Alternative algebraic definitions of the Hessenberg natural operations in the ordinal numbers.

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Abstract

This paper proves prerequisite results for the theory of Ordinal Real Numbers. In this paper, is proved that any field-inherited abelian operations and the Hessenberg operations ,in the ordinal numbers coincide.It is given an algebraic characterisation of the Hessenberg operations ,that can be described as an abelian, well-ordered,double monoid with cancelation laws.

Key words: *Hessenberg natural operations,ordinal numbers***,** *semirings***,***inductive rules***,***transfinite induction*

Subject Classification of AMS *03,04,08,13,46*

§0.   Introduction   This is a series of five papers that have as goal the definition of topological complete linearly ordered fields (continuous numbers) that include the real numbers and are obtained from the ordinal numbers in a method analogous to the way that Cauchy derived the real numbers from the natural numbers. We may call them linearly ordered Newton-Leibniz numbers. The author started and completed this research in the island of Samos during 1990-1992 . Seven years later (1999) he discovered how such numbers can be interpreted as fields of random variables (*stochastic real numbers*) that links them to applications of Bayesian statistics, stochastic processes and computer procedures. From this point of view it turns out **that the ontology of infinite is the phenomenology of changes of the finite**. In particular *the phenomenology of stochastic changes of the finite can be formulated as ontology of the infinite* .He hopes that in future papers he shall be able to present this perspective in detail. Another application is in the speeds of convergence to infinite of real functions, therefore with applications in the measurement of computational complexity of Turing machines and algorithms.

 It is a wonderful perspective to try to define the Dirac's deltas as natural entities of such stochastic real numbers. If in the completion of rational numbers to real numbers we ramify the equivalence relation of convergent sequences to others that include not only where the sequences converge (if they converge at the same point) but also how fast (if they converge in the same way ,an attribute related also to computer algorithms complexity) ,then we get non-linearly ordered topological fields that contain ordinal numbers (certainly up to ω^a , a= ω^ω) and are closer to practical applications. This approach does not involve the random variables at all, but involves directly sequences of rational numbers as "*Newtonian fluxions*". This creation can be considered as a model of such linearly ordered fields (up-to-characteristic ω^a , a= ω^ω) ,when these linearly ordered fields are defined axiomatically. This gives also a construction of the real numbers with a set which is countable. This does not contradict that all models of the real numbers are isomorphic as the field- isomorphism is not

in this case also an ∈-isomorphism so the Cauchy-real numbers and such a model still have different cardinality. If we want to define in this way *all* the Ordinal real numbers , then it is still possible but then this would give also a device for a model of all the ZFC-set theory! And such a model is indeed possible: By taking again sequences of non-decreasing (in the inclusion) finite sets of ZFC , and requiring that any property ,relation or operation if it is to hold for this set-sequence it must hold finally for each term of the sequence and finite set. In other words we take a minimality relative to the axiom of infinite for every set of it. It is easy to prove that the (absolute) cardinality of such a model is at most 2^ω , that is at most the cardinality of the continuum. We could conceive such a model as the way a computer with its algorithms , data bases tables etc would represents sets of ZFC in a logically consistent way. There is no contradiction with the 2nd-incompletness theorem of Gödel as the argument to prove that it is a model of ZFC-set theory is already outside ZFC-set theory (as are also the arguments of Gödel ,or of Lowenheim-Skolem that gives a countable model of ZFC-set theory).

In this paper are studied the Hessenberg operations in the ordinal numbers, from an algebraic point of view.

The main results are the characterisation theorems 9,10 . They are characterisations of the Hessenberg operations as

a) field-inherited operations in the ordinal numbers,that satisfy two inductive properties .(see proposition 10)

b) operations that satisfy a number of purely algebraic properties, that could be called in short operations of a well-ordered commutative semiring with unit ;(see Lemma 0, proposition 9).

In a next paper I shall give two more algebraic characterisations of the Hessenberg operations as

c) operations defined by transfinite induction in the ordinal numbers and by two recursive rules ,

d) operations of the free semirings in the category of abelian semirings; or as the operations of the formal polynomial algebras of the category of abelian semirings. These characterisations of the Hessenberg operations are independed of the standard non-commutative operations of the ordinal numbers and can be considered as alternative and simpler definitions of them (especially the c),d)).

