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Quantum corrections to the Einstein-Hilbert term in Type IIB Superstring Theories compactified on $K3 \times K3$ surfaces

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Abstract

This work will examine aspects of perturbative string theory in the one-loop level, utilising methods of conformal field theory on the Riemann surfaces defined by the corresponding cosmic string surfaces. By non-renormalisation theorems, it is known that the Planck mass does not admit quantum corrections in any order of perturbation theory, apart from a possible correction in the one-loop level, that may appear in type II string theories of which the compactification manifold is characterised by a non-trivial value of the Euler characteristic, regardless of the existence or not of unbroken supersymmetry. The proof of these theorems relies on the background field method, which however is not well defined in the case of two-dimensional spacetimes, which result after compactification of the 8 out of 10 dimensions of the superstring, with use of non trivial 8-dimensional subspaces, such as the $K3 \times K3$ surface. In the present work, after inspecting the problem through the known background field method, we will generalise the calculation of quantum corrections of the Einstein-Hilbert term to the two-dimensional case as well, sidestepping the background field method and performing directly the calculation of the corresponding scattering amplitude of gravitons. This method will succeed in the reconstruction of the respective term in the effective action of the string and will prove that the quantum correction to the Planck mass in type II theories is proportional to the modified elliptic genus of the 8-dimensional compactification manifold, receives contributions only from its topological characteristics (and in particular the dimensions of the Dolbeault cohomology) and is finally expressed through the Euler characteristic.

Πανεπιστήμιο Ιωαννίνων
Σχολή Θετικών Επιστημών
Τμήμα Φυσικής

Πρόγραμμα Μεταπτυχιακών Σπουδών στη Φυσική
Διπλωματική Εργασία

Κβαντικές διορθώσεις στον όρο Einstein-Hilbert σε Θεωρίες Υπερχορδών Τύπου IIB συμπαγοποιημένες σε επιφάνειες $K3 \times K3$

Γεώργιος Σταυρόπουλος

Διπλωματική Εργασία για το
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Περίληψη

Στην εργασία αυτή θα μελετηθούν πτυχές της διαταρακτικής θεωρίας χορδών στο επίπεδο ενός βρόχου με χρήση μεθόδων σύμμορφης θεωρίας πεδίου επί των επιφανειών Riemann που ορίζουν οι αντίστοιχες κοσμικές επιφάνειες των χορδών. Από θεωρήματα μη επανακανονικοποίησης είναι γνωστό ότι η μάζα του Planck δε λαμβάνει κβαντικές διορθώσεις σε καμία τάξη της θεωρίας διαταραχών εκτός από μια ενδεχόμενη διόρθωση στο επίπεδο ενός βρόχου η οποία μπορεί να εμφανισθεί σε θερίες χορδών τύπου II των οποίων η πολλαπλότητα συμπαγοποίησης χαρακτηρίζεται από μη τετριμμένη τιμή της χαρακτηριστικής Euler, ασχέτως του αν υπάρχει ή όχι άσπαστη υπερσυμμετρία. Η απόδειξη των εν λόγω θεωρημάτων στηρίζεται στη μέθοδο υποβάθρου (background field method), η οποία όμως δεν είναι καλώς ορισμένη στην περίπτωση διδιάστατων χωρόχρονων, οι οποίοι προκύπτουν κατόπιν συμπαγοποίησης των 8 εκ των 10 διαστάσεων της υπερχορδής με χρήση μη τετριμμένων 8- διάστατων υποχώρων, όπως η επιφάνεια $K3 \times K3$. Στην παρούσα εργασία, κατόπιν μελέτης του προβλήματος μέσω της γνωστής μεθόδου υποβάθρου, θα γενικεύσουμε τον υπολογισμό των κβαντικών διορθόσεων του όρου Einstein-Hilbert και στη διδιάστατη περίπτωση, παρακάμπτωντας τη μέθοδο υποβάθρου και εκτελώντας απευθείας τον υπολογισμό του αντίστοιχου πλάτους σκέδασης βαρυτονίων. Με τη μέθοδο αυτή θα επιτευχθεί η ανακατασκευή του αντίστοιχου όρου στην ενεργό δράση της χορδής και θα αποδειχθεί ότι η κβαντική διόρθωση στη μάζα του Planck σε θεωρίες τύπου II είναι ανάλογη του τροποποιημένου ελλειπτικού γένους της 8- διάστατης πολλαπλότητας συμπαγοποίησης, λαμβάνει συνεισφορές μόνο από τα τοπολογικά χαρακτηριστικά αυτής (και συγκεκριμένα τις διαστάσεις της συνομολογίας Dolbeault) και τελικώς εκφράζεται μέσω της χαρακτηριστικής Euler.

Contents

Chapter 1

The Case for String Theory

1.1 Motivation for String Theory

1.1.1 The Quest of Unification in Theoretical Physics

The main ambition of modern theoretical physics has been, undoubtedly, the unification of the fundamental interactions in Nature. The Standard Model (SM) of particle physics does indeed provide the unified mathematical framework of gauge Quantum Field Theories (QFTs) as a general method to describe the fundamental interactions of Nature and their properties as well; however, it seems to fall short of fulfilling the quest of their unification. Before anything else, the attempt of unifying the gravitational interaction with the gauge interactions in the SM yields a non-renormalisable QFT. In addition, the SM alone, containing roughly twenty five free physical parameters, is an experimentally successful, yet highly theoretically arbitrary physical theory. Throughout the years, experience has shown that both the non-renormalisability as well as the theoretical arbitrariness of a physical theory may be interpeted as signs pointing to the existence of a new kind of Physics, that is manifest in the high-energy regime. Consequently, there is a need of new principles in Physics, which will prove to be succesful in resolving both the problem of physical divergences, that are always present in a non-renormalisable theory, as well as in providing a theoretically well-reasoned structure, in which the goal of the unification of the fundamental interactions of Nature can be achieved.
In order to attend to this enterprise, several ideas have been proposed. On the one hand, Grand Unification Theories (GUTs) attempt to extend, rationalise, and simplify the gauge structure of the SM. Several GUTs have been succesful towards this end, up to an extent, even providing correct physical predictions in certain cases. Nonetheless, all GUTs continue to be plagued by non-renormalisability problems similar to those of the SM, in the sense that they fail to incorporate a renormalisable QFT that can accurately describe the gravitational interaction. On the other hand, the physical conjecture of the existence of extra spacetime dimensions, embedded in arbitrary manifolds of high curvature, that is, being practically undetectable at typical low-energy regimes, seems to be a logical idea, since it appears to be a generalisation of the dynamical spacetime of the General Theory of Relativity (GTR). In QFTs of extra spacetime dimensions, the possibility of the unification of the gravitational interaction with the gauge interactions of the SM remains open, via several natural mechanisms regarding higher-dimensional quantum fields. Finally, the concept of Supersymmetry (SUSY) relates fields of different spin and statistics, and may be useful in resolving the problem of physical divergences in a QFT. It is reasonable to think that a complete, unified Theory Of Everything (TOE), will contain elements of every idea that has been proposed until today, in order to resolve the problems of the SM.

1.1.2 An Overview of String Theory

Over the past few decades, string theory has been the most prominent candidate of a physical theory that is successful in unifying all the fundamental interactions in Nature, including the gravitational interaction, accomplishing this endeavour both in a theoretically well-reasoned fashion, as well as in a structure that seems to be relieved of any physical divergences. In the case of string theory, all elementary particles are assumed to be extended, one-dimensional objects, that is, strings, rather than point-like, zero-dimensional objects, as they are supposed to be in the generic QFTs of the SM. This very fact makes string theory much more complicated, but at the same time much richer in structure, as well as much less theoretically arbitrary, in terms of free physical parameters, than the SM; in the case of String Theory, the only relevant free physical parameter left in the theory is the length of the strings. The strings of string theory may be open, or closed strings, and their different modes of vibration may be translated as generating particles of various masses. In particular, a remarkable property of any consistent string theory is the fact that it must always predict the existence of a massless, spin-two particle, the interactions of which necessarily reduce, in the low-energy limit, to these of the GTR; in this sense, String Theory is guaranteed to always contain a quantum theory of the gravitational interaction, along with any other interaction that it may describe. The rich structure of string theory means precisely that even some of the simplest string theories lead to gauge structures large enough to contain the SM itself, or gauge groups that arise in several GUTs. Moreover, with the addition of the concept of SUSY to string theory, a consistent superstring theory requires a definite number of extra spacetime dimensions, namely, six spacetime dimensions in number, which are embedded in a highly curved, arbitrary manifold, so that the total number of spacetime dimensions is now ten; as a consequence, the field equations of string theory possess string solutions in four ”large” flat, and six ”small” curved spacetime dimensions, and result in four-dimensional physics that resemble to the SM. Additionally, string theory appears to provide a finite theory of the gravitational interaction, at least as far as perturbation theory is concerned, even though there is yet to be a complete, explicit proof of this conjecture. However, there are mainly two important reasons for string theory to be rid of UV divergences. The first reason is the fact that the string dynamics and interactions are inextricably linked to the geometry of two-dimensional surfaces, in contrast to the particle dynamics and interactions that are present in the SM. The generalisation from zero-dimensional particles to one-dimensional strings smears out the interactions from spacetime points to surfaces and seems to have the result of effectively smoothening the divergent behaviour of the amplitudes of the interactions. The second reason is the very presence of an infinite number of particles of various masses, in the sense of the aforementioned different modes of vibrations of the strings, that can be regarded as forming an ”infinite tower” of states. The interactions between these particles of different masses are tuned carefully, so that they become soft, and thus finite, at distances of length larger than the length of the string.

The first attempts to form a quantised and relativistic string theory yielded bosonic string theory. Bosonic string theory, however, did not include fermions in its physical spectrum. The addition of fermions in the physical spectrum of string theory ensued from superstring theory, which naturally incorporated the concept of SUSY. There exist five kinds of consistent, stable superstring theories in the so-called ”critical”, ten-dimensional case; namely, these are the type I theory, of open and closed strings, the type IIA and type IIB theories of closed srings, as well as the $E8 \times E8$ and $SO(32)$ heterotic theories of closed strings. Despite the apparent differences between these kinds of superstring theories, it was eventually realised that they are all relevant to each other, in terms of dualities; this hinted to the fact that all of these kinds of superstring theories are, essentially, different vacua of a single underlying theory, that is referred to as the M-theory. Presently, string theory and its non-perturbative extension, the M-theory, seem to be the most promising candidates for a consistent, and unified, quantum theory of gravity. In summary, the reason why string theory in particular, and not any other physical theory, must be successful in providing a unified TOE, is, of course, not at all obvious a priori. However, experience has shown that problems regarding physical divergences in QFTs cannot be easily solved, especially so in a manner that is as elegant, and rational, as that of string theory. What is a certain fact, then, is the fact that string theory should be seriously taken into consideration.

Chapter 2

A Short Introduction to Bosonic String Theory

Bosonic string theory constitutes the first attempt to a quantised and relativistic string theory, although the absence of fermions from its spectrum, along with its other complications, renderred it physically irrelevant. However, we will see that the study of the quantisation procedure of bosonic string theory is quite instructive, in the sense that it elucidates the general methods of quantisation of a theory, and produces interesting results, that are similar in a large class of theories; after a swift glimpse of the general action of classical bosonic string theory, and of its symmetries, we will very briefly identify this class of theories, and describe their elements, using the example of bosonic string theory itself. We shall then move on to the quantisation procedure of bosonic string theory, in the path integral formulation, with a short presentation of its basic common characteristics, as well as its unique properties, with relation to a generic QFT. We will, in all respects, highlight several qualitative physical features of any string theory in general, pinpointing some of the deficiencies or complications of bosonic string theory.

2.1 Classical Bosonic String Theory

2.1.1 The Polyakov Action

A general and physically acceptable action $S[X, g]$ that may describe a string X of string length $l_s = \sqrt{\alpha'}$, which is embedded in a D-dimensional spacetime manifold M of metric field $G(X)$, which is alternatively called the target-space, is given by the following non-linear sigma model:

$$S[X, g] = -\frac{1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{|\det(g)|}(G_{\mu\nu}(X)g^{ab} + iB_{\mu\nu}(X)\frac{\epsilon^{ab}}{\sqrt{|\det(g)|}})\partial_a X^\mu \partial_b X^\nu$$
$$-\frac{\lambda}{4\pi} \int_W d^2\sigma \sqrt{|\det(g)|}\Phi(X)R_W, \tag{2.1}$$

where the integration takes place on the closed 2-dimensional surface W of metric g, which is called the world-sheet, and R_W signifies the Ricci scalar of the world-sheet; ϵ^{ab} is, of course, the antisymmetric Levi-Civita symbol or tensor density. The antisymmetric $(0, 2)$-tensor field $B(X)$ is called the Kalb-Ramond field, or, simply, the B-field, and it represents an electromagnetic potential, while the scalar field $\Phi(X)$ is called the dilaton field. By requiring a Minkowski, electromagnetically-neutral spacetime background, in the sense of imposing that $G(X) = \eta$, $B(X) = 0$, and $\Phi(X) = \langle\Phi\rangle$, all hold identically, where, of course, η is the Minkowski metric, and $\langle\Phi\rangle$ is the vacuum expectation value (vev) of the dilaton field, $\Phi(X)$, the non-linear sigma model of *eq.*(2.1) reduces to what is known as the Brink-Di Vecchia-Howe-Deser-Zumino action, which is, by virtue of its subsequent analysis by Polyakov, more commonly called the Polyakov action, for short:

$$S_P[X,g] = -\frac{1}{4\pi\alpha'}\int_W d^2\sigma\,\sqrt{|\det(g)|}\eta_{\mu\nu}g^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{\lambda\langle\Phi\rangle}{4\pi}\int_W d^2\sigma\,\sqrt{|\det(g)|}R_W, \tag{2.2}$$

where the Gauss-Bonnet term, which is relevant to the Ricci scalar of the world-sheet, is now proportional to the Euler characteristic of the world-sheet, $\chi_W = \frac{1}{4\pi}\int_W d^2\sigma\,\sqrt{|\det(g)|}R_W$, that is a topological invariant; as such, the Gauss-Bonnet term cannot influence the local dynamics of the string, and the Polyakov action of *eq.*(2.2) can be equivalently expressed as:

$$S_P[X,g] = -\frac{1}{4\pi\alpha'}\int_W d^2\sigma\,\sqrt{|\det(g)|}\eta_{\mu\nu}g^{ab}\partial_a X^\mu \partial_b X^\nu - \lambda\langle\Phi\rangle\chi_W. \tag{2.3}$$

For subsequent use, we recall that the Gauss-Bonnet theorem, for a closed world-sheet, W, relates its Euler characteristic, χ_W, to its genus, g_W, according to the following formula:

$$\chi_W = 2 - 2g_W. \tag{2.4}$$

The Polyakov action of *eq.*(2.3) displays the symmetries of target-space Poicaré invariance and world-sheet diffeomorphism invariance, as well as the symmetry of Weyl invariance, which consists of local rescalings of the world-sheet metric. The direct product of the symmetry groups of world-sheet diffeomorphisms and Weyl transformations is referred to as the gauge symmetry group of the Polyakov action. In fact, it can be easily seen that the Polyakov action is indeed the most general action that exhibits all three of these aforementioned types of symmetries. Specifically, as far as Weyl invariance is concerned, it can be shown that the group of Weyl transformations on a specific world-sheet metric can be used to define a relevant equivalence relation, and so an equivalence class of metrics, the elements of which are called conformally equivalent metrics; Weyl invariance of the Polyakov action, then, simply states that the Polyakov action is invariant for conformally equivalent metrics.

Assuming a closed world-sheet and a closed string X, the extrema of the Polyakov action, with respect to the closed string coordinates, X^μ, yield the equations of motion:

$$\nabla^2 X^\mu = 0, \tag{2.5}$$

while, the variation of the Polyakov action, with respect to the world-sheet metric, yields the energy-momentum tensor, T_P, which has the following components:

$$T_P^{ab} = -4\pi\sqrt{|\det(g)|}\frac{\delta S_P}{\delta g_{ab}} = -\frac{1}{\alpha'}\eta_{\mu\nu}(\partial^a X^\mu \partial^b X^\nu - \frac{1}{2}g^{ab}g_{cd}\partial^c X^\mu \partial^d X^\nu). \tag{2.6}$$

Obviously, Noether's theorem implies that, when subject to the constraints of the equations of motion, the energy-momentum tensor is conserved, that is:

$$\nabla_a T_P^{ab} = 0, \tag{2.7}$$

as a direct consequence of the diffeomorphism invariance of the Polyakov action. In addition, the energy-momentum tensor is traceless:

$$tr(T_P) = 0, \tag{2.8}$$

as an immediate result of the Weyl invariance of the Polyakov Action. Furthermore, considering the extrema of the Polyakov action, with respect to the world-sheet metric, implies the following constraints:

$$T_P = 0, \tag{2.9}$$

that is, the constraints of a null energy-momentum tensor. These constraints of the above *eq.*(9) are known as the Virasoro constraints, and their physical significance is demonstrated by the following reasoning:

According to the standard principle of simplicity, the most natural candidate for the action that describes a target-space string should be no other than the total area of the 2-dimensional world-sheet of its parametrisation. By imposing the Virasoro constraints on the Polyakov action, it reduces to the so-called Nambu-Goto action:

$$S_{NG}[X] = -\frac{1}{2\pi\alpha'}\int_W d^2\sigma\sqrt{|\det(\gamma)|} - \lambda\langle\Phi\rangle\chi_W = -\frac{1}{2\pi\alpha'}\int_W d^2\sigma\sqrt{\left|\det\Big(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\Big)\right|} - \lambda\langle\Phi\rangle\chi_W, \tag{2.10}$$

which simply expresses just the total area of the 2-dimensional world-sheet, that now possesses the naturally induced metric $\gamma_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$. Of course, the Nambu-Goto action describes the correct dynamical theory of a classical string; what is more, it does so in the simplest target-space Poincaré invariant and world-sheet diffeomorphism invariant way that is possible; however, the Nambu-Goto action is rarely used for the study of bosonic string theory, mainly due to issues of ambiguity in its quantisation procedure. Hence, the Polyakov action is regarded as the preferrable starting point of bosonic string theory.

Elements of Conformal Field Theory

As any 2-dimensional surface is always diffeomorphic to a surface that is conformally equivalent to a flat space, the 2-dimensional world-sheet is always conformally flat. By additionally assuming a closed world-sheet W of a topology of genus g_W equal to zero, that is $g_W = 0$, the Euler characteristic χ_W is consequently equal to 2, that is $\chi_W = 2$, according to the preceding *eq.*(2.4); then, the conformally flat world-sheet metric, g, can, without loss of generality, be set equal to the Minkowski metric, η, that is $g = \eta$, due to Weyl invariance of the the Polyakov action. The Polyakov action is, then, expressed in the so-called conformal gauge, as in the following:

$$S_P[X, g] = -\frac{1}{4\pi\alpha'}\int_W d^2\sigma \eta_{\mu\nu}\eta^{ab}\partial_a X^\mu \partial_b X^\nu - 2\lambda\langle\Phi\rangle. \tag{2.11}$$

Equivalently, the light-cone coordinates, σ^+ and σ^-, defined as:

$$\sigma^+ = \sigma^0 + \sigma^1, \tag{2.12}$$

and,

$$\sigma^- = \sigma^0 - \sigma^1, \tag{2.13}$$

may also be used, in order to alternatively express the Polyakov action in the conformal gauge, as:

$$S_P[X, g] = -\frac{1}{2\pi\alpha'}\int_W d\sigma^+ d\sigma^- \eta_{\mu\nu}\partial_+ X^\mu \partial_- X^\nu - 2\lambda\langle\Phi\rangle. \tag{2.14}$$

However, for reasons relevant to the correct quantisation of the Polyakov action, in the path-integral formulation, the appropriate Wick rotations must be performed, both on the world-sheet time-like variable, $\sigma^0 = c\tau$, as well as on the target-space time-like coordinate, X^0. These Wick rotations, which are simply the redefinitions $\sigma^0 = i\sigma^2$, and, $X^0 = iX^D$, respectively, result to the emergence of an extra, overall imaginary unit factor, i, in the Polyakov action, a factor which will be ommitted for the moment, as well as to the effective transformation of the target-space Minkowski metric to the Kronecker metric, respectively. Henceforth, a natural step further is the use of the complex coordinates, defined as:

$$w = \sigma^1 + i\sigma^2, \tag{2.15}$$

and,

$$\bar{w} = \sigma^1 - i\sigma^2, \tag{2.16}$$

so that the powerful methods and techniques of complex analysis can be utilised in full, for the study of the Polyakov action. Using complex coordinates, the Polyakov action in the conformal gauge is expressed as:

$$S_P[X, g] = -\frac{1}{\pi\alpha'}\int_W d\,\mathrm{Im}(w) d\,\mathrm{Re}(w)\delta_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu - 2\lambda\langle\Phi\rangle = -\frac{1}{2\pi\alpha'}\int_W d^2 w \delta_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu - 2\lambda\langle\Phi\rangle. \tag{2.17}$$

The equations of motion, then, reduce to:

$$\partial\bar{\partial}X^{\mu} = 0, \tag{2.18}$$

which is simply the wave equation, in the 2 dimensions of a conformally flat world-sheet, expressed in complex coordinates. For the case of a closed string, the equations of motion must be solved for the closed string boundary conditions, which may be expressed as:

$$X^{\mu} \in \mathbb{R}, \tag{2.19}$$

and:

$$X^{\mu}(w, \bar{w}) = X^{\mu}(w + 2\pi, \bar{w} + 2\pi), \tag{2.20}$$

by which it is required that the string coordinates X^{μ} be real, 2π-periodic functions of the world-sheet space-like variable σ^1. The solution to the Cauchy problem of *eqs*.(2.18), (2.19), and (2.20), is a harmonic function, so it can be expressed as a sum of a holomorphic, or "left-moving", and an anti-holomorphic, or "right-moving", part, as in:

$$X^{\mu}(w, \bar{w}) = X^{\mu}_L(w) + X^{\mu}_R(\bar{w}), \tag{2.21}$$

subject, of course, to the reality and periodicity conditions of *eqs*.(2.19) and (2.20). The left-moving and right-moving string coordinate functions, X^{μ}_L and X^{μ}_R, are, then, expressed by the following mode expansions:

$$X^{\mu}_L(w) = \frac{x^{\mu}_0}{2} - \frac{\alpha'}{2}p^{\mu}w - i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{a^{\mu}_n}{n}e^{-inw}, \tag{2.22}$$

and:

$$X^{\mu}_R(\bar{w}) = \frac{x^{\mu}_0}{2} + \frac{\alpha'}{2}\tilde{p}^{\mu}\bar{w} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\tilde{a}^{\mu}_n}{n}e^{-in\bar{w}}, \tag{2.23}$$

respectively, while the reality and periodicity conditions of *eqs*.(2.19) and (2.20), impose the following constraints:

$$p^{\mu} = \tilde{p}^{\mu}, \tag{2.24}$$

and:

$$a^{\mu}_n = \bar{a}^{\mu}_{-n}; \tilde{a}^{\mu}_n = \bar{\tilde{a}}^{\mu}_{-n}, \tag{2.25}$$

respectively. So, the full string coordinates, X^{μ}, are expressed as:

$$X^{\mu}(w, \bar{w}) = x^{\mu}_0 - \frac{\alpha'}{2}p^{\mu}(w - \bar{w}) - i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{a^{\mu}_n}{n}e^{-inw} - \frac{\tilde{a}^{\mu}_n}{n}e^{in\bar{w}}, \tag{2.26}$$

with derivatives:

$$\partial X^{\mu}(w, \bar{w}) = \partial X^{\mu}_L(w) = -\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}a^{\mu}_n e^{-inw}, \tag{2.27}$$

and:

$$\bar{\partial}X^{\mu}(w, \bar{w}) = \bar{\partial}X^{\mu}_R(\bar{w}) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\tilde{a}^{\mu}_n e^{-in\bar{w}}. \tag{2.28}$$

In the above relations the definitions:

$$p^{\mu} = \sqrt{\frac{2}{\alpha'}}a^{\mu}_0 = \sqrt{\frac{2}{\alpha'}}\tilde{a}^{\mu}_0, \tag{2.29}$$

are implied. It can be seen, now, that the world-sheet is naturally compactified topologically into a cylinder, which is formed by the motion and by the vibrations of the string. Obviously, the motion of the center-of-mass of the string, $X_{CM}(w, \bar{w})$, is described by its initial position, x_0, and its canonical momentum, p, as in the following:

$$X_{CM}(w, \bar{w}) = x_{CM}(\tau) = x_0 - \frac{\alpha'}{2} p(w - \bar{w}) = x_0 + \alpha' p c \tau. \tag{2.30}$$

Additionally, the energy-momentum tensor now satisfies the relations:

$$T_{P;ww} = -\frac{1}{\alpha'} \delta_{\mu\nu} \partial X^\mu \partial X^\nu = -\frac{1}{\alpha'} \delta_{\mu\nu} \partial X_L^\mu \partial X_L^\nu = T_P(w), \tag{2.31}$$

$$T_{P;\bar{w}\bar{w}} = -\frac{1}{\alpha'} \delta_{\mu\nu} \bar{\partial} X^\mu \bar{\partial} X^\nu = -\frac{1}{\alpha'} \delta_{\mu\nu} \partial X_R^\mu \partial X_R^\nu = \tilde{T}_P(\bar{w}), \tag{2.32}$$

and:

$$tr(T_P) = 4 T_{P;w\bar{w}} = 0, \tag{2.33}$$

that is, the energy-momentum tensor consists of a holomorphic component, $T_P(w)$, and an anti-holomorphic component, $\tilde{T}_P(\bar{w})$, while the trace of the energy-momentum tensor, $tr(T_P)$, obviously vanishes, due to the Weyl invariance of the Polyakov action.

The expression of the Polyakov action in the conformal gauge, by use of the complex coordinates, $(w, \bar{w})$, that were defined above, manifests explicitly both the full Poincaré invariance and gauge invariance of the Polyakov action, as well as the invariance of the conformal gauge choice, under the gauge subgroup which consists of holomorphic diffeomorphisms, combined with corresponding, appropriate Weyl transformations, such that the world-sheet metric remains invariant. This gauge subgroup is called the conformal group, while its, finite-dimensional, $PSL(2;\mathbb{C})$-subgroup, is known as the restricted conformal group. Therefore, the Polyakov action describes a Conformal Field Theory (CFT) on a 2-dimensional world-sheet. Finally, the following conformal transformation may be considered:

$$w \to z = e^{-iw} \Leftrightarrow \bar{w} \to \bar{z} = e^{i\bar{w}}, \tag{2.34}$$

which has the effect of a conformal mapping of the topologically cylindrical world-sheet onto the complex plane plus the point at infinity, that is topologically equivalent to the Riemann sphere. The Polyakov action in the conformal gauge may then be expressed as:

$$S_P[X, g] = -\frac{1}{\pi\alpha'} \int_W d\,\mathrm{Im}(z) d\,\mathrm{Re}(z) \delta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu - 2\lambda\langle\Phi\rangle = -\frac{1}{2\pi\alpha'} \int_W d^2 z \delta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu - 2\lambda\langle\Phi\rangle. \tag{2.35}$$

The expression of *eq.*(2.35) for the Polyakov action demonstrates the following significant advantage: The equations of motion for the string X, as they were expressed in *eq.*(2.18), are invariant, while the boundary periodicity contitions for the string X are now transformed into the relatively more trivial single-valuedness conditions; what is of even greater importance, however, is the consequential fact that the string coordinates X^μ are now expressed by a mode expansion which is identical to their respective Laurent series:

$$X^\mu(z, \bar{z}) = X_L(z) + X_R(\bar{z}) = x_0^\mu - i\frac{\alpha'}{2} p^\mu \log|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{a_n^\mu}{n} z^{-n} + \frac{\tilde{a}_n^\mu}{n} \bar{z}^{-n}, \tag{2.36}$$

with derivatives:

$$\partial X^\mu(z, \bar{z}) = \partial X_L^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} a_n^\mu z^{-n-1}, \tag{2.37}$$

and:

$$\partial X^\mu(z, \bar{z}) = \partial X_R^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \tilde{a}_n^\mu \bar{z}^{-n-1}. \tag{2.38}$$

Furthermore, the holomorphic and anti-holomorphic components of the energy-momentum tensor can be expressed as the following Laurent series:

$$T_P(z) = -\frac{1}{\alpha'}\delta_{\mu\nu}\partial X^\mu_L \partial X^\nu_L = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{2.39}$$

and:

$$\tilde{T}_P(\bar{z}) = -\frac{1}{\alpha'}\delta_{\mu\nu}\bar{\partial} X^\mu_R \bar{\partial} X^\nu_R = \sum_{n\in\mathbb{Z}} \tilde{L}_n \bar{z}^{-n-2}, \tag{2.40}$$

respectively, where the Laurent coefficients L_n and $\tilde{L}_n$ are known as the Virasoro coefficients; it can be proved that the Virasoro coefficients, L_n and $\tilde{L}_n$, are expressed as the following infinite series:

$$L_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} \delta_{\mu\nu} a^\mu_{n-m} a^\nu_m, \tag{2.41}$$

and:

$$\tilde{L}_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} \delta_{\mu\nu} \tilde{a}^\mu_{n-m} \tilde{a}^\nu_m, \tag{2.42}$$

respectively, with the use of simple calculations. The Virasoro coefficients are of great importance to the quantisation of the Polyakov action. In the quantum theory of the Polyakov action, the Virasoro coefficients are the generators, that is, the Noether charges, of the conformal group. Additionally, in the quantum theory of the Polyakov action, the Virasoro coefficients are subject to an appropriate notion of normal ordering; it has been proved that they generally result in an anomaly, which is related to the target-space dimension of the bosonic string X. It has been also proved that this anomaly always persists, but for the special case of the so-called critical target-space dimension, $D = 26$, of the bosonic string X.

