A refinement of a Hardy type inequality for negative exponents, and sharp applications to Muckenhoupt weights on $\mathbb{R}$

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Abstract

We prove a sharp integral inequality that generalizes the well known Hardy type integral inequality for negative exponents. We also give sharp applications in two directions for Muckenhoupt weights on $\mathbb{R}$. This work refines the results that appear in [9].

1 Introduction

In 1920, Hardy has proved (as one can see in [2] or [3]) the following inequality which is known as Hardy's inequality

Theorem A. *If $p > 1$, $a_n \geq 0$ and $A_n = a_1 + a_2 + \ldots + a_n$, $n \in \mathbb{N}^*$, then*

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.1}$$

Moreover, inequality (1.1) *is best possible, that is the constant on the right side cannot be decreased.*

In 1926, Copson generalized in [1] Theorem A, by replacing the arithmetic mean of a sequence by a weighted arithmetic mean. More precisely, he proved the following

Theorem B. *Let $p > 1$, $a_n, \lambda_n > 0$ for $n = 1, 2, \ldots$. Further suppose that $\Lambda_n = \sum_{i=1}^n \lambda_i$ and $A_n = \sum_{i=1}^n \lambda_i a_i$. Then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p, \tag{1.2}$$

where the constant involved in (1.2) *is best possible.*

Certain generalizations of (1.1) have been given in [6], [7] and elsewhere. For example, one can see in [8] further generalizations of Hardy's and Copson's inequalities be replacing means by more general linear transforms. Theorem A has a continued analogue which is the following

Theorem C. *If $p > 1$ and $f : [0, +\infty) \to \mathbb{R}^+$ is L^p-integrable, then*

$$\int_0^\infty \left(\frac{1}{t} \int_0^t f(u)\,\mathrm{d}u \right)^p \mathrm{d}t \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(t)\,\mathrm{d}t. \tag{1.3}$$

The constant in the right side of (1.3) *is best possible.*

It is easy to see that Theorems A and C are equivalent, by standard approximation arguments which involve step functions. Now as one can see in [4], there is a continued analogue of (1.3) for negative exponents, which is presented there without a proof. This is described in the following

Theorem D. *Let $f : [a, b] \to \mathbb{R}^+$. Then for every $p > 0$ the following is true*

$$\int_a^b \left(\frac{1}{t-a} \int_a^t f(u)\,\mathrm{d}u \right)^{-p} \mathrm{d}t \le \left(\frac{p+1}{p} \right)^p \int_a^b f^{-p}(t)\,\mathrm{d}t. \tag{1.4}$$

Moreover (1.4) *is best possible.*

In [9], a generalization of (1.4) has been given, which can be seen in the following

Theorem E. *Let $p \ge q > 0$ and $f : [a, b] \to \mathbb{R}^+$. The following inequality is true and sharp*

$$\int_a^b \left(\frac{1}{t-a} \int_a^t f(u)\,\mathrm{d}u \right)^{-p} \mathrm{d}t \le \left(\frac{p+1}{p} \right)^q \int_a^b \left(\frac{1}{t-a} \int_a^t f(u)\,\mathrm{d}u \right)^{-p+q} f^{-q}(t)\,\mathrm{d}t. \tag{1.5}$$

What is proved in fact in [9] is a more general weighted discrete analogue of (1.5) which is given in the following

Theorem F. *Let $p \ge q > 0$ and $a_n, \lambda_n > 0$ for $n = 1, 2, \ldots$. Define A_n and Λ_n as in Theorem B. Then*

$$\sum_{n=1}^\infty \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^{-p} \le \left(\frac{p+1}{p} \right)^q \sum_{n=1}^\infty \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^{-p+q} a_n^{-q}. \tag{1.6}$$

Certain applications exist for the above two theorems. One of them can be seen in [9], concerning Muckenhoupt weights. In this paper we generalize and refine inequality (1.5) by specifying the integral of f over $[a, b]$. We also assume, for simplicity reasons, that f is Riemann integrable on $[a, b]$. More precisely we will prove the following

Theorem 1. *Let $p \geq q > 0$ and $f : [a,b] \to \mathbb{R}^+$ with $\frac{1}{b-a}\int_a^b f = \ell$. Then the following inequality is true*

$$\int_a^b \left(\frac{1}{t-a}\int_a^t f(u)\,\mathrm{d}u\right)^{-p} \mathrm{d}t \leq \left(\frac{p+1}{p}\right)^q \int_a^b \left(\frac{1}{t-a}\int_a^t f(u)\,\mathrm{d}u\right)^{-p+q} f^{-q}(t)\,\mathrm{d}t - \\ - \frac{q}{p+1}(b-a)\cdot \ell^{-p}. \quad (1.7)$$

Moreover, inequality (1.7) is sharp if one considers all weights f that have mean integral average over $[a,b]$ equal to ℓ.

