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The Cauchy Problem for the Navier-Stokes and Euler Equations in Three-Dimensional Space and the Vanishing Viscosity Limit

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Statutory Declaration
I lawfully declare here with statutorily that the present dissertation thesis was carried out under the international ethical and academical rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis.

Theodora Syntaka

Dedicated to my grandfather,
Athanasios Syntakas.

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Περίληψη

Στο Κεφάλαιο 1, το οποίο είναι η εισαγωγή της παρούσας εργασίας, πρώτα παρουσιάζουμε τις εξισώσεις Navier-Stokes και Euler καθώς και κάποιες αναφορές προηγούμενων εργασιών σε αυτές και έπειτα τα κύρια αποτελέσματα που πραγματεύεται αυτή η εργασία.

Στο Κεφάλαιο 2, εισάγουμε κάποιες προκαταρκτικές έννοιες οι οποίες χρησιμοποιούνται στα επόμενα κεφάλαια.

Στο Κεφάλαιο 3, παρουσιάζουμε το πρώτο αποτέλεσμα, το οποίο είναι η ύπαρξη και μοναδικότητα ισχυρής λύσης u_ν, τοπικά στον χρόνο, του προβλήματος αρχικών τιμών για την εξίσωση Navier-Stokes στον $\mathbb{R}^3$ σε χώρους Sobolev, με συντελεστή ιξώδους $\nu > 0$ καθώς και η σύγκλιση των λύσεων u_ν προς τη μοναδική λύση u_0 του προβλήματος αρχικών τιμών για την εξίσωση του Euler, με την ίδια αρχική τιμή, όταν το ν τείνει στο 0. Βλέπε Θεώρημα 1.2.1. Το αποτέλεσμα αυτό οφείλεται στον T. Kato [15].

Στο Κεφάλαιο 4, περιέχεται το δεύτερο αποτέλεσμα, το οποίο είναι η ύπαρξη μοναδικής ισχυρής λύσης, σε χώρους Lebesgue $L^p(\mathbb{R}^3)$ για ορισμένες γενικότερες τιμές $p \in [1, \infty]$, του προβλήματος αρχικών τιμών για την εξίσωση Navier-Stokes, τοπικά στον χρόνο. Επιπλέον, στην περίπτωση μικρών αρχικών δεδομένων, προκύπτει η ύπαρξη μοναδικής ισχυρής λύσης ολικά στον χρόνο. Βλέπε Θεώρημα 1.2.2. Αυτό το αποτέλεσμα εξασφαλίζεται εφαρμόζοντας μία μέθοδο η οποία οφείλεται επίσης στον T. Kato [16].

Τέλος, στο Κεφάλαιο 5, αποδεικνύουμε το τρίτο αποτέλεσμα της παρούσας εργασίας, το οποίο βασίζεται στο διάσημο άρθρο των L. Caffarelli, R. Kohn, L. Nirenberg [5]. Παρουσιάζουμε, δηλαδή, την απόδειξη ότι το σύνολο των ιδιαζόντων σημείων κατάλληλων ασθενών λύσεων της εξίσωσης Navier-Stokes στον $\mathbb{R}^3$ (δηλαδή σημείων στα οποία οι λύσεις αυτές είναι μη φραγμένες) έχει μέτρο Hausdorff διάστασης 1 ίσο με το 0. Βλέπε Θεώρημα 1.2.3.

Abstract

In Chapter 1, which is the introduction of the present thesis, we present first, the Navier-Stokes and Euler equations as well as some references of previous works on these equations and, second, the main results of this thesis.

In Chapter 2, we introduce some preliminary concepts which will be used in the chapters that follow.

In Chapter 3, we present the first result, which is the existence and uniqueness of strong solutions u_ν, locally-in-time, of the Cauchy problem of the Navier-Stokes equations in $\mathbb{R}^3$ in Sobolev spaces, with viscosity coefficient $\nu > 0$ and the vanishing viscosity limit of the solutions u_ν to the unique solution u_0 of the Cauchy problem of the Euler equation, with the same initial data, as ν tends to 0. See Theorem 1.2.1. This result is due to T.Kato [15].

Chapter 4 contains the second result, which is the existence of a unique strong solution, in Lebesque spaces $L^p(\mathbb{R}^3)$ for certain more general values of $p \in [1, \infty]$, of the Cauchy problem of the Navier-Stokes equations, locally-in-time. Moreover, in the case of small initial data, we obtain existence of a unique strong solution, globally-in-time. See Theorem 1.2.2. This result is obtained by applying a method which is due to T. Kato [16].

Finally, in Chapter 5, we prove the third result of the present thesis, which is based on the celebrated article of L. Caffarelli, R. Kohn and L. Nirenberg [5]. That is to say, we present the proof that the set of singular points of suitable weak solutions of the Navier-Stokes equations in $\mathbb{R}^3$ (i.e., of points where these solutions are unbounded) has Hausdorff measure of dimension 1 equal to 0. See Theorem 1.2.3.

Contents

CHAPTER 1

Introduction

1.1 The Navier-Stokes and Euler Equations

The Navier-Stokes equations, named after the French engineer and physicist Claude-Louis Navier (1785-1835) and the Anglo-Irish physicist and mathematician George Gabriel Stokes (1819-1903), are a system of equations which govern the motion of viscous fluids, such as, e.g., oil and - to a much lesser extent - water, air etc., where for the first example the viscosity is much higher than the one of the latter examples. These equations are derived from Newton's second law, that is, mass times acceleration equals force ($m \cdot \alpha = F$). They are used in fluid mechanics, meteorology, thermodynamics etc, and they are considered to be the most basic equations which describe a fluid flow.

The Navier-Stokes equations are second order non-linear equations of parabolic type, and the non-linear term is given by $(u \cdot \nabla)u$. Due to this nonlinearity, their mathematical study is rather difficult.
The (incompressible) Navier-Stokes equations in $\mathbb{R}^n$ are given by

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, & \text{in } \mathbb{R}^n \times (0,\infty) \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) = a, & \text{on } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$, $u(x,t) = (u_1(x,t), ..., u_n(x,t))$ is the velocity field of the fluid, $p\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, p is the pressure, $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is the given initial data, $f\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$, $f(x,t) = (f_1(x,t), ..., f_n(x,t))$ is the given external force, $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$ is the gradient vector in the space variables, and $\Delta = \nabla \cdot \nabla = \sum_{i=1}^{n} \frac{\partial^2}{\partial xi^2}$ is the Laplacian in the space variables. In addition, the equation $\operatorname{div} u = \nabla \cdot u = 0$ is the incompressibility constraint. This condition enforces "constant density" within an infinitesimal volume that moves with the flow velocity. Finally, $\nu > 0$ is the given kinematic constant viscosity. In the case of $\nu = 0$ (1.1) becomes the (incompressible) Euler equations. We point out that $(u,p) \in \mathbb{R}^{n+1}$ are the unknown functions of the system (1.1) of $n+1$ equations and that we have only the n initial conditiions $u(\cdot,0) = a$.

The Navier-Stokes equations are of particular interest, not only for their utility, but also for the reason that there is no proof or counter-example yet for the existence of globally smooth solutions in $\mathbb{R}^3 \times [0,\infty)$, see Chapter 3. This is one of the seven important open problems in Mathematics for the solution of which the Clay Mathematics Institute has offered a prize of one-million US-Dollars, see, e.g., [12] and [25]. Standard reference works on the mathematical analysis for the Navier-Stokes and Euler equations are, e.g., [23], [3], [33], [32]. In the present thesis, we consider solely the three-dimensional case (in particular for the Navier-Stokes equations) which is the physically more relevant one and, also the mathematically more interesting. For the questions related to the Cauchy problem of the Navier-Stokes and Euler equations in the two-dimensional case, see [20].

1.2 The main results

In the following, we give a short overview of the results presented in this thesis. These results concern the existence and uniqueness of local-in-time strong solutions in L^2-based Sobolev-spaces and more generally L^p-spaces obtained by T. Kato in 1972 [15], 1984 [16], respectively, and moreover the celebrated partial regularity result of Caffarelli-Kohn-Nirenberg in 1982 [5].

The system (1.1) can be transformed into an abstract evolution equation of the form

$$\begin{cases} \frac{du}{dt} + \nu A u = F(u,u) + b, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) = a, & \text{on } \mathbb{R}^n, \end{cases} \tag{1.2}$$

where F is the bilinear operator

$$F(u,v) = -P(u\cdot\nabla)v. \quad F\colon L^2(\mathbb{R})^3 \times H_\sigma \to H_\sigma.$$

Here, P is the orthogonal projection of $L^2(\mathbb{R})^3$ onto H_σ, which consists of all solenoidal vectors (i.e. $\operatorname{div} u = 0$) of the space $L^2(\mathbb{R}^3)$, that is to say H_σ is the closure of $\{\phi \in C_0^\infty(\mathbb{R}^3)^3 : \operatorname{div}\phi = 0\}$ in $L^2(\mathbb{R}^3)^3$, see [15]. Furthermore, $b(t) \in H_\sigma$ with $b(t) = Pf(t)$ is the projection of the given external force f to H_σ, $a \in H_\sigma$ is the initial velocity and $A = -\Delta$ is a nonnegative self-adjoint operator in H_σ. We denote by $(1.2)_0$ the equation (1.2) for $\nu = 0$, which is the Euler equation. Moreover, we denote by $H^m_\sigma = H^m \cap H_\sigma$, where $H^m(\mathbb{R}^3)^3 = W^{m,2}(\mathbb{R}^3)^3$ the Sobolev space of order m in $L^2(\mathbb{R}^3)^3$. Also, the space $C([0,T_0];H^m_\sigma)$ consists of all continuous functions $u\colon [0,T_0] \to H^m_\sigma$ with finite norm, $AC([0,T_0];H^m_\sigma)$ consists of all absolutely continuous functions $u\colon [0,T_0] \to H^m_\sigma$ with finite norm and $L^1([0,T_0];H^m_\sigma)$ consists of all integrable functions $u\colon [0,T_0] \to H^m_\sigma$ with finite norm. See Chapter 2.

In Chapter 3, we present the local-in-time existence and uniqueness of strong solutions of Navier-Stokes and Euler equations in $\mathbb{R}^3$ and the vanishing viscosity limit of solutions of the Navier-Stokes equation to the solutions of the Euler equation, which is based on the article [15] of T. Kato. More precisely, we want to prove that the Cauchy problem for the Navier-Stokes equation in $\mathbb{R}^3$ has a unique solution u_ν on a time interval $[0,T_0]$ independent of the viscosity ν, if the initial velocity field and the external force field are sufficiently smooth (regular) in space and decay sufficiently fast at (spatial) infinity.

The exact result of Kato [15], which we prove in Chapter 3 is the following.

Theorem 1.2.1. *Let $a \in H^m_\sigma$ and $b \in L^1([0,T];H^m_\sigma)$ with $m \geq 3$ and $T > 0$. Then,*

1. *There exists $T_0 > 0$, with $T_0 \leq T$, where $T_0 := T_0(\|a\|_{H^m}, \|b\|_{L^1([0,T];H^m_\sigma)})$ such that* (1.2) *has a unique solution*

$$u_\nu \in C([0,T_0];H^m_\sigma) \cap AC([0,T_0];H^{m-1}_\sigma) \cap L^1([0,T_0];H^{m+1}_\sigma). \tag{1.3}$$

Furthermore,

$$u_\nu \text{ is bounded in } C([0,T_0];H^m_\sigma), \ \forall\, \nu > 0.$$

2. $\forall\, t \in [0,T_0]$,

$$\lim_{\nu\to 0} u_\nu(t) = u_0(t)$$

exists strongly in H^{m-1}_σ *and weakly in* H^m_σ*, uniformly in* t*, where* u_0 *is the unique solution of* $(1.2)_0$ *satisfying*

$$u_0 \in C([0,T_0];H^m_\sigma) \cap AC([0,T_0];H^{m-1}_\sigma). \tag{1.4}$$

The proof of this theorem is based on techniques for abstract nonlinear evolution equations in Hilbert spaces, see [18]. More precisely, first we give some estimates for the nonlinear operator F, in order to restrict it by norms of functions of the Hilbert space H^m_σ, see Lemma 3.1.3. Secondly, we prove two lemmas in order to normalise the viscosity as $\nu = 1$ and to show the time reversibility of the Euler equation. After that, we begin the proof of the theorem. In the first step of the proof, we construct a local solution of the abstract evolution equation of the Navier-Stokes equation. Note, that the existence time depends on the viscosity parameter $\nu = 1$. To do so, we take the equivalent integral equation (see (3.7)) of the abstract evolution equation (1.2) and construct a local solution of this, using *Banach's fixed point theorem.* Then, we show that this local solution satisfies (1.3) locally in time. Furthermore, we show that this local solution is unique.
In the second step of the proof, we show that this local solution can be extended to a solution to a time interval independent of the viscosity ν. To do this, we derive a uniform bound for the H^m-norm of the solution, which is independent of the viscosity ν. Thus, this local solution can be extended continuously in a time interval independent of ν.
In the third and final step of the proof, we show that the solution of the Navier-Stokes equation is uniformly convergent to the solution of the Euler equation, as the viscosity $\nu \to 0$. Then, we show that the solution of the Euler equation is unique and furthermore belongs to the space (1.4).

In Chapter 4 we present some results for the Navier-Stokes equation (1.1) with $\nu = 1$ and $f \equiv 0$ in the Banach spaces $L^q(\mathbb{R}^m)^m$, for $q \geq m$. We prove the local-in-time existence and uniqueness of strong solutions of the Navier-Stokes equation in $L^m_\sigma := L^m(\mathbb{R}^m)$, $m = 2,3,...$ which denotes the subspace of $L^m := L^m(\mathbb{R}^m;\mathbb{R}^m) = L^m(\mathbb{R}^m)^m$ in which the functions are divergence free (note that here, m denotes the space dimension). Moreover, for initial data $a \in L^m_\sigma$ with $\|a\|_{L^m}$ sufficiently small, we show that there exists a unique solution in L^m_σ-space globally in time for which we can also estimate its decay rate in time. See article [16] (in particular the Theorems 1 and 2 there), on which this part of the thesis is based. We point out that in the three-dimensional case $m = 3$, in which we are interested here, these results apply for solutions in L^q_σ with $q \geq 3$.

The theorem which we present in Chapter 4 is the following. Here we denote by $BC([0,T);L^q_\sigma)$ the space of all bounded and continuous functions $u\colon [0.T) \to L^q_\sigma$.

Theorem 1.2.2. *Let* $a \in L^m_\sigma$.

1. *Then there exists a* $T > 0$ *and a unique solution* u*, such that*

$$t^{\frac{1-\frac{m}{q}}{2}} u \in BC([0,T);L^q_\sigma), \ \text{for } m \leq q \leq \infty \tag{1.5}$$

$$t^{1-\frac{m}{2q}} \nabla u \in BC([0,T);L^q_\sigma), \ \text{for } m \leq q < \infty \tag{1.6}$$

both with values equal to zero, when $t = 0$*, except for* $q = m$ *in* (1.5)*, where* $u(\cdot, 0) = a$*. Furthermore, there exists some* $0 < T_1 \leq T$*, such that*

$$u \in L^r((0, T_1); L^q_\sigma), \text{ with } \frac{1}{r} = \frac{1 - \frac{m}{q}}{2}, \; m < q < \frac{m^2}{m-2}. \tag{1.7}$$

2. *There exist* $\lambda > 0$ *such that, if* $\|a\|_{L^m} \leq \lambda$*, the unique solution* u *of 1. is global in time, i.e.* $T = T_1 = \infty$*. More precisely,* $\|u(t)\|_{L^q}$ *decays like* $t^{\frac{-1+\frac{m}{q}}{2}}$*, as* $t \to \infty$*, including* $q = \infty$ *and* $\|\nabla u\|_{L^q}$ *decays like* $t^{-1+\frac{m}{2q}}$*, as* $t \to \infty$*, including* $q = m$*.*

Specifically, the first part of the theorem gives some decay properties that depend on the space dimension m and the order q of integrability, for the unique solution of the Navier-Stokes equation. The second part establishes the global-in-time existence of L^p-strong solutions and some properties of the solution's behavior.

The steps of the proof are as follows. First, we consider the abstract evolution equation (1.2) of the Navier-Stokes equations and as in Chapter 3, rescale it to the case $\nu = 1$, since ν is fixed. Then we transform it into a nonlinear integral equation. Furthermore, we give some estimates for the heat kernel e^{-tA}, its spatial derivatives and the nonlinear operator F. Secondly, we derive some estimates for the integral equation. Thirdly, we solve equivalently the integral equation with successive approximation. By choosing small initial data, one can show that this approximation is valid on the time interval $[0, \infty)$.

Finally, in Chapter 5, we study the partial regularity of a special case of weak solutions of the Navier-Stokes equations in $\mathbb{R}^3$ with $\nu = 1$, which are called suitable weak solutions. The main result of this chapter, which is based on [5], is the following.

Theorem 1.2.3. *For any suitable weak solution of the Navier-Stokes equation in an open set* $D \subset \mathbb{R}^3 \times \mathbb{R}$*, the associated singular set* S *has* $\mathcal{P}^1(S) = 0$*.*

Note 1.2.4. The importance of this theorem is that it establishes that for any suitable weak solution, the set S of singular points (that is, of points (x, t) such that the solution u is essentially unbounded in any neighborhood of (x, t)) has a "parabolic" Hausdorff measure of dimension 1, equal to 0. This means, roughly speaking, that the singular set consists essentially of points in $\mathbb{R}^4$. In particular, a possible singularity in some point $x \in \mathbb{R}^3$ in space does not persist over time. This is of course better explained by Caffarelli himself in [4].

The exposition of Chapter 5 is the following. First, in Section 5.1 we give some needed definitions, such as the definition of *suitable weak solutions* and the definition of *Hausdorff Measure*. Then we introduce the main Theorem 1.2.3 and the proof of it. For this proof, we need two main propositions which are Propositions 5.1.12 and 5.1.14. These two propositions are proven in Section 5.2 and 5.3 respectively. Lastly, we present the proof of the existence of suitable weak solutions of the Navier-Stokes equations in $\mathbb{R}^3$, for which the result of the main theorem holds true.

CHAPTER 2

Preliminaries

Throughout this thesis we consider functions in Sobolev spaces. Sobolev spaces are vector spaces of functions with a norm, which is a combination of the L^p-norm of the function itself together with the L^p-norms of its partial derivatives up to a certain order. These derivatives are understood in a weak sense.
The importance of these spaces derives from the fact that weak solutions of some PDEs exists in some Sobolev spaces even if there is no strong solutions in the space of continuous functions with strong derivatives.

2.1 The Lebesgue space L^p

In this subsection we introduce some elements of Measure Theory. See, e.g., [10].

Definition 2.1.1. We call as measurable set of $\mathbb{R}^n$, the set in which we correspond a number to represent its volume (content).

Note 2.1.2. We define as $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R}^n)$ the set of all measurable subsets of $\mathbb{R}^n$.

Definition 2.1.3. A function $u\colon \mathbb{R}^n \to \mathbb{R}$ is a measurable function if

$$u^{-1}(U) \in \mathcal{M}, \ \forall\, U \subseteq \mathbb{R} \text{ open set.}$$

e.g.

$$u\colon \mathbb{R}^n \to \mathbb{R} \quad \text{continuous} \Rightarrow u\colon \mathbb{R}^n \to \mathbb{R} \quad \text{measurable.}$$

Note 2.1.4. We say that a property holds almost everywhere, and write "a.e.", if the property holds everywhere on $\mathbb{R}^n$, except for a measurable set with measure zero.

Let $1 \leq p < \infty$ and X denote a real Banach space. We set as

$$\mathcal{L}^p(X) := \{u\colon X \to \mathbb{R} \text{ measurable function with} \int_X |u|^p < \infty\},$$

with a norm denoted as

$$\|u\|_{L^p} = \left(\int |u|^p\right)^{\frac{1}{p}}.$$

Now, let $p = \infty$. Then we define by

$$\mathcal{L}^\infty(X) := \{u\colon X \to \mathbb{R} \text{ measurable function with } \|u\|_{L^\infty} < \infty\},$$

where

$$\|u\|_{L^\infty} := \text{esssup } u := \inf\{\mu \in \mathbb{R} \ : \ |u > \mu| = 0\}.$$

We define the equivalence relation in $\mathcal{L}^p(X)$

$$f \sim g \Leftrightarrow f = g \ \ a.e.$$

Then, the space $L^p(X)$ is denoted as the set of all equivalence classes, that is $L^p(X) = \mathcal{L}^p(X)/\sim$.

Definition 2.1.5. Let X denote a real Banach space. The space $L^p([0,T];X)$ is defined as

$$L^p([0,T];X) := \{u : [0,T] \to X \text{ measurable function with } \|u\|_{L^p([0,T];X)} < \infty\},$$

where
i) for $1 \le p < \infty$,

$$\|u\|_{L^p([0,T];X)} := \left(\int_0^T \|u(t)\|^p \ \mathrm{d}t\right)^{\frac{1}{p}}.$$

ii) for $p = \infty$,

$$\|u\|_{L^\infty([0,T];X)} := \text{esssup}_{t\in[0,T]}\|u(t)\|.$$

Note that the norm $\|\cdot\|$ is the norm of the space X.

2.2 Weak derivatives

We define now the notion of a weak derivative. For this we first introduce the space $C_c^\infty(U)$ consists of all functions $\phi\colon U \to \mathbb{R}$ infinitely differentiable with compact support in U. These functions are called test functions. The motivation of weak derivatives came from the following. Let a function $u \in C^1(U)$, then $\forall\ \phi \in C_c^\infty(U)$ the integration by parts gives

$$\int_U u\phi_{x_i} \ \mathrm{d}x = -\int_U u_{x_i}\phi \ \mathrm{d}x + \underbrace{\int_{\partial V} u\phi n_i \ \mathrm{d}s}_{=0,\forall\phi\in C_c^\infty(U)},$$

since supp $\phi \subset V \subset\subset U$ $\partial V \in C^1$,

where supp $\phi := \overline{\{u \in U : \ f(u) \neq 0\}}$ and $V \subset\subset U :\Leftrightarrow V$ open, $\overline{V}$ compact with $\overline{V} \subset U$.

We introduce the space

$$L^p_{loc}(U) := \{f\colon \mathbb{R}^n \to \mathbb{R} \text{ measurable with } f|_V \in L^p(V) : \ \forall\ V \subset\subset U\}.$$

Definition 2.2.1. Let u,v$\in L^1_{loc}(U)$. We say that v is the a-th weak derivative of u and write $D^a u = v$, if

$$\int_U uD^a\phi \ \mathrm{d}x = (-1)^{|a|}\int_U v\phi \ \mathrm{d}x, \ \ \forall\ \phi \in C_c^\infty(U).$$

For the definition of the multiidex notation see, e.g., section 3.8 in [2].

Definition 2.2.2 (Sobolev space $H^m(\mathbb{R}^n)$)**.** We define as

$$H^m(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \ : \forall a \in \mathbb{N}_0^n \ : |a| \le m \ \ D^a u \in L^2(\mathbb{R}^n)\}.$$

The H^m-norm denoted as $\|u\|_{H^m}$, is

$$\|u\|_{H^m} := \left(\sum_{|a|\le m} \|D^a u\|_{L^2}^2 \right)^{\frac{1}{2}} .$$

Note 2.2.3. The Sobolev space $H^m(\mathbb{R}^n)$ is a Hilbert space. See, e.g., [3] and $\|\cdot\|$ is the norm of the real Banach space X. See, e.g., [21].

Definition 2.2.4. The space

$$H^m_\sigma(\mathbb{R}^3) := \{v \in H^m(\mathbb{R}^3) : \ \operatorname{div} v = 0\}$$

is a Hilbert space, with the $\|\cdot\|_{H^m}$-norm, consisting of all solenoidal vectors of the Hilbert space $H^m(\mathbb{R}^3)$.

We define as

$$(u, v)_{H^m} := \left(\sum_{0\le|a|\le m} (D^a u, D^a v)_{L^2} \right)^{\frac{1}{2}} ,$$

where

$$(u, v)_{L^2} = \left(\int_{\mathbb{R}^3} uv \ \mathrm{d}x \right)^{\frac{1}{2}} .$$

We will need also the following results.

Theorem 2.2.5. *(Banach's Fixed Point Theorem)*
Let X be a Banach space. Assume that $A \colon X \to X$ is a nonlinear mapping and suppose that

$$\exists\ 0 \le \gamma < 1 : \quad \|A[u] - A[\tilde{u}]\| \le \gamma \|u - \tilde{u}\|, \quad \forall u, \tilde{u} \in X. \tag{2.1}$$

Then A has a unique fixed point.

Note 2.2.6. We say that A is a strict contraction, if (2.1) holds true.

Proof. Fix a point $u_0 \in X$ and define $u_{k+1} = A[u_k], \ k = 0, 1, 2, ...$ Then

$$\|A[u_{k+1}] - A[u_k]\| \le \gamma \|u_{k+1} - u_k\| = \gamma \|A[u_k] - A[u_{k-1}]\|$$

$$\le \gamma^2 \|A[u_{k-1}] - A[u_{k-2}]\|$$

$$\le ... \le \gamma^k \|A[u_0] - A[u_{-1}]\|, \quad k = 1, 2, ...$$

$\Rightarrow if \ k \ge l$

$$\|u_k - u_l\| = \|A[u_{k-1}] - A[u_{l-1}]\| = \Big\| \sum_{j=l-1}^{k-2} \big[A[u_{j+1}] - A[u_j]\big] \Big\|$$

$$\leq \sum_{j=l-1}^{k-2} \|A[u_{j+1}] - A[u_j]\| \leq \sum_{j=l-1}^{k-2} \gamma^j \|A[u_0] - u_0\|$$
$$= \|A[u_0] - u_0\| \sum_{j=l-1}^{k-2} \gamma^j = \|A[u_0] - u_0\| \big(\gamma^{l-1} + \gamma^l + ... + \gamma^{k-2}\big)$$
$$= \|A[u_0] - u_0\| \gamma^{l-1} \big(1 + \gamma + \gamma^2 + ... + \gamma^{k-2-l+1}\big)$$
$$\leq \gamma^{l-1} \sum_{n=0}^{\infty} \gamma^n \|A[u_0] - u_0\| = \gamma^{l-1} \frac{1}{1-\gamma} \|A[u_0] - u_0\|$$

and $\gamma < 1$

$$\Rightarrow \lim_{l \to \infty} \gamma^{l-1} = 0$$

$\Rightarrow (u_k)_{k=1,...}$ is a Cauchy sequence in X, and so

$$\exists u \in X : u_k \to u \;\; inX.$$

That is

$$\lim u_k = u \;\; in\ X \quad\;\; and \;\; A[u_k] = u_{k+1}$$

$$\Rightarrow \lim u_{k+1} = \lim A[u_k] = u \Rightarrow A[u_k] \to u$$

$$\begin{cases} \text{A is a contraction } \Rightarrow \text{ A is continuous} \\ \text{and we have that } \; u_k \to u \end{cases} \Longrightarrow A[u_k] \to A[u].$$

But, by the uniqueness of the limit $\Rightarrow A[u] = u$.

$$\big[\text{let } u, v \in X \;\; : \;\; u \neq v \quad \text{and} \;\; A[u] = u, \;\; A[v] = v.$$

Then

$$\|A[u] - A[v]\| \leq \gamma \|u - v\|$$
$$\Rightarrow \|u - v\| \leq \gamma \|u - v\| \Rightarrow (1 - \gamma)\|u - v\| \leq 0$$

$$\Rightarrow \|u - v\| \leq 0 \Rightarrow \Rightarrow \|u - v\| = 0 \Rightarrow u = v\big].$$

Hence, u is the unique fix point for A.

□

Theorem 2.2.7. *(Gronwall's lemma)*
If v, ϕ *are continuous, non-negative functions such that*

$$v(t) \leq K + \int_{t_0}^{t} \phi(s) v(s) \; ds, \quad t \geq t_0.$$

Then,

$$0 \leq v(t) \leq K e^{\int_{t_0}^{t} \phi(s) ds}.$$

Proof. Set
$$w(t) := K + \int_{t_0}^{t} \phi(s)v(s)\ \mathrm{d}s,$$
then $w(t_0) = K$ and $v(t) \leq w(t)$,$\forall t \geq t_0$ by the assumptions.

$$\Rightarrow w'(t) = \phi(t)v(t) \leq \phi(t)w(t)$$

$$\Rightarrow w'(t) - \phi(t)w(t) \leq 0$$

$$\Rightarrow w'(t)e^{-\int_{t_0}^{t} \phi(s)ds} - \phi(t)e^{-\int_{t_0}^{t} \phi(s)ds}w(t) \leq 0$$
$$\Rightarrow \left(w(t)e^{-\int_{t_0}^{t} \phi(s)ds}\right)' \leq 0$$

$\Rightarrow w(t)e^{-\int_{t_0}^{t} \phi(s)ds}$ is decreasing function.

$$\Rightarrow \ \forall t \geq t_0, \quad w(t)e^{-\int_{t_0}^{t} \phi(s)ds} \leq w(t_0)e^{-\int_{t_0}^{t_0} \phi(s)ds} = K$$
$$\Rightarrow w(t) \leq e^{\int_{t_0}^{t} \phi(s)ds} \ \mathit{and}\ v(t) \leq w(t)$$

$$\Rightarrow 0 \leq v(t) \leq e^{\int_{t_0}^{t} \phi(s)ds}.$$

□

2.3 Dual space and weak convergence

Definition 2.3.1. Let X and Y be real Banach spaces. A mapping $T\colon X \to Y$ is called linear operator if
$$T(\lambda u + \mu v) = \lambda Tu + \mu Tv, \quad \forall u, v \in X, \lambda, \mu \in \mathbb{R}.$$

Definition 2.3.2. Let X and Y be real Banach spaces. A linear operator $T\colon X \to Y$ is bounded if
$$\|T\| := \sup\{\|Tu\|_Y \ :\ \|u\|_X \leq 1\} < \infty$$
$$\Leftrightarrow \|T\| := \inf\{M > 0 \ :\ \|Tu\|_Y \leq M\|u\|_X,\ \forall u \in X\}.$$

See, e.g., [1].

Remark 2.3.3. Every linear, bounded operator $T\colon X \to Y$ is continuous.

Proof. Let $\epsilon > 0$
If $u, v \in X$ with $\|u - v\|_X \leq \delta$. Then
$$\|Tu - Tv\|_Y = \|T(u - v)\|_Y \leq \|T\|\|u - v\|_X \leq M\|u - v\|_X < M\delta.$$
It is enough to set $\delta = \frac{\epsilon}{M}$.

It follows that, $T\colon X \to Y$ is uniformly continuous $\Rightarrow T\colon X \to Y$ is continuous.

□

Definition 2.3.4. 1. A bounded, linear operator $f\colon X \to \mathbb{R}$ is called bounded, linear functional on X.

2. We define as

$$X^* := \{f\colon X \to \mathbb{R} \text{ bounded, linear functional on } X\}.$$

The space X^* is called the dual space of X.

Definition 2.3.5. Let X be a real Banach space. We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ convergence weakly to an element $x \in X$ and write $x_n \rightharpoonup x$ if

$$(f, x_n) \to (f, x), \quad \forall f \in X^*.$$

Remark 2.3.6. 1. Strong convergence $\Rightarrow$ Weak convergence.

