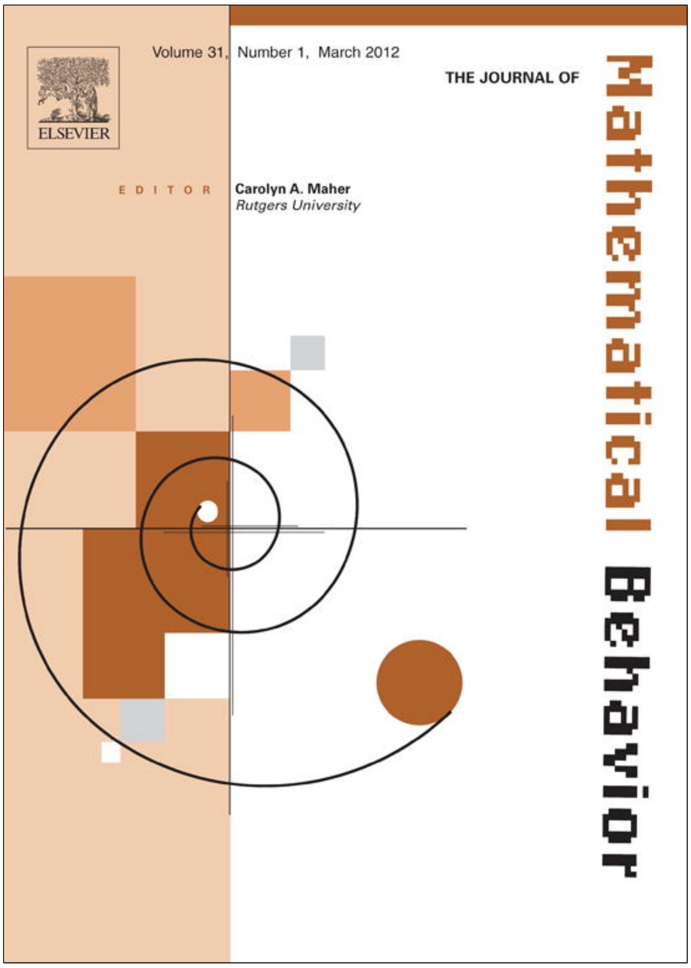

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Naturally biased? In search for reaction time evidence for a natural number bias in adults

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ABSTRACT

A major source of errors in rational number tasks is the inappropriate application of natural number rules. We hypothesized that this is an instance of intuitive reasoning and thus can persist in adults, even when they respond correctly. This was tested by means of a reaction time method, relying on a dual process perspective that differentiates between intuitive and analytic reasoning. We measured fifty-eight educated adults' accuracies and reaction times in a variety of rational number tasks. In half of the items (congruent), the correct response was compatible with natural number properties (thus intuitive reasoning led to a correct answer). In contrast, in the incongruent items, intuitive reasoning would lead to an incorrect answer. In comparing two numbers, there were hardly any natural-number-based errors but correct responses to incongruent items took longer. Regarding the effect of operations, more mistakes were made in incongruent items, and correct responses required longer reaction time. Incongruent items about density elicited considerably more errors than congruent items. These findings can be considered as evidence that the natural number bias is an instance of intuitive reasoning.

1. Introduction

Research in cognitive-developmental psychology and mathematics education has repeatedly shown that a major source of difficulty in the learning of rational numbers[1] is the inappropriate application of natural number rules. This phenomenon is associated with the so-called "whole number bias"[2] (Ni & Zhou, 2005). Systematic errors arise when the "behavior" of rational numbers differs from that of natural numbers (Lamon, 1999; Moss, 2005; Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005; Resnick et al., 1989; Vamvakoussi & Vosniadou, 2010). Evidence for this phenomenon typically comes from paper-and-pencil tests, interviews, and classroom observations. The majority of studies in this area—and all of the above-mentioned ones—have been conducted with primary and secondary school students. Some, however, also targeted adults (e.g., Post, Harel, Behr, & Lesh, 1988; Stacey et al., 2001; Tirosh, Fischbein, Graeber, & Wilson, 1999). It appears that some of these systematic errors diminish or even disappear with age and level of instruction, whereas others are remarkably persistent.

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[1] In this article, the term "rational numbers" refers to numbers that are or can be expressed in the form a/b, where a and b are integers and b is not zero.

[2] Ni and Zhou (2005) coined the term "whole number bias" to refer to the "tendency in children to use the single-unit counting scheme applied to whole numbers to interpret instructional data on fractions" (p. 27). The term "whole numbers" customarily refers to natural numbers (the counting numbers) and zero. Because negative integers are in some sense "whole", we will use the term "natural number bias" throughout the text to stress that it is not merely "integrity" but also the positive character of natural numbers that interferes with people's attempts to deal with non-natural numbers.

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The central hypothesis in the present study is that natural number reasoning still interferes with educated adults'[3] reasoning when dealing with rational numbers tasks, even when they give correct responses and thus *appear* unaffected by a natural number bias. To test this hypothesis, we applied, besides error analysis, a new method to investigate the natural number bias, namely reaction-time methodology. This way, the interference of natural number reasoning could not only manifest itself through erroneous responses, but also through the increased time that subjects need in order to provide a *correct* response. The use of reaction time data and the underlying assumptions are inspired by dual-process accounts of reasoning (Epstein, 1994; Evans & Over, 1996; Kahneman, 2000; Sloman, 1996; Stanovich, 1999).