What we mean by noting that (1.7) is sharp is the following: The constant in front of the integral on the right side cannot be decreased, while the one in front of ℓ^{-p} cannot be increased. These facts will be proved below. In fact more is true as can be seen in the following

Theorem 2. *Let $p \geq q > 0$ and $a_n, \lambda_n > 0$, for every $n = 1, 2, \ldots$. Define A_n and Λ_n as above. Then the following inequality holds for every $N \in \mathbb{N}$.*

$$\sum_{n=1}^N \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p} \leq \left(\frac{p+1}{p}\right)^q \sum_{n=1}^N \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p+q} a_n^{-q} - \frac{q}{p+1}\Lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p}. \quad (1.8)$$

In Section 2 we describe the proof of Theorem 2 and we also prove the validity and the sharpness of (1.7). Moreover if one wants to study the whole topic concerning generalization of inequalities (1.1) or (1.2), can see [5] and [10].
In the last section we prove an application of Theorem 1. More precisely we prove the following

Theorem 3. *Let $\varphi : [0,1) \to \mathbb{R}^+$ be non-decreasing satisfying the following Muckenhoupt type inequality*

$$\left(\frac{1}{t}\int_0^t \varphi(y)\,\mathrm{d}y\right)\left(\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}(y)\,\mathrm{d}y\right)^{q-1} \leq M, \quad (1.9)$$

for every $t \in (0,1]$, where $q > 1$ is fixed and $M \geq 1$ is given. Let now $p_0 \in (1,q)$ be defined as the solution of the following equality:

$$\frac{q-p_0}{q-1}(M\,p_0)^{1/(q-1)} = 1\,. \quad (1.10)$$

Then for every $p \in (p_0, q]$ the following inequality

$$\frac{1}{t}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s \leq \left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)} \frac{1}{K'}\,c\,\frac{q}{p}\left(\frac{p-1}{q-1}\right)^2 \quad (1.11)$$

is true, for every $t \in (0,1]$, where $c = M^{1/(q-1)}$ and $K' = K'(p,q,c) = \dfrac{1}{p^{1/(q-1)}} - c\,\dfrac{q-p}{q-1}$. It is also sharp for $t = 1$.

The above theorem implies immediately the following

Corollary. *Let φ be as in Theorem 3. Then the following inequality is true for every $t \in (0,1]$ and every $p \in (p_0, q]$.*

$$\left(\frac{1}{t}\int_0^t \varphi^{-1/(p-1)}\right)^{p-1} \left(\frac{1}{t}\int_0^t \varphi\right) \le \left[\frac{1}{K'}\, c\, \frac{q}{p}\left(\frac{p-1}{q-1}\right)^2\right]^{p-1}$$

This gives us the best possible range of p's for which the Muckenhoupt condition (1.9) *still holds, under the hypothesis of* (1.9).

The above corollary is the content of [9] but with another constant. Thus by proving Theorem 3 we refine the results in [9] by improving the constants that appear there and by giving certain sharp inequalities that involve Muckenhoupt weights on $\mathbb{R}$.

2 The Hardy inequality

Proof of Theorem 2.
Let $p \ge q > 0$ and $a_n, \lambda_n > 0$, for every $n \in \mathbb{N}^*$. We define $\Lambda_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n$, $A_n = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n$, for $n = 1, 2, \ldots$. We shall prove inequality (1.8). In order to do this we will give two Lemmas that are stated below. We follow [9].

Lemma 1. *Under the above notation the following inequality holds for every $n \in \mathbb{N}^*$.*

$$\left(\frac{p+1}{p}\right)^q a_n^{-q}\left(\frac{A_n}{\Lambda_n}\right)^{-p+q} + p\left(\frac{p}{p+1}\right)^{q/p} a_n^{q/p}\left(\frac{A_n}{\Lambda_n}\right)^{-p-q/p} \ge (p+1)\left(\frac{A_n}{\Lambda_n}\right)^{-p}. \tag{2.1}$$

Proof. It is well known that the following inequality holds

$$y_1^{-p} + p\, y_1\, y_2^{-p-1} - (p+1)\, y_2^{-p} \ge 0, \tag{2.2}$$

for every $y_1, y_2 > 0$.
This is in fact an immediate consequence of the inequality

$$y^{-p} + p\, y \ge (p+1), \text{ for every } y, p \ge 0. \tag{2.3}$$

Inequality (2.3) is true in view of Young's inequality which asserts that for every t, s nonnegative the following inequality is true

$$\frac{1}{q}t^q + \frac{1}{q'}s^{q'} \ge ts \tag{2.4}$$

whenever q is greater than 1, and q' is such that $\frac{1}{q} + \frac{1}{q'} = 1$. Then by choosing $q = p + 1$ in Young's inequality, and setting $t = \frac{1}{y}$ we obtain (2.3).