In particular it is proved that the Hessenberg natural operations are free finitary operations ; We make use of rudimentary techniques relevant to K-theory and Universal Algebra .

The main application of the present results is in the definition of the **ordinal real numbers**.(see [Kyritsis C.E.1991] .By making use of the present results and techniques it is proved in [Kyritsis C.E.1991] that all the three techniques and Hierarchies of *transfinite real numbers* ,see [Glayzal A 1937],of *surreal numbers*, see [Conway J.H. 1976], of *ordinal*

real numbers see [Kyritsis C.E.1991], give by inductive limit or union, the same class of numbers, already known as the class No. We refer to the class No as **"the totally ordered Newton-Leibniz realm of numbers"**).

1. Two algebraic characterisations of the Hessenberg operations in the ordinal numbers.

Let us denote by F a linearly ordered field of characteristic ω (also said of characteristic 0).Let us denote by h a mapping of an initial segment of ordinal numbers, denoted by W(a),in F such that it is 1-1, order preserving and h(0)=0, h(1)=1, h(s(b))=b+1,where s(b) is the sequent of the b , b<a, the b+1 is in the field operations the set h(W(a)) is closed in the field addition and multiplication . We shall call field-inherited operations in the ordinal numbers of W(a), the operations induced by the field, in the initial segment W(a).

(For a reference to standard symbolisms and definitions for ordinal numbers, see [Cohn P.M. 1965] p.1-36 also [Kutatowski K.-Mostowski A. 1968]).

The following properties hold for these field-inherited abelian operations for the ordinals of w(a) (in that case, it is needless to say that a is a limit ordinal).

Lemma 1. For the field-inherited abelian operations in the initial segment w(a) of ordinal numbers, the followings hold (x,y,z,c,x',y', ∈w(a)).

0) $s(x) = x+1$ for every x ∈ w(a)

1) $x+y = y+x$, $x.y = y.x$

2) $x+(y+z) = (x+y)+z$, $x.(y.z) = (x.y).z$

3) $x+0 = 0+x = x$,

 $x.0 = 0$

 $x.1 = 1.x = x$

4) $x(y+z) = xy+xz$

5) $\left. \begin{array}{ll} x + c = y + c \Rightarrow x = y & x \cdot c = y \cdot c \\ c \in w(a) & c \neq 0 \end{array} \right\} \Rightarrow x = y$

6) If x>y, x'>y' then x+x'>y+y', and xx'+yy'>xy'+yx'

The proof of the previous lemma is direct from the properties of a linearly ordered field.

We mention two more properties that they will be of significance in the followings paragraphs.

7) The $w(x+y)$ is a cofinal set with the $\{w(x)+x \cup x+w(y)\}$ and we write $cf(w(x+y)) = cf\{w(x)+y \cup x+w(y)\}$.

8) The $W(x.y)$ is cofinal set with the

$h^{-1}(\{h(y)h(w(x))+h(x)h(w(y))-h(w(x)).h(w(y))\})$

and we write

$cf(w(x.y)) = cf\ h^{-1}\ (\{h(y)h(w(x))+h(x)h(w(y))-h(w(x)).h(w(y))\})$

To continue our argument we need a many-variables form of transfinite induction.

Let a_i $i = 1,...,n$ $n \in N$ ordinal numbers and $(b_1,...b_n) \in w(\alpha_i)x...xw(\alpha_n)$. We define as simultenous initial segment of n-variables defined by $(b_1,...b_n)$, the set $w((b_1,...b_n)) = \cup$ $w(b_1)x...x\{b_1\}x...xw(b_n)$ for every

$$i \in I \subseteq \{1,...,n\}$$

$I \overset{\subseteq}{\neq} \{1,...n\}$, or $w((b_1,...,b_n)) = \cup w\{b_1\}x...xw\{b_i\}x...x\{b_n\}$ for

$$i \in I \subseteq \{1,...,n\}$$

every $I \subseteq \{1,...n\}$ with $I \neq \varnothing$.

Lemma 2. (many-variables transfinite induction)

Let $A \subseteq w(a_1)x...xw(a_n)$ such that

1. $(0,...,0) \in A$

2. For every $(b_1,...b_n) \in w(a_1)x...xw(a_n)$ it holds that $w((b_1,...,b_n)) \subseteq A \Rightarrow (b_1,...,b_n) \in A$.