1.1. Natural number bias

Mastery of the rational numbers is considered an important aspect of mathematical literacy. It is, however, widely documented that learning about rational numbers presents students with many difficulties. In particular, students are found to make systematic errors when a task requires reasoning that is not in line with their prior knowledge and experience about natural numbers (Moss, 2005; Ni & Zhou, 2005; Smith et al., 2005; Vamvakoussi & Vosniadou, 2010). On the other hand, students deal effectively with rational number tasks when these are compatible with natural number strategies (e.g., Nunes & Bryant, 2008; Stafylidou & Vosniadou, 2004). New information about rational numbers that is presented in a context that allows for natural number reasoning is deemed easier for students. One such example is the part-whole aspect of fraction, which is typically used in instruction as an introduction to fractions. Students are asked to count the number of parts in a shape that represents the whole (e.g., a pie). Then they count the number of parts shaded. Subsequently, they use these counts as the basis for naming and symbolically representing fractions. This practice allows students to treat the parts of the whole as discrete objects and apply familiar counting strategies, which in turn helps them perform simple operations on fractions (e.g., to add two fractions with the same denominator). Interestingly, it is documented that over-reliance on the part whole aspect of fraction eventually becomes an obstacle to further understanding of rational number concepts (Mamede, Nunes, & Bryant, 2005; Moss, 2005). Another example is that presentation of information in terms of frequencies instead of rates facilitates probabilistic reasoning. Butterworth (2007) explains this finding as preference for natural numbers. Thus students' reliance on natural number reasoning when dealing with rational number tasks is pervasive, facilitates reasoning when it is appropriate, but has an adverse effect when it is not. Hence the term *bias*, first introduced by Ni and Zhou (2005) in relation to this phenomenon, seems justified.

There is no consensus regarding the origins of the natural number bias. Although some researchers argue that early quantitative representation is limited to discrete quantities—and thus the natural number concept is privileged—this issue is still controversial (Ni & Zhou, 2005; Rips, Blomfield, & Asmuth, 2008). It appears that some pre-instructed ideas pertaining to rational number are also present in young children, even pre-schoolers (e.g., Moss & Case, 1999). However, it is clear that the externalization and systematization of such intuitions about rational numbers is typically much less socially supported in the first years of a child's life (Greer, 2004). In contrast, the development of the natural number concept is mediated from early on by cultural representational tools [e.g., language (Carey, 2004)] and practices such as finger counting (Andres, Di Luca, & Pesenti, 2008) as well as counting songs, rhymes, and cardboard games. In addition, early instruction focuses on natural number arithmetic, thus supporting the systematization and validation of children's initial understandings of number as natural numbers.

It thus appears that, before students are introduced to rational numbers through instruction, they have already constructed rich understandings of number, which are mainly tied around their formal as well as informal knowledge of natural numbers. These understandings sustain students' beliefs about what counts as a number and how numbers are supposed to "behave" (Gelman, 2000; Smith et al., 2005; Vamvakoussi & Vosniadou, 2010). Several expectations related to the behavior of number are violated when rational numbers—in the form of decimals and fractions—are introduced in the curriculum. To make sense of these new mathematical constructs that are presented as numbers, it appears that students draw heavily on natural number knowledge. This explains why systematic errors occur precisely where the behavior of rational numbers deviates from that of the natural numbers.

In the following we discuss three kinds of rational number tasks that are found to elicit such systematic errors. These tasks will lie at the basis of our research instrument.

1.2. Comparison

Comparison tasks have been widely used to assess understanding of rational numbers. In the comparison of decimals, the literature reports a salient intrusion of prior knowledge about natural numbers underlying the judgment that "*longer decimals are larger*", which may be correct in cases such as 2.15 > 2.1, but not in cases such as 2.12 > 2.2 (Resnick et al., 1989).

[3] For the purposes of this article, the term "educated adults" is used to refer to people who posses a secondary education diploma—meaning that they have completed twelve years of formal math instruction successfully—and continue or have completed their studies (outside the domain of mathematics) at the tertiary level. Thus, they should be, at least in principle, capable of dealing with rational numbers tasks that require the knowledge and skills taught in primary and secondary school. In this sense, "educated adults" are not deemed novices—like primary school children who are beginning to learn about rational numbers—nor experts, who have been trained in mathematics at the tertiary level.

This judgment may be grounded in the observation that one of the numbers has more digits than the other—which for natural numbers characterizes a larger number. Another plausible explanation is that students compare the fractional parts of the two numbers as if they were natural numbers (i.e., 2.12 is deemed larger than 2.2, because 12 is larger than 2). It appears that this type of errors is typical at the early stages of students' encounter with decimal numbers, but tends to decrease with age (Desmet, Grégoire, & Mussolin, 2010; Stacey & Steinle, 1998, 1999) and is not common in educated adults (Stacey et al., 2001). An error that is more likely to be found even in educated adults is the "*shorter is larger*" one (Stacey et al., 2001). This error is usually explained as an intrusion of knowledge about fractions. For example, 2.3 is deemed larger than 2.32, because the fractional parts of the two numbers refer to "tenths" and "hundredths', respectively, and "*tenths are always larger than hundredths*" (see also Peled & Awawdy-Shahbari, 2009; Resnick et al., 1989).

In the comparison of fractions, a crucial factor is students' difficulty to understand that the magnitude of a fraction depends on the relation between its terms (Moss, 2005; Ni & Zhou, 2005; Smith et al., 2005). Instead, students initially tend to interpret the symbol a/b as two independent natural numbers, separated by a bar (e.g., Stafylidou & Vosniadou, 2004). This leads them to conclude that a fraction increases when its numerator, its denominator, or both increase. Focusing on each term of the fraction separately can result in correct judgments in cases such as $2/5 < 3/5$, but also in incorrect ones in cases such as $2/5 < 2/7$. Recently, there has been an interest in educated adults' processing of fractions. Neuro-psychological studies indicate that when comparing fractions, adults rely mainly on *componential* strategies, in the sense that they access the terms of the fractions separately (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Kallai & Tzelgov, 2009). Meert, Grégoire, and Noël's findings (2009, 2010) suggested that even when students (10 and 12 year olds) and also adults process fractions *holistically* (i.e., by accessing the magnitude of the whole fraction), there are still indications of componential processing. For instance, in the case of fractions with the same numerators (such as 2/5 and 2/7), there was interference of the relative magnitude of the denominators. Componential processing in the comparison of fractions may arguably trigger natural-number based reasoning, which has to be inhibited for making a correct comparison.