2. Every weakly convergent sequence is bounded.

3. If $x_n \rightharpoonup x$, then $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.

Proof. 1. By applying the C-S inequality we get

$$|(f, x_n) - (f, x)| = |(f, x_n - x)| \le \underbrace{\|f\|}_{bdd} \ \underbrace{\|x_n - x\|}_{\to 0} \longrightarrow 0\ , \quad \forall f \in X^*.$$

2. $\forall\ f \in X^*$, the $(f, x_n)_{n\in\mathbb{N}}$ is bounded, since $\forall\ f \in X^*$, the $(f, x_n)_{n\in\mathbb{N}}$ is convergent.

3. We observe that

$$|(f, x_n)| \le \|f\|\ \|x_n\|$$

$$\Rightarrow \liminf_{n\to\infty} |(f, x_n)| \le \|f\| \liminf_{n\to\infty} \|x_n\|$$

$$\Rightarrow |(f, x)| \le \|f\| \liminf_{n\to\infty} \|x_n\| \tag{2.2}$$

and

$$\forall x \in X,\ \|x\| = \sup_{f\in X^*, \|f\|\le 1} |(f, x)|.$$

Hence,

$$(2.2) \Rightarrow \|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

□

See, e.g., [3], Proposition 3.5, p.58.

Theorem 2.3.7. *Let X be a normed space with finite dimension. Then*

$$x_n \textit{ converges weakly to } x \in X \Rightarrow x_n \textit{ converges strongly to } x \in X.$$

Proof. Let $dimX = k$ and $x_n \rightharpoonup x$.

If $\{e_1, ..., e_k\}$ be a basis of X, we write

$$x_n = a_1^{(n)} e_1 + a_2^{(n)} e_2 + ... + a_k^{(n)} e_k \ , \ n = 1, 2, ...$$

and

$$x = a_1 e_1 + a_2 e_2 + ... + a_k e_k.$$

Now, by hypothesis we have

$$(f, x_n) \to (f, x) \quad , \forall f \in X^*.$$

We consider the functionals $f_1, f_2, ..., f_k \in X^*$ such, that

$$f_j(e_i) := \begin{cases} 1 & , j = i \\ 0 & , j \neq i \end{cases}$$

$$\begin{aligned} &\Rightarrow f_j(x_n) \to f_j(x) \\ &\Rightarrow a_j^{(n)} \to a_j \ , \ for \ j = 1, 2, ..., k. \end{aligned}$$

Hence,

$$\begin{aligned} \|x_n - x\| &= \Big\| \sum_{j=1}^{k} (a_j^{(n)} - a_j) e_j \Big\| \\ &\leq \sum_{j=1}^{k} \|(a_j^{(n)} - a_j) e_j\| = \sum_{j=1}^{k} \underbrace{|a_j^{(n)} - a_j|}_{\to 0} \ \underbrace{\|e_j\|}_{bdd} \ \to 0. \end{aligned}$$

Therefore,

$$\|x_n - x\| \to 0.$$

That is, $x_n \to x$ strongly in X.

□

See, e.g. [9], Proposition 7.1.3(iii), p.102, for a proof.

Definition 2.3.8. Let a function $u \colon [0, T] \to X$, where X is a Hilbert space. We say that the mapping $[0, T] \ni t \mapsto u(t)$ in X is weakly continuous if

$\forall f \in X^*$ the mapping $t \mapsto (f, u(t))_X$ is strongly continuous.

That is,

$$\forall f \in X^* : \forall (t_n)_{n \in \mathbb{N}} \in [0, T] \text{ with } t_n \to t \Rightarrow (f, u(t_n))_X \to (f, u(t))_X$$

$$:\Leftrightarrow \forall (t_n)_{n \in \mathbb{N}} \in [0, T] \text{ with } t_n \to t \Rightarrow u(t_n) \rightharpoonup u(t) \text{ in } X.$$

Proposition 2.3.9. *If $u\colon [0,T] \to X$ is strongly continuous $\Rightarrow$ $u\colon [0,T] \to X$ is weakly continuous.*

Proof. Let $u\colon [0,T] \to X$ strongly continuous

$$\Rightarrow \forall t \in [0,T]: \ \|u(t) - u(t_0)\|_X \longrightarrow 0, \text{ as } t \to t_0.$$

It suffices to show that

$$\forall f \in X^*: \ \left|(f, u(t))_X - (f, u(t_0))_X\right| \longrightarrow 0, \text{ as } t \to t_0.$$

Hence,

$$\left|(f, u(t) - u(t_0))_X\right| \le \underbrace{\|f\|_X}_{bdd} \underbrace{\|u(t) - u(t_0)\|_X}_{\longrightarrow 0} \longrightarrow 0, \text{ as } t \to t_0.$$

Thus, strong continuity $\Rightarrow$ weak continuity.

□

Note 2.3.10. 1) In the proof we use the Cauchy Schwarz inequality.
2) In definition of weak continuity the $(\cdot,\cdot)_X$ is the inner product of the space X.

2.4 Orthogonal projection

Definition 2.4.1. A projection is a linear transformation P from a vector space to itself such that, $P^2 = P$. See, e.g., [7], p.480.

Definition 2.4.2. A projection P on a Hilbert space H is called an orthogonal projection if

$$(Px, y) = (x, Py), \quad \forall x, y \in H.$$

Note 2.4.3. $\|P\| = 1$ on a Hilbert space, since

$$\|P^2\| = \|P\| \Leftrightarrow \|P\| = 1.$$

Theorem 2.4.4 (The Hodge Decomposition in H^m)**.** *Every vector field $u \in H^m(\mathbb{R}^n)$, $m \in \mathbb{N}_0$, has a unique orthogonal decomposition*

$$u = v + \nabla\phi,$$

such that the Leray's projection $Pu = v$ onto divergence-free vector-fields has the following properties:

i) $Pu,\ \nabla\phi \in H^m$, $\int_{\mathbb{R}^n} Pu \cdot \nabla\phi \mathrm{d}x = 0$, $\operatorname{div} Pu = 0$, $\|Pu\|^2_{H^m} + \|\nabla\phi\|^2_{H^m} = \|u\|^2_{H^m}$.

ii) P commutes with the distribution derivatives

$$PD^a u = D^a Pu, \ \forall u \in H^m, \ |a| \le m.$$

iii) P is symmetric, i.e.

$$(Pw, u)_{H^m} = (w, Pu)_{H^m}, \ w, u \in H^m.$$

See, e.g., Lemma 3.6 in [23].

Definition 2.4.5. A function f: $[a, b] \to X$ is said to be absolutely continuous if: $\forall\ \epsilon > 0$ $\exists\ \delta > 0$: if (a_k, b_k), $1 \le k \le n$ disjoint subsets of [a,b] with

$$\sum_{k=1}^{n}(b_k - a_k) < \delta\ \Rightarrow \sum_{k=1}^{n}\operatorname{osc}_{(a_k,b_k)}(f) < \epsilon$$

where,

$$\operatorname{osc}_{(a_k,b_k)}(f) := \inf\{\operatorname{diam}\big(f(U)\big) :\ U \text{ is a neighborhood of } x\}.$$

Definition 2.4.6. The space $AC([0,T];X)$ is defined as

$$AC([0,T];X) := \{f\colon [0,T] \to X :\ \text{f absolutely continuous function}\}.$$

See, e.g. page 493 in [24].

Definition 2.4.7. The space $C([0,T];X)$ is defined as

$$C([0,T];X) := \{f\colon [0,T] \to X :\ \text{f continuous function} : \|u\|_{C([0,T];X)} < \infty\},$$

where

$$\|u\|_{C([0,T];X)} := \max_{0\le t\le T} \|u(t)\|.$$

CHAPTER 3

Strong solutions and the vanishing viscosity limit

In this chapter, we prove the first main result of this thesis, that is to say, Theorem 1.2.1. This means we prove, first, that the Cauchy problem for the Navier-Stokes equation in $\mathbb{R}^3$ has a unique solution u_ν on a time interval $[0, T_0]$ independent of the viscosity ν, if the initial velocity field and the external force field are sufficiently smooth and decay sufficiently fast at spatial infinity. Second, we will show that the solutions u_ν of the Navier-Stokes equations converge, uniformly in time, to the unique solution u_0 of the Euler equation. In this section, our main source is the paper [15] of T.Kato.

3.1 Preliminary results

First, we give the transformation of the Navier-Stokes system into an abstract evolution equation.

Let the Navier-Stokes equation

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, & \text{in } \mathbb{R}^n \times (0, \infty) \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(\cdot, 0) = a, & \text{on } \mathbb{R}^n, \end{cases} \tag{1.1}$$

and let the (Leray's) orthogonal projection $P\colon L^2(\mathbb{R}^3) \to H_\sigma$ on the space of divergence-free vector fields.. Then, by applying this projection to the Navier-Stokes equation we get

$$P(\frac{\partial u}{\partial t}) + P(u \cdot \nabla)u = -P\nabla p + \nu P\Delta u + Pf$$
$$\frac{\partial u}{\partial t} + \nu A u = F(u, u) + b,$$

where $P(\frac{\partial u}{\partial t}) = \frac{\partial u}{\partial t}$, since $u \in H_\sigma$, $P(u \cdot \nabla)u =: -F(u, u)$, $P\Delta =: -A$, $Pf =: b$ as already defined in the introduction, see after equation (1.2).
Furthermore, $P\nabla p = 0$, by the properties of the projection. See, Theorem 2.4.4, Section 2.4.
Thus, we managed to transform the nonlinear Navier-Stokes equation into an abstract evolution equation of the form

$$\begin{cases} \frac{du}{dt} + \nu A u = F(u, u) + b, & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(\cdot, 0) = a, & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

In this way, while the Navier-Stokes equations (1.1) contain the time derivatives of only three out of the four unknown functions, the equation (1.2) consists of three evolution equations for

u and the fourth equation $\operatorname{div} u = 0$ of (1.1) is satisfied via its inclusion in the solution space. Moreover, due to this process the term ∇p appearing in (1.1) is eliminated through Leray's projection on the space of divergence-free vector fields H_σ. From the solution u of (1.2) we can then recover p by solving a Poisson equation which is obtained from (1.1). See, e.g., Section 1.8 in [23].

Definition 3.1.1 (Strong solution of the Navier-Stokes equation)**.** The solution u of Navier-Stokes system is called "strong" in $\Omega \times (0,T) \subset \mathbb{R}^3 \times \mathbb{R}$, with initial data $u_0 \in H^1_\sigma(\Omega)$ if it is true that
i) (regularity)

$$u \in L^\infty((0,T); H^1_\sigma(\Omega)) \cap L^2((0,T); H^2(\Omega)).$$

ii) u satisfies

$$\int_0^T \int_\Omega (-u \cdot \phi_t + \nu \nabla u : \nabla \phi + (u \otimes u) : \nabla \phi) \, \mathrm{d}x\mathrm{d}t = \int_\Omega u_0 \cdot \phi(0) \mathrm{d}x,$$

$\forall\ \phi \in C_c^\infty(\Omega \times [0,T); \mathbb{R}^3)$ such that ϕ is divergence free. For this definition see, [26].

To prove the Theorem 1.2.1 we need some preliminary results. We start with the following lemma. For the definition of the multiindex notation see, e.g., section 3.8 in [2].

Lemma 3.1.2. *Let* $0 < \beta \le a$, *then*

$$\|(D^\beta u \cdot \nabla) D^{a-\beta} v)\|_{L^2} \le \begin{cases} c\|u\|_{H^3} \|v\|_{H^{|a|}}, & |\beta| = 1,2 \\ c\|u\|_{H^{|\beta|}} \|v\|_{H^{|a|-|\beta|+3}}, & |\beta| \ge 3. \end{cases}$$

Proof. We use the estimates

$$\|fg\|_{L^2} \le c\|f\|_{H^2}\|g\|_{L^2} \tag{3.1}$$

and

$$\|fg\|_{L^2} \le c\|f\|_{H^1}\|g\|_{H^1}. \tag{3.2}$$

If $|\beta| = 1$ we have

$$\|D^\beta u\|_{H^2} \le \|u\|_{H^3} \quad \text{and} \quad \|\nabla D^{a-\beta} v\|_{L^2} \le \|v\|_{H^{|a|}}.$$

Thus,

$$\|(D^\beta u \cdot \nabla) D^{a-\beta} v)\|_{L^2} \le c\|D^\beta u\|_{H^2} \|\nabla D^{a-\beta} v\|_{L^2} \le c\|u\|_{H^3}\|v\|_{H^{|a|}}$$

if $|\beta| = 2$ we have

$$\|D^\beta u\|_{H^1} \le \|u\|_{H^3} \quad \text{and} \quad \|\nabla D^{a-\beta} v\|_{H^1} \le \|v\|_{H^{|a|}}.$$

Hence,

$$\|(D^\beta u \cdot \nabla) D^{a-\beta} v)\|_{L^2} \le c\|D^\beta u\|_{H^1} \|\nabla D^{a-\beta} v\|_{H^1} \le c\|u\|_{H^3}\|v\|_{H^{|a|}}.$$

If $|\beta| \ge 3$ we have

$$\|D^\beta u\|_{L^2} \le \|u\|_{H^{|\beta|}} \quad \text{and} \quad \|\nabla D^{a-\beta} v\|_{H^2} \le \|v\|_{H^{|a|-|\beta|+3}}.$$

Therefore,

$$\|(D^\beta u \cdot \nabla)D^{a-\beta}v)\|_{L^2} \leq c\|D^\beta u\|_{L^2}\|\nabla D^{a-\beta}v\|_{H^2} \leq c\|u\|_{H^{|\beta|}}\|v\|_{H^{|a|-|\beta|+3}}.$$

□

Lemma 3.1.3. *For the bilinear operator F the following estimates hold*

$$\|F(u,v)\|_{H^m} \leq c\|u\|_{H^m}\|v\|_{H^{m+1}}, \; m \geq 2, \; u \in H^m_\sigma, \; v \in H^{m+1}_\sigma \tag{3.3}$$

$$|(F(u,v),v)_{H^m}| \leq c'\|u\|_{H^m}\|v\|^2_{H^m}, \; m \geq 3, \; u \in H^m_\sigma, \; v \in H^{m+1}_\sigma \tag{3.4}$$

$$|(F(u,v),v)_{H^2}| \leq c'\|u\|_{H^3}\|v\|^2_{H^2}, \; u,v \in H^3_\sigma \tag{3.5}$$

where, c,c′ may depend on m.

Proof. We first prove (3.3). For the projection P we have in any $H^m(\mathbb{R}^3)$ that $\|P\| = 1$. Moreover, $H^m(\mathbb{R}^3)$ is a Banach algebra, that is to say for any $u, v \in H^m(\mathbb{R}^3)$ with $m \geq 2$ it holds $u \cdot v \in H^m(\mathbb{R}^3)$ and there exists a positive constant c depending on m such that $\|u \cdot v\|_{H^m} \leq c\|u\|_{H^m}\|v\|_{H^m}$. (Indeed, this holds true in general for $H^s(\mathbb{R}^d)$ with $s > d/2$, see, e.g.,

$$\begin{aligned}\|F(u,v)\|_{H^m} &= \|-P(u\cdot\nabla)v\|_{H^m} \\ &\leq \underbrace{\|P\|}_{=1}\|(u\cdot\nabla)v\|_{H^m} \\ &\leq c\|u\|_{H^m}\|\nabla v\|_{H^m} \\ &\leq c\|u\|_{H^m}\|v\|_{H^{m+1}}.\end{aligned}$$

We turn now to the proofs of (3.4) and (3.5) where we first assume additionally. We assume that $u, v \in C^\infty_c(\mathbb{R}^3)$ in view of (3.3).
Then, using the symmetry of P and the fact that $v \in H^m_\sigma$, we have

$$\begin{aligned}(F(u,v),v)_{H^m} &= -(P(u\cdot\nabla)v,v)_{H^m} \\ &= -((u\cdot\nabla)v,Pv)_{H^m} \\ &= -((u\cdot\nabla)v,v)_{H^m} \\ &= -\sum_{|a|\leq m}(D^a((u\cdot\nabla)v),D^a v)_{L^2},\end{aligned}$$

where D^a are multiindexed derivatives in space. By Leibniz's formula we have

$$D^a(u\cdot\nabla)v = (u\cdot\nabla)D^a v + \sum_{0<\beta\leq a} c_{a,\beta}(D^\beta u\cdot\nabla)D^{a-\beta}v,$$

where the Leibniz's formula is

$$D^a(uv) = \sum_{\beta\leq a}\binom{a}{\beta}D^\beta u D^{a-\beta}v, \quad \text{with } u,v\colon \mathbb{R}^n \to \mathbb{R}.$$

Thus,

$$\begin{aligned}(F(u,v),v)_{H^m} &= -\sum_{|a|\le m}(D^a((u\cdot\nabla)v),D^a v)_{L^2}\\ &= -\sum_{|a|\le m}((u\cdot\nabla)D^a v+\sum_{0<\beta\le a}c_{a,\beta}(D^\beta u\cdot\nabla)D^{a-\beta}v,D^a v)_{L^2}\\ &= -\sum_{1\le|a|\le m}((u\cdot\nabla)D^a v,D^a v)_{L^2}\\ &\quad -\sum_{1\le|a|\le m}\sum_{0<\beta\le a}c_{a,\beta}((D^\beta u\cdot\nabla)D^{a-\beta}v,D^a v)_{L^2}\end{aligned}$$

and

$$(F(u,v),v)_{H^m} = -\sum_{1\le|a|\le m}\sum_{0<\beta\le a}c_{a,\beta}((D^\beta u\cdot\nabla)D^{a-\beta}v,D^a v)_{L^2}. \tag{3.6}$$

In order to complete the proof we first apply the Cauchy-Schwarz inequality to the above equation to obtain

$$|(F(u,v),v)_{H^m}| \le \sum_{1\le|a|\le m}\sum_{0<\beta\le a}c_{a,\beta}\|(D^\beta u\cdot\nabla)D^{a-\beta}v\|_{L^2}\|D^a v\|_{L^2}.$$

Hence

$$|(F(u,v),v)_{H^m}| \le \|D^a v\|_{L^2}\sum_{1\le|a|\le m}\sum_{0<\beta\le a}c_{a,\beta}\|(D^\beta u\cdot\nabla)D^{a-\beta}v\|_{L^2}.$$

Moreover, for $m=2$ we obtain by the first case of the following lemma

$$\begin{aligned}\sum_{1\le|\alpha|\le 2}\sum_{0<\beta\le\alpha}c_{\alpha,\beta}\|(D^\beta u\cdot\nabla)D^{a-\beta}v)\|_{L^2} &\le \|u\|_{H^3}\sum_{1\le|\alpha|\le 2}\sum_{0<\beta\le\alpha}c_{\alpha,\beta}\tilde{c}_{\alpha,\beta}\|v\|_{H^{|\alpha|}}\\ &\le c'\|u\|_{H^3}\|v\|_{H^2},\end{aligned}$$

with

$$c' = \sum_{1\le|\alpha|\le 2}\sum_{0<\beta\le\alpha}c_{\alpha,\beta}\tilde{c}_{\alpha,\beta}.$$

This is the estimate (3.5).

For $m\ge 3$ we split the sum in (3.6)

$$\sum_{1\le|\alpha|\le m}\sum_{0<\beta\le\alpha} = \sum_{1\le|\alpha|\le 2}\sum_{0<\beta\le\alpha} + \sum_{3\le|\alpha|\le m}\sum_{0<\beta\le\alpha,|\beta|\le 2} + \sum_{3\le|\alpha|\le m}\sum_{\beta\le\alpha,|\beta|\ge 3}.$$

For the first of these three sums we obtain as above the estimate

$$c_1'\|u\|_3\|v\|_2.$$

For the second sum we get similarly

$$c_1'\|u\|_3\|v\|_m.$$

Finally, for the third sum we get the estimate

$$\begin{aligned}\sum_{3\le|\alpha|\le m}\sum_{\beta\le\alpha,|\beta|\ge3} c_{\alpha,\beta}\|(D^\beta u\cdot\nabla)D^{a-\beta}v)\|_{L^2} &\le \sum_{3\le|\alpha|\le m}\sum_{\beta\le\alpha,|\beta|\ge3} c_{\alpha,\beta}\tilde{c}_{\alpha,\beta}\|u\|_{|\beta|}\|v\|_{|\alpha|-|\beta|+3}\\ &\le \|u\|_m\sum_{3\le|\alpha|\le m}\sum_{\beta\le\alpha,|\beta|\ge3} c_{\alpha,\beta}\tilde{c}_{\alpha,\beta}\|v\|_{|\alpha|-|\beta|+3}\\ &\le c_3'\|u\|_m\|v\|_m\end{aligned}$$

with

$$c_3' = \sum_{3\le|\alpha|\le m}\sum_{\beta\le\alpha,|\beta|\ge3} c_{\alpha,\beta}\tilde{c}_{\alpha,\beta}$$

since for the multi-indices appearing in this sum we have

$$3\le|\alpha|-|\beta|+3\le|\alpha|\le m.$$

Considering that $m\ge3$, we obtain from the three estimates the desired estimate (3.4).

□

Next, we normalize the kinematic constant viscosity in the Navier-Stokes equation, see [**?**].

Lemma 3.1.4. *For fixed kinematic constant viscosity $\nu>0$ in the Navier-Stokes equation* (1.1) *we may normalize to $\nu=1$, by applying the rescaling $\tilde{u}(x,t)=\frac{1}{\nu}u(x,\frac{1}{\nu}t)$, $\tilde{p}(x,t)=\frac{1}{\nu^2}(x,\frac{1}{\nu}t)$.*

Proof. Assume we have

$$\tilde{u}_t+(\tilde{u}\cdot\nabla)\tilde{u}=\nu\Delta\tilde{u}-\nabla\tilde{p}, \qquad \text{where} \ \ \tilde{u}=\tilde{u}(x,t).$$

Now, by setting $\tilde{u}(x,t)=\nu u(x,\nu t)$ and $\tilde{p}(x,t):=\nu p(x,\nu t)$
we get,

$$\tilde{u}_t(x,t)=\nu u_t(x,\nu t)\frac{\partial(\nu t)}{\partial t}=\nu^2u_t(x,\nu t) \quad \text{and} \quad \nabla\tilde{p}(x,t)=\nu^2\nabla p(x,\nu t).$$

Thus,

$$\begin{aligned}&\tilde{u}_t+(\tilde{u}\cdot\nabla)\tilde{u}=\nu\Delta\tilde{u}-\nabla\tilde{p}\\ \Leftrightarrow&\nu^2u_t(x,\nu t)+(\nu u(x,\nu t)\cdot\nabla)\nu u(x,\nu t)=\nu^2\Delta u(x,\nu t)-\nu^2\nabla p(x,\nu t)\\ \Leftrightarrow&\nu^2u_t(x,\nu t)+\nu^2(u(x,\nu t)\cdot\nabla)u(x,\nu t)=\nu^2\Delta u(x,\nu t)-\nu^2\nabla p(x,\nu t)\\ \Leftrightarrow&u_t(x,\nu t)+(u(x,\nu t)\cdot\nabla)u(x,\nu t)=\Delta u(x,\nu t)-\nabla p(x,\nu t).\end{aligned}$$

□

Finally, we address the time-reversibility of the incompressible Euler equations.

Lemma 3.1.5. *The three-dimensional incompressible Euler equations are time reversible.*

Proof. We have that

$$\tilde{u}_t+(\tilde{u}\cdot\nabla)\tilde{u}=-\nabla\tilde{p} \quad \text{where,} \ \ \tilde{u}=\tilde{u}(x,t).$$

Now, by setting $\tilde{u}(x,t)=-u(x,-t)$ and $\tilde{p}(x,t):=p(x,-t)$
we get,

$$\tilde{u}_t(x,t)=-u_t(x,-t)\frac{\partial(-t)}{\partial t}=u_t(x,-t).$$

Thus,

$$\tilde{u}_t + (\tilde{u} \cdot \nabla)\tilde{u} = -\nabla \tilde{p}$$

$$\begin{aligned}
&\Leftrightarrow u_t(x,-t) + (-u(x,-t) \cdot \nabla)(-u(x,-1t)) = -\nabla p(x,-t) \\
&\Leftrightarrow u_t(x,-t) + (u(x,-t) \cdot \nabla)u(x,-1t) = -\nabla p(x,-t).
\end{aligned}$$

□

3.2 The main theorem

We are now ready to begin the proof of Theorem 1.2.1, which we restate here for the convenience of the reader. The general strategy of the proof has been outlined in the Introduction.

Theorem. 1.2.1. *Let* $a \in H^m_\sigma$ *and* $b \in L^1([0,T]; H^m_\sigma)$ *with* $m \geq 3$ *and* $T > 0$. *Then,*

1. *There exists* $T_0 > 0$, *with* $T_0 \leq T$, *where* $T_0 := T_0(\|a\|_{H^m}, \|b\|_{L^1([0,T];H^m_\sigma)})$ *such that* (1.2) *has a unique solution*

$$u_\nu \in C([0,T_0]; H^m_\sigma) \cap AC([0,T_0]; H^{m-1}_\sigma) \cap L^1([0,T_0]; H^{m+1}_\sigma). \tag{1.3}$$

   Furthermore,

$$u_\nu \text{ is bounded in } C([0,T_0]; H^m_\sigma),\ \forall\ \nu > 0.$$

2. $\forall\ t \in [0,T_0]$,

$$\lim_{\nu \to 0} u_\nu(t) = u_0(t)$$

   exists strongly in H^{m-1}_σ *and weakly in* H^m_σ, *uniformly in t, where* u_0 *is the unique solution of* $(1.2)_0$ *satisfying*

$$u_0 \in C([0,T_0]; H^m_\sigma) \cap AC([0,T_0]; H^{m-1}_\sigma).$$

Proof. We split the proof into three steps.

1st step. Construction of a local solution u_ν of (1.2) with an existence time $T_\nu > 0$.

We can assume that $\nu = 1$, since $\nu > 0$ is fixed. Set $F(u,u) =: Fu$ throughout this section. Then we have, written formally,

$$\frac{du}{dt} + Au = Fu + b(t)\ , t > o\ ,\quad u(\cdot,0) = a$$

$$\begin{aligned}
&\Rightarrow e^{At}\frac{du}{dt} + e^{At}Au = e^{At}[Fu + b(t)] \\
&\Rightarrow \frac{d}{dt}[e^{At}u] = e^{At}[Fu + b(t)] \\
&\Rightarrow e^{At}u - a = \int_0^t e^{As}[Fu(s) + b(s)]ds \\
&\Rightarrow u = e^{-At}a + e^{-At}\int_0^t e^{As}[Fu(s) + b(s)]ds
\end{aligned}$$

$$\Rightarrow u(t) \equiv Gu(t) = e^{-At}a + \int_0^t e^{-(t-s)A}[Fu(s) + b(s)]ds. \tag{3.7}$$

Now, we want to construct a local solution of the integral equation (3.7). To do so, we will use the contraction mapping principle (Banach's fixed-point theorem). See, Theorem 2.2.5 in Chapter 2.
For more convenience, we introduce the spaces.

$$X_j := C([0,T^{'}]; H^j_\sigma)$$
$$Y_j := L^1([0,T^{'}]; H^j_\sigma) \quad ,j = 1,2,...$$

and

$$Z := X_m \cap Y_{m+1}.$$

and equip the space Z with the norm

$$\|v\|_Z := \max(K^{-1}\|v\|_{X_m}, L^{-1}\|v\|_{Y_{m+1}}) \tag{3.8}$$

in order to make it a Banach space. The values $T', K, L > 0$ are unknown at this stage and will be determined later in such a way as to provide a contraction.
We next note that $u, v \in Z$ implies $F(u,v) \in Y_m$ with

$$\|F(u,v)\|_{Y_m} \le c\|u\|_{X_m}\|v\|_{Y_{m+1}}. \tag{3.9}$$

This follows directly from the inequality (3.3), since integrating over $[0,T']$ and applying Hölder inequality gives

$$\begin{aligned}\|F(u,v)\|_{Y^m} &= \int_0^{T'} \|F(u,v)\|_{H^m}dt \\ &\le c\int_0^{T'} \|u\|_{H^m}\|v\|_{H^{m+1}}dt \\ &\le c\|u\|_{X^m}\int_0^{T'} \|v\|_{H^{m+1}}dt \\ &= c\|u\|_{X^m}\|v\|_{Y^{m+1}}.\end{aligned}$$

We next compute $Gu(t) - Gv(t)$. For $u, v \in Z$ we have, by the bilinearity of $F(u,v)$ that

$$\begin{aligned}Gu(t) - Gv(t) &= e^{-tA}a + \int_0^t e^{-(t-s)A}[Fu(s)+b(s)]\ \mathrm{d}s - e^{-tA}a - \int_0^t e^{-(t-s)A}[Fv(s)+b(s)]\ \mathrm{d}s \\ &= \int_0^t e^{-(t-s)A}[F(u,u) - F(v,v)]\ \mathrm{d}s \\ &= \int_0^t e^{-(t-s)A} \underbrace{[F(u,u) - F(v,u) + F(v,u) - F(v,v)]}_{=F(u-v,u)+F(v,u-v)}\ \mathrm{d}s\end{aligned}$$

$$\Rightarrow Gu(t) - Gv(t) = \int_0^t e^{-(t-s)A}[F(u-v,u) + F(v,u-v)]\ \mathrm{d}s. \tag{3.10}$$

Since, by standard results of the theory of semigroups, $\|e^{-tA}\|_{Y_m} = 1$ as an operator in H^m_σ, we have

$$\Rightarrow \|Gu - Gv\|_{X_m} = \max_{t\in[0,T']} \|\int_0^t e^{-(t-s)A}[F(u-v,u) + F(v,u-v)]\ \mathrm{d}s\|_{H_m},$$

where

$$\| \int_0^t e^{-(t-s)A}[F(u-v,u)+F(v,u-v)]ds\|_{H^m} \leq \int_0^t \|e^{-(t-s)A}[F(u-v,u)+F(v,u-v)]\|_{H^m}ds,$$

with

$$\begin{aligned}\|e^{-(t-s)A}[F(u-v,u)+F(v,u-v)]\|_{H^m} &\leq \|[F(u-v,u)+F(v,u-v)]\|_{H^m}\\ &\leq \|F(u-v,u)\|_{H^m}+\|F(v,u-v)\|_{H^m}.\end{aligned}$$

Thus,

$$\|Gu-Gv\|_{X^m} \leq \|F(u-v,u)\|_{Y^m}+\|F(v,u-v)\|_{Y^m}$$

and from (3.9) and (3.8) we obtain

$$\begin{aligned}\|Gu-Gv\|_{X^m} &\leq c\|u-v\|_{X^m}\|u\|_{Y^{m+1}}+c\|v\|_{X^m}\|u-v\|_{Y^{m+1}}\\ &= cK(K^{-1}\|u-v\|_{X^m})L(L^{-1}\|u\|_{Y^{m+1}})+cK(K^{-1}\|v\|_{X^m})L(L^{-1}\|u-v\|_{Y^{m+1}})\\ &\leq cKL(\|u\|_Z+\|v\|_Z)\|u-v\|_Z.\end{aligned}$$

On the other hand,

$$e^{-tA}: H^m_\sigma \to H^{m+1}_\sigma \quad \Rightarrow \|e^{-tA}\|_{H^m} = t^{-\frac{1}{2}}e^t$$

and by (3.10)

$$\begin{aligned}\|Gu(\cdot)-Gv(\cdot)\|_{H^{m+1}} &= \Big\| \int_0^t e^{-(t-s)A}[F(u-v,u)+F(v,u-v)]\ \mathrm{d}s\Big\|_{H^{m+1}}\\ &\leq \int_0^t \|e^{-(t-s)A}[F(u-v,u)+F(v,u-v)]\|_{H^{m+1}}\ \mathrm{d}s\\ &\leq \int_0^t t^{-\frac{1}{2}}(t-s)^{-1/2}e^{t-s}\big[\|F(u-v,u)\|_{H^m}+\|F(v,u-v)\|_{H^m}\big]\ \mathrm{d}s.\end{aligned}$$

Hence,

$$\|Gu-Gv\|_{Y_{m+1}} \leq 2e^{T'}T'^{\frac{1}{2}}cKL(\|u\|_Z+\|v\|_Z)\|u-v\|_Z. \tag{3.11}$$

Now, we assume that

$$L=\gamma K,\ \text{where}\ \gamma = 2e^{T'}T'^{\frac{1}{2}}.$$

Thus, by relations (3.8), (3.11) we have that

$$\|Gu-Gv\|_Z \leq cL(\|u\|_Z+\|v\|_Z)\|u-v\|_Z. \tag{3.12}$$

We observe now that

$$G0 = e^{-tA}a + \int_0^t e^{-(t-a)A}b(s)\ \mathrm{d}s.$$

Then

$$\|G0\|_{X_m} = \big\|e^{-tA}a+\int_0^t e^{-(t-a)A}b(s)\ \mathrm{d}s\big\|_{X_m} \leq \|e^{-tA}a\|_{Y_m} + \big\|\int_0^t b(s)\ \mathrm{d}s\big\|_{Y_m}$$

$$\|G0\|_Z \leq \big(\|a\|_{H^m} + B\big)\gamma L^{-1} \text{ where, } B = \int_0^T \|b(t)\|_{H^m} \, \mathrm{d}t. \tag{3.13}$$

By the previous, it follows that $G\colon Z \to Z$, with

$$\|Gu\|_Z = \|Gu - G0 + G0\|_Z \leq \|Gu - G0\|_Z + \|G0\|_Z \leq cL\|u\|_Z\|u - 0\|_Z + \big(\|a\|_{H^m} + B\big)\gamma L^{-1}$$

$$\Rightarrow \|Gu\|_Z \leq cL\|u\|_Z^2 + \big(\|a\|_{H^m} + B\big)\gamma L^{-1}. \tag{3.14}$$

Now, we want to show that $G\colon S_Z \to S_Z$, where S_Z is the unit ball of Z, in order to prove that G is a strict contraction on S.

