ORDINAL REAL NUMBER 3. The techniques of transfinite real , surreal, ordinal real, numbers ; unification .

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Abstract

In this last paper on the theory of the ordinal real numbers, is proved ,that the three different techniques and hierarchies of transfinite real numbers , of the surreal numbers ,of the ordinal real numbers, give by inductive limit or union the same class of numbers.

Key words: Linearly ordered commutative fields,transfinite real numbers,surreal numbers,formal power series fields

Subject Classification of AMS *03,04,08,13,46*

§ **0 Introduction** . In this third paper on ordinal real numbers, it is proved that the three different techniques and Hierarchies of transfinite real number-fields, of surreal numbers, and of ordinal real numbers , give by inductive limit ,or union, the same class of numbers ,already known as the class No .It can be characterized, simply, as the smallest (linearly ordered field which is a) class and contains every linearly ordered set- field as a subfield . This class ,and also the category of linearly ordered set-fields, we call **" the linearly ordered Newton-Leibniz realm of numbers".** It is obvious that without the set theory of G. Cantor as it is formalized, for instance, by Zermelo-Frankel and a correct thinking about the infinite, this "realm of numbers" would not be definable.

Seven years later (1999) the author discovered how such numbers can be interpreted as fields of random variables and completions of them in appropriate stochastic limits (***stochastic real numbers***), that links them to applications of statistics, stochastic processes and computer procedures. Thus, for instance, the ordinal natural numbers (including ω) can be interpreted appropriately as stochastic limit of normal random variables. This requires Bayesian statistics for higher ordinals. This interpretation permits stochastic differential and integral calculus that succeeds exactly where the known stochastic calculi fail! (The known stochastic calculi are : a) that in signal theory which is based on the spectral representation of stationary processes, b) that of Ito's usually with applications in Economics and c) that of Heiseberg-Schrondinger with applications in microphysical reality and based on operators in Hilbert spaces) From this point of view it turns out **that the ontology of infinite is the phenomenology of changes of the finite**. In particular *the phenomenology of stochastic changes of the finite can be formulated as ontology of the infinite* .He hopes that in future papers he shall be able to present this perspective in detail. It is a wonderful perspective to try to define the Dirac's deltas as entities of such stochastic real numbers. Another application is in the speeds of convergence to infinite of real functions, therefore with applications in the measurement of computational complexity of Turing machines and algorithms.

Nevertheless we could follow a still different approach. If in the completion of rational numbers to real numbers we ramify the equivalence relation of convergent sequences to others that include not only where the sequences converge (if they converge at the same point) but also how fast (if they converge in the same way e.g finally equal ,an attribute related also to computer algorithms complexity) ,then we get non-linearly ordered topological fields that contain ordinal numbers (certainly up to ω^a , a= ω^ω) and are closer to practical applications. This approach does not involve the random variables at all, but involves directly sequences of rational numbers as "*Newtonian fluxions*". This creation can be considered as a model of such linearly ordered fields (up-to-characteristic ω^a , a= ω^ω) ,when these linearly ordered fields are defined axiomatically. This gives also a construction of the real numbers with a set which is countable. This does not contradict that all models of the real numbers are isomorphic as the field- isomorphism is not in this case also an $\in$-isomorphism so the Cauchy-real numbers and such a model still have different cardinality.

If we want to define in this way *all* the Ordinal real numbers , then it is still possible but then this would give also a device for a model of all the ZFC-set theory! And such a model is indeed possible: By taking again sequences of non-decreasing (in the inclusion) finite sets of ZFC , and requiring that any property ,relation or operation if it is to hold for this set-sequence it must hold finally for each term of the sequence and finite set. In other words we take a minimality for every set of it, relative to the axiom of infinite. It is easy to prove that the (absolute) cardinality of such a model is at most 2^ω , that is at most the cardinality of the continuum. We could conceive such a model as the way a computer with its algorithms , data bases tables etc would represents sets of ZFC in a logically consistent way. Needles to say that a similar model of ZFC set theory could be given within Euclidean geometry! (e.g. after the Hilbert's axiom system, and in addition accepting only the natural numbers). There is no contradiction with the 2nd-incompletness theorem of Gödel as the argument to prove that it is a model of ZFC-set theory is already outside ZFC-set theory (as are also the arguments of Gödel ,or of Lowenheim-Skolem that gives a countable model of ZFC-set theory).