2.2 Quantisation of the Polyakov Action

2.2.1 Vacuum Expectation Values

As it is usual, in a QFT, we are concerned with the time-ordered vacuum expectation values of local observables, $O(X(z,\bar{z}))$, which, for an interaction-free QFT, such as the QFT of the Polyakov action, are defined, in the path-integral formulation, by the following functional integral:

$$\langle R\{\prod_{n\in\mathbb{N}_N} O_n(X(z_n,\bar{z}_n))\}\rangle_g = \int_{F_{X_g}} [dX]_g e^{-S_P[X,g]} \prod_{n\in\mathbb{N}_N} O_n(X(z_n,\bar{z}_n)), \tag{2.43}$$

where the overall extra imaginary unit factor that emerges in the action as a result of the Wick rotation of the world-sheet time-like variable, $\sigma^0 = i\sigma^2$, is now cancelled out against the positive imaginary unit factor that multiplies the action in the definition of the path-integral, leaving an overall negative sign in the action; the overall extra imaginary unit factor, that emerges in the functional integration measure $[dX]_g$ of the function space F_X of the closed string X, as a result of the Wick rotation of the target-space time-like coordinate, $X^0 = iX^D$, is regarded to be of no physical significance, and can be freely omitted. Of course, the respective Minkowski functional integral of the Euclidean functional integral in the above *eq.*(2.43) can neither be defined, or be practically calculated, without considering, in one way or another, the Wick rotations that have been performed in it; rather, it is only defined in terms of the appropriate analytical continuation of its respective Euclidean functional integral in the above *eq.*(2.43). Additionally, the symbolism of $R\{\dots\}$, signifies the radial ordering operation on the respective operator product, which is equivalent to the world-sheet Wick-rotated "time" $i\sigma^0 = \sigma^2 = \mathrm{Im}(w) = \log|z|$ -ordering operation on the operator product; time ordering emerges naturally in the path-integral formalism. Formally, the radial ordering operation, $R\{O_1(X(z_1,\bar{z}_1))O_2(X(z_2,\bar{z}_2))\}$, on an operator product of two local operators, $O_1(X(z_1,\bar{z}_1))O_2(X(z_2,\bar{z}_2))$, can be expressed as:

$$R\{O_1(X(z_1,\bar{z}_1))O_2(X(z_2,\bar{z}_2))\} =$$

$$= O_1(X(z_1,\bar{z}_1))O_2(X(z_2,\bar{z}_2))H(|z_1|-|z_2|) + O_2(X(z_2,\bar{z}_2))O_1(X(z_1,\bar{z}_1))H(|z_2|-|z_1|), \tag{2.44}$$

while, in general, the radial ordering operation, $R\{\prod_{n\in\mathbb{N}_N} O_n(X(z_n,\bar{z}_n))\}$, on an operator product of N local operators, $\prod_{n\in\mathbb{N}_N} O_n(X(z_n,\bar{z}_n))$, can be expressed recursively, as in the following formula:

$$R\{\prod_{n\in\mathbb{N}_N} O_n(X(z_n,\bar{z}_n))\} = \sum_{n\in\mathbb{N}_N} O_n(X(z_n,\bar{z}_n))R\{\prod_{m\in\mathbb{N}_N\setminus\{n\}} H(|z_n|-|z_m|)O_m(X(z_m,\bar{z}_m))\}. \tag{2.45}$$

This symbolism, $R\{\dots\}$, for the radial ordering operation on any operator product, will be ommitted from here on out, and it shall be implicitly assumed that every operator product is radially-ordered. In the conformal gauge, now, Ehrenfest's theorem and the Schwinger-Dyson equations imply the following differential equations:

$$\partial\bar{\partial}X^\mu(z,\bar{z}) = 0, \tag{2.46}$$

and:

$$\partial\bar{\partial}X^\mu(z,\bar{z})X^\nu(z_0,\bar{z}_0) = -\pi\alpha'\delta^{\mu\nu}\delta^{(2)}(z-z_0,\bar{z}-\bar{z}_0), \tag{2.47}$$

respectively; these equations, of course, hold as operator equations. Now, assuming that the worldsheet W is topologically equivalent to the Riemann sphere, we can solve for the Green's functions, or propagators:

$$\langle X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)\rangle = -\frac{\alpha'}{2}\delta^{\mu\nu}\log|z_1-z_2|^2. \tag{2.48}$$

Then, we can find the following correlation functions:

$$\langle\partial X^\mu(z_1)\partial X^\nu(z_2)\rangle = -\frac{\alpha'}{2}\frac{\delta^{\mu\nu}}{(z_1-z_2)^2}, \tag{2.49}$$

and similarly:

$$\langle\bar{\partial}X^\mu(\bar{z}_1)\bar{\partial}X^\nu(\bar{z}_2)\rangle = -\frac{\alpha'}{2}\frac{\delta^{\mu\nu}}{(\bar{z}_1-\bar{z}_2)^2}, \tag{2.50}$$

and so on and so forth, for the correlators of higher derivatives of the string functions. We note that the correlation functions of *eqs*.(2.49) and (2.50), as well as the correlators of higher derivatives of the string coordinates, are conformally covariant only under the restricted conformal group, while, the propagators of *eq*.(2.48) have rather poor conformal properties, as they are invariant only under the $U(1)$ subrgoup of the conformal group. We can see, then, that the string X itself cannot be considered the natural field of interest in the QFT of the Polyakov action, while the restricted conformal covariance of the correlation functions of its derivatives indicates an anomaly in the quantum theory, regarding the subgroup of non-restricted conformal transformations; this anomaly will be calculated shortly.

2.2.2 The Weyl Anomaly

In general, now, Ward's identity for the Noether current $(J(z),\tilde{J}(\bar{z}))$, and any operator, $O(z,\bar{z})$, implies that:

$$res_{z\to z_0}J(z)O(z_0,\bar{z}_0) + \overline{res}_{\bar{z}\to z_0}\tilde{J}(\bar{z})O(z_0,\bar{z}_0) = -i\delta O(z_0,\bar{z}_0), \tag{2.51}$$

holds as an operator equation. However, one must ensure that the vacuum expectation value of the energy-momentum tensor is indeed zero. To that purpose, we can define the normal ordering operation on the operators $T_P(z)$ and $\tilde{T}_P(\bar{z})$ respectively:

$$:T_P(z_0) := -\frac{1}{\alpha'}\lim_{z\to z_0}\delta_{\mu\nu}(\partial X^\mu(z)\partial X^\nu(z_0) - \langle\partial X^\mu(z)\partial X^\nu(z_0)\rangle), \tag{2.52}$$

and:

$$: \tilde{T}_P(z_0) := -\frac{1}{\alpha'} \lim_{\bar{z}\to\bar{z}_0} \delta_{\mu\nu}(\bar{\partial}X^\mu(\bar{z})\bar{\partial}X^\nu(\bar{z}_0) - \langle\bar{\partial}X^\mu(\bar{z})\bar{\partial}X^\nu(\bar{z}_0)\rangle). \tag{2.53}$$

The normal ordering operation, $:\cdots:$, on any arbitrary operator-valued functional, $O[X]$, is then defined accordingly, as $: O[X] :$; in general, we have the following definition:

$$: O[X] := e^{-\frac{1}{2}\int d^2z_1 d^2z_2 \langle X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)\rangle \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta X^\nu(z_2,\bar{z}_2)}} O[X], \tag{2.54}$$

and so we can see that the definition of the above *eq.*(2.54) ensures that Wick's theorem is automatically satisfied, that is:

$$: O_1[X] :: O_2[X] := e^{\int d^2z_1 d^2z_2 \langle X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)\rangle \frac{\delta_1}{\delta_1 X^\mu(z_1,\bar{z}_1)}\frac{\delta_2}{\delta_2 X^\nu(z_2,\bar{z}_2)}} : O_1[X]O_2[X] : . \tag{2.55}$$

It is obvious from the above that normal ordering operation comes with the following property:

$$\partial\bar{\partial} : X^\mu(z,\bar{z})X^\nu(z_0,\bar{z}_0) := 0, \tag{2.56}$$

which may well be used alternatively as an equivalent, yet quicker and more concise definition for the normal ordering operation. The symbolism $:\cdots:$, for the normal ordering operation on any operator, will be ommitted from here on out, and it shall be implicitly assumed that every operator is normally-ordered. Now, using Wick's theorem, one can deduce the following Operator Product Expansions (OPEs), for the energy-momentum tensor, where, the symbol $\sim$ denotes equality up to regular terms:

$$T_P(z)X^\mu(z_0) \sim \frac{1}{z-z_0}\partial X^\mu(z_0), \tag{2.57}$$

$$\tilde{T}_P(\bar{z})X^\mu(\bar{z}_0) \sim \frac{1}{\bar{z}-\bar{z}_0}\bar{\partial} X^\mu(\bar{z}_0), \tag{2.58}$$

$$T_P(z)\partial X^\mu(z_0) \sim \frac{1}{(z-z_0)^2}\partial X^\mu(z_0) + \frac{1}{z-z_0}\partial^2 X^\mu(z_0), \tag{2.59}$$

$$\tilde{T}_P(\bar{z})\bar{\partial} X^\mu(\bar{z}_0) \sim \frac{1}{(\bar{z}-\bar{z}_0)^2}\bar{\partial} X^\mu(\bar{z}_0) + \frac{1}{\bar{z}-\bar{z}_0}\bar{\partial}^2 X^\mu(\bar{z}_0), \tag{2.60}$$

$$T_P(z)T_P(z_0) \sim \frac{D}{2(z-z_0)^4} + \frac{2}{(z-z_0)^2}T_P(z_0) + \frac{1}{z-z_0}\partial T_P(z_0), \tag{2.61}$$

and:

$$\tilde{T}_P(\bar{z})\tilde{T}_P(\bar{z}_0) \sim \frac{D}{2(\bar{z}-\bar{z}_0)^4} + \frac{2}{(\bar{z}-\bar{z}_0)^2}\tilde{T}_P(\bar{z}_0) + \frac{1}{\bar{z}-\bar{z}_0}\bar{\partial}\tilde{T}_P(\bar{z}_0), \tag{2.62}$$

while, any OPE between a holomporphic and an anti-holomorphic operator is purely regular (non-singular). By comparing the above OPEs with Ward's identity, for an infinitesimal conformal transformation of the form $\delta z = v(z); \delta\bar{z} = \bar{v}(\bar{z})$, which results to the conserved Noether currents $J(z) = iv(z)T(z)$ and $\tilde{J}(\bar{z}) = i\bar{v}(\bar{z})\tilde{T}(\bar{z})$, one ends up with the following operator equations, for infinitesimal conformal transformations:

$$\delta X^\mu(z,\bar{z}) = -v\partial X^\mu(z) - \bar{v}\bar{\partial}X^\mu(\bar{z}), \tag{2.63}$$

$$\delta\partial X^\mu(z) = -\partial v\partial X^\mu(z) - v\partial^2 X^\mu(z), \tag{2.64}$$

$$\delta\bar{\partial} X^\mu(\bar{z}) = -\bar{\partial}\bar{v}\bar{\partial} X^\mu(\bar{z}) - \bar{v}\bar{\partial}^2 X^\mu(\bar{z}), \tag{2.65}$$

$$\delta T(z) = -\frac{c}{12}\partial^3 v - 2T(z)\partial v - v\partial T(z), \tag{2.66}$$

and:

$$\delta\tilde{T}(\bar{z}) = -\frac{\tilde{c}}{12}\bar{\partial}^3\bar{v} - 2\tilde{T}(\bar{z})\bar{\partial}\bar{v} - \bar{v}\bar{\partial}\tilde{T}(\bar{z}), \tag{2.67}$$

as well as their corresponding operator equations, for finite conformal transformations of the form $z \to z'(z); \bar{z} \to \bar{z}'(\bar{z})$:

$$X'(z',\bar{z}') = X(z,\bar{z}), \tag{2.68}$$

$$\partial' X'(z') = (\partial z')^{-1} \partial X(z), \tag{2.69}$$

$$\bar{\partial}' X'(\bar{z}') = (\bar{\partial} \bar{z}')^{-1} \bar{\partial} X(\bar{z}), \tag{2.70}$$

$$T'(z') = (\partial z')^{-2}(T(z) - \frac{c}{12} S_z(z')), \tag{2.71}$$

and:

$$\tilde{T}'(\bar{z}') = (\bar{\partial} \bar{z}')^{-2}(\tilde{T}(\bar{z}) - \frac{\tilde{c}}{12} S_{\bar{z}}(\bar{z}')), \tag{2.72}$$

where the constant $c = \tilde{c} = D$ is called the central charge of the CFT described by the Polyakov action, and the Schwarzian derivative:

$$S_z(f) = \frac{\partial^3 f}{\partial f} - \frac{3}{2} \frac{(\partial^2 f)^2}{(\partial f)^2}, \tag{2.73}$$

has been used, due to its property of having the correct infinitesimal form, as well as preserving the group structure of the conformal group:

$$S_z((f_1 \circ f_2)) = S_{f_2}(f_1)(\partial f_2)^2 + S_z(f_2). \tag{2.74}$$

The Schwarzian derivative of a conformal transformation, which, by definition, consists of a particular product combination of a diffeomorphism, and a corresponding, appropriate Weyl transformation, such that the world-sheet metric remains invariant, refers only to the Weyl part of the conformal transformation, precisely due to the commutativity property between the diffeomorphism group and the Weyl group. As a result, the Schwarzian derivative may define a specific representation of the Weyl group, with respect to its action on the energy-momentum tensor, T, and it can be easily proved that the Schwarzian derivative must be equal to the following:

$$S_z(z') = 2(\partial^2 \phi - (\partial \phi)^2), \tag{2.75}$$

for a Weyl transformation of the form $g' = e^{2\phi} g$. That is, the Weyl group acts on the components $T(z)$ and $\tilde{T}(\bar{z})$ of the energy-momentum tensor T in the following manner:

$$T(z) \to T(z) - \frac{c}{6}(\partial^2 \phi - (\partial \phi)^2), \tag{2.76}$$

and:

$$\tilde{T}(\bar{z}) \to \tilde{T}(\bar{z}) - \frac{\tilde{c}}{6}(\bar{\partial}^2 \phi - (\bar{\partial} \phi)^2), \tag{2.77}$$

for any Weyl transformation of the world-sheet metric that has the form of $g' = e^{2\phi} g$. The transformations of *eqs*.(2.68) through (2.72) imply that, under conformal transformations, the string X is a scalar, while its derivative $\partial_a X$ is a $(0,1)$-world-sheet tensor; fields with tensorial conformal transformation properties such as these fields, are called primary fields, so the string X, and its derivatives $\partial_a X$, are primary fields, with respective conformal weights 0, $(1,0)$ and $(0,1)$. The energy-momentum tensor T is not a conformal tensor, due to the central charges $c = D$ and $\tilde{c} = D$ of the CFT that is described by the Polyakov action. Still, the energy momentum tensor has at least some tensorial properties; in particular, it is a restricted conformal group tensor, of conformal dimension equal to 2, as $S_z(f) = 0$ only for the, finite-dimensional, $PGL(2;\mathbb{C})$ -subgroup of the conformal group, which is known as the Möbius group. Fields with quasi-tensorial conformal transformation properties such as this field, are called quasi-primary fields, so the components $T(z)$ and $\tilde{T}(\bar{z})$ of the energy-momentum tensor T, are quasi-primary fields of respective conformal weights $(2,0)$ and $(0,2)$. Furthermore, fields that are neither primary nor quasi-primary are called descendant fields; the descendant fields can usually be expressed as derivatives of primary or quasi-primary fields. The fact that the components $T(z)$ and $\tilde{T}(\bar{z})$ of the energy-momentum tensor are quasi-primary fields, results in a quantum CFT with the following anomaly:

$$tr(T) = -\frac{c + \tilde{c}}{24} R_W = -\frac{D}{12} R_W, \tag{2.78}$$

which is known as the Weyl anomaly of the quantum theory of the Polyakov action. The above *eq*.(2.78) can be proved easily, starting from the useful relations of *eqs*.(2.76) and (2.77), and considering the conservation of the energy-momentum tensor. Moreover, from the definition of the energy-momentum tensor, it can be proved that in any CFT defined on a 2-dimensional surface, the energy-momentum tensor must be a quasi-primary field of conformal dimension equal to 2; the relevant *eqs*.(2.71) and (2.72), that describe its transformation properties under the conformal group, hold, in general, for any such CFT, of central charges c and $\tilde{c}$. Then, the energy-momentum tensor must transform in accordance with *eqs*.(2.76) and (2.77), under a Weyl transformation of the appropriate form; it follows, in a straightforward manner, that the form of the Weyl anomaly in *eq*.(2.78), is universal, and holds for any such CFT, of central charges c and $\tilde{c}$. This fact is related to the more general Wess-Zumino consistency condition, which is a very powerful restriction on the form of possible anomalies that may be contained in a QFT.

2.2.3 Scattering Amplitudes I: The State-Operator Correspondence and Vertex Operators

Contrary to an arbitrary, generic QFT, CFTs possess the property of the state-operator correspondence; this remarkable property consists of the fact that any quantum state of a CFT bijectively corresponds to a local operator, and can then be expressed as this local operator acting on the vacuum state. In other words, the entire Fock space of a CFT is local; in fact, it can be regarded that a copy of the entire Fock space of the CFT exists on each and any point of the world-sheet Riemann sphere. The state-operator correspondence property of CFTs can be explained and understood shortly: The symmetry of a CFT under the conformal transformation $z = e^{-iw}$ means that the world-sheet cylinder of the w-conformal coordinate system can be mapped to the world-sheet Riemann sphere of the z-conformal coordinate system, so any asymptotic string state can be defined to exist only locally; this fact simply means, in turn, that any asymptotic string state can be expressed as a local operator, which is evaluated on an arbitrary point of the world-sheet Riemann sphere, acting on the vacuum state. The local operators, in terms of which the state-operator correspondence is defined, are called vertex operators, they are denoted as $V^{(n)}(p; X(z, \bar{z}))$, and they constitute the asymptotic, local Fock space of the CFT. It is clear from the aforementioned that the state-operator correspondence is a property of CFTs due to their symmetry under the conformal mapping of the world-sheet cylinder to the Riemann sphere; on the contrary, in non-CFTs, a typical local operator can create many different states. Among other uses, the state-operator correnspondence property of CFTs can also justify the fact that the OPEs are exact and convergent statements in CFTs.
A scattering amplitude is, by definition, a transition amplitude between asymptotic states; as a result, then, of the state-operator correspondence, any scattering amplitude in string theory can be expressed as integrated vacuum expectation values of vertex operators. That is, a generic scattering amplitude of N vertex operators, $V_n(p_n; X(z_n, \bar{z}_n))$, can be expressed as the following functional integral:

$$\langle \prod_{n\in\mathbb{N}_N} \int_W d^2 z_n V_n(p_n; X(z_n, \bar{z}_n))\rangle_g = \int_{F_{X_g}} [dX]_g e^{-S_P[X,g]} \prod_{n\in\mathbb{N}_N} \int_W d^2 z_n V_n(p_n; X(z_n, \bar{z}_n)). \tag{2.79}$$

The scattering amplitudes in string theory, can then be expressed in a fashion that is similar to the LSZ-reduction formula of an ordinary QFT, where scattering amplitudes are also expressed as integrated vacuum expectation values of local functions of field operators. However, there are two crucial differences, between scattering amplitudes in an ordinary QFT and in string theory: First and foremost, it can be proved that the vertex operators must be, in fact, on -shell quantities; this is to be expected, as the vertex operators express the external, asymptotic string states of the scattering amplitude, which are, by definition, on-shell quantum states. On the contrary, in the LSZ-reduction formula of an ordinary QFT, scattering amplitudes are expressed as integrated vacuum expectation values of local functions of off-shell field operators; these off-shell field operators do not, strictly, express the external states of the scattering amplitude, but they merely serve as a mathematical convenience. Secondly, in string theory, there can obviously be no additional, gauge invariant interaction terms in the Polyakov action, as the Polyakov action is, in fact, the most general physically acceptable and gauge invariant action, regarding its gauge symmetry group, of course. Then, an interaction between vertex operators cannot be defined via an interaction Lagrangian; in string theory, vertex operators are considered to interact with each other exclusively by coupling to the world-sheet. Each vertex operator can couple to the world-sheet only via the creation of a respective local puncture on the world-sheet. This way of coupling of vertex operators to the world-sheet has the effect of the contribution of a positive unit to the original genus of the world-sheet, so string theory must necessarily come with a totally natural definition of the string coupling constant in terms of the genus of the world-sheet; the precise definition and form of the string coupling constant will be derived shortly. On the contrary, in an ordinary QFT, the interaction between functions of field operators is defined via interaction terms in the action of the theory, which constitute the interaction Lagrangian; then, the coupling constant is just a free parameter in this interaction Lagrangian.

In principle, now, a scattering amplitude has a physical meaning and interpretation only if it is a gauge invariant quantity; evidently, this requirement imposes severe constraints on the functional form of the vertex operators, $V^{(n)}(p;X(z,\bar{z}))$, for the case of such physical scattering amplitudes, as this requirement is, in general, not satisfied, for the generic scattering amplitude of *eq*.(2.79). It follows that, in a physical scattering amplitude, the integrated vertex operators, which can be termed as $V^{(n)}[X](p)$, must all be conformally invariant operators. Equivalently, the vertex operators, $V^{(n)}(p;X(z,\bar{z}))$, must all be primary fields of conformal weights $(1,1)$; these vertex operators are called physical vertex operators. Undeniably, then, the requirement of conformal invariance of the physical scattering amplitudes means precisely that the physical spectrum of the theory must be identified, via the state-operator correspondence, to be the spectrum of the physical vertex operators. In fact, it can be easily seen that this definition of the physical spectrum of the theory also ensures that the Virasoro constraints are satisfied automatically, for any and all physical string states; this follows from the fact that the vacuum expectation value of the product of the energy-momentum tensor with primary fields always vanishes, as long as the Weyl anomaly of the theory is be cancelled, which will be the case, as we shall show in the next section. It can be easily proved, now, that the simplest physical vertex operators are the following:

$$V^{(0)}(p;X(z,\bar{z})) = g_s e^{ip_\mu X^\mu(z,\bar{z})}, \tag{2.80}$$

with the additional requirement $p^2 = -m^2 = \frac{4}{\alpha'}$ satisfied, and:

$$V^{(1)}(p;X(z,\bar{z})) = \frac{2}{\alpha'} g_s f_{\mu\nu} \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) e^{ip_\lambda X^\lambda(z,\bar{z})}, \tag{2.81}$$

with the additional requirements $p^\mu f_{\mu\nu} = p^\nu f_{\mu\nu} = 0$ and $p^2 = -m^2 = 0$ satisfied. The normalisation of the above vertex operators is justified purely by virtue of dimensional analysis, and it also contains a g_s-factor. This g_s-factor is just the aforementioned string coupling constant, which is induced from to the coupling of the string states to the world-sheet. As it has been mentioned, the coupling of string states to the world-sheet can only happen via the creation of a local puncture on the world-sheet, so, the string coupling constant, g_s, must have the effect of the contribution of a positive unit to the original genus of the world-sheet; we shall just keep this in mind for now, and postpone the precise definition and form of the string coupling constant, g_s, for the following section. Obviously, the identity operator, $\mathbb{I}$, can also be regarded as an integrated physical vertex operator, due to its trivial properties. On the one hand, now, the vertex operator of *eq*.(2.80) describes a target-space scalar string state, with negative mass-squared; this is a physical tachyonic string state, and is a complication of bosonic string theory. However, this complication can be resolved when one moves on to the superstring theory, as we shall see in later chapters. On the other hand, the traceless symmetric, the antisymmetric, and the trace parts of the vertex operator of *eq*.(2.81) describe the graviton, the Kalb-Ramond, and the dilaton string states, respectively; these states are all spacetime tensor massless states.

It would be practical, for subsequent use, to also have the definition of *eq*.(2.79), for a generic scattering amplitude, expressed in a form which is manifestly gauge invariant, that is, in a form where the integrated vertex operators are expressed in a general and arbitrary coordinate system. The integrated vertex operators, which are themselves gauge invariant, can be expressed as:

$$V^{(n)}[X,g](p) = \int_W d^2z V^{(n)}(p;X(z,\bar{z})) = \int_W d^2\sigma V^{(n)}(p;X(\sigma),g(\sigma)), \tag{2.82}$$

where the vertex operators, $V^{(n)}(p;X(\sigma),g(\sigma))$, must be of the form:

$$V^{(n)}(p;X(\sigma),g(\sigma)) = \sqrt{|\det(g(\sigma))|}v^{(n)}(p,X(\sigma)), \tag{2.83}$$

with $v^{(n)}(p,X(\sigma))$ being, of course, a diffeomorphism invariant operator, of conformal weights $(1,1)$, so that the overall gauge invariance of the integrated vertex operators, $V^{(n)}[X](p)$ is ensured. It follows that, a generic scattering amplitude of N vertex operators, $V_n(p_n, X(\sigma_n))$, can be expressed, in a manifestly gauge invariant form, as the following functional integral:

$$\langle \prod_{n\in\mathbb{N}_N} \int_W d^2\sigma_n V_n(p_n;X(\sigma_n))\rangle_g = \int_{F_{X_g}} [dX]_g e^{-S_P[X,g]} \prod_{n\in\mathbb{N}_N} \int_W d^2\sigma_n V_n(p_n;X(\sigma_n)), \tag{2.84}$$

or, equivalently, as:

$$\langle \prod_{n\in\mathbb{N}_N} V_n[X,g](p)\rangle_g = \int_{F_{X_g}} [dX]_g e^{-S_P[X,g]} \prod_{n\in\mathbb{N}_N} V_n[X,g](p), \tag{2.85}$$

more concisely, in terms of the integrated vertex operators, $V^{(n)}[X,g](p)$, as they are expressed in *eq*.(2.82).

Now, as far as gravity is concerned, it can be seen, from the vertex operator of *eq*.(2.79), that the graviton states can only have a number of $\frac{(D-2)(D-3)}{2}$ different spacetime polarisations, so, they can define a $(D-2)$-dimensional subspace of the D-dimensional spacetime, which is called the transverse spacetime, as well as its orthogonal space, of 2 dimensions, which are called the lightcone dimensions of the spacetime. This provides us with a hint, that the very existence of the target-space string vibration modes, or string excitations, in the 2 lightcone dimensions of the spacetime, might as well be a redundancy, as an artifact of the gauge invariance of the theory. This hint means that it would be possible that these string states, which correspond to the 2 lightcone dimensions of the target-space, can be excluded entirely from the analysis of the spectrum of the theory. This supposed elimination of these string states is indeed a fact that can be proved rigorously; a short, descriptive summary for this proof shall be given in a next section, where the partition function of bosonic string theory will be calculated.

2.2.4 Scattering Amplitudes II: The Polyakov Expansion of the String S-Matrix

Despite the seemingly proper definition of arbitrary, physical scattering amplitudes, that was given in the previous section, one can see that the scattering amplitudes of *eq*.(2.85) are not, in fact, fully conformally invariant quantities, but merely invariant quantities only under the restricted conformal group, even in the case of physical vertex operators; this fact is precisely due to the Weyl anomaly of the theory. The Weyl anomaly of the theory is contained in the scattering amplitudes of *eq*.(2.85), through the functional integration measure, $[dX]_g$, so the Weyl anomaly of the theory will, inevitably, emerge as a part of any Weyl transformation of these scattering amplitudes, due to the Weyl transformation properties of the functional integration measure, $[dX]_g$, as it will be explained shortly. In other words, the vacuum state of the theory itself is not fully gauge invariant, but merely invariant under the restricted conformal group. Now, in order to form a fully gauge invariant scattering amplitude, one must also integrate the scattering amplitudes of *eq*.(2.85) over the function space F_g, which consists of all the possible, different metrics, g_{g_W}, and for all the possible closed topologies, that is, values for the genus, g_W, of the closed world-sheet, W. This functional integration must, necessarily, be defined along with an appropriate functional integration measure,

$[dg]_g$, that automatically imposes the essential normalisations, that is, the multiplicative weight factors $\frac{1}{V^{gauge}_{g_W}}$, for each such possible, different world-sheet metric; the factor $V^{gauge}_{g_W}$ simply expresses the volume of the gauge symmetry group of the theory on the corresponding topology of the closed world-sheet. These multiplicative weight factors $\frac{1}{V^{gauge}_{g_W}}$ are the appropriate multiplicative weight factors that can, essentially, account for the treatment of all the various such world-sheet metrics, that can be related through a transformation of the gauge symmetry group of the theory, as gauge equivalent. Then, the fully gauge invariant quantity that is formed in this way is called the string S-matrix; it is termed as $S(\{V_n[X,g](p_n)\})$, and it is expressed as in the following:

$$S(\{V_n[X,g](p_n)\}) = \int_{F_g}[dg]_g\langle \prod_{n\in\mathbb{N}_N} V_n[X,g](p_n)\rangle_g = \int_{F_g}[dg]_g\int_{F_{X_g}}[dX]_g e^{-S_P[X,g]}\prod_{n\in\mathbb{N}_N} V_n[X,g](p_n) =$$
$$= \sum_{g_W\in\mathbb{N}}\int_{F_{g_{g_W}}}[dg]_{g_{g_W}}\frac{1}{V^{gauge}_{g_W}}\langle \prod_{n\in\mathbb{N}_N} V_n[X,g_{g_W}](p_n)\rangle_{g_{g_W}} =$$
$$= \sum_{g_W\in\mathbb{N}}\int_{F_{g_{g_W}}}[dg]_{g_{g_W}}\frac{1}{V^{gauge}_{g_W}}\int_{F_{X_{g_W}}}[dX]_{g_{g_W}} e^{-S_P[X,g_{g_W}]}\prod_{n\in\mathbb{N}_N} V_n[X,g_{g_W}](p_n), \tag{2.86}$$

where $F_{g_{g_W}}$ is, of course, the function space of all the possible, different metrics, g_{g_W}, of a closed world-sheet, W, of a topology of genus g_W. The scattering amplitudes of string theory must, then, be considered as the elements of the string S -matrix, in the above *eq.*(2.86). The functional integration measures, $[dX]_g$ and $[dg]_g$, for the integration over the function space F, of all the various metrics, g, of a closed world-sheet, W, of a topology of a specific, yet arbitrary genus, may be respectively defined from the following norms:

$$\|\delta X\|^2_g = \int_W d^2\sigma\sqrt{|\det(g)|}\delta_{\mu\nu}\delta X^\mu\delta X^\nu, \tag{2.87}$$

and:

$$\|\delta g\|^2_g = \int_W d^2\sigma\sqrt{|\det(g)|}M^{abcd}(g)\delta g_{ab}\delta g_{cd}, \tag{2.88}$$

where $M^{abcd}(g) = g^{ac}g^{bd}$ is the normalised metric on the space of the symmetric $(0,2)$-tensor fields on the 2-dimensional world-sheet, W. The norms of *eqs.*(2.87) and (2.88) can, equivalently, be expressed in the conformal gauge, where the world-sheet metric g is conformally flat, that is, where $g = e^{2\phi}\gamma$, with the metric γ being a constant, fiducial, flat world-sheet metric, as in the following:

$$\|\delta X\|^2_g = \int_W d^2\sigma\sqrt{|\det(\gamma)|}e^{2\phi}\delta_{\mu\nu}\delta X^\mu\delta X^\nu, \tag{2.89}$$

and:

$$\|\delta g\|^2_g = \int_W d^2\sigma\sqrt{|\det(\gamma)|}e^{-2\phi}M^{abcd}(\gamma)\delta g_{ab}\delta g_{cd}. \tag{2.90}$$

