Free algebrae and alternative definitions of the Hessenberg operations in the ordinal numbers .

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Abstract.

It is proved and is given, in this paper, two alternative algebraic definitions of the Hessenberg natural numbers in the ordinal numbers: a) by definition with transfinite induction and two inductive rules , b) by the free algebras of the polynomial symbols of the commutative semirings with unit.

Key Words: Hessenberg natural operations, ordinal numbers, free algebras, semirings

Subject Classification of AMS *03,04,08,13,46*

§0. Introduction This is the second paper of a series of five papers that have as goal the definition of topological complete linearly ordered fields (continuous numbers) that include the real numbers and are obtained from the ordinal numbers in a method analogous to the way that Cauchy derived the real numbers from the natural numbers. We may call them linearly ordered Newton-Leibniz numbers. The author started and completed this research in the island of Samos during 1990-1992 . Seven years later (1999) he discovered how such numbers can be interpreted as fields of random variables (*stochastic real numbers*) that links them to applications of statistics, stochastic processes and computer procedures. From this point of view it turns out **that the ontology of infinite is the phenomenology of changes of the finite**. In particular *the phenomenology of stochastic changes of the finite can be formulated as ontology of the infinite* .He hopes that in future papers he shall be able to present this perspective in detail. Another application is in the speeds of convergence to infinite of real functions therefore with applications in the measurement of computational complexity of Turing machines and algorithms.

It is a wonderful perspective to try to define the Dirac's deltas as natural entities of such stochastic real numbers. If in the completion of rational numbers to real numbers we ramify the equivalence relation of convergent sequences to others that include not only where the sequences converge (if they converge at the same point) but also how fast (if they converge in the same way ,an attribute related also to computer algorithms complexity) ,then we get non-linearly ordered topological fields that contain ordinal numbers (certainly up to ω^a , a= ω^ω) and are closer to practical applications. This approach does not involve the random variables at all, but involves directly sequences of rational numbers as "*Newtonian fluxions*". This creation can be considered as a model of such linearly ordered fields (up-to-characteristic ω^a , a= ω^ω) ,when these linearly ordered fields are defined axiomatically. This gives also a construction of the real numbers with a set which is countable. This does not contradict that all models of the real numbers are isomorphic as the field- isomorphism is not in this case also an $\in$-isomorphism so the Cauchy-real numbers and such a model still have different cardinality.

If we want to define in this way *all* the Ordinal real numbers, then it is still possible but then this would give also a device for a model of all the ZFC-set theory (ZFC=Zermelo-Frankel set theory with axiom of choice)! And such a model is indeed possible: By taking again sequences of non-decreasing (in the inclusion) finite sets of ZFC , and requiring that any property ,relation or operation if it is to hold for this set-sequence it must hold finally for each term of the sequence and finite set. In other words we take a minimality for every set of it, relative to the axiom of infinite. It is easy to prove that the (absolute) cardinality of such a model is at most 2^{ω} , that is at most the cardinality of the continuum. We could conceive such a model as the way a computer with its algorithms , data bases tables etc would represents sets of ZFC in a logically consistent way. There is no contradiction with the 2nd-incompletness theorem of Gödel as the argument to prove that it is a model of ZFC-set theory is already outside ZFC-set theory (as are also the arguments of Gödel ,or of Lowenheim-Skolem that gives a countable model of ZFC-set theory).

In this second paper on the same subject, I shall give two more ,and even simpler, algebraic characterizations of the Hessenberg natural operations in the ordinal numbers. These characterizations are actually alternative and direct definitions of the Hessenberg natural operations; independent from the standard non-commutative operations in the ordinal numbers .The main results are the characterization theorems 4.7.

.These characterisations of the Hessenberg natural operations are :

a) *As operations defined by transfinite induction through two inductive rules that are already satisfied by the usual operations in the natural numbers .*

b) *An initial segment of a principal ordinal* $\omega^{\omega^{\alpha}}$ *in the Hessenberg natural operations is isomorphic with the free semiring of* α *many generators in the category of abelian semirings ;or isomorphic with the algebra of polynomial symbols of* α *inderminates of the type of algebra of semirings with constants the natural numbers .*

The previous characterizations proves that the Hessenberg natural operations are the natural extensions in the ordinal numbers, of the usual operations in the natural numbers,. This turns out to be so, if we approach this subject from whatever aspect. Thus the Hessenberg natural operations should be coined as the standard abelian operations in the ordinal numbers, for all practical algebraic purposes .There are already extended applications of this . (see [Conway J.H.])

The main application of the previous results is in the theory of ordinal real numbers (see [Kyritsis C.E.]).The final result is that the three Hierarchies and different techniques of transfinite real numbers (see [Gleyzal A.]), of the surreal numbers (see [Conway J.H.])) ,of the ordinal real numbers (see [Kyritsis C.E.]) give by inductive limit or union the same class of numbers ,already known as the class No and to which we make reference in [Kyritsis C.E.] as the "***totally ordered Newton-Leibniz realm of numbers*** ".