If we apply (2.3) when $y = y_1/y_2$ we obtain (2.2). Now we apply (2.2) when $y_1 = \left(\frac{p}{p+1}\right)^{1+q/p} a_n^{q/p} \left(\frac{A_n}{\Lambda_n}\right)^{1-q/p}$ and $y_2 = \left(\frac{p}{p+1}\right) \frac{A_n}{\Lambda_n}$.
Then as it is easily seen (2.1) is immediately proved. Our proof of Lemma 1 is now complete. $\square$

As a consequence of Lemma 1 we have (by summing the respective inequalities) that:

$$\left(\frac{p+1}{p}\right)^q \sum_{n=1}^{N} \lambda_n a_n^{-q} \left(\frac{A_n}{\Lambda_n}\right)^{-p+q} + p\left(\frac{p}{p+1}\right)^{q/p} \sum_{n=1}^{N} \lambda_n a_n^{q/p} \left(\frac{A_n}{\Lambda_n}\right)^{-p-q/p} \geq (p+1) \sum_{n=1}^{N} \left(\frac{A_n}{\Lambda_n}\right)^{-p} \lambda_n, \quad (2.5)$$

for every $N \in \mathbb{N}^*$.
We proceed to the proof of

Lemma 2. *Under the above notation the following inequality is true for every* $N \in \mathbb{N}^*$

$$\sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p} - \left(\frac{p}{p+1}\right) \sum_{n=1}^{N} \lambda_n a_n \left(\frac{A_n}{\Lambda_n}\right)^{-p-1} \geq \frac{\Lambda_n}{p+1} \left(\frac{A_n}{\Lambda_n}\right)^{-p}. \quad (2.6)$$

Proof. We follow [9].
For $N = 1$, inequality (2.6) is in fact equality. We suppose now that it is true with $N-1$ in place of N. We will prove that it is also true for the choice of N.

$$\text{Define } S_N = \sum_{n=1}^{N} \left[\lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p} - \left(\frac{p}{p+1}\right) \lambda_n\, a_n \left(\frac{A_n}{\Lambda_n}\right)^{-p-1}\right] = \\ = \sum_{n=1}^{N-1} \left[\lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p} - \left(\frac{p}{p+1}\right) \lambda_n\, a_n \left(\frac{A_n}{\Lambda_n}\right)^{-p-1}\right] + \\ + \lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p} - \left(\frac{p}{p+1}\right) (A_N - A_{N-1}) \left(\frac{A_N}{\Lambda_N}\right)^{-p-1}. \quad (2.7)$$

By our induction step we obviously see that

$$S_N \geq \frac{\Lambda_{N-1}}{p+1} \left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p} + \lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p} - \left(\frac{p}{p+1}\right) (A_N - A_{N-1}) \left(\frac{A_N}{\Lambda_N}\right)^{-p-1} = \\ = \frac{\Lambda_{N-1}}{p+1} \left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p} + \lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p} - \frac{p}{p+1} \Lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p} + \\ + \frac{\Lambda_{N-1}}{p+1} \left[p \frac{A_{N-1}}{\Lambda_{N-1}} \left(\frac{A_N}{\Lambda_N}\right)^{-p-1}\right]. \quad (2.8)$$

We use now inequality (2.2) in order to find a lower bound for the expression in brackets in (2.8). We thus have

$$p\left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)\left(\frac{A_N}{\Lambda_N}\right)^{-p-1} \geq -\left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p} + (p+1)\left(\frac{A_N}{\Lambda_N}\right)^{-p}. \tag{2.9}$$

We use (2.9) in (2.8) and obtain that

$$\begin{aligned} S_N \geq \frac{\Lambda_{N-1}}{p+1}\left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p} + \lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p} - \left(\frac{p}{p+1}\right)\Lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p} + \\ + \frac{\Lambda_{N-1}}{p+1}\left[(p+1)\left(\frac{A_N}{\Lambda_N}\right)^{-p} - \left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p}\right] = \\ = \left(\frac{A_N}{\Lambda_N}\right)^{-p}\left(\lambda_N - \frac{p}{p+1}\Lambda_N + \Lambda_{N-1}\right) = \frac{\Lambda_N}{p+1}\left(\frac{A_N}{\Lambda_N}\right)^{-p} \end{aligned}$$

that is (2.6) holds. In this way we derived inductively the proof of our Lemma. □

We consider now the quantity

$$y = \sum_{n=1}^{N} \lambda_n a_n^{q/p}\left(\frac{A_n}{\Lambda_n}\right)^{-p-q/p}. \tag{2.10}$$

