subsets of $[0, 1]$ of measure 0. Under pointwise multiplication, $L^\infty(\mu)$ *is obviously a commutative Banach algebra.*

If $f \in L^\infty(\mu)$ and G_f is the union of all open sets $G \subset \mathbb{C}$ with $\mu(f^{-1}(G)) = 0$, then the complement of G_f (called the *essential range* of f) coincides with the spectrum $\sigma(f)$ of f and, hence, with the range of its Gelfand transform $\hat{f}$. It follows that $\hat{f}$ is real, if f is real. Hence, $\widehat{L^\infty(\mu)}$ is closed under complex conjugation. By the Stone–Weierstrass Theorem, $\widehat{L^\infty(\mu)}$ is dense in $C(\Delta)$, where Δ is the maximal ideal space of $L^\infty(\mu)$. It also follows that $f \mapsto \hat{f}$ is an isometry, so that $\widehat{L^\infty(\mu)}$ is closed in $C(\Delta)$. *We conclude that $f \mapsto \hat{f}$ is an isometry of $L^\infty(\mu)$ onto $C(\Delta)$.*

Next, we prove that $\hat{f} \mapsto \int f d\mu$ is a bounded linear functional on $C(\Delta)$. By the Riesz representation theorem (Theorem 1.6.23), there exists a regular Borel probability measure $\hat{\mu}$ on Δ that satisfies

$$\int_\Delta \hat{f} d\hat{\mu} = \int_0^1 f d\mu. \tag{1.11}$$

By a *Borel measure*, we mean a measure μ that is defined on all open (hence in all Borel) sets of a topological space. It is called *regular*, if it is both inner and outer regular. A *probability measure* is a measure that must assign the value 1 to the entire topological space. Now, if Ω is a nonempty open set in Δ, Urysohn lemma implies that there exists $\hat{f} \in C(\Delta)$, with $\hat{f} \geq 0$, such that $\hat{f} = 0$ outside of Ω and $\hat{f}(p) = 1$ at some $p \in \Omega$. Hence, f is not the zero element of $L^\infty(\mu)$ and the integrals in (1.11) are positive. *Thus, $\hat{\mu}(\Omega) > 0$ if Ω is open and nonempty.*

Assume next that ϕ is a Borel function on Δ with $|\phi| \leq 1$. By Lusin theorem (Theorem 1.6.24), there are functions $\hat{f_n} \in C(\Delta)$, with $|\hat{f_n}| \leq 1$, that converge to ϕ in the norm of $L^2(\hat{\mu})$. Since $f \mapsto \hat{f}$ preserves complex conjugation and is a homomorphism, applying (1.11) to $(f_i - f_j)(\overline{f}_i - \overline{f}_j)$, we get that

$$\int_\Delta |\hat{f}_i - \hat{f}_j|^2 d\hat{\mu} = \int_0^1 |f_i - f_j|^2 d\mu. \tag{1.12}$$

Thus, $\{f_n\}$ is a Cauchy sequence in $L^2(\mu)$. Also, $|f_n| \leq 1$ almost everywhere. Hence, there exist an $f \in L^\infty(\mu)$ such that $f_n \longrightarrow f$ in $L^2(\mu)$. Now, (1.12) implies that $\hat{f}_n \longrightarrow \hat{f}$ in $L^2(\mu)$. The conclusion is that $\phi = \hat{f}$ almost everywhere. *Thus, every bounded Borel function ϕ on Δ coincides with some $\hat{f} \in C(\Delta)$ almost everywhere and $C(\Delta)$, $L^\infty(\hat{\mu})$ are identical as Banach spaces.*

Remark 1.6.26. As we mentioned throughout the proof of Theorem 1.6.17, it is a fact that for any Banach algebra A, any $\epsilon > 0$ and any $x \in A$, with $\|x\| < \epsilon$, we have that $|\phi(x)| < \epsilon$, for any complex homomorphism $\phi : A \to \mathbb{C}$. We proved this last statement in Lemma 1.6.13, from which we deduce the following result

"*In any Banach algebra (unital and over the field $\mathbb{C}$),*
every complex homomorphism is continuous."

This is true, since if X, Y are normed linear spaces and $T : X \to Y$ is linear, then T is bounded if, and only if, T is continuous.

Remark 1.6.27. Theorem 1.6.17 and Corollary 1.6.18 are special cases of the following theorem.

"*Suppose that X is a complex topological vector space and Y is a subspace of X,*
with $\dim Y = n$*, where n is a fixed natural number.*
Then every isomorphism of Y onto $\mathbb{C}^n$ is a homeomorphism."

The interested reader should look for Theorem 1.21 in [33] for more details.

Chapter 2

Basics of C^*-Algebras

We begin this chapter with several definitions and properties about C^*-algebras. We prove Dini Theorem (Theorem 2.1.9), which helps us distinguish pointwise convergence of a sequence of functions from uniform convergence, under specific circumstances. We prove that the closure of a real algebra is also an algebra, which we use to prove the Stone-Weierstrass Theorem over the field $\mathbb{R}$ (Theorem 2.1.15) and over the field $\mathbb{C}$ (Theorem 2.1.16). The latter is a consequence of the former. We make a construction about unitizing C^*-algebras that are not already unital (Theorem 2.2.8). This construction allows us to unitize any C^*-algebra by just adjoining an identity and it will be useful in the proof of nonunital Gelfand Theorem (Theorem 2.2.14). The unital case of Gelfand Theorem can be easily approached by using the tools that we presented in the previous sections. We close up this chapter with the notion of an approximate identity.

2.1 Introduction to C^*-Algebras

For this section, all algebras are taken over the field $\mathbb{C}$, unless explicitly stated otherwise.

Definition 2.1.1 ([29], p.35)**.** An **involution** on an algebra A is a map from A into itself, such that for all $x, y \in A$ and all $\alpha \in \mathbb{C}$

(i) $(x+y)^* = x^* + y^*$

(ii) $(\alpha x)^* = \overline{\alpha} x^*$

(iii) $(x^*)^* = x$

(iv) $(xy)^* = y^* x^*$

Definition 2.1.2 ([29], p.36)**.** A ***C*-algebra*** A is a Banach algebra with an involution, that satisfies

$$\|a^*a\| = \|a\|^2, \quad \text{for all } a \in A.$$

This last property of the norm is usually referred to as the ***C*-condition*** and an algebra norm that satisfies this condition is called a ***C*-norm***.

Definition 2.1.3 ([29], p.36)**.** A ***C*-subalgebra*** of a C^*-algebra A is a closed *-subalgebra of A.

Remark 2.1.4. Certain properties follow immediately from the aforementioned definitions. For example, the involution map $x \mapsto x^*$ from A into itself is a bijection, that is, injective and surjective.

Indeed, we have, for any $x, y \in A$ with $x^* = y^*$, that $(x^*)^* = (y^*)^*$, which implies that $x = y$. This means that the map $x \mapsto x^*$ is injective. Also, we can easily see that the map $x \mapsto x^*$ is surjective, since $(x^*)^* = x$. Thus, the involution map is a bijection.

Remark 2.1.5. We observe that the involution map is isometric. Indeed, if we apply the C^*-condition to the element $a^* \in A$, we have that

$$\|a^*\|^2 = \|(a^*)^*(a^*)\| = \|aa^*\| \leq \|a\|\|a^*\|.$$

Thus $\|a^*\| \leq \|a\|$. Following the same procedure as above, we can see that the converse inequality holds too, that is

$$\|a\| \leq \|a^*\|, \quad \text{for all } a \in A.$$

Hence, $\|a\| = \|a^*\|$, for all $a \in A$, which means that the involution map is isometric.

Definition 2.1.6 ([20], p.272)**.** Let X be any topological space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions, with $f_n : X \to \mathbb{R}$, for all $n \in \mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges **pointwise** to the function $f : X \to \mathbb{R}$, if

$$f_n(x) \longrightarrow f(x), \quad \text{for all } x \in X.$$

Definition 2.1.7 ([20], p.267)**.** Let X be any topological space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions, with $f_n : X \to \mathbb{R}$, for all $n \in \mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges **uniformly** to the function $f : X \to \mathbb{R}$, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ with $n \geq N$, it holds that

$$|f_n(x) - f(x)| < \epsilon, \quad \text{for all } x \in X.$$

Remark 2.1.8. Uniform convergence implies pointwise convergence, as it can be seen by the definitions above. But, the converse does not always hold as we shall see in Dini Theorem (Theorem 2.1.9) that follows.

Theorem 2.1.9 (Dini Theorem, [18], p.277)**.** *Let X be a compact topological space and let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence of functions $f_n : X \to \mathbb{R}$, that converges pointwise to a continuous function $f : X \to \mathbb{R}$. Then, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.*

Proof. We define the sequence of functions $(g_n)_{n\in\mathbb{N}}$, with $g_n : X \to \mathbb{R}$, such that $g_n = f_n - f$. For each $n \in \mathbb{N}$, g_n is continuous. The sequence $(g_n)_{n\in\mathbb{N}}$ converges pointwise to 0, since $f_n \longrightarrow f$ pointwise, and is decreasing, since $(f_n)_{n\in\mathbb{N}}$ is decreasing. We, now, prove that $g_n \longrightarrow 0$ uniformly.

Let $\epsilon > 0$ and define

$$K_n = \{x \in X : g_n(x) \geq \epsilon\}, \quad \text{for all } n \in \mathbb{N}.$$

Since g_n is continuous for any $n \in \mathbb{N}$, we have that K_n is closed hence compact, as a closed subset of a compact space. Since $(g_n)_{n\in\mathbb{N}}$ is decreasing, that is, $g_n \geq g_{n+1}$, for all $n \in \mathbb{N}$, we have that $K_n \supset K_{n+1} \supset \ldots$. Now, let $x \in X$ be any element of X. Then, $g_n(x) \longrightarrow 0$ pointwise, which means that $x \notin K_n$, for all $n \in \mathbb{N}$. Thus $x \notin \cap_{n\in\mathbb{N}} K_n$. This means that $\cap_{n\in\mathbb{N}} K_n = \emptyset$. So, there exists $N \in \mathbb{N}$, for which $K_N = \emptyset$ and since $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of compact sets, we have, for all $n \in \mathbb{N}$, with $n \geq N$, that $0 \leq g_n(x) < \epsilon$, for all $x \in X$. Thus, $g_n \longrightarrow 0$ uniformly, which means that $f_n \longrightarrow f$ uniformly. □

Lemma 2.1.10. *If A is a real algebra, that is, an algebra over the field $\mathbb{R}$, then so is its closure $\overline{A}$.*

Proof. The fact that $\overline{A}$ is a vector space is trivial. Now, let x, y, z be arbitrary elements of $\overline{A}$. Then, there should exist sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ in A, such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $z_n \longrightarrow z$. Since A is a real algebra, we deduce the following properties

(i) $(x_n y_n)z_n = x_n(y_n z_n)$.

(ii) $(x_n + y_n)z_n = x_n z_n + y_n z_n$.

(iii) $x_n(y_n + z_n) = x_n y_n + x_n z_n$.

(iv) $r(x_n y_n) = (rx_n)y_n = x_n(ry_n), \quad$ for all $n \in \mathbb{N}$ and all $r \in \mathbb{R}$.

Since $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are convergent and since the limit of any sequence is unique, we deduce the following

(v) $(xy)z = x(yz)$.

(vi) $(x + y)z = xz + yz$.

(vii) $x(y + z) = xy + xz$.

(viii) $r(xy) = (rx)y = x(ry), \quad$ for all $r \in \mathbb{R}$.

Thus, $\overline{A}$ is a real algebra. □

Lemma 2.1.11 ([18], p.277)**.** *Suppose that A is a subalgebra of $C_0^{\mathbb{R}}(X)$, where*

$$C_0^{\mathbb{R}}(X) = \{f : X \to \mathbb{R} : \ f \text{ continuous and vanishes at } \infty\}$$

and let X be a compact topological space.

(i) If $f \in \overline{A}$, then $|f| \in \overline{A}$.

(ii) If $f, g \in \overline{A}$, then $\max(f, g) \in \overline{A}$ and $\min(f, g) \in \overline{A}$.

Proof. (i) If $f = 0 \in A$, then it is obvious that $|f| = 0 \in \overline{A}$. Now, suppose that $0 \neq f \in A$. Let $\|f\|_{sup} = \sup\{f(x) : x \in A\}$, so that $\frac{f}{\|f\|_{sup}} \in A$. Then, $f(X) \subset [-1, 1]$ and $f(x)^2 \in [0, 1]$, for all $x \in X$. We construct a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$, such that $p_1 \equiv 0$ and

$$p_{n+1}(t) = p_n(t) - \frac{1}{2}(p_n^2(t) - t), \quad \text{for all } t \in [0, 1].$$

We will show, using induction, that $0 \leq p_n(t) \leq \sqrt{t}$ and $p_n(0) = 0$, for all $n \in \mathbb{N}$ and all $t \in [0, 1]$. For $n = 1$, we get $p_1(t) = 0$ for all $t \in [0, 1]$. Thus, the result is true for $n = 1$.

Now, suppose that the result is true for some $n \in \mathbb{N}$. We will show that it is also true for $n + 1$. Indeed

$$\begin{aligned} p_{n+1}(t) - \sqrt{t} &= p_n(t) - \sqrt{t} - \frac{1}{2}(p_n^2(t) - t) \\ &= (p_n(t) - \sqrt{t}) - \frac{1}{2}(p_n(t) - \sqrt{t})(p_n(t) + \sqrt{t}) \\ &= (p_n(t) - \sqrt{t}) \left[1 - \frac{1}{2}(p_n(t) + \sqrt{t})\right] \leq 0. \end{aligned}$$

The last inequality is true, since $p_n(t) \leq \sqrt{t}$ and $p_n(t) + \sqrt{t} \leq 2\sqrt{t} \leq 2$, for all $t \in [0, 1]$. Also, for $t = 0$, we have

$$p_{n+1}(0) = p_n(0) - \frac{1}{2}p_n^2(0) = 0.$$

Hence, $0 \leq p_n(t) \leq \sqrt{t}$ and $p_n(0) = 0$, for all $n \in \mathbb{N}$. Furthermore, we have that

$$p_{n+1}(t) - p_n(t) = \frac{1}{2}(t - p - n(t)^2) \geq 0.$$

This is true, since $0 \leq p_n(t) \leq \sqrt{t}$, for all $t \in [0, 1]$. Thus, the sequence $(p_n)_{n\in\mathbb{N}}$ is increasing and bounded by $\sqrt{t}$. So, it converges to a function $g \in C_0^{\mathbb{R}}(X)$, such that

$$0 \leq g(t) \leq \sqrt{t}, \quad \text{for all } t \in [0, 1].$$

Thus

$$\begin{aligned} 0 = g(t) - g(t) &= \lim_n (p_{n+1}(t) - p_n(t)) \\ &= \lim_n \frac{1}{2}(t - p_n(t)^2) \\ &= \frac{1}{2}(t - g(t)^2). \end{aligned}$$

Thus $g(t) = \sqrt{t}$, for all $t \in [0, 1]$.

To sum up, we proved so far that the sequence of polynomial $(p_n)_{n\in\mathbb{N}}$ is increasing and that it converges to the continuous function

$$g(t) = \sqrt{t}, \quad \text{for all } t \in [0, 1].$$

Utilizing the Dini Theorem (Theorem 2.1.9), we deduce that the sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ converges uniformly to g, in $[0,1]$. We define the sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n : X \to \mathbb{R}$ and $f_n(x) = p_n(f^2(x))$, for all $n \in \mathbb{N}$ and we observe that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the function $\sqrt{f^2} = |f|$ in X. Since $(f_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of powers of f, that is, a linear combination of powers of f, we have that

$$\{f_n : n \in \mathbb{N}\} \subset A.$$

Hence

$$\lim_n f_n = |f| \in \overline{A}.$$

So, in any case, we have shown that if $f \in \overline{A}$, then $|f| \in \overline{A}$.

(ii) We will make use of the formulas

$$\max(f,g) = \frac{1}{2}(f + g + |f - g|)$$

and

$$\min(f,g) = \frac{1}{2}(f + g - |f - g|), \quad \text{for all } f, g \in \overline{A}.$$

Since A is a real algebra, by Lemma 2.1.10 we have that $\overline{A}$ is a real algebra, so we conclude that $\max(f,g) \in \overline{A}$ and $\min(f,g) \in \overline{A}$. □

Definition 2.1.12 ([18], p.276)**.** Let A be a family of functions on a topological space X.

(i) A is said to **separate points on** X, if for any $x, y \in X$, there exists a function $f \in A$, such that $f(x) \neq f(y)$.

(ii) If for each $x \in X$, there exists a function $g \in A$, such that $g(x) \neq 0$, then we say that A **does not vanish on** X.

Definition 2.1.13 ([18], p.275)**.** Let A be a real algebra of functions on a topological space X and let B be a subalgebra of A. Then B **is dense in** A, if $\overline{B} = A$ or, equivalently, if for all $f \in A$ and all $\epsilon > 0$ there exists a $g \in B$ such that

$$|f(x) - g(x)| < \epsilon, \quad \text{for all } x \in X.$$

Lemma 2.1.14 ([18], p.278)**.** *Suppose A is a real algebra of functions on a topological space X, A separates points on X and A vanishes at no point of X. Also, suppose that $x_1, x_2 \in X$ are distinct points of A and $c_1, c_2 \in \mathbb{R}$. Then, there exists a function $f \in A$, such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

Proof. There exist functions $g, h, k \in A$, such that $g(x_1) \neq g(x_2)$ (A separates the points $x_1, x_2 \in X$), $h(x_1) \neq 0$ (A does not vanish at $x_1 \in X$) and $k(x_2) \neq 0$ (A does not vanish at $x_2 \in X$). Let $u = (g - g(x_1))k$ and $v = (g - g(x_2))h$. Since $u(x_2) \neq 0$, $v(x_1) \neq 0$ and $u, v \in A$, we define the function $f : X \to \mathbb{R}$, with

$$f(x) = \frac{c_1 v(x)}{v(x_1)} + \frac{c_2 u(x)}{u(x_2)}.$$

Then $f(x_1) = c_1$ and $f(x_2) = c_2$ and the proof is complete. □

Now, we are in place to state and prove the Stone-Weierstrass Theorem, over $\mathbb{R}$ and $\mathbb{C}$.

Theorem 2.1.15 (Stone–Weierstrass Theorem over $\mathbb{R}$, [18], p.277)**.** *Let X be a locally compact Hausdorff space and suppose that the set $A \subset C_0^{\mathbb{R}}(X) = C_0(X, \mathbb{R})$ is a subalgebra of $C_0^{\mathbb{R}}(X)$, such that all of the following hold.*

(i) A separates the points of X.

(ii) A vanishes at no point of X.

Then A is dense in $C_0^{\mathbb{R}}(X)$.

Proof. Let $h \in C_0^{\mathbb{R}}(X)$ and let $\epsilon > 0$ be arbitrary. We need to show that there exists $f \in \overline{A}$, such that

$$|h(x) - f(x)| < \epsilon, \quad \text{for all } x \in X.$$

We note here that the algebra A in dense in $\overline{A}$. For any $x, y \in X$, with $x \neq y$, we choose $g_{x,y} \in A$, such that $h(x) = g_{x,y}(x)$ and $h(y) = g_{x,y}(y)$. By applying Lemma 2.1.14, we get that $h(x)$ and $h(y)$ exist.

For a fixed element $y \in X$ we define

$$U_x = \{z \in X : (h - g_{x,y})(z) < \epsilon\}, \quad \text{for all } x \in X \text{ with } x \neq y.$$

Then, U_x is an open neighborhood of x and since $(h - g_{x,y})$ vanishes at ∞, we deduce that the set

$$X \setminus U_x = \{z \in X : (h - g_{x,y})(z) \geq \epsilon\}$$

is compact. Thus, we fix an element $x_1 \in X$, and there exist elements $x_2, x_3, ..., x_k \in X \setminus U_{x_1}$, such that

$$X \setminus U_{x_1} \subset \cup_{i=2}^{k} U_{x_i},$$

which implies that

$$X \subset \cup_{i=1}^{k} U_{x_i}.$$

If we let

$$f_y = \max(g_{x_1,y}, g_{x_2,y}, \dots, g_{x_k,y}),$$

we can see from Lemma 2.1.11 that $f_y \in \overline{A}$, because every $g_{x_i,y}$ lies in $\overline{A}$, for each $i \in \{1, 2, \dots, k\}$. Also, for each $z \in U_{x_i}$, we have that $h(z) - g_{x_i,y}(z) < \epsilon$, which implies that $h(z) - f_y(z) \leq \epsilon$. Thus $h(z) - f_y(z) < \epsilon$, for all $z \in X$.

Now, for any $y \in X$, we define the set

$$V_y = \{z \in X : (f_y - h)(z) < \epsilon\}.$$

Since $f_y(y) = \max(g_{x_1,y}, \dots, g_{x_k,y}) = h(y)$, we have that V_y is an open neighborhood of Y and, similarly as above, we deduce that there exist elements $y_1, y_2, \dots, y_\lambda \in X$, such that

$$X \subset \cup_{j=2}^{\lambda} V_{y_j}.$$

If we let $f = \min(f_{y_1}, \dots, f_{y_\lambda})$, then, by Lemma 2.1.11, we have that $f \in \overline{A}$, since each f_{y_j}, $j \in \{1, 2, \dots, \lambda\}$, lies in $\overline{A}$, as proven earlier. Also, for any $z \in X$, we observe that

$$f(z) - h(z) = \min(f_{y_1}, \dots, f_{y_\lambda})(z) - h(z) = (f_{y_j}(z) - h(z)) < \epsilon,$$

for some $j \in \{1, 2, \dots, \lambda\}$. As stated previously, we also have that $h(z) - f_y(z) < \epsilon$. Thus, $|h(z) - f(z)| < \epsilon$, for all $z \in X$ and the proof is complete. □

So far, we proved the version of the Stone-Weierstrass Theorem for real-valued functions. We will, now, show that the version for complex-valued functions follows from the real one.

Theorem 2.1.16 (Stone-Weierstrass Theorem over $\mathbb{C}$, [18], p.277)**.** *Let X be a locally compact Hausdorff space and suppose that $A \subset C_0(X)$ is a subalgebra of $C_0(X)$ (the algebra of all continuous complex-valued functions $f : X \to \mathbb{C}$ that vanish at infinity) such that all of the following hold.*

(i) A separates the points of X.

(ii) For all $x \in X$ there exists an $f \in A$, such that $f(x) \neq 0$.

(iii) A is closed under complex conjugation.

Then, A is dense in $C_0(X)$.

Proof. Let $f \in A$ and let $A^{\mathbb{R}} = A \cap C_0^{\mathbb{R}}(X)$. Then, the function $f \in A$ can be decomposed as

$$f = Re(f) + iIm(f),$$

where $Re(f) = \dfrac{1}{2}(f + \overline{f})$ and $Im(f) = \dfrac{1}{2i}(f - \overline{f})$, since A is closed under complex conjugation. We also observe that $Re(f), Im(f) \in A^{\mathbb{R}}$. Thus $A = A^{\mathbb{R}} + iA^{\mathbb{R}}$. If A satisfies the conditions of Theorem 2.1.15, then $A^{\mathbb{R}} = A \cap C_0^{\mathbb{R}}(X)$ satisfies the conditions of Dini Theorem (Theorem 2.1.9), as a real subalgebra of $C_0^{\mathbb{R}}(X)$. Thus $\overline{A}^{\mathbb{R}} = C_0^{\mathbb{R}}(X)$ and

$$\overline{A} = \overline{A}^{\mathbb{R}} + i\overline{A}^{\mathbb{R}} = C_0^{\mathbb{R}}(X) + iC_0^{\mathbb{R}}(X).$$

This means that

$$\overline{A} = C_0^{\mathbb{R}}(X)$$

and the proof is complete. □

2.2 Gelfand Theorems

In this section, all algebras are commutative, unless explicitly stated otherwise.

Definition 2.2.1 ([16], p.234)**.** Let A be a C^*-algebra.

(i) An element $a \in A$ is called **hermitian** or **self-adjoint**, if $a^* = a$.

(ii) An element $a \in A$ is called **normal**, if $a^*a = aa^*$.

(iii) A hermitian element $p \in A$ is called **positive**, denoted by $p \geq 0$, if $\sigma(p) \geq 0$. The set of all positive elements of A is denoted by A^+.

(iv) An element $u \in A$ is called **unitary**, if $u^*u = uu^* = 1_A$.

Proposition 2.2.2 ([33], p.275)**.** *Let A be a C^*-algebra and $a \in A$. Then all of the following are true.*

(i) The elements $a + a^*$, $i(a - a^*)$ and aa^* are hermitian.

(ii) The element $a \in A$ has a unique representation, as $a = b + ic$, with $b, c \in A$ being both hermitian.

(iii) If A is unital, with unit 1_A, then 1_A is hermitian.

(iv) An element $x \in A$ is invertible if, and only if, x^* is invertible. In this case we have that $(x^*)^{-1} = (x^{-1})^*$.