To achive this, it suffices to choose L, such that

$$\big(\|a\|_{H^m} + B\big)\gamma L^{-1} + cL = 1.$$

Such $L > 0$ exists if

$$4c\gamma\big(\|a\|_{H^m} + B\big) < 1. \tag{3.15}$$

Then, we can take

$$L = (2c)^{-1}\big[1 - \big(1 - 4\gamma c(\|a\|_{H^m} + B)\big)^{\frac{1}{2}}\big] < (2c)^{-1} \tag{3.16}$$

since $\gamma = 2e^{T^{'}}T^{'\frac{1}{2}}$, condition (3.15) is satisfied by $T^{'}$ sufficiently small.

Therefore, we get

$$\begin{aligned} \|Gu - Gv\|_Z &\leq cL(\|u\|_Z + \|v\|_Z)\|u - v\|_Z \\ &< c(2c)^{-1}(\|u\|_Z + \|v\|_Z)\|u - v\|_Z \\ &= \frac{1}{2}\big(\underbrace{\|u\|_Z}_{\leq 1} + \underbrace{\|v\|_Z}_{\leq 1}\big)\|u - v\|_Z \leq \frac{1}{2}\|u - v\|_Z \end{aligned}$$

$$\Longrightarrow \|Gu - Gv\|_Z < \frac{1}{2}\|u - v\|_Z \ , \ \forall u, v \in S.$$

Hence, G is a strict contraction on S and, by the Banach fixed-point theorem, we get that G has a unique fixed point u in S, which is a solution of the integral equation (3.7).

For this u, we have $Fu \in Y_m$ by (3.9). That is

$$\|Fu\|_{Y_m} = \|F(u,u)\|_{Y_m} \leq \|u\|_{Y_m}\|u\|_{Y_{m+1}} \leq \|u\|_{Y_m}\|u\|_{Y_m} < \infty,$$

since $u, v \in Y_m$ and, since $b \in Y_m \Longrightarrow Fu + b \in Y_m$.

Now, if we define

$$v(t) := e^{-tA}a + \int_0^t e^{-(t-s)A}[Fu(s) + b(s)] \, \mathrm{d}s,$$

we take that,

$$Av(t) := \int_0^t \big(A^{\frac{1}{2}}e^{-(t-s)A}\big)\big(A^{\frac{1}{2}}[Fu(s)+b(s)]\big)\ \mathrm{d}s$$

is in Y_{m-1}, since $Fu+b \in Y_m \Rightarrow A^{\frac{1}{2}}[Fu+b] \in Y_{m-1}$ and $\|A^{\frac{1}{2}}e^{-tA}\| \leq t^{-\frac{1}{2}}$. See, e.g., [17], L.2. Thus,

$$\frac{dv}{dt} = -Av + Fu + b \in Y_{m-1} \implies v \in AC([0,T^{'}];H_\sigma^{m-1}).$$

The consequence above follows from the fact that, see, e.g., [10],

$$\begin{aligned}
&\frac{dv}{dt}\colon [0,T^{'}] \to H_\sigma^{m-1} \text{ is integrable in t}\\
&\Rightarrow \int_0^x \frac{dv}{dt}(t)\ \mathrm{d}t \text{ is absolutely continuous in t.}\\
&\Leftrightarrow v(x) - v(0) = v(x) \text{ is absolutely continuous in } t.
\end{aligned}$$

Also it is true that $e^{-tA}a \in AC([0,T^{'}];H_\sigma^{m-1})$, with

$$\begin{aligned}
\left\|\frac{d}{dt}e^{-tA}a\right\|_{H^{m-1}} &= \left\|Ae^{-tA}a\right\|_{H^{m-1}} \leq \|A^{\frac{1}{2}}e^{-tA}a\|_{H^{m-1}}\|A^{\frac{1}{2}}a\|_{H^{m-1}}\\
&\leq t^{-\frac{1}{2}}\|A^{\frac{1}{2}}a\|_{H^{m-1}} \leq t^{-\frac{1}{2}}\|a\|_{H^m}. \text{ See, e.g.,[17].}
\end{aligned}$$

Thus, u is a solution of (1.2) satisfying (1.3) locally in time. [i.e. with T_0 replaced by $T^{'}$]. Consequently, we constructed a local solution of (1.2).

To prove the uniqueness of this solution, see [17], we define as $S := S[0,T]$ the set of all functions $u\colon [0,T] \to H_\sigma$ with the following properties.

i) $u(t)$ is continuous on [0,T].

ii) $A^{\frac{1}{2}}u(t)$ exists and is continuous for $t \in (0,T]$ and $\|A^{\frac{1}{2}}u(t)\| = o(t^{-\frac{1}{4}})$, $as\ t \to 0$.

iii) $A^{\frac{3}{4}}u(t)$ is continuous on (0,T] and $\|A^{\frac{3}{4}}\| = o(t^{-\frac{1}{2}})$, $t \to 0$.

Now, let $v(t) \in S[0,T_1]$ be another solution of (1.2) with the same initial value a. Then,

$$w(t) := v(t) - u(t) = \int_0^t e^{-(t-s)A}[Fv(s) - Fu(s)]\ \mathrm{d}s, \quad 0 < t \leq T_0$$

and let

$$w_{n+1}(t) := u_{n+1}(t) - u_n(t) = \int_0^t e^{-(t-s)A}[Fu_n(s) - Fu_{n-1}(s)]\ \mathrm{d}s, \quad n = 0,1,...$$

with $u_{-1} := 0$.
Then,

$$\|A^a w_{n+1}(t)\| \leq \int_0^t \|A^a e^{-(t-s)A}\|\ \|Fu_n(s) - Fu_{n-1}(s)\|\ \mathrm{d}s,$$

by L2, L3 in [17].

$$\leq \int_0^t (t-s)^{-a}M\big[\|A^{\frac{3}{4}}u_n(s)\|\ \|A^{\frac{1}{2}}(u_n(s)-u_{n-1}(s))\| + \|A^{\frac{3}{4}}(u_n(s)-u_{n-1}(s))\|\ \|A^{\frac{1}{2}}u_{n-1}(s)\|\big]\ \mathrm{d}s$$

$$= M \int_0^t (t-s)^{-a} \left[\|A^{\frac{3}{4}} u_n(s)\| \ \|A^{\frac{1}{2}} w_n(s)\| + \|A^{\frac{3}{4}} w_n(s)\| \ \|A^{\frac{1}{2}} u_{n-1}(s)\| \right] \, \mathrm{d}s$$

and since, $\|A^a u_n(t)\| \le K t^{\frac{1}{4}-q}$, $0 < t \le T$, $\frac{1}{4} \le a < 1$

$$\le KM \int_0^t (t-s)^{-a} \left[s^{\frac{1}{4}-\frac{3}{4}} \ \|A^{\frac{1}{2}} w_n(s)\| + \|A^{\frac{3}{4}} w_n(s)\| \ s^{\frac{1}{4}-\frac{3}{4}} \right] \, \mathrm{d}s$$

$$\Rightarrow \|A^a w_{n+1}(t)\| \le KM \int_0^t (t-s)^{-a} \left[s^{-\frac{1}{2}} \ \|A^{\frac{1}{2}} w_n(s)\| + s^{-\frac{1}{4}} \ \|A^{\frac{3}{4}} w_n(s)\| \right] \, \mathrm{d}s.$$

Thus, by induction we get

$$\|A^{\frac{1}{2}} w_n(t)\| \le \frac{(2\beta MK)^n K}{2t^{\frac{1}{4}}}$$

and

$$\|A^{\frac{3}{4}} w_n(t)\| \le \frac{(2\beta MK)^n K}{2t^{\frac{1}{2}}} \ , \ 0 < t \le T.$$

With the same way we get

$$\|A^{\frac{1}{2}} w(t)\| \le \frac{(2\beta MK)^n K}{2t^{\frac{1}{4}}} \ , \ n = 1, 2, ... \tag{3.17}$$

and

$$\|A^{\frac{3}{4}} w(t)\| \le \frac{(2\beta MK)^n K}{2t^{\frac{1}{2}}} \ , \ 0 < t \le T \tag{3.18}$$

where K may be made arbitrary small by choosing T_0 sufficiently small.

By (3.18) $\Rightarrow A^{\frac{1}{2}} w(t) = 0 \Rightarrow w(t) = 0$ as $n \to \infty$.

To show that $v(t) = u(t)$ for $0 \le t \le T' := min(T, T_1)$, it is enough to observe that any solution $v(t) \in S$ of (3.7) is also a solution of (3.7) with the initial time $t_0 > 0$.
That is,

$$v(t) = e^{-(t-t_0)A} v(t_0) + \int_{t_0}^t e^{-(t-s)A} b(s) \, \mathrm{d}s + \int_{t_0}^t e^{-(t-s)A} F v(s) \, \mathrm{d}s, \ t_0 < t \le T_1. \tag{3.19}$$

This is obvious since (1.2) $\Leftrightarrow$ (3.7). Thus $u(t)$ and $v(t)$ satisfies (3.19).

If $u(t_0) = v(t_0) \Rightarrow u(t) = v(t)$ in $t \in [t_0, t_0 + h]$, $h > 0$. Therefore, the local solution u of (1.2) is unique.

Thus, this local solution ($\nu > 0$) satisfies the following

$$u_\nu \in C([0, T_\nu]; H_\sigma^m) \cap AC([0, T_\nu]; H_\sigma^{m-1}) \cap L^1([0, T_\nu]; H_\sigma^{m+1}), \tag{3.20}$$

where

$$T_\nu > 0, \quad T_\nu := T(\nu, \|a\|_{H_m}, \|b(\cdot)\|_{H^m}).$$

2nd step. Now, we want to show that u_ν can be extended to a solution on [0,T_0], $T_0 > 0$ independent of ν.

To this end, we export an estimate for $\|u_\nu(t)\|_{H^m}$.

Since u_ν is a local solution of (1.2), we have

$$\frac{du_\nu}{dt} + \nu A u_\nu = F u_\nu + b(t).$$

Taking the H^m inner product with $u_\nu(t)$ we get

$$\begin{aligned}
&\big(\frac{du_\nu}{dt}, u_\nu(t)\big)_{H^m} + \big(\nu A u_\nu, u_\nu(t)\big)_{H^m} = \big(F u_\nu + b(t), u_\nu(t)\big)_{H^m} \\
\Rightarrow &\frac{1}{2}\frac{d}{dt}\|u_\nu\|^2_{H^m} + \nu (A u_\nu, u_\nu)_{H^m} \le \big|(F u_\nu, u_\nu(t))_{H^m}\big| + \big|(b(t), u_\nu(t))_{H^m}\big| \\
\Rightarrow &\frac{1}{2}\frac{d}{dt}\|u_\nu\|^2_{H^m} + \nu (A u_\nu, u_\nu)_{H^m} \le c^{'}\|u_\nu\|^3_{H^m} + \|b(t)\|_{H^m}\|u_\nu\|_{H^m}
\end{aligned}$$

by applying (3.4) in the second inequality and, since $\nu > 0$, we have

$$\begin{aligned}
\Rightarrow \frac{1}{2} 2\|u_\nu\|_{H^m}\frac{d}{dt}\|u_\nu\|_{H^m} &\le c^{'}\|u_\nu\|^3_{H^m} + \|b(t)\|_{H^m}\|u_\nu\|_{H^m} \\
\Rightarrow \frac{d}{dt}\|u_\nu\|_{H^m} &\le c^{'}\|u_\nu\|^2_{H^m} + \|b(t)\|_{H^m}
\end{aligned}$$

and

$$\|u_\nu(0)\|_{H^m} = \|a\|_{H^m}.$$

Thus we have

$$\|u_\nu(t)\|_{H^m} \le \phi(t), \tag{3.21}$$

where ϕ is the solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\phi(t) = c^{'}\phi(t)^2 + \|b(t)\|_{H^m} \\ \phi(0) = \|a\|_{H^m}. \end{cases} \tag{3.22}$$

We observe that ϕ exists as a continuous function on $[0,T_0]$, $T_0 > 0$ since, $b \in L^1([0,T_0], H^m_\sigma)$. That is, $\|b(t)\|_{H^m}$ is integrable in $t \in [0,T_0]$. T_0 and ϕ are independent on ν.

The inequality $\|u_\nu(t)\|_{H^m} \le \phi(t)$ holds for $t \in [0,T_\nu]$ by (3.17).
If $T_\nu < T_0$ then, we can solve the Cauchy problem (1.2) for $t \ge T_\nu$ with initial data $u_\nu(T_\nu) \in H^m_\sigma$, in order to continue the solution u_ν in a time interval $[0, T_\nu + T^{'}_\nu]$ in which (3.18) holds true. Now, since $T^{'}_\nu > 0$, can be defined as $T^{'}_\nu := T^{'}_\nu\big(\nu, \|u_\nu(T_\nu)\|_{H^m}, \|b(\cdot)\|_{H^m}\big)$.
After a finite number of such extensions, u_ν is continuous as a solution on $[0,T_0]$, where T_0 is independent of ν, with the estimate (3.18). Therefore, u_ν is bounded in the space $C([0,T_0]; H^m_\sigma)$. This completes the 2nd step of the proof.

3rd step. We will show that $\lim_{\nu\to 0} \underbrace{u_\nu(t)}_{\in H^{m-1}_\sigma} = u_0(t)$ uniformly for $t \in [0,T_0]$, where we'll show that u_0 is a solution of the Euler equation.

To achieve that, we set as $u_1 := u_{\nu_1}$ and $u_2 := u_{\nu_2}$ with, $\nu_1 < \nu_2$ and u_1, u_2 are solutions of (1.2). That is,

$$\frac{du_1}{dt} + \nu_1 A u_1 = F u_1 + b(t)$$

and

$$\frac{du_2}{dt} + \nu_2 A u_2 = F u_2 + b(t)$$

$$\Rightarrow \frac{du_1}{dt} - \frac{du_2}{dt} + \nu_1 Au_1 - \nu_2 Au_2 = Fu_1 - Fu_2$$

and, since $\frac{d}{dt}$ is a linear operator

$$\Rightarrow \frac{d}{dt}(u_1 - u_2) + \nu_1 Au_1 - \nu_2 Au_2 + \nu_1 Au_2 - \nu_1 Au_2 = \underbrace{F(u_1, u_1)}_{=Fu_1} - F(u_2, u_1) + F(u_2, u_1) - \underbrace{F(u_2, u_2)}_{=Fu_2}$$

$$\Rightarrow \frac{d}{dt}\underbrace{(u_1 - u_2)}_{=:w} + \nu_1 A(u_1 - u_2) + (\nu_1 - \nu_2)Au_2 = F(u_1 - u_2, u_1) + F(u_2, u_1 - u_2)$$

$$\Rightarrow \frac{dw}{dt} + \nu_1 Aw + (\nu_1 - \nu_2)Au_2 = F(w, u_1) + F(u_2, w).$$

Now, taking the $(m-1)$ inner product with $w(t)$, it follows that

$$\Big(\frac{dw}{dt}, w\Big)_{H^{m-1}} + \nu_1\big(Aw, w\big)_{H^{m-1}} + (\nu_1 - \nu_2)\big(Au_2, w\big)_{H^{m-1}} = \big(F(w, u_1), w\big)_{H^{m-1}} + \big(F(u_2, w), w\big)_{H^{m-1}}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{m-1}} + \underbrace{\nu_1}_{\geq 0}\big(Aw, w\big)_{H^{m-1}} = (\nu_2 - \nu_1)\big(Au_2, w\big)_{H^{m-1}} + \big(F(w, u_1), w\big)_{H^{m-1}} + \big(F(u_2, w), w\big)_{H^{m-1}}$$

$$\begin{aligned}\Rightarrow \frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{m-1}} &\leq (\nu_2 - \nu_1)\big(Au_2, w\big)_{H^{m-1}} + \big|\big(F(w, u_1), w\big)_{H^{m-1}}\big| + \big|\big(F(u_2, w), w\big)_{H^{m-1}}\big| \\ &\leq (\nu_2 - \nu_1)\big(Au_2, w\big)_{H^{m-1}} + c\|u_1\|_{H^{m-1}}\|w\|^2_{H^{m-1}} + c'\|u_2\|_{H^{m-1}}\|w\|^2_{H^{m-1}} \\ &\leq (\nu_2 - \nu_1)\big(Au_2, w\big)_{H^{m-1}} + c\|u_1\|_{H^m}\|w\|^2_{H^{m-1}} + c'\|u_2\|_{H^m}\|w\|^2_{H^{m-1}}\end{aligned}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{m-1}} \leq (\nu_2 - \nu_1)\big(Au_2, w\big)_{H^{m-1}} + \big(c\|u_1\|_{H^m} + c'\|u_2\|_{H^m}\big)\|w\|^2_{H^{m-1}}.$$

We observe that, for the solutions u_1 and u_2, the relation (3.21) holds true. Hence, $\|u_1\|_{H^m}$ and $\|u_2\|_{H^m}$ are uniformly bounded.

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{m-1}} \leq \underbrace{(\nu_2 - \nu_1)}_{\leq \nu_2}\big(Au_2, w\big)_{H^{m-1}} + \big(\underbrace{c\|u_1\|_{H^m} + c'\|u_2\|_{H^m}}_{\leq K}\big)\|w\|^2_{H^{m-1}}$$

$$\Rightarrow \frac{1}{2}2\|w\|_{H^{m-1}}\frac{d}{dt}\|w\|_{H^{m-1}} \leq \nu_2\big|\big(Au_2, w\big)_{H^{m-1}}\big| + K\|w\|^2_{H^{m-1}}$$

$$\leq \nu_2\|Au_2\|_{H^{m-1}}\|w\|_{H^{m-1}} + K\|w\|^2_{H^{m-1}}$$

$$\Rightarrow \frac{d}{dt}\|w\|_{H^{m-1}} \leq \nu_2\|Au_2\|_{H^{m-1}} + K\|w\|_{H^{m-1}}.$$

Integrating over time t each sides, we get

$$\Rightarrow \int_0^t \frac{d}{ds}\|w(s)\|_{H^{m-1}}ds \leq \int_0^t \nu_2\|Au_2(s)\|_{H^{m-1}}ds + \int_0^t K\|w(s)\|_{H^{m-1}}ds$$

$$\Rightarrow \|w(t)\|_{H^{m-1}} - \|w(0)\|_{H^{m-1}} \leq \int_0^t \nu_2\|Au_2(s)\|_{H^{m-1}} \text{ d}s + \int_0^t K\|w(s)\|_{H^{m-1}} \text{ d}s, \tag{3.23}$$

where $\|w(0)\|_{H^{m-1}} = 0$ since, $w(0) = u_1(0) - u_2(0) = a - a = 0$.

Now, by using Gronwall's lemma, since (3.20) is true, we take

$$\Rightarrow \|w(t)\|_{H^{m-1}} \leq \nu_2 \int_0^t \|Au_2(s)\|_{H^{m-1}}ds \cdot e^{\int_0^t Kds}$$

$$\Rightarrow \|w(t)\|_{H^{m-1}} \leq \nu_2 e^{Kt} \int_0^t \|Au_2(s)\|_{H^{m-1}}ds$$

$$\leq \nu_2 e^{Kt} \left(\int_0^t 1 \text{ d}s\right)^{\frac{1}{2}} \left(\int_0^t \|Au_2(s)\|^2_{H^{m-1}} \text{ d}s\right)^{\frac{1}{2}} \text{ by Hölder's inequality}$$

$$\Rightarrow \|w(t)\|_{H^{m-1}} \leq \left(\nu_2 t\right)^{\frac{1}{2}} e^{Kt} \left(\nu_2 \int_0^t \|Au_2(s)\|^2_{H^{m-1}} \text{ d}s\right)^{\frac{1}{2}}. \tag{3.24}$$

At this point, we claim that the following inequality holds true.

$$\nu \int_0^t \left(Au_\nu(s), u_\nu(s)\right)_{H^m} \text{ d}s \leq \psi(t), \quad t \in [0, T_0], \tag{3.25}$$

where $\psi\colon \mathbb{R} \to \mathbb{R}$ is a continuous function depending on $\|a\|_{H^m}$, $\|b(\cdot)\|_{H^m}$ and not on ν.

To show this, we integrate (3.4) over time to get

$$\frac{1}{2}\int_0^t \frac{d}{dt}\|u_\nu(s)\|^2_{H^m} \text{ d}s + \nu \int_0^t (Au_\nu(s), u_\nu(s))_{H^m} \text{ d}s \leq c' \int_0^t \|u_\nu(s)\|^3_{H^m} \text{ d}s + \int_0^t \|b(t)\|_{H^m}\|u_\nu(s)\|_{H^m} \text{ d}s$$

$$\Rightarrow \frac{1}{2}\|u_\nu(t)\|^2_{H^m} - \frac{1}{2}\|u_\nu(0)\|^2_{H^m} + \nu \int_0^t (Au_\nu(s), u_\nu(s))_{H^m} \text{ d}s \leq c' \int_0^t \|u_\nu(s)\|^3_{H^m} \text{ d}s + \|b(t)\|_{H^m} \int_0^t \|u_\nu(s)\|_{H^m} \text{ d}s$$

$$\Rightarrow \nu \int_0^t (Au_\nu(s), u_\nu(s))_{H^m} \text{ d}s \leq \frac{1}{2}\|a\|^2_{H^m} - \frac{1}{2}\|u_\nu(t)\|^2_{H^m} + c' \int_0^t \|u_\nu(s)\|^3_{H^m} \text{ d}s + \|b(t)\|_{H^m} \int_0^t \|u_\nu(s)\|_{H^m} \text{ d}s$$

$$\leq \underbrace{\frac{1}{2}\|a\|^2_{H^m} + \frac{1}{2}\phi^2(t) + c' \int_0^t \phi^3(s) \text{ d}s + \|b(t)\|_{H^m} \int_0^t \phi(s) \text{ d}s}_{=:\psi(t)}, \ t \in [0, T_0]$$

$$\Rightarrow \nu \int_0^t (Au_\nu(s), u_\nu(s))_{H^m} \text{ d}s \leq \psi(t), \ t \in [0, T_0]. \tag{3.26}$$

Now, since

$$\|Au_2\|^2_{H^{m-1}} = \|(A+I)^{-\frac{1}{2}}Au_2\|^2_{H^m} \leq \|A^{\frac{1}{2}}\|^2_{H^m} = (Au_2, u_2)_{H^m}$$

$$\begin{aligned}\Rightarrow \|w(t)\|_{H^{m-1}} &\leq (\nu_2 t)^{\frac{1}{2}} e^{Kt}\left(\nu_2\int_0^t \|Au_2(s)\|^2_{H^{m-1}}\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &\leq (\nu_2 t)^{\frac{1}{2}} e^{Kt}\left(\nu_2\int_0^t (Au_2(s), u_2)_{H^m}\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &\leq \underbrace{(\nu_2 t)^{\frac{1}{2}} e^{Kt}\psi^2(t)}_{\longrightarrow 0} \quad \textit{as } \nu_2 \to 0,\ t\in[0,T_0].\end{aligned}$$

Thus, $\|u_{\nu_1} - u_{\nu_2}\|_{H^{m-1}} \longrightarrow 0$ as $\nu_2 \to 0$ and, since H^{m-1} is a Banach space.

$$\Longrightarrow \lim_{\nu\to 0} u_\nu(t) = u_0(t) \in H^{m-1}_\sigma \text{ exists in } (m-1) \text{ norm uniformly for } t\in[0,T_0].$$

Now, we observe that $u_0(\cdot)$ is continuous in $(m-1)$ norm and, since

$$\|u_\nu(t)\|_{H^m} \leq \phi(t)$$

$$\Rightarrow \lim_{\nu\to 0}\|u_\nu(t)\|_{H^m} \leq \lim_{\nu\to 0}\phi(t) = \phi(t),$$

and $\|\cdot\|_{H^m}$ is a continuous function

$$\Rightarrow \|\lim_{\nu\to 0} u_\nu(t)\|_{H^m} \leq \phi(t)$$

$$\Rightarrow \|u_0(t)\|_{H^m} \leq \phi(t) \Rightarrow u_0(t) \in H^m_\sigma.$$

Moreover, $u_\nu(t) \rightharpoonup u_0(t)$ in H^m_σ, uniformly in t
since,
if $f \in (H^m_\sigma)^* \simeq H^m_\sigma$ (See, e.g., Riesz representation, e.g., in [3])
then

$$\lim_{\nu\to 0}(f, u_\nu(t))_{H^m} = (\lim_{\nu\to 0} f, \lim_{\nu\to 0} u_\nu(t))_{H^m} = (f, u_0(t))_{H^m}, \quad \forall t\in[0,T_0]$$

$\Rightarrow u_\nu(t) \rightharpoonup u_0(t)$ in H^m_σ, uniformly in t.

Additionally, $u_0(\cdot)$ is continuous in $(m-1)$ norm, since if we take $(t_n)_{n\in\mathbb{N}} \subset H^{m-1}$ with $t_n \to t$, we want to show that $u_0(t_n) \to u_0(t)$.
We estimate

$$\begin{aligned}\|u_0(t_n) - u_o(t)\|_{H^{m-1}} &= \|\lim_{\nu\to 0} u_\nu(t_n) - \lim_{\nu\to 0} u_\nu(t)\|_{H^{m-1}}\\ &= \lim_{\nu\to 0}\|u_\nu(t_n) - u_\nu(t)\|_{H^{m-1}}\\ &\leq \lim_{\nu\to 0}\|t_n - t\|_{H^{m-1}}??\\ &= \|t_n - t\|_{H^{m-1}} \to 0 \textit{ as } n\to\infty.\end{aligned}$$

Thus, $u_0(\cdot)$ is continuous in $(m-1)$ norm. And, $u_0(\cdot)$ is weakly continuous in H^m_σ.
Furthermore, we have

$$Fu_\nu(t) \rightharpoonup Fu_0(t) \textit{ in } H^{m-1}_\sigma \text{ uniformly for } t\in[0,T_0]$$

and

$$Fu_0(\cdot) \text{ is weakly continuous in } H_\sigma^{m-1}.$$

Since,

$Fu_\nu(\cdot)$ is strongly continuous and uniformly bounded in H_σ^{m-1} by (1.3) and (3.3)

$\Rightarrow \forall f \in H_\sigma^{m-1}$ sufficiently smooth function. I want to show

$$(Fu_\nu(t), f)_{H^{m-1}} \to (Fu_0(t), f)_{H^{m-1}} \text{uniformly for } t \in [0, T_0].$$

But, we observe that

$$\big(Fu, f\big)_{H^{m-1}} = -\big((u \cdot \nabla)u, f\big)_{H^{m-1}} \overset{*}{=} \big(uu, \nabla f\big)_{H^{m-1}}$$

and

$$\begin{aligned} \lim_{\nu \to 0} \big(Fu_\nu(t), f\big)_{H^{m-1}} &= \lim_{\nu \to 0} \big(u_\nu^2(t), \nabla f\big)_{H^{m-1}} \\ &= \big(\lim_{\nu \to 0} u_\nu^2(t), \nabla f\big)_{H^{m-1}} = \big(u_0^2(t), \nabla f\big)_{H^{m-1}} \\ &= (Fu_0(t), f)_{H^{m-1}} \text{ uniformly for } t \in [0, T_0] \end{aligned} \tag{3.27}$$

to show (*), it suffices to show

$$-\big((u \cdot \nabla)u, f\big)_{H^{m-1}} = \big(u^2, \nabla f\big)_{H^{m-1}}.$$

Hence,

$$\big((u \cdot \nabla)u, f\big)^2_{H^{m-1}} = \sum_{0 \le |a| \le m-1} \int_{\mathbb{R}^n} D^a (u \cdot \nabla) u D^a f \, \mathrm{d}x$$

applying integration by parts, we get

$$= \sum_{0 \le |a| \le m-1} \left(\underbrace{\int_{\partial \mathbb{R}^n} D^a (u \cdot u) D^a f \, \mathrm{d}x}_{=0} - \int_{\mathbb{R}^n} D^a (u \cdot u) D^a (\nabla f) \, \mathrm{d}x \right)$$

$$= - \sum_{0 \le |a| \le m-1} \left(\int_{\mathbb{R}^n} D^a (u^2) D^a (\nabla f) \, \mathrm{d}x \right) = -\big(u^2, \nabla f\big)^2_{H^{m-1}}.$$

To show that u_0 is a solution of $(1.2)_0$, we take the integral of (1.2) over the time interval $[t', t''] \subseteq (0, T_0]$.