Then $y = \sum_{n=1}^{N} \lambda_n \left[a_n^{q/p}\left(\frac{A_n}{\Lambda_n}\right)^{-q-q/p}\right]\left[\frac{A_n}{\Lambda_n}\right]^{-p+q}$. Suppose that $p > q$. The case $p = q$ will be discussed in the end of the proof. Applying Hölder's inequality now in the above sum with exponents $r = \frac{p}{q}$ and $r' = \frac{p}{p-q}$, we have as a consequence that

$$\begin{aligned} y &\leq \left\{\sum_{n=1}^{N} \lambda_n a_n \left(\frac{A_n}{\Lambda_n}\right)^{-p-1}\right\}^{\frac{q}{p}} \left\{\sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p}\right\}^{1-\frac{q}{p}} \\ &\leq \left\{\frac{p+1}{p}\sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p} - \frac{1}{p}\Lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p}\right\}^{\frac{q}{p}} \cdot \left\{\sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p}\right\}^{1-\frac{q}{p}}, \end{aligned} \tag{2.11}$$

in view of Lemma 2.
We set now $z = \sum_{n=1}^{N} \lambda_n a_n^{-q}\left(\frac{A_n}{\Lambda_n}\right)^{-p+q}$ and $x = \sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p}$. Because of (2.11) we have that

$$y \leq \left\{\frac{p+1}{p}x - \frac{1}{p}\Lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p}\right\}^{\frac{q}{p}} \cdot x^{1-\frac{q}{p}}. \tag{2.12}$$

By setting now $c = \Lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p}$, we have because of (2.12),

$$y \leq \left(\frac{p+1}{p}x - \frac{c}{p}\right)^{\frac{q}{p}} \cdot x^{1-\frac{q}{p}} = \left(\frac{p+1}{p}\right)^{\frac{q}{p}} \left[x - \frac{c}{p+1}\right]^{\frac{q}{p}} x^{1-\frac{q}{p}}. \tag{2.13}$$

Note that by (2.13) the quantity $x - \frac{c}{p+1}$ is positive, that is $x > \frac{c}{p+1}$. Now because of Lemma 1, it is immediate that

$$\left(\frac{p+1}{p}\right)^{q} z + p\left(\frac{p}{p+1}\right)^{\frac{q}{p}} y \geq (p+1)x \overset{(2.13)}{\Longrightarrow}$$

$$\left(\frac{p+1}{p}\right)^{q} z + p\left(\frac{p}{p+1}\right)^{\frac{q}{p}} \left(\frac{p+1}{p}\right)^{\frac{q}{p}} \left[x - \frac{c}{p+1}\right]^{\frac{q}{p}} x^{1-\frac{q}{p}} \geq (p+1)x \implies$$

$$\left(\frac{p+1}{p}\right)^{q} z \geq (p+1)x - p\left[x - \frac{c}{p+1}\right]^{\frac{q}{p}} x^{1-\frac{q}{p}} =$$

$$= x + \left\{p\,x - p\left[x - \frac{c}{p+1}\right]^{\frac{q}{p}} x^{1-\frac{q}{p}}\right\} = x + p\,G(x), \tag{2.14}$$

where $G(x)$ is defined for $x > \frac{c}{p+1}$, by $G(x) = x - \left[x - \frac{c}{p+1}\right]^{\frac{q}{p}} x^{1-\frac{q}{p}}$.
By (2.14) now we obtain

$$\left(\frac{p+1}{p}\right)^{q} z - x \geq p\,G(x) \geq p \inf\left\{G(x) : x > \frac{c}{p+1}\right\}, \tag{2.15}$$

We will now find the infimum in the above relation. Note that

$$G'(x) = 1 - \left(1 - \frac{q}{p}\right) x^{-\frac{q}{p}} \left(x - \frac{c}{p+1}\right)^{\frac{q}{p}} - x^{1-\frac{q}{p}} \left(\frac{q}{p}\right) \left(x - \frac{c}{p+1}\right)^{\frac{q}{p}-1} =$$

$$= 1 - \left(1 - \frac{q}{p}\right) \left(1 - \frac{c}{(p+1)\,x}\right)^{\frac{q}{p}} - \frac{q}{p} \left(1 - \frac{c}{(p+1)\,x}\right)^{\frac{q}{p}-1}. \tag{2.16}$$

We consider now the function

$$H(t) = 1 - \left(1 - \frac{q}{p}\right) t^{\frac{q}{p}} - \frac{q}{p}\, t^{\frac{q}{p}-1}, \quad t \in (0,1).$$

Then $H'(t) = -t^{q/p-2}\left(1 - \frac{q}{p}\right) \frac{q}{p}\,(t-1) > 0$, for every $t \in (0,1)$. Thus $H(t)$ is strictly increasing $\implies$ $H(t) \leq H(1) = 0$, $\forall t \in (0,1)$. By setting now $t = 1 - \frac{c}{(p+1)x}$, we conclude that the expression in the right of (2.16) is negative, that is $G'(x) \leq 0$, $\forall x > \frac{c}{p+1}$ $\implies$ G is decreasing in $\left(\frac{c}{p+1}, +\infty\right)$. Thus $G(x) \geq \lim_{x\to+\infty} G(x) = \ell$.