By 1.2. we infer that $A = w(a_1)x...xw(a_n)$.

Lemma 3. (many-variables definition by transfinite

   induction)

Let a set A and ordinal numberss $a_1,...,a_n$. Let a set denoted by B, such that it is sufficient for inductive rules $h\ B \rightarrow A$:

This means that :

a) $$B \underset{(b_1,...b_n)}{\subseteq} \cup A^{w((b_1,...b_n))}, \quad (b_1,...,b_n) \in w(a_1)x...xw(\alpha_n)$$

The set B is a set of functions, denoted by $f_{(b_1,...,b_n)}$ and defined on simulteneous initial segments with values in A. $f_{(b_1,...,b_n)}$: $w((b_1,...,b_n)) \to A$.

If $f_{(b_1,...,b_n)} \in B$ and $c_1 < b_1,...,c_n < b_n$ then $\dfrac{f(b_1,...,b_n)}{w((c_1,...c_n))} \in B$

b) For every $(c_1,...,c_n) \in w(\alpha_1)x...xw(\alpha_n)$ there is a $f_{(c_1,...c_n)}$ such that $f_{(c_1,...c_n)} \in B$

c) Let $f \in B^{w((\alpha_1,...,\alpha_n))}$ and let us denote the value of f at $(b_1,...,b_n)$ with $f_{(b_1,...,b_n)}$. Let us suppose that it holds that whenever $c_1 \le b_1,b'_1,...,c_n \le b_n,b'_n$, $(b_1,...,b_n),(b'_1,...,b'_n) \in$

$w(\alpha_1)x...xw(\alpha_n)$ then $\dfrac{f(b_1,...b_n)}{w((c_1,...c_n))} = f_{(b'_1,...b'_n)_{w((c_1,...,c_n))}}$

Then let us suppose that we get as a consequence that the function defined by

$g(b_1,...,b_n) = f_{(b_1,...,b_n)}(b_1,...,b_n),(b_1,...,b_n) \in w((\alpha_1,...,\alpha_n))$, belongs to B

It holds that for every function $h:B \to A$

(called many -variables transfinite inductive rule) there is one and only one function f defined on $w(a_1)x...xw(a_n)$ with values in A such that for every $(b_1,...,b_n) \in$

$w(\alpha_1)x...xw(\alpha_n)$ it holds that $f(b_1,...,b_n) = h\left(\dfrac{f}{w((b_1,...,b_n))}\right)$

Remark:We notice that even for one variable this version of the definition by transfinite induction is somehow different from that which appears usually in the bibliography (e.g.see [Kutayowski K. –Mostowski A 1968] §4 pp 233).It uses not all the set $A^{w(\alpha)}$,but only a subset of it, sufficient for recursive rules .The proof ,for one variable, is nevertheless exactly the same as with the ordinary version .

In order to save space and because the proofs are not directly relevant to our subject we will not give the proofs of lemma 2 and 3 but we will mention that they are analogous, without serious difficulties, to the ones with one-variable only (see e.g. [Kutayowski K. – Mostowski A 1968],[Lang S.1984]).

Proposition 4. (Uniqueness)

Any two pairs of field-inherited operations in the initial segment $w(a)$ of ordinals, satisfying properties 7,8 of lemma 1 (a is a limit ordinal) are isomorphic .

Proof: Let a monomorphic embedding denoted by h of $w(a)$, as is described in the beggining of the paragraph in two linearly ordred fields denoted by F_1, F_2.

Let the two pairs of field inherited operations in $w(a)$ be denoted by $((+,.)$ $(\oplus,\circ)$ respectively. They satisfy the properties 0.1.2.3.4.5.6.7.8.of lemma 1 . Suppose that the operations $+$, $\oplus$ coincide for the set $w((b_1,b_2)) \subseteq w(\alpha)^2$ where $b_1,b_2 \in w(\alpha)$ $w((b_1,b_2)) = w(b_1) \times w(b_2) \cup (\{b_1\} \times w(b_2)) \cup (w(b_1) \times \{b_2\})$. Then by property 7 $b_1 + b_2 = s(b_1 + w(b_2) \cup w(b_1) + b_2) =$ (by the hypothesis of transfinite induction) $=$

$$s(b_1 \oplus w(b_2) \cup w(b_1) \oplus b_2) = b_1 \oplus b_2;$$

Where by S(A) we denote the sequent of the set A .Thus by lemma 2 the operations $+$, $\oplus$ coincide on $w(a) \times w(a)$.

