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Department of Mathematics

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# Mean Curvature Flow and Isotopy Problems

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The present dissertation thesis was carried out under the postgraduate program of the Department of Mathematics of the University of Ioannina in order to obtain the master degree.

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The acceptance of this dissertation by the Department of Mathematics of the University of Ioannina does not imply the approval of the opinions of the author (section 202 par. 2 Law 5343/1932 and section 50 par. 8 Law 1268/1982).

**Statutory Declaration**

I lawfully declare here with statutorily that the present dissertation thesis was carried out under the international ethical and academical rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis.

*Dedicated to my parents Dimitris and Vivi and our beloved dog Hera.*

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# Abstract

Let $f\colon M \to N$ be a smooth map between two manifolds $M$ and $N$. It is an interesting problem to find *canonical representatives* in the homotopy class of $f$. By a canonical representative is usually meant a map in the homotopy class of the given map $f$ which is critical point of a suitable functional. One possible approach is the *harmonic map heat flow* that was defined by Eells and Sampson in [19]. If $M$ and $N$ are compact and carry appropriate Riemannian metrics, they proved long-time existence and convergence of the heat flow, showing that under these assumptions one finds harmonic representatives in a given homotopy class. This approach is applicable usually when the target space is negatively curved. However, in general one can neither expect long-time existence nor convergence of the flow, in particular for maps between spheres, since the flow usually develops singularities.

There is another interesting functional that we may consider in the space of smooth maps. Namely, given a map $f : M \to N$ between Riemannian manifolds $M$ and $N$, let us denote its *graph* in the product space $M \times N$ by

$$\Gamma(f) := \big\{\big(x, f(x)\big) \in M \times N : x \in M\big\}.$$

Following the terminology introduced by Schoen [38], a map whose graph is minimal submanifold is called *minimal map*. Therefore, minimal maps are critical points of the volume functional.

Another way to deform a smooth map $f\colon M \to N$ between Riemannian manifolds $M$ and $N$ is by deforming its corresponding graph $\Gamma(f)$ in the product space $M \times N$ via the mean curvature flow. A graphical solution of the mean curvature flow can be described completely in terms of a smooth family of maps $f_t\colon M \to N$, $t \in [0, T)$, $f_0 = f$, where $0 < T \leq \infty$ is the maximal time for which the smooth graphical solution exists. In the case of long-time existence of a graphical solution and convergence we would thus obtain a smooth homotopy from the initial map $f$ to a minimal map $f_\infty\colon M \to N$ as time $t$ tends to infinity.

The first result in this direction is due to K. Ecker and G. Huisken [18]. They proved long-time existence of entire graphical hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$. Moreover, they proved convergence to a flat subspace, if the growth rate at infinity of the initial graph is linear. The complexity of the normal bundle in higher codimensions makes the situation much more complicated. Results analogous to [18] are not available any more without further assumptions. However, the ideas developed in [18] opened a new era for the study of the mean curvature flow of submanifolds in Riemannian manifolds of arbitrary dimension and codimension; see for example the papers [6, 7, 9, 28, 30, 34–37, 41, 43–45, 47–49] and the references therein.

The goal of this thesis is to show that the deformation of area preserving maps between Riemann surfaces under its mean curvature gives the following result:

**Theorem.** *Suppose $M$ and $N$ are compact Riemann surfaces with the same constant scalar curvature $\sigma$ and $f\colon M \to N$ be an area preserving diffeomorphism. Then, there exists a family of smooth area preserving maps $f_t\colon M \to N$, $t \in [0,\infty)$, $f_0 = f$, such that the graphs $\Gamma(f_t)$ of $f_t$ move by mean curvature flow in the Riemannian product space $M \times N$. Moreover, depending on the sign of $\sigma$ we have the following behaviour:*

(a) *If $\sigma > 0$, then the family of the graphs $\Gamma(f_t)$ smoothly converges to the graph of an isometry.*

(b) *If $\sigma = 0$, then $\Gamma(f_t)$ smoothly converges to a totally geodesic graph $\Gamma(f_\infty)$ of the product $M \times N$.*

(c) *If $\sigma < 0$, then the graphs $\Gamma(f_t)$ smoothly converges to a minimal surface $M_\infty$ of the Riemannian product $M \times N$.*

The above theorem is due to K. Smoczyk [41] and M.-T. Wang [46, 49]. The proof that we present here follows closely also ideas developed in [34].

Since any diffeomorphism is isotopic to an area preserving diffeomorphism, this gives another proof of Smale's theorem [39] that the orthogonal group $\mathbb{O}(3)$ *is the deformation retract of the diffeomorphism group of the sphere* $\mathbb{S}^2$.

The organisation of the thesis is as follows: In Chapter 1 we will review the geometric structure equations for immersions in Riemannian manifolds and we will introduce most of our terminology and notations that will be used throughout the manuscript. In Chapter 2 we discuss complex and Lagrangian submanifolds of Kähler manifolds. In Chapter 3 we will outline facts from the theory of differential operators in vector bundles. Additionally, we present the comparison maximum principle for parabolic partial differential equations. In Chapter 4 we shall introduce the mean curvature flow. We will show that the mean curvature flow is a quasilinear (degenerate) parabolic system and we will treat the existence and uniqueness problem. Moreover, we will derive the evolution equations of the most important geometric quantities in the general situation, i.e. for immersions in arbitrary Riemannian manifolds. Finally, in Chapter 5 we introduce the graphical mean curvature flow and prove the main result of this thesis.

# ΠΕΡΙΛΗΨΗ

Έστω $f\colon M \to N$ μία λεία απεικόνιση μεταξύ δύο πολυπτυγμάτων $M$ και $N$. Ένα ενδιαφέρον ερώτημα είναι να βρεθούν *κανονικοί εκπρόσωποι* στην κλάση ομοτοπίας της απεικόνισης $f$. Με την έννοια κανονικό εκπρόσωπο εννοούμε μία απεικόνιση στην κλάση ομοτοπίας της απεικόνισης $f$, η οποία είναι κρίσιμο σημείο ενός κατάλληλου συναρτησιακού. Μία πιθανή προσέγγιση είναι είναι η *ροή θερμότητας*, που μελετήθηκε από τους Eells και Sampson στο [19]. Αν το πολυπτύγμα $M$ είναι συμπαγές και το $N$ είναι επίσης συμπαγές με αρνητική καμπυλότητα τομής, οι Eells και Sampson [19] απέδιξαν σύγκλιση της ροής θερμότητας σε μια *αρμονική απεικόνιση*. Κάτω λοιπόν από τέτοιες υποθέσεις μπορούν να βρεθούν αρμονικοί εκπρόσωποι μίας δοσμένης κλάσης ομοτοπίας. Αυτή η προσέγγιση όμως είναι δυνατή όταν το πεδίο τιμών έχει αρνητική καμπυλότητα. Ωστόσο, δεν αναμένεται σύγκλιση της ροής θερμότητας στην περίπτωση που το πεδίο τιμών είναι θετικά καμπυλομένο, όπως για παράδειγμα για απεικονίσεις μεταξύ σφαιρών, αφού η ροή θερμότητας συνήθως αναπτύσσει ιδιομορφίες.

Υπάρχει ένα άλλο ενδιαφέρον συναρτησιακό, που μπορεί να θεωρηθεί στον χώρο των λείων συναρτήσεων. Δοσμένης μίας απεικόνισης $f\colon M \to N$ μεταξύ πολυπτυγμάτων Riemann $M$ και $N$, συμβολίζουμε με

$$\Gamma(f) := \big\{\big(x, f(x)\big) \in M \times N : x \in M\big\}$$

το *γράφημα* στο πολύπτυγμα γινόμενο $M \times N$. Ακολουθώντας την ορολογία που εισήχθη από τον Schoen [38], μία απεικόνιση λέγεται *ελαχιστική* όταν το γράφημά της είναι ελαχιστικό υποπολύπτυγμα. Έτσι, οι ελαχιστικές απεικονίσεις είναι τα κρίσιμα σημεία του συναρτησιακού του όγκου γραφημάτων σε πολυπτύγματα γινόμενο.

Ένας τρόπος παραμόρφωσης μίας απεικόνισης $f\colon M \to N$ μεταξύ συμπαγών πολυπτυγμάτων Riemann $M$ και $N$ είναι παραμορφώνοντας το γράφημα $\Gamma(f)$ της $f$ στο πολύπτυγμα γινόμενο $M \times N$ μέσω της *ροής μέσης καμπυλότητας*. Εάν η παραμόρφωση μέσω της ροής μέσης καμπυλότητας εξακολουθεί να παραμένει γράφημα μέχρι κάποια χρονική στιγμή $T > 0$, τότε λαμβάνουμε μία μονοπαραμετρική οικογένεια διαφορίσιμων απεικονίσεων $f_t\colon M \to N$, $t \in [0, T)$, τέτοια ώστε $f_0 = f$. Στην περίπτωση που η εξέλιξη του γραφήματος υπάρχει για όλους τους χρόνους και εξακολουθεί να παραμένει γράφημα, τότε προκύπτει μια ομοτοπία απο την αρχική απεικόνιση $f$ σε μία ελαχιστική απεικόνιση $f_\infty\colon M \to N$, καθώς το $t$ τείνει στο άπειρο.

Το πρώτο αποτέλεσμα σχετικά με εξέλιξη γραφημάτων στον ευκλείδειο χώρο μέσω της ροής μέσης καμπυλότητας, οφείλεται τους Ecker και Huisken [18]. Συγκεκριμένα, απέδειξαν ότι η ρόη μέσης καμπυλότητας ένος ολικού γραφήματος συνδιάστασης $1$ στον ευκλείδειο χώρο υπάρχει για όλους τους χρόνους. Επιπλέον, κάτω από υποθέσεις για τη γεωμετρία του γραφήματος στο άπειρο, απέδειξαν και σύγκλιση της ρόης σε υπερεπίπεδο. Όμως, η πολυπλοκότητα της κάθετης δέσμης, περιπλέκει την διαδικασία σε μεγαλύτερες συνδιαστάσεις. Κατά συνέπεια, αποτελέσματα ανάλογα με το [18] δεν είναι διαθέσιμα χωρίς επιπλέον υποθέσεις. Ωστόσο, οι ιδέες που αναπτύχθηκαν στο [18] άνοιξαν μία νέα εποχή στη μελέτη της ροής μέσης καμπυλότητας υποπολυπτυγμάτων οποιαδήποτε διάστασης και συνδυάστασης. Για παράδειγμα αναφέρουμε τα άρθρα [6, 7, 9, 28, 30, 34–37, 41, 43–45, 47–49].

Στόχος της παρούσας μεταπτυχιακής διατριβής είναι να μελετήσουμε την εξέλιξη μέσω της ροής μέσης καμπυλότητας ισεμβαδικών απεικονίσεων μεταξύ συμπαγών επιφανειών Riemann με σταθερή καμπυλότητα. Πιο συγκεκριμένα, θα αποδείξουμε το ακόλουθο αποτέλεσμα:

**Θεώρημα:** *Έστω M και N δύο συμπαγείς επιφάνειες Riemann με σταθερή καμπυλότητα τομής $\sigma$ και $f\colon M \to N$ ένας ισεμβαδικός διαφορομορφισμός. Τότε, υπάρχει μια οικογένεια λείων ισεμβαδικών απεικονίσεων $f_t\colon M \to N$, $t \in [0, \infty)$, $f_0 = f$, τέτοια ώστε τα γραφήματα $\Gamma(f_t)$ της $f$ να εξελίσονται μέσω της ροής μέσης καμπυλότητας στο πολύπτυγμα γινόμενο $M \times N$. Επιπλέον, ανάλογα με το πρόσημο του αριθμού $\sigma$ έχουμε τα ακόλουθα αποτελέσματα:*

(a) *Αν $\sigma > 0$, τότε η οικογένεια γραφημάτων $\Gamma(f_t)$ συγκλίνει ομαλά στο γράφημα $\Gamma(f_\infty)$ μιας ισομετρίας $f_\infty : M \to N$.*

(b) *Αν $\sigma = 0$, τότε η οικογένεια γρφημάτων $\Gamma(f_t)$ συγκλίνει ομαλά στο γράφημα $\Gamma(f_\infty)$ μιας αφινικής απεικόνισης $f_\infty : M \to N$.*

(c) *Αν $\sigma < 0$, τότε η οικογένεια $\Gamma(f_t)$ συγκλίνει ομαλά σε μία ελαχιστική επιφάνεια $M_\infty$ στο πολύπτυγμα γινόμενο $M \times N$.*

Το παραπάνω αποτέλεσμα οφείλεται στους K. Smoczyk [41] και M.-T. Wang [46, 49]. Η απόδειξη που παρουσιάζουμε σε αυτή τη διατριβή ακολουθεί ιδέες που αναπτύχθηκαν στο άρθρο [34].

Κάθε διαφορομορφισμός είναι ισοτοπικός με έναν ισεμβαδικό διαφορομορφισμό. Συνεπώς, το προηγούμενο θεώρημα δίνει μία εναλλακτική απόδειξη του θεωρήματος του Smale [39] που λέει ότι *κάθε διαφορομορφισμός δύναται να παραμορωθεί με συνεχή τρόπο σε μια ισομετρία.*

Η οργάνωση της διατριβής είναι ως εξής: Στο Κεφάλαιο 1 κάνουμε ανασκόπηση των βασικών εννοιών από τη θεωρία υποπολυπτυγμάτων. Στο Κεφάλαιο 2 κάνουμε μια σύντομη επισκόπιση των μιγαδικών και υποπολυπτυγμάτων Lagrange σε πολυπτύγματα Kähler. Στο Κεφάλαιο 3 παρουσιάζουμε στοιχεία της θεωρίας διαφορικών τελεστών σε διανυσματικές δέσμες. Επίσης, θα παρουσιάσουμε την αρχή μεγίστου για παραβολικές μερικές διαφορικές εξισώσεις. Στο Κεφάλαιο 4 εισάγουμε τη ροή μέσης καμπυλότητας. Θα δείξουμε ότι η ροή μέσης καμπυλότητας είναι ένα ιωνό εκφυλισμένο παραβολικό σύστημα μερικών διαφορικών εξισώσεων και μεταξύ άλλων θα εξετάσουμε την ύπαρξη, την μοναδικότητα και την ομαλότητα των λύσεων. Τέλος, στο Κεφάλαιο 5 εισάγουμε τη γραφική ροή μέσης καμπυλότητας και αποδεικνύουμε το κύριο θεώρημα της διατριβής.

# Contents

# CHAPTER 1

# Riemannian Submanifolds

In this section we set up the notation and will review some basic facts from Riemannian and submanifold geometry. We closely follow the exposition in [1], [14], [16], [25] and [51].

## 1.1 Riemannian vector bundles

Let $M$ be a smooth manifold of dimension $m$. The set of all smooth functions of $M$ will be denoted by $C^\infty(M)$. Moreover, the tangent space of the manifold $M$ at a point $x \in M$ will be denoted by the symbol $T_x M$.

**Definition 1.1.1** *Let $E$ be a manifold and $\pi : E \to M$ a smooth surjective map. The triple $(E, \pi, M)$ is called a differentiable (real) vector bundle of rank $n$ over $M$, if it satisfies the following conditions:*

($C_1$) *For any $x \in M$, the fiber $E_x := \pi^{-1}(x)$ possesses a (real) vector space structure of dimension $n$.*

($C_2$) *For any $x_0 \in M$, there exists a neighbourhood $U \subset M$ around the point $x_0$ and a diffeomorphism $\Phi : U \times \mathbb{R}^n \to \pi^{-1}(U)$, with the property $\Phi(x, \xi) \in E_x$, for any $(x, \xi) \in U \times \mathbb{R}^n$.*

($C_3$) *The map $\Phi_x : \mathbb{R}^n \to E_x$ given by*

$$\Phi_x(\xi) := \Phi(x, \xi)$$

*is an $\mathbb{R}$-linear isomorphism.*

The manifold $E$ is called the *total manifold*, the map $\pi$ is said to be the *projection map* and $M$ the *base manifold*. The pair $(U, \Phi)$ is called a *local trivialisation* of $E$. Vector bundles of rank one are called *line bundles*.

In the sequel, we shall omit the word "differentiable" for a vector bundle. Often, a vector bundle will simply be denoted by its total space.

**Definition 1.1.2** *Let $(E, \pi, M)$ be a vector bundle over $M$. A $k$-dimensional submanifold $V$ of $E$ is called a subbundle of rank $k$ if the triple $(V, \pi|_V, M)$ is an $\mathbb{R}$-vector bundle of rank $k$ over $M$, where $\pi|_V$ is the restriction of the projection $\pi$ on $V$.*

A *section* $\sigma$ of $(E, \pi, M)$ is a smooth map $\sigma : M \to E$ with the property $\pi \circ \sigma = I$, where $I$ is the identity map. The space of sections of $(E, \pi, M)$ is denoted by $\Gamma(E)$. If $\phi, \phi_1, \phi_2$ are elements of $\Gamma(E)$ and $f \in C^\infty(M)$ is a smooth function, then we can form the sections $\phi_1 + \phi_2$ and $f\phi$ given by

$$(\phi_1 + \phi_2)(x) := \phi_1(x) + \phi_2(x) \text{ and } (f\phi)(x) := f(x)\,\phi(x),$$

obtained from point wise addition and multiplication, respectively. Thus, the set of smooth sections $\Gamma(E)$ of $E$ becomes a module over $C^\infty(M)$.

The most simple vector bundle over a manifold $M$ is the so called *trivial vector bundle* $M \times \mathbb{R}^k$. However, there is a plethora of non trivial vector bundles. One of the most fundamental vector bundles over a manifold $M$ is the tangent bundle $TM$. As a matter of fact, the rank of $TM$ is equal to the dimension of the manifold. The space of its sections is usually denoted by $\mathfrak{X}(M)$. One can use the operations of Linear Algebra to produce new vector bundles from given ones. For example, if $E$ and $V$ are vector bundles over a manifold $M$, then the spaces $E \times V$, $E \otimes V$, $E \oplus V$, $\mathrm{Hom}(E \times V)$, $\wedge^r(V)$ and $\mathrm{Sym}^r(V)$ can be considered as vector bundles over $M$.

Let now $(E_1, \pi_1, M_1)$ and $(E_2, \pi_2, M_2)$ be vector bundles. A pair $(f, T)$ of smooth maps $f : M_1 \to M_2$ and $T : E_1 \to E_2$ is called a *bundle map* if

$$f \circ \pi_1 = \pi_2 \circ T$$

and if

$$T_x = T|_{(E_1)_x} : (E_1)_x \to (E_2)_{f(x)}$$

is a linear map for all $x \in M$. If $M = M_1 = M_2$, then a map $T : E_1 \to E_2$ is called a *morphism* if the pair $(I, T)$ is a bundle map, where $I : M \to M$ is the identity map. A morphism is called *isomorphism* if it is invertible.

There is a natural way to introduce a notion of differentiation on bundles.

**Definition 1.1.3** *A linear connection on a vector bundle $E$ over the manifold $M$ is a map $\nabla^E : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, written*

$$\nabla^E(v, \phi) = \nabla^E_v \phi,$$

*satisfying the following properties:*

(C$_1$) *For any $v_1, v_2 \in \mathfrak{X}(M)$ and $\phi \in \Gamma(E)$, it holds*

$$\nabla^E_{v_1+v_2}\phi = \nabla^E_{v_1}\phi + \nabla^E_{v_2}\phi.$$

($C_2$) *For any* $v \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ *and* $\phi \in \Gamma(E)$, *it holds*

$$\nabla^E_{fv}\phi = f\nabla^E_v\phi.$$

($C_3$) *For any* $v \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ *and* $\phi_1, \phi_2 \in \Gamma(E)$, *it holds*

$$\nabla^E_v(\phi_1 + \phi_2) = \nabla^E_v\phi_1 + \nabla^E_v\phi_2.$$

($C_4$) *For any* $v \in \mathfrak{X}(M)$, $\phi \in \Gamma(E)$ *and* $f \in C^\infty(M)$, *it holds*

$$\nabla^E_v(f\phi) = (vf)\,\phi + f\nabla^E_v\phi.$$

It turns out that for any section $\phi$ and any point $x_0 \in M$, the value of the quantity $\nabla^E_v\phi$ at a point $x_0 \in M$ depends only on the value of $v$ at $x_0$ and on the restriction of $\phi$ on a neighbourhood of $x_0$. More precisely, if $\phi_1$ and $\phi_2$ are sections which coincide on a neighbourhood of $x_0 \in M$, then

$$\nabla^E_{v_1}\phi_1(x_0) = \nabla^E_{v_2}\phi_2(x_0),$$

for any pair of vector fields $v_1, v_2 \in \mathfrak{X}(M)$ with $v_1(x_0) = v_2(x_0)$.

**Definition 1.1.4** *A section* $\phi \in \Gamma(E)$ *is said to be parallel with respect to the connection* $\nabla^E$ *if, for any vector field* $v$ *on* $M$, *holds* $\nabla^E_v\phi = 0$.

We can define higher derivatives of sections of a vector bundle over a manifold $M$ whose tangent bundle $TM$ is already equipped with a connection.

**Definition 1.1.5** *Suppose that* $M$ *is a smooth manifold and* $(E, \pi, M)$ *a vector bundle over* $M$. *Let* $\nabla^M$ *be a connection of* $TM$ *and* $\nabla^E$ *a connection of* $E$. *For any pair of vector fields* $v_1, v_2 \in \mathfrak{X}(M)$, *the map*

$$\nabla^2_{v_1,v_2} : \Gamma(E) \to \Gamma(E),$$

*defined by*

$$\nabla^2_{v_1,v_2}\phi := \nabla^E_{v_1}\nabla^E_{v_2}\phi - \nabla^E_{\nabla^M_{v_1}v_2}\phi,$$

*is called the second covariant derivative of* $\phi$, *with respect to the directions* $v_1$ *and* $v_2$. *By coupling the connections* $\nabla^M$ *and* $\nabla^E$, *one may define similarly, the* $k$*-th derivative* $\nabla^k$ *of a section* $\phi$ *in* $\Gamma(E)$.

To each linear connection there is associated an important operator, which measures the non commutativity of the covariant derivatiation.