Then $\ell = \lim\limits_{x\to+\infty} \left[x - x^{1-\frac{q}{p}} \left(x - \frac{c}{p+1}\right)^{\frac{q}{p}}\right] = \lim\limits_{x\to+\infty} \dfrac{1 - \left(1 - \frac{c}{(p+1)x}\right)^{\frac{q}{p}}}{\frac{1}{x}} =$
$= \lim_{y\to 0^+} \frac{1-\left(1-\frac{c}{p+1}y\right)^{\frac{q}{p}}}{y} = -\frac{q}{p}\left(-\frac{c}{p+1}\right) = \frac{q\,c}{p\,(p+1)}$, by applying the De L'Hospital

rule. Thus we have by (2.15) that $\left(\frac{p+1}{p}\right)^q z - x \geq p\frac{q\,c}{p\,(p+1)} = \frac{q\,c}{p+1}$, which gives inequality (1.8), by the definitions of x, z and c.

The proof of Theorem 2 in the case $p > q$ is complete. The case $p = q$ is also true by continuity reasons, that is by letting $p \to q^+$ in (1.8). $\square$

Proof of Theorem 1. We first prove the validity of (1.7). We simplify the proof by considering the case where $a = 0$ and $b = 1$. We consider also the case where $f : [0,1] \to \mathbb{R}^+$ is continuous. The general case for Riemann integrable functions can be handled by using approximation arguments which involve sequences of continuous functions. We suppose that $\int_0^1 f = \ell$. We define $F : (0,1] \to \mathbb{R}^+$ by $F(t) = \frac{1}{t}\int_0^t f(u)\,\mathrm{d}u$. Then

$$\int_0^1 \left(\frac{1}{t}\int_0^t f(u)\,\mathrm{d}u\right)^{-p} \mathrm{d}t = \int_0^1 (F(t))^{-p}\mathrm{d}t.$$

The integral above can be approximated by Riemann sums of the following type:

$$\sum_{n=1}^{2^k} \frac{1}{2^k}(F(\frac{n}{2^k}))^{-p} = \frac{1}{2^k}\sum_{n=1}^{2^k}(\frac{\sum_{i=1}^n a_i^{(k)}}{n})^{-p}.$$

where the quantities $a_i^{(k)}$ are defined as follows:

$$a_i^{(k)} = 2^k \int_{\frac{i-1}{2^k}}^{\frac{i}{2^k}} f$$

for $i = 1, ..., 2^k$. We use now inequality (1.8). Thus the sum that appears above is less or equal than

$$(\frac{p+1}{p})^q \frac{1}{2^k}\sum_{n=1}^{2^k}(\frac{\sum_{i=1}^n a_i^{(k)}}{n})^{-p+q}(a_n^{(k)})^{-q} - \frac{q}{p+1}(\frac{\sum_{n=1}^{2^k} a_n^{(k)}}{2^k})^{-p}.$$

Now we obviously have that $\frac{\sum_{n=1}^{2^k} a_n^{(k)}}{2^k} = \ell$, while since f is continuous, for every $n = 1, ..., 2^k$ there exists $b_n^{(k)} \in [\frac{n-1}{2^k}, \frac{n}{2^k}]$, such that $a_n^{(k)} = f(b_n^{(k)})$. Thus the quantity that appears above equals

$$(\frac{p+1}{p})^q \frac{1}{2^k}\sum_{n=1}^{2^k}(F(\frac{n}{2^k}))^{-p+q}(f(b_n^{(k)}))^{-q} - \frac{q}{p+1}\ell^{-p}.$$

