ORDINAL REAL NUMBERS 2. The arithmetization of order types .

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Abstract

In this paper the main results are :Proofs that the ordinal real numbers are real closed fields and complete up-to-characteristic .They are also Dedekind ,and Archemedean complete fields .They are real formal power series fields and Pythagorean complete fields It is proved and discussed the K-fundamental arithmetisation and the binary arithmetisation of the order types .

Key words: Real closed commutative fields,Grothendick group,Archemidean complete fields,linearly ordered commutative fields,full binary trees

Subject Classification of AMS *03,04,08,13,46*

§0 **Introduction** .

The author initiated and completed this research in the island of Samos in Greece during 1990-1992 .One years later (1993) he discovered how sequences of rational numbers with equivalent relation based not only that they converge to the same point but also with the same "speed" lead directly to a partially ordered topological complete field that is probably nothing more that the **Newtonian Fluxes**.The author gave it an other name*: Floware of the rational numbers* . Such a field contains fields of ordinal real numbers. Seven years later (1999) he discovered how such numbers can be interpreted as fields of random variables and completions of them in appropriate stochastic limits (***stochastic real numbers***), that links them to applications of statistics, stochastic processes and computer procedures. Thus, for instance, the ordinal natural numbers (including ω) can be interpreted appropriately as stochastic limit of normal random variables. This requires Bayesian statistics for higher ordinals. This interpretation permits stochastic differential and integral calculus that succeeds exactly where the known stochastic calculi fail! (The known stochastic calculi are : a) that in signal theory which is based on the spectral representation of stationary processes, b) that of Ito's usually with applications in Economics and c) that of Heiseberg-Schrondinger with applications in microphysical reality and based on operators in Hilbert spaces) From this point of view it turns out **that the ontology of infinite is the phenomenology of changes of the finite**. In particular *the phenomenology of stochastic changes of the finite can be formulated as ontology of the infinite* .He hopes that in future papers he shall be able to present this perspective in detail. It is a wonderful perspective to try to define the Dirac's deltas as entities of such stochastic real numbers. Another application is in the speeds of convergence to infinite of real functions, therefore with applications in the measurement of computational complexity of Turing machines and algorithms.

In this second paper on ordinal real numbers are proved, the main (elementary) properties of them. It is proved that the ordinal real numbers R_{α} of characteristic α, is the maximal field of characteristic α (maximality) and that it is , according to the theory of Artin-Screier, a real closedfield. (It turned out ,after the work was completed and by thinking aside, that they are also Archimedean complete (see [Glayzal A. 1937]),formal power series fields with real coefficients ,Dedekind complete (see [Massaza,Carla 1971]), and Pythagorean fields).

It is also proved a classification theorem which is analogous to the Hölder theorem for the Archimedean linearly ordered fields.In particular it is proved that any linearly ordered field of characteristic α contains the field Q_α of ordinal rational numbers of characteristic α, as a dense subfield and it is contained in the field R_α of ordinal real numbers of characteristic α, as a subfield ..As it is known, the linear segments of elementary euclidean geometry can be defined as special order-types with Archimedean property, and Archimedean (Hilbert) completness through axioms (see e.g. for a not ancient approach the Hilbert axiomatisation in [Hilbert D.1977] ch 1).It can be

proved to be order isomorphic with subsets of the real number field R. This is well-known and it can be called , the elementary arithmetisation of the order-types of Euclidean linear segments . On this fact is based the Cartesian idea of analytic geometry. This was an important turning point in the developments of the ideas and techniques of mathematics , of the discrete nature of numbers and continuous nature of geometry. The basic principle is that *the continuum is developed from the discrete* and not vice versa! An instance of this principle is the development of images and animation in computers through pixels and bits! It is surprising that in one of the consequences of the theory of ordinal real numbers, it is proved a far more advanced and complete result for the whole category of order types that has as corollary the previous important and elementary arithmetisation. Although more advanced, the result remains in the context of elementary theory of ordinal real numbers .In this result any order type can be "discretised" or "arithmetised" through the ordinal numbers.