§1.The third algebraic characterizations of the Hessenberg natural operations.

Let a initial segment of an ordinal number of type $\beta=\omega^{\alpha}$. Let us define a binary operation ,denoted by +, in W(β) by definition by transfinite induction (see e.g. [Kuratowski K.-Mostoeski A.] §4 pp 233, [Enderton B.H.], [Frankel A.A.], [Kyritsis C.E.] Lemma 2, 3) and the inductive rule $p_+ : \bigcup_{x,y<\alpha} w(\gamma)^{W(\omega^x)\times W(\omega^y)} \to W(\gamma+1)$ defined by $p_+(f)=S(\{f(W(x),y)\}\cup\{f(x,W(y))\})$ for an $f \in W(\gamma)^{W(\omega^x)\times W(\omega^y)}$.

The definition by transfinite induction is supposed of two variables as is also the inductive rule (see [Kyritsis C.E] Lemma 2,3). Thus, there exists a unique function denoted by $(+):W(\omega^\alpha)^2 \to W(\gamma+1)$, where the γ is an ordinal number with $\omega^\alpha < \gamma$ such that it satisfies the inductive rule p_+; thus it holds :

p_+ $x+y=S(\{x+W(y)\}\cup\{\{W(x)+y\})$.

The restriction of this operation in $W(\omega)=\omega$ coincides with the usual operations of the natural numbers, since the addition of the natural numbers satisfies also the inductive rule p_+ :

Lemma 0. *The Hessenberg natural sum in a initial segment* $W(\omega^\alpha)$, *satisfies the inductive rule* p_+ .

Proof: See [Kyritsis C.E] Proposition 8; the arguments hold also for the initial segments of type $W(\omega^\alpha)$; if we are concerned only for the natural sum . Q.E.D.

Thus by the uniqueness ,the natural sum coincides with the operation defined with the inductive rule p_+.

Corollary 1. *The operation defined as before by the inductive rule* p_+, *satisfies the properties* 0.1.2.3.5.6.7.*(See [* Kyritsis C.E*]* lemma 1, *the part of the properties that refer only to the sum).*

Proof: See again the [Kyritsis C.E] proposition 8. Q.E.D.

Since the commutative monoid $W(\omega^\alpha)$ relative the Hessenberg natural sum satisfies the cancellation law, it has a monomorphic embedding in the Universal group of it, which at this case is called also the Grothendick group and it is denoted by $K(W(\omega^\alpha))$. Thus the difference x-y is definable in $W(\omega^\alpha)$ with values in $K(W(\omega^\alpha))$.

See also [Kyritsis C.E] the remark before the proof of the proposition 8. Let an initial segment $W(\omega^\alpha)$ of a principal ordinal number ω^{ω^α}. Let the binary operation denoted by $(.):W(\omega^{\omega^\alpha})^2 \to W(\gamma)$,where the γ is an ordinal number with $\omega^{\omega^\alpha} < \gamma$ defined with definition by transfinite induction and with inductive rule the function

p_*

$p : \bigcup_{x,y<\alpha} W(\gamma)^{W(\omega^{\omega^x})\times W(\omega^{\omega^y})} \to W(\gamma+1)$ such that for every

$f \in W(\gamma)^{W(\omega^{\omega^x})\times W(\omega^{\omega^y})}$

$p_*(f)=S(\{f(x,W(y))+f(W(x),y)-f(W(x),W(y)\}\cap W(\gamma+1))$. Thus there is a unique function $(.):W(\omega^{\omega^\alpha})^2 \to W(\gamma+1)$ such that it satisfies the inductive rule

p_* $x.y=S(\{W(x).y+x.W(y)-W(x).W(y)\}\cap W(\gamma+1))$.

Lemma 2. *The Hessenberg natural product satisfies the inductive rule* p_* .

Proof: See [Kyritsis C.E]Proposition 8 . Q.E.D.

Therefore by the uniqueness of the function (.) this operation coincides with the Hessenberg natural product .

Corollary 3. *Let an initial segment of a principal ordinal number* $\omega^{\omega^{\alpha}}$. *The operations that are defined as before with the inductive rules* p_+, p_* *satisfy the properties* 0.1.2.3.4.5.6.7.8. *(See [* Kyritsis C.E*]* lemma 1*)and coincide with the Hessenberg natural sum and product .*

Proof: See again [Kyritsis C.E] proposition 8.

Corollary 4. (third characterisation)

Let an initial segment of a principal ordinal number $\omega^{\omega^{\alpha}}$. Two operations in W($\omega^{\omega^{\alpha}}$) are the Hessenberg natural operations if and only if they satisfy the inductive rules p_+, p_*.

Proof: Direct from lemma 2 and corollary 3. Q.E.D.