Now, as far as the Weyl anomaly of the theory is concerned, it is obvious that, similarly to the Polyakov action, both of the norms of the above *eqs.*(2.87) and (2.88) manifestly exhibit target-space Poincaré invariance and world-sheet diffeomorphism invariance, while, contrary to the Polyakov action, they fail the test of Weyl invariance; consequently, the respective functional integration measures, $[dX]_g$ and $[dg]_g$, are also not Weyl invariant, in general. This symmetry breaking of Weyl invariance of the functional integration measures, $[dX]_g$ and $[dg]_g$, is, essentially, the very cause of the Weyl anomaly of the theory, which has been described in a previous section. The Weyl anomaly for a CFT of a conformally flat world-sheet metric $g = e^{2\phi}\gamma$, where γ is a fiducial, flat world-sheet metric, must necessarily imply, then, the existence of a Weyl factor, $A[\phi,\gamma]$; this Weyl factor, $A[\phi,\gamma]$, is coupled to the functional integration measures, $[dX]_\gamma$ and $[dg]_\gamma$, which are defined in terms of the fiducial, flat world-sheet metric, γ, so that the functional integration measures, $[dX]_g$ and $[dg]_g$, which are defined in terms of the conformally flat world-sheet metric, $g = e^{2\phi}\gamma$, can be formed, in the definition of *eq*.(2.86), for the functional integral of the S-matrix of the CFT. Consequently, the functional integration measures, $[dX]_g$ and $[dg]_g$, can, then, be expressed as:

$$[dX]_g[dg]_g = A[\phi,\gamma][dX]_\gamma[dg]_\gamma. \tag{2.91}$$

Evidently, due to the overall gauge invariance of the S -matrix of the CFT, the Weyl factor, $A[\phi,\gamma]$, must also be precisely the factor which completely describes the Weyl dependence of the functional integration measures. Considering the universality of the form of the Weyl anomaly of a CFT, as it was expressed in *eq*.(2.78), it can be straightforwardly proved that the Weyl factor must be of the form:

$$A[\phi,\gamma] = e^{c_{tot}S_L[\phi,\gamma]}, \tag{2.92}$$

where c_{tot} is the total central charge of the CFT, that is described by the functional integral $\int_{F_g}[dg]_\gamma \int_{F_{X_g}}[dX]_\gamma e^{-S_P[X,\gamma]}$, and the functional $S_L[\phi,\gamma]$, which is equal to the following:

$$S_L[\phi,\gamma] = -\frac{1}{24\pi}\int_{W_f} d^2\sigma\,\sqrt{|\det(\gamma)|}((\partial\phi)^2 + R_W\phi), \tag{2.93}$$

has, apparently, the form of an action, for the dynamical, conformal factor, ϕ, of the conformally flat world-sheet metric $g = e^{2\phi}\gamma$, on a world-sheet, W_f, of the fiducial, flat metric, γ. This action, $S_L[\phi,\gamma]$, was named the Liouville action, by Polyakov. The Weyl factor, as it is expressed in *eq*.(2.92), completely describes, then, the Weyl transformation of the functional integration measures, $[dX]_g$ and $[dg]_g$, as in the following:

$$[dX]_\gamma[dg]_\gamma \to [dX]_g[dg]_g = [dX]_\gamma[dg]_\gamma e^{c_{tot}S_L[\phi,\gamma]}, \tag{2.94}$$

under a Weyl transformation of the form of $\gamma \to g = e^{2\phi}\gamma$, for the fiducial, flat world-sheet metric, γ. However, as the process of quantisation of the Liouville action of *eq*.(2.93), and of the succeeding calculation of a generic string S-matrix element, as it is expressed from the functional integral of *eq*.(2.86), with the Weyl factor included, are no trivial matters, we shall restrict our attention to the case of a specific class of quantum CFTs, in which, by definition, there is no Weyl anomaly. Such a quantum CFT would then be a truly conformally invariant CFT, and it would have to incorporate a natural way, so that its Weyl anomaly may be automatically cancelled. From the universal form of the Weyl anomaly of a CFT, as it was expressed in *eq*.(2.78), it is evident that the automatic cancellation of the Weyl anomaly of a CFT can happen only if the total central charge of the CFT is equal to zero, that is $c_{tot} = 0$. As it has been shown in a previous section, the total central charge of a generic CFT can depend only on the dimension, or, number, D, as well as the various conformal weights, of the fields in the CFT. Then, a CFT of total central charge that is equal to zero is said to exist in its critical dimension, $D_{crit.}$; the critical dimension, $D_{crit.}$, of a CFT be solved for, using precisely its definition, that is: $c_{tot}(D_{crit.}) = 0$. By requiring, hence, that the theory described by the functional integral $\int_{F_g}[dg]_\gamma \int_{F_{X_g}}[dX]_\gamma e^{-S_P[X,\gamma]}$, is a CFT that exists in its critical dimension, $D_{crit.}$, the total central charge of the CFT, c_{tot}, is imposed to be equal to zero; accordingly, the Weyl factor is enforced to be equal to the unit, and it effectively decouples completely from the functional integration measures, $[dX]_g$ and $[dg]_g$, in the functional integral of *eq*.(2.86), for the string S-matrix. Consequently, the theory that is described by the functional integral $\int_{F_g}[dg]_\gamma \int_{F_{X_g}}[dX]_\gamma e^{-S_P[X,\gamma]}$, is now a quantum CFT that is truly conformally invariant, as the functional integration measures, $[dX]_g$ and $[dg]_g$, are now Weyl invariant; then, the expression of *eq*.(2.86), for the string S-matrix, reduces to the following:

$$S(\{V_n[X,g](p_n)\}) = \sum_{g_W\in\mathbb{N}} \int_{F_{g_{gW}}} [dg]_{\gamma_{g_W}} \frac{1}{V^{gauge}_{g_W}} \langle \prod_{n\in\mathbb{N}_N} V_n[X,\gamma_{g_W}](p_n)\rangle_{\gamma_{g_W}} =$$

$$= \sum_{g_W\in\mathbb{N}} \int_{F_{g_{gW}}} [dg]_{\gamma_{g_W}} \frac{1}{V^{gauge}_{g_W}} \int_{F_{X_{g_W}}} [dX]_{\gamma_{g_W}} e^{-S_P[X,\gamma_{g_W}]} \prod_{n\in\mathbb{N}_N} V_n[X,\gamma_{g_W}](p_n). \tag{2.95}$$

What remains to be calculated, in the above expression of *eq.*(2.95), for the string S-matrix, is the effect of the functional integrations over the different function spaces, $F_{g_{g_W}}$. In general, it is known that a metric, g_{g_W}, of a closed world-sheet, W, may have a number of μ_{g_W} real metric moduli, or Teichmüller parameters, which are denoted as $\tau_{\mu'}$, as well as a number of κ_{g_W} real Conformal Killing Vectors (CKVs), which are simply the generators of its Conformal Killing Group (CKG); these numbers, μ_{g_W} and κ_{g_W}, are related to the value of the genus, g_W, of the closed world-sheet, W, via a lemma of the Riemann -Roch theorem:

$$\mu_{g_W} - \kappa_{g_W} = 6g_W - 6, \tag{2.96}$$

while, furthermore, μ_{g_W} vanishes for values $g_W < 1$ of the genus, and κ_{g_W} vanishes for values $g_W > 1$ of the genus. As a reminder, the CKG of a metric, g_{g_W}, is defined as the subgroup of the diffeomorphism group on the closed world-sheet W, which relates all of its conformally equivalent metrics; that is, the CKG has the same effect that the Weyl group has, on the metric, g_{g_W}, of the closed world-sheet, W. Hence, an arbitrary metric, of a closed world-sheet, W, of a topology of a specific genus, g_W, shall be denoted as $g_{g_W}(\tau)$, where (τ) is an arbitrary element of the space of its metric moduli, which may be, in general, the space $\mathbb{R}^{\mu_{g_W}}$; that is, $(\tau) \in \mathbb{R}^{\mu_{g_W}}$. In addition, it is also known that the set of all the various metrics, $\{g_{g_W}(\tau)\}$, is the disjoint union of the equivalence classes $[g_{g_W}(\tau)]$, each of which consists of world-sheet metrics that can be related to each other via a diffeomorphism on the closed world-sheet W; such world-sheet metrics are called equivalent to each other. Now, a generic diffeomorphism on a closed world-sheet W typically results in the transformation of the world-sheet metric, $g_{g_W}(\tau)$, yet its moduli element, (τ), typically remains invariant under any such diffeomorphism, with the exception of the so-called large diffeomorphisms; the large diffeomorphisms on a closed world-sheet W, act on the world-sheet metric, $g_{g_W}(\tau)$, and have, by definition, the property of keeping its functional form invariant, while transforming only the value of its moduli element, (τ). That is, the world-sheet metric, $g_{g_W}(\tau)$, is simply transformed as $g_{g_W}(\tau) \to g'(\tau) = g_{g_W}(\tau')$, under a large diffeomorphism. It is evident, then, that the large diffeomorphisms on a closed world-sheet W form a subgroup of the diffeomorphism group on the closed world-sheet W, which is simply called the large diffeomorphism subgroup, or, the Teichmüller modular group, or, just the modular group, M_{g_W}, of the closed world-sheet, W, of a topology of genus g_W. The name of the modular group can obviously be justified from the fact that the large diffeomorphisms on the closed world-sheet, W, can be considered as acting only on the moduli element, (τ), of the respective metric, $g_{g_W}(\tau)$, as they keep the functional form of the metric invariant. Evidently, the modular group is a discrete subgroup of the respective diffeomorphism group. In fact, the modular group constitutes the disconnected subroup of the respective diffeomorphism group; by denoting the respective diffeomorphism group and its connected subgroup as D_{g_W} and $D^c_{g_W}$ respectively, we can see that the modular group, M_{g_W}, is equal to their relative quotient group, that is, $M_{g_W} = D_{g_W}/D^c_{g_W}$. Then, it can be deduced that there are values of the moduli element, (τ), that can be related to each other via a transformation element of the modular group; such moduli elements, (τ), can form their respective equivalence class, $[(\tau)]$, and, so, they are called equivalent to each other, while moduli elements that belong to different equivalence classes, cannot be related to each other via a transformation element of the modular group, and so they are called inequivalent to each other. Accordingly, the space of the metric moduli, $\mathbb{R}^{\mu_{g_W}}$, is simply the disjoint union of these equivalence classes, $[(\tau)]$, and the quotient set $F_{g_W} = \mathbb{R}^{\mu_{g_W}}/M_{g_W}$, which is called the fundamental domain, F_{g_W}, of the space of the metric moduli, $\mathbb{R}^{\mu_{g_W}}$, is the set that consists of these equivalence classes, $[(\tau)]$. Notice how the very definition of the fundamental domain is invariant, under the action of the respective modular group, so the fundamental domain, F_{g_W}, due to its definition as a quotient set, cannot have a unique representation, as a subset of the space of the metric moduli, $\mathbb{R}^{\mu_{g_W}}$; naturally, all the possible, different such representations of the fundamental domain, F_{g_W}, must be physically equivalent. As a result of all the above, it can be now shown, via a subsequent implementation of a standard Faddeev -Popov procedure, that the functional integration over the function space $F_{g_{g_W}}$ can be expressed in an elegant way, with the use of the respective Faddeev-Popov determinant,

$\Delta_{FP}(\gamma_{g_W}(\tau))$, which can be defined as in the following:

$$\int_{F_{g_{g_W}}} [dg(\tau)]_{\gamma_{g_W}(\tau)} \frac{1}{V_{g_W}^{gauge}} = \int_{F_{g_W}} d^{\mu_{g_W}}\tau \Delta_{FP}(\gamma_{g_W}(\tau)), \tag{2.97}$$

where it is noted that the respective modular integration is over all the moduli elements, (τ), that are inequivalent to each other, that is, over the corresponding fundamental domain, F_{g_W}. It is also noted, for subsequent use, that, the integration measure, $d^{\mu_{g_W}}\tau$, can, without loss of generality, be set to be invariant, under the action of the respective modular group, along with an appropriate rescaling, or redefinition, of the Faddeev-Popov determinant. Then, using *eq*.(2.97), the expression of *eq*.(2.95), for the string S-matrix, reduces to the following:

$$S(\{V_n[X,g](p_n)\}) = \sum_{g_W \in \mathbb{N}} \int_{F_{g_W}} d^{\mu_{g_W}}\tau \Delta_{FP}(\gamma_{g_W}(\tau)) \langle \prod_{n \in \mathbb{N}_N} V_n[X, \gamma_{g_W}(\tau)](p_n) \rangle_{\gamma_{g_W}(\tau)} =$$

$$= \sum_{g_W \in \mathbb{N}} \int_{F_{g_W}} d^{\mu_{g_W}}\tau \Delta_{FP}(\gamma_{g_W}(\tau)) \int_{F_{X_{g_W}}} [dX]_{\gamma_{g_W}(\tau)} e^{-S_P[X,\gamma_{g_W}(\tau)]} \prod_{n \in \mathbb{N}_N} V_n[X, \gamma_{g_W}(\tau)](p_n). \tag{2.98}$$

Furthermore, it can be proved that each of the Faddeev-Popov determinants, $\Delta_{FP}(\gamma_{g_W}(\tau))$, can be expressed in a simple way, by the following Berezin functional integral:

$$\Delta_{FP}(\gamma_{g_W}(\tau)) = \frac{1}{V_{g_W}^{CKG}(\tau)} \int_{F_{b_{g_W}}} [db]_{\gamma_{g_W}(\tau)} \int_{F_{c_{g_W}}} [dc]_{\gamma_{g_W}(\tau)} G[b, c, \gamma_{g_W}(\tau)] e^{-S_{gh}[b,c,\gamma_{g_W}(\tau)]}, \tag{2.99}$$

where the factor $V_{g_W}^{CKG}(\tau)$ expresses the volume of the respective CKG, of course. The functional $G[b, c, \gamma_{g_W}(\tau)]$ is proved to be equal to:

$$G[b, c, \gamma_{g_W}(\tau)] = \prod_{\mu' \in \mathbb{N}_{\mu_{g_W}}} \frac{1}{4\pi} \int_W d^2\sigma \sqrt{|\det\big(\gamma_{g_W}(\tau)\big)|} b_{mn} \gamma_{g_W}^{mk}(\tau) \gamma_{g_W}^{nl}(\tau) \frac{\partial \gamma_{g_W}^{kl}(\tau)}{\partial \tau^{\mu'}}$$

$$\prod_{\kappa' \in \mathbb{N}_{\frac{\kappa_{g_W}}{2}}} \frac{1}{V_W} \int_W d^2\sigma_{\kappa'} \frac{1}{2} \epsilon_{nm} c^n(\sigma_{\kappa'}) c^m(\sigma_{\kappa'}), \tag{2.100}$$

where, of course, the factor V_W expresses the volume of the world-sheet, and the functional $S_{gh}[b, c, \gamma_{g_W}(\tau)]$ is called the ghost action, or, simply, the *b*-*c* system, and it is equal to the following:

$$S_{gh}[b, c, \gamma_{g_W}(\tau)] = \frac{1}{2\pi} \int_W d^2\sigma \sqrt{|\det\big(\gamma_{g_W}(\tau)\big)|} \gamma_{g_W}^{km}(\tau) b_{mn} \nabla_k(\tau) c^n, \tag{2.101}$$

with the ghost field b being symmetric and traceless, by its definition, while $F_{b_{g_W}}$ and $F_{c_{g_W}}$ are the function spaces for the, Grassmann-valued, that is, anti-commuting, ghost fields, b and c, respectively. Eventually, it can be easily proved, in a fashion similar to the procedure that was used for the case of the Polyakov action itself, that the *b*-*c* system describes a CFT of central charges $c_{gh} = \tilde{c}_{gh} = -26$. The total central charge of the CFT described by the functional integral $\int_{F_{g_{g_W}}} [dg]_{\gamma_{g_W}(\tau)} \int_{F_{X_{g_W}}} [dX]_{\gamma_{g_W}(\tau)} e^{-S_P[X,\gamma_{g_W}(\tau)]}$ is, then, $c_{tot}(D) = D - 26$, and the condition so that this CFT exists in the critical dimension, $D_{crit.}$, is simply $c_{tot}(D_{crit.}) = D_{crit.} - 26 = 0$, that is, $D_{crit.} = 26$; it can be easily seen, now, that this condition is, essentially, a constraint on the number D of the spacetime dimensions. In other words, it has been shown that the Weyl anomaly of the CFT of the Polyakov action always persists, but in the special case of the critical target-space dimension, $D = 26$, as it has been mentioned previously in this chapter. Additionally, by considering the action redefinition $S[X, \gamma_{g_W}(\tau)] = S_P[X, \gamma_{g_W}(\tau)] + \lambda\langle\Phi\rangle\chi_W$, as well as the naturally motivated definition $g_s = e^{-\lambda\langle\Phi\rangle}$, for the string coupling constant, g_s, along with the reminder that, for the closed world-sheet, W, the Euler characteristic, χ_W, is related to the genus, g_W, by the Gauss-Bonnet theorem for closed surfaces, $\chi_W = 2 - 2g_W$, it can be concluded that the expression of *eq*.(2.98), for the string S-matrix, is equal to the following power-series expansion, with respect to the string coupling constant, g_s:

$$S(\{V_n[X,g](p_n)\}) = \sum_{g_W \in \mathbb{N}} g_s^{2g_W - 2} S(\{V_n[X,g](p)\})_{g_W} =$$
$$= \sum_{g_W \in \mathbb{N}} g_s^{2g_W-2} \int_{F_{g_W}} d^{\mu_{g_W}}\tau \frac{1}{V_{g_W}^{CKG}(\tau)} \int_{F_{b_{g_W}}} [db]_{\gamma_{g_W}(\tau)} \int_{F_{c_{g_W}}} [dc]_{\gamma_{g_W}(\tau)} G[b,c,\gamma_{g_W}(\tau)] e^{-S_{gh}[b,c,\gamma_{g_W}(\tau)]}$$
$$\langle \prod_{n \in \mathbb{N}_N} V_n[X,\gamma_{g_W}(\tau)](p_n)\rangle_{\gamma_{g_W}(\tau)} =$$
$$= \sum_{g_W \in \mathbb{N}} g_s^{2g_W-2} \int_{F_{g_W}} d^{\mu_{g_W}}\tau \frac{1}{V_{g_W}^{CKG}(\tau)} \int_{F_{b_{g_W}}} [db]_{\gamma_{g_W}(\tau)} \int_{F_{c_{g_W}}} [dc]_{\gamma_{g_W}(\tau)} G[b,c,\gamma_{g_W}(\tau)] e^{-S_{gh}[b,c,\gamma_{g_W}(\tau)]}$$
$$\int_{F_{X_{g_W}}} [dX]_{\gamma_{g_W}(\tau)} e^{-S[X,\gamma_{g_W}(\tau)]} \prod_{n \in \mathbb{N}} V_n[X,\gamma_{g_W}(\tau)](p_n). \tag{2.102}$$

We note that, by this well-reasoned definition of the the string coupling constant, g_s, as $g_s = e^{-\lambda\langle\Phi\rangle}$, it is ensured that any additional factor of the string coupling constant, in each term of the power-series expression of *eq*.(2.102), for the string S -matrix, which may emerge exclusively through the integrated vertex operators, $V_n[X,\gamma_{g_W}(\tau)](p_n)$, simply results in the effective contribution of a positive unit to the original genus, g_W, of the world-sheet, W. It can be seen then, that, the coupling of each vertex operator to the closed world-sheet, which can only happen via the creation of a local puncture on the world-sheet, has indeed the effect of the contribution of a positive unit to the original genus of the closed world-sheet, as it has been mentioned in the previous section. By taking, now, the small g_s-limit of the power-series expression of *eq*.(2.102), for the string S -matrix, we can define the Polyakov expansion of the string S -matrix; that is, the Polyakov expansion is defined as the perturbative expansion of the string S-matrix, with respect to the string coupling constant, g_s, so, the Polyakov epansion is, naturally, the cornerstone of the perturbation theory of the bosonic string. Of course, the value of the genus of a closed world-sheet, which is simply the number of punctures on the closed world-sheet, can, in essence, be equivalently considered as the number of loops of the world-sheet. Accordingly, the term that corresponds to a value $g_W = N$, for the genus, g_W, of a closed world-sheet, W, in the Polyakov expansion of the string S-matrix, and by which the S-matrix of a string theory, in a world-sheet of a topology of genus N, is, essentially, defined, can equivalently be considered as the term by which the S-matrix of a respective, ordinary QFT, in the N-loop level, is described; this simple reasoning defines the correspondence between the S-matrix of a string theory, in a world-sheet of a topology of genus N, and the S-matrix, in the N-loop level, of a respective, ordinary QFT. In particular, the scattering amplitudes of string theory, in the world-sheet topology of genus 0, which is the world-sheet topology of the Riemann sphere, S^2, correspond to the scattering amplitudes of a respective, ordinary QFT, in the zero-loop, "tree"-level, while, the scattering amplitudes of string theory, in the world-sheet topology of genus 1, which is the world-sheet topology of the Riemann torus, T^2, correspond to the scattering amplitudes of a respective, ordinary QFT, in one-loop level, and so on and so forth. It can be seen, then, that the sum of all the, factorially growing in number, with the increase of the loop level, different Feynman diagrams, with a number of N loops, which defines the S-matrix, in the N-loop level, of an ordinary QFT, corresponds to, and is expressed as, a single term, in the Polyakov expansion of the string S-matrix. This fact constitutes, then, a formal, as well as a computational, advantage of the S-matrix of a string theory, in comparison to the S-matrix of a respective, ordinary QFT, in each loop level, as the need for the calculation of all the respective different, and factorially growing in number with the increase of the loop level, Feynman diagrams, the sum of which defines the S-matrix in the latter case, is eliminated, in the calculation of the string S-matrix, of the former case, precisely due to its form. As a result, it appears that the overall calculation of the string S-matrix, via the Polyakov expansion, is significantly simpler, than the overall calculation of the S-matrix, in a respective, ordinary QFT.

Modular Invariance and Unitarity

Taking a closer look at the power-series expression of *eq*.(2.102), for the string S-matrix, we can deduce, for subsequent use, two important facts, for the terms $S(\{V_n[X,g](p_n)\})_{g_W}$ of the string S -matrix, each of which defines the S-matrix of a string theory in a world-sheet of a topology of genus g_W. Such a term, $S(\{V_n[X,g](p_n)\})_{g_W}$, of the string S-matrix, for a specific value g_W of the genus of a closed world-sheet, W, can, evidently, be expressed as:

$$\begin{aligned}
&S(\{V_n[X,g](p_n)\})_{g_W} = \\
&= \int_{F_{g_W}} d^{\mu_{g_W}}\tau \frac{1}{V^{CKG}_{g_W}(\tau)} \int_{F_{b_{g_W}}} [db]_{\gamma_{g_W}(\tau)} \int_{F_{c_{g_W}}} [dc]_{\gamma_{g_W}(\tau)} G[b,c,\gamma_{g_W}(\tau)] e^{-S_{gh}[b,c,\gamma_{g_W}(\tau)]} \\
&\langle \prod_{n\in\mathbb{N}_N} V_n[X,\gamma_{g_W}(\tau)](p_n)\rangle_{\gamma_{g_W}(\tau)} = \\
&= \int_{F_{g_W}} d^{\mu_{g_W}}\tau \frac{1}{V^{CKG}_{g_W}(\tau)} \int_{F_{b_{g_W}}} [db]_{\gamma_{g_W}(\tau)} \int_{F_{c_{g_W}}} [dc]_{\gamma_{g_W}(\tau)} G[b,c,\gamma_{g_W}(\tau)] e^{-S_{gh}[b,c,\gamma_{g_W}(\tau)]} \\
&\int_{F_{X_{g_W}}} [dX]_{\gamma_{g_W}(\tau)} e^{-S[X,\gamma_{g_W}(\tau)]} \prod_{n\in\mathbb{N}} V_n[X,\gamma_{g_W}(\tau)](p_n) = \\
&= \int_{F_{g_W}} d^{\mu_{g_W}}\tau z_{g_W}(\tau; \prod_{n\in\mathbb{N}} V_n[X,\gamma_{g_W}(\tau)](p_n)),
\end{aligned} \tag{2.103}$$

in accordance with *eq*.(2.102). First of all, it is noted that the form of the functional $G[b,c,\gamma_{g_W}(\tau)]$, of *eq*.(2.100), implies that the symmetry of the respective CKG can be used in such a way, so that the world-sheet position arguments of its number of κ_{g_W} ghost fields, c, are fixed to be the same as the world-sheet position arguments of a number of $\frac{\kappa_{g_W}}{2}$ vertex operators, which may exist, in the above term $S(\{V_n[X,g](p_n)\})_{g_W}$ of the string S-matrix. However, the multiplicative factor of $\frac{1}{V^{CKG}_{g_W}(\tau)}$, which also appears in the above *eq*.(2.103), for the term $S(\{V_n[X,g](p_n)\})_{g_W}$ of the string S-matrix, and, which expresses the volume of the respective CKG, imposes automatically a formal constraint, on the value of this term, regarding the total number of its integrated vertex operators, in the specific case where the volume of the respective CKG tends to infinity. In particular, it can be easily seen that the terms $S(\{V_n[X,g](p_n)\})_{g_W}$ of the string S-matrix, of a total number of N integrated vertex operators, that is less than the number $\frac{\kappa_{g_W}}{2}$, formally vanish, in the case where the volume of the respective CKG tends to infinity, precisely on account of suppression due to the infinite volume of the respective CKG. Secondly, the integration measure, $d^{\mu_{g_W}}\tau$, in the above expression of *eq*.(2.103), for the term $S(\{V_n[X,g](p_n)\})_{g_W}$ of the string S -matrix, can, without loss of generality, be simply set to be invariant, under the action of the respective modular group, M_{g_W}, along with an appropriate redefinition, or rescaling, of the Faddeev-Popov determinant, as it has been formerly mentioned. In addition, it is a fact that each term $S(\{V_n[X,g](p_n)\})_{g_W}$ of the string S-matrix is also invariant, under the action of the respective modular group, M_{g_W}, for any one of the, naturally equivalent, different representations of the fundamental domain, F_{g_W}, as a consequence of the overall gauge invariance of this term. It follows that, the, appropriately redefined, in accordance with the respective redefinition, or rescaling, of the Faddeev-Popov determinant, quantity $z_{g_W}(\tau; \prod_{n\in\mathbb{N}} V_n[X,\gamma_{g_W}(\tau)](p_n))$, which was defined in *eq*.(2.103), must also be invariant, under the action of the respective modular group, M_{g_W}, for any product $\prod_{n\in\mathbb{N}} V_n[X,\gamma_{g_W}(\tau)](p_n)$, of integrated vertex operators, $V_n[X,\gamma_{g_W}(\tau)](p_n)$. It is clear, from the aforementioned argument, that this modular invariance of the quantity $z_{g_W}(\tau; \prod_{n\in\mathbb{N}} V_n[X,\gamma_{g_W}(\tau)](p_n))$ is simply the consequence of the overall gauge invariance of the S-matrix of a string theory, in a closed world-sheet, W, of a topology of genus g_W, that is, in turn, the consequence of the overall gauge invariance of the string S-matrix. The modular group, M_{g_W}, is the symmetry group of the respective quantity, $z_{g_W}(\tau; \prod_{n\in\mathbb{N}} V_n[X,\gamma_{g_W}(\tau)](p_n))$, as a remnant result of the overall gauge invariance of the string S-matrix, so, the modular group, M_{g_W}, can, then, be considered as the remnant symmetry group of the respective term, $S(\{V_n[X,g](p_n)\})_{g_W}$, of the string S-matrix, that, essentially, characterises the the S-matrix of a string theory, in a closed world-sheet, W, of a topology of genus g_W. Eventually, it can be proved that, the overall gauge invariance of the string S-matrix, is a necessary, and sufficient condition, for the unitarity of the string S-matrix. Consequently, the modular invariance of the quantities $z_{g_W}(\tau; \prod_{n\in\mathbb{N}} V_n[X, \gamma_{g_W}(\tau)](p_n))$, which are contained in the respective terms, $S(\{V_n[X,g](p_n)\})_{g_W}$, of the expression of *eq*.(2.102), for the string S-matrix, is, then, a necessary condition for the unitarity of the string S-matrix; it can also be proved that this modular invariance also determines the very finiteness of the string S-matrix, as well as being essential for the cancellation of various spacetime anomalies of string theory.

2.2.5 The Perturbative Vacuum Amplitude and the Partition Function of Bosonic String Theory

We are, now, in the position to further proceed to the analysis of the string S-matrix, with the calculation of the perturbative vacuum amplitude of bosonic string theory. The calculation of the perturbative vacuum amplitude is, of course, nothing more than the calculation of the terms of the various orders, with respect to the string coupling constant, g_s, in the power-series of the Polyakov expansion, with no insertions of vertex operators. The tree-level vacuum amplitude of bosonic string theory, which, obviously, corresponds to the term of the value $g_W = 0$, for the genus of the world-sheet, W, of the power-series of the Polyakov expansion, with no insertions of vertex operators, is formally equal to zero, as it automatically vanishes, due to the suppression induced by the infinite volume of the CKG of the metrics of a world-sheet of the topology of the Riemann sphere, S^2, as it has already been mentioned in the previous section. For the very same reason, all the tree-level scattering amplitudes, of a number N of vertex operators, are, also, formally, equal to zero, when this number N is less than 3, that is, half the number of the 6 real CKVs of the metrics of a world-sheet of the topology of the Riemann sphere, S^2; on these grounds, it can be concluded that all the tree-level scattering amplitudes of 1 and 2 vertex operators automatically vanish. We may proceed, then, to the calculation of the one-loop level vacuum amplitude of bosonic string theory. The one-loop level vacuum amplitude of bosonic string theory corresponds, obviously, to the term of the value $g_W = 1$, for the genus of the world-sheet, W, of the power-series of the Polyakov expansion, with no insertions of vertex operators. The one-loop level vacuum amplitude is the simplest (non-zero) perturbative quantum correction to the tree-level vacuum amplitude, and it can be proved that it also is the simplest case of a (non-zero) perturbative quantum correction, in bosonic string theory; accordingly, the field-theoretic interpretation of the one-loop vacuum amplitude, is that it constitutes a representation of the simplest quantum correction to the vacuum energy of the corresponding effective field theory level of bosonic string theory. This fact means precisely that the original vacuum of bosonic string theory is unstable; this result is already known, due to the presence of physical tachyonic states in the spectrum of bosonic string theory, as it has been mentioned in a previous section. However, as we shall see in the next chapters, tachyonic states are not present in the physical spectrum of superstring theory: there, a non-zero value for the respective one-loop vacuum amplitude, would mean the destabilisation of the vacuum state, and it would constitute an instance of the cosmological constant problem; it can be proved, eventually, that the cosmological constant problem is, in general, as serious in string theory, as it is in ordinary QFTs [2]. The calculation of the one-loop level vacuum amplitude is considered to be additionally useful because, apart from providing the vacuum energy of the theory, it also contains, in particular, the partition function of the theory; this fact can be easily deduced, from the very definition of the partition function itself, which can be naturally expressed as an appropriate trace over the entire Fock space of the theory, and so, equivalently, as a corresponding functional integral, which ought to be defined on the genus-one topology of the world-sheet torus [1]. The one-loop vacuum amplitude contains and encodes, then, all the information about the spectrum of the theory; however, it does not contain information about the various interactions between the states of the spectrum of the theory, as this information is, similarly, contained and encoded in the respective two-loop level vacuum amplitude. An attempt at the calculation of the two-loop level vacuum amplitude of bosonic string theory is beyond the purposes of this report.