1.3. Arithmetical operations

Within the natural numbers set, whenever two numbers are added or multiplied, the outcome is always bigger than the initial numbers. Similarly, when two numbers are subtracted or divided, the outcome is smaller than the minuend and the dividend, respectively. None of the above is necessarily true within the rational numbers set: The effect of operations depends on the numbers involved. For example, $3 + (-5)$ is smaller than 3; and $8 \div 0.5$ is larger than 8.

In a seminal paper, Fischbein, Deri, Nello, & Marino (1985) argued that there are intuitive models of the four operations, associating addition with *putting together*, subtraction with *taking away*, multiplication with *repeated addition*, and division with *equal sharing*. These intuitive models are assumed to be implicit and to shape students' expectations about the effect of operations, as well as the role of the numbers involved. Later research challenged some of Fischbein et al.'s claims (see for example De Corte & Verschaffel, 1996; De Corte, Verschaffel, & Van Coillie, 1988; Mulligan & Mitchelmore, 1997). Nevertheless, there is plenty of evidence showing that in extending the meaning of operations from natural to non-natural numbers, the idea that "*multiplication makes bigger*" and "*division makes smaller*" is difficult to overcome (Greer, 1994); and that it might be present in adults as well (Graeber, Tirosh, & Glover, 1989). There is also evidence showing that students associate "more" and "less" with addition and subtraction, respectively, when solving word problems (De Corte, Verschaffel, & Pauwels, 1990), which is compatible with the idea that "*addition makes bigger*" and "*subtraction makes smaller*". Some researchers point out the possibility that, similarly to multiplication and division, students also hold intuitive beliefs about the effect of addition and subtraction (e.g., Tirosh, Tsamir, & Hershkovitz, 2008). Nevertheless, so far this phenomenon has—to the best of our knowledge—never been explicitly and systematically investigated.

1.4. Density property

Between any two non-equal natural numbers there is a finite number of numbers. On the contrary, between any two non-equal rational numbers there are infinitely many intermediates. It is amply documented that the density property of rational numbers is difficult to grasp for elementary and secondary students (Hannula, Pehkonen, Maijala, & Soro, 2006; Hartnett & Gelman, 1998; Merenluoto & Lehtinen, 2002; Smith et al., 2005; Vamvakoussi & Vosniadou, 2004, 2010; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011). In younger ages, the response that there are no other numbers between two given numbers that are pseudosuccessive, such as 0.005 and 0.006 or 1/2 and 1/3, is quite frequent, but it decreases with age. Older students usually refer to some intermediate numbers between such pseudosuccessive numbers, but limit their answers to numbers such as 0.0051, 0.0052, . . ., 0.0059, and therefore do not accept that there are infinitely many. There is evidence that university students also face difficulties with the notion of density (Giannakoulias, Souyoul, & Zachariades, 2007; Tirosh et al., 1999).

To sum up, erroneous judgments that can be attributed to the natural number bias occur with respect to the comparison, the operations, and the dense ordering of rational numbers. There is evidence that some of the errors decrease substantially with age, while others are more persistent and present even in educated adults. Thus it appears that the manifestation of the natural number bias in terms of errors might not occur across different mathematical tasks. Comparison tasks, in particular the comparison of decimals, appear to be less challenging for educated adults. On the other hand, tasks related to the density property are more likely to elicit erroneous responses even in educated adults. This is an indication that the over-reliance on natural number reasoning is not a one-dimensional bias that either manifests itself through erroneous responses across all kinds of rational number tasks, or does not exert any influence at all. There is a vast literature on the acquisition of rational number concepts. Some studies investigate rational number competence using various tasks simultaneously (e.g., Tirosh et al., 1999), but they do not focus explicitly on natural number interference. There is also quite some work on the way in which the natural number bias affects rational number reasoning, but these studies focus only on specific tasks such as the comparison of fractions (Stafylidou & Vosniadou, 2004) or the density property (Hartnett & Gelman, 1998; Smith et al., 2005; Vamvakoussi & Vosniadou, 2010; Vamvakoussi et al., 2011). To the best of our knowledge, there are no studies investigating the natural number bias across different rational number tasks, in particular at the individual level. Moreover, we are not aware of any study investigating whether the natural number bias can still manifest itself in the reasoning process even when errors are no longer committed. To this end, we looked at the natural number bias from a different perspective.

1.5. The natural number bias as an instance of intuitive reasoning

It is arguably unavoidable—and to some extent also recommendable—for students to build on their prior knowledge of natural numbers in their early attempts to make sense of rational numbers (Steffe & Olive, 2010). Several authors, however, stress that learning about rational numbers eventually calls for substantial reorganization of students' prior knowledge about number, namely conceptual change (Ni & Zhou, 2005; Smith et al., 2005; Vamvakoussi & Vosniadou, 2010). This is a difficult and time consuming process because—as explained above—the conception of number as *counting* number is firmly established on the basis of in and out of school experiences. In addition, this process is typically impeded by the fact that students remain unaware of the discrepancy between what they already know about numbers and what is to be learned. Thus they tend to regard their beliefs as necessarily correct and do not challenge them in view of the new information regarding rational numbers presented at school. In the words of Greer (2009), students do not recognize that this is a situation where they need to "stop and think".

We suggest that this problem can be examined from the perspective of the distinction between intuitive and analytical reasoning. In fact, reasoning that leads to erroneous responses based on natural numbers bears many similarities to intuitive reasoning as described in science and mathematics in the influential work of Fischbein (1987) (see Merenluoto and Lehtinen (2004) for a similar observation). Fischbein described intuitions as (cognitive) beliefs characterized by self evidence, intrinsic certainty, and coerciveness. The latter means that intuitions are taken to be necessarily true beyond the need for any further justification while any possible alternatives are readily discarded as unacceptable. They are also characterized by globality (i.e., they allow for an immediate and integrated grasp of a situation, via the selection of features that are deemed relevant). Furthermore, in contrast to analytical reasoning, intuitions are implicit (i.e., they are not under the conscious control of the individual). Intuitions have a (mini) theory status, in the sense that they are not isolated, unitary perceptions, skills, or beliefs and that are characterized by extrapolativeness (i.e., they provide the basis upon which inferences are made that go beyond the information at hand). Finally, they are characterized by perseverance (i.e., once established they are robust and therefore not easily eradicated by instruction). Fischbein made the rather strong claim that some intuitions are never completely abandoned, but survive—and may coexist with scientific accounts—throughout a person's life.