It is not directly apparent that so different techniques and ideas would have such an underlying unity. It is, also , surprising that, although the Hessenberg operations were very early known in the theory of ordinal numbers, (at least since 1906, see [Gleyzal A. 1937]) no one went far enough to define through them, fields in a way similar to the way that the real numbers are defined from the natural numbers. Although G.Cantor, himself was conceiving the ordinals as a natural continuation of the natural numbers (see [Frankell A. A. 1953] introduction pp 3) ,as it is kmown, he rejected the attempts to define infinitesimals through them . (see [Frankell A. A. 1953] ch ii § 7.7 pp 120). We could speculate that un underlying reason for this, might be that, his set-theory was already strongly attacked and was facing the danger of final rejection ,and these were good enough reasons to avoid the additional charge that his theory " opened the door" to infinitesimals . In spite of this, there are many who might consider that although the present results are coming now, nevertheless it is too late and ,they might speculate ,for this long delay (more than eighty years) and diversion of ideas and technique, nevertheless on the same subject, we could suspect systematic obstructions, that came outside the mathematics. Nevertheless, there are others who consider that it is too early for such a development ,and especially for an analysis on such numbers. It seems that it has never been published any **"partially ordered Newton-Leibniz realm of numbers"** (in other words a category of transcendental extensions of the real numbers ,that are partially ordered fields and complete in the order topology) with reasonably "good" properties for a classification.

In this paper we use the surreal numbers, as they are definable in the Zermelo-Frankel set theory, through the binary trees, directly as a class, and not as union of some set fields.(The original technique of J.H.Conway).I met J.H.Conway during 1992 at Philadelphia in the USA, I talked to him about the new developments in this area of research and I gave to him the present work but as he told me he had more than a decade that for the last time he had active interest in the subject. I is somehow necessary to make use of classes instead of sets; since, for the kind of "induction" that the J.H.Conway uses, we prove that it is reduced to the usual transfinite induction on the height of the elements of the trees; but in their union as a class and not for each one of them separately as a set; in the latter case in which the trees are sets the induction fails .The key-point is to prove that for every cut that J.H.Conway uses it does really exist a unique element of the trees of least height . "simplest number" as it is used to be called). This is a very crucial point, for the whole technique of the surreal numbers, to work, and it seems that it has been obscured, by not paying sufficient attention to it

The author has initially included also the non-standard real numbers in the classification. As they are also linearly ordered fields and the present classification is of all linearly ordered fields it was natural to include them. There were experts in non-standard analysis that were glad about it. Nevertheless there were experts that insisted that according to the initial definition of A.Robinson and not of later definitions, it was not claimed that the non-standard real numbers were sets inside Zermelo-Frankel system. Only if Zermelo-Frankel system was used to model meta-mathematics also the they would be also sets. This was nevertheless different as such sets would models of meta-mathematical entities different than the sets that are models of mathematical and not meta-mathematical entities. Because of their arguments and in spite the fact that this made some other researchers of non-standard mathematics unhappy, the author prefers in this first publication about ordinal real numbers not to include the non-standard real numbers in the unification. Any definition nevertheless that has the non-standard real numbers as ordinary sets of Zermelo-Frankel set theory, would naturally lead to a straightforward proof that such fields are always subfields of some field of ordinal real numbers! The author has already produced pages with this proof that is based on the premise that I mentioned.

§ 2. The surreal numbers .