(v) It holds that $\lambda \in \sigma(x)$ if, and only if, $\overline{\lambda} \in \sigma(x^*)$.

Proof. (i) For $a \in A$, we have

$$(a + a^*)^* = a^* + (a^*)^* = a^* + a = a + a^*.$$

Hence, the element $(a + a^*)$ is hermitian. Also

$$(i(a - a^*))^* = (a^* - a)(-i) = i(a - a^*).$$

Hence, the element $i(a - a^*)$ is hermitian. Finally

$$(aa^*)^* = (a^*)^*a^* = aa^*.$$

Thus, the element aa^* is hermitian.

(ii) Since $(a+a^*)$ and $i(a-a^*)$ are hermitian, we construct the hermitian elements

$$b = \frac{a + a^*}{2} \text{ and } c = \frac{a - a^*}{2i}.$$

We can, now, see that

$$b + ic = \frac{a + a^*}{2} + \frac{a - a^*}{2} = a.$$

The uniqueness part of the proof is immediate.

(iii) Since $1_A 1_A^* = 1_A$, by applying (i), we deduce that the element 1_A is hemirian.

(iv) If $a \in A$ is invertible, then

$$aa^{-1} = a^{-1}a = 1_A \text{ if, and only if, } (1_A)^* = 1A = (aa^{-1})^*$$

and

$$(1_A)^* = 1A = (aa^{-1})^* \text{ if, and only if, } 1_A = (a^{-1})^*a^* = a^*(a^{-1})^*.$$

Thus, a^* is invertible. As a consequence, we have that

$$(a^*)^{-1} = (a^{-1})^*.$$

(v) We apply (iv) to the element $\lambda 1_A - a$, for $\lambda \in \mathbb{C}$, and we have that

$$\lambda 1_A - a \in \operatorname{inv}(A) \text{ if, and only if, } \overline{\lambda} 1_A - a^* \in \operatorname{inv}(A).$$

Thus

$$\lambda \in \sigma(a) \text{ if, and only if, } \overline{\lambda} \in \sigma(a^*).$$ □

Following (v) of Proposition 2.2.2, we get the following corollary.

Corollary 2.2.3. *Let A be a C^*-algebra and let $a \in A$. Then*

$$\sigma(a^*) = \overline{\sigma(a)} = \{\overline{\lambda} \in \mathbb{C} : \lambda \in \sigma(a)\}.$$

Theorem 2.2.4 ([16], p.234)**.** *If a is a hermitian element of a C^*-algebra A, then the spectral radius of a is equal to $\|a\|$.*

Proof. Since a is hermitian, we have that

$$\|a\|^2 = \|aa^*\| = \|aa\| = \|a^2\|.$$

By induction, we have that

$$\|a\|^{2^n} = \|a^{2^n}\|, \quad \text{for all } n \in \mathbb{N}.$$

Hence, by the spectral radius formula (Theorem 1.2.14) we have that the spectral radius of a is equal to

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2^n}\|^{1/(2^n)} = \|a\|.$$ □

Corollary 2.2.5. *In any *-algebra A, there exists at most one norm making A a C^*-algebra.*

Proof. If $\|\cdot\|_1, \|\cdot\|_2$ are norms on A such that it is a C^*-algebra, then

$$\|a\|_i^2 = \|a^2\|_i = \|a^*a\|_i = p(a^*a) = \sup\{|\lambda| : \lambda \in \sigma(a^*a)\}, \text{ for } i = 1, 2.$$

Thus, $\|a\|_1 = \|a\|_2$. □

Proposition 2.2.6 ([16], p.235). *Let A be a unital commutative C^*-algebra, $a \in A$ and $h \in \Delta_A$, that is, $h : A \to \mathbb{C}$ is a nonzero complex homomorphism.*

(i) If a is hermitian, then $h(a) \in \mathbb{R}$.

(ii) It holds that $h(a^*) = \overline{h(a)}$.

(iii) It holds that $h(a^*a) \geq 0$.

(iv) If $u \in A$ is unitary, then $|h(u)| = 1$.

Proof. (i) If a is hermitian, then $a^* = a$. Now, consider $t \in \mathbb{R}$. Then, by Lemma 1.6.14, we have that

$$|h(a+it)|^2 \leq \|a+it\|^2 \leq \|a\|^2 + |t|^2.$$

So, if $h(a) = x + iy$, with $x, y \in \mathbb{R}$ and $y \neq 0$, then

$$\begin{aligned} \|a\|^2 + |t|^2 &\geq |h(a+it)| \\ &= |h(a) + h(it)| = |x + iy + ith(1_A)| \\ &= |x + i(y+t)|^2 = x^2 + y^2 + 2yt \\ &\geq y^2 + 2yt. \end{aligned}$$

So, if $t \in \mathbb{R}$ tends to infinity, we have a contradiction, unless $y = 0$. Hence $h(a) = x \in \mathbb{R}$.

(ii) According to Proposition 2.2.2, we can write the element $a \in A$ as $a = x + iy$, where both x and y are hermitian. So, by (i), we have that $h(x), h(y) \in \mathbb{R}$. Thus

$$h(a^*) = h(x - iy) = h(x) - ih(y) = \overline{h(x) + ih(y)} = \overline{h(a)}.$$

(iii) By (ii) we have

$$h(a^*a) = h(a^*)h(a) = \overline{h(a)}h(a) = |h(a)|^2 \geq 0.$$

(iv) Using the conclusion of (ii), we have that

$$h(u)^2 = h(u^*)h(u) = h(u^*u) = h(1_A) = 1.$$

Hence, $|h(u)| = 1$. $\square$

Our next step is to prove the Gelfand Theorems, that characterize an arbitrary commutative C^*-algebra, in terms of its maximal ideal space. In order to prove both the unital and nonunital case, we need to construct a method of unitizing nonunital C^*-algebras, so that we may confine our attention to unital C^*-algebras only. While this construction is very useful in many cases, it does not always work, as we shall see in the next section.

Remark 2.2.7. We recall that if V is a finite dimensional vector space and $U_1, U_2, ..., U_m$ are vector subspaces of V, such that

$$V = U_1 \bigoplus U_2 \bigoplus ... \bigoplus U_m,$$

then

$$\dim V = \dim U_1 + \dim U_2 + ... + \dim U_m.$$

Theorem 2.2.8 (Unitization of a C*-algebra, [17], p.2)**.** *Every nonunital C^*-algebra A is contained in a unital C^*-algebra $\tilde{A}$, being a maximal ideal of codimension* 1.

Proof. Let $\tilde{A} = A \bigoplus \mathbb{C}$. We know that $\tilde{A}$ is an algebra, since the direct sum of vector spaces is also a vector space and the binary operation

$$((a, \lambda), (b, \mu)) \mapsto (a, \lambda)(b, \mu),$$

defined by

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu), \quad \text{for all } a, b \in A,\ \lambda\mu \in \mathbb{C},$$

satisfies all the properties of a multiplication between vector spaces.

Since both algebras A and $\mathbb{C}$ are normed algebras, we define the norm $\|\cdot\|_{\tilde{A}} : \tilde{A} \to \mathbb{R}$, with

$$\|(a, \lambda)\|_{\tilde{A}} = \sup_{\|c\| \leq 1} \|ac + \lambda c\|.$$

This is a norm on $\tilde{A}$ since, for all $a, b \in A$ and all $\lambda, \mu \in \mathbb{C}$, we have that

$$\|(a, \lambda)\|_{\tilde{A}} = \sup_{\|c\| \leq 1} \|ac + \lambda c\| \geq 0.$$

Also

$$\|(a, \lambda)\|_{\tilde{A}} = 0 \Leftrightarrow \sup_{\|c\| \leq 1} \|ac + \lambda c\| = 0 \Leftrightarrow (a, \lambda) = (0, 0).$$

Additionally

$$\begin{aligned} \|\mu(a, \lambda)\|_{\tilde{A}} = \|(\mu a, \mu\lambda)\|_{\tilde{A}} &= \sup_{\|c\| \leq 1} \|\mu ac + \mu\lambda c\| \\ &= |\mu| \sup_{\|c\| \leq 1} \|ac + \lambda c\| \\ &= |\mu| \|(a, \lambda)\|_{\tilde{A}}. \end{aligned}$$

Furthermore

$$\begin{aligned} \|(a, \lambda) + (b, \mu)\|_{\tilde{A}} = \|(a + b, \lambda + \mu)\|_{\tilde{A}} &= \sup_{\|c\| \leq 1} \|(a + b)c + (\lambda + \mu)c\| \\ &= \sup_{\|c\| \leq 1} \|(a + \lambda)c + (b + \mu)c\| \\ &\leq \sup_{\|c\| \leq 1} \|ac + \lambda c\| + \sup_{\|c\| \leq 1} \|bc + \mu c\| \\ &= \|(a, \lambda)\|_{\tilde{A}} + \|(b, \mu)\|_{\tilde{A}}. \end{aligned}$$

Let $(a_n, \lambda_n)_{n\in\mathbb{N}}$, be a Cauchy sequence in $\tilde{A}$. Then $(a_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ are Cauchy sequences in A and $\mathbb{C}$, respectively. Since A and $\mathbb{C}$ are Banach spaces, there exist elements $a \in A$ and $\lambda \in \mathbb{C}$, such that

$$a_n \xrightarrow{\|\cdot\|} a \quad \text{and} \quad \lambda_n \xrightarrow{|\cdot|} \lambda.$$

This means, equivalently, that for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, with $n \geq n_0$, it holds that

$$\|a_n - a\| < \frac{\epsilon}{2} \quad \text{and} \quad |\lambda_n - \lambda| < \frac{\epsilon}{2}.$$

Thus

$$\begin{aligned}
\|(a_n, \lambda_n) - (a, \lambda)\|_{\tilde{A}} &= \|(a_n - a, \lambda_n - \lambda)\|_{\tilde{A}} \\
&= \sup_{\|c\|\leq 1} \|(a_n - a)c + (\lambda_n - \lambda)c\| \\
&\leq \sup_{\|c\|\leq 1} [\|c\|(\|a_n - a\| + |\lambda_n - \lambda|)] \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, the sequence $(a_n, \lambda_n)_{n\in\mathbb{N}} \subset \tilde{A}$ converges to the element $(a, \lambda) \in \tilde{A}$, which implies that $\tilde{A}$ is a Banach space.

Furthermore, for any $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$, we have that

$$\begin{aligned}
\|(a, \lambda)(b, \mu)\|_{\tilde{A}} &= \|(ab + \mu a + \lambda b, \lambda\mu)\| \\
&= \sup_{\|c\|\leq 1} \|abc + \mu ac + \lambda bc + \lambda\mu c\| \\
&\leq \|ab + \mu a + \lambda b + \lambda\mu\| \sup_{\|c\|\leq 1} \|c\| \\
&\leq \|a + \lambda\|\|b + \mu\| \\
&= \|(a, \lambda)\|_{\tilde{A}}\|(b, \mu)\|_{\tilde{A}}.
\end{aligned}$$

Thus, $\tilde{A}$ is a Banach algebra. Now, define the map $^* : \tilde{A} \to \tilde{A}$, by $(a, \lambda)^* = (a^*, \overline{\lambda})$. We will show that this map defines an involution on $\tilde{A}$. For any pairs $(a, \lambda)(b, \mu) \in \tilde{A}$ and any $c \in \mathbb{C}$, the following hold

- $((a, \lambda) + (b, \mu))^* = (a + b, \lambda + \mu)^* = (a^* + b^*, \overline{\lambda} + \overline{\mu}) = (a, \lambda)^* + (b, \mu)^*$.
- $c(a, \lambda))^* = (ca, c\lambda)^* = (\overline{c}a^*, \overline{c}\overline{\lambda}) = \overline{c}(a, \lambda)^*$.
- $((a, \lambda)^*)^* = (a^*, \overline{\lambda})^* = (a^{**}, \lambda) = (a, \lambda)$.
- $((a, \lambda)(b, \mu))^* = (ab + \mu a + \lambda B, \lambda\mu)^* = (b^*a^* + \overline{\mu}a^* + \overline{\lambda}b^*, \overline{\lambda\mu}) = (b^*, \overline{\mu})(a^*, \overline{\lambda}) = (b, \mu)^*(a, \lambda)^*$.

Thus, the map $^* : \tilde{A} \to \tilde{A}$ defines an involution on $\tilde{A}$.

Our next step is to prove the C^*-condition for $\tilde{A}$, assuming that both A and $\mathbb{C}$ are C^*-algebras. For any $(a,\lambda) \in \tilde{A}$, we have that

$$\begin{aligned}
\|(a,\lambda)\|_{\tilde{A}}^2 &= \sup_{\|c\|\le 1} \|ac+\lambda c\| \sup_{\|c\|\le 1} \|ac+\lambda c\| \\
&\le \sup_{\|c\|\le 1} \|c\|(\|a+\lambda\|^2) \\
&= \|a+\lambda\|^2 = \|(a+\lambda)^*(a+\lambda)\| \\
&= \|a^*a+\lambda a^*+\overline{\lambda}a+|\lambda|^2\| \\
&= \|(a^*a+\lambda a^*+\overline{\lambda}a, |\lambda|^2)\|_{\tilde{A}} = \|(a,\lambda)^*(a,\lambda)\|_{\tilde{A}} \\
&\le \|(a,\lambda)^*\|_{\tilde{A}}\|(a,\lambda)\|_{\tilde{A}}.
\end{aligned}$$

So

$$\|(a,\lambda)\|_{\tilde{A}} \le \|(a,\lambda)^*\|_{\tilde{A}}.$$

Following the same procedure for the element $(a^*,\overline{\lambda}) = (a,\lambda)^* \in \tilde{A}$, we deduce that

$$\|(a,\lambda)^*\|_{\tilde{A}} = \|(a,\lambda)\|_{\tilde{A}}.$$

Thus

$$\|(a,\lambda)\|_{\tilde{A}}^2 = \|(a,\lambda)^*(a,\lambda)\|_{\tilde{A}}.$$

So $\tilde{A}$ is a C^*-algebra.

The algebra $\tilde{A}$ is also unital, with unit $(0,1) \in \tilde{A}$, since for any $(a,\lambda) \in \tilde{A}$, we have

$$(a,\lambda)(0,1) = (a,\lambda) = (0,1)(a,\lambda).$$

Next, we define the map $\phi : \tilde{A} \to \mathbb{C}$, by $\phi(a,\lambda) = \lambda$. Observe that

$$\ker\phi = \{(a,\lambda) \in \tilde{A} : \phi(a,\lambda) = \lambda = 0\} = \{(a,0) : a \in A\}.$$

Since $\mathbb{C}$ is a field, we have that $\ker\phi$ is a maximal ideal of $\tilde{A}$. Using the map $\gamma : A \to \ker\phi$, with $\gamma(a) = (a,0)$, we can easily see that this map defines an algebra isomorphism. Hence, the algebra A is a maximal ideal of $\tilde{A}$. From the fact that $\tilde{A} = A \bigoplus \mathbb{C}$, we deduce that

$$\dim\tilde{A} = \dim A + \dim\mathbb{C} = \dim A + 1. \qquad \square$$

Before stating and proving Gelfand Theorems, we gather some useful definitions and properties, concerning Banach algebras, that were introduced in Chapter 1.

Remark 2.2.9. Let Δ be the set of all nonzero complex homomorphisms of a commutative Banach algebra A. For each $x \in A$, we define the Gelfand transform of x, by the map

$$\hat{x} : \Delta \to \mathbb{C}, \text{ with } \hat{x}(h) = h(x), \quad \text{for all } h \in \Delta.$$

Remark 2.2.10. Let $\hat{A}$ denote the set of all $\hat{x} : \Delta \to \mathbb{C}$. The map $\gamma : A \to \hat{A}$, with

$$\gamma(x)(h) = \hat{x}(h) = h(x), \quad \text{for all } h \in \Delta,$$

is an algebra homomorphism, whose kernel is the radical of A, denoted by $\mathrm{rad}A$.

Remark 2.2.11. The maximal ideal space Δ of a commutative Banach algebra A is a locally compact Hausdorff space. Additionally, it is compact if, and only if, A is unital.

Remark 2.2.12. For any commutative Banach algebra A, a norm can be defined on $\hat{A}$ by the map $\|\cdot\|_\infty : \hat{A} \to \mathbb{R}$, with

$$\|\hat{x}\|_\infty = \max_{h\in\Delta}\{|\hat{x}(h)|\}.$$

Now, we can state and prove the Gelfand Theorems for commutative C^*-algebras.

Theorem 2.2.13 (Unital case of Gelfand Theorem, [16], p.236)**.** *Let A be a unital commutative C^*-algebra, with norm $\|\cdot\|$. Then, the Gelfand transform, which is a map $\gamma : A \to C(\Delta)$, is an isometric *-isomorphism.*

Proof. Using Theorem 1.6.15, we have that $\|\hat{x}\|_\infty = p(x) \leq \|x\|$, for all $x \in A$. Also, from Theorem 2.2.4, we have that $\|a\| = p(a)$, for any hermitian element $a \in A$. So, we conclude that $\|x^*x\| = \|\widehat{x^*x}\|_\infty$, for all $x \in A$, since the element $x^*x \in A$ is, itself, hermitian. Now, for any $a \in A$ and $h \in \Delta$, we have from Proposition 2.2.6, that

$$\widehat{a^*}(h) = h(a^*) = \overline{h(a)} = \overline{\hat{a}}(h)$$

from which we derive that $\widehat{a^*} = \overline{\hat{a}}$. Since the involution in $\mathbb{C}$ is the complex conjugation, we deduce that

$$\gamma(a^*) = \widehat{a^*} = \overline{\hat{a}} = \hat{a}^* = \gamma(a)^*.$$

As we mentioned in Remark 2.2.12, the map $\gamma : A \to C(\Delta)$ is a homomorphism. Thus, the Gelfand transform is a *-homomorphism. Also, we observe, for any $a \in A$, that

$$\begin{aligned}
\|a\|^2 &= \|a^*a\| = \|\widehat{a^*a}\|_\infty \\
&= \max_{h\in\Delta}\{|\widehat{a*a}(h)|\} = \max_{h\in\Delta}\{|h(a)^*h(a)|\} \\
&= \max_{h\in\Delta}\{|h^2(a)|\} = \max_{h\in\Delta}\{|\hat{a}^2(h)|\} \\
&= \||\hat{a}|^2\|_\infty = \|\hat{a}\|_\infty^2.
\end{aligned}$$

Therefore, we have that $\|a\| = \|\hat{a}\|_\infty$, for all $a \in A$. Hence, γ is an isometry.

We, now, prove that $\gamma : A \to C(\Delta)$ is injective. For this, let $\gamma(a) = \gamma(b)$, for $a, b \in A$. This means, for any $h \in \Delta$, that $\gamma(a)(h) = \gamma(b)(h)$ and, thus, that $h(a) = h(b)$. So

$$\|\gamma(a) - \gamma(b)\|_\infty = 0 = \|\gamma(a-b)\|_\infty = \|a-b\|.$$

Since γ is an isometry, we deduce that $a = b$ and that γ is injective.

The last part of the proof is to show that γ is surjective, that is, $\gamma(A) = C(\Delta)$. For this, we will use the Stone-Weierstrass Theorem (Theorem 2.1.16). We can do this, since A is unital and, so, Δ is a compact Hausdorff space. Since A is a Banach space, hence complete, we immediately have that $\gamma(A)$ is complete, because γ is continuous, being an isometry. So $\gamma(A)$ is closed in $C(\Delta)$.

The fact that every isometry is continuous is evident, since for any sequence $(a_n)_{n\in\mathbb{N}}$ in A, such that $a_n \longrightarrow a \in A$, we have

$$\|a_n - a\| < \epsilon, \quad \text{for any } \epsilon > 0.$$

Thus

$$\|\gamma(a_n - a)\| = \|\gamma(a_n) - \gamma(a)\| < \epsilon$$

and γ is continuous.

We, now, show that $\gamma(A)$ separates points in Δ. Let $\phi, h \in \Delta$ with $\phi \neq h$. Then there exists an $a \in A$, such that $\phi(a) \neq h(a)$. This means that $\hat{a}(\phi) \neq \hat{a}(h)$, which implies that $\gamma(a)(\phi) \neq \gamma(a)(h)$. Thus, $\gamma(A)$ separates points in Δ. Since γ preserves involution, that is, $\gamma(a^*) = \gamma(a)^*$, for all $a \in A$, we have that $\gamma(A)$ is closed under complex conjugation. Indeed, if $h \in \gamma(A)$, then $h = \gamma(a)$, for some $a \in A$, which implies that

$$\overline{h} = \overline{\gamma(a)} = \gamma(a)^* = \gamma(a^*) \in \gamma(A).$$

Finally, for any $h \in \Delta$, we have by the definition of the maximal ideal space Δ, that $h \neq 0$. This means that there exists an $a \in A$, such that $h(a) \neq 0$. By the definition of the Gelfand transform, we have that $\hat{a}(h) \neq 0$, which implies that $\gamma(a)(h) \neq 0$. Thus, $\gamma(A)$ vanishes at no point of Δ. Hence, we can apply the Stone-Weierstrass Theorem (Theorem 2.1.16) and deduce that $\overline{\gamma(A)} = C(\Delta)$. Since $\gamma(A)$ is closed in $C(\Delta)$, we also have that $\overline{\gamma(A)} = \gamma(A) = C(\Delta)$ and so γ is surjective. Thus, the Gelfand transform is an isometric *-isomorphism. □

Theorem 2.2.14 (Nonunital case of Gelfand Theorem, [17], p.7)**.** *Let A be a nonunital commutative C^*-algebra. Then, the Gelfand transform, which is a map $\gamma : A \to C_0(\Delta)$, is an isometric *-isomorphism.*

Proof. We apply Theorem 2.2.8, to the C^*-algebra A, in order to get a unital commutative C^*-algebra $\tilde{A}$, which has A as a maximal ideal of codimension 1. Since $\tilde{A}$ is unital, we deduce that $\tilde{A}$ is isometrically *-isomorphic to $C(\Delta_{\tilde{A}})$. From this we derive that A is isometrically *-isomorphic to $C(\Delta_{\tilde{A}}/\{\infty\})$. Here, $\{\infty\}$ is the point at infinity which made Δ_A into the compact Hausdorff space $\Delta_{\tilde{A}}$, via the one-point compactification.

Indeed, by identifying $C(\Delta_{\tilde{A}}/\{\infty\})$ with $C_0(\Delta_A)$, using the obvious *-isomorphism between these algebras, we deduce that A is isometrically *-isomorphic to $C_0(\Delta_A)$, since the Gelfand transform maps the ideal A of $\tilde{A}$, to the ideal $C_0(\Delta_A)$ of $C(\Delta_{\tilde{A}})$. The isomorphism maps each function $f : \Delta_A \to \mathbb{C}$, to itself, that is, to the function $f : \Delta_{\tilde{A}} \to \mathbb{C}$. Thus, we have that the Gelfand transform, regarded as a map $\gamma : A \to C_0(\Delta)$, is an isometric *-isomorphism. □

An immediate application of Gelfand Theorems is the Spectral Mapping Theorem (Theorem 2.2.17) which states that the spectrum of the functional calculus of a normal element in a C^*-algebra A is exactly the functional calculus of the spectrum of this element. First, we need to define the notion of the functional calculus of a normal element in a C^*-algebra.

Definition 2.2.15 ([29], p.41)**.** Let A be a unital C^*-algebra, with unit 1_A, and $a \in A$ be a normal element of A. The commutative C^*-algebra $C^*(a) = \overline{\text{span}}\{1_A, a\}$ is called the C^*-algebra **generated by $\boldsymbol{a}$ and $\mathbf{1_A}$**. It can be easily proved that $C^*(a)$ is the smallest C^*-subalgebra of A that contains a and 1_A.

Definition 2.2.16 ([29], p.43)**.** Let A be a unital C^*-algebra, with unit 1_A and $a \in A$ be a normal element of A. We define the map $r : C(\sigma(a)) \to C^*(a)$, by

$$r(f) = f(a), \quad \text{for all } f \in C(\sigma(a)).$$

The map r is called the **functional calculus for $\boldsymbol{a}$**.