**Definition 1.1.6** *The operator* $R^E : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, *defined by the formula*

$$R^E(v_1, v_2, \phi) := \nabla^2_{v_1,v_2}\phi - \nabla^2_{v_2,v_1}\phi,$$

*for any vectors* $v_1, v_2 \in \mathfrak{X}(M)$ *and* $\phi \in \Gamma(E)$, *is called the curvature operator of the linear connection* $\nabla^E$.

Now let us discuss Riemannian vector bundles. A *Riemannian metric* on a vector bundle $E$ of rank $k$ over the manifold $M$ is a smooth map

$$g_E : \Gamma(E) \times \Gamma(E) \to C^\infty(M),$$

such that its restriction to the fibers is a positive definite inner product. Note that a Riemannian metric is just a positive definite section of $\mathrm{Sym}^2(E)$. It is a well known fact that any vector bundle admits a Riemannian metric. Quite often, Riemannian metrics are denoted by the symbol $\langle \cdot\,, \cdot \rangle$.

A connection $\nabla^E$ is called *compatible* with the Riemannian metric $g_E$ if it satisfies the product or Leibniz rule of differentiation

$$v g_E(\phi_1, \phi_2) = g_E\big(\nabla^E_v \phi_1, \phi_2\big) + g_E\big(\phi_1, \nabla^E_v \phi_2\big),$$

for any $v \in \mathfrak{X}(M)$ and $\phi_1, \phi_2 \in \Gamma(E)$. A vector bundle $E$ endowed with these structures is called *Riemannian vector bundle endowed with a compatible linear connection*.

We say that a set of sections $\{\phi_1, \dots, \phi_k\}$ forms an *orthonormal frame*, with respect to $g_E$ if and only if

$$g_E(\phi_i, \phi_j) = \delta_{ij},$$

for any pair of indices $i, j \in \{1, \dots, m\}$. The existence of such frames, at least locally, is guaranteed from the Gram-Schmidt process. As a matter of fact, around a fixed point $x_0 \in M$, we can always find an orthonormal frame $\{\phi_1, \dots, \phi_k\}$ such that specifically at $x_0$, it holds

$$\nabla^E_v \phi_1(x_0) = \cdots = \nabla^E_v \phi_k(x_0) = 0,$$

for any vector field $v \in \mathfrak{X}(M)$. Such orthonormal frames of sections are used to simplify lengthy computations.

Let us restrict ourselves at the tangent bundle $TM$ of a manifold $M$. Given a Riemannian metric $g$ on the tangent bundle $TM$ of a manifold $M$, there is a unique connection $\nabla$, referred as the *Levi-Civita connection*, which is compatible to the Riemannian metric. The Levi-Civita connection $\nabla$ is given by the so called Koszul formula

$$\begin{aligned} 2g\big(\nabla_{v_1} v_2, v_3\big) \;&=\; v_1\big(g(v_2, v_3)\big) + v_2\big(g(v_1, v_3)\big) - v_3\big(g(v_1, v_2)\big) \\ &\quad + g\big([v_1, v_2], v_3\big) - g\big([v_1, v_3], v_2\big) - g\big([v_2, v_3], v_1\big), \end{aligned}$$

for all $v_1, v_2, v_3 \in \mathfrak{X}(M)$. The Levi-Civita is also torsion free, that is

$$\nabla_{v_1} v_2 - \nabla_{v_2} v_1 = [v_1, v_2],$$

for any pair of vector fields $v_1, v_2 \in \mathfrak{X}(M)$.

Denote now by $R$ the curvature operator with respect to the connection $\nabla$. Combining $R$ with $g$ we obtain a (4,0)-tensor which, for simplicity we again denote with the letter $R$. More precisely,

$$R(v_1, v_2, v_3, v_4) := -g(R(v_1, v_2, v_3), v_4),$$

for any $v_1, v_2, v_3, v_4 \in TM$. If $v_1$, $v_2$ are linearly independent vectors, then the quantity

$$\sec(v_1, v_2) := \frac{R(v_1, v_2, v_1, v_2)}{g(v_1, v_1)g(v_2, v_2) - g(v_1, v_2)^2},$$

is called the *sectional curvature* of the space spanned by $v_1$ and $v_2$. Note that the denominator of the above expression is the squared area of the region created by the vectors $v_1$ and $v_2$, that is

$$|v_1 \wedge v_2|^2 := g(v_1, v_1)g(v_2, v_2) - g(v_1, v_2)^2.$$

By contracting the curvature operator $R$, with respect to the metric, we obtain the *Ricci operator* Ric and *scalar curvature* scal of $M$, namely

$$\mathrm{Ric}(v_1, v_2) := \sum_i R(v_1, e_i, v_2, e_i) \quad \text{and} \quad \mathrm{scal} := \sum_i \mathrm{Ric}(e_i, e_i),$$

where $v \in TM$ and $\{e_1, \ldots, e_m\}$ is a local orthonormal frame of the tangent bundle.

Let us complete this section with an important example.

**Example 1.1.7** Let $(M, g, \nabla)$ be a Riemannian manifold and let us consider the homomorphism bundle

$$\mathrm{Hom}\,\big(\underbrace{TM \times \cdots \times TM}_{k-times}\big).$$

Sections of the homomorphism bundle are usually called $(k, 0)$-*tensors*. We can equip the homomorphism bundle with a natural Riemannian metric. Indeed, if $T$ and $S$ are two $(k, 0)$-tensors, then

$$\langle T, S\rangle = \sum_{i_1,\ldots,i_k} T(e_{i_1}, \ldots, e_{i_k})S(e_{i_1}, \ldots, e_{i_k}),$$

where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame, with respect to the metric $g$ on $M$. Of course, the definition does not depend on the choice of the frame. There is a natural connection $\nabla^{\mathrm{Hom}}$ on this bundle which is compatible with the metric we just defined. Namely, if $T$ is a $(k, 0)$-tensor, then we define $\nabla^{\mathrm{Hom}}$ by

$$\begin{aligned}\big(\nabla^{\mathrm{Hom}}_v T\big)(v_1, \ldots, v_k) &= vT(v_1, \ldots, v_k)\\ &\quad -T(\nabla_v v_1, \ldots, v_k) - \cdots - T(v_1, \ldots, \nabla_v v_k),\end{aligned}$$

for $v, v_1, \ldots, v_k \in \mathfrak{X}(M)$. In a similar manner, we define a covariant derivative for tensors with values on $TM$, i.e., if $T$ is a $(k, 1)$-tensor, we define

$$\begin{aligned}\big(\nabla^{\mathrm{Hom}}_v T\big)(v_1, \ldots, v_k) &= \nabla_v T(v_1, \ldots, v_k)\\ &\quad -T(\nabla_v v_1, \ldots, v_k) - \cdots - T(v_1, \ldots, \nabla_v v_k).\end{aligned}$$

As usual, in order to simplify the notation we denote $\nabla^{\mathrm{Hom}}$ again by $\nabla$.

## 1.2 The pull-back bundle

In this section, we will introduce the notion of the *pull-back bundle* that is induced by a map of the base manifold of a given vector bundle. Let $M$ and $N$ be two manifolds, assume that $(E, \pi, N)$ is a given vector bundle over $N$ and suppose that $f\colon M \to N$ is a smooth map. The map $f$ induces a new vector bundle of rank $k$ over $M$. In particular, take as total space the set

$$f^*E := \big\{(x, \xi) : x \in M,\, \xi \in E_{f(x)}\big\}$$

and as projection the map $\pi_f : f^*E \to M$ given by

$$\pi_f(x, \xi) := x.$$

The space $\Gamma(f^*E)$ contains all sections of $E$ with base point at $f(M)$ and inherit naturally a vector space structure from $E_{f(x)}$, given by

$$(x, \xi) + (x, \eta) := (x, \xi + \eta) \quad \text{and} \quad \lambda(x, \xi) := (x, \lambda\xi).$$

Furthermore, for each local trivialisation $\Phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ define the smooth map $(f^*\Phi) : V \times \mathbb{R}^k \to \pi_f^{-1}(V)$ given by

$$(f^*\Phi)(x, \xi) := \big(x, \Phi(f(x), \xi)\big),$$

where

$$V = f^{-1}(U).$$

The map $f^*\Phi$ is a diffeomorphism and, for a fixed point $x_0$, the map

$$(f^*\Phi)_{x_0} := f^*\Phi(x_0, \xi)$$

is a linear isomorphism. Consequently, the triple $(f^*E, \pi_f, M)$ carries the structure of a vector bundle over $M$. This bundle is called the *pull-back* or the *induced by $f$ vector bundle* on $M$.

Suppose now that $\nabla^E$ is a connection on the vector bundle $E$. We would like to introduce a natural linear connection on the pull-back bundle. In other words, the problem we face here is how to differentiate a section along $f(M)$. We can use the connection $\nabla^E$ if we could extend locally a section along $f(M)$. However, such an extension does not always exist. For example, consider the Neil's parabola $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, given by

$$\gamma(t) = (t^2, t^3).$$

Then, the smooth section

$$\gamma'(t) = (2t, 3t^2)$$

does not admit a local smooth extension in a neighbourhood of

$$\gamma(0) = (0, 0).$$

To show this, assume that $\gamma'$ could be extended to an open neighbourhood $U$ of the origin of $\mathbb{R}^2$ to a smooth vector field
$$F\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2.$$
Then, we should have that
$$F(t^2, t^3) = (2t, 3t^2),$$
for $t$ sufficiently small. Differentiating with respect to $t$, we get
$$2tF_x(t^2, t^3) + 3t^2 F_y(t^2, t^3) = (2, 6t).$$
Estimating the above relation for $t = 0$, we obtain a contradiction.

In order to define a natural connection $\nabla^{f^*E}$ on the pull-back bundle $f^*E$, we make the following construction: Let $\{\varphi_1, \dots, \varphi_k\}$ be a local orthonormal frame field of $E$ in a neighbourhood of the point $f(x) \in N$. Then, any section $\phi \in \Gamma\big(f^*E\big)$ can be written in the form
$$\phi(x) = \Big(x, \sum_\alpha \phi^\alpha(x)(\varphi_\alpha \circ f)(x)\Big) \cong \sum_\alpha \phi^\alpha(x)(\varphi_\alpha \circ f)(x),$$
where $\phi^\alpha$, $\alpha \in \{1, \dots, k\}$, are the components of $\phi$ with respect to the given frame field. These functions are defined in a neighbourhood of $M$ and they are smooth. Define now,
$$\nabla^{f^*E}_v \phi := \sum_\alpha (v\phi^\alpha)\, \varphi_\alpha \circ f + \sum_\alpha \phi^\alpha \big(\nabla^E_{df(v)} \varphi_\alpha\big) \circ f,$$
for any $v \in TM$. The above definition of the pull-back connection is independent of the choice of the frame field. The curvature operator $R^{f^*E}$ of the pull-back bundle is given by
$$R^{f^*E}\big(v_1, v_2, \phi(x)\big) = R^E\big(df(v_1), df(v_2), \phi(x)\big),$$
for any $x \in M$, $v_1, v_2 \in T_xM$ and $\phi \in \Gamma(f^*E)$.

Let us complete this discussion with the special case, where $E$ is the tangent bundle $TN$ of $N$. Assume further that $N$ is equipped with a Riemannian metric $h$ with Levi-Civita connection $\nabla^h$. Sections of the vector bundle $f^*TN$ are often called *vector fields along* $f$. For example, if $v$ is a vector field on $M$, then $df(v)$ is a vector field along $f$. The restriction of the metric $h$ on vector fields along $f$, induces a natural Riemannian metric on $f^*TN$, which is compatible with the Riemannian metric, that is
$$vh(\phi_1, \phi_2) = h\big(\nabla^{f^*TN}_v \phi_1, \phi_2\big) + h\big(\phi_1, \nabla^{f^*TN}_v \phi_2\big),$$
for any $v \in \mathfrak{X}(M)$ and $\phi_1, \phi_2 \in f^*TN$. Moreover, for any pair $v_1, v_2$ of vector fields on $M$, it holds
$$\nabla^{f^*TN}_{v_1} df(v_2) - \nabla^{f^*TN}_{v_2} df(v_1) = df\big([v_1, v_2]\big).$$
The *Hessian* of a map $f : (M, g, \nabla^g) \to (N, h, \nabla^h)$ between two Riemannian manifolds is given by
$$B(v_1, v_2) = \nabla^{f^*TN}_{v_1} df(v_2) - df(\nabla^g_{v_1} v_2),$$
for any $v_1, v_2 \in \mathfrak{X}(M)$. It turns out that $B$ is symmetric with values on the pull-back bundle of $f$. The trace of $B$ with respect to $g$ is denoted by $\Delta_{g,h} f$ and is called the *Laplacian* or the *tension field* of the map $f$. If the Laplacian of $f$ is zero, then $f$ is called a *harmonic map.*

## 1.3 The second fundamental form

Consider Riemannian manifolds $(M, g, \nabla^g)$ and $(N, h, \nabla^h)$ of dimension $m$ and $n$, respectively. A smooth map $f : M \to N$ is called an *isometric immersion* if and only if

$$f^*h = g.$$

For simplicity, we often denote both metrics $g$ and $h$ by $\langle \cdot\,, \cdot \rangle$. In this case, $f(M)$ is said to be an *immersed Riemannian submanifold* of $N$. Locally, an immersion is an embedding, i.e., $f$ is additionally homeomorphism into its image. So, practically, we may view the subset $f(M)$ as a submanifold of $N$ with possible self-intersections. At every $x \in M$, we have the orthogonal decomposition

$$T_{f(x)}N = df_x(T_xM) \oplus N_{f(x)}M,$$

where $N_{f(x)}M$ is the orthogonal complement of $df_x(T_xM)$ with respect to $h$.

The space

$$NM = \bigcup_{x \in M} N_{f(x)}M,$$

is a real vector bundle over $M$ of dimension $n - m$ and is called the *normal bundle of the submanifold*. According to the above decomposition, any section $v \in f^*TN$ can be decomposed in a unique way in the form

$$v = v^\top + v^\perp,$$

where $v^\top$ is the tangent and $v^\perp$ is normal part along the submanifold. It is a well known fact in submanifold theory that

$$\big(\nabla^{f^*TN}_{v_1} df(v_2)\big)^\top = df\big(\nabla^g_{v_1} v_2\big), \tag{1.1}$$

for any $v_1, v_2 \in \mathfrak{X}(M)$. This means that the Hessian of an isometric immersion takes values on the normal bundle of the immersion. In submanifold theory, the Hessian of an isometric immersion $f$ is denoted by the letter $A$, that is

$$A(v_1, v_2) := \nabla^{f^*TN}_{v_1} df(v_2) - df(\nabla^g_{v_1} v_2)$$

and is called the *second fundamental form* or the *shape operator* of $f$. The Laplacian of $f$ or, equivalently, the trace of $A$ in submanifold theory is called the *mean curvature* of $f$ and is denoted by the letter $H$.

Let us turn now at the normal bundle $NM$ of the isometric immersion $f$. The restriction of the Riemannian metric of $N$ on $NM$ gives rise to a Riemannian metric on the normal bundle. Moreover, the restriction of the Levi-Civita connection of $N$ induces a linear connection $\nabla^\perp$ on $NM$; just define

$$\nabla^\perp_v \xi = \big(\nabla^N_v \xi\big)^\perp,$$

for any $v \in \mathfrak{X}(M)$ and any normal section $\xi$. One can easily check that $\nabla^\perp$ is compatible with the Riemannian metric on $NM$. The curvature tensor of the normal bundle is denoted by $R^\perp$ and is given by

$$R^\perp(v_1, v_2, \xi) = \nabla^\perp_{v_1}\nabla^\perp_{v_2}\xi - \nabla^\perp_{v_2}\nabla^\perp_{v_1}\xi - \nabla^\perp_{[v_1,v_2]}\xi,$$

for any $v_1, v_2 \in \mathfrak{X}(M)$ and normal section $\xi$. Contracting with the metric on the normal bundle, we can form from $R^\perp$ a $C^\infty(M)$-valued tensor which we denote again by $R^\perp$, namely

$$R^\perp(v_1, v_2, \xi, \eta) = -\langle R^\perp(v_1, v_2, \xi), \eta\rangle.$$

It turns out, that the Riemann curvature operator $R$ of the metric on $M$, the curvature operator $\widetilde{R}$ of the ambient space and the normal curvature $R^\perp$ are related to the second fundamental form $A$ through the Gauss-Codazzi-Ricci equations, namely:

(a) **Gauss equation:**

$$\begin{aligned} R(v_1, v_2, v_3, v_4) &= \widetilde{R}\big(df(v_1), df(v_2), df(v_3), df(v_4)\big) \\ &\quad + \langle A(v_1, v_3), A(v_2, v_4)\rangle - \langle A(v_2, v_3), A(v_1, v_4)\rangle. \end{aligned}$$

(b) **Codazzi equation:**

$$\big(\nabla^\perp_{v_1} A\big)(v_2, v_3) - \big(\nabla^\perp_{v_2} A\big)(v_1, v_3) = \big(\widetilde{R}\big(df(v_1) df(v_2), df(v_3)\big)\big)^\perp.$$

(c) **Ricci equation:**

$$\begin{aligned} R^\perp(v_1, v_2, \xi, \eta) &= \widetilde{R}\big(df(v_1), df(v_2), \xi, \eta\big) \\ &\quad + \sum_k \big(\langle A(v_1, e_k), \xi\rangle\langle A(v_2, e_k), \eta\rangle - \langle A(v_1, e_k), \eta\rangle\langle A(v_2, e_k), \xi\rangle\big), \end{aligned}$$

where $v_1, v_2, v_3, v_4 \in \mathfrak{X}(M)$, $\xi, \eta \in NM$ and $\{e_1, \dots, e_m\}$ is a local orthonormal frame field with respect to $g$.

## 1.4 Local representations

Suppose that $(M, g)$ is a Riemannian manifold of dimension $m$ and let $\nabla^g$ be its associated Levi-Civita connection. For computations, we will often need local expressions of tensors. There are two ways to consider local expressions. We can express coordinates with respect to basic vector fields or with respect to local orthonormal frames.

Let us discuss at first the notation with respect to a local coordinate system. Choose a coordinate system $(U, \varphi, \Omega)$ around a point in $M$, where the chart $\varphi : U \to \Omega$ is given in the form $\varphi = (x_1, \dots, x_m)$. We denote the corresponding basic vector fields by $\{\partial_{x_1}, \dots, \partial_{x_m}\}$ and their dual forms by $\{dx_1, \dots, dx_m\}$. With respect to this system the Riemannian metric $g$ can be written in the form

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

Denote by $\Gamma_{ij}^k$ the *Christoffel symbols* of the metric $g$, which are defined by the formula

$$\nabla^g_{\partial_{x_i}} \partial_{x_j} = \sum_k \Gamma_{ij}^k \partial_{x_k}$$

and they can be expressed in terms of the metric as

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \big( -\partial_{x_l} g_{ij} + \partial_{x_i} g_{jl} + \partial_{x_j} g_{li} \big),$$

where $g^{ij}$ are the components of the inverse of the matrix of the metric $g$, with respect to the basis $\{\partial_{x_1}, \dots, \partial_{x_m}\}$.

Let $(N, h)$ be another Riemannian manifold of dimension $n$ and $f\colon (M, g) \to (N, h)$ is an isometric immersion. Choose a coordinate system $(U, \varphi, \Omega)$ in a neighborhood $U$ around a point $x$ in $M$ and a coordinate system $(V, \psi, \Lambda)$ in a neighborhood $V$ around the point $f(x) \in N$, such that $f|_U\colon U \to f(U) \subset V$ is an embedding. Here, $\varphi : U \to \Omega$ is represented as $\varphi = (x_1, \dots, x_m)$ and $\psi : V \to \Lambda$ as $\psi = (y_1, \dots, y_n)$. We make the following general assumptions and notations:

(a) All indices referring to $M$ will be denoted by latin minuscules and those related to $N$ by greek minuscules.

(b) From the local coordinate functions

$$(x_i)_{i\in\{1,\dots,m\}} : U \to \mathbb{R} \quad \text{and} \quad (y_\alpha)_{\alpha\in\{1,\dots,n\}} : V \to \mathbb{R},$$

we obtain the local expression

$$\psi \circ f \circ \varphi^{-1} = (f^1, \dots, f^n),$$

where

$$f^\alpha = y^\alpha \circ f \circ \varphi^{-1}.$$

(c) The Christoffel symbols of the Levi-Civita connections on the manifold $M$ will be denoted by $\Gamma_{ij}^k$ and the Christoffel symbols of the Levi-Civita connection on the Riemannian manifold $N$ by $\Gamma_{\alpha\beta}^\gamma$.