In this paragraph we define the class No of surreal numbers inside the ZF-set theory.We use the binary trees (see [Conway J.H. 1976] appendix to part zero pp 65 and [Kuratwski K.-Mostowski A. 1968] Ch IX §1, §2).The crucial point is to prove that for the cuts defined by J.H.Conway in these trees it does really exist a unique element strictly greater than all the elements of the left section and strictly smaller than all the elements of the right section (the "simplest number").Through this the Conway-induction us reduced to the usual transfinite induction on the height of the elements of the tree .As we shall see this works for the union of all trees as a class but fails for each one set-tree .For the definition of the tree, binary tree, height, levels of the tree ,H_ξ-set see [Kuratwski K.-Mostowski A. 1968] Ch IX §1, §2 Theorem 2, . The binary tree of height α we denote by D_α . More precisely we are interested for the trees of the next definition.

Definition 1. *Let α be an ordinal . We define* $\dot{D}_\alpha$ *= {x|x* ∈ *Da such that there is β<α such that for the element x as a zero-one sequence x =* $\{x_\xi|\xi<a\}$ *holds that* $x_\beta = 1$ *and* $x_\xi = 0$ *for* $\xi>\beta$*}.*

We call the set $\mathring{D}_\alpha$ the open full-binary tree of height α.

We also remind that if for the height α, holds that $\aleph(\alpha)$ is a cofinal to α regular aleph: $\aleph(\alpha) = \aleph_{cf(\alpha)} = \aleph_\xi$ the open full-binary tree is an H_ξ set, (see [Kuratwski K.-Mostowski A. 1968] ChIX §2 Theorem 2, the proof works also for trees $\mathring{D}_\alpha$ where $\aleph(\alpha) = \aleph_{cf(\alpha)}$)

Lemma 2. *For every pair of subsets L, R of the open-full-binary tree* $\mathring{D}_\alpha$ *of height the ordinal α, such that* $\aleph(\alpha)$ *is a regular aleph, and holds that: for every* $l \in L$, $r \in R$, $l < r$, *and* $\aleph(L)$, $\aleph(R) < \aleph(\alpha)$, *there is exactly one element* x_0 *of least height in* $\mathring{D}_\alpha$ *such that* $l < x_0 < r$ *for every* $l \in L, r \in R$.

Proof. Let D(L) = {x|x $\in \mathring{D}_\alpha$ such that there exists $l \in L$ with $x \le l$} and I(R) = {x|x $\in \mathring{D}_\alpha$ such that there exists $r \in R$ with $r \in x$} that is D(L), I(R) are the decreasing and increasing lower and upper half subsets of $\mathring{D}_\alpha$ determined by L, R, in the linear ordering of $\mathring{D}_\alpha$ as a tree (see [Kuratwski K.-Mostowski A. 1968] Ch IX §1 Lemma A). Let the set M = {x|x e $\mathring{D}_\alpha$ and for every $\not\subset l \in D(L)$, $r \in I(R)$ it holds that $l < x < r$}. By the H_ξ property of $\mathring{D}_\alpha$ it holds that $M \neq \varnothing$. Let A = {β|β is an ordinal number such that there is $x \in M$ with $x \in T_\beta$ where T_β is the β-level of Dα in other words there is $x \in M$ of height β}. Let $\alpha_0 = \min A$. Let $D\alpha_0(L)$ $I\alpha_0(R)$ the subsets of D(L) R(L) of elements of height less than α_0, and let $M\alpha_0 \subseteq M$ the subset of M that consists of elements of height α_0. Suppose that the set $M\alpha_0$ contains two elements x, y with e.g. $x \le y$. We will prove that $M\alpha_0$ contains only one element.