The process of definition of the maximal fields R_α, from the minimal (double well ordered) monoids $N_\alpha=\alpha$, of principal ordinal numbers,we call K- fundamental densification. It is proved that any order-type is order isomorphic to a subset of some field R_α. Thus any order-type is constuctible by K-fundamental densification from ordinal numbers .This is called the K-arithmetisation of the order-types. Although in the way it is presented, this result is softly obtained, throws new light to the relation of ordinal numbers and order-types ,this relation turns out to be similar to the elementary relation of numbers and line segments in geometry.
Also it, holds a second kind of arithmetisation ,the binary arithmetisation which we state in the same paragraph .

§1 On the topology of linearly ordered fields. Local deepness, α-sequences.

The ordering of any linearly ordered field F defines a well known topology : the order-topology denoted by $T_{<;}$ In this topology, as it is known, the field F is a topological field.

This topology has very good separation properties; it is a T_1-T_5 topology, that is a completely normal topology (see for instance [Lynn A.Steen-Seebach J.A. Jr 1970] § 39 p. 66-68, also see [Munkress J.R. 1975] Chapters I, II)

The previously described order topology is also called the locally convex topology compatible with the order (see [Nachbin L. 1976] Ch I, II). (The convexity defined by the order).

Definition 1. Let X be a topological space. Let $p \in X$. The least ordinal α such that it exists a (local) base denoted by Bp of open neighbourhoods of the point p which is an α-sequence such that if $x<y<\alpha$, $U_x \subset U_y$, is called local deepness of X at p.

We notice that the concept of local deepness is very close to the concept of local weight of a topological space, where instead of ordinal we have an initial ordinal that is a cardinal number (see [Kuratowski K. 1966] V-I p. 53-54).

Examples of topological spaces such that every point has local deepness, are the ξ^*-uniform topological spaces as they are defined in [Cohen L.W.Goffman C. 1949] pp 66 conditions 1.2.3.4.

As in the case of fields that are classes, we may permit topological spaces that are classes and the open sets is a class of subclasses closed to union and finite intersection. For such spaces, the local deepness may be Ω_1 that is the class of all ordinal numbers.

Proposition 2. Let X a topological space and α, a limit ordinal such that every point has local-deepness α let $A \subseteq X$. It holds that $x \in \overline{A} \Leftrightarrow$ there is an β-sequence $\{x_s | s<\beta\}$ from elements of A such that $\lim_{s<\beta} x_s = x$. In other words topological convergence in X can be treated with β-sequences where β=car(α) is the upper character of α (see [N.L.Alling 1987] ch 1 §1.30 pp 29)

The proof is almost direct and to save space we shall not give it here.

Proposition 3. Let a field denoted by F of ordinal characteristic α, where α is a limit ordinal. Then every point $x \in F$ in the order-topology has local-deepness car(α), where car(α) is the upper character of α (see [N.L.Alling 1987] ch 1 §1.30 pp 29).

The proof is again direct and outside the scope of the paper.

Corollary 4. Convergence in the order-topology of a field of ordinal characteristic α, can be treated with β-sequences β≥car(α)

Needless to say, that in the case in which the topological space is a class and the local deepness is Ω_1, then convergence can be treated with Ω_1-sequences.

§2 The Holder- type classification .

Lemma 5. In every field of characteristic αthe field **Q**α is a dense subset.

Proof. Let a field of characteristic α, which we denote by Fα. By the theorem 17 of [Kyritsis C. OR1] the field Qα is a subfield of Fα. Let us suppose that it is not dense in Fα. Then there are two elements $x,y \in F\alpha$ $x<y$, such that there is no element of Qα in the internal [x,y]. Then the element z = y-x is Qα-infinitesimal.