Now, by continuity reasons, and by the choice of $b_n^{(k)}$, the quantity above approximates

$$(\frac{p+1}{p})^q \frac{1}{2^k}\sum_{n=1}^{2^k}(F(b_n^{(k)}))^{-p+q}(f(b_n^{(k)}))^{-q} - \frac{q}{p+1}\ell^{-p}$$

as $k \to \infty$. It is clear now that this last quantity approximates the right side of (1.7), as $k \to \infty$.
We now prove the sharpness of (1.7). Let $\ell > 0$ be fixed and $p \geq q > 0$. We consider for any $a \in \left(-\frac{1}{p}, 0\right)$ the following function $g_a(t) = \ell\,(1-a)\,t^{-a}$, $t \in [0,1]$. It is easy to see that $\int_0^1 g_a = \ell$, $\frac{1}{t}\int_0^t g_a = \frac{1}{1-a} g_a(t)$ for every $t \in (0,1]$ and that $\int_0^1 g_a^{-p} = \frac{\ell^{-p}(1-a)^{-p}}{1+ap}$. We consider now the difference

$$L_a = \int_0^1 \left(\frac{1}{t}\int_0^t g_a\right)^{-p} \mathrm{d}t - \left(\frac{p+1}{p}\right)^q \int_0^1 \left(\frac{1}{t}\int_0^t g_a\right)^{-p+q} g_a^{-q}(t)\,\mathrm{d}t.$$

It equals to (because of the above properties that g_a satisfy)

$$L_a = \ell^{-p} \frac{\left[1 - (1-a)^{-q}\left(\frac{p+1}{p}\right)^q\right]}{1 + a\,p}.$$

We let $a \to -\frac{1}{p}^+$ and we conclude that

$$\lim_{a \to -\frac{1}{p}^+} L_a = \ell^{-p}\, q\,(1-a)^{-q-1}\Big]_{a=-\frac{1}{p}} (-1)\left(\frac{p+1}{p}\right)^q = -\frac{q}{p+1}\ell^{-p}.$$

In this way we derived the sharpness or (1.7).
The proof of Theorem 1 is complete. □

3 Proof of Theorem 3

Let $\varphi : [0,1) \to \mathbb{R}^+$ be non decreasing satisfying the inequality

$$\left(\frac{1}{t}\int_0^t \varphi\right)\left(\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}\right)^{q-1} \leq M, \tag{3.1}$$

for every $t \in (0,1]$, where q is fixed such that $q > 1$ and $M > 0$. We assume also that there exists an $\varepsilon > 0$ such that $\varphi(t) \geq \varepsilon > 0$, $\forall t \in [0,1)$. The general case can be handled using this one, by adding a small constant $\varepsilon > 0$ to φ.
We need the following from [9].

Lemma A. *Let $\psi : (0,1) \to [0,+\infty)$, such that $\lim_{t\to 0} t\,[\psi(t)]^a = 0$, where $a \in \mathbb{R}$, $a > 1$ and $\psi(t)$ is continuous and monotone on $(0,1)$. Then the following is true for any $a \in (0,1)$.*

$$a\int_0^u \psi^{a-1}(t)\,[t\,\psi(t)]'\mathrm{d}t = u\,\psi^a(u) + (a-1)\int_0^u \psi^a(t)\,\mathrm{d}t. \tag{3.2}$$

We refer to [9] for the proof.

We continue the proof of Theorem 3. We set $h : [0,1) \to \mathbb{R}^+$ by $h(t) = \varphi^{-1/(q-1)}(t)$. Then obviously h satisfies $h(t) \le \varepsilon^{-1/(q-1)}$, $\forall t \in [0,1)$. Let also $p_0 \in [1,q]$ be defined such that

$$\frac{q-p_0}{q-1}\,(M\,p_0)^{1/(q-1)} = 1.$$

Let also $p \in (p_0, q]$. Define ψ by $\psi(t) = \frac{1}{t}\int_0^t \varphi^{-1/(q-1)}$. Then by Lemma A, we get for $a = \frac{q-1}{p-1} > 1$, the following:

$$\frac{q-1}{p-1}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-p}{p-1}} \mathrm{d}s - \\ -\left(\frac{q-p}{p-1}\right)\int_0^t \left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}} \mathrm{d}s = t\left(\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}}. \quad (3.3)$$

Define for every $y > 0$ the following function of the variable of $x \in [y, +\infty)$

$$g_y(x) = \frac{q-1}{q-p}y\,x^{(q-p)/(p-1)} - x^{(q-1)/(p-1)}. \quad (3.4)$$

Then $g_y'(x) = \frac{q-1}{p-1}x^{[(q-1)/(p-1)]-2}(y-x) \le 0$, $\forall x \ge y$. Then g_y is strictly decreasing on $[y, +\infty)$.
So if $y \le x \le w \implies g_y(x) \ge g_y(w)$. For every $s \in (0,t]$ set now

$$x = \frac{1}{s}\int_0^s \varphi^{-1/(q-1)},\ y = \varphi^{-1/(q-1)}(s),\ c = M^{1/(q-1)}, \text{ and } z = \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{q-1}}.$$