With respect to these considerations, the Riemannian metrics $g$ and $h$ can be written in the form

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j \quad \text{and} \quad h = \sum_{\alpha,\beta} h_{\alpha\beta} dy_\alpha \otimes dy_\beta.$$

Moreover, the components of a tensor

$$T : \underbrace{TM \times \cdots \times TM}_{m-times} \times \underbrace{TN \times \cdots \times TN}_{n-times} \to C^\infty(\mathbb{R})$$

will be denoted by $T_{i_1\dots i_m\alpha_1\dots\alpha_n}$, i.e.,

$$T_{i_1\dots i_m\alpha_1\dots\alpha_n} = T(\partial_{x_{i_1}},\dots,\partial_{x_{i_m}},\partial_{y_{\alpha_1}},\dots,\partial_{y_{\alpha_n}})$$

and the components of a tensor

$$T\colon \underbrace{TM\times\cdots\times TM}_{m-times}\times\underbrace{TN\times\cdots\times TN}_{n-times}\to TN$$

will be denoted by $T^{\beta}_{i_1\dots i_m\alpha_1\dots\alpha_n}$, that is

$$T^{\beta}_{i_1\dots i_m\alpha_1\dots\alpha_n} = T(\partial_{x_{i_1}},\dots,\partial_{x_{i_m}},\partial_{y_{\alpha_1}},\dots,\partial_{y_{\alpha_n}}).$$

Consequently,

$$df = \sum_{\alpha} f^{\alpha}_{x_i}\partial_{y_\alpha}\otimes dx_i,\ \ g = f^*h = \sum_{\alpha,\beta} h_{\alpha\beta} f^{\alpha}_{x_i} f^{\beta}_{x_j}$$

and

$$A = \sum_{i,j} A_{ij}dx_i\otimes dx_j = \sum_{i,j,\alpha} A^{\alpha}_{ij}\partial_{y_\alpha}\otimes dx_i\otimes dx_j.$$

Let us compute the second fundamental form $A$ in terms of second derivatives of $f$. We have

$$\begin{aligned} A_{ij} &= \nabla^{f^*TN}_{\partial_{x_i}} df(\partial_{x_j}) - df(\nabla^{g}_{\partial_{x_i}}\partial_{x_j}) \\ &= \sum_{\alpha}\nabla^{f^*TN}_{\partial_{x_i}} f^{\alpha}_{x_j}\partial_{y_\alpha} - \sum_k df(\Gamma^k_{ij}\partial_{x_k}). \end{aligned}$$

Hence,

$$\begin{aligned} A_{ij} &= \sum_{\alpha}\left(f^{\alpha}_{x_ix_j}\partial_{y_\alpha} + f^{\alpha}_{x_j}\nabla^{f^*TN}_{\partial_{x_i}}\partial_{y_\alpha}\right) - \sum_k \Gamma^k_{ij} df(\partial_{x_k}) \\ &= \sum_{\alpha}\left(f^{\alpha}_{x_ix_j}\partial_{y_\alpha} + f^{\alpha}_{x_j}\nabla^{h}_{df(\partial_{x_i})}\partial_{y_\alpha} - \sum_k \Gamma^k_{ij} f^{\alpha}_{x_k}\partial_{y_\alpha}\right). \end{aligned}$$

But,

$$\sum_{\alpha} f^{\alpha}_{x_j}\nabla^{h}_{df(\partial_{x_i})}\partial_{y_\alpha} = \sum_{\alpha,\beta} f^{\alpha}_{x_j}\nabla^{N}_{f^{\beta}_{x_i}\partial_{y_\beta}}\partial_{y_\alpha} = \sum_{\alpha,\beta} f^{\alpha}_{x_j} f^{\beta}_{x_i}\nabla^{N}_{\partial_{y_\beta}}\partial_{y_\alpha}$$

and

$$\nabla^{N}_{\partial_{y_\beta}}\partial_{y_\alpha} = \sum_{\delta}\Gamma^{\delta}_{\alpha\beta}\partial_{y_\delta}.$$

Therefore, rearranging the indices, we obtain

$$A^{\alpha}_{ij} = f^{\alpha}_{x_ix_j} - \sum_k \Gamma^k_{ij} f^{\alpha}_{x_k} + \sum_{\beta,\gamma}\Gamma^{\alpha}_{\beta\gamma} f^{\beta}_{x_i} f^{\gamma}_{x_j}$$

and since

$$H = \sum_{i,j} g^{ij}A_{ij},$$

we have

$$H = \sum_{i,j} g^{ij} \big( f^{\alpha}_{x_i x_j} - \sum_k \Gamma^k_{ij} f^{\alpha}_{x_k} + \sum_{\beta,\gamma} \Gamma^{\alpha}_{\beta\gamma} f^{\beta}_{x_i} f^{\gamma}_{x_j} \big). \tag{1.2}$$

Lastly, let us express the curvature quantities in local orthonormal frames. Let us suppose that $\{e_1, \ldots, e_m\}$ is a local orthonormal frame of the tangent bundle and $\{\xi_{m+1}, \ldots, \xi_n\}$ a local orthonormal frame of the normal bundle. In this case, we use latin indices for components on the tangent bundle and greek indices for components on the normal bundle. The following notation should be obvious:

$$\begin{aligned} A^{\alpha}_{ij} &= \langle A(e_i, e_j), \xi_\alpha \rangle = \langle A_{ij}, \xi_\alpha \rangle, \\ \widetilde{R}_{ijkl} &= \widetilde{R}\big(df(e_i), df(e_j), df(e_k), df(e_l)\big), \\ \widetilde{R}_{ij\alpha\beta} &= \widetilde{R}\big(df(e_i), df(e_j), \xi_\alpha, \xi_\beta\big). \end{aligned}$$

Now the fundamental equations of submanifold theory can be written as:

(a) **Gauss equation:**

$$R_{ijkl} = \widetilde{R}_{ijkl} + \sum_\alpha \big( A^{\alpha}_{ik} A^{\alpha}_{jl} - A^{\alpha}_{jk} A^{\alpha}_{il} \big). \tag{1.3}$$

(b) **Codazzi equation:**

$$\big(\nabla^{\perp}_{e_i} A\big)^{\alpha}_{jk} - \big(\nabla^{\perp}_{e_j} A\big)^{\alpha}_{ik} = -\sum_\alpha \widetilde{R}_{ijk\alpha}. \tag{1.4}$$

(c) **Ricci equation:**

$$R^{\perp}_{ij\alpha\beta} = \widetilde{R}_{ij\alpha\beta} + \sum_k \big( A^{\alpha}_{ik} A^{\beta}_{jk} - A^{\beta}_{ik} A^{\alpha}_{jk} \big). \tag{1.5}$$

## 1.5 Time-dependent metrics on bundles over $M \times \mathbb{R}$

We would like now to apply the machinery that we developed above in a setting adapted to time-dependent immersions. Let us begin with a rather general setting. Suppose that $I$ is an open interval of the real line and let $\{g(t)\}_{t\in I}$ be an arbitrary smooth family of Riemannian metrics on a manifold $M$. This means that for any $(x,t) \in M \times I$ we have an inner product $g_{(x,t)}$ on $T_x M$. We can regard $\{g(t)\}_{t\in I}$ this as a metric $g$ acting on the *spatial tangent bundle* $\mathcal{H}$, defined by

$$\mathcal{H} = \{v \in T(M \times \mathbb{R}) : d\pi_2(v) = 0\},$$

where $\pi_2 : M \times I \to I$ is the natural projection map given by

$$\pi_2(x,t) = t.$$

Observe that each $g(t)$ is a Riemannian metric on $\mathcal{H}$ since $\mathcal{H}_{(x,t)}$ is isomorphic to $T_xM$ via $\pi_2$. We can even extend naturally $g$ into a Riemannian metric on $M \times I$, with respect to which we have the orthogonal decomposition

$$T(M \times I) = \mathcal{H} \oplus \mathbb{R}\partial_t.$$

Since $\mathcal{H}$ is a vector subbundle of $T(M \times I)$, any section of $\mathcal{H}$ is also a section of the tangent bundle $T(M \times I)$. We call the elements of $\Gamma(\mathcal{H})$ *spatial vector fields*. There is a natural connection $\nabla$ on $M \times I$. As a matter of fact, define $\nabla$ by

$$\nabla_v w = \nabla^t_v w,\ \nabla_v \partial_t = 0,\ \nabla_{\partial t}\partial_t = 0 \text{ and } \nabla_{\partial_t} v = [\partial_t, v], \tag{1.6}$$

for any spatial vector fields $v$ and $w$, where $\nabla^t$ stand for the Levi-Civita connection of the metric $g(t)$. One can readily check that $\nabla$ is compatible with $g$, i.e.,

$$vg(w_1, w_2) = g(\nabla_v w_1, w_2) + g(w_1, \nabla_v w_2),$$

for any $v \in \mathfrak{X}(M \times \mathbb{R})$ and spatial vector fields $w_1, w_2 \in \Gamma(\mathcal{H})$. Moreover, the connection $\nabla$ is spatially symmetric, that is

$$\nabla_{w_1} w_2 - \nabla_{w_2} w_1 = [w_1, w_2],$$

for any $w_1, w_2 \in \Gamma(\mathcal{H})$.

A very interesting example, where the situation we discussed above occurs, is when we have a time-dependent family of immersions $F : M \times I \to N$ into a Riemannian manifold $(N, h)$. In this case, $F^*h$ gives a time-dependent family of Riemannian metrics on $M$. Endowing $M \times I$ with the natural connection $\nabla$, we have

$$\nabla^{F^*TN}_{\partial_t} dF(v) - \nabla^{F^*TN}_{v} dF(\partial_t) = dF([\partial_t, v]) = dF(\nabla_{\partial_t} v),$$

for any spatial vector field $v$.

CHAPTER 2

# Complex and Lagrangian Submanifolds

The purpose of this section is to set up the notation and to review the basic facts from the theory of Kähler manifolds. A classic reference is the book by Kobayashi and Nomizu [27, Chapter IX].

## 2.1 Kähler manifolds

Let $M$ be a smooth manifold of dimension $2m$ endowed with a Riemannian metric $g$ and a metric connection $\nabla$. An *almost complex structure* on the manifold $M$ is by definition an isomorphism $J : TM \to TM$, satisfying

$$J^2 = J \circ J = -I,$$

where $I$ stands for the identity map on $TM$. The pair $(M, J)$ is called an *almost complex manifold*. Each tangent space $T_xM$ of an almost complex manifold admits a basis of the form

$$\{e_1, \dots, e_m, Je_1, \dots, Je_m\}.$$

Any such two bases differ by an isomorphism with positive determinant and, consequently, such a manifold is orientable. If the almost complex structure $J$ on $M$ is an isometry with respect to $g$, that is

$$g\big(Jv_1, Jv_2\big) = g(v_1, v_2),$$

for any $v_1, v_2 \in \mathfrak{X}(M)$, then the triple $\big(M, g, J\big)$ is called *Hermitian manifold*. If, in addition, $J$ is parallel with respect to $\nabla$ the triple $(M, g, J)$ is called *Kähler manifold*.

Suppose from now on that $(M, g, J)$ is a Kähler manifold. In this case, the form $\Omega$ given by

$$\Omega(v_1, v_2) := g\big(Jv_1, v_2\big),$$

where $v_1, v_2 \in \mathfrak{X}(M)$, is closed and is called the *Kähler form* of the manifold $M$. Using this additional structure, we define the *Ricci form* by

$$\mathcal{R}(v_1, v_2) := \text{Ric}(Jv_1, v_2).$$

The curvature and the Ricci tensor of a Kähler manifold satisfies some special properties; see for example [27]. We recall them in the following proposition.

**Proposition 2.1.1** *Let $(M, g, J)$ be a Kähler manifold. Then, the following facts are true:*

(a) *The curvature operator $R$ satisfies the following identities*

$$R\big(v_1, v_2, Jv_3\big) = J\big(R(v_1, v_2, v_3)\big) \text{ and } R\big(Jv_1, Jv_2, v_3\big) = R(v_1, v_2, v_3),$$

*for any $v_1, v_2, v_3 \in \mathfrak{X}(M)$.*

(b) *The Ricci tensor* Ric *satisfies the relation*

$$\mathrm{Ric}(v_1, v_2) = \mathrm{Ric}(Jv_1, Jv_2) = -\frac{1}{2}\sum_k R(Jv_1, v_2, e_k, Je_k),$$

*where $v_1, v_2 \in \mathfrak{X}(M)$ and $\{e_1, \ldots, e_{2m}\}$ is an orthonormal frame with respect to the metric $g$.*

(c) *The Ricci form $\mathcal{R}$ is given by the formula*

$$\mathcal{R}(v_1, v_2) = \mathrm{Ric}(Jv_1, v_2) = \frac{1}{2}\sum_k R(v_1, v_2, e_k, Je_k),$$

*where $v_1, v_2 \in \mathfrak{X}(M)$ and $\{e_1, \ldots, e_{2m}\}$ is an orthonormal frame with respect to the metric $g$.*

The sectional curvature is a function defined on the Grassmannian bundle of the 2-planes in the tangent space of $M$. A 2-plane is said to be *holomorphic* if it is invariant by the complex structure $J$. The set of holomorphic planes consists a holomorphic bundle over $M$ with fiber the complex projective space $\mathbb{CP}^{m-1}$, where here $m$ is the dimension of $M$. The restriction of the sectional curvature to this complex bundle is called the *holomorphic sectional curvature* and will be denoted by hol. That is

$$\mathrm{hol}(v) := \sec\big(v \wedge Jv\big),$$

for any non-zero vector field $v \in \mathfrak{X}(M)$. It turns out, that the holomorphic sectional curvature determines the whole curvature tensor. If at a point $x$ of $M$ all the holomorphic sectional curvatures are equal to $\sigma$, then for any 2-plane $\Pi$ in $T_xM$ we have that

$$\sec(\Pi) = \frac{\sigma}{4}\big(1 + 3\cos^2(\alpha)\big),$$

where $\alpha$ is the smallest angle between the planes $\Pi$ and $J(\Pi)$.

**Proposition 2.1.2** *Suppose that $(M, g, J)$ is a Kähler manifold with constant holomorphic sectional curvature $\sigma$. Then,*

$$\begin{aligned} R(v_1, v_2, v_3, v_4) &= \frac{\sigma}{4}\Big\{g(v_1, v_3)g(v_2, v_4) - g(v_1, v_4)g(v_2, v_3) \\ &\quad +\Omega(v_1, v_3)\Omega(v_2, v_4) - \Omega(v_1, v_4)\Omega(v_2, v_3) \\ &\quad +2\Omega(v_1, v_2)\Omega(v_3, v_4)\Big\}, \end{aligned}$$

*for any $v_1, v_2, v_3, v_4 \in \mathfrak{X}(M)$.*

There are many examples of Kähler manifolds. For example, the complex space $\mathbb{C}^n$, the complex projective space $\mathbb{CP}^n$ and every orientable 2-dimensional manifold. Also, the product of two Kähler manifolds is a Kähler manifold.

## 2.2 Complex submanifolds

We will present now an interesting category of Kähler submanifolds the so called complex submanifolds.

**Definition 2.2.1** *A map $f\colon (M^{2m}, J_M) \to (N^{2n}, J_N)$ between Kähler manifolds is called holomorphic if*

$$df \circ J_M = J_N \circ df.$$

*If the map $f$ is a holomorphic isometric immersion, then $f(M)$ will be called an immersed complex submanifold of $N$.*

**Theorem 2.2.2** *Let $f\colon (M^{2m}, J_M) \to (N^{2n}, J_N)$ be a holomorphic isometric immersion. Then, $f$ is minimal.*

*Proof.* For any $\xi \in NM$ and $v \in \mathfrak{X}(M)$, we have

$$g_N\big(J_N\xi, df(v)\big) = -g_N\big(\xi, J_N \circ df(v)\big) = -g_N\big(\xi, df \circ J_M v\big) = 0.$$

Hence, the normal space is invariant under $J_N$. Moreover, for any pair of vector fields $v_1, v_2 \in \mathfrak{X}(M)$, we have

$$\begin{aligned} A(v_1, J_M v_2) &= \big(\nabla^N_{df(v_1)} df \circ J_M v_2\big)^\perp = \big(\nabla^N_{df(v_1)} J_N \circ df(v_2)\big)^\perp \\ &= \big(J_N \nabla^N_{df(v_1)} df(v_2)\big)^\perp. \end{aligned}$$

Since the tangent and the normal space of $f$ are invariant under $J_N$, we deduce that

$$A(v_1, J_M v_2) = J_N\big(\nabla^N_{df(v_1)} df(v_2)\big)^\perp = J_N A(v_1, v_2).$$

Choose a local Hermitian frame field $\{e_i, J_M e_i\}$ on $TM$, where $i \in \{1, \dots, m\}$. Then,

$$H = \sum_i \big(A(e_i, e_i) + A(J_M e_i, J_M e_i)\big) = \sum_i \big(A(e_i, e_i) + J_N^2 A(e_i, e_i)\big) = 0.$$

This completes the proof. □

## 2.3 Lagrangian submanifolds

We will present now another interesting category of submanifolds the so called Lagrangian submanifolds.

**Definition 2.3.1** *Let $M^m$ be a Riemannian manifold, $(N^{2m}, g_N, J, \Omega)$ be a Kähler manifold and $f\colon M^m \to N^{2m}$ an isometric immersion. The immersion $f$ will be called Lagrangian if and only if $f^*\Omega = 0$.*

Note that if $f\colon M^m \to N^{2m}$ is a Lagrangian submanifold, then the complex structure of the ambient space maps the tangent bundle of $f$ onto the normal bundle. Hence, we may associate to the map $f$ the trilinear form $C\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$, given by

$$C(v_1, v_2, v_3) = g_N\big(A(v_1, v_2), Jdf(v_3)\big).$$

The trilinear form $C$ is called the *fundamental cubic* of the Lagrangian submanifold. Obviously, the fundamental cubic is symmetric on the first two coordinates. The next lemma shows that the fundamental cubic is actually fully symmetric.

**Proposition 2.3.2** *The fundamental cubic $C$ of Lagrangian submanifold is fully symmetric. If, in addition, the Lagrangian is minimal, then $C$ is traceless.*

*Proof.* We will show that $C$ is symmetric in the last two components. Indeed,

$$\begin{aligned} C(v_1, v_2, v_3) &= g_N\big(A(v_1, v_2), Jdf(v_3)\big) = g_N\big(\nabla^{f^*TN}_{v_2} df(v_1), Jdf(v_3)\big) \\ &= g_N\big(df(v_1), -\nabla^N_{df(v_2)} Jdf(v_3)\big) = g_N\big(df(v_1), -J\nabla^{f^*TN}_{v_2} df(v_3)\big) \\ &= g_N\big(Jdf(v_1), \nabla^{f^*TN}_{v_2} df(v_3)\big) = g_N\big(Jdf(v_1), A(v_2, v_3)\big) \\ &= C(v_1, v_3, v_2), \end{aligned}$$

for any $v_1, v_2, v_3 \in \mathfrak{X}(M)$. This completes the proof. □

**Lemma 2.3.3** *The fundamental cubic $C$ satisfies the identity*

$$\big(\nabla_{v_1} C\big)(v_2, v_3, v_4) = \big(\nabla_{v_2} C\big)(v_1, v_3, v_4) - \widetilde{R}\big(df(v_1), df(v_2), df(v_3), Jdf(v_4)\big),$$

*for any $v_1, v_2, v_3, v_4 \in \mathfrak{X}(M)$.*

*Proof.* Without loss of generality, we may assume that $\{v_1, v_2, v_3\}$ is part of a normal frame at a fixed point on $M$. We compute

$$\begin{aligned}\big(\nabla_{v_1} C\big)(v_2, v_3, v_4) &= v_1 C(v_2, v_3, v_4) = v_1 g_N\big(A(v_2, v_3), Jdf(v_4)\big)\\ &= g_N\big(\nabla^{\perp}_{v_1} A(v_2, v_3), Jdf(v_4)\big) + g_N\big(A(v_2, v_3), \nabla^{f^*TN}_{v_1} Jdf(v_4)\big)\\ &= g_N\big((\nabla^{\perp}_{v_1} A)(v_2, v_3), Jdf(v_4)\big).\end{aligned}$$

Consequently, from Codazzi equation we deduce

$$\begin{aligned}&\big(\nabla_{v_1} C\big)(v_2, v_3, v_4) - \big(\nabla_{v_2} C\big)(v_1, v_3, v_4)\\ &\qquad= g_N\big((\nabla^{\perp}_{v_1} A)(v_2, v_3) - (\nabla^{\perp}_{v_2} A)(v_1, v_3), Jdf(v_4)\big)\\ &\qquad= -\widetilde{R}\big(df(v_1), df(v_2), df(v_3), Jdf(v_4)\big).\end{aligned}$$

This completes the proof. □

**Definition 2.3.4** *Let $f\colon M^m \to N^{2m}$ be a Lagrangian submanifold. The form $\theta$ given by*

$$\theta(v) = g_N\big(JH, df(v)\big) = -\operatorname{tr}_g C(v, \cdot, \cdot),$$

*is called the Maslov form of the Lagrangian.*

**Lemma 2.3.5** *The Maslov form of a Lagrangian submanifold in a Kähler-Einstein manifold is a closed form.*

*Proof.* For simplicity, we assume that $\sigma = 4$. Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame on $M$. Then,

$$d\theta(e_i, e_j) = e_i\theta(e_j) - e_j\theta(e_i) - \theta([e_i, e_j]).$$

Therefore,

$$\begin{aligned}d\theta(e_i, e_j) &= \sum_k e_i C(e_k, e_k, e_j) - \sum_k e_j C(e_k, e_k, e_i) - \sum_k C(e_k, e_k, [e_i, e_j])\\ &= \sum_k (\nabla_{e_i} C)(e_k, e_k, e_j) + \sum_k C(\nabla_{e_i} e_k, e_k, e_j) + \sum_k C(e_k, \nabla_{e_i} e_k, e_j)\\ &+ \sum_k C(e_k, e_k, \nabla_{e_i} e_j) - \sum_k (\nabla_{e_j} C)(e_k, e_k, e_i) - \sum_k C(\nabla_{e_j} e_k, e_k, e_i)\\ &- \sum_k C(e_k, \nabla_{e_j} e_k, e_i) - \sum_k C(e_k, e_k, \nabla_{e_j} e_i) - \sum_k C(e_k, e_k, [e_i, e_j])\\ &= \sum_k (\nabla_{e_i} C)(e_k, e_k, e_j) - \sum_k (\nabla_{e_j} C)(e_k, e_k, e_i).\end{aligned}$$

Since the fundamental cubic C is trilinear and using Lemma 2.3.3, we obtain

$$d\theta(e_i, e_j) = -\sum_k \widetilde{R}\big(df(e_i), df(e_j), df(e_k), Jdf(e_k)\big).$$

Then,

$$\begin{aligned} d\theta(e_i,e_j) &= -\sum_k \langle df(e_i),df(e_k)\rangle\langle df(e_j),Jdf(e_k)\rangle \\ &\quad -\sum_k \langle df(e_i),Jdf(e_k)\rangle\langle df(e_j),df(e_k)\rangle \\ &\quad +\sum_k \Omega\big(df(e_i),df(e_k)\big)\Omega\big(df(e_k),Jdf(e_k)\big) \\ &\quad -\sum_k \Omega\big(df(e_i),Jdf(e_k)\big)\Omega\big(df(e_j),df(e_k)\big) \\ &\quad +2\sum_k \Omega\big(df(e_i),df(e_j)\big)\Omega\big(df(e_k),Jdf(e_k)\big) \end{aligned}$$

and so

$$\begin{aligned} d\theta(e_i,e_j) &= -\sum_k \langle Jdf(e_i),df(e_k)\rangle\langle Jdf(e_j),Jdf(e_k)\rangle \\ &\quad -\sum_k \langle Jdf(e_i),Jdf(e_k)\rangle\langle Jdf(e_j),df(e_k)\rangle \\ &\quad +2\sum_k \langle Jdf(e_i),df(e_j)\rangle\langle Jdf(e_k),Jdf(e_k)\rangle \\ &= 0. \end{aligned}$$

This completes the proof. □

In the next lemma we will present a characterisation of graphical Lagrangian submanifolds in Euclidean spaces.