Then the set $w(a)$ is an ordered abelian monoid relative to addition, with cancelation law.

The Grothendieck groups of $w(a)$ for both $+$, and $\oplus$ coincide, and we denote it by $k(w(a))$ (see for the definition of Grothendieck group [Lang S. 1984] Ch1 §4 p. 44 or [Cohn P.M. 1965] ch vii §3 pp 263). Thus also the opposite $-x$ of an element x of $w(a)$ is the same in the Grothendieck group $k(w(a))$ of $w(a)$ for both the two operations $+$ and $\oplus$.

Suppose also that the operations $\bullet,\circ$ coincide for the set $w((b_1, b_2))$. Then by property 8

$$b_1 \bullet b_2 = s\left(h^{-1}\left(\{h(b_1)h(w(b_2)) + h(w(b_1)) \bullet h(b_2) - h(w(b_1)) \bullet h(w(b_2))\}\right)\right) =$$

(because $+$, and $\oplus$ are isomorphic and the hypothesis of transfinite induction for $\bullet,\circ) = s(h^{-1}(\{h(b_1) \circ h(w(b_2)) \oplus h(w(b_1)) \circ h(b_2) - h(w(b_1)) \circ h(w(b_2))\})) = b_1 \circ b_2$. Hence by lemma 2 the two operations $\bullet,\circ$ coincide on the whole set $w(a) \times w(a)$ Q.E.D.

The next step is to find the relation of field-inherited operations in an initial segment of ordinals with the Hessenberg operations . It will turn out that, if they satisfy the properties 7.8.of lemma 1 ,then they are nothing else than the Hessenberg-Conway natural sum and product (see [Kutatoski K.Mostowski A. 1968] ch VII §7 p. 252-253 exercises 1. 2. 3.) and [Frankel A.A.1953] pp. 591-594 also [Conway J.H. 1976] ch2 p. 27-28).

The way in which the Hessenberg operations are defined, traditionally , depends on the standard non-commutative operation on ordinals.

In order to define the Hessenberg-Conway operations in the traditional way ,we remind that :

Lemma 6 (Cantor normal form).

For every ordinal a there exists a natural number n and finite sequences : $b_1,...,b_2$ of natural numbers and ordinal numbers $a_1, a_2, ... a_n$ with $a_1 > ... > a_n$ such that $\alpha = \omega^{a_1} b_1 + ... + \omega^{a_n} b_n$ (For a proof see for instance [Kutatowski K.-Mostowski A. 1968] ch VII §7 p. 248-251).

Then we get for the two ordinal numbers α, b, by adding terms with zero coefficients, to make their Cantor normal forms of equal length , that $\alpha = \omega^{\xi_1} p_1 + ... + \omega^{\xi_n} p_n$ $b = \omega^{\xi_1} q_1 + ... \omega^{\xi_n} q_n$;

we define the natural sum (we denote it by (+)) with $\alpha(+)b = \omega^{\xi_1}(p_1 + q_1) + ... + \omega^{\xi_n}(p_n + q_n).$ The natural product, denoted by $\alpha(.)b$ is defined to be the ordinal arising by multiplication (using distributive and associative laws) from the Cantor normal forms of a and b and by using the rule: $\omega^x(.)\omega^y = \omega^{x(+)y}$ to multiply powers of ω. As a result we get for instance that

Remark 7

1)   The normal form of a can also be written in the standard Hessenberg-Conway operations that is

$$\alpha = \omega^{\xi_1}(.)p_1(+)...(+)\omega^{\xi_n}(.)p_n.$$

2)   The sum a(+)b is an increasing function of a and b.