Thus, we have

$$\int_{t'}^{t''} \frac{du_\nu(t)}{dt} \, \mathrm{d}t = \int_{t'}^{t''} (-\nu A u_\nu + F u_\nu + b) \, \mathrm{d}t$$

$$\Rightarrow u_\nu(t'') - u_\nu(t') = \int_{t'}^{t''} (-\nu A u_\nu + F u_\nu + b) \, \mathrm{d}t$$

take the $(m-1)$ inner product with $f \in H_\sigma^{m-1}$ smooth function, we have

$$\Rightarrow \big(u_\nu(t'') - u_\nu(t'), f\big)_{H^{m-1}} = \big(\int_{t'}^{t''} (-\nu A u_\nu + F u_\nu + b) \, \mathrm{d}t, f\big)_{H^{m-1}}$$

$$= \int_{t'}^{t''} (-\nu A u_\nu + F u_\nu + b, f)_{H^{m-1}} \text{ dt}$$

by (3.24) and the fact that

$$\nu(Au_\nu, f)_{H^{m-1}} = \nu(u_\nu, Af)_{H^{m-1}} \longrightarrow 0 \text{ , as } \nu \to 0$$

uniformly in t .
We go to the limit $\nu \to 0$ to get

$$(u_0(t'') - u_0(t'), f)_{H^{m-1}} = \int_{t'}^{t''} (Fu_0 + b, f)_{H^{m-1}} \text{ dt}$$

$$\Rightarrow u_0(t'') - u_0(t') = \int_{t'}^{t''} (Fu_0 + b) \text{ dt}. \tag{3.28}$$

Previously, we have shown that $Fu_0(t) \in H_\sigma^{m-1}$ is weakly continuous in t.
Furthermore, $b \in L^1([0,T]; H_\sigma^m)$ from the assumptions and, since $H_\sigma^m \subset H_\sigma^{m-1}$, it follows that $b \in L^1([0,T]; H_\sigma^{m-1})$. That is, $b(t) \in H_\sigma^{m-1}$ is integrable functiom in t.

By (3.25) and for $t' \to 0$

$$\Rightarrow u_0(t'') - u_0(0) = \int_0^{t''} (Fu_0 + b) \text{ dt}.$$

Thus,

$$u_0 \in AC([0,T_0]; H_\sigma^{m-1}) \cap L^\infty([0,T_0]; H_\sigma^m) \tag{3.29}$$

and u_0 is a solution of $(1.2)_0$. (Euler)

Now, we will show the uniqueness of solution of $(1.2)_0$ within the class of (3.29).
To this ends, let u,v be two solutions of $(1.2)_0$ and set $w := u - v$.
Then

$$\begin{aligned} \frac{dw}{dt} &= \frac{d}{dt}(u-v) = \frac{du}{dt} - \frac{dv}{dt} = Fu + b(t) - Fv - b(t) \\ &= Fu - Fv = F(u,u) - F(v,v) \\ &= F(u,u) - F(u,v) + F(u,v) - F(v,v) \\ &= F(u-v,v) + F(u,u-v) = F(w,v) + F(u,w) \end{aligned}$$

$$\Rightarrow \frac{dw}{dt} = F(w,v) + F(u,w).$$

Taking the $(m-1)$ inner product with w, we have

$$\Big(\frac{dw}{dt}, w\Big)_{H^{m-1}} = \big(F(w,v), w\big)_{H^{m-1}} + \big(F(u,w), w\big)_{H^{m-1}}$$

$$\begin{aligned} \Rightarrow \frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{m-1}} &\le \big|(F(w,v),w)_{H^{m-1}}\big| + \big|(F(u,w),w)_{H^{m-1}}\big| \\ &\le c\|v\|_{H^m}\|w\|^2_{H^{m-1}} + c'\|u\|_{H^m}\|w\|^2_{H^{m-1}} \end{aligned}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{m-1}} \le \big(c\|v\|_{H^m} + c'\|u\|_{H^m}\big)\|w\|^2_{H^{m-1}}. \tag{3.30}$$

Now, since $c\|v\|_{H^m} + c'\|u\|_{H^m} \le$ constant, by the assumptions, and $w(0) = 0$.

$$(3.30) \Rightarrow 2\|w\|_{H^{m-1}} \frac{1}{2}\frac{d}{dt}\|w\|_{H^{m-1}} \le \tilde{c}\|w\|^2_{H^{m-1}}$$

$$\Rightarrow \frac{d}{dt}\|w\|_{H^{m-1}} \leq \tilde{c}\|w\|_{H^{m-1}} \Rightarrow \frac{d}{dt}\|w\|_{H^{m-1}} - \tilde{c}\|w\|_{H^{m-1}} \leq 0$$
$$\Rightarrow e^{-\int \tilde{c}dt}\frac{d}{dt}\|w\|_{H^{m-1}} - \tilde{c}e^{-\int \tilde{c}dt}\|w\|_{H^{m-1}} \leq 0$$
$$\Rightarrow \frac{d}{dt}\left(e^{-\int \tilde{c}dt}\|w\|_{H^{m-1}}\right) \leq 0$$

$\Rightarrow e^{-\int \tilde{c}dt}\|w\|_{H^{m-1}}$ is a decreaising function in $[0, T_0]$ and, since $\|w(0\|_{H^{m-1}} = 0$

$$\Rightarrow \|w\|_{H^{m-1}} = 0 \Rightarrow w = 0 \Rightarrow v = w.$$

Thus, the solution of $(1.2)_0$ is unique.

To end the proof, it suffices to show that the solution of the Euler equation u_0 is in the class $C([0, T_0]; H^m_\sigma)$. Then, in addition to (3.29), the result of the part 2 of the theorem will have been proven.

We have shown that $\|u_0(t)\|_{H^m} \leq \phi(t)$ and that $\phi(0) = \|a\|_{H^m}$

$$\Rightarrow \limsup_{t\to 0}\|u_0(t)\|_{H^m} \leq \limsup_{t\to 0}\phi(t) = \phi(0) = \|a\|_{H^m} = \|u_0(0)\|_{H^m}$$

and since, furthermore, we have that u_0 is weakly continuous in H^m_σ, it follows that u_0 is strongly continuous in H^m_σ at $t = 0$.
But, we want to prove that u_0 is strongly continuous in H^m_σ, *for all* $t \in [0, T_0]$. Let v the solution of $(1.2)_0$, i.e. a solution of the Euler equation, for $t \geq t_0$ with the initial data $u_0(t_0)$.
By the previous ones, v is continuous in H^m_σ at $t = 0$, and $u_0 \equiv v$, for $t \geq t_0$ by uniqueness. It follows that u_0 is right continuous at t_0.
In addition, $(1.2)_0$ (Euler) is reversible in time t $\Rightarrow u_0$ is left continuous at t_0.
Therefore,

$$u_0 \in C([0, T_0]; H^m_\sigma), \tag{3.31}$$

and thus, by (3.29) and (3.31), we have the result that

$$u_0 \in C([0, T_0]; H^m_\sigma) \cap AC([0, T_0]; H^{m-1}_\sigma).$$

This ends the proof of the main theorem of the part 1. □

CHAPTER 4

Strong L^p-solutions for the Navier-Stokes equations

In this chapter, we present a result concerning the existence and uniqueness of strong solutions of the Navier-Stokes equations (1.1) (with $n = m$, $\nu = 1$ and $f \equiv 0$) globally in time, provided that the initial data are small enough and they are given in space L^p_σ which denotes the subspace of $L^p(\mathbb{R}^m; \mathbb{R}^m) = L^p(\mathbb{R}^m)^m$ in which the functions are divergence free. This chapter is based on the article [16] of T. Kato.
The main theorem in this chapter is the following. See [16, Theorems 1 and 2].

Theorem. 1.2.2. *Let $a \in L^m_\sigma$.*

1. *Then there exists a $T > 0$ and a unique solution u, such that*

$$t^{\frac{1-\frac{m}{q}}{2}} u \in BC([0,T); L^q_\sigma), \text{ for } m \leq q \leq \infty \tag{1.5}$$

$$t^{1-\frac{m}{2q}} \nabla u \in BC([0,T); L^q_\sigma), \text{ for } m \leq q < \infty \tag{1.6}$$

   both with values equal to zero, when $t = 0$, except for $q = m$ in (1.5), where $u(\cdot, 0) = a$. Furthermore, there exists some $0 < T_1 \leq T$, such that

$$u \in L^r((0,T_1); L^q_\sigma), \text{ with } \frac{1}{r} = \frac{1-\frac{m}{q}}{2}, \ m < q < \frac{m^2}{m-2}. \tag{1.7}$$

2. *There exist $\lambda > 0$ such that, if $\|a\|_{L^m} \leq \lambda$, the unique solution u of 1. is global in time, i.e. $T = T_1 = \infty$. More precisely, $\|u(t)\|_{L^q}$ decays like $t^{\frac{-1+\frac{m}{q}}{2}}$, as $t \to \infty$, including $q = \infty$ and $\|\nabla u(t)\|_{L^q}$ decays like $t^{-1+\frac{m}{2q}}$, as $t \to \infty$, including $q = m$.*

Proof. We will give here only a sketch of the proof. For more details we refer the interested reader to the original article [16] and the references given therein, in particular [17].
We consider the abstract evolution equation of the Navier-Stokes equation

$$u_t + Au + Fu = 0, \quad u(0) = a, \quad \text{where A} = -\text{P}\Delta = -\Delta\text{P} \tag{4.1}$$

and

$$Fu = F(u,u) \quad \text{and} \quad F(u,v) = P(u \cdot \nabla)v.$$

P is the orthogonal projection of L^2 into L^2_σ extended to a bounded operator on L^p to L^p_σ, $1 < p < \infty$. Furthermore, since P commutes with Δ, $A \equiv -\Delta$. In addition, e^{-tA} is essentially the heat operator. We first write (4.1) as an integral equation.

$$\begin{aligned}
& u_t(t) + Au(t) + Fu(t) = 0 \\
\Leftrightarrow\ & e^{At}u_t(t) + e^{At}Au(t) + e^{At}Fu(t) = 0 \\
\Leftrightarrow\ & \frac{d}{dt}\left[e^{At}u(t)\right] = -e^{At}Fu(t) \\
\Leftrightarrow\ & e^{At}u(t) - a = -\int_0^t e^{As}Fu(s)ds \\
\Leftrightarrow\ & u(t) = e^{-At}a - e^{-At}\int_0^t e^{As}Fu(s)ds \\
\Leftrightarrow\ & u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}Fu(s)\mathrm{d}s.
\end{aligned}$$

Thus

$$(4.1) \ \Leftrightarrow \ u = u_0 + Gu, \tag{4.2}$$

where

$$u_0(t) = e^{-tA}a \quad \text{and} \quad Gu(t) = -\int_0^t e^{-(t-s)A}Fu(s)\mathrm{d}s.$$

We will without proof some estimates, see [16, p. 474], which are the following:

$$\|e^{-tA}u\|_{L^q} \le ct^{-\frac{\frac{m}{p}-\frac{m}{q}}{2}}\|u\|_{L^p}, \quad 1 < p \le q < \infty \tag{4.3}$$

$$\|\nabla e^{-tA}u\|_{L^q} \le ct^{-\frac{1+\frac{m}{p}-\frac{m}{q}}{2}}\|u\|_{L^p}, \quad 1 < p \le q < \infty \tag{4.4}$$

$$\|F(u,v)\|_{L^p} \le c\|u\|_{L^r}\|\nabla v\|_{L^s}, \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s}, \tag{4.5}$$

where the constants $c > 0$ are independent of the functions u and v. Notice that (4.5) is the Hölder inequality.

Now, we estimate $\|Gu(t)\|_{L^{\frac{m}{\gamma}}}$ and $\|\nabla Gu(t)\|_{L^{\frac{m}{\gamma}}}$ using (4.3)-(4.5).

$$\begin{aligned}
\|Gu(t)\|_{L^{\frac{m}{\gamma}}} &= \left\|\int_0^t e^{-(t-s)A}Fu(s)\mathrm{d}s\right\|_{L^{\frac{m}{\gamma}}} \\
&\le c\int_0^t (t-s)^{-\frac{\alpha+\beta-\gamma}{2}}\|F(u(s),u(s))\|_{L^{\frac{m}{\alpha+\beta}}}\mathrm{d}s \\
&\le c\int_0^t (t-s)^{-\frac{\alpha+\beta-\gamma}{2}}\|u(s)\|_{L^{\frac{m}{\alpha}}}\|\nabla u(s)\|_{L^{\frac{m}{\beta}}}\mathrm{d}s,
\end{aligned} \tag{4.6}$$

where in the first inequality we use (4.3) with $q = m/\gamma,\ p = m/(\alpha+\beta)$, where $\alpha, \beta, \gamma > 0$ and $\gamma \le \alpha + \beta < m$, and in the second one (4.5) with $r = m/\alpha,\ s = m/\beta$,

$$\begin{aligned}
\|\nabla Gu(t)\|_{L^{\frac{m}{\gamma}}} &= \left\|\nabla\int_0^t e^{-(t-s)A}Fu(s)\mathrm{d}s\right\|_{L^{\frac{m}{\gamma}}} \\
&\le c\int_0^t (t-s)^{-\frac{1+\alpha+\beta-\gamma}{2}}\|F(u(s),u(s))\|_{L^{\frac{m}{\alpha+\beta}}}\mathrm{d}s \\
&\le c\int_0^t (t-s)^{-\frac{1+\alpha+\beta-\gamma}{2}}\|u(s)\|_{L^{\frac{m}{\alpha}}}\|\nabla u(s)\|_{L^{\frac{m}{\beta}}}\mathrm{d}s,
\end{aligned} \tag{4.7}$$

where in the first inequality we use (4.4) and in the second one (4.5) with the same values for q, p, r, s as before. Furthermore, in order for the estimates (4.6) and (4.7) to be useful, we require $\alpha + \beta - \gamma < 2$ and $\alpha + \beta - \gamma < 1$, respectively.

Now, we will solve the integral equation (4.2) by consecutive approximation.
Let u_0 be given by

$$u_0(t) = e^{-tA}a$$

and set

$$u_{n+1} = u_0 + Gu_n, \quad n = 0, 1, 2, ... \tag{4.8}$$

We will first show by induction that these u_n exist and that they satisfy

$$t^{\frac{1-\delta}{2}} u_n \in BC([0,\infty); L_\sigma^{\frac{m}{\delta}}), \quad \text{with norm } \le K_n \tag{4.9}$$

and

$$t^{\frac{1}{2}} \nabla u_n \in BC([0,\infty); L_\sigma^m), \quad \text{with norm } \le K_n^{'} \tag{4.10}$$

where, $0 < \delta < 1$ is fixed, and also that the functions in (4.9) and (4.10) vanish at $t = 0$.

For $n = 0$, by (4.3) and (4.4) for $q = \frac{m}{\delta}$ and $q = m$, respectively, and $p = m$, $u = a$ we have

$$\|e^{-tA}a\|_{L^{\frac{m}{\delta}}} \le ct^{-\frac{\frac{m}{m} - \frac{m}{\frac{m}{\delta}}}{2}} \|a\|_{L^m} = ct^{-\frac{1-\delta}{2}} \|a\|_{L^m}, \quad a \in L_\sigma^m,$$

and thus

$$t^{\frac{1-\delta}{2}} \|u_0\|_{L^{\frac{m}{\delta}}} = t^{\frac{1-\delta}{2}} \|e^{-tA}a\|_{L^{\frac{m}{\delta}}} \le c\|a\|_{L^m},$$

and

$$\|\nabla e^{-tA}a\|_{L^m} \le ct^{-\frac{1+\frac{m}{m} - \frac{m}{m}}{2}} \|a\|_{L^m} = ct^{-\frac{1}{2}} \|a\|_{L^m}, \quad a \in L_\sigma^m,$$

which yields

$$t^{\frac{1}{2}} \|\nabla u_0\|_{L^m} = t^{\frac{1}{2}} \|\nabla e^{-tA}a\|_{L^m} \le c\|a\|_{L^m}.$$

Therefore, we can take as

$$K_0 = K_0^{'} = c\|a\|_{L^m}. \tag{4.11}$$

The continuity at $t = 0$, with values zero, of the functions in (4.9) and (4.10) for $n = 0$, follows from the fact that the operators $t^{\frac{1-\delta}{2}} e^{-tA} : L_\sigma^m \to L_\sigma^{\frac{m}{\delta}}$ and $t^{\frac{1}{2}} \nabla e^{-tA} : L_\sigma^m \to L_\sigma^m$ are uniformly bounded with respect to t, as we have just proven, and, moreover, from their strong convergence to 0 as $t \to 0$. (Cf., e.g., with [17, p. 251].)

Suppose now that (4.9) and (4.10) hold true for n. We will show that they hold true for $n + 1$. We will estimate the second term of u_{n+1} for $\gamma = \alpha = \delta$, $\beta = 1$ in (4.6):

$$\begin{aligned}
\|Gu_n(t)\|_{L^{\frac{m}{\delta}}} &\le c \int_0^t (t-s)^{-1} \|u_n(s)\|_{L^{\frac{m}{\delta}}} \|\nabla u_n(s)\|_{L^m} \mathrm{d}s \\
&\le cK_n K_n^{'} \int_0^t (t-s)^{-\frac{1}{2}} s^{-1+\frac{\delta}{2}} \mathrm{d}s \\
&= cK_n K_n^{'} t^{-\frac{1-\delta}{2}}
\end{aligned}$$

Moreover, the first term of u_{n+1} has been estimated already. Thus, (4.9) has been proved for $n + 1$ with

$$K_{n+1} \le K_0 + cK_n K_n^{'}. \tag{4.12}$$

Similarly, differentiating (4.8) and taking (4.7) with $\alpha = \delta$, $\beta = \gamma = 1$, we obtain (4.10) for u_{n+1} with

$$K'_{n+1} \leq K'_0 + cK_n K'_n. \tag{4.13}$$

Now, since the continuity of $t = 0$ with values zero of (4.9) and (4.10) holds true by the induction hypothesis, the constants K_n and K'_n can be made arbitrarily small, if we restrict ourselves to a sufficiently small time interval $[0, \tau)$. Moreover, as noted above, the same is true for the constants K_0 and K'_0. Then, it follows by (4.12) and (4.13), that K_{n+1} and K'_{n+1} can be also made arbitrarily small if we restrict ourselves to a time interval $[0, \tau)$.

The system of recurrence inequalities (4.12)-(4.13) can be treated as in [17, p. 249]. Accordingly, there exists a $\lambda > 0$ such that if $K_0 \leq \lambda$, then K_n and K'_n are bounded by a fixed constant K. But, by (4.11), this is true if $\|a\|_{L^m}$ is sufficiently small. In this case, the sequences in (4.9) and (4.10) are uniformly bounded, and they uniformly converge on $[0, \infty)$. (See [17] for the method of proof.)

We still need to obtain a limit function $u \in BC([0, \infty); L^m_\sigma)$. To obtain $u(t) \in L^m_\sigma$, we need to control $\|u_n(t)\|_{L^m}$. We will do this for the norm $\|u_n(t)\|_{L^q}$ with $m \leq q < \infty$. From (4.8) we obtain with $\gamma = \frac{m}{q}$, $\alpha = \delta$, $\beta = 1$ in (4.6) and with (4.9), (4.10) (and $p = m$, $u = a$ in (4.3))

$$\begin{aligned}
\|u_{n+1}(t)\|_{L^q} &\leq \|u_0(t)\|_{L^q} + c\int_0^t (t-s)^{-\frac{1+\delta-\frac{m}{q}}{2}} \|u_n(s)\|_{L^{\frac{m}{\delta}}} \|\nabla u_n(s)\|_{L^m} \mathrm{d}s \\
&\leq \|u_0(t)\|_{L^q} + cK_n K'_n \int_0^t (t-s)^{-\frac{1+\delta-\frac{m}{q}}{2}} s^{-(1-\frac{\delta}{2})} \mathrm{d}s \\
&\leq Kt^{-\frac{1-\frac{m}{q}}{2}}
\end{aligned}$$

for $K_0 \leq \lambda$, since in this case the sequences K_n, K'_n are bounded by K, as shown above. Thus

$$\|u_{n+1}(t)\|_{L^q} \leq Kt^{-\frac{1-\frac{m}{q}}{2}}. \tag{4.14}$$

This yields, by the method cited above, the uniform convergence of u_n to a limit function $u \in BC([0, T); L^m_\sigma)$ for $q = m$, which satisfies the properties in (1.5) for $m < q < \infty$. Moreover, the continuity of the functions in (1.5) at $t = 0$, with values zero in the case $m < q < \infty$, is obtained with the same arguments as above.

Similarly, one can prove (1.6), by considering first the case $q < \frac{m}{\delta}$ and using then a bootstrap argument with a smaller β, which yields the validity for any $q < \infty$. This is used in turn to obtain (1.5) in the case $q = \infty$ by use of the Gagliardo-Nirenberg inequality, which is the following

$$\|u(t)\|^2_{L^\infty} \leq C\|u(t)\|_{L^{2m}} \|\nabla u(t)\|_{L^{2m}}.$$

Indeed, using this inequality and (1.5), (1.6) for $q = 2m < \infty$ we obtain

$$\begin{aligned}
\|u(t)\|^2_{L^\infty} &\leq C\|u(t)\|_{L^{2m}} \|\nabla u(t)\|_{L^{2m}} \\
&\leq \tilde{c}\, t^{-\frac{1-\frac{m}{2m}}{2}} t^{-1+\frac{m}{4m}} \\
&= \tilde{c}\, t^{-\frac{1}{4}} t^{-\frac{3}{4}}
\end{aligned}$$

and thus

$$\|u(t)\|_{L^\infty} \leq \tilde{c}^{\frac{1}{2}}\, t^{-\frac{1}{2}}. \tag{4.15}$$

If $\|a\|_{L^m}$ is not small enough, such that $K_0 = K'_0 \leq \lambda$ holds globally in time, we have to restrict ourselves to a finite time interval $[0, T]$, such that (4.9) and (4.10) for $n = 0$ hold true with $K_0, K_0' \leq \lambda$ on this time interval. If $a \in PL^m$, such a $T > 0$ always exists, since the functions in (4.9) and (4.10) for $n = 0$ are continuous at $t = 0$, taking there the value zero, as already shown. As a consequence, we obtain as above that there exists a local solution u on the time interval $[0, T)$ for which the stated results hold true. (For more details and also for the proof of uniqueness we refer the reader to the similar case in [17].)

To conclude the proof, it suffices to show (1.7). To this end, we use a result of [13], which is the following

$$u_0 \in L^{q,r}(0,\infty) \equiv L^r((0,\infty); L^q_\sigma), \text{ for } \frac{1}{r} = \frac{1 - \frac{m}{q}}{2} \text{ and } m < q < \frac{m^2}{m-2}, \tag{4.16}$$

where the associated norm is arbitrarily small, if $\|a\|_{L^m}$ is sufficiently small. Now, one has to estimate the $L^{q,r}(0,\infty)$-norm of u. To avoid the singularity of $u(t)$ at $t = 0$, one can first estimate it on the time interval (ε, ∞) and then sent $\varepsilon \to 0$.

Taking into account (4.2), one has to estimate the $L^{q,r}(0,\infty)$-norm of Gu. Formally, utilizing an analogue of (4.7), which results from the observation that $(u \cdot \nabla)u = \nabla \cdot (uu^T)$ (due to $\nabla \cdot u = 0$), we obtain

$$\|Gu(t)\|_{L^{\frac{m}{\gamma}}} \leq c \int_0^t (t-s)^{-\frac{1+\alpha+\beta-\gamma}{2}} \|u(s)\|_{L^{\frac{m}{\alpha}}} \|u(s)\|_{L^{\frac{m}{\beta}}} \mathrm{d}s.$$

For $\alpha = \beta = \gamma = \frac{m}{q}$, we get

$$\|Gu(t)\|_{L^q} \leq c \int_0^t (t-s)^{-\frac{1+\frac{m}{q}}{2}} \|u(s)\|^2_{L^q} \mathrm{d}s. \tag{4.17}$$

Applying the Hardy-Littlewood inequality (see [14, Theorem 3]) to (4.17), and using the fact that $u = u_0 + Gu$, one gets (see [13])

$$\|u\|_{L^{q,r}} \leq \|u_0\|_{L^{q,r}} + c\|u\|^2_{L^{q,r}}$$

on any time interval $(0, T)$, $T > 0$. Therefore, $\|u\|_{L^{q,r}}$ can be estimated in terms of $\|u_0\|_{L^{q,r}}$, provided that $\|u_0\|_{L^{q,r}}$ is sufficiently small, which can be achieved by (4.16), if T is small enough. Moreover, if $\|a\|_{L^m}$ is small enough, the same is true for $\|u_0\|_{L^{q,r}}$ on $(0, \infty)$, by (4.16). Thus, in that case, we can choose $T = \infty$.

□

CHAPTER 5

Partial regularity of suitable weak solutions for the Navier-Stokes equations

In this chapter we introduce the *suitable weak solutions* of the Navier-Stokes equation in $\mathbb{R}^3$, prove their existence and study their regularity. In addition, we will present the definition of Hausdorff measure and a variant of it, especially useful in the present setting, which uses parabolic cylinders instead of Euclidean balls.
The main result of this section is to give the proof of a celebrated theorem of Caffarelli, Kohn and Nirenberg in 1982, see, [5].

5.1 Hausdorff measure and partial regularity

We first give the definition of suitable weak solutions of the Navier-Stokes equation.

Definition 5.1.1 (Suitable weak solution of the Navier-Stokes equation)**.** The pair (u, p) is called *suitable weak solution* of Navier-Stokes on an open set $D \subset \mathbb{R}^3 \times \mathbb{R}$ with force f, if

1. (Integrability hypotheses) u, p and f are measurable functions on D and
   (a) $f \in L^q(D)$ for some $q > \frac{5}{2}$, and $\nabla \cdot f = 0$,
   (b) $p \in L^{\frac{5}{4}}(D)$,
   (c) for some constants $E_0, E_1 < \infty$,
   $$\int_{D_t} |u|^2 dx \leq E_0, \quad D_t = D \cap (\mathbb{R}^3 \times \{t\}).$$
   For almost every t such that $D_t \neq \emptyset$, and
   $$\int_D |\nabla u|^2 dxdt \leq E_1.$$
2. (Equations) u, p, and f satisfy the incompressible Navier-Stokes equation in the sense of distributions on D. See, e.g., [34].
3. (Generalized energy inequality) For each real-valued $\phi \in C_c^\infty(D)$ with $\phi \geq 0$, the following inequality holds

$$2\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \le \int_0^t \int_{\mathbb{R}^3} |u|^2(\phi_t + \Delta\phi) + \int_0^t \int_{\mathbb{R}} (|u|^2 + 2p)(u\cdot\nabla)\phi + 2(u\cdot f)\phi.$$

The Hausdorff measure is a "lower dimensional" measure on $\mathbb{R}^n \times \mathbb{R}$, which allows us to measure "sufficiently small" subsets of $\mathbb{R}^n \times \mathbb{R}$. The precise definition is given as follows.

Definition 5.1.2. i) Let $A \subset \mathbb{R}^n \times \mathbb{R}$, $k > 0$ and $0 < \delta \le \infty$.
Then,

$$\mathcal{H}^k_\delta(A) := \inf\{\sum_{i=1}^{\infty} r_i^k \ : \ A \subset \bigcup_{i=1}^{\infty} C_i \ : \ r_i < \delta\},$$

where $r_i := \text{diam}C_i$ and C_i are euclidean balls on $\mathbb{R}^n \times \mathbb{R}$.

ii) For the subset A in i) we define

$$\mathcal{H}^k(A) := \lim_{\delta\to 0} \mathcal{H}^k_\delta(A) := \sup_{\delta>0} \mathcal{H}^k_\delta(A)$$

and it is called "k-dimensional" Hausdorff measure on $\mathbb{R}^n \times \mathbb{R}$.

Remark 5.1.3. The requirement $\delta \to 0$ in the definition of Hausdorff measure, forces the coverings to "follow the local geometry" of the set A.

The importance of the following theorem is that, in space $\mathbb{R}$, the Hausdorff and Lebesgue measures are equivalent.

Theorem 5.1.4. *The equality $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}^1$ holds true, where $\mathcal{L}^1$ is the Lebesgue measure.*

Proof. Let $A \subset \mathbb{R}$ and $\delta > 0$. Then

$$\mathcal{L}^1(A) := \inf\{\sum_{i=1}^{\infty} r_i \ : \ A \subset \bigcup_{i=1}^{\infty} C_i\} \le \inf\{\sum_{i=1}^{\infty} r_i \ : \ A \subset \bigcup_{i=1}^{\infty} C_i : r_i < \delta\} = \mathcal{H}^1_\delta(A)$$
$$\Longrightarrow \mathcal{L}^1(A) \le \mathcal{H}^1_\delta(A).$$

Let us now define as $I_s := [s\delta, (s+1)\delta]$, $s \in \mathbb{Z}$. Then,

$$\text{diam}(C_i \cap I_s) < \delta \text{ and } \sum_{s=-\infty}^{\infty} \text{diam}(C_i \cap I_s) \le \text{diam}C_i.$$

Thus,

$$\mathcal{L}^1(A) = \inf\{\sum_{i=1}^{\infty} r_i \ : \ A \subset \bigcup_{i=1}^{\infty} C_i\} \ge \inf\{\sum_{i=1}^{\infty}\sum_{s=-\infty}^{\infty} \text{diam}(C_i \cap I_s) \ : \ A \subset \bigcup_{i=1}^{\infty} C_i\}$$
$$\ge \inf\{\sum_{i=1}^{\infty} r_i \ : \ A \subset \bigcup_{i=1}^{\infty} C_i \ , \ r_i < \delta\} = \mathcal{H}^1_\delta(A)$$
$$\Longrightarrow \mathcal{L}^1(A) \ge \mathcal{H}^1_\delta(A).$$

Hence,

$$\mathcal{L}^1(A) = \mathcal{H}^1_\delta(A), \ \forall\delta > 0$$
$$\Longrightarrow \mathcal{L}^1(A) = \mathcal{H}^1(A) \ \text{ on } \mathbb{R}^1.$$

□

Remark 5.1.5. The equivalent of Theorem 5.1.4 holds true even if we have "n-dimensional" Hausdorff measure on $\mathbb{R}^n$, but this is not so trivial. See, e.g, [11] for a proof. However, for our work we will use the analogous of Hausdorff measure with parabolic cylinder which is defined as follows.

Definition 5.1.6. 1. Let $A \subset \mathbb{R}^n \times \mathbb{R}$, $k > 0$ and $0 < \delta \leq \infty$. Then

$$\mathcal{P}^k_\delta(A) := \inf\{\sum_{i=1}^{\infty} r_i^k \ : \ A \subset \bigcup_{i=1}^{\infty} Q_{r_i} \ , \ r_i < \delta\}$$

where $r_i := \text{diam} Q_{r_i}$ and Q_r the parabolic cylinder on $\mathbb{R}^n \times \mathbb{R}$.

2. For the subset A in 1. we define

$$\mathcal{P}^k(A) := \lim_{\delta \to 0} \mathcal{P}^k_\delta(A) := \sup_{\delta > 0} \mathcal{P}^k_\delta(A)$$

and it is called "k-dimensional parabolic" Hausdorff measure on $\mathbb{R}^n \times \mathbb{R}$.

Remark 5.1.7. 1. We define as parabolic cylinder

$$Q_r(x,t) := \{(y,\tau) : \ |y-x| < r \ , t - r^2 < \tau < t\},$$

and when we write Q_r, we mean

$$Q_r := Q_r(0,0) = \{(x,t) : \ |x| < r \ , -r^2 < \tau < 0\}.$$

In other words, with parabolic cylinder on $\mathbb{R}^n \times \mathbb{R}$ we mean one with radius r in space and r^2 in time.