This holds because $Q\alpha = \overline{L}(x) \cup \overline{R}(x) = \overline{L}(y) \cup \overline{R}(y)$ where $\overline{L}(x) = \{v | v \in Q\alpha \ \ v \leq x\}$ $\overline{R}(x) = \{v | v \in Q\alpha \ \ x \leq v\}$ and $\overline{L}(y)$ $\overline{R}(y)$ similarly. But by the hypothesis $\overline{L}(x) = \overline{L}(y), \overline{R}(x) = \overline{R}(y)$. *Thus* $Q\alpha = \overline{L}(x) \cup \overline{R}(y)$ and every element of Qα can be written as $r_2 - r_1$ where $r_2 \in \overline{R}(y)$ and $r_1 \in \overline{L}(x)$. Also we have that $0<y-x<r_2-r_1$. Then y-x is a Qα-infinitesimal and $\frac{1}{y-x}$ is a Qα-infinite element of Fα, thus $\frac{1}{y-x} >$ >α, contradiction since Char Fα = α.

Then there are not two element $y,x \in F\alpha$ $x<y$ with no element of Qα in [x,y], and Qα is dense in Fα. Q E D

Remark. Thus every field Fα of characteristic α is a <u>Weil completion</u> of the field Qα of ordinal rational numbers (see [Weil A.] ChIII Definition 2 but applied not only to local fields).

Theorem 6 (<u>Maximality</u> or <u>completness up-to-characteristic</u>).

The field Rα is <u>the maximal</u> field, of characteristic α. In the sense that every field of characteristic α is contained as subfield of Rα (more precisely Rαcontains an order preserving monomorphic image of the field).The field R_α is the <u>unique</u> fundamentally complete field of characteristic α.

Remark. This theorem is analogous to the well-known Holders theorem theorem ςηιψη στατες that every linearly ordered Archimedean field is a subfield of the field of real numbers. In other words the field of real numbers is the maximal Archimedean linearly ordered field. The previous property of the ordinal real numbers R_α relative to their characteristic ,we call <u>maximality</u> or <u>completness up-to characteristic</u> .

But as an erroneous application of terms R is also the minimal Cauchy complete field of characteristic ω and this also applies for the fields Rα in the sense that a completion of a linearly ordered field of characteristic α must be the field Rα .

Proof. Let any field of ordinal characteristic α denoted by Fα. By theorem 17 of [Kyritsis C. 1991], the field Qα is contained in Fα: $Q\alpha\subseteq F\alpha$. Let $x \in F\alpha$. Let (L(x), R(x)) be the cut that x defines on Qα ($L(x) = \{v|v \in Q\alpha\ v<x\}$, $R(x) = \{v|v \in Q\alpha\ x<v\}$). Since Qα is dense in Fα (Lemma **5**). There is a Cauchy α-sequence $\{x_n|n \in w(\alpha)\}$ of elements of Qα that converges in Fα to x (all topologies are the order-topologies). Hence $Q\alpha\subseteq F\alpha\subseteq R\alpha$ and the field Rα is a maximal field of characteristic α ; but also the field Rα is actually a minimal Cauchy complete field of characteristic α in the sense that the (strong) Cauchy completion $\hat{F}_\alpha$ of any field F_α of characteristic α contains the field R_α:$Q_\alpha\subseteq F_\alpha$ has as a concequence that $R_\alpha\subseteq \hat{F}_\alpha$. Thus if F_α is complete then $R_\alpha\subseteq F_\alpha$, $F_\alpha\subseteq R_\alpha$ hence $F_\alpha = R_\alpha$ Q.E.D.

The theory of Artin-Schreier of real closed fields has an excellant application to the ordinal real numbers .

Corollary 7. The fields of ordinal real numbers Rα are real closed fields.

Proof. Direct from Theorem 6 , and remark 5 of [Kyritsis C.1991] Q.E.D.

