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ΕΠΕΚΤΑΣΕΙΣ ΤΗΣ ΘΕΩΡΙΑΣ PERRON-FROBENIUS
(EXTENSIONS OF PERRON-FROBENIUS THEORY)

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ΔΙΔΑΚΤΟΡΙΚΗ ΔΙΑΤΡΙΒΗ

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To my family

The present dissertation was carried out under the Ph.D. program of the Department of Mathematics at the University of Ioannina in order to obtain the degree of Doctor of Philosophy.

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Thaniporn Chaysri

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Περιληψη

Από το 1907, ο Oskar Perron απέδειξε ένα θεώρημα για θετικούς πίνακες, το οποίο επεκτάθηκε από τον Georg Frobenius το 1912 για μη αναγώγιμους μη αρνητικούς πίνακες. Στη συνέχεια αναπτύχθηκε η γνωστή θεωρία Perron-Frobenius για μη αρνητικούς πίνακες. Ένας M_{V}-πίνακας γράφεται στην μορφή $A = sI - B$, όπου $0 \leq \rho(B) \leq s$ και B είναι τελικά μη αρνητικός πίνακας. Ένας GM-πίνακας γράφεται στην μορφή $A = sI - B$, όπου οι B και B^T έχουν την ιδιότητα Perron-Frobenius (Perron-Frobenius property). Αυτές οι κλάσεις πινάκων είναι επεκτάσεις των γνωστών M-πινάκων. Στην διδακτορική διατριβή, διατυπώνουμε αρχικά τους ορισμούς και τα θεωρήματα που χρειάζονται για να γίνει κατανοητή η Θεωρία Perron-Frobenius σε σχέση και με τις επεκτάσεις των M-πινάκων. Στην συνέχεια, στο κεφάλαιο 2, μελετούμε τους M_{V}-πίνακες σε σχέση με τη Θεωρία Perron-Frobenius. Ειδικότερα, δίνουμε και αποδεικνύουμε ικανές και αναγκαίες συνθήκες τέτοιες ώστε ένας M_{V}-πίνακας να έχει θετικό αριστερό και δεξιό ιδιοδιάνυσμα που αντιστοιχεί στην απόλυτα μικρότερη πραγματική ιδιοτιμή, λαμβάνοντας υπόψη ότι $\mathrm{index}_0 B \leq 1$ ή όχι. Επιπλέον, μελετώνται ανάλογες συνθήκες για τελικά μη αρνητικούς πίνακες ή M_{V}-πίνακες ώστε όλα τα υπόλοιπα ιδιοδιανύσματα ή γενικευμένα ιδιοδιανύσματα, εκτός από το ιδιοδιάνυσμα Perron, να μην είναι μη αρνητικά. Στη συνέχεια, παρουσιάζονται ισοδύναμες ιδιότητες τελικά εκθετικά μη αρνητικών πινάκων και M_{V}-πινάκων. Στο κεφάλαιο 3, μελετούμε ιδιότητες για M_{V}- και GM-πίνακες σε σχέση με το συμπλήρωμα Schur. Συγκεκριμένα, μελετούμε ικανές και αναγκαίες συνθήκες ώστε το συμπλήρωμα Schur διαφόρων τύπων M_{V}-πινάκων, να έχουν την M_{V}-ιδιότητα. Επίσης, μελετούμε το συμπλήρωμα Schur για διαταραγμένους M_{V}-πίνακες. Στη συνέχεια, αποδεικνύουμε την M_{V}-ιδιότητα του συμπληρώματος Schur οποιουδήποτε πίνακα, όταν ο υποπίνακας A_{22} είναι M_{V}-πίνακας. Μελετούμε επίσης ανάλογες συνθήκες για το συμπλήρωμα Schur των GM-πινάκων ώστε να έχουν την GM-ιδιότητα. Στο κεφαλαίο 4, παρουσιάζουμε εφαρμογές των επεκτάσεων της θεωρίας Perron-Frobenius σε άλλες επιστήμες, όπως η Θεωρία Δικτύου, η Βιολογία, η Οικονομία κ.λ.π. Στα κεφάλαια 2 και 3, παρουσιάζονται πολλά αριθμητικά παραδείγματα που υποστηρίζουν και επιβεβαιώνουν τα θεωρητικά αποτελέσματα.

Abstract

The foundations of what today is called Perron-Frobenius theory were laid by Oscar Perron in 1907 with a result on positive matrices and Georg Frobenius in 1912, who extended that result to the case of irreducible nonnegative matrices. An M_{V}-matrix is a matrix of the form $A = sI - B$, where $0 \leq \rho(B) \leq s$ and B is an eventually nonnegative matrix. A GM-matrix denotes a matrix of the form $A = sI - B$, when both B and B^T possess the Perron-Frobenius property. These classes of matrices are extensions of the well-known M-matrices. In this thesis, we first provide all the definitions and theorems that are necessary to understand the Perron-Frobenius theory and extensions of M-matrices. We then study, in chapter 2, the M_{V}-matrices concerning the Perron-Frobenius theory. Specifically, sufficient and necessary conditions for an M_{V}-matrix to have positive left and right eigenvectors corresponding to its eigenvalue with smallest real part without considering or not if $\mathrm{index}_0 B \leq 1$ are stated and proven. Moreover, analogous conditions for eventually nonnegative matrices or M_{V}-matrices to have all the non Perron eigenvectors or generalized eigenvectors not being nonnegative are studied. Then, equivalent properties of eventually exponentially nonnegative matrices and M_{V}-matrices are presented. In chapter 3, we study the main result for M_{V}- and GM-matrices focusing on the properties their Schur complements inherit. Specifically, we study sufficient and necessary conditions for the Schur complement of various types of M_{V}-matrices that have the M_{V}-property. Also, the Schur complements of perturbed M_{V}-matrices are studied. Then, we present the M_{V}-property of the Schur complement of any matrix when its A_{22} block is an M_{V}-matrix. We also study analogous conditions for the Schur complement of GM-matrices to have the GM-property. In chapter 4, we present applications of the Perron-Frobenius theory in other fields such as Network Theory, Biology, Economy etc. In chapter 2 and 3, numerous numerical examples are presented to support our theoretical findings.

Symbols and Abbreviations

Symbols

$\det A$	Determinant of the matrix A
A^T	Transpose of the matrix A
$\|A\|$	Euclid norm of the matrix A
I_n	$n \times n$ Identity matrix
$\mathbb{Z}$	Set of integers
$\mathbb{Z}_+$	Set of positive integers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n_+$	Nonnegative orthant or set of nonnegative vector in $\mathbb{R}^n$ (Nonnegative cone)
$\mathbb{R}^{n,n}$	Set of $n \times n$ real matrices
$\mathrm{Re}(\lambda)$	Real part of λ
$\mathbb{C}$	Set of complex numbers
$\mathrm{Im}(\lambda)$	Imaginary part of λ
x^T	Transpose of the vector $x \in \mathbb{R}^n$
$\|x\|$	Euclid norm of the vector x
e^A	Exponential of A

Abbreviations

iff	if and only if

Contents

CHAPTER 1

Introduction, notation, definitions and preliminaries

The aim of this thesis is to study further results of Perron-Frobenius theory, focusing on the class of M-matrices and their extensions. In this chapter, we will give the necessity definitions and notations used for this thesis. We follow with some historical result of Perron-Frobenius theory and conclude with some preliminary results concerning the extensions of Perron-Frobenius theory.

Definition 1.1. Let a matrix $A \in \mathbb{R}^{n,n}$ and $\lambda_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, denote its eigenvalues, then

- $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is called the *spectrum* of the matrix A;
- $\rho(A) = \max_{i=1,2,\ldots,n} |\lambda_i|$ is called the *spectral radius* of the matrix A;
- λ is called a *dominant eigenvalue* of the matrix A if $|\lambda| = \rho(A)$;
- $\lambda \in \sigma(A)$ is called the *strictly dominant eigenvalue* of the matrix A if $|\lambda| > |\mu|$, $\forall \mu \in \sigma(A), \mu \neq \lambda$;
- $\text{index}_\lambda(A)$ denotes the degree of λ as a root of the minimal polynomial of the matrix A;
- $E_\lambda(A)$ is called the *ordinary eigenspace* for eigenvalue λ of the matrix A and denotes by $E_\lambda(A) = \mathscr{N}(A - \lambda I)$ where $\mathscr{N}(A - \lambda I)$ is the nullspace of $A - \lambda I$.

Definition 1.2. Let $A \in \mathbb{R}^{n,n}$ be a square matrix partitioned as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{1.1}$$

Then, by

$$\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}$$

we denote a block matrix of A^k partitioned conformably to (1.1).

Definition 1.3. A matrix $A \in \mathbb{C}^{n,n}$ is called *reducible* matrix if there exists a permutation matrix $P \in \mathbb{R}^{n,n}$ such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{1.2}$$

where $A_{11} \in \mathbb{C}^{r,r}$, $A_{22} \in \mathbb{C}^{n-r,n-r}$ and $A_{12} \in \mathbb{C}^{r,n-r}$, $0 < r < n$. Otherwise, A is called *irreducible*.

Definition 1.4. A matrix $A \in \mathbb{R}^{n,n}$ is called

- *positive*, denoted by $A > 0$, if A is entrywise positive;
- *nonnegative*, denoted by $A \geq 0$, if A is entrywise nonnegative;
- *primitive* if $A \geq 0$ and there exists a positive integer k such that $A^k > 0$;
- *weakly cyclic of index* $k\,(>1)$ [53] if there exists a permutation matrix $P \in \mathbb{R}^{n,n}$ such that PAP^T is partitioned in the form:

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{1.3}$$

  where all the diagonal blocks are square zero matrices. If it is already in the form (1.3) it is simply called *cyclic of index k*;
- *eventually nonnegative (positive)*, denoted by $A \overset{\vee}{\geq} 0$ $(A \overset{\vee}{>} 0)$, if there exists an integer $k_0 > 0$ such that $A^k \geq 0$ $(A^k > 0)$ for all $k \geq k_0$. The smallest such positive integer is called the *power index* of A;
- *exponentially nonnegative (positive)* if for all $t > 0$, $e^{tA} = \sum\limits_{k=0}^{\infty} \frac{t^k A^k}{k!} \geq 0$ $(e^{tA} > 0)$;

- *eventually exponentially nonnegative (positive)* if there exists $t_0 \in [0, \infty)$ such that for all $t > t_0$, $e^{tA} \geq 0$ ($e^{tA} > 0$). The smallest such nonnegative number is called the *exponential index* of A;

- *nilpotent* if there exists an integer $k > 0$ such that $A^k = 0$. The smallest such positive integer is called the *index of nilpotence* of A.

The Perron-Frobenius theory was established by Perron [48] in 1907, who proved that the dominant eigenvalue of an entry-wise positive matrix is positive and its corresponding eigenvector is positive, and later by Frobenius [24] in 1912, who extended it to irreducible nonnegative matrices. We recall the classical well-known Perron-Frobenius theory for irreducible nonnegative matrices.

Theorem 1.5 (Perron-Frobenius)**.** *Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,*

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.

(ii) To $\rho(A)$ there corresponds an eigenvector $x > 0$.

(iii) $\rho(A)$ increases when any entry of A increases.

(iv) $\rho(A)$ is a simple eigenvalue of A.

(v) All nonnegative eigenvectors of A are multiples of x.

We give now the preliminary results for the extensions of Perron-Frobenius theory, given by Noutsos ([42]).

Definition 1.6. A matrix $A \in \mathbb{R}^{n,n}$ possesses

- the *Perron-Frobenius property* if it has a positive dominant eigenvalue $\lambda_1 = \rho(A) > 0$ and the corresponding eigenvector $x^{(1)} \geq 0$;

- the *strong Perron-Frobenius property* if it has a positive strictly dominant eigenvalue $\lambda_1 = \rho(A) > 0$, $\lambda_1 > |\lambda_i|, i = 2, 3, \ldots, n$, and the corresponding eigenvector $x^{(1)} > 0$.

The following theorem gives the equivalence of the strong Perron-Frobenius property of a matrix and the eventually positive property.

Theorem 1.7 ([42], Theorem 2.2)**.** *For a matrix $A \in \mathbb{R}^{n,n}$ the following are equivalent:*

(i) Both matrices A and A^T possess the strong Perron-Frobenius property.

(ii) A is an eventually positive matrix.

(iii) A^T is an eventually positive matrix.

The following theorem gives the result that the Perron-Frobenius property of a matrix is weaker than the eventually nonnegative property.

Theorem 1.8 ([42], Theorem 2.3)**.** *Let $A \in \mathbb{R}^{n,n}$ be an eventually nonnegative matrix which is not nilpotent. Then, both matrices A and A^T possess the Perron-Frobenius property.*

The term M-matrix (Metzler matrix) was first introduced by Ostrowski [46, 47] to honor his Professor Minkowski [39, 40], who proved that: "The determinant of the matrix $A \in \mathbb{R}^{n,n}$, whose off-diagonal elements are non-positive, is positive if all of its row sums are positive". Many extensions of the class of M-matrices have been introduced in the last seventeen years or so. First, in 2004, the class of "pseudo M-matrices" was introduced by Johnson and Tarazaga [34]. Next, in 2006, "M_{V}-matrices" were introduced and studied by Olesky et al. [45]. Finally, in 2008, the class of "Generalized M-matrices" or "GM-matrices" was studied by Elhashash and Szyld [17].

Definition 1.9. A matrix $A \in \mathbb{R}^{n,n}$ is called

- *M-matrix*, if $A = sI - B$ where $B \geq 0$ and $s \geq \rho(B) \geq 0$;
- *Pseudo M-matrix*, if $A = sI - B$ where $B \overset{\mathrm{v}}{>} 0$ and $s > \rho(B) > 0$;
- *M_{V}-matrix*, if $A = sI - B$ where $B \overset{\mathrm{v}}{\geq} 0$ and $s \geq \rho(B) \geq 0$;
- *generalized M-matrix or GM-matrix*, if $A = sI - B$ where both B and B^T possess the Perron-Frobenius property and $s \geq \rho(B) > 0$.

Based on the corresponding definitions of the extensions of the M-matrices, we have that the class of M-matrices is a subclass of M_{V}-matrices and the class of pseudo M-matrices is also a subclass of M_{V}-matrices; however, an M-matrix may not be a pseudo M-matrix and vice versa. The class of M_{V}-matrices is also a subclass of GM-matrices because for every eventually nonnegative matrix B, both B and B^T possess the Perron-Frobenius property (see [42, Theorem 2.3]).

We recall the definitions of inverse M-matrices and their extensions.

Definition 1.10. A nonsingular matrix $A \in \mathbb{R}^{n,n}$ is called

- *inverse M-matrix* if A^{-1} is an M-matrix;
- *inverse M_V-matrix* if A^{-1} is an M_V-matrix;
- *inverse GM-matrix* if A^{-1} is a GM-matrix.

The term Schur Complement was named and given a notation by Haynsworth in 1968 to honor the famous mathematician Issai Schur. We give the definition of the Schur complement (see, e.g., [57, 14]).

Definition 1.11 (Schur Complement)**.** Let $A \in \mathbb{R}^{n,n}$ be partitioned in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1.4}$$

where $A_{11} \in \mathbb{R}^{r,r}$ is a nonsingular submatrix of A and $A_{22} \in \mathbb{R}^{n-r,n-r}$. Then, the *Schur complement* of A_{11} in A is

$$(A/A_{11}) = A_{22} - A_{21}{A_{11}}^{-1}A_{12} \tag{1.5}$$

and the *Schur's formula* is

$$\det A = \det A_{11} \det(A/A_{11}).$$

Remark 1.12. *It is pointed out that from now onwards the dimensions of the unit matrices I, the zero matrices (or vectors), and the A_{11} matrices will be omitted as indices but they will be made clear from the context.*

Remark 1.13. *From the definition of the Schur complement, the matrix A_{11} is nonsingular. Hence, the matrices A_{11} in all theorems for the Schur complement in Chapter 3 are considered to be nonsingular.*

CHAPTER 2

The Perron-Frobenius theory of M_{V}-matrices

The Perron-Frobenius theory was established by Perron [48] in 1907 for positive matrices and was extended by Frobenius [24] in 1912 for irreducible nonnegative matrices. Since then the well-known Perron-Frobenius theory was studied by many researchers. Extensions and generalizations to the Perron-Frobenius theory were given by Friedland [23], Eschenbach and Johnson [19], Tarazaga et al. [52], Naqvi and McDonald [41], Maroulas et al. [38], Johnson and Tarazaga [34], Le and McDonald [36], Elhashash and Szyld [18], Gao [25], etc..

In 2006, Noutsos [42] extended the Perron-Frobenius theory by introducing the definitions of the Perron-Frobenius property and the strong Perron-Frobenius property and connected matrices having these properties with eventually positive and eventually nonnegative matrices. Later in 2012, this theory was extended into complex matrices by Noutsos and Varga [44].

Extensions of M-matrices are applied in many scientific fields such as in mathematics (iterative methods, discretizations of differential operators), economics (gross substitutability, stability of a general equilibrium and Leontief's input-output analysis in economic systems), optimization, Markov chains in the field of probability theory and operation research like queuing theory, engineering (control theory) and also biology (population dynamics). Many equivalent properties that characterize M-matrices were stated and proven by many researchers. In the book of Berman and Plemmons [6], over than 70 such properties are presented not all of which are valid for M_{V}-matrices. In this chapter, we study the M_{V}-matrices in connection with the Perron-Frobenius theory. Specifically, sufficient conditions for an M_{V}-matrix with $\mathrm{index}_0\, B \leq 1$ to have positive left and right eigenvectors corresponding to its eigenvalue with smallest real part are studied. Also, sufficient and necessary conditions are proven without considering that $\operatorname{index}_0 B \leq 1$. Then analogous properties of such class of matrices having all non Perron eigenvectors and generalized eigenvectors not being nonnegative are presented and proven. Finally, we give equivalent properties of eventually exponentially nonnegative matrices and M_{V}-matrices.

2.1 Eigenvectors of M_{V}-matrices

In 2006, Noutsos [42] studied the eigenvectors of matrices that have some negative entries (eventually nonnegative matrices and eventually positive matrices) and gave the result that every eventually nonnegative matrix has positive dominant eigenvalue correspoding to nonnegative eigenvector. This brings up the question about the eigenvalues and eigenvectors of M_{V}-matrices, e.g., matrices that are based on eventually nonnegative matrices. First, we will study the eigenpair of an irreducible M_{V}-matrix with $\operatorname{index}_0 B \leq 1$.

Theorem 2.1. *Let A be an irreducible M_{V}-matrix, written in the form $sI - B$ with $B \overset{\mathrm{V}}{\geq} 0$, $0 \leq \rho(B) \leq s$ and $\operatorname{index}_0 B \leq 1$. Then, to the smallest real eigenvalue $\lambda_1 \geq 0$ of A there correspond positive right and left eigenvectors. Moreover, $\lambda_1 < \operatorname{Re} \lambda_i$, $i = 2, 3, \dots, n$.*

Proof. Suppose that $\mu_1 = \rho(B) \geq |\mu_2| \geq \cdots \geq |\mu_n|$ are the eigenvalues of B and let k_0 is the power index of B. Since B is irreducible and $\operatorname{index}_0 B \leq 1$, from [41, Theorem 3.4] we obtain that there exist integers $k \geq k_0$ such that B^k is irreducible and nonnegative. This means, see also [8, Proposition 2.1], either:

1. B^k is a primitive matrix, for all $k \geq k_0$, which means that $\rho(B^k)$ is a simple eigenvalue of B^k, for all $k \geq k_0$, implying that $\rho(B)$ is a simple one of B.

2. B^k is a nonnegative cyclic matrix of index r, for $k \geq k_0$, $k \neq mr$, m integer, implying that $\rho(B^k)$ is a simple eigenvalue of B^k and therefore $\rho(B)$ is a simple one of B.

In both cases the right and left Perron eigenvectors of B^k are positive and so are the ones of B. For the other eigenvalues of B there hold $\operatorname{Re} \mu_i < \rho(B) = \mu_1$, $i = 2, 3, \dots, n$. Thus, for the eigenvalues of $A = sI - B$ there hold

$\lambda_1 = s - \rho(B) < \operatorname{Re}\lambda_i = s - \operatorname{Re}\mu_i$, $i = 2, 3, \ldots, n$. Obviously, to λ_1 there correspond the same right and left eigenvectors. □

Corollary 2.2. *Let A be an irreducible symmetric M_V-matrix. Then, its smallest real eigenvalue $\lambda_1 \geq 0$ is a simple one, and the corresponding eigenvector is positive.*

Proof. In view of the symmetry, we get that $\operatorname{index}_0 B \leq 1$ and the assumptions of Theorem 2.1 hold true. □

We will give examples to show the validity of Theorem 2.1. First, we consider an M_V-matrix with $\operatorname{index}_0 B = 1$ and then, an M_V-matrix with $\operatorname{index}_0 B = 0$ with complex eigenvalues.

Example 2.3. *Consider the matrix A given by*

$$A = 7I - B, \text{ where } B = \begin{bmatrix} -6 & 11 & -19 & -23 \\ -8 & 17 & -19 & -29 \\ 7 & -10 & 22 & 24 \\ -8 & 13 & -22 & -27 \end{bmatrix} \overset{\text{v}}{\geq} 0.$$

We can easily see that B is an irreducible matrix with $\operatorname{index}_0 B = 1$. *The matrix B^k is positive for $k \geq 6$. The spectrum of A is given by $\sigma(A) = \{1.8292,\ 4.7254,\ 7,\ 8.4454\}$ and the right and left eigenvectors of A corresponding to the smallest real eigenvalue* 1.8292 *are* $[0.2337\ \ 0.9339\ \ 0.0975\ \ 0.2526]^T > 0$ *and* $[0.1470\ 0.1366\ \ 0.8808\ \ 0.4289]^T > 0$, *respectively, which shows that Theorem* 2.1 *holds.*

Example 2.4. *Consider the matrix A given by*

$$A = 13I - B, \text{ where } B = \begin{bmatrix} 5 & 4 & 6 & -4 \\ 6 & 1 & 4 & 4 \\ 0 & 3 & 6 & 1 \\ 5 & 3 & 2 & 4 \end{bmatrix} \overset{\text{v}}{\geq} 0.$$

We can easily see that B is an irreducible matrix with $\operatorname{index}_0 B = 0$. *The matrix B^k is positive for $k \geq 3$. The spectrum of A is given by $\sigma(A) = \{1.3482,\ 9.4830 \pm 3.8545i,\ 15.6857\}$ and the right and left eigenvectors of A corresponding to the smallest real eigenvalue* 1.3482 *are* $[0.3760\ \ 0.5872\ \ 0.4151\ \ 0.5844]^T > 0$ *and* $[0.4403\ \ 0.4136\ \ 0.7919\ \ 0.0895]^T > 0$, *respectively. Moreover,* 1.3482 *is the smallest real part, e.g.,* $1.3482 < \operatorname{Re}(9.4830 \pm 3.8545i) = 9.4830$, *which shows that Theorem* 2.1 *holds.*

We can use the following example to demonstrate that Theorem 2.1 does not hold if $\mathrm{index}_0 B = 2$.

Example 2.5. *(see [43, Example 3.11]) We consider the matrix*

$$A = 3I - B, \text{ where } B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \overset{\mathrm{v}}{\geq} 0.$$

It can be easily seen that B is an irreducible matrix with $\mathrm{index}_0 B = 2$. *All powers $B^k, k \geq 2$ become reducible and Theorem 2.1 does not hold: $\rho(B) = 2$ is a double eigenvalue and the right and left eigenvectors* $[1\ 1\ 0\ 0]^T$ *and* $[0\ 0\ 1\ 1]^T$, *respectively, are both nonnegative and not positive.*

We have to remark that the assumption $\mathrm{index}_0 B \leq 1$ is sufficient and not necessary. To be specific, there are M_{V}-matrices with $\mathrm{index}_0 B > 1$ that have positive right and left eigenvectors corresponding to their smallest eigenvalue and this is shown in the following examples.

Example 2.6. *Consider the matrix A given by $A = 3I - B$, where*

$$B = \begin{bmatrix} 0.0163 & -0.2113 & 0.6667 & 0.2887 & 0.5163 \\ 0.183 & 0.5 & 0 & 1 & 0.6830 \\ 0.6667 & 0.5774 & 0.3335 & 0.5774 & 0.6667 \\ 0.3943 & 0.5 & 1.1547 & 0 & -0.1057 \\ -0.3497 & 0.7887 & 0.6667 & 0.2887 & -0.8497 \end{bmatrix} \overset{\mathrm{v}}{\geq} 0.$$

It can be shown that B is irreducible with $\mathrm{index}_0 B = 2$. *However, $\rho(B) = 2$ is a simple eigenvalue. Let x and y denote the right and left eigenvectors of B (or right and left eigenvectors corresponding to the smallest eigenvalue* 1 *of A), respectively. Then, we can easily see that* $x = y = [0.2887\ \ 0.5\ \ 0.5774\ \ 0.5\ \ 0.2887]^T$ *are positive eigenvectors. The matrix B^k is positive for $k \geq 4$, indeed,*

$$B^4 = \begin{bmatrix} 0.7622 & 2.56 & 3.4887 & 2.06 & 0.2622 \\ 1.4047 & 4.2504 & 5.7741 & 3.7504 & 0.9047 \\ 2.3339 & 4.6201 & 5.6683 & 4.6201 & 2.3339 \\ 3.2144 & 3.7506 & 3.465 & 4.2506 & 3.7144 \\ 2.5717 & 2.0598 & 1.1792 & 2.5598 & 3.0717 \end{bmatrix}.$$

Example 2.7. *Consider the matrix A given by*

$$A = 6I - B, \text{ where } B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 1 & 1 & 3 \\ -1 & 0.5 & -1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix} \overset{\mathrm{v}}{\geq} 0.$$

It can be shown that B *is irreducible with* $\operatorname{index}_0 B = 2$. *The spectrum of* A *is given by* $\sigma(A) = \{1.063,\ 8.937,\ 6,\ 6\}$ *and the right and left eigenvectors of* A *corresponding to the smallest real eigenvalue* 1.063 *are* $[0.6429\ \ 0.571\ \ 0.1079\ \ 0.499]^T > 0$ *and* $[0.4653\ \ 0.444\ \ 0.1237\ \ 0.7557]^T > 0$, *respectively, which shows that the necessity of Theorem* 2.1 *does not hold. The matrix* B^k *is positive for* $k \geq 6$, *indeed,*

$$B^6 = \begin{bmatrix} 4761.5 & 4317.625 & 1440.25 & 7269.5 \\ 4088.75 & 3888.625 & 1097.5 & 6613.75 \\ 1009.125 & 643.5 & 514.125 & 983.625 \\ 3416 & 3459.625 & 754.750 & 5958 \end{bmatrix}.$$

The following theorem shows equivalent conditions for irreducible M_{V}-matrices which may have positive right and left eigenvectors corresponding to the smallest eigenvalue.

Theorem 2.8. *Let* A *be an irreducible* M_{V}*-matrix, written in the form* $A = sI - B$ *with* $B \overset{\mathrm{V}}{\geq} 0$, $0 \leq \rho(B) \leq s$. *Then, the following statements are equivalent:*

(i) *There exists* $\alpha > 0$ *such that* $B + \alpha I \overset{\mathrm{V}}{\geq} 0$.

(ii) $B + \alpha I \overset{\mathrm{V}}{>} 0$ *for all* $\alpha > 0$.