2. Generally, the following relation holds

$$\mathcal{H}^k \leq c(k)\mathcal{P}^k,$$

see, e.g., [29], p.107 for a proof of this relation.

Now, we will give some explanation of the dimensional analysis. Let $u(x,t)$ and $p(x,t)$ be the solution of the system:

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p - \Delta u = f, & \text{in } \mathbb{R}^n \times (0,\infty) \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) = a, & \text{on } \mathbb{R}^n, \end{cases} \tag{5.1}$$

with force $f(x,t)$, then $\forall \lambda > 0 : \ u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$ and $p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t)$ also solve (5.1) with force $f_\lambda(x,t) := \lambda^3 f(\lambda x, \lambda^2 t)$.
Indeed, It suffices to show that

$$\begin{cases} (u_\lambda)_t + (u_\lambda \cdot \nabla)u_\lambda + \nabla p_\lambda - \Delta u_\lambda = f_\lambda, & \text{in } \mathbb{R}^n \times (0,\infty) \\ \operatorname{div} u_\lambda = 0, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u_\lambda(\cdot,0) = \lambda a, & \text{on } \mathbb{R}^n, \end{cases} \tag{5.2}$$

With some computations we take

$$u_\lambda(x,0) = \lambda u(\lambda x, t) = \lambda a(\lambda x).$$

$$\nabla \cdot u_\lambda(x,t) = \nabla \cdot (\lambda u(\lambda x, \lambda^2 t)) = \lambda(\nabla \cdot u)(\lambda x, \lambda^2 t)\lambda = \lambda^2 (\nabla \cdot u)(\lambda x, \lambda^2 t) = 0,$$

since u is divergence free vector field.

$$(u_\lambda)_t(x,t) = (\lambda u(\lambda x, \lambda^2 t))_t = \lambda^3 u_t(\lambda x, \lambda^2 t).$$

$$(\Delta u_\lambda)(x,t) = \begin{pmatrix} \Delta u_\lambda^1 \\ \vdots \\ \Delta u_\lambda^n \end{pmatrix},$$

with

$$\begin{aligned} \Delta u_\lambda^i(x,t) &= \lambda \Delta u^i(\lambda x, \lambda^2 t) = \lambda \nabla \cdot \nabla u^i(\lambda x, \lambda^2 t) = \lambda^2 \nabla \cdot (\nabla u^i)(\lambda x, \lambda^2 t) = \lambda^3 \nabla \cdot \nabla u^i(\lambda x, \lambda^2 t) \\ &= \lambda^3 \Delta u^i(\lambda x, \lambda^2 t) \end{aligned}$$

and

$$\nabla p_\lambda(x,t) = \nabla(\lambda^2 \nabla p(\lambda x, \lambda^2 t)) = \lambda^3 \nabla p(\lambda x, \lambda^2 t)$$

and, furthermore

$$\begin{aligned} (u_\lambda \cdot \nabla) u_\lambda^i(x,t) &= (\lambda u(\lambda x, \lambda^2 t) \cdot \nabla)(\lambda u^i(\lambda x, \lambda^2 t)) \\ &= \lambda u(\lambda x, \lambda^2 t) \cdot \lambda^2 \nabla u^i(\lambda x, \lambda^2 t) \\ &= \lambda^3 (u(\lambda x, \lambda^2 t) \cdot \nabla u^i)(\lambda x, \lambda^2 t) \end{aligned}$$

Thus,

$$\begin{aligned} (u_\lambda)_t(x,t) + (u_\lambda \cdot \nabla) u_\lambda(x,t) - \Delta u_\lambda(x,t) + \nabla p_\lambda(x,t) &= \lambda^3 u_t(\lambda x, \lambda^2 t) + \lambda^3 (u(\lambda x, \lambda^2 t) \cdot \nabla) u(\lambda x, \lambda^2 t) \\ &\quad - \lambda^3 \Delta u(\lambda x, \lambda^2 t) + \lambda^3 \nabla p(\lambda x, \lambda^2 t) \\ &= \lambda^3 [u_t + (u \cdot \nabla) u - \Delta u + \nabla p](\lambda x, \lambda^2 t) \\ &= \lambda^3 f(\lambda x, \lambda^2 t) \\ &= f_\lambda(x,t) \end{aligned}$$

Therefore, u_λ and p_λ solve the Navier-Stokes equations with force f_λ.

Since time has dimension 2 and space has dimension 1, we work with parabolic (since Navier-Stokes are of parabolic type) cylinders instead of Euclidean balls. Due to this fact, we defined the parabolic Hausdorff measure.

Dimensional analysis is a very important topic on understanding the theory of Caffarelli-Kohn-Nirenberg in [5], due to the fact that the Navier-Stokes equations model fluids and with this rescaling, they model them in a very small area, such as flow of blood in the arteries and in large areas, such as flow of water in a river.

The proposition that we will use for the proof of the main result is the following.

Proposition 5.1.8. *For every $A \subset \mathbb{R}^n \times \mathbb{R}$, $\mathcal{P}^k(A) = 0 \iff \forall \delta$ the set A can be covered with parabolic cylinders $\{Q_{r_i}\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} < \delta$.*

We state here two inequalities which are the key ingredients for the proof of the partial regularity theorem.

Lemma 5.1.9 (Interpolation Inequality)**.** *For* $u \in H^1(\mathbb{R}^3)$ *one has*

$$\int_{B_r} |u|^q \le c\Big(\int_{B_r} |\nabla u|^2\Big)^a \Big(\int_{B_r} |u|^2\Big)^{\frac{q}{2}-a} + \frac{c}{r^{2a}}\Big(\int_{B_r} |u|^2\Big)^{\frac{q}{2}} \tag{5.3}$$

where, c is a constant independent on r, B_r *is a ball of radius r and*

$$2 \le q \le 6 \ \ \text{and} \ \ a = \frac{3}{4}(q-2).$$

Lemma 5.1.10 (Generalized Energy Inequality)**.** *A weak solution* (u,p) *of* (1.1) *is said to satisfy the generalized energy inequality if*

$$2\int_0^T \int_\Omega |\nabla u|^2 \phi \ \mathrm{d}x\mathrm{d}t \le \int_0^T \int_\Omega |u|^2(\phi_t + \nu\Delta\phi) \ \mathrm{d}x\mathrm{d}t + \int_0^t \int_\Omega (|u|^2 + 2p)(u\cdot\nabla)\phi \ \mathrm{d}x\mathrm{d}t, \tag{5.4}$$

$\forall \phi \in C_c^\infty\big(\Omega \times (0,T); [0,\infty)\big)$.

Definition 5.1.11 (Singular point)**.** A point (x,t) is called **singular** if u is not locally essentially bounded in any neighborhood of (x,t).
On the other hand, the points (x,t) in which u is locally essentially bounded, called **regular** points.
The set of all singular points is symbolized with S.

Now, we are ready to write down the main theorem of this part of the thesis, which is the main result of Caffarelli-Kohn-Niremberg [5]. This is the following.

Theorem. 1.2.3 *For any suitable weak solution of the Navier-Stokes equation in an open set* $D \subset \mathbb{R}^3 \times \mathbb{R}$*, the particular singular set S has* $\mathcal{P}^1(S) = 0$*.*

To "use" Theorem 1.2.3 we want to know if there exist such suitable weak solutions. See, e.g., [29] and section 5.5 for a proof.

To prove Theorem 1.2.3 we have to prove first two Propositions.

Proposition 5.1.12. *There exist absolute constants* ϵ *and* $C > 0$ *such that, if the pair* (u,p) *is a suitable weak solution of the Navier-Stokes system on* Q_1 *and if*

$$\int_{Q_1} \big(|u|^3 + |u||p|\big) \ \mathrm{d}(x,t) + \int_{-1}^0 \Big(\int_{|x|<1} |p| dx\Big)^{\frac{5}{4}} \mathrm{d}t \le \epsilon.$$

Then

$$\big|u(x,t)\big| \le C,$$

for Lebesgue almost every $(x,t) \in Q_{\frac{1}{2}}$*.That is, u is regular on* $Q_{\frac{1}{2}}$*.*

Note 5.1.13. Proposition 5.1.12 "says" that, if u, p are "sufficiently small" on a cylinder $Q_1 = Q_1(0,0)$, then u is regular on a smaller cylinder $Q_{\frac{1}{2}} = Q_{\frac{1}{2}}(0,0)$.

The other proposition that we want to prove is the following.

Proposition 5.1.14. *There exist an absolute constant $\epsilon_1 > 0$ such that, if the pair (u, p) is a suitable weak solution of the Navier-Stokes system near the point (x, t) and if*

$$\limsup_{r\to 0} r^{-1} \int_{Q_r^*(x,t)} |\nabla u|^2 \le \epsilon_1.$$

Then, (x, t) is regular point.

The cylinder here is defined as

$$Q_r^*(x,t) := \{(y,\tau) : |y-x| < r : \ t - \frac{7}{8} < \tau < t + \frac{1}{8}r^2\}.$$

Suppose for the moment that the two propositions have been proved. Then, provided that, we will be ready to give the proof of the main result of Caffarelli-Kohn-Nirenberg, [5]. This will be achieved by applying the Proposition 5.1.14 and a covering lemma, analogous to Vitali's Lemma for balls. See, e.g., [30] p.9 for a proof.
We state the analogous of Vitali's lemma.

Lemma 5.1.15. *Let $\mathcal{F}$ the family of parabolic cylinder $Q_r^*(x,t)$ which contained in a bounded subset of $\mathbb{R}^3 \times \mathbb{R}$. Then, there exists a finite or countable subfamily $\mathcal{F}_* := \{Q_i^* := Q_{r_i}^*(x_i, t_i)\}$ such that,*

$$Q_i^* \cap Q_j^* = \varnothing \quad \text{for} \quad i \ne j, \tag{5.5}$$

$$\forall\ Q^* \in \mathcal{F}\ , \exists\ Q_{r_i}^*(x_i,t_i) \in \mathcal{F}_*\ :\ Q^* \subset Q_{5r_i}^*(x_i,t_i). \tag{5.6}$$

Proof. We are going to choose the elements of $\mathcal{F}_*$ by induction.
Choose $\{Q_k^*\}_{k=1}^n = \{Q_1^*, ..., Q_n^*\}$ and let

$$\mathcal{F}_0 := \mathcal{F}$$

and

$$\mathcal{F}_n := \{Q^* \in \mathcal{F}\ :\ Q^* \cap Q_k^* = \varnothing,\ 1 \le k \le n\}.$$

If $\mathcal{F}_n = \varnothing$, then $\mathcal{F}_*$ is finite.

If, now $\mathcal{F}_n \ne \varnothing$, then choose

$$Q_{n+1}^* \in \mathcal{F}_n\ :\ \forall Q^* := Q_r^*(x,t) \in \mathcal{F}_n\ ,\ r \le \frac{3}{2} r_{n+1}. \tag{5.7}$$

That is, we choose one with maximal radius, if there exists one. Else, if $R < \infty$ is the supremum of the radii, choose one with radius $\ge (2/3)R$. [$R < \infty$ is satisfied since the parabolic cylinders are contained in a bounded subset, let's say D, by assumption].

We notice that (5.5) holds true, since

$$Q_r^* \cap\ Q_k^* = \varnothing\ ,\ 1 \le k \le r.$$

Hence, it suffices to show (5.6).

If $\mathcal{F}_*$ is infinite, then $r \to 0$ as $n \to \infty$. Since the elements of $\mathcal{F}_*$ are disjoint subsets of the bounded D with $|D| < \infty$, we have $\sum_{n=1}^{\infty} |Q_{r_n}^*| \le |D|$. Hence the series converges, and thus $|Q_{r_n}^*| = (4\pi/3) r_n^3 \times r_n^2 \to 0$, which implies $r_n \to 0$.

Therefore, for

$$Q^* = Q^*(x,t) \in \mathcal{F} \setminus \mathcal{F}_* \ , \ \exists n \geq 0 : \quad Q^* \in \mathcal{F}_n \text{ and } Q^* \notin \mathcal{F}_{n+1}.$$

We have that $r_{n+2} < (2/3)r$ and hence $Q^* \notin \mathcal{F}_{n+1}$ by (5.7).

But $Q^* \in \mathcal{F} = \mathcal{F}_0$, and since $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ for all n, there exists some $0 \leq m \leq n$ with $Q^* \in \mathcal{F}_m \setminus \mathcal{F}_{m+1}$.

Let's denote this m by n.

The same applies for finite $\mathcal{F}_*$: If $\mathcal{F}_n = \emptyset$, then for each $Q^* \in \mathcal{F}$ there exists an $1 \leq \ell \leq n$, such that $Q^* \cap Q^*_\ell \neq \emptyset$. Take the smallest of such ℓ. Then $Q^* \in \mathcal{F}_{\ell-1} \setminus \mathcal{F}_\ell$ and, by (5.7), $r \leq (3/2)r_\ell$.

Wlog let $Q^*_{r_{n+1}}$ be centered at $(0,0)$.
Since $Q *_r \cap Q^*_{r_{n+1}} \neq \emptyset$, in order to obtain $Q*_r \subset Q^*_{\alpha r_{n+1}}$ for some $\alpha \in \mathbb{N}$ to be determined, we need $Q^*_{\alpha r_{n+1}} \supset B(0, r_{n+1} + 2r) \times [-(7/8)r^2_{n+1} - r^2, (1/8)r^2_{n+1} + r^2]$. Since $r \leq (3/2)s$ by (6.3), this is surely satisfied if $Q^*_{\alpha r_{n+1}} \supset B(0, 4r_{n+1}) \times [-(7/8 + 9/4)r^2_{n+1}, (1/8 + 9/4)r^2_{n+1}]$. Thus $\alpha \in \mathbb{N}$ must satisfy the conditions $\alpha r_{n+1} \geq 4r_{n+1}$, $(7/8)(\alpha r_{n+1})^2 \geq (25/8)r^2_{n+1}$, and $(1/8)(\alpha r_{n+1})^2 \geq (19/8)r^2_{n+1}$, or, equivalenty, $\alpha \geq 4$, $\alpha^2 \geq 25/7$, and $\alpha^2 \geq 19$. Since we want $\alpha \in \mathbb{N}$, these conditions are satisfied for $\alpha \geq 5$.
Therefore,

$$Q^* \subset Q^*_{5r_{n+1}}(x_{n+1}, t_{n+1}).$$

□

In this moment, we have the tools that are needed to prove the main result of Caffarelli-Kohn-Nirenberg [5]. We restate the main result here for more convenience.

Theorem. 1.2.3. *For any suitable weak solution of Navier-Stokes equation in an open set* $D \subset \mathbb{R}^3 \times \mathbb{R}$*, the particular singular set* S *has* $\mathcal{P}^1(S) = 0$.

Proof. Let (u,p) be a suitable weak solution defined on an open set D. Without loss of generality we suppose that D is bounded. By Proposition 5.1.14,

$$(x,t) \in S \Rightarrow \limsup_{r \to 0} \int_{Q^*_r(x,t)} |\nabla u|^2 dxdt > \epsilon_3.$$

Let V be a neighborhood of S in D independent of $\delta > 0$. [e.g. constructed as the union of balls (or parabolic cylinders) around each $(x,t) \in S$. Since $S \subset D$ and D is open, we can choose balls that are $\subset D$, and hence this V satisfies $V \subset D$.] For each $(x,t) \in S$. we choose $Q^*_r(x,t)$ with $r < \delta$ such that

$$\frac{1}{r} \int_{Q^*_r(x,t)} |\nabla u|^2 dxdt > \epsilon_3 \quad \text{and} \quad Q^*_r(x,t) \subset V.$$

Using now Lemma 5.1.15 to this family of cylinders, we obtain a disjoint subfamily $\{Q^*_{r_i}(x_i,t_i)\}$ such that

$$S \subset \bigcup_i Q_{5r_i}(x_i,t_i) \Rightarrow \quad |S| \leq \sum_i \frac{4\pi}{3}(5r_i)^3(5r_i)^2 = \frac{4\pi 5^5}{3} \sum_i r_i^5 \leq \delta^4 \frac{4\pi 5^5}{3} \sum_i r_i.$$

Since the L^1-norm on the right hand side is bounded, independently of δ, and since the above estimate holds for all δ, we obtain $|S| = 0$. This follows directly from the definition of the

Lebesgue outer measure and the Lebesgue measure in $\mathbb{R}^n$ (see, [19], Definition. 3.5 and Definition. 3.8, Theorem 3.9, Definition 3.11 and the comment before Theorem 3.9.)

Furthermore,

$$\sum_i r_i \leq \frac{1}{\epsilon_3} \sum_i \int_{Q^*_{r_i}} |\nabla u|^2 dxdt = \frac{1}{\epsilon_3} \int_{\bigcup Q^*_{r_i}(x_i,t_i)} |\nabla u|^2 dxdt \leq \frac{1}{\epsilon_3} \int_V |\nabla u|^2 dxdt,$$

since these Q are disjoint. We have also that,

$$\frac{5}{\epsilon_3} \int_V |\nabla u|^2 dxdt \geq \mathcal{P}^1_{5\delta}(S),$$

for every neighborhood V of S, by the definition of the Lebesgue measure, for all $\delta > 0$, where the integral over V is independent of $\delta > 0$.

Thus $\mathcal{P}^1(S) = \sup_{\delta>0} \mathcal{P}^1(S)$ is bounded by this integral. [for =, see e.g. [?], Def. 2.1, p. 81, and note that $\delta \mapsto \mathcal{P}^1_\delta(S)$ is by definition strictly decreasing.]

Since $|\nabla u|^2$ is an integrable function, by definition of suitable weak solutions and since S is Lebesgue-measurable with $|S| = 0$ (see above), there exists a sequence of open neighborhoods of S in D ($S \subset V_n \subset D$), such that $|V_n| \to 0$. (see, e.g., [19], Proposition 3.12).

From that we obtain that $W_n := \bigcap_{i=1}^n W_i$ is a decreasing sequence of neighborhoods of S in D with $|W_1| = |V_1| \leq |D| < \infty$ (by assumption), $S \subset W := \bigcap_{i=1}^\infty W_i$, $|W_n| \leq |V_n| \to 0$, and, thus, $|W| = \lim_{n\to 0} |W_n| = 0$ (see, e.g., [19], Proposition 2.5 (ii)).

Moreover, since, for $V \subset D$ Lebesgue-measurable, $\nu(V) := \int_V |\nabla u|^2$ with $\nu(D) < \infty$ (by assumption) is a finite measure, absolutely continuous with respect to (the Lebesgue-measure) $|\cdot|$ (see, e.g., [19], Corollary 6.12, (i), (ii)), we obtain (together with $\nu(W_1) = \nu(V_1) \leq \nu(D) < \infty$) $\nu(W) = 0 = \lim n \to \nu(W_n)$ (again by [19], Proposition 2.5 (ii)).

Thus, recallling that W_n are open neighborhoods of S in D, we have $0 \leq \mathcal{P}^1(S) \leq \frac{5}{\epsilon_3}\nu(W_n) \to 0$ and, hence, $\mathcal{P}^1(S) = 0$.

□

5.2 Proof of Proposition 5.1.12

Let

$$Q_\rho = Q_\rho(0,0) = \{(x,t) : |x| < \rho,\ \ \rho < t < 0\}$$

and the functions u, p which are measurable on Q_ρ.
We define, for $r < \rho$, the following.

$$A(r) := \sup_{-r^2<t<0} r^{-1} \int_{B_r \times \{t\}} |u|^2, \tag{5.8}$$

$$G(r) := r^{-2} \int_{Q_r} |u|^3, \tag{5.9}$$

$$\delta(r) := r^{-1} \int_{Q_r} |\nabla u|^2, \tag{5.10}$$

$$L(r) := r^{-2} \int_{Q_r} |u||p - \bar{p}_r|, \tag{5.11}$$

$$K(r) := r^{-\frac{13}{4}} \int_{-r^2}^0 \left(\int_{B_r} |p| dx \right)^{\frac{5}{4}} \mathrm{d}t, \tag{5.12}$$

where $Q_r = Q_r(0,0)$, $B_r := \{x : |x| < r\}$ *and* $\bar{p}_r = \bar{p}_r(t) = \fint_{B_r} p(y,t)\,dy$, which denotes the average of p over the ball B_r.

From now on, we will use the fact that the previous quantities are finite and the following hold true

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j) \tag{5.13}$$

and

$$\nabla \cdot u = 0, \quad on \quad B_\rho \times \{t\}, \tag{5.14}$$

for almost every t, such that $-\rho < t < 0$.

The main goal of this section is to bound $G(r)$ *and* $L(r)$ in terms of A, δ and K. Let us start this process with the lemma bellow.

Lemma 5.2.1.

$$G(r) \leq cA^{\frac{3}{4}}(r)\big(A^{\frac{3}{4}}(r) + \delta^{\frac{3}{4}}(r)\big) \tag{5.15}$$

Proof. Applying the interpolation inequality (4.2) for

$$q = 3 \Rightarrow a = \frac{3}{4}(3-2) = \frac{3}{4}$$

we have

$$\int_{B_r} |u|^3 \leq c\Big(\int_{B_r} |\nabla u|^2\Big)^{\frac{3}{4}}\Big(\int_{B_r} |u|^2\Big)^{\frac{3}{4}} + \frac{c}{r^{2\frac{3}{4}}}\Big(\int_{B_r} |u|^2\Big)^{\frac{3}{2}}.$$

Now integrating over $-r^2 < t < 0$ we get

$$\begin{aligned}
\int_{-r^2}^{0}\int_{B_r} |u|^3 \,\mathrm{d}x\mathrm{d}t &\leq c\Big(\int_{-r^2}^{0}\int_{B_r} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}t\Big)^{\frac{3}{4}}\Big(\int_{-r^2}^{0}\Big(\int_{B_r} |u|^2 \,\mathrm{d}x\Big)^{3}\mathrm{d}t\Big)^{\frac{1}{4}} \\
&\quad + \frac{c}{r^{\frac{3}{2}}}\int_{-r^2}^{0}\Big(\int_{B_r} |u|^2 \,\mathrm{d}x\Big)^{\frac{3}{2}}\mathrm{d}t \\
&\leq c\Big(\int_{Q_r} |\nabla u|^2 \,\mathrm{d}(x,t)\Big)^{\frac{3}{4}}\Big(r^5 \sup_{-r^2<t<0}\Big(r^{-1}\int_{B_r\times\{t\}} |u|^2 \,\mathrm{d}x\Big)^{3}\Big)^{\frac{1}{4}} \\
&\quad + cr^{-\frac{3}{2}}r^2 r^{\frac{1}{2}} r^{-\frac{1}{2}}\Big(\sup_{-r^2<t<0}\int_{B_r\times\{t\}} |u|^2 \,\mathrm{d}x\Big)^{\frac{3}{2}} \\
\Rightarrow \int_{Q_r} |u|^3 \,\mathrm{d}(x,t) &= cr^{\frac{3}{4}}r^{-\frac{3}{4}}\Big(\int_{Q_r} |\nabla u|^2 \,\mathrm{d}(x,t)\Big)^{\frac{3}{4}}\big(r^5 A^3(r)\big)^{\frac{1}{4}} + cr^{\frac{1}{2}}\big(rA(r)\big)^{\frac{3}{2}} \\
\Rightarrow \int_{Q_r} |u|^3 \,\mathrm{d}(x,t) &= cr^{\frac{3}{4}}\delta^{\frac{3}{4}}(r)\big(r^5 A^3(r)\big)^{\frac{1}{4}} + cr^{\frac{1}{2}}\big(rA(r)\big)^{\frac{3}{2}}.
\end{aligned}$$

Multiplying each term by r^{-2}

$$\begin{aligned}
&\Rightarrow r^{-2}\int_{Q_r} |u|^3 \,\mathrm{d}(x,t) = cr^{-2}r^{\frac{3}{4}}\delta^{\frac{3}{4}}(r)r^{\frac{5}{4}}A^{\frac{3}{4}}(r) + cr^{-2}r^{\frac{1}{2}}r^{\frac{3}{2}}A^{\frac{3}{2}}(r) \\
&\Rightarrow r^{-2}\int_{Q_r} |u|^3 \,\mathrm{d}(x,t) = c\delta^{\frac{3}{4}}(r)A^{\frac{3}{4}}(r) + cA^{\frac{3}{2}}(r),
\end{aligned}$$

which ends the proof of lemma. □

Lemma 5.2.2. *Let* $r \leq \frac{1}{2}\rho$ *Then,*

$$\begin{aligned} L(r) \leq c\left(\frac{r}{\rho}\right)^{\frac{7}{5}} A^{\frac{1}{5}}(r) G^{\frac{1}{5}}(r) K^{\frac{4}{5}}(\rho) \\ + c\left(\frac{r}{\rho}\right)^{\frac{5}{3}} G^{\frac{1}{3}}(r) G^{\frac{2}{3}}(\rho) \\ + c G^{\frac{1}{3}}(r) G^{\frac{2}{3}}(2r) + c r^3 G^{\frac{1}{3}}(r) \sup_{-r^2<t<0} \int_{2r<|y|<\rho} \frac{|u|^2}{|y|^4}. \end{aligned} \tag{5.16}$$

Proof. We choose a function ϕ, such that

$$\phi(y) = \begin{cases} 1, & \text{if } |y| \leq \frac{3}{4}\rho \\ 0, & \text{if } |y| \geq \rho, \end{cases}$$

with the relations

$$|\nabla_i \phi| \leq C\rho^{-1} \quad \textit{and} \quad |\nabla_{ij}\phi| \leq C\rho^{-2}$$

and $\tilde{p} = p_1 + p_2$, where

$$\begin{aligned} p_1 &= \frac{3}{4\pi} \int_{|y|<2r} \nabla_{y_i y_j} \left(\frac{1}{|x-y|}\right) \cdot \phi u_i u_j \, \mathrm{d}y, \\ p_2 &= \frac{3}{4\pi} \int_{|y|>2r} \nabla_{y_i y_j} \left(\frac{1}{|x-y|}\right) \cdot \phi u_i u_j \, \mathrm{d}y. \end{aligned}$$

We observe that

$$|p - \bar{p}_r| = \left| p - \frac{1}{|B_r|}\int_{B_r} p \right| \leq \sum_{i=1}^{4} |p_i - \bar{p}_i|,$$

where $\bar{p}_i = ⨍_{B_r} p_i = \frac{1}{|B_r|}\int_{B_r} p_i$, where $|B_r|$ is the volume of the ball B_r.
For the p_1, it is known (from Calderón-Zygmund operator theorem, see, e.g., [28]) the fact that every operator of the form

$$T_{ij}(\psi) = \left(\nabla_{ij}\frac{1}{|x|}\right) * \psi$$

is bounded operator in the space $L^q(\mathbb{R}^3)$ for $1 < q < \infty$.

Now, we set $\psi := \phi u_i u_j\big|_{B_2}$ and $q = \frac{3}{2}$ to get

$$\|p_1\|_{L^{\frac{3}{2}}(B_r)} \leq C\left(\int_{B_{2r}} |u|^{2\frac{3}{2}}\right)^{\frac{2}{3}}. \tag{5.17}$$

Thus,

$$\begin{aligned} \int_{B_r} |u||p_1 - \bar{p}_1| &\leq C\left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \left(\int_{B_r} |p_1 - \bar{p}_1|^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ &\leq C\left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \left(\int_{B_{2r}} |u|^3\right)^{\frac{2}{3}} \end{aligned} \tag{5.18}$$

where, in the first inequality we use the Hölder inequality and in the second we use (5.17).

Now, for p_2, p_3, p_4 we bound $|p_i - \bar{p}_i|$ uniformly on B_r.
That is, for $|x| < r$

$$|\nabla p_2(x)| \le C \int_{2r<|y|<\rho} \left(\frac{|u|^2}{|y|^4} \right) \, \mathrm{d}y,$$
$$|\nabla p_3(x)| \le C\rho^{-4} \int_{B_\rho} |u|^2,$$
$$|\nabla p_4(x)| \le C\rho^{-4} \int_{B_\rho} |p|.$$

For the cases $i = 2, 3$ we do the estimates

$$\begin{aligned} \int_{B_r} |u||p_i - \bar{p}_i| &\le Cr^2 \left(\int_{B_r} |u|^3 \right)^{\frac{1}{3}} \sup_{x \in B_r} |p_i(x) - \bar{p}_i| \\ &\le Cr^3 \left(\int_{B_r} |u|^3 \right)^{\frac{1}{3}} \sup_{x \in B_r} |\nabla p_i|. \end{aligned}$$

Here, in the second inequality we use the Poincare inequality, see, e.g, Theorem 4.9, in [11]. Thus, we see that

$$\int_{B_r} |u||p_2 - \bar{p}_2| \le Cr^3 \left(\int_{B_r} |u|^3 \right)^{\frac{1}{3}} \int_{2r<|y|<\rho} \left(\frac{|u|^2}{|y|^4} \right) \, \mathrm{d}y \tag{5.19}$$

and

$$\begin{aligned} \int_{B_r} |u||p_3 - \bar{p}_3| &\le Cr^3 \left(\int_{B_r} |u|^3 \right)^{\frac{1}{3}} \rho^{-4} \int_{B_\rho} |u|^2 \\ &\le C\frac{r^3}{\rho^3} \left(\int_{B_r} |u|^3 \right)^{\frac{1}{3}} \left(\int_{B_\rho} |u|^3 \right)^{\frac{2}{3}} \end{aligned} \tag{5.20}$$

It suffices to find an estimate for p_4.

$$\begin{aligned} \int_{B_r} |u||p_4 - \bar{p}_4| &\le C \left(\int_{B_r} |u| \right) \sup_{x \in B_r} |p_4(x) - \bar{p}_4| \\ &\le Cr \left(\int_{B_r} |u| \right) \sup_{x \in B_r} |\nabla p_4| \\ &\le Cr \left(\int_{B_r} |u| \right) \rho^{-4} \int_{B_\rho} |p| \\ &\le C\frac{r^{\frac{14}{5}}}{\rho^4} \left(\int_{B_r} |u|^2 \right)^{\frac{1}{5}} \left(\int_{B_r} |u|^3 \right)^{\frac{1}{5}} \int_{B_\rho} |p| \\ &\le C\frac{r^{\frac{15}{5}}}{\rho^4} \left(\sup_{-r^2<t<0} r^{-1} \int_{B_r} |u|^2 \right)^{\frac{1}{5}} \left(\int_{B_r} |u|^3 \right)^{\frac{1}{5}} \int_{B_\rho} |p|. \end{aligned}$$

Therefore,

$$\int_{B_r} |u||p_4 - \bar{p}_4| \le C\frac{r^3}{\rho^4} A^{\frac{1}{5}}(r) \left(\int_{B_r} |u|^3 \right)^{\frac{1}{5}} \int_{B_\rho} |p|. \tag{5.21}$$

Here, in the second inequality we use the Poincare inequality, see, e.g, Theorem 4.9 in [11].