As it has already been mentioned before, the one-loop level vacuum amplitude of bosonic string theory, corresponds, obviously, to the term of the value $g_W = 1$, for the genus of the world-sheet, W, of the power-series of the Polyakov expansion, with no insertions of vertex operators. So, in accordance with *eq.*(2.102) of the previous section, the one-loop level vacuum amplitude of bosonic string theory is equal to the following functional integral:

$$S(\{\emptyset\})_1 = \int_{F_{T^2}} d\tau_1 d\tau_2 \frac{1}{V^{CKG}_{T^2}(\tau)} \int_{F_{b_{T^2}}} [db]_{\gamma_{T^2}(\tau)} \int_{F_{c_{T^2}}} [dc]_{\gamma_{T^2}(\tau)} G[b, c, \gamma_{T^2}(\tau)] e^{-S_{gh}[b,c,\gamma_{T^2}(\tau)]}$$

$$\int_{F_{X_{T^2}}} [dX]_{\gamma_{T^2}(\tau)} e^{-S[X,\gamma_{T^2}(\tau)]}, \tag{2.104}$$

on a world-sheet of the topology of the Riemann torus, T^2. From the Riemann-Roch theorem, it is concluded that, the metric of the Riemann torus, T^2, has two real moduli, τ_1 and τ_2, where $\tau_2 > 0$; these two real moduli can be encoded, without loss of generality, into one complex modulus, which is denoted as τ, in the above *eq.*(2.104), where τ is equal to $\tau = \tau_1 + i\tau_2$, and $\tau \in \mathbb{H}$, with the space $\mathbb{H} = \{\tau = \tau_1 + i\tau_2 \in \mathbb{C} | \tau_1 \in \mathbb{R}; \tau_2 > 0\}$, being, of course, the so-called upper-half plane. We shall, principally, consider the following functional integral:

$$\int_{F_{X_{T^2}}} [dX]_{\gamma_{T^2}(\tau)} e^{-S[X,\gamma_{T^2}(\tau)]}, \tag{2.105}$$

and work, for convenience, in the conformal gauge, with the real coordinate system (σ^1, σ^2), which satisfies $\sigma^{1,2} \in [0, 2\pi]$. In this coordinate system, (σ^1, σ^2), the topology of the world-sheet torus, T^2, can be defined by the following boundary conditions:

$$X^\mu(\sigma^1, \sigma^2) = X^\mu(\sigma^1 + 2\pi, \sigma^2), \tag{2.106}$$

and:

$$X^\mu(\sigma^1, \sigma^2) = X^\mu(\sigma^1, \sigma^2 + 2\pi), \tag{2.107}$$

where, the former boundary condition refers to the nature and properties of the closed string coordinates, X^μ, as it is the very definition of a closed string X, while, the latter boundary condition refers to the topology of the Riemann torus, T^2, itself. From its definition, the length element, ds_{T^2}, on a Riemann torus, T^2, of complex modulus τ, is equal to $ds_{T^2}(\tau) = \frac{1}{2\pi}|d\sigma^1 + \tau d\sigma^2|$, so, its modular group, M_{T^2}, is the group $M_{T^2} = PSL(2;\mathbb{Z})$. Consequently, the fundamental domain, F_{T^2}, of the space $\mathbb{H}$, of its complex modulus, τ, is the quotient set $F_{T^2} = \mathbb{H}/M_{T^2} = \mathbb{H}/PSL(2;\mathbb{Z})$; the conventional, and most convenient, representation, of this fundamental domain, $F_{T^2} = \mathbb{H}/PSL(2;\mathbb{Z})$, is the region $\mathbb{F}_{T^2} = \{\tau \in \mathbb{H} | \operatorname{Re}\{\tau\} \in (-\frac{1}{2}, +\frac{1}{2}); |\tau| > 1\}$, of the upper -half plane, $\mathbb{H} = \{\tau = \tau_1 + i\tau_2 \in \mathbb{C} | \tau_1 \in \mathbb{R}; \tau_2 > 0\}$. In addition, $\det(\gamma_{T^2}(\tau)) = \frac{1}{4\pi^2}\tau_2^2$, and the area element, $dA_{T^2}(\tau)$, on the Riemann torus, T^2, of complex modulus τ, is equal to $dA_{T^2}(\tau) = d^2\sigma \sqrt{|\det(\gamma_{T^2}(\tau)|)} = d^2\sigma \frac{1}{4\pi^2}\tau_2$, so, the respective volume, $V_{T^2}(\tau)$, of the torus, is, then, equal to $V_{T^2}(\tau) = \int_{T^2} d^2\sigma \sqrt{|\det(\gamma_{T^2}(\tau)|)} = \tau_2$. Alternatively, an appropriate, global Weyl transformation, can yield a conformally equivalent, yet normalised length element, $ds^{(N)}_{T^2}(\tau)$, on the Riemann torus, T^2, of complex modulus τ, which is defined to be equal to $ds^{(N)}_{T^2}(\tau) = \frac{1}{\sqrt{\tau_2}}|d\sigma^1 + \tau d\sigma^2|$, so that, now, $\det\left(\gamma^{(N)}_{T^2}(\tau)\right) = 1$, and the respective normalised area element, $dA^{(N)}_{T^2}(\tau)$, is simply equal to $dA^{(N)}_{T^2}(\tau) = d^2\sigma \sqrt{|\det\left(\gamma^{(N)}_{T^2}(\tau)|\right)} = d^2\sigma$; the respective normalised volume, $V^{(N)}_{T^2}(\tau)$, is, then, equal to $V^{(N)}_{T^2}(\tau) = \int_{T^2} d^2\sigma \sqrt{|\det\left(\gamma^{(N)}_{T^2}(\tau)|\right)} = 4\pi^2$. Accordingly, the normalised Laplacian, $\nabla^2_{T^2(N)}(\tau)$, on a Riemann torus, T^2, of complex modulus τ, is equal to the following:

$$\nabla^2_{T^2(N)}(\tau) = \frac{1}{\tau_2}|\tau\partial_1 - \partial_2|^2, \tag{2.108}$$

and, the action $S[X, \gamma^{(N)}_{T^2}(\tau)]$, on the Riemann torus, T^2, of complex modulus τ, can be expressed as in the following:

$$S[X, \gamma_{T^2}(\tau)] = S[X, \gamma^{(N)}_{T^2}(\tau)] = -\frac{1}{4\pi\alpha\prime} \int_{T^2} d^2\sigma \sqrt{|\det\left(\gamma^{(N)}_{T^2}(\tau)|\right)} \gamma^{(N)ab}_{T^2}(\tau)\delta_{\mu\nu}\partial_a X^\mu \partial_a X^\nu =$$

$$= \frac{1}{4\pi\alpha\prime} \int_{T^2} d^2\sigma \frac{1}{\tau_2} \delta_{\mu\nu} X^\mu \nabla^2_{T^2(N)}(\tau) X^\nu. \tag{2.109}$$

Consequently, the functional integral of *eq*.(2.105) is equal to:

$$\int_{F_{X_{T^2}}} [dX]_{\gamma_{T^2}(\tau)} e^{-S[X,\gamma_{T^2}(\tau)]} = \int_{F_{X_{T^2}}} [dX]_{\gamma^{(N)}_{T^2}(\tau)} e^{-S[X,\gamma^{(N)}_{T^2}(\tau)]} =$$

$$= \int_{F_X} [dX]_{\gamma^{(N)}_{T^2}(\tau)} \delta_{\sigma^2}[X(2\pi) - X(0)] \delta_{\sigma^1}[X(2\pi) - X(0)] e^{-S[X,\gamma^{(N)}_{T^2}(\tau)]} =$$

$$= \int_{F_X} [dX]_{\gamma^{(N)}_{T^2}(\tau)} \delta_{\sigma^2}[X(2\pi) - X(0)] \delta_{\sigma^1}[X(2\pi) - X(0)] e^{-\frac{1}{4\pi\alpha\prime} \int_{T^2} d^2\sigma \frac{1}{\tau_2} \delta_{\mu\nu} X^\mu \nabla^2_{T^2(N)}(\tau) X^\nu} =$$

$$= V_{26} (\det_{T^2}{}'(\frac{V^{(N)}_{T^2}(\tau)}{\pi} \frac{1}{4\pi\alpha'} \nabla^2_{T^2(N)}(\tau)))^{-13}, \tag{2.110}$$

where, V_{26} is the volume of the 26-dimensional target-space, and, $V^{(N)}_{T^2}(\tau)$ is the normalised volume of the world-sheet torus, T^2, that was given above, as $V^{(N)}_{T^2}(\tau) = 4\pi^2$; of course, in the above functional integral of *eq*.(2.110), the very topology of the world-sheet torus, T^2, in the function space $F_{X_{T^2}}$, was obviously expressed with the manifest appearance of the appropriate delta-functional factors, $\delta_{\sigma^2}[X(2\pi) - X(0)]$, and $\delta_{\sigma^1}[X(2\pi) - X(0)]$, multiplying the respective functional integration measure, $[dX]_{\gamma^{(N)}_{T^2}(\tau)}$, that correspond to the real, conformal world-sheet coordinate system, (σ^1, σ^2), of the Riemann torus, T^2. Now, the determinant $\det'_{T^2}(\frac{V^{(N)}_{T^2}(\tau)}{\pi} \frac{1}{4\pi\alpha'} \nabla^2_{T^2(N)}(\tau))$ is, of course, the regularised determinant of the operator $\frac{V^{(N)}_{T^2}(\tau)}{\pi} \frac{1}{4\pi\alpha'} \nabla^2_{T^2(N)}(\tau)$, on the torus T^2 of complex modulus τ, with the eigenvalues that correspond to its zero-mode eigenfunctions excluded, obviously; the proof that this operator is self-adjoint is trivial, and it can be easily shown that the the spectrum of the eigenfunctions of this operator consists of the following, normalised eigenfunctions:

$$x_{nm}(\sigma^1, \sigma^2) = \frac{1}{V^{(N)}_{T^2}(\tau)} e^{i(n\sigma^1 + m\sigma^2)}, \tag{2.111}$$

which correspond to their respective eigenvalues:

$$\lambda_{nm}(\tau) = \frac{1}{\tau_2} |n\tau - m|^2, \tag{2.112}$$

where, of course, $(n, m) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$, so, the determinant $\det'_{T^2}(\frac{V^{(N)}_{T^2}(\tau)}{\pi} \frac{1}{4\pi\alpha'} \nabla^2_{T^2(N)}(\tau))$ can, then, be expressed as the regularised, infinite product, of the respective eigenvalues, that is:

$$\det_{T^2}{}'(\frac{V^{(N)}_{T^2}(\tau)}{\pi} \frac{1}{4\pi\alpha'} \nabla^2_{T^2(N)}(\tau)) = \prod_{(n,m) \in \mathbb{Z}^2 \backslash \{(0,0)\}} \frac{1}{\alpha' \tau_2} |n\tau - m|^2. \tag{2.113}$$

The most natural and convenient regularisation scheme, for the calculation of the above infinite product of *eq*.(2.113) is, perhaps, the zeta-function regularisation scheme. In this scheme, the infinite products are regularised with utilising the special values of the anayltical continuation of the zeta-function, $\zeta(s)$, and its derivative, $\zeta'(s)$. We cite the special values $\zeta(-1) = -\frac{1}{12}$, $\zeta(0) = -\frac{1}{2}$, and $\zeta'(0) = -\frac{1}{2} \log(2\pi)$, that are needed for our purposes. Accordingly, the above infinite product of *eq*.(2.113) is equal to the following:

$$\prod_{(n,m) \in \mathbb{Z}^2 \backslash \{(0,0)\}} \frac{1}{\alpha' \tau_2} |n\tau - m|^2 = 4\pi^2 \alpha' \tau_2 |\eta(q(\tau))|^4, \tag{2.114}$$

where, the product formula $\prod_{n\in\mathbb{N}}(n+c) = 2i\sin(\pi c)$ has also been applied, and the nome, $q(\tau)$, has been defined as $q(\tau) = e^{2\pi i\tau}$; the function $\eta(q(\tau))$ is the Dedekind eta-function, and it can be expressed as in the following relation:

$$\eta(q(\tau)) = q^{\frac{1}{24}}(\tau)\prod_{n\in\mathbb{N}\backslash\{0\}}(1-q^n(\tau)). \tag{2.115}$$

Then, the functional integral of *eq*.(2.110) is equal to:

$$\int_{F_{X_{T^2}}}[dX]_{\gamma_{T^2}(\tau)}e^{-S[X,\gamma_{T^2}(\tau)]} = V_{26}(4\pi^2\alpha'\tau_2|\eta(q(\tau))|^4)^{-13} = V_{26}(4\pi^2\alpha'\tau_2)^{-13}|\eta(q(\tau))|^{-52}. \tag{2.116}$$

In the very same fashion as all of the above, the, regularised, Berezin functional integral:

$$\frac{1}{V^{CKG}_{T^2}(\tau)}\int_{F_{b_{T^2}}}[db]_{\gamma_{T^2}(\tau)}\int_{F_{c_{T^2}}}[dc]_{\gamma_{T^2}(\tau)}G[b,c,\gamma_{T^2}(\tau)]e^{-S_{gh}[b,c,\gamma_{T^2}(\tau)]}, \tag{2.117}$$

which corresponds to the *b*-*c* system, can be calculated; it can be proved that the result of this calculation is simply [1]:

$$\frac{1}{V^{CKG}_{T^2}(\tau)}\int_{F_{b_{T^2}}}[db]_{\gamma_{T^2}(\tau)}\int_{F_{c_{T^2}}}[dc]_{\gamma_{T^2}(\tau)}G[b,c,\gamma_{T^2}(\tau)]e^{-S_{gh}[b,c,\gamma_{T^2}(\tau)]} = \frac{1}{V^{CKG}_{T^2}(\tau)}|\eta(q(\tau))|^4, \tag{2.118}$$

where $V^{CKG}_{T^2}(\tau)$ is the volume of the CKG of the metrics of the torus T^2 of complex modulus τ. From the Riemann-Roch theorem, it is concluded that the CKG of the metrics on the torus T^2 has 2 real CKVs, as well as a discrete subgoup, which is the transformation group, $\mathbb{Z}_2$, that is, in general, a subgroup of the CKG on any oriented Riemann surface, of any topology. Then, the CKG of the metrics on the torus T^2 of complex modulus τ, is, obviously, the disconnected group $U(1)\times U(1)\times\mathbb{Z}_2$, so, its volume is simply equal to twice the original volume of the the torus T^2 of complex modulus τ, that is $V^{CKG}_{T^2}(\tau) = V_{\mathbb{Z}_2}V_{T^2}(\tau) = 2\int_{T^2}d^2\sigma\sqrt{|\det(\gamma_{T^2}(\tau)|)} = 2\tau_2$. Consequently, by multiplying the two preceding *eqs*.(2.116) and (2.118), the one-loop level vacuum amplitude of bosonic string theory, that is expressed in *eq*.(2.104), is equal to:

$$\begin{aligned} S(\{\emptyset\})_1 = &\int_{\mathbb{F}_{T^2}}d^2\tau\frac{1}{2}\frac{1}{V^{CKG}_{T^2}(\tau)}\int_{F_{b_{T^2}}}[db]_{\gamma_{T^2}(\tau)}\int_{F_{c_{T^2}}}[dc]_{\gamma_{T^2}(\tau)}G[b,c,\gamma_{T^2}(\tau)]e^{-S_{gh}[b,c,\gamma_{T^2}(\tau)]} \\ &\int_{F_{X_{T^2}}}[dX]_{\gamma_{T^2}(\tau)}e^{-S[X,\gamma_{T^2}(\tau)]} = \\ = &V_{26}\int_{\mathbb{F}_{T^2}}d^2\tau\frac{1}{2}\frac{1}{2\tau_2}(2\pi\sqrt{\alpha'\tau_2})^{-26}|\eta(q(\tau))|^{-48} = \\ = &\frac{V_{26}}{4(2\pi\sqrt{\alpha'})^{26}}\int_{\mathbb{F}_{T^2}}\frac{d^2\tau}{\tau_2^2}(\tau_2)^{-12}|\eta(q(\tau))|^{-48}. \end{aligned} \tag{2.119}$$

The modular group, M_{T^2}, of the metrics on the torus T^2, is the group $M_{T^2} = PSL(2;\mathbb{Z})$, as it was shown before; it can be proved that this modular group can be generated by the following transformations:

$$T:\tau\to\tau+1, \tag{2.120}$$

and:

$$S:\tau\to-\frac{1}{\tau}. \tag{2.121}$$

Then, focusing on the one-loop vacuum amplitude, as it is expressed in *eq*.(2.119), it is evident that its integration measure, $\frac{d^2\tau}{\tau_2^2}$, which may be sometimes referred to as the Poincaré measure, is invariant under the action of the modular group $PSL(2;\mathbb{Z})$; additionally, the Dedekind eta function, $\eta(q(\tau))$, has, by its definition, the following transformation properties, under the action of the modular group $PSL(2;\mathbb{Z})$:

$$(T\eta(q))(\tau) = \eta(q(\tau+1)) = e^{\frac{\pi i}{12}}\eta(q(\tau)), \tag{2.122}$$

and:

$$(S\eta(q))(\tau) = \eta(q(-\frac{1}{\tau})) = \sqrt{-i\tau}\eta(q(\tau)), \tag{2.123}$$

as it can be deduced from its expression as in *eq*.(2.115), so, it is also evident that the product $(\tau_2)^{-12}|\eta(\tau)|^{-48}$, that is also contained in the expression of *eq*.(2.119), for the one-loop vacuum amplitude, is invariant as well, under the action of the modular group, $PSL(2;\mathbb{Z})$. As a result, the one-loop vacuum amplitude, $S(\emptyset)_1$, is an overall modular invariant quantity, under the action of the modular group, $PSL(2;\mathbb{Z})$, of the metrics on the torus T^2, in accordance with the fact that all the scattering amplitudes, in the topology of an arbitrary genus, are expected to be invariant quantities, under the action of the respective modular group, as it has been mentioned in the previous section. Furthermore, the expression of the one-loop level vacuum amplitude of bosonic string theory as in *eq*.(2.119), allows for the deduction of the partition function of bosonic string theory; it is clear that the factor $|\eta(q(\tau))|^{-48}$, in *eq*.(2.119), has the typical form of a partition function, as it can be expanded in a power-series, with respect to the nome $q(\tau) = e^{2\pi i\tau}$. Consequently, the term $|\eta(q(\tau))|^{-48}$ must be interpreted as the partition function of bosonic string theory, in an "inverse temperature" that is equal to $2\pi\tau_2\sqrt{\frac{\alpha'}{2}}$; hence, this term shall be denoted as:

$$Z_{\{X,b,c\}}(\tau,\bar{\tau}) = |\eta(q(\tau))|^{-48}, \tag{2.124}$$

where it can be seen, now, that the contribution to the partition function, of each one of the $D = 26$ target-space string coordinates, X^μ, is a factor of $|\eta(q(\tau))|^{-2}$, while the contribution to the partition function, of the b and c ghost fields is simply to cancel out the contribution of these 2 target-space string coordinates that are in the 2 lightcone dimensions of the $D = 26$-dimensional spacetime, leaving, then, only the contribution of these $(D-2) = 24$ target-space string string coordinates that are in a $(D-2) = 24$-dimensional subspace of the $D = 26$-dimensional spacetime, which was previously called the transverse spacetime. Now, it is clear that the contribution of the b and c ghost fields to the partition function of bosonic string theory is due to the Faddeev-Popov determinant, which, in turn, is nothing more than a consequence of the overall gauge invariance of bosonic string theory, as it was mentioned in the previous section. Then, it can be seen that the the contribution to the partition function, of the 2 target-space string coordinates that are in the 2 lightcone dimensions of the $D = 26$-dimensional spacetime, is, indeed, a redundancy, or an artifact of the overall gauge invariance of the theory. That is, the hint that was mentioned in the previous chapter, that the very existence of the target-space vibration modes, or string excitations, in the 2 lightcone dimensions of the spacetime might as well be a redundancy, or an artifact of the overall gauge invariance of the theory, which is a hint that means precisely that it would be possible that these string states, which correspond to the 2 lightcone dimensions of the target-space, can be excluded entirely, from the analysis of the spectrum of the theory, has, effectively, been proved. Additionally, it can be proved that the following power-series expansion holds:

$$Z_{\{X,b,c\}}(\tau,\bar{\tau}) = \sum_{n,m\in\mathbb{Z}} Z^{(n,m)}_{\{X,b,c\}} q^n(\tau)\bar{q}^m(\bar{\tau}) = \sum_{n,m\in\mathbb{Z}} Z^{(n,m)}_{\{X,b,c\}} e^{2\pi i\tau_1(n-m)} e^{-2\pi\tau_2(n+m)}, \tag{2.125}$$

for the partition function of *eq*.(2.124), where the numbers $Z^{(n,m)}_{\{X,b,c\}}$, are the positive integers, that is, $Z^{(n,m)}_{\{X,b,c\}} \geqq 0$, which, naturally, express the level of degeneracy of the states of bosonic string theory with energy, that is, mass, equal to $M = \sqrt{(n+m)}\sqrt{\frac{2}{\alpha'}}$; these numbers, $Z^{(n,m)}_{\{X,b,c\}}$, may be accordingly called the degeneracy numbers, or multiplicities, of their respective states. It is also evident, from the form of the power-series expression of the above *eq*.(2.125), for the partition function of bosonic string theory, that only the states of degeneracy numbers $Z^{(n,m)}_{\{X,b,c\}}$, with $n = m$, are the states that constitute the physical spectrum of bosonic string theory, whereas, the states of degeneracy numbers $Z^{(n,m)}_{\{X,b,c\}}$, with $n \neq m$, are unphysical. Of course, in the general form of the power-series expression of the partition function of an arbitrary theory, it is naturally required that the degeneracy numbers of the bosonic states are positive integers, whereas the degeneracy numbers of the fermionic states are negative integers. As superstring theory naturally exhibits a spectrum that contains both bosonic states, as well as their corresponding superpartners, which are fermionic states, the relative degeneracy numbers of states, in the power-series expression of the partition function of superstring theory, will, eventually, be either positive or negative integers, for the cases of bosonic or fermionic states, respectively, as we shall see in the next chapter, where the partition function of superstring theory will be calculated.

Chapter 3

A Glance at Superstring Theory

After a lightning presentation of the conventional formalism for the inclusion of fermions in string theory, we shall subjoin a short presentation of superstring theory, as a theory which can naturally emerge as the result of the need of an appropriate, well-reasoned physical extension of bosonic string theory. This nod to superstring theory follows along with a very brief qualitative description of the physical essence of superstring theory, identifying the various types of superstring theory, as well as highlighting several of their aspects, and specifically pinpointing the basic features of their physical spectra.

3.1 Fermions in Superstring Theory

Bosonic string theory, as it has been described until now, appears to suffer from several deficiencies, one of which is the complications of the presence of a tachyonic state in its physical spectrum. Additionally, the target-space string coordinates, X^μ, in the Polyakov action of the bosonic string theory, may transform under tensor, rather than spinor, representations of the spacetime Poincaré group; this results to the total absence of spacetime fermionic states, from the spectrum of bosonic string theory. There is, then, a need for an appropriate extension of bosonic string theory, that is, for an extension which contains spacetime fermionic states in its physical spectrum; the full theory that comes as a result of this extension is superstring theory, and the conventional approach to its description is given in terms of the Ramond-Neuveu-Schwarz (RNS) formulation. In the RNS formulation, the spacetime fermionic states can arise from the action of the fields ψ^μ, where μ is a spacetime index, on a degenerate vacuum state, which transforms under the spinorial representation of the little group of the spacetime Poincaré group. However, the introduction of an action of these new, D in number, world-sheet fermion fields, ψ^μ, results, in general, in quantum states of negative norms; naturally, there is also a need for the introduction of a new, appropriate gauge symmetry of the overall action of the theory, so that these quantum states of negative norms automatically decouple, from the spectrum of the physical states of the theory. It turns out that this new, appropriate gauge symmetry of the overall action of superstring theory, must be the super-diffeomorphism invariance; the full gauge group of superstring theory, then, consists of the diffeomorphism group and the Weyl group, as well as local supersymmetry, super-Weyl transformations, and 2-dimensional local Lorentz transformations, which are precisely the transformations of the fundamental spinor representation of the Lorentz group. The in-depth description of this full gauge group of superstring theory, as well as the proof for the general form of the full action of superstring theory, are beyond the purposes of this presentation. It can be proved that, after the appropriate gauge fixing, the action of the D world-sheet fermion fields takes the following form, in the conformal gauge, expressed in the complex coordinate system, $(z, \bar{z})$ [2]:

$$S_F[\psi, \tilde{\psi}] = -\frac{1}{2\pi} \int_W d^2z (\psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu), \tag{3.1}$$

where the left-handed and right-handed world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, respectively, are Majorana-Weyl spinors, and so they can be taken to be real; the equations of motion for the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, are, obviously, the following:

$$\bar{\partial}\psi^\mu = 0, \tag{3.2}$$

and:

$$\partial\tilde{\psi}^\mu = 0, \tag{3.3}$$

respectively, while, the holomorphic and anti-holomorphic components of the relevant energy-momentum tensor, T_F, are respectively denoted as $T_F(z)$ and $\tilde{T}_F(\bar{z})$, and are equal to the following:

$$T_F(z) = -\frac{1}{2}\psi^\mu\partial\psi_\mu(z), \tag{3.4}$$

and:

$$\tilde{T}_F(\bar{z}) = -\frac{1}{2}\tilde{\psi}^\mu\bar{\partial}\tilde{\psi}_\mu(\bar{z}). \tag{3.5}$$

The fact that the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, are real fields, restricts their possible boundary conditions, up to a real phase; from the global Poincaré invariance of their action, it can be seen that the boundary conditions for the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, can be expressed, conventionally, as in the following:

$$\psi^\mu(e^{2\pi i}z) = e^{i\pi(1-a)}\psi(z), \tag{3.6}$$

and, similarly:

$$\tilde{\psi}^\mu(e^{2\pi i}\bar{z}) = e^{i\pi(1-\tilde{a})}\tilde{\psi}(\bar{z}), \tag{3.7}$$

respectively, where $a, \tilde{a} = 0, 1$, of course. The case of the periodic boundary conditions of a world-sheet fermion field is referred to as the Ramond (R) sector, whereas, the case of the anti-periodic boundary conditions of a world-sheet fermion field is referred to as the Neveu -Schwarz (NS) sector; accordingly, the full, composed sector of the action of the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, can be any one of the composed R-R, R-NS, NS-R, or NS-NS sectors. It can be seen that a world-sheet fermion field is single-valued in the NS sector, whereas a world-sheet fermion field has a $\mathbb{Z}_2$ branch cut in the R sector: this fact implies that the action of the world-sheet fermion field on a degenerate vacuum state is only possible in the R sector, so, accordingly, a world-sheet fermion field can produce spacetime fermionic states only in the R sector, whereas it produces exclusively spacetime bosonic states in the NS sector; consequently, fermionic states can be produced only in the R-NS and the NS-R sectors, whereas bosonic states can be produced only in the R-R and the NS-NS sectors, of the action of the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$.
It can be seen, now, that the action of the $2D$ world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, describes a CFT of central charges $c = \tilde{c} = \frac{D}{2}$. The residual gauge symmetry of the the full action of the Super-CFT (SCFT) of superstring theory is invariance under superconformal transformations, which are generated by the holomorphic and anti-holomorphic Noether world-sheet supercurrents, $T_S(z)$, and $\tilde{T}_S(\bar{z})$, as in the following [2]:

$$T_S(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X_\mu(z), \tag{3.8}$$

and:

$$\tilde{T}_S(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu\bar{\partial}X_\mu(\bar{z}), \tag{3.9}$$

respectively, where the target-space string coordinates, ∂X^μ and $\bar{\partial}X^\mu$, act like the respective super-partners of the worldsheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, so they are appropriately referred to as such; these world-sheet supercurrents obviously have the same boundary conditions as their respective worldsheet fermion fields, while the current algebra of these world-sheet supercurrents, with the energy-momentum tensor of the full action of the SCFT of superstring theory, is called the $N = 1$

superconformal algebra.
Considering, now, the scattering amplitudes of superstring theory, it can be proved that the standard Faddeev-Popov procedure, for the full action of superstring theory, results to the emergence of the appropriate "superpartners" for the b and c ghost fields, which are the, commuting, β and γ superghost fields, respectively [2]; the corresponding action of the β and γ superghost fields is called the superghost action, or, simply, the β-γ system, and it can be proved to eventually emerge, from the standard Faddeev-Popov prodecure, as an action which is identical, in form, to the b-c system, yet which describes a CFT of central charges $c = \tilde{c} = 11$ [2]. Consequently, the total central charge, c_{tot}, of the SCFT of the superstring theory is $c_{tot} = D + \frac{D}{2} - 26 + 11$, and the critical dimension, $D_{crit.}$, of the SCFT of superstring theory, which can be solved for, by the requirement that $c_{tot}(D_{crit.}) = 0$, is accordingly found to be $D_{crit.} = 10$. Then, similarly to the case of bosonic string theory, which was presented in the previous chapter, it must be, eventually, imposed that the number D of the spacetime dimensions be equal to $D = D_{crit.} = 10$, in order that the Weyl and the respective super-Weyl anomalies of the SCFT of superstring theory be cancelled, in general.