3)   If $\xi < \omega^{\omega^\alpha}$ and $\eta < \omega^{\omega^\alpha}$ then $\xi.\eta < \omega^{\omega^\alpha}$ for ordinals ζ, η, α and conversely if an ordinal j satisfies the condition: "if $\xi<j$ and $\eta<j$ then $\xi.\eta<j$" then there exists an ordinal number α such that $\xi = \omega^{\omega^\alpha}$;we call ordinal numbers of the type ω^{ω^α} **principal ordinals** of the Hessenberg operations.(see [Kutatowski K.-Mostowski A. 1968] ch vii paragraph 7,p 253) This has also as a consequence that we define the Hessenberg-Conway natural operations only for initial segments of the type $W^{\left(\omega^{\omega^\alpha}\right)}$ for some ordinal number α (we will call them principal initial segments).

4)   The Hessenberg-Conway natural operation restricted on the set of Natural numbers coincide with the ordinary sum and product of natural numbers.

5)   The operation "powers of ω" , through the Hessendberg-Conway natural operation, can be defined as follows:

a) $\omega^{(0)} = 1$ $\omega^{(1)} = \omega$ if ξ is a limit ordinal $\omega^{(\xi)} = \sup_{\eta<\xi} \omega^{(\eta)}$

b) If ξ is not a limit ordinal then there exists an ordinal η with $\eta(+)1 = s(\eta) = \xi$ and we define $\omega^{(\xi)} = \omega^{\eta}(.)\omega$.

It holds, (this happens especially for the base ω), that these "natural powers" of ω coincide with the standard powers of ω defined through the standard non-commutative multiplication of ordinal numbers (this can be proved with transtinite induction since $\omega^{\eta}.\omega = \omega^{(\eta)} \circ \omega$. This gives us the right to express any ordinal number α in Cantor normal form, exclusively with natural operations:

$$\alpha = \omega^{(\xi_1)}(.)p_1(+)...(+)\omega^{(\xi_n)}(.)p_n$$

6) Also we notice that, the natural difference denoted by a (-) b, of two ordinals a, b in Cantor normal forms $a = \omega^{(\xi_1)}(.)p_1(+)...(+)\omega^{(\xi_n)}(.)p_n$

$$b = \omega^{(\xi_1)}(.)q_1(+)...(+)\omega^{(\xi_n)}(.)q_n$$, is defined only if $p_1 \geq q_1 ... p_n \geq q_n$.

7) We notice that if $\xi_i < \xi_j$ for two ordinals then $\omega^{\xi_i} < \omega^{\xi_j}$ but also $\omega^{\xi_i}.a < \omega^{\xi_j}.b$

for every pair of non-zero natural numbers a,b. (in the standard non-commutative operations on ordinals). But this has as consequence that the ordering of a finite set of ordinal numbers in Cantor normal form (normalizing the Cantor normal forms by adding terms with zero coefficients so that all of them have the same set of exponents) is isomorphic (similar) to the lexicographical ordering of the coefficients of the normal forms.

Proposition 8. For every principal initial segment of ordinal numbers, the Hessenberg natural operations satisfy the properties 0.1.2.3.4.5.6.7.8. of lemma 1.

Remark. From the moment we have proved the properties 0.1.2.3.4.5.6. for the natural operations in the principal initial segment w(a), there is the Grothendieck group k(w(a)) of the monoid relative to sum, w(a) such that the w(a) is monomorphicaly embedded in k(w(a) (because of cancelation law) and also there is an ordering in k(w(a)) that restricted on w(a) coincides with the standard ordering in w(a).

Then the difference that occurs in property 8 has meaning and also the statement of property 8 itself has meaning (see [Lang S. 1984] Ch I §9 p. 44).We denote by h the monomorphism of the W(a) in the K(W(a)) .

Proof. The properties 0.1.2.3.4. are directly proved from the definition of the natural operations. Let us check the property 5. Namely , the cancelation laws. Let us suppose that y,x,c, are ordinal numbers with $y,x,c \in w\left(\omega^{\omega^{\alpha}}\right)$ for some ordinal a and their Cantor normal form , in natural operations , are

$$x = \omega^{(\xi_1)}(.)p_1(+)...(+)\omega^{(\xi_n)}(.)p_n$$

$$c = \omega^{(\xi_1)}(.)c_1(+)...(+)\omega^{(\xi_n)}(.)c_n$$

$$y = \omega^{(\xi_1)}(.)q_1(+)...(+)\omega^{(\xi_n)}(.)q_n \quad p_i, c_i, y_1 \in No$$

then

$$x(+)c = \omega^{(\xi_1)}(.)\left(p_1 + c_1\right)(+)...(+)\omega^{(\xi_n)}(.)\left(p_n + c_n\right)$$

$$y(+)c = \omega^{(\xi_1)}(.)\left(q_1 + c_1\right)(+)...(+)\omega^{(\xi_n)}(.)\left(q_n + c_n\right)$$

hence

$$x(+)c = y(+)c \Rightarrow p_i + c_i = q_i + c_i \qquad i = 1,...,n$$

and by cancelation law for addition in natural numbers we deduce that $p_i = q_i$ $i=1,...,n$ hence $x = y$.