Thus, combining (5.18)-(5.21) and integrating over t we obtain

$$\begin{aligned}\int_{Q_r}|u||p-\bar{p}_r| \leq\, & C\left(\int_{Q_r}|u|^3\right)^{\frac{1}{3}}\left(\int_{Q_{2r}}|u|^3\right)^{\frac{2}{3}} \\ & + Cr^{\frac{13}{3}}\left(\int_{Q_r}|u|^3\right)^{\frac{1}{3}}\sup_{-r^2<t<0}\int_{2r<|y|<\rho}\left(\frac{|u|^2}{|y|^4}\right)\mathrm{d}y \\ & + C\frac{r^3}{\rho^3}\left(\int_{Q_r}|u|^3\right)^{\frac{1}{3}}\left(\int_{Q_\rho}|u|^3\right)^{\frac{2}{3}} \\ & + C\frac{r^3}{\rho^4}A^{\frac{1}{5}}(r)\left(\int_{Q_r}|u|^3\right)^{\frac{1}{5}}\left[\int_{-\rho^2}^{0}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}\right]^{\frac{4}{5}}.\end{aligned}$$

□

In this section, we give the proof of Proposition 5.1.12, in which we use induction.

Definition 5.2.3. Let $f \in L_{loc}(\mathbb{R}^n)$. A point x in the domain of f is said to be a Lebesgue point, if

$$\lim_{r\to 0^+}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f(x)|\,\mathrm{d}y = 0.$$

It is known that, If $0 \leq \phi \in C^\infty(Q_1(0,0))$ and ϕ decay near $\{|x|=1\}\cup\{t=-1\}$. Then, for $-1<s<0$

$$\int_{B_1\times\{s\}}|u|^2\phi + 2\int_{-1}^{s}\int_{B_1}|\nabla u|^2\phi \leq \int_{-1}^{s}\int_{B_1}|u|^2(\phi_t+D\phi) + \int_{-1}^{s}\int_{B_1}(|u|^2+2p)u\cdot\nabla\phi. \quad (5.22)$$

Before we start the proof, we state here the Proposition 5.1.12 for convenience.

Proposition. 5.1.12. *There exist absolute constants ϵ and $C>0$ such that, if the pair (u,p) is a suitable weak solution of the Navier-Stokes system on Q_1 and if*

$$\int_{Q_1}(|u|^3+|u||p|)\mathrm{d}(x,t) + \int_{-1}^{0}\left(\int_{|x|<1}|p|\,\mathrm{d}x\right)^{\frac{5}{4}}dt \leq \epsilon.$$

Then

$$\big|u(x,t)\big| \leq C$$

for Lebesgue almost every $(x,t)\in Q_{\frac{1}{2}}$. That is, u is regular on $Q_{\frac{1}{2}}$.

Proof. Let

$$\int_{Q_1}(|u|^3+|u||p|)d(x,t) + \int_{-1}^{0}\left(\int_{|x|<1}|p|dx\right)^{\frac{5}{4}}dt \leq \epsilon_1,$$

where $B_1 := \{|x|<1\}$ and $Q_1 = Q_1(0,0)$.

It suffices to show that

$$⨍_{|x-a|<r_n}|u|^2(x,s)\,\mathrm{d}x \leq C_0\epsilon_1^{\frac{2}{3}}, \quad (5.23)$$

$\forall (a,s) \in Q_{\frac{1}{2}}$ and $\forall u \geq 2$ such that $r_n = 2^{-n}$. Then, we will have that

$$|u|^2(a,s) \leq C_0 \epsilon_1^{\frac{2}{3}} =: C_1,$$

where (a,s) is a Lebesgue point for the function u, that is, almost everywhere on $Q_{\frac{1}{2}}(0,0)$.

Fix a point $(a,s) \in Q_{\frac{1}{2}}(0,0) = \{(x,t) : |x| < \frac{1}{2} . -\frac{1}{4} < t < 0\}$, we observe that $Q_{\frac{1}{2}}(a,s) = \{(x,t) : |x-a| < \frac{1}{2} .s - \frac{1}{4} < t < s\} \subset Q_1(0,0)$ such that

$$\int_{Q_{\frac{1}{2}}(a,s)} \big(|u|^3 + |u||p|\big) \mathrm{d}(x,t) + \int_{s-\frac{1}{4}}^{s} \left(\int_{|x-a|<\frac{1}{2}} |p| \ \mathrm{d}x \right)^{\frac{5}{4}} \mathrm{d}t \leq \epsilon_1. \tag{5.24}$$

Define $Q^n := Q_{r_n}(a,s), \ \ n = 1, 2, ...$
I want to show that, for $n \geq 3$

$$⨍_{Q^n} |u|^3 + r_n^{\frac{3}{5}} ⨍_{Q^n} |u||p - \bar{p}_n| \leq \epsilon_1^{\frac{2}{3}}, \tag{5.25}$$

and for $n \geq 2$

$$\sup_{s-r_n^2<t\leq s} ⨍_{|x-a|<r_n} |u|^2 \ \mathrm{d}x + r_n^{-3} \int_{Q^n} |\nabla u|^2 \leq C_0 \epsilon_1^{\frac{2}{3}}, \tag{5.26}$$

where, $\bar{p}_n = \bar{p}_n(t) := ⨍_{|x-a|<r_n} p dx$.

We assume that $\epsilon_1 \leq 1$. Note that, (5.26) contains (5.23). Thus, it suffices to show the relation (5.26).

First step: for $n = 2$ we want to show

$$\sup_{s-r_2^2<t\leq s} ⨍_{|x-a|<r_2} |u|^2 \ \mathrm{d}x + r_2^{-3} \int_{Q^2} |\nabla u|^2 \leq C_0 \epsilon_1^{\frac{2}{3}}. \tag{5.27}$$

Let a function $\phi \geq 0$ with $\phi = 1$ on Q^2 and supp $\phi \subset Q^1$, then by (5.22)

$$\sup_{s-r_2^2<t\leq s} ⨍_{|x-a|<r_2} |u|^2 \ \mathrm{d}x + r_2^{-3} \int_{Q^2} |\nabla u|^2 \leq C \left(\int_{Q_1} |u|^2 + \int_{Q_1} (|u|^3 + |u||p|) \right) < C_0 \epsilon_1^{\frac{2}{3}},$$

where we use Hölder inequality and (5.24).

Second step: Assume now that (5.27) holds for $n = k$ for $2 \leq k \leq n$. We will prove that (5.25) holds true for $n+1$, for $n \geq 2$. To this ends, we will use the previous lemmas.
We know that,

$$A(r_k) + \delta(r_k) \leq C \epsilon_1^{\frac{2}{3}} r_k^2 \quad 2 \leq k \leq n \tag{5.28}$$

and, by (5.24) follows,

$$G(r_1) + K(r_1) \leq C \epsilon_1. \tag{5.29}$$

Now, by Lemma 5.2.1 and (5.28),

$$(4.25) \Rightarrow G(r_n) = r_n^{-2} \int_{Q^n} |u|^3 \leq C\epsilon_1 r_n^3, \tag{5.30}$$

such that

$$⨍_{Q^{n+1}} |u|^3 \leq C ⨍_{Q^n} |u|^3 \leq C^\star \epsilon_1. \tag{5.31}$$

Therefore, if ϵ_1 is sufficiently small such that C^* in (5.31) satisfies $C^* \epsilon_1^{\frac{1}{3}} \leq \frac{1}{2}$, then

$$⨍_{Q^{n+1}} |u|^3 \leq C^\star \epsilon_1^{\frac{1}{3}} \epsilon_1^{\frac{2}{3}} \leq \frac{1}{2} \epsilon_1^{\frac{2}{3}}$$

$$\Longrightarrow ⨍_{Q^{n+1}} |u|^3 \leq \frac{1}{2} \epsilon_1^{\frac{2}{3}}. \tag{5.32}$$

For the other part of $(5.25)_{n+1}$ we use Lemma 5.2.2 for $\rho = \frac{1}{4}$ and $r = r_n$.

From the previous one, the following are hold true

$$G(r_{n+1}) \leq CG(r_n) \leq C\epsilon_1 r_n^3,$$
$$A(r_{n+1}) \leq CA(r_n) \leq C\epsilon_1^{\frac{2}{3}} r_n^2,$$

and

$$(5.29) \Rightarrow K(\frac{1}{4}) \leq C\epsilon_1.$$

For $\epsilon_1 \leq 1$ in (5.16)

$$r_{n+1}^{\frac{7}{5}} A^{\frac{1}{5}}(r_{n+1}) G^{\frac{1}{5}}(r_{n+1}) K^{\frac{4}{5}}(\frac{1}{4}) \leq r_{n+1}^{\frac{7}{5}} C \epsilon_1^{\frac{2}{3}\frac{1}{3}} r_n^{2\frac{1}{5}} \epsilon_1^{\frac{1}{5}} r_n^{\frac{3}{5}} \epsilon_1^{\frac{4}{5}} = C r_n^{\frac{12}{5}} \epsilon_1,$$

and

$$r_{n+1}^{\frac{5}{3}} G^{\frac{1}{3}}(r_{n+1}) G^{\frac{2}{3}}(\frac{1}{4}) \leq r_n^{\frac{5}{3}} C \epsilon_1^{\frac{1}{3}} r_n \epsilon_1^{\frac{2}{3}} = C r_n^{\frac{8}{3}} \epsilon_1.$$

Furthermore,

$$G^{\frac{1}{3}}(r_{n+1}) G^{\frac{2}{3}}(r_n) \leq C \epsilon_1^{\frac{1}{3}} r_n \epsilon_1^{\frac{2}{3}} r_n^2 = C r_n^3 \epsilon_1$$

and, finally

$$r_{n+1}^3 G^{\frac{1}{3}}(r_{n+1}) \sup_{s - r_{n+1}^2 < t < s} \int_{r_n < |y| < \frac{1}{4}} \frac{|u|^2}{|y|^4} \leq r_n^3 C \epsilon_1^{\frac{1}{3}} r_n \sum_{k=2}^{n} r_k^{-3} A(r_k)$$

$$\leq C \epsilon_1^{\frac{1}{3}} r_n^4 \sum_{k=2}^{n} r_k^{-3} \epsilon_1^{\frac{2}{3}} r^2 = C r_n^4 \epsilon_1 \sum_{k=2}^{n} r_k^{-1} \leq C r_n^3 \epsilon_1,$$

since $r_n \leq 1$. Then, for $\epsilon_1 \leq 1$ we have

$$L(r_{n+1}) \leq C \epsilon_1 r_n^{\frac{12}{5}}. \tag{5.33}$$

Therefore,

$$r_{n+1}^{\frac{3}{5}} ⨍_{Q^{n+1}} |u||p - \bar{p}_{n+1}| \leq C r_n^{-\frac{12}{5}} L(r_{n+1}) \leq C^{**} \epsilon_1, \tag{5.34}$$

by lemma 5.2.2 and (5.33).
We claim that ϵ_1 is small such that the constant C^{**} satisfies

$$C^{**}\epsilon_1^{\frac{1}{3}} \leq \frac{1}{2}.$$

Hence,

$$r_{n+1}^{\frac{3}{5}} \fint_{Q^{n+1}} |u||p - \bar{p}_{n+1}| \leq \frac{1}{2}\epsilon_1^{\frac{2}{3}}. \tag{5.35}$$

Consequently, combining (5.32) and (5.35) we have

$$\fint_{Q^{n+1}} |u|^3 + r_{n+1}^{\frac{3}{5}} \fint_{Q^{n+1}} |u||p - \bar{p}_{n+1}| \leq \frac{1}{2}\epsilon_1^{\frac{2}{3}} + \frac{1}{2}\epsilon_1^{\frac{2}{3}} = \epsilon_1^{\frac{2}{3}} \tag{5.36}$$

which is $(5.25)_{n+1}$.

Third step: Assume that $(5.25)_k$, $3 \leq k \leq n$ holds. I want to show that $(5.26)_n$ holds too, for $n \geq 3$. We use the inequality (5.22) for $\phi = \phi_n$. For simplicity, we will take the point (0,0) at the origin of the cylinder, instead of (a,s).
Define as

$$\phi_n := \chi\psi_n \quad \text{with,} \quad \psi_n = c\Phi, \tag{5.37}$$

where Φ is the fundamental solution of the heat equation

$$\Phi_t + \Delta\Phi = 0, \tag{5.38}$$

with singularity at the point $(0, r_n^2)$ and χ a cutoff function.
More specific,

$$\psi_n = \frac{1}{(r_n^2 - t)^{\frac{3}{2}}} \exp\{\frac{-|x|^2}{4(r_n^2 - t)}\} \ , \quad r_n^2 - t > 0$$

and the cutoff function $\chi \in C^\infty(\{t \leq 0\})$ with $0 \leq \chi \leq 1$ and

$$\chi = \begin{cases} 1 \ , & \text{on } Q^2 = Q_{\frac{1}{4}}(0,0) \\ 0 \ , & \text{off } Q_{\frac{1}{3}}(0,0). \end{cases}$$

Hence

$$\phi_n = \begin{cases} \psi_n \ , & \text{on } Q^2 = Q_{\frac{1}{4}}(0,0) \\ 0 \ \ , & \text{off } Q_{\frac{1}{3}}(0,0). \end{cases}$$

Therefore, $\phi_n \geq 0$, since $\psi_n \geq 0$ and

$$\frac{\partial\phi_n}{\partial t} + \Delta\phi_n = 0 \ \ on \ \ Q^2, \tag{5.39}$$

since ψ_n has defined as the fundamental solution of (5.38).

$$\left|\frac{\partial\phi_n}{\partial t} + \Delta\phi_n\right| \leq C \text{ everywhere} \tag{5.40}$$

$$\frac{1}{C}r_n^{-3} \leq \phi_n \leq Cr_n^{-3} \quad , |\nabla\phi_n| \leq Cr_n^{-4}, \text{ on } \ Q^2, \ n \geq 2 \tag{5.41}$$

and

$$\phi_n \leq Cr_k^{-3}, \quad |\nabla\phi_n| \leq Cr_k^{-4}, \text{ on } \ Q^{k-1}, \text{off } \ Q^k, \ 1 < k \leq n, \tag{5.42}$$

for an absolute constant C independent on n.

Now, by (5.22), where $\phi =: \phi_n$ using the previous inequality (5.41) we get

$$\begin{aligned}\sup_{-r_n^2<t\leq 0} \fint_{|x|<r_n} |u|^2(x,t)\ \mathrm{d}x + r_n^{-3}\int_{Q^n} |\nabla u|^2 &\leq \int_{B_1\times\{s\}} |u|^2\phi_n + 2\int_{-1}^{s}\int_{B_1} |\nabla u|^2\phi_n \\ &\leq C(I+II+III),\end{aligned}$$

where

$$\begin{aligned} I &= \int_{Q^1} |u|^2 \left(\frac{\partial \phi_n}{\partial t} + \Delta\phi_n\right), \\ II &= \int_{Q^1} |u|^3 |\nabla \phi_n|, \\ III &= \left|\int_{Q^1} p(u\cdot\nabla\phi_n)\right|. \end{aligned}$$

We will bound the I, II, III.

$$\begin{aligned} I = \int_{Q^1} |u|^2 \left(\frac{\partial \phi_n}{\partial t} + \Delta\phi_n\right) &\leq \left|\int_{Q^1} |u|^2 \left(\frac{\partial \phi_n}{\partial t} + \Delta\phi_n\right)\right| \\ &\leq \int_{Q^1} |u|^2 \left|\frac{\partial \phi_n}{\partial t} + \Delta\phi_n\right| \\ &\leq C\int_{Q^1} |u|^2 \leq C\epsilon_1^{\frac{2}{3}}. \end{aligned}$$

Here, we used (5.24) and Hölder inequality.

$$\begin{aligned} II = \int_{Q^1} |u|^3|\nabla\phi_n| &\leq C\sum_{k=1}^{n} r_k^{-4}\int_{Q^1}|u|^3 \\ &\leq C\sum_{k=1}^{n} r_k^{-4} r_k^5 \epsilon_1^{\frac{2}{3}} \\ &= C\sum_{k=1}^{n} r_k \epsilon_1^{\frac{2}{3}} \\ &\leq C\epsilon_1^{\frac{2}{3}}, \end{aligned}$$

where in the first inequality we used (5.42), in the second inequality we used $(5.25)_k$ for $3 \leq k \leq n$ and in the last, the fact that $\sum\limits_{k=1}^{n} r_k \leq 1$ because, for all $n \geq 2$, $r_n = \frac{1}{2^n}$.

Now, to bound III, we will use the fact that the function u is divergence free and we will transfer the problem to one that involves the oscillation of p.
Let, the cutoff function $0 \leq \chi_k \leq 1$, $\forall k \geq 1$, $\chi_k \in C^\infty(Q^1)$, $Q^1 := Q_{\frac{1}{2}}(0,0)$, such that

$$\chi_k = \begin{cases} 1\ , & \text{on } Q_{\frac{7r_k}{8}}(0,0) := \{(x,t):\ |x| < \frac{7r_k}{8}\ ,\ -\left(\frac{7r_k}{8}\right)^2 < t < 0\} \\ 0\ , & \text{off } Q_{r_k}(0,0) := \{(x.t):\ |x| < r_k,\ -r_k^2 < t < 0\}, \end{cases}$$

and

$$|\nabla\chi_k| \le \frac{C}{r_k}.$$

Then $\chi_1\phi_n = \phi_n$. Thus,

$$\begin{aligned} III = \left|\int_{Q^1} p(u\cdot\nabla\phi_n)\right| &= \int_{Q^1} p(u\cdot\nabla\phi_n) \\ &= \int_{Q^1} p\big(u\cdot\nabla(\chi_1-\chi_n+\chi_n)\phi_n\big) \\ &= \sum_{k=1}^{n-1}\int_{Q^1} pu\cdot\nabla\big((\chi_k-\chi_{k+1})\phi_n\big) + \int_{Q_1} pu\cdot\nabla(\chi_n\phi_n). \end{aligned}$$

We observe that, $\chi_k-\chi_{k+1}$ is supported on Q^k and that $\nabla\cdot u = 0$. Thus, we have, for $k\ge 3$, that

$$\int_{Q^1} pu\cdot\nabla\big((\chi_k-\chi_{k+1})\phi_n\big) = \int_{Q^k}(p-\bar p_k)u\cdot\nabla\big((\chi_k-\chi_{k+1})\phi_n\big),$$

where $\bar p_k = \bar p_k(t) = \int_{|x|<r_k} p$.
Similarly,

$$\int_{Q^1} pu\cdot\nabla(\chi_n\phi_n) = \int_{Q^n}(p-\bar p_n)u\cdot\nabla(\chi_n\phi_n).$$

Now, for $k=1,2$

$$\left|\int_{Q^1} pu\cdot\nabla\big((\chi_k-\chi_{k+1})\phi_n\big)\right| \le C\int_{Q^1}|p||u|, \quad \text{by (5.42).}$$

Hencefore,

$$\begin{aligned} III &= \sum_{k=1}^{n-1}\int_{Q^1} pu\cdot\nabla\big((\chi_k-\chi_{k+1})\phi_n\big) + \int_{Q_1} pu\cdot\nabla(\chi_n\phi_n) \\ &\le C\int_{Q^1}|p||u| + \sum_{k=3}^{n-1}\int_{Q^k}|p-\bar p_k||u|\cdot\big|\nabla\big((\chi_k-\chi_{k+1})\phi_n\big)\big| \\ &+ \int_{Q^n}|p-\bar p_n||u|\cdot\big|\nabla(\chi_n\phi_n)\big| \\ &\le C\int_{Q^1}|p||u| + C\sum_{k=3}^{n}\int_{Q^k}|p-\bar p_k||u|r_k^{-4} \\ &\le C\epsilon_1^{\frac{2}{3}} + C\sum_{k=3}^{n} r_k^{\frac{2}{5}}\epsilon_1^{\frac{2}{3}} \le C\epsilon_1^{\frac{2}{3}}, \end{aligned}$$

in which we used (5.24), $(5.25)_k$, $3\le k\le n$ and the fact that $\sum_{k=1}^n r_k^{\frac{2}{5}} \le 1$.
Furthermore, we use the estimate

$$|\nabla(\chi_n\phi)| \le Cr_n^{-4}$$

and

$$|\nabla((\chi_k-\chi_{k+1})\phi)| \le Cr_n^{-4}.$$

More specific,

$$\begin{aligned}|\nabla(\chi_n\phi)| &= |\nabla\chi_n\phi + \chi_n\nabla\phi| \\ &\le |\nabla\chi_n\phi| + |\chi_n\nabla\phi| \\ &\le \frac{C}{r_n} r_n^{-3} + C r_n^{-4} =: \tilde{C} r_n^{-4}.\end{aligned}$$

The other estimate follows similarly. Note that each $C > 0$ may be different. Hence, we get the following estimate

$$\sup_{-r_n^2 < t \le 0} \fint_{|x|<r_n} |u|^2(x,t)\ \mathrm{d}x + r_n^{-3} \int_{Q^n} |\nabla u|^2 \le C\epsilon_1^{\frac{2}{3}}.$$

Thus, the third step is complete which is what it was needed to show. Therefore, the proof of Proposition 5.1.12 is complete.

□

5.3 Proof of Proposition 5.1.14

Let

$$Q_r^* := Q_r^*(0,0) = \{(x,t) : |x| < r, -\frac{7}{8}r^2 < t < \frac{1}{8}r^2\}$$

and the functions u,p which are measurable on Q_r^*
We consider the following quantities which some of these are analogous of these in the section 5.3.

$$A_*(r) := \sup_{-\frac{7}{8}r^2 < t < \frac{1}{8}r^2} r^{-1} \int_{B_r \times \{t\}} |u|^2, \tag{5.43}$$

$$G_*(r) := r^{-2} \int_{Q_r^*} |u|^3, \tag{5.44}$$

$$\delta_*(r) := r^{-1} \int_{Q_r^*} |\nabla u|^2, \tag{5.45}$$

$$K_*(r) := r^{-\frac{13}{4}} \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \left(\int_{B_r} |p| dy \right)^{\frac{5}{4}} \mathrm{d}t, \tag{5.46}$$

$$J_*(r) := r^{-2} \int_{Q_r^*} |u||p|, \tag{5.47}$$

$$H_*(r) := r^{-2} \int_{Q_r^*} |u| \big| |u|^2 - \bar{|u|}_r^2 \big|, \tag{5.48}$$

where $\bar{|u|}_r^2 = \bar{|u|}_r^2(t) = \fint_{B_r} |u|^2$.
Consider, furthermore, a combination of these quantities

$$M_*(r) := G_*^{\frac{2}{3}}(r) + H_*(r) + J_*(r) + K_*^{\frac{8}{5}}(r). \tag{5.49}$$

We state, now, the following Proposition 5.3.1, which suffices to prove in order to prove Proposition 5.1.14.

Proposition 5.3.1. *Let (u,p) a suitable weak solution of Navier-Stokes equation with $\delta_*(\rho) \le 1$. Then, for $r \le \frac{1}{4}\rho$ it follows*

$$M_*(r) \le C\Big[\big(\frac{r}{\rho}\big)^{\frac{1}{5}} M_*(\rho) + \big(\frac{\rho}{r}\big)^2 \big[M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}} + M_*(\rho)\delta_*(\rho) \big] \Big]. \tag{5.50}$$

Proposition. 5.1.14. *There exist an absolute constant $\epsilon_1 > 0$ such that, if the pair (u, p) is a suitable weak solution of the Navier-Stokes system near the point (x, t) and if*

$$\limsup_{r\to 0} r^{-1} \int_{Q_r^*(x,t)} |\nabla u|^2 \leq \epsilon_1.$$

Then, (x, t) is regular point.

Remark 5.3.2. Proposition 5.1.14 is an estimation which estimate the region in which a singularity could be developed. Notice that, $\frac{r}{\rho}$ is small and multiplied by quantities that are not include $\delta_*(\rho)$ which is small. On the other hand, $\frac{\rho}{r}$ is large and multiplied by positive powers of $\delta_*(\rho)$ which is small. Furthermore, if u is regular in $(0, 0)$, then $M_*(r) \to 0$, as $r \to 0$.

Without loss of generality, we consider that $(x, t) = (0, 0)$ and that (u, p) defined in a region D of $(0, 0)$.
By Corollary 1, page 776 of [5], $(0, 0)$ is regular if

$$\liminf_{r\to 0}\{r^{-2} \int_{Q_r^*} (|u|^3 + |u||p|) + r^{-\frac{13}{4}} \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \left(\int_{B_r} |p|\right)^{\frac{5}{4}} \mathrm{d}t\} \leq \epsilon. \tag{5.51}$$

Combining (5.49) and (5.51)

$$\Rightarrow \liminf_{r\to 0} M_*(r) \leq \bar{\epsilon},$$

for a suitable uniform constant $\bar{\epsilon} \Rightarrow u$ is regular. By Corollary 1, page 776 of [5]. Now, we may define uniform constants

$$0 < \epsilon_1 \leq 1 \text{ and } \ 0 < \gamma \leq \frac{1}{4},$$

such that

$$M_*(\rho) > \bar{\epsilon} \text{ and } \ \delta_*(\rho) \leq \epsilon_1 \tag{5.52}$$

$$\Rightarrow M_*(\gamma\rho) \leq \frac{1}{2} M_*(\rho). \tag{5.53}$$

Now, with such a choice of ϵ_1, we claim that

$$\limsup_{r\to 0} r^{-1} \int_{Q_r^*(x,t)} |\nabla u|^2 \leq \epsilon_1 \tag{5.54}$$

$$\Rightarrow \liminf_{r\to 0} M_*(r) \leq \bar{\epsilon} \tag{5.55}$$

Proof of claim
Indeed, for $r_0 > 0$ we have $\delta_*(r) \leq \epsilon_1$, $\forall r < r_0$. By using (5.54) repeatedly,

$$\Rightarrow M_*(\gamma^n r_0) \leq \bar{\epsilon} \ \ , \text{ for } \ n \in \mathbb{N}.$$

Therefore, (5.54) $\Rightarrow$ (5.55) That is, for this choice of $\bar{\epsilon}$, we conclude that, if (5.54) holds true, then $(0, 0)$ is a regular point. Now, we will choose these uniform constants ϵ_1 and γ. If (5.52) holds true, then

$$M_*^{\frac{1}{2}}(\rho) < \bar{\epsilon}^{-\frac{1}{2}} M_*(\rho).$$

By using (5.50)

$$\begin{aligned} M_*(r) &\le C\Big[(\frac{r}{\rho})^{\frac{1}{5}} M_*(\rho) + (\frac{\rho}{r})^2 \big[M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}} + M_*(\rho)\delta_*(\rho) + \delta_*(\rho)\big]\Big] \\ &\le C\Big[(\frac{r}{\rho})^{\frac{1}{5}} M_*(\rho) + (\frac{\rho}{r})^2 \big[\epsilon_1 M_*(\rho) + \bar{\epsilon}^{-\frac{1}{2}} M_*(\rho)\epsilon_1^{\frac{1}{2}}\big] + (\frac{\rho}{r})^2 \epsilon_1\Big] \\ &= C\Big[(\frac{r}{\rho})^{\frac{1}{5}} M_*(\rho) + (\frac{\rho}{r})^2 \big[\epsilon_1 + \Big(\frac{\epsilon_1}{\bar{\epsilon}}\Big)^{\frac{1}{2}}\big] M_*(\rho) + (\frac{\rho}{r})^2 \epsilon_1\Big] \end{aligned}$$

$$\Rightarrow M_*(r) \le C\Big[(\frac{r}{\rho})^{\frac{1}{5}} M_*(\rho) + (\frac{\rho}{r})^2 \big[\epsilon_1 + \Big(\frac{\epsilon_1}{\bar{\epsilon}}\Big)^{\frac{1}{2}}\big] M_*(\rho) + (\frac{\rho}{r})^2 \epsilon_1\Big], \quad \text{for } r \le \frac{1}{4}\rho. \tag{5.56}$$

Choosing, now, γ such that

$$C\gamma^{\frac{1}{5}} \le \frac{1}{6} \text{ and } \gamma \le \frac{1}{4}.$$

Thereafter, for fixed γ, we choose $\epsilon_1 \le 1$ such that

$$C\gamma^{-2}\left(\epsilon_1 + \Big(\frac{\epsilon_1}{\bar{\epsilon}}\Big)^{\frac{1}{2}}\right) \le \frac{1}{6} \text{ and } C\gamma^{-2}\epsilon_1 \le \frac{1}{6}\bar{\epsilon}.$$

Applying (5.56) with $r =: \gamma\rho$ we get

$$\begin{aligned} M_*(\gamma\rho) &\le C\Big[(\frac{\gamma\rho}{\rho})^{\frac{1}{5}} M_*(\rho) + (\frac{\rho}{\gamma\rho})^2 \big[\epsilon_1 + \Big(\frac{\epsilon_1}{\bar{\epsilon}}\Big)^{\frac{1}{2}}\big] M_*(\rho) + (\frac{\rho}{\gamma\rho})^2 \epsilon_1\Big] \\ &= C\Big[\gamma^{\frac{1}{5}} M_*(\rho) + \gamma^{-2} \big[\epsilon_1 + \Big(\frac{\epsilon_1}{\bar{\epsilon}}\Big)^{\frac{1}{2}}\big] M_*(\rho) + \gamma^{-2} \epsilon_1\Big] \\ &\le C\gamma^{\frac{1}{5}} M_*(\rho) + \frac{1}{6}M_*(\rho) + \frac{1}{6}\bar{\epsilon} \\ &\le \frac{1}{6}M_*(\rho) + \frac{1}{6}M_*(\rho) + \frac{1}{6}\bar{\epsilon} \\ &\le \frac{1}{3}M_*(\rho) + \frac{1}{6}M_*(\rho). \end{aligned}$$

Here, in the latter inequality we used (5.52). Therefore,

$$M_*(\gamma\rho) \le \frac{1}{2}M_*(\rho).$$

Thus, the proof of relation (5.53) is done.
We observe that, we made use of Proposition 5.3.1. Now, it remains to prove it. This is going to achieve as follows. First, we are going to bound u in terms of ∇u by interpolation and secondly, to bound p in terms of u.We will succeed it by solving the equation

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j).$$

We will need some lemmas in order to prove the proposition.