3.1.1 The Partition Function of the fermionic part of Superstring Theory

The partition function, for the action of the $D = 10$ "doublets" of the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, which has been expressed in the conformal gauge in the preceding *eq*.(3.1), is given by a fashion kindred to the case of bosonic string theory, that is, from the calculation of the respective, regularised, Berezin functional integral, as in the following:

$$\begin{aligned} Z_{\{\psi,\tilde{\psi}\}}\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\tau,\bar{\tau}) &= \int_{F_{\psi_{T^2}}} [d\psi]_{\gamma_{T^2}(\tau)} \int_{F_{\tilde{\psi}_{T^2}}} [d\tilde{\psi}]_{\gamma_{T^2}(\tau)} e^{S_F[\psi,\tilde{\psi}]} = \\ &= \int_{F_{\psi_{T^2}}} [d\psi]_{\gamma_{T^2}(\tau)} \int_{F_{\tilde{\psi}_{T^2}}} [d\tilde{\psi}]_{\gamma_{T^2}(\tau)} e^{-\frac{1}{2\pi}\int_W d^2z(\psi^\mu\bar{\partial}\psi_\mu + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu)} = \\ &= \left(\int_{F_{\psi_{T^2}}} [d\psi]_{\gamma_{T^2}(\tau)} e^{-\frac{1}{2\pi}\int_W d^2z\psi^\mu\bar{\partial}\psi_\mu}\right)\left(\int_{F_{\tilde{\psi}_{T^2}}} [d\tilde{\psi}]_{\gamma_{T^2}(\tau)} e^{-\frac{1}{2\pi}\int_W d^2z\tilde{\psi}^\mu\partial\tilde{\psi}_\mu}\right), \end{aligned} \tag{3.10}$$

in the world-sheet topology of the Riemann torus, T^2, of the complex modulus, τ, which is defined by the following boundary conditions, for the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$:

$$\begin{aligned} \psi(z + 2\pi) &= e^{i\pi(1-a)}\psi(z), \\ \tilde{\psi}(\bar{z} + 2\pi) &= e^{i\pi(1-\tilde{a})}\tilde{\psi}(\bar{z}), \end{aligned} \tag{3.11}$$

and:

$$\begin{aligned} \psi(z + 2\pi\tau) &= e^{i\pi(1-b)}\psi(z), \\ \tilde{\psi}(\bar{z} + 2\pi\tau) &= e^{i\pi(1-\tilde{b})}\tilde{\psi}(\bar{z}), \end{aligned} \tag{3.12}$$

where $a, \tilde{a}, b, \tilde{b} = 0, 1$, of course. Then, by the usual methods of zeta-function regularisation, it can be proved that the following holds [2]:

$$Z_{\{\psi,\tilde{\psi}\}}\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\tau,\bar{\tau}) = \int_{F_{\psi_{T^2}}} [d\psi]_{\gamma_{T^2}(\tau)} \int_{F_{\tilde{\psi}_{T^2}}} [d\tilde{\psi}]_{\gamma_{T^2}(\tau)} e^{S_F[\psi,\tilde{\psi}]} = \left(\frac{\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^5 \left(\frac{\bar{\theta}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^5, \tag{3.13}$$

where the function $\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau))$ is the Jacobi theta-function of the characteristics a and b, and it can be expressed as in the following relation [2]:

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau)) = \lim_{\upsilon\to 0}\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau);\upsilon) = \lim_{\upsilon\to 0}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{a}{2})^2}(\tau)e^{2\pi i(\upsilon-\frac{b}{2})(n-\frac{a}{2})} =$$

,

$$= e^{i\pi\frac{ab}{2}} q^{\frac{1}{8}a^2}(\tau) \prod_{n\in\mathbb{N}} (1 - q^n(\tau))(1 + q^{n+\frac{a-1}{2}}(\tau) e^{i\pi b})(1 + q^{n-\frac{a+1}{2}}(\tau) e^{-i\pi b}) =$$

$$= e^{i\pi\frac{ab}{2}} q^{\frac{1}{8}a^2-\frac{1}{24}}(\tau) \eta(q(\tau)) \prod_{n\in\mathbb{N}} (1 + q^{n+\frac{a-1}{2}}(\tau) e^{i\pi b})(1 + q^{n-\frac{a+1}{2}}(\tau) e^{-i\pi b}), \tag{3.14}$$

and $q(\tau)$, is the nome, $q(\tau) = e^{2\pi i\tau}$, of course. The Jacobi theta-functions, $\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau))$, have the following, trivial, periodicity properties, relating to their characteristics, a and b:

$$\theta\begin{bmatrix} a+2 \\ b \end{bmatrix}(q(\tau)) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau)) \tag{3.15}$$

and:

$$\theta\begin{bmatrix} a \\ b+2 \end{bmatrix}(q(\tau)) = e^{i\pi a}\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau)), \tag{3.16}$$

respectively, as well as the following transformation properties, under the transformations that are defined by the generators T and S of the modular group, $PSL(2;\mathbb{Z})$, of the metrics on the topology of the torus, T^2:

$$\left(T\theta\begin{bmatrix} a \\ b \end{bmatrix}(q)\right)(\tau) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau+1)) = e^{-\frac{i\pi}{4}a(a-2)}\theta\begin{bmatrix} a \\ a+b-1 \end{bmatrix}(q(\tau)), \tag{3.17}$$

and:

$$\left(S\theta\begin{bmatrix} a \\ b \end{bmatrix}(q)\right)(\tau) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(q(-\frac{1}{\tau})) = \sqrt{-i\tau}\, e^{\frac{i\pi}{2}ab}\theta\begin{bmatrix} b \\ -a \end{bmatrix}(q(\tau)), \tag{3.18}$$

respectively. Similarly to the aforementioned, it can be proved that the contribution of the respective functional integral, which corresponds to the β and γ superghost fields, is given by the following [2]:

$$Z_{\{\beta,\gamma\};m,\tilde{m}}\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\tau,\bar{\tau}) = (-1)^{b+mab}\frac{\eta(q(\tau))}{\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau))}(-1)^{\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\frac{\bar{\eta}(\bar{q}(\bar{\tau}))}{\bar{\theta}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\bar{q}(\bar{\tau}))}, \tag{3.19}$$

where the numbers m and $\tilde{m}$, are called the left and right chirality projections, respectively, and they may take the values $m, \tilde{m} = 0, 1$. Consequently, by mutiplying the two preceding *eqs*.(3.13), and (3.19), the consolidated partition function, which may be denoted as $Z_{\{\psi,\tilde{\psi},\beta,\gamma\};m,\tilde{m}}\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\tau,\bar{\tau})$, for the action of the $D = 10$ pairs of the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, combined with the contribution of the functional integral that corresponds to the β and γ superghost fields, is equal to the following:

$$Z_{\{\psi,\tilde{\psi},\beta,\gamma\};m,\tilde{m}}\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\tau,\bar{\tau}) = Z_{\{\psi,\tilde{\psi}\}}\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\tau,\bar{\tau}) Z_{\{\beta;\gamma\};m,\tilde{m}}\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\tau,\bar{\tau})$$

$$= (-1)^{b+mab}\left(\frac{\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^4 (-1)^{\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^4. \tag{3.20}$$

It can be easily seen, now, in the partition function of the above *eq*.(3.20), that the contribution of each of the $D = 10$ pairs of the world-sheet fermion fields, ψ^μ and $\tilde{\psi}^\mu$, of the appropriate boundary conditions, is a factor of $\left(\frac{\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^{\frac{1}{2}}\left(\frac{\bar{\theta}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^{\frac{1}{2}}$, while, the contribution of the functional integral,

that corresponds to the β and γ superghost fields, is simply to cancel out the contributions of these 2 doublets of world-sheet fermion fields, which are the superpartners of these target-space string coordinates that are in the 2 lightcone dimensions of the $D = 10$-dimensional spacetime, up to the phase factor of $(-1)^{b+mab}(-1)^{\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}$, leaving, then, only the effects of exclusively these $(D-2) = 8$ doublets of world-sheet fermion fields, which are the superpartners of these target-space string coordinates that are in the $(D-2) = 8$-dimensional, transverse spacetime.
However, there still remains the additional complication of the variety of the aforementioned possible boundary conditions of the world-sheet fermion fields, ψ^{μ} and $\tilde{\psi}^{\mu}$, by which the world-sheet topology of the Riemann torus T^2 of the complex modulus τ is defined; from reasons which can be regarded as being relevant to the consistency of superstring theory, such as, for instance, the proper definition and calculation of functional integrals, in the world-sheet topology of the Riemann torus, T^2, as well as the modular invariance of the full partition function of superstring theory, it can be concluded that an appropriately weighted summation, over all the various possible boundary conditions of the world-sheet fermion fields, ψ^{μ} and $\tilde{\psi}^{\mu}$, must be additionally included, in the definition of the full partition function of superstring theory. It can be proved, accordingly, that the appropriate form of the weight factor for the summation over all the possible boundary conditions of the world-sheet fermion fields, ψ^{μ} and $\tilde{\psi}^{\mu}$, is given by the weight factor $C_{a,\tilde{a}} = \frac{1}{2}(-1)^{a}\frac{1}{2}(-1)^{\tilde{a}}$; consequently, the appropriate partition function, which is denoted as $Z_{\{\psi,\tilde{\psi},\beta,\gamma\};m,\tilde{m}}(\tau,\bar{\tau})$, for the theory of the $D = 10$ pairs of the world-sheet fermion fields, ψ^{μ} and $\tilde{\psi}^{\mu}$, along with the β and γ superghost fields, is the partition function of the fermionic part of superstring theory and it is, eventually, equal to the following:

$$Z_{\{\psi,\tilde{\psi},\beta,\gamma\};m,\tilde{m}}(\tau,\bar{\tau}) = \sum_{a,b=0}^{1}\sum_{\tilde{a},\tilde{b}=0}^{1} C_{a,\tilde{a}} Z_{\{\psi,\tilde{\psi},\beta,\gamma\};m,\tilde{m}}\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\tau,\bar{\tau}) =$$

$$= \frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a}\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}} Z_{\{\psi,\tilde{\psi}\}}\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\tau,\bar{\tau}) Z_{\{\beta,\gamma\};m,\tilde{m}}\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\tau,\bar{\tau}) =$$

$$= \frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+mab}\left(\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^{4}\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^{4}, \tag{3.21}$$

where, the different choices, $m, \tilde{m} = 0, 1$, for the value of the left and right chirality projections, m, and $\tilde{m}$, respectively, distinguish, between the various types of superstring theories, as it will be explained in the next section. Evidently, now, from the aforementioned properties of the Jacobi theta-functions, $\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))$, and the Dedekind eta-function, $\eta(q(\tau))$, under the transformations of the modular group, $PSL(2;\mathbb{Z})$, on the topology of the Riemann torus, T^2, the partition function $Z_{\{\psi,\tilde{\psi},\beta,\gamma\};m,\tilde{m}}(\tau,\bar{\tau})$ of the above *eq*.(3.21), is, indeed, a quantity that exhibits modular invariance; this fact holds precisely due to the appropriate form of its weight factor, $C_{a,\tilde{a}} = \frac{1}{2}(-1)^{a}\frac{1}{2}(-1)^{\tilde{a}}$. Additionally, it can be proved that this specific form of the weight factor, $C_{a,\tilde{a}} = \frac{1}{2}(-1)^{a}\frac{1}{2}(-1)^{\tilde{a}}$, in the partition function $Z_{\{\psi,\tilde{\psi},\beta,\gamma\};m,\tilde{m}}(\tau,\bar{\tau})$ of the above *eq*.(3.21), can also be deduced, from the requirement of the unitarity, that is, of the modular invariance, of the string S-matrix elements, in the topology of a higher genus, along with the requirement of their appropriate factorisation, as products of string S-matrix elements, in the world-sheet topology of the Riemann torus, T^2; it can be shown that, this specific form of the weight factor ensures the correct spin-statistics, as well as the correct particle interpretation, for the partition function $Z_{\{\psi,\tilde{\psi},\beta,\gamma\};m,\tilde{m}}(\tau,\bar{\tau})$ of *eq*.(3.21), and, consequently, for the full partition function of superstring theory [3].

3.2 The Partition Function of Superstring Theory

We are, now, in the position to calculate the full partition function of superstring theory. According to the previous section, superstring theory, existing in its critical dimension, $D = D_{crit.} = 10$, is a theory that contains $D = 10$ target-space string coordinates, X^{μ}, and the anti-commuting ghost fields, b and c, along with their superpartners, which are the $D = 10$ pairs of the world-sheet fermion fields, ψ^{μ}, and $\bar{\psi}^{\mu}$, and the commuting superghost fields, β and γ, respectively. As a consequence, the full partition function of superstring theory can, then, be readily caclulated, by assembling all the appropriate preceding equations, which respectively express the functional integral contributions of all these fields, that are contained in superstring theory, to its partition function; the result is the following:

$$Z_{II;m,\tilde{m}}(\tau,\bar{\tau}) = Z_{\{X,b,c,\psi,\bar{\psi},\beta,\gamma\};m,\tilde{m}}(\tau,\bar{\tau}) =$$

$$= |\eta(q(\tau))|^{-16}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+mab}\left(\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^{4}\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^{4}, \tag{3.22}$$

where two different cases, concerning the values of the left and right chirality projections, m and $\tilde{m}$, respectively, can already be distinguished: namely, the case where $m = \tilde{m}$, that is conventionally taken to be the case where $m = \tilde{m} = 1$, refers to the so-called type IIB superstring theory, which is, obviously, a chiral theory, whereas, the case where $m \neq \tilde{m}$, that is conventionally taken to be the case where $m = 0$, and $\tilde{m} = 1$, refers to the so-called type IIA superstring theory, which is, obviously, a non-chiral theory. Then, the partition function of the chiral, type IIB superstring theory can be expressed as:

$$Z_{IIB}(\tau,\bar{\tau}) = Z_{II;m,\tilde{m}}(\tau,\bar{\tau})|_{m=\tilde{m}=1} =$$

$$= |\eta(q(\tau))|^{-16}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\left(\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^{4}\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^{4}, \tag{3.23}$$

whereas, the partition function of the non-chiral, type IIA superstring theory can be expressed as:

$$Z_{IIA}(\tau,\bar{\tau}) = Z_{II;m,\tilde{m}}(\tau,\bar{\tau})|_{m=0,\tilde{m}=1} =$$

$$= |\eta(q(\tau))|^{-16}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\left(\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^{4}\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^{4}. \tag{3.24}$$

We additionally mention that the type I superstring theory, which is a superstring theory that contains both open strings and closed strings, as well as the two distinct "flavours" of the heterotic theory of closed strings, that contains an uneven number of left ψ^{μ} and right $\bar{\psi}^{\mu}$ world-sheet fermion fields, may also stand as possibilities for a consistent superstring theory; we will not be occupied with these other kinds of supertsring theory, as they are beyond the scope of this report, where the range of the analysis of the superstring theory shall be restricted to enclose exclusively the cases of the type IIA and type IIB superstring theory. Before we move on to the analysis of the respective physical spectra of the type IIA and type IIB superstring theory, we note, for expressional convenience, that the summations over the numbers b and $\tilde{b}$, which appear explicitly in the partition functions of *eqs.*(3.23) and (3.24), for the type IIB and type IIA superstring theories, respectively, are called the left and right Gliozzi-Scherk-Olive (GSO) projections, respectively. As it was mentioned in the previous section, all the space-time bosonic states are encoded exclusively in the NS-NS and R-R sectors, whereas all the space-time fermionic states are encoded exclusively in the NS-R and R-NS sectors, of the theory of the world-sheet fermion fields, ψ^{μ} and $\bar{\psi}^{\mu}$. The physical spectra of the type IIA and type IIB superstring theory may then be presented in an orderly fashion, relating to the sectors of the respective theories; we will see that, in each of these sectors, the respective GSO projections serve as a filter for the spectrum, prohibiting the controversial states, such as the physical tachyonic states, while permitting exclusively the existence of appropriate, non-tachyonic states, in the physical spectrum of the theory. Of course, in order to deduce the respective physical spectra of the type IIB and type IIA superstring theory, their respective partition functions of *eqs*.(3.23) and (3.24) must be expressed as their respective power-series expansions, in terms of the nome, $q = e^{2\pi i\tau}$, just like in the case of the partition function of bosonic string theory; similarly to the previous chapter, we will not write these respective power-series expansions explicitly, but we shall rather consider them implicitly, and mention their corresponding massless physical spectra only qualitatively.

On the one hand, concerning the spacetime bosonic states of the physical spectra of the type IIB and type IIA superstring theory, it can be shown that they are identical in their respective NS-NS sectors; there, the corresponding GSO projections, in the respective partition functions of *eqs*.(3.23) and (3.24), ensure that they contain massless string states, all of which can transform under tensor representations of the little group, $SO(8)$, of the spacetime Poincaré group, $SO(10)$, that corresponds to the $D = 10$-dimensional target-space: namely, these massless string states can form a traceless symmetric spacetime tensor and an antisymmetric spacetime tensor, as well as a, pure trace, spacetime scalar; these massless states are, of course, the graviton, the Kalb-Ramond, and the dilaton states, respectively. The most remarkable fact, however, for the GSO projections, in the partition functions of *eqs*.(3.23) and (3.24), for the type IIB and type IIA superstring theory, respectively, is that these GSO projections also ensure that the tachyonic state, of negative mass-squared, that was contained in the physical spectrum of bosonic string theory, is now removed; the spectra of the type IIA and type IIB superstring theories now contain no physical tachyonic states, in their respective NS-NS sectors. Of course, from the very form of the power-series expansions, in terms of the nome, $q = e^{2\pi i\tau}$, of the partition functions of the type IIA and type IIB superstring theories, it is clear that their respective physical spectra can contain no such tachyonic states, of negative mass-squared, in their respective R-R, NS-R, and R-NS sectors, precisely due to the properties of the Dedekind eta-function and of the Jacobi theta-functions. Therefore, it is evident that any such tachyonic states, of negative mass-squared, that may possibly be contained in the physical spectra of the type IIA and type IIB superstring theories, must necessarily be spacetime bosonic states that are included exclusively in their respective NS-NS sectors. Thus, it can be concluded that the physical spectra of the type IIA and type IIB superstring theories, contain no tachyonic states at all, in any one of their respective NS-NS, R-R, NS-R and R-NS sectors whatsoever. It has been shown, then, that as the physical spectra of the type IIA and type IIB superstring theories contain no tachyonic states at all, the complications of the presence of the tachyonic state in the physical spectrum of the bosonic string theory can be completely resolved, when one moves on to the superstring theory, as it was stated in the previous chapter, by the effects of the GSO projections in the partition function of superstring theory. Additionally, it can be shown that the physical spectra of the type IIB and type IIA superstring theory are different in their respective R-R sectors; there, the corresponding GSO projections, in the respective partition functions of *eqs*.(3.23) and (3.24), result to the fact that the physical spectrum of the type IIB superstring theory contains massless states that can form a spacetime scalar, a two-index antisymmetric spacetime tensor, and a self-dual, four-index antisymmetric spacetime tensor, whereas the physical spectrum of the type IIA superstring theory contains massless states that can form a spacetime vector, and a three-index antisymmetric spacetime tensor.

On the other hand, concerning the spacetime fermionic states of the physical spectra of the type IIB and type IIA superstring theory, it can be shown that they are identical in their respective NS-R sectors; there, the corresponding GSO projections, in the respective partition functions of *eqs*.(3.23) and (3.24), ensure that they contain massless spacetime fermionic states, which correspond to a spin-$\frac{3}{2}$ gravitino, and to a massless, spin-$\frac{1}{2}$ spacetime fermion, of opposite chiralities. In particular, the spin-$\frac{3}{2}$ gravitino and the massless, spin-$\frac{1}{2}$ spacetime fermion are conventionally taken to have right and left chiralities, respectively. Additionally, it can be shown that the physical spectra of the type IIB and type IIA superstring theory are different, in their chirality, in their respective

R-NS sectors; there, the corresponding GSO projections, in the respective partition functions of *eqs*.(3.23) and (3.24), result to the fact that that the respective physical spectra of the type IIB and type IIA superstring theory contain massless spacetime fermionic states of the same chirality, which correspond to a spin-$\frac{3}{2}$ gravitino, and to a massless, spin-$\frac{1}{2}$ spacetime fermion. In particular, the spin-$\frac{3}{2}$ gravitino and the massless, spin-$\frac{1}{2}$ spacetime fermion are conventionally taken to have left and right chiralities, in the case of the physical spectra of the type IIA and type IIB superstring theory, respectively. In summary, it can be seen that, each of the physical spectra of the type IIA and type IIB superstring theory contains two spin-$\frac{3}{2}$ gravitini; it is said, then, that both of the type IIA and type IIB superstring theories, exhibit $\mathcal{N}_{D=10} = 2$ local supersymmetry. However, in the type IIB superstring theory, the two spin-$\frac{3}{2}$ gravitini have the same chiralities, whereas the two spin-$\frac{1}{2}$ massless spacetime fermions have opposite chiralities, and the theory is then chiral; on the contrary, in the type IIA superstring theory, the two spin-$\frac{3}{2}$ gravitini and the two spin-$\frac{1}{2}$ massless spacetime fermions can have any one of the two possible chiralities, and the theory is then non-chiral, as it was mentioned before.

Moreover, in both of the cases of the type IIA and type IIB superstring theory, it can be easily seen that the contribution of the bosonic part of their respective partition functions, which corresponds to the factor of $|\eta(q(\tau))|^{-16}$, that appears in their respective partition functions of *eqs*.(3.23) and (3.24), refers exclusively to the massive states that are contained in their respective spectra. Furthermore, it can be also seen that both the partition functions of *eqs*.(3.23) and (3.24), for the type IIB and type IIA superstring theory, respectively, are identically equal to zero; this fact is to be expected, as both of the type IIA and type IIB superstring theories are supersymmetric theories. The supersymmetric nature of a theory means precisely that the spectrum of the theory contains an equal number of spacetime bosonic states and spacetime fermionic states of a specific mass, as for each of these spacetime bosonic states there corresponds exactly one spacetime fermionic state, of the same specific mass, which is dubbed as its superpartner state; this bijective correspondence means precisely that the net degeneracy numbers of all the states of any specific mass, in the partition function of the theory, are zero. To elaborate even more, in the power-series expression of the partition function for the case of a superstring theory the negative degeneracy number of a spacetime fermionic state of any specific mass is cancelled out against the positive degeneracy number of its superpartner, bosonic spacetime state, yielding, eventually, a zero result. This fact is in accordance with what was mentioned in the previous chapter, where the partition function of bosonic string theory was calculated, concerning the general form and the signs of the degeneracy numbers in the power-series expression of the partition function of an arbitrary physical theory.

Chapter 4

Space-Time Compactification

The notion of spacetime compactification is nothing more than the identification of regions of the spacetime, by means of enforcing an specific equivalence relation on the spacetime. Of course, the motivation for spacetime compactification is perfectly natural, as, in accordance with the current knowledge and observations, it is expected that the 10-dimensional spacetime of a consistent superstring theory must necessarily contain an exclusively 4-dimensional, uncompactified spacetime, in addition to a 6-dimensional, arbitrarily curved and compactified spacetime, so that superstring theory may constitute a physically acceptable theory. Spacetime compactification in string theory is a vast subject, as vast as is the number of the various possible compactified spacetimes, in which a string theory can exist. However, string theory in an arbitrary compactified spacetime is not known to constitute, in general, an exactly solvable model, apart from some special cases. The presentation that follows consists of the description of superstring theory in two rather simple cases, which constitute exactly solvable models; namely, these cases are primarily the more generic case of spacetime T^d-type toroidal compactification, and, after the introduction of the notion of the orbifold, the more specific case of spacetime $T^d/\mathbb{Z}_N$-type orbifold compactification.

4.1 Space-Time T^d-type Toroidal Compactification

We now return to the general, non-linear sigma model of *eq*.(2.1), where we now impose the typically less restricting background of a flat, constant spacetime metric field, G, of a constant Kalb-Ramond field, B, and of a constant dilaton field, Φ, that is equal to its vacuum expectation value, $\Phi = \langle\Phi\rangle$, assuming again, as always, that the string field, X, expresses a closed string; then, the general, non-linear sigma model of *eq*.(2.1), is reduced to the following:

$$S[X,g] = -\frac{1}{4\pi\alpha'}\int_W d^2\sigma\,\sqrt{|\det(g)|}(G_{\mu\nu}g^{ab} + iB_{\mu\nu}\frac{\epsilon^{ab}}{\sqrt{|\det(g)|}})\partial_a X^\mu \partial_b X^\nu - \lambda\langle\Phi\rangle\chi_W, \tag{4.1}$$

where we can require, without loss of generality, that the action of the above *eq*.(4.1) is expressed in the conformal gauge, with the real world-sheet coordinate system, (σ^1, σ^2), which satisfies $\sigma^{1,2} \in [0, 2\pi]$. The spacetime T^d-type toroidal compactification means precisely the enforcement of the following equivalence relation on the spacetime:

$$x^M \equiv x^M + 2\pi; \forall M \in \mathbb{N}_d, \tag{4.2}$$

that is, with regard to a specific, d-dimensional subspace of the D-dimensional spacetime, which is called the compactified, or, internal, space T^d. Accordingly, while the extrema of the action of *eq*.(4.1), with respect to the closed string coordinates, X^μ, yield the same equations of motion as those in *eq*.(2.5), that is:

$$\nabla^2 X^\mu = 0, \tag{4.3}$$

the boundary conditions for the closed target-space string coordinates, X^M, which are in the compactified space T^d, as it was defined in *eq*.(4.2), and which define the very topology of the compactified target-space, are, now, the following:

$$X^M(\sigma^1 + 2\pi, \sigma^2) \equiv X^M(\sigma^1, \sigma^2); \forall M \in \mathbb{N}_d, \tag{4.4}$$

contrary to the case of the boundary conditions for the closed target-space string coordinates, X^μ, which are not in the compactified space T^d, that are simply the normal closed string boundary conditions, that is:

$$X^\mu(\sigma^1 + 2\pi, \sigma^2) = X^\mu(\sigma^1, \sigma^2); \forall \mu \in \mathbb{N}_D \backslash \mathbb{N}_d, \tag{4.5}$$

as usual. It is evident, from what was mentioned in the previous chapters, that the partition function of the theory of the action of *eq*.(4.1), taking into consideration only these X^μ target-space string coordinates, which are not in the compactified spacetime T^d, is precisely the following:

$$Z^{(D-d)}_{\{X\}}(\tau, \bar{\tau}) = |\eta(q(\tau))|^{-2(D-d)}, \tag{4.6}$$

so, in what follows, we shall give prominence to the calculation of the partition function of the theory of the action of the preceding *eq*.(4.1), taking into consideration only those X^M target-space string coordinates, which are in the compactified space T^d. In general, we can always express any such target-space string coordinate, X^M, as $X^M = X^M_{c.} + X^M_{q.}$, that is, as the sum of a classical field part, $X^M_{c.}$, which is a solution to the Cauchy problem of the equations of motion of *eq*.(4.3), along with the boundary conditions of *eq*.(4.4), and a quantum field part, $X^M_{q.}$, which just satisfies the usual, normal boundary conditions of *eq*.(4.5); then, taking into consideration only these X^M string fields, the action of *eq*.(4.1) is partitioned as in the following:

$$\begin{aligned} S[X, g] = &-\frac{1}{4\pi\alpha'}\int_W d^2\sigma \sqrt{|\det(g)|}(G_{MN}g^{ab} + iB_{MN}\frac{\epsilon^{ab}}{\sqrt{|\det(g)|}})\partial_a X^M_{c.}\partial_b X^N_{c.} \\ &-\frac{1}{4\pi\alpha'}\int_W d^2\sigma \sqrt{|\det(g)|}(G_{MN}g^{ab} + iB_{MN}\frac{\epsilon^{ab}}{\sqrt{|\det(g)|}})\partial_a X^M_{q.}\partial_b X^N_{q.} - \lambda\langle\Phi\rangle\chi_W = \\ &= S[X_{c.}, g] + S[X_{q.}, g] + \lambda\langle\Phi\rangle\chi_W. \end{aligned} \tag{4.7}$$