Post written Remark A. The author developed the theory of ordinal real numbers during 1990-1992 He had used the name “transfinite real numbers” without being aware that this term had been introduced by A.Glayzal during 1937 for his theory of linearly ordered fields. From the moment he fell upon the work of A.Glayzal (see [Glayzal A. 1937]) in the bibliography of the Book of N.L alling (see [N.L.Alling 1987]) he changed the title to “Ordinal Real Numbers” . After the work had been completed ,the author realised , by thinking aside, a quite unexpected and not unhappy fact :That the fields of ordinal real numbers are algebraically and order isomorphic to the fields of transfinite real numbers of Galyzal .This can be deduced by the fact that the fields of transfinite real numbers are exactly all the Archemidean complete fields (see [Glayzal A. 1937] theorems 4,8,9) and by the maximality of the ordinal real numbers (theorem 6). Thus if R_α is a field of ordinal real numbers of characterisic α, any Archemidean (linearly ordered field) extension of it ,it shall have the same characteristic with R_α. It seems that it can be proved , that any cofinal (coterminal) linearly ordered field extension ,is of the same characteristic . By the maximality of R_α (theorem 6) it shall have to coinside with R_α. In other words the fields of ordinal real numbers are Archemidean complete fields (although they may be non-Archemidean).But this is a characteristic property of the fields of transfinite real numbersb of Glayzal.

Thus they are order and field isomorphic with fields of transfinite real numbers .Conversely ,let any field R(λ) of transfinite real numbers of Archemidean base λ. Let us denote by α its ordinal characteristic .Let us suppose that there is an order and field extension of it with the same characteristic .Then it has to be an Archemidean extension of R(λ). By the Archemidean completness of the transfinite real numbers ,it has to coinside with the R(λ). Thus the transfinite real numbers are also complete up-to-characteristic .

But this is a characteristic property of the fields of the ordinal real numbers.Hence they are order and field isomorphic with fields of ordinal real numbers .Thus the ordinal real numbers should be considered as a different technique ,nevertheless indispensable and more far reaching .It is the technique that everyone would like to work.

Post written Remark B .Let a field R_α of ordinal real numbers of ordinal characteristic α. It is also a field of transfinite real numbers of archemidean base λ. The set of all elements of R_α that as formal power series have support of ordinality less than $\beta \leq o(\lambda)$=maximum ordinality of well ordered set of λ, and which we denote

by $R_{\alpha,\beta}$ is a field ,subfield of R_α .Indeed $R_{\alpha,o(\lambda)} = R_\alpha$. For the applications and especially with measurment proceeses ,the fields $R_{\alpha,\omega}$ are of prime interest and indispensable .

Post written remark C .The facts of the previous remark ,have as a concequence that the fields of ordinal real numbers are <u>formal power series fields</u> with coefficients in the real numbers and exponents in some order types.Thus the n-roots of their positive elements are contained in them (see [Neumann B.H. 1949] pp 211 ,4.91 Corollary).In other words they are <u>Pythagorean complete fields</u>.

Theorem 8. <u>(The Holder-type classification theorem)</u>.

Every field of ordinal characteristic α, denoted by Fα (where α is a principal ordinal) is contained between the fields Qα and Rα :$Q_\alpha \subseteq F_\alpha \subseteq R_\alpha$.

Proof. Contained in the proofs of the theorem 7 and lemma 5 Q.E.D.

Remark.9 The previous theorem gives that the hierarchy of ordinal real numbers has *universal embedding property* for the category of linearly ordered fields, that is every linearly ordered field has an monomorphic image in some field of the hierarchy.The hierarchy of transfinite real numbers is known to have, also, this property .Such hierarchies we call <u>universal embedding hierarchies</u>. Especially the hierarchy of ordinal real numbers after the classification theorem 8 ,we call also, <u>universal classification hierarchy</u>.

Remark.10 We notice that since every order type λ is order-embeddable in some transfinite real number field R(λ) (see [Glayzal A. 1937])as Archemidean base which in its turn is embeddable in some ordinal real number field R_α ,the above two hierarchies as hierarhies of order-types are <u>universal embedding hierarchies</u> for the category of order-types .Let an order type λ ; the least principal ordinal number α such that λ is order-embeddable (by a monomorphism) in the order-type and field R_α, is called the <u>fundamental density</u> of the order type λ and is denoted by df(λ).