§ **2 .The definition of the Hessenberg natural operations with finitary free algebras** .

In this paragraph we shall prove a key result with respect to the Hessenberg natural operations. We shall prove that the Hessenberg operations are actually free finitary operations definable by the operations of the Natural numbers .

(see [Graetzer G.] about operations of polynomial symbols ch 1 etc.)

Proposition 5. Let an initial segment $W(\omega^{\alpha})$ of an ordinal number of type ω^{α}. The commutative monoid $W(\omega^{\alpha})$ relative to the Hessenberg natural sum is isomorphic with the commutative free monoid $\coprod_{\alpha} N_0$ in the category of commutative monoids .

Remark: The free monoid $\coprod_{\alpha} N_0$ coincides with the polynomial algebra of polynomial symbols of the algebra of type (N_0,.),in other words of commutative monoids with nullary operations the constants of N_0. Since the commutative monoids is an equational class (variety) there are free commutative monoids ;(see [Graetzer G.] ch 4 §25 corollary 2 pp 167).

Proof: Let us define a function h : $\coprod_{\alpha} N_0 \rightarrow W(\omega^{\alpha})$ by $h(x)=\omega^x$ for $x<\alpha$ and $h(n_1x_1 +...+n_kx_k)=n_1\omega_1^x +...+ n_k\omega_k^x$ for any $y=n_1x_1 +...+ n_kx_k \in \coprod_{\alpha} N_0$. The operations in the second part of the defining equation of h are the Hessenberg natural operations .By the definition of $\coprod_{\alpha} N_0$ and the Cantor normal form of ordinal numbers in the Hessenberg operations we get that the h is 1-1 on-to and homomorphism of abelian monoids .Thus an isomorphism of commutative monoids . Q.E.D.

Remark 6. We deduce from the previous proposition that two initial segments $W(\omega^{\alpha})$, $W(\omega^{\beta})$ are algebraically isomorphic as commutative monoids if and only if $\aleph(\alpha)$=

$\aleph(\beta)$, in other words the ordinals α, β have the same cardinality .

Proposition 7. (Fourth characterisation) *Let an initial segment W(* $\omega^{\omega^{\alpha}}$ *) of a principal ordinal number* $\omega^{\omega^{\alpha}}$ *. The commutative semiring W(* $\omega^{\omega^{\alpha}}$ *) relative to the Hessenberg natural operations is isomorphic with the commutative free semiring*

N_0 $\left(\coprod_{\alpha} N_0\right)$ *in the category of commutative semirings with unit .*

Remark : The free commutative semiring with unit N_0 $\left(\coprod_{\alpha} N_0\right)$ coincides with the polynomial algebra of polynomial symbols of the algebrae of type (N_0,+,.) in other words of the commutative semirings with nullary operations the constants of N_0. The commutative semirings with unit are an equational class thus they have free semirings;

(see again [Graetzer G.] ch 4 § 25 corollary 2 pp 167).The semiring N_0 $\left(\coprod_{\alpha} N_0\right)$ is constructed as the semigroup semiring of the semigroup $\left(\coprod_{\alpha} N_0\right)$ written multiplicativelly;(in a way analogous to the construction of the semigroup ring of a semigroup).

Proof: Let us define a function as in the proof of proposition 5

h_2 :N_0 $\left(\coprod_{\alpha} N_0\right)$ $\rightarrow$W($\omega^{\omega^{\alpha}}$) by h_2 (x)=$\omega^{h(x)}$ for $x \in \coprod_{\alpha} N_0$ where the h is as in the proof of the proposition 5 h : $\left(\coprod_{\alpha} N_0\right)$ $\rightarrow$W(ω^{α}) and the $\left(\coprod_{\alpha} N_0\right)$ is written multiplicatively; y $\in N_0$ $\left(\coprod_{\alpha} N_0\right)$ y= n_1x_1 +...+n_kx_k with $x_1,...,x_k$ $\in$ $\left(\coprod_{\alpha} N_0\right)$ h(y)=

$n_1\omega^{h(x_1)} + \dots + n_k\omega^{h(x_k)}$. Again by the definition of the N_0 $\left(\coprod_{\alpha} N_0\right)$ and the uniqueness of the Cantor normal form in the Hessenberg natural operations (see [Kyritsis C.E.] Remark 7,5),b)) we get that the function h is an homomorphism of semirings,1-1 and on-to ;thus an isomorphism of abelian semigroups with unit . Q.E.D.

Remark 8. From the previous proposition and the dependence of the free semiring $N_0^{\left(\coprod_{\alpha} N_0\right)}$, up-to-isomorphism,on the cardinality of the set α, we deduce that two initial segments W($\omega^{\omega^{\alpha}}$) , W($\omega^{\omega^{\beta}}$) are algebraically isomorphic relative to the Hessenberg natural operations if and only if ℵ(α)=ℵ (β); n other words if the ordinal numbers α, β are of the same cardinality .

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