Naturally, the partition function of the theory of the action of the above *eq*.(4.7), can be calculated from the respective functional integral, of the form of *eq*.(2.105), that is:

$$Z_{T^d}(\tau, \bar{\tau}) = \int_{F_{X_{T^2}}/\equiv} [dX]_{\gamma_{T^2}(\tau)} e^{-S[X, \gamma_{T^2}(\tau)]}, \tag{4.8}$$

where, now, the function space $F_{X_{T^2}}/\equiv$, of volume that is equal to:

$$\begin{aligned} V_{F_{X_{T^2}}/\equiv} &= \int_{F_{X_{T^2}}/\equiv} [dX] \sqrt{\underset{(d)}{\det}(G)} = V_{\mathbb{R}^d/\equiv} = \int_{\mathbb{R}^d/\equiv} d^dX \sqrt{\underset{(d)}{\det}(G)} = \\ &= \int_{R^d} d^dX \sqrt{\underset{(d)}{\det}(G)} H(X)H(X - 2\pi) = \int_0^{2\pi} d^dX \sqrt{\underset{(d)}{\det}(G)} = (2\pi)^d \sqrt{\underset{(d)}{\det}(G)}, \end{aligned}$$

is the quotient space of the function space $F_{X_{T^2}}$, of target-space string fields, X, on the world-sheet of the topology of the Riemann torus, T^2, with respect to the equivalence relation of the spacetime T^d-type toroidal compactification, as it was defined in *eq*.(4.2). Of course, in the functional integral of the preceding *eq*.(4.8), the action $S[X, \gamma_{T^2}(\tau)]$ is just the action of *eq*.(4.7), evaluated in the world-sheet metric $\gamma_{T^2}(\tau)$ of the Riemann torus T^2 of the complex modulus τ, so the topology of the world-sheet Riemann torus T^2 itself is, now, obviously not exclusively defined by the boundary conditions of the *eqs*.(2.106) and (2.107), but it is naturally defined by the boundary conditions of the *eqs*.(2.106) and (2.107) for the quantum field parts, $X^M_{q.}$, along with the appropriate boundary conditions for the classical field parts, $X_{c.}^M$, of the target-space string fields, X^M, that are in the compactified space T^d. These appropriate boundary conditions must naturally enforce the equivalence relation of the spacetime T^d-type toroidal compactification, as it was defined in *eq*.(4.2), so the topology of the world-sheet torus, T^2, may, now, be defined by none other than the following boundary conditions:

$$X^M(\sigma^1,\sigma^2) \equiv X^M(\sigma^1+2\pi,\sigma^2), \tag{4.9}$$

and:

$$X^M(\sigma^1,\sigma^2) \equiv X^M(\sigma^1,\sigma^2+2\pi), \tag{4.10}$$

or, equivalently, with employment of the natural function for toroidal compactification, $f(x)=e^{ix}$:

$$f(X^M(\sigma^1,\sigma^2)-X^M(\sigma^1+2\pi,\sigma^2))=1, \tag{4.11}$$

and:

$$f(X^M(\sigma^1,\sigma^2)-X^M(\sigma^1+2\pi,\sigma^2))=1, \tag{4.12}$$

where, again, the former boundary condition refers to the nature and properties of the closed string, as it is none other than the very definition of a closed string, of the preceding *eq*.(4.4), while, the latter boundary condition refers to the topology of the world-sheet torus T^2 itself, all similarly to the boundary conditions of *eqs*.(2.106) and (2.107), respectively, yet, now, for the target-space string fields, X^M, that are in the compactified space T^d. Consequently, and similarly to *eq*.(2.110), the functional integral of *eq*.(4.8), is equal to the following:

$$Z_{T^d}(\tau,\bar\tau)=\int_{F_{X_{T^2}}/\equiv}[dX]_{\gamma_{T^2}(\tau)}e^{-S[X,\gamma_{T^2}(\tau)]}=$$

$$=\int_{F_X/\equiv}[dX]_{\gamma_{T^2}^{(N)}(\tau)}\delta_{\sigma^2}[f(X(2\pi)-X(0))]\delta_{\sigma^1}[f(X(2\pi)-X(0))]e^{-S[X,\gamma_{T^2}^{(N)}(\tau)]}=$$

$$=\int_{F_X/\equiv}[dX]_{\gamma_{T^2}^{(N)}(\tau)}\sum_{n,m\in\mathbb{Z}^d}\delta_{\sigma^2}[X_{c.}(2\pi)-X_{c.}(0)+2\pi n]\delta_{\sigma^1}[X_{c.}(2\pi)-X_{c.}(0)+2\pi m]e^{-S[X,\gamma_{T^2}^{(N)}(\tau)]}, \tag{4.13}$$

where, now, the topology of the world-sheet torus, T^2, in the function space $F_{X_{T^2}/\equiv}$, was evidently expressed with the manifest appearance of the appropriate delta-functional factors, $\delta_{\sigma^2}[f(X(2\pi)-X(0))]$ and $\delta_{\sigma^1}[f(X(2\pi)-X(0))]$, multiplying the respective functional integration measure $[dX]_{\gamma_{T^2}^{(N)}(\tau)}$, that correspond to the real, conformal world-sheet coordinate system (σ^1,σ^2) of the Riemann torus, T^2. Of course, it can be easily seen that, in the functional integral of the above *eq*.(4.13), the classical field, $X_{c.}$, cannot but satisfy that:

$$X_{c.}(\sigma^1,\sigma^2)=X_0+n\sigma^1+m\sigma^2, \tag{4.14}$$

where the integer elements, $n,m\in\mathbb{Z}^d$, are called its winding numbers and its momentum numbers, respectively; hence, in the functional integral of *eq*.(4.13), the action $S[X,\gamma_{T^2}^{(N)}(\tau)]$ can, obviously, be expressed as:

$$\begin{aligned}S[X,\gamma_{T^2}^{(N)}(\tau)]&=-\frac{\pi}{\tau_2\alpha'}(n\tau-m)^{(T)}(G+B)(n\bar\tau-m)+\\&+\frac{1}{4\pi\alpha'}\int_{T^2}d^2\sigma\sqrt{|\det\big(\gamma_{T^2}^{(N)}(\tau)\big)|}G_{MN}X_{q.}^M\gamma_{T^2}^{(N)ab}(\tau)\partial_a\partial_bX_{q.}^N=\\&=-\frac{\pi}{\tau_2\alpha'}(n\tau-m)^{(T)}(G+B)(n\bar\tau-m)+S[X_{q.},\gamma_{T^2}^{(N)}(\tau)],\end{aligned} \tag{4.15}$$

and the functional integration measure, $[dX]_{\gamma_{T^2}^{(N)}(\tau)}$, can, obviously, be factorised, as in $[dX]_{\gamma_{T^2}^{(N)}(\tau)}=d^dX_0\sqrt{\det_{(d)}(G)}[dX_{q.}]_{\gamma_{T^2}^{(N)}(\tau)}$. Then, the functional integral of *eq*.(162), is reduced to the following:

$$Z_{T^d}(\tau,\bar\tau)=\int_{F_{X_{T^2}}/\equiv}[dX]_{\gamma_{T^2}(\tau)}e^{-S[X,\gamma_{T^2}(\tau)]}=$$

$$= \int_{\mathbb{R}^d/\equiv} d^d X_0 \sqrt{\underset{(d)}{\det}(G)} \sum_{n,m \in \mathbb{Z}^d} e^{\frac{\pi}{\tau_2 \alpha'}(n\tau - m)^{(T)}(G+B)(n\bar{\tau}-m)} \int_{F_{X_{q.}}} [dX_{q.}]_{\gamma^{(N)}_{T^2}(\tau)} e^{-S[X_{q.},\gamma^{(N)}_{T^2}(\tau)]} =$$

$$= V_{F_{X_{T^2}}/\equiv} \sum_{n,m \in \mathbb{Z}^d} e^{\frac{\pi}{\tau_2 \alpha'}(n\tau - m)^{(T)}(G+B)(n\bar{\tau}-m)} \int_{F_{X_{q.}}} [dX_{q.}]_{\gamma^{(N)}_{T^2}(\tau)} e^{-S[X_{q.},\gamma^{(N)}_{T^2}(\tau)]} =$$

$$= \sqrt{\underset{(d)}{\det}(G)} \sum_{n,m \in \mathbb{Z}^d} e^{\frac{\pi}{\tau_2 \alpha'}(n\tau - m)^{(T)}(G+B)(n\bar{\tau}-m)} (\alpha' \tau_2)^{-\frac{d}{2}} (|\eta(q(\tau))|)^{-2d}. \tag{4.16}$$

Now, the Poisson resummation formula can be used:

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \mathcal{F}[f](n), \tag{4.17}$$

where $\mathcal{F}$ is the operator of the Fourier transformation; it can be proved that, the result of the appropriate use of the Poisson ressumation formula, with respect to the momentum numbers m of the classical field $X_{c.}$, in the preceding *eq*.(4.16), is equal to the following:

$$Z_{T^d}(\tau, \bar{\tau}) = \int_{F_{X_{T^2}}/\equiv} [dX]_{\gamma_{T^2}(\tau)} e^{-S[X,\gamma_{T^2}(\tau)]} = \Gamma_{(d,d)}(\tau, \bar{\tau}; G, B)(|\eta(q(\tau))|)^{-2d}, \tag{4.18}$$

where the factor $\Gamma_{(d,d)}(q(\tau), \bar{q}(\bar{\tau}); G, B)$ is called the d-dimensional Narain lattice, and it is equal to the following:

$$\Gamma_{(d,d)}(q(\tau), \bar{q}(\bar{\tau}); G, B) = \sum_{n,m \in \mathbb{Z}^d} q^{\frac{P^2_L(G,B)}{2}}(\tau) \bar{q}^{\frac{P^2_R(G,B)}{2}}(\bar{\tau}), \tag{4.19}$$

with:

$$P^2_{L,R}(G,B) = P^{(T)}_{L,R}(G,B) G^{(-1)} P_{L,R}(G,B), \tag{4.20}$$

and:

$$P_{L,R}(G,B) = \frac{1}{\sqrt{2}}(\sqrt{\alpha'} m \pm \frac{G \pm B}{\sqrt{\alpha'}} n), \tag{4.21}$$

respectively. Then, by multiplying the preceding *eqs*.(4.6) and (4.18), we arrive at the following result:

$$Z_{D;T^d}(\tau, \bar{\tau}) = Z^{(D-d)}_{\{X\}}(\tau, \bar{\tau}) Z_{T^d}(\tau, \bar{\tau}) = \Gamma_{(d,d)}(q(\tau), \bar{q}(\bar{\tau}); G, B)(|\eta(q(\tau))|)^{-2D}, \tag{4.22}$$

which can be interpreted as the contribution of D closed target-space string coordinates, in the case of a D-dimensional spacetime with a T^d-type toroidal compactification, as it was defined in *eq*.(4.2), to the partition function of a bosonic string theory. Now, by additionally assuming that this bosonic string theory is part of an arbitrary, general CFT, which exists in its critical dimension, $D_{crit.} = D$, the contribution of the b and c ghost fields to the partition function of the above *eq*.(4.22), leads to the following result:

$$Z_{\mathbb{R}^{D-d} \times T^d}(\tau, \bar{\tau}) = \Gamma_{(d,d)}(q(\tau), \bar{q}(\bar{\tau}); G, B)(|\eta(q(\tau))|)^{-2D+4}, \tag{4.23}$$

which can be interpreted as the partition function for a bosonic string theory, in the case of a D-dimensional spacetime with a T^d-type toroidal compactification, as it was defined in *eq*.(4.2). That is, the result of the above *eq*.(4.23) yields the full partition function of the bosonic string theory part of an arbitrary, general theory, which exists in its critical dimension, D, for the D-dimensional spacetime with a T^d-type toroidal compactification. Evidently, from the above *eq*.(4.23), it can be seen that the sole effect of the spacetime T^d-type toroidal compactification is the emergence of a factor of $\Gamma_{(d,d)}(q(\tau), \bar{q}(\bar{\tau}); G, B)$, of the d-dimensional Narain lattice, in the generic partition function of bosonic string theory. Of course, the spacetime T^d-type toroidal compactification, as it was defined in *eq*.(4.2) has, obviously, no effect whatsoever on any world-sheet fermion fields that may possibly be incorporated into the theory, although we note that, in the context of superstring theory, the spacetime states that correspond to the, d in number, pairs of world-sheet fermion fields, which are the superpartners of the, d in number, target-space string coordinates, that are in the compactified spacetime T^d, have a different interpretation, in a group-theoretical sense. Consistently, the partition function of the theory of the D pairs of world-sheet fermion fields, which are the superpartners of the D target-space string coordinates of bosonic string theory, included in a general, arbitrary theory, which exists in its critical dimension, $D_{crit.} = D$, in the case of a D-dimensional spacetime with a T^d-type toroidal compactification, is still identical in form to the partition function of the fermionic part of superstring theory, in the sense that it is equal to the following:

$$Z^{(F)}_{\mathbb{R}^{D-d}\times T^d}(\tau,\bar{\tau}) = \sum_{a,b=0}^{1}\sum_{\tilde{a},\tilde{b}=0}^{1} Z^{(F)}_{\mathbb{R}^{D-d}\times T^d}\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\tau,\bar{\tau}) =$$

$$= \frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+mab}\left(\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^{\frac{D}{2}-1}\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^{\frac{D}{2}-1}, \tag{4.24}$$

where the contribution of the β and γ superghost fields, $Z_{\{\beta,\gamma\};m,\tilde{m}}(\tau,\bar{\tau})$, as it has been expressed in *eq*.(2.144), has been already included. Eventually, by multiplying the *eqs*.(4.23) and (4.24), for the special case of the critical dimension of superstring theory, $D_{crit.} = D = 10$, the result can, obviously, be expressed as in the following:

$$Z_{II;m,\tilde{m}|\mathbb{R}^{10-d}\times T^d}(\tau,\bar{\tau}) = Z_{\mathbb{R}^{10-d}\times T^d}(\tau,\bar{\tau})Z^{(F)}_{\mathbb{R}^{10-d}\times T^d}(\tau,\bar{\tau}) =$$

$$= \Gamma_{(d,d)}(q(\tau),\bar{q}(\bar{\tau});G,B)(|\eta(q(\tau))|)^{-16}$$

$$\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+mab}\left(\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^{4}\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^{4}, \tag{4.25}$$

where, of course, $0 \leq d \leq 8$; The result of the above *eq*.(4.25) is, evidently, the partition function of a type II superstring theory, in the case of a $D = 10$-dimensional spacetime with a T^d-type toroidal compactification.

4.1.1 Orbifolds and Space-Time $T^d/\mathbb{Z}_N$-type Orbifold Compactification

Considering a discrete symmetry group, G, of a manifold M, an orbifold can be defined as the quotient set M/G. If the manifold M has no fixed points under the action of its symmetry group G, then the symmetry group G is said to be freely acting, and M/G is smooth, whereas if the manifold M has fixed points under the action of its symmetry group G, then the symmetry group G is non-freely acting, and M/G is no longer smooth, but has conical singularities at these fixed points, which are known as orbifold singularities. Orbifolds are interesting in the context of CFTs and superstring theory, as they provide new spaces for spacetime string compactification, and a way of explicitly reducing the number of supercharges, while also admitting an exact CFT description, meaning that strings propagate smoothly on them, in the sense that the correlation functions of the respective CFT are finite.

We shall principally consider the theory of the orbifold $T^2/\mathbb{Z}_N$. The spacetime $T^2/\mathbb{Z}_N$-type orbifold compactification means precisely the enforcement of the following equivalence relation:

$$z \sim e^{\frac{2\pi i}{N}}z + 2\pi, \tag{4.26}$$

or:

$$z \equiv e^{\frac{2\pi i c}{N}}z; c \in \mathbb{N}_N \tag{4.27}$$

that is, concerning a specific, 2-dimensional complexified subspace, $z = z(x^1, x^2)$, of the D-dimensional spacetime, which may be called the twisted compactified space, or, simply, the internal space of the orbifold $T^2/\mathbb{Z}_N$; then, the topology of the orbifold $T^2/\mathbb{Z}_N$ is defined by a complex target-space string coordinate, Z, which, on a world-sheet of the topology of the torus, T^2, satisfies the following boundary conditions:

$$Z(\sigma^1 + 2\pi, \sigma^2) \sim Z(\sigma^1, \sigma^2) \equiv e^{\frac{2\pi i h}{N}} Z(\sigma^1, \sigma^2), \tag{4.28}$$

or:

$$Z(\sigma^1 + 2\pi, \sigma^2) \equiv e^{\frac{2\pi i h}{N}} Z(\sigma^1, \sigma^2), \tag{4.29}$$

and:

$$Z(\sigma^1, \sigma^2 + 2\pi) \sim Z(\sigma^1, \sigma^2) \equiv e^{\frac{2\pi i g}{N}} Z(\sigma^1, \sigma^2), \tag{4.30}$$

or:

$$Z(\sigma^1, \sigma^2 + 2\pi) \equiv e^{\frac{2\pi i h}{N}} Z(\sigma^1, \sigma^2), \tag{4.31}$$

which are simply the boundary conditions of the preceding *eqs*.(4.9) and (4.10), now for a complex target-space string coordinate, Z, and "twisted" by the factors $e^{\frac{2\pi i h}{N}}$ and $e^{\frac{2\pi i g}{N}}$, respectively, where $h, g \in \mathbb{N}_N$, of course. The action, $S[Z, \gamma_{T^2}(\tau)]$, of the theory of the orbifold $T^2/\mathbb{Z}_N$, is given by the non-linear sigma model of the preceding *eq*.(4.1) of the previous section, yet now for the complex target-space string coordinate, $Z = G_1 X^1 + G_2 X^2$, and its complex conjugate, of the topology of the orbifold $T^2/\mathbb{Z}_N$, where $G_1 = \sqrt{G_{11}}$ and $G_2 = \frac{G_{12} + i\sqrt{det(G)}}{\sqrt{G_{11}}}$, and evaluated in the world-sheet metric $\gamma_{T^2}(\tau)$ of the world-sheet torus, T^2 of the complex modulus τ, as in the following:

$$S[Z, \bar{Z}; \gamma_{T^2}(\tau)] = -\frac{1}{4\pi\alpha'} \int_{T^2} d^2\sigma \sqrt{|\det(\gamma_{T^2}(\tau))|}\gamma^{ab}_{T^2}(\tau)\partial_a Z \partial_b \bar{Z} - \frac{1}{4\pi\alpha'} \int_{T^2} d^2\sigma i \sqrt{|\det(B)|}\epsilon^{ab}\partial_a Z \partial_b \bar{Z}. \tag{4.32}$$

Naturally, the partition function of the theory of the orbifold $T^2/\mathbb{Z}_N$ can be calculated from the following functional integral:

$$Z_{T^2/\mathbb{Z}_N}(\tau, \bar{\tau}; G, B) = \frac{1}{2} \int_{F_{X_{T^2}}/\sim} [d^2 Z]_{\gamma_{T^2}(\tau)} e^{-S[Z, \bar{Z}; \gamma_{T^2}(\tau)]}, \tag{4.33}$$

where, evidently, the functional integration measure is equal to $[d^2 Z]_{\gamma_{T^2}(\tau)} = 2[dX]\sqrt{\det_{(2)}(G)} = 2[dX^1][dX^2]\sqrt{\det_{(2)}(G)}$, and over the functional space $F_{X_{T^2}}/\sim = (F_{X_{T^2}}/\equiv)/\mathbb{Z}_N$, of volume that is equal to:

$$V_{F_{X_{T^2}}/\sim} = V_{(F_{X_{T^2}}/\equiv)/\mathbb{Z}_N} = \frac{1}{V_{\mathbb{Z}_N}} V_{F_{X_{T^2}}/\equiv} = \frac{1}{N} V_{F_{X_{T^2}}/\equiv} = \frac{1}{N} 4\pi^2 \sqrt{\det_{(2)}(G)}.$$

Evidently, for the case $(h, g) = (0, 0)$, which is called the $(0, 0)$-sector of the theory of the orbifold $T^2/\mathbb{Z}_N$, the calculation of the partition function of *eq*.(4.33) can be done simply by replicating the procedure that was presented in the previous chapter, for the case of the value $d = 2$, yet, now, using the boundary conditions of the topology of the orbifold $T^2/\mathbb{Z}_N$, instead of the boundary conditions of the preceding *eqs*.(4.9) and (4.10); the result is, then, given by the partition function of the theory of $d = 2$ target-space string coordinates with spacetime T^2-type toroidal compactification, as it can be expressed by the preceding *eq*.(4.18), of the previous section, for the case of the value $d = 2$:

$$Z_{T^2/\mathbb{Z}_N}\begin{bmatrix}0\\0\end{bmatrix}(\tau, \bar{\tau}; G, B) = \Gamma_{(2,2)}(q(\tau), q(\bar{\tau}); G, B)(|\eta(q(\tau))|)^{-4}. \tag{4.34}$$

Similarly, for the cases $(h, g) \neq (0, 0)$, which are respectively called the (h, g)-sectors of the theory of the orbifold $T^2/\mathbb{Z}_N$, the calculation of the partition function of *eq*.(4.33) can also be done in a simple way, again by replicating the procedure that was presented in the previous chapter, for the case of the value $d = 2$, yet, now, using the boundary conditions of the topology of the orbifold $T^2/\mathbb{Z}_N$, instead of the boundary conditions of the preceding *eqs.*(4.9) and (4.10); it can be straightforwardly proved, then, that the result is:

$$Z_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g\end{bmatrix}(\tau,\bar{\tau};G,B)=\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g\end{bmatrix}\frac{\eta(q(\tau))}{\theta\begin{bmatrix}1-\frac{2h}{N}\\1-\frac{2g}{N}\end{bmatrix}(q(\tau))}\frac{\bar{\eta}(\bar{q}(\bar{\tau}))}{\bar{\theta}\begin{bmatrix}1-\frac{2h}{N}\\1-\frac{2g}{N}\end{bmatrix}(\bar{q}(\bar{\tau}))},\tag{4.35}$$

where the numbers $\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g\end{bmatrix}$ just express the number of fixed points of the torus T^2, under the action of its symmetry group $\mathbb{Z}_N$, in the respective (h,g)-sector of the theory of the orbifold $T^2/\mathbb{Z}_N$. We note that the partition function of *eq.*(4.33), of the theory of the orbifold $T^2/\mathbb{Z}_N$ in its (h,g)-sectors, where $(h,g)\neq(0,0)$, does not depend at all on the spacetime background fields G and B; it can be proved that this is a general characteristic of the cases of non-freely acting orbifolds, such as the orbifold $T^2/\mathbb{Z}_N$ that is presently studied, while the situation is different in the cases of freely acting orbifolds [2]. The full partition function of the theory of the orbifold $T^2/\mathbb{Z}_N$, which is denoted as $Z_{T^2/\mathbb{Z}_N}(\tau,\bar{\tau};G,B)$, as it is expressed by the functional integral of *eq.*(4.33), is naturally given by the weighted, that is, divided by the volume of the discrete symmetry group $\mathbb{Z}_N$ of the torus T^2, summation of the partition functions of the preceding *eqs.*(4.34) and (4.35), over all the cases of its (h,g)-sectors, and it can expressed in a concise manner, as in the following:

$$Z_{T^2/\mathbb{Z}_N}(\tau,\bar{\tau};G,B)=\frac{1}{N}\sum_{h,g\in\mathbb{N}_N}Z_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g\end{bmatrix}(\tau,\bar{\tau};G,B).\tag{4.36}$$

The $(h=0,g)$-sectors and $(h\neq0,g)$-sectors of the theory of the orbifold $T^2/\mathbb{Z}_N$ are conventionally called its untwisted sectors and twisted sectors, respectively. It can be proved that the numbers $\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g\end{bmatrix}$, that express the number of fixed points of the torus T^2, under the action of its symmetry group $\mathbb{Z}_N$, in the respective (h,g)-sector of the theory of the orbifold $T^2/\mathbb{Z}_N$, satisfy the following modular transformation requirements:

$$\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g\end{bmatrix}=\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}g\\h\end{bmatrix},\tag{4.37}$$

and:

$$\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g\end{bmatrix}=\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g+h\end{bmatrix},\tag{4.38}$$

in accordance with the property of modular invariance of the full partition function of the theory of the orbifold $T^2/\mathbb{Z}_N$, as it is expressed in *eq.*(4.36); in particular, it can be proved, using Pick's theorem, as a special case of the Lefschetz fixed-point theorem, that in the partition function of *eq.*(4.35), for the specific case of the untwisted $(0,g)$-sector of the theory of the orbifold $T^2/\mathbb{Z}_N$, the corresponding numbers $\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}h=0\\g\end{bmatrix}$ can be given by the following formula:

$$\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}0\\g\end{bmatrix}=4\sin^2(\frac{\pi g}{N});\tag{4.39}$$

of course, the number $\chi_{T^2/\mathbb{Z}_N}\begin{bmatrix}h\\g\end{bmatrix}$, for the case of an arbitrary (h,g)-sector of the theory of the orbifold $T^2/\mathbb{Z}_N$, can then be calculated simply by using the above *eq.*(4.39), along with its modular transformation requirements, of *eqs.*(4.37) and (4.38). In the very same fashion as above, it is evident that the spacetime $T^2/\mathbb{Z}_N$-type orbifold compactification, has a rather more trivial effect on the 2 pairs of world-sheet fermion fields, which are the superpartners of the above complex target-space string coordinate Z, than it has on the complex target-space string coordinate Z itself, simply twisting their two distinct kinds of possible boundary conditions on the world-sheet torus, T^2, by the factors $e^{\frac{2\pi i h}{N}}$ and $e^{\frac{2\pi i g}{N}}$, respectively, where $h, g \in \mathbb{N}_N$, of course; the partition function of the (h, g)-sector of the $T^2/\mathbb{Z}_N$-orbifold theory of the 2 pairs of world-sheet fermion fields, which are the superpartners of the above complex target-space string coordinate Z, the boundary conditions of which were twisted by the equivalence relation of the spacetime $T^2/\mathbb{Z}_N$-type orbifold compactification, is, then, equal to:

$$Z^{(F)}_{T^2/\mathbb{Z}_N}\begin{bmatrix} a, h \\ b, g \end{bmatrix}\begin{bmatrix} \tilde{a}, h \\ \tilde{b}, g \end{bmatrix}(\tau, \bar{\tau}) = \left(\frac{\theta\begin{bmatrix} a+\frac{2h}{N} \\ b+\frac{2g}{N} \end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)\left(\frac{\bar{\theta}\begin{bmatrix} \tilde{a}+\frac{2h}{N} \\ \tilde{b}+\frac{2g}{N} \end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right). \tag{4.40}$$

In accordance with the previous section, we can see that, unsurprisingly, in the $(h, g) = (0, 0)$-sector of the fermionic partition function of the above *eq*.(4.40), the spacetime T^d-type toroidal compactification alone has no effect whatsoever on the world-sheet fermion fields, apart from the different interpretation, in a group-theoretical sense, of the spacetime states that correspond to these, d in number, pairs of world-sheet fermion fields which are the superpartners of the, d in number, target-space string coordinates that are in the compactified space of T^d-type toroidal compactification; evidently, only the spacetime $T^d/\mathbb{Z}_N$-type orbifold compactification, in its $(h, g) \neq (0, 0)$-sectors, can essentially affect the world-sheet fermion fields, in the sense that was described above, concerning the specific case of the value of $d = 2$. Eventually, as the higher-dimensional torus, T^{2d}, includes, and may be factorised into lower-dimensional tori T^2, the partition function of a type II superstring theory, compactified on the appropriately factorised orbifold $\mathbb{R}^4 \times T^2 \times (T^4/\mathbb{Z}_2) = \mathbb{R}^4 \times T^2 \times ((T^2 \times T^2)/\mathbb{Z}_2)$, for example, is, evidently, equal to the following:

$$Z_{II;m,\tilde{m}|\mathbb{R}^4\times T^2\times(T^4/\mathbb{Z}_2)}(\tau, \bar{\tau}) = Z_{\mathbb{R}^4\times T^2\times((T^2\times T^2)/\mathbb{Z}_2)}(\tau, \bar{\tau}) =$$

$$= \frac{1}{2}\sum_{h,g\in\mathbb{N}_2}\Gamma_{(2,2)}(q(\tau), \bar{q}(\bar{\tau}); G, B)(|\eta(q(\tau))|)^{-8}\left(Z_{T^2/\mathbb{Z}_2}\begin{bmatrix} h \\ g \end{bmatrix}(\tau, \bar{\tau}; G, B)\right)^2$$

$$\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+mab}\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\left(Z^{(F)}_{T^2/\mathbb{Z}_2}\begin{bmatrix} a, h \\ b, g \end{bmatrix}\begin{bmatrix} \tilde{a}, h \\ \tilde{b}, g \end{bmatrix}(\tau, \bar{\tau})\right)^2\left(\frac{\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^2\left(\frac{\bar{\theta}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^2 =$$

$$= \frac{1}{2}\sum_{h,g\in\mathbb{N}_2}\Gamma_{(2,2)}(q(\tau), \bar{q}(\bar{\tau}); G, B)(|\eta(q(\tau))|)^{-8}Z_{T^2/\mathbb{Z}_2}\begin{bmatrix} h \\ g \end{bmatrix}(\tau, \bar{\tau}; G, B)Z_{T^2/\mathbb{Z}_2}\begin{bmatrix} -h \\ -g \end{bmatrix}(\tau, \bar{\tau}; G, B)$$

$$\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+mab}\left(\frac{\theta\begin{bmatrix} a \\ b \end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^2\frac{\theta\begin{bmatrix} a+h \\ b+g \end{bmatrix}(q(\tau))}{\eta(q(\tau))}\frac{\theta\begin{bmatrix} a-h \\ b-g \end{bmatrix}(q(\tau))}{\eta(q(\tau))}$$

$$\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^2\frac{\bar{\theta}\begin{bmatrix} \tilde{a}+h \\ \tilde{b}+g \end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\frac{\bar{\theta}\begin{bmatrix} \tilde{a}-h \\ \tilde{b}-g \end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}, \tag{4.41}$$

and the partition functions for the cases of a type II superstring theory compactified on various appropriately factorised orbifolds of the above kind can be calculated similarly.

Chapter 5

The One-Loop Level Correction to the Einstein-Hilbert term

As it was mentioned throughout this report, the string length, $l_s = \sqrt{\alpha'}$, or, equivalently, the string mass, $M_s = \frac{1}{\sqrt{\alpha'}}$, is the sole dimensionful, physical parameter of string theory; in the level of the corresponding effective field theory, the string mass, $M_s = \frac{1}{\sqrt{\alpha'}}$, is related to the dimensionless string coupling constant, g_s, through the definition of the Planck mass, as in the following:

$$M_P^2 = \frac{M_s^2}{g_s^2}, \tag{5.1}$$

which is to be identified with the coefficient of the spacetime Einstein-Hilbert term, $\int_{\mathbb{R}^n} d^n x \sqrt{|\det(G)|}\mathcal{R}$, in the effective action of the string. In principle, then, the Planck mass is subject both to perturbative and non-perturbative corrections; it has been proved in [5] that it receives no radiative corrections in the case of a heterotic superstring, although it might receive such constant corrections in the type I and type IIB superstring theory. In general, as it is conjectured that superstring theory is a UV-finite theory, the radiative corrections to the various, running gauge coupling constants, which are to be identified with the coefficient of the gauge group kinetic terms of the corresponding effective field theory, may exhibit at most IR divergencies. In what follows, we will briefly present some results that capture the spirit of the renormalisation of the gauge coupling constants of the effective field theory with respect to attempts at their string unification. We will argue that possible quantum corrections to the Einstein-Hilbert term may modify *eq*.(5.1) and so affect the string unification mass scale. We will also shortly present the essence of the background field method approach to the calculation of the one-loop level perturbative corrections to the gauge coupling constants, as well as its limitations; in particular, we will see that the background field method seems to fail in superstring theories of 8-dimensional orbifolds, where the whole spacetime but its 2 lightcone dimensions is twisted and compactified. We will then resort to the conventional approach to the calculation of the one-loop level perturbative correction to the Planck mass, which is through the explicit calculation of the scattering amplitude of two gravitons, in the one-loop level, that corresponds to the one-loop level perturbative correction in the spacetime Einstein-Hilbert term of the effective field theory. Eventually, we will present a proof of the general form of the one-loop level perturbative correction to the Planck mass in the type II superstring theory, as well as an application of this proof to the specific case of the calculation of the quantum correction to the Einstein-Hilbert term, in a type IIB superstring theory, compactified on a $K3 \times K3$-orbifold.