Remark. In the [Massaza Carla, 1971] Definition I , is defined which cuts are the Dedekind cuts in linearly ordered fields .It is proved also that the Dedekind completion D(F) of a linaerly ordered field F is also its Cauchy completion (in the order topology).If we take the Dedekind completion $D(R_\alpha)$ of a field of ordinal real numbers R_α, it has to be its Cauchy completion which is again the R_α. Thus the fields of the ordinal real numbers are alsoDedekind complete . Conversely ,let any Dedekind complete linearly field F .Let us denote with α its ordinal characteristic .Then by the Holder type classification (theorem 8) it is a subfield of the field R_α of ordinal real numbers of characteristic α .Since the Dedekind completion D(F) =F coincides with the F and also with its Cauchy completion ,we get that $F=R_\alpha$, because the Cauchy completion of F is the R_α. In other words the class of Dedekind complete fields coincides with the class of the fields of ordinal real numbers .

Summarising we mention that the fields of ordinal real numbers have at least four kinds of completnesses that characterise them : Cauchy completness ,Dedekind completness,completness up-to-characteristic, Archemidean completness .It seems that he previous four completnesses can be summarised by saying that there is no cofinal (coterminal) order field extensions of them ;in short they are cofinally complete ,or cofinally maximal .They are also realcomplete (closed ,Artin-Shreier) and Pythagorean complete.

Remark. By corollary 7 we get that the field C_α is the algebraic closure of R_α :$C_\alpha = R_\alpha^a$.

We close this paragraph by mentioning that an axiomatic definition of the field Rα (α is a principal ordinal) would be the following:

First axiomatic definition of Rα.

The field of ordinal real numbers Rα is the unique Funtamental (Canchy)-complete, in the order-topology, field of characteristic α.

Second axiomatic definition of R_α .

The field of ordinal real numbers R_α is the unique complete (up-to-characteristic) field of characteristic α . These definitions apply even in the case of the field of real numbers (a = ω).

§ 3 The arithmetisation of order-types .

Remark.As it is known the linear segments of elementary Euclidean geometry can be defined as special order-types with Archimedean property and Hilbert completness through axioms (see e.g. for a not ancient approach the Hilbert axiomatisation in [Hilbert D 1977] ch 1).Then ,they can be proved to be order isomorfic with subsets of the real number field R. This is known as the elementary arithmetisation of the order-types of Euclidean linear segments.

Proposition 10.(the K- funtamental arithmetisation theorem of order-types.)

Every order-type λ is K-arithmetisable with ordinal numbers and has a fundamental density df(λ) which is a principal ordinal number .

In the next paper ,after the unification theorem of the transfinite real ,surreal ,ordinal real numbers ,a second arithmetisation theorem shall be proved. Two more universal hierarchies of formal power series fields shall be, also, proved that they are universal embedding hierarchies .We state these results here. For the definition of tree ,height of a tree, level of a tree, binary tree e.t.c.see [Kuratowski K.-Mostowski A. 1968] ch ii § 1, § 2 . The binary tree of height the ordinal α we denote with D_α. After the previously mentioned unification theorem 17 of the next paper we get that the hierarchy of binary trees is a universal embedding hierarchy for the order- types . Since the binary trees are subsets of linearly ordered fields and their elements consisting exclusively from 1's in the binary sequence, correspond to the ordinal numbers with the Hessenberg operations (see also [Conway J.H. 1976] ch 3 note pp 28 and also [Kyritsis C. 1991Alt] the characterisation theorem) this universal embedding property we call also binary arithmetisation .The least ordinal αsuch that an order-type λ is order embeddable in the binary tree D_α ,we call the binary density of λ ,and we denote it by db(λ).

Theorem 11 (The binary arithmetisation theorem of order-types)

Every order-type λ is binary arithmetisable and has a binary density db(λ) which is an ordinal number .