**Lemma 2.3.6** *Let $U$ be an open and simply connected domain of the Euclidean space $\mathbb{R}^m$. Let $f\colon U \to \mathbb{R}^m$ be a smooth map and $F\colon U \to \mathbb{R}^m \times \mathbb{R}^m$ be the graph of $f$. Then, $F$ is a Lagrangian immersion if and only if $f$ is the gradient of a potential $u\colon U \to \mathbb{R}$, i.e., $f = \nabla u$.*

*Proof.* Recall that for any $v_1, v_2 \in \mathbb{R}^m$, we have that

$$J(v_1,v_2) = (-v_2,v_1).$$

Hence,

$$\begin{aligned} 0 &= F^*\Omega(v_1,v_2) = \Omega\big(dF(v_1),dF(v_2)\big) = \Omega\big((v_1,df(v_1)),(v_2,df(v_2))\big) \\ &= \langle J\big(v_1,df(v_1)\big),\big(v_2,df(v_2)\big)\rangle = \langle\big(-df(v_1),v_1\big),\big(v_2,df(v_2)\big)\rangle \\ &= -\langle df(v_1),v_2\rangle + \langle v_1,df(v_2)\rangle. \end{aligned}$$

Hence, the differential $df$ of $f$ is symmetric. Since the domain $U \subset \mathbb{R}^m$ is simply connected, by Frobenius' Theorem it follows that there exists a smooth function $u\colon U \to \mathbb{R}$, such that $f = \nabla u$. The converse of the above argument is trivial. □

# CHAPTER 3

# Parabolic PDEs

In this chapter we will review some basic facts about existence, uniqueness and regularity of solutions of differential parabolic systems.

## 3.1 Differential operators

We will outline facts from the theory of differential operators in vector bundles. We shall discuss, at first, linear smooth differential operators and their characterisations. Then, we shall investigate the case of non-linear operators, which is the most important and interesting case. We will focus mostly on non-linear parabolic PDEs. In our exposition we follow closely the notes by Kazdan [26].

### 3.1.1 Linear differential operators

Let $M$ be a manifold and $E, F$ two vector bundles over $M$. Suppose that $TM$ and $E$ are endowed with linear connections $\nabla^M$ and $\nabla^E$, respectively, where $\nabla^M$ is the Levi-Civita connection of a Riemannian metric $g_M$. Let $\phi \in \Gamma(E)$. By coupling the connections $\nabla^M$ and $\nabla^E$, one can form repeated derivatives of $\phi$. As usual, we write $\nabla^k$ for the $k$-th derivative of $\phi$.

**Definition 3.1.1** *A map $P\colon \Gamma(E) \to \Gamma(F)$ of the form*

$$(P\phi)(x) := Q\big(x, \nabla^1\phi(x), \dots, \nabla^k\phi(x)\big) \in F_x,$$

*where $Q$ is smooth in all its variables, will be called differential operator of order $k$. In the case where $P$ is $\mathbb{R}$-linear, we say that $P$ is a linear differential operator of order $k$. Otherwise, we say that $P$ is non-linear.*

Suppose that $P\colon \Gamma(E) \to \Gamma(F)$ is a linear differential operator of degree $k$. Then, in index notation, it can be written in the form

$$P\phi = \sum_{i_1,\dots,i_k} A^{i_1\cdots i_k}\nabla^k_{\partial_{x_{i_1}}\cdots\partial_{x_{i_k}}}\phi + \cdots + \sum_{i_1} A^{i_1}\nabla^1_{\partial_{x_{i_1}}}\phi + A^0\phi,$$

where for each $x \in M$,

$$A^{i_1\cdots i_k}(x)\colon E_x \to F_x$$

is linear map. These maps are called the *coefficients* of the linear operator $P$.

**Definition 3.1.2** *Let $P\colon \Gamma(E) \to \Gamma(F)$ be a linear differential operator of order k, let $x$ be an arbitrary point of $M$ and $\xi \in T_xM$. The linear map $\sigma_\xi(P;x)\colon E_x \to F_x$, given by*

$$\sigma_\xi(P;x)\phi := \xi_{i_1}\cdots\xi_{i_k}A^{i_1\dots i_k}\phi,$$

*is called the principal symbol of the operator $P$ at the point $x$ and in the direction $\xi$. In particular, the operator $P$ is called:*

(a) *Elliptic, if its principal symbol is an isomorphism, for every point $x$ and every non-zero direction $\xi$. A necessary condition is $\dim E_x = \dim F_x$, for any $x \in M$.*

(b) *Underdetermined elliptic, if its principal symbol is surjective but not injective, for every point $x$ and every non-zero direction $\xi$. This can happen only if $\dim E_x > \dim F_x$, for any $x \in M$.*

(c) *Overdetermined elliptic, if its principal symbol is injective but not surjective, for every point $x \in M$ and every non-zero direction $\xi$. This can happen only if $\dim E_x < \dim F_x$, for any $x \in M$.*

**Remark 3.1.1** Let us explain the terms elliptic, underdetermined and overdetermined in the definition. Consider the equation $P(\phi) = \theta$, in a sufficiently small neighbourhood of $M$. Locally, we can consider $\phi$ as a vector valued map $(\phi_1,\dots,\phi_l)$, where $l$ is the dimension of the vector bundle $E$ and $\theta$ as the vector valued map $(\theta_1,\dots,\theta_d)$, where $d$ stands for the dimension of $F$. Now, guided by Linear Algebra, we except that a system of $d$-equations in $l$-variables is likely to have many solutions if $l > d$ (underdetermined), one solution if $l = d$ (elliptic) and no solutions at all if $l < d$ (overdetermined).

Let us illustrate now the above mentioned notions with concrete examples.

**Example 3.1.3** Let $M$ be a Riemannian manifold and $E$ a Riemannian vector bundle endowed with a bundle metric $\mathrm{g}_E$ and compatible metric connection $\nabla^E$. Consider the linear operator $\mathscr{L}\colon \Gamma(E) \to \Gamma(E)$ given by

$$\mathscr{L} := \sum_{i,j} A^{ij}\nabla^2_{\partial_{x_i}\partial_{x_j}},$$

where $A^{ij} = A^{ji}$, $i, j \in \{1, \ldots, m\}$, are linear operators of the form

$$A^{ij} = a^{ij} I^E,$$

where $I^E$ stands for the identity bundle map on $E$. For any $\xi \in T_x M$, we see that

$$\sigma_\xi(\mathscr{L}; x) = \sum_{i,j} a^{ij} \xi_i \xi_j I^E.$$

Hence, $\mathscr{L}$ is elliptic if and only if the symmetric 2-tensor $a$ given by

$$a := \sum_{i,j} a^{ij} dx_i \otimes dx_j,$$

is positive or negative definite.

**Example 3.1.4** Let $M$ be a Riemannian manifold. Recall that any section of the bundle $M \times \mathbb{R}$ represents a smooth function and vice-versa. Consider the operator $\operatorname{div}\colon \mathfrak{X}(M) \to C^\infty(M)$, given by

$$\operatorname{div} v = \sum_{i,j} g^{ij} g\big(\nabla^M_{\partial_{x_i}} v, \partial_{x_j}\big).$$

For fixed point $x \in M$ and direction $\xi$, we have

$$\sigma_\xi(\operatorname{div}; x) = \sum_{i,j} g^{ij} \xi_i \xi_j dx_i.$$

From the above description one can see that div is underdetermined elliptic.

**Example 3.1.5** Let $M$ be a Riemannian manifold. Consider $\operatorname{grad}\colon C^\infty(M) \to \mathfrak{X}(M)$, given by

$$\operatorname{grad}(f) = \sum_{i,j} g^{ij} \big(\partial_{x_j} f\big) \partial_{x_i}.$$

One can readily see that for any fixed point $x$ and direction $\xi$,

$$\sigma_\xi(\operatorname{grad}; x) f = \sum_{i,j} f(x) g^{ij} \xi_i \partial_{x_j}.$$

Consequently, the operator grad is overdetermined elliptic.

### 3.1.2 Linearisation of differential operators

When faced with a non-linear PDE, one attempts to *linearise* in such a way that linear theory can be applied. We would like to introduce a notion of *differential* $DP$ of $P$ at $\phi_0$. This can be achieved by using the linearity of the vector bundles.

**Definition 3.1.6** *The differential or the linearisation of $P$ at $\phi_0$, if it exists, is defined to be the linear map $DP|_{\phi_0}\colon \Gamma(E) \to \Gamma(F)$, given by the expression*

$$DP|_{\phi_0}(\psi) := \lim_{s\to 0} \frac{P(\phi_0 + s\psi) - P(\phi_0)}{s},$$

*for any $\psi \in \Gamma(E)$.*

Note that if the operator $P$ is linear, then $DP|_{\phi_0} \equiv P$. One can immediately see that $DP|_{\phi_0}(\psi)$ is a directional derivative of $P$. Hence, we equivalently may write

$$DP|_{\phi_0}(\psi) = \frac{\partial P\big(\phi(s)\big)}{\partial s},$$

where $\phi(s)$ is a one parameter family of sections with

$$\phi(0) = \phi_0 \quad \text{and} \quad \phi'(0) = \psi.$$

Observe that for any fixed $x \in M$, the curve

$$\gamma(s) := \phi_x(s)$$

sits in $E_x$. Hence, $\phi'_x(s)$ is just the velocity of $\gamma$.

**Example 3.1.7** Let $f\colon M \to N$ be a smooth map between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Let us consider the Laplacian operator $\Delta\colon C^\infty(M) \to C^\infty(M)$, given by

$$\Delta_{g_M,g_N} f = \mathrm{tr}_{g_M} B,$$

where $B$ is the Hessian of $f$. In local coordinates, we have

$$\Delta_{g_M,g_N} f = \sum_{i,j} g_M^{ij}\big(f^\alpha_{x_i x_j} - \sum_k \Gamma^k_{ij} f^\alpha_{x_k} + \sum_{\gamma,\delta} \Gamma^\alpha_{\gamma\delta} f^\gamma_{x_i} f^\delta_{x_j}\big).$$

By a straight-forward computation we see that the linearisation of $\Delta_{g_M,g_N} f$ is

$$\begin{aligned} D\Delta_{g_M,g_N}|_f(G) &= \lim_{s\to\infty} \frac{\Delta_{g_M,g_N}(f + sG) - \Delta_{g_M,g_N}(f)}{s} \\ &= \sum_{i,j} g_M^{ij} G^\alpha_{x_i x_j} \partial_{y_\alpha} + \text{lower order terms.} \end{aligned}$$

Therefore, the principal symbol is given by

$$\sigma_\xi(D\Delta_{g_M,g_N}, x)G = \sum_{i,j} g_M^{ij} \xi_i \xi_j G(x),$$

and so the Laplacian operator is elliptic.

### 3.1.3 Solvability of initial value problems

Let $(E, \pi_E, M)$ and $(F, \pi_F, M)$ be vector bundles over a Riemannian manifold $M$. Suppose that $\{\phi(t)\}_{t\in[0,T)}$ is a smooth one parameter family of sections of the bundle $E$. In the matter of fact we may regard $E$ as a vector bundle over the spatial vector sub-bundle $\mathcal{H}$ of $M \times \mathbb{R}$. Therefore, we can view the family $\{\phi(t)\}_{t\in[0,T)}$ as a section $\phi$ on $(E, \pi_E, \mathcal{H})$. We are interested now in expressions of the form:

$$\frac{\partial \phi}{\partial t}(x,t) = (P\phi)(x,t) = Q\big(x, t, \nabla^1\phi(x,t), \dots, \nabla^k\phi(x,t)\big), \tag{3.1}$$

where again $P\colon \Gamma(E) \to \Gamma(F)$ is a time-dependent differentiable operator of order $k$. If for each fixed $t$ the operator $P$ is linear elliptic, we say that (3.1) is a *linear parabolic differential equation*. We say that (3.1) is a *non-linear parabolic differential equation* if and only if, for any $\phi \in \Gamma(E)$, its linearisation

$$\frac{\partial \psi}{\partial s} = DP|_\phi(\psi)$$

is linear parabolic.

**Theorem 3.1.8** *If the differential operator $P$ is parabolic at $\phi_0 \in \Gamma(E)$, then there exist a $T > 0$ and a smooth family $\phi(t) \in \Gamma(E)$, for $t \in [0,T]$, such that there exists a unique smooth solution for the initial value problem*

$$\begin{cases} \frac{\partial \phi}{\partial t} = P\phi \\ \phi(0) = \phi_0 \end{cases},$$

*for $t \in [0,T]$, where $T$ depends on the initial data $\phi_0$.*

**Definition 3.1.9** *Let $(M, g_M)$ and $(N, g_N)$ be Riemannian manifolds. We say that a family of smooth maps $F\colon M \times [0,T) \to N$ evolves by (harmonic) heat flow, with initial data $F_0\colon M \to N$, if it satisfies the initial value problem*

$$\begin{cases} \frac{\partial F}{\partial t} = \Delta_{g_M,g_N} F \\ F(\cdot, 0) = F_0 \end{cases}. \tag{3.2}$$

**Theorem 3.1.10** *Suppose that $(M, g_M)$ is a compact Riemannian manifold. Consider a smooth map $F_0\colon (M, g_M) \to (N, g_N)$ into a Riemannian manifold $(N, g_N)$. Then, the system (3.2) admits a unique, smooth solution on a maximal time interval $[0,T)$, where $0 < T \le \infty$.*

*Proof.* From Example 3.1.7 the principal symbol of the linearisation of $\Delta_{g_M,g_N}$ is given by

$$\sigma_\xi(D\Delta_{g_M,g_N}, x)G = \sum_{i,j} g_M^{ij}\xi_i\xi_j G(x).$$

Therefore, the parabolic theory can be used to ensure short-time existence. This completes the proof. □

## 3.2 Parabolic maximum principle

An important tool in order to control the behaviour of solutions to parabolic differential equations is the *Maximum Principle*. In this section, we use the Maximum Principle to show that pointwise inequalities based on initial data of parabolic differential equations are preserved in time.

Throughout this section, we will assume that $M$ is a compact manifold endowed with a continuous time-dependent family of Riemannian metrics $\{g(t)\}_{t\in[0,T)}$. Consider the parabolic linear operator $\mathscr{L}$, given by

$$\mathscr{L}u = u_t - \Delta u - g(X, \nabla u) - \Psi(u,t),$$

where $X$ is a time-dependent continuous vector field, $\Delta$ is the Laplacian with respect to $g(t)$, $\nabla$ is the corresponding Levi-Civita connection of $g(t)$ and $\Psi\colon \mathbb{R}\times[0,T)\to\mathbb{R}$ Lipschitz in the first and continuous in the second variable. In local coordinates, the above equation can be written in the form

$$\Delta = \sum_{i,j} a^{ij}\partial_{x_i}\partial_{x_j} + \sum_i \beta^i \partial_{x_i},$$

where $a^{ij}, \beta^i$ are time-dependent continuous functions.

**Proposition 3.2.1** *Let $u \in C^\infty\big(M\times[0,T)\big)$ be a solution of the inequality*

$$u_t - \Delta u \geq 0,$$

*Let $c\in\mathbb{R}$, such that $u(x,0)\geq c$, for all $x\in M$. Then $u(x,t)\geq c$, for any $(x,t)\in M\times[0,T)$.*

*Proof.* Let $\varepsilon>0$ and define the auxiliary function $u^\varepsilon \in C^\infty\big(M\times[0,T)\big)$ given by

$$u^\varepsilon(x,t) = u(x,t) + \varepsilon(1+t).$$

Note that,

$$u^\varepsilon(x,0) = u(x,0)+\varepsilon \geq c+\varepsilon > c.$$

We will show $u^\varepsilon(x,t)>c$, for every $(x,t)\in M\times[0,T)$. Arguing by contradiction, suppose that there exists an $\varepsilon>0$, such that

$$u^\varepsilon(x,t)\leq c,$$

for every $(x,t)\in M\times[0,T)$. Since the $M$ is compact, there exists a point $(x',t')\in M\times[0,T)$ such that

$$u^\varepsilon(x',t') = c \text{ and } u^\varepsilon(x,t)\leq c,$$

for every $(x,t)\in M\times[0,t']$. Therefore, at $(x',t')$ we have

$$u^\varepsilon_t(x',t')\leq 0 \text{ and } \Delta u^\varepsilon(x',t')\geq 0.$$

Thus, from hypothesis, we have

$$0\geq u^\varepsilon_t - \Delta u^\varepsilon = u_t + \varepsilon - \Delta u \geq \varepsilon > 0,$$

which is a contradiction. Hence,

$$u^{\varepsilon}(x,t) > c,$$

for $(x,t) \in M \times [0,T)$ and since $\varepsilon$ is arbitrary, we have $u(x,t) \geq c$, for $(x,t) \in M \times [0,T)$. This completes the proof. □

**Proposition 3.2.2** *Let $u, v\colon M \times [0,T) \to \mathbb{R}$ be two smooth functions, which satisfy the differential inequality*

$$\mathscr{L}v \leq \mathscr{L}u.$$

*If $v(x,0) \leq u(x,0)$, for every $x \in M$, then*

$$v(x,t) \leq u(x,t),$$

*for every $(x,t) \in M \times [0,T)$.*

*Proof.* Consider the function $w = u - v$. Then, we have

$$0 \leq w_t - \Delta w - g(X, \nabla w) - \big(\Psi(u,t) - \Psi(v,t)\big). \tag{3.3}$$

Since $\Psi$ is Lipschitz in the first variable, there exists a constant $c > 0$, such that

$$\big|\Psi\big(u(x,t),t\big) - \Psi\big(v(x,t),t\big)\big| \leq c|u(x,t) - v(x,t)| = c|w(x,t)|,$$

for every $(x,t) \in M \times [0,t_1]$ where $t_1 < T$. For any $\varepsilon > 0$, define the auxiliary function $w^{\varepsilon}$ given by

$$w^{\varepsilon}(x,t) = w(x,t) + \varepsilon e^{2ct}.$$

Observe that, for every $x \in M$, it holds

$$w^{\varepsilon}(x,0) = w(x,0) + \varepsilon = u(x,0) - v(x,0) + \varepsilon > 0.$$

On the other hand,

$$\mathscr{L}w^{\varepsilon} = \mathscr{L}w + \varepsilon\mathscr{L}(e^{2ct}) = \mathscr{L}w + 2c\varepsilon e^{2ct} \geq 2c\varepsilon e^{2ct}.$$

Therefore,

$$w^{\varepsilon}_t \geq \Delta w + g(X, \nabla w) - c|w| + 2c\varepsilon e^{2ct}.$$

Let $(x', t') \in M \times [0,T)$ be the point where $w^{\varepsilon}(x',t') = 0$. Then, at that point we have

$$w = -\varepsilon e^{2ct'}.$$

Moreover, since the spatial derivatives vanish at $(x', t')$, we obtain

$$\nabla w = 0 \ \text{ and } \ \Delta w = 0.$$

On the other hand, at the point $(x', t')$ we have

$$\partial_t w = -2c\varepsilon e^{2ct'} \leq 0.$$

Thus, at the point $(x', t') \in M \times [0, T)$, we obtain

$$0 \geq w_t^\varepsilon \geq \Delta w + g(X, \nabla w) - c\varepsilon e^{2ct'} + 2c\varepsilon e^{2ct'} \geq c\varepsilon e^{2ct'} > 0,$$

which is a contradiction. Therefore, for every $\varepsilon > 0$ we have $w^\varepsilon > 0$ which implies $w \geq 0$ on $M \times [0, t_1]$. Since $t_1 \in [0, T)$ is arbitrary, we have $w \geq 0$ on $M \times [0, T)$. Hence,

$$v(x, t) \leq u(x, t),$$

for every $(x, t) \in M \times [0, T)$. This completes the proof. □

The following theorem is an immediate consequence of the above proposition.

**Theorem 3.2.3 (Comparison principle-I)** *Let $u \colon M \times [0, T) \to \mathbb{R}$ be a smooth function, which satisfies the differential inequality*

$$u_t - \Delta u \geq g(X, \nabla u) + \Psi(u, t).$$

*Let $\phi$ be the solution of the associated ODE*

$$\begin{cases} \phi'(t) = \Psi(\phi(t), t) \\ \phi(0) = \min_{x \in M} u(x, 0) \end{cases}.$$

*Then, the solution $u$ of the partial differential inequality is bounded from below by the solution $\phi$ of the ODE, that is*

$$u(x, t) \geq \phi(t),$$

*for every $(x, t) \in M \times [0, T)$.*

Similarly, we prove the following version of the above theorem.

**Theorem 3.2.4 (Comparison principle-II)** *Let $u \colon M \times [0, T) \to \mathbb{R}$ be a smooth function, which satisfies the differential inequality*

$$u_t - \Delta u \leq g(X, \nabla u) + \Psi(u, t).$$

*Let $\rho$ be the solution of the associated ODE*

$$\begin{cases} \rho'(t) = \Psi(\rho(t), t) \\ \rho(0) = \max_{x \in M} u(x, 0) \end{cases}.$$

*Then, the solution $u$ of the partial differential inequality is bounded from below by the solution $\rho$ of the ODE, that is*

$$u(x, t) \leq \rho(t),$$

*for every $(x, t) \in M \times [0, T)$.*

CHAPTER 4

# Mean Curvature Flow

In this chapter we introduce the notion of the mean curvature flow. Later, we will examine how various geometric quantities evolve under the mean curvature flow. Suppose that $M$ is a manifold of dimension $m$, let $T > 0$ be a real number and $F\colon M \times [0,T) \to N$ a smooth time-dependent family of immersions of $M$ into a Riemannian manifold $N$ of dimension $m+n$. We follow the exposition in [17, 29, 42].