Also

$$x(.)c = \sum_{1 \le i \ j \le n} \omega^{\xi_i(+)\xi_j}(.)\left(p_i.c_j\right) \qquad \text{and}$$

$$y(.)c = \sum_{1 \le i \ j \le n} \omega^{\xi_i(+)\xi_j}(.)\left(q_i.c_j\right) \qquad \text{and if } c \ne 0$$

and $x(.)c = y(.)c$ then $p_i.c_j = q_i.c_j$ with not all of c_j equal to zero. Say $c_{j_0} \ne 0$, then $p_i.c_{j_0} = q_i.c_{j_0}$ for every $i = 1,...,n$ hence $p_i = q_i$ and $x=y$.

Let us check the property 6. The first part of property 6 is immediate from Remark 7, 2. Let, furthermore, $x',y' \subset w\left(\omega^{\omega^\alpha}\right)$ with Cantor normal form (changing the ξ_i, in order to have the same exponents for all x, y, x', y')

$$x' = \sum_1^n \omega^{(\xi_i)}(.)p_i' \qquad\qquad y' = \sum_1^n \omega^{(\xi_i)}(.)q_i'$$

with $p_i', q_i' \in No$ and with summation interpreted as natural sum. By hypothesis $x'>y'$, $x>y$.

Then

$$x(.)x' = \sum_{ij} \omega^{\xi_i(+)\xi_j}(.)\left(p_i.p_j'\right) \qquad \text{and}$$

$$y(.)y' = \sum_{ij} \omega^{\xi_i(+)\xi_j}(.)\left(q_i.q_j'\right)$$

and the coefficient of the monomial of greatest exporent of $x(.)x'(+)y(.)y'$ is $p_1p_1'+ q_1q_1'$ and of $x(.)y'(+)y(.)x'$ is $p_1q_1'+q_1p_1'$. But $p_1p_1' + q_1q_1' - p_1q_1' - q_1p_1' = p_1(p_1'-q_1') - q_1(p_1'-q_1') = (p_1-q_1).(p_1'-q_1') > 0$ which is a product of the positive factors p_1-q_1, $(p_1'-q_1')$ hence it is positive. By Remark 7,7 because $p_1 q_1' + q_1 q_1' > p_1 q_1' + q_1 p_1$ we deduce that $x(.)x'(+)y(.)y'>x(.)y'(+)x'(.)y$. Next we prove the property 7. Let x' as before but also satisfying $x' \in w(x)$ that is $x'<x$. Then by property 5 we deduce that $w(x)+y \subseteq w(x+y)$.

Conversely let $z<x+y$ $z \in w(x+y)$. Let the Cantor normal form of z be $z = \sum_1^n \omega^{\xi_1}(.)r_i$ (we rearrange appropriately the normal forms of x, x', y, y', Z) with $r_i \in$No. From the last inequality we get that in the lexiographical ordering it holds that $(r_i,...,r_n)<(p_1+q_1,...,p_n+q_n)(*)$

Let $k_i = \max_{1\le i\le n}\{r_i,q_i\}$ and $\lambda_i = \max_{1\le i\le n}\{r_i,p_i\}$

Let $z_1' = \sum_i^n \omega^{\xi_i} k_i$ and $z_2' = \sum_i^n \omega^{\xi_i} \lambda_i$ Then the following ordinals are defined: $z_1'(-)z$, $z_1'(-)y$, $z_2'(-)z$, $z_2'(-)y$, and also by Remark 7,7. It holds that $z\le z_1'$ $z\le z_2'$, $y\le z_1'$ $x\le z_2'$. From the inequality (*) and the inequality (**) $q_i \le p_i + q_i$ $i = 1,...,n$ and the definition of k_i we infer that it holds in the lexicographical ordering, the inequality $(k_1,...,k_n) \le (p_1 +q_1,...,p_n + q_n)$ similarly $(\lambda_i,...,\lambda_n) \le (p_1 + q_1,...,p_n + q_n)$. Hence by Remark 7.7. it holds that $z_1'\le x(+)y$ and $z_2'\le x(+)y$ If for both z_1', z_2' holds that $z_1'=x(+)y=z_2'$.