(iii) *The smallest real eigenvalue* $\lambda_1 \geq 0$ *of* A *is simple, to* λ_1 *correspond positive right and left eigenvectors and* $\lambda_1 < \operatorname{Re}\lambda_i$, $i = 2, 3, \ldots, n$.

Proof. $(ii) \Rightarrow (i)$: Holds trivially.

$(ii) \Rightarrow (iii)$: Since $B + \alpha I \overset{\mathrm{V}}{>} 0$ for all $\alpha > 0$, $(B + \alpha I)^k > 0$ for some k. Both $B + \alpha I$ and $(B + \alpha I)^k$ have the same eigenvectors, thus $B + \alpha I$ has positive right and left eigenvectors and obviously statement (iii) holds true.

$(iii) \Rightarrow (ii)$: Since the right and left eigenvectors of λ_1 are positive, so are the Perron eigenvectors of B and therefore of $B + \alpha I$ for all $\alpha > 0$. Since $B \overset{\mathrm{V}}{\geq} 0$ and λ_1 is simple, for the eigenvalues of B there hold $\rho(B) = \mu_1 \geq |\mu_2| \geq |\mu_3| \geq \cdots \geq |\mu_n|$ and $\mu_1 > \operatorname{Re}\mu_2$. For any $\alpha > 0$, the eigenvalues of $B + \alpha I$ are $\mu_i + \alpha$, $i = 1, 2, 3, \ldots, n$. Then, $|\mu_2 + \alpha|^2 = (\operatorname{Re}\mu_2 + \alpha)^2 + (\operatorname{Im}\mu_2)^2 = (\operatorname{Re}\mu_2)^2 + 2\alpha \operatorname{Re}\mu_2 + \alpha^2 + (\operatorname{Im}\mu_2)^2 = |\mu_2|^2 + 2\alpha \operatorname{Re}\mu_2 + \alpha^2 < \mu_1^2 + 2\alpha\mu_1 + \alpha^2 = (\mu_1 + \alpha)^2$. Thus, $\mu_1 + \alpha > |\mu_2 + \alpha|$ which means that $B + \alpha I$ and $B^T + \alpha I$ have the strong Perron-Frobenius property, implying that $B+\alpha I \overset{\mathrm{V}}{>} 0$, see Theorem 2.2 in [42].

To complete the proof we only need to show (i) implies (iii). Suppose (i) holds. Then we distinguish between two cases.

Case 1: $\mathrm{index}_0\, B \le 1$ or $\mathrm{index}_0(B+\alpha I) \le 1$.

Then, $B^k \ge 0$ or $(B+\alpha I)^k \ge 0$ remains irreducible for all $k = rm + 1 \ge k_0$, where r is the index of cyclicity of B ($r = 1$ if B is primitive) and $k_0 = \max\{k_0(B),\ k_0(B+\alpha I)\}$. This means that the right and left Perron eigenvectors of B^k or $(B+\alpha I)^k$ are positive. But these eigenvectors are also the Perron eigenvectors of B. Therefore, statement (*iii*) holds true.

Case 2: $\mathrm{index}_0\, B \ge 2$ and $\mathrm{index}_0(B+\alpha I) \ge 2$.

Let $r_0 = \mathrm{index}_0(B)$, $r_\alpha = \mathrm{index}_0(B+\alpha I)$, k_0 is the power index of B and k_α the power index of $B+\alpha I$. If B^k is an irreducible matrix for some $k \ge k_0$ or $(B+\alpha I)^k$ is irreducible for some $k \ge k_\alpha$, then, obviously B has positive right and left Perron vectors and statement (*iii*) holds true. Thus, we suppose that B^k and $(B+\alpha I)^k$ are reducible matrices for all $k \ge k_0$ and $k \ge k_\alpha$, respectively. First, we will prove that B^k and $(B+\alpha I)^k$ do not have the same Frobenius normal form. Looking for a contradiction, we suppose these two matrices have the same Frobenius normal form. For simplicity we assume that B^k and $(B+\alpha I)^k$ are in their reducible form: $B^k = \begin{bmatrix} B_{11}^{(k)} & B_{12}^{(k)} \\ 0 & B_{22}^{(k)} \end{bmatrix}$ and $(B+\alpha I)^k = \begin{bmatrix} (B_\alpha)_{11}^{(k)} & (B_\alpha)_{12}^{(k)} \\ 0 & (B_\alpha)_{22}^{(k)} \end{bmatrix}$,
where $B_{11}^{(k)}, (B_\alpha)_{11}^{(k)} \in \mathbb{R}^{m,m}$ and $B_{22}^{(k)}, (B_\alpha)_{22}^{(k)} \in \mathbb{R}^{n-m,n-m}$. On the other hand we have that

$$(B+\alpha I)^k = \alpha^k I + \binom{k}{1}\alpha^{k-1}B + \binom{k}{2}\alpha^{k-2}B^2 + \cdots + B^k. \tag{2.1}$$

For each row index $i = m+1, m+2, \ldots, n$ and column index $j = 1, 2, \ldots, m$, that correspond to zero entries of both matrices, relation (2.1) takes the form

$$\begin{aligned}\left((B+\alpha I)^k\right)_{ij} &= \binom{k}{1}\alpha^{k-1}b_{ij} + \binom{k}{2}\alpha^{k-2}(B^2)_{ij} \\ &\quad + \cdots + \binom{k}{r_0-1}\alpha^{k-r_0+1}(B^{r_0-1})_{ij} = 0,\end{aligned} \tag{2.2}$$

for all $k \ge \max\{r_0, k_\alpha\}$.

Taking r_0-1 successive values of k; i.e.: $k, k+1, \ldots, k+r_0-2$ we get the linear system

$$\begin{array}{ccccccc} \binom{k}{1}\alpha^{k-1}b_{ij} & + & \binom{k}{2}\alpha^{k-2}(B^2)_{ij} & +\cdots+ & \binom{k}{r_0-1}\alpha^{k-r_0+1}(B^{r_0-1})_{ij} & = & 0 \\ \binom{k+1}{1}\alpha^{k}b_{ij} & + & \binom{k+1}{2}\alpha^{k-1}(B^2)_{ij} & +\cdots+ & \binom{k+1}{r_0-1}\alpha^{k-r_0+2}(B^{r_0-1})_{ij} & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ \binom{k+r_0-2}{1}\alpha^{k+r_0-3}b_{ij} & + & \binom{k+r_0-2}{2}\alpha^{k+r_0-4}(B^2)_{ij} & +\cdots+ & \binom{k+r_0-2}{r_0-1}\alpha^{k-1}(B^{r_0-1})_{ij} & = & 0 \end{array}, \tag{2.3}$$

considering as unknown vector : $\begin{pmatrix} b_{ij} & (B^2)_{ij} & \cdots & (B^{r_0-1})_{ij} \end{pmatrix}^T$. The coefficient matrix is a Vandermonde type matrix, and thus, it is a nonsingular one. Obviously, this system has the unique solution of zeros. This means that $b_{ij}=0$, and this happens for all $i=m+1, m+2, \ldots, n$, and $j=1,2,3,\ldots,m$. Thus, the matrix B is a reducible matrix which constitutes a contradiction.

Now we consider the matrix

$$C^{(k)} = B^k + (B+\alpha I)^k, \tag{2.4}$$

for some $k \geq \max\{r_0, r_\alpha, k_0, k_\alpha\}$. This matrix is a nonnegative irreducible one, since otherwise B^k and $(B+\alpha I)^k$ should have the same reducible form and we arrive at the same contradiction. Since $C^{(k)}$ is a polynomial of B, it has the same eigenvectors of B. Thus, the Perron right and left eigenvectors of B are those of $C^{(k)}$, which are positive vectors, proving the validity of statement (iii), and the proof is complete. □

The above theorem does not hold for GM-matrices. From the definition of GM-matrices and [42] we obtain that any GM-matrix (which is not an $M_{\rm V}$-matrix) may have nonnegative eigenvector corresponding to its smallest eigenvalue (see [17], Example 2.2).

From Example 2.6 and 2.7, there exists $\alpha > 0$ such that $B+\alpha I \overset{\rm v}{\geq} 0$ which confirm the validity of Theorem 2.8. Also, the following examples show that if a matrix has $\text{index}_0\, B \geq 2$ but there exists $\alpha > 0$ such that $B+\alpha I \overset{\rm v}{\geq} 0$, then Theorem 2.8 is valid.

Example 2.9. *Consider the matrix A given by*

$$A = 14I - B, \text{ where } B = \begin{bmatrix} 4 & -3 & 15 & 1 & 2 & 4 \\ 1 & -1 & 7 & 1 & 1 & 1 \\ 1.5 & 1 & 3 & 1.5 & 1.5 & 1.5 \\ 2 & 1 & 16 & 2 & 2 & 1 \\ 1.5 & -1 & 1 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 \end{bmatrix} \overset{\rm v}{\geq} 0.$$

We can easily see that B is irreducible with $\text{index}_0\, B = 2$. *The matrix B^k is positive for $k \geq 3$, indeed,*

$$B^3 = \begin{bmatrix} 554 & 21.75 & 1757.25 & 414.5 & 461 & 519.5 \\ 233.75 & 10.75 & 756.25 & 178.25 & 196.75 & 218.25 \\ 254.375 & 34.25 & 756.25 & 197.375 & 216.375 & 239.875 \\ 543.75 & 42.75 & 1722.25 & 419.25 & 460.75 & 508.25 \\ 179.375 & 21.75 & 528.75 & 137.375 & 151.375 & 169.875 \\ 235.125 & 37.875 & 680.625 & 183.375 & 200.625 & 222.375 \end{bmatrix}.$$

The spectral radius $\rho(B) = 12.8955$ is a simple eigenvalue of B. The spectrum of A is given by $\sigma(A) = \{1.1045,\ 12.064,\ 14,\ 14,\ 14.4812, 17.3504\}$, both right and left eigenvectors corresponding to the smallest real eigenvalue 1.1045 *of A:* $[0.6137 \quad 0.2621 \quad 0.2807 \quad 0.6087 \quad 0.1965 \quad 0.2582]^T$ *and* $[0.2825 \quad 0.0282 \quad 0.8631 \quad 0.2168 \quad 0.2387 \quad 0.2657]^T$ *are positive since there exist $\alpha = 4$, such that $B + 4I$ is an eventually nonnegative matrix with power index* 4 $((B + \alpha I)^k > 0,\ \forall k \geq 4)$.

Example 2.10. *Consider the matrix A given by $A = 9I - B$, where*

$$B = \frac{1}{155} \begin{bmatrix} 2021 & 4346 & 3318 & -8517 & 9414 & -5835 \\ -2810 & -3895 & -2225 & 9325 & -9060 & 6120 \\ 2402 & 3642 & 2591 & -7699 & 8438 & -5055 \\ -318 & -628 & -394 & 1591 & -1392 & 1270 \\ -877 & -2272 & -1536 & 5289 & -5193 & 3405 \\ -1224 & -2464 & -1227 & 6583 & -5896 & 3815 \end{bmatrix} \overset{\text{v}}{\geq} 0.$$

We can easily see that B is irreducible. The spectrum of B is given by $\sigma(B) = \{7,\ 5,\ 0,\ 0,\ -3,\ -3\}$ with $\text{index}_0\, B = 2$ *and* $\text{index}_{-3}\, B = 2$. *The matrix B^k is positive for $k \geq 15$. The right and left eigenvectors corresponding to the smallest real eigenvalue* 2 *of A:* $[0.3123 \quad 0.4685 \quad 0.3123 \quad 0.3123 \quad 0.3123 \quad 0.6247]^T$ *and* $[0.3041 \quad 0.3041 \quad 0.5744 \quad 0.4054 \quad 0.5406 \quad 0.1689]^T$ *are positive since there exists $\alpha = 3$, such that $B + 3I$ is an eventually nonnegative matrix with power index* 23 $((B + \alpha I)^k > 0,\ \forall k \geq 23)$.

According to Theorem 2.8, the following example shows that if the statement (i) is not valid (there is no $\alpha > 0$ such that $B + \alpha I \overset{\text{v}}{\geq} 0$), then the right and left eigenvectors corresponding to the smallest real eigenvalue may be nonnegative and not positive.

Example 2.11. *Consider the matrix A given by*

$$A = 4I - B, \text{ where } B = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \overset{\mathrm{v}}{\geq} 0.$$

We can easily see that B is irreducible. The spectrum of B is given by $\sigma(B) = \{2,\ 1,\ 0,\ 0\}$ with $\operatorname{index}_0 B = 2$. The right and left eigenvectors corresponding to the smallest real eigenvalue 2 of A are $(1\ 1\ 1\ 1)^T > 0$ and $[0\ 0\ 1\ 1]^T \geq 0$, respectively. However, the left eigenvector $[0\ 0\ 1\ 1]^T$ is a nonnegative vector. The matrix B^k is reducible for $k \geq 2$,

$$B^k = \begin{bmatrix} 0.5 & 0.5 & 2^{k-1} - 0.5 & 2^{k-1} - 0.5 \\ 0.5 & 0.5 & 2^{k-1} - 0.5 & 2^{k-1} - 0.5 \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix}.$$

If $\alpha > 0$, then $B + \alpha I$ is not an eventually nonnegative matrix because $(B + \alpha I)^k$ has the submatrix

$$(B + \alpha I)^{(k)}_{21} = \begin{bmatrix} k\alpha^{k-1} & -k\alpha^{k-1} \\ -k\alpha^{k-1} & k\alpha^{k-1} \end{bmatrix}$$

that always has negative entries $(-k\alpha^{k-1})$.

There is no $\alpha > 0$ such that $B + \alpha I \overset{\mathrm{v}}{\geq} 0$. Properties (i) and (ii) of Theorem 2.8 do not hold true and the left Perron vector of B is nonnegative and not positive.

From Theorem 2.8, we have the property of right and left eigenvectors corresponding to the smallest real eigenvalue of an M_{V}-matrix. In the next two theorems, we study the eigenvectors and generalized eigenvectors corresponding to other eigenvalues which are not the dominant eigenvalue of an irreducible eventually nonnegative matrix, or equivalent, the smallest one of an M_{V}-matrix. First, we will state and prove one theorem for an irreducible eventually nonnegative matrix and then we will give the equivalent version for an M_{V}-matrix in remark below.

Theorem 2.12. *The eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ of an irreducible eventually nonnegative matrix B with $\operatorname{index}_0 B \leq 1$ are not nonnegative vectors (have positive and negative, or complex entries).*

Proof. Let $\lambda \neq \rho(B)$ and y be a nonnegative right eigenvector corresponding to λ. Let also $w > 0$ be the left eigenvector corresponding to $\rho(B)$. Then

$$w^T By = \rho(B) w^T y > 0 \quad \text{and} \quad w^T By = \lambda w^T y > 0,$$

which means that $\lambda = \rho(B)$ and constitutes a contradiction.

Let y_s be a nonnegative generalized eigenvector corresponding to $\lambda \neq \rho(B)$ of order s. Let $[y_1\ y_2\ \cdots\ y_s\ \cdots\ y_q]$ be the chain of the generalized eigenspace of λ. Then, y_s is an eigenvector of $(B - \lambda I)^s$ and w is a left eigenvector of $B - \lambda I$ corresponding to the eigenvalue $(\rho(B) - \lambda)^s$. Hence,

$$w^T(B - \lambda I)^s y_s = 0 \quad \text{and} \quad w^T(B - \lambda I)^s y_s = (\rho(B) - \lambda)^s w^T y_s > 0,$$

which constitutes a contradiction.

The proof for the left eigenvector is analogous. □

Remark 2.13. Theorem 2.12 could be stated equivalently as:

The eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \lambda_{\min}(A)$ of an irreducible $M_{\rm V}$-matrix $A = sI - B$ with $\operatorname{index}_0 B \leq 1$ *are not nonnegative vectors.*

We give the example to show the validity of the theorem above.

Example 2.14. *Consider the matrix A given by*

$$A = 12I - B, \text{ where } B = \begin{bmatrix} 0 & 3 & 6 & -2 \\ 4 & 7 & 3 & 1 \\ -2 & 3 & -1 & 4 \\ 5 & 4 & 5 & -4 \end{bmatrix} \overset{\rm v}{\geq} 0.$$

We can easily see that B is an irreducible matrix. The spectrum of B is given by $\sigma(B) = \{10.3514,\ -3.9723 \pm 3.6657i,\ -0.4067\}$ with $\operatorname{index}_0 B = 0$. *The right and left eigenvectors of A corresponding to the smallest eigenvalue* 1.6486 *of A are* $[0.3196\ \ 0.7842\ \ 0.3046\ \ 0.4360]^T > 0$ *and* $[0.3035\ \ 0.8328\ \ 0.4417\ \ 0.1388]^T > 0$*, respectively.*

The right and left eigenvectors corresponding to $\lambda \neq \lambda_{\min}(A)$ of A are, respectively, as follows: the eigenvectors $[0.6947\ \ -0.1763 + 0.0072i\ \ -0.2551 - 0.3482i\ \ 0.3501 + 0.4046i]^T$ *and* $[0.6719\ \ -0.0202 - 0.0051i\ \ 0.056 + 0.4531i\ \ -0.4953 - 0.3073i]^T$ *corresponding to* $-3.9723 + 3.6657i$*; the eigenvectors* $[0.6947\ \ -0.1763 - 0.0072i\ \ -0.2551 + 0.3482i\ \ 0.3501 - 0.4046i]^T$

and $[0.6719\ \ -0.0202+0.0051i\ \ 0.056-0.4531i\ \ -0.4953+0.3073i]^T$ *corresponding to* $-3.9723-3.6657i$*; the eigenvectors* $[0.4196\ \ -0.4855\ \ 0.4267\ \ 0.6372]^T$ *and* $[-0.2287\ \ -0.4776\ \ 0.5697\ \ 0.6285]^T$ *corresponding to* 12.4067 *which are not nonnegative vectors.*

We have to remark that the assumption $\text{index}_0\, B \leq 1$ is sufficient and not necessary. To be specific, there are $M_{\rm V}$-matrices with $\text{index}_0\, B > 1$ that have eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \lambda_{\min}(A)$ not to be nonnegative as is shown in the following example.

Example 2.15. *Consider the matrix A given by*

$$A = 7I - B, \text{ where } B = \begin{bmatrix} 3 & 4 & 2 & -3 \\ 7 & 8 & 2 & -11 \\ -9 & -14 & -2 & 27 \\ 5 & 6 & 2 & -7 \end{bmatrix} \overset{\rm V}{\geq} 0.$$

We can easily see that B is an irreducible matrix. The spectrum of B is given by $\sigma(B) = \{5.1231,\ \ -3.1231,\ \ 0,\ \ 0\}$ *with* $\text{index}_0\, B = 2$*. The right and left eigenvectors of A corresponding to the smallest eigenvalue* 1.8769 *of A are* $[1\ \ 1\ \ 0.5616\ \ 1]^T > 0$ *and* $[1\ \ 0.8256\ \ 0.6213\ \ 0.3871]^T > 0$*, respectively.*

The eigenvectors or generalized right and left eigenvectors corresponding to $\lambda \neq \lambda_{\min}(A)$ of A are, respectively, as follows: the eigenvectors $[0.2808\ \ 0.2808\ \ -1\ \ 0.2808]^T$ *and* $[0.3464\ \ 0.58\ \ 0.1312\ \ -1]^T$ *corresponding to* 10.1231*; the eigenvectors* $[-1\ \ 3\ \ -3\ \ 1]^T$ *and* $[-0.5\ \ -0.5\ \ 0\ \ 1]^T$ *corresponding to* 7*; the generalized eigenvectors* $[1.5\ \ -0.5\ \ -1.75\ \ 0]^T$ *and* $[0.125\ \ -0.125\ 0\ 0]^T$ *corresponding to* 7 *which are not nonnegative vectors.*

Even $\text{index}_0\, B = 2$*, the eigenvectors corresponding to $\lambda \neq \lambda_{\min}(A)$ of A are not nonnegative vectors.*

The assumption $\text{index}_0\, B \leq 1$ is sufficient since otherwise the matrix B^k may be reducible and the Perron eigenvectors should be nonnegative and not positive. But this assumption is not necessary. We now state and prove sufficient and necessary conditions in the following theorem.

Theorem 2.16. *The right and left eigenvectors and generalized eigenvectors corresponding to the eigenvalue $\lambda \neq \rho(B)$ of an irreducible eventually nonnegative matrix B are not nonnegative vectors iff there exists $\alpha > 0$ such that* $B + \alpha I \overset{\rm V}{\geq} 0$.

Proof. In the proof of Theorem 2.8 we have proven that the Perron eigenvector of B is positive iff there exists $\alpha > 0$ such that $B+\alpha I \overset{\mathrm{V}}{\geq} 0$, even if $\mathrm{index}_0\, B \geq 2$ and $\mathrm{index}_0(B+\alpha I) \geq 2$. Thus, from Theorem 2.12, we get our result. $\square$

Remark 2.17. As in Remark 2.13, Theorem 2.16 could be stated in an analogous way for M_{V}-matrices.

From Example 2.15, Theorem 2.16 is valid : the eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ of B are not nonnegative vectors because there exists $\alpha = 8$ such that $B + \alpha I$ is an eventually nonnegative matrix with power index 9 ($(B + \alpha I)^k \geq 0$, $\forall k \geq 9$).

We now give examples which support the result of Theorem 2.16. The following example shows that if there is no $\alpha > 0$ such that $B + \alpha I \overset{\mathrm{V}}{\geq} 0$, the right and left eigenvectors and generalized eigenvectors corresponding to the eigenvalue $\lambda \neq \rho(B)$ may be nonnegative.

Example 2.18. *Consider the matrix A given by*

$$A = 5I - B, \text{ where } B = \begin{bmatrix} 1 & 1 & 0.5 & 0.5 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & -1 & 0.5 & 0.5 \\ -1 & 1 & 0.5 & 0.5 \end{bmatrix} \overset{\mathrm{V}}{\geq} 0.$$

We can easily see that B is an irreducible matrix. The spectrum of B is given by $\sigma(B) = \{2,\ 1,\ 0,\ 0\}$ with $\mathrm{index}_0\, B = 2$. *The right and left eigenvectors of B corresponding to $\rho(B) = 2$ are $[1\ 1\ 0\ 0]^T \geq 0$ and $[1\ 1\ 1\ 1]^T > 0$, respectively. The right and left eigenvectors of B corresponding to 1 are $[-1\ \ -1\ 1\ 1]^T$ and $[0\ 0\ 1\ 1]^T \geq 0$, respectively. The right and left eigenvectors of B corresponding to 0 are $[0\ \ 0\ \ -1\ \ 1]^T$ and $[-1\ \ 1\ \ 0\ \ 0]^T$, respectively. The generalized right and left eigenvectors of B corresponding to 0 are $[-0.5\ \ 0.5\ \ 0\ \ 0]^T$ and $[0\ \ 0\ \ -0.5\ \ 0.5]^T$, respectively.*

However, the left eigenvector corresponding to $\lambda = 1$: $[0\ 0\ 1\ 1]^T$ is a nonnegative vector. The power of B:

$$B^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & 2^{k-1} - 0.5 & 2^{k-1} - 0.5 \\ 2^{k-1} & 2^{k-1} & 2^{k-1} - 0.5 & 2^{k-1} - 0.5 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

is a reducible nonnegative matrix for $k \geq 2$.

If $\alpha > 0$, then $B + \alpha I$ is not an eventually nonnegative matrix because $(B + \alpha I)^k$ has a submatrix

$$(B + \alpha I)_{21}^{(k)} = \begin{bmatrix} k\alpha^{k-1} & -k\alpha^{k-1} \\ -k\alpha^{k-1} & k\alpha^{k-1} \end{bmatrix}$$

that always has negative entries $(-k\alpha^{k-1})$.

There is no $\alpha > 0$ such that $B + \alpha I \overset{\text{v}}{\geq} 0$. The assumption of Theorem 2.16 *does not hold and the left eigenvector corresponding to the eigenvalue $\lambda = 1 \neq 2 = \rho(B)$ of B is a nonnegative vector.*

The following example shows that if we can find $\alpha > 0$ such that $B+\alpha I \overset{\text{v}}{\geq} 0$, according to Theorem 2.16, the eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ of B are not nonnegative vectors without considering the $\text{index}_0\, B$.

Example 2.19. *Consider the matrix A given by*

$$A = 9I - B, \text{ where } B = \begin{bmatrix} 2 & 4 & 2 & 5 & -1 \\ 3 & 2 & 4 & 4 & 3 \\ 1 & 5 & -2 & 4 & -5 \\ 1 & -2 & 2 & -1 & 4 \\ 2 & -4 & 4 & -2 & 8 \end{bmatrix} \overset{\text{v}}{\geq} 0.$$

We can easily see that B is an irreducible matrix. The spectrum of B is given by $\sigma(B) = \{7.7572,\ 2.8635,\ 0,\ 0,\ -1.6207\}$ with $\text{index}_0\, B = 2$. The right and left eigenvectors of B corresponding to $\rho(B)$ are $[0.6250\ \ 0.6735\ \ 0.3794\ \ 0.0485\ \ 0.0970]^T > 0$ *and* $[0.4692\ \ 0.2632\ \ 0.4945\ \ 0.5105\ \ 0.4532]^T > 0$, *respectively. The eigenvectors or generalized right and left eigenvectors corresponding to $\lambda \neq \rho(B)$ of B are, respectively, as follows: the eigenvectors* $[0.3886\ \ 0.0581\ \ 0.5473\ \ -0.3305\ \ -0.6610]^T$ *and* $[-0.2212\ \ 0.4338\ \ -0.2465\ \ 0.3106\ \ -0.7782]^T$ *corresponding to* 2.8635*; the eigenvectors* $[-0.4584\ \ 0.6845\ \ -0.1432\ \ -0.1991\ \ 0.4854]^T$ *and* $[-0.3756\ \ 0.3756\ \ 0\ \ 0.6676\ \ -0.5216]^T$ *corresponding to* 0*; the generalized eigenvectors* $[0.5416\ \ -0.1301\ \ 0.4765\ \ -0.4115\ \ -0.5416]^T$ *and* $[0.4973\ \ -0.4973\ \ 0\ \ -0.5044\ \ 0.5009]^T$ *corresponding to* 0*; the eigenvectors* $[-0.0107\ \ 0.2317\ \ -0.8077\ \ 0.2424\ \ -0.4848]^T$ *and* $[-0.2566\ \ -0.2869\ \ 0.8064\ \ 0.1163\ \ 0.4336]^T$ *corresponding to* -1.6207 *which are not nonnegative vectors.*

Let $\alpha = 2$, then $B + 2I$ is an eventually nonnegative matrix with power index 6 *($(B + \alpha I)^k \geq 0$, $\forall k \geq 6$). From Theorem* 2.16*, the eigenvectors and* *generalized eigenvectors corresponding to* $\lambda \neq \rho(B)$ *of* B *are not nonnegative vectors because there exists* $\alpha = 2$ *such that* $B + \alpha I \overset{\mathrm{v}}{\geq} 0$.

2.2 Equivalence of eventually exponentially nonnegative and M_{V}-matrices

Properties connecting eventually nonnegative matrices and eventually exponentially nonnegative matrices have been proven in [43, Theorem 3.7] when $\mathrm{index}_0\, B \leq 1$. In this section, we give results connecting M_{V}-matrices and eventually exponentially nonnegative matrices for every case of $\mathrm{index}_0\, B$.

First, we prove a lemma for series of the inverse of a matrix $A = sI - B$.