Note that by (3.1) the following is true $y \le x \le c\,z =: w$. Thus

$$g_y(x) \ge g_y(w) \implies \\ \frac{q-1}{q-p}\varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{(q-p)}{(p-1)}} - \left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{(q-1)}{(p-1)}} \ge \\ \ge \frac{q-1}{q-p}\varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi\right)^{\frac{1}{q-1}-\frac{1}{p-1}} c^{\frac{q-p}{p-1}} - c^{\frac{q-1}{p-1}}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{(p-1)}} \quad (3.5)$$

Integrating (3.5) on $s \in (0,t]$ we get

$$\frac{q-1}{q-p}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s \cdot c^{\frac{q-p}{p-1}} \le \\ \le \frac{q-1}{q-p}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-p}{p-1}} \mathrm{d}s - \\ -\int_0^t \left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}} \mathrm{d}s + c^{\frac{q-1}{p-1}}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s \quad (3.6)$$

Now because of (3.3) we get

$$\frac{q-1}{q-p}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-p}{p-1}} \mathrm{d}s - \int_0^t \left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}} \mathrm{d}s$$
$$= \frac{p-1}{q-p}\,\frac{1}{t^{(q-p)/(p-1)}}\left(\int_0^t \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}} \qquad (3.7)$$

Thus (3.6) gives

$$c^{\frac{q-p}{p-1}}\frac{q-1}{q-p}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s \leq$$
$$\leq c^{\frac{q-1}{p-1}}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s + \frac{p-1}{q-p}\,t\left(\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}\right)^{(q-1)/(p-1)}. \qquad (3.8)$$

But

$$\left[\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}\right]^{(q-1)/(p-1)} \leq M^{1/(p-1)}\left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)} \overset{(3.8)}{\Longrightarrow}$$
$$c^{\frac{q-p}{p-1}}\,\frac{q-1}{q-p}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s \leq$$
$$\leq c^{\frac{q-1}{p-1}}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s + \frac{p-1}{q-p}\,t\,M^{1/(p-1)}\left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)} \Longrightarrow$$
$$A_1 := \frac{q-1}{q-p}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s \leq$$
$$\leq c\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s + \frac{p-1}{q-p}\,\frac{M^{1/(p-1)}}{c^{(q-p)/(p-1)}}\,t\left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)}. \qquad (3.9)$$

Now by using Theorem 1 we get

$$\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}} \mathrm{d}s \leq$$
$$\left(\frac{1+\frac{1}{p-1}}{\frac{1}{p-1}}\right)^{\frac{1}{q-1}}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \varphi^{-\frac{1}{q-1}}(s)\,\mathrm{d}s - \frac{\frac{1}{q-1}}{1+\frac{1}{p-1}}\,t\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}} =$$
$$= p^{\frac{1}{q-1}}A_1\,\frac{q-p}{q-1} - \frac{p-1}{(q-1)\,p}\,t\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}}. \qquad (3.10)$$

Thus in view of (3.10), (3.9) becomes

$$
\begin{aligned}
A_1 \le c\,p^{1/(q-1)} A_1 \,\frac{q-p}{q-1} - c\,\frac{p-1}{(q-1)\,p}\, t\left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)} &+ \\
+\,\frac{p-1}{q-p}\,\frac{M^{1/(p-1)}}{c^{(q-p)/(p-1)}}\, t\left(\frac{1}{t}\int_0^1 \varphi\right)^{-1/(p-1)} &\Longrightarrow \\
\left[1 - c\,p^{1/(q-1)}\frac{q-p}{q-1}\right] A_1 \le \left[\frac{M^{1/(p-1)}}{c^{(q-p)/(p-1)}}\,\frac{p-1}{q-p} - c\,\frac{p-1}{(q-1)\,p}\right]\cdot & \\
\cdot\, t\left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)} &\Longrightarrow \\
\Longrightarrow K(p,q,c)\left[\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)+1/(q-1)} \mathrm{d}s\right] \le & \\
\le \left[\frac{p-1}{q-1}\,\frac{M^{1/(p-1)}}{c^{(q-p)/(p-1)}} - c\,\frac{(p-1)(q-p)}{p\,(q-1)^2}\right]\left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)} &
\end{aligned}
\tag{3.11}
$$

where $K = K(p,q,c) = 1 - c\,p^{1/(q-1)}\frac{q-p}{q-1} > 0$, $\forall p \in (p_0,\, q]$.
As a consequence (3.11) gives

$$
K\left[\frac{1}{t}\int_0^t \varphi^{-1/(p-1)}(s)\left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)+1/(q-1)} \mathrm{d}s\right] \le \le \left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)}\left(\frac{p-1}{q-1}\right)^2 c\,\frac{q}{p}. \tag{3.12}
$$