5.1 The Renormalisation of the Gauge Couplings and String Unification

In the effective field theory level of a superstring theory, the renormalisation group flow for the gauge coupling $g_F(\mu)$ that refers to the gauge group F is given by the following relation, which holds up to the one-loop level of perturbation theory [4]:

$$\frac{1}{g_F^2(\mu)} = \frac{1}{g_s^2} + \frac{\beta_F}{16\pi^2}\log\left(\frac{M_s^2}{\mu^2}\right) + \frac{\Delta_F(G,B)}{16\pi^2}, \tag{5.2}$$

where G and B are the scalar moduli arising from the toroidal compactification of the 10-dimensional spacetime metric and Kalb-Ramond field respectively. In the above *eq*.(5.2), the beta-function term $\frac{\beta_F}{16\pi^2}\log\left(\frac{M_s^2}{\mu^2}\right)$ expresses the logarithmic dependence of the gauge coupling $g_F(\mu)$ on the mass scale μ, which exists in a QFT independently of the superstring theory; it is exclusively due to the massless spectrum of the theory, and it exhibits an IR divergence, as it is expected to in a superstring theory. Separately, the moduli-dependent term $\frac{\Delta_F(G,B)}{16\pi^2}$, of the above *eq*.(5.2), is called the threshold correction to the gauge coupling $g_F(\mu)$; it expresses its dependence on the moduli G and B, it is exclusively due to the massive spectrum of the theory, and it is purely an effect of string theory. In theories preserving an unbroken $N = 2$ supersymmetry, a special universality structure often emerges, and it can be proved that the threshold correction takes the following form [2]:

$$\Delta_F(G,B) = \beta_F\Delta(G,B) - \Upsilon(G,B), \tag{5.3}$$

where the gauge group dependence is, now, solely in the beta-function factor, β_F, while the moduli-dependent factor $\Delta(G,B)$ is exclusively due to the charged massive spectrum of the theory, that is, due to the Kaluza-Klein momenta and winding modes of the string, and the term $\Upsilon(G,B)$ is a universal moduli-dependent contribution. The result of the above *eq*.(5.3) is commonly referred to as the $N = 2$ universality. We note that the functions $\Delta(G,B)$ and $\Upsilon(G,B)$ are known functions of the compactification moduli and their form can be almost entirely fixed by general principles of holomorphy and unitarity. Then, the $N = 2$ universality of *eq*.(5.3), combined with the the renormalisation group flow for the gauge coupling $g_F(\mu)$ of *eq*.(5.2) result in the following:

$$\frac{1}{g_F^2(\mu)} = \frac{1}{g_U^2(G,B)} + \frac{\beta_F}{16\pi^2}\log\left(\frac{M_U^2(G,B)}{\mu^2}\right), \tag{5.4}$$

where the moduli-dependent quantities:

$$g_U(G,B) = \frac{g_s}{\sqrt{1 - \frac{\Upsilon(G,B)}{16\pi^2}}}, \tag{5.5}$$

and:

$$M_U(G,B) = M_s e^{\frac{\Delta(G,B)}{2}}, \tag{5.6}$$

are respectively called the unification gauge coupling constant and the unification mass scale. Evidently, the renormalisation group flow of *eq*.(5.4) implies an automatic string unification of all gauge couplings into the unified value $g_U(G,B)$, of *eq*.(5.5), at the string unification scale $\mu = M_U(G,B)$, of *eq*.(5.6). Assuming the desert scenario from a bottom-up approach, it would be tempting to identify the string unification data, $(g_U(G,B), M_U(G,B))$, with the corresponding values arising in GUTs. This possibility, however, relies on the existence of specific moduli configurations, (G,B), consistent with the aforementioned identification, $\frac{g_U^2(G,B)}{4\pi} = \frac{g_{GUT}^2}{4\pi} \approx \frac{1}{25}$, and $M_U(G,B) = M_{GUT} \approx 10^{16}GeV$. Assumming, additionally, a constant, one-loop level perturbative correction c to the Einstein-Hilbert term, we will have:

$$M_P^2 = \frac{M_s^2}{g_s^2} + M_s^2 c, \tag{5.7}$$

and combining the above relation with the renormalisation group flow of *eq*.(5.4), the unification mass of *eq*.(5.6) can be expressed as:

$$M_U(G,B) = \frac{M_P}{\sqrt{1+g_U^2(G,B)(\frac{Y(G,B)}{16\pi^2}+c)}} g_U(G,B) e^{\frac{\Delta(G,B)}{2}}. \tag{5.8}$$

As it has been formerly mentioned, it has been proved in [5] that the constant radiative correction factor c, in the definition of of *eq*.(5.7) for the Planck mass is zero, and that the Planck mass recieves no radiative corrections whatsoever, in the case of a heterotic superstring theory. Unfortunately, it can be shown that, in the most simple cases, no (G,B) moduli configurations exists, such that the string unification data matches the values of the GUT scale parameters. Rather, the best one can do in heterotic theories is $M_U \approx 20 M_{GUT}$. In the case of type IIB orientifolds or type I superstrings, where gauge theory lives on D-branes, this discrepancy can be usually lifted by a suitable choice of the D-brane volume. However, there can be, in principle, radiative corrections to the Planck mass, which again modify the relation between g_s and M_s, as well as the metric of Kähler moduli in the string effective action (after a suitable Weyl rescaling to go to the Einstein frame). One such contribution is common to type I and type IIB orientifolds, and arises from the closed string oriented topology of the world-sheet torus.
In what follows, we will give the general form of the one-loop level perturbative correction to the spacetime Einstein-Hilbert term of the corresponding effective field theory, in the case of a type IIB superstring theory. We will principally attend to some special cases of a type IIB superstring theory, with various examples that demonstrate the benefits and limitations of the background field method approach.

5.2 The Background Field Method

The background field method in string theory was introduced in [5], where it is motivated with respect to the calculation of one-loop level perturbative corrections to coupling constants; a comprehensive summary of the background field method is also presented in [6], where it is used for the calculation of gravitational threshold corrections in the case of non-supersymmetric heterotic strings. The essence of the background field method is the introduction of a marginal, that is, an exactly solvable, deformation to the general non-linear sigma model of *eq*.(2.1); this deformation corresponds to a small, constant spacetime curvature, $\mathcal{R}$, and the correlation functions of interest are then calculated as the corresponding derivatives of the deformed partition function, $Z(\tau,\bar{\tau};\mathcal{R})$ [5]. It is proved in [5] that the deformed partition function, $Z(\tau,\bar{\tau};\mathcal{R})$, corresponds, in essence, to a Lorenzian boost of the left and right moving R-symmetry lattices, preserving $L_0 - \bar{L}_0 = const.$. This amounts to the introduction of complex, z and $\bar{z}$ -deformations, to the zero-valued Jacobi parameters, $\upsilon = 0$ and $\bar{\upsilon} = 0$, of the Jacobi theta-functions which correspond to a pair of untwisted left and right world-sheet fermions, $\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau);\upsilon=0)$ and $\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau});\bar{\upsilon}=0)$, respectively, as in:

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau);\upsilon=0) \to \theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau);\upsilon=z),$$

and:

$$\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau});\bar{\upsilon}=0) \to \theta\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau});\bar{\upsilon}=\bar{z}),$$

in the original, undeformed partition function $Z(\tau,\bar{\tau}) = Z(\tau,\bar{\tau};\mathcal{R}=0)$; consequently, the correlation functions of interest can be calculated from the corresponding derivatives of the deformed partition function, $Z(\tau,\bar{\tau};z,\bar{z})$, with respect to the z and $\bar{z}$ -deformed Jacobi parameters of these Jacobi theta-functions, $\theta\begin{bmatrix}a\\b\end{bmatrix}(\tau;z)$ and $\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau});\bar{z})$, evaluated at the point $z=\bar{z}=0$. It is also proved in [5] that the correlation function of two gravitons, which can give the one-loop level perturbative correction to the corresponding spacetime Einstein-Hilbert term of the effective field theory, is related to the following derivatives of the deformed partition function, $Z(\tau,\bar{\tau};z,\bar{z})$:

$$\langle\mathcal{R}\rangle_{one-loop} = \int_{\mathbb{F}_{T^2}} \frac{d^2\tau}{\tau_2^3} 2\pi\tau_2 \lim_{\bar{z}\to 0}\left(\frac{1}{2\pi i}\bar{\partial}_{\bar{z}}\right)\lim_{z\to 0}\left(\frac{1}{2\pi i}\partial_z\right) Z(z,\bar{z};\tau,\bar{\tau}). \tag{5.9}$$

For example, in the case of a type II superstring theory compactified on the factorised orbifold $\mathbb{R}^4\times T^2\times(T^4/\mathbb{Z}_2) = \mathbb{R}^4\times T^2\times((T^2\times T^2)/\mathbb{Z}_2)$, the partition function was given in *eq*.(4.41) of the previous chapter, and it is now deformed as in the following:

$$\begin{aligned}
&Z_{II;m,\tilde{m}|\mathbb{R}^4\times T^2\times(T^4/\mathbb{Z}_2)}(\tau,\bar{\tau};z,\bar{z}) = Z_{\mathbb{R}^4\times T^2\times((T^2\times T^2)/\mathbb{Z}_2)}(\tau,\bar{\tau};z,\bar{z}) = \\
&= \frac{1}{2}\sum_{h,g\in\mathbb{N}_2} \Gamma_{(2,2)}(q(\tau),\bar{q}(\bar{\tau});G,B)(|\eta(q(\tau))|)^{-8} Z_{T^2/\mathbb{Z}_2}\begin{bmatrix}h\\g\end{bmatrix}(\tau,\bar{\tau};G,B) Z_{T^2/\mathbb{Z}_2}\begin{bmatrix}-h\\-g\end{bmatrix}(\tau,\bar{\tau};G,B) \\
&\quad \frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+mab} \frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau);z)}{\eta(q(\tau))} \frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \frac{\theta\begin{bmatrix}a+h\\b+g\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \frac{\theta\begin{bmatrix}a-h\\b-g\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \\
&\quad \frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{m}\tilde{a}\tilde{b}} \frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau});\bar{z})}{\bar{\eta}(\bar{q}(\bar{\tau}))} \frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))} \frac{\bar{\theta}\begin{bmatrix}\tilde{a}+h\\\tilde{b}+g\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))} \frac{\bar{\theta}\begin{bmatrix}\tilde{a}-h\\\tilde{b}-g\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}.
\end{aligned} \tag{5.10}$$

The Jacobi theta-functions of a z-deformed Jacobi parameter, $\theta\begin{bmatrix}a\\b\end{bmatrix}(\tau;z)$, were given in *eq*.(3.14), by which one can deduce that the following holds:

$$\lim_{z\to 0}\left(\frac{1}{2\pi i}\partial_z\right)\theta\begin{bmatrix}a\\b\end{bmatrix}(\tau;z) = -i\eta^3(q(\tau))\delta_{a1}\delta_{b1}, \tag{5.11}$$

so, only the term $a = b = \tilde{a} = \tilde{b} = 1$, which can be called the odd spin-structure, of the partition function of the above *eq*.(5.10), may contribute to the the one-loop level perturbative correction in the corresponding spacetime Einstein-Hilbert term; we can single out this odd spin-structure as in the following:

$$\begin{aligned}
&Z_{II;m,\tilde{m}|\mathbb{R}^4\times T^2\times(T^4/\mathbb{Z}_2)}(\tau,\bar{\tau};z,\bar{z})_{odd\ spin} = Z_{\mathbb{R}^4\times T^2\times((T^2\times T^2)/\mathbb{Z}_2)}(\tau,\bar{\tau};z,\bar{z})_{odd\ spin} = \\
&= \frac{1}{2}\sum_{h,g\in\mathbb{N}_2} \Gamma_{(2,2)}(q(\tau),\bar{q}(\bar{\tau});G,B)(|\eta(q(\tau))|)^{-8} Z_{T^2/\mathbb{Z}_2}\begin{bmatrix}h\\g\end{bmatrix}(\tau,\bar{\tau};G,B) Z_{T^2/\mathbb{Z}_2}\begin{bmatrix}-h\\-g\end{bmatrix}(\tau,\bar{\tau};G,B) \\
&\quad \frac{1}{2}(-1)^{m} \frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(q(\tau);z)}{\eta(q(\tau))} \frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \frac{\theta\begin{bmatrix}1+h\\1+g\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \frac{\theta\begin{bmatrix}1-h\\1-g\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \\
&\quad \frac{1}{2}(-1)^{\tilde{m}} \frac{\bar{\theta}\begin{bmatrix}1\\1\end{bmatrix}(\bar{q}(\bar{\tau});\bar{z})}{\bar{\eta}(\bar{q}(\bar{\tau}))} \frac{\bar{\theta}\begin{bmatrix}1\\1\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))} \frac{\bar{\theta}\begin{bmatrix}1+h\\1+g\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))} \frac{\bar{\theta}\begin{bmatrix}1-h\\1-g\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))} = \\
&= \frac{1}{2}(|\eta(q(\tau))|)^{-16}\Gamma^3_{(2,2)}(q(\tau),\bar{q}(\bar{\tau});G,B)\frac{1}{4}(-1)^{m+\tilde{m}} \frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(q(\tau);z)}{\eta(q(\tau))}\left(\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^3 \frac{\bar{\theta}\begin{bmatrix}1\\1\end{bmatrix}(\bar{q}(\bar{\tau});\bar{z})}{\bar{\eta}(\bar{q}(\bar{\tau}))}\left(\frac{\bar{\theta}\begin{bmatrix}1\\1\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^3 +
\end{aligned}$$

$$+3\frac{16}{2}(|\eta(q(\tau))|)^{-8}\Gamma_{(2,2)}(q(\tau),\bar{q}(\bar{\tau});G,B)\frac{1}{4}(-1)^{m+\tilde{m}}\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(q(\tau);z)}{\eta(q(\tau))}\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\frac{\bar{\theta}\begin{bmatrix}1\\1\end{bmatrix}(\bar{q}(\bar{\tau});\bar{z})}{\bar{\eta}(\bar{q}(\bar{\tau}))}\frac{\bar{\theta}\begin{bmatrix}1\\1\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}. \tag{5.12}$$

Then, using the formula of the preceding *eq.*(5.9), we can see that the one-loop level perturbative correction to the corresponding spacetime Einstein-Hilbert term vanishes:

$$\langle\mathcal{R}\rangle_{one-loop|\mathbb{R}^4\times T^2\times(T^4/\mathbb{Z}_2)} =$$

$$= \int_{\mathbb{F}_{T^2}}\frac{d^2\tau}{\tau_2^3}2\pi\tau_2\lim_{\bar{z}\to 0}\left(\frac{1}{2\pi i}\bar{\partial}_{\bar{z}}\right)\lim_{z\to 0}\left(\frac{1}{2\pi i}\partial_z\right)Z_{II;m,\tilde{m}|\mathbb{R}^4\times T^2\times(T^4/\mathbb{Z}_2)}(\tau,\bar{\tau};z,\bar{z};)_{odd\ spin} = 0, \tag{5.13}$$

due to the Jacobi theta-function identity $\theta\begin{bmatrix}1\\1\end{bmatrix}(q(\tau)) = 0$. The above example demonstrates the formal and calculative elegance of the background field method for the calculation of the one-loop level perturbative correction in the spacetime Einstein-Hilbert term of the effective field theory, in the case of a type II superstring theory compactified on an orbifold. For the case of a type II superstring theory compactified on an orbifold $\mathbb{R}^4\times T^2\times T^4/\mathbb{Z}_2$, this correction vanishes due to the presence of fermionic zero modes in the odd spin-structure of the path integral. In the corresponding scattering amplitude, this is manifested in the presence of an untwisted theta-function $\theta\begin{bmatrix}1\\1\end{bmatrix}(q(\tau)) = 0$, that is associated with fermions which are periodic in both cycles of the world-sheet torus. In essence, the presence of these fermionic zero modes, in the odd spin-structure, reflect the fact that the T^2-torus directions are untwisted by the orbifold action. It is therefore clear that a non-vanishing contribution to the 4-dimensional spacetime Einstein-Hilbert term can only occur in the odd spin-structures of a type II superstring theory, and on those orbifold compactifications leaving no internal directions untwisted [5].

However, there are extreme cases, which consist of type II superstring theory compactified on an 8-dimensional orbifold, where the background method fails. In order to clarify the deficiency of the background field method, we may consider the partition function of a type II superstring theory compactified on the factorised orbifold $\mathbb{R}^2\times T^4/\mathbb{Z}_2\times T^4/\mathbb{Z}_2 = \mathbb{R}^2\times((T^2\times T^2)/\mathbb{Z}_2)\times((T^2\times T^2)/\mathbb{Z}_2) = \mathbb{R}^2\times K3\times K3$, which can be expressed as in the following:

$$Z_{II;m,\tilde{m}|\mathbb{R}^2\times K3\times K3}(\tau,\bar{\tau}) =$$

$$= \frac{1}{2}\sum_{h_1,g_1\in\mathbb{N}_2}\frac{1}{2}\sum_{h_2,g_2\in\mathbb{N}_2}Z_{T^2/\mathbb{Z}_2}\begin{bmatrix}h_1\\g_1\end{bmatrix}(\tau,\bar{\tau};G,B)Z_{T^2/\mathbb{Z}_2}\begin{bmatrix}-h_1\\-g_1\end{bmatrix}(\tau,\bar{\tau};G,B)Z_{T^2/\mathbb{Z}_2}\begin{bmatrix}h_2\\g_2\end{bmatrix}(\tau,\bar{\tau};G,B)Z_{T^2/\mathbb{Z}_2}\begin{bmatrix}-h_2\\-g_2\end{bmatrix}(\tau,\bar{\tau};G,B)$$

$$\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+mab}\frac{\theta\begin{bmatrix}a+h_1\\b+g_1\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\frac{\theta\begin{bmatrix}a-h_1\\b-g_1\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\frac{\theta\begin{bmatrix}a+h_2\\b+g_2\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\frac{\theta\begin{bmatrix}a-h_2\\b-g_2\end{bmatrix}(q(\tau))}{\eta(q(\tau))}$$

$$\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{m}\tilde{a}\tilde{b}}\frac{\bar{\theta}\begin{bmatrix}\tilde{a}+h_1\\\tilde{b}+g_1\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\frac{\bar{\theta}\begin{bmatrix}\tilde{a}-h_1\\\tilde{b}-g_1\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\frac{\bar{\theta}\begin{bmatrix}\tilde{a}+h_2\\\tilde{b}+g_2\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\frac{\bar{\theta}\begin{bmatrix}\tilde{a}-h_2\\\tilde{b}-g_2\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}, \tag{5.14}$$

where the background field method amounts to deformations to the Jacobi parameters of the Jacobi theta-functions that correspond to the world-sheet fermion superpartners of the lightcone target-space coordinates. In the odd spin-structure, the $z,\bar{z}$-deformation in the Jacobi theta-functions that correspond to the lightcone world-sheet fermions effectively computes:

$$\langle Q\bar{Q}\rangle_{one-loop} = \lim_{\bar{z}\to 0}\bar{\partial}_{\bar{z}}\lim_{z\to 0}\partial_z\langle e^{-z\int_W d^2w\Psi\bar{\Psi}-\bar{z}\int_W d^2w\tilde{\Psi}\bar{\tilde{\Psi}}}\rangle_{one-loop} = \langle\int_W d^2w\Psi\bar{\Psi}\int_W d^2w\tilde{\Psi}\bar{\tilde{\Psi}}\rangle_{one-loop}, \tag{5.15}$$

where Q is the $SO(2) \simeq U(1)$ charge associated to the left-moving complexified lightcone fermions $\Psi, \bar{\Psi}$, and similarly for the right-movers. The above one-loop level correlator function receives contributions only from zero modes, and gives:

$$\langle Q\bar{Q}\rangle_{one-loop} = \langle \Psi_0 \bar{\Psi}_0 \tilde{\Psi}_0 \bar{\tilde{\Psi}}_0 \rangle_{one-loop} = |\eta(q(\tau))|^4. \tag{5.16}$$

In the trivial orbifold sector $h_1 = g_1 = h_2 = g_2 = 0$, all left-moving fermions in the internal directions develop a total of 8 zero modes, two of which cancel against the zero mode contribution of the superghosts, and so the contribution vanishes, and similarly for the right-movers. Likewise, in the sectors where $h_1 = g_1 = 0; (h_2, g_2) \neq (0,0)$ or $h_2 = g_2 = 0; (h_1, g_1) \neq (0,0)$, there are 4 fermionic zero modes, two of which cancel against the superghosts, and the contribution vanishes as well. In all other orbifold sectors, the internal world-sheet fermions are twisted, so there are no fermionic zero modes. In those cases, the zero modes of the superghosts no longer cancel, and have to be removed by hand, in order to avoid a formally divergent path integral. After properly removing the zero modes, the contribution of the superghosts gives $|\eta(q(\tau))|^{-4}$, cancelling the lightcone correlator $\langle Q\bar{Q}\rangle$ of the above *eq.*(5.16). As a result, with this prescription, we expect the one-loop level contribution to the corresponding Einstein-Hilbert term to be given by:

$$\langle \mathcal{R}\rangle_{one-loop|\mathbb{R}^2\times K3\times K3} \propto \int_{\mathbb{F}_{T^2}} \frac{d^2\tau}{\tau_2^2} Z_{II;m,\tilde{m}|\mathbb{R}^2\times K3\times K3}(\tau,\bar{\tau})_{odd\ spin}. \tag{5.17}$$

The apparent failure of the supeghosts to cancel against fermionic zero modes in the odd spin-structure, reflects the fact that the background field method is ill-defined at face value, for superstring theories compactified on these exotic 2-dimensional spacetime constructions. In what follows we will justify the above prescription leading to *eq.*(5.17) by explicitly calculating the corresponding one-loop, two-graviton scattering amplitude.

5.3 The One-Loop Level Two-Graviton Scattering Amplitude

We shall now consider the scattering amplitude of two gravitons, on the Riemann surface of the world-sheet torus, T^2, in the odd spin-structure of a superstring theory. Since we are interested in the one-loop level perturbative correction to the corresponding spacetime Einstein-Hilbert term, we need to extract exactly two powers of momenta p. Here, we take spacetime to be 2-dimensional and assume that the theory is compactified on a general, 8-dimensional space, K. Our result will be quite general and valid both in the case of smooth compactification manifolds, as well as their singular (orbifold) limits, provided that the internal CFT description is well defined. In the odd spin-structure of the theory, world-sheet fermions in the spacetime directions contribute zero modes, and so do the left and right superghost fields. In order to obtain an non-vanishing result the latter must be soaked up by appropriate superghost correlators, necessitating a specific choice of superghost picture [7]. In particular, we may choose the one graviton in the $(-1,0)$-picture and the other graviton in the $(0,-1)$-picture. Furthermore, we need to cancel one unit of superghost charge in the left-moving sector, and similarly for the right-movers; this is achieved by inserting two Picture Changing Operators (PCOs) into the amplitude. The vertex operators of the two gravitons in the above asymmetric pictures read:

$$V_{(-1,0)}(p;z,\bar{z}) = h_{\mu\nu} e^{-\phi}\psi^\mu(\bar{\partial}X^\nu + i(p\cdot\tilde{\psi})\tilde{\psi}^\nu)e^{ip\cdot X}(z,\bar{z}), \tag{5.18}$$

and:

$$V_{(0,-1)}(p;z,\bar{z}) = h_{\mu\nu}(\partial X^\mu + i(p\cdot\psi)\psi^\mu)e^{-\tilde{\phi}}\tilde{\psi}^\nu e^{ip\cdot X}(z,\bar{z}), \tag{5.19}$$

while the vertex operators of the PCOs read:

$$V_{PCO;(1,0)}(z) = \oint_{C_z} \frac{dw}{2\pi i} e^{\phi}\psi_\mu \partial X^\mu(w), \tag{5.20}$$

and:

$$V_{PCO;(0,1)}(\bar{z}) = \oint_{C_z} \frac{d\bar{w}}{2\pi i} e^{\tilde{\phi}} \tilde{\psi}_\mu \bar{\partial} X^\mu(\bar{w}). \tag{5.21}$$

Here, the fields ϕ and $\tilde{\phi}$ arise from the bosonisation of the $\beta, \tilde{\beta}; \gamma, \tilde{\gamma}$-superghost system, p is the spacetime momentum, and $h_{\mu\nu} = h_{\mu\nu}(p)$ is the graviton polarisation. Normal ordering is implicitly assummed throughout. The scattering amplitude takes the following form:

$$\begin{aligned}
&\int_{\mathbb{F}_{T^2}} d^2\tau \frac{1}{2\tau_2} \int_W d^2z \frac{1}{\tau_2} \langle V_{PCO;(1,0)}(z) V^{(1)}_{(-1,0)}(p^{(1)}; z, \bar{z}) V_{PCO;(0,1)}(0) V^{(2)}_{(0,-1)}(p^{(2)}; 0, 0)\rangle_{T^2|odd} = \\
&= h^{(1)}_{\mu\nu} h^{(2)}_{\rho\sigma} \int_{\mathbb{F}_{T^2}} d^2\tau \frac{1}{2\tau_2} \int_W d^2z \frac{1}{\tau_2} \oint_{C_z} \frac{dw}{2\pi i} \oint_{C_0} \frac{d\bar{w}}{2\pi i} \\
&\langle e^{-\phi} \psi^\mu (\bar{\partial} X^\nu + i(p^{(1)} \cdot \tilde{\psi}) \tilde{\psi}^\nu) e^{ip^{(1)} \cdot X}(z, \bar{z}) (\partial X^\rho + i(p^{(2)} \cdot \psi) \psi^\rho) \\
&e^{-\tilde{\phi}} \tilde{\psi}^\sigma e^{ip^{(2)} \cdot X}(0, 0) e^{\phi} \psi_\alpha \partial X^\alpha(w) e^{\tilde{\phi}} \tilde{\psi}_\beta \bar{\partial} X^\beta(\bar{w})\rangle_{T^2|odd},
\end{aligned} \tag{5.22}$$

where the factor $\frac{1}{2\tau_2}$, in the modular integral over the fundamental domain $\mathbb{F}_{T^2}$, in the left-hand side of the above *eq*.(5.22) is, of course, the inverse of the volume of the CKG of the metrics on the world-sheet torus, T^2, of the complex modulus τ, as it has been shown that $V^{CKG}_{T^2} = 2\tau_2$. Making use of the symmetry of the CKG, we may fix the position of one graviton to zero, whereas the position $(z, \bar{z})$ of the other will be integrated over the Riemann surface of the world-sheet torus. By charge conservation, the left-moving world-sheet fermions and superghosts must necessarily contract with the corresponding fields in the PCOs, and similarly for the right-movers. Clearly, the bosons may only contract among themselves. Then, the scattering amplitude of the above *eq*.(5.22) is factorised as in the following:

$$\begin{aligned}
&\int_{\mathbb{F}_{T^2}} d^2\tau \frac{1}{2\tau_2} \int_W d^2z \frac{1}{\tau_2} \langle V_{PCO;(1,0)}(z) V^{(1)}_{(-1,0)}(p^{(1)}; z, \bar{z}) V_{PCO;(0,1)}(0) V^{(2)}_{(0,-1)}(p^{(2)}; 0, 0)\rangle_{T^2|odd} = \\
&= h^{(1)}_{\mu\nu} h^{(2)}_{\rho\sigma} \int_{\mathbb{F}_{T^2}} d^2\tau \frac{1}{2\tau_2} \int_W d^2z \frac{1}{\tau_2} \oint_{C_z} \frac{dw}{2\pi i} \oint_{C_0} \frac{d\bar{w}}{2\pi i} \langle e^{-\phi(z)} e^{\phi(w)}\rangle_{T^2|odd} \langle e^{-\tilde{\phi}(0)} e^{\tilde{\phi}(\bar{w})}\rangle_{T^2|odd} \\
&\langle \psi^\mu(z) \psi_\alpha(w)\rangle_{T^2|odd} \langle \tilde{\psi}^\sigma(0) \tilde{\psi}_\beta(\bar{w})\rangle_{T^2|odd} \langle \bar{\partial} X^\nu e^{ip^{(1)} \cdot X}(z, \bar{z}) \partial X^\rho e^{ip^{(2)} \cdot X}(0, 0) \partial X^\alpha(w) \bar{\partial} X^\beta(\bar{w})\rangle_{T^2}.
\end{aligned} \tag{5.23}$$

The fermionic and superghost correlators receive contributions only from zero modes, and are therefore constant. A direct calculation gives:

$$\langle e^{-\phi(z)} e^{\phi(w)}\rangle_{T^2} = \eta^{-2}(q(\tau)), \tag{5.24}$$

$$\langle e^{-\tilde{\phi}(0)} e^{\tilde{\phi}(\bar{w})}\rangle_{T^2} = \bar{\eta}^{-2}(\bar{q}(\bar{\tau})), \tag{5.25}$$

$$\langle \psi^\mu(z) \psi_\alpha(w)\rangle_{T^2} = \eta_{\alpha\kappa} \epsilon^{\mu\kappa} \eta^2(q(\tau)) = \epsilon^\mu{}_\alpha \eta^2(q(\tau)), \tag{5.26}$$

and:

$$\langle \tilde{\psi}^\sigma(0) \tilde{\psi}_\beta(\bar{w})\rangle_{T^2} = \eta_{\beta\lambda} \epsilon^{\sigma\lambda} \bar{\eta}^2(\bar{q}(\bar{\tau})) = \epsilon^\sigma{}_\beta \bar{\eta}^2(\bar{q}(\bar{\tau})). \tag{5.27}$$

Note that the $\tau, \bar{\tau}$-superghost contributions precisely cancel those of the spacetime fermions, as expected in a covariant description of the superstring. The contour integrals in w and $\bar{w}$ can only give non-vanishing contributions from appropriate simple poles; therefore, $\partial X^\alpha(w)$ can only contract with the exponential $e^{ip^{(1)} \cdot X}(z, \bar{z})$, and, similarly, $\bar{\partial} X^\beta(\bar{w})$ can only contract with the exponential $e^{ip^{(2)} \cdot X}(0, 0)$. Thus, $\bar{\partial} X^\nu(\bar{z})$ must necessarily contract with $\partial X^\rho(0)$. As a result, we have:

$$\begin{aligned}
&\int_W d^2z \frac{1}{\tau_2} \oint_{C_z} \frac{dw}{2\pi i} \oint_{C_0} \frac{d\bar{w}}{2\pi i} \langle \bar{\partial} X^\nu e^{ip^{(1)} \cdot X}(z, \bar{z}) \partial X^\rho e^{ip^{(2)} \cdot X}(0, 0) \partial X^\alpha(w) \bar{\partial} X^\beta(\bar{w})\rangle_{T^2} = \\
&= -p^{(1)}_\gamma p^{(2)}_\delta \int_W d^2z \frac{1}{\tau_2} \oint_{C_z} \frac{dw}{2\pi i} \oint_{C_0} \frac{d\bar{w}}{2\pi i} \langle X^\gamma(z, \bar{z}) \partial X^\alpha(w)\rangle_{T^2} \langle X^\delta(0, 0) \bar{\partial} X^\beta(\bar{w})\rangle_{T^2} \langle \bar{\partial} X^\nu(\bar{z}) \partial X^\rho(0)\rangle_{T^2} + O(p^3) =
\end{aligned}$$

$$= -p^{(1)}_{\gamma} p^{(2)}_{\delta} \eta^{\gamma\alpha}\eta^{\delta\beta} \int_W d^2 z \frac{1}{\tau_2} \langle \bar{\partial} X^{\nu}(\bar{z}) \partial X^{\rho}(0) \rangle_{T^2} + O(p^3). \tag{5.28}$$

In going from the first line to the second, we extracted the $O(p^2)$ terms that are relevant to our scattering, and in the third line we used the free scalar propagator on the world-sheet torus, T^2, of the complex modulus τ [2]:

$$\langle X^{\mu}(z,\bar{z}) X^{\nu}(0,0) \rangle = -\eta^{\mu\nu} \log \left(e^{-\frac{2\pi \operatorname{Im}^2(z)}{\tau_2}} \left| \frac{\theta_1(q(\tau);z)}{\partial_z \theta_1(q(\tau);0)} \right|^2 \right). \tag{5.29}$$