Lemma 2.20. *Let* $A \in \mathbb{R}^{n,n}$ *be an invertible matrix, written in the form* $A = sI - B$. *Then,* $(A^{-1})^k$ *is given in the following series form:*

$$\begin{aligned}(A^{-1})^k &= \frac{1}{s^k}I + \frac{1}{s^{k+1}}\binom{k}{1}B + \frac{1}{s^{k+2}}\binom{k+1}{2}B^2 + \frac{1}{s^{k+3}}\binom{k+2}{3}B^3 \\ &\quad + \cdots + \frac{1}{s^{k+m}}\binom{k+m-1}{m}B^m + \cdots . \end{aligned} \tag{2.5}$$

Proof. By induction, for $k = 1$, the statement is true because

$$\begin{aligned}(A^{-1})^1 &= \frac{1}{s}(I - \frac{1}{s}B)^{-1} = \frac{1}{s}I + \frac{1}{s^2}B + \frac{1}{s^3}B^2 + \frac{1}{s^4}B^3 + \cdots + \frac{1}{s^{m+1}}B^m + \cdots \\ &= \frac{1}{s}I + \frac{1}{s^2}\binom{1}{1}B + \frac{1}{s^3}\binom{2}{2}B^2 + \frac{1}{s^4}\binom{3}{3}B^3 \\ &\quad + \cdots + \frac{1}{s^{m+1}}\binom{m}{m}B^m + \cdots\end{aligned}$$

Assume (2.5) holds. Then,

$$\begin{aligned}(A^{-1})^{k+1} &= (A^{-1})^k(A^{-1})^1 \\ &= \left(\frac{1}{s^k}I + \frac{1}{s^{k+1}}\binom{k}{1}B + \frac{1}{s^{k+2}}\binom{k+1}{2}B^2 + \frac{1}{s^{k+3}}\binom{k+2}{3}B^3\right. \\ &\quad \left. + \cdots + \frac{1}{s^{k+m}}\binom{k+m-1}{m}B^m + \cdots\right) \\ &\quad \times \left(\frac{1}{s}I + \frac{1}{s^2}B + \frac{1}{s^3}B^2 + \cdots + \frac{1}{s^{m+1}}B^m + \cdots\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{s^{k+1}}I + \frac{1}{s^{k+2}}\left(\binom{k}{1} + \binom{k-1}{0}\right)B \\
&\quad + \frac{1}{s^{k+3}}\left(\binom{k+1}{2} + \binom{k}{1} + \binom{k-1}{0}\right)B^2 \\
&\quad + \cdots + \frac{1}{s^{k+m+1}}\left(\binom{k+m-1}{m} + \binom{k+m-2}{m-1}\right. \\
&\quad \left. + \cdots + \binom{k}{1} + \binom{k-1}{0}\right)B^m + \cdots
\end{aligned}$$

We have to prove that

$$\binom{k+i-1}{i} + \binom{k+i-2}{i-1} + \cdots + \binom{k-1}{0} = \binom{k+i}{i}, \quad i = 1, 2, \ldots, m. \tag{2.6}$$

For this, we use induction :
For $i = 1$, we have $\binom{k}{1} + \binom{k-1}{0} = k + 1 = \binom{k+1}{1}$, thus (2.6) holds true.
Suppose that (2.6) holds true for $i = j$, we prove it for $i = j + 1$:

$$\begin{aligned}
& \binom{k+j}{j+1} + \binom{k+j-1}{j} + \binom{k+j-2}{j-1} + \cdots + \binom{k}{1} + \binom{k-1}{0} \\
= \; & \binom{k+j}{j+1} + \binom{k+j}{j} \\
= \; & \frac{(k+j)(k+j-1)\cdots k}{(j+1)!} + \frac{(k+j)(k+j-1)\cdots(k+1)}{j!} \\
= \; & \frac{(k+j)(k+j-1)\cdots(k+1)}{j!}\left(\frac{k}{j+1} + 1\right) \\
= \; & \frac{(k+j)(k+j-1)\cdots(k+1)}{j!}\,\frac{k+j+1}{j+1} \\
= \; & \binom{k+j+1}{j+1}
\end{aligned}$$

and the proof is complete. □

Le and McDonald ([36]) studied the inverse of an eventually nonnegative matrix and gave the result as the following theorem.

Theorem 2.21 ([36])**.** *Let $B \in \mathbb{R}^{n,n}$ be an irreducible eventually nonnegative matrix with* $\text{index}_0 B \leq 1$ *then $\exists \lambda > \rho(B)$ such that $\lambda > s > \rho(B)$, then $(sI - B)^{-1} > 0$.*

From the above theorem and Theorem 2.1, we obtain the following result

Theorem 2.22. *Let A be an irreducible $M_{\rm V}$-matrix, written in the form $sI-B$ with $B \overset{\rm V}{\geq} 0$, $0 \leq \rho(B) \leq s$ and* $\mathrm{index}_0\, B \leq 1$. *Then, $A^{-1} \overset{\rm V}{>} 0$.*

Proof. Since the inverse of A share the same right and left Perron eigenvectors, then A^{-1} has positive right and left Perron eigenvectors and from Theorem 2.1 and [42, Theorem 2.2], we get that A^{-1} is eventually positive. □

Now we will study the inverse of eventually nonnegative matrix without concerning the $\mathrm{index}_0\, B$ and give the following result.

Theorem 2.23. *Let $B \in \mathbb{R}^{n,n}$ be an eventually nonnegative matrix. Let $A \in \mathbb{R}^{n,n}$, of the form $A = sI - B$, be the associated $M_{\rm V}$-matrix and $0 \leq \rho(B) < s$. Then, the following statements are equivalent:*

(i) There exists $\alpha > 0$ such that $(B + \alpha I) \overset{\rm V}{\geq} 0$.

(ii) B is an eventually exponentially nonnegative matrix.

(iii) $A^{-1} \overset{\rm V}{\geq} 0$.

Proof. Statement (i) means that there exists $k_\alpha > 0$ such that $(B + \alpha I)^k \geq 0$ for all $k \geq k_\alpha$. Let B has power index k_0 and we choose $k > \max\{k_0, k_\alpha\}$. Then,

$$\begin{aligned}(B+\alpha I)^k &= \alpha^k \left(I + \binom{k}{1}\left(\frac{B}{\alpha}\right) + \binom{k}{2}\left(\frac{B}{\alpha}\right)^2 \right. \\ &\quad + \cdots + \binom{k}{k_0-1}\left(\frac{B}{\alpha}\right)^{k_0-1} + \binom{k}{k_0}\left(\frac{B}{\alpha}\right)^{k_0} \\ &\quad \left. + \cdots + \binom{k}{k}\left(\frac{B}{\alpha}\right)^{k} \right) \geq 0. \end{aligned} \tag{2.7}$$

Statement (ii) means that there exists $t_0 > 0$ such that $e^{tB} \geq 0$ for all $t > t_0$. Thus,

$$\begin{aligned} e^{tB} &= I + tB + \frac{t^2}{2!}B^2 + \frac{t^3}{3!}B^3 + \cdots + \frac{t^{k_0-1}}{(k_0-1)!}B^{k_0-1} \\ &\quad + \frac{t^{k_0}}{k_0!}B^{k_0} + \cdots \geq 0. \end{aligned} \tag{2.8}$$

Statement (*iii*) means that there exists $m_0 > 0$ such that $(A^{-1})^m \geq 0$ for all $m > m_0$. Taking into account the expansion proven in Lemma 2.20 we get that

$$\begin{aligned}(A^{-1})^m &= \frac{1}{s^m}\left(I + \binom{m}{1}\left(\frac{B}{s}\right) + \binom{m+1}{2}\left(\frac{B}{s}\right)^2\right. \\ &\quad + \cdots + \binom{m+k_0-2}{k_0-1}\left(\frac{B}{s}\right)^{k_0-1} \\ &\quad \left. + \binom{m+k_0-1}{k_0}\left(\frac{B}{s}\right)^{k_0} + \cdots\right) \geq 0.\end{aligned} \tag{2.9}$$

We observe that in (2.7), we have a polynomial of B which should be nonnegative, while in (2.8) and (2.9) we have series expansion in B to be nonnegative. Since $B \overset{\mathrm{V}}{\geq} 0$, the first k_0 terms may have negative entries in all cases. These entries should be the same for the three cases, because all the coefficients in the powers of B are positive.

Case 1: B is irreducible and $\mathrm{index}_0\, B \leq 1$.

Suppose first that B is not a weakly cyclic matrix. Then, both right and left Perron vectors of B are positive, and thus, B should be eventually positive and the validity of (*i*) is guaranteed from Theorem 2.8. This means that the last $k - k_0 + 1$ terms dominate the first k_0 ones in order to eliminate the negative entries. We observe that in statement (*ii*) we can choose a large enough t such that the $(k_0 + 1)$st term (monomial in t of degree k_0) should dominate all the previous sum (polynomial in t of degree $k_0 - 1$). Thus, (*i*) $\Rightarrow$ (*ii*) is proven. We observe also that in statement (*iii*) we can choose large enough m such that the $(k_0 + 1)$st term should dominate all the previous sum, since the coefficient of this term is a polynomial in m of degree k_0 while the coefficients of the previous terms, are polynomials in m of smaller degrees. Thus, (*i*) $\Rightarrow$ (*iii*) is proven.

The proof in the opposite directions is exactly the same. Indeed, the validity of (*ii*) means that the series of $(k_0 + 1)$st term and thereafter, dominates the first sum. Then we can choose a large enough k such that the $(k_0 + 1)$st term of polynomial (2.7) should dominate all the previous sum, proving that (*ii*) $\Rightarrow$ (*i*). Similarly, we prove that (*ii*) $\Rightarrow$ (*iii*), (*iii*) $\Rightarrow$ (*i*) and (*iii*) $\Rightarrow$ (*ii*).

In case where B is a weakly cyclic matrix of index r, then we consider r sums of (2.7), taking in each sum the terms of modulus r, i.e.

$$\alpha^k\left(\binom{k}{i}\left(\tfrac{B}{\alpha}\right)^i+\binom{k}{r+i}\left(\tfrac{B}{\alpha}\right)^{r+i}+\binom{k}{2r+i}\left(\tfrac{B}{\alpha}\right)^{2r+i}+\cdots\right), i=0,1,\ldots,r-1.$$

Each term in this sum has the same cyclic structure. Analogously, we consider r subseries of (2.8) and (2.9) taking in each subseries the powers of modulus r, as in (2.7). Then, the proof follows the same steps as before, connecting each polynomial of (2.7) with each subseries of (2.8) and (2.9) having the same cyclic structure.

Case 2: B is irreducible and $\operatorname{index}_0 B \geq 2$.

Suppose first that (i) holds true. Then, from Theorem 2.8 we obtain that both the right and left Perron vectors of B are positive. Thus there exists k_0 such that B^k is irreducible and $B^k \geq 0$ for all $k \geq k_0$. Then, to prove $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ we follow the same arguments as in Case 1.

Now suppose that (ii) holds true. Then, from (2.8), since B is irreducible, e^{tB} is irreducible even if we consider that B^k maybe reducible for all $k \geq r_0$ ($r_0 = \operatorname{index}_0 B$). Otherwise, supposing e^{tB} is reducible, we arrive at the same contradiction following the proof of Case 2 in Theorem 2.8, where in (2.2) we consider the associated terms of $\left(e^{tB}\right)_{ij}$ instead of $\left((B+\alpha I)^k\right)_{ij}$, and system (2.3) is taken by choosing different values of t. Thus, e^{tB}, and therefore B, has positive right and left Perron vectors. Now, following the same steps as in the proof of Case 1, we prove that $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$.

Finally, we suppose that (iii) holds true. Then, from (2.9), following the same steps previously, we obtain that $\left(A^{-1}\right)^m$ is irreducible and thus, $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.

Case 3: B is reducible.

For simplicity, and without loss of generality, suppose that B is in its Frobenius normal form

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1q} \\ & B_{22} & \cdots & B_{2q} \\ & & \ddots & \vdots \\ & & & B_{qq} \end{bmatrix} \tag{2.10}$$

where B_{ii}, $i = 1, 2, \ldots, q$ are square irreducible matrices or 1×1 zero ones.

Since $B \overset{\mathrm{v}}{\geq} 0$, we have that

$$B^k = \begin{bmatrix} B_{11}^k & B_{12}^{(k)} & \cdots & B_{1q}^{(k)} \\ & B_{22}^k & \cdots & B_{2q}^{(k)} \\ & & \ddots & \vdots \\ & & & B_{qq}^k \end{bmatrix} \geq 0, \tag{2.11}$$

for all $k \geq k_0$. If B_{ii} is 1×1 zero matrix then so is B_{ii}^k, $\forall k \geq 0$. Thus, $(B+\alpha I)_{ii}^k = \alpha^k > 0$ for relation (2.7). For relation (2.8), we have $e^{tB} = I + \sum_{j=1}^{\infty} \frac{(tB)^j}{j!}$, thus $(e^{tB})_{ii} = 1 > 0$ and for relation (2.9), $(A^{-1})^m = \frac{1}{s^m}\left(I + \sum_{j=1}^{\infty} \binom{m+j-1}{j}\left(\frac{B}{s}\right)^j\right)$, we get $(A^{-1})_{ii}^m = \frac{1}{s^m} > 0$.

If B_{ii} is irreducible, $B_{ii} \overset{\mathrm{v}}{\geq} 0$, then we follow the proof of case 1, if $\mathrm{index}_0\, B_{ii} \leq 1$ or of case 2, if $\mathrm{index}_0\, B_{ii} \geq 2$, where we consider the matrix B_{ii} in the places of B. Thus, the proof of theorem concerning the diagonal blocks is complete.

We consider the (i, j) off diagonal block, $i < j$. Then, (2.7) gives us

$$(B+\alpha I)_{ij}^k = \alpha^k \left(\binom{k}{1}\left(\frac{B_{ij}}{\alpha}\right) + \binom{k}{2}\left(\frac{B_{ij}^{(2)}}{\alpha^2}\right) + \cdots + \binom{k}{k}\left(\frac{B_{ij}^{(k)}}{\alpha^k}\right)\right),$$

(2.8) presents

$$\left(e^{tB}\right)_{ij} = tB_{ij} + \frac{t^2}{2!}B_{ij}^{(2)} + \cdots + \frac{t^{k_0}}{k_0!}B_{ij}^{(k_0)} + \cdots,$$

while (2.9)

$$\begin{aligned}(A^{-1})_{ij}^m &= \frac{1}{s^m}\left(\binom{m}{1}\left(\frac{B_{ij}}{s}\right) + \binom{m+1}{2}\left(\frac{B_{ij}^{(2)}}{s^2}\right)\right. \\ &\quad \left. + \cdots + \binom{m+k_0-1}{k_0}\left(\frac{B_{ij}^{(k_0)}}{s^{k_0}}\right) + \cdots\right).\end{aligned} \tag{2.12}$$

Let $B_{ij} \in \mathbb{R}^{i_n, j_n}$. We consider the (μ, ν) entry of B_{ij}, $1 \leq \mu \leq i_n$, $1 \leq \nu \leq j_n$. Then, the sequence of matrices $\left\{B_{ij}^{(k)}\right\}_{k=1}^{\infty}$ defines a sequence of real numbers for the associated (μ,ν) entries: $\left\{ \left(B_{ij}^{(k)}\right)_{\mu,\nu} \right\}_{k=1}^{\infty}$. For simplicity, we symbolize this sequence by $\{b_k\}_{k=1}^{\infty}$. From the fact that $B \overset{\mathrm{v}}{\geq} 0$, we have that $b_k \geq 0$, $\forall k \geq k_0$. Relations (2.7), (2.8) and (2.9) for this entry, are given us

$$\left((B+\alpha I)_{ij}^{k}\right)_{\mu,\nu} = \alpha^{k-1}\binom{k}{1}b_1 + \alpha^{k-2}\binom{k}{2}b_2 + \cdots + \binom{k}{k}b_k,$$

$$\left((e^{tB})_{ij}\right)_{\mu,\nu} = tb_1 + \frac{t^2}{2!}b_2 + \cdots + \frac{t^{k_0}}{k_0!}b_{k_0} + \cdots ,$$

and

$$\begin{aligned}\left((A^{-1})_{ij}^{m}\right)_{\mu,\nu} &= \frac{1}{s^{m+1}}\binom{m}{1}b_1 + \frac{1}{s^{m+2}}\binom{m+1}{2}b_2 \\ &\quad + \cdots + \frac{1}{s^{m+k_0}}\binom{m+k_0-1}{k_0}b_{k_0} + \cdots .\end{aligned} \tag{2.13}$$

Suppose first that $\{b_k\}_{k=1}^{\infty}$ is the zero sequence: $b_k = 0$, $\forall k \geq 1$, then the relations above give all zeros. Thus, the equivalence of the three statements of the Theorem, concerning this entry, is trivially proven.

Let $b_k = 0$ for all $k > k_1 > 0$. The validity of statement (i) means that $b_{k_1} > 0$ and k is chosen large enough, such that the last nonzero term $\alpha^{k-k_1}\binom{k}{k_1}b_{k_1}$ dominates all the previous sum. Then, the same hold true for the terms $\frac{t^{k_1}}{k_1!}b_{k_1}$ and $\frac{1}{s^{m+k_1}}\binom{m+k_1-1}{k_1}b_{k_1}$ for large enough t and m, respectively. This because the k_1 terms are polynomials in k, in t or in m, respectively, of degree k_1, while all the previous sums are polynomials of smaller degree.

Finally, suppose that b_k has nonzero entries as k tends to infinity. Then, we choose $k_1 > k_0$ such that $b_{k_1} > 0$. We follow the same argument of the previous case, for such k_1, to prove the equivalence of (i), (ii) and (iii), for the associated entry.

Applying the same argument for any entry of B_{ij}, and every off-diagonal block, the theorem is proven. □

The following examples show the validity of Theorem 2.23 for all cases.

Example 2.24. *Consider the M_V-matrix A given by*

$$A = 7I - B, \text{ where } B = \begin{bmatrix} 3 & 2 & -1 & -2 \\ 1 & 2 & 5 & -1 \\ 1 & 3 & 1 & 3 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

The spectral radius of B is given by $\rho(B) = 5.9389$ and B is an irreducible eventually nonnegative matrix with power index 8 and $\operatorname{index}_0 B = 0$. *The right and left Perron eigenvectors are* $[0.3251 \quad 0.7714 \quad 0.5467 \quad 0.0203]^T$ *and* $[0.4452 \quad 0.6676 \quad 0.5950 \quad 0.0459]^T$*, respectively.*

If $\alpha = 2$, then $B + 2I$ is an eventually nonnegative matrix with power index 11 $((B + \alpha I)^k \geq 0, \forall k \geq 11)$. *From Theorem* 2.23*, the matrix*

$$A^{-1} = \begin{bmatrix} 0.3042 & 0.1993 & 0.1014 & -0.0839 \\ 0.2483 & 0.5420 & 0.4161 & 0.0350 \\ 0.1958 & 0.3007 & 0.3986 & 0.0839 \\ 0.0420 & -0.0070 & 0.0140 & 0.1608 \end{bmatrix}$$

is an eventually nonnegative matrix, $(A^{-1})^k \geq 0$, $\forall k \geq 3$.

Also, the matrix B is an eventually exponentially nonnegative matrix. Choosing $t = 1.8275$, we get

$$e^{tB} = \begin{bmatrix} 7536.4060 & 11383.9671 & 10170.9115 & 796.2579 \\ 18034.4030 & 26999.1535 & 24064.8506 & 1877.0609 \\ 12781.4358 & 19145.2405 & 17045.1966 & 1290.6572 \\ 468.1169 & 723.8724 & 623.3201 & 0.0124 \end{bmatrix} > 0.$$

Example 2.25. *Consider the M_V-matrix A given by*

$$A = 8I - B, \text{ where } B = \begin{bmatrix} 6 & 2 & -2 & 5 & -1 \\ 2 & 4 & 1 & 1 & 2 \\ 2 & 4 & 1 & -1 & 3 \\ -3 & 1 & 2 & -4 & 1 \\ -3 & 1 & 2 & -4 & 1 \end{bmatrix}.$$

The matrix B is an irreducible eventually nonnegative matrix with power index 8 *and* $\sigma(B) = \{6.6286, \ 3.4354, \ 0, \ 0, \ -2.064\}$ *with* $\operatorname{index}_0 B = 2$. *The right and left Perron eigenvectors are* $[0.4311 \quad 0.6396 \quad 0.6301 \quad 0.0630 \quad 0.0630]^T$ *and* $[0.5254 \quad 0.7679 \quad 0.1213 \quad 0.2026 \quad 0.2802]^T$*, respectively, even if* $\operatorname{index}_0 B = 2$.

If $\alpha = 1$*, then* $B + I$ *is an eventually nonnegative matrix with power index* 8 $((B + \alpha I)^k \geq 0, \forall k \geq 8)$*. From Theorem* 2.23*, the matrix*

$$A^{-1} = \begin{bmatrix} 0.3988 & 0.2262 & -0.0238 & 0.1793 & 0.0231 \\ 0.2202 & 0.4881 & 0.0714 & 0.0766 & 0.1496 \\ 0.2262 & 0.3571 & 0.1905 & 0.0551 & 0.1592 \\ -0.0476 & 0.0476 & 0.0476 & 0.0476 & 0.0476 \\ -0.0476 & 0.0476 & 0.0476 & -0.0774 & 0.1726 \end{bmatrix}$$

is an eventually nonnegative matrix, $(A^{-1})^k \geq 0$*,* $\forall k \geq 4$*.*

The matrix B *is also an eventually exponentially nonnegative matrix. Choosing* $t = 1.1925$*, we get*

$$e^{tB} = \begin{bmatrix} 809.6378 & 1078.1401 & 134.7511 & 359.9745 & 348.2426 \\ 1084.3121 & 1617.5316 & 266.1097 & 403.9465 & 603.3538 \\ 1078.1401 & 1591.0872 & 257.4735 & 407.0276 & 588.6072 \\ 72.1968 & 164.3321 & 46.0677 & 1.0062 & 85.2909 \\ 72.1968 & 164.3321 & 46.0677 & 1.0062 & 85.2909 \end{bmatrix} > 0.$$

Example 2.26. *Consider the reducible* M_{V}*-matrix* A *given by*

$$A = 7I - B, \text{ where } B = \left[\begin{array}{cc|cccc} 4 & 3 & 1 & -1 & -2 & 1 \\ 5 & -2 & 4 & 4 & 2 & -4 \\ \hline 0 & 0 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 & -2 & -1 \\ 0 & 0 & 1 & -1 & -2 & -2 \end{array}\right].$$

The spectral radius of B *is given by* $\rho(B) = 5.8990$ *and* B *is an eventually nonnegative matrix with power index* 18 *and the right and left Perron eigenvectors are* $[0.8499\ \ 0.5348\ \ 0\ \ 0\ \ 0\ \ 0]^T$ *and* $[0.5270\ \ 0.2002\ \ 0.5559\ \ 0.5801\ \ 0.0920\ \ 0.1679]^T$*, respectively.*

The block matrix $B_{11} = \begin{bmatrix} 4 & 3 \\ 5 & -2 \end{bmatrix}$ *is an irreducible eventually nonnegative matrix with power index* 4 *and the block matrix* $B_{22} = \begin{bmatrix} 2 & 3 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & -1 & -2 & -1 \\ 1 & -1 & -2 & -2 \end{bmatrix}$ *is also an irreducible eventually nonnegative matrix with power index* 18 *and* $\sigma(B_{22}) = \{4.4051, -3.4051,\ 0,\ 0\}$*, with* $\mathrm{index}_0\, B_{22} = 2$*.*

If $\alpha = 5$*, then* $B + 5I$ *is an eventually nonnegative matrix with power index* 8 $((B + \alpha I)^k \geq 0, \forall k \geq 8)$*. From Theorem* 2.23*, the matrix*

$$A^{-1} = \left[\begin{array}{rr|rrrr} 0.75 & 0.25 & 0.4456 & 0.3696 & -0.0045 & 0.1128 \\ 0.4167 & 0.25 & 0.3927 & 0.4049 & 0.0836 & 0.0581 \\ \hline 0 & 0 & 0.2562 & 0.1625 & 0.0491 & 0.0695 \\ 0 & 0 & 0.0771 & 0.2819 & 0.062 & 0.0416 \\ 0 & 0 & 0.0181 & -0.0121 & 0.1126 & -0.0098 \\ 0 & 0 & 0.0159 & -0.0106 & -0.0265 & 0.1164 \end{array}\right]$$

is an eventually nonnegative matrix, $(A^{-1})^k \geq 0,\ \forall k \geq 5$.

The matrix B *is also an eventually exponentially nonnegative matrix. Choosing* $t = 1.0584$*, we get*

$$e^{tB} = \left[\begin{array}{rr|rrrr} 414.9039 & 157.5727 & 349.3079 & 327.0651 & 30.9242 & 90.9654 \\ 262.6211 & 99.7586 & 261.8324 & 267.2673 & 39.594 & 76.0199 \\ \hline 0 & 0 & 49.6756 & 70.932 & 22.2564 & 23.3148 \\ 0 & 0 & 37.1674 & 56.5734 & 18.406 & 17.3476 \\ 0 & 0 & 1.989 & 1.6553 & 0.6663 & 0.7247 \\ 0 & 0 & 1.5059 & 1.5061 & 0.0002 & 1.0002 \end{array}\right] \geq 0.$$

2.3 Summary

To study the eigenvectors of an M_V-matrix, we categorize into M_V-matrices with $\text{index}_0\, B \leq 1$ and M_V-matrices with $\text{index}_0\, B > 1$ and we obtain results as follows:

1. For an irreducible M_V-matrix with $\text{index}_0\, B \leq 1$, to the smallest real eigenvalue $\lambda_1 \geq 0$ of A there correspond positive right and left eigenvectors.

2. We gave equivalent statements for M_V-matrices with $\text{index}_0\, B > 1$ to have positive right and left Perron eigenvectors.

3. For an irreducible eventually nonnegative matrix B with $\text{index}_0\, B \leq 1$, its eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ are not nonnegative vectors.

4. For an irreducible eventually nonnegative matrix B (with $\text{index}_0 B \leq 1$ or $\text{index}_0 B > 1$), its right and left eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ are not nonnegative vectors iff there exists $\alpha > 0$ such that $B + \alpha I \overset{\text{v}}{\geq} 0$.

Finally, we gave and proved equivalent properties of eventually exponentially nonnegative matrices and M_{V}-matrices.

It is trivial from the definition of GM-matrices and [42] that any GM-matrix (which is not an M_{V}-matrix) may have nonnegative eigenvector corresponding to its spectral radius. Hence, Theorems 2.1 and 2.8 do not hold for GM-matrices.

We note that the results of this chapter were published in [11].

CHAPTER 3

The Schur Complement of M_V- and GM-matrices

In this chapter, we will study the extensions of M-matrices as regards their Schur complement. First, we present the result that the Schur complement of inverse M_V- and inverse GM-matrices inherit the inverse M_V- and inverse GM-property. Then, the Schur complement of M_V- and GM-matrices associated with the matrix B being cyclic of index k is studied. Following by one of our results for the Schur complement of irreducible M_V-matrices and matrices with their A_{22} block being an M_V-matrix and also the corresponding results for the Schur complement of irreducible GM-matrices. Finally, we analyze and study the reducible M_V- and GM-matrices with respect to their Schur complements.