From the previous theorem ,by denoting a level of height α of a binary tree ,by T_α ,and giving to the Cartesian product $\prod_{\beta<\alpha} T_\beta$ the lexicographical ordering ,we also get the next :

Corollary 12. The formal power series hierarchies $R((D_\alpha))$, $R_{\beta<\alpha}\left(\left(\prod T_\beta\right)\right)$, are universal embedding hierarchies for the linearly ordered fields .

§ **4 Some general results on linearly ordered fields .**

In this paragraph we give some results generally for the category of linearly ordered fields. To save space we shall not give the proofs, since they do not have serious dificulties,nevertheless we shall indicate how they can be obtained .

Lemma 13 (On the rank and characteristic)

Let us suppose that the characteristic of the field F is ω^{ω^α} *where* α, *is* α *limit ordinal. It holds that the rank of the extension F/K is a cofinal order-type with the characteristic of the field F. That is cf(r(F/K))= cf(charF)=cf(char F-char K).*

Remark.For the definition of the rank of an extension see [Kyritsis C. 1991] § 4. For the proof of the previous theorem we use the

existence for any principal ordinal ω^{ω^α} of the ordinal real numbers fields $R_{\omega^{\omega^\alpha}}$ of characteristic ω^{ω^α} .

Let F be a linearly ordered field. If $x \in F$ by L(x) we denote the set L(x) = {y| $y \in F$ y<x} and by R(x) the set R(x) ={y| $y \in F$ x<y}.

By elementary arguments on linearly ordered fields the following identities can be proved.

Lemma 14

Let x, y $\in$ F. The following hold

1. L(-x) = - L(x) R(-x) = -R(x)
2. L(x+y) = L(x)+y = x+L(y)

   R(x+y) = R(x)+y = x+R(y)
3. L(x.y) = L(x).y + xL(y) - L(x).L(y) = R(x).y + xR(y) - R(x)R(y)

   R(x.y) = L(x).y+xR(y) - L(x).R(y) = R(x).y + xL(y) - R(x).L(y)

4. $\left.\begin{array}{l} y < R(x) \\ L(y) < x \end{array}\right\} \Leftrightarrow y \leq x$

$$L\left(x^{-1}\right) = \frac{1+(R(x)-x)L(y)}{R(x)} = \frac{1+(L(x)-X)R(y)}{L(x)}$$

5. $R\left(x^{-1}\right) = \frac{1+(L(x)-x)L(y)}{L(x)} = \frac{1+(R(x)-x)R(y)}{R(x)}$

The previous identities show also that the definition of operations used to define the surreal number fields are not something peculiar to these fields but hold in any linearly ordered field .

In the next paper of this work we will understand the true peculiarity of the technique of the surreal numbers.

Lemma 15 If F/k is an extension of two linearly ordered fields , it holds that

tr.d.(F/k) ≤ $2^{\aleph(\text{Char.F})}$ where tr.d.is the transandental degree of the extension .

Remark. For the definition of the transandental degree, base e.t.c see for instance [Zariski O.-Samuel P. 1958] vol. I pp. 95-102 also [Kyritsis C. 1991] § 4). The proof is obtained by using the Holder-type classification for F :$Q_\alpha \subseteq F \subseteq R_\alpha$ where α=char(F).

The next proposition shows that all the information of an extension of linearly ordered fields is to be found in the ideal of infinitesimals (or in the infinite elements). **Proposition 16**. Let F/k, F'/k two (ordered) extensions of the same linearly ordered field k. If the ideals of K-infinitesimals of the extension denoted respectively by m_F and m_F' are isomorphic as ordered integral domains ,then this isomorphism is extendable to an algebraic and order isomorphism of the fields F, F'.

Remark .The proof is direct from the definitions.

Remark. An extension F/k of the linearly ordered field k to F, is transcendental if Char F>char k and then the field F is an infinite dimensional vector space over k.

Proposition 17 . Let F be a linearly ordered field of characteristic char(F)= ω^{ω^α} where α is a limit ordinal . It holds that the field F in the order topology is totally disconnected .