Then, the only remaining $z, \bar{z}$-dependence is in the factor $\langle \bar{\partial} X^{\nu}(\bar{z}) \partial X^{\rho}(0) \rangle_{T^2}$, which must be integrated over the world-sheet torus, giving:

$$\int d^2 z \frac{1}{\tau_2} \langle \bar{\partial} X^{\nu}(\bar{z}) \partial X^{\rho}(0) \rangle_{T^2} = -\frac{\pi}{\tau_2} \eta^{\nu\rho}. \tag{5.30}$$

Putting everything together, we have the following result:

$$\int_{\mathbb{F}_{T^2}} d^2\tau \frac{1}{2\tau} \int d^2 z \frac{1}{\tau_2} \langle V_{PCO;(1,0)}(z) V^{(1)}_{(-1,0)}(p^{(1)}; z, \bar{z}) V_{PCO;(0,1)}(0) V^{(2)}_{(0,-1)}(p^{(2)}; 0, 0) \rangle_{T^2|odd} =$$

$$= h^{(1)}_{\mu\nu} h^{(2)}_{\rho\sigma} \eta_{\alpha\kappa} \epsilon^{\mu\kappa} \eta_{\beta\lambda} \epsilon^{\sigma\lambda} p^{(1)}_{\gamma} p^{(2)}_{\delta} \eta^{\gamma\alpha} \eta^{\delta\beta} \eta^{\nu\rho} \int_{\mathbb{F}_{T^2}} \frac{d^2\tau}{2\tau_2} \frac{\pi}{\tau_2} Z_K(\tau, \bar{\tau})_{odd\ spin} + O(p^3). \tag{5.31}$$

By going into the lightcone coordinates:

$$x^{+} = \frac{x^0 + x^1}{\sqrt{2}}, \tag{5.32}$$

$$x^{-} = \frac{x^0 - x^1}{\sqrt{2}}, \tag{5.33}$$

the kinematics factor, which appears in the scattering amplitude of *eq*.(5.31), becomes:

$$h^{(1)}_{\mu\nu} h^{(2)}_{\rho\sigma} \eta_{\alpha\kappa} \epsilon^{\mu\kappa} \eta_{\beta\lambda} \epsilon^{\sigma\lambda} p^{(1)}_{\gamma} p^{(2)}_{\delta} \eta^{\gamma\alpha} \eta^{\delta\beta} \eta^{\nu\rho} = 2 p_{1;+} p_{2;-} h^{(1);+-} h^{(2);+-} \to \int d^2 x (-2 h^{+-} \partial_{+} \partial_{-} h^{+-}), \tag{5.34}$$

where in the last step we performed a Fourier transformation, in order to convert the above kinematics factor from the momentum space to the position space, taking also into account an additional overall momentum conserving delta function, $(2\pi)^2 \delta^{(2)}(p^{(1)} + p^{(2)})$, which arises from the integration over the X-zero modes. We can now check that the above kinematics factor of our scattering amplitude precisely reproduces the Einstein-Hilbert term, to linear order. Indeed, by considering a linear perturbation, $h_{\mu\nu}$, around the 2-dimensional Minkowski background, $\eta_{\mu\nu}$, the 2-dimensional spacetime metric, $g_{\mu\nu}$, can be expressed as:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}, \tag{5.35}$$

making it straightforward to compute the linear order correction to the determinant:

$$\sqrt{|\det(g)|} \approx 1 + \frac{1}{2} \det(h), \tag{5.36}$$

and to the Ricci scalar:

$$R \approx \partial_{\mu} \partial_{\nu} h^{\mu\nu} - \eta^{\mu\nu} \partial_{\mu} \partial_{\nu} \det(h); \tag{5.37}$$

therefore, modulo trivial total derivative terms, the spacetime Einstein-Hilbert term of interest becomes:

$$\int d^2 x \sqrt{|\det(g)|} R \approx \int d^2 x \frac{1}{2} \det(h) (\partial_{\mu} \partial_{\nu} h^{\mu\nu} - \eta^{\mu\nu} \partial_{\mu} \partial_{\nu} \det(h)) = \int d^2 x (-2 h^{+-} \partial_{+} \partial_{-} h^{+-}), \tag{5.38}$$

matching precisely the kinematics factor which appears in the scattering amplitude of *eq*.(5.31). Thus, factoring out the kinematics in the scattering amplitude of *eq*.(5.31), the one-loop level contribution to the spacetime Einstein-Hilbert term is the following:

$$\langle \mathcal{R} \rangle_{one-loop|\mathbb{R}^2\times K} = \frac{\pi}{2}\int_{\mathbb{F}_{T^2}} \frac{d^2\tau}{\tau_2^2} Z_K(\tau,\bar{\tau})_{odd\ spin}, \tag{5.39}$$

where K is an appropriate, internal CFT, associated to the 8-dimensional internal space, and $Z_K(\tau,\bar{\tau})_{odd\ spin}$ is the corresponding partition function in the odd spin-structure. The above result in *eq*.(5.39) holds for any consistent choice of the internal CFT, K. For the aforementioned special case of the $K3\times K3$ compactification of a type II superstring theory, the corresponding partition function, $Z_{K3\times K3}(\tau,\bar{\tau})_{odd\ spin}$, is given by *eq*.(5.14), of the previous section, evaluated only in the odd-spin structure. It might be interesting to investigate the contribution to the partition function $Z_K(\tau,\bar{\tau})_{odd\ spin}$ of individual target-space degrees of freedom in the string spectrum; however, it turns out that this quantity is actually topological, and computes the Euler characteristic, $\chi(K)$, of the internal space K. In what follows, we will prove that this is indeed the case, by relating the partition function $Z_K(\tau,\bar{\tau})_{odd\ spin}$ to the modified elliptic genus of K, which, in turn, contains information about the dimensions of the Dolbeault cohomologies of K.

Topological Interpretation of the Partition Function $Z_K(\tau,\bar{\tau})_{odd\ spin}$

We now focus on a general symmetric orbifold compactification, $K = T^8/\Gamma$, preserving at least $\mathcal{N} = (1,1)$ target-space supersymmetries, implying the presence of an enhanced $N = (2,2)$ world-sheet SCFT [8]. The orbifold is assumed to act non-freely, by appropriate crystallographic rotations of the four complexified internal coordinates parametrising $T^8 = T^2\times T^2\times T^2\times T^2$. The orbifold action must necessarily induce non-trivial twists on all four complex supercoordinates, otherwise the fermions in the untwisted directions contain zero modes and the partition function $Z_K(\tau,\bar{\tau})_{odd\ spin}$ vanishes trivially. The contribution of all internal supercoordinates to the partition function $Z_K(\tau,\bar{\tau})_{odd\ spin}$, in the sector $\{h_n\},\{g_n\}; n = 1,2,3,4$ is the following:

$$\prod_{n=1}^{4} Z_{T^2/\Gamma}\begin{bmatrix} h_n(\Gamma) \\ g_n(\Gamma)\end{bmatrix}(\tau,\bar{\tau}) \left| \frac{\theta\begin{bmatrix} 1+h_n(\Gamma) \\ 1+g_n(\Gamma)\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right|^2 = \prod_{n=1}^{4} \chi_{T^2/\Gamma}\begin{bmatrix} h_n(\Gamma) \\ g_n(\Gamma)\end{bmatrix}. \tag{5.40}$$

Note that the contributions of twisted bosons precisely cancel the contribution of twisted fermions in the odd spin-structure. This is not surprising, since, in the odd spin-structure, the path integral contribution of twisted fermions is the inverse of the contribution of their bosonic superpartners, aside from the zero mode contribution of the latter. As a result, all $\tau,\bar{\tau}$-dependence drops out, in the left-hand side of the above *eq*.(5.40), and we are left with a constant, zero mode contribution $\chi_{T^2/\Gamma}\begin{bmatrix} h_n \\ g_n\end{bmatrix}$. As we have already discussed, this constant, zero mode contribution, precisely counts the number of simultaneous fixed points of the given orbifold sector. Putting everything together, we have:

$$Z_K(\tau,\bar{\tau})_{odd\ spin} = \frac{1}{card(\Gamma)}\sum_{\Gamma}\prod_{n=1}^{4} \chi_{T^2/\Gamma}\begin{bmatrix} h_n(\Gamma) \\ g_n(\Gamma)\end{bmatrix}. \tag{5.41}$$

Taking into account the operatorial representation of the Euclidean path integral [1] over (anti-)periodic paths, the contribution of the internal SCFT can be alternatively expressed as a trace over the Hilbert space, in the RR sector of the theory, with the insertion of the left- and right- moving world-sheet fermion parity:

$$Z_K(\tau,\bar{\tau})_{odd\ spin} = Tr_{RR}[(-1)^{f+\bar{f}} q^{L_0-\frac{c}{24}}(\tau)\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}(\bar{\tau})], \tag{5.42}$$

and it is now clear, from the above trace representation, that all coefficients in this $q, \bar{q}$-series of the partition function $Z_K(\tau, \bar{\tau})_{odd\ spin}$ are integers. Moreover, the preceding *eq*.(5.41) implies that the partition function $Z_K(\tau, \bar{\tau})_{odd\ spin}$ is independent of the compactification moduli, and receives contributions only from the ground states in the RR sector. Its precise numerical value is directly related to the number of fixed points in the various orbifold sectors and must, then, be topological in origin.

The role of the enhancement into an $N = (2, 2)$ SCFT was crucial in the cancellation of string oscillator states in *eq*.(5.40), indicating that the partition function $Z_K(\tau, \bar{\tau})_{odd\ spin}$ is, in fact, an index. To show that this is the case, we shall rewrite *eq*.(5.42) in a form that only involves the operators of a general $N = (2, 2)$ SCFT. We focus on the left-moving $N = 2_L$ SCFT, involving the energy-momentum tensor $T(z)$, a $U(1)$ R-symmetry current $J(z)$ and two supercurrents $G_+(z), G_-(z)$ with $U(1)$-charges ± 1 respectively. Similar arguments hold for the right-movers. Let us call $\{Z^n, \Psi^n\}$ the complexified internal supercoordinates in the left-moving sector. The $N = 2_{(L)}$ SCFT can be realised in terms of free fields, as:

$$G_+(z) = \sqrt{2} \sum_n \Psi^n \partial \bar{Z}^n(z), \tag{5.43}$$

$$G_-(z) = \sqrt{2} \sum_n \bar{\Psi}^n \partial Z^n(z), \tag{5.44}$$

and:

$$J(z) = \sum_n \Psi^n \bar{\Psi}^n(z). \tag{5.45}$$

Taking the contour integral of $J(z)$ around the point $z = 0$ defines the $U(1)_{(L)}$ left R-symmetry charge:

$$J_0 = \oint_{C_0} \frac{dz}{2\pi i} \sum_n \Psi^n \bar{\Psi}^n(z) = \sum_n \sum_{\nu>0} \Psi^n_{-\nu} \bar{\Psi}^n_{\nu} = f, \tag{5.46}$$

which is nothing but the total internal world-sheet fermion number, f, in the left-moving sector. With this identification, *eq*.(5.42) becomes:

$$Z_K(\tau, \bar{\tau})_{odd\ spin} = Tr_{RR-ground\ states}[(-1)^{J_0 - \tilde{J}_0}]. \tag{5.47}$$

Since the trace is only over the RR-ground states, it will involve only a finite sum. We can equivalently let this sum run over RR-ground states with all possible $U(1)_{(L)} \times U(1)_{(R)}$ charges:

$$Z_K(\tau, \bar{\tau})_{odd\ spin} = \sum_{J_0, \tilde{J}_0} (-1)^{J_0 - \tilde{J}_0} h^{(J_0, \tilde{J}_0)}, \tag{5.48}$$

where $h^{(J_0, \tilde{J}_0)}$ are the appropriate multiplicities of the RR-ground states with charges $(J_0, \tilde{J}_0)$.

It is known that $N = 2$ SCFTs admit a spectral flow, which induces an bijective map of RR-ground states, with $U(1)_{(L)}$ charge $q - \frac{c}{6}$, to chiral primary states in the NS sector, which are states of $U(1)_{(L)}$ charge q and conformal weight $\frac{q}{2}$. The shift in the $U(1)_{(L)}$ charges is proportional to the central charge of the internal SCFT only. A similar spectral flow exists in the right-moving sector, and therefore *eq*.(5.48) can be rewritten as:

$$Z_K(\tau, \bar{\tau})_{odd\ spin} = \sum_{p,q} (-1)^{p+q} h^{p,q}, \tag{5.49}$$

where the sum is now over the chiral primary states, in the NS sector, of the $N = (2, 2)$ internal SCFT, with $U(1)_{(L)} \times U(1)_{(R)}$ charges p, q, and $h^{p,q}$ being their multiplicities. It has been shown in [9] that the $N = 2$ chiral ring is in bijective correspondence with the Dolbeault cohomology classes $\Omega(K)$ of the manifold K. In particular, chiral primaries with charges p, q are mapped to p, q-forms in the cohomology $\Omega^{p,q}(K)$, which implies that the $h^{p,q}$ defined in *eq*.(5.49) are identified with the Hodge numbers, $h^{p,q} = \dim_{\mathbb{C}}(\Omega^{p,q}(K))$, of the internal manifold K. In a complex manifold K, however, the alternating sum of Hodge numbers precisely reproduces the topological (homological) definition of the Euler characteristic $\chi(K)$. Our result for the one-loop correction to the Einstein-Hilbert term for such compactifications is, accordingly, the following:

$$\langle\mathcal{R}\rangle_{one-loop|\mathbb{R}^2\times K} = \zeta(2)\chi(K), \tag{5.50}$$

where the volume of the fundamental domain $V_{\mathbb{F}_{T^2}} = \frac{\pi}{3}$, from the modular integral in *eq.*(5.39), has been combined with the $\frac{\pi}{2}$-prefactor of the modular integral, to consequently produce a factor $\frac{\pi^2}{6} = \zeta(2)$. Our result in the above *eq.*(5.50) is the 2-dimensional extention of known results in the literature, obtained in 4 dimensions, both by the background field method and as a limit of four-graviton scattering [10],[11]. In all these works, the one-loop correction to the Einstein-Hilbert term is also found to be proportional to $\zeta(2)$ and to the Euler characteristic $\chi(K)$. It is interesting to observe that this precise structure persists also in the 2-dimensional case.

The case of the $K3\times K3$ orbifold

The partition function of a type IIB supestring theory, compactified on a $K3\times K3$ orbifold, can be given by setting the values $m=\tilde{m}=0,1$, for the left and right chirality projectors, respectively, in the preceding *eq.*(5.14). On the grounds of *eq.*(5.49) of the previous paragraph, we have but to count the multiplicities, $h_{p,q}$, of chiral primary states of specific $U(1)_{(L)}\times U(1)_{(R)}$ charges p,q, in the NS sector of the theory, which are in bijective correspondence to the RR-ground states. As the $K3\times K3$ orbifold is explicitly factorised, we may freely consider the much simpler case of a $K3$ orbifold; the topological nature of our result, in *eq.*(5.50), along with the factorisation property of the Euler characteristic: $\chi(K_1\times K_2)=\chi(K_1)\chi(K_2)$, justifies this simplification.
The partition function of a type IIB supestring theory, compactified on a $K3$ orbifold, can be given in accordance with the preceding *eq.*(5.14), as in the following:

$$\begin{aligned} &Z_{IIB|\mathbb{R}^6\times K3}(\tau,\bar{\tau}) = \\ &= \frac{1}{2}\sum_{h,g\in\mathbb{N}_2}(|\eta(q(\tau))|)^{-8} Z_{T^4/\mathbb{Z}_2}\begin{bmatrix}h\\g\end{bmatrix}(\tau,\bar{\tau};G,B) \\ &\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\left(\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^2 \frac{\theta\begin{bmatrix}a+h\\b+g\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\frac{\theta\begin{bmatrix}a-h\\b-g\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \\ &\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{a}\tilde{b}}\left(\frac{\bar{\theta}\begin{bmatrix}\tilde{a}\\\tilde{b}\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\right)^2 \frac{\bar{\theta}\begin{bmatrix}\tilde{a}+h\\\tilde{b}+g\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}\frac{\bar{\theta}\begin{bmatrix}\tilde{a}-h\\\tilde{b}-g\end{bmatrix}(\bar{q}(\bar{\tau}))}{\bar{\eta}(\bar{q}(\bar{\tau}))}. \end{aligned} \tag{5.51}$$

We can decompose the partition function of the above *eq.*(5.51) as a sum of $SO(4)$ Kac-Moody character product terms; the Kac-Moody characters of the four distinct, adjoint O_4, vector V_4, spinor S_4, and conjugate spinor C_4 representations of $SO(4)$ are defined as in [2]:

$$\chi_r = r = Tr_{(r)}[q^{L_0-\frac{c}{24}}(\tau)], \tag{5.52}$$

where c is the central charge relating to two complex fermions (that is, c=2), while $r=O_4,V_4,S_4,C_4$, and a straightforward calculation results in the following expressions:

$$O_4 = \frac{1}{2}\sum_{b=0}^{1}\left(\frac{\theta\begin{bmatrix}0\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))}\right)^2, \tag{5.53}$$

$$V_4 = \frac{1}{2}\sum_{b=0}^{1}(-1)^b \left(\frac{\theta\begin{bmatrix}0\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \right)^2 , \tag{5.54}$$

$$S_4 = \frac{1}{2}\sum_{b=0}^{1}(-1)^b \left(\frac{\theta\begin{bmatrix}1\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \right)^2 , \tag{5.55}$$

and:

$$C_4 = \frac{1}{2}\sum_{b=0}^{1} \left(\frac{\theta\begin{bmatrix}1\\b\end{bmatrix}(q(\tau))}{\eta(q(\tau))} \right)^2 , \tag{5.56}$$

respectively. Focusing, now, on the RR sector of the partition function of *eq.*(5.51), we find, after some algebra, that it can be expressed as in the following:

$$\begin{aligned} &Z_{IIB|\mathbb{R}^6\times K3}(\tau,\bar{\tau})_{RR-sector} = \\ &= Z_{T^4/\mathbb{Z}_2}\begin{bmatrix}0\\+\end{bmatrix}(\tau,\bar{\tau};G,B)(|\eta(q(\tau))|)^{-8}(S_4S_4\bar{S}_4\bar{S}_4 + C_4C_4\bar{C}_4\bar{C}_4)+ \\ &+Z_{T^4/\mathbb{Z}_2}\begin{bmatrix}1\\+\end{bmatrix}(\tau,\bar{\tau};G,B)(|\eta(q(\tau))|)^{-8}(O_4C_4\bar{O}_4\bar{C}_4 + V_4S_4\bar{V}_4\bar{S}_4)+ \\ &+Z_{T^4/\mathbb{Z}_2}\begin{bmatrix}0\\-\end{bmatrix}(\tau,\bar{\tau};G,B)(|\eta(q(\tau))|)^{-8}(S_4S_4\bar{C}_4\bar{C}_4 + C_4C_4\bar{S}_4\bar{S}_4)+ \\ &+Z_{T^4/\mathbb{Z}_2}\begin{bmatrix}1\\-\end{bmatrix}(\tau,\bar{\tau};G,B)(|\eta(q(\tau))|)^{-8}(O_4C_4\bar{V}_4\bar{S}_4 + V_4S_4\bar{O}_4\bar{C}_4), \end{aligned} \tag{5.57}$$

where:

$$Z_{T^4/\mathbb{Z}_2}\begin{bmatrix}k\\\pm\end{bmatrix}(\tau,\bar{\tau};G,B) = \frac{1}{2}Z_{T^4/\mathbb{Z}_2}\begin{bmatrix}k\\0\end{bmatrix}(\tau,\bar{\tau};G,B) \pm \frac{1}{2}Z_{T^4/\mathbb{Z}_2}\begin{bmatrix}k\\1\end{bmatrix}(\tau,\bar{\tau};G,B); k=0,1. \tag{5.58}$$

The power series expansion of the partition function of *eq.*(5.57), with respect to $q(\tau), \bar{q}(\bar{\tau})$, reveals that only the terms of the Kac-Moody character products $S_4S_4\bar{S}_4\bar{S}_4$, $C_4C_4\bar{C}_4\bar{C}_4$, and $O_4C_4\bar{O}_4\bar{C}_4$, correspond to ground states; specifically, the former two terms of the untwisted sector respectively describe 4 distinct, internal RR-ground states of $U(1)_{(L)} \times U(1)_{(R)}$ charges $(1,1)$, $(-1,1)$, $(1,-1)$, and $(-1,-1)$, as well as 4 $U(1)_{(L)} \times U(1)_{(R)}$-neutral internal RR-ground states, whereas the latter term of the twisted sector describes 16 $U(1)_{(L)} \times U(1)_{(R)}$-neutral internal RR-ground states, the multiplicity of which being simply a result of the total number of 16 fixed points, in the twisted sector, of the $T^4/\mathbb{Z}_2$ orbifold. The spectral flow that is admitted by the theory bijectively maps the preceding internal RR-ground states to internal chiral primary states of $U(1)_{(L)} \times U(1)_{(R)}$ charges shifted by $(\frac{c}{6}, \frac{\tilde{c}}{6})$, in the NS sector; the internal central charges are evidently $c = \tilde{c} = 6$, so the possible $U(1)_{(L)} \times U(1)_{(R)}$ charges, p, q, of the internal chiral primary states may accordingly be $(0,0)$, $(2,0)$, $(0,2)$, $(2,2)$, and $(1,1)$, while the Hodge square may be constructed from their respective multiplicities, $h_{p,q}$, and found to be:

$$\begin{pmatrix} h_{0,0} & h_{0,1} & h_{0,2} \\ h_{1,0} & h_{1,1} & h_{1,2} \\ h_{2,0} & h_{2,1} & h_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{5.59}$$

We underline that the elements, $h_{p,q}$, of the Hogde square, refer exclusively to the multiplicities of internal states, in the NS sector of the theory. Using, then, the preceding *eq.*(5.49), along with the topological definition of the Euler characteristic, we can straightforwardly compute the Euler characterictic of the $K3$ orbifold, $\chi(K3) = 24$, meaning that the Euler characteristic of the $K3 \times K3$ factorised orbifold of interest is simply found to be $\chi(K3 \times K3) = 24^2 = 576$, a result that is well-known and established, in the mathematics literature. As a consequence, our derived general result, in *eq*.(5.50), yields the following:

$$\langle \mathcal{R} \rangle_{one-loop|\mathbb{R}^2 \times K3 \times K3} = 96\pi^2, \tag{5.60}$$

for the one-loop level correction to the Einstein-Hilbert term, in a type IIB superstring theory, compactified on a $K3 \times K3$ orbifold. It is simple to see that the result of the above *eq*.(5.60) may be explicitly verified by the straightforward calculation of the odd spin-structure of the partition function of *eq*.(5.14), which corresponds to a type II superstring theory compactified on a $K3 \times K3$ orbifold, along with the use of the final result of the one-loop level two-graviton scattering, in *eq*.(5.39). Nevertheless, it would be much more instructive to explicitly show exactly how our derived general result, in *eq*.(5.50), incorporates the topological definition of the Euler characteristic. To that purpose, we can define the modified elliptic genus of a topological space K, by performing a $(t, \bar{t})$-deformation in the partition function of the preceding *eq*.(5.42), that corresponds to the RR sector of the respective internal SCFT, as in the following:

$$Z_K(\tau, \bar{\tau}; t, \bar{t})_{odd\ spin} = Tr_{RR}[(-1)^{J_0 - \bar{J}_0} q^{L_0 - \frac{c}{24}}(\tau) \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}(\bar{\tau}) t^{J_0} \bar{t}^{\bar{J}_0}]. \tag{5.61}$$

We may also define the following quantity:

$$P_K(t, \bar{t}) = Tr_{RR-ground\ states}[(t\bar{t})^{\frac{c}{6}} t^{J_0} \bar{t}^{\bar{J}_0}], \tag{5.62}$$

which, by virtue of the spectral flow, that is admitted by the SCFT, as it was described in the previous paragraph, can be redefined as in the following:

$$P_K(t, \bar{t}) = Tr_{NS-chiral\ primary\ states}[t^p \bar{t}^q], \tag{5.63}$$

where the trace of the above *eq*.(5.63) is, now, over the chiral primary states of $U(1)_{(L)} \times U(1)_{(R)}$ charges p, q, in the NS sector of the SCFT. We can see that the quantity $P_K(t, \bar{t})$ is essentially the partition function relative to the chiral primary states in the NS sector of the SCFT, and can naturally be expressed as a $(t, \bar{t})$-polynomial:

$$P_K(t, \bar{t}) = \sum_{p,q} h^{p,q} t^p \bar{t}^q, \tag{5.64}$$

where the coefficients $h^{p,q}$ are, evidently, the respective multiplicities of the chiral primary states of $U(1)_{(L)} \times U(1)_{(R)}$ charges p, q, in the NS sector of the SCFT. As it has been mentioned in the previous paragraph, it has been shown in [9] that the coefficients $h^{p,q}$ are identified with the Hodge numbers of the topological space K, so that the $(t, \bar{t})$-polynomial $P_K(t, \bar{t})$ is identified with the Poincaré polynomial of the topological space K. Noting, now, that only the ground states may have a non-zero contribution to the modified elliptic genus of *eq*.(5.61), in its limit where $\tau_2 \to +\infty \Rightarrow q(\tau), \bar{q}(\bar{\tau}) \to 0$, the Poincaré polynomial $P_K(t, \bar{t})$ may also be rewritten as:

$$P_K(t, \bar{t}) = (t\bar{t})^{\frac{c}{6}} \lim_{\tau_2 \to +\infty} Z_K(\tau, \bar{\tau}; -t, -\bar{t})_{odd\ spin}, \tag{5.65}$$

and we can essentially reproduce the *eqs*.(5.47) – (5.49), of the previous paragraph, explicitly in the form of the topological definition, $\chi(K) = P_K(-1, -1)$, for the Euler characteristic, $\chi(K)$, of the topological space K, as in the following:

$$\chi(K) = P_K(-1, -1) = \lim_{\tau_2 \to +\infty} Z_K(\tau, \bar{\tau}; 1, 1)_{odd\ spin} =$$

$$= Tr_{RR-ground\ states}[(-1)^{J_0 - \bar{J}_0}] = Tr_{NS-chiral\ primary\ states}[(-1)^p (-1)^q] = \sum_{p,q} h^{p,q} (-1)^p (-1)^q = Z_K(\tau, \bar{\tau})_{odd\ spin}, \tag{5.66}$$

on the grounds of the topological interpretation of the partition function $Z_K(\tau,\bar{\tau})_{odd\ spin}$, of the previous paragraph. Focusing, then, on the case of the $K3$ orbifold, we can straightforwardly calculate the Poincaré polynomial $P_{K3}(t,\bar{t})$, using *eq*.(5.61), along with the odd spin-structure of the partition function of *eq*.(5.51); after some algebra, we find that the Poincaré polynomial $P_{K3}(t(z),\bar{t}(\bar{z}))$ is equal to:

$$P_{K3}(t,\bar{t}) = (t\bar{t})^{\frac{c}{6}} \lim_{\tau_2\to+\infty} Z_{IIB|\mathbb{R}^6\times K3}(\tau,\bar{\tau};-t,-\bar{t})_{odd\ spin} = 1 + t^2 + \bar{t}^2 + 20t\bar{t} + t^2\bar{t}^2 = \sum_{p=0}^{2}\sum_{q=0}^{2} h^{p,q}t^p\bar{t}^q, \quad (5.67)$$

reading off the corresponding Hodge numbers $h^{p,q}$ exactly as they appear in the Hodge square of *eq*.(5.59), so that the topological definition, $P_{K3}(-1,-1) = \chi(K3)$, of the Euler characteristic, $\chi(K3)$, gives $\chi(K3) = 24$, and the Euler characteristic of the $K3\times K3$ orbifold is again found to be equal to $\chi(K3\times K3) = 24^2 = 576$, as it is expected. It is evident, then, that the result in *eq*.(5.60), for the one-loop level correction to the Einstein-Hilbert term, in a type IIB superstring theory, compactified on a $K3\times K3$ orbifold, may also be reproduced and verified via the above calculation of the Poincaré polynomial, $P_{K3}(t,\bar{t})$, of the $K3$ orbifold.

Bibliography

[1] Polchinski, J.(1988).*String Theory*(Vols. I-II).Cambridge University Press.

[2] Kiritsis, E.(2007).*String Theory in a Nutshell*.Princeton University Press.

[3] Antoniadis I., Bachas C.P., Kounnas C.(1986).Four-Dimensional Superstrings.*Nucl.Phys.B 289 (1987) 87*.https://doi.org/10.1016/0550-3213(87)90372-5.

[4] Kaplunovsky, V., Louis, J.(1995).On Gauge couplings in string theory.*Nucl.Phys.B 444 (1995) 191-244*.https://doi.org/10.1016/0550-3213(95)00172-O.

[5] Kiritsis, E., Kounnas, C.(1994).Infrared regularization of superstring theory and the one loop calculation of coupling constants.*Nucl.Phys.B 442 (1995) 472-493*.https://doi.org/10.1016/0550-3213(95)00156-M.

[6] Florakis, I.(2016).Gravitational Threshold Corrections in Non-Supersymmetric Heterotic Strings.*Nucl.Phys.B 916 (2017) 484-509*.https://doi.org/10.1016/j.nuclphysb.2017.01.016.

[7] Friedan, D., Martinec, E.J., Shenker, S.H.(1985).Conformal Invariance, Supersymmetry and String Theory.*Nucl.Phys.B 271 (1986) 93-165*.https://doi.org/10.1016/S0550-3213(86)80006-2.

[8] Banks, T., Dixon, L.J.(1988).Constraints on String Vacua with Space-Time Supersymmetry. *Nucl.Phys.B 307 (1988) 93-108*.https://doi.org/10.1016/0550-3213(88)90523-8.

[9] Lerche, W., Vafa, C., Warner, N.P.(1989).Chiral Rings in N=2 Superconformal Theories.*Nucl.Phys.B 324 (1989) 427-474*.https://doi.org/10.1016/0550-3213(89)90474-4.

[10] Antoniadis, I., Minasian, R., Vanhove, P.(2002).Noncompact Calabi-Yau manifolds and localized gravity.*Nucl.Phys.B 648 (2003) 69-93*.https://doi.org/10.1016/S0550-3213(02)00974-4.

[11] Berg, M., Haack, M., Kang, J.U., Sjörs, S.(2014)*JHEP 12 (2014) 077*.doi:10.1007/JHEP12(2014)077.