**Definition 4.0.1** *Let $N$ be a Riemannian manifold. We will say that a family of immersions $F\colon M \times [0,T) \to N$ evolves by mean curvature flow (MCF for short) with initial data the immersion $F_0\colon M \to N$ if it satisfies the initial value problem*

$$\begin{cases} F_t(x,t) = H\big(F(x,t)\big) \\ F(x,0) = F_0(x) \end{cases},$$

*for any $(x,t) \in M \times [0,T)$, where $H\big(F(x,t)\big)$ denotes the mean curvature vector of the immersion $F(\cdot\,,t) : M \to N$ at the point $x \in M$.*

## 4.1 Short-time existence of the flow

Consider the map $F\colon M \times [0,T) \to N$, where $M$ and $N$ are of dimension $m$ and $m+n$, respectively, and let represent it in local coordinates by

$$F(x_1,\dots,x_m,t) = \big(F^1(x_1,\dots,x_m,t),\dots,F^{m+n}(x_1,\dots,x_m,t)\big).$$

Then, from (1.2) we have

$$F_t^\alpha = \sum_{i,j} g^{ij}\big(F^\alpha_{x_ix_j} - \sum_k \Gamma^k_{ij}F^\alpha_{x_k} + \sum_{\gamma,\delta}\Gamma^\alpha_{\gamma\delta}F^\gamma_{x_i}F^\delta_{x_j}\big),$$

where

$$g_{ij} = \sum_{\alpha,\beta} h_{\alpha\beta}F^\alpha_{x_i}F^\beta_{x_j} \ \text{ and } \ \Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}(\partial_{x_i}g_{jl} + \partial_{x_j}g_{il} - \partial_{x_l}g_{ij}).$$

We compute

$$\partial_{x_i} g_{jl} = \sum_{\alpha,\beta} \big( h_{\alpha\beta} F^\alpha_{x_i x_j} F^\beta_{x_l} + h_{\alpha\beta} F^\alpha_{x_j} F^\beta_{x_i x_l} + \partial_{x_i} h_{\alpha\beta} F^\alpha_{x_j} F^\beta_{x_l} \big),$$

$$\partial_{x_j} g_{il} = \sum_{\alpha,\beta} \big( h_{\alpha\beta} F^\alpha_{x_i x_j} F^\beta_{x_l} + h_{\alpha\beta} F^\alpha_{x_i} F^\beta_{x_j x_l} + \partial_{x_j} h_{\alpha\beta} F^\alpha_{x_i} F^\beta_{x_l} \big),$$

$$\partial_{x_l} g_{ij} = \sum_{\alpha,\beta} \big( h_{\alpha\beta} F^\alpha_{x_i x_l} F^\beta_{x_j} + h_{\alpha\beta} F^\alpha_{x_i} F^\beta_{x_l x_j} + \partial_{x_l} h_{\alpha\beta} F^\alpha_{x_i} F^\beta_{x_j} \big).$$

Therefore, we obtain

$$\Gamma^k_{ij} = \sum_l g^{kl} F^\alpha_{x_i x_j} F^\beta_{x_l} + \text{lower order terms.} \tag{4.1}$$

Combining the formula (4.1) with equation (1.2), we have that mean curvature flow in local coordinates is of the form

$$F^\alpha_t = \sum_{i,j,\beta} g^{ij} \big( \delta_{\alpha\beta} - \sum_{k,l,\gamma} g^{kl} h_{\beta\gamma} F^\alpha_{x_k} F^\gamma_{x_l} \big) F^\beta_{x_i x_j} + \text{lower order terms.}$$

By a straightforward computation, we obtain

$$\begin{aligned} DH|_F(G) &= \lim_{s\to 0} \frac{H(F+sG) - H(F)}{s} \\ &= \sum_{i,j,\beta} g^{ij} \big( \delta_{\alpha\beta} - \sum_{k,l,\gamma} g^{kl} h_{\beta\gamma} F^\alpha_{x_k} F^\gamma_{x_l} \big) G^\beta_{x_i x_j} + \text{lower order terms.} \end{aligned}$$

Observe now that the projection of $\mathbb{R}^{m+n}$ onto $dF(T_x M)$ is given by

$$\pi_{TM}(\xi) = \sum_{k,l} g^{kl} \langle \xi, dF(\partial_{x_k}) \rangle dF(\partial_{x_l}) = \sum_{k,l,\alpha,\beta} g^{kl} \xi_\alpha F^\alpha_{x_k} F^\beta_{x_l} \partial_{y_\beta},$$

for any $\xi \in \mathbb{R}^{m+n}$. Therefore, the principal symbol is given by

$$\sigma_\xi(DH; x)G = |\xi|^2 \pi_{NM}(\xi) G(x),$$

where $\pi_{NM}$ is the projection on the normal bundle of the immersion $F$.

Observe that the principal symbol is zero for tangent directions. Therefore, the mean curvature flow is a degenerate quasilinear parabolic evolution equation and we can't obtain information from the standard theory of parabolic partial differential equations about short-time existence. Before giving the proof of short-time existence, let us discuss at first the behavior of the flow under the action of diffeomorphism group.

**Theorem 4.1.1 (Invariance under diffeomorphisms)** *Let $F\colon M \times [0,T) \to N$ be a solution of the mean curvature flow and that $\psi$ a fixed diffeomorphism of $M$. Then $\widehat{F}\colon M \times [0,T) \to N$ given by*

$$\widehat{F}(x,t) = F\big(\psi(x), t\big),$$

*for any $(x,t) \in M \times [0,T)$, is another solution of the flow.*

*Proof.* At first notice that the induced by $\widehat{F}$ time-dependent metrics $\widehat{g}$ are related with the induced by $F$ time-dependent metrics $g$ on $M$ through the relation $\widehat{g} = \psi^* g$. Let us fix the time parameter. Denote by $\widehat{\nabla}$ the associated with $\widehat{g}$ Levi-Civita connection. Since the map $\psi : (M, \widehat{g}) \to (M, g)$ is an isometry, we have

$$d\psi(\widehat{\nabla}_v w) = \nabla_{d\psi(v)} d\psi(w),$$

for any $v, w \in \mathfrak{X}(M)$. The second fundamental form $\widehat{A}$ of $\widehat{F}$ is given by

$$\begin{aligned} \widehat{A}(v, w) &= \nabla^N_{d\widehat{F}(v)} d\widehat{F}(w) - d\widehat{F}(\widehat{\nabla}_v w) \\ &= \nabla^N_{dF\left(d\psi(v)\right)} dF\big(d\psi(w)\big) - dF\big(d\psi(\widehat{\nabla}_v w)\big) \\ &= \nabla^N_{dF\left(d\psi(v)\right)} dF\big(d\psi(w)\big) - dF\big(\nabla_{d\psi(v)} d\psi(w)\big) \\ &= A\big(d\psi(v), d\psi(w)\big), \end{aligned}$$

for any $v, w \in \mathfrak{X}(M)$, where $\nabla^N$ stands for the Levi-Civita connection of $N$. Hence, the mean curvatures $\widehat{H}$ and $H$ are related by the expression

$$\widehat{H}\big(\widehat{F}(x, t)\big) = H\big(F(\psi(x), t)\big),$$

for any $(x, t) \in M \times [0, T)$. Thus,

$$\widehat{F}_t(x, t) = F_t\big(\psi(x), t\big) = H\big(F(\psi(x), t)\big) = \widehat{H}\big(\widehat{F}(x, t)\big),$$

for any $(x, t) \in M \times [0, T)$. Therefore, $\widehat{F}$ is another solution of the mean curvature flow. This completes the proof. ☐

**Theorem 4.1.2 (Invariance under tangential variations)** *Let $F\colon M \times [0, T) \to N$ be a family of immersions satisfying the system of PDEs*

$$\begin{cases} F_t(x, t) = H\big(F(x, t)\big) + dF_{(x,t)}\big(V(x, t)\big) \\ F(x, 0) = F_0(x) \end{cases}, \tag{4.2}$$

*where $(x, t) \in M \times [0, T)$, the manifold $M$ is compact and $V$ is a time-dependent family of smooth vector fields. Then, there exists a unique family of time-dependent diffeomorphisms $\psi\colon M \times [0, T) \to M$, such that the map $\widehat{F}\colon M \times [0, T) \to N$ given by*

$$\widehat{F}(x, t) = F\big(\psi(x, t), t\big),$$

*is a solution of the mean curvature flow*

$$\begin{cases} \widehat{F}_t(x, t) = H\big(\widehat{F}(x, t)\big) \\ \widehat{F}(x, 0) = F_0\big(\psi(x, 0)\big) \end{cases}.$$

*Conversely, if the map $F\colon M \times [0, T) \to N$ is a solution of the mean curvature flow and $\psi\colon M \times [0, T) \to M$ is a time-dependent family of diffeomorphisms, then the smooth map $\widehat{F}\colon M \times [0, T) \to N$ satisfies a system of the form (4.2).*

*Proof.* Let us show the first part of the Theorem. Consider for the moment an arbitrary family a time-dependent of diffeomorphisms $\psi\colon M \times [0,T) \to M$ and define $\widehat{F}\colon M \times [0,T) \to N$ given by

$$\widehat{F}(x,t) = F\big(\psi(x,t),t\big),$$

for any $(x,t) \in M \times [0,T)$. From the chain rule, we have

$$\begin{aligned}\widehat{F}_t(x,t) &= dF_{(\psi(x,t),t)}(\partial_t) + dF_{(\psi(x,t),t)}\big(d\psi_{(x,t)}(\partial_t)\big)\\ &= H\big(\psi(x,t),t\big) + dF_{(\psi(x,t),t)}\big(V(\psi(x,t),t) + d\psi_{(x,t)}(\partial_t)\big)\\ &= H\big(\widehat{F}(x,t)\big) + dF_{(\psi(x,t),t)}\big(V(\psi(x,t),t) + d\psi_{(x,t)}(\partial_t)\big).\end{aligned}$$

Therefore, it suffices to find a one-parameter family of time-dependent diffeomorphisms $\psi\colon M \times [0,T) \to M$ solving the initial value problem

$$\begin{cases} d\psi_{(x,t)}(\partial_t) = -V\big(\psi(x,t),t\big) \\ \psi(x,0) = I \end{cases},$$

for any $(x,t) \in M \times [0,T)$, where $I\colon M \to M$ is the identity map. This is an initial value problem for a system of ODE's and by Picard-Lindelöf theorem always there exists a unique solution. Moreover, due to the fact that the initial data is the identity, taking $T > 0$ small enough we can assume that for any $t \in [0,T]$ the map $\psi(\cdot\,,t)\colon M \to M$ is a diffeomorphism. This completes the proof. □

**Theorem 4.1.3 (Short-time existence)** *Suppose $M$ is a compact Riemannian manifold and $F_0\colon M \to N$ is an immersion into a Riemannian manifold $N$. Then, the mean curvature flow with initial data the immersion $F_0$ admits a smooth solution on a maximal time interval $[0,T)$, where $0 < T \leq \infty$.*

*Proof.* As we already observed the mean curvature flow is degenerate. The idea is to search for an equivalent flow which is parabolic. Motivated by the computations in Theorem 4.1.2, we will try to modify the mean curvature flow by adding some tangential component in order to make it parabolic. So, let us suppose that $F : M \times [0,T) \to N$ satisfies the mean curvature flow system. Fix a Riemannian metric $\widehat{g}$ on $M$, denote its Levi-Civita connection by $\widehat{\nabla}$ and consider the time-dependent vector field $V_{DT}$ on $M$ given by

$$V_{DT} = \operatorname{tr}_g(\nabla - \widehat{\nabla}). \tag{4.3}$$

Note that in local coordinates, $V_{DT}$ has the form

$$V_{DT} = \sum_{i,j,k} g^{ij}(\Gamma^k_{ij} - \widehat{\Gamma}^k_{ij})\partial_{x_k},$$

where $\Gamma^k_{ij}$ and $\widehat{\Gamma}^k_{ij}$ are the Christoffel symbols of the connections $\nabla$ and $\widehat{\nabla}$, respectively. Consider now the initial value problem,

$$\begin{cases} F_t = H + dF(V_{DT}), \\ F(\cdot,0) = F_0. \end{cases} \tag{4.4}$$

The first equation of (4.4) in local coordinates takes the form

$$\begin{aligned} F_t &= \sum_{i,j,\alpha} g^{ij}\big(F^{\alpha}_{x_i x_j} - \sum_k \Gamma^k_{ij} F^{\alpha}_{x_k} + \sum_{\gamma,\delta} \Gamma^{\alpha}_{\gamma\delta} F^{\gamma}_{x_i} F^{\delta}_{x_j}\big)\partial_{y_\alpha} \\ &\quad + \sum_{i,j,k} g^{ij}(\Gamma^k_{ij} - \widehat{\Gamma}^k_{ij}) dF(\partial_{x_k}) \\ &= \sum_{i,j,\alpha} g^{ij}\big(F^{\alpha}_{x_i x_j} - \sum_k \widehat{\Gamma}^k_{ij} F^{\alpha}_{x_k} + \sum_{\gamma,\delta} \Gamma^{\alpha}_{\gamma\delta} F^{\gamma}_{x_i} F^{\delta}_{x_j}\big)\partial_{y_\alpha}. \end{aligned}$$

Since $\widehat{\Gamma}^k_{ij}$ does not depend on time, the principal symbol of (4.4) is

$$\sigma_\xi\big(D(H + V_{DT}), \cdot\big) = |\xi|^2 I$$

and so the problem is parabolic. Thus, the parabolic theory can be used to ensure short-time existence of the modified flow. According to Theorem 4.1.2, from a solution of (4.4) we obtain a solution of the standard mean curvature flow. This completes the proof. □

**Definition 4.1.4** *Let $F\colon M \times [0,T) \to N$ be a solution of the mean curvature flow. Fix a metric $\widehat{g}$ and let $V_{DT}$ be the vector field given by* (4.3)*. The modified flow* (4.4) *is called DeTurck mean curvature flow.*

**Lemma 4.1.5** *The vector field $V_{DT}$ defined in equation* (4.3) *is minus the Laplacian of the identity map $I\colon (M,g) \to (M,\widehat{g})$.*

*Proof.* The Hessian $B$ of the map $I$ is given by

$$B(v_1, v_2) = \widehat{\nabla}_{dI(v_1)} dI(v_2) - dI(\nabla_{v_1} v_2) = \widehat{\nabla}_{v_1} v_2 - \nabla_{v_1} v_2,$$

for any vector fields $v_1, v_2 \in \mathfrak{X}(M)$. Therefore,

$$\Delta_{g,\widehat{g}} I = \operatorname{tr}_g B = -V_{DT}.$$

This completes the proof. □

**Theorem 4.1.6 (Uniqueness)** *Let $M$ be a compact Riemannian manifold and let $F_0\colon M \to N$ an immersion into a Riemannian manifold $N$. Then, the solution of the mean curvature flow, with initial data the immersion $F_0\colon M \to N$, is unique up to diffeomorphisms.*

*Proof.* Suppose that $\widetilde{F}\colon M \times [0,T) \to N$ is a solution of the mean curvature flow, with initial data the given immersion $F_0$, and denote the associated the induced metric by $\widetilde{g}$. As in the existence part, fix a metric $\widehat{g}$ and denote by $\widehat{\nabla}$ its associated Levi-Civita connection. Consider now the initial value problem

$$\begin{cases} \phi_t = \Delta_{\widetilde{g},\widehat{g}} \phi, \\ \phi(\cdot, 0) = I. \end{cases}$$

Observe that the above problem is a parabolic and thus its solution gives rise to a unique one parameter family of diffeomorphisms $\phi : M \times [0, \varepsilon) \to M$, for at least some short time $\varepsilon > 0$. Denote by $\psi \colon M \times [0, \varepsilon) \to M$ the one parameter family of diffeomorphisms with the property that, for each $t$, the map $\psi(\cdot\,, t)$ is the inverse of $\phi(\cdot\,, t)$, i.e.,

$$\psi(\phi(x,t),t) = x = \phi(\psi(x,t),t)$$

for any $(x, t)$ in space-time. From the chain rule, we have

$$d\psi_{(\phi(x,t),t)}(\partial_t) = -d\psi_{(\phi(x,t),t)}\big((\Delta_{\widetilde{g},\widehat{g}}\phi)(x)\big). \tag{4.5}$$

Define now the map $F \colon M \times [0, \varepsilon) \to N$ given by

$$F(x,t) = \widetilde{F}(\psi(x,t),t),$$

for any $(x, t) \in M \times [0, T)$. The induced time-dependent metric on $M$ is $g = \psi^*\widetilde{g}$. Moreover, the map $F$ satisfies the evolution equation

$$F_t = H + d\widetilde{F}(W), \tag{4.6}$$

where for any point $(x, t)$ in space-time, we have

$$W(\psi(x,t),t) = d\psi_{(x,t)}(\partial_t).$$

Taking into account (4.5) and the composition formula for the Laplacian (see for example [12, page 116, equation (2.56)]), we have

$$\begin{aligned} W(\psi(x,t),t) &= d\psi_{(x,t)}\big((\Delta_{\widetilde{g},\widehat{g}}\phi)(\psi(x,t))\big) \\ &= d\psi_{(x,t)}\big((\Delta_{\psi^*\widetilde{g},\widehat{g}}I)(x)\big) = d\psi_{(x,t)}\big((\Delta_{g,\widehat{g}}I)(x)\big) \\ &= d\psi_{(x,t)}(V_{DT}(x)), \end{aligned} \tag{4.7}$$

for any $(x, t) \in M \times [0, \varepsilon)$. From (4.6) and (4.7), we see that $F$ satisfies the DeTurck mean curvature flow

$$F_t = H + dF(V_{DT}),$$

with initial data the immersion $F_0$.

Suppose now that $\widetilde{F}_1, \widetilde{F}_2 \colon M \times [0, T) \to N$ are two solutions of the mean curvature flow, with the same initial condition $F_0 \colon M \to N$. As before fix a metric $\widehat{g}$ on the manifold $M$ and denote by $\widetilde{g}_1$ and $\widetilde{g}_2$ the induced time-dependent metrics on $M$ by $\widetilde{F}_1$ and $\widetilde{F}_2$, respectively. Denote by $\phi^i \colon M \times [0, \varepsilon) \to N$, $i \in \{1, 2\}$, the one-parameter family of diffeomorphisms solving the initial value problem

$$\begin{cases} \eta_t = \Delta_{\widetilde{g}_i,\widehat{g}}\eta \\ \eta(\cdot, 0) = I \end{cases}.$$

Then, as we already saw above, the maps $F_i : M \times [0, \varepsilon) \to N$, $i \in \{1, 2\}$, satisfying

$$\widetilde{F}_i(x,t) = F_i\big(\phi^i(x,t),t\big),$$

for any $(x, t) \in M \times [0, \varepsilon)$, form solutions of the DeTurck mean curvature flow, with common initial data the immersion $F_0 \colon M \to N$. Since the DeTurck mean curvature flow is parabolic, it follows that its solution is unique. This completes the proof. □

**Remark 4.1.1** Short-time existence and uniqueness of the mean curvature flow was originally proven using results of Hamilton [20, 21] based on the Nash-Moser iteration method. The proof that we presented was adapting a variant of the so called DeTurck's trick which was first used in Ricci flow [15].

In the following theorem we give a characterisation of the maximal time of solutions of the mean curvature flow.

**Theorem 4.1.7 (Huisken)** *Let $M$ be a compact manifold and let $F_0\colon M \to N$ be a smooth immersion into a complete Riemannian manifold $N$. Then, the maximal time $T$ of the solution of the mean curvature flow, with initial data the immersion $F_0$, is finite if and only if*

$$\limsup_{t\to T}\big(\max_{M\times[0,t]}|A|\big) = \infty.$$

The above theorem is also of crucial importance in our analysis. Its proof has been done by Huisken in [23, 24] and is based on the parabolic maximum principle. The key observation is that all higher derivatives $\nabla^k A$, $k \in \mathbb{N}$, of the second fundamental tensor are uniformly bounded, once $A$ is uniformly bounded.

We complete this subsection with the following theorem.

**Theorem 4.1.8 (Invariance under ambient isometries)** *Let $F\colon M\times[0,T)\to N$ be a smooth solution of the mean curvature flow and assume that $\Phi$ is an isometry of the ambient space $N$. Then, the family*

$$\widehat{F} := \Phi \circ F : M \times [0,T) \to N$$

*is another smooth solution of the mean curvature flow. In particular, if the initial immersion is invariant under $\Phi$, then the solution will stay invariant as long the flow exists.*

*Proof.* Suppose that $F$ is a solution of the mean curvature flow and that $\Phi$ is an isometry of the ambient space. From the chain rule, we have

$$\begin{aligned} d\widehat{F}_{(x,t)}(\partial_t) &= d\Phi_{F(x,t)}\big(dF_{(x,t)}(\partial_t)\big) = d\Phi_{F(x,t)}\big(H(F(x,t))\big) \\ &= \widehat{H}\big(\widehat{F}(x,t)\big), \end{aligned}$$

for any $(x,t) \in M \times [0,T)$, where $\widehat{H}$ is the mean curvature vector of $\widehat{F}$. Therefore, the family $\widehat{F}$ is another smooth solution of the mean curvature flow. This completes the proof. □

## 4.2 Evolution equations under MCF

We start by computing the evolution of first order quantities and geometric quantities involving the second fundamental form.

### 4.2.1 Evolution of first order quantities

Let us start by computing evolution of first order quantities i.e., quantities that arise from the differential of the immersions.

**Lemma 4.2.1** *Suppose that* $F\colon M \times [0,T) \to N$ *is a solution of the mean curvature flow. Then, the following facts are true:*

(a) *The induced metrics* $g$ *evolve in time under the equation*

$$\big(\nabla_{\partial_t} g\big)(v_1, v_2) = -2\langle H, A(v_1, v_2)\rangle := -2A^H(v_1, v_2),$$

*for any* $v_1, v_2 \in \mathfrak{X}(M)$.