Theorem 2.2.17 (Spectral Mapping Theorem, [29], p.43)**.** *If A is a C^*-algebra and $a \in A$ is a normal element of A, then $\sigma(f(a)) = f(\sigma(a))$, for all $f \in C(\sigma(a))$.*

Proof. Let $r : C(\sigma(a)) \to C^*(a)$ be defined by $r(f) = f(a)$. Then, r is a *-isomorphism. The proof that r is a *-isomorphism is trivial and will be omitted. Hence

$$\sigma(f(a)) = \sigma(r(f)) = \sigma(f).$$

Applying Gelfand Theorem (Theorem 2.2.13), we have that $\sigma(f) = f(\sigma(a))$. Hence $\sigma(f(a)) = f(\sigma(a))$. □

Proposition 2.2.18 ([16], p.240)**.** *If A is a C^*-algebra and $a \in A$ is a hermitian element of A, then there are positive elements $u, v \in A$ such that $a = u - v$ and $uv = vu = 0$.*

Proof. We define the functions $f, g : A \to \mathbb{R}$ by $f(t) = \max(t, 0)$ and $g(t) = \min(t, 0)$. Then, $f, g \in C(\mathbb{R})$, $f(t) - g(t) = t$ and $f(t)g(t) = 0$. Using the definition of functional calculus, we find elements $u, v \in A$, with $u = f(a)$ and $v = g(a)$. Then, u and v are hermitian, since $u^* = f(a)^* = f(a^*) = f(a) = u$ and $v^* = g(a)^* = g(a^*) = g(a) = v$. Using the Spectral Mapping Theorem (Theorem 2.2.17), we have that $\sigma(u) = \sigma(f(a)) = f(\sigma(a)) \geq 0$ and that $\sigma(v) = \sigma(g(a)) = g(\sigma(a)) \geq 0$. Thus, the elements u and v are positive and satisfy $u - v = f(a) - g(a) = a$ and $uv = vu = f(a)g(a) = 0$. □

Proposition 2.2.19 ([16], p.241)**.** *If A is a C^*-algebra and $a \in A$ is such, that $a = x^*x$, then $a \geq 0$.*

Proof. We observe that $a^* = x^*x = a$. Hence, by Proposition 2.2.18, there exist positive elements $u, v \in A$, such that $a = u - v$ and $uv = vu = 0$. We need to show that $v = 0$. Let $b + ic = xv^{1/2}$, where $b, c \in A$ are hermitian elements of A. The existence of the elements b and c is due to Proposition 2.2.2. Then

$$(xv^{1/2})^*(xv^{1/2}) = (b - ic)(b + ic) = b^2 + c^2 + i(bc - cb)$$

and

$$(xv^{1/2})^*(xv^{1/2}) = v^{1/2}x^*xv^{1/2} = v^{1/2}av^{1/2} = v^{1/2}(u - v)v^{1/2} = uv - v^2 = -v^2.$$

Hence $i(bc - cb) = -(v^2 + b^2 + c^2) \leq 0$ and since $(xv^{1/2})^*(xv^{1/2}) = -v^2$, we observe that $(xv^{1/2})^*(xv^{1/2}) \leq 0$. If we let $y = -(xv^{1/2})(xv^{1/2})^*$, we can see that $y \in A^+$. So $-y = (b + ic)(b - ic) = b^2 + c^2 - i(bc - cb)$ and thus $i(bc - cb) = b^2 + c^2 + y \geq 0$. So

$$0 \leq i(bc - cb) \leq 0.$$

Hence, $i(bc - cb) = 0$ and $-v^2 = b^2 + c^2 \geq 0$. Thus, $v = 0$ and the proof is complete. □

2.3 Approximate Identities

We finish this chapter with some useful definitions concerning C^*-algebras.

Definition 2.3.1 ([29], p.77)**.** Let A be a C^*-algebra. An **approximate identity** of A is a net $(x_\lambda)_{\lambda\in\Lambda}$ of elements of A, such that all of the following hold.

(i) The net $(x_\lambda)_{\lambda\in\Lambda}$ is increasing.

(ii) The net $(x_\lambda)_{\lambda\in\Lambda}$ is contained in the closed unit ball of A.

(iii) The net $(x_\lambda)_{\lambda\in\Lambda}$ contains only positive elements of A.

(iv) $\lim\limits_{\lambda} \|x_\lambda - x\| = 0 = \lim\limits_{\lambda} \|xx_\lambda - x\|$, for all $x \in A$.

Remark 2.3.2. If A is a unital C^*-algebra, then an approximate identity of A is the constant net $(x_\lambda)_{\lambda\in\Lambda}$ with $x_\lambda = 1_A$, for all $\lambda \in \Lambda$. If A is a nonunital C^*-algebra, then A admits an approximate identity, as we shall see in Theorem 2.3.7.

Remark 2.3.3. If A and B are both nonunital commutative C^*-algebras and $\phi : A \to B$ is a *-homomorphism from A into B, then ϕ always respects the approximate identity $(a_\lambda)_{\lambda\in\Lambda}$ of A, that is, $\phi(a_\lambda) = (b_\mu)_{\mu\in M}$, where $(b_\mu)_{\mu\in M}$ is the approximate identity of B.

Remark 2.3.4. The unitization of a nonunital C^*-algebra A does not always come with no drawbacks. For example, if J is a right ideal of A and $\tilde{J} = J \bigoplus \mathbb{C}$ is the unitization of J, then $\tilde{J}$ is no longer an ideal of A, for if $(j, c) \in \tilde{J}$ and $a \in A$, then

$$(j, c)a = (j, c)(a, 0) = (ja + ca, 0) \notin \tilde{J},$$

since $ca \notin J$.

Remark 2.3.5. The problem of showing that every closed ideal J of a C^*-algebra A is self-adjoint, is solved by using approximate identities. Indeed, if $(x_\lambda)_{\lambda\in\Lambda}$ is an approximate identity of J and if $x \in J$ is an arbitrary element of J, then

$$\lim_{\lambda} \|x_\lambda x - x\| = 0 = \lim_{\lambda} \|x_\lambda x^* - x^*\| = \lim_{\lambda} \|x^* x_\lambda - x^*\|.$$

Since $x^* x_\lambda \in J$ for all $\lambda \in \Lambda$ and since J is closed, we have that

$$\lim_{\lambda}(x^* x_\lambda) = x^* \in J.$$

Thus, J is self-adjoint.

Lemma 2.3.6 ([8], p.36)**.** *Let A be a unital C^*-algebra, with unit 1_A, and x, y be positive invertible elements of A, satisfying $x \leq y$. Then $y^{-1} \leq x^{-1}$.*

Proof. Since both x and y are positive, we can find positive roots for both, which are denoted by $x^{1/2}$ and $y^{1/2}$, respectively. Multiplying $x \leq y$, left and right, with $y^{-1/2}$, we get

$$y^{-1/2}xy^{-1/2} \leq 1_A.$$

If we let $z = x^{1/2}y^{-1/2} \in A$, we get

$$z^*z = y^{-1/2}xy^{-1/2} \leq 1_A.$$

Thus, we have that

$$\begin{aligned} \|z^*z\| = \|z\|^2 \leq 1 &\Rightarrow \|z\| \leq 1 \\ &\Rightarrow \|x^{1/2}y^{-1/2}\| \leq 1 \\ &\Rightarrow \|(x^{1/2}y^{-1/2})^*\| = \|y^{-1/2}x^{1/2}\| \leq 1. \end{aligned}$$

So, we deduce that $zz^* \leq 1_A$, since $\|z\| \leq 1$ and $\|z^*\| \leq 1$. Hence

$$x^{1/2}y^{-1}x^{1/2} \leq 1_A,$$

which implies that $y^{-1} \leq x^{-1}$. □

Theorem 2.3.7 ([8], p.36)**.** *Every C^*-algebra A admits an approximate identity.*

Proof. If A is unital, we immediately have that an approximate identity is $(x_\lambda)_{\lambda\in\Lambda}$, with $x_\lambda = 1_A$, for all $\lambda \in \Lambda$. So, we assume that A has no unit and let $\tilde{A}$ be the unitization of A, with unit $1_{\tilde{A}} = (0, 1)$. Also, under the usual set inclusion, we define the partially ordered set

$$\Lambda = \{U_i : i \in I\}$$

where

$$U_i = \{\{x_1, \dots, x_i\} : x_j \in A, \quad \text{for all } j = 1, 2, \dots, i,\ i \in I\}.$$

For each integer $n \geq 1$, let f_n be the function $f_n : [0, +\infty) \to [0, 1]$, defined by $f_n(t) = nt(1 + nt)^{-1}$. Since f_n is continuous, for every integer $n \geq 1$, it can operate on the set of all positive elements of A, denoted by A^+, instead of $\mathbb{R}$. In other words, for each integer $n \geq 1$, we can define a function $f_n : A^+ \to A^+$. The domain of this function contains the positive elements of A whose spectrum is contained in $[0, +\infty)$ and its image consists of those positive elements of A whose spectrum is contained in $[0, 1]$. So, for each $\lambda \in \Lambda$, say $\lambda = \{x_1, \dots, x_n\}$, we define the element $e_\lambda \in A$, by

$$e_\lambda = f_n(x_1^2 + \cdots + x_n^2).$$

The element $x_1^2 + \cdots + x_n^2$ is positive, since each x_i^2 is positive, for any $i \in \{1, \dots, n\}$, because $\sigma(x_i^2) = \sigma(x_i)^2 \geq 0$. Hence $x_1^2 + ... + x_n^2 \geq 0$, which implies that $\sigma(e_\lambda) \subset [0, 1]$. Thus, $(e_\lambda)_{\lambda\in\Lambda}$ is contained in the closed unit ball of A and is positive, so $e_\lambda \geq 0$ and $\|e_\lambda\| \leq 1$, for all $\lambda \in \Lambda$.

Now, let $\lambda \leq \mu$, say $\lambda = \{x_1, \ldots, x_m\}$ and $\mu = \{x_1, \ldots, x_n\}$, where $m \leq n$. Since $1 - f_n(t) = (1 + nt)^{-1}$, we have that

$$1_{\tilde{A}} + m(x_1^2 + \cdots + x_m^2) \leq 1_{\tilde{A}} + n(x_1^2 + \cdots + x_n^2).$$

This implies that

$$[1_{\tilde{A}} + n(x_1^2 + \cdots + x_n^2)]^{-1} \leq [1_{\tilde{A}} + m(x_1^2 + \cdots + x_m^2)]^{-1}.$$

Furthermore

$$1_{\tilde{A}} - f_n(x_1^2 + \cdots + x_n^2) \leq 1_{\tilde{A}} - f_m(x_1^2 + \cdots + x_m^2).$$

Thus $e_\lambda \leq e\mu$. Consequently, the net $(e_\lambda)_{\lambda \in \Lambda}$ is increasing.

Finally, we need to prove that

$$\lim_\lambda \|e_\lambda x - x\| = 0 = \lim_\lambda \|x e_\lambda - x\|,$$

for any hermitian element $x \in A$, since any other element $z \in A$ can be decomposed into the form $z = z_1 + z_2$, where z_1, z_2 are hermitian. So, we fix an $x \in A$, for which $x = x^*$. Let m be a positive integer and let $\lambda = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n\}$ be any finite set of hermitian elements that contains x and that has at least m elements. We claim that

$$\|x e_\lambda - x\|^2 \leq \frac{1}{4m}.$$

Indeed, since $x^2 \leq x_1^2 + \cdots + x_n^2$, we have that

$$(1_{\tilde{A}} - e_\lambda) x^2 (1_{\tilde{A}} - e_\lambda) \leq (1_{\tilde{A}} - e_\lambda)(x_1^2 + \cdots + x_n^2)(1_{\tilde{A}} - e_\lambda).$$

We define the function $g_n : [0, +\infty) \to [0, 1]$, by

$$g_n(t) = (1 - f_n(t)) t (1 - f_n(t)).$$

We observe that

$$g_n(t) = t(1 - f_n(t))^2 = t(1 + nt)^{-2}$$

and that

$$|g_n(t)| = |t(1 + nt)^{-2}| \leq \frac{1}{4n}.$$

Thus

$$(1_{\tilde{A}} - e_\lambda) x^2 (1_{\tilde{A}} - e_\lambda) \leq g_n(x_1^2 + \cdots + x_n^2) \leq \frac{1}{4n} 1_{\tilde{A}},$$

where g_n is considered as $g_n : A^+ \to A^+$. So, the element $z = x - e_\lambda = x(1_{\tilde{A}} - e_\lambda)$, satisfies

$$z^* z = (1_{\tilde{A}}^* - e_\lambda^*) x^* x (1_{\tilde{A}} - e_\lambda) = (1_{\tilde{A}} - e_\lambda) x^2 (1_{\tilde{A}} - e_\lambda) \leq \frac{1}{4n} 1_{\tilde{A}}.$$

Thus

$$\|z^* z\| = \|z\|^2 \leq \frac{1}{4n}.$$

The above relation implies that

$$\|xe_\lambda - x\|^2 \le \frac{1}{4n} \le \frac{1}{4m}.$$

We, also, observe that

$$\|xe_\lambda - x\| = \|(xe_\lambda - x)^*\| = \|e_\lambda x^* - x^*\| = \|e_\lambda x - x\|.$$

Since m is arbitrary, we deduce that

$$\lim_\lambda \|xe_\lambda - x\| = 0 = \lim_\lambda \|e_\lambda x - x\|.$$

Thus, $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate identity of A. $\square$

Chapter 3

Gelfand Duality and Applications

This chapter is about category theory and the Gelfand Duality. The first section consists of some preliminary notions of category theory such as the notion of a category and that of a morphism. We state some terminology about certain categories of topological spaces and C^*-algebras. In the second section we define the notion of a covariant and a contravariant functor and we give some examples of special kinds of functors that will be used throughout this chapter. The third section is about equivalences of categories. We provide a theorem (Theorem 3.3.10) that relates certain properties of functors with the equivalence of the categories associated to that functor. Finally, the last section is about Gelfand Duality. Specifically, we prove that the category $\mathcal{LCS}$ of locally compact Hausdorff spaces is dual to the category of nonunital commutative C^*-algebras $C^*Alg_{com,nu}$ and the category $\mathcal{CS}$ of compact Hausdorff spaces is dual to the category of unital commutative C^*-algebras $C^*Alg_{com,u}$. We close up this chapter with some interesting consequences of Gelfand Duality.

3.1 Categories

In this section, we will analyze some basic notions of category theory, that is, the notions of a category $\mathcal{C}$ and that of a morphism, which is a structure preserving map, alongside with the notion of a functor between two categories. We will introduce the notion of a category equivalence and that of category duality, which we will make use of in the proofs of Gelfand Theorems.

Definition 3.1.1 ([10], p.71). A **category** $\mathcal{C}$ consists of a collection $\mathcal{O}$ of **objects** and a collection $\mathcal{M}$ of **morphisms**, which are structure preserving maps between same types of structures, such that all of the following conditions hold

(i) For all $A, B \in \mathcal{O}$, there is a (possibly empty) set, denoted by $\mathrm{Mor}(A, B)$, called the **set of morphisms** $f : A \to B$, such that

$$\mathrm{Mor}(A, B) \cap \mathrm{Mor}(A', B') = \emptyset, \quad \text{whenever } (A, B) \neq (A', B').$$

(ii) If $A, B, C \in \mathcal{O}$, we can define an operation, called **composition**, from $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C)$ to $\mathrm{Mor}(A, C)$, with $(f, g) \to g \circ f$, such that

- (Associativity) If $f : A \to B$, $g : B \to C$ and $h : C \to D$ are morphisms in $\mathcal{C}$, then $(h \circ g) \circ f = h \circ (g \circ f)$.
- (Existence of identity) For each $A \in \mathcal{O}$, there exists an identity morphism $id_A : A \to A$, such that $f \circ id_A = f$ and $id_A \circ g = g$, for any morphisms $f : A \to B$, $g : E \to A$ of $\mathcal{C}$ and any $B, E \in \mathcal{O}$.

Definition 3.1.2 ([10], p.71)**.** A morphism $f : A \to B$ in $\mathcal{C}$ is said to be an **isomorphism**, if there exists a morphism $g : B \to A$ in $\mathcal{C}$, such that $f \circ g = id_B$ and $g \circ f = id_A$.

Definition 3.1.3 ([10], p.71)**.** A category $\mathcal{C}$ is a **subcategory** of a category $\mathcal{D}$, if all of the following conditions are satisfied.

(i) Every object of $\mathcal{C}$ is an object of $\mathcal{D}$.

(ii) If A and B are objects of $\mathcal{C}$, then $\mathrm{Mor}_{\mathcal{C}}(A, B) \subseteq \mathrm{Mor}_{\mathcal{D}}(A, B)$.

(iii) The composition of morphism in $\mathcal{C}$ coincides with the composition of morphisms in $\mathcal{D}$.

(iv) For every object A of $\mathcal{C}$, the identity morphism $id_A : A \to A$ of $\mathcal{C}$ coincides with the identity morphism $id_A : A \to A$ of $\mathcal{D}$.

Definition 3.1.4 ([10], p.72)**.** If $\mathcal{C}$ is a subcategory of $\mathcal{D}$ and $\mathrm{Mor}_{\mathcal{C}}(A, B) = \mathrm{Mor}_{\mathcal{D}}(A, B)$, for all objects A, B of $\mathcal{C}$, then $\mathcal{C}$ is called a **full subcategory of** $\mathcal{D}$.

Definition 3.1.5 ([10], p.72)**.** A category $\mathcal{C}$ is called **additive**, if

(i) $\mathrm{Mor}(A, B)$ has the structure of an additive abelian group, for each pair of objects A, B of $\mathcal{C}$ and

(ii) $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$ and $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$, for all $f, f_1, f_2 \in \mathrm{Mor}(A, B)$ and $g, g_1, g_2 \in \mathrm{Mor}(B, C)$, where A, B, C are objects of $\mathcal{C}$.

Definition 3.1.6 ([10], p.72)**.** An object O of a category $\mathcal{C}$ is said to be a **zero object** of $\mathcal{C}$, if $\mathrm{Mor}(0, A)$ and $\mathrm{Mor}(A, 0)$ contain a single morphism each, for all objects A of $\mathcal{C}$.

Remark 3.1.7. If $\mathcal{C}$ is an additive category, then $\mathrm{Mor}(A, B) \neq \emptyset$, since the zero morphism $O_{AB} : A \to B$ is in $\mathrm{Mor}(A, B)$, for any pair of objects A, B of $\mathcal{C}$.

Remark 3.1.8. For the remainder of this chapter, we will denote any morphism $f \in \mathrm{Mor}(A, B)$ by $f : A \to B$ or $A \xrightarrow{f} B$, for any pair of objects A, B of $\mathcal{C}$.

Definition 3.1.9. A map $f : X \to Y$ between topological spaces, is called **proper**, if the preimage of every compact set in Y is compact in X, that is, for any $K \subset Y$, with K compact in Y, we have that $f^{-1}(K)$ is compact in X.

Definition 3.1.10 ([27], p.5). Let A and B be any C^*-algebras. A *-homomorphism $\phi : A \to B$ is called **proper**, if for any approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ of A, $(\phi(e_\lambda))_{\lambda\in\Lambda}$ is an approximate identity of B.

Remark 3.1.11. The category $\boldsymbol{\mathcal{TOP}}$ consists of all topological spaces and all continuous maps. The category $\boldsymbol{\mathcal{CS}}$ consists of all compact Hausdorff spaces and all continuous maps. Note that any $f \in \mathrm{Mor}_{\mathcal{CS}}(X, Y)$ is proper, by definition. The category $\boldsymbol{\mathcal{LCS}}$ consists of all locally compact Hausdorff spaces and all continuous proper maps between these spaces. We observe that $\mathcal{CS}$ is a full subcategory of $\mathcal{LCS}$, since every continuous map $f \in \mathrm{Mor}_{\mathcal{CS}}(X, Y)$ between compact spaces, is proper and every compact Hausdorff topological space is locally compact. Indeed, if X is a Hausdorff topological space, then for each $x \in X$, we have

$$\{x\} = \bigcap_{i\in I} U_i, \text{ for every closed neighborhood } U_i \text{ of } x.$$

If X is assumed to be compact, then these closed neighborhoods are compact themselves. Thus, X is locally compact, since any $x \in X$ has a neighborhood basis, consisting of compact sets. The category $\boldsymbol{C^*Alg_u}$ consists of all unital C^*-algebras and all *-homomorphisms, while the category $\boldsymbol{C^*Alg_{nu}}$ consists of all nonunital C^*-algebras and all proper *-homomorphisms. In the same manner, we define the categories $\boldsymbol{C^*Alg_{com,u}}$ and $\boldsymbol{C^*Alg_{com,nu}}$, as the commutative analogues of the categories mentioned above. Note that $C^*Alg_{com,u}$ and $C^*Alg_{com,nu}$ are full subcategories of C^*Alg_u and C^*Alg_{nu}, respectively, since every commutative C^*-algebra, either unital or nonunital, is a C^*-algebra by itself. Also

$$\mathrm{Mor}_{C^*Alg_u}(A, B) = \mathrm{Mor}_{C^*Alg_{com,u}}(A, B)$$

and

$$\mathrm{Mor}_{C^*Alg_{nu}}(X, Y) = \mathrm{Mor}_{C^*Alg_{com,nu}}(X, Y),$$

for any pair of objects A, B of $C^*Alg_{com,u}$ and X, Y of $C^*Alg_{com,nu}$. The category $\boldsymbol{\mathcal{SET}}$ consists of all sets and if A and B are objects of $\mathcal{SET}$, then $\mathrm{Mor}(A, B)$ is the set of all function $f : A \to B$. Note that each of the examples mentioned above, are subcategories of the category $\mathcal{SET}$. This statement is true, since any topological space and any C^*-algebra are sets by themselves and any morphism between topological spaces or C^*-algebras is a morphism between sets. We will refer to a category, whose objects are sets, as a **concrete category**. So, all the categories that were mentioned in the examples above are concrete categories.

3.2 Functors

In this section, we will discuss the notion of functors, which provide a method of transferring information from one category to another in a way that is, in some sense, structure preserving.

Definition 3.2.1 ([28], p.34). If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a pair of maps, that consists of

(i) An **object map** F, which associates each object A of $\mathcal{C}$ with exactly one object $F(A)$ of $\mathcal{D}$.

(ii) A **morphism map** (also denoted by F), which associates each morphism $f : A \to B$ of $\mathcal{C}$ with exactly one morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$, such that all of the following hold

- $F(id_A) = id_{F(A)}$ for each object A of $\mathcal{C}$.
- If $f : A \to B$ and $g : B \to C$ are morphisms in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$ in $\mathcal{D}$.

Definition 3.2.2 ([10], p.76)**.** A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ has exactly the same properties as a covariant functor, except that the morphism map F associates each morphism $f : A \to B$ in $\mathcal{C}$ with a morphism $F(f) : F(B) \to F(A)$ in $\mathcal{D}$. More precisely, if $f : A \to B$ and $g : B \to C$ are morphisms in $\mathcal{C}$, then $F(f) : F(B) \to F(A)$, $F(g) : F(C) \to F(B)$ and $F(g \circ f) : F(C) \to F(A)$ are morphisms in $\mathcal{D}$. Hence $F(g \circ f) = F(f) \circ F(g)$.

Definition 3.2.3 ([10], p.76)**.** A functor $F : \mathcal{C} \to \mathcal{D}$, either covariant or contravariant, between additive categories $\mathcal{C}$ and $\mathcal{D}$ is said to be **additive** if $F(f + g) = F(f) + F(g)$, for all $f, g \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and all $A, B \in \mathcal{O}_{\mathcal{C}}$.

Definition 3.2.4 ([10], p.75)**.** If $\mathcal{C}$ is a category, then the **opposite category** $\mathcal{C}^{op}$ of $\mathcal{C}$ is defined to be the category which satisfies all of the following.

(i) The objects of $\mathcal{C}^{op}$ are the objects of $\mathcal{C}$.

(ii) For any pair of objects A, B of $\mathcal{C}^{op}$, $\mathrm{Mor}^{op}(B, A)$ is the set of morphisms $\mathrm{Mor}(A, B)$ of $\mathcal{C}$, meaning that if $f^{op} : B \to A$ and $g^{op} : C \to B$ are morphisms in $\mathcal{C}^{op}$, then $f : A \to B$ and $g : B \to C$ are morphisms in $\mathcal{C}$. The morphism f^{op} is called the **opposite of** f and the morphisms in $\mathcal{C}^{op}$ are obtained from those of $\mathcal{C}$ by reversing the arrows. The composition of morphisms $f : A \to B$, $g : B \to C$ in $\mathcal{C}$ gives $g \circ f : A \to C$, while the composition in $\mathcal{C}^{op}$ gives $f^{op} \circ g^{op} : C \to A$. Hence

$$(g \circ f)^{op} = f^{op} \circ g^{op}.$$

Remark 3.2.5. It can be easily proved that $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Remark 3.2.6. For the remainder of this thesis, the terms functor and cofunctor will refer to a covariant and a contravariant functor, respectively.

Definition 3.2.7 ([10], p.76)**.** If $\mathcal{C}$ is an additive category, then the **identity functor** $id_{\mathcal{C}}$ is defined by $id_{\mathcal{C}}(A) = A$, for each object A of $\mathcal{C}$. If $f : A \to B$ is a morphism of $\mathcal{C}$, between objects A, B of $\mathcal{C}$, then

$$id_{\mathcal{C}}(f) : id_{\mathcal{C}}(A) \to id_{\mathcal{C}}(B)$$

is the morphism $f : A \to B$. It is an additive functor, since for any morphism $f, g \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ we have that

$$id_{\mathcal{C}}(f + g) = f + g = id_{\mathcal{C}}(f) + id_{\mathcal{C}}(g).$$

Definition 3.2.8 ([10], p.76)**.** If $\mathcal{C}$ is an additive subcategory of an additive category $\mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$ is a functor, such that $F(A) = A$ and $F(f) = f$, for any object A and any morphism f of $\mathcal{C}$, then F is called the **canonical embedding functor**. F is an additive functor, since

$$F(f + g) = f + g = F(f) + F(g),$$

for any morphism f of $\mathcal{C}$. We also observe that the identity functor $id_{\mathcal{C}}$ of a category $\mathcal{C}$ is a canonical embedding functor, since any category $\mathcal{C}$ is a subcategory of itself.