Now we use the inequality

$$
\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}(s)\left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)+1/(q-1)} \mathrm{d}s \ge \ge \left[\frac{1/(p-1)}{1+(1/(p-1))}\right]^{1/(q-1)} \cdot \frac{1}{t}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s
$$

which is true because of Theorem E. Thus (3.12) gives

$$
\frac{K'}{t}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s \le \left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)}\left(\frac{p-1}{q-1}\right)^2 c\,\frac{q}{p} \tag{3.13}
$$

where $K' = \frac{K}{p^{1/(q-1)}}$, $K = 1 - c\,p^{1/(q-1)}\frac{q-p}{q-1}$.
Thus the inequality stated in Theorem 3 is proved.
We need to prove the sharpness of (3.13). We consider a such that $0 < a < q-1$ and the function $\varphi_a : (0,1] \to \mathbb{R}^+$ defined by $\varphi_a(t) = t^a$, $t \in (0,1]$. The function

φ_a is strictly increasing and $\frac{1}{t}\int_0^t \varphi_a = \frac{1}{t}\frac{t^{a+1}}{a+1} = \frac{1}{a+1}\varphi_a(t)$, $\forall t \in (0,1]$, while $\int_0^t \varphi_a^{-1/(q-1)} = \frac{1}{1-a/(q-1)} t^{1-a/(q-1)}$. Thus

$$\left(\frac{1}{t}\int_0^t \varphi_a\right)\left[\frac{1}{t}\int_0^t \varphi_a^{-1/(q-1)}\right]^{q-1} = \left[\frac{q-1}{q-1-a}\right]^{q-1}\left[t^{-a/(q-1)}\right]^{q-1}\cdot$$
$$\cdot\left(\frac{1}{t}\int_0^t \varphi_a\right) = \frac{1}{a+1}\left(\frac{q-1}{q-1-a}\right)^{q-1} =: M(q,a)$$

and

$$c_a = c(q,a) = [M(q,a)]^{1/(q-1)} = \left[\frac{q-1}{(q-1)-a}\right]\frac{1}{(1+a)^{1/(q-1)}}.$$

Let now $p \in (p_0, q]$ and suppose additionally that $a < p-1$ so that $\int_0^1 \varphi_a^{-1/(p-1)} = (p-1)/(p-1-a)$. We prove the sharpness of (1.11) for $t=1$. That is we prove that the inequality

$$\frac{K'}{t}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s \le \left(\frac{1}{t}\int_0^t \varphi\right)^{-1/(p-1)} c\,\frac{q}{p}\left(\frac{p-1}{q-1}\right)^2$$

becomes sharp for $t=1$. Obviously if $I_a = \displaystyle\int_0^1 \left(\frac{1}{s}\int_0^s \varphi_a\right)^{-1/(p-1)} \mathrm{d}s$, then $I_a = \dfrac{1}{(1+a)^{-1/(p-1)}}\cdot\cdot\int_0^1 \varphi_a^{-1/(p-1)} = (1+a)^{1/(p-1)}\frac{1}{1-a/(p-1)}$ while $\left(\displaystyle\int_0^1 \varphi_a\right)^{-1/(p-1)} = \left(\dfrac{1}{a+1}\right)^{-1/(p-1)}$. Thus in order to prove the sharpness of the above inequality we just need to prove that the following is true

$$\left[\frac{1}{p^{1/(q-1)}} - \frac{q-p}{q-1}c_a\right]\left(\frac{p-1}{(p-1)-a}\right) \cong c_a\,\frac{q}{p}\left[\frac{(p-1)}{(q-1)}\right]^2 \text{ as } a \to (p-1)^- \iff$$
$$\left[\frac{1}{p^{1/(q-1)}} - \frac{q-p}{q-1}\frac{1}{(1+a)^{1/(q-1)}}\left(\frac{q-1}{(q-1)-a}\right)\right]\frac{1}{(p-1)-a} \cong$$
$$\cong \frac{q}{p}\,\frac{p-1}{(q-1)^2}\,\frac{1}{(1+a)^{1/(p-1)}}\,\frac{q-1}{(q-1)-a}, \text{ as } a \to (p-1)^-. \tag{3.14}$$

Let then $a \to (p-1)^-$ or equivalently $x := (a+1) \to p^-$. Then for the proof of (3.14) we just need to note that

$$\frac{\left[p^{-\frac{1}{1/(q-1)}} - \frac{q-p}{q-x}\frac{1}{x^{1/(q-1)}}\right]}{p-x} \cong \frac{q}{p}\,\frac{p-1}{q-1}\,\frac{1}{p^{1/(q-1)}}\,\frac{1}{q-p}, \quad \text{as} \quad x \to p^-,$$

which is a simple application of De L'Hospitals rule.
The proof of Theorem 3 is now complete.

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