(b) *The induced volume form* $d\mu$ *on* $(M, g)$ *evolves according to*

$$\partial_t(d\mu) = -|H|^2 d\mu.$$

(c) *There exists local smooth time-dependent tangent orthonormal frame field and local smooth time-dependent orthonormal frame field along the normal bundle of the evolving submanifolds.*

*Proof.* We have:

(a) Let $v_1, \ldots, v_m$ be time independent tangent vector fields. Then,

$$\nabla^F_{\partial_t} dF(v_i) = \nabla^F_{v_i} dF(\partial_t) + dF\big([\partial_t, v_i]\big) = \nabla^F_{v_i} H,$$

for any $i \in \{1, \ldots, m\}$. Therefore, for any $i, j \in \{1, \ldots, m\}$, we have

$$\begin{aligned}
\big(\nabla_{\partial_t} g\big)(v_i, v_j) &= \partial_t g(v_i, v_j) \\
&= \partial_t \langle dF(v_i), dF(v_j)\rangle \\
&= \langle \nabla^F_{v_i} H, dF(v_j)\rangle + \langle \nabla^F_{v_j} H, dF(v_i)\rangle \\
&= -\langle H, \nabla^F_{v_i} dF(v_j)\rangle - \langle H, \nabla^F_{v_j} dF(v_i)\rangle \\
&= -2\langle H, A(v_i, v_j)\rangle.
\end{aligned}$$

(b) We compute

$$\begin{aligned}
\partial_t \sqrt{\det g_{ij}} &= \frac{\big(g^{kl}\partial_t g_{kl}\big)\det g_{ij}}{2\sqrt{\det g_{ij}}} = -\sum_{k,l}\langle H, g^{kl} A_{kl}\rangle \sqrt{\det g_{ij}} \\
&= -|H|^2 \sqrt{\det g_{ij}}.
\end{aligned}$$

Integrating the above equation, we obtain

$$\sqrt{\det g_{ij}}(x,t) = \sqrt{\det g_{ij}}(x,0)e^{-\rho(x,t)},$$

where

$$\rho(x,t) = \int_0^t |H|^2(x,t)dt.$$

But, for $\tau < T$, the second fundamental form is bounded. Hence, $|H|$ is also bounded and so

$$\sqrt{\det g_{ij}}(x,t) > 0,$$

for all $(x,t) \in M \times [0,\tau]$.

(c) Recall that $g^{ij}$ are elements of the inverse matrix of $g$ with respect to the basis $\{v_1,\dots,v_m\}$, i.e.,

$$(g^{ij}) = (g_{ij})^{-1},$$

where

$$g_{ij} = g(v_i, v_j),$$

for any $i,j \in \{1,\dots,m\}$. Since $A^H$ is a symmetric 2-tensor, its associated adjoint operator $P\colon (TM,g) \to (TM,g)$ satisfies

$$A^H(v_1,v_2) = g(Pv_1,v_2) = g(v_1,Pv_2), \tag{4.8}$$

for $v_1, v_2 \in \mathfrak{X}(M)$. Consider the time-dependent bundle isomorphism

$$\iota(t) : \big(TM, g(0)\big) \to \big(TM, g(t)\big),$$

given as the solution of the following initial value problem

$$\begin{cases} \nabla_{\partial_t}\iota(t) = P \circ \iota(t) \\ \iota(0) = I. \end{cases}$$

We claim that

$$\iota^*(t)g(t) = g(0).$$

By a straightforward computation we see that

$$\begin{aligned}\partial_t\big(\iota^*(t)g(t)(\partial_{x_i},\partial_{x_j})\big) &= \partial_t\big(g(t)(\iota(t)\partial_{x_i},\iota(t)\partial_{x_j})\big)\\ &= \big(\nabla_{\partial_t}g(t)\big)\big(\iota(t)\partial_{x_i},\iota(t)\partial_{x_j}\big) + g(t)\big(\nabla_{\partial_t}\iota(t)\partial_{x_i},\iota(t)\partial_{x_j}\big)\\ &\quad + g(t)\big(\iota(t)\partial_{x_i},\nabla_{\partial_t}\iota(t)\partial_{x_j}\big),\end{aligned}$$

which using part (a) becomes

$$\begin{aligned}\partial_t\big(\iota^*(t)g(t)(\partial_{x_i},\partial_{x_j})\big) = &-2A^H\big(\iota(t)\partial_{x_i},\iota(t)\partial_{x_j}\big) + g(t)\big(P\iota(t)\partial_{x_i},\iota(t)\partial_{x_j}\big)\\ &+ g(t)\big(\iota(t)\partial_{x_i},P\iota(t)\partial_{x_j}\big).\end{aligned}$$

Thus, from (4.8), we have

$$\begin{aligned}\partial_t\big(\iota^*(t)g(t)(\partial_{x_i},\partial_{x_j})\big) &= -2A^H(\iota(t)\partial_{x_i},\iota(t)\partial_{x_j}) + A^H(\iota(t)\partial_{x_i},\iota(t)\partial_{x_j})\\ &\quad + A^H(\iota(t)\partial_{x_i},\iota(t)\partial_{x_j})\\ &= 0.\end{aligned}$$

Consequently,

$$\iota^*(t)g(t) = \iota^*(0)g(0) = g(0).$$

Therefore, if $\{e_1(0), \ldots, e_m(0)\}$ is a local orthonormal frame of $\big(TM, g(0)\big)$, then

$$\{e_1(t) = \iota(t)e_1(0), \ldots, e_m(t) = \iota(t)e_m(0)\}$$

is a local orthonormal frame of $\big(TM, g(t)\big)$. By taking the complement of the frame $\{e_1, \ldots, e_m\}$, we get the desired time-dependent frame field on the normal bundles of the evolving submanifolds.

This completes the proof. □

**Lemma 4.2.2** *The Christoffel symbols $\Gamma^k_{ij}$ of the Levi-Civita connection $\nabla$ evolve in time according to the equation*

$$\partial_t\big(\Gamma^k_{ij}\big) = -\sum_l g^{kl}\big((\nabla_{\partial_{x_i}} A^H)_{jl} + (\nabla_{\partial_{x_j}} A^H)_{il} - (\nabla_{\partial_{x_l}} A^H)_{ij}\big),$$

*for any $i, j, k \in \{1, \ldots, m\}$.*

*Proof.* Differentiating in time the identity

$$\sum_s g^{is} g_{sl} = \delta_{il},$$

we obtain

$$\sum_s \big(\partial_t(g^{is})g_{sl} + g^{is}\partial_t(g_{sl})\big) = 0.$$

Consequently, we get

$$\sum_{s,l} \partial_t(g^{is})g_{sl}g^{lj} = \sum_s \partial_t(g^{is})\delta_{sj} = -\sum_{s,l} g^{is}g^{lj}\partial_t(g_{sl}).$$

According to Lemma 4.2.1, we obtain

$$\partial_t(g^{ij}) = \sum_{s,l} 2g^{is}g^{lj}A^H_{sl}. \tag{4.9}$$

From the Koszul formula we have that

$$2\Gamma^k_{ij} = \sum_l g^{kl}(\partial_{x_j} g_{il} + \partial_{x_i} g_{jl} - \partial_{x_l} g_{ij}).$$

By differentiating, we obtain

$$\begin{aligned} 2\partial_t(\Gamma^k_{ij}) &= \sum_l g^{kl}\big(\partial_t(\partial_{x_j} g_{il}) + \partial_t(\partial_{x_i} g_{jl}) - \partial_t(\partial_{x_l} g_{ij})\big) \\ &\quad + \sum_l \partial_t(g^{kl})(\partial_{x_j} g_{il} + \partial_{x_i} g_{jl} - \partial_{x_l} g_{ij}). \end{aligned}$$

Using Lemma 4.2.1 and the equation (4.9), we get

$$\begin{aligned}\partial_t(\Gamma^k_{ij}) &= -\sum_l g^{kl}\big(\partial_{x_j}A^H_{il} + \partial_{x_i}A^H_{jl} - \partial_{x_l}A^H_{ij}\big) \\ &\quad + \sum_{l,s,z} g^{ks}g^{zl}A^H_{sz}\big(\partial_{x_j}g_{il} + \partial_{x_i}g_{jl} - \partial_{x_l}g_{ij}\big).\end{aligned}$$

Hence,

$$\begin{aligned}\partial_t(\Gamma^k_{ij}) &= -\sum_l g^{kl}\big((\nabla_{\partial_{x_j}}A^H)_{il} + (\nabla_{\partial_{x_i}}A^H)_{jl} - (\nabla_{\partial_{x_l}}A^H)_{ij}\big) \\ &\quad -\sum_{l,z} g^{kl}\big(\Gamma^z_{ij}A^H_{zl} + \Gamma^z_{jl}A^H_{iz} + \Gamma^z_{ij}A^H_{zl} + \Gamma^z_{il}A^H_{jz} - \Gamma^z_{il}A^H_{zj} - \Gamma^z_{lj}A^H_{iz}\big) \\ &\quad + \sum_{l,s,z} g^{ks}g^{zl}A^H_{sz}\big(\partial_{x_j}g_{il} + \partial_{x_i}g_{jl} - \partial_{x_l}g_{ij}\big).\end{aligned}$$

Consequently,

$$\begin{aligned}\partial_t(\Gamma^k_{ij}) &= -\sum_l g^{kl}\big((\nabla_{\partial_{x_j}}A^H)_{il} + (\nabla_{\partial_{x_i}}A^H)_{jl} - (\nabla_{\partial_{x_l}}A^H)_{ij}\big) - \sum_{l,z} 2g^{kl}\Gamma^z_{ij}A^H_{zl} \\ &\quad + \sum_{l,s,z} g^{ks}g^{zl}A^H_{sz}\big(\partial_{x_j}g_{il} + \partial_{x_i}g_{jl} - \partial_{x_l}g_{ij}\big) \\ &= -\sum_l g^{kl}\big((\nabla_{\partial_{x_j}}A^H)_{il} + (\nabla_{\partial_{x_i}}A^H)_{jl} - (\nabla_{\partial_{x_l}}A^H)_{ij}\big) \\ &\quad -\sum_{l,s,z} g^{kl}g^{zs}A^H_{zl}\big(\partial_{x_j}g_{is} + \partial_{x_i}g_{js} - \partial_{x_s}g_{ij}\big) \\ &\quad + \sum_{l,s,z} g^{ks}g^{zl}A^H_{sz}\big(\partial_{x_j}g_{il} + \partial_{x_i}g_{jl} - \partial_{x_l}g_{ij}\big).\end{aligned}$$

Rearranging the indices, we obtain

$$\partial_t(\Gamma^k_{ij}) = -\sum\nolimits_l g^{kl}\big((\nabla_{\partial_{x_i}}A^H)_{jl} + (\nabla_{\partial_{x_j}}A^H)_{il} - (\nabla_{\partial_{x_l}}A^H)_{ij}\big).$$

This completes the proof. □

### 4.2.2 Evolution of second order quantities

In this subsection we compute the evolution of the mean curvature and the second fundamental form of the evolved submanifolds.

**Lemma 4.2.3** *The time derivative of the second fundamental form $A$ is given by the formula*

$$\big(\nabla^\perp_{\partial_t}A\big)^\alpha_{ij} = \big(\nabla^{\perp 2}H\big)^\alpha_{ij} - \sum_{k,\beta} H^\beta A^\beta_{jk}A^\alpha_{ik} - \sum_\beta H^\beta \widetilde{R}_{\beta ij\alpha},$$

*where the indices are with respect to a local orthonormal frame.*

*Proof.* Fix a point $(x_0, t_0)$ in space-time and choose $\{\partial_{x_1}, \ldots, \partial_{x_m}\}$, such that at $x_0$ it forms an orthonormal frame with respect to $g(t_0)$. We have

$$(\nabla_{\partial_t} A)_{ij} = \nabla_{\partial_t} \nabla_{\partial_{x_i}} dF(\partial_{x_j}) - \nabla_{\partial_t} dF(\nabla_{\partial_{x_i}} \partial_{x_j}).$$

Now, from the identity

$$\widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_j})\big) = \nabla_{dF(\partial_t)} \nabla_{dF(\partial_{x_i})} dF(\partial_{x_j}) - \nabla_{dF(\partial_{x_i})} \nabla_{dF(\partial_t)} dF(\partial_{x_j})$$

and Lemma 4.2.2, we obtain

$$\begin{aligned}
(\nabla_{\partial_t} A)_{ij} &= \nabla_{\partial_{x_i}} \nabla_{\partial_t} dF(\partial_{x_j}) + \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_j})\big) \\
&\quad -\nabla_{\nabla_{\partial_{x_i}} \partial_{x_j}} H - dF\big([\partial_t, \nabla_{\partial_{x_i}} \partial_{x_j}]\big) \\
&= \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} H - \nabla_{\nabla_{\partial_{x_i}} \partial_{x_j}} H + \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_j})\big) \\
&\quad -\sum\nolimits_k \partial_t(\Gamma^k_{ij}) dF(\partial_{x_k}) \\
&= \nabla^2_{\partial_{x_i}, \partial_{x_j}} H + \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_j})\big) \\
&\quad +\sum\nolimits_{k,l} g^{kl} \big((\nabla_{\partial_{x_i}} A^H)_{jl} + (\nabla_{\partial_{x_j}} A^H)_{il} - (\nabla_{\partial_{x_l}} A^H)_{ij}\big) dF(\partial_{x_k}).
\end{aligned}$$

Hence,

$$\begin{aligned}
(\nabla^\perp_{\partial_t} A)_{ij} &= \sum_\alpha \langle (\nabla^\perp_{\partial_t} A)_{ij}, \xi_\alpha \rangle \xi_\alpha = \sum_\alpha \langle (\nabla_{\partial_t} A)_{ij}, \xi_\alpha \rangle \xi_\alpha \\
&= \sum_\alpha \langle \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} H - \nabla_{\nabla_{\partial_{x_i}} \partial_{x_j}} H, \xi_\alpha \rangle \xi_\alpha + \sum_{\alpha,\beta} H^\beta \widetilde{R}_{\beta ij\alpha} \xi_\alpha,
\end{aligned}$$

where $\{\xi_\alpha\}$ is an orthonormal frame. But,

$$\begin{aligned}
\sum_\alpha \langle \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} H - \nabla_{\nabla_{\partial_{x_i}} \partial_{x_j}} H, \xi_\alpha \rangle \xi_\alpha &= \sum_\alpha \langle \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} H - \nabla^\perp_{\nabla_{\partial_{x_i}} \partial_{x_j}} H, \xi_\alpha \rangle \xi_\alpha \\
&= \sum_\alpha \langle \nabla^\perp_{\partial_{x_i}} \big(\nabla^\perp_{\partial_{x_j}} H + \sum_k \langle \nabla_{\partial_{x_j}} H, dF(\partial_{x_k}) \rangle dF(\partial_{x_k})\big), \xi_\alpha \rangle \\
&\quad - \sum_\alpha \langle \nabla^\perp_{\nabla_{\partial_{x_i}} \partial_{x_j}} H, \xi_\alpha \rangle \xi_\alpha \\
&= \sum_\alpha \langle \nabla^\perp_{\partial_{x_i}} \big(\nabla^\perp_{\partial_{x_j}} H - \sum_k \langle H, A_{jk} \rangle dF(\partial_{x_k})\big), \xi_\alpha \rangle \\
&\quad - \sum_\alpha \langle \nabla^\perp_{\nabla_{\partial_{x_i}} \partial_{x_j}} H, \xi_\alpha \rangle \xi_\alpha.
\end{aligned}$$

Estimating at $x_0$, we get the result. □

**Lemma 4.2.4** *The mean curvature $H$ evolves in time under the equation*

$$(\nabla^\perp_{\partial_t} H)^\alpha = (\Delta^\perp H)^\alpha + \sum_{i,\beta} H^\beta \widetilde{R}_{\beta ii\alpha} + 2 \sum_{i,j} A^H_{ij} A^\alpha_{ij}.$$

*Moreover,*

$$\partial_t |H|^2 = \Delta |H|^2 - 2|\nabla^{\perp} H|^2 + 2|A^H|^2 + 2\sum_{i,\alpha,\beta} H^\alpha H^\beta \widetilde{R}_{\alpha i \beta i},$$

*where the indices are with respect to a local orthonormal frame.*

*Proof.* Fix a point $(x_0, t_0)$ in space-time and choose $\{\partial_{x_1}, \dots, \partial_{x_m}\}$, such that at $x_0$ it forms an orthonormal frame with respect to $g(t_0)$. From Lemma 4.2.3, we have

$$\begin{aligned}
\nabla_{\partial_t} H &= \sum_{i,j} \big( g^{ij} (\nabla_{\partial_t} A)_{ij} + \partial_t (g^{ij}) A_{ij} \big) \\
&= \sum_{i,j} \big( g^{ij} \nabla^2_{\partial_{x_i}, \partial_{x_j}} H + g^{ij} \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_j})\big) + \partial_t (g^{ij}) A_{ij} \big) \\
&\quad + \sum_{i,j,k,l} g^{ij} g^{kl} \big( (\nabla_{\partial_{x_j}} A^H)_{il} + (\nabla_{\partial_{x_i}} A^H)_{jl} - (\nabla_{\partial_{x_l}} A^H)_{ij} \big) dF(\partial_{x_k}).
\end{aligned}$$

Consequently, from (4.9) we obtain

$$\begin{aligned}
\nabla_{\partial_t} H &= \Delta H + \sum_{i,j} g^{ij} \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_j})\big) + 2\sum_{i,j,k,s} g^{ik} g^{sj} A^H_{sk} A_{ij} \\
&\quad + \sum_{i,j,k} g^{ij} g^{kl} \big( (\nabla_{\partial_{x_j}} A^H)_{il} + (\nabla_{\partial_{x_i}} A^H)_{jl} - (\nabla_{\partial_{x_l}} A^H)_{ij} \big) dF(\partial_{x_k}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla^{\perp}_{\partial_t} H &= \sum_\alpha \langle (\nabla^{\perp}_{\partial_t} H), \xi_\alpha \rangle \xi_\alpha = \sum_\alpha \langle \nabla_{\partial_t} H, \xi_\alpha \rangle \xi_\alpha \\
&= \sum_\alpha \langle \Delta H + \sum_i \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_i})\big) + 2\sum_{i,j} A^H_{ij} A_{ij}, \xi_\alpha \rangle \xi_\alpha \\
&= \Delta^{\perp} H + \sum_{i,\alpha} \langle \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_i})\big), \xi_\alpha \rangle \xi_\alpha + 2\sum_{i,j,\alpha} \langle A^H_{ij} A_{ij}, \xi_\alpha \rangle \xi_\alpha
\end{aligned}$$

and thus,

$$(\nabla^{\perp}_{\partial_t} H)^\alpha = (\Delta^{\perp} H)^\alpha + \sum_i \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_i}), \xi_\alpha\big) + 2\sum_{i,j} A^H_{ij} A^\alpha_{ij}.$$

We compute

$$\begin{aligned}
\partial_t |H|^2 &= \partial_t \langle H, H \rangle = 2\langle \nabla_{\partial_t} H, H \rangle \\
&= 2\langle \Delta H, H \rangle - 2\textstyle\sum_{i,j} g^{ij} \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_j}), H\big) \\
&\quad + 4\textstyle\sum_{i,j,k,s} g^{ik} g^{sj} A^H_{sk} A^H_{ij} \\
&= \Delta |H|^2 - 2|\nabla H|^2 - 2\textstyle\sum_{i,j} g^{ij} \widetilde{R}\big(H, dF(\partial_{x_i}), dF(\partial_{x_j}), H\big) \\
&\quad + 4\textstyle\sum_{i,j,k,s} g^{ik} g^{sj} A^H_{sk} A^H_{ij}.
\end{aligned}$$

Since

$$\begin{aligned}\nabla_{\partial_{x_j}} H &= \sum_{i,j} g^{ij}\langle \nabla_{\partial_{x_j}} H, dF(\partial_{x_i})\rangle dF(\partial_{x_j}) + \nabla^{\perp}_{\partial_{x_j}} H \\ &= -\sum_{i,j} g^{ij}\langle H, \nabla_{\partial_{x_j}} dF(\partial_{x_i})\rangle dF(\partial_{x_j}) + \nabla^{\perp}_{\partial_{x_j}} H \\ &= -\sum_{i,j} g^{ij} A^H_{ij} dF(\partial_{x_j}) + \nabla^{\perp}_{\partial_{x_j}} H,\end{aligned}$$

we have

$$\begin{aligned}\partial_t |H|^2 &= \Delta|H|^2 - 2|\nabla^{\perp} H|^2 + 2\sum_{i,j,k,s} g^{ik}g^{sj}A^H_{sk}A^H_{ij} \\ &\quad +2\sum_{i,j} g^{ij}\widetilde{R}\big(H, dF(\partial_{x_i}), H, dF(\partial_{x_j})\big).\end{aligned}$$

Estimating at $x_0$, we get the result. □

Let us compute now the evolution of the second fundamental form $A$. The proof is based on the well known Simons' formula [50] and can be found in [2], [4], [42] or [48]. For reader's convenience let us provide some details.