This analysis of mathematical intuitions is obviously relevant to the natural number bias. Fischbein (1987) himself offered examples related to the number concept throughout his book, in particular the above-mentioned successor principle, as well as the conceptualization of multiplication as repeated addition. The question arises: Is there empirical evidence that the natural number bias is indeed an instance of intuitive reasoning?

This question can be placed into the more general frame of intuitive reasoning in the domain of mathematics. Recently, it has been argued that the dual-process theories in cognitive psychology and their accompanying methodologies could be a valuable tool in establishing the intuitive nature of erroneous reasoning in various mathematical domains (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009a; Leron & Hazzan, 2006, 2009). Dual-process accounts of reasoning (e.g., Epstein, 1994; Evans & Over, 1996; Kahneman, 2000; Sloman, 1996; Stanovich, 1999) were originally developed to account for poor performance in reasoning and decision-making by individuals who otherwise possessed the knowledge and skills necessary to deal with the tasks at hand. In these theories, it is assumed that humans have an intuitive/heuristic (S1) and an analytic processing system (S2). S1 is deemed fast, automatic, associative and undemanding of working memory capacity, whereas S2 is deemed slow, controlled, deliberate and effortful. Fast S1-heuristics often lead to correct responses, but sometimes they do not. In these latter cases, either an incorrect response is provided, or S2 needs to intervene and override the initial response. Hence, errors may be attributed to S1's pervasiveness and S2's failure to intervene. There are at least two processing claims within the dual-process framework that may serve as a basis to empirically identify whether a response is the result of a heuristic or an analytic process. The first is the differential processing speed—S1 is faster than S2—and the second is the differential involvement of resources—S1 is less demanding in working memory resources than S2.

The dual-process accounts of reasoning are not without criticism (see for example Osman, 2004; Osman & Stavy, 2006). Nevertheless, reaction time data and manipulation of working memory capacity are less contested, and have been extensively used to study the underlying reasoning processes in cases of interference, such as in the Stroop Color-Word Task, namely the well-known Stroop effect (see McLeod, 1991 for a comprehensive review). Recently, such methods have been fruitfully employed in the case of mathematical tasks that are known to systematically elicit erroneous responses (Gillard et al., 2009a). For example, Babai, Levyadun, Stavy, and Tirosh (2006) studied a widely documented error that students make when comparing two polygons with respect to their perimeter, consistent with the assumption that "*the larger the area, the* *larger the perimeter*", which these researchers take to be an instance of a more general intuitive rule termed *more A–more B* (Stavy & Tirosh, 2000). They showed that incorrect responses in line with the *more A–more B* rule were provided faster than correct answers. Similar results were obtained in a probability task (Babai, Brecher, Stavy, & Tirosh, 2006). Gillard, Van Dooren, Schaeken, & Verschaffel (2009b) studied in two experiments the overuse of proportional solution methods in arithmetic word problems from a dual-process perspective. They (a) restricted the solution time and (b) manipulated participants' available working memory capacity by increasing working memory load using a dual-task method. In both experimental conditions, there was an increase of the errors based on inappropriate proportional reasoning when compared to a control condition without time pressure or working memory load, respectively. On the other hand, appropriate proportional reasoning remained unaffected under time pressure or working memory load.

From a dual-process framework perspective, these findings support the hypothesis that systematic errors in a variety of mathematical tasks are the result of intuitive, heuristic reasoning. Applying this framework in the case of the natural number bias can further provide methods to test for its intuitive character that go beyond the mere observation of errors.

1.6. Hypotheses

The central hypothesis underlying this study is that errors due to the natural number bias can be characterized as intuitive and thus are persistent and present even in educated adults, who otherwise have the knowledge and skills necessary to respond correctly. We assumed that when an individual, even an educated adult, is confronted with a rational number task—such as comparing fractions—an intuitive response grounded in natural number reasoning comes to mind first. When an item is compatible with natural number properties—hereafter called a congruent item—(e.g., "*1/3 < 2/3, True or false?*"), the intuitive response leads to a correct answer. For incongruent items (e.g., "*1/5 < 1/9, True or false?*") this intuitive response needs to be inhibited for a correct answer to be given. We thus predicted that incongruent items would trigger more incorrect responses (because the intuitive response was not inhibited) than congruent ones; and also that correct responses for incongruent items would have a longer reaction time than congruent items (because inhibiting the intuitive response required time).

We also hypothesized that there would be differences across the different kinds of tasks involved in this study (comparison, operations, and density tasks). Given the above literature review, we predicted that the comparison tasks would elicit fewer errors, and the density tasks would elicit more errors, than the other tasks, while even in the absence of errors the bias would still be manifested in terms of reaction times. Moreover, we expected the difference in accuracy between the tasks to be present not only at the group but also at the individual level.

2. Method

2.1. Participants

The participants were 58 students at the Department of Educational Sciences of the Katholieke Universiteit Leuven, Belgium. Their age ranged from 18 to 28 years and about two thirds were female. Students participated in the experiment in return for course credit. The students involved in our study were trained to become educational researchers, rather than elementary or secondary school mathematics teachers. None of the university courses they had followed so far has explicitly dealt with the issues referred to in our study.

2.2. Materials

Each student solved 7 item-blocks that addressed different aspects of rational number understanding (see Appendix A). Each item presented a statement the correctness of which had to be judged (True/False) by the participants. Each block contained 8 items (50% congruent and 50% incongruent) and for half of the congruent and incongruent items, the correct response was "true" and for half of them, it was "false". We thus included all possible combinations of congruent/incongruent and true/false statements, namely (a) congruent/true, (b) congruent/false, (c) incongruent/true, and (d) incongruent/false (see Appendix A for examples per block). Within each block, all items were made equal in length. No such manipulation was necessary across blocks, because between blocks comparisons with respect to reaction times were not of interest in the present study.