Let $x'=\{x_\beta|\beta<\alpha_0\}$ that is that part of the α_0-sequence x with terms of indifes less than α_0. And the same also with $y' = \{y_\beta|\beta < \alpha_0\}$. Then there is l_x or r_x and l_y or r_y respectively in $D\alpha_0(L)$, $I\alpha_0(R)$ such that they are equal with x', y'. If $x=r_x$ then, if the α_0-term of x is 0 or 1, in both cases $x > r_x$, contradiction. Hence there is no such r_x and also such r_y. Then $l_x=x'$ $l_y=y'$ and $l_x \le l_y$. The α_0-term of x and y might be 0 or 1. The only possible cases are $\{x = (l_x,0), y = (l_y,0)\}$, $\{x = (l_x,0), y = (l_y,1\}$ $\{x = (l_x,1), (l_y,1)\}$, $\{x = (l_x,1), y = (l_y,0\}$ where with the parenthesis we symbolize the α_0-sequence which is the elements x, y. Let us suppose that $x \neq y$ and $\underset{\beta<\delta}{D\beta}(x) = \{x_\beta|\beta < \delta\}$, the part of the α_0-sequence with terms with indices less than δ, with $\delta \le \alpha$. Let the least value of δ, be denoted by δ_0 such that $\underset{b<\delta}{D\beta}(x) = \underset{\beta<\delta}{D\beta}(y)$, $\delta_0 \le a_0$ and $X\delta_0 \neq Y\delta_0$. If holds that $0 = X\delta_0 \neq Y\delta_0 \neq 1$ because x<y. In the sequent, let $z=(D\beta(x)=D\beta(y)\ \beta<\delta_0, 1)$. Then $x < z \le y$. If $\delta_0 = a_0$ then x=y because $X\alpha_0 = Y\alpha_0 = 1$. Then $\delta_0 < \alpha_0$ and also $z < y$ and $x < z < y$ and the height of z is $\delta_0 < a_0$ contradiction. Hence x= y, and $M\alpha_0$ contains only one element. It also holds that if we restrict to $D_c(L)$, $I_c(R)$ where $D_c(L) = DC(L) \cap \mathring{D}_c$ $I_c(R) = I(R) \cap \mathring{D}_c$ (and L, R have height <c), then $a_0 \le c$ by the H_ξ-property of $\mathring{D}_c$ if c is also such that $\aleph(\alpha) = \aleph_{cf(\alpha)}$ Q.E.D.

Definition 3. *The open full binary tree* $\dot{D}\alpha$ *of height* α*, such that* $\aleph(\alpha)$ *is a cofinal to* α*, regular aleph , I call* *regular open full- binary tree.*

The property of the previous lemma of a regular open full-binary tree I call H_ξ-leveled Dedekind completness.

We remark that the class of regular alephs is unbounded (see [Kuratwski K.-Mostowski A. 1968] p. 275 relation 5) Thus the class of ordinals α such that $\aleph(\alpha)= \aleph_{cf(\alpha)}$ is unbounded.

The next definition is the definition of the class of surreal numbers in the ZF-set theory and it depends as we mentioned on the lemma 2 .As it is seen ,in the hypotheses of the lemma 2 the cardinality of halfs of the cut is bounded by $\aleph(\alpha)$. If it is to include all possible cuts of the tree $\dot{D}\alpha$ then the lemma 2 will give the element x_o in some tree $\dot{D}\beta$, of sufficient greater height,thus outside the original tree D_α. This is why we mentioned that the definition of surreal numbers (with the original technique of J.H.Conway) does not apply to the trees $\dot{D}\alpha$ separately .

Definition 4. Let U $\dot{D}\alpha$ =No be the union of all regular open full-binary trees. It is a class (after axiom A2.(see [Cohn P.M. 1965] p1-36)) Operations may be defined in this linearly ordered class according to the formulae of Lemma 2 in [Kyritsis C.1991 Alt. or Free etc.)] II, that hold for every linearly ordered field that is:

1. let α be an ordinal with $\aleph(\alpha)= \aleph_{cf(\alpha)}$ and L,R subsets of $\dot{D}\alpha$ such that for every $l \in L$, $r \in R$ holds that $l < r$. Then there exists a regular aleph β such that $L,R \subseteq \dot{D}_\beta$ and $\aleph(L)$, $\aleph(R)< \aleph(\beta)$. Then there is by lemma 2 a unique element $x_0 \in \dot{D}_\beta$ of least height such that $l < x_0 < r$ for every $l \in L$, $r \in R$, we denote this element by {L|R} and we write $x_0=$ {L|R}. We note that although $L,R \subseteq \dot{D}_\alpha$, it holds that $x_0 \in \dot{D}_\beta$ and $\alpha <\beta$.