**Lemma 4.2.5 (Simons' formula)** *The Laplacian and the time derivative of the second fundamental form satisfies the following equations*

$$\begin{aligned}(\Delta^{\perp}A)^{\alpha}_{ij} &= (\nabla^{\perp 2}H)^{\alpha}_{ij} - \sum_k (\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha} - \sum_k (\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha} \\ &- 2\sum_{k,\beta} A^{\beta}_{ik}\widetilde{R}_{kj\beta\alpha} - 2\sum_{k,\beta} A^{\beta}_{jk}\widetilde{R}_{ki\beta\alpha} + 2\sum_{k,l} A^{\alpha}_{kl}\widetilde{R}_{kijl} \\ &- \sum_{k,\beta} A^{\beta}_{ij}\widetilde{R}_{k\beta k\alpha} + \sum_{k,l} A^{\alpha}_{il}\widetilde{R}_{kjkl} + \sum_{k,l} A^{\alpha}_{jl}\widetilde{R}_{iklk} - \sum_{\beta} H^{\beta}\widetilde{R}_{\beta ij\alpha} \\ &+ \sum_{k,l,\beta} A^{\alpha}_{kl}\big(A^{\beta}_{kj}A^{\beta}_{il} - A^{\beta}_{ij}A^{\beta}_{kl}\big) + \sum_{k,l,\beta} A^{\alpha}_{jl}\big(A^{\beta}_{kk}A^{\beta}_{il} - A^{\beta}_{ik}A^{\beta}_{kl}\big) \\ &+ \sum_{k,l,\beta} A^{\beta}_{jk}\big(A^{\alpha}_{kl}A^{\beta}_{il} - A^{\alpha}_{il}A^{\beta}_{kl}\big),\end{aligned}$$

*where the indices are with respect to a local orthonormal frame.*

*Proof.* Since the formula is tensorial, we may perform all the computations at a fixed point $x_0$. Moreover, we may assume that at this point we have an orthonormal frame, such that

$$\nabla_{e_j} e_i = 0.$$

Consequently, at the point $x_0$, we have

$$A_{ij} = \nabla_{e_j} dF(e_i) - dF(\nabla_{e_j} e_i) = \nabla_{e_j} dF(e_i).$$

From the Codazzi equation (1.4) we have

$$\sum_\alpha \widetilde{R}_{kij\alpha}\xi_\alpha = (\nabla^\perp_{e_i} A)_{kj} - (\nabla^\perp_{e_k} A)_{ij}.$$

Differentiating once more, we get

$$(\nabla^\perp_{e_k}\nabla^\perp_{e_k} A)_{ij} = \nabla^\perp_{e_k}\big((\nabla^\perp_{e_k} A)_{ij}\big) = \nabla^\perp_{e_k}\big((\nabla^\perp_{e_i} A)_{kj}\big) - \sum_\alpha \nabla^\perp_{e_k}\big(\widetilde{R}_{kij\alpha}\xi_\alpha\big).$$

Thus,

$$\begin{aligned}(\nabla^\perp_{e_k}\nabla^\perp_{e_k} A)_{ij} &= \nabla^\perp_{e_k}\big(\nabla^\perp_{e_i} A_{kj} - A(\nabla_{e_i} e_k, e_j) - A(e_k, \nabla_{e_i} e_j)\big)\\ &\quad - \sum_\alpha e_k\big(\widetilde{R}_{kij\alpha}\big)\xi_\alpha - \sum_\alpha \widetilde{R}_{kij\alpha}\nabla^\perp_{e_k}\xi_\alpha.\end{aligned}$$

But,

$$\nabla_{e_k}\xi_\alpha = -\sum_l A^\alpha_{kl} e_l - \sum_\beta \omega_{\alpha\beta}(e_k)\xi_\beta,$$

where with $\omega_{\alpha\beta}$ we denote the associated forms of the normal connection, i.e.,

$$\omega_{\alpha\beta}(e_k) = \langle \nabla_{e_k}\xi_\alpha, \xi_\beta\rangle = -\omega_{\beta\alpha}(e_k).$$

Therefore,

$$\sum_{l,\alpha} A^\alpha_{lk}\widetilde{R}_{kijl}\xi_\alpha = -\sum_\alpha \widetilde{R}(e_k, e_i, e_j, \nabla_{e_k}\xi_\alpha)\xi_\alpha - \sum_\alpha \widetilde{R}_{kij\alpha}\nabla^\perp_{e_k}\xi_\alpha$$

and

$$\nabla^\perp_{e_k}\nabla^\perp_{e_i} A_{jk} = \sum_\alpha A^\alpha_{jk}\widetilde{R}^\perp_{ki\alpha} + \nabla^\perp_{e_i}\nabla^\perp_{e_k} A_{jk}.$$

Therefore,

$$\begin{aligned}(\nabla^\perp_{e_k}\nabla^\perp_{e_k} A)_{ij} &= \widetilde{R}^\perp\big(e_k, e_i, A_{kj}\big) + \nabla^\perp_{e_i}\nabla^\perp_{e_k} A_{kj} - A(\nabla_{e_k}\nabla_{e_i} e_k, e_j)\\ &\quad - A(e_k, \nabla_{e_k}\nabla_{e_i} e_j) - \textstyle\sum_\alpha (\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^\beta_{kk}\widetilde{R}_{\beta ij\alpha}\xi_\alpha\\ &\quad - \textstyle\sum_{\alpha,\beta} A^\beta_{ki}\widetilde{R}_{k\beta j\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^\beta_{kj}\widetilde{R}_{ki\beta\alpha}\xi_\alpha\\ &\quad + \textstyle\sum_{l,\alpha} A^\alpha_{kl}\widetilde{R}_{kijl}\xi_\alpha.\end{aligned} \tag{4.10}$$

Again, using the Codazzi equation (1.4), we deduce that

$$\begin{aligned}\nabla^\perp_{e_i}\nabla^\perp_{e_k} A_{jk} &= \nabla^\perp_{e_i}\big((\nabla^\perp_{e_k} A)_{jk}\big) + \nabla^\perp_{e_i}\big(A(\nabla_{e_k} e_j, e_k) + A(e_j, \nabla_{e_k} e_k)\big)\\ &= \nabla^\perp_{e_i}\nabla^\perp_{e_j} A_{kk} - 2A(\nabla_{e_i}\nabla_{e_j} e_k, e_k) - \textstyle\sum_\alpha (\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha}\xi_\alpha\\ &\quad - \textstyle\sum_{\alpha,\beta} A^\beta_{ki}\widetilde{R}_{\beta jk\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^\beta_{ij}\widetilde{R}_{k\beta k\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^\beta_{ik}\widetilde{R}_{kj\beta\alpha}\xi_\alpha\\ &\quad + \textstyle\sum_{l,\alpha} A^\alpha_{il}\widetilde{R}_{kjkl}\xi_\alpha + A(\nabla_{e_i}\nabla_{e_k} e_j, e_k)\\ &\quad + A(e_j, \nabla_{e_i}\nabla_{e_k} e_k).\end{aligned} \tag{4.11}$$

Using Ricci equations (1.5), we have

$$\sum_{\alpha} A^{\alpha}_{kj} \widetilde{R}^{\perp}_{ki\alpha} = -\sum_{\alpha,\beta} A^{\beta}_{kj} \widetilde{R}_{ki\beta\alpha}\xi_\alpha + \sum_{l,\alpha} A^{\alpha}_{kl} A^{\alpha}_{il} A_{kl} - \sum_{l,\alpha} A^{\alpha}_{kj} A^{\alpha}_{kl} A_{il}. \tag{4.12}$$

Combining (4.11),(4.12) with (4.10), we obtain

$$\begin{aligned}
(\nabla^{\perp}_{e_k}\nabla^{\perp}_{e_k} A)_{ij} &= \nabla^{\perp}_{e_i}\nabla^{\perp}_{e_j} A_{kk} - 2A(\nabla_{e_i}\nabla_{e_j} e_k, e_k) + A(\widetilde{R}_{ikk}, e_j) + A(\widetilde{R}_{ikj}, e_k) \\
&\quad + \sum_{l,\alpha} A^{\alpha}_{kj} A^{\alpha}_{il} A_{kl} - \sum_{l,\alpha} A^{\alpha}_{kj} A^{\alpha}_{kl} A_{il} - \sum_{\alpha} (\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha}\xi_\alpha \\
&\quad - \sum_{\alpha,\beta} A^{\beta}_{kk} \widetilde{R}_{\beta ij\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^{\beta}_{ik} \widetilde{R}_{k\beta j\alpha}\xi_\alpha - 2\sum_{\alpha,\beta} A^{\beta}_{kj} \widetilde{R}_{ki\beta\alpha}\xi_\alpha \\
&\quad + \sum_{l,\alpha} A^{\alpha}_{kl} \widetilde{R}_{kijl}\xi_\alpha - \sum_{\alpha} (\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^{\beta}_{ki} \widetilde{R}_{\beta jk\alpha}\xi_\alpha \\
&\quad - \sum_{\alpha,\beta} A^{\beta}_{ij} \widetilde{R}_{k\beta k\alpha}\xi_\alpha - \sum_{\alpha,\beta} A^{\beta}_{ik} \widetilde{R}_{kj\beta\alpha}\xi_\alpha + \sum_{l,\alpha} A^{\alpha}_{il} \widetilde{R}_{kjkl}\xi_\alpha.
\end{aligned}$$

Now, if $\omega_{ij}$ are the connection forms of the Levi-Civita connection, then note that at the point $x$ we have that

$$\nabla_{e_j} e_k = \sum_l \langle \nabla_{e_j} e_k, e_l \rangle e_l = \sum_l \omega_{kl}(e_j) e_l.$$

Differentiating once more, and estimating at $x_0$, we get

$$\nabla_{e_i}\nabla_{e_j} e_k = \sum_l e_i\omega_{kl}(e_j) e_l + \sum_l \omega_{kl}(e_j)\nabla_{e_i} e_l = \sum_l e_i\omega_{kl}(e_j) e_l.$$

Thus,

$$\begin{aligned}
\sum_k A(\nabla_{e_i}\nabla_{e_j} e_k, e_k) &= \sum_{k,l} e_i\omega_{kl}(e_j) A_{lk} = \sum_k e_i\omega_{kk}(e_j) A_{kk} + \sum_{k>l} e_i\omega_{kl}(e_j) A_{lk} \\
&\quad + \sum_{k<l} e_i\omega_{kl}(e_j) A_{lk} \\
&= 0.
\end{aligned}$$

Now, using the Gauss equation (1.3) and taking a trace, we obtain

$$
\begin{aligned}
(\Delta^{\perp}A)_{ij} \quad = \quad & \sum_{k}\nabla^{\perp}_{e_i}\nabla^{\perp}_{e_j}H-\sum_{k,\alpha}(\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha}\xi_{\alpha}-\sum_{k,\alpha}(\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha}\xi_{\alpha}\\
&-\sum_{k,\alpha,\beta}A^{\beta}_{ki}\widetilde{R}_{\beta jk\alpha}\xi_{\alpha}-\sum_{k,\alpha,\beta}A^{\beta}_{ij}\widetilde{R}_{k\beta k\alpha}\xi_{\alpha}-\sum_{k,\alpha,\beta}A^{\beta}_{ik}\widetilde{R}_{kj\beta\alpha}\xi_{\alpha}\\
&-\sum_{k,\alpha,\beta}H^{\beta}\widetilde{R}_{\beta ij\alpha}\xi_{\alpha}-\sum_{k,\alpha,\beta}A^{\beta}_{ki}\widetilde{R}_{k\beta j\alpha}\xi_{\alpha}-2\sum_{k,\alpha,\beta}A^{\beta}_{kj}\widetilde{R}_{ki\beta\alpha}\xi_{\alpha}\\
&+\sum_{k,l,\alpha}A^{\alpha}_{il}\widetilde{R}_{kjkl}\xi_{\alpha}+\sum_{k,l,\alpha}A^{\alpha}_{kl}\widetilde{R}_{kijl}\xi_{\alpha}-\sum_{k,l,\alpha}A^{\alpha}_{lj}\widetilde{R}_{ikkl}\xi_{\alpha}\\
&-\sum_{k,l,\alpha,\beta}A^{\alpha}_{lj}A^{\beta}_{ik}A^{\beta}_{kl}\xi_{\alpha}+\sum_{k,l,\alpha,\beta}A^{\alpha}_{lj}H^{\beta}A^{\beta}_{il}\xi_{\alpha}-\sum_{k,l,\alpha}A^{\alpha}_{lk}\widetilde{R}_{ikjl}\xi_{\alpha}\\
&-\sum_{k,l,\alpha,\beta}A^{\alpha}_{lk}A^{\beta}_{ij}A^{\beta}_{kl}\xi_{\alpha}+\sum_{k,l,\alpha,\beta}A^{\alpha}_{lk}A^{\beta}_{kj}A^{\beta}_{il}\xi_{\alpha}+\sum_{k,l,\alpha,\beta}A^{\beta}_{kj}A^{\beta}_{il}A^{\alpha}_{kl}\xi_{\alpha}\\
&-\sum_{k,l,\alpha,\beta}A^{\beta}_{kj}A^{\beta}_{kl}A^{\alpha}_{il}\xi_{\alpha}.
\end{aligned}
$$

Now, from the 1st Bianchi identity, we deduce that

$$
\begin{aligned}
\big(\Delta^{\perp}A)^{\alpha}_{ij} \quad = \quad & \big(\nabla^{\perp 2}H\big)^{\alpha}_{ij}-\sum_{k}(\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha}-\sum_{k}(\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha}\\
-\quad & 2\sum_{k,\beta}A^{\beta}_{ik}\widetilde{R}_{kj\beta\alpha}-2\sum_{k,\beta}A^{\beta}_{jk}\widetilde{R}_{ki\beta\alpha}+2\sum_{k,l}A^{\alpha}_{kl}\widetilde{R}_{kijl}\\
-\quad & \sum_{k,\beta}A^{\beta}_{ij}\widetilde{R}_{k\beta k\alpha}+\sum_{k,l}A^{\alpha}_{il}\widetilde{R}_{kjkl}+\sum_{k,l}A^{\alpha}_{jl}\widetilde{R}_{iklk}-\sum_{\beta}H^{\beta}\widetilde{R}_{\beta ij\alpha}\\
+\quad & \sum_{k,l,\beta}A^{\alpha}_{kl}\big(A^{\beta}_{kj}A^{\beta}_{il}-A^{\beta}_{ij}A^{\beta}_{kl}\big)+\sum_{k,l,\beta}A^{\alpha}_{jl}\big(A^{\beta}_{kk}A^{\beta}_{il}-A^{\beta}_{ik}A^{\beta}_{kl}\big)\\
+\quad & \sum_{k,l,\beta}A^{\beta}_{jk}\big(A^{\alpha}_{kl}A^{\beta}_{il}-A^{\alpha}_{il}A^{\beta}_{kl}\big).
\end{aligned}
$$

This completes the proof. □

**Lemma 4.2.6** *The second fundamental form evolves in time under the equation*

$$\begin{aligned}(\nabla^{\perp}_{\partial_t} A \quad - \quad \Delta^{\perp} A)^{\alpha}_{ij} &= \sum_k (\widetilde{\nabla}_{e_k} \widetilde{R})_{kij\alpha} + \sum_k (\widetilde{\nabla}_{e_i} \widetilde{R})_{kjk\alpha} \\ &+2 \sum_{k,\beta} A^{\beta}_{ik} \widetilde{R}_{kj\beta\alpha} + 2 \sum_{k,\beta} A^{\beta}_{jk} \widetilde{R}_{ki\beta\alpha} - 2 \sum_{k,l} A^{\alpha}_{kl} \widetilde{R}_{kijl} \\ &+ \sum_{k,\beta} A^{\beta}_{ij} \widetilde{R}_{k\beta k\alpha} - \sum_{k,l} A^{\alpha}_{il} \widetilde{R}_{kjkl} - \sum_{k,l} A^{\alpha}_{jl} \widetilde{R}_{kikl} \\ &- \sum_{k,l,\beta} A^{\alpha}_{kl} \big(A^{\beta}_{kj} A^{\beta}_{il} - A^{\beta}_{ij} A^{\beta}_{kl}\big) - \sum_{k,l,\beta} A^{\alpha}_{jl} \big(A^{\beta}_{kk} A^{\beta}_{il} - A^{\beta}_{ik} A^{\beta}_{kl}\big) \\ &- \sum_{k,l,\beta} A^{\beta}_{jk} \big(A^{\alpha}_{kl} A^{\beta}_{il} - A^{\alpha}_{il} A^{\beta}_{kl}\big) - \sum_{k,\beta} H^{\beta} A^{\beta}_{jk} A^{\alpha}_{ik},\end{aligned}$$

*where the indices are with respect to a local orthonormal frame.*

*Proof.* Using Lemma 4.2.3 and Lemma 4.2.5, we immediately get the result. □

**Lemma 4.2.7** *The squared norm $|A|^2$ of the second fundamental form evolves in time under the equation*

$$\begin{aligned}\partial_t |A|^2 \quad &= \quad \Delta |A|^2 - 2 |\nabla^{\perp} A|^2 \\ &+2 \sum_{i,j,k,l} \Big( \sum_{\alpha} A^{\alpha}_{ij} A^{\alpha}_{kl} \Big)^2 + 2 \sum_{i,j,\alpha,\beta} \Big( \sum_k (A^{\alpha}_{ik} A^{\beta}_{jk} - A^{\beta}_{ik} A^{\alpha}_{jk}) \Big)^2 \\ &+4 \sum_{i,j,k,l,\alpha} \Big( A^{\alpha}_{ij} A^{\alpha}_{kl} - \delta_{kl} \sum_p A^{\alpha}_{ip} A^{\alpha}_{jp} \Big) \widetilde{R}_{kilj} \\ &+2 \sum_{i,j,k,\alpha,\beta} \Big( 4 A^{\alpha}_{jk} A^{\beta}_{ik} \widetilde{R}_{\alpha\beta ji} + A^{\alpha}_{jk} A^{\beta}_{jk} \widetilde{R}_{\alpha i\beta i} \Big) \\ &+2 \sum_{i,j,k,\alpha} A^{\alpha}_{jk} \Big( (\widetilde{\nabla}_{e_i} \widetilde{R})_{\alpha jki} + (\widetilde{\nabla}_{e_k} \widetilde{R})_{\alpha iji} \Big),\end{aligned}$$

*where the indices are with respect to a local orthonormal frame.*

*Proof.* Let us compute at first the time derivative of $|A|^2$. We have,

$$\begin{aligned}\partial_t |A|^2 \quad &= \quad \sum_{i,j} \partial_t \langle A_{ij}, A_{ij} \rangle = 2 \sum_{i,j} \langle \partial_t A_{ij}, A_{ij} \rangle \\ &= \quad 2 \sum_{i,j} \langle (\nabla_{\partial_t} A)_{ij} + A(\nabla_{\partial_t} e_i, e_j) + A(e_i, \nabla_{\partial_t} e_j), A_{ij} \rangle.\end{aligned}$$

Recall from Lemma 4.2.1 (c) that

$$\nabla_{\partial_t} e_i = \sum_{k,s} A^H_{ik} g^{ks} e_s.$$

Therefore, the previous equation becomes

$$\partial_t |A|^2 = 2\sum_{i,j}\langle(\nabla_{\partial_t} A)_{ij}, A_{ij}\rangle + 2\sum_{i,j,l,\alpha,\beta} A^\alpha_{ij}A^\alpha_{lj}H^\beta A^\beta_{il} + 2\sum_{i,j,l,\alpha,\beta} A^\alpha_{ij}H^\beta A^\beta_{jl}A^\alpha_{il}.$$

Fix a point $(x_0, t_0)$ in space-time and assume that at $x_0$ we have

$$\nabla_{e_i} e_j = 0.$$

Then, at the point $x_0$ we have

$$\Delta|A|^2 = \sum_k e_k e_k |A|^2 = 2\sum_{i,j,k} e_k\langle\nabla^\perp_{e_k} A_{ij}, A_{ij}\rangle.$$

Therefore,

$$\begin{aligned}\Delta|A|^2 &= 2\sum_{i,j,k}\langle(\nabla^\perp_{e_k}\nabla^\perp_{e_k} A)_{ij}, A_{ij}\rangle + 2\sum_{i,j,k}|\nabla^\perp_{e_k} A_{ij}|^2\\ &= 2\sum_{i,j,k}\langle(\Delta^\perp A)_{ij}, A_{ij}\rangle + 2\sum_{i,j,k}|\nabla^\perp_{e_k} A_{ij}|^2,\end{aligned}$$

since

$$\begin{aligned}&\sum_{i,j,k}\langle A(\nabla_{e_k}\nabla_{e_k} e_i, e_j) + A(e_i, \nabla_{e_k}\nabla_{e_k} e_j), A_{ij}\rangle\\ &\qquad= \sum_{i,j,k}\langle A\big(\nabla_{e_k}\omega_{il}(e_k)e_l, e_j\big) + A\big(e_i, \nabla_{e_k}\omega_{jl}(e_k)e_l\big), A_{ij}\rangle\\ &\qquad= \sum_{i,j,k,l} e_k\omega_{il}(e_k)\langle A_{lj}, A_{ij}\rangle + \sum_{i,j,k,l} e_k\omega_{jl}(e_k)\langle A_{il}, A_{ij}\rangle\\ &\qquad= 0.\end{aligned}$$

Thus,

$$\begin{aligned}\partial_t|A|^2 - \Delta|A|^2 &= 2\sum_{i,j,k}\langle(\nabla^\perp_{\partial_t} A - \Delta^\perp A)_{ij}, A_{ij}\rangle + 2\sum_{i,j,l,\alpha,\beta} A^\alpha_{ij}A^\alpha_{lj}H^\beta A^\beta_{il}\\ &\quad+2\sum_{i,j,l,\alpha,\beta} A^\alpha_{ij}H^\beta A^\beta_{jl}A^\alpha_{il} - 2|\nabla^\perp A|^2.\end{aligned}$$

Using Lemma 4.2.6, we obtain

$$\begin{aligned}
\partial_t|A|^2 &= \Delta|A|^2 - 2|\nabla^\perp A|^2 + 2\sum_{i,j,k,\alpha}\big((\widetilde{\nabla}_{e_k}\widetilde{R})_{kij\alpha} + (\widetilde{\nabla}_{e_i}\widetilde{R})_{kjk\alpha}\big)A^\alpha_{ij} \\
&\quad +4\sum_{i,j,k,\alpha,\beta} A^\alpha_{ij}A^\beta_{ik}\widetilde{R}_{kj\beta\alpha} + 4\sum_{i,j,k,\alpha,\beta} A^\alpha_{ij}A^\beta_{jk}\widetilde{R}_{ki\beta\alpha} \\
&\quad -4\sum_{i,j,k,l,\alpha} A^\alpha_{ij}A^\alpha_{kl}\widetilde{R}_{kijl} + 2\sum_{i,j,k,\alpha,\beta} A^\alpha_{ij}A^\beta_{ij}\widetilde{R}_{k\beta k\alpha} \\
&\quad -2\sum_{i,j,k,l,\alpha} A^\alpha_{ij}A^\alpha_{il}\widetilde{R}_{kjkl} - 2\sum_{i,j,k,l,\alpha} A^\alpha_{ij}A^\alpha_{jl}\widetilde{R}_{kikl} \\
&\quad -2\sum_{i,j,k,l,\alpha,\beta} A^\alpha_{ij}A^\beta_{kj}A^\alpha_{kl}A^\beta_{il} + 2\sum_{i,j,k,l,\alpha,\beta} A^\alpha_{ij}A^\alpha_{kl}A^\beta_{ij}A^\beta_{kl} \\
&\quad +2\sum_{i,j,k,l,\alpha,\beta} A^\alpha_{ij}A^\alpha_{jl}A^\beta_{ik}A^\beta_{kl} - 2\sum_{i,j,k,l,\alpha,\beta} A^\alpha_{ij}A^\beta_{jk}A^\alpha_{kl}A^\beta_{il} \\
&\quad +2\sum_{i,j,k,l,\alpha,\beta} A^\alpha_{ij}A^\beta_{jk}A^\alpha_{il}A^\beta_{kl}.
\end{aligned}$$