Definition 3.2.9 ([10], p.77)**.** If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, then we can define the functor $GF : \mathcal{C} \to \mathcal{E}$ in the obvious way. We send each object A of $\mathcal{C}$ to an object $F(A)$ in $\mathcal{D}$ and, via G, we get an object $(GF)(A)$ in $\mathcal{E}$. Also, for any morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ of $\mathcal{C}$ we have that

$$GF(g \circ f) = G(F(g) \circ F(f)) = (GF(g)) \circ (GF(f))$$

and

$$GF(id_A) = G(F(id_A)) = G(id_{F(A)}) = id_{GF(A)}.$$

Hence, the functor $GF : \mathcal{C} \to \mathcal{R}$ is well defined and is called the **composition** of F and G. It is additive if, and only if, the functors F, G and the categories in question are additive. Indeed, if $f, g \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ are additive morphisms in the category $\mathcal{C}$, we have that

$$GF(f + g) = G(F(f + g)) = G(F(f) + F(g)) = GF(f) + GF(g).$$

Definition 3.2.10 ([10], p.76)**.** The **forgetful functor** from a concrete category $\mathcal{C}$ into the category $\mathcal{SET}$, assigns to each object A of $\mathcal{C}$, the set A, with no additional structure, and to any morphism $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ the morphism $f \in \mathrm{Mor}_{\mathcal{SET}}$, where A and B are objects of $\mathcal{C}$, regarded as sets.

Example 3.2.11. Let $\mathcal{CS}$ be the category of compact Hausdorff spaces and continuous maps. Then the functor $F : \mathcal{CS} \to \mathcal{SET}$, sending each compact Hausdorff topological space to itself and each continuous map $f \in \mathrm{Mor}_{\mathcal{CS}}(X, Y)$ to itself is forgetful. $\square$

Definition 3.2.12 ([10], p.79)**.** If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{R}$ are cofunctors, then we define the composition of F and G, via the map $GF : \mathcal{C} \to \mathcal{R}$.

Remark 3.2.13. Following Definition 3.2.12, we observe that $GF : \mathcal{C} \to \mathcal{R}$ remains a functor, since for any morphisms $f : A \to B$ and $g : B \to C$ of $\mathcal{C}$, we have that

$$GF(g \circ f) = G(F(f) \circ F(g)) = GF(g) \circ GF(f).$$

On the other hand, if either $F : \mathcal{C} \to \mathcal{D}$ or $G : \mathcal{D} \to \mathcal{R}$ is a cofunctor, then the composition of F and G is a cofunctor, for if $f : A \to B$ and $g : B \to C$ are morphisms in $\mathcal{C}$, then

$$GF(g \circ f) = (GF(f)) \circ (GF(g)).$$

3.3 Natural Transformations

In this section we will analyze the notions of equivalent and dual categories, which are key components of this thesis.

Definition 3.3.1 ([10], p.97)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a **category isomorphism**, if there exists a functor $G : \mathcal{D} \to \mathcal{C}$, such that $FG = id_{\mathcal{D}}$ and $GF = id_{\mathcal{C}}$. If such a functor F exists between $\mathcal{C}$ and $\mathcal{D}$, then $\mathcal{C}$ and $\mathcal{D}$ are called **isomorphic categories** and we denote this relation by $\mathcal{C} \cong \mathcal{D}$.

Example 3.3.2. The identity functor $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a category isomorphism, since the existence of $id_{\mathcal{C}}$ implies that $id_{\mathcal{C}} id_{\mathcal{C}} = id_{\mathcal{C}}$. $\square$

Remark 3.3.3. Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories. We form the category $\mathcal{C} \times \mathcal{D}$, called the product category, as follows. The objects of $\mathcal{C} \times \mathcal{D}$ are pairs (A, B), where A, B are objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. A morphism in $\mathrm{Mor}_{\mathcal{C}\times\mathcal{D}}((A_1, B_1), (A_2, B_2))$ is a pair (f, g), where $f : A_1 \to A_2$ is a morphism in $\mathcal{C}$ and $g : B_1 \to B_2$ is a morphism in $\mathcal{D}$. Composition of morphisms is given by

$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1),$$

where $f_2 \circ f_1$ is defined in $\mathcal{C}$ and $g_2 \circ g_1$ is defined in $\mathcal{D}$. So, suppose that $F : \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{C}$ is such that $F(A, B) = (B, A)$ and $F(f, g) = (g, f)$. Next, let $G : \mathcal{D} \times \mathcal{C} \to \mathcal{C} \times \mathcal{D}$ be defined by $G(B, A) = (A, B)$ and $G(g, f) = (f, g)$. Then, F is a category isomorphism, since

$$FG = id_{\mathcal{D}\times\mathcal{C}} \text{ and } GF = id_{\mathcal{C}\times\mathcal{D}}.$$

Definition 3.3.4 ([10], p.98)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors and suppose that for each object A of $\mathcal{C}$, there exists a morphism $\eta_A : F(A) \to G(A)$ in $\mathcal{D}$ such, that for each morphism $f : A \to B$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"] & G(B)
\end{tikzcd}$$

is commutative.

(i) The class of morphisms $\eta = \{\eta_A : F(A) \to G(A)\}$, indexed over the objects of $\mathcal{C}$, is said to be a **natural transformation** from F to G. Such a transformation will be denoted by $\eta : F \to G$.

(ii) If η_A is an **isomorphism** in $\mathcal{D}$, for each object A of $\mathcal{C}$, then $\eta : F \to G$ is said to be a **natural isomorphism** and F, G are called **naturally equivalent functors**, denoted by $F \approx G$.

Definition 3.3.5 ([10], p.98)**.** If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, such that $GF \approx id_{\mathcal{C}}$ and $FG \approx id_{\mathcal{D}}$, then $\mathcal{C}$ and $\mathcal{D}$ are said to be **equivalent categories**, denoted by $\mathcal{C} \approx \mathcal{D}$ and we say that the pair (F, G) gives a **category equivalence** between $\mathcal{C}$ and $\mathcal{D}$.

Definition 3.3.6 ([10], p.98)**.** If F and G are cofunctors (i.e. contravariant functors), such that $GF \approx id_{\mathcal{C}}$ and $FG \approx id_{\mathcal{D}}$, then $\mathcal{C}$ and $\mathcal{D}$ are said to be **dual categories**, denoted by $\mathcal{C} \approx \mathcal{D}^{op}$, and we say that the pair (F, G) forms a **duality** between $\mathcal{C}$ and $\mathcal{D}$.

Definition 3.3.7 ([32], p.30)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is called **full**, if for each pair of objects A, B of $\mathcal{C}$, the map

$$\mathrm{Mor}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(A), F(B))$$

is surjective.

Definition 3.3.8 ([32], p.30)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is called **faithful**, if for each pair of objects A, B of $\mathcal{C}$, the map

$$\mathrm{Mor}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(A), F(B))$$

is injective.

Definition 3.3.9 ([32], p.31)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is called **essentially surjective**, if for each object B of $\mathcal{D}$, there is exists an object A of $\mathcal{C}$, such that $F(A) \cong B$.

Following the definitions stated above, we get a theorem which can be used to clarify the notion of a category equivalence.

Theorem 3.3.10 ([32], p.31)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories $\mathcal{C}$ and $\mathcal{D}$. Then F is an equivalence if, and only if, F is full, faithful and essentially surjective.*

Proof. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence. Then, there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : 1_{\mathcal{C}} \to GF$ and $\mu : 1_{\mathcal{D}} \to FG$, such that $GF \approx 1_{\mathcal{C}}$ and $FG \approx 1_{\mathcal{D}}$, meaning that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & C' \\ \big\downarrow \eta_C & & \big\downarrow \eta_{C'} \\ GF(C) & \xrightarrow{GF(f)} & GF(C') \end{array} \qquad\qquad \begin{array}{ccc} D & \xrightarrow{\quad g \quad} & D' \\ \big\downarrow \mu_D & & \big\downarrow \mu_{D'} \\ FG(D) & \xrightarrow{FG(g)} & FG(D') \end{array}$$

commute, for objects C, C' of $\mathcal{C}$, D, D' in $\mathcal{D}$ and morphisms $f \in \mathrm{Mor}_{\mathcal{C}}(C, C')$ and $g \in \mathrm{Mor}_{\mathcal{D}}(D, D')$. From this, we derive that the map $\alpha : \mathrm{Mor}_{\mathcal{C}}(C, C') \to \mathrm{Mor}_{\mathcal{C}}(GF(C), GF(C'))$, with $\alpha(f) = GF(f)$ is a bijection, that is, injective and surjective. Indeed, if $f_1, f_2 \in \mathrm{Mor}_{\mathcal{C}}(C, C')$ with $GF(f_1) = GF(f_2)$, then $GF(f_1) = GF(f_2)$, which means that $GF(f_1) \circ \eta_C = GF(f_2) \circ \eta_C$. This implies that $\eta_{C'} \circ f_1 = \eta_{C'} \circ f_2$. Consequently $f_1 = f_2$. Also, for any $g \in \mathrm{Mor}_{\mathcal{C}}(GF(C), GF(C'))$, we have that the morphism $f : C \to C'$ with $f = \eta_{C'}^{-1} \circ g \circ \eta_C$ such that $g = GF(f) = \alpha(f)$. Hence, the map

$$\alpha : \mathrm{Mor}_{\mathcal{C}}(C, C') \to \mathrm{Mor}_{\mathcal{C}}(GF(C), GF(C'))$$

is bijective. Now, for any morphism $f : C \to C'$ in the category $\mathcal{C}$ we define the maps

$$\beta : \mathrm{Mor}_{\mathcal{C}}(C, C') \to \mathrm{Mor}_{\mathcal{D}}(F(C), F(C')),$$

with $\beta(f) = F(f)$ and

$$\gamma : \mathrm{Mor}_{\mathcal{D}}(F(C), F(C')) \to \mathrm{Mor}_{\mathcal{C}}(GF(C), GF(C')),$$

with $\gamma(F(f)) = GF(f)$. The map γ is well defined, since for any morphism $f : C \to C'$ in $\mathcal{C}$, there exists a map $F(f) : F(C) \to F(C')$ in $\mathcal{D}$, for which $\gamma(F(f)) = GF(f)$. Now, we can see that the map β is injective and the map γ is surjective. Indeed, if $f_1, f_2 \in \mathrm{Mor}_{\mathcal{C}}(C, C')$, with $\beta(f_1) = \beta(f_2)$, then $F(f_1) = F(f_2)$, which means that $GF(f_1) = GF(f_2)$. This implies that $\alpha(f_1) = \alpha(f_2)$. Thus, $f_1 = f_2$. We have previously proved that for any $g \in \mathrm{Mor}_{\mathcal{C}}(GF(C), GF(C'))$, there exists an $f \in \mathrm{Mor}_{\mathcal{C}}(C, C')$, with $f = \eta_{C'}^{-1} \circ g \circ \eta_C$ such that $GF(f) = g$. So, by the definition of the map γ, we get $\gamma(F(f)) = g$. Thus, β is injective and γ is surjective. Additionally, concerning the map

$$\beta : \mathrm{Mor}_{\mathcal{C}}(C, C') \to \mathrm{Mor}_{\mathcal{D}}(F(C), F(C')),$$

we observe that for any $h \in \mathrm{Mor}_{\mathcal{D}}(F(C), F(C'))$, there exists $f \in \mathrm{Mor}_{\mathcal{C}}(C, C')$, such that $F(f) = h$. So, $h = \beta(f)$. Hence, the map $\beta : \mathrm{Mor}_{\mathcal{C}}(C, C') \to \mathrm{Mor}_{\mathcal{D}}(F(C), F(C'))$ is surjective. Thus, we deduce that the functor F is full and faithful.

Since $\mu : 1_{\mathcal{D}} \to FG$ is a natural isomorphism, we have, for any object D of $\mathcal{D}$, that $D \approx FG(D)$. Hence, the functor F is essentially surjective.

For the converse, suppose that the functor F is full, faithful and essentially surjective. We have to construct a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\phi : 1_{\mathcal{D}} \to FG$ and $\psi : 1_{\mathcal{C}} \to GF$. Since F is essentially surjective, we have that for any object D of $\mathcal{D}$, there exists an object C of $\mathcal{C}$, such that $F(C) \approx D$. This gives rise to an isomorphism $\phi_D : D \to F(C)$. We set $G(D) = C$. Then $\phi_D : D \to FG(D)$ is also an isomorphism. Now, for each morphism $g : D \to D'$ in $\mathcal{D}$, there exists a morphism $f : C = G(D) \to C' = G(D')$ (since F is full), such that the diagram

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "\phi_D"] & D' \arrow[d, "\phi_{D'}"] \\ FG(D) \arrow[r, "FG(g)=F(f)"] & FG(D') \end{tikzcd}$$

commutes. Let $G(g) = f$. Then, for each object D of $\mathcal{D}$, we have the identity morphism $g = id_D : D \to D$ and the commutative diagram

$$\begin{tikzcd} D \arrow[r, "id_D"] \arrow[d, "\phi_D"] & D \arrow[d, "\phi_D"] \\ FG(D) \arrow[r, "FG(id_D)"] & FG(D) \end{tikzcd}$$

Hence

$$\phi_D \circ id_D(D) = FG(id_D) \circ \phi_D(D) = \phi_D(D),$$

which implies that

$$FG(id_D) = id_{FG(D)}.$$

Thus, $G(id_D) = id_{G(D)}$. Now, for any pair of morphisms $g : D \to D'$, $g' : D' \to D''$ in $\mathcal{D}$, there exist morphisms $f : C \to C'$, $f' : C' \to C''$ in $\mathcal{C}$, such that

$$G(g' \circ g) = f' \circ f = G(g') \circ G(g).$$

Thus $G : \mathcal{D} \to \mathcal{C}$ is a functor and $\phi : 1_{\mathcal{D}} \to FG$ is a natural isomorphism. Now, let C be any object of $\mathcal{C}$. Define an isomorphism $\phi_{F(C)} : F(C) \to FGF(C)$. Since F is full, there exists a morphism $\psi_C : C \to GF(C)$ in $\mathcal{C}$, such that $F(\phi_C) = \phi_{F(C)}$. Since $\phi_{F(C)}$ is an isomorphism, there exists a morphism $g : FGF(C) \to F(C)$ in $\mathcal{D}$, such that $g \circ \phi_{F(C)} = id_{F(C)}$ and $\phi_{F(C)} \circ g = id_{FGF(C)}$. Since g is a morphism in $\mathrm{Mor}_{\mathcal{D}}(FGF(C), F(C))$ and F is full, we get a morphism $h : GF(C) \to C$, for which $F(h) = g$. Thus

$$F(h \circ \psi_C) = F(h) \circ F(\psi_C) = g \circ \phi_{F(C)} = id_{F(C)} = F(id_C)$$

and also

$$F(\psi_C \circ h) = F(\psi_C) \circ F(h) = \phi_{F(C)} \circ g = id_{FGF(C)} = F(id_{GF(C)}.$$

From hypothesis, F is faithful, so $h \circ \psi_C = id_C$ and $\psi_C \circ h = id_{GF(C)}$. Hence, $\psi_C : C \to GF(C)$ is an isomorphism. Now, let $f : C \to C'$ be any morphism in $\mathcal{C}$ and consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & C' \\ \Big\downarrow{\scriptstyle \psi_C} & & \Big\downarrow{\scriptstyle \psi_{C'}} \\ GF(C) & \xrightarrow{GF(f)} & GF(C') \end{array}$$

By constructing the diagram (via the functor F)

$$\begin{array}{ccc} F(C) & \xrightarrow{\quad F(f) \quad} & F(C') \\ \Big\downarrow{\scriptstyle \psi_{F(C)}} & & \Big\downarrow{\scriptstyle \psi_{F(C')}} \\ FGF(C) & \xrightarrow{FGF(f)} & FGF(C') \end{array}$$

and by setting $F(C) = D$ and $F(C') = D'$, we get the diagram

$$\begin{array}{ccc} D & \xrightarrow{\quad F(f) \quad} & D' \\ \Big\downarrow{\scriptstyle \psi_D} & & \Big\downarrow{\scriptstyle \psi_{D'}} \\ FG(D) & \xrightarrow{FGF(f)} & FG(D') \end{array}$$

which we have already proved to be commutative. So, we have

$$\phi_{D'} \circ F(f) = FGF(f) \circ \phi_D \Rightarrow F(\psi_{C'} \circ f) = F(GF(f) \circ \psi_C).$$

But F is faithful, so we have $\psi_{C'} \circ f = GF(f) \circ \psi_C$, which implies that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & C' \\ \Big\downarrow{\scriptstyle \psi_C} & & \Big\downarrow{\scriptstyle \psi_{C'}} \\ GF(C) & \xrightarrow{GF(f)} & GF(C') \end{array}$$

is commutative. Hence $\psi : 1_{\mathcal{C}} \to GF$ is a natural isomorphism and the proof is complete. $\square$

Remark 3.3.11. A similar argument holds for cofunctors, that is

"If $F : \mathcal{C} \to \mathcal{D}$ is a cofunctor between categories $\mathcal{C}$ and $\mathcal{D}$, then F is a duality if, and only if, F is full, faithful and essentially surjective".

The proof will be omitted, as it is similar to that of Theorem 3.3.10, except that $G : \mathcal{D} \to \mathcal{C}$ is proved to be a cofunctor, instead of a functor.

3.4 Gelfand Duality

In this paragraph, we present the two basic theorems of Gelfand Duality, followed by some applications. These theorems establish a duality, that is, a contravariant equivalence between categories of commutative C^*-algebras and algebras of continuous functions. Intuitively, we need a way to transfer information from one category to another, without the concern of the structure of the aforementioned algebras.

Theorem 3.4.1 ([27], p.5)**.** *The category of locally compact Hausdorff spaces $\mathcal{LCS}$ is dual to the category of nonunital commutative C^*-algebras $C^*Alg_{com,nu}$.*

Proof. We need to construct two contravariant functors $\Delta : C^*Alg_{com,nu} \to \mathcal{LCS}$ and $C_0 : \mathcal{LCS} \to C^*Alg_{com,nu}$, such that $\Delta C_0 \approx id_{\mathcal{LCS}}$ and $C_0\Delta \approx id_{C^*Alg_{com,nu}}$. We will first define the cofunctor $C_0 : \mathcal{LCS} \to C^*Alg_{com,nu}$. To each locally compact Hausdorff space X, we associate the algebra $C_0(X)$ which is nonunital and commutative. If $f : X \to Y$ is a continuous and proper map between locally compact Hausdorff spaces X and Y, we define

$$C_0(f) = f^* : C_0(Y) \to C_0(X),$$

with $f^*(g) = g \circ f$, which is the pullback of f. We just need to show that f^* is a proper *-homomorphism between nonunital commutative C^*-algebras. Indeed, for any $g, h \in C_0(Y)$, we have

- $f^*(g \cdot h) = (g \cdot h) \circ f = (g \circ f) \cdot (h \circ f) = f^*(g) \cdot f^*(h)$,
- $f^*(g + h) = (g + h) \circ (f) = g \circ f + h \circ f = f^*(g) + f^*(h)$,
- $f^*(g^*) = g^* \circ f = (g \circ f)^* = (f^*(g))^*$.

Hence, f^* is a *-homomorphism. Now, let $(g_\lambda)_{\lambda\in\Lambda}$ be an approximate identity of $C_0(Y)$. This means that

(i) $g_\lambda \geq 0$ and $\|g_\lambda\| \leq 1$, for all $\lambda \in \Lambda$,

(ii) $g_\lambda \leq g_\mu$, for all $\lambda, \mu \in \Lambda$, with $\lambda \leq \mu$,

(iii) $\lim\limits_{\lambda} \|g_\lambda \cdot g - g\| = 0 = \lim\limits_{\lambda} \|g \cdot g_\lambda - g\|$

Now, we observe that

- $f^*(g_\lambda) = g_\lambda \circ f \geq 0$, for all $\lambda \in \Lambda$ and $\|f^*(g_\lambda)\| = \|g_\lambda \circ f\| \leq 1$, since $\|g_\lambda\| \leq 1$, for all $\lambda \in \Lambda$.

- For any $\lambda, \mu \in \Lambda$, with $\lambda \leq \mu$, we have $g_\lambda \leq g_\mu$, which implies that $g_\lambda \circ f \leq g_\mu \circ f$. Consequently, we have $f^*(g_\lambda) \leq f^*(g_\mu)$.

- Since $(g_\lambda)_{\lambda\in\Lambda}$ is an approximate identity of $C_0(Y)$ from (iii), we have that

$$\lim_\lambda \|f^*(g_\lambda) \cdot f^*(g) - f^*(g)\| = \lim_\lambda \|f^*(g_\lambda \cdot g - g)\| = 0.$$

  and, also, that

$$\lim_\lambda \|f^*(g) \cdot f^*(g_\lambda) - f^*(g)\| = 0.$$

Thus $(f^*(g_\lambda))_{\lambda\in\Lambda}$ is an approximate identity of $C_0(X)$ and f^* is a proper *-homomorphism. Hence, the cofunctor C_0 is defined.

To define the cofunctor Δ, let A be a nonunital commutative C^*-algebra. Now, let $\Delta(A)$ denote the maximal ideal space Δ_A of A. As we have shown in Chapter 1, $\Delta(A)$ is a locally compact Hausdorff space (see Theorem 1.6.15). For a proper *-homomorphism $f : A \to B$ between nonunital commutative C^*-algebras, let

$$\Delta(f) = \hat{f} : \Delta(B) = \Delta_B \to \Delta(A) = \Delta_A,$$

with $\hat{f}(g) = g \circ f$, for any $g \in \Delta(B)$. We will show that $\hat{f}$ is continuous and proper. For this, let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $\Delta(B)$, with $h_n \longrightarrow h \in \Delta(B)$. Then

$$\hat{f}(h_n) = h_n \circ f \longrightarrow h \circ f = \hat{f}(h).$$

Thus, $\hat{f}$ is continuous. Now, let K be a compact subset of $\Delta(A)$ and define $\hat{f}^{-1} : \Delta(A) \to \Delta(B)$, by $\hat{f}^{-1}(g) = g \circ f^{-1}$. We will show that the set $\hat{f}^{-1}(K) \subset \Delta(B)$ is compact. For this, let $(z_n)_{n\in\mathbb{N}}$ be a sequence of complex homomorphisms in $\hat{f}^{-1}(K)$. Then, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in K, such that

$$z_n = \hat{f}^{-1}(g_n) = g_n \circ f^{-1}, \quad \text{for all } n \in \mathbb{N}.$$

Since K is compact, there exists a convergent subsequence $(g_{k_n})_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$, meaning that there exists a function $g \in \Delta_A$, for which $g_{k_n} \longrightarrow g$, as $n \longrightarrow +\infty$. Hence, we can construct a subsequence $(z_{k_n})_{n\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$, for which

$$z_{k_n} = \hat{f}^{-1}(g_{k_n}) = g_{k_n} \circ f^{-1} \longrightarrow g \circ f^{-1}.$$

Thus, if we let $z = \hat{f}^{-1}(g) = g \circ f^{-1} \in \hat{f}^{-1}(K) \subset \Delta_B$, we deduce that $z_{k_n} \longrightarrow z \in \Delta_B$ and, hence, that $\hat{f}$ is proper. So, the cofunctor Δ is defined.