2. If $x,y \in \dot{D}\alpha$ and we denote the height of x, y by h(x), h(y) and by L(x), L(y), R(x), R(y) the sets

$$L(x)=\left\{v \middle| v \in \dot{D}\alpha \;\; h(v)<h(x) \text{ and } v<x\right\} L(y)=\left\{V \middle| V \in \dot{D}\alpha, h(v)<h(y) \text{ and } v<y\right\},$$

$$R(x)=\left\{v \middle| v \in \dot{D}\alpha, h(v)<h(x) \text{ and } x<v\right\} \;\; R(y)=\left\{v \middle| v \in \dot{D}\alpha, h(y) \text{ and } y<v\right\}$$

Then the operations are defined through simultaneous two-variable transfinite induction in the form of the lemma 2,3 in [Kyritsis C. 1991 Free etc.], for the heights of the trees $\dot{D}\alpha$ where for the initial segments of ordinals we substitute the corresponding trees of No (For every ordinal $\beta<\alpha$ such that $N(\beta)=N_{cf(\beta)}$ corresponds a tree $\dot{D}\beta$). Thus the function of operation is defined not on $w(\alpha)^2$ but on $\dot{D}\alpha^2$. For the addition, the next rule is used $x+y=\{L(x)+y \cup x+L(y) | x+R(y)\} \cup R(x)+y\}$.

3. The opposite is defined by:

$-x = \{-R(x)|-L(x)\}$

4. Multiplication is defined by

$x.y=\{L(x).y+xL(y)-L(x).L(y)\cup R(x).y+xR(y)-R(x)R(y) \mid$

$|L(x).y+x.R(y)-L(x).R(y) \cup R(x).y+x.L(x)-R(x).L(x)\}$.

This definition presupposes the definition of addition.

5. Inverse is defined by

$$x^{-1} = \left\{ 0, \frac{1+(R(x)-x)L(x)}{R(x)} \cup \frac{1+(L(x)-x).R(x)}{L(x)} \middle| \frac{1+(L(x)-x)L(y)}{L(x)} \cup \frac{1+(R(x)-x).R(y)}{R(x)} \right\}$$

As it is proved in [Conway J.H. 1976] Ch0, 1 the set No is a linearly ordered c-field. The characteristic of No is easily proved to be Ω_1, we call this c-field, c-field of surreal numbers. According to Definition 3 No is an H_ξ -leveled Dedekind complete field.

§ **3 The unification** .

In this paragraph we prove that all the three different techniques and hierarchies of transfinite real ,of surreal ,of ordinal real numbers give by inductive limit or union the same class of numbers .We have already proved that $CR=\Omega_1R=C^*R$. (see corollary 10) and it remains to prove No=CR.

Lemma 5 . *It holds that* $CR=\Omega_1R=C^*R\subseteq No$.

Proof .Let an open full binary tree $\dot{D}\alpha$ of height the principal ordinal a .Then $\dot{D}\alpha \subseteq$ No ,and the field-inherited operations in the initial segment $W(\alpha)$ are the Hessenberg operations (see [Conway J.H. 1976] ch 2 § ""containment of the ordinals "note pp 28 and also [Kyritsis C.1991 Alt] the characterisation theorem).If α was not a principal ordinal, the $W(\alpha)$ would not be closed to the Hessenberg operations .Thus the N_α, Z_α, Q_α are contained in No ,since what it is used to define them from $W(\alpha)$ is only the field operations .The Q_α is a field and from the fact that No is closed to extensions of its set-subfields (see[Conway J.H. 1976] ch 4 theorem 28)we deduce that the field of ordinal real numbers R_α is contained in No, for every principal ordinal number α .Thus $\cup R_\alpha=\Omega_1R\subseteq No$ .Q.E.D.