Then $\max_i \{r_i, q_i\}= \max_i \{r_i, p_i\}=p_i + q_i$, hence $r_i = p_i + q_i$ $i=1,...,n$.

But then $z=x(+)y$,contradiction.

Let us suppose then, that $z_1'<x(+)y$.

Then if $z'' = z_1'(-)y$ by the last inequality we get that $z''(+)y = z_1'(-)y(+)y = z_1' < x(+)y$ or $z''(+)y < x(+)y$.

That is we proved that for every $z \in w(x+y)$ there is z'' an other ordinal with $z\le z''(+)y<x+y$. If $z''\ge x$ then $z''(+)y\ge x(+)y$ contradiction, hence $z''<x$ that is $z'' \in w(x)$ and $z''(+)y \in w(x)+y$. From this and also that $w(x)+y\subseteq w(x+y)$,that we have already proved ,we deduce that $w(x+y)$ and $\{w(x)+y\cup x+w(y)\}$ are cofinal sets; we write $cf(w(x+y))=cf(\{w(x)+y\cup x+w(y)\})$. In other words we haved proved the property 7.

Let us prove the property 8. As we have already remarked the difference is to be understood in the extension of the additive monoid $w(a)$ into the linearly ordered Grothendick group $k(w(a))$. The ordering in $k(w(a))$ is defined by: $(x,y)\le(x',y')\Leftrightarrow x+y'\le x'+y$.

Where by (x,y) we denote the equivalence class of the free abelian group generated by w(a), which is denoted by $F_{a,b}$ (w(a)) (k(w(a)) = $F_{a,b}$ (w(a))/ ([x+y]-[x]-[y])), in the process of taking the quotient by the normal subgroup generated by the elements of the form [x+y]-[x]-[y] in $F_{a,b}$ (w(a)) (the corresponding generator of x $\in$ w(a), in $F_{a,b}$ (w(a) we denote by [x]), that is defined by the representative x+(-y). Needless to mention that the natural difference in w(a), isn't but an instance of difference in k(w(a)).

We make clear that h^{-1}({h(x)(.)h(w(y))(+)h(w(x))(.)h(y) -h(w(x))(.)h(w(y))}={v|v $\in$ w(α) and v =h^{-1} (h(x)(.)h(y)'(+)h(x')(.)h(y) - h(x')(.)h(y')) with x' $\in$ w(x) y' $\in$ w(y) and x,y $\in$ w(a)}. By the property 6 we get that h(x)(.)h(y')(+)h(x')(.)h(y) <h(x)(.)h(y)(+)h(x')(.)h(y') hence

h(x)(.)h(y')(+)h(x')(.)h(y) - h(x')(.)h(y') < h(x)(.)h(y)

hence h^{-1}({h(x)(.)h(w(y))(+)h(w(x))(.)h(y) - h(w(x))(.)h(w(y))}} $\subseteq$ w(x(.)y).

Conversely, let, z $\in$ w(x(.)y), that is z<x(.)y.

If x(.)y = $\sum_{1\leq i,j\subseteq n} \omega^{\xi_i(+)\xi_j}(.)\left(p_i.q_i\right)$ then we also write for the normal form of

z: z = $\sum_{1\leq i,j\subseteq n} \omega^{\xi_i(+)\xi_j}(.)r_{ij}$ and $r_{ij} \in$ No. By Remark 7,7. We deduce that in the lexicographical ordering it holds that $(r_{11},...,r_{ij},...,r_{n,n}$) < $(p_1p_1,...,p_ip_j,...,p_n.p_n)$. It is sufficient to prove that for every $(r_{11},...,r_{ij},...,r_{n,n})$< $(p_1p_1,...,p_ip_j,...,p_n.p_n)$ there are $(p_1',...,p_n')$ and $(q_1',...,q_n')$, p_i, q_j $\in$ No with $(p_1',...,p_n')$< $(p_1,...,p_n)$ and $(q_1',...,q_n')$< $(q_1,...,q_n)$ such that $(r_{11},...,r_{ij},...,r_{n,n})$ $\leq$ $(p_1q_1'+p_1'q_1 - p_1'.q_1',...,p_n.q_n'+p_n'.q_n - p_n'.q_n')$ < $p_1.q_1,..., p_n.q_n)$.