The first and second block targeted the comparison of fractions. The first consisted of pairs of fractions with either common denominators (congruent items) or common numerators (incongruent items). The expected intuitive response is the one based on the comparison of the common terms, which may lead to an incorrect answer in the incongruent case, such as in the comparison 1/6 < 1/9. The fractions included in the second block had a salient difference with respect to the size of both their terms (e.g., 1/2 < 89/90). The expected intuitive response is the one that reflects the idea that "*the larger the terms, the larger the fraction*" and neglects the relation between the numerator and the denominator.

The third block addressed the comparison of decimals and targeted the intuitive response reflecting the idea that "*longer decimals are larger*", which may lead in an incorrect response in a comparison such as 1.4 < 1.198.

The fourth and fifth blocks targeted the effect of addition/subtraction and multiplication/division, respectively. The items presented an operation with a known and an unknown quantity (e.g., 1 + 10y) and asked whether the outcome is possibly or

Table 1
Accuracies for congruent and incongruent tasks, per block.

Block	Correct responses (%)				Congruence Wald χ^2 (df = 1)
	Congruent items		Incongruent items		
	%	n	%	n	
1 (Comparison, fractions I)	95.0%	220	92.9%	211	.547
2 (Comparison, fractions II)	89.7%	223	92.6%	215	1.441
3 (Comparison, decimals)	90.7%	214	96.8%	222	9.251**
4 (Addition/subtraction)	98.7%	223	73.3%	217	36.830***
5 (Multiplication/division)	97.7%	222	83.0%	218	14.507***
6 (Density, decimals)	91.1%	214	54.8%	210	63.236***
7 (Density, fractions)	88.3%	222	49.5%	214	46.739***

** $p < .01$.
*** $p < .001$.

necessarily larger (for addition and multiplication) or smaller (for subtraction and division) than the known quantity (e.g., *1 + 10y is always greater than 1*, *True or False?*). Eight additional items were used as buffers to counterpart the repeated use of the expressions "always" and "can be" in false and correct statements, respectively. In these buffer items the expression "always" was used in true statements and "can be" in false statements. For instance, "*10y + 1 is always greater than 10y*" is a true statement.[4]

The sixth and seventh blocks targeted the number of numbers in an interval defined by decimals and fractions, respectively. A difficulty that emerged while designing the items was that correct reasoning in this kind of task is always "*there are infinitely many numbers between any two given numbers*". Participants might notice that this expression is linked to the truth of a statement, and thus a correct response might be based merely on the superficial recognition of a true statement. Therefore, we opted for items asking whether the number of intermediate numbers is larger or smaller than a given, finite number of numbers. In the incongruent correct cases (e.g., "*Between 0.6 and 0.8 there are more than 3.000 numbers*") we took care to refer to a number of numbers exceeding an estimate made by students aware of the existence of intermediate numbers up to the order of "thousandths". Finally, to counterpart the repeated use of the expressions "more" and "less" in correct and false statements, respectively, we again used buffer items. For example, the statement "*Between 1 and 1.0 there are more than 3.000 numbers*" is false and uses the term "more".

2.3. Procedure

The items were presented on a computer screen and the participants had to press one of two buttons ("f" or "j", corresponding to the words "fout and "juist", that is, "false" and "true" in Dutch, respectively). Each item remained on the screen until one of the buttons was pressed. Participants' responses as well as their reaction times were registered. The order of the blocks as well as the order of the items within each block was randomized.

The experiment started with general instructions explaining that the participants would have to decide about the truth of a statement, and clarifying the meaning of the buttons. Each block was preceded by a screen that introduced the kind of items that would appear in the block by means of a congruent example item, and reminded the correspondence between the two options ("false", "true") and the buttons ("f", "j"). The presentation of the items in each block started after students chose to press the space button. Between items, a fixation cross was shown for 500 ms.

3. Results

For each block, outliers were omitted by removing trials that took longer than the group mean reaction time plus and minus three standard deviations. Also responses shorter than 600 ms were deleted because based on pilot trials it was assumed that the item could not be processed and responded in less than 600 ms. This led to the elimination of 273 out of 4640 trials (5.9%).

The data were first analyzed separately for each block using a Generalized Estimating Equations approach (GEE) in SPSS that accounted for repeated measurements. The analysis of accuracies (a binary outcome) was done using a logistic regression approach. A linear regression approach was used to analyze reaction times. Congruency of the task (incongruent vs. incongruent) was the factor in both cases.

The results with respect to accuracy are presented in Table 1. Accuracy was rather high across all blocks, except for the incongruent items from blocks 6 and 7. This indicates, as could be expected, that the participants had the necessary mathematical knowledge and skills to deal with the tasks. Nevertheless, across all blocks—except for blocks 2 and 3—there

[4] Such buffers are possible to create in the case of addition and subtraction, but not for multiplication and division. Thus, the items of the fourth and fifth blocks were mixed in the administration to the students. The results, however, will be presented separately.

Table 2
Mean reaction times in correct answers and standard errors between parentheses (in ms).

Block	Congruent items			Incongruent items			Congruence Wald χ^2
	M (SE)	*n*	95% Wald CIs	*M* (SE)	*n*	95% Wald CIs	
1 (Comparison, fractions I)	4488 (249)	209	(3999, 4976)	5529 (229)	196	(5081, 5977)	17.309***
2 (Comparison, fractions II)	6011 (316)	200	(5392, 6629)	6196 (323)	199	(5563, 6829)	.404
3 (Comparison, decimals)	3096 (116)	194	(2867, 3324)	3080 (105)	215	(2874, 3286)	.025
4 (Addition/subtraction)	6548 (310)	220	(5940, 7157)	7798 (371)	159	(7072, 8525)	13.541***
5 (Multiplication/division)	4879 (190)	217	(4507, 5251)	5437 (250)	181	(4948, 5927)	4.590*
6 (Density, decimals)	4126 (139)	195	(3854, 4399)	4253 (273)	115	(3717, 4789)	.308
7 (Density, fractions)	4938 (245)	196	(4458, 5418)	5154 (358)	106	(4451, 5857)	.478

* $p < .05$.
*** $p < .001$.

were fewer correct responses given to the incongruent items than to the congruent ones, the difference being significant for blocks 4–7. On the other hand, in block 3 there was a significant difference in favor of the incongruent items.