Lemma 6 . *For every regular open full binary tree* $\dot{D}\alpha$, *it holds that* $\dot{D}\alpha\subseteq R_\beta$, *for some sufficiently big principal ordinal number* β . *(With the inclusion is meant that the restriction of ordering of* R_α *in the tree, coincides with the ordering of the tree).*

Proof . We shall prove it by transfinite induction .It holds for the trees of finite height. The transfinite induction shall be on the transfinite sequence of all ordinal numbers such that $\aleph(\alpha)=\aleph_{cf(\alpha)}$ and $\aleph(\alpha)$ is a regular aleph. Let us suppose that it holds for all such ordinal numbers of $W(\alpha)$, and $\aleph(\alpha)=\aleph_{cf(\alpha)}$and $\aleph(\alpha)$ is a regular aleph .Then : $\dot{D}\alpha\subseteq \bigcup_{s\in W(\alpha)} \dot{D}_s \bigcup_{s\in W(\alpha)} R_{\beta(s)} \subseteq R_{\beta(\alpha)}$ hence $\dot{D}_\alpha \subseteq R_{\beta(\alpha)}$ where $\beta(\alpha)$ is a principal ordinal with $\beta(\alpha) > \lim_{s\in W(\alpha)} \beta(s)$. Q.E.D.

From the previous lemma we get that $\cup \dot{D}\alpha$ = No $\subseteq \Omega_1R$,thus :

The unification theorem 7

It holds that the classes of transfinite real numbers CR , of surreal numbers No, of ordinal real numbers $\Omega_1 R$ *,coincide ,and it is the smallest class (and linearly ordered c-field) ,that contains all linearly ordered set-fields as subfields.*

We can have obviously analogous statements for the other classes of numbers (complex , quaternion e.t.c.). After the previous theorem, the binary arithnetisation of the order-types, stated in [Kyritsis C. 1991] II ,theorem 11, is directly provable. We remark that because the levels of the open full binary trees have the property that any upper (lower bounded set has supremum (infimum) ,(see [Kuratowski K. -Mostowski A 1968] ch ix §1, § 2 theorem 2),and after the Hilbert and fundamental (Cauchy) completness of the ordinal real numbers, and remark after definition 13 and ω-normal form according to [Frankel A.A. 1953] ch 3 theorem 21 ,and after corollary 21 in [Kyritsis C. 1991] ,II , we also get:

Theorem 8 . *The class of numbers* $CR=\Omega_1 R=No$ *has leveled formal power series representation, leveled Hilbert completeness, leveled fundamental (Cauchy) completeness, leveled* H_ξ *Dedekind completeness ,leveled supremum completeness and representation with ω-normal forms.*

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List of special symbols

α,β,ω : Small Greek letters

Ω_1 : Capital Greek letter omega with the subscript 1

F^a : Capital letter F with superscript a.

N : Capital Aleph ,the first letter of the hebrew alphabet . In the text is used a capital script. letter n .

$\oplus,\odot$: cross in a circle, point in a circle .

$N_\alpha, Z_\alpha, Q_\alpha, R_\alpha$,: Roman capital letters with subscript small Greek letters

C_α, H_α

$^*X$, $^*R$ et.c : Capital standard or roman letters with left superscript a star.

CN,CZ,CQ, :Capital standard letter c followed by capital letters, with possibly

$C^{*}R$ a left superscript a star

$\hat{X}$: Capital tstandard letter with a cap.

Σ : Capital Greek letter sigma

$\mathring{D}_{\alpha}$: Capital standard D with subscript a small Greek letter and in upper place a small zero.