3.1 Preliminaries Result

The study of the Schur complement of M-matrices was introduced by Crabtree [12, 13] and Ky Fan [21] and their main result was that: The "Schur com-plement of any M-matrix is also an M-matrix. Moreover, the block A_{22} of an original matrix A is greater than its Schur complement." The questions that arise are:

1. Do the Schur complement of M_V- and GM-matrices preserve the M_V- and GM-property, respectively?
2. If the original matrix A is an M_V-matrix or a GM-matrix, then the block A_{22} of an original matrix is greater than its Schur complement?

To answer those questions, we conducted the experiment using MATLAB to observe the behaviour of the Schur complement of M_V- and GM-matrices. We found that the Schur complement of nonsingular M_V- and GM-matrices does not preserve the M_V- or GM-property. To be specific, the Schur complement of a nonsingular M_V- or GM-matrix may not be an M_V- nor a GM-matrix. Moreover, the block A_{22} is not always greater than its Schur complement as we can show by following examples:

• Non-symmetric nonsingular M_V-matrix where its Schur complement is an M_V-matrix.

Example 3.1. *Consider the irreducible nonsingular M_V-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rrrr|rrrr} 14 & -3 & -2 & 0 & -1 & -4 & -8 & 1 \\ -3 & 24 & -4 & -1 & -2 & -5 & -1 & 4 \\ -2 & -4 & 21 & -6 & -8 & 2 & -1 & -4 \\ -3 & -4 & -8 & 21 & -1 & 1 & -4 & -2 \\ \hline -2 & -1 & 1 & -2 & 21 & -4 & -2 & -4 \\ -1 & -2 & -3 & -1 & -2 & 20 & -3 & -1 \\ -3 & 1 & -5 & -7 & -8 & -4 & 22 & 0 \\ -23 & -3 & 14 & -5 & -4 & -3 & 2 & 34 \end{array}\right]$$

with right and left eigenvectors $x = [0.49\ \ 0.201\ \ 0.3821\ \ 0.4038\ \ 0.2457\ \ 0.2529\ \ 0.4477\ \ 0.2928]^T$ *and* $y = [0.5372\ \ 0.2337\ \ 0.3216\ \ 0.3384\ \ 0.4197\ \ 0.3217\ \ 0.3915\ \ 0.0777]^T$*, respectively, corresponding to its smallest eigenvalue* 1.933. *The Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 20.3042 & -5.0305 & -4.0551 & -3.7869 \\ -4.5138 & 19.1086 & -5.2284 & -1.3703 \\ -13.2287 & -5.1036 & 15.9855 & -1.8257 \\ -3.2079 & -11.8274 & -11.8675 & 38.1622 \end{bmatrix}$$

is an M_V-matrix with right and left eigenvectors $x = [0.4046\ \ 0.408\ \ 0.6993\ \ 0.4252]^T$ *and* $y = [0.6356\ \ 0.5075\ \ 0.5689\ \ 0.1221]^T$*, respectively, corresponding to the smallest eigenvalue* 4.2434.

Moreover,

$$A_{22} - (A/A_{11}) = \begin{bmatrix} 0.6958 & 1.0305 & 2.0551 & -0.2131 \\ 2.5138 & 0.8914 & 2.2284 & 0.3703 \\ 5.2287 & 1.1036 & 6.0145 & 1.8257 \\ -0.7921 & 8.8274 & 13.8675 & -4.1622 \end{bmatrix}$$

is not a positive matrix. That is, $A_{22} \ngeq (A/A_{11})$.

• Symmetric nonsingular M_V-matrix where its Schur complement is not an M_V-matrix.

Example 3.2. *Consider the irreducible symmetric nonsingular M_V-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rr|rrr} 31.2 & -17.35 & -16.35 & 3.95 & 2.95 \\ -17.35 & 51.1 & 4.35 & -9.85 & -8.5 \\ \hline -16.35 & 4.35 & 41.9 & 3.35 & -4.1 \\ 3.95 & -9.85 & 3.35 & 42 & -13.65 \\ 2.95 & -8.5 & -4.1 & -13.65 & 39.9 \end{array}\right]$$

with eigenvector $[0.8103 \;\; 0.3635 \;\; 0.4464 \;\; 0.0134 \;\; 0.1084]^T > 0$ *corresponding to its smallest eigenvalue* 14.8692*.*

The Schur complement

$$(A/A_{11}) = \begin{bmatrix} 32.7895 & 4.5444 & -3.3388 \\ 4.5444 & 40.0868 & -15.29 \\ -3.3388 & -15.29 & 38.4859 \end{bmatrix}$$

is not M_V- or GM-matrix because the eigenvector $x = [-0.0895 \;\; 0.6974 \;\; 0.7111]^T$ *corresponding to the smallest eigenvalue* 23.912 *has a negative entry.*

Moreover,

$$A_{22} - (A/A_{11}) = \begin{bmatrix} 9.1105 & -1.1944 & -0.7612 \\ -1.1944 & 1.9132 & 1.6400 \\ -0.7612 & 1.6400 & 1.4141 \end{bmatrix}$$

is not a positive matrix. That is, $A_{22} \not\geq (A/A_{11})$*.*

• Non-symmetric nonsingular GM-matrix where its Schur complement is an M_V-matrix.

Example 3.3. *Consider the irreducible non-symmetric non-singular GM-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rr|rr} 5 & -3 & 1 & -1 \\ -3 & 5 & -1 & 1 \\ \hline -5 & -3 & 7 & -1 \\ -3 & -5 & -1 & 7 \end{array}\right]$$

with right and left eigenvectors $x = [0.3162 \;\; 0.3162 \;\; 0.6325 \;\; 0.6325]^T$ *and* $y = [1 \;\; 1 \;\; 0 \;\; 0]^T$*, respectively, corresponding to its smallest eigenvalue* 2*. The Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 7.25 & -1.25 \\ -1.25 & 7.25 \end{bmatrix}$$

is a symmetric M_V-matrix with eigenvector $[1 \;\; 1]^T$ corresponding to the smallest eigenvalue 6*.*

Moreover,

$$A_{22} - (A/A_{11}) = \begin{bmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{bmatrix}$$

is not a positive matrix. That is, $A_{22} \ngeq (A/A_{11})$.

• Non-symmetric nonsingular GM-matrix where its Schur complement is not an M_V- or GM-matrix.

Example 3.4. *Consider the irreducible symmetric singular GM-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 10.9 & 4.55 & -6.65 & -6.69 \\ -1.88 & 15.14 & -3.98 & -4.15 \\ \hline -3.77 & -3.49 & 11.11 & 4.56 \\ -0.68 & -2.61 & 2.11 & 14.35 \end{array}\right]$$

with right and left eigenvectors $x = [0.5657 \;\; 0.4243 \;\; 0.7071 \;\; 0]^T$ and $y = [0.6175 \;\; 0.0126 \;\; 0.7831 \;\; 0.0733]^T$, respectively, corresponding to its smallest eigenvalue 6*.*

The Schur complement

$$(A/A_{11}) = \begin{bmatrix} 15.9248 & -5.1270 & -5.3039 \\ -1.9163 & 8.8100 & 2.2461 \\ -2.3261 & 1.6951 & 13.9326 \end{bmatrix}$$

is not an M_V- or GM-matrix because the eigenvector $[-0.5008 \;\; -0.8643 \;\; 0.0472]^T$ corresponding to the smallest eigenvalue 7.5768 *has negative entries.*

Moreover,

$$A_{22} - (A/A_{11}) = \begin{bmatrix} -0.7848 & 1.1470 & 1.1539 \\ -1.5737 & 2.3000 & 2.3139 \\ -0.2839 & 0.4149 & 0.4174 \end{bmatrix}$$

is not a positive matrix. That is, $A_{22} \ngeq (A/A_{11})$.

However, despite the nonsingular case, we are able to study and present the result for the Schur complement of singular M_V- or GM-matrices and their perturbed form with some additional requirements. Meanwhile, we give various numerical examples to confirm our theoretical findings.

3.2 Schur complement of inverse M_{V}-matrices and GM-matrices

The study of Schur complement of inverse M-matrices was introduced by Imam [32] in 1984 and the result was that: "The Schur complement of inverse M-matrix is also an inverse M-matrix". In this section, we will give the result for the Schur complement of inverse M_{V}- and inverse GM-matrices. These are given in the following theorems.

Theorem 3.5. *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{n,n}$ *be an inverse* M_{V}*-matrix, partitioned as in (1.4), and* $C = A^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ *be the associated* M_{V}*-matrix, partitioned as in* A*. Then, the Schur complement* (A/A_{11}) *of* $A_{11} \in \mathbb{R}^{m,m}$ *in* A *is an inverse* M_{V}*-matrix, iff* C_{22} *is an* M_{V}*-matrix.*

Theorem 3.6. *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{n,n}$ *be an inverse* GM*-matrix, partitioned as in the previous theorem, and* $C = A^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ *be the associated* GM*-matrix, partitioned as in* A*. Then, the Schur complement* (A/A_{11}) *of* $A_{11} \in \mathbb{R}^{m,m}$ *in* A *is an inverse* GM*-matrix, iff* C_{22} *is a* GM*-matrix.*

The same proof applies to both inverse M_{V}- and inverse GM-matrices. So, we will give the proof only in the latter case.

Proof. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $C = A^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ be the inverse GM- and the GM-matrices, respectively.

From the Partitioned Inversion Formula of A (see, e.g., [5]):

$$C = A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A/A_{11})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix},$$

we obtain that $C_{22} = (A/A_{11})^{-1}$. Thus, (A/A_{11}) is an inverse GM-matrix *iff* C_{22} is a GM-matrix. □

Here are the examples for both cases.

Example 3.7. *Consider the matrix*

$$A = \left[\begin{array}{c|ccc} 0.2593 & 0.0926 & 0.0185 & -0.1111 \\ \hline 0.7037 & 1.0370 & 0.9074 & 0.0556 \\ 0.7407 & 0.9074 & 0.9815 & 0.1111 \\ 0.4444 & 0.4444 & 0.3889 & 0.1667 \end{array}\right].$$

The matrix $C = A^{-1} = \left[\begin{array}{c|ccc} 3 & -2 & 1 & 2 \\ \hline -1 & 5 & -5 & 1 \\ -1 & -3 & 5 & -3 \\ -3 & -1 & -1 & 5 \end{array}\right]$ *is written as* $6I - B$*, where* B *is an eventually nonnegative matrix (* $B^k \geq 0 \;\; \forall \;\; k \geq 11$*). Thus* C *is an* M_{V}*-matrix and* A *is an inverse* M_{V}*-matrix.* C_{22} *is written as* $6I - B_{22}$*, where* B_{22} *is an eventually nonnegative matrix (*$B_{22}^k \geq 0 \;\; \forall \;\; k \geq 2$*). Thus* C_{22} *is an* M_{V}*-matrix.*

The Schur complement

$$(A/A_{11}) = \begin{bmatrix} 0.7857 & 0.8571 & 0.3571 \\ 0.6429 & 0.9286 & 0.4286 \\ 0.2857 & 0.3571 & 0.3571 \end{bmatrix} = C_{22}^{-1}$$

is an inverse M_{V}*-matrix and the validity of Theorem 3.5 is confirmed.*

Example 3.8. *Consider the matrix*

$$A = \left[\begin{array}{c|ccc} 0.6667 & 0.3333 & 1 & 1 \\ \hline 0.3333 & 0.6667 & 1 & 1 \\ -0.1111 & 0.1111 & 0.6667 & 0.3333 \\ 0.1111 & -0.1111 & 0.3333 & 0.6667 \end{array}\right],$$

where $C = A^{-1} = \left[\begin{array}{c|ccc} 2 & -1 & -1 & -1 \\ \hline -1 & 2 & -1 & -1 \\ 1 & -1 & 2 & -1 \\ -1 & 1 & -1 & 2 \end{array}\right]$*. The matrix* A *is an inverse* GM*-matrix with left and right Perron eigenvectors* $[0\ 0\ 1\ 1]^T$ *and* $[1\ 1\ 0\ 0]^T$*, respectively.* C_{22} *is a* GM*-matrix with eigenvectors* $[0\ 1\ 1]^T$ *and* $[1\ 1\ 0]^T$*.*

The Schur complement

$$(A/A_{11}) = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.1667 & 0.8333 & 0.5 \\ -0.1667 & 0.1667 & 0.5 \end{bmatrix}$$

is an inverse GM*-matrix, with eigenvectors* $[0\ 1\ 1]^T$ *and* $[1\ 1\ 0]^T$*, respectively, and the validity of Theorem 3.6 is confirmed.*

3.3 Schur complement of M_V- and GM-matrices associated with B being block "cyclic of index k" matrix

We begin with a statement concerning the simplest case when the index $k = 2$.

Lemma 3.9. *Let B be an eventually nonnegative cyclic of index 2 matrix. Its form will be:*

$$B = \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix}. \tag{3.1}$$

Then, the matrices $B_{12}B_{21}$ as well as $B_{21}B_{12}$ are eventually nonnegative matrices.

Proof. From (3.1) we have

$$B^{2l} = \begin{bmatrix} (B_{12}B_{21})^l & 0 \\ 0 & (B_{21}B_{12})^l \end{bmatrix} \text{ and } B^{2l+1} = \begin{bmatrix} 0 & (B_{12}B_{21})^l B_{12} \\ (B_{21}B_{12})^l B_{21} & 0 \end{bmatrix}.$$

In the spectrum $\sigma(B)$, the nonzero eigenvalues appear in $\pm$ pairs of the same multiplicity. So, $\sigma(B) := \{\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots\}$ and the associated right and left Perron eigenvectors $x^{(1)}, y^{(1)}$ are nonnegative.

The matrix B^{2l} has double its nonzero eigenvalues $(\lambda_1^{2l}, \lambda_2^{2l}, \ldots)$ and its right and left Perron eigenvectors nonnegative. The nonzero eigenvalues of the matrix B^{2l+1} are pairwise opposite in sign $\sigma(B^{2l+1}) = \{\lambda_1^{2l+1}, -\lambda_1^{2l+1}, \lambda_2^{2l+1}, -\lambda_2^{2l+1}, \ldots\}$ and its Perron eigenvectors are nonnegative.

Since B is eventually nonnegative, there exists $l_0 > 0$ such that $B^{2l} \geq 0$ for all $l \geq l_0$. Thus, $(B_{12}B_{21})^l \geq 0$ and $(B_{21}B_{12})^l \geq 0$ for all $l \geq l_0$, which completes the proof. □

The results above can be extended for cyclic of index k matrices. Let $A = sI - B$ be a cyclic of index k matrix and B is of the form

$$\begin{bmatrix} 0 & B_{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & B_{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & B_{k-1,k} \\ B_{k1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{3.2}$$

Lemma 3.9 is generalized as follows.

Lemma 3.10. *Let B be an eventually nonnegative cyclic of index k matrix. Then, all the matrices formed by the cyclic products: $B_{12}B_{23}\cdots B_{k1}$, $B_{23}B_{34}\cdots B_{k1}B_{12}$, ..., $B_{k1}B_{12}\cdots B_{k-1,k}$ are eventually nonnegative matrices.*

Proof. From (3.2) we have

$$B^{kl} = \begin{bmatrix} (B_{12}B_{23}\cdots B_{k1})^l & 0 & \cdots & 0 \\ 0 & (B_{23}B_{34}\cdots B_{k1}B_{12})^l & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (B_{k1}B_{12}\cdots B_{k-1,k})^l \end{bmatrix}$$

In the spectrum of B, the nonzero eigenvalues appear as k-tuple: $\lambda_i e^{i\frac{2j\pi}{k}}$, $j = 0, 1, \ldots, k-1$. The eigenvalues corresponding to the spectral radius are $\lambda_1 e^{i\frac{2j\pi}{k}}$, $j = 0, 1, \ldots, k-1$, $\lambda_1 > 0$ and the right and left Perron eigenvectors $x^{(1)}, y^{(1)}$ are nonnegative.

Since B is eventually nonnegative, there exists $l_0 > 0$ such that $B^{kl} \geq 0$ for all $l \geq l_0$. Thus, $(B_{12}B_{23}\cdots B_{k1})^l \geq 0$, $(B_{23}B_{34}\cdots B_{k1}B_{12})^l \geq 0$, ..., $(B_{k1}B_{12}\cdots B_{k-1,k})^l \geq 0$, which completes the proof. □

Theorem 3.11. *The Schur complement of an M_{V}-matrix $A = sI - B$, where B is a non-nilpotent cyclic of index 2 matrix of the form (3.1), is an M_{V}-matrix.*

Proof. Let $A = sI - B$ be an M_{V}-matrix where B is a non-nilpotent eventually nonnegative cyclic of index 2 matrix with $s \geq \rho(B) > 0$. A can be written as follows:

$$A = sI - \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix} = \begin{bmatrix} sI & -B_{12} \\ -B_{21} & sI \end{bmatrix}.$$

Then, the Schur complement of A_{11} in A is

$$(A/A_{11}) = sI - (-B_{21}(sI)^{-1}(-B_{12})) = sI - \frac{1}{s}B_{21}B_{12}.$$

Thus, the Schur complement is written as $(A/A_{11}) = sI - B'$, where $B' = \frac{1}{s}B_{21}B_{12}$ which is eventually nonnegative from Lemma 3.9.

Since $\rho(\frac{1}{s}B_{21}B_{12}) = \frac{1}{s}\rho^2(B) \leq \rho(B) \leq s$, the Schur complement (A/A_{11}) is an M_{V}-matrix. □

In the following theorem we state a stronger property.

Theorem 3.12. *The Schur complement matrix $A' = sI - B'$, of an irreducible M_{V}-matrix $A = sI - B$, where B is cyclic of index 2 of the form (3.1) with $\operatorname{index}_0 B \le 1$, is an irreducible M_{V}-matrix where B' is an eventually positive matrix.*

Proof. From Theorem 3.11, the Schur complement matrix $A' = sI - B'$ is an M_{V}-matrix, where B' is an eventually nonnegative matrix. Since $A = sI - B$ is an irreducible M_{V}-matrix and $\operatorname{index}_0 B \le 1$, from Theorem 2.1 we know that to the smallest real eigenvalue of A there correspond positive right and left eigenvectors. Hence, $x_2 > 0$ and $y_2 > 0$ which are the right and left Perron eigenvectors of B', respectively. Thus, B' is eventually positive. □

The following example shows the validity of Theorems 3.11 and 3.12.

Example 3.13. *Consider the irreducible symmetric M_{V}-matrix*

$$A = 16I - B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rr|rr} 16 & 0 & -6 & -1.7 \\ 0 & 16 & -8.4 & 1 \\ \hline -6 & -8.4 & 16 & 0 \\ -1.7 & 1 & 0 & 16 \end{array}\right]$$

with eigenvector $[0.4129 \;\; 0.5740 \;\; 0.7070 \;\; 0.0124]^T$ *corresponding to the smallest eigenvalue* 5.6757*, where B is cyclic of index* 2 *matrix and* $\operatorname{index}_0 B = 0$*.*

Then, the Schur complement

$$(A/A_{11}) = \begin{bmatrix} 9.3400 & -0.1125 \\ -0.1125 & 15.7569 \end{bmatrix}$$

is a symmetric M_{V}-matrix with eigenvector $[0.9998 \;\; 0.0175]^T$ *corresponding to the smallest eigenvalue* 9.338 *and*

$$B' = 16I - (A/A_{11}) = \begin{bmatrix} 6.6600 & 0.1125 \\ 0.1125 & 0.2431 \end{bmatrix}$$

is a positive matrix (obviously, it is also an eventually positive matrix).

Lemma 3.14. *Let B be a non-nilpotent cyclic of index 2 matrix of the form (3.1) and both B and B^T possess the Perron-Frobenius property. Then, the matrices $B_{12}B_{21}$ and its transpose, as well as $B_{21}B_{12}$ and its transpose, possess the Perron-Frobenius property.*

Proof. Let $x \geq 0$ be the Perron eigenvector of B. We partition it as in (3.1): $x = [x_1^T \quad x_2^T]^T$. Thus,

$$\begin{aligned} \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \rho(B) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} B_{12}B_{21} & 0 \\ 0 & B_{21}B_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \rho(B) \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \rho^2(B) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned} \tag{3.3}$$

It is easily proved that $x_1 \neq 0$ and $x_2 \neq 0$, since otherwise if $x_1 = 0$ then $0 = B_{21}x_1 = \rho(B)x_2$ which constitutes a contradiction due to the non-nilpotent property of B. We have the same contradiction supposing that $x_2 = 0$. Thus, $x_1 \neq 0$ and $x_2 \neq 0$. From (3.3) we obtain that $B_{12}B_{21}$ and $B_{21}B_{12}$ possess the Perron-Frobenius property with Perron pairs $(\rho^2(B), x_1)$ and $(\rho^2(B), x_2)$, respectively.

The same result is obtained for the transpose of the above matrices, following the same analysis for the left eigenvector. □

Theorem 3.15. *Let $A = sI - B$ be a GM-matrix, where B is non-nilpotent cyclic of index 2 matrix of the form (3.1). Then, its Schur complement is a GM-matrix.*

Proof. Following exactly the same analysis in the proof of Theorem 3.11, we obtain that the Schur complement is given by $sI - \frac{1}{s}B_{21}B_{12}$ where $\rho(\frac{1}{s}B_{21}B_{12}) \leq s$. It remains to prove that $B_{21}B_{12}$ and $(B_{21}B_{12})^T$ possess the Perron-Frobenius property, which was proved in Lemma 3.14. □

We can state the general theorem for M_{V}-matrices associated with B being eventually nonnegative cyclic of index k matrices.

Theorem 3.16. *Let $A_{11} = sI - B_{11}$ be the first $r \times r$ principal submatrix of the irreducible M_{V}-matrix $A = sI - B$, where B is cyclic of index k of the form (3.2) with* $\mathrm{index}_0\, B \leq 1$. *Then, the Schur complement* $(A/A_{11}) = sI - B'$ *is* *an irreducible M_{V}-matrix where B' is an eventually nonnegative cyclic matrix of index $k-r$, if $k > r+1$, or an eventually positive one if $k = r+1$.*

Proof. The Schur complement is given by

$$
\begin{aligned}
(A/A_{11}) &= \begin{bmatrix} sI & -B_{r+1,r+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & sI & -B_{k-1,k} \\ 0 & 0 & \cdots & 0 & sI \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ -B_{k1} & \cdots & 0 \end{bmatrix} \\
&\times \begin{bmatrix} sI & -B_{12} & & 0 \\ & \ddots & \ddots & \vdots \\ & & sI & -B_{r-1,r} \\ & & & sI \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -B_{r,r+1} & 0 & \cdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} sI & -B_{r+1,r+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & sI & -B_{k-1,k} \\ 0 & 0 & \cdots & 0 & sI \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ -B_{k1} & \cdots & 0 \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{s}I & \frac{1}{s^2}B_{12} & \cdots & \frac{1}{s^r}B_{12}B_{23}\cdots B_{r-1,r} \\ & \ddots & \ddots & \vdots \\ & & \frac{1}{s}I & \frac{1}{s^2}B_{r-1,r} \\ & & & \frac{1}{s}I \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ -B_{r,r+1} & \cdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} sI & -B_{r+1,r+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & sI & -B_{k-1,k} \\ -\frac{1}{s^r}B_{k1}B_{12}B_{23}\cdots B_{r-1,r}B_{r,r+1} & 0 & \cdots & 0 & sI \end{bmatrix}.
\end{aligned}
$$

Thus, B' is cyclic of index $k-r$. Taking the power ${B'}^{k-r}$ we obtain a diagonal matrix whose elements are products, the same ones with $k-r$ elements of B^k, multiplied by $\frac{1}{s^r} > 0$. Thus, B' is eventually nonnegative cyclic of index $k-r$ and the proof is complete. □

The following example shows the validity of Theorem 3.16. We consider when B is cyclic of index 5 matrix with $\mathrm{index}_0\, B \leq 1$ and the block matrix A_{11} is 2×2 principal submatrix.

Example 3.17. *Consider the irreducible M_{V}-matrix*

$$A = 8I - B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
$$= \left[\begin{array}{cccc|cccccc} 8 & 0 & -7 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & -5 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & -1 & -7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & -11 & 5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 8 & 0 & -7 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 2 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & -0.4 & -0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & -3 & -1.7 \\ -11 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 6 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{array}\right],$$

where B is eventually nonnegative cyclic of index 5 *matrix. The spectrum of B is given by $\sigma(B) = \{6.7952,\ -3.5530,\ 2.8745 \pm 2.0884i,\ 2.0998 \pm 6.4626i,$ $-1.0980 \pm 3.3791i,\ -5.4974 \pm 3.9941i\}$, with* $\operatorname{index}_0 B = 0$.

Then, the Schur complement

$$(A/A_{11}) = \begin{bmatrix} 8 & 0 & -7 & -6 & 0 & 0 \\ 0 & 8 & 2 & -7 & 0 & 0 \\ 0 & 0 & 8 & 0 & -0.4 & -0.1 \\ 0 & 0 & 0 & 8 & -3 & -1.7 \\ 1.4688 & -10.2188 & 0 & 0 & 8 & 0 \\ -9.1719 & 4.9844 & 0 & 0 & 0 & 8 \end{bmatrix}$$

is an M_{V}-matrix. The right and left eigenvectors of (A/A_{11}) corresponding to the smallest eigenvalue 1.9055 *are* $[0.4345\ \ 0.4310\ \ 0.0455\ \ 0.3882\ \ 0.6179\ \ 0.3014]^T$ *and* $[0.2243\ \ 0.4535\ \ 0.1088\ \ 0.7417\ \ 0.3722\ \ 0.2087]^T$, *respectively. The matrix $B' = 8I - (A/A_{11})$ is an eventually nonnegative cyclic of index* 3 *with power index* 4 *($B^k \geq 0,\ \forall k \geq 4$).*

The results of this section can be extended by reducing the index of cyclicity via permutation transformation. A characteristic application is the one of B being *cyclic of index* $2k$, with $k \geq 2$. It is then easily proved that the matrix B can be transformed by permutation transformation, following the block "*red-black ordering*" (see [55]) into the block *cyclic of index* 2 form (3.1).

For example if $k = 3$ (cyclic of index 6), given the matrix

$$B = \begin{bmatrix} 0 & B_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{56} \\ B_{61} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

considering the block red-black ordering and using the block permutation $(1\ \ 3\ \ 5\ \ 2\ \ 4\ \ 6)$, we obtain that

$$B' = PBP^T = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & B_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{56} \\ \hline 0 & B_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{45} & 0 & 0 & 0 \\ B_{61} & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Note that if B is *weakly cyclic of index* $2k$, $k \geq 2$, then its block graph is a cycle, [31], suggesting what the permutation matrix P of (1.3) should be.

After this analysis we can state without proof the following theorems.

Theorem 3.18. *Let A be an irreducible $M_{\rm V}$-matrix, written as $A = sI - B$, where B is a non-nilpotent cyclic of index $2k$ with $k \geq 2$ a positive integer, of the form*

$$B = \begin{bmatrix} 0 & B_{12} & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_{23} & 0 & \cdots & 0 \\ 0 & 0 & 0 & B_{34} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & B_{2k-1,2k} \\ B_{2k,1} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.4}$$

Then, the block red-black ordering permutation transformation of A, $A' = PAP^T$, makes A' be written as

$$A' = \left[\begin{array}{c|c} sI & -B'_{12} \\ \hline -B'_{21} & sI \end{array}\right] \tag{3.5}$$

and the associated Schur complement is an $M_{\rm V}$-matrix.