The results with respect to reaction times in correct responses are presented in Table 2. The mean reaction times were longer for incongruent than congruent items across all blocks, except for block 3. The effect of congruence was significant for blocks 1, 4, 5, and 8.

To sum up, the effect of congruence on accuracy was significant and in the expected direction in all blocks, except the ones corresponding to the comparison tasks (blocks 1, 2, and 3). Specifically, congruence had no effect in the comparison of fractions (blocks 1 and 2). On the other hand, it had a significant effect on the comparison of decimals (block 3), but not in the expected direction. This suggests that the participants selected shorter decimals as the bigger ones, which cannot be explained as an intrusion of natural number reasoning. We will elaborate on this finding in the discussion section.

The effect of congruence on reaction time was significant and in the expected direction with respect to the blocks addressing the effect of operations (blocks 4 and 5) and the comparison of fractions with one common term (block 1). No significant effect was found for the comparison of fraction without a common term (block 2), the comparison of decimals (block 3) and the density tasks (blocks 6 and 7).

These results with respect to accuracy *within* a block already point to the differences *between* the blocks. To further establish these differences, we conducted a logistic regression on students' responses on the incongruent items, with block as independent and accuracy as dependent variable. The analysis showed a significant main effect, Wald χ^2 (1, 58) = 127.904, $p < .001$. Pairwise comparisons are presented in Table 3 (for the sake of simplicity, comparisons between blocks pertaining to the same kind of task—for example, blocks 2 and 3—are excluded).

Table 3 shows that the density tasks (blocks 6 and 7) elicited significantly more errors than the operations (blocks 4 and 5), as well as the comparison tasks (blocks 1, 2 and 3); and the operations tasks elicited significantly more errors than the comparison tasks.

In the following we look more closely at the consistency in students' answers in the four incongruent items that were included within each block. Table 4 presents a categorization of the students, according to the number of correct responses provided by each individual in the incongruent items within a block.

Table 3
Pairwise comparisons on accuracies, between blocks.

(I) Block	(J) Block	Mean difference (I – J)	SE	df	Sig.	95% Wald confidence interval for difference	
						Lower	Upper
1st	4th	.20	.049	1	.000	.10	.29
	5th	.10	.037	1	.008	.03	.17
	6th	.38	.039	1	.000	.31	.46
	7th	.43	.064	1	.000	.31	.56
2nd	4th	.19	.046	1	.000	.10	.28
	5th	.10	.040	1	.018	.02	.17
	6th	.38	.037	1	.000	.31	.45
	7th	.43	.058	1	.000	.32	.54
3rd	4th	.24	.045	1	.000	.15	.32
	5th	.14	.037	1	.000	.07	.21
	6th	.42	.036	1	.000	.35	.49
	7th	.47	.059	1	.000	.36	.59
4th	6th	.19	.050	1	.000	.09	.28
	7th	.24	.066	1	.000	.11	.37
5th	6th	.28	.040	1	.000	.20	.36
	7th	.33	.061	1	.000	.22	.45

Table 4
Frequencies and percents of student categories, based on the number of correct responses provided for incongruent items, within block.

Block	Max 1 Correct		Intermediate		Min 3 Correct	
	n	%	*n*	%	*n*	%
1 (Comparison, fractions I)	5	8.6%	2	3.5%	51	87.9%
2 (Comparison, fractions II)	3	5.7%	4	6.9%	51	87.9%
3 (Comparison, decimals)	1	1.7%	2	3.5%	55	94.8%
4 (Addition/subtraction)	10	17.2%	13	22.4%	35	60.3%
5 (Multiplication/division)	5	8.6%	9	15.5%	44	75.9%
6 (Density, decimals)	20	34.5%	14	24.1%	24	41.4%
7 (Density, fractions)	27	46.6%	9	15.5%	22	37.9%

The first category (*Max 1 Correct*) comprises the students who provided at most one correct response on the four incongruent items of the block. The third category (*Min 3 Correct*) comprises the students who answered correctly to at least three incongruent items. The remaining students (who gave 2 correct responses on the 4 incongruent items) were placed in the *Intermediate* category. Although it is admittedly a rather rough categorization, it provides a clearer picture of the individual students' performance in the incongruent items and allows for assessing consistency across blocks. As could be expected by the results regarding accuracy for blocks 1 to 3 (that were already presented before), the great majority of students were found at the *Min 3 Correct* category and hardly any students were found in the *Max 1 correct* category for the comparison tasks. As far as the effect of operations tasks (blocks 4 and 5) is concerned, the majority of the students was also found in *the Min 3 correct* category; a substantial percent, however, was found in the *Max 1 Correct* and the *Intermediate* categories. Finally, for the blocks addressing the number of intermediates (blocks 6 and 7), most students were placed in the *Max 1 correct* and *Intermediate* categories. For block 7, in particular, students who provided at most 1 correct answer in the incongruent items formed the dominant category.

To assess consistency across tasks at the individual level, we assigned each student the score 1 for a kind of task (i.e., comparison, operations, and density tasks), if she was found at the *Min 3 Correct* category over all blocks targeting the same kind of task, and 0 otherwise. This means, for instance, that a student would be assigned 1 for operations if she was placed in the *Min 3 Correct* category in both blocks 4 and 5. Table 5 presents the profiles corresponding to all possible combinations of 0's and 1's across the three kinds of tasks, and the frequency and percent of students in each profile. It can be seen that the great majority of students—amounting to 87.9%—belonged to four out of eight possible profiles.