Lemma 5.3.3.

$$H_*(r) \le C\big(G_*^{\frac{2}{3}}(r) + A_*(r)\delta_*(r)\big). \tag{5.57}$$

Proof. For almost every $t \in \left(-\frac{7}{8}r^2, \frac{1}{8}r^2\right)$, we compute

$$\begin{aligned}\int_{B_r} |u|\left||u|^2 - \bar{|u|}^2_r\right| &\leq \left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \left(\int_{B_r} \left||u|^2 - \bar{|u|}^2_r\right|^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ &\leq C\left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \left(\int_{B_r} \left|\nabla|u|^2\right|\right)^{\frac{3}{2}\frac{2}{3}} \\ &\leq \left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \int_{B_r} |u||\nabla u| \\ &\leq \left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \left(\int_{B_r} |u|^2\right)^{\frac{1}{2}} \left(\int_{B_r} |\nabla u|^2\right)^{\frac{1}{2}},\end{aligned}$$

where in the first and the latter inequality we made use of Hölder inequality and in the second inequality we made use of Poincare inequality, see, e.g., Theorem 4.9, in [11].

$$\begin{aligned}\Rightarrow \int_{B_r} |u|\left||u|^2 - \bar{|u|}^2_r\right| &\leq \left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \left(rr^{-1}\int_{B_r} |u|^2\right)^{\frac{1}{2}} \left(\int_{B_r} |\nabla u|^2\right)^{\frac{1}{2}} \\ &\leq C\big(rA_*(r)\big)^{\frac{1}{2}} \left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \left(\int_{B_r} |\nabla u|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Integrating over $\left(-\frac{7}{8}r^2, \frac{1}{8}r^2\right)$, we take

$$\Rightarrow \int_{Q_r^*} |u|\left||u|^2 - \bar{|u|}^2_r\right| \leq C\big(rA_*(r)\big)^{\frac{1}{2}} \left(\int_{Q_r^*} |u|^3\right)^{\frac{1}{3}} \left(\int_{Q_r^*} |\nabla u|^2\right)^{\frac{1}{2}} r^{\frac{1}{3}}.$$

$$\begin{aligned}\Rightarrow \frac{1}{r^2}\int_{Q_r^*} |u|\left||u|^2 - \bar{|u|}^2_r\right| &\leq Cr^{\frac{1}{2}}A_*^{\frac{1}{2}}(r)r^{\frac{2}{3}}\frac{1}{r^{\frac{2}{3}}} \left(\int_{Q_r^*} |u|^3\right)^{\frac{1}{3}} r^{-\frac{1}{2}}r^{\frac{1}{2}} \left(\int_{Q_r^*} |\nabla u|^2\right)^{\frac{1}{2}} \frac{r^{\frac{1}{3}}}{r^2} \\ &= CA_*^{\frac{1}{2}}(r)G_*^{\frac{1}{3}}(r)\delta_*^{\frac{1}{2}}(r)\underbrace{r^{\frac{2}{3}}r^{\frac{1}{2}}r^{\frac{1}{2}}r^{\frac{1}{3}}r^{-2}}_{=1} \\ \Rightarrow H_*(r) &\leq CA_*^{\frac{1}{2}}(r)G_*^{\frac{1}{3}}(r)\delta_*^{\frac{1}{2}}(r) \\ &\leq C\big(G_*^{\frac{2}{3}}(r) + A_*(r)\delta_*(r)\big)\end{aligned}$$

□

Lemma 5.3.4. *For* $r \leq \rho$,

$$G_*(r) \leq C\left[\left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + \left(\frac{\rho}{r}\right)^3 A_*^{\frac{3}{4}}(\rho)\delta_*^{\frac{3}{4}}(\rho)\right]$$

Proof. We want to show

$$\begin{aligned}r^{-2}\int_{Q_r^*} |u|^3 \leq C\Bigg[&\left(\frac{r}{\rho}\right)^3 \left(\sup_{-\frac{7}{8}\rho^2<t<\frac{1}{8}\rho^2} \rho^{-1}\int_{B_\rho\times\{t\}} |u|^2\right)^{\frac{3}{2}} \\ &+ \left(\frac{\rho}{r}\right)^3 \left(\sup_{-\frac{7}{8}\rho^2<t<\frac{1}{8}\rho^2} \rho^{-1}\int_{B_\rho\times\{t\}} |u|^2\right)^{\frac{3}{4}} \left(\rho^{-1}\int_{Q_\rho^*} |\nabla u|^2\right)^{\frac{3}{4}}\Bigg].\end{aligned}$$

Now, for almost every $t \in \left(-\frac{7}{8}r^2, \frac{1}{8}r^2\right)$, we compute

$$\begin{aligned}
\int_{B_r} |u|^2 &= \int_{B_r} \left(|u|^2 - \bar{|u|_\rho}^2\right) + \int_{B_r} \bar{|u|_\rho}^2 \\
&\leq \int_{B_\rho} \left||u|^2 - \bar{|u|_\rho}^2\right| + \int_{B_r} \bar{|u|_\rho}^2 \\
&\leq C\rho \int_{B_\rho} \left|\nabla |u|^2\right| + C|B_r| \fint_{B_\rho} |u|^2 \\
&= 2C\rho \int_{B_\rho} |u|\left|\nabla u\right| + C \underbrace{\frac{|B_r|}{|B_\rho|} \int_{B_\rho} |u|^2}_{=\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2} \\
&= \tilde{C}\rho \int_{B_\rho} |u|\left|\nabla u\right| + C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2,
\end{aligned}$$

where in the third inequality we use Poincare inequality and the change rule.

$$\begin{aligned}
\int_{B_r} |u|^2 &\leq \tilde{C}\rho \int_{B_\rho} |u|\left|\nabla u\right| + C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\leq \tilde{C}\rho \frac{\rho^{\frac{1}{2}}}{\rho^{\frac{1}{2}}} \left(\int_{B_\rho} |u|^2\right)^{\frac{1}{2}} \left(\int_{B_\rho} \left|\nabla u\right|^2\right)^{\frac{1}{2}} + C\left(\frac{r}{\rho}\right)^3 \frac{\rho}{\rho} \int_{B_\rho} |u|^2,
\end{aligned}$$

by using Hölder inequality in the second inequality. Therefore,

$$\Rightarrow \int_{B_r} |u|^2 \leq \tilde{C}\rho^{\frac{3}{2}} A_*^{\frac{1}{2}}(\rho) \left(\int_{B_\rho} \left|\nabla u\right|^2\right)^{\frac{1}{2}} + C\frac{r^3}{\rho^2} A_*(\rho). \tag{5.58}$$

Now, by using the interpolation inequality (5.3) for $q = 3$ and $a = \frac{3}{4}$ we get

$$\begin{aligned}
\int_{B_r} |u|^3 &\leq c\left(\int_{B_r} |\nabla u|^2\right)^{\frac{3}{4}} \left(\int_{B_r} |u|^2\right)^{\frac{3}{4}} + \frac{c}{r^{\frac{3}{2}}}\left(\int_{B_r} |u|^2\right)^{\frac{3}{2}} \\
&\leq c\rho^{\frac{3}{4}} A_*^{\frac{3}{4}}(\rho)\left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{3}{4}} + \frac{c}{r^{\frac{3}{2}}}\left(\int_{B_r} |u|^2\right)^{\frac{3}{2}} \\
&\leq c\left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + c\left[\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right] A_*^{\frac{3}{4}}(\rho)\left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{3}{4}}.
\end{aligned}$$

By integrating over $t \in \left(-\frac{7}{8}r^2, \frac{1}{8}r^2\right)$

$$\int_{Q_r^*} |u|^3 \leq cr^2\left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + c\left[\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right] A_*^{\frac{3}{4}}(\rho)\left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{3}{4}}$$

and, now, using Hölder inequality we get

$$\Rightarrow \int_{Q_r^*} |u|^3 \leq cr^2\left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + c\left[\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right] r^{\frac{1}{2}} A_*^{\frac{3}{4}}(\rho)\left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{3}{4}}.$$

Dividing by r^2

$$\Rightarrow \frac{1}{r^2}\int_{Q_r^*}|u|^3 \le c\left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + c\left[\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right] r^{\frac{1}{2}} A_*^{\frac{3}{4}}(\rho)\frac{\rho^{\frac{3}{4}}}{r^2}\left(\rho^{-1}\int_{B_\rho}|\nabla u|^2\right)^{\frac{3}{4}}$$
$$= c\left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + c\left[\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right] r^{\frac{1}{2}} r^{-2} A_*^{\frac{3}{4}}(\rho)\rho^{\frac{3}{4}}\delta_*^{\frac{3}{4}}(\rho)$$
$$= c\left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + c\left[\left(\frac{\rho}{r}\right)^{\frac{3}{2}} + \left(\frac{\rho}{r}\right)^3\right] A_*^{\frac{3}{4}}(\rho)\delta_*^{\frac{3}{4}}(\rho).$$

Now, by using the fact that $\frac{\rho}{r} \ge 1$ we take

$$G_*(r) \le C\left[\left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + c\left(\frac{\rho}{r}\right)^3 A_*^{\frac{3}{4}}(\rho)\delta_*^{\frac{3}{4}}(\rho)\right].$$

Therefore, the lemma has been proved. □

Until now, we have bound the terms H_* and G_*. In what follows, we are going to find bounds of J_* and K_*. It will be achieved this, by using the representation of p. See the relation (2.14) in [5].
We will use the same methods as in Lemma 5.2.2.
Let a cutoff function such that,

$$\phi(y) = \begin{cases} 1, & \text{if } |y| \le \frac{3}{4}\rho \\ 0, & \text{if } |y| \ge \rho, \end{cases}$$

with the relations

$$|\nabla_i\phi| \le C\rho^{-1} \quad \text{and} \quad |\nabla_{ij}\phi| \le C\rho^{-2}.$$

Let $\bar{U}_1 \subset U$ and $\phi \in C_c^\infty(U)$ with $\phi = 1$ in a region of $\bar{U_1}$, then $\forall t$ we have

$$\phi(x)p(x,t) = -\frac{3}{4\pi}\int_{\mathbb{R}^n}\frac{1}{|x-y|}\Delta_y(\phi p)\ \mathrm{d}y$$
$$= -\frac{3}{4\pi}\int_{\mathbb{R}^n}\frac{1}{|x-y|}\left[p\Delta\phi + 2(\nabla\phi,\nabla p) + \phi\Delta p\right]\ \mathrm{d}y.$$

Then, by integration by parts, we have

$$p(x) = p_4(x) + p_5(x)\ , \quad \text{for} \quad |x| \le \frac{3}{4}\rho,$$

where

$$p_4(x) = \frac{3}{4\pi}\int_{\mathbb{R}^n}\frac{1}{|x-y|}p(y)\Delta\phi(y)\ \mathrm{d}y + \frac{3}{2\pi}\int_{\mathbb{R}^n}\frac{x_i-y_i}{|x-y|^3}\nabla_i\phi(y)p(y)\ \mathrm{d}y$$

and

$$p_5(x) = \frac{3}{4\pi}\int_{\mathbb{R}^n}\frac{1}{|x-y|}\phi(y)\nabla_i u_j(y)\nabla_j u_i(y)\ \mathrm{d}y.$$

We observe, for $|x| \le \frac{1}{2}\rho$ that

$$|p_4(x)| \le C \fint_{B_\rho}|p|. \tag{5.59}$$

Furthermore, we observe that p_5 is convolution of the form $\frac{1}{|x|} * g$, where g is in the ball B_ρ. Hence, by Young's inequality

$$\int_{B_r} |p_5|^2 \leq C\rho \left(\int_{B_\rho} g \right)^2$$

$$\Rightarrow \left(\int_{B_r} |p_5|^2 \right)^{\frac{1}{2}} \leq C\rho^{\frac{1}{2}} \int_{B_\rho} |\nabla u|^2, \tag{5.60}$$

where $g = |\nabla u|^2$.

Lemma 5.3.5. *If $r \leq \frac{1}{2}\rho$, then*

$$J_*(r) \leq C \left[\left(\frac{r}{\rho} \right)^{\frac{1}{5}} A_*^{\frac{1}{5}}(\rho) G_*^{\frac{1}{5}}(r) K_*^{\frac{4}{5}}(\rho) + \left(\frac{\rho}{r} \right)^2 A_*^{\frac{1}{2}}(\rho) \delta_*(\rho) \right]. \tag{5.61}$$

Proof. Fix a $t \in \big(-\frac{7}{8}r^2, \frac{1}{8}r^2 \big)$. By the relation (5.59) we have

$$\begin{aligned}
\int_{B_r} |u||p_4| &\leq C \left(\int_{B_r} |u| \right) \left(⨍_{B_\rho} |p| \right) \\
&\leq C r^{\frac{9}{5}} \left(\int_{B_r} |u|^2 \right)^{\frac{1}{5}} \left(\int_{B_r} |u|^3 \right)^{\frac{1}{5}} \left(⨍_{B_\rho} |p| \right) \\
&\leq C r^{\frac{9}{5}} \rho^{\frac{1}{5}} A_*^{\frac{1}{5}}(\rho) \left(\int_{B_r} |u|^3 \right)^{\frac{1}{5}} \left(⨍_{B_\rho} |p| \right).
\end{aligned}$$

Now, integrating over $\big(-\frac{7}{8}r^2, \frac{1}{8}r^2 \big)$

$$\begin{aligned}
\Rightarrow \int_{Q_r^*} |u||p_4| &\leq C r^{\frac{9}{5}} \rho^{\frac{1}{5}} A_*^{\frac{1}{5}}(\rho) \big[r^2 G_*(r) \big]^{\frac{1}{5}} \frac{1}{|B_\rho|} \int_{Q_\rho^*} |p| \\
&\leq C r^{\frac{9}{5}} \rho^{\frac{1}{5}} A_*^{\frac{1}{5}}(\rho) \big[r^2 G_*(r) \big]^{\frac{1}{5}} \big[\rho^{-\frac{1}{2}} K_*(\rho) \big]^{\frac{4}{5}}.
\end{aligned}$$

Where, in the last inequality we use Hölder inequality with $p = 5$ and $q = \frac{5}{4}$. Then, dividing by r^2

$$\Rightarrow r^{-2} \int_{Q_r^*} |u||p_4| \leq C r^{-2} r^{\frac{9}{5}} \rho^{\frac{1}{5}} r^{\frac{2}{5}} \rho^{-\frac{4}{10}} A_*^{\frac{1}{5}}(\rho) G_*^{\frac{1}{5}}(r) K_*^{\frac{4}{5}}(\rho)$$

$$\Rightarrow r^{-2} \int_{Q_r^*} |u||p_4| \leq C \left(\frac{r}{\rho} \right)^{\frac{1}{5}} A_*^{\frac{1}{5}}(\rho) G_*^{\frac{1}{5}}(r) K_*^{\frac{4}{5}}(\rho). \tag{5.62}$$

Now, for the second part of (5.61), using Hölder and (5.60) we have

$$\begin{aligned}
\int_{B_r} |u||p_5| &\leq \left(\int_{B_r} |u|^2 \right)^{\frac{1}{2}} \left(\int_{B_r} |p_5|^2 \right)^{\frac{1}{2}} \\
&\leq C \rho^{\frac{1}{2}} A_*^{\frac{1}{2}}(\rho) \rho^{\frac{1}{2}} \int_{B_\rho} |\nabla u|^2.
\end{aligned}$$

And, again, integrating by t to take

$$\int_{Q_r^*} |u||p_5| \leq C\rho A_*^{\frac{1}{2}}(\rho) \int_{Q_\rho^*} |\nabla u|^2.$$

Dividing by r^2

$$r^{-2}\int_{Q_r^*}|u||p_5| \leq Cr^{-2}\rho A_*^{\frac{1}{2}}(\rho)\rho\rho^{-1}\int_{Q_\rho^*}|\nabla u|^2$$

$$\Rightarrow r^{-2}\int_{Q_r^*}|u||p_5| \leq C\left(\frac{\rho}{r}\right)^2 A_*^{\frac{1}{2}}(\rho)\delta_*(\rho). \tag{5.63}$$

Therefore, adding (5.62) and (5.63) to take the result.

□

Lemma 5.3.6. *If* $r \leq \frac{1}{2}\rho$, *then*

$$K_*(r) \leq C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{2}} K_*(\rho) + \left(\frac{\rho}{r}\right)^{\frac{5}{4}} A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho)\right]. \tag{5.64}$$

Proof. By the relation (5.59) we have

$$\int_{B_r}|p_4| \leq C\fint_{B_\rho}|p| \leq \frac{1}{|B_\rho|}|B_r|\int_{B_\rho}|p| \tag{5.65}$$

$$\Rightarrow \int_{B_r}|p_4| \leq C\left(\frac{r}{\rho}\right)^3\int_{B_\rho}|p|$$

$$\Rightarrow \left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} \leq C\left(\frac{r}{\rho}\right)^{\frac{15}{4}}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}.$$

Integrating by t we take

$$\Rightarrow \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} \leq C\left(\frac{r}{\rho}\right)^{\frac{15}{4}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}$$

$$\Rightarrow r^{\frac{13}{4}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} \leq C\left(\frac{r}{\rho}\right)^{\frac{15}{4}} r^{-\frac{13}{4}}\rho^{\frac{13}{4}}\rho^{-\frac{13}{4}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}$$

$$\leq C\left(\frac{r}{\rho}\right)^{\frac{15}{4}}\left(\frac{r}{\rho}\right)^{-\frac{13}{4}}K_*(\rho)$$

$$\Rightarrow r^{\frac{13}{4}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} \leq C\left(\frac{r}{\rho}\right)^{\frac{15}{4}}\left(\frac{r}{\rho}\right)^{-\frac{13}{4}}K_*(\rho). \tag{5.66}$$

Now, we want to take an estimate for p_5. We have that, p_5 have the following representation,

$$p_5 = p_6 + p_7,$$

where

$$p_6(x) = -\frac{3}{4\pi}\int \frac{x_i - y_i}{|x-y|^3}\phi(y)(u\cdot\nabla u_i)(y)\ \mathrm{d}y$$

and

$$p_7(x) = -\frac{3}{4\pi}\int \frac{1}{|x-y|}\big(\nabla_i\phi(y)\big)\big(u\cdot\nabla u_i(y)\big)\ \mathrm{d}y.$$

For the p_6 we have

$$\begin{aligned}\int_{B_r}|p_6(x)|\ \mathrm{d}x &\le C\int_{B_r}\int\frac{1}{|x-y|^2}|u(y)||\nabla u(y)|\ \mathrm{d}y\mathrm{d}x\\ &= C\int_{|y|<\rho}\int_{|x|<r}\frac{1}{|x-y|^2}|u(y)||\nabla u(y)|\ \mathrm{d}x\mathrm{d}y\\ &\le Cr\int_{B_\rho}|u||\nabla u|\\ &\le Cr\left(\int_{B_\rho}|u|^2\right)^{\frac12}\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12}\\ &\le Cr\rho^{\frac12}A_*^{\frac12}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12},\end{aligned}$$

where in the second equality we use Fubini's Theorem and, in the forth inequality we use Hölder.

$$\Rightarrow \int_{B_r}|p_6(x)| \le Cr\rho^{\frac12}A_*^{\frac12}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12}. \tag{5.67}$$

For an estimate of p_7 we have

$$\begin{aligned}|p_7(x)| &\le C\rho^{-2}\int_{B_\rho}|u||\nabla u|\\ &\le C\rho^{-2}\left(\int_{B_\rho}|u|^2\right)^{\frac12}\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12}\\ &= C\rho^{-2}\rho^{\frac12}\left(\rho^{-1}\int_{B_\rho}|u|^2\right)^{\frac12}\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12}.\end{aligned}$$

Here, in the second inequality we used Hölder.

$$\begin{aligned}\Rightarrow \int_{B_r}|p_7(x)| &\le Cr^2\rho^{-\frac32}A_*^{\frac12}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12}\\ &\le Cr\rho^{\frac12}A_*^{\frac12}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12}.\end{aligned}$$

We use the fact that $r \le \frac12\rho$, when $|x|\le r$

$$\Rightarrow \int_{B_r}|p_7(x)| \le Cr\rho^{\frac12}A_*^{\frac12}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12}. \tag{5.68}$$

And, since $p_5 = p_6 + p_7$, by adding (5.67) and (5.68) we take

$$\int_{B_r}|p_6| + \int_{B_r}|p_7| \le Cr\rho^{\frac12}A_*^{\frac12}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac12}$$

$$\Rightarrow \left(\int_{B_r} |p_5|\right)^{\frac{5}{4}} \leq C r^{\frac{5}{4}} \rho^{\frac{5}{8}} A_*^{\frac{5}{8}}(\rho) \left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{5}{8}}.$$

Integrating by t, we get

$$\Rightarrow \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \left(\int_{B_r} |p_5|\right)^{\frac{5}{4}} \leq C r^2 \rho^{\frac{5}{8}} A_*^{\frac{5}{8}}(\rho) \left(\int_{Q_\rho^*} |\nabla u|^2\right)^{\frac{5}{8}}$$

and dividing by $r^{-\frac{13}{4}}$

$$\Rightarrow r^{-\frac{13}{4}} \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \left(\int_{B_r} |p_5|\right)^{\frac{5}{4}} \leq C r^2 r^{-\frac{13}{4}} \rho^{\frac{5}{8}} A_*^{\frac{5}{8}}(\rho) \left(\int_{Q_\rho^*} |\nabla u|^2\right)^{\frac{5}{8}}$$

$$\Rightarrow r^{-\frac{13}{4}} \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \left(\int_{B_r} |p_5|\right)^{\frac{5}{4}} \leq C \left(\frac{\rho}{r}\right)^{\frac{5}{4}} A_*^{\frac{5}{8}}(\rho) \delta_*^{\frac{5}{8}}(\rho). \tag{5.69}$$

Thus, by adding (5.66) and (5.69)

$$\Rightarrow r^{-\frac{13}{4}} \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \left(\int_{B_r} |p|\right)^{\frac{5}{4}} \leq C \left[\left(\frac{r}{\rho}\right)^{\frac{1}{2}} K_*(\rho) + \left(\frac{\rho}{r}\right)^{\frac{5}{4}} A_*^{\frac{5}{8}}(\rho) \delta_*^{\frac{5}{8}}(\rho)\right].$$

This completes the proof of Lemma 5.3.6. □

It remains to find an estimate for $A_*(r)$. This will be accomplished with the lemma below.

Lemma 5.3.7. *If $r \leq \rho$, we have*

$$A_*(r) \leq C \left(\frac{\rho}{r}\right) \left[G_*^{\frac{2}{3}}(\rho) + H_*(\rho) + J_*(\rho)\right]. \tag{5.70}$$

Proof. Using generalized energy inequality for a test function ϕ such that, $\phi \in C_c^\infty(Q_\rho^*)$, with $0 \leq \phi \leq 1$ and $\phi = 1$ on Q_r^* with

$$|\nabla \phi| \leq \frac{c}{\rho} \quad \text{and} \quad |\phi_t| + |\Delta \phi| \leq \frac{c}{\rho^2}. \tag{5.71}$$

Then, for $-\frac{7}{8}r^2 < t \leq \frac{1}{8}r^2$ we have

$$\int_{B_r \times \{t\}} |u|^2 \phi \leq \int_{Q_\rho^*} |u|^2 (\phi_t + \Delta \phi) + \int_{Q_\rho^*} (|u|^2 + 2p) u \cdot \nabla \phi \tag{5.72}$$

and since u is divergence free $\Leftrightarrow \nabla \cdot u = 0$ we have that

$$\int_{Q_\rho^*} |u|^2 u \cdot \nabla \phi = \int_{Q_\rho^*} (|u|^2 + |\bar{u}|_\rho^2) u \cdot \nabla \phi. \tag{5.73}$$

Thus, the equation (5.72) becomes

$$\begin{aligned}\int_{B_r \times \{t\}} |u|^2 \phi &\leq \int_{Q_\rho^*} |u|^2 (|\phi_t| + |\Delta \phi|) + \int_{Q_\rho^*} (|u|^2 u \cdot \nabla \phi + \int_{Q_\rho^*} 2|u||p||\nabla \phi| \\ &\leq \int_{Q_\rho^*} \frac{c}{\rho^2} |u|^2 + \int_{Q_\rho^*} \left||u|^2 - |\bar{u}|_\rho^2\right| |u| \underbrace{|\nabla \phi|}_{\leq \frac{c}{\rho}} + \int_{Q_\rho^*} \frac{2c}{\rho} |u||p|\end{aligned}$$

$$\Rightarrow \int_{B_r\times\{t\}} |u|^2\phi \le c\left[\rho^{-2}\int_{Q^*_\rho}|u|^2 + \rho\rho^{-2}\int_{Q^*_\rho}\left||u|^2-\bar{|u|}^2_\rho\right||u| + \rho\rho^{-2}\int_{Q^*_\rho}|u||p|\right]$$

$$\Rightarrow \int_{B_r\times\{t\}} |u|^2\phi = c\left[\rho^{-2}\int_{Q^*_\rho}|u|^2 + \rho H_*(\rho) + \rho J_*(\rho)\right].$$

But, by using the Hölder inequality we get

$$\rho^{-2}\int_{Q^*_\rho}|u|^2 \le c\rho^{-\frac{1}{3}}\left(\int_{Q^*_\rho}|u|^3\right)^{\frac{2}{3}} = c\rho G_*^{\frac{2}{3}}(\rho).$$

Therefore, by the two previous relations

$$\begin{aligned}\Rightarrow \int_{B_r\times\{t\}} |u|^2\phi &\le c\left[\rho^{-2}\int_{Q^*_\rho}|u|^2 + \rho H_*(\rho) + \rho J_*(\rho)\right]\\ &\le c\left[\rho G_*^{\frac{2}{3}}(\rho) + \rho H_*(\rho) + \rho J_*(\rho)\right]\end{aligned}$$

$$\Rightarrow r^{-1}\int_{B_r\times\{t\}} |u|^2\phi \le c\left(\frac{\rho}{r}\right)\left[G_*^{\frac{2}{3}}(\rho) + H_*(\rho) + J_*(\rho)\right].$$

Thus,

$$\Rightarrow A^*(r) \le c\left(\frac{\rho}{r}\right)\left[G_*^{\frac{2}{3}}(\rho) + H_*(\rho) + J_*(\rho)\right],$$

which is the conclusion of the lemma. □

Now, having estimates for H_*, G_*, J_*, K_* and A_*, we are ready to prove the Proposition 5.3.1. For convenience, we state again the proposition.

Proposition. 5.3.1. *Let* (u,p) *a suitable weak solution of Navier-Stokes equation with* $\delta_*(\rho)\le 1$. *Then, for* $r \le \frac{1}{4}\rho$ *it follows that*

$$M_*(r) \le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2\left[M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}} + M_*(\rho)\delta_*(\rho) + \delta_*(\rho)\right]\right]. \tag{5.74}$$

Proof. (Proposition 5.3.1)

By Lemma 5.3.7 we have

$$\begin{aligned}A_*\left(\frac{1}{2}\rho\right) &\le C\left(\frac{\rho}{\frac{1}{2}\rho}\right)\left[G_*^{\frac{2}{3}}(\rho) + H_*(\rho) + J_*(\rho)\right]\\ &\le CM_*(\rho),\end{aligned}$$

since $\frac{1}{2}\rho \le \rho$. If $r \le \frac{1}{4}\rho$, we can use Lemmas 5.57-5.3.7. We will do so, in order to estimate $M_*(r)$ in terms of $M_*(\frac{1}{2}\rho)$, $A_*(\frac{1}{2}\rho)$ and $\delta_*(\frac{1}{2}\rho)$.

Let $\delta_*(\rho)\le 1$ by the hypothesis of proposition. Then,

$$M_*\left(\frac{1}{2}\rho\right) \le CM_*(\rho) \quad \text{and} \quad \delta_*\left(\frac{1}{2}\rho\right) \le C\delta_*(\rho).$$

By Lemma 5.3.4, we have

$$G_*^{\frac{2}{3}}(r) \leq C\left[\left(\frac{r}{\rho}\right)^{3\frac{2}{3}} A_*^{\frac{3}{2}\frac{2}{3}}(\frac{1}{2}\rho) + \left(\frac{\rho}{r}\right)^{3\frac{2}{3}} A_*^{\frac{3}{4}\frac{2}{3}}(\frac{1}{2}\rho)\delta_*^{\frac{3}{4}\frac{2}{3}}(\rho)\right]$$
$$= C\left[\left(\frac{r}{\rho}\right)^{2} A_*(\frac{1}{2}\rho) + \left(\frac{\rho}{r}\right)^{2} A_*^{\frac{1}{2}}(\frac{1}{2}\rho)\delta_*^{\frac{1}{2}}(\rho)\right]$$

$$\Rightarrow G_*^{\frac{2}{3}}(r) \leq C\left[\left(\frac{r}{\rho}\right)^{2} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right]. \tag{5.75}$$

Furthermore, by Lemma 5.57

$$H_*(r) \leq C\big(G_*^{\frac{2}{3}}(r) + A_*(r)\delta_*(r)\big)$$

and, by Lemma 5.3.7

$$A_*(r) \leq C\left(\frac{\rho}{r}\right)\left[G_*^{\frac{2}{3}}(\rho) + H_*(\rho) + J_*(\rho)\right]$$
$$\leq C\left(\frac{\rho}{r}\right)M_*(\rho)$$

and

$$\delta_*(r) = r^{-1}\int_{Q_r^*} |\nabla u|^2 \leq r^{-1}\rho\rho^{-1}\int_{Q_\rho^*} |\nabla u|^2 = \left(\frac{\rho}{r}\right)\delta_*(\rho)$$
$$\Rightarrow \delta_*(r) \leq \left(\frac{\rho}{r}\right)\delta_*(\rho).$$

Therefore,

$$H_*(r) \leq C\left[G_*^{\frac{2}{3}}(r) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)\delta_*(\rho)\right]. \tag{5.76}$$

Adding (5.75) and (5.76)

$$G_*^{\frac{2}{3}}(r) + H_*(r) \leq C\left[\left(\frac{r}{\rho}\right)^{2} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2}\left[M_*(\rho)\delta_*(\rho) + M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right]\right]. \tag{5.77}$$

We will use Lemma 5.3.5 to find an estimate for J_*. We have that

$$A_*^{\frac{1}{5}}(\frac{1}{2}\rho)G_*^{\frac{1}{5}}(r)K_*^{\frac{4}{5}}(\rho) \leq C\left[A_*(\frac{1}{2}\rho) + G_*^{\frac{2}{3}}(r) + K_*^{\frac{8}{5}}(\rho)\right]$$

and since $\frac{r}{\rho} \leq 1$

$$\Rightarrow \left(\frac{r}{\rho}\right)^{\frac{1}{5}} A_*^{\frac{1}{5}}(\frac{1}{2}\rho)G_*^{\frac{1}{5}}(r)K_*^{\frac{4}{5}}(\rho) \leq C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + G_*^{\frac{2}{3}}(r)\right]. \tag{5.78}$$

For the rest bound of J_* we have

$$\Rightarrow \left(\frac{\rho}{r}\right)^{2} A_*^{\frac{1}{2}}(\frac{1}{2}\rho)\delta_*(\frac{1}{2}\rho) \leq C\left(\frac{\rho}{r}\right)^{2} M_*^{\frac{1}{2}}(\rho)\delta_*(\frac{1}{2}\rho)$$
$$\leq C\left(\frac{\rho}{r}\right)^{2} M_*^{\frac{1}{2}}(\rho)\delta_*(\rho).$$

By Lemma 5.3.5 we have

$$
\begin{aligned}
J_*(r) &\le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} A_*^{\frac{1}{5}}(\rho) G_*^{\frac{1}{5}}(r) K_*^{\frac{4}{5}}(\rho) + \left(\frac{\rho}{r}\right)^2 A_*^{\frac{1}{2}}(\rho)\delta_*(\rho)\right] \\
&\le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + G_*^{\frac{2}{3}}(r) + \left(\frac{\rho}{r}\right)^2 M_*^{\frac{1}{2}}(\rho)\delta_*(\rho)\right] \\
&\le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho) + \left(\frac{\rho}{r}\right)^2 M_*^{\frac{1}{2}}(\rho)\delta_*(\rho)\right].
\end{aligned}
$$

Hence,

$$
J_*(r) \le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right],
$$

since $\frac{r}{\rho} \le 1$. To finish the proof, it remains to estimate $K_*^{\frac{8}{5}}$. By Lemma 5.3.6 we get

$$
\begin{aligned}
K_*^{\frac{8}{5}}(r) &\le C\left[\left(\frac{r}{\rho}\right)^{\frac{4}{5}} K_*^{\frac{8}{5}}(\rho) + \left(\frac{\rho}{r}\right)^2 A_*(\frac{1}{2}\rho)\delta_*(\rho)\right] \\
&\le C\left[\left(\frac{r}{\rho}\right)^{\frac{4}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*(\rho)\delta_*(\rho)\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
M_*(r) &= G_*^{\frac{2}{3}}(r) + H_*(r) + J_*(r) + K_*^{\frac{8}{5}}(r) \\
&\le C\left[\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left[M_*(\rho)\delta_*(\rho) + M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right]\right] \\
&+ C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right] + C\left[\left(\frac{r}{\rho}\right)^{\frac{4}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*(\rho)\delta_*(\rho)\right] \\
&= \tilde{C}\left[\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{r}{\rho}\right)^{\frac{4}{5}} M_*(\rho)\right] + \left(\frac{\rho}{r}\right)^2 \left[M_*(\rho)\delta_*(\rho) + M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right] \\
&\le \bar{C}\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left[M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho) + M_*(\rho)\delta_*(\rho)\right]\right],
\end{aligned}
$$

since $\frac{r}{\rho} \le 1 \Rightarrow \left(\frac{r}{\rho}\right)^2 \le \left(\frac{r}{\rho}\right)^{\frac{1}{5}}$ and $\left(\frac{r}{\rho}\right)^{\frac{4}{5}} \le \left(\frac{r}{\rho}\right)^{\frac{1}{5}}$. Where, the constants C may be different at each step. This completes the proof of Proposition 5.3.1. □

The proof of Proposition 5.1.14 follows by the Proposition 5.3.1 as we showed previously.