Theorem 3.19. *Let $A = sI - B$ be a GM-matrix, where B is a non-nilpotent cyclic of index $2k$ matrix, $k \geq 2$ positive integer, of the form* (3.4)*. Then, following the block red-black ordering, with a permutation transformation matrix P, we have $A' = PAP^T$, where A' has been written as in* (3.5)*. Then, the associated Schur complement is a GM-matrix.*

The following example shows the validity of Theorem 3.18.

Example 3.20. *Consider the irreducible M_V-matrix $A = 12I - B$, where B is a cyclic of index* 4 *eventually nonnegative matrix. For the block red-black ordering permutation, we get that*

$$A' = PAP^T = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix} = \left[\begin{array}{rrrr|rrrr} 12 & 0 & 0 & 0 & -7 & 2 & 0 & 0 \\ 0 & 12 & 0 & 0 & -6 & -6 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & -7 & -6 \\ 0 & 0 & 0 & 12 & 0 & 0 & 2 & -7 \\ \hline 0 & 0 & -1 & -7 & 12 & 0 & 0 & 0 \\ 0 & 0 & -11 & 5 & 0 & 12 & 0 & 0 \\ -11 & -11 & 0 & 0 & 0 & 0 & 12 & 0 \\ -6 & -11 & 0 & 0 & 0 & 0 & 0 & 12 \end{array}\right]$$

is an M_V-matrix with right and left eigenvectors $[0.0326 \;\; 0.3619 \;\; 0.5123 \;\; 0.19 \;\; 0.1771 \;\; 0.4504 \;\; 0.4171 \;\; 0.4015]^T$ *and* $[0.3506 \;\; 0.5118 \;\; 0.2919 \;\; 0.2479 \;\; 0.5311 \;\; 0.2278 \;\; 0.1488 \;\; 0.3352]^T$*, respectively, corresponding to the smallest eigenvalue* 1.5975*.*

Then, the Schur complement

$$(A/A_{11}) = \begin{bmatrix} 12 & 0 & 0.5833 & -4.5833 \\ 0 & 12 & -7.25 & -2.5833 \\ -11.9167 & -3.6667 & 12 & 0 \\ -9 & -4.5 & 0 & 12 \end{bmatrix}$$

is an M_V-matrix with right and left eigenvectors $[0.2347 \;\; 0.5969 \;\; 0.5528 \;\; 0.5321]^T$ *and* $[0.7760 \;\; 0.3328 \;\; 0.2173 \;\; 0.4897]^T$*, respectively, corresponding to the smallest eigenvalue* 2.9824*.*

3.4 Schur complement of irreducible M_V-matrices

The study of the property of M_V-matrices is distinguished into two cases: the symmetric and the non-symmetric matrices.

3.4.1 Symmetric M_{V}-matrices

Here we provide a result for irreducible symmetric singular M_{V}-matrices concerning their Schur complement.

Theorem 3.21. *Let $A \in \mathbb{R}^{n,n}$ be an irreducible symmetric singular M_{V}-matrix. Then, the Schur complement (A/A_{11}) is a symmetric singular M_{V}-matrix.*

Proof. Let $A = sI - B$ be an irreducible singular M_{V}-matrix of the form (1.4). Then, there exists a positive vector x such that $Ax = 0$. The last equality can be written in the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.6}$$

Then, we have

$$\begin{array}{r} A_{11}x_1 + A_{12}x_2 = 0 \\ A_{12}^T x_1 + A_{22}x_2 = 0 \end{array} \Leftrightarrow \begin{array}{r} x_1 = -A_{11}^{-1}A_{12}x_2 \\ (A_{22} - A_{12}^T A_{11}^{-1} A_{12})x_2 = 0 \end{array}.$$

Thus, (A/A_{11}) has a positive eigenvector x_2 corresponding to zero. On the other hand, A is a symmetric M_{V}-matrix which means that it is positive semidefinite. Then, (A/A_{11}) is also a positive semidefinite matrix with all its eigenvalues being nonnegative, with the note that when $A_{11} \in \mathbb{R}^{n-1,n-1}$ then $(A/A_{11}) = 0$. Hence, (A/A_{11}) is a singular M_{V}-matrix, which completes the proof. □

The following theorems show the validity of the above result for a class of symmetric nonsingular M_{V}-matrices using the matrix perturbation theory.

Theorem 3.22. *Let $A \in \mathbb{R}^{n,n}$ be an irreducible symmetric singular M_{V}-matrix. Then, there exists an $\alpha > 0$ such that for the nonsingular M_{V}-matrices $C = A + \lambda I$, $0 < \lambda < \alpha$, there holds: The Schur complement (C/C_{11}) is a symmetric nonsingular M_{V}-matrix.*

Proof. Obviously, $C = A + \lambda I$ is a symmetric non-singular M_{V}-matrix for $\lambda > 0$. Then, C and A have the same eigenvectors and assume that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the Perron eigenvector of A. From Theorem 3.21, (A/A_{11}) is a symmetric singular M_{V}-matrix with Perron eigenvector x_2. The Schur complement of

C_{11} in C is $(C/C_{11}) = ((A+\lambda I)/(A_{11}+\lambda I))$ with Perron eigenvector x_2'. For "small" enough λ, the Schur complement is a perturbation of (A/A_{11}). Since 0 is a simple eigenvalue of (A/A_{11}), the entries of the corresponding eigenvector are continuous functions of the entries of the matrix (A/A_{11}). Using the continuity argument and the fact that $x_2 > 0$, the eigenvector x_2' corresponding to the smallest eigenvalue of (C/C_{11}) would be $x_2' > 0$ for "small" enough λ. Thus, (C/C_{11}) is a symmetric nonsingular $M_{\rm V}$-matrix for λ belonging to a "small" interval $(0, \alpha)$. □

Theorem 3.23. *Let $A \in \mathbb{R}^{n,n}$ be an irreducible symmetric singular $M_{\rm V}$-matrix. Then, there exist symmetric positive definite matrices $E \in \mathbb{R}^{n,n}$ such that the perturbed matrices $C = A + E$ are $M_{\rm V}$-matrices and their Schur complement (C/C_{11}) is a symmetric nonsingular $M_{\rm V}$-matrix.*

Proof. Consider a symmetric positive definite matrix $E \in \mathbb{R}^{n,n}$ with "small" norm $(\|E\| \leq \varepsilon)$. Then, the eigenvalues of the perturbed matrix $A+E$ will be positive. Since, 0 is a simple eigenvalue of A and the corresponding eigenvector x is positive, we use the continuity argument to get that the corresponding eigenvector x_E of the smallest eigenvalue λ_E of $A+E$ is positive for "small" enough ε. Thus, there exists E such that $A+E$ is an $M_{\rm V}$-matrix. Since $A+E$ is a positive definite matrix, so is the Schur complement $((A+E)/(A_{11}+E_{11}))$. We recall again the continuity argument to prove that the eigenvector x_2' of $((A+E)/(A_{11}+E_{11}))$ corresponding to the smallest eigenvalue is positive because $x_2 > 0$ and $((A+E)/(A_{11}+E_{11}))$ is a "small" perturbation of (A/A_{11}) for "small" enough ε. Thus, $((A+E)/(A_{11}+E_{11}))$ is an $M_{\rm V}$-matrix. □

In the following theorem we give, without proof, the property concerning "big" shifts of symmetric singular matrices. The proof is the same as in the nonsymmetric case. (See Theorem 3.35 in the sequel).

Theorem 3.24. *Let $A \in \mathbb{R}^{n,n}$ be an irreducible symmetric singular matrix. Then, for the nonsingular matrices $C = A + \lambda I$, there exists a "big" enough $\beta > 0$ such that for all $\lambda > \beta$ the Schur complement is an $M_{\rm V}$-matrix if A_{22} is an $M_{\rm V}$-matrix.*

The following example shows the result for a symmetric $M_{\rm V}$-matrix with different partitions of A. At first, we consider that A_{22} is an $M_{\rm V}$-matrix, and then that A_{22} is **not** an $M_{\rm V}$-matrix. Then, the validity of Theorems 3.21, 3.22 and 3.24 is shown. Furthermore, perturbations with a positive definite matrix E are taken to show the validity of Theorem 3.23.

Example 3.25. *Consider the irreducible symmetric singular $M_{\rm V}$-matrix*

$$A = \begin{bmatrix} 71.691 & 12 & -11 & -8 & -18 & -8 & -1 \\ 12 & 63.691 & 18 & -4 & -20 & 19 & -15 \\ -11 & 18 & 38.691 & -9 & -16 & 13 & -19 \\ -8 & -4 & -9 & 48.691 & -4 & 7 & -4 \\ -18 & -20 & -16 & -4 & 45.691 & -10 & -16 \\ -8 & 19 & 13 & 7 & -10 & 39.691 & -7 \\ -1 & -15 & -19 & -4 & -16 & -7 & 41.691 \end{bmatrix}$$

with eigenvector corresponding to its zero eigenvalue $[0.2368 \;\; 0.1066 \;\; 0.5542 \;\; 0.2374 \;\; 0.5398 \;\; 0.0023 \;\; 0.5269]^T$. *Let* $A_{11} = \begin{bmatrix} 71.691 & 12 & -11 \\ 12 & 63.691 & 18 \\ -11 & 18 & 38.691 \end{bmatrix}$, *then we observe that A_{22} is an $M_{\rm V}$-matrix. The Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 44.8154 & -10.8112 & 8.5854 & -9.1532 \\ -10.8112 & 30.7188 & -4.1361 & -26.5817 \\ 8.5854 & -4.1361 & 31.1984 & 0.2358 \\ -9.1532 & -26.5817 & 0.2358 & 31.3552 \end{bmatrix}$$

is a symmetric singular $M_{\rm V}$-matrix with eigenvector corresponding to its zero eigenvalue $[0.3002 \;\; 0.6826 \;\; 0.0028 \;\; 0.6663]^T$.

For the "small" shift $\lambda = 10$, $C = A + 10I$:

$$(C/C_{11}) = \begin{bmatrix} 55.7275 & -9.4882 & 8.4364 & -8.0349 \\ -9.4882 & 43.0595 & -4.7947 & -24.7376 \\ 8.4364 & -4.7947 & 42.2638 & -0.6181 \\ -8.0349 & -24.7376 & -0.6181 & 43.1763 \end{bmatrix}$$

is a symmetric $M_{\rm V}$-matrix with eigenvector corresponding to the smallest eigenvalue 14.5804, $[0.2791 \;\; 0.6846 \;\; 0.0485 \;\; 0.6717]^T$.

For a "big" shift $\lambda = 100$, $C = A + 100I$:

$$(C/C_{11}) = \begin{bmatrix} 147.6338 & -6.2410 & 7.7273 & -5.5261 \\ -6.2410 & 139.9919 & -7.4581 & -19.7376 \\ 7.7273 & -7.4581 & 136.1982 & -3.9467 \\ -5.5261 & -19.7376 & -3.9467 & 138.1019 \end{bmatrix}$$

is a symmetric $M_{\rm V}$-matrix with eigenvector corresponding to the smallest eigenvalue 115.2386, $[0.17 \;\; 0.6631 \;\; 0.2985 \;\; 0.6651]^T$.

Consider a new partitioning and let $A_{11} = \begin{bmatrix} 71.691 & 12 \\ 12 & 63.691 \end{bmatrix}$*, and note that* A_{22} *is not an* M_{V}*-matrix, then the Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 30.6210 & -9.3716 & -13.2977 & 5.2301 & -14.3823 \\ -9.3716 & 47.6835 & -6.7414 & 6.9847 & -4.7515 \\ -13.2977 & -6.7414 & 36.4935 & -6.4073 & -20.3359 \\ 5.2301 & 6.9847 & -6.4073 & 32.0917 & -2.2207 \\ -14.3823 & -4.7515 & -20.3359 & -2.2207 & 38.1103 \end{bmatrix}$$

is a symmetric singular M_{V}*-matrix with eigenvector* $[0.5739\ \ 0.2458\ \ 0.559\ \ 0.0023\ \ 0.5456]^T$ *corresponding to its zero eigenvalue.*

For small shift $\lambda = 2$*,* $C = A + 2I$ *:*

$$(C/C_{11}) = \begin{bmatrix} 32.9036 & -9.3566 & -13.3868 & 5.5015 & -14.5421 \\ -9.3566 & 49.7084 & -6.6764 & 6.9907 & -4.7365 \\ -13.3868 & -6.6764 & 38.7233 & -6.5170 & -20.2177 \\ 5.5015 & 6.9907 & -6.5170 & 34.3553 & -2.3823 \\ -14.5421 & -4.7365 & -20.2177 & -2.3823 & 40.2236 \end{bmatrix}$$

is a symmetric M_{V}*-matrix with eigenvector* $[0.5751\ \ 0.2455\ \ 0.5574\ \ 0.0017\ \ 0.5462]^T$ *corresponding to the smallest eigenvalue* 2.1409.

Now, for the shift $\lambda = 10$*,* $C = A + 10I$ *:*

$$(C/C_{11}) = \begin{bmatrix} 41.8603 & -9.3069 & -13.6906 & 6.4207 & -15.0840 \\ -9.3069 & 57.7967 & -6.4444 & 7.0089 & -4.6813 \\ -13.6906 & -6.4444 & 47.5370 & -6.8946 & -19.8043 \\ 6.4207 & 7.0089 & -6.8946 & 43.2487 & -2.9318 \\ -15.0840 & -4.6813 & -19.8043 & -2.9318 & 48.6117 \end{bmatrix}$$

is not an M_{V}*-matrix because the eigenvector* $[0.5795\ \ 0.2442\ \ 0.5516\ \ -0.0007\ \ 0.5479]^T$ *corresponding to the smallest eigenvalue* 10.6373 *has a negative entry. Also, for every* $\lambda > 10$*, each eigenvector corresponding to its smallest eigenvalue always has negative entries.*

Let

$$E = \begin{bmatrix} 1.62 & 0.545 & -1.19 & 1.65 & -0.415 & 0.535 & -0.425 \\ 0.545 & 2.8 & -0.195 & 0.06 & 0.295 & -0.415 & -1.475 \\ -1.19 & -0.195 & 3.79 & 0.16 & -0.545 & -0.815 & 0.745 \\ 1.65 & 0.06 & 0.16 & 0.59 & 1.12 & -0.17 & 0.935 \\ -0.415 & 0.295 & -0.545 & 1.12 & 1.45 & -0.57 & -0.09 \\ 0.535 & -0.415 & -0.815 & -0.17 & -0.57 & 2.51 & -0.605 \\ -0.425 & -1.475 & 0.745 & 0.935 & -0.09 & -0.605 & 0.45 \end{bmatrix}$$

and $C = A + E$ *be a symmetric* M_V*-matrix with eigenvector* $[0.2477\ \ 0.1142\ \ 0.5252\ \ 0.2025\ \ 0.5593\ \ 0.049\ \ 0.5418]^T$ *corresponding to its smallest eigenvalue* 1.8764.

Let $C_{11} \in \mathbb{R}^{3,3}$*, then*

$$(C/C_{11}) = \begin{bmatrix} 46.2361 & -9.0164 & 8.3264 & -7.4486 \\ -9.0164 & 32.1365 & -5.3062 & -26.315 \\ 8.3264 & -5.3062 & 34.6370 & -0.853 \\ -7.4486 & -26.3150 & -0.8530 & 32.7265 \end{bmatrix}$$

is a symmetric M_V*-matrix with eigenvector* $[0.2464\ \ 0.6933\ \ 0.0692\ \ 0.6737]^T$ *corresponding to its smallest eigenvalue* 2.833.

Let $C_{11} \in \mathbb{R}^{2,2}$*, then*

$$(C/C_{11}) = \begin{bmatrix} 34.3052 & -9.0138 & -14.4897 & 4.8036 & -13.4744 \\ -9.0138 & 48.6045 & -5.2091 & 7.0642 & -3.9082 \\ -14.4897 & -5.2091 & 38.2566 & -7.3351 & -20.6237 \\ 4.8036 & 7.0642 & -7.3351 & 35.3096 & -2.7397 \\ -13.4744 & -3.9082 & -20.6237 & -2.7397 & 38.0190 \end{bmatrix}$$

is a symmetric M_V*-matrix with eigenvector* $[0.5468\ \ 0.2103\ \ 0.58\ \ 0.0507\ \ 0.5637]^T$ *corresponding to its smallest eigenvalue* 2.0236.

Let x *be the Perron eigenvector of* A *and let* $E = \alpha x x^T$ *be such that* $C = A + E$ *is a symmetric* M_V*-matrix with the same Perron eigenvector as that of* A *corresponding to its smallest eigenvalue* α.

For $\alpha = 2$, $C = A + 2xx^T$*, suppose that* $C_{11} \in \mathbb{R}^{3,3}$*. Then,*

$$(C/C_{11}) = \begin{bmatrix} 45.1954 & -10.0657 & 8.5073 & -8.4518 \\ -10.0657 & 32.1811 & -4.2893 & -25.2059 \\ 8.5073 & -4.2893 & 31.2145 & 0.0917 \\ -8.4518 & -25.2059 & 0.0917 & 32.6494 \end{bmatrix}$$

is a symmetric M_V*-matrix with eigenvector* $[0.295\ \ 0.6833\ \ 0.0128\ \ 0.6677]^T$ *corresponding to its smallest eigenvalue* 3.1232.

Let $C_{11} \in \mathbb{R}^{2,2}$*, then*

$$(C/C_{11}) = \begin{bmatrix} 31.2674 & -9.0685 & -12.5959 & 5.2376 & -13.7616 \\ -9.0685 & 47.8256 & -6.4125 & 6.9882 & -4.4605 \\ -12.5959 & -6.4125 & 37.2553 & -6.3992 & -19.6621 \\ 5.2376 & 6.9882 & -6.3992 & 32.0918 & -2.2135 \\ -13.7616 & -4.4605 & -19.6621 & -2.2135 & 38.7063 \end{bmatrix}$$

is a symmetric M_{V}*-matrix with eigenvector* $[0.5752 \;\; 0.2455 \;\; 0.5573 \;\; 0.0016 \;\; 0.5462]^T$ *corresponding to its smallest eigenvalue* 2.1407*.*

The following example shows that the Schur complement of a singular symmetric M_{V}-matrix is not an M_{V}-matrix for large λ but it holds the M_{V}-property for small λ.

Example 3.26. *Consider the irreducible symmetric singular* M_{V}*-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rr|rr} 48.8818 & -1.2 & -2.1 & 6.3 \\ -1.2 & 16.8818 & -15.9 & -13.4 \\ \hline -2.1 & -15.9 & 15.3818 & 10.35 \\ 6.3 & -13.4 & 10.35 & 30.2818 \end{array}\right]$$

with eigenvector $[0.0384 \;\; 0.7137 \;\; 0.6959 \;\; 0.07]^T$ *corresponding to the zero eigenvalue.* A_{22} *is not an* M_{V}*-matrix because the eigenvector* $[-0.89 \;\; 0.456]^T$ *corresponding to its smallest eigenvalue* 10.0794 *has a negative entry.*

The Schur complement

$$(A/A_{11}) = \begin{bmatrix} 0.1927 & -1.9167 \\ -1.9167 & 19.0595 \end{bmatrix}$$

is a symmetric singular M_{V}*-matrix with eigenvector* $[0.995 \; 0.1001]^T$ *corresponding to the zero eigenvalue.*

For the "small" shift $\lambda = 2$*,* $C = A + 2I$ *:*

$$(C/C_{11}) = \begin{bmatrix} 3.8023 & -0.6003 \\ -0.6003 & 22.1879 \end{bmatrix}$$

is a symmetric M_{V}*-matrix with eigenvector* $[0.9995 \;\; 0.0326]^T$ *corresponding to the smallest eigenvalue* 3.7828*.*

For the "big" shift $\lambda = 10$ *(which we consider as a big shift in this case),* $C = A + 10I$*, then the Schur complement*

$$(C/C_{11}) = \begin{bmatrix} 15.8431 & 2.6965 \\ 2.6965 & 33.0496 \end{bmatrix}$$

is not an M_{V}*-matrix because the eigenvector* $[-0.9885 \;\; 0.1513]^T$ *corresponding to the smallest eigenvalue* 15.4305 *has a negative entry.*

Let $E = \left[\begin{array}{cc|cc} -3.9 & 0.4 & 0.6 & -1.95 \\ 0.4 & 2.1 & 1.1 & 0.65 \\ \hline 0.6 & 1.1 & 1.4 & -0.8 \\ -1.95 & 0.65 & -0.8 & 4.5 \end{array}\right]$ *and* $C = A + E$ *is a symmetric* M_V*-matrix, then the Schur complement*

$$(C/C_{11}) = \begin{bmatrix} 5.1420 & -0.2109 \\ -0.2109 & 25.8943 \end{bmatrix}$$

is a symmetric M_V*-matrix with eigenvector* $[0.9999 \;\; 0.0102]^T$ *corresponding to its smallest eigenvalue* 5.1399*, thus the validity of Theorem 3.23 is confirmed.*

However, there are cases where the Schur complements (C/C_{11}), when $C = A + \lambda I$ and A is a symmetric singular M_V-matrix, are M_V-matrices for **all** λ as this is shown in the following example.

Example 3.27. *Consider the irreducible symmetric singular* M_V*-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{r|rrrr} 33.5796 & -9.5 & -3.5 & -17.5 & -6 \\ \hline -9.5 & 31.5796 & -19.5 & -2 & -13.5 \\ -3.5 & -19.5 & 42.5796 & -11.5 & 6 \\ -17.5 & -2 & -11.5 & 37.5796 & -10.5 \\ -6 & -13.5 & 6 & -10.5 & 39.5796 \end{array}\right]$$

with eigenvector $[0.4931 \;\; 0.5460 \;\; 0.3695 \;\; 0.4634 \;\; 0.3279]^T$ *corresponding to the zero eigenvalue. The Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 28.8920 & -20.4902 & -6.9509 & -15.1975 \\ -20.4902 & 42.2148 & -13.3240 & 5.3746 \\ -6.9509 & -13.3240 & 28.4595 & -13.6269 \\ -15.1975 & 5.3746 & -13.6269 & 38.5076 \end{bmatrix}$$

is a singular M_V*-matrix with eigenvector* $[0.6276 \;\; 0.4247 \;\; 0.5326 \;\; 0.3769]^T$ *corresponding to its zero eigenvalue.*

For the "small" shift $\lambda = 10$, $C = A + 10I$:

$$(C/C_{11}) = \begin{bmatrix} 39.5087 & -20.2630 & -5.8149 & -14.8080 \\ -20.2630 & 52.2985 & -12.9055 & 5.5181 \\ -5.8149 & -12.9055 & 40.5523 & -12.9094 \\ -14.8080 & 5.5181 & -12.9094 & 48.7536 \end{bmatrix}$$

is an M_V*-matrix with eigenvector* $[0.6416 \;\; 0.4380 \;\; 0.5059 \;\; 0.3749]^T$ *corresponding to the smallest eigenvalue* 12.4372.

We observe that A_{22} *is also an* M_{V}*-matrix. For a "big" shift* $\lambda = 800$, $C = A + 800I$*, then*

$$(C/C_{11}) = \begin{bmatrix} 831.4714 & -19.5399 & -2.1994 & -13.5684 \\ -19.5399 & 842.5649 & -11.5735 & 5.9748 \\ -2.1994 & -11.5735 & 837.2122 & -10.6260 \\ -13.5684 & 5.9748 & -10.6260 & 839.5364 \end{bmatrix}$$

is also an M_{V}*-matrix with eigenvector* $[0.6935 \quad 0.4846 \quad 0.3955 \quad 0.3576]^T$ *corresponding to the smallest eigenvalue* 809.5665*.*

We checked the associated Schur complements for intermediate shiftings $\lambda = 11, 12, \ldots, 799$ *and found out that all Schur complements are* M_{V}*-matrices. We claim that the Schur complement is an* M_{V}*-matrix for* **all** $\lambda > 0$*. In this example it seems that* α *of Theorem 3.22 is greater than* β *of Theorem 3.24. Thus, the union of the sets* $(0, \alpha) \cup (\beta, \infty) = (0, \infty)$*.*

3.4.2 Non-symmetric M_{V}-matrices

The following theorem concerns the Schur complement of non-symmetric singular M_{V}-matrices.

Theorem 3.28. *Let* $A \in \mathbb{R}^{n,n}$*, partitioned in the form (1.4), be an irreducible non-symmetric singular* M_{V}*-matrix and can be written as* $A = sI - B$*, where* B *and* $B + \beta I$*, for some* $\beta > 0$*, are irreducible eventually nonnegative matrices. Then, the Schur complement* (A/A_{11}) *is a non-symmetric singular matrix, with a simple zero eigenvalue and its associated eigenvector is positive.*

Proof. A is an irreducible non-symmetric singular M_{V}-matrix written as $A = sI - B$. From Theorem 2.8, there exists a vector $x > 0$ such that $Ax = 0 \Leftrightarrow Bx = sx$. Thus, $\rho(B) = s$ is a simple eigenvalue of B corresponding to the Perron eigenvector $x > 0$ and 0 is a simple eigenvalue of A. For any other eigenvalue λ_i of A there will hold $\lambda_i = s - \mu_i$, where μ_i, $i = 2, 3, \ldots, n$, denote the remaining eigenvalues of B and $|\mu_i| < s$, thus $Re\lambda_i = s - Re\mu_i \geq s - |Re\mu_i| \geq s - |\mu_i| > 0$ for $i = 2, 3, \ldots, n$. We have

$$Ax = 0 \Leftrightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{array}{c} x_1 = -A_{11}^{-1} A_{12} x_2, \\ (A_{22} - A_{21} A_{11}^{-1} A_{12}) x_2 = 0. \end{array}$$

Hence, $x_2 > 0$ is a right eigenvector of (A/A_{11}) corresponding to the zero eigenvalue.

Also, there exists $y > 0$ such that

$$y^T A = 0 \Leftrightarrow \begin{bmatrix} y_1^T & y_2^T \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0^T & 0^T \end{bmatrix}$$

$$\Leftrightarrow \begin{array}{c} y_1^T = -y_2 A_{21} A_{11}^{-1} \\ y_2^T (A_{22} - A_{21} A_{11}^{-1} A_{12}) = 0 \end{array}.$$

Thus, $y_2 > 0$ is a left eigenvector of (A/A_{11}) corresponding to the zero eigenvalue.

We will prove that 0 is a simple eigenvalue. Looking for a contradiction, we suppose that 0 is a multiple eigenvalue of (A/A_{11}). Let $\text{index}_0(A/A_{11}) = 1$ and 0 be a double eigenvalue of (A/A_{11}). Then, there exists $z_2 \neq 0$ linearly independent with x_2 such that

$$(A/A_{11})z_2 = 0 \quad \Leftrightarrow \quad (A_{22} - A_{21}A_{11}^{-1}A_{12})z_2 = 0 \Leftrightarrow \begin{array}{c} A_{11}z_1 + A_{12}z_2 = 0 \\ A_{21}z_1 + A_{22}z_2 = 0 \end{array}$$

$$\Leftrightarrow \quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ where } z_1 = -A_{11}^{-1}A_{12}z_2.$$

The vectors $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$ are linearly independent because z_2 and x_2 are linearly independent. This constitutes a contradiction because 0 is a simple eigenvalue of A.