Lastly, we will show the equivalence of the categories $\mathcal{LCS}$ and $C^*Alg_{com,nu}$. For this, it is enough to show that $C_0\Delta \approx id_{C^*Alg_{com,nu}}$ and $\Delta C_0 \approx id_{\mathcal{LCS}}$. We define the map $\gamma : X \to \Delta(C_0(X))$, by

$$\gamma(x)(f) = e_x(f) = f(x),$$

called the **evaluation map**, which we will show to be a homeomorphism. Indeed

$$\ker\gamma = \{x \in X : e_x(f) = \gamma(x)\,(f) = 0\} = \{x \in X : f(x) = x \cdot f(1) = 0\} = \{0\}.$$

Thus, γ is injective. Also, for any complex number $\lambda \in \mathbb{C}$, let $f \in C_0(X)$ be the constant function $f(x) = \lambda$. Then, the evaluation map sends the function $f \in C_0(X)$ to the complex number $f(x) = \lambda$. Thus, the evaluation map is surjective and so is the map γ. Hence, γ is a homeomorphism. Using the Gelfand transform, regarded as a map from A into $C_0(\Delta(A))$, we get the homeomorphism in question. Hence, for any morphism $f : X \to Y$ of $\mathcal{LCS}$, we have that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma"] & Y \arrow[d, "\gamma"] \\ \Delta C_0(X) \arrow[r, "\Delta C_0(f)"] & \Delta C_0(Y) \end{tikzcd}$$

is commutative, since for any $x \in X$ and any $g \in C_0(Y)$ we have

$$\begin{aligned}(\Delta C_0(f) \circ \gamma)(x) &= [\Delta C_0(f) \circ e_x](g) = e_x \circ (\Delta C_0(f))(g) \\ &= e_x \circ (\Delta f^*(g)) = e_x \circ (\Delta(g \circ f)) \\ &= e_x \circ (\Delta(g(f))) = e_x \circ (\hat{g}(f)) \\ &= e_x \circ (f \circ g) = (f \circ g)(x) \\ &= (\gamma \circ f)(x)(g).\end{aligned}$$

Thus $id_{\mathcal{LCS}} \approx \Delta C_0$. In the same way we can prove, using the Gelfand transform regarded as a map $\hat{}: A \to C_0\Delta(A)$, that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\hat{}"] & B \arrow[d, "\hat{}"] \\ C_0\Delta(A) \arrow[r, "C_0\Delta(f)"] & C_0\Delta(A) \end{tikzcd}$$

is commutative, for any morphism $f : A \to B$ in the category $C^*Alg_{com,nu}$. Hence $id_{C^*Alg_{com,nu}} \approx C_0\Delta$. So, we established the duality

$$\{\text{locally compact Hausdorff spaces}\} \approx \{\text{nonunital commutative } C^*\text{-algebras}\}^{op}$$

and the proof is complete. $\square$

Theorem 3.4.2 ([27], p.4)**.** *The category of compact Hausdorff spaces $\mathcal{CS}$ is dual to the category of unital commutative C^*-algebras $C^*Alg_{com,u}$.*

Proof. We need to construct two contravariant functors $\Delta : C^*Alg_{com,u} \to \mathcal{CS}$ and $C : \mathcal{CS} \to C^*Alg_{com,u}$, such that $\Delta C \approx id_{\mathcal{CS}}$ and $C\Delta \approx id_{C^*Alg_{com,u}}$.

We will, first, define the cofunctor C. To each compact Hausdorff space X, we associate the algebra $C(X)$ which is unital and commutative. If $f : X \to Y$ is a morphism in $\mathcal{CS}$, between compact Hausdorff spaces X and Y, we define

$$C(f) = f^* : C(Y) \to C(X),$$

with $f^*(g) = g \circ f$, which is the pullback of f. We, just, need to show that f^* is a *-homomorphism between unital commutative C^*-algebras. Indeed, for any $g, h \in C(Y)$ we have that

- $f^*(g \cdot h) = (g \cdot h) \circ f = (g \circ f) \cdot (h \circ f) = f^*(g) \cdot f^*(h)$,
- $f^*(g + h) = (g + h) \circ f = g \circ f + h \circ f = f^*(g) + f^*(h)$,
- $f^*(g^*) = g^* \circ f = (g \circ f)^* = (f^*(g))^*$.

We, also, observe that

$$f^*(1_{C(Y)}) = f^*(1) = 1 = 1_{C(X)}.$$

Thus, f^* is a *-homomorphism and, hence, the cofunctor C is defined.

To define the cofunctor Δ, let A be a unital commutative C^*-algebra and let $\Delta(A)$ denote the maximal ideal space of A, that is, Δ_A. By Theorem 1.6.15, Δ_A is a compact Hausdorff space, since the C^*-algebra A is unital. For a *-homomorphism $f : A \to B$, between unital commutative C^*-algebras, let

$$\Delta(f) = \hat{f} : \Delta(B) = \Delta_B \to \Delta(A) = \Delta_A,$$

with

$$\hat{f}(g) = g \circ f, \text{ for any } g \in \Delta_B.$$

We will show that $\hat{f}$ is continuous. Indeed, for any sequence $(g_n)_{n\in\mathbb{N}} \subset \Delta(B)$, with $g_n \longrightarrow g \in \Delta_B$, we have that

$$\hat{f}(g_n) = g_n \circ f \longrightarrow g \circ f = \hat{f}(g).$$

Thus, $\hat{f}$ is continuous and the cofunctor Δ is well defined. Lastly, we will show the equivalence of the categories $\mathcal{CS}$ and $C^*Alg_{com,u}$. For this, it is enough to show that $C\Delta \approx id_{C^*Alg_{com,u}}$ and $\Delta C \approx id_{\mathcal{CS}}$. As we did in Theorem 3.4.1, we can define the map $\gamma : X \to \Delta(C(X))$, by

$$\gamma(x)(f) = e_x(f) = f(x),$$

which is proved to be a homeomorphism. Using the Gelfand transform, regarded as a map from A into $C(\Delta(A))$, with $\hat{a}(h) = h(a)$, for any $h \in \Delta_A$, we get the homeomorphism in question. Hence, for any morphism $f : X \to Y$ in $\mathcal{CS}$, we have that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma"] & Y \arrow[d, "\gamma"] \\ \Delta C(X) \arrow[r, "\Delta C(f)"] & \Delta C(Y) \end{tikzcd}$$

is commutative, since for any $x \in X$ and any $g \in C(Y)$, we have

$$\begin{aligned} (\Delta C(f) \circ \gamma)(x) &= [\Delta C(f) \circ e_x](g) = e_x \circ (\Delta f^*(g)) \\ &= e_x \circ (\Delta(g(f))) = e_x \circ (\hat{f}(g)) \\ &= (g \circ f)(x) = (\gamma \circ f)(x)(g). \end{aligned}$$

Hence $id_{\mathcal{CS}} \approx \Delta C$. Similarly, we can prove that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \Big\downarrow{\scriptstyle \hat{\cdot}} & & \Big\downarrow{\scriptstyle \hat{\cdot}} \\ C\Delta(A) & \xrightarrow{C\Delta(f)} & C\Delta(A) \end{array}$$

is commutative, for any morphism $f : A \to B$ in the category $C^*Alg_{com,u}$. Indeed, for any $g \in \Delta_B$ and any $x \in A$, we have that

$$\begin{aligned} (C\Delta(f) \circ \hat{x})(g) &= \hat{x} \circ (C(\hat{f}(g))) = \hat{x} \circ (C(g \circ f)) \\ &= \hat{x} \circ (C(g))(f) = \hat{x} \circ (f \circ g) \\ &= (f \circ g)(x) = (\hat{x} \circ f)(g). \end{aligned}$$

Hence $id_{C^*Alg_{com,u}} \approx C\Delta$. So, we establised the duality

$$\{\text{compact Hausdorff spaces}\} \approx \{\text{unital commutative } C^*\text{-algebras}\}^{op}$$

and the proof is complete. □

3.5 Consequences of Gelfand Duality

As a consequence of Gelfand Duality, we have the following corollary.

Corollary 3.5.1 ([36], p.17)**.** *Two unital C^*-algebras are isomorphic if, and only if, their maximal ideal spaces are homeomorphic.*

Proof. Let A and B be any two unital C^*-algebras that are isomorphic, that is, there exists an isomorphism $f : A \to B$. Using Theorem 2.2.13, we can associate both algebras A and B, with the algebras $C(\Delta_A)$ and $C(\Delta_B)$ respectively. Furthermore, there exist isometric *-isomorphisms $\gamma_A : A \to C(\Delta_A)$ and $\gamma_B : B \to C(\Delta_B)$, whose existence we get from Theorem 2.2.13, again. Using Theorem 3.4.2, we derive the existence of a homeomorphism $g = F(f)$, where F is the contravariant functor from the category $C^*Alg_{com,u}$ to the category $\mathcal{CS}$.

The converse is trivial, if we observe, for any two unital C^*-algebras A and B with homeomorphic maximal ideal spaces Δ_A and Δ_B, respectively, that there exist isometric *-isomorphisms $\gamma_A : A \to C(\Delta_A)$ and $\gamma_B : B \to C(\Delta_B)$ by Theorem 2.2.13. Again, using Theorem 3.4.2, we deduce that A and B are isomorphic and the proof is complete. □

Remark 3.5.2. It is a fact that a categorical equivalence maintains isomorphisms in the categories involved, while a categorical duality reflects them (see [10]). Thus, the corollary above can be seen as a consequence of this remark, applied to Gelfand Duality. Moreover, Gelfand Duality established a whole new method of passing information from topological objects to algebraic ones and vice-versa. The next two propositions are examples of how this transition is interpreted, in terms of Gelfand Duality.

Proposition 3.5.3 ([36], p.18)**.** *A compact topological space X is metrizable if, and only if, the algebra $C(X)$ is separable.*

Proof. First, we assume that the space X is metrizable. Then, there exists a countable family of open balls $\{B_n\}_{n=1}^{\infty}$ that generates its topology. For any $n \in \mathbb{N}$, define the functions

$$f_n(x) = \operatorname{dist}(x, X \setminus B_n), \quad \text{for all } x \in X,$$

where $\operatorname{dist}(x, X \setminus B_n) = \inf_{z \in X/B_n} \{d(x,z)\}$ and $d : X \times X \to \mathbb{R}$ is the metric of X that we get by hypothesis. Let $x \neq y$ be any two elements of X. Consider the set $W = X \setminus \{y\}$. Since X is compact and since W is a neighborhood of $\{x\}$, there exists and open set U, such that $\overline{U} \subset W$. Since U is open, there exists an element B_i of $\{B_n\}_{n=1}^{\infty}$, such that $\{x\} \subset B_i \subset U \subset \overline{U} \subset W$. Hence, we found an element B_i of $\{B_n\}_{n=1}^{\infty}$, such that

$$f_i(x) = \operatorname{dist}(x, X \setminus B_i) \neq 0 = \operatorname{dist}(y, X \setminus B_i) = f_i(y).$$

Thus, the family of functions $\{f_n\}_{n=1}^{\infty}$ separates the points of X and belongs to the algebra $C(X)$. Hence, the algebra generated by the functions f_n, for all $n \in \mathbb{N}$, and the constant functions of $C(X)$ is dense in $C(X)$ by the Stone-Weierstrass Theorem (Theorem 2.1.15). Consequently, the algebra $C(X)$ is separable, since it admits a countable and dense subset.

Conversely, if $C(X)$ is separable, it contains a countable and dense family of continuous functions $\{f_n\}_{n=1}^{\infty}$. We may suppose that all of them have norm less than 1. Otherwise, if some f_n had norm greater than 1, we could replace it with the function $\dfrac{f_n}{1+|f_n|}$. Since the family $\{f_n\}_{n=0}^{\infty}$ is dense in $C(X)$, it approximates all continuous functions, hence the ones that separate the points of X. Thus, $\{f_n\}_{n=0}^{\infty}$ separates the points of X. We define the function $p : X \times X \to \mathbb{R}$, with

$$p(x,y) = \sum_{n=0}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n}.$$

The function p is well defined, since we assumed that each f_n has norm less that 1, for all $n \in \mathbb{N}$, therefore, the series $\sum_{n=0}^{\infty} \frac{|f_n(x)-f_n(y)|}{2^n}$ converges. It, also, defines a metric on X, since for any $x, y, z \in X$ we have that

$$p(x,y) = \sum_{n=0}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n} \geq 0$$

and

$$p(x,x) = \sum_{n=0}^{\infty} \frac{|f_n(x) - f_n(x)|}{2^n} = 0.$$

Additionally

$$p(x,y) = \sum_{n=0}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n} = \sum_{n=0}^{\infty} \frac{|f_n(y) - f_n(x)|}{2^n} = p(y,x).$$

Furthermore

$$\begin{aligned} p(x,z) &= \sum_{n=0}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n} \\ &\leq \sum_{n=0}^{\infty} \frac{|f_n(x) - f_n(y)| + |f_n(y) - f_n(z)|}{2^n} \\ &\leq p(x,y) + p(y,z). \end{aligned}$$

Consequently, the map p defines a metric on X. Also, the identity map $i : (X, \mathcal{T}) \to (X, \mathcal{T}_p)$, where $\mathcal{T}$ is the original topology of X and $\mathcal{T}_p$ is the induced topology by p, is a homeomorphism. Thus, X is metrizable. □

Proposition 3.5.4 ([36], p.18). *A compact topological space X is connected if, and only if, $C(X)$ has no nontrivial idempotents.*

Proof. Recall that an idempotent in an algebra A is an element $e \in A$, such that $e^2 = e$. Now, suppose that there exists a nontrivial idempotent $e \in C(X)$. For $e \in C(X)$ to be nontrivial is equivalent to e not being constant since, otherwise, the condition $e^2 = e$ would never hold. Using the condition $e^2 = e$, we deduce that $e(x) = 0$ or $e(x) = 1$, for all $x \in X$. Thus, we can write X as the disjoint union of the sets $\{x : e(x) = 0\}$ and $\{x : e(x) = 1\}$. Hence, X is not connected.

For the converse, suppose that X is not connected. Then, there exist open subsets A, B of X, such that $A \cap B = \emptyset$ and $A \cup B = X$. Using Urysohn Lemma, we can find an $f \in C(X)$, such that $f(a) = 0$, for all $a \in A$ and $f(b) = 1$, for all $b \in B$. Now, it is clear that the function $f \in C(X)$ is a nontrivial idempotent of $C(X)$, since it is not constant, for any $x \in X$, and the proof is complete. □

Chapter 4

A Noncommutative Gelfand Theorem

We begin this chapter with some basics of functional analysis, such as the notion of the Hilbert space, the notions of the algebras $L(H)$ and $B(H)$ and that of a representation. The aim of this chapter is to generalize Theorem 2.2.13, which was introduced in Chapter 2. The term “generalize” refers to the withdrawal of the word “commutative” in Theorem 2.2.13. There have been many attempts to generalize this notion in various directions. The direction we will focus on heavily depends on the theory of von Neumann algebras.

4.1 Representations of C^*-Algebras

We begin this section with some definitions.

Definition 4.1.1 ([35], p.75)**.** If V is a vector space over the field F, then a map $\langle \cdot, \cdot \rangle : V \times V \to F$ is called an **inner product** if for all vectors $x, y, z \in V$ and all $a \in F$, the following conditions are satisfied

(i) $\langle ax, y \rangle = a\langle x, y \rangle$ and $\langle x + y, z \rangle = \langle x + z \rangle + \langle y + z \rangle$ (Linearity).

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (Conjugate symmetry).

(iii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$, if $x \neq 0$ (Positive definiteness).

In this case, V is called an **inner product space**.

Remark 4.1.2. We defined an inner product space over any field F. In most cases, F will be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.1.3 ([33], p.293)**.** An inner product space H is called a **Hilbert space** if it is complete, with respect to the norm

$$\|x\|_H = \langle x, x \rangle^{1/2}, \quad \text{for all } x \in H.$$

A Hilbert space can also be defined as the **completion** of the inner product space H, with respect to the norm induced by the inner product.

Definition 4.1.4 ([16], p.19)**.** If H and V are Hilbert spaces, an **isomorphism** between H and V is a linear surjection $f : H \to V$, such that

$$\langle fx, fy\rangle_V = \langle x, y\rangle_H$$

for all $x, y \in H$. In this case, H and V are said to be **isomorphic**.

Definition 4.1.5 ([33], p.292)**.** Let H be a Hilbert space and S be a subset of H. Then the set of vectors **orthogonal** to S is defined by

$$S^{\perp} = \{x \in H : \langle x, s\rangle = 0, \text{ for all } s \in S\}.$$

Remark 4.1.6. We observe that $S^{\perp}$ is a closed subspace of H and, hence, a Hilbert space itself. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $S^{\perp}$, with $x_n \longrightarrow x \in H$, then by the continuity of the inner product of H we have that $\langle x, s\rangle = 0$, for all $s \in S$. Thus, $s \in S^{\perp}$ and $S^{\perp}$ is a Hilbert space.

Definition 4.1.7 ([33], p.294)**.** Let H be a Hilbert space and V be a closed subspace of H. Then $V^{\perp}$ is called the **orthogonal complement of $\boldsymbol{V}$**.

Remark 4.1.8. Let's recall some definitions from operator theory. $L(H)$ is the algebra of linear operators on a Hilbert space H, namely

$$L(H) = \{T : T \text{ is a linear operator from H into H}\},$$

whereas $B(H)$ is the algebra of bounded linear operators on a Hilbert space H, namely

$$B(H) = \{T : T \text{ is a bounded and linear operator from H into H}\}.$$

It is obvious that both algebras are noncommutative and that $B(H)$ is a *-subalgebra of $L(H)$. If $T : H \to H$ is a bounded operator, then the **adjoint of $\boldsymbol{T}$** is the operator $T^* : H \to H$, for which $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for all $x, y \in H$.

Theorem 4.1.9 ([16], p.10)**.** *Let V be a closed linear subspace of the Hilbert space H. For any $h \in H$, let Ph be the unique point in V for which $(h - Ph)\perp V$. Then*

(i) *P is a linear transformation in H.*

(ii) *$\|Ph\| \leq \|h\|$, for every $h \in H$.*

(iii) *$P^2 = P$.*

(iv) *$\ker P = V^{\perp}$ and $\operatorname{Im}P = V$.*

Proof. (i) We observe that the relation $h - Ph\perp V$ implies that $\langle h - Ph, v\rangle = 0$, for any $v \in V$. This means that $h - Ph \in V^{\perp}$ and that $\|h - Ph\| = \operatorname{dist}(h, V)$. So, if we choose any element $v \in V$, we immediately get that $\|v - Pv\| = 0$ or, equivalently, $Pv = v$. This means that the correspondence $h \to Ph$ can be extended to an operator $P : H \to H$. Now, let $h_1, h_2 \in H$ and $a_1, a_2 \in \mathbb{R}$. If $f \in V$, then

$$\langle (a_1h_1 + a_2h_2) - (a_1Ph_1 + a_2Ph_2), f\rangle = a_1\langle h_1 - Ph_1, f\rangle + a_2\langle h_2 - Ph_2, f\rangle = 0.$$

Hence, $P(a_1h_1 + a_2h_2) = a_1Ph_1 + a_2Ph_2$.

(ii) For any $h \in H$ we have, by hypothesis, that $Ph \in V$ and $h - Ph \in V^{\perp}$. Thus, the relation $h = (h - Ph) + Ph$ implies that $\|h\| \geq \|h - Ph\| + \|Ph\| \geq \|Ph\|$.

(iii) For any $h \in H$ we have that $Ph \in V$. Therefore, we have that $\|Ph - P(Ph)\| = \text{dist}(Ph, V) = 0$, hence $P(Ph) = Ph$. Thus, $P^2 = P$.

(iv) If $Ph = 0$, then $h = h - Ph \in V^{\perp}$. This means that $\ker P \subseteq V^{\perp}$. Conversely, if $h \in V^{\perp}$, then 0 is the unique vector in V, such that $h - 0 = h \perp V$. Therefore, $Ph = 0$, which means that $h \in \ker P$. Hence, $\ker P = V^{\perp}$. Now, for any $v \in V$, we have that $0 = \text{dist}(v, V) = \|v - Pv\|$. Hence, $Pv = v$ and the image $\text{Im}P$ of P is V. □

Definition 4.1.10 ([16], p.10)**.** Let V be a closed linear subspace of a Hilbert space H. Then the linear operator $P : H \to H$ defined in the preceding theorem is called the **orthogonal projection of H onto V**.

Remark 4.1.11. Throughout this chapter, the term "**projection**" will be an abbreviation of "orthogonal projection". Also, we will consider a C^*-algebra A as lying in its second dual A^{**} (regarded as Banach spaces), through the canonical embedding $T : A \to A^{**}$, with $(T(x))(f) = f(x)$.

Definition 4.1.12 ([29], p.93)**.** A **representation** of a C^*-algebra A is a pair (π, H), where H is a Hilbert space and $\pi : A \to B(H)$ is a *-homomorphism. In particular, if the C^*-algebra A is unital, with unit 1_A, we require that $\pi(1_A) = 1$.

Definition 4.1.13 ([29], p.93)**.** Let A be a C^*-algebra and (π, H) be a representation of A. We say that π is **faithful**, if π is injective.

Remark 4.1.14. We must not confuse the notion of a faithful representation with the notion of a faithful functor, as introduced in Definition 3.3.8. Nevertheless, the context is similar, meaning that the property of injectivity is required in both terms.

Remark 4.1.15. For the remainder of this thesis we will say that π is a representation of A, instead of (π, H), when no confusion is made.

Definition 4.1.16 ([16], p.249)**.** If $\{(\pi_i, H_i) : i \in I\}$ is a family of representations of A, let $H = \oplus_{i \in I} H_i = \left\{x = (x_i)_{i \in I} \in \Pi_{i \in I} H_i : \sum_{i \in I} \|x_i\|_{H_i}^2 < \infty\right\}$ and $\pi(a)(x_i)_{i \in I} = (\pi_i(a)x_i)_{i \in I}$, for all $i \in I$, all $(x_i)_{i \in I} \in H$ and all $a \in A$. Then, (π, H) is called the **direct sum** of this family of representations.

Example 4.1.17. If H is a Hilbert space and A is a C^*-subalgebra of $B(H)$, then the identity map $id_A : A \to B(H)$, with $id_A(1_A) = 1$ is a representation. □

Definition 4.1.18 ([16], p.249)**.** A representation π of a C^*-algebra A is called **cyclic**, if there exists a vector $e \in H$ such that

$$\overline{\pi(A)e} = H,$$

with respect to the norm of H. In other words, π is a cyclic representation, if the set of vectors

$$\{\pi(x)e : x \in A\}$$

is dense in H. In that case, we call the vector $e \in H$ a **cyclic vector** for the representation π.

Definition 4.1.19 ([16], p.249). If A is a C^*-algebra and (π_1, H_1), (π_2, H_2) are two representations of A, then π_1 and π_2 are **equivalent**, if there exists an isomorphism $f : H_1 \to H_2$, such that

$$f \circ \pi_1(a) \circ f^{-1} = \pi_2(a), \quad \text{for all } a \in A.$$

For a better intuition, we provide the following commutative diagram

$$\begin{tikzcd} H_1 \arrow[r, leftrightarrow, "f", "f^{-1}"'] \arrow[d, "\pi_1(a)"] & H_2 \arrow[d, "\pi_2(a)"] \\ H_1 \arrow[r, leftrightarrow, "f", "f^{-1}"'] & H_2 \end{tikzcd}$$

The importance of cyclic representations arises from the fact that every representation is equivalent to the direct sum of cyclic representations, as presented in the next theorem.

Theorem 4.1.20 ([16], p.249). *If π is a representation of the C^*-algebra A, then there is a family of cyclic representation $\{\pi_i\}$ of A, such that π and $\oplus_{i\in I}\pi_i$ are equivalent.*

Proof. We define the set

$$S = \{E \subset H : 0 \notin E \text{ and } \langle \pi(A)e, \pi(A)f\rangle = 0, \text{ for all } e, f \in E \text{ with } e \neq f\}.$$

We partially order S by set inclusion and apply Zorn Lemma (Lemma 1.3.9) in order to get a maximal element E_0 of S. Now, let

$$H_0 = \oplus_{e\in E_0}\overline{\pi(A)e} \subset H.$$

If $h \in H_0^\perp$ then $\langle \pi(a)e, h\rangle = 0$, for all $a \in A$ and all $e \in E_0$. So, for any $a, b \in A$ and any $e \in E_0$ we have that

$$0 = \langle \pi(b^*a)e, h\rangle = \langle \pi(b)^*\pi(a)e, h\rangle = \langle \pi(a)e, \pi(b)h\rangle.$$

Hence, $\langle \pi(a)e, \pi(b)h\rangle = 0$, for all $e \in E_0$ and, thus, $E_0 \cup \{h\} \in S$. Since E_0 is maximal, the relation above implies that $h = 0$. Hence $H = H_0$.

Now, for any $e \in E_0$, we define the set

$$H_e = \overline{\pi(A)e}.$$

If $a \in A$, then $\pi(a)H_e \subseteq H_e$ and since $a^* \in A$ and $\pi(a^*) = \pi(a)^*$, we deduce that the operator $\pi(a) : H \to H$ can be reduced to an operator $\pi(a)|_{H_e} : H_e \to H_e$. If we define the operator $\pi_e : A \to \underline{B(H_e)}$ by $\pi_e(a) = \pi(a)|_{H_e}$, we can see that π_e is a cyclic representation of A, since $\overline{\pi(A)e} = H_e$, and that $\pi = \oplus_{e\in E_0}\pi_e$. Thus, the representations $\oplus_{e\in E_0}\pi_e$ and π are equivalent. □

Definition 4.1.21 ([16], p.250). Let A be a C^*-algebra and $f : A \to \mathbb{C}$ be a linear functional. Then f is called **positive** if $f(a) \geq 0$, for all $a \in A^+$.

Definition 4.1.22 ([16], p.250)**.** Let A be a C^*-algebra and $f : A \to \mathbb{C}$ be a positive linear functional. Then f is called a **state** if $\|f\|_{sup} = \sup\{|f(x)| : x \in A\} = 1$. The space of all states of A is denoted by $S(A)$.