5.4 Existence of suitable weak solutions

In this section we will prove the existence of suitable weak solutions of the initial value problem see, the appendix in *[5]*.

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \Delta u & , \mathbb{R}^3 \times (0, \infty) \\ \qquad \nabla \cdot u = 0 & \\ \qquad u(x,0) = u_0(x) & , x \in \mathbb{R}^3 \end{cases} \tag{N.S}$$

We will use the following spaces.

$H_0^1(\mathbb{R}^3) :=$ the closure of $C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ in the norm

$$\|u\|_{H^1} := (\|u\|_{L^2} + \|Du\|_{L^2})^{\frac{1}{2}}$$

That is,

$$u \in H_0^1(\mathbb{R}^3) :\Leftrightarrow \exists u_m \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3) \text{ s. th. } \underbrace{u_m \to u \quad \text{with the norm } \|\cdot\|_{H^1}}_{:\Leftrightarrow \|u_m - u\|_{H^1} \to 0}$$

$H^{-1}(\mathbb{R}^3) := \{f : H_0^1(\mathbb{R}^3) \to \mathbb{R} : \ f \text{ linear and bounded}\}$, which is the dual space of $H_0^1(\mathbb{R}^3)$

with norm

$$\|f\|_{H_0^1(\mathbb{R}^3)} := \sup\{(f,u) \ : \ u \in H_0^1(\mathbb{R}^3) \text{ and } \|u\|_{H^1} \leq 1\}$$

$\mathcal{V} := C_c^\infty(\mathbb{R}^3; \mathbb{R}^3) \cap \{u : \ \operatorname{div} u = 0\} = \{u \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3) \ : \ \operatorname{div} u = 0\}$

$H :=$ the closure of $\mathcal{V}$ in the space $L^2(\mathbb{R}^3)$

That is,

$$\begin{aligned} u \in H &:\Leftrightarrow \exists u_m \in \mathcal{V} \text{ s. th. } u_m \to u \text{ in } L^2(\mathbb{R}^3) \\ &\Leftrightarrow \exists u_m \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3) \text{ with } \operatorname{div} u = 0 \text{ s. th. } \|u_m - u\|_{L^2(\mathbb{R}^3)} \to 0 \end{aligned}$$

$V :=$ the closure of $\mathcal{V}$ in the norm

$$\|u\|_{H^1} := (\|u\|_{L^2} + \|Du\|_{L^2})^{\frac{1}{2}}$$

$V^* := \{f : V \to \mathbb{R} : \ f \text{ linear and bounded}\}$, which is the dual space of V

with norm

$$\|f\|_V := \sup\{(f,u) \ : \ u \in V \text{ and } \|u\|_{H^1} \leq 1\}$$

Furthermore, we will define

$$D := \mathbb{R}^3 \times (0,T),$$
$$E_0(u) := \mathrm{esssup}_{0<t<T} \int_{\mathbb{R}^3} |u|^2,$$
$$E_1(u) := \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2.$$

The theorem of existence that we want to prove is the following

Theorem 5.4.1. *Let $u_0 \in H$, then, there exists a weak solution (u,p) of the Navier-Stokes system on D such that,*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \tag{5.79}$$

$$u(t) \rightharpoonup u_0 \quad \text{as } t \to 0, \text{ in } H \tag{5.80}$$

$$p \in L^{\frac{5}{3}}(D) \tag{5.81}$$

If $\phi \in C^\infty(\bar{D})$, $\phi \geq 0$ and $\phi(\cdot,t) \in C_c^\infty(\mathbb{R}^3)$ for $0<t<T$, then for $0<t<T$

$$\int_{\mathbb{R}^3\times\{t\}} |u|^2\phi + 2\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2\phi \leq \int_{\mathbb{R}^3} |u_0|^2\phi(x,0) + \int_0^t \int_{\mathbb{R}^3} |u|^2(\phi_t + \Delta\phi) + \int_0^t \int_{\mathbb{R}^3} (|u|^2 + 2p)u \cdot \nabla\phi \tag{5.82}$$

Remark 5.4.2. This inequality is a form of the generalized energy inequality.

Now, we will state a lemma for whose proof we refer to Lemma 1.2, Ch.3, in [33].

Lemma 5.4.3. *Let $V, H, V^{'}$ three Hilbert spaces such that, $V \subset H \subset V^{'}$. If $u \in L^2(0,T;V)$ and its derivative $v_t \in L^2(0,T;V^{'})$. Then $u \equiv v$, where $v \in C([0,T];H)$ and*

$$\frac{d}{dt}|u|^2 = 2(u_t, u)$$

holds true in the scalar distribution on $(0,T)$.

Lemma 5.4.4. *Let $u \in L^2(0,T;V)$, $f \in L^2(0,T;V^{'})$, p a distribution and*

$$u_t - \Delta u + \nabla p = f \tag{5.83}$$

in the sense of distributions on D. Then

$$u_t \in L^2(0,T;V^{'}) \tag{5.84}$$

$$\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 = 2\int_{\mathbb{R}^3} (u_t, u) \tag{5.85}$$

and

$$u \in C([0,T];H) \tag{5.86}$$

after a modification in a set of measure zero. Furthermore, the solution of (5.83) is unique in the space $L^2(0,T;V)$, for a given $u_0 \in H$.

Proof. Let $u \in L^2(0,T;V)$ and let (5.83) holds true on D, in the sense of distribution. We want to prove that $u_t \in L^2(0,T;V^{'})$.

By the assumptions we have that $u \in L^2(0,T;V)$, $f \in L^2(0,T;V^{'})$ and $u_t = \Delta u - \nabla p + f$ on D. Thus, it follows that $u_t \in L^2(0,T;V^{'})$.
Therefore, having in addition (5.84), the relations (5.85) and (5.86) follows by the previous lemma.

For the uniqueness of the solution,

let u, v be two solutions of (5.83) with the same initial data $u(0) = u_0 \in H$ and define $w := u - v$. Then,

$$w \in L^2(0,T;V)$$

and

$$w_t - \Delta w = 0$$

with

$$w(0) = 0.$$

Taking the inner product with $w(t)$ we get

$$\begin{aligned} &(w_t(t), w(t)) + \|w(t)\|^2 = 0 \quad \text{a.e.} \\ &\Rightarrow \frac{d}{dt}|w(t)|^2 + 2\|w(t)\|^2 = 0 \\ &\Rightarrow \frac{d}{dt}|w(t)|^2 = -2\|w(t)\|^2 \le 0. \end{aligned}$$

Hence, $|w(t)|^2$ is a decreasing function of t.

$$\Rightarrow |w(t)|^2 \le |w(0)|^2 = 0, \quad t \in [0,T].$$

Thus, $u \equiv v$. □

Lemma 5.4.5. *Let $u_0 \in H$ and $w \in C^\infty(\bar{D};\mathbb{R}^3)$ with $\nabla \cdot w = 0$. Then, there exist a unique function u and a distribution p such that,*

$$u \in C([0,T];H) \cap L^2(0,T;V) \tag{5.87}$$

$$u_t + w \cdot \nabla u - \Delta u + \nabla p = 0 \tag{5.88}$$

in the sense of distributions on D and

$$u(0) = u_0. \tag{5.89}$$

Proof. Suppose that there exists a function u with the following properties

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \tag{5.90}$$

and $\forall v \in V$

$$\frac{d}{dt}\int_{\mathbb{R}^3}(u,v) + \int_{\mathbb{R}^3}(w \cdot \nabla u) + \int_{\mathbb{R}^3}(\nabla u, \nabla v) = 0. \tag{5.91}$$

in the sense of distributions on $(0,T)$, and $u(0) = u_0$.

To prove the uniqueness of the function u, it suffices to proof that if $u_0 = 0$ implies $u \equiv 0$. Let $u_0 = 0$. Then, by (5.85)

$$\Rightarrow \frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 = 2\int_{\mathbb{R}^3}(u_t, u) = 2\int_{\mathbb{R}^3}(u, -w\cdot\nabla u + \Delta u)$$
$$\Rightarrow \frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 = -2\int_{\mathbb{R}^3}(u, -\Delta u),$$

since $\nabla \cdot w = 0$. By integration by parts it follows

$$\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 = -2\left(\int_{\partial\mathbb{R}^3}(u, -\nabla u) - \int_{\mathbb{R}^3}(\nabla u, -\nabla u)\right)$$
$$= -2\int_{\mathbb{R}^3} |\nabla u|^2 \le 0.$$

Hence, $\int_{\mathbb{R}^3} |u|^2$ is a decreasing function of t.

$$\Rightarrow \int_{\mathbb{R}^3} |u|^2 \le \int_{\mathbb{R}^3} |u_0|^2 = 0.$$

Thus, $u \equiv 0$.

□

Lemma 5.4.6. *Let $u_0 \in H$. Then, in the context of Lemma 5.4.5 the pressure satisfies the equation*

$$\Delta p = -\sum_{i,j} \nabla_{i,j}(w^i u^j) \tag{5.92}$$

whence it follows

$$\int_D |p|^{\frac{5}{3}} \le C\int_D |w|^{\frac{5}{3}}|u|^{\frac{5}{3}}. \tag{5.93}$$

Proof. Taking the divergence of equation (5.88) it follows

$$\nabla\cdot u_t + \nabla\cdot(w\cdot\nabla u) - \nabla\cdot\Delta u + \nabla\cdot\nabla p = 0,$$

where

$$\nabla\cdot(w\cdot\nabla u) = \operatorname{div}(w\cdot\nabla u) = \partial_j(w\cdot\nabla u^j) = \partial_j(w^i\partial_i u^j) = \partial_j((\partial_i w^i)u^j + w^i\partial_i u^j) = \partial_j\partial_i(w^j u^i)$$

since $\operatorname{div} w = 0$, but also

$$= (\partial_j w^i)\partial_i u^j + w^i\partial_i\partial_j u^j = (\partial_j w^i)\partial_i u^j,$$

since $\operatorname{div} u = 0$.

Thus, we have

$$\Leftrightarrow \Delta p = -\sum_{i,j}\nabla_{i,j}(w^i u^j),$$

which is (5.92). From which we deduce (5.93), since the integral which solve the (5.92) is bounded in the space $L^{\frac{5}{3}}(\mathbb{R}^3)$. □

Lemma 5.4.7. *Let $u, w \in L^2(0,T;H^1(\mathbb{R}^3))$. Then*

$$\|w\cdot\nabla u\|_{L^{\frac{5}{4}}} \le CE_1^{\frac{1}{2}}(u)E_1^{\frac{3}{10}}(w)E_0^{\frac{1}{5}}(w) \tag{5.94}$$

$$\|u\|_{L^{\frac{10}{3}}} \leq C E_1^{\frac{3}{10}}(u) E_0^{\frac{1}{5}}(u) \tag{5.95}$$

$$\|u\|_{L^5(0,T;L^{\frac{5}{2}})} \leq C T^{\frac{1}{20}} E_0^{\frac{7}{20}}(u) E_1^{\frac{3}{20}}(u) \tag{5.96}$$

Proof. By using the interpolation inequality (5.3). □

Lemma 5.4.8. *Let $u_0 \in H$ and $w \in C^\infty(\bar{D}; \mathbb{R}^3)$ with $\nabla \cdot w = 0$. Furthermore, let that (u,p) is a solution of (5.87), (5.88), (5.89). Then, $\forall \phi \in C^\infty(\bar{D}; \mathbb{R})$ with $\phi \in C_c^\infty(\mathbb{R}^3)$ and $\forall t : 0 < t \leq T$*

$$\int_{\mathbb{R}^3\times\{t\}} |u|^2\phi + 2\int_D |\nabla u|^2\phi = \int_{\mathbb{R}^3} |u_0|^2\phi(x,0) + \int_D |u|^2(\phi_t + \Delta\phi) + \int_D (|u|^2 w + 2pu)\cdot\nabla\phi. \tag{5.97}$$

Proof. Let $u_0 \in H$ and $\phi \in C_c^\infty(\mathbb{R}^3)$. Define as $F := -w \cdot \nabla u$. Thus, we have

$$u_t - \Delta u + \nabla p = F, \quad \text{on } D. \tag{5.98}$$

Mollifying equation (5.98), see, e.g., [10], Thm 6, Appendix C, then we get smooth functions $\{u_m\}$, $\{p_m\}$, $\{F_m\}$, $m \in \mathbb{N}$ which approach each term of equation (5.98) and they are such that

$$\begin{cases} \frac{d}{dt}u_m - \Delta u_m + \nabla p_m &= F_m \\ \nabla \cdot u_m &= 0, \end{cases} \tag{5.99}$$

in a neighborhood of suppϕ and

$$\begin{aligned} u_m &\to u \text{ in } L^5(0,T; L^{\frac{5}{2}}(\mathbb{R})^3) \cap L^2(D), \\ \nabla u_m &\to \nabla u \text{ in } L^{\frac{5}{4}}(D), \\ p_m &\to p \text{ in } L^{\frac{5}{3}}(D), \\ F_m &\to F \text{ in } L^2(D). \end{aligned}$$

Now, take inner product in (5.99) with $2u_m\phi$ to get

$$\Big(\frac{d}{dt}u_m, 2u_m\phi\Big) - (\Delta u_m, 2u_m\phi) + (\nabla p_m, 2u_m\phi) = (F_m, 2u_m\phi).$$

Integrating over D and then doing integration by parts to get

$$2\int_D \Big(\frac{d}{dt}u_m, u_m\phi\Big) - 2\int_D (\Delta u_m, u_m\phi) + 2\int_D (\nabla p_m, u_m\phi) = 2\int_D (F_m, u_m\phi)$$

$$\Rightarrow 2\int_D |\nabla u_m|^2\phi = \int_D |u_m|^2(\phi_t + \Delta\phi) + 2\int_D p_m(u_m\cdot\nabla\phi) + 2\int_D (u_m \cdot F_m)\phi. \tag{5.100}$$

Now, take the limit $n \to \infty$ and using Dominated Convergence Theorem we have

$$2\int_D |\nabla u|^2\phi = \int_D |u|^2(\phi_t + \Delta\phi) + 2\int_D p(u\cdot\nabla\phi) + 2\int_D (u \cdot F)\phi$$

and, since $F := -w \cdot \nabla u$, we get

$$2\int_D (u \cdot F)\phi = -2\int_D u \cdot (w \cdot \nabla u)\phi = -2\int_{\partial D} |u|^2 w\phi + \int_D |u|^2 w \cdot \nabla\phi$$
$$\Rightarrow 2\int_D (u \cdot F)\phi = \int_D |u|^2 w \cdot \nabla\phi.$$

Therefore,

$$(5.100) \Rightarrow 2\int_D |\nabla u|^2\phi = \int_D |u|^2(\phi_t + \Delta\phi) + 2\int_D p(u \cdot \nabla\phi) + \int_D |u|^2 w \cdot \nabla\phi$$

which satisfies (5.97), if $\phi \in C_c^\infty(D)$ and $t = T$. □

Now, we will define a so called "retarded" mollification $\Psi_\delta(u)$ of u. Let $\psi(x,t) \in C^\infty(D;\mathbb{R}^3)$ such that, $\psi \geq 0$ and

$$\int_D \psi = 1,$$

with

$$\text{supp}\ \psi \subset \{(x,t) :\ |x|^2 < t,\ 1 < t < 2\}.$$

Now, for $u \in L^2(0,T;V)$ define $\tilde{u} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ as

$$\tilde{u} = \begin{cases} u(x,t) & (x,t) \in D, \\ 0 & \text{otherwise}, \end{cases} \tag{5.101}$$

and define the convolution $\Psi_\delta(u) := \delta^{-4}\psi(\frac{\cdot}{\delta}, \frac{\cdot}{\delta}) * u$, that is

$$\Psi_\delta(u)(x,t) := \delta^{-4}\int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta})\tilde{u}(x-y, t-\tau)dyd\tau. \tag{5.102}$$

Remark 5.4.9. The values of $\Psi_\delta(u)$ as a function of t, depend only on the values of u as a function of $\tau \in (t-2\delta, t-\delta)$. Indeed, we integrate only over $|y|^2 < \delta\tau,\ \delta < \tau < 2\delta$ and, thus, for (x,t) only the values of $\tilde{u}$ for $t-\delta > t-\tau > t-2\delta$ are taken into account. Note also that of course $\Psi_\delta(u)$ is a smooth function of (x,t).

Lemma 5.4.10. *Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$. Then*

$$\text{div}\, \Psi_\delta(u) = 0, \tag{5.103}$$

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^3} |\Psi_\delta(u)|^2 \leq CE_0(u), \tag{5.104}$$

and

$$\int_D |\nabla\Psi_\delta(u)|^2 \leq CE_1(u), \tag{5.105}$$

where C is a universal constant.

Proof. We have that $u \in L^2(0,T;V)$ and $\tilde{u}$: $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & (x,t) \in D \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\nabla \cdot \tilde{u} = 0$ on $\mathbb{R}^4 \setminus \partial D$, since $\nabla \cdot u = 0$ on D. Therefore, we deduce that $\nabla \cdot \Psi_\delta(u) = 0$, since

$$\begin{aligned}\nabla \cdot \Psi_\delta(u)(x,t) &= \delta^{-4} \nabla \cdot (\psi(\frac{\cdot}{\delta}, \frac{\cdot}{\delta}) * \tilde{u})(x,t) \\ &= \delta^{-4}(\psi(\frac{\cdot}{\delta}, \frac{\cdot}{\delta}) * \nabla \cdot \tilde{u})(x,t) = 0.\end{aligned}$$

Notice that we pass the divergence only on the function u. For a proof, see, e.g., [10], Appendix C, Theorem 6. We begin with

$$\begin{aligned}|\Psi_\delta(u)(x,t)| &= \left| \frac{1}{\delta^4} \int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) \tilde{u}(x-y, t-\tau) dy d\tau \right| \\ &\leq \frac{1}{\delta^4} \int_{\mathbb{R}^4} \psi^{1-\frac{1}{2}}(\frac{y}{\delta}, \frac{\tau}{\delta}) \psi^{\frac{1}{2}}(\frac{y}{\delta}, \frac{\tau}{\delta}) |\tilde{u}(x-y, t-\tau)| dy d\tau \\ &\leq \frac{1}{\delta^4} \left(\int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) dy d\tau \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) |\tilde{u}(x-y, t-\tau)|^2 dy d\tau \right)^{\frac{1}{2}},\end{aligned}$$

where in the second inequality we used the Hölder's inequality in $\mathbb{R}^4$.

Thus,

$$|\Psi(u)(x,t)|^2 \leq \frac{1}{\delta^8} \left(\int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) dy d\tau \right) \left(\int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) |\tilde{u}(x-y, t-\tau)|^2 dy d\tau \right).$$

Hence, by integration over $\mathbb{R}^3$ we get

$$\int_{\mathbb{R}^3} |\Psi_\delta(u)(x,t)|^2 dx \leq \frac{1}{\delta^4} \left(\int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) dy d\tau \right) \frac{1}{\delta^4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) \| \tilde{u}(x-y, t-\tau)|^2 dy d\tau dx,$$

where

$$\frac{1}{\delta^4} \left(\int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) dy d\tau \right) = \frac{1}{\delta^4} \frac{1}{4} \delta^4 \underbrace{\int_{\mathbb{R}^4} \psi(y, \tau) dy d\tau}_{=1} = \frac{1}{4}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} \int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) \| \tilde{u}(x-y, t-\tau)|^2 dy d\tau dx &= \int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) \underbrace{\int_{\mathbb{R}^3} |\tilde{u}(x-y, t-\tau)|^2 dx dy d\tau}_{\leq E_0(u), \ \forall t} \\ &\leq E_0(u) \int_{\mathbb{R}^4} \psi(\frac{y}{\delta}, \frac{\tau}{\delta}) dy d\tau.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^3} |\Psi_\delta(u)(x,t)|^2 dx \leq E_0(u) \frac{1}{4} \underbrace{\int_{\mathbb{R}^4} \psi(y, \tau) dy d\tau}_{=1},$$

and, therefore (5.104) has been proved.

Similarly, we have

$$\begin{aligned}|\nabla\Psi_\delta(u)(x,t)|^2 &\leq \left(\frac{1}{\delta^4}\int_{\mathbb{R}^4}\psi(\frac{y}{\delta},\frac{\tau}{\delta})|\nabla\tilde{u}(x-y,t-\tau)|dyd\tau\right)^2\\ &\leq \frac{1}{\delta^8}\int_{\mathbb{R}^4}\psi(\frac{y}{\delta},\frac{\tau}{\delta})dyd\tau\int_{\mathbb{R}^4}\psi(\frac{y}{\delta},\frac{\tau}{\delta})|\nabla\tilde{u}(x-y,t-\tau)|^2dyd\tau.\end{aligned}$$

Hence, by integrating over D we get

$$\begin{aligned}\int_D|\nabla\Psi_\delta(u)|^2 &\leq \frac{1}{\delta^4}\int_{\mathbb{R}^4}\psi(\frac{y}{\delta},\frac{\tau}{\delta})dyd\tau\int_{\mathbb{R}^4}\frac{1}{\delta^4}\int_{\mathbb{R}^4}\psi(\frac{x-y}{\delta},\frac{t-\tau}{\delta})dxdt|\nabla\tilde{u}(y,\tau)|^2dyd\tau\\ &\leq \frac{1}{16}E_1(u),\end{aligned}$$

with the same arguments as above and, the relation (5.105) has been proved.

□

Now, we are ready to give the proof of the existence theorem. More precisely, the idea of the proof is the following: For fixed $N>0$ we set $\delta := \frac{T}{N}$ and we find $u=u_N$ and $p=p_N$ which satisfy the equation

$$u_t + \Psi_\delta(u)\cdot\nabla u - \Delta u + \nabla p = 0 \quad \text{on } D. \tag{5.106}$$

To solve (5.106), it is equivalent to solve a linear equation in each strip $\mathbb{R}^3\times(m\delta,(m+1)\delta)$, for $0\leq m\leq N-1$. We will show that the solution of (5.106) satisfies an inequality, analogous to (5.82) and we will find estimates for u_N and p_N which are independent of N. These inequalities will assert the existence of a convergent subsequence the limit of which has the properties (5.79)-(5.82). We will start now with the proof of Theorem 5.4.5 which has been outlined above.

Proof. Let $\delta := \frac{T}{N}$, $\forall N\in\mathbb{N}$ and solve the system

$$\frac{d}{dt}u_N + \Psi_\delta(u_N)\cdot\nabla u_N - \Delta u_N + \nabla p_N = 0 \tag{5.107}$$

$$u_N \in L^2(0,T;V)\cap C([0,T];H) \tag{5.108}$$

$$u_N(0) = u_0. \tag{5.109}$$

We observe that u_N and p_N exist, by Lemma 5.4.5, since $w =: \Psi_\delta(u_N)\in C^\infty(\mathbb{R}^3\times[0,T];\mathbb{R})$, with $\nabla\cdot\Psi_\delta(u_N)=0$ and $u_0\in H$, and in fact, these are unique, using Lemma 5.4.5 in each time interval $(m\delta,(m+1)\delta)$, $0\leq m\leq N-1$.
Taking inner product of (5.107) with u_N and integrating over $\mathbb{R}^3$ we get

$$\begin{aligned}&\int_{\mathbb{R}^3}(\frac{d}{dt}u_N,u_N)+\int_{\mathbb{R}^3}(\Psi_\delta(u_N)\cdot\nabla u_N,u_N)-\int_{\mathbb{R}^3}(\Delta u_N,u_N)+\int_{\mathbb{R}^3}(\nabla p_N,u_N)=0\\ \Rightarrow&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|u_N|^2+\int_{\mathbb{R}^3}(\Psi_\delta(u_N)\cdot\nabla u_N,u_N)-\int_{\mathbb{R}^3}(\Delta u_N,u_N)+\int_{\mathbb{R}^3}(\nabla p_N,u_N)=0.\end{aligned}$$

By applying the relation (5.85). On the other hand

$$-\int_{\mathbb{R}^3}(\Delta u_N,u_N) = \underbrace{-\int_{\partial\mathbb{R}^3}(\nabla u_N,u_N)}_{=0}+\int_{\mathbb{R}^3}|\nabla u_N|^2.$$

Thus, the equation becomes

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|u_N|^2+\int_{\mathbb{R}^3}|\nabla u_N|^2=0.$$

Integrating, now, over time

$$\begin{aligned}
&\Rightarrow \frac{1}{2}\int_0^t \frac{d}{dt}\int_{\mathbb{R}^3}|u_N|^2+\int_0^t\int_{\mathbb{R}^3}|\nabla u_N|^2=0\\
&\Rightarrow \frac{1}{2}\int_{\mathbb{R}^3}|u_N(t)|^2-\frac{1}{2}\int_{\mathbb{R}^3}|u_N(0)|^2+\int_0^t\int_{\mathbb{R}^3}|\nabla u_N|^2=0\\
&\Rightarrow \int_{\mathbb{R}^3\times\{t\}}|u_N|^2+2\int_0^t\int_{\mathbb{R}^3}|\nabla u_N|^2=\int_{\mathbb{R}^3}|u_0|^2,
\end{aligned}$$

for $t\in(0,T)$. Also, the integral on the right hence side is finite, since $u_0\in H$. Therefore, we conclude that u_N remains bounded in the space

$$L^\infty(0,T;H)\cap L^2(0,T;V). \tag{5.110}$$

Define now, for simplicity, the space V_2 := the closure of $\mathcal{V}$ in the space $H^2(\mathbb{R}^3)$ and let the V_2^* := the dual. space of V_2. Hence, by Lemma Temam (see below) and the relations (5.107) and (5.110) we deduce that $\frac{d}{dt}u_N$ remains bounded in the space

$$L^2(0,T;V_2^*). \tag{5.111}$$

Therefore, applying Theorem 2.1 in Temam (see below), since $u_N\in L^2(0,T;V)$ and $\frac{d}{dt}u_N\in L^2(0,T;V_2^*)$, we conclude that $\{u_N\}$ remains in a compact subset of $L^2(D)$. That fact, with addition (5.110) and Lemmas 5.4.6, 5.4.7 and 5.4.10, where $u=u_N$ and $w=\Psi_\delta(u_N)$, give the conclution that $\{p_N\}$ remains bounded in the space $L^{\frac{5}{3}}(D)$. And since $\{u_N\}$, $\{p_N\}$ are bounded, it follows that there exist convergent subsequences in each space respectively. That is, there exist $\{u_{k_N}\}$ and $\{p_{k_N}\}$ subsequences of $\{u_N\}$ and $\{p_N\}$ such that

$$u_{k_N}\to u_* \tag{5.112}$$

strongly in $L^2(D)$, weakly in $L^2(0,T;V)$ and weakly* in $L^\infty(0,T;H)$ and

$$p_{k_N}\to p_* \tag{5.113}$$

weakly in $L^{\frac{5}{3}}(D)$.

Remark 5.4.11. We known, see, e.g., Lemma 3.4 in weak convergence file, that if $1\le q<r$ with $u_{k_N}\to u_*$ strongly in $L^q(D)$ and $\{u_{k_N}\}$ is bounded in $L^r(D)$. Then $u_{k_N}\to u_*$ strongly in $L^s(D)$ for $q<s<r$.

Now, using the remark above for $q=2$ and $r=\frac{10}{3}$ we have that

$$u_{k_N}\to u_* \tag{5.114}$$

strongly in $L^s(D)$, for $2\le s<\frac{10}{3}$ and, by the definition of Ψ_δ it holds that

$$\Psi_\delta(u_{k_N})\to u_* \tag{5.115}$$

strongly in $L^s(D)$, $2\le s<\frac{10}{3}$.
Therefore, we conclude, by the relations (5.112)-(5.115) that u_* is a weak solution of the Navier-Stokes system.

Now, since by (5.111) we have that $\frac{d}{dt}u_N$ is bounded in the space $L^2(0,T;V_2')$, it follows that $\{u_N\}$ is a Lipschitz function. Thus, $\{u_N\}$ is uniformly continuous as a function of time t. That is,

$$u_*(0) = \lim_{N\to\infty} u_N(0) = \lim_{N\to\infty} u_0 = u_0.$$

Recall that a well-known property of a weak solution u is the weak continuity of u as a function of time. That is,

$$\int_\Omega u(x,t)\cdot W(x)dx \to \int_\Omega u(x,t_0)\cdot W(x)dx,$$

$\forall\ W \in L^2(\Omega)$, as $t \to t_0 \in [0,T]$.

Therefore, $u(t) \rightharpoonup u_0$ as $t \to 0$ in the space H, since by the above, weak solutions are weak continuous in H. This completes the prove of (5.1).
To prove (5.82), it suffices to show the case where $t = 0$ and $\phi = 0$ on $\mathbb{R}^3 \times \{T\}$. Then the general case follows.

Lets consider that ϕ is smooth function, $\phi \geq 0$ and that $\phi = 0$ on $\{|x| > R\} \cup \{t = T\}$. With the same process as in (5.100),

$$(5.99) \Rightarrow 2\int_D |\nabla u_N|^2\phi = \int_{\mathbb{R}^3} |u_0|^2\phi + \int_D |u_N|^2(\phi_t+\Delta\phi) + 2\int_D p_N(u_N\cdot\nabla\phi) + \int_D |u_N|^2\Psi_\delta(u_N)\cdot\nabla\phi. \tag{5.116}$$

Sending $N \to \infty$,

$$\int_D |\nabla u_N|^2\phi$$

is lower semicontinuous function. That is

$$2\int_D |\nabla u_*|^2\phi \leq \int_{\mathbb{R}^3} |u_0|^2\phi + \int_D |u_*|^2(\phi_t + \Delta\phi) + 2\int_D p_*(u_*\cdot\nabla\phi) + \int_D |u_*|^2\Psi_\delta(u_*)\cdot\nabla\phi$$

$$\Rightarrow 2\int_D |\nabla u_*|^2\phi \leq \int_{\mathbb{R}^3} |u_0|^2\phi + \int_D |u_*|^2(\phi_t + \Delta\phi) + \int_D (|u_*|^2 + 2p_*)u_*\cdot\nabla\phi,$$

which proves (5.82) and this completes the proof of Theorem 5.4.5.

□

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