Let $\text{index}_0(A/A_{11}) \geq 2$. Then, there exists a generalized right eigenvector z_2 of (A/A_{11}) and a generalized left eigenvector w_2 of (A/A_{11}). Thus $(A/A_{11})z_2 = x_2$ and $w_2^T(A/A_{11}) = y_2^T$. Obviously, z_2 and w_2 are right and left eigenvectors of $(A/A_{11})^2$, respectively, corresponding to 0. Hence,

$$w_2^T (A/A_{11})^2 z_2 = 0 \Leftrightarrow w_2^T (A/A_{11})(A/A_{11})z_2 = 0 \Leftrightarrow y_2^T x_2 = 0.$$

This constitutes a contradiction because $x_2 > 0$ and $y_2 > 0$. □

If we prove that $Re\lambda > 0$ for any eigenvalue λ of (A/A_{11}) different from zero, then it could be proved that (A/A_{11}) is an M_{V}-matrix. However, this can **not** happen for any class of real M_{V}-matrices and we can show in the following example.

Example 3.29. *Consider the irreducible non-symmetric singular M_{V}-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|ccc} 8.3917 & -5 & -2 & 5 & -3 \\ -1 & 6.3917 & 3 & -3 & -7 \\ \hline -6 & 4 & 3.3917 & -10 & 9 \\ 2 & -9 & 10 & 2.3917 & -1 \\ -1 & -6 & -7 & 5 & 8.3917 \end{array}\right]$$

with right and left eigenvectors $x = [0.2263 \;\; 0.5686 \;\; 0.3687 \;\; 0.5752 \;\; 0.3983]^T$ *and* $y = [0.2685 \;\; 0.7656 \;\; 0.202 \;\; 0.1288 \;\; 0.5333]^T$*, respectively, corresponding to the zero eigenvalue. The Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 1.7592 & -6.2487 & 7.3946 \\ 14.1972 & -2.0388 & -10.1971 \\ -4.0955 & 2.8598 & -0.3387 \end{bmatrix}$$

is a singular matrix with right and left eigenvectors $x = [0.4662 \;\; 0.7273 \;\; 0.5037]^T$ *and* $y = [0.3456 \;\; 0.2203 \;\; 0.9122]^T$*, respectively, corresponding to its zero eigenvalue. However, it is not an M_{V}-matrix because it has eigenvalues with negative real part :* $\sigma(A/A_{11}) = \{0, -0.3091 \pm 12.0239i\}$.

Analogous perturbation properties for non-symmetric M_{V}-matrices can be stated below.

Theorem 3.30. *Let $A \in \mathbb{R}^{n,n}$ be an irreducible non-symmetric singular M_{V}-matrix partitioned in the form (1.4) and can be written as $A = sI - B$, where B and $B + \beta I$, for some $\beta > 0$, are irreducible eventually nonnegative matrices. Then, there exists $\alpha > 0$ such that for the nonsingular M_{V}-matrices $C = A + \lambda I$, $0 < \lambda < \alpha$, there exists a simple positive eigenvalue $\tilde{\lambda}$ of the Schur complement (C/C_{11}), which is the smallest in modulus, with the corresponding right and left eigenvectors being positive. Moreover, (C/C_{11}) has eventually positive inverses for all $0 < \lambda < \alpha$.*

Proof. Since there exists $\beta > 0$ such that $B + \beta I$ is an eventually nonnegative matrix then, from Theorem 2.8, A has positive Perron left and right eigenvectors corresponding to its smallest eigenvalue. The matrix A is an irreducible singular M_{V}-matrix, so there exists a positive vector x such that $Ax = 0$. Suppose $\lambda > 0$ is a “small” real number. Obviously, $(A + \lambda I)$ is a nonsingular M_{V}-matrix.

Let $\tilde{x}_2$ be an eigenvector of the Schur complement of $A_{11} + \lambda I$ in $A + \lambda I$ corresponding to the absolutely smallest eigenvalue $\tilde{\lambda}$. Thus,

$$(A_{22} + \lambda I - A_{21}(A_{11} + \lambda I)^{-1}A_{12})\tilde{x}_2 = \tilde{\lambda}\tilde{x}_2.$$

Let $\tilde{x}_1 = -(A_{11} + \lambda I)^{-1} A_{12} \tilde{x}_2$. Then we have,

$$\begin{aligned} (A_{11} + \lambda I)\tilde{x}_1 + A_{12}\tilde{x}_2 &= 0 \\ A_{21}\tilde{x}_1 + (A_{22} + \lambda I)\tilde{x}_2 &= \tilde{\lambda}\tilde{x}_2 \end{aligned}$$

$$\Leftrightarrow \begin{bmatrix} (A_{11} + \lambda I) & A_{12} \\ A_{21} & (A_{22} + \lambda I) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} 0 \\ \tilde{x}_2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} (A_{11} + \lambda I) & A_{12} \\ A_{21} & (A_{22} + \lambda I - \tilde{\lambda} I) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = 0 \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

Let $A' = \begin{bmatrix} (A_{11} + \lambda I) & A_{12} \\ A_{21} & (A_{22} + \lambda I - \tilde{\lambda} I) \end{bmatrix}$, then the above relation can be written in the form $A'\tilde{x} = 0 \Leftrightarrow ((\rho(B) + \lambda)I - (B + \tilde{\lambda} J))\tilde{x} = 0$, where $J = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$. There exists a positive eigenvalue μ of $(B + \tilde{\lambda} J)$ such that

$$\rho(B) < \rho(B) + \lambda = \mu \leq \rho(B + \tilde{\lambda} J). \tag{3.7}$$

Since B is an eventually positive matrix, from [42, Theorem 2.7], $B + \tilde{\lambda} J$ is also an eventually positive matrix for "small" $\tilde{\lambda} > 0$. The smallness of $\tilde{\lambda}$ implies from (3.7) the smallness of λ.

Hence, using the continuity argument, the Schur complement has a positive eigenvector $\tilde{x}_2$ corresponding to the smallest in modulus eigenvalue $\tilde{\lambda}$ and it should be positive for "small" enough λ. We prove now that $(C/C_{11})^{-1} \overset{\text{V}}{>} 0$: (C/C_{11}) has the smallest in modulus eigenvalue $\tilde{\lambda}$ being positive. Then, $(C/C_{11})^{-1}$ has its spectral radius $\dfrac{1}{\tilde{\lambda}} > 0$, which is a simple eigenvalue, with positive right and left eigenvectors. Thus, $(C/C_{11})^{-1}$ and its transpose possess the strong Perron-Frobenius property. From [42, Theorem 2.2] we get that $(C/C_{11})^{-1} \overset{\text{V}}{>} 0$. □

Here is an example for a non-symmetric singular M_{V}-matrix, where A_{22} is **not** an M_{V}-matrix. The Schur complement (C/C_{11}), where $C = A + \lambda I$, is an M_{V}-matrix for "small" λ and its inverse is an eventually positive matrix and the validity of Theorem 3.30 is confirmed.

Example 3.31. *Consider the irreducible non-symmetric singular* M_V*-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|ccc} 75.9289 & -6 & -37 & -23 & -20 \\ -23 & 57.9289 & -2 & -35 & -21 \\ \hline -19 & -40 & 47.9289 & -17 & -6 \\ -9 & 11 & -24 & 46.9289 & 13 \\ -22 & 2 & -2 & 15 & 70.9289 \end{array}\right]$$

with right and left eigenvectors $x = [0.4859 \quad 0.4390 \quad 0.6816 \quad 0.3136 \quad 0.0912]^T$ *and* $y = [0.3667 \quad 0.3365 \quad 0.6056 \quad 0.6042 \quad 0.1435]^T$*, respectively, corresponding to the zero eigenvalue. Note that* A_{22} *is not an* M_V*-matrix, however the Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 28.9014 & -53.7953 & -31.0177 \\ -25.9638 & 51.8979 & 15.5909 \\ -12.6590 & 8.5315 & 65.2601 \end{bmatrix}$$

is a singular M_V*-matrix with eigenvectors* $x = [0.9018 \quad 0.4149 \quad 0.1207]^T$ *and* $y = [0.6982 \quad 0.6965 \quad 0.1654]^T$ *corresponding to its zero eigenvalue.*

For the "small" shift $\lambda = 10$, $C = A + 10I$ *:*

$$(C/C_{11}) = \begin{bmatrix} 42.3305 & -47.7306 & -26.8433 \\ -26.0138 & 60.9559 & 15.0264 \\ -11.3897 & 9.3992 & 75.9927 \end{bmatrix}$$

is an M_V*-matrix with right and left eigenvectors* $x = [0.8815 \quad 0.4631 \quad 0.0924]^T$ *and* $y = [0.7051 \quad 0.6957 \quad 0.1377]^T$*, respectively, corresponding to the smallest eigenvalue* 14.4399*. Its inverse*

$$(C/C_{11})^{-1} = \begin{bmatrix} 0.0489 & 0.0367 & 0.0100 \\ 0.0197 & 0.0317 & 0.0007 \\ 0.0049 & 0.0016 & 0.0146 \end{bmatrix}$$

is a positive matrix.

For the nonsingular M_V*-matrices* $C = A + \lambda I, 0 < \lambda \leq 52$*, there holds: The Schur complement* (C/C_{11}) *is a non-symmetric nonsingular* M_V*-matrix. Also, there holds that: The Schur complement has an eventually positive inverse for all* $0 < \lambda \leq 52$*. Thus, Theorem 3.30 is confirmed.*

The following example shows that the Schur complement of the nonsingular M_V-matrices $C = A + \lambda I$ has an eventually positive inverse for λ belonging to only a small interval $(0, \alpha)$ from Theorem 3.30. For larger λ, some entry of the eigenvector corresponding to the smallest in modulus eigenvalue of the Schur complement may be negative, thus its inverse is not an eventually positive matrix.

Example 3.32. *Consider the irreducible non-symmetric singular M_V-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{r|rrrr} 12.3991 & -3.81 & -1.33 & -1.67 & -0.45 \\ \hline 0.20 & 5.5191 & -2.69 & -2.23 & -0.09 \\ -5.16 & 0.84 & 2.4591 & -1.84 & 1.35 \\ 0.77 & -3.45 & -2.60 & 9.4091 & -3.75 \\ 2.89 & -5.48 & -0.82 & -1.71 & 5.2191 \end{array}\right]$$

with right and left eigenvectors $x = [0.2669 \;\; 0.4272 \;\; 0.4804 \;\; 0.4803 \;\; 0.5337]^T$ *and* $y = [0.3195 \;\; 0.3116 \;\; 0.8422 \;\; 0.3009 \;\; 0.0313]^T$*, respectively, corresponding to the zero eigenvalue,* A_{22} *is not an* M_V*-matrix. The Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 5.5805 & -2.6685 & -2.2031 & -0.0827 \\ -0.7456 & 1.9056 & -2.5350 & 1.1627 \\ -3.2134 & -2.5174 & 9.5128 & -3.7221 \\ -4.5920 & -0.5100 & -1.3208 & 5.3240 \end{bmatrix}$$

is a singular M_V*-matrix with eigenvector* $x = [0.4433 \;\; 0.4985 \;\; 0.4983 \;\; 0.5537]^T$ *and* $y = [0.3288 \;\; 0.8888 \;\; 0.3176 \;\; 0.0330]^T$ *corresponding to its zero eigenvalue.*

For the "small" shift $\lambda = 3$, $C = A + 3I$ *:*

$$(C/C_{11}) = \begin{bmatrix} 8.5686 & -2.6727 & -2.2083 & -0.0842 \\ -0.4367 & 5.0134 & -2.3996 & 1.1992 \\ -3.2595 & -2.5335 & 12.4926 & -3.7275 \\ -4.7650 & -0.5704 & -1.3966 & 8.3035 \end{bmatrix}$$

is an M_V*-matrix with right and left eigenvectors* $x = [0.4371 \;\; 0.4126 \;\; 0.5168 \;\; 0.6096]^T$ *and* $y = [0.2815 \;\; 0.9087 \;\; 0.3079 \;\; 0.0164]^T$*, respectively, corresponding to the smallest eigenvalue* 3.3169*.*

Its inverse

$$(C/C_{11})^{-1} = \begin{bmatrix} 0.1431 & 0.1001 & 0.0453 & 0.0074 \\ 0.0234 & 0.2347 & 0.0479 & -0.0122 \\ 0.0706 & 0.1007 & 0.1162 & 0.0383 \\ 0.0956 & 0.0905 & 0.0488 & 0.1303 \end{bmatrix}$$

is an eventually positive matrix with power index 5*.*

For $\lambda > 6.25$, the inverse of the Schur complement is not an eventually positive matrix. Even if the smallest in modulus eigenvalue is always positive, its left eigenvector has negative and positive entries. So, for this example, the Schur complement of the nonsingular $M_{\rm V}$-matrices $C = A + \lambda I$ has an eventually positive inverse for all $0 < \lambda \leq 6.25$ which confirms Theorem 3.30.

Despite the previous example, we give another example with a non-symmetric singular $M_{\rm V}$-matrix where the Schur complement is **not** an $M_{\rm V}$-matrix (it has a negative eigenvalue) but to its zero eigenvalue there correspond positive right and left eigenvectors. Also, the validity of Theorem 3.30 is shown, i.e., the Schur complement of the nonsingular $M_{\rm V}$-matrices $C = A + \lambda I$ has an eventually positive inverse for λ belonging to a "small" interval $(0, \alpha)$.

Example 3.33. *Consider the irreducible non-symmetric singular $M_{\rm V}$-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{r|rrr} -37.8973 & -92.47 & 131.53 & 136.57 \\ \hline 25.06 & 58.5327 & -85.47 & -86.21 \\ -15.8 & -34.94 & 53.0627 & 51.15 \\ 15.07 & 32.84 & -51.16 & -47.6873 \end{array}\right]$$

with right and left eigenvectors $x = [0.2832 \;\; 0.8227 \;\; 0.1861 \;\; 0.4564]^T$ and $y = [0.2288 \;\; 0.6654 \;\; 0.6854 \;\; 0.1873]^T$, respectively, corresponding to the zero eigenvalue. Although A_{22} is an $M_{\rm V}$-matrix, the Schur complement

$$(A/A_{11}) = \begin{bmatrix} -2.6141 & 1.5056 & 4.0984 \\ 3.6122 & -1.7743 & -5.7883 \\ -3.9310 & 1.1434 & 6.6203 \end{bmatrix}$$

is a singular matrix with right and left eigenvectors $x = [0.8578 \;\; 0.1941 \;\; 0.4759]^T$ and $y = [0.6836 \;\; 0.7041 \;\; 0.1924]^T$, respectively, corresponding to its zero eigenvalue. However, it is not an $M_{\rm V}$-matrix because one of its eigenvalues is negative: $\sigma(A/A_{11}) = \{0, 4.0089, -1.777\}$.

let $\lambda = 1$ and $C = A + I$, then

$$(C/C_{11}) = \begin{bmatrix} -3.2713 & 3.8629 & 6.5460 \\ 4.6571 & -2.2605 & -7.3314 \\ -4.9276 & 2.5609 & 9.0921 \end{bmatrix}$$

is a matrix with right and left eigenvectors $x = [0.8637 \;\; 0.1493 \;\; 0.4813]^T$ and $y = [0.4435 \;\; 0.8121 \;\; 0.3791]^T$, respectively, corresponding to the smallest *in modulus eigenvalue* 1.0445 *but it is not an* M_{V}*-matrix because one of its eigenvalues is negative*: $\sigma(C/C_{11}) = \{1.0445, 5.0089, -2.4931\}$. *Its inverse*

$$(C/C_{11})^{-1} = \begin{bmatrix} 0.1363 & 1.4075 & 1.0368 \\ 0.4766 & -0.1927 & -0.4985 \\ -0.0604 & 0.8171 & 0.8123 \end{bmatrix}$$

is an eventually positive matrix with power index 2*.*

We observe that for $0 < \lambda < 37.8973$*, there exists a negative eigenvalue of the Schur complement, but the smallest in modulus eigenvalue is always positive and the corresponding left and right eivenvectors are positive. Thus the Schur complement has an eventually positive inverse, confirming the validity of Theorems 3.30. For* $\lambda = 37.8973$*, the Schur complement is not defined, since* C_{11} *is singular. For* $\lambda > 37.8973$*, all eigenvalues of the Schur complement are positive and it is an* M_{V}*-matrix.*

We may note that for some matrices, the Schur complement (C/C_{11}), where $C = A + \lambda I$, with A a non-symmetric singular M_{V}-matrix, are M_{V}-matrices and their inverses are eventually positive ones for all $\lambda > 0$. This is shown in the following example.

Example 3.34. *Consider the irreducible non-symmetric singular* M_{V}*-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{r|rrrr} 28.2447 & -5 & -11 & -5 & -4 \\ \hline -7 & 37.2447 & -3 & -16 & -5 \\ -12 & -14 & 44.2447 & 5 & -12 \\ -4 & -18 & -5 & 28.2447 & -9 \\ -15 & -19 & -17 & -7 & 40.2447 \end{array}\right]$$

with right and left eigenvectors $x = [0.3691 \;\; 0.4021 \;\; 0.3168 \;\; 0.5416 \;\; 0.5554]^T$ *and* $y = [0.5175 \;\; 0.5706 \;\; 0.3396 \;\; 0.4342 \;\; 0.3207]^T$*, respectively, corresponding to the zero eigenvalue, then the Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 36.0055 & -5.7262 & -17.2392 & -5.9913 \\ -16.1243 & 39.5712 & 2.8757 & -13.6994 \\ -18.7081 & -6.5578 & 27.5366 & -9.5665 \\ -21.6554 & -22.8418 & -9.6554 & 38.1204 \end{bmatrix}$$

is a singular M_{V}*-matrix with right and left eigenvectors* $x = [0.4326 \;\; 0.3408 \;\; 0.5827 \;\; 0.5976]^T$ *and* $y = [0.6668 \;\; 0.3969 \;\; 0.5074 \;\; 0.3748]^T$*, respectively, corresponding to its zero eigenvalue.*

For the "small" shift $\lambda = 10$, $C = A + 10I$:

$$(C/C_{11}) = \begin{bmatrix} 46.3295 & -5.0134 & -16.9152 & -5.7321 \\ -15.5688 & 50.7932 & 3.4312 & -13.2551 \\ -18.5229 & -6.1505 & 37.7217 & -9.4184 \\ -20.9611 & -21.3143 & -8.9611 & 48.6758 \end{bmatrix}$$

is an $M_{\rm V}$-matrix with right and left eigenvectors $x = [0.4396\ \ 0.3210\ \ 0.6033\ \ 0.5829]^T$ and $y = [0.6733\ \ 0.3722\ \ 0.5196\ \ 0.3717]^T$, respectively, corresponding to the smallest eigenvalue 11.8534. Its inverse

$$(C/C_{11})^{-1} = \begin{bmatrix} 0.0373 & 0.0104 & 0.0183 & 0.0108 \\ 0.0170 & 0.0268 & 0.0078 & 0.0108 \\ 0.0283 & 0.0142 & 0.0415 & 0.0152 \\ 0.0287 & 0.0189 & 0.0189 & 0.0327 \end{bmatrix}$$

is a positive matrix.

3.4.3 Matrices with the block A_{22} being an irreducible $M_{\rm V}$-matrix

In this section we will study matrices which are not necessarily $M_{\rm V}$- or GM-matrices but their block A_{22} is an irreducible $M_{\rm V}$-matrix.

Theorem 3.35. *Let $A \in \mathbb{R}^{n,n}$ be an irreducible singular matrix partitioned in the form (1.4) and let that A_{22} is an irreducible $M_{\rm V}$-matrix. Then, for the nonsingular matrices $C = A + \lambda I$, there exists a "big" enough $\beta > 0$ such that for all $\lambda > \beta$, there holds: The Schur complement (C/C_{11}) is a nonsingular $M_{\rm V}$-matrix.*

Proof. Suppose A_{22} is an $M_{\rm V}$-matrix and, obviously, $A_{22} + \lambda I$ is also an $M_{\rm V}$-matrix. The Schur complement of C_{11} in C is

$$(C/C_{11}) = A_{22} + \lambda I - A_{21}(A_{11} + \lambda I)^{-1} A_{12}.$$

When λ tends to infinity, the Schur complement (C/C_{11}) tends to $A_{22} + \lambda I$ which is an $M_{\rm V}$-matrix. Hence, from the continuity argument, we get that (C/C_{11}) is a nonsingular $M_{\rm V}$-matrix for λ belonging to an interval (β, ∞). $\square$

Here is an example for a singular matrix A which is **not** an $M_{\rm V}$-matrix, where A_{22} is an $M_{\rm V}$-matrix and the Schur complement (C/C_{11}) is a nonsingular $M_{\rm V}$-matrix for sufficiently "big" λ, confirming Theorem 3.35.

Example 3.36. *Consider the irreducible singular matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rr|rrr} 28.9012 & -3.7 & -3 & 2.3 & 0.4 \\ -3.8 & 28.8012 & -0.8 & -0.7 & 1.9 \\ \hline -4.9 & 7.5 & 10.7012 & 11 & -18 \\ 7.1 & -9.3 & -12 & 12.7012 & 20 \\ -8.2 & -8.5 & -11 & -27 & 20.7012 \end{array}\right].$$

The matrix A is not an M_V-matrix while A_{22} is. The Schur complement

$$(A/A_{11}) = \begin{bmatrix} 10.4825 & 11.4865 & -18.4061 \\ -11.617 & 12.0183 & 20.4804 \\ -12.254 & -26.4815 & 21.4732 \end{bmatrix}$$

is not an M_V-matrix because the right eigenvector $x = [0.8694 \quad -0.0016 \quad 0.4941]^T$ corresponding to the zero eigenvalue has a negative entry. Let $\lambda = 100$ and $C = A + 100I$, then

$$(C/C_{11}) = \begin{bmatrix} 110.6380 & 11.1236 & -18.0941 \\ -11.8976 & 112.5298 & 20.1131 \\ -11.2511 & -26.8966 & 120.8564 \end{bmatrix}$$

is an M_V-matrix with right and left eigenvectors $x = [0.8654 \;\; 0.0186 \;\; 0.5007]^T$ and $y = [0.8382 \;\; 0.2754 \; 0.4708]^T$, respectively, corresponding to the smallest eigenvalue 100.4089. Thus, Theorem 3.35 is confirmed.

The following examples show that for any M_V-matrix with block A_{22} being an M_V-matrix, Theorem 3.35 is valid.

Example 3.37. *From Example 3.31, we observe that A_{22} is not an M_V-matrix. For the "big" shift $\lambda = 52.5$, $C = A + 52.5I$, the Schur complement*

$$(C/C_{11}) = \begin{bmatrix} 91.7297 & -35.0293 & -18.1500 \\ -25.7583 & 101.6017 & 13.9766 \\ -8.2614 & 11.4079 & 120.2214 \end{bmatrix}$$

is not an M_V-matrix because the right eigenvector $x = [0.8083 \;\; 0.5888 \; - 0.0007]^T$ corresponding to the smallest eigenvalue 66.2269 has a negative entry. Its inverse

$$(C/C_{11})^{-1} = \begin{bmatrix} 0.0122 & 0.0040 & 0.0014 \\ 0.0030 & 0.0110 & -0.0008 \\ 0.0006 & -0.0008 & 0.0085 \end{bmatrix}$$

is not an eventually positive matrix.

Since A_{22} is not an M_V-matrix, we have that the Schur complement (C/C_{11}) for a "big" shift is not an M_V-matrix which confirms Theorem 3.35.

Example 3.38. *From Example 3.33, we observe that A_{22} is an M_V-matrix. For $\lambda = 37.9$ and $C = A + 37.9I$, the Schur complement*

$$(C/C_{11}) = \begin{bmatrix} 858355.0253 & -1220878.7293 & -1267658.1359 \\ -541155.6807 & 769785.0368 & 799238.5574 \\ 516152.4326 & -734183.4193 & -762272.7132 \end{bmatrix}$$

has right and left eigenvectors $x = [0.8677 \;\; 0.1057 \;\; 0.4857]^T$ and $y = [0.2310 \;\; 0.8380 \;\; 0.4944]^T$, respectively, corresponding to the smallest eigenvalue 37.993 and it is an M_V-matrix because all of its eigenvalues are positive: $\sigma(C/C_{11}) = \{37.983, 41.9089, 865787.4469\}$, confirming Theorem 3.35. Its inverse

$$(C/C_{11})^{-1} = \begin{bmatrix} 0.0011 & 0.0370 & 0.0369 \\ 0.0150 & 0.0031 & -0.0218 \\ -0.0137 & 0.0221 & 0.0460 \end{bmatrix}$$

is an eventually positive matrix with power index 24, confirming Theorem 3.30.

Example 3.39. *From Example 3.34, we observe that A_{22} is an M_V-matrix. For a "big" shift $\lambda = 800$, $C = A + 800I$, the Schur complement*

$$(C/C_{11}) = \begin{bmatrix} 837.2024 & -3.0930 & -16.0423 & -5.0338 \\ -14.0724 & 844.0853 & 4.9276 & -12.0580 \\ -18.0241 & -5.0531 & 828.2205 & -9.0193 \\ -19.0906 & -17.1992 & -7.0906 & 840.1722 \end{bmatrix}$$

is also an M_V-matrix with right and left eigenvectors $x = [0.4600 \;\; 0.2562 \;\; 0.6632 \;\; 0.5319]^T$ and $y = [0.6871 \;\; 0.2960 \;\; 0.5583 \;\; 0.3586]^T$, respectively, corresponding to the smallest eigenvalue 806.5336. Its inverse

$$(C/C_{11})^{-1} = \begin{bmatrix} 0.001195202 & 0.000004671 & 0.000023187 & 0.000007477 \\ 0.000020169 & 0.001185097 & -0.000006514 & 0.000017059 \\ 0.000026436 & 0.000007598 & 0.001207988 & 0.000013235 \\ 0.000027794 & 0.000024430 & 0.000010588 & 0.001190863 \end{bmatrix}$$

is also an eventually positive matrix with power index 24.

We checked the associated Schur complements for intermediate shifts $10 < \lambda < 800$ and found that all of them are M_V-matrices and their inverses remain *eventually positive matrices. We claim that the Schur complements are M_V-matrices and their inverses remain eventually positive matrices for* **all** $\lambda > 0$. *In this example it seems that α of Theorem 3.30 is greater than β of Theorem 3.35. Thus, the union of the sets $(0, \alpha) \cup (\beta, \infty) = (0, \infty)$.*

3.5 Schur complement of irreducible GM-matrices

The study of the property of GM-matrices is also distinguished into two cases: the symmetric and the non-symmetric matrices.

3.5.1 Symmetric GM-matrices

Theorem 3.40. *Let $A \in \mathbb{R}^{n,n}$ be an irreducible symmetric singular GM-matrix. Then, the Schur complement (A/A_{11}) is a symmetric singular GM-matrix.*

Proof. Let $A = sI - B$ be an irreducible singular GM-matrix and $x \geq 0$ its right and left Perron eigenvectors, respectively. We partition A, x as
$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, respectively. Obviously, $x_1, x_2 \geq 0$.

We exclude the case $x_2 = 0$, since then A_{11} would be singular and the Schur complement would not be defined. In case $x_1 = 0$, from $Ax = 0$, we obtain that $A_{12}x_2 = 0$ and $A_{22}x_2 = 0$. Thus, A_{22} is a singular matrix with $x_2 \geq 0$ being its right eigenvector corresponding to zero. Then, for the Schur complement there holds $\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right) x_2 = A_{22}x_2 = 0$, where $x_2 \geq 0$ is its right eigenvector.