Students in the first profile (17.2%) were scored by 0 across all kinds of tasks. Students in the second profile (24.1%) were consistently successful in the comparison tasks only, and not in any of the other tasks. Students in the third profile (25.9%) were consistently successful in the operations and comparison tasks, but not in the density tasks. Finally, students in the fourth profile (20.7%) were consistently successful across all three kinds of tasks. Interestingly, there were no students who succeeded in the density tasks only, and no students who succeeded in the density and operation tasks but not in the comparison tasks. These findings indicate quite clearly that success in the comparison tasks does not guarantee success in the operations and the density tasks. Moreover, a student who deals with both comparison and operations tasks successfully does not necessarily answer the density tasks correctly. On the other hand, a student who succeeds in the later tasks could be reasonably expected to deal successfully with the other two kinds of tasks, as well.

4. Discussion

This study investigated a well known phenomenon, namely the interference of natural number knowledge in rational number tasks, drawing on the distinction between intuitive and analytic reasoning. This allowed for applying a validated

Table 5
Student profiles, based on the number of correct responses provided for incongruent items, within and across kinds of tasks (density, operations, and comparisons).

Density	Operations	Comparisons	*n*	%
0	0	0	10	17.2%
0	0	1	14	24.1%
0	1	0	2	3.4%
0	1	1	15	25.9%
1	0	0	0	0%
1	0	1	5	8.6%
1	1	0	0	0%
1	1	1	12	20.7%
		Total	58	100%

but (in this domain) new methodology, namely the use of reaction time data. Building on prior research on the challenges facing the students in the shift from natural to rational numbers, it contributes to the domain by systematically targeting the natural number bias using a variety of tasks that are known to be challenging in this respect.

In the following we summarize and discuss the results, first by item block (1–7) and then by kind of task (comparisons, operations, and density property). Finally, we present our overall conclusion and suggestions for further research.

4.1. Results by block

In the comparison of fractions with one common term (block 1) the effect of congruence was significant on reaction time (for correct answers), but not on accuracy. In other words, participants did not make more errors to incongruent than to congruent items, but correct responses for incongruent items took significantly longer. Keeping in mind that the incongruent items were fractions with common numerators, it appears that the participants' sensitivity to the relative magnitude of the denominators triggered an incorrect response first, which had to be inhibited (see also Meert et al., 2009, 2010).

In the comparison of fractions in which one fraction had clearly larger terms than the other (block 2), congruency had no significant effect on accuracy nor on reaction time. It thus appears that the participants were not prone to the "*bigger terms*, *bigger fraction*" errors and they presumably treated both congruent and incongruent items with *holistic* strategies, for example, by assessing the relation between numerator and denominator (see also Meert et al., 2009).

In the comparison of decimals (block 3), congruence had no effect on reaction time, but slightly more correct responses were given in the incongruent cases than in the congruent ones. This shows that participants were not prone to "*longer is larger*" errors that reflect the intrusion of natural number reasoning; on the contrary, they rather seem to have provided "*shorter is larger*" responses. This finding is in line with evidence showing that "*longer is larger*" errors are present in younger children but decrease with age, while the "*shorter is larger*" one is persistent for older children and present also in adults (Desmet et al., 2010; Stacey et al., 2001; Stacey & Steinle, 1998, 1999). This error is usually interpreted as the result of an intrusion of knowledge about fractions (see also Peled & Awawdy-Shahbari, 2009; Resnick et al., 1989). An alternative explanation for this finding could be that students became quickly aware of the "tricky" nature of the congruent items, which may have an adverse effect on the congruent cases where students would be suspicious too. This phenomenon has also been repeatedly shown in research on non-proportional reasoning: As soon as students were made aware of the challenges involved in the non-proportional items, their performance on proportional items dropped. This finding that could be interpreted to suggest that students also became "suspicious" when solving proportional problems (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004).

For both blocks addressing the effect of operations (4 and 5), there was a significant effect of congruence on accuracy as well as on reaction time: Accuracy decreased considerably, but remained quite high for the incongruent items—at the cost of time, since the correct responses took significantly longer. The expectations that "*multiplication always makes bigger*" and "*division always makes smaller*" seems indeed intuitively appealing and challenging to discard, in line with evidence coming from word problem posing and solving research area (De Corte & Verschaffel, 1996; De Corte et al., 1988; Fischbein et al., 1985; Greer, 1994). An important finding in our study is that that similar expectations hold for addition and subtraction as well: "*addition makes bigger*" and "*subtraction makes smaller*".

In both blocks addressing the number of intermediate numbers (6 and 7), congruence had a significant effect on accuracy. From congruent to incongruent items, correct responses decreased by more than 35% both for decimal and fraction items, more than in any other block. Moreover, although not significant, the effect of congruence on reaction time was also in the expected direction: Correct responses took longer in the incongruent items. It should be taken into consideration that the items required from students merely to decide if the number of intermediates is smaller or larger than a given (finite) number of numbers and did not involve the notion of infinity as such (i.e., "*Between 1/13 and 3/13 there are more than 3.000 numbers*, *True or False?*" as opposed to "*Between 1/13 and 3/13 there are infinitely many numbers*, *True or False*"). This finding confirms that the idea of infinitely many numbers in an interval is difficult for students at various levels of instruction (Giannakoulias et al., 2007; Hannula et al., 2006; Hartnett & Gelman, 1998; Merenluoto & Lehtinen, 2002; Smith et al., 2005; Tirosh et al., 1999; Vamvakoussi & Vosniadou, 2004, 2010).

4.2. Differences between tasks

The categorization based on the individual students' responses in the incongruent items gives a clear picture of how the different tasks were responded by our participants in terms of accuracy. It is apparent that the comparison tasks triggered very few errors in general, and very few students made more than two errors in the four incongruent items within block. On the other hand, a substantial percent of students consistently provided incorrect answers to the operations tasks. As expected, the density tasks elicited incorrect responses more consistently than any other task, supporting the claim that the problem of the number of the intermediates is a challenging one.

The observed differences in difficulty between the three kinds of tasks, present at the group as well as at the individual level, indicate that the natural number bias is not an "all or nothing" condition, causing errors over all possible tasks, or no errors at all (for a similar observation regarding a different phenomenon, namely the over-reliance on linear reasoning, see Van Dooren, De Bock, Janssens, & Verschaffel, 2008).