But the property 8 holds for a=ω, that is for the natural numbers. Hence there are p_1', q_1' with $p_1' < p_1$ $q_1' < q_1$ and $r_{11} \leq p_1$ $q_1'+p_1'q_1 - p_1'.q_1' < p_1'$ q_1 and completing with arbitrary p_i', q_i' i = 2,...,n) that give positive the terms p_i q_j' + $p_i'q_j$ - $p_i'q_j'$ (by elementary arithmetic of natural numbers this is always possible) we define

x' = $\sum_1^n \omega^{\xi_i}(.)p_i'$ and y' = $\sum_1^n \omega^{\xi_i}(.)q_i'$.

By the lexicographical ordering it holds that x' $\in$ w(x), y' $\in$ w(y) and h(z) $\leq$ h(x)(.)h(y')(+)h(x')(.)h(y) - h(x')(.)h(y') <h(x)(.)h(y). Hence the sets W(x(.)y) and h^{-1}({h(x)(.)h(w(y)) (+) h(w(x))(.)h(y) - h(w(x))(.)h(w(y))}) are cofinal and we write

cf(w(x(.)y)) = cfh^{-1}({h(x)(.)h(w(y)) (+) h(w(x))(.)h(y) - h(w(x))(.)h(w(y))}).

This is the end of the proof of the property 8.   Q.E.D.

Corollary 9 (first characterisation)

Every pair of operations in a principal initial segment of ordinal that satisfy the properties 0.1.2.3.4.5.6.7.8 of lemma 1 ,.is unique up-to-isomorphism and coincides with the Hessenberg natural operations .

Remark: The difference that appears in the property 8 is defined as in the remark after the proposition 8 .

Proof:Direct after the proposition 4 and 8   Q.E.D.

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Corollary 10.(Second characterisation)

Every pair of field -inherited operations in a principal initial segment of ordinals, $w(a)$ that satisfy the properties 7. 8. coincides with the natural sum and product of Hessenberg .

(For the existence of field-inherited operations in the ordinal numbers see [C Conway J.H.1976] ch note pp 28 .)

Proof: The proof is immediate from proposition 4 and 8.Q.E.D.

Remark.11 It seems that N.L.Alling in his publications:

a)On the existence of real closed Fields that are η_α -sets of power ω_α Transactions Amer.Math.Soc. 103 (1962) pp 341-352.

b)Conway's field of surreal numbers. Trans.Amer.Math.Soc.287 (1985) pp.365-386.

c)Fountations of Analysis over Surreal number Fields .Math. studies 141 North-Holland 1987.

he is unaweare that if an initial segment of ordinals is contained in a set-field and it is cofinal with the field ,(and it induces the Hessenbeg operations in it) then it has to be an initial segment of a principal ordinal that is of type $\omega^{\omega^{\omega^\alpha}}$ (see [Kutatowski K-Mostowski A.1968] ch VII §7 p. 252-253 exercises 1. 2. 3.)

Thus properties 0.1.2.3.4.5.6.7.8. can be taken as an axiomatic definition of the Hessenberg operations without having to mention the non-commutative ordinal operations.

In a forthcoming paper, I will be able to prove the non-contradictory of properties 0.1.2.3.4.5.6.7.8. (actually the existence in Zermelo-Frankel set theory, of the operations +,.) without using the non-commutative ordinal operations,neither field-inherited operations. but through transfinite induction and other methods of universal algebra .

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List of special symbols

ω : Small Greek letter omega, the first infinit number.

α, b : Small Greek letter alfa, an ordinal.

Ω_1 : Capital Greek letter omega with   the subscript one.

$\aleph(x)$: Aleph of x, the cardinality of the set X.

 N: the fisrt capital letter of the Hebrew alfabet.

$\oplus$,$\circ$: Natural sum and product of G. Hessenberg plus and point in parenthesis.