We consider now the case where $x_1 \geq 0, x_1 \neq 0$. The proof is analogous to the proof of Theorem 3.21 for M_V-matrices. Hence, it is omitted. □

In general, we cannot state perturbation properties for GM-matrices as we did for M_V-matrices. This is due to the fact that the eigenvector x_2 may have zero entries. In such a case a "small" perturbation of the zero entry may push the corresponding zero entry of x_2 to the negative direction as this is shown in the following example.

Example 3.41. *Consider the irreducible symmetric singular GM-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \left[\begin{array}{rr|rrr} 5.0302 & 0.5535 & -1.0971 & -0.8499 & 0.6939 \\ 0.5535 & 4.8857 & -1.9036 & -2.3750 & -1.0429 \\ \hline -1.0971 & -1.9036 & 5.2868 & -2.2632 & 0.3527 \\ -0.8499 & -2.3750 & -2.2632 & 3.6911 & 0.3442 \\ 0.6939 & -1.0429 & 0.3527 & 0.3442 & 6.6062 \end{array}\right]$$

with the eigenvector $[0.25 \;\; 0.75 \;\; 0.75 \;\; 1 \;\; 0]^T$ *corresponding to its zero eigenvalue. The Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 4.3888 & -3.2916 & 0.0904 \\ -3.2916 & 2.4687 & -0.0678 \\ 0.0904 & -0.0678 & 6.2508 \end{bmatrix}$$

is a symmetric singular GM-matrix with eigenvector $[0.75 \;\; 1 \;\; 0]^T$ *corresponding to its zero eigenvalue.*

Let $\lambda = 1$ *and* $C = A + I$*, then*

$$(C/C_{11}) = \begin{bmatrix} 5.5302 & -3.1275 & 0.1370 \\ -3.1275 & 3.6671 & 0.0064 \\ 0.1370 & 0.0064 & 7.3165 \end{bmatrix}$$

is not an M_{V}*- nor a GM-matrix. This is due to the fact that the eigenvector* $x = [0.5978 \;\; 0.8015 \;\; -0.0145]^T$ *corresponding to the smallest eigenvalue* 1.3341 *has a negative entry.*

However, we can derive perturbation properties, under a further assumption, namely for $x_2 > 0$.

Theorem 3.42. *Let* $A \in \mathbb{R}^{n,n}$ *be an irreducible symmetric singular GM-matrix, partitioned in the form (1.4) and* $x \geq 0$ *its Perron eigenvector partitioned as* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$*. If* $x_2 > 0$ *and the zero eigenvalue is simple, then there exists* $\alpha > 0$ *such that for the nonsingular GM-matrix* $C = A + \lambda I$, $0 < \lambda < \alpha$, *there holds: The Schur complement* (C/C_{11}) *is a symmetric nonsingular GM-matrix (and also an* M_{V}*-matrix).*

Proof. It is analogous to the proof of Theorem 3.22. We obtained that the Schur complement is an M_{V}-matrix, despite the fact that we started with a

GM-matrix. This is due to the fact that $x_2 > 0$ and that the zero eigenvalue is simple. □

The following example shows the validity of Theorem 3.42.

Example 3.43. *Let the GM-matrix of Example 3.41, where using a permutation matrix P, we interchange the first and the last rows and columns. Let $A^* = PAP^T$ be the permuted matrix written as*

$$A^* = \begin{bmatrix} A^*_{11} & A^*_{12} \\ A^*_{21} & A^*_{22} \end{bmatrix} = \left[\begin{array}{rr|rrr} 6.6062 & -1.0429 & 0.3527 & 0.3442 & 0.6939 \\ -1.0429 & 4.8857 & -1.9036 & -2.3750 & 0.5535 \\ \hline 0.3527 & -1.9036 & 5.2868 & -2.2632 & -1.0971 \\ 0.3442 & -2.3750 & -2.2632 & 3.6911 & -0.8499 \\ 0.6939 & 0.5535 & -1.0971 & -0.8499 & 5.0302 \end{array}\right]$$

with the eigenvector $x = [0 \quad 0.75 \quad 0.75 \quad 1 \quad 0.25]^T$ corresponding to its zero eigenvalue and $x_2 = [0.75 \quad 1 \quad 0.25]^T > 0$. The Schur complement

$$(A^*/A^*_{11}) = \begin{bmatrix} 4.5447 & -3.1899 & -0.8746 \\ -3.1899 & 2.5324 & -0.5601 \\ -0.8746 & -0.5601 & 4.8642 \end{bmatrix}$$

is a symmetric singular M_V-matrix with eigenvector $[0.75 \quad 1 \quad 0.25]^T$ corresponding to its zero eigenvalue.

Let $\lambda = 1$ and $C = A^* + I$, then

$$(C/C_{11}) = \begin{bmatrix} 5.6711 & -3.0311 & -0.9197 \\ -3.0311 & 3.7320 & -0.6184 \\ -0.9197 & -0.6184 & 5.8937 \end{bmatrix}$$

is an M_V-matrix with eigenvector $x = [0.5852 \quad 0.7801 \quad 0.2213]^T$ corresponding to the smallest eigenvalue 1.2827.

Remark 3.44. *Under the same assumptions as those of Theorem 3.42, a more general property could be stated if we assume a positive definite matrix perturbation E, instead of λI, as in Theorem 3.23.*

3.5.2 Non-symmetric GM-matrices

Unlike the symmetric case, the Schur complement of an irreducible non-symmetric singular GM-matrix is **not** always a singular GM-matrix.

Theorem 3.45. *Let $A \in \mathbb{R}^{n,n}$ be partitioned in the form (1.4) and be irreducible non-symmetric singular GM-matrix. Then, the Schur complement (A/A_{11}) is a non-symmetric singular matrix, where to the zero eigenvalue there correspond right and left nonnegative eigenvectors.*

Proof. A is written as $sI - B$, where B and B^T possess the Perron-Frobenius property. Thus, there exist $x \geq 0, y \geq 0, x \neq 0, y \neq 0$ such that $Ax = 0 \Leftrightarrow Bx = sx$ and $y^T A = 0 \Leftrightarrow y^T B = sy^T$. The cases $x_2 = 0$ or $y_2 = 0$ are excluded for otherwise A_{11} would be singular. Following the proof of Theorem 3.28, we may obtain that x_2 and y_2 are nonnegative right and left eigenvectors of (A/A_{11}), respectively, corresponding to the zero eigenvalue. □

Remark 3.46. *We cannot conclude anything general as regards the multiplicity of the zero eigenvalue, since for the GM-matrix A, the multiplicity of the zero eigenvalue may be greater than one. It seems that what we can prove, provided that* 0 *is a simple eigenvalue of A, is that the geometric multiplicity of* 0 *in the Schur complement is one. The algebraic multiplicity of* 0 *may be greater than one because the right and left Perron eigenvectors x_2 and y_2 may be orthogonal. This issue is made clear in the following example.*

Example 3.47. *Consider the non-symmetric singular GM-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{r|rr} 1 & -1.5 & -1 \\ \hline -1 & 1.5 & 1 \\ -0.5 & -1.25 & 0.5 \end{array}\right]$$

with right and left eigenvectors $x = [1 \;\; 0 \;\; 1]^T$ and $y = [1 \;\; 1 \;\; 0]^T$, respectively, corresponding to its simple zero eigenvalue. The Schur complement

$$(A/A_{11}) = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}$$

is a GM-matrix with right and left eigenvectors $x = [0 \;\; 1]^T$ and $y = [1 \;\; 0]^T$, respectively, corresponding to its zero eigenvalue. The generalized right and left eigenvectors are $x = [-0.5 \;\; 1]^T$ and $y = [1 \; -0.5]^T$, respectively, corresponding to zero. We observe that the algebraic multiplicity of 0 *is* 2 *but the geometric one is* 1 *because $x_2 = [0 \;\; 1]^T$ and $y_2 = [1 \;\; 0]^T$ are orthogonal.*

The following example shows that the Schur complement (A/A_{11}) is **not** a GM-matrix, since it has a negative eigenvalue, but its Perron eigenvector, corresponding to its zero eigenvalue, is nonnegative. Furthermore, it is shown that using another partitioning of the same matrix the Schur complement is a GM-matrix.

Example 3.48. *Consider the non-symmetric singular GM-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rrrr|rrr} 8 & -1 & -6 & 6 & 1 & 15 & -18 \\ 2 & 5 & -4 & 1 & -2 & 5 & -6 \\ 2 & 1 & -7 & 7 & -1 & 14 & -16 \\ -2 & -1 & -4 & 6 & -1 & 3 & -3 \\ \hline -3 & -1 & 14 & -13 & 0 & -22 & 22 \\ 5 & 2 & -24 & 19 & 3 & 37 & -34 \\ 2 & 1 & -16 & 13 & 2 & 22 & -17 \end{array}\right]$$

with right and left eigenvectors $x = [0 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 0]^T$ *and* $y = [0.1443 \;\; 0.097 \;\; 0.3673 \;\; 0.3524 \;\; 0.8136 \;\; 0.1464 \;\; 0.1651]^T$, *respectively, corresponding to its zero eigenvalue. Then, the Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 0 & 30.9268 & -37.8902 \\ 0 & -95.3171 & 115.4756 \\ 0 & -67.8780 & 84.3171 \end{bmatrix}$$

is not an M_{V}*- nor a GM-matrix because one of its eigenvalues is negative* : $\sigma(A/A_{11}) = \{0, 9.6281, -20.6281\}$. *However, its right and left eigenvectors* $x = [1 \;\; 0 \;\; 0]^T$ *and* $y = [0.9651 \;\; 0.1737 \;\; 0.1958]^T$, *respectively, corresponding to its zero eigenvalue, are nonnegative.*

If the matrix A is partitioned as is shown below

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rr|rrrrr} 8 & -1 & -6 & 6 & 1 & 15 & -18 \\ 2 & 5 & -4 & 1 & -2 & 5 & -6 \\ \hline 2 & 1 & -7 & 7 & -1 & 14 & -16 \\ -2 & -1 & -4 & 6 & -1 & 3 & -3 \\ -3 & -1 & 14 & -13 & 0 & -22 & 22 \\ 5 & 2 & -24 & 19 & 3 & 37 & -34 \\ 2 & 1 & -16 & 13 & 2 & 22 & -17 \end{array}\right],$$

then the Schur complement

$$(A/A_{11}) = \begin{bmatrix} -4.9048 & 5.619 & -0.7143 & 9.9524 & -11.1429 \\ -6.0952 & 7.381 & -1.2857 & 7.0476 & -7.8571 \\ 11.0952 & -10.881 & -0.2143 & -16.0476 & 14.8571 \\ -19 & 15.5 & 3.5 & 27 & -22 \\ -13.9048 & 11.619 & 2.2857 & 17.9524 & -12.1429 \end{bmatrix}$$

is a GM-matrix with right and left eigenvectors $x = [1 \quad 1 \quad 1 \quad 0 \quad 0]^T$ *and* $y = [0.373 \quad 0.3579 \quad 0.8262 \quad 0.1487 \quad 0.1676]^T$*, respectively, corresponding to its zero eigenvalue.*

We state the following theorem analogous to Theorem 3.30 for M_V-matrices.

Theorem 3.49. *Let* $A \in \mathbb{R}^{n,n}$*,* $A = sI - B$*, be an irreducible non-symmetric singular GM-matrix partitioned in the form (1.4) and* $x \geq 0, y \geq 0$ *with its right and left Perron eigenvectors, respectively, being partitioned as* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$*. If* $x_2 > 0, y_2 > 0$ *and the zero eigenvalue is simple, then there exists an* $\alpha > 0$ *such that for the nonsingular matrix* $C = A + \lambda I$*,* $0 < \lambda < \alpha$*, there holds: The Schur complement* (C/C_{11}) *has an eventually positive inverse.*

Proof. Let A be a GM-matrix with its right and left Perron eigenvectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq 0$, respectively, corresponding to the simple zero eigenvalue and $x_2 > 0, y_2 > 0$. Suppose $\lambda > 0$ is a "small" real number. Obviously, $C = A + \lambda I$ is also a non-singular GM-matrix. Following the proof of Theorem 3.28, we obtain that $x_2 > 0$ and $y_2 > 0$ are also the right and left Perron eigenvectors, respectively, corresponding to the zero eigenvalue of (A/A_{11}). Using the perturbation argument of Theorem 3.30, the smallest in modulus eigenvalue $\tilde{\lambda}$ of (C/C_{11}) is positive and therefore (C/C_{11}) has an eventually positive inverse. $\square$

We show the validity of the above theorem by the following example.

Example 3.50. *Consider the irreducible singular GM-matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{rr|rr} 10.53 & 44.07 & 3.07 & -23.47 \\ -0.67 & -6.83 & -1.33 & 4.33 \\ \hline -3.8 & -6.1 & 1.4 & 2.2 \\ -1.6 & -18.2 & -3.2 & 10.4 \end{array}\right]$$

with right and left eigenvectors $x = [0.4867 \quad 0 \quad 0.8111 \quad 0.3244]^T$ *and* $y = [0.3164 \quad 0.6328 \quad 0.6317 \quad 0.3169]^T$*, respectively, corresponding to its zero eigenvalue. Then, the Schur complement*

$$(A/A_{11}) = \begin{bmatrix} -0.2552 & 0.6380 \\ 0.5087 & -1.2717 \end{bmatrix}$$

is not a singular GM-matrix since it has a negative eigenvalue, but the right and left eigenvectors $x = [0.8111 \;\; 0.3244]^T$ *and* $y = [0.6317 \;\; 0.3169]^T$*, respectively, corresponding to its zero eigenvalue are positive.*

Let $\lambda = 1$ *and* $C = A + I$*, then*

$$(C/C_{11}) = \begin{bmatrix} 0.4442 & 2.1086 \\ 1.4830 & -2.8215 \end{bmatrix}$$

is not a GM-matrix because one eigenvalue is negative: $\sigma(C/C_{11}) = \{1.2182, -3.5956\}$*. However,* $x = [0.9387 \;\; 0.3446]^T$ *and* $y = [0.8865 \;\; 0.4627]^T$ *are positive right and left eigenvectors, respectively, corresponding to the eigenvalue* 1.2182*.*

Its inverse matrix

$$(C/C_{11})^{-1} = \begin{bmatrix} 0.6442 & 0.4814 \\ 0.3386 & -0.1014 \end{bmatrix}$$

is an eventually positive matrix with power index 2 *because* $x_2 = [0.8111 \;\; 0.3244]^T$ *and* $y_2 = [0.6317 \;\; 0.3169]^T$ *of* A *are positive and the zero eigenvalue is simple.*

3.6 Schur complement of reducible M_V-matrices and GM-matrices

In this section, we will study the Schur complement of reducible M_V- and GM-matrices. First, we present and analyse the reducible ones partitioned into following cases, where the Frobenius normal form is considered to be a block 2×2 matrix.

Trivial case: Let $A = sI - B$ be a reducible M_V- or GM-matrix and A be partitioned in the block form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is a nonsingular principal submatrix of A. Then, the Schur complement of A_{11} in A is $(A/A_{11}) = A_{22} - 0 \; {A_{11}}^{-1} A_{12} = A_{22}$. It is easily proved that, since A is an M_V-matrix so is A_{22}. Thus, the M_V-property of A is inherited by its Schur complement. If A_{22} is a GM-matrix, then it is trivially seen that its Schur complement is also a GM-matrix.

Now, we will consider three cases of non-trivial partitionings of reducible M_V- and GM-matrices.

Case 1: Let $A = sI - B$ be an M_V-matrix (resp. GM-matrix) partitioned in block form as follows

$$A = \left[\begin{array}{c|cc} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{array}\right].$$

Then, the Schur complement of A_{11} in A is

$$\begin{aligned}(A/A_{11}) &= \begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix} - \begin{bmatrix} A_{21} \\ 0 \end{bmatrix} A_{11}^{-1} \begin{bmatrix} A_{12} & A_{13} \end{bmatrix} \\ &= \begin{bmatrix} A_{22} - A_{21}A_{11}^{-1}A_{12} & A_{23} - A_{21}A_{11}^{-1}A_{13} \\ 0 & A_{33} \end{bmatrix}.\end{aligned}$$

Thus, the Schur complement (A/A_{11}) is an M_V-matrix (resp. GM-matrix) if the Schur complement $\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} /A_{11}\right)$ and A_{33} are M_V-matrices (resp. GM-matrices). Furthermore, if certain conditions on the Schur complement-like matrix $A_{23} - A_{21}A_{11}^{-1}A_{13} = \left(\begin{bmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{bmatrix} /A_{11}\right)$ hold then the whole matrix (A/A_{11}) is an M_V-matrix (resp. GM-matrix).

Example 3.51. *Consider the reducible M_V-matrix*

$$A = \left[\begin{array}{c|cc} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{array}\right] = \left[\begin{array}{rr|r|rr} 4 & -4 & -5 & 1 & -4 \\ -3 & 10 & 5 & -3 & -4 \\ \hline -1 & -6 & 9 & -6 & 0 \\ \hline 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & -1 & 8 \end{array}\right]$$

with right and left eigenvectors $[0.9445\ \ 0.1761\ \ 0.2774\ \ 0\ \ 0]^T$ *and* $[0.2076\ \ 0.1369\ \ 0.0490\ \ 0.5756\ \ 0.7775]^T$*, respectively, corresponding to the smallest eigenvalue* 1.7858*.*

Then, the Schur complement of A_{11} in A

$$(A/A_{11}) = \begin{bmatrix} \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} /A_{11}\right) & A_{23} - A_{21}A_{11}^{-1}A_{13} \\ 0 & A_{33} \end{bmatrix} = \left[\begin{array}{r|rr} 9 & -8 & -8 \\ \hline 0 & 4 & -6 \\ 0 & -1 & 8 \end{array}\right]$$

is a reducible M_{V}-matrix with right and left eigenvectors $[0.8357\ \ 0.5392\ \ 0.1045]^T$ *and* $[0\ \ 0.6522\ \ 0.758]^T$, *respectively, corresponding to the smallest eigenvalue* 2.8377.

Case 2: Let $A = sI - B$ be written in block form as follows

$$A = \left[\begin{array}{cc|c} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ \hline 0 & A_{32} & A_{33} \end{array}\right].$$

Then, the Schur complement of $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ in A is

$$\begin{aligned}
\left(A/\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\right) &= A_{33} - \begin{bmatrix} 0 & A_{32} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} \\
&= A_{33} - \begin{bmatrix} 0 & A_{32} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & A_{12}' \\ 0 & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} \\
&= A_{33} - A_{32}A_{22}^{-1}A_{23} = \left(\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} / A_{22}\right),
\end{aligned}$$

which is the Schur complement of A_{22} in $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$. The last matrix is irreducible. Thus, the problem is reduced to studying the Schur complement of the irreducible M_{V}-matrix (resp. GM-matrix) above.

Example 3.52. *Consider the reducible M_{V}-matrix*

$$A = \left[\begin{array}{cc|c} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ \hline 0 & A_{32} & A_{33} \end{array}\right] = \left[\begin{array}{rr|r|rr} 12 & -3 & 4 & -3 & -7 \\ -5 & 6 & 5 & -6 & -4 \\ \hline 0 & 0 & 16 & -4 & 2 \\ \hline 0 & 0 & -3 & 5 & -4 \\ 0 & 0 & 0 & -5 & 8 \end{array}\right]$$

with right and left eigenvectors $[0.4116\ \ 0.8637\ \ 0.0390\ \ 0.2293\ \ 0.1748]^T$ *and* $[0\ \ 0\ \ 0.1779\ \ 0.8633\ \ 0.4722]^T$, *respectively, corresponding to the smallest eigenvalue* 1.4408.

Then, the Schur complement of $A_{11}' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ *in A is the Schur Complement of* $A_{22} = [16]$ *in* $A' = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} = \left[\begin{array}{r|rr} 16 & -4 & 2 \\ \hline -3 & 5 & -4 \\ 0 & -5 & 8 \end{array}\right]$:

$(A/A'_{11}) = (A'/A_{22}) = \begin{bmatrix} 4.25 & -3.625 \\ -5 & 8 \end{bmatrix}$ *is an* M_V*-matrix with right and left eigenvectors* $[0.7938 \quad 0.6081]^T$ *and* $[0.8742 \quad 0.4855]^T$*, respectively, corresponding to the smallest eigenvalue* 1.4731*.*

Case 3: Let $A = sI - B$ be written in the block partitioned form below

$$A = \left[\begin{array}{cc|cc} A_{11} & A_{13} & A_{12} & A_{14} \\ 0 & A_{33} & 0 & A_{34} \\ \hline A_{21} & A_{23} & A_{22} & A_{24} \\ 0 & A_{43} & 0 & A_{44} \end{array}\right] = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}.$$

The matrix A is reducible and its reducible form is given by block permutation transformation of A, which is $\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}$.

Then, the Schur complement of A'_{11} in A is

$$(A/A'_{11}) = A'_{22} - A'_{21}{A'_{11}}^{-1}A'_{12}$$

$$= \begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix} - \begin{bmatrix} A_{21} & A_{23} \\ 0 & A_{43} \end{bmatrix} \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}^{-1} \begin{bmatrix} A_{12} & A_{14} \\ 0 & A_{34} \end{bmatrix}$$

$$= \begin{bmatrix} A_{22} - A_{21}A_{11}^{-1}A_{12} & A_{24} - A_{21}A_{11}^{-1}A_{14} - (A_{23} - A_{21}A_{11}^{-1}A_{13})A_{33}^{-1}A_{34} \\ 0 & A_{44} - A_{43}A_{33}^{-1}A_{34} \end{bmatrix}.$$

The problem will be considered as the Schur complement of the irreducible forms $\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} / A_{11}\right)$ and $\left(\begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix} / A_{33}\right)$. Furthermore, the upper right block term must be such that the whole Schur complement (A/A'_{11}) is an M_V-matrix (resp. GM-matrix).

Example 3.53. *Consider the reducible* M_V*-matrix*

$$A = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix} = \left[\begin{array}{cccc|cccc} 10 & -3 & -5 & -1 & -4 & 0 & 1 & -6 \\ 0 & 11 & -3 & -7 & 2 & -4 & -1 & 2 \\ 0 & 0 & 11 & -6 & 0 & 0 & -4 & -4 \\ 0 & 0 & 2 & 13 & 0 & 0 & 0 & -7 \\ \hline -6 & -2 & 5 & -4 & 9 & 1 & 4 & -2 \\ -4 & -4 & -6 & 4 & -4 & 17 & -3 & -3 \\ 0 & 0 & -5 & -7 & 0 & 0 & 10 & 0 \\ 0 & 0 & -3 & -1 & 0 & 0 & -6 & 10 \end{array}\right]$$

with right and left eigenvectors $[0.5792\ \ 0.2923\ \ 0.3118\ \ 0.1272\ \ 0.2583\ \ 0.4796\ \ 0.2754\ \ 0.3052]^T$ *and* $[0\ \ 0\ \ 0.3345\ \ 0.5303\ \ 0\ \ 0\ \ 0.5334\ \ 0.5678]^T$*, respectively, corresponding to the smallest eigenvalue* 1.1051*.*

The problem will be considered as the Schur complement of the irreducible forms $\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}/A_{11}\right) = \begin{bmatrix} 7.2909 & -0.3818 \\ -4.6545 & 15.1091 \end{bmatrix}$ *which is an* M_{V}*-matrix with right and left eigenvectors* $[0.8654\ \ 0.5011]^T$ *and* $[0.9989\ \ 0.0474]^T$*, respectively, corresponding to the smallest eigenvalue* 7.0698 *and of the form* $\left(\begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix}/A_{33}\right) = \begin{bmatrix} 8.6839 & -6.1484 \\ -6.9548 & 7.7355 \end{bmatrix}$ *which is an* M_{V}*-matrix with right and left eigenvectors* $[0.6583\ \ 0.7528]^T$ *and* $[0.7033\ \ 0.7109]^T$*, respectively, corresponding to the smallest eigenvalue* 1.6533*. Then, the Schur complement*

$$(A/A'_{11}) = \begin{bmatrix} 7.2909 & -0.3818 & 4.9401 & -7.4489 \\ -4.6545 & 15.1091 & -6.2474 & -10.0367 \\ 0 & 0 & 8.6839 & -6.1484 \\ 0 & 0 & -6.9548 & 7.7355 \end{bmatrix}$$

is an M_{V}*-matrix with right and left eigenvectors* $[0.3209\ \ 0.6814\ \ 0.4330\ \ 0.4952]^T$ *and* $[0\ \ 0\ \ 0.7033\ \ 0.7109]^T$*, respectively, corresponding to the smallest eigenvalue* 1.6533*.*

From the analysis presented in previous sections concerning irreducible matrices, we can give general theorems for the Schur complement of reducible M_{V}- and GM-matrices. We begin by presenting the result for the symmetric case.

Theorem 3.54. *Let* $A \in \mathbb{R}^{n,n}$*,* $A = sI - B$*, be a reducible symmetric singular* M_{V}*-matrix. Then, the Schur complement* (A/A_{11}) *is a symmetric singular* GM*-matrix.*

Proof. Since the matrix A is symmetric, the Frobenius normal form of A is a direct sum : $A' = P^TAP = \mathrm{diag}(A'_{11}, A'_{22}, \ldots, A'_{rr})$ with all diagonal blocks of A' being irreducible M_{V}-matrices, and at least one of them is a singular matrix. It is easily seen that the eigenvector of the zero eigenvalue can have positive entries at the places corresponding to the singular diagonal blocks and zeros at the places corresponding to the nonsingular ones. If all the blocks are singular matrices, then there exists a positive eigenvector. Following the proof of Theorem 3.21, we obtain that (A/A_{11}) is a symmetric singular M_{V}-matrix. In the case where exist nonsingular diagonal blocks, the parts x_1, x_2 of the eigenvector $x = [x_1^T \ x_2^T]^T$ are nonnegative vectors. Then, we can follow exactly the proof of Theorem 3.40 concerning GM-matrices to get our result. □

Remark 3.55. *As in the case where $x > 0$, if $x_2 > 0$, the obtained result of the theorem above is that the Schur complement is an M_V-matrix. In this case, some perturbation properties can be stated here as Theorem 3.42 concerning GM-matrices.*

For GM-matrices the proof of theorem above holds true except the case where $x > 0$. Thus, this result holds true also for GM-matrices. This is just Theorem 3.40 supposing that A is reducible instead of irreducible. Also, Theorem 3.42 can be stated in case $x_2 > 0$.

Let A be a reducible non-symmetric singular M_V-matrix, then its Frobenius normal form is

$$A' = P^T A P = \begin{bmatrix} A'_{11} & A'_{12} & \cdots & A'_{1r} \\ & A'_{22} & \cdots & A'_{2r} \\ & & \ddots & \vdots \\ & & & A'_{rr} \end{bmatrix}, \tag{3.8}$$

where all diagonal blocks A'_{jj} are irreducible M_V-matrices with at least one of them being singular. If m of the diagonal blocks are singular, then the algebraic multiplicity is at least m. Consider now the associated eventually nonnegative matrix B', where $A' = sI - B'$. If there exists $\beta > 0$ such that $B_{jj} + \beta I$, $j = 1, 2, \dots, r$, are also eventually nonnegative matrices, then the algebraic multiplicity is exactly m. This is obtained by [11, Theorem 3.5]: each singular diagonal block has a simple zero eigenvalue. The geometric multiplicity depends also on the structure of the Frobenius normal form. It is known from the Perron-Frobenius theory that it depends on how the singular diagonal blocks are connected to each other and to the nonsingular ones, via the directed graph of the block matrix A' in (3.8). In any case, the geometric multiplicity is at least one. Thus, there exist nonnegative right and left eigenvectors $x \geq 0, y \geq 0$, respectively.