Definition 4.1.23 ([27], p.206)**.** Let A be a C^*-algebra. Then a state $f \in S(A)$ is called **pure**, if $f \in \mathrm{Ext}(S(A))$.

Proposition 4.1.24 ([16], p.250)**.** *If f is a positive linear functional of a C^*-algebra A, then for all $x, y \in A$ it holds that*

$$|f(y^*x)|^2 \leq f(y^*y)f(x^*x).$$

Proof. Since f is a positive linear functional of A we have, for any $\lambda, \mu \in \mathbb{C}$ and all $x, y \in A$, that $f((\lambda x + \mu y)^*(\lambda x + \mu y)) \geq 0$. Thus

$$\begin{aligned} 0 \leq f((\overline{\mu} y^* + \overline{\lambda} x^*)(\lambda x + \mu y)) &= f(\lambda\overline{\mu} y^*x + |\mu|^2 y^*y + |\lambda|^2 x^*x + \overline{\lambda}\mu x^*y) \\ &= |\lambda|^2 f(x^*x) + f(\lambda\overline{\mu} y^*x) + f(\overline{\lambda\overline{\mu} y^*x}) + |\mu|^2 f(y^*y) \\ &= |\lambda|^2 f(x^*x) + 2Re(\lambda\overline{\mu} f(y^*x)) + |\mu|^2 f(y^*y). \end{aligned}$$

Hence

$$4Re(\overline{\mu} f(y^*x))^2 - 4|\mu|^2 f(x^*x)f(y^*y) \leq 0$$

and

$$|f(y^*x)|^2 \leq f(x^*x)f(y^*y). \qquad \square$$

Corollary 4.1.25. *If f is a positive linear functional on a unital C^*-algebra A, with unit 1_A, then f is bounded and $\|f\|_{sup} = f(1_A)$.*

Proof. We apply Proposition 4.1.24, with $y = 1_A$, and for any $x \in A$ we have that $|f(x)|^2 \leq f(1_A)f(x^*x)$. So,

$$|f(x)|^2 \leq f(1_A)|f(x^*x)| \leq f(1_A)^2\|x^*x\| = f(1_A)^2\|x\|^2.$$

Hence, $|f(x)| \leq f(1_A)\|x\|$, which means that f is bounded, with $\|f\|_{sup} = f(1_A)$. $\square$

Theorem 4.1.26 (Gelfand–Naimark–Segal Construction, [16], p.250)**.** *Let A be a unital C^*-algebra with unit 1_A.*

(i) If f is a positive linear functional on A, then there exists a cyclic representation (π_f, H_f) of A, with cyclic vector e, such that $f(a) = \langle \pi_f(a)e, e\rangle$.

(ii) If (π, H) is a cyclic representation of A, with cyclic vector e, and $f(a) = \langle \pi(a)e, e\rangle$, then (π, H) and (π_f, H_f) are equivalent.

Proof. (i) Define $L = \{x \in A : f(x^*x) = 0\}$. Clearly L is closed in A. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in L such that $x_n \longrightarrow x \in A$, then $x_n^* \longrightarrow x^*$. This implies that $x_n^*x_n \longrightarrow x^*x$, which means that $f(x_n^*x_n) \longrightarrow f(x^*x)$, since f is bounded, hence continuous, by Corollary 4.1.25. Thus $f(x^*x) = 0$ and $x \in L$.

Also, if $a \in A$ and $x \in L$, then by Proposition 4.1.24 we have that

$$\begin{aligned} f((ax)^*(ax))^2 &= f(x^*a^*ax)^2 \\ &\leq f(x^*x)f((a^*ax)^*a^*ax) \\ &= f(x^*x)f(x^*a^*aa^*ax) \\ &= 0. \end{aligned}$$

Hence $ax \in L$ which means that L is a left ideal of A. Thus, A/L is a vector space.

Now, for any $x, y \in A$, we define the map $\langle \cdot, \cdot \rangle : A/L \times A/L \to \mathbb{C}$ by $\langle x + L, y + L \rangle = f(y^*x)$. This map defines an inner product on A/L. Indeed, for any $x, y, z \in A$ and any $\lambda, \mu \in \mathbb{C}$, we have that

$$\langle x + L, x + L \rangle = f(x^*x) \geq 0$$

and

$$\begin{aligned} \langle x + L, x + L \rangle = 0 &\Leftrightarrow f(x^*x) = 0 \\ &\Leftrightarrow x \in L \\ &\Leftrightarrow x + L = 0 + L = L. \end{aligned}$$

Additionally

$$\langle x + L, y + L \rangle = f(y^*x) = f(\overline{x^*y}) = \overline{\langle y + L, x + L \rangle}.$$

Also

$$\begin{aligned} \langle \lambda(x + L) + \mu(y + L), z + L \rangle &= \langle (\lambda x + \mu y) + L, z + L \rangle \\ &= f(z^*(\lambda x + \mu y)) \\ &= \lambda f(z^*z) + \mu f(z^*y) \\ &= \lambda \langle x + L, z + L \rangle + \mu \langle y + L, z + L \rangle. \end{aligned}$$

Now, for $a \in A$, we define the map $h : A/L \to A/L$ by $h(x + L) = ax + L$. We can see that it is linear since for any $x, y \in A$ and any $\lambda, \mu \in \mathbb{C}$ we have

$$\begin{aligned} h(\lambda(x + L) + \mu(y + L)) &= h((\lambda x + \mu y) + L) \\ &= a(\lambda x + \mu y) + L \\ &= a(\lambda x) + a(\mu y) + L \\ &= ah(x + L) + ah(y + L). \end{aligned}$$

Furthermore

$$\|ax + L\|^2 = \langle ax + L, ax + L \rangle = f((ax)^*ax) = f(x^*a^*ax).$$

We, also, observe that the element $\|a^*a\| - a^*a$ is positive, since a^*a is hermitian, by Proposition 2.2.2, and since

$$\sigma(\|a^*a\| - a^*a) = \|a^*a\| - \sigma(a^*a) \geq 0.$$

The above relation holds since $\|a^*a\| = p(a^*a)$ by Theorem 2.2.4. Additionally, if $a \in A^+$ and $x, y \in A$, with y being hermitian, then $x^*ax = x^*y^*yx = (xy)^*(xy)$. By Proposition 2.2.19, we observe that $x^*ax \geq 0$. We, also, observe that $0 \leq x^*(\|a^*a\| - a^*a)x = \|a^2\|x^*x - x^*a^*ax$ and, hence, $x^*a^*ax \leq \|a\|^2 x^*x$. Consequently, we have that

$$\|ax + L\|^2 = f(x^*a^*ax) \leq \|a^2\| f(x^*x) = \|a\|^2 \|x + L\|^2.$$

Now, let H_f be the completion of A/L, with respect to the aforementioned inner product, and define the map $\pi_f(a) : A/L \to A/L$ by $\pi_f(a)(x + L) = ax + L$. The map $\pi_f(a)$ is a linear operator on A/L which is bounded, since

$$\begin{aligned}
\|\pi_f(a)\| &= \sup\{\|\pi_f(a)(x + L)\| : \|x + L\| \leq 1\} \\
&= \sup\{\|ax + L\| : \|x + L\| \leq 1\} \\
&\leq \sup\{\|a\|\|x + L\| : \|x + L\| \leq 1\} \\
&\leq \|a\|.
\end{aligned}$$

Thus, $\pi_f(a) \in B(H_f)$, for all $a \in A$. By the definition of π_f, we can see that it is a *-homomorphism. Indeed, for any $a \in A$, we have that

$$\begin{aligned}
\pi_f(a)(x_1 + L + x_2 + L) &= \pi_f(a)(x_1 + x_2 + L) \\
&= a(x_1 + x_2) + L = ax_1 + L + ax_2 + L \\
&= \pi_f(a)(x_1 + L) + \pi_f(a)(x_2 + L), \text{ for all } x_1, x_2 \in A.
\end{aligned}$$

Also

$$\begin{aligned}
\pi_f(a)(\lambda(x_L)) &= \pi_f(a)(\lambda x + L) \\
&= a\lambda x + L = \lambda ax + L = \lambda(ax + L) \\
&= \lambda \pi_f(a)(x + L), \text{ for all } x \in A \text{ and all } \lambda \in \mathbb{C}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\pi_f(a)((x + L)^*) &= \pi_f(a)(x^* + L) \\
&= ax^* + L = a^*x^* + L = (ax + L)^* \\
&= (\pi_f(a)(x + L))^*.
\end{aligned}$$

Hence, π_f is a *-homomorphism. Furthermore, we observe that $\pi_f(1_A)(1_A + L) = 1_A 1_A + L = 1_A + L$. The element $1_A + L$ is the unit of A/L and, thus, π_f is a representation of A. Let $e = 1_A + L \in H_f$. Then $\pi_f(A)e = \{a + L : a \in A\} = A/L$ is dense in H_f, by the definition of H_f. Hence, e is a cyclic vector of π_f, for which $\langle \pi_f(a)e, e\rangle = \langle a + L, 1_A + L\rangle = f(1_A^* a) = f(a)$.

(ii) Let e_f be the cyclic vector for π_f, such that $f(a) = \langle \pi_f(a)e_f, e_f\rangle$, for all $a \in A$. Then $\langle \pi(a)e, e\rangle = f(a) = \langle \pi_f(a)e_f, e_f\rangle$, for all $a \in A$. We define the map

$g : \pi_f(A) \to H$ by $g\pi_f(a)e_f = \pi(f)e$, where $\pi_f(A)e_f$ is dense in H_f by the definition of H_f. Since

$$\begin{aligned}\|\pi(a)e\|^2 &= \langle \pi(a)e, \pi(a)e\rangle = \langle \pi(a)\pi(a)^*e, e\rangle \\ &= \langle \pi(aa^*)e, e\rangle = \langle \pi(aa^*)e_f, e_f\rangle \\ &= \|\pi_f(a)e_f\|^2,\end{aligned}$$

we have that g is an isometry, since $\|g\pi_f(a)e_f\| = \|\pi_f(a)e_f\|$. Thus, g can be extended to an isomorphism $g : H_f \to H$. So, if we let $x, a \in A$ be any elements of A, then

$$\begin{aligned}g\pi_f(a)\pi_f(x)e_f = g\pi_f(ax)e_f &= \pi(ax)e \\ &= \pi(a)\pi(x)e = \pi(a)g\pi_f(x)e_f.\end{aligned}$$

Thus, $\pi(a)g = g\pi_f(a)$ or, equivalently, $g^{-1}\pi(a)g = \pi_f(a)$. Hence, the representations π and π_f of A are equivalent. □

Remark 4.1.27. The representation (π_f, H_f) constructed in Theorem 4.1.26 is the Gelfand-Naimark-Segal representation (or GNS representation) of A, associated to f.

Remark 4.1.28. Let A be a C^*-algebra. If A is nonzero, then its **universal representation** is defined to be the representation (π, H), where $\pi = \oplus_{f\in S(A)}\pi_f$ and $H = \oplus_{f\in S(A)}H_f$.

Before proceeding to the Gelfand-Naimark Theorem, we state and prove some theorems. The Hahn-Banach Theorem's proof will be omitted.

Theorem 4.1.29 (Hahn–Banach, [33], p.56)**.** *Let X be a linear space. If Y is a linear subspace of X, $f : Y \to \mathbb{R}$ is a linear map and $p : X \to \mathbb{R}$ is a nonnegative sublinear functional such that $f(y) \le p(y)$, for all $y \in Y$, then there exists a linear map $g : X \to \mathbb{R}$, such that*

$$g(y) = f(y), \text{ for all } y \in Y$$

and

$$g(x) \le p(x), \text{ for all } x \in X.$$

Remark 4.1.30. The difference between linear and sublinear functionals is that a linear functional satisfies $f(x + y) = f(x) + f(y)$, while a sublinear one satisfies $f(x + y) \le f(x) + f(y)$.

Theorem 4.1.31 ([29], p.88)**.** *Let f be a bounded linear functional on a C^*-algebra A. The following are equivalent*

(i) f is positive.

(ii) For each approximate identity $(u_\lambda)_{\lambda\in\Lambda}$ of A, $\|f\| = \lim_\lambda f(u_\lambda)$.

(iii) For some approximate identity $(u_\lambda)_{\lambda\in\Lambda}$ of A, $\|f\| = \lim_\lambda f(u_\lambda)$.

Proof. Since f is bounded, we may suppose that $\|f\| = 1$. First, we show that (i) implies (ii). Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate identity of A. Then, $f(u_\lambda)_{\lambda\in\Lambda}$ is an increasing net in $\mathbb{R}$ and, hence, $\lim_\lambda f(u_\lambda) \leq 1$. Now, let $a \in A$ be an element of A with $\|a\| \leq 1$. Then, by Proposition 4.1.24 we have

$$\begin{aligned} |f(u_\lambda a)|^2 &\leq f(u_\lambda^2)f(a^*a) \\ &\leq f(u_\lambda)f(a^*a) \\ &\leq f(u_\lambda)\|f\|\|a^*a\| \\ &\leq f(u_\lambda) \\ &\leq \lim_\lambda f(u_\lambda). \end{aligned}$$

Hence, $|f(u_\lambda a)|^2 \leq \lim_\lambda f(u_\lambda)$ and, thus, $1 \leq \lim_\lambda f(u_\lambda)$. Consequently, $\lim_\lambda f(u_\lambda) = 1$.

The implication (ii)$\Rightarrow$(iii) is obvious. Now, we show that (iii) implies (i). Suppose that $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate identity of A, such that $\lim_\lambda f(u_\lambda) = 1$. Let $a \in A$ be a hermitian element of A, such that $\|a\| \leq 1$ and write $f(a) \in \mathbb{C}$ as $f(a) = x + iy$. In order to show that $f(a) \in \mathbb{R}$, suppose that $y \leq 0$. If we let $n \in \mathbb{N}$ be arbitrary, then

$$\begin{aligned} \|a - inu_\lambda\|^2 &= \|(a + inu_\lambda)(a - inu_\lambda)\| \\ &\leq \|a^2 + n^2u_\lambda^2 - in(au_\lambda - u_\lambda a)\| \\ &\leq \|a^2\| + n^2\|u_\lambda\|^2 + n\|au_\lambda - u_\lambda a\| \\ &\leq 1 + n^2 + \|au_\lambda - u_\lambda a\|. \end{aligned}$$

Hence

$$|f(a - inu_\lambda)|^2 \leq \|a - inu_\lambda\|^2 \leq 1 + n^2 + n\|au_\lambda - u_\lambda a\|.$$

However, $\lim_\lambda f(a - inu_\lambda) = f(a) - in$ and $\lim_\lambda(au_\lambda - u_\lambda a) = 0$. Thus, by taking limits as $\lambda \longrightarrow +\infty$, we have that $|x + iy - in|^2 \leq 1 + n^2$, which implies that $x^2 + y^2 - 2ny + n^2 \leq 1 + n^2$. Consequently, $-2ny \leq 1 - y^2 - x^2$. Since $n \in \mathbb{N}$ is arbitrary, we demand that $y = 0$ in order for the above inequality to hold. Thus, $f(a) \in \mathbb{R}$ whenever $a \in A$ is hermitian.

If $a \in A$ is positive, with $\|a\| \leq 1$, then the element $(u_\lambda - a)$ is hermitian, for any $\lambda \in \Lambda$, since $(u_\lambda - a)^* = u_\lambda^* - a^* = (u_\lambda - a)$. The last assertion follows from Corollary 2.2.3. Also, we have that $\|u_\lambda - a\| \leq 1$, which implies that $f(u_\lambda - a) \leq 1$. However, $1 - f(a) = \lim_\lambda f(u_\lambda - a) \leq 1$ and, therefore, $f(a) \geq 0$. Thus, f is positive and the proof is complete. □

Corollary 4.1.32. *If f is a bounded linear functional on a unital C^*-algebra A, with unit 1_A, then f is positive if, and only if, $f(1_A) = \|f\|$.*

Proof. Consider the approximate identity $u_\lambda = 1_A$, for all $\lambda \in \Lambda$. Then, by Theorem 4.1.31 we have that $\|f\| = \lim_\lambda f(1_A)$. Thus, f is positive if, and only if, $\|f\| = f(1_A)$. □

Corollary 4.1.33. *If f, g are positive linear functionals on a C^*-algebra A, then $\|f+g\| = \|f\| + \|g\|$.*

Proof. If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate identity of A, then

$$\|f+g\| = \lim_{\lambda}(f+g)(u_\lambda) = \lim_{\lambda} f(u_\lambda) + \lim_{\lambda} g(u_\lambda) = \|f\| + \|g\|. \qquad \square$$

Theorem 4.1.34 ([29], p.90)**.** *If a is a normal element of a nonzero C^*-algebra A, then there is a state f of A, such that $\|a\| = |f(a)|$.*

Proof. We assume that $a \neq 0$. Let $C^*(a)$ be the C^*-algebra generated by $1_{\tilde{A}}$ and a in $\tilde{A}$. Since $C^*(a)$ is commutative and the Gelfand transform $\hat{a}$ is continuous on $\Delta_{C^*(a)}$, there exists an element $f_2 \in \Delta_{C^*(a)}$, such that $\|a\| = \|\hat{a}\|_\infty = |f_2(a)|$. By the Hahn-Banach Theorem (Theorem 4.1.29), there is a linear functional f_1 extending f_2 to the whole algebra $\tilde{A}$ and preserving the norm. Hence $\|f_1\| = 1$. Since $f_1(1_A) = f_2(1_A) = 1$, by Lemma 1.6.14, we deduce that f_1 is positive, by Corollary 4.1.32. If we define the linear functional f, as $f = f_1|_A$, then we can see that f is a positive linear functional on A, such that $\|a\| = |f(a)|$. Hence $\|a\| = |f(a)| \leq \|f\|\|a\|$, which implies that $\|f\| \geq 1$. The converse inequality is obvious. Therefore, $\|f\| = 1$ and f is a state of A, by definition. $\square$

The following theorem played a key role in the development of the theory of C^*-algebras since it established the possibility of considering an abstract C^*-algebra as a C^*-subalgebra, up to an isometric *-isomorphism, of the C^*-algebra $B(H)$ of bounded operators on a Hilbert space. This result was first formulated and proven by Israel Gelfand and Mark Naimark in 1943 and played a major role for the development of the theory of operator algebras since then. For the original proof, we refer to [24].

Theorem 4.1.35 (Gelfand–Naimark, [29], p.94)**.** *If A is a C^*-algebra, then it admits a faithful representation. Specifically, its universal representation is faithful.*

Proof. Let (π, H) be the universal representation of A and suppose that $a \in A$ is such, that $\pi(a) = 0$. By Theorem 4.1.34, there exists a state f of A, such that $\|a^*a\| = f(a^*a)$, since the element a^*a is normal. Using the GNS construction (Theorem 4.1.26) and letting $L = \{x \in A : f(x^*x) = 0\}$ (as in Theorem 4.1.26), we have that $\|a\|^2 = f(a^*a) = \|\pi_f(a^*a)(a^*a+1)\| = 0$, since $\pi_f(a^*a) = 0$. Hence $a = 0$ and π is injective. Thus, the universal representation (π, H) of A is faithful. $\square$

Before closing up this section, we state some definitions and the Krein-Milman Theorem, omitting its proof.

Definition 4.1.36 ([33], p.70)**.** Let X be a real vector space and K be a convex subset of X. Then an element $x \in K$ is called an **extreme point of $\boldsymbol{K}$**, if there do not exist elements $y, z \in K$, with $y \neq z$, and $\lambda \in \mathbb{R}$, with $0 < \lambda < 1$, such that

$$x = \lambda y + (1-\lambda)z.$$

The set of all extreme points of K is denoted by $\mathrm{Ext}(K)$.

Definition 4.1.37 ([33], p.8)**.** A topological vector space X is called **locally convex**, if the origin of X has a neighborhood basis consisting of convex sets.

Definition 4.1.38 ([29], p.272)**.** Let X be a topological vector space and K be a subset of X. Then, the **convex hull of K** is the intersection of all convex sets containing K and is denoted by $\text{conv}(K)$. Equivalently

$$\text{conv}(K) = \bigcap_{i \in I} \{K_i \supset K : K_i \text{ is convex, for all } i \in I\}.$$

Remark 4.1.39. If X is a topological vector space and K is a convex subset of X, then $\text{conv}(K) = K$.

Theorem 4.1.40 (Krein–Milman, [16], p.142)**.** *Let X be a locally convex Hausdorff topological space. If K is a compact convex subset of X, then K is equal to the closed convex hull of its extreme points or, equivalently*

$$K = \overline{\text{conv}(\text{Ext}(K))}.$$

4.2 Von Neumann Algebras

There are several topologies on $B(H)$ that are weaker that the norm topology. We introduce two of them that will be used to define the notion of a von Neumann algebra.

Definition 4.2.1 ([17], p.16)**.** The **weak operator topology** (abbreviated as **WOT**) on $B(H)$ is defined as the weakest topology on $B(H)$, such that, for any $x, y \in H$ and any $T \in B(H)$, the sets of the form

$$W(T, x, y) = \{A \in B(H) : |\langle (T - A)x, y\rangle| < 1\}$$

are open.

Definition 4.2.2 ([17], p.16)**.** A net $(T_\lambda)_{\lambda \in \Lambda}$ in $B(H)$ **converges WOT** to an operator T, if for any $x, y \in H$, it holds that $\lim_\lambda(\langle T_\lambda x, y\rangle) = \langle Tx, y\rangle$. We denote this convergence by $T_\lambda \xrightarrow{WOT} T$.

Definition 4.2.3 ([17], p.16)**.** The **strong operator topology** (abbreviated as **SOT**) on $B(H)$ is the topology defined by the open sets

$$S(T, x) = \{A \in B(H) : \|(T - A)x\| < 1\},$$

for all $T \in B(H)$ and all $x \in H$, where $\|\cdot\| : H \to \mathbb{R}$ is the norm on H induced by its inner product.

Definition 4.2.4 ([17], p.16)**.** A net $(T_\lambda)_{\lambda \in \Lambda}$ in $B(H)$ converges SOT to T if for any $x \in H$, it holds that $\lim_\lambda(T_\lambda x) = Tx$, for any $x \in H$. We denote this convergence by $T_\lambda \xrightarrow{SOT} T$.

Remark 4.2.5. Left and right multiplication by a fixed operator $A \in B(H)$ is continuous in both SOT and WOT topologies. In other words, if $T_\lambda \xrightarrow{WOT} T$, then $AT_\lambda \xrightarrow{WOT} AT$ and $T_\lambda A \xrightarrow{WOT} TA$ and if $T_\lambda \xrightarrow{SOT} T$, then $AT_\lambda \xrightarrow{SOT} AT$ and $T_\lambda A \xrightarrow{SOT} TA$.

Remark 4.2.6. Multiplication is SOT-continuous on the unit ball. Suppose that $S_\lambda \xrightarrow{SOT} S$ and $T_\lambda \xrightarrow{SOT} T$ and that $\|S_\lambda\| \leq 1$, for all $\lambda \in \Lambda$. Then, for any $x \in H$, we have that

$$\|(ST - S_\lambda T_\lambda)x\| \leq \|(S - S_\lambda)Tx\| + \|S_\lambda\|\|(T - T_\lambda)x\|.$$

Consequently, if we let λ tend to infinity, we deduce that $S_\lambda T_\lambda \xrightarrow{SOT} ST$. Thus, the assertion is true.

Definition 4.2.7 ([17], p.19)**.** A C^*-subalgebra of $B(H)$ which contains the identity operator and is closed in the weak operator topology, is called a **von Neumann algebra**.

Definition 4.2.8 ([17], p.19)**.** If S is a subset of $B(H)$, we define the **commutant of S** to be

$$S' = \{T \in B(H) : AT = TA, \text{ for all } A \in S\}.$$

Remark 4.2.9. We observe by Definition 4.2.8 that S' is a unital algebra with unit being the identity operator of $B(H)$. Furthermore, S' is self-adjoint, if S is self-adjoint. Indeed, for any $A \in S$ and any $T \in S'$, we have that

$$AT^* = (TA^*)^* = (A^*T)^* = T^*A.$$

Moreover, S' is WOT-closed. Indeed, if $(T_\lambda)_{\lambda\in\Lambda}$ is a net in S' such that $T_\lambda \xrightarrow{WOT} T \in B(H)$, then for any $A \in S$, we have that

$$AT = WOT - \lim_\lambda AT_\lambda = WOT - \lim_\lambda T_\lambda A = TA.$$

Hence, S' is a von Neumann algebra. In fact, there exists an alternative definition for a von Neumann algebra that contains the aforementioned notions.

Definition 4.2.10 ([16], p.281)**.** A **von Neumann algebra** A is a C^*-subalgebra of $B(H)$, such that $A = A''$.

Next, we define a topology that will be used through the rest of this chapter.