Remark. The proof uses the existence, for every principal ordinal ω^{ω^α} , of the fields of ordinal real numbers R ω^{ω^α} .

Theorem 18 The classes of transfinite real numbers CR, and of ordinal real numbers Ω_1R, coinside.

Proof. Since both Hierarchies of transfinite real and ordinal real numbers have the universal embedding property (see remark 9) ,every transfinite real number-field is contained in some ordinal real number-field and every ordinal real-number field in some transfinite real number-field.Thus CR $\subseteq\Omega_1$R and Ω_1R $\subseteq$ CR, and CR = Ω_1R . Q.E.D.

§ 5 The A-Archimedeanity

The, at least two different, definitions of archemideanity, that can be found for instance in [Glayzal A. 1937] and in other authors as in [Conway J.H. 1976] or [Arin E. Schreier O.

1927] give us the opportunity to treat them in unified way through the concept of archemideanity relative to a monoid.

The fact that the linearly ordered field F has characteristic ω (the least infinite ordinal) is equivalent with the statement that the field F is Archimedean according to any (classical) known definition.

Let us denote by G a linearly ordered group and by A a monoid of endomorphisms of G as a group.

It is said that x is A-Archimedean to y where x,y ∈G iff there are a,b ∈A with a(x)≥y and b(y)≥x. If A is the domain Z of integers (the endomorphisms are multiplication with an integer)we simply say that x is Archimedean to y. If for every pair x,y of elements of G holds that x is A-Archimedean to y, it is said that G is A-Archimedean

Let F be a linearly ordered field .If we consider it as an additive group, and we denote by A_1 a monoid of endomorphisms of the additive group , we get the concept of x being A-additively Archimedean to y. If we consider the multiplicative group F^* and we take a monoid, denoted by A_2, of edomorphisms of the multiplicative group, we get the concept of x being A-multiplicatively Archimedean to y.

Let $A=A_1VA_2$ be the monoid of mappings from F to F generated by the previous monoids . It is said that x is A-field-Archimedean to y iff there are a, b ∈A such that a(x)≥y, b(y)≥x.

In any extension F/k of a field K by a field F, where F,k are fields of ordinal characteristic with char F>Char K, if we take as A_1, to be the multiplication with elements from the field K (considering the field F as a linear space over K), we get the concept of x being K-additively Archimedean to y.(For K=R this is also known as "x is commensurate to y " see [Conway J.H. 1976] ch 3 pp 31).

If A_1 is the multiplication with integers and A_2 is power with integral exponents ,then it is simply said that x is field Archimedean to y (Known also from the A. Gleyzal's definition of Archimedeanity)

A non-Archimedean linearly ordered field denoted by F is simply a linearly ordered field for which not all pairs (x,y) of its elements are mutually additively Archimedean. (Thus charF>ω) But it can be very well A-additively Archimedean for other monoids A.In particular if charF=α and A is the monoid of endomorphisms of the additive group of F defined by (field) multiplication with ordinals less than α, then it is A-additively Archimedean and we denote it by writing that it is α-additively Archimedean

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List of special symbols

α,β,ω : Small Greek letters

Ω_1 : Capital Greek letter omega with the subscript 1

F^a : Capital letter F with superscript a.

N : Capital Aleph ,the first letter of the hebrew alphabet . In the text is used a capital script. letter n .

$\oplus, \circ$: cross in a circle, point in a circle .

$N\alpha, Z\alpha, Q\alpha, R\alpha$,: Roman capital letters with subscript small Greek letters

$C\alpha, H\alpha$

$^*X$, $^*R$ et.c : Capital standard or roman letters with left superscript a star.

CN,CZ,CQ, :Capital standard letter c followed by capital letters

C^*R, with possibly a left superscript a star

$\hat{X}$: Capital tstandard letter with a cap.

Σ : Capital Greek letter sigma

$\mathring{D}_\alpha$: Capital standard D with subscript a small Greek letter and in upper place a small zero.