Now we can state the main theorem, concerning reducible non-symmetric singular M_V-matrix analogous to Theorem 3.28.

Theorem 3.56. *Let $A \in \mathbb{R}^{n,n}$ partitioned in the form (1.4), be a reducible non-symmetric singular M_V-matrix. Then, the Schur complement (A/A_{11}) is a non-symmetric singular matrix and to zero eigenvalue there exist at least one right and left eigenvectors, $x_2 \geq 0, y_2 \geq 0$, respectively.*

Proof. The proof is given according the lines of the proof of Theorem 3.28. It is not needed to check the uniqueness of the zero eigenvalue, since it may be a multiple one. The case where $x_2 = 0$ or $y_2 = 0$ is excluded, since otherwise A_{11} would be singular. □

Remark 3.57. *In the case where $x_2 > 0$ and $y_2 > 0$, the analogous perturbation properties can be stated here as in Theorem 3.30, without of the uniqueness of the zero eigenvalue.*

For reducible GM-matrices, taken all the consideration above, we can see that the theorem above holds true, as it is stated.

We will show the validity of the theorems above by considering $A = A'$ in its Frobenius normal form as in Case 1.

$$A' = \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix} = A = \left[\begin{array}{c|cc} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{array}\right]. \tag{3.9}$$

Then, the Schur complement

$$\begin{aligned}(A/A_{11}) &= \begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix} - \begin{bmatrix} A_{21} \\ 0 \end{bmatrix} A_{11}^{-1} \begin{bmatrix} A_{12} & A_{13} \end{bmatrix} \\ &= \begin{bmatrix} A_{22} - A_{21}A_{11}^{-1}A_{12} & A_{23} - A_{21}A_{11}^{-1}A_{13} \\ 0 & A_{33} \end{bmatrix}. \end{aligned} \tag{3.10}$$

The first diagonal block of (A/A_{11}) is the Schur complement $\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} / A_{11}\right) = (A'_{11}/A_{11})$.

Let that A is a singular M_V-matrix. First, suppose that both blocks A'_{11} and A'_{22} are singular matrices: Then, if $A'_{12} = 0$, there exists a positive right and left Perron eigenvector of $A : x = [x_1^T \;\; x_2^T \;\; x_3^T]^T$ and $y = [y_1^T \;\; y_2^T \;\; y_3^T]^T$ partitioned as A in (3.9). Hence, the associated right and left Perron eigenvector of the Schur complement are the positive ones $[x_2^T \;\; x_3^T]^T$ and $[y_2^T \;\; y_3^T]^T$, respectively, thus the Schur complement has positive right and left Perron vectors. If $A'_{12} \neq 0$, then the right Perron vector is $x = [x_1^T \;\; x_2^T \;\; 0^T]^T$ while the left one $[0^T \;\; 0^T \;\; y_3^T]^T$. Then, the Schur complement has nonnegative right and left Perron vectors $[x_2^T \;\; 0^T]^T$ and $[0^T \;\; y_3^T]^T$.

Second, suppose that A'_{11} is a singular matrix and A'_{22} a nonsingular one: Then, the right and left Perron vectors of A are $x = [x_1^T \;\; x_2^T \;\; 0^T]^T$ and

$y = [y_1^T \;\; y_2^T \;\; y_3^T]^T$, respectively. (The left one is $y = [y_1^T \;\; y_2^T \;\; 0^T]^T \geq 0$ if $A'_{12} = 0$.) The Schur complement has as associated eigenvectors $[x_2^T \;\; 0^T]^T \geq 0$ and $[y_2^T \;\; y_3^T]^T > 0$. Thus, the Schur complement has nonnegative the right and positive the left eigenvectors.

Finally, suppose that A'_{11} is a nonsingular matrix and $A'_{22} = A_{33}$ is a singular one. Then, the right and left Perron eigenvectors of A are $x = [x_1^T \;\; x_2^T \;\; x_3^T]^T > 0$ and $y = [0^T \;\; 0^T \;\; y_3^T]^T \geq 0$, respectively. Thus, the Schur complement has the associated Perron eigenvectors $[x_2^T \;\; x_3^T]^T > 0$ and $[0^T \;\; y_3^T]^T \geq 0$. Thus, Theorem 3.56 is confirmed.

Analogous results are obtained under the assumption that A is a GM-matrix.

3.7 Summary

- The Schur complement of an inverse M_V-matrix (resp. inverse GM-matrix) preserves the inverse M_V-property (resp. inverse GM-property) *iff* the block C_{22} of the inverse of A, $C = A^{-1}$, is an M_V-matrix (resp. GM-matrix).

- The Schur complement of a non-singular block 2-cyclic M_V-matrix (resp. GM-matrix) is also an M_V-matrix (resp. GM-matrix). The same is true for a $2k$-*cyclic* matrix, with the same properties, provided the off-diagonal blocks are in the *cyclic of index* $2k$ form (1.3). Then, a block *red-black ordering* permutation transformation makes A be *block* 2-*cyclic*.

- The Schur complement of a non-singular block k-cyclic M_V-matrix (resp. GM-matrix) is also an M_V-matrix (resp. GM-matrix).

- The Schur complement of an irreducible symmetric singular M_V-matrix (resp. GM-matrix) is also a symmetric singular M_V-matrix (resp. GM-matrix). The Schur complement of a perturbed symmetric M_V-matrix is a symmetric nonsingular M_V-matrix if the shift is "small".

- The Schur complement of an irreducible non-symmetric singular M_V-matrix may **not** be a singular M_V-matrix but it has a simple zero eigenvalue corresponding to a positive eigenvector. For a perturbed M_V-matrix, the Schur complement has a simple positive, absolutely smallest, eigenvalue associated with a positive eigenvector; also, it has an eventually positive inverse if the shift is "small".

- The Schur complement of a perturbed matrix with the block A_{22} being an M_{V}-matrix is also an M_{V}-matrix, if the shift is very "big".

- The perturbation properties for the Schur complement are preserved for symmetric GM-matrices only under the assumption that the lower part x_2 of the partitioned eigenvector x is positive and the zero eigenvalue is simple.

- The Schur complement of an irreducible non-symmetric singular GM-matrix may **not** be a singular GM-matrix but it has (not necessarily a simple) zero eigenvalue corresponding to right and left nonnegative eigenvectors.

- The Schur complement of a reducible symmetric singular M_{V}-matrix or GM-matrix is a symmetric singular GM-matrix. If $x_2 > 0$ then it is a singular M_{V}-matrix and the Schur complement of the perturbed nonsingular matrix $A + \lambda I$ is a nonsingular M_{V}-matrix, for sufficiently "small" λ.

- The Schur complement of a reducible non-symmetric M_{V}-matrix or GM-matrix, is a non-symmetric singular matrix and to the zero eigenvalue there exist right and left Perron vectors $x_2 \geq 0, y_2 \geq 0$. If $x_2 > 0$ and $y_2 > 0$, the Schur complement of the perturbed matrix has positive absolutely smallest eigenvalue associated with positive eigenvectors. Also, it has an eventually positive inverse if the shift is "small".

CHAPTER 4

Applications of Perron-Frobenius theory in other fields

In the book of Berman and Plemmons [6, Chapter 11], some applications of Perron-Frobenius theory are presented. Nowadays, many researchers study and apply the well-known Perron-Frobenius theory in many fields. For example, in the fields of Algebra and specifically in Linear Algebra (e.g., eventual positivity of matrices and matrix semigroups [33, 54]). Also, in the fields of Network theory (e.g., [2, 3, 26]), Wireless network optimization (e.g., [51, 59, 56, 58]), Biology (e.g., [4, 9, 16, 22, 27, 28, 35]), Economy (e.g., [7, 15, 49]), etc. In this chapter, first, we give definitions related to Graph theory and stochastic process, that are used in this chapter. Then, we present the results of many research works done in the last seven years and show how the Perron-Frobenius theory and its extensions are applied.

We begin with the basic definitions of Graph theory.

Definition 4.1. A graph G can be written in the form $G = (V, E)$, where

- V is the *vertex set* of the graph G with v_i denoting the *node (or vertex)* i of G;
- $E \subseteq V \times V$ is the *edge set* of the graph G with $e_{ij} = (v_i, v_j)$ denoting the *edge* connecting the nodes i and j of G;

If there exist nodes $v_i, v_{i_1}, v_{i_2}, \ldots, v_{i_{r-1}}, v_j$ such that $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots,$ $(v_{i_{r-1}}, v_j) \in E$, we say that the pair (v_i, v_j) is a *path* of order r.

Definition 4.2. The graph $G = (V, E)$ is called

- *digraph (or directed graph)*, if for every edge, there is an ordered pair of two connecting nodes.
- *signed digraph*, if it is a digraph with positive or negative sign on each edge.
- *weighted (signed) digraph*, if it is a digraph with a weight assigned on each edge.
- *strongly connected*, if for any $v_i, v_j \in V$, there exists the path from node v_i to v_j in the graph G.

Definition 4.3. The *adjacency matrix* $A = a_{ij} \in \mathbb{R}^{n,n}$ is defined by

$$a_{ij} = \begin{cases} 1, & \text{if there is a connection between nodes } i \text{ and } j \\ 0, & \text{if there is no connection between nodes } i \text{ and } j \end{cases}.$$

If the graph is weighted signed digraph, the entry a_{ij} of adjacency matrix is the weight on the edge from node i to j. If $a_{ii} = 0$ for all i, it means that the graph has no loop.

Definition 4.4. Let $A \in \mathbb{R}^{n,n}$ be an adjacency matrix, then

- $G(A)$ denotes a *digraph* of the matrix A;
- $\text{adj}(i)$ denotes the *set of vertices adjacent to* i, and $j \in \text{adj}(i)$ iff $a_{ij} \neq 0$;
- $S = \text{sgn}(A)$ denotes a *signature matrix (or signed matrix)* with entries in $\{0, +1, -1\}$;
- $Q(A)$ denotes the *qualitative class* of all matrices having the same sign pattern as A.

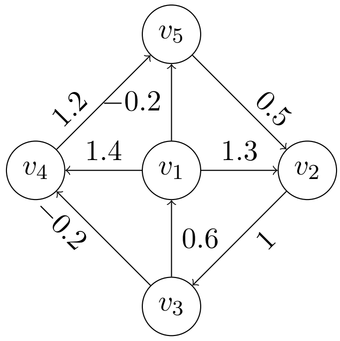
We can give an example to demonstrate the concept of Graph theory that will use in Network theory and Ecological networks.

Example 4.5. *Consider the adjacency matrix*

$$A = \begin{bmatrix} 0 & 1.3 & 0 & 1.4 & -0.2 \\ 0 & 0 & 1 & 0 & 0 \\ 0.6 & 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & 1.2 \\ 0 & 0.5 & 0 & 0 & 0 \end{bmatrix}.$$

The digraph $G(A)$ is a strongly connected weighted signed digraph, because it can be shown that for any pair (v_i, v_j) there exists a path from node v_i to node v_j. From the definition, it can be easily seen that $\text{adj}(1) = \{2, 4, 5\}$, $\text{adj}(2) = \{3\}$, $\text{adj}(3) = \{1, 4\}$, $\text{adj}(4) = \{5\}$ *and* $\text{adj}(5) = \{2\}$.

The corresponding graph $G(A)$ is shown as



The signature matrix S of $G(A)$ can be written as

$$\text{sgn}(A) = \begin{bmatrix} 0 & +1 & 0 & +1 & -1 \\ 0 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 \end{bmatrix}.$$

The qualitative class $Q(A)$ is the class of all matrices having the same sign pattern as A, e.g.,

$$A' = \begin{bmatrix} 0 & 3 & 0 & 0.4 & -0.5 \\ 0 & 0 & 1.2 & 0 & 0 \\ 0.3 & 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.7 \\ 0 & 0.3 & 0 & 0 & 0 \end{bmatrix} \text{ or } A'' = \begin{bmatrix} 0 & 2 & 0 & 3 & -1 \\ 0 & 0 & 1.5 & 0 & 0 \\ 0.1 & 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0.7 & 0 & 0 & 0 \end{bmatrix}.$$

We recall the definitions of stochastic process from [9] and [1].

Definition 4.6. Let $A \in \mathbb{R}^{n,n}$ be a non-negative matrix, then A is called

- *stochastic*, if $\sum_{j=1}^{n} a_{ij} = 1$ for all i;

- *sub-stochastic*, if $\sum_{j=1}^{n} a_{ij} \leq 1$ and the inequality is strict for at least one value of i.
- *eventually stochastic*, if $A \overset{\vee}{>} 0$ and $\sum_{j=1}^{n} a_{ij} = 1$ for all i;

Definition 4.7. Let A be an $n \times n$ matrix and $A_i = \sum_{k=1}^{n} a_{ik}$ denote the sum of elements in the i-th row of A. Then, A is called

- *stochastically ordered*, if $A_i \geq A_j$, for all $i > j$;
- *strictly stochastically ordered*, if $A_i > A_j$, for all $i > j$;
- *eventually stochastically ordered*, if there exists $k_0 \in \mathbb{Z}_+$ such that A^k is stochastically ordered for $k \geq k_0$;
- *eventually strictly stochastically ordered*, if there exists $k_0 \in \mathbb{Z}_+$ such that A^k is strictly stochastically ordered for $k \geq k_0$.

We can give examples to show the definitions above that will use in Population dynamics.

Example 4.8. *Consider the stochastic matrix*

$$A = \begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0.2 & 0.5 & 0 & 0.3 \\ 0 & 0.4 & 0.6 & 0 \\ 0.1 & 0.3 & 0 & 0.6 \end{bmatrix},$$

where all row sums are equal to one. Obviously, it is also an eventually positive matrix ($A^k > 0$, $\forall k \geq 3$). From the definition, this matrix is called eventually stochastic matrix.

Example 4.9. *Consider the sub-stochastic matrix*

$$A = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 \\ 0.2 & 0.4 & 0 & 0.1 \\ 0 & 0.4 & 0.4 & 0 \\ 0.1 & 0.3 & 0 & 0.6 \end{bmatrix}.$$

We can see that $A_1 = 0.6 < A_2 = 0.7 < A_3 = 0.8 < A_4 = 1$ and from the definition, it is a strictly stochastically ordered. The matrix A is also an *eventually strictly stochastically ordered because A^k is strictly stochastically ordered for $k \geq 3$, indeed,*

$$A^3 = \begin{bmatrix} 0.04 & 0.16 & 0.112 & 0.016 \\ 0.074 & 0.138 & 0.084 & 0.079 \\ 0.084 & 0.204 & 0.096 & 0.056 \\ 0.127 & 0.253 & 0.072 & 0.264 \end{bmatrix}$$

and $A_1^{(3)} = 0.328 < A_2^{(3)} = 0.375 < A_3^{(3)} = 0.440 < A_4^{(3)} = 0.716$.

4.1 Applications of Perron-Frobenius theory in Network Theory

Let $A \in \mathbb{R}^{n,n}$ be an adjacency matrix, where $a_{ii} = 0$, and let $G(A)$ be a strongly connected signed digraph of the matrix A. Assume that a distributed process of opinion forming has a mathematical model as

$$\dot{x}_i = -\sigma_i x_i + \sum_{j \in \mathrm{adj}(i)} a_{ij} x_j, \qquad i = 1, 2, \ldots, n, \tag{4.1}$$

where $\sigma_i > 0, i = 1, 2, \ldots, n$, are called the degradation terms of the interconnected systems and represent forgetting factors for the opinions. Assume $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, then (4.1) is written as

$$\dot{x}_i = -Ex, \; E = \Sigma - A. \tag{4.2}$$

Suppose $\sigma_i^{\mathrm{in}} = \sum\limits_{j=1}^{n} a_{ij}$ denotes the weight in-degree of node i in graph $G(A)$ and let $\Sigma^* = \mathrm{diag}(\sigma_1^{\mathrm{in}}, \ldots, \sigma_n^{\mathrm{in}})$, then Altafini [1] defined the Laplacian as

$$L = \Sigma^* - A. \tag{4.3}$$

Assume $\dot{x}_i = u_i, \;\; i = 1, 2, \ldots, n$, and $u_i = -\sum\limits_{j=1}^{n} a_{ij}(x_i - x_j)$, then (4.1) can be written as

$$\dot{x} = -Lx. \tag{4.4}$$

In 2014-2015, Altafini and Lini [2, 3] published their works in social network theory. The authors modeled the non-interactions of the opinion forming process as negative weights on the adjacency matrix of the social network. If the adjacency matrix A of the system (4.2) is eventually positive, the authors applied the Perron-Frobenius property from [42] to predict the pattern of unanimous opinions. Later in 2017, Shi, Altafini and Baras [50] combined generalized Perron-Frobenius theory, graph theory and elementary algebraic recursion to define an algebraic-graphical method for signed networks. Recently in 2019, Altafini [1] extended his former research and studied the Laplacian matrix of the form (4.3) which represents the negative weights on the adjacency matrix. If the adjacency matrix A of the system (4.3) is eventually positive, the author applied [43, Theorem 3.3] to prove the stability of signed Laplacian matrix L.

Another research related to network theory can be founded in [26]. In 2015, Gao, Liu and Baras studied the social network service and the problem of global trust evaluation. The authors defined a *trust network* as a directed weighted signed graph and formulated the problem as bipartite consensus. Then, they extended the result more generally, using the Perron-Frobenius property [42] in the concept of eventually positivity, and found some conditions for bipartite consensus on global trust.

4.2 Applications of Perron-Frobenius theory in Biology

4.2.1 Applications of Perron-Frobenius theory in Ecological network

Consider the nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \tag{4.5}$$

where $x(t) \in \mathbb{R}^n$ is a solution of the system at time t with initial condition $x(0)$. Assume that the system admits an asymptotically stable equilibrium point x^*, such that $f(x^*) = 0$. Let $x_i(t)$ of $x(t) = [x_1(t)\ x_2(t)\ \ldots\ x_n(t)]^T$ represents the population density of species i and $f(x(t)) = [f_1(x(t))\ f_2(x(t))\ \ldots\ f_n(x(t))]^T$ be the function of the species densities, where $f_i(x(t))$ is the corresponding overall growth rate of species i. The system (4.5) represents the evolution of an ecological system of an n-species community. The *community matrix* associated with the dynamical system (4.5) can be derived from the Jacobian matrix, evaluated at the equilibrium point x^*, and can be written as

$$J(x) = \frac{\partial f(x)}{\partial x}\Big|_{x=x^*}, \tag{4.6}$$

where J_{ij} is the direct effect of species j on the growth rate of species i and $sgn(J)$ represents the positive or negative direct influence, or no direct influence on each of the other species, if $sgn(J_{ij}) = +1,\ -1,\ 0$, respectively.

Let $\sigma = (\sigma_1, \dots, \sigma_n)$, where $\sigma_i \in \{\pm 1\}$ is a partial order for the axes of $\mathbb{R}^n$ and $\Sigma = \text{diag}(\sigma)$ be the associated *gauge matrix* (see [20]). If $\sigma_i = +1\ \forall i$, the system (4.5) is called *cooperative* iff its Jacobian matrix of the form (4.6) is an M-matrix.

Let A denote the *steady-state influence matrix (SSIM)*, where entries a_{ij} predicts the signed steady-state response of species i to a positive step perturbation on species j. At the new equilibrium, the density of species i (the x_i^*) will be higher if $a_{ij} > 0$, lower if $a_{ij} < 0$ and unchanged if $a_{ij} = 0$. The SSIM is the sign pattern of negative adjoint matrix of the community matrix J which is non-singular, since $\det(-J) > 0$, then SSIM can be written as

$$A = \text{sgn}[\text{adj}(-J)] = \text{sgn}[(-J)^{-1}]. \tag{4.7}$$

The entry a_{ij} of SSIM is said to be *qualitatively signed* if it always has the same sign, $+1$, -1, 0, for any choice of parameter values in the system (see [29]); otherwise it can have different sign for particular parameter values.

Giordano and Altafini [27, 28] published their works in 2014 and 2017 on qualitative and quantitative criteria for biological and ecological networks. For qualitative criteria, their works showed the result that, if f has its associated Jacobian matrix $J(x)$ of the form (4.6) being an M-matrix, then $a_{ij} = +1$ for all $i, j \in \{1, \dots, n\}$. If the SSIM is undetermined, the authors assumed that $J(x)$ is an M_{V}-matrix and adapted [36, Theorem 4.2] to confirm that the SSIM has a positive inverse, then proceeded to their result on quantitative criteria.

4.2.2 Applications of Perron-Frobenius theory in Neural systems

In 2014, Karamintziou et al. [35] published their work on deep brain stimulation (DBS) of the subthalamic nucleus (STN) for the treatment of advanced Parkinson's disease. During surgery, the medical team applied the microelectrode recording (MER) in conjunction with functional stimulation techniques to securely implant the DBS electrode. The authors tried to improve the entire electrophysiological procedure, by optimizing clinical decision making and decreasing total surgical time. Starting from applying MER data related to ten DBS precedures, then they studied the behavior of measurement noise and formulated the multivariate phase synchronization index into a nonlinear stochastic model (see Equation (12) of [35]). Since their stochastic matrix has all positive entries and a spectral radius equal to one, the authors employed the Perron-Frobenius theory and reachability and holdability of nonnegative states from [43] to clarify that their stochastic matrix possessed the strong Perron-Frobenius property.

Another research related to neural system was published in 2020 by Franci, O'Leary, and Golowasch [22]. The authors studied positive dynamical networks in neuronal regulation and used theory of $M-$matrices as well as the Perron-Frobenius property from [42, Theorem 2.2], to prove and show that their proposed generic post-transcriptional molecular regulatory network dynamics are stable.

4.2.3 Applications of Perron-Frobenius theory in Population dynamics

Chalub and Souza [9, 10] studied the discrete Markov chains that are used to describe two-types of evolution population and published their research in 2017 and 2019. We recall the definitions from [9, 10].

Let **A** and **B** denote the two types of population and the size is $n \in \mathbb{Z}_+$. The authors assumed that the population dynamics is described by a discrete time Markov chain, with time-homogeneous transition probabilities, and where the chain is in state j, if there are j individuals of type **A** in the population.

Let $M = m_{ij}, i, j = 0, 1, \ldots, n$, denote the stochastic matrix for the probability of going from state j to state i.

Definition 4.10 (The Kimura class of matrices)**.** Let M be an $(n+1)\times(n+1)$ stochastic matrix. The matrix M is called *Kimura*, if M can be written as

$$M = \begin{bmatrix} 1 & 0_{1\times(n-1)} & 0 \\ a_{(n-1)\times 1} & C_{(n-1)\times(n-1)} & b_{(n-1)\times 1} \\ 0 & 0_{1\times(n-1)} & 1 \end{bmatrix}, \tag{4.8}$$

where $C_{(n-1)\times(n-1)}$ is an $(n-1)\times(n-1)$ sub-stochastic irreducible matrix, $0_{1\times(n-1)} \in \mathbb{R}^{n-1}$ is the zero row vector, and $a_{(n-1)\times 1}, b_{(n-1)\times 1} \in \mathbb{R}^{n-1} \setminus \{0\}$ are nonnegative column vectors.

Let $p \in \mathbb{R}^{n+1}$ be a vector of $n+1$ type selection probabilities. Given a population with n individuals at state i, then p_i represents the probability that an individual of type **A** is chosen for reproduction in a population with i individuals of this type and $1 - p_i$ represents the probability for individuals of type **B**. We recall the definition of regular Kimura matrix as follows.

Definition 4.11. The vector $x \in \mathbb{R}^n$ is increasing (non-decreasing) iff $x_i > x_j$ for all $i > j$.

Definition 4.12 ([9])**.** An evolution process such that the stochastic matrix belong to the Kimura class is *regular* (*weakly regular*), if the associated fixation vector (see [9]) is increasing (non-decreasing, respectively).

The Perron-Frobenius property from [42] and [52] was applied to prove the following theorems.

Theorem 4.13 ([9], Theorem 1)**.** *Let M be an $(n+1) \times (n+1)$ Kimura matrix. Then, M is regular iff it is eventually strictly stochastically ordered.*

Theorem 4.14 ([9], Theorem 2)**.** *Let M be a Kimura stochastic matrix and let P be a permutation matrix such that*

$$P^{-1}MP = \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & L_{n\times n} \end{bmatrix}.$$

Then, M is regular iff L possesses the Perron-Frobenius property.

4.3 Applications of Perron-Frobenius theory in E-conomy

In 2018, Riposo [49] studied the asset allocation method and provided a decomposition method from Matrix Theory for characterizing in-sample each term of the return decomposition from a matrix of returns. We can define terms used in portfolio management as follows:

- the *Information Coefficient (IC)* measures the global time correlation of the returns of the portfolio assets;
- the *Information Ratio (IR)* measures the active management opportunities;
- the *Breadth (BR)* is the number of times per year that the manager can use his skills to reallocate his portfolio.

The fundamental law of active management (see, e.g., [30]) can be written in a mathematical model as

$$\mathrm{IR} = \mathrm{IC} \cdot \sqrt{\mathrm{BR}} \tag{4.9}$$

and the fundamental of the above equation is called *Decomposition of Returns.*

Let $\mathcal{R}$ be the expected excess return of an asset (or a portfolio) and M represents the market. Then, from [37], equation (4.9) can be written as:

$$\mathcal{R} = \beta \mathcal{R}_M + \alpha, \tag{4.10}$$

where the *benchmark* $\mathcal{R}_M$ is the expected excess return following the global tendency of the market; β is the linear regression of estimator of $\mathcal{R}$ on $\mathcal{R}_M$; and α represents the component orthogonal with the market, which is unknown.

Riposo supposed that the portfolio manager has a matrix of returns and he is interested in formulating a decomposition of returns similar to equation (4.10). We recall definitions from [49]: The price returns of an asset are the elements of the vector $B \in \mathbb{R}^{T,1}$, where the i-th component is the return of the price at time t, $\forall t \in 1, 2, ..., T$. Let $A \in \mathbb{R}^{n,T}$ be the *matrix of returns*, where the element (i, t) is the return of the i-th asset at time t, $\forall i \in 1, 2, ..., n$ and $t \in 1, 2, ..., T$, and can be written as

$$A = \begin{bmatrix} \mathcal{R}_1(1) & \mathcal{R}_1(2) & \cdots & \mathcal{R}_1(T-1) & \mathcal{R}_1(T) \\ \mathcal{R}_2(1) & \mathcal{R}_2(2) & \cdots & \mathcal{R}_2(T-1) & \mathcal{R}_2(T) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{R}_n(1) & \mathcal{R}_n(2) & \cdots & \mathcal{R}_n(T-1) & \mathcal{R}_n(T) \end{bmatrix}. \tag{4.11}$$

The *decomposition of returns* obtained by applying the eigen-decomposition to the matrix $AA^T \in \mathbb{R}^{n,n}$, where the element (i, j) represents the covariance of assets i-th and j-th for the given time period $1, 2, ..., T$. Then, consider the largest eigenvalue of AA^T and its corresponding eigenvector. By implementing the Perron-Frobenius property to the matrix AA^T, the author proceeded to his result that the study of decomposition of returns can be provided from a matrix of returns. From the economic point of view, this is profitable for the manager strategy.

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