Definition 4.2.11 ([29], p.59)**.** Let u be an operator on a Hilbert space H and suppose that E is an orthonormal basis of H. We define the map $\|\cdot\|_2 : H \to \mathbb{R}$, with $\|u\|_2 = \left(\sum_{x\in E} \|u(x)\|^2\right)^{1/2}$, where $\|u(x)\| = \langle u(x), x\rangle^{1/2}$. If $\|u\|_2 < +\infty$, then $\|u\|_2$ is called the **Hilbert-Schmidt norm** of u and u is called a **Hilbert-Schmidt operator**.

Definition 4.2.12 ([29], p.63)**.** Let u be an operator on a Hilbert space H and E be an orthonormal basis of H. We define the map $\|\cdot\|_1 : H \to \mathbb{R}$, with $\|u\|_1 = \sum_{x\in E} \langle |u|(x), x\rangle$. If $\|u\|_1 < +\infty$, then $\|u\|_1$ is called the **trace-class norm** of u and u is called a **trace-class operator**. It holds that $\|u\|_1 = \||u|^{1/2}\|_2^2$. The set of trace-class operators is denoted by $L^1(H)$.

Definition 4.2.13 ([29], p.63)**.** Let H be a Hilbert space and E be an orthonormal base of H. We define the **trace** of an operator $u \in B(H)$ to be

$$\operatorname{tr}(u) = \sum_{x\in E} \langle u(x), x\rangle.$$

Remark 4.2.14. Concerning Definitions 4.2.11, 4.2.12 and 4.2.13 the choice of the orthonormal basis is arbitrary, since the results remain the same.

Definition 4.2.15 ([29], p.126)**.** The **ultraweak** or **σ-weak** topology on $B(H)$ is the Hausdorff locally convex topology on $B(H)$ generated by the family of seminorms $\{p_i : i \in I\}$, with $p_i(u) = |\operatorname{tr}(uv)|$, for any $v \in L^1(H)$. It is the weakest topology on $B(H)$, such that all elements of the continuous predual of $B(H)$ are continuous, considered as functions on $B(H)$.

Remark 4.2.16. The ultraweak topology is similar to the WOT topology of $B(H)$. For example, on any norm-bounded set, the WOT topology and the ultraweak topology are the same and, in particular, the unit ball is compact on both topologies.

Remark 4.2.17. The WOT topology on $B(H)$ is weaker than the ultraweak topology. Indeed, let E be an orthonormal basis of H and $(u_\lambda)_{\lambda\in\Lambda}$ be a net on $B(H)$ converging ultraweakly to an operator $u \in B(H)$. Then for any $\lambda \in \Lambda$, we have that $|\langle u_\lambda(x) - u(x), y\rangle| \le |\sum_{x\in E} \langle (u_\lambda - u)(x), y\rangle| = |\operatorname{tr}((u_\lambda - u)(x \otimes y))|$, which converges to 0, for any $x, y \in H$. So, the net $(u_\lambda)_{\lambda\in\Lambda}$ converges WOT to $u \in B(H)$. The notation $\otimes$ is of a rank-one operator and must not be confused with the usual tensor product.

Definition 4.2.18 ([16], p.33)**.** Let H be a Hilbert space and $T \in B(H)$. Then T is called **hermitian** or **self–adjoint**, if $T^* = T$. It is called **normal**, if $TT^* = T^*T$.

Remark 4.2.19. Observe that the definitions of normal and hermitian elements in operator algebras are similar to those of C^*-algebras. In fact, the theory of hermitian and normal elements in a C^*-algebra arises from the theory of operator algebras.

Remark 4.2.20. For a deeper understanding of the following measure theoretic notions, we suggest [22], [23] and [35].

Definition 4.2.21 ([35], p.12)**.** Let X be a topological space. We form the class $\mathcal{B}$ of **Borel sets** to be the smallest collection of sets that includes the open and closed sets, such that if $B_1, B_2, \ldots$ are in $\mathcal{B}$ then so are the sets $\bigcup_{i=1}^{\infty} B_i$, $\bigcap_{i=1}^{\infty} B_i$ and $X \setminus B_i$, for $i = 1, 2, \ldots$.

Definition 4.2.22 ([35], p.47)**.** Let X be a topological space and let Σ be a σ-algebra on X. Also, let μ be a positive measure on (X, Σ). A measurable subset A of X is said to be **inner regular**, if $\mu(A) = \sup\{\mu(F) : F \subseteq A, F \text{ is compact and measurable}\}$ and **outer regular** if $\mu(A) = \inf\{\mu(G) : G \supseteq A, G \text{ is open and measurable}\}$. If every Borel set in X is both outer and inner regular, then μ is called **regular**.

Definition 4.2.23 ([29], p.66)**.** Let Ω be a compact Hausdorff space and H be a Hilbert space. A **spectral measure** $\boldsymbol{E}$ is a map from the σ-algebra of all Borel sets of Ω to the set of projections in $B(H)$, such that

(i) $E(\emptyset) = 0$ and $E(\Omega) = 1$.

(ii) $E(S_1 \cap S_2) = E(S_1)E(S_2)$, for all Borel sets S_1, S_2 of Ω.

(iii) for all $x, y \in H$, the function $E_{x,y} : S \mapsto \langle E(S)x, y\rangle$ is a regular Borel measure in Ω, for all Borel sets S of Ω.

Remark 4.2.24. Let $T \in B(H)$ be a normal operator. If we apply Definition 4.2.23 for the special case of $\Omega = \sigma(T)$, we get, for any Borel subset $A \subset \sigma(T)$, a projection $E(A)$ in $B(H)$. The projection $E(A)$ is called the **spectral projection** of $\boldsymbol{T}$.

Remark 4.2.25. If p, q are projections on a Hilbert space H, then we can define an order $\leq$, such that $p \leq q$ if, and only if, the element $q - p$ is positive, meaning that its spectrum lies in $[0, +\infty)$.

Theorem 4.2.26 ([29], p.50)**.** *Let p, q be projections on a Hilbert space H. Then, the following are equivalent*

(i) $p \leq q$.

(ii) $pq = p$.

(iii) $qp = p$.

(iv) $p(H) \subseteq q(H)$.

(v) $\|p(x)\| \leq \|q(x)\|$, *for all* $x \in H$.

(vi) $q - p$ *is a projection.*

Proof. The equivalence of Conditions (ii), (iii) and (iv) is clear, as are the implications (ii)⇒(vi)⇒(i). So, we just need to prove the implications (i)⇒(v)⇒(ii) and the proof will be complete.

If we assume that Condition (i) holds, then $\|q(x)\|^2 - \|p(x)\|^2 = \langle (q-p)(x), x\rangle = \|(q-p)(x)\|^2 \geq 0$, hence Condition (v) holds. Now, if we assume that Condition (v) holds, then $\|p(1-q)(x)\| \leq \|(q-q^2)(x)\| = 0$, since q is a projection. Hence, $p = pq$, that is, Condition (ii) holds. □

Definition 4.2.27 ([39], p.51)**.** A projection p in a C^*-algebra A is called **minimal**, if $pAp = \mathbb{C}p$.

Definition 4.2.28 ([29], p.138)**.** Let A be a C^*-algebra. Then a **central** projection is an element $p \in A$, such that $p = p^*$, $p^2 = p$ and $px = xp$, for all $x \in A$.

The next definition distinguishes the notions of the support projection and the support of a projection, when the latter is seen as a positive linear functional. In an analogous way, we can define the support of an ideal and of an order ideal, as presented in Definition 4.2.33.

Definition 4.2.29 ([39], p.140,126)**.** Let M be a von Neumann algebra. Given an element $x \in M$, the smallest projection $p \in M$ (see Remark 4.2.25) with $px = x$ is called the **left support** of x. The **right support** of x is the smallest projection $q \in M$, such that $xq = x$. Now, let $f \in M^*$ be arbitrary. Then the **left** (resp. **right**) **support projection** w of f is the smallest projection in M, such that $f = wf$ (resp. $f = fw$).

Remark 4.2.30. Recall that a **positive cone** of a real linear space X is a subset K satisfying the following conditions

- If $x, y \in K$ and $a, b \geq 0$, then $ax + by \in K$.
- $K \cap (-K) = \{0\}$.

As an example we can take the Banach space of all continuous real-valued functions on $\mathbb{R}$ with a positive cone being the subset of all continuous non-negative functions on $\mathbb{R}$.

Definition 4.2.31 ([21], p.391)**.** An **order ideal** in a partially ordered Banach space X is a subset I of the positive cone that satisfies the following conditions

(i) If $x, y \in I$, then $x + y \in I$.

(ii) If $x \in I$ and $\lambda \geq 0$, then $\lambda x \in I$.

(iii) If $x \in I$ and $y \in X$ with $0 \leq y \leq x$, then $y \in I$.

Example 4.2.32. Lets consider the field of complex numbers $\mathbb{C}$ with the partial order $(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \leq y_2$. A positive cone I can be constructed by taking the positive values of any $(x, y) \in \mathbb{C}$, meaning that $I = \{(x, y) \in \mathbb{C} : x \geq 0 \text{ and } y \geq 0\}$. It is easily verified that I is an order ideal of $\mathbb{C}$ with the partial order defined above.

Definition 4.2.33 ([2], p.278)**.** If A is a C^*-algebra and I is an order ideal in A^*, then the **support of *I*** is the supremum of the supports of the elements of I in A^{**}. If I is taken to be a left ideal of A, then the **support of *I*** is a projection $p \in A^{**}$, such that $\overline{I} = A^{**}p$. Here $\overline{I}$ denotes the $\mathcal{W}^*$ closure of I in A^{**}.

The following definitions concern the theory of projections in von Neumann algebras and are very useful in the theory of operator algebras. Particularly, in what follows, we will analyze certain types of projections and will state some of their properties.

Definition 4.2.34 ([2], p.279,282)**.** A projection $p \in A^{**}$ is **open**, if there exists an increasing net $(a_\lambda)_{\lambda\in\Lambda}$ of positive elements of A, such that $a_\lambda \xrightarrow{w*} p$. If p is open, we say that the projection $p' = 1 - p$ is **closed**. The **closure** of p, denoted by $\overline{p}$, is the smallest closed projection that majorizes p, meaning that $\overline{p} \geq p$ (see Remark 4.2.25). The projection p is called **regular**, if $\|ap\| = \|a\overline{p}\|$, for all $a \in A$.

For the following two definitions, M will denote the von Neumann algebra zA^{**}, where A is a C^*-algebra and z is a central projection of A^{**}.

Definition 4.2.35 ([5], p.1,2)**.** A projection p in M is called **q-open** if there exists a closed left ideal I of A, such that the $\mathcal{W}^*$ closure $\overline{I}$ of I in M is of the form Mp or, equivalently, if $\overline{I} = zA^{**}p$. If p is q-open, then its complement $p' = 1 - p$ (in M) is called **q-closed**. The projection p is called **q-compact** if p is q-closed and if there exists an element $b \in A^+ = \{a \in A : a \geq 0\}$, with $bp = p$.

Definition 4.2.36 ([5], p.6)**.** A self-adjoint operator $b \in M$ ($b^* = b$) is **q-continuous** if each spectral projection of b, corresponding to an open subset of $\sigma(b)$, is also q-open.

4.3 A Noncommutative Gelfand Theorem

Now we will present a series of theorems that will be used to prove our general Gelfand Theorem. By Theorem 4.1.35, we can view any C^*-algebra as a C^*-subalgebra of $B(H)$. Thus, from now on, any C^*-algebra will be considered as an algebra of operators. Also, for the remainder of this thesis, we will consider the von Neumann algebra $M = zA^{**}$, with A being a C^*-algebra and z being a central projection of A^{**}.

Theorem 4.3.1 ([3], p.545)**.** *Let A be a C^*-algebra and B be the C^*-subalgebra of A that contains a positive, increasing approximate identity $(a_\lambda)_{\lambda\in\Lambda}$. Then, given $a \in A$ with $a \geq 0$ and $\epsilon > 0$, there exists $b \in B$ with $b \geq 0$, such that $b \geq a$ and $\|b\| \leq \|a\| + \epsilon$.*

Proof. First, we observe that if the theorem is true for all $a \in A$ with $\|a\| = 1$, then it is true for all $a \in A$. Indeed, if we chose any element $a \in A$, we could normalize it by dividing with its norm and get the desirable element whose norm would be equal to 1. So, let's assume that $\|a\| = 1$. We choose an element a_{λ_1} of $(a_\lambda)_{\lambda\in\Lambda}$, such that $\|a - a_{\lambda_1} a a_{\lambda_1}\| < \frac{\epsilon}{2}$. Since $(a_\lambda)_{\lambda\in\Lambda}$ is an increasing, positive approximate identity and $\|a\| = 1$, we have that $a_{\lambda_1} \geq a_{\lambda_1} a a_{\lambda_1}$. Set $a_1 = a - a_{\lambda_1} a a_{\lambda_1}$. Then, we can find an element a_{λ_2} of $(a_\lambda)_{\lambda\in\Lambda}$, such that $\|a_1 - a_{\lambda_2} a_1 a_{\lambda_2}\| < \frac{\epsilon}{4}$. Since $\|a_\lambda\| \leq 1$, for all $\lambda \in \Lambda$, and $\|\frac{2}{\epsilon} a_1\| \leq 1$, we have that

$$a_{\lambda_2} \geq a_{\lambda_2}^2 \geq a_{\lambda_2} \left(\frac{2}{\epsilon} a_1\right) a_{\lambda_2}.$$

Thus, $\frac{\epsilon}{2}a_{\lambda_2} \geq a_{\lambda_2}a_1a_{\lambda_2}$. Set $a_2 = a_1 - a_{\lambda_2}a_1a_{\lambda_2}$. By induction, we can find sequences $(\lambda_n)_{n\in\mathbb{N}} \in \Lambda$ and $(a_n)_{n\in\mathbb{N}} \in A$, such that $a_n = a_{n-1} - a_{\lambda_n}a_{n-1}a_{\lambda_n}$, for all $n \in \mathbb{N}$, $\|a_n\| \leq \frac{\epsilon}{2^n}$ and $\frac{\epsilon}{2^{n-1}}a_{\lambda_n} \geq a_{\lambda_n}a_{n-1}a_{\lambda_n}$. Hence, the series $\sum_{n=1}^{\infty} a_{\lambda_n}a_{n-1}a_{\lambda_n}$ is convergent to a_0. Indeed, for a fixed $N \in \mathbb{N}$, we have that

$$\left\|\sum_{n=1}^{N} a_{\lambda_n}a_{n-1}a_{\lambda_n} - a_0\right\| = \left\|\sum_{n=1}^{N}(a_{n-1} - a_n) - a_0\right\| = \|a_0 - a_N - a_0\| < \frac{\epsilon}{2^N}.$$

We, also, observe that $a_0 = \sum_{n=1}^{\infty} a_{\lambda_n}a_{n-1}a_{\lambda_n} \leq a_{\lambda_1} + \sum_{n=2}^{\infty} \frac{\epsilon}{2^{n-1}}a_{\lambda_n}$. The right hand side of the above inequality converges absolutely to an element $b \in B$. Indeed, we have that

$$\sum_{n=2}^{\infty}\left\|\frac{\epsilon}{2^{n-1}}a_{\lambda_n}\right\| \leq \sum_{n=2}^{\infty}\left\|\frac{\epsilon}{2^{n-1}}\right\| = \epsilon.$$

Hence

$$\|b\| = \left\|a_{\lambda_1} + \sum_{n=2}^{\infty}\frac{\epsilon}{2^{n-1}}a_{\lambda_n}\right\| \leq \|a_{\lambda_1}\| = \left\|\sum_{n=2}^{\infty}\frac{\epsilon}{2^{n-1}}a_{\lambda_n}\right\| \leq 1 + \epsilon.$$

Thus $\|b\| \leq \|a\| + \epsilon$. □

Theorem 4.3.2 ([1], p.222). *A projection p in A^{**} supports a $\mathcal{W}^*$ closed order ideal in A^* if, and only if, $p = \lim_{\lambda} a_\lambda$, where $(a_\lambda)_{\lambda\in\Lambda}$ is a decreasing net of positive elements of A.*

Proof. If I is a $\mathcal{W}^*$ closed order ideal with support p, we can see that $p'A^{**}p'$ is the $\mathcal{W}^*$ closure of $A_{p'} = \{a \in A : p'ap' = a\}$ in A^{**}, where $p' = 1 - p$. Thus, there is an increasing approximate identity $(b_\lambda)_{\lambda\in\Lambda}$ in $A_{p'}$, with $b_\lambda \geq 0$, for all $\lambda \in \Lambda$. Since multiplication is $\mathcal{W}^*$ continuous in A^{**}, we have that $b_\lambda \xrightarrow{w^*} p'$. If we set $a_\lambda = 1 - b_\lambda$, for all $\lambda \in \Lambda$, we get a decreasing net $(a_\lambda)_{\lambda\in\Lambda}$ of positive elements of A, such that $a_\lambda \xrightarrow{w^*} p$.

Now, let $(a_\lambda)_{\lambda\in\Lambda}$ be a decreasing net of positive elements of A, with $a_\lambda \xrightarrow{w^*} p$. We define, for all $\lambda \in \Lambda$, the sets

$$I_\lambda = \{f \in A^* : f \geq 0 \text{ and } f(a'_\lambda) = 0\}.$$

We observe that each I_λ is a $\mathcal{W}^*$ closed order ideal, so that the set $I = \cap_{\lambda\in\Lambda}I_\lambda$ is also a $\mathcal{W}^*$ closed order ideal. Let q be the support of I. Now, if $f \geq 0$ in A^*, then we have that $f(p') = 0$ if, and only if, $f(a'_\lambda) = 0$, for all $\lambda \in \Lambda$ or, equivalently if, and only if, $f \in I_\lambda$. Thus, $f(p') = 0$ if, and only if, $f(q') = 0$. Hence, $p' = q'$ which means that $p = q$. □

Corollary 4.3.3 ([1], p.223). *If p is a minimal projection in A^{**}, then there exists a decreasing net $(a_\lambda)_{\lambda\in\Lambda}$ in A, such that $a_\lambda \xrightarrow{w^*} p$.*

Proof. By Theorem 4.3.2, we only need to show that p supports a $\mathcal{W}^*$ closed order ideal in A^*. If we define the set $I = \{f \in A^* : f \geq 0 \text{ and } f(p') = 0\}$, then we can see that p supports I. Let $f, g \in I$ with $g \neq 0$. Then $f(a) = f(pap)$ and $g(a) = g(pap)$, for all $a \in A^{**}$. Since p is minimal, there exists a scalar $\mu \geq 0$ in $\mathbb{C}$, such that $f(pap) = \mu g(pap)$, for all $a \in A^{**}$. Thus, $f(a) = \mu g(a)$, for all $a \in A^{**}$. This means that the set I can be written as

$$I = \{\mu g : \mu \in \mathbb{C} \text{ with } \mu \geq 0\}$$

and, hence, I is $\mathcal{W}^*$ closed. □

Theorem 4.3.4 ([2], p.279)**.** *A projection $p \in A^{**}$ is closed if, and only if, p supports a $\mathcal{W}^*$ closed order ideal in A^*.*

Proof. By applying Theorem 4.3.2, we have that p supports a $\mathcal{W}^*$ closed order ideal if, and only if, there exists a decreasing net $(a_\lambda)_{\lambda \in \Lambda}$ of positive elements of A, such that $a_\lambda \xrightarrow{w^*} p$. We set $a'_\lambda = 1 - a_\lambda \geq 0$ and we get $a'_\lambda \xrightarrow{w^*} p'$. Thus, p' is open and, hence, p is closed. □

Theorem 4.3.5 ([2], p.282)**.** *Let $p \in A^{**}$ be a projection and $\phi : A \to A^{**}$, defined by $\phi(a) = ap$. Let K be the kernel of ϕ. Then, p is regular if, and only if, $\|a + K\| = \|ap\|$, for all $a \in A$.*

Proof. We define the map $\phi_0 : A \to A^{**}$ by $\phi_0(a) = a\overline{p}$ and we let $K_0 = \ker\phi_0$. We observe that both K and K_0 are norm closed left ideals of A and that $K \supset K_0$. We will show that $K_0 = K$.

Since $\overline{p}$ is closed, $\overline{p}' = 1 - \overline{p}$ is the support of K_0. Let q be the support of K. Then q' is closed and $q' \leq \overline{p}$. On the other hand, $q' \geq p$, since $q \leq p'$. This last statement is true, since $a_\lambda \xrightarrow{w^*} q$, for an increasing net $(a_\lambda)_{\lambda \in \Lambda}$ of A. Thus, $a_\lambda p = 0$, for all $\lambda \in \Lambda$ and all $a \in A$. Hence, $qp = 0$. We deduce that $q' = \overline{p}$ and, also, that $K_0 = K$. Since $\overline{p}$ is closed, we have that $\|a\overline{p}\| = \|a + K_0\| = \|a + K\|$. The converse statement is trivial and its proof will be omitted. □

Theorem 4.3.6 ([2], p.283)**.** *If A is a von Neumann algebra, then each projection in A^{**} is regular.*

Proof. Let $p \in A^{**}$ be open and let $(a_\lambda)_{\lambda \in \Lambda}$ be an increasing net of positive elements of A, such that $a_\lambda \xrightarrow{w^*} p$. Then, there exists a projection $q \in A$, such that $a_\lambda \xrightarrow{w^*} q \in A$, since A is a von Neumann algebra. Clearly, $q \geq p$. Since $q' \in A$, we have that q' is open and, hence, q is closed. Now, we will prove that $q = \overline{p}$. We observe that $ap = 0$ if, and only if, $aa_\lambda = 0$, for all $a \in A$, which holds if, and only if, $aq = 0$, for all $a \in A$. Furthermore, $\|ap\| = \lim\limits_{\lambda} \|aa_\lambda\| = \|aq\| = \|a\overline{p}\|$ and, thus, p is regular. □

Theorem 4.3.7 ([2], p.283)**.** *Let $p \in A^{**}$ be a closed projection in A^{**}. Then, $\overline{zp} = p$, where z is a central projection of A^{**}, and zp is regular.*

Proof. Let I be a $\mathcal{W}^*$ closed order ideal in A^* with support p. Also, let I_1 be the $\mathcal{W}^*$ closed unit ball of A^*. It is clear that I_1 is convex, compact and that its extreme points are pure states, by definition. Thus, by the Krein-Milman Theorem (Theorem 4.1.40), I_1 is the closed convex hull of the pure states it contains. But, the ideal zI_1 contains all the pure states of I_1 and is convex, since I_1 is convex. Thus, I_1 is the $\mathcal{W}^*$ closure of zI_1. On the other hand, zI is a norm closed order ideal of A^* which can be easily proved, since the projection z is central. Hence, by Theorem 4.3.6, we have that $\|\overline{zp}\| = \|\overline{zp} + I_1\| = \|zp + I\| = \|zp\|$. So, zp is regular and since I_1 is the $\mathcal{W}^*$ closure of zI_1, we deduce that $\overline{zp} = p$. □

Theorem 4.3.8 ([2], p.284)**.** *If $p, q \in A$ are either open or closed projections and $zp \geq zq$, where z is a central projection of A^{**}, then $p \geq q$.*

Proof. If both p and q are closed, then $p = \overline{zp} \geq \overline{zq} = q$. If both are open, we have that $p' = 1 - p$ and $q' = 1 - q$ are closed, by definition, and that $p' = \overline{zp'} \leq \overline{zq'} \leq q'$. Hence $p \geq q$. If p is closed and q is open, then there exist nets $(a_\lambda)_{\lambda \in \Lambda}$ and $(b_\mu)_{\mu \in M}$ of elements of A, with $a_\lambda \xrightarrow{w^*} p$ and $b_\mu \xrightarrow{w^*} q$. Then, $z(a_\lambda - b_\mu) \geq 0$, for all $\lambda \in \Lambda$ and all $\mu \in M$, since $za_\lambda \geq zp \geq zq \geq zb_\mu$. Also, $za_\lambda - zb_\mu = z(a_\lambda - b_\mu) \geq 0$, since the map $\gamma : A \to zA$ with $\gamma(a) = za$ is a *-isomorphism. Thus $\lim\limits_{\mu,\lambda} |a_\lambda - b_\mu| = |p - q| \geq 0$ and $p \geq q$.