There were of course participants who were consistently either unsuccessful or successful across all three kinds of tasks. However, more than half of the participants were successful in some, but not all kinds of tasks. Moreover, the three kinds of tasks (comparison, operations, and density) are rather clearly ordered from the less to the most probable to elicit natural-number-based errors. This pattern can be traced back, to a great extent, to participants' familiarity with the tasks, either through formal and/or informal learning. One has to keep in mind that the comparison of decimals and fractions is a topic that is treated explicitly and systematically in instruction, starting already at Grade 3. In addition, comparing decimals is a skill that is essential in everyday life (consider, for example, the use of money). Operations with rational numbers are also practiced extensively at school, but the focus is typically on their procedural, rather than on their conceptual aspect, as it is commonly the case with rational number instruction (Moss & Case, 1999). The target of the operation tasks (i.e., the effect of operations) was arguably more on the conceptual than on the procedural side. Similarly, the density tasks targeted a major conceptual difficulty in the shift from natural to rational numbers relating to the change from discrete to dense ordering (Gelman, 2000; Smith et al., 2005; Vamvakoussi & Vosniadou, 2010). Additionally, as far as the tasks addressing density are concerned, everyday experiences with numbers do not support grasping of this property. Rather, a quite abstract, symbolic understanding of numbers is required, which is typically not fostered at the secondary level (Kilpatrick, Swafford, & Findell, 2001).

5. Overall conclusion and perspectives for further research

The results of this study suggest that natural-number-biased reasoning is still present in educated adults, pointing to its intuitive character. They nevertheless indicate that, in some contexts—an example being the comparison of decimals—there may be hardly any traces of the natural number bias left in the majority of adults. These findings suggest several opportunities for further research.

An issue to be investigated in future studies is the development of competence in rational number reasoning using the tasks from the current study, comparing between elementary and secondary students of various ages and adults of various levels of expertise. An option could be to investigate whether younger students, for instance in the last year of elementary school or the first of secondary school, would be affected by congruence, also in terms of accuracy, for blocks where adults were only affected in terms of reaction times. Expert—rather than just educated—adults (e.g., mathematics graduates) are also an interesting group that would allow for investigating which aspects of the natural number bias can disappear with instruction, and whether there are yet some other aspects that may persist, as Fischbein (1987) claimed, throughout a person's life, even despite a high level of domain specific expertise.

It should be noted that the dual-process framework offers more methodological tools than we have taken advantage of in this study (Gillard et al., 2009a). In particular, it would be interesting to investigate whether restricting solution time (assuming that the time-demanding analytic processes cannot initiate or successfully come to an end), or increasing cognitive load (again hindering analytic reasoning) would result in an increase of errors in the incongruent items while performance on congruent items remains unaffected (or less affected). These measures could be used to further establish the heuristic-intuitive nature of these errors.

Another question that cannot be answered by our study or by reaction time studies in general, is whether the individuals—consciously or unconsciously—experience a conflict between their intuitions and the valid mathematical knowledge. This is an interesting question because—depending on the answer—errors due to the natural number bias can be explained to result from a failure to detect the conflict between the intuitive, erroneous response and the mathematically correct answer, or from a failure to discard the initial, tempting intuition (De Neys, Moyens, & Vansteenwegen, 2010; Gillard et al., 2009b). In case of conscious conflict experience, quantitative data about people's self-assessment of certitude of correctness, or qualitative data based on think-aloud protocols could be useful. Methodologies coming from neuroscientific research could be valuable in case of unconscious conflict detection. In particular, an option would be to monitor participants' skin conductance responses while they solve congruent and incongruent tasks. This methodology, which has already been successfully applied (De Neys et al., 2010), focuses on the autonomic nervous system modulation during biased reasoning and can provide further help to resolve the issue of conflict detection.

Finally, we would like to highlight what we see as the educational relevance of the present study. First, we note that we do not attribute to the term "natural number bias" a negative connotation. Rather, we consider it as a natural product of active and constructive learning that is cumulative, that is, learning that builds on people's prior knowledge, partly leading to appropriate constructions and partly leading to misconceptions and buggy strategies. We nevertheless believe that this phenomenon should be taken into consideration in rational number instruction. Second, we stress that, although we believe that reaction time methodologies can provide information about people's thinking, we by no means consider faster reaction times as a valuable educational goal. On the contrary, we suggest that a possible way for instruction to address the issue of the natural number bias is by raising students' awareness of the discrepancy between the intuitively compelling conception of number as *counting* number and the notion of rational number and by helping them engage in intentional, controlled thinking (e.g., to apply the "stop and think" strategy), which may lead to slower instead of faster reaction times.

Appendix A. Examples of items used by block

Blocks	Congruent, true	Congruent, false	Incongruent, true	Incongruent, false
1 (Comparison, fractions I)	2/5 < 3/5	6/7 < 3/7	1/8 < 1/3	1/6 < 1/9
2 (Comparison, fractions II)	1/4 < 19/20	35/50 < 1/2	28/90 < 1/3	1/2 < 22/50
3 (Comparison, decimals)	1.3 < 1.859	2.899 < 2.6	3.479 < 3.6	1.4 < 1.198
4 (Addition/subtraction)	5 + 2*x* can be greater than 5	1 + 10*y* is always greater than 1	3 + 12*z* can be smaller than 3	2 + 4*y* is always smaller than 2
5 (Multiplication/division)	10 ÷ *x* can be smaller than 10	6 ÷ *a* is always greater than 6	8 ÷ *z* can be greater than 8	2 ÷ *y* is always smaller than 2
6 (Density, decimals)	Between 2.32 and 2.39 there are more than three numbers	Between 1.51 and 1.57 there are fewer than three numbers	Between 5.12 and 5.14 there are more than 4.000 numbers	Between 5.21 and 5.24 there are fewer than 4.000 numbers
7 (Density, fractions)	Between 1/7 and 6/7 there are more than 2 numbers	Between 1/11 and 9/11 there are fewer than 5 numbers	Between 1/13 and 3/13 there are more than 3.000 numbers	Between 1/7 and 5/7 there are fewer than 5.000 numbers

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