Now, suppose that p is open and q is closed. By the definition of open projections, there exists an increasing net $(a_\lambda)_{\lambda \in \Lambda}$ of positive elements of A, with $a_\lambda \xrightarrow{w^*} p$. In order to conclude the proof, we need to show that $g(a_\lambda) \xrightarrow{w^*} 1$, for each state g of A, with $g(q) = 1$. We define the set $\{\lim\limits_{\lambda} g(a_\lambda) : g \text{ is a state and } g(q) = 1\}$. Let $\epsilon = \inf\limits_{g}\{\lim\limits_{\lambda} g(a_\lambda) : g \text{ is a state and } g(q) = 1\}$. If $\epsilon \geq 1$, we immediately have that $g(p) = \lim\limits_{\lambda} g(a_\lambda) \geq \epsilon \geq g(q)$ and, thus, $p \geq q$. If $\epsilon < 1$, we define, for every $\lambda \in \Lambda$, the set

$$K_\lambda = \{g : g \text{ is a state, } \ g(a_\lambda) \leq \epsilon \text{ and } g(q) = 1\}.$$

Since q is closed, the set S of all states of g of A^* for which $g(q) = 1$ is $\mathcal{W}^*$ compact. Thus, for every $\lambda \in \Lambda$, we have that the function $\gamma : S \to \mathbb{R}^+$, with $\gamma(g) = g(a_\lambda)$ attains a minimum which must be less than ϵ. Hence, the family $\{K_\lambda\}_{\lambda \in \Lambda}$ is a decreasing family of nonempty, compact, convex sets which is directed by set inclusion. Thus, the set $K = \cap_{\lambda \in \Lambda} K_\lambda$ is a nonempty, compact and convex set and, thus, by the Krein-Milman Theorem (Theorem 4.1.40), it has an extreme point, say f. We need to prove that f is also an extreme point of S and, hence, a pure state, since S contains the states of norm 1 in the order ideal with support q.

Suppose that f is not an extreme point of S. Then, there should exist states f_1, f_2 of S and a $\mu \in (0, 1)$, such that $f = \mu f_1 + (1 - \mu) f_2$. If $f_1 \notin K$, then there would exist a $\lambda_0 \in \Lambda$, for which $f_1(a_{\lambda_0}) > \epsilon$ and $f_1(a_\lambda) > \epsilon$, for all $\lambda \geq \lambda_0$. Thus, $f_2(a_\lambda) < \epsilon$, for all $\lambda \geq \lambda_0$, since $\epsilon \geq f(a_\lambda) = \mu f_1(a_\lambda) + (1 - mu) f_2(a_\lambda)$. Since $\lim\limits_{\lambda} f(a_\lambda) = \epsilon$, we deduce that $\lim\limits_{\lambda} f_2(a_\lambda) \geq \epsilon$ and, thus, $\lim\limits_{\lambda} f_1(a_\lambda) \leq \epsilon$, a contradiction. Hence, f is a pure state of S.

Now, by hypothesis, we have that $zp \geq zq$. Thus

$$\lim_{\lambda} f(a_\lambda) = \lim_{\lambda} f(za_\lambda) = f(zp) \geq f(zq) = 1,$$

since f is a pure state of S. But this contradicts the assumption that $\epsilon < 1$. Thus, $\epsilon \geq 1$ and, hence, $p \geq q$. $\square$

Theorem 4.3.9 ([2], p.285)**.** *Let p_1 be closed and p_2 be one-dimensional projection in A^{**}, with $p_1 p_2 = 0$. Then, there exist open projections q_1 and q_2 in A^{**}, such that $q_1 q_2 = 0$, $q_1 \geq p_1$ and $q_2 \geq p_2$.*

Proof. Let I be a norm closed left ideal of A with support p_1. Also, let $B = I \cap I^*$. Then B is a C^*-subalgebra of A. Now, let f be a pure state of A with support p_2. We will show that f is pure on B as well.

Suppose that $f|_B$ is not pure. Then, there exist states f_1, f_2 of A and a scalar $\mu \in (0,1)$, such that $f|_B = \mu f_1|_B + (1-\mu) f_2|_B$. By definition, we have that the condition for p_1 being closed implies that p_1' is open. So, there exists an increasing net $(b_\lambda)_{\lambda\in\Lambda}$ of positive elements of B, such that $b_\lambda \xrightarrow{w^*} p_1'$. Since $p_1 p_2 = 0$, we have that $1 - (p_1 p_2)' = 0$ and, hence, $f(p_1') = 1$, which implies that $f_1(p_1') = 1 = f_2(p_1')$. Since f is pure, there exists a positive element $b \in B^+$, with $\|b\| \leq 1$ and $f(b) \neq \mu f_1(b) + (1-\mu) f_2(b)$. As B^+ is an order ideal, we have, for each $b \in B^+$, that $0 \leq b_\lambda b b_\lambda \leq b_\lambda^2 \in B^+$ and, hence, $b_\lambda b b_\lambda \in B^+$. Thus, we have

$$f(b_\lambda b b_\lambda) = \mu f_1(b_\lambda b b_\lambda) + (1-\mu) f_2(b_\lambda b b_\lambda)$$

and

$$\begin{aligned} f(b) &= f(p_1' b p_1') \\ &= \lim_\lambda f(b_\lambda b b_\lambda) \\ &= \mu f_1(p_1' b p_1') + (1-\mu) f_2(p_1' b p_1') \\ &= \mu f_1(b) + (1-\mu) f_2(b), \end{aligned}$$

which is a contradiction by the choice of the element $b \in B^+$. Hence, f is pure on B.

Since f is pure, there exists an element $b_0 \in B$, with $\|b_0\| \leq 1$, $b_0^* = b_0$ and $f(b_0) = 1$. We define the characteristic functions

$$q_1 = \chi_{(-2,\frac{1}{2})}(b_0) \quad \text{and} \quad q_2 = \chi_{(\frac{1}{2},2)}(b_0)$$

and we observe that both q_1 and q_2 are open projections and that $q_1 q_2 = 0$. We, also, observe that $f(b_0) = 1 = f(b_0^2)$, which implies that $q_2 \geq p_2$, since p_2 is, by definition, the support projection of f. Furthermore, we have that $b_0 \in B = I \cap I^*$ and that p_1 is the support of I. Hence, $b_0 p_1 = 0$ and $q_1 \geq p_1$. $\square$

Theorem 4.3.10 ([4], p.306)**.** *Let p and q be closed projections in A^{**}, with $pq = 0$. Then, there exists $a \in A$ with $0 \leq a \leq 1$, $ap = 0$ and $aq = q$.*

Proof. Let $A_0 = \{a \in A : ap = pa = 0\}$. We observe that A_0 is a C^*-algebra without unit since, otherwise, the identity $ap = pa = 0$ would never hold. Also, let $\tilde{A}_0$ denote the C^*-subalgebra of A generated by A_0 and 1_A. Since p is closed, $p' = 1 - p$ is open and, hence, there exists an increasing net $(a_\lambda)_{\lambda\in\Lambda}$ of positive elements of A_0, such that $a_\lambda \xrightarrow{w^*} p'$. Since A_0^{**} is isomorphic to $cl_{w^*}(A_0)$ (the $\mathcal{W}^*$ closure of A_0), we have that $A_0^{**} \cong p' A^{**} p'$. We, also, observe that $q \leq p'$, since from Theorem 4.2.26 we have that $qp' = q(1-p) = q - qp = q$. Thus, we may consider $q \in A_0^{**} \subset \tilde{A_0}^{**}$. Now, we will prove that the projection q is closed in $\tilde{A_0}^{**}$. This would follow from Theorem 4.3.4, if we could show that $(q\tilde{A_0}^{*}q)^+$ is a $\mathcal{W}^*$ closed order ideal. Since $\tilde{A}_0$ is the algebra generated by A_0 and 1_A, we can identify $\tilde{A_0}^{*}$ with A^* and, hence, the set $(q\tilde{A_0}^{*}q)^+$ with $(qA^*q)^+$. We, also, have that the set $K = \{f \in (qA^*q)^+ : \|f\| \leq 1\}$ is compact in the ultraweak topology of A. Since the ultraweak topology in $\tilde{A}_0$ is weaker than the ultraweak topology in A, they must coincide on K. Thus, the set $K = \{f \in (q\tilde{A_0}^{*}q)^+ : \|f\| \leq 1\}$ is compact in the ultraweak topology of $\tilde{A}_0$. Hence, $(q\tilde{A_0}^{*}q)$ is $\mathcal{W}^*$ closed and, thus, the projection q is closed in $\tilde{A_0}^{**}$, by Theorem 4.3.4.

Now, let f be the pure state of $\tilde{A}_0$ defined by $f(a + \lambda 1_A) = \lambda$, for every $a \in A_0$ and any $\lambda \in \mathbb{C}$. Let x be the support of f in $\tilde{A_0}^{**}$. We can see that $xq = 0$. Indeed, we have that $x = xp$ implies that $x(1-p) = xp' = 0$, which means that $xq \leq xp' = 0$. Consequently, $xq = 0$. Hence, we can apply Theorem 4.3.9 in order to get an element $a \in \tilde{A}_0$ with $0 \leq a \leq 1$, $ax = 0$ and $aq = q$. From the fact that $ax = 0$, we have that $a \in A_0$ and, thus, $ap = 0$. This concludes the proof. □

Theorem 4.3.11 ([4], p.315)**.** *Let $a \in A^{**}$ be a hermitian element of A^{**} and suppose that every spectral projection of $a \in A^{**}$, corresponding to an open set in $\sigma(a)$, is also an open projection for A. Then, $a \in A$.*

Proof. Suppose that the conclusion is false. Then, there exists a hermitian element $\phi \in A^{***}$, such that $\phi(a) \neq 0$ and $\phi|_A = 0$. Let $C^*(a)$ be the C^*-algebra generated by a and 1_A. Then, $C^*(a)$ is isomorphic to $C(\sigma(a))$, by Theorem 2.2.17. This isomorphism is the functional calculus of a, as introduced in Definition 2.2.16. Since $\phi(a) \neq 0$, there exists an open set $(\beta, \gamma) \cap \sigma(a) = Q$ in $\sigma(a)$, such that $\phi(q) \neq 0$. By the definition of a spectral projection, we can identify the Borel subset Q of $\sigma(a)$ with the projection q. Let $\delta = \beta - \gamma$ and consider, for $j = 1, 2, \ldots$, the open sets $Q_n^j = (\beta + \delta 2^{-n-j}, \gamma - \delta 2^{-n-j}) \cap \sigma(a)$, for all $n \in \mathbb{N}$. Again, these open sets correspond to spectral projections q_n^j, for $j = 1, 2, \ldots$ and for all $n \in \mathbb{N}$, which are open in A by hypothesis.

Next, we choose, for every $n \in \mathbb{N}$, elements $a_n, c_n \in C^*(a)$, with $q_n^1 \leq a_n \leq q_n^2$ and $q_n^3 \leq c_n \leq q$. Let p_n be the projection in A^{**} corresponding to the closure of Q_n^2 in $\sigma(a)$. Then, $p_n \leq q_n^3$ and since q_n^3 is open (by hypothesis), Theorem 4.3.10 is applied in order to get, for any $n \in \mathbb{N}$, elements $b_n \in A$, such that $q_n^2 \leq p_n \leq b_n \leq q_n^3$. Thus, we obtain

$$q_n^1 \leq a_n \leq q_n^2 \leq p_n \leq b_n \leq q_n^3 \leq c_n \leq q.$$

When $n \in \mathbb{N}$ tends to infinity, we have that $\lim_{n\to\infty} b_n = q$, in the $\mathcal{W}^*$ topology, which is a contradiction, since $\phi|_A = 0$ and, hence, $\phi(b_n) = 0$ and $\phi(q) \neq 0$. □

The next theorem's proof will be omitted, as it is trivially followed by Theorems 4.3.8 and 4.3.11.

Theorem 4.3.12 ([5], p.2)**.** *A self-adjoint operator $b \in M$ lies in A if, and only if, each spectral projection of b which corresponds to an open subset of $\mathbb{R}$ is also a q-open projection.*

Lemma 4.3.13 ([5], p.3)**.** *Suppose B is a C^*-algebra, $b \in B^+$, $p \in B^{**}$ a projection and let $(a_\lambda)_{\lambda\in\Lambda}$ be an increasing net of positive elements of B, with $\|b^{1/2}-b^{1/2}a_\lambda\| \longrightarrow 0$. If $b \geq p$, then $\|p - a_\lambda p\| \longrightarrow 0$.*

Proof. Since $\|b^{1/2} - b^{1/2}a_\lambda\| \longrightarrow 0$, we can see that

$$\|(1-a_\lambda)b(1-a_\lambda)\| = \|b^{1/2} - b^{1/2}a_\lambda\|^2 \longrightarrow 0.$$

Since $(1-a_\lambda)b(1-a_\lambda) \geq (1-a_\lambda)p(1-a_\lambda)$, for all $\lambda \in \Lambda$, we get

$$\begin{aligned}\|p - a_\lambda p\|^2 &= \|(1-a_\lambda)p\|^2 \\ &= \|(1-a_\lambda)p(1-a_\lambda)\| \\ &\leq \|(1-a_a\lambda)b(1-a_\lambda)\| \longrightarrow 0.\end{aligned}$$

Hence, $\|p - a_\lambda p\| \longrightarrow 0$, as λ tends to infinity. □

Lemma 4.3.14 ([5], p.3)**.** *If p is a q-closed projection in A, $b \in A^+$ with $b \geq p$ and $\tilde{A}$, $\tilde{M}$ are the unitizations of A and M, respectively, then p is q-closed in $\tilde{M}$.*

Proof. Let $K = (pA^*p)^+$. By Theorem 4.3.4, K is closed in the ultraweak topology of A. If K is not closed in the ultraweak topology of $\tilde{A}$, then there exists a net $(f_\lambda)_{\lambda\in\Lambda}$ in K, with $\|f_\lambda\| = 1$ and $f_\lambda \longrightarrow f$ in the ultraweak topology of $\tilde{A}$, for some $f \in \tilde{A}^*$, with $\|f\| = f(1_{\tilde{A}}) = 1$. Since $\tilde{A}^* = A^* \oplus \{\mu f_\infty\}$, where f_∞ is a pure state of $\tilde{A}$ which vanishes on A, we have that $f = f_0 + \mu f_\infty$, where $f_0 \in (A^*)^+$ and $\mu \geq 0$ is a complex scalar. For any $q \geq p$, we get

$$f_0(q) = f(q) = \lim_\lambda f_\lambda(q) \geq \lim_\lambda f_\lambda(p) = 1, \tag{4.1}$$

since each $f_\lambda \in K$, for every $\lambda \in \Lambda$.

Now, if $1_{\tilde{A}} \in A$, then A^* is closed in the ultraweak topology of $\tilde{A}$ and the proof is complete. If $1_{\tilde{A}} \notin A$, let $(a_n)_{n\in\mathbb{N}}$ be an increasing approximate identity of positive elements of A. Then, by Lemma 4.3.13, we have that $\|p - a_n p\| \longrightarrow 0$, as n tends to infinity. Thus, for a given $\epsilon > 0$, there exists a $c \in A$, with $c \geq p$ and $\|c\| \leq 1+\epsilon$, by Theorem 4.3.1. Hence, we have that $f_\lambda(c) \geq 1$ for $c = q$ in the relation (4.1). Since $f = f_0 + \lambda f_\infty$, we have that $\|f_0\| = 1$ and $\lambda = 0$, because $\|f\| = \|f_0\| + |\lambda|\|f_\infty\| = \|f_0\| + |\lambda| = 1$. Thus, $f \in A^*$. Since $(f_\lambda)_{\lambda\in\Lambda}$ belongs to K, we deduce that K is closed in the ultraweak topology of A and that $f_\lambda \longrightarrow f$ in the ultraweak topology of $\tilde{A}$. Hence, $f \in K$ and, so, K is closed in the ultraweak topology of $\tilde{A}$. □

Theorem 4.3.15 (Noncommutative Gelfand Theorem, [5], p.6)**.** *The hermitian elements of A are exactly those q-continuous elements b of M, such that the spectral projections of b corresponding to closed subsets of the spectrum of b, which don't contain 0, are q-compact, that is, b "vanishes at ∞".*

Proof. Let $\tilde{A}, \tilde{M}$ be the unitizations of A and M, respectively, as introduced in Lemma 4.3.14. Since $b \in M = zA^{**}$, we have that $b \in \tilde{A}$ implies $b \in A$. Let p be the spectral projection of b, corresponding to an open subset U of $\sigma(b)$. We consider two cases.

If $0 \notin U$, then $p \in M$. Since $b \in M$ is q-continuous, we have that p is q-open for A and, thus, for $\tilde{A}$.

If $0 \in U$, then the complement of U is closed and does not contain 0. Thus, the projection p' is q-closed for A and, thus, q-compact for A. By applying Lemma 4.3.14, we have that the projection p' is q-closed for $\tilde{A}$.

Thus, in any case, we have that a spectral projection of b, corresponding to closed subsets that do not contain 0, is q-compact for $\tilde{A}$. Hence, b is q-continuous for $\tilde{A}$ and by applying Theorem 4.3.12, we deduce that $b \in \tilde{A}$ and, thus, that $b \in A$ by the above discussion. This completes the proof. □

Remark 4.3.16. Theorem 4.3.15 is a generalized analogue of Theorem 2.2.14, meaning that we associated all hermitian elements of a C^*-algebra A, with certain types of projections that are q-continuous and "vanish at ∞". This is similar to Theorem 2.2.14, except the fact that we withdrew the hypothesis of commutativity and we made a construction of recovering a general C^*-algebra from the space of all spectral projections b that correspond to open subsets of the spectrum of b (analogous to the maximal ideal space that was presented in Theorem 2.2.14).

4.4 Comments on Further Research

In this section, we will present some subjects incorporated in the field of Noncommutative Geometry like the *Noncommutative K-theory* and the *Noncommutative Measure theory*. We will comment on certain topics in the field of Noncommutative Geometry and will state some references for the interested researcher. Finally, we will provide the reader with some other approaches regarding Chapter 4.

4.4.1 Physical Origin of Noncommutative Geometry

To control the divergences which from the very beginning had plagued quantum electrodynamics, Heisenberg already in the 1930's proposed to replace the space-time continuum by a lattice structure. A lattice however breaks Lorentz invariance and can hardly be considered as fundamental. It was Snyder who first had the idea of using a noncommutative structure at small length scales to introduce an effective cut-off in field theory similar to a lattice but at the same time maintaining Lorentz invariance. Some time later von Neumann introduced the term "noncommutative geometry" to refer in general to a geometry in which an algebra of functions is replaced by a noncommutative algebra. As in the quantization of classical phase-space, coordinates are replaced by generators of the algebra. Since these do not commute they cannot be simultaneously diagonalized and the space disappears. One can argue that, just as Bohr cells replace classical phase-space points, the appropriate intuitive notion to replace a "point" is a Planck cell of dimension given by the Planck area. If a coherent description could be found for the structure of space-time which were pointless on small length scales, then the ultraviolet divergences of quantum field theory could be eliminated.

4.4.2 K-Theory

Now, we will briefly present a subject that is commonly seen around the field of noncommutative Geometry, but demands a deep mathematical background for a proper understanding. K-theory, is a topological invariant, under Gelfand Duality, which can be recovered by the algebraic counterpart of Gelfand Duality, that is, by the algebra $C(X)$, over the compact topological space X. Indeed, the 2-graded abelian group $K(X)$ (meaning that $K(X)$ is the direct sum of 2 different abelian groups) is generated, on its first part, by the stable isomorphism classes of complex vector bundles over the compact topological space X has a very simple description in terms of the C^*-algebra $C(X)$. The second part of this group, provided to us by a result of Serre [37] and Swan [38], is the abelian group generated by the stable isomorphism classes of finite projective modules over $C(X)$, a purely algebraic notion which, also, makes no use of the commutativity of $C(X)$. A key result of K-theory is the Bott Periodicity Theorem [13], [14]. Thanks to the work of Atiyah and Bott [9], this result, once formulated in the algebraic context, has a very simple proof and holds for any (not necessarily commutative) Banach algebra and, in particular, any C^*-algebra.

4.4.3 Noncommutative Measure Theory

The next subject that we briefly present is the *Noncommutative Measure theory.* The relation between von Neumann algebras and Measure spaces is similar to the relation between commutative C^*-algebras and locally compact Hausdorff spaces. Given a measure space (X, μ), let $L^\infty(X, \mu)$ denote the *-algebra of essentially bounded, measurable and complex valued functions on X. This algebra acts on a Hilbert space $L^2(X, \mu)$, as multiplication operators, and its image in $L(H)$ is closed in the weak operator topology (WOT). Hence, it is a commutative von Neumann algebra. Conversely, any commutative von Neumann algebra can be shown to be algebraically isomorphic to $L^\infty(X, \mu)$, for some measure space (X, μ).

In a general framework, a construction of a Hilbert space with a countable basis provides specific automorphisms (unitary operators) of that space. The algebra of operators on the Hilbert space which commute with these particular automorphisms, form a von Neumann algebra and all von Neumann algebras are obtained in that manner. The theory of not necessarily commutative von Neumann algebras was initiated by Murray and von Neumann [30] and is considerably more difficult than the commutative case. The center of a von Neumann algebra is a commutative von Neumann algebra and, thus, dual to an essentially unique measure space. The general case, thus, decomposes over the center as a direct integral of, so called, **factors**, that is, von Neumann algebras with trivial center. Using the above construction, Murray [30] managed to classify the factors of a von Neumann algebra intro three types I, II and III, with varying degree of complexity. Because of the above correspondences and constructions, the theory of von Neumann algebras is often regarded as *Noncommutative Measure theory.*

Concerning the discussion that was presented in the previous paragraph, Pavlov [31] managed to prove a categorical equivalence between the following five categories.

(i) The opposite category of commutative von Neumann algebras.

(ii) The category of compact strictly localizable enhanced measurable spaces.

(iii) The category of measurable locales.

(iv) The category of hyperstonean locales.

(v) The category of hyperstonean spaces.

This result was a measure-theoretic counterpart of the Gelfand Duality between commutative unital C^*-algebras and compact Hausdorff topological spaces. We give some terminology for a better understanding of the aforementioned approach.

Definition 4.4.1 ([31])**.** An **enhanced measurable space** is a triple (X, M, N), where X is a set, M is a σ-algebra on X and N is a σ-ideal on X, such that $N \subset M$.

Definition 4.4.2 ([31])**.** An enhanced measurable space (X, M, N) is **strictly localizable** if it is isomorphic to the coproduct of a small family of σ-finite enhanced measurable spaces in the category ***PreEMS*** of enhanced measurable spaces (as objects) and premaps of enhanced measurable spaces (as morphisms).

Definition 4.4.3 ([31])**.** A **Stonean space** is a compact, Hausdorff, extremely disconnected topological space. A **Hyperstonean space** is a Stonean space such that the union of supports of all normal measures is everywhere dense. A **normal measure** is a Radon measure that vanishes on nowhere dense subsets. A **Radon measure** on a Hausdorff topological space X is a Borel measure μ such that μ is locally finite and inner regular on all Borel subsets.

Definition 4.4.4 ([11])**.** A **Heyting algebra H** is a lattice, with top and bottom elements, in which for every element $b \in H$ the functor

$$\cdot \wedge b : H \to H, \quad a \mapsto a \wedge b$$

has a right adjoint.

Definition 4.4.5 ([11])**.** A **locale L** is a complete Heyting algebra in which arbitrary joins distribute over finite meets, i.e. the distributivity law

$$a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i)$$

holds, where I is an arbitrary indexing set and a, b_i are elements of L.

4.4.4 Noncommutative Geometry of Schemes

Noncommutative Algebraic Geometry is the study of "spaces" represented in terms of algebras or categories. Commutative Algebraic Geometry restricts its attention to spaces, whose local description is made, via commutative rings and algebras, while Noncommutative Algebraic Geometry allows for more general local (or affine) models, like **schemes**. The categories involved, are viewed as categories of **quasicoherent modules** on noncommutative locally affine spaces and are, typically, abelian, triangulated or DG-categories. In the best case scenario, given a scheme X, one can construct a spectrum $S(X)$ by utilizing algebraic tools. This spectrum is proved to be equivalent (either geometrically or topologically) to X, but that is not always the case. For more details, one should look for [6] and [7].

4.4.5 Derived Noncommutative Algebraic Geometry

In Noncommutative Algebraic Geometry one represents a scheme by an abelian category of quasicoherent sheaves on the scheme and looks at more general abelian categories as categories of quasicoherent sheaves on a noncommutative space. In Derived Noncommutative Algebraic Geometry one, instead, considers the derived category of quasicoherent sheaves, or more precisely its DG-enhancement, since there exist many spaces, theories and problems around the field of Theoretical Physics (Quantum Mechanics, String and Superstring Theory, etc.), that can not be solved by using commutative tools. Derived Noncommutative Algebraic Geometry has been informally introduced by Kapranov-Bondal 1990, although the full framework belongs to Kontsevich and van den Bergh.

4.4.6 Topos-Theoretic Gelfand Duality

Another approach to generalize the classical Gelfand Duality was made by Simon Henry [26], which is topos-theoretic and states that any Boolean locally separated topos can be reconstructed as the classifying topos of "non-degenerate" monoidal normal *-representations of both its category of internal Hilbert spaces and its category of square integrable Hilbert spaces. In both cases, these categories suggest a symmetric monoidal monotone complete C^*-category. Specifically Simon Henry proves that if T is a boolean locally separated topos, then T is the classifying topos for non-degenerate normal symmetric monoidal representations of either $H^{red}(T)$ and $H(T)$, which are defined to be the symmetric monoidal C^*-categories of Hilbert bundles and square-integrable Hilbert bundles, respectively. We suggest [26] and [12] for the elementary definitions needed and for a better understanding of this approach.

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