Rearranging the indices, the third term becomes

$$2\sum_{i,j,k,\alpha} A^\alpha_{jk}\big((\widetilde{\nabla}_{e_i}\widetilde{R})_{\alpha jki} + (\widetilde{\nabla}_{e_k}\widetilde{R})_{\alpha iji}\big).$$

Interchanging $i$ with $j$ to the fourth and fifth term and rearranging the terms, we have

$$4\sum_{i,j,k,\alpha,\beta} A^\alpha_{ij}A^\beta_{ik}\widetilde{R}_{kj\beta\alpha} + 4\sum_{i,j,k,\alpha,\beta} A^\alpha_{ij}A^\beta_{jk}\widetilde{R}_{ki\beta\alpha} = 8\sum_{i,j,k,\alpha,\beta} A^\alpha_{jk}A^\beta_{ik}\widetilde{R}_{\alpha\beta ji}. \tag{4.13}$$

Similarly, interchanging $i$ with $k$ to the seventh term, we obtain

$$2\sum_{i,j,k,\alpha,\beta} A^\alpha_{ij}A^\beta_{ij}\widetilde{R}_{k\beta k\alpha} = 2\sum_{i,j,k,\alpha,\beta} A^\alpha_{jk}A^\beta_{jk}\widetilde{R}_{\alpha i\beta i}. \tag{4.14}$$

Adding (4.13) and (4.14), we have

$$\begin{aligned}
8\sum_{i,j,k,\alpha,\beta} A^\alpha_{jk}A^\beta_{ik}\widetilde{R}_{\alpha\beta ji} \quad &+ \quad 2\sum_{i,j,k,\alpha,\beta} A^\alpha_{jk}A^\beta_{jk}\widetilde{R}_{\alpha i\beta i} \\
&= \quad 2\sum_{i,j,k,\alpha,\beta}\big(4A^\alpha_{jk}A^\beta_{ik}\widetilde{R}_{\alpha\beta ji} + A^\alpha_{jk}A^\beta_{jk}\widetilde{R}_{\alpha i\beta i}\big).
\end{aligned}$$

Interchanging $i$ with $j$ to the eighth and ninth term and adding them together, we have

$$\begin{aligned}
-2\sum_{i,j,k,l,\alpha} A^\alpha_{ij}A^\alpha_{il}\widetilde{R}_{kjkl} \quad - \quad 2\sum_{i,j,k,l,\alpha} A^\alpha_{ij}A^\alpha_{jl}\widetilde{R}_{kikl} &= -4\sum_{i,j,k,\alpha}\Big(\sum_p A^\alpha_{ip}A^\alpha_{jp}\widetilde{R}_{kikj}\Big) \\
&= -4\delta_{kl}\sum_{i,j,k,l,\alpha}\Big(\sum_p A^\alpha_{ip}A^\alpha_{jp}\widetilde{R}_{kilj}\Big).
\end{aligned} \tag{4.15}$$

Therefore, adding the sixth term and (4.15), we obtain

$$-4\sum_{i,j,k,l,\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\widetilde{R}_{kijl} \quad - \quad 4\delta_{kl}\sum_{i,j,k,l,\alpha}\Big(\sum_{p} A^{\alpha}_{ip}A^{\alpha}_{jp}\widetilde{R}_{kilj}\Big)$$
$$= 4\sum_{i,j,k,l,\alpha}\big(A^{\alpha}_{ij}A^{\alpha}_{kl} - \delta_{kl}\sum_{p} A^{\alpha}_{ip}A^{\alpha}_{jp}\big)\widetilde{R}_{kilj}.$$

The tenth and thirteenth term add up to

$$-2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ij}A^{\beta}_{kj}A^{\alpha}_{kl}A^{\beta}_{il} - 2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ij}A^{\beta}_{jk}A^{\alpha}_{kl}A^{\beta}_{il} = -4\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ik}A^{\beta}_{kj}A^{\alpha}_{jk}A^{\beta}_{ik}. \tag{4.16}$$

For $\alpha = \beta$ on the eleventh term, we obtain

$$2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ij}A^{\alpha}_{kl}A^{\beta}_{ij}A^{\beta}_{kl} = 2\sum_{i,j,k,l}\Big(\sum_{\alpha} A^{\alpha}_{ij}A^{\alpha}_{kl}\Big)^2.$$

Adding the twelfth term, fourteenth term and (4.16), we have

$$2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ij}A^{\alpha}_{jl}A^{\beta}_{ik}A^{\beta}_{kl} \quad + \quad 2\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ij}A^{\beta}_{jk}A^{\alpha}_{il}A^{\beta}_{kl} - 4\sum_{i,j,k,l,\alpha,\beta} A^{\alpha}_{ik}A^{\beta}_{kj}A^{\alpha}_{jk}A^{\beta}_{ik}$$
$$= 2\sum_{i,j,\alpha,\beta}\Big(\sum_{k}(A^{\alpha}_{ik}A^{\beta}_{jk} - A^{\beta}_{ik}A^{\alpha}_{jk})\Big)^2.$$

This completes the proof. □

CHAPTER 5

# Graphical MCF

In this chapter we provide the proof of the main theorem of this thesis.

## 5.1 Graphical surfaces

Let $M$ and $N$ be two Riemann surfaces endowed with Riemannian metrics $g_M$ and $g_N$ and suppose that $f : M \to N$ is a smooth map. The induced metric on the product manifold will be denoted by

$$\mathrm{g}_{M\times N} := \langle \cdot\,,\cdot\rangle = g_M \times g_N.$$

Define now the embedding $F\colon\, M \to M \times N$, given by $F := I \times f$. The graph of $f$ is defined to be the submanifold $\Gamma(f) := F(M)$. Since $F$ is an embedding, it induces another Riemannian metric $g := F^*\mathrm{g}_{M\times N}$ on $M$. The natural projections $\pi_M :\, M \times N \to M$, $\pi_N :\, M \times N \to N$ are submersions. The metrics $g_M$, $\mathrm{g}_{M\times N}$ and $g$ are related by

$$\mathrm{g}_{M\times N} = \pi_M^* g_M + \pi_N^* g_N \quad \text{and} \quad g = g_M + f^* g_N\,.$$

The Levi-Civita connection $\widetilde{\nabla}$ of the product manifold is related to the Levi-Civita connections $\nabla^{g_M}$ and $\nabla^{g_N}$ by

$$\widetilde{\nabla} = \pi_M^* \nabla^{g_M} \oplus \pi_N^* \nabla^{g_N}\,.$$

The corresponding curvature operator $\widetilde{R}$ is related to the curvature operators $R_M$ and $R_N$ by

$$\widetilde{R} = \pi_M^* R_M \oplus \pi_N^* R_N.$$

The Levi-Civita connection of $g$ will be denoted by $\nabla$, its curvature tensor by $R$ and it sectional curvature by $\sigma_g$. We denote the sectional curvatures of $(M, g_M)$ and $(N, g_N)$ by $\sigma_M$ and $\sigma_N$, respectively.

## 5.2 Singular value decomposition

Let us recall here some Linear Algebra constructions. Fix a point $x \in M$. Let $\lambda^2 \le \mu^2$ be the eigenvalues at $x$ of $f^*g_N$, with respect to $g_M$ and denote by $\{\alpha_1, \alpha_2\}$ the corresponding eigenvectors. Without loss of generality, we may assume that $\{\alpha_1, \alpha_2\}$ is oriented, since otherwise we consider $\{-\alpha_1, \alpha_2\}$. The corresponding values $0 \le \lambda \le \mu$ are called *singular values* of $f$ at $x$. Then, there exists an orthonormal basis $\{\alpha_1, \alpha_2\}$ of $T_xM$, with respect to $g_M$ and $\{\beta_1, \beta_2\}$ of $T_{f(x)}N$, with respect to the metric $g_N$, such that

$$df(\alpha_1) = \lambda\beta_1 \quad \text{and} \quad df(\alpha_2) = \mu\beta_2.$$

Indeed, in the case where the values $\lambda$ and $\mu$ are positive, define as

$$\beta_1 := \frac{df(\alpha_1)}{|df(\alpha_1)|} \quad \text{and} \quad \beta_2 := \frac{df(\alpha_2)}{|df(\alpha_2)|}.$$

In the case where $\lambda$ vanishes and $\mu$ is positive, define first $\beta_2$ by

$$\beta_2 := \frac{df(\alpha_2)}{|df(\alpha_2)|}$$

and take as $\beta_1$ a unit vector perpendicular to $\beta_2$. In the special case, where both $\lambda$ and $\mu$ are zero, we may consider as $\{\beta_1, \beta_2\}$ an arbitrary base of $T_{f(x)}N$. This procedure is called the *singular value decomposition* of $df$. Observe that

$$e_1 := \frac{\alpha_1}{\sqrt{1+\lambda^2}} \quad \text{and} \quad e_2 := \frac{\alpha_2}{\sqrt{1+\mu^2}}$$

are orthonormal, with respect to the metric $g$. Hence,

$$dF(e_1) = \frac{\alpha_1 \oplus \lambda\beta_1}{\sqrt{1+\lambda^2}} \quad \text{and} \quad dF(e_2) = \frac{\alpha_2 \oplus \mu\beta_2}{\sqrt{1+\mu^2}}$$

is orthonormal basis of $dF(T_xM)$. Moreover, the vectors

$$\xi_3 := \frac{1}{\sqrt{1+\lambda^2}}\big(-\lambda\alpha_1 \oplus \beta_1\big) \quad \text{and} \quad \xi_4 := \frac{1}{\sqrt{1+\mu^2}}\big(-\mu\alpha_2 \oplus \beta_2\big)$$

form an orthonormal basis, with respect to $\mathrm{g}_{M\times N}$ of the normal space $N_xM$ of the graph $\Gamma(f)$ at the point $f(x)$. Observe that $\{\xi_3, \xi_4\}$ is an oriented basis of the normal space of the graph if and only if $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ is an oriented basis of $T_xM \times T_{f(x)}N$.

The *area functional* $A(f)$ of the graph is given by

$$A(f) := \int_M \sqrt{\det(g_M + f^*g_N)}d\mu = \int_M \sqrt{(1+\lambda^2)(1+\mu^2)}d\mu,$$

where $d\mu$ stands for the volume form of $(M, g_M)$. Then, the map $f$ is *minimal* if it is a critical point of the area functional, or equivalently, if $\Gamma(f)$ is a minimal surface of the Riemannian product $M \times N$.

By *Gauss equation*, we get that the curvature $\sigma_g$ of the graph is given by

$$2\sigma_g = 2u_1^2\sigma_M + 2u_2^2\sigma_N + |H|^2 - |A|^2.$$

From the *Ricci equation*, we see that the curvature $\sigma_n$ of the normal bundle of $\Gamma(f)$ is given by the formula

$$\sigma_n := \mathrm{R}^{\perp}_{1234} = \widetilde{R}_{1234} + A^3_{11}A^4_{12} - A^3_{12}A^4_{11} + A^3_{12}A^4_{22} - A^3_{22}A^4_{12}.$$

The sum of the last four terms in the above formula is equal to minus the commutator $\sigma^{\perp}$ of the matrices

$$A^3 = (A^3_{ij}) \quad \text{and} \quad A^4 = (A^4_{ij}),$$

that is,

$$\sigma^{\perp} := \langle [A^3, A^4]e_1, e_2\rangle = -A^3_{11}A^4_{12} + A^3_{12}A^4_{11} - A^3_{12}A^4_{22} + A^3_{22}A^4_{12}.$$

Note that

$$[A^3, A^4] = -[A^4, A^3].$$

Moreover, the quantity $\sigma^{\perp}$ does not depend on the choice of the oriented frame of the normal bundle.

## 5.3 Evolution of geometric quantities

### 5.3.1 Kähler angles

Denote by $J_M$ and $J_N$ the corresponding complex structures on $M$ and $N$ and by $\Omega_M$ and $\Omega_N$ their associated Kähler forms. Since $M$ and $N$ are two dimensional, the forms $\Omega_M$ and $\Omega_N$ coincide with the volume forms of $(M, g_M)$ and $(N, g_N)$. In the Riemannian product manifold $(M \times N, g_{M\times N})$, there are two natural complex structures. These are

$$J_1 := \pi_M^* J_M - \pi_N^* J_N \;\text{ and }\; J_2 := \pi_M^* J_M + \pi_N^* J_N.$$

Consider now on the product the forms

$$\Omega_1 := \pi_M^*\Omega_M \;\text{ and }\; \Omega_2 := \pi_N^*\Omega_N.$$

Using the Hodge star operator, with respect to the induced metric $g$, we define the functions $u_1$ and $u_2$, given by

$$u_1 := *(F^*\Omega_1) = *\big((\pi_M \circ F)^*\Omega_M\big) \;\text{ and }\; u_2 := *(F^*\Omega_2) = *\big((\pi_N \circ F)^*\Omega_N\big).$$

Observe now that the functions $u_1$ and $u_2$ are the Jacobian determinants of the natural projection maps $\pi_1 : \Gamma(f) \to M$ and $\pi_2 : \Gamma(f) \to N$, respectively. As a matter of fact, in terms of the singular values of $f$, we have that

$$u_1 = \frac{1}{\sqrt{(1+\lambda^2)(1+\mu^2)}} \quad \text{and} \quad |u_2| = \frac{\lambda\mu}{\sqrt{(1+\lambda^2)(1+\mu^2)}}.$$

We will give another geometric interpretation of the functions $u_1$ and $u_2$. Since $F : M \to M \times N$ is an isometric immersion and $J_1$ is complex structure of $(M \times N, \mathrm{g}_{M\times N})$, there exists a function $a_1$ with values in $[0, \pi]$, such that

$$\cos a_1 = \varphi := g_{M\times N}\big(J_1 dF(e_1), dF(e_2)\big).$$

The function $a_1$ is called the *Kähler angle of F, with respect to* $J_1$. By a direct computation, we see that

$$\cos a_1 = u_1 - u_2.$$

Note that the angle $a_1$ is not everywhere smooth. Moreover, if $\alpha_1(x) = 0$, for a point $x \in M$, then $dF(T_x M)$ is a complex line of $T_{f(x)}(M \times N)$. For that reason, a point $x$ where $a$ vanishes is called a *complex point* of $F$. Similarly, if $a_1(x) = \pi$, then $dF(T_x M)$ is an anti-complex line of $T_{f(x)}(M \times N)$ and $x$ is called an *anti-complex point* of $F$. Additionally, if $a_1(x) = \pi/2$, then the point $x$ is called *Lagrangian point* of $F$.

Similarly, we define the *Kähler angle of F, with respect to* $J_2$. In this case,

$$\cos a_2 = \vartheta := g_{M\times N}\big(J_2 dF(e_1), dF(e_2)\big) = u_1 + u_2.$$

From the Ricci equation, we obtain

$$\sigma_n = u_1 u_2(\sigma_M + \sigma_N) - \sigma^{\perp}.$$

In the case where $u_1 = u_2$ and $\sigma_M = \sigma_N = \sigma$, the immersion $F$ is Lagrangian and $\sigma_g = \sigma_n$. In this case, we deduce that

$$\sigma^{\perp} = \frac{|A|^2 - |H|^2}{2}. \tag{5.1}$$

### 5.3.2 Evolution of Jacobians

In the next lemma, we provide the derivative of the pullback of a parallel $k$-tensor $\Phi$ on the Riemannian manifold $N$.

**Lemma 5.3.1** *The covariant derivative of the tensor $F^*\Phi$ is given by*

$$(\nabla_{e_s} F^*\Phi)_{ij} = \sum_{\alpha} \big(A^{\alpha}_{si}\Phi_{\alpha j} + A^{\alpha}_{sj}\Phi_{i\alpha}\big),$$

*for any adapted orthonormal frame field* $\{e_1, e_2; \xi_3, \xi_4\}$.

*Proof.* For simplicity, let us suppose that $\{e_1, e_2\}$ is a normal frame at the point $x_0 \in M$ i.e.,

$$\nabla_{e_1} e_2(x_0) = \nabla_{e_2} e_1(x_0) = 0.$$

By a direct computation we get that

$$\begin{aligned}(\nabla_{e_s}F^*\Phi)_{ij} &= e_s\Phi\big(dF(e_i),dF(e_j)\big)\\ &= \Phi\big(\nabla^F_{e_s}dF(e_i),dF(e_j)\big)+\Phi\big(dF(e_i),\nabla^F_{e_s}dF(e_j)\big)\\ &= \Phi\big(A(e_s,e_i),dF(e_j)\big)+\Phi\big(dF(e_i),A(e_s,e_j)\big)\\ &= \sum_\alpha\big(A^\alpha_{si}\Phi_{\alpha j}+A^\alpha_{sj}\Phi_{i\alpha}\big).\end{aligned}$$

This completes the proof. □

Again by a direct computation we can show the expression of the Laplacian of the pullback of a parallel $k$-tensor on $N$.

**Lemma 5.3.2** *The Laplacian of the tensor $F^*\Phi$ is given by*

$$\begin{aligned}(\Delta F^*\Phi)_{ij} = &\,\Phi\big(\nabla^\perp_{e_i}H,dF(e_j)\big)+\Phi\big(dF(e_i),\nabla^\perp_{e_j}H\big)+2\sum_{k,\alpha,\beta}A^\alpha_{ki}A^\beta_{kj}\Phi_{\alpha\beta}\\ &-\sum_{k,l,\alpha}\big(A^\alpha_{ki}A^\alpha_{kl}F^*\Phi_{lj}+A^\alpha_{kj}A^\alpha_{kl}F^*\Phi_{il}\big)-\sum_{k,\alpha}\big(\widetilde{R}_{kik\alpha}\Phi_{\alpha j}+\widetilde{R}_{kjk\alpha}\Phi_{i\alpha}\big).\end{aligned}$$

*where $\{e_1,e_2;\xi_3,\xi_4\}$ is an arbitrary adapted local orthonormal frame field.*

*Proof.* For simplicity, let us suppose that $\{e_1,e_2\}$ is a normal frame at the point $x\in M$. We compute,

$$\begin{aligned}\big(\nabla_{e_k}\nabla_{e_k}F^*\Phi\big)_{ij} &= e_k\big(\Phi\big(A_{ik},dF(e_j)\big)+\Phi\big(dF(e_i),A_{jk}\big)\big)\\ &= \Phi\big((\nabla_{e_k}A)_{ik},dF(e_j)\big)+2\Phi(A_{ik},A_{jk})+\Phi\big(dF(e_i),(\nabla_{e_k}A)_{jk}\big)\\ &= \Phi\big((\nabla^\perp_{e_k}A)_{ik},dF(e_j)\big)+2\Phi(A_{ik},A_{jk})+\Phi\big(dF(e_i),(\nabla^\perp_{e_k}A)_{jk}\big)\\ &\quad-\sum_l\langle A_{ik},A_{kl}\rangle F^*\Phi_{lj}-\sum_l\langle A_{jk},A_{kl}\rangle F^*\Phi_{il}.\end{aligned}$$

Using the Codazzi equation, we obtain

$$\begin{aligned}(\nabla_{e_k}\nabla_{e_k}F^*\Phi)_{ij} &= \Phi\Big((\nabla^\perp_{e_i}A)_{kk},dF(e_j)\Big)+\Phi\Big(dF(e_i),(\nabla^\perp_{e_j}A)_{kk}\Big)\\ &\quad+2\sum_{\alpha,\beta}A^\alpha_{ki}A^\beta_{kj}\Phi_{\alpha\beta}-\sum_{l,\alpha}(A^\alpha_{ki}A^\alpha_{kl}F^*\Phi_{lj}+A^\alpha_{kj}A^\alpha_{kl}F^*\Phi_{il})\\ &\quad-\sum_\alpha(\widetilde{R}_{kik\alpha}\Phi_{\alpha j}+\widetilde{R}_{kjk\alpha}\Phi_{i\alpha}).\end{aligned}$$

By summing over $k$ we get the result. This completes the proof. □

**Lemma 5.3.3** *Suppose that $F\colon M\times[0,T)\to N$ is a solution of the mean curvature flow and let $\Phi$ be a parallel 2-form on $N$. Then, the function $u := *(F^*\Phi)$ evolves in time under the equation*

$$\partial_t u = \Delta u + |A|^2 u - 2\sum_{k,\alpha,\beta} A^\alpha_{k1}A^\beta_{k2}\Phi_{\alpha\beta} + \sum_\alpha \big(\widetilde{R}_{212\alpha}\Phi_{\alpha 2} + \widetilde{R}_{121\alpha}\Phi_{1\alpha}\big),$$

*where $\{e_1, e_2; \xi_3, \xi_4\}$ is an arbitrary adapted local orthonormal frame field.*

*Proof.* Let us make our computations again, with respect to a time-dependent orthonormal frame field $\{e_1, e_2\}$, as in Lemma 4.2.1. Taking into account from (1.6) that

$$\nabla_{e_1}\partial_t = \nabla_{e_2}\partial_t = 0,$$

we have

$$\begin{aligned}
\partial_t u &= \partial_t\big((F^*\Phi)(e_1,e_2)\big)\\
&= \Phi\big(\nabla^F_{\partial_t} dF(e_1), dF(e_2)\big) + \Phi\big(dF(e_1), \nabla^F_{\partial_t} dF(e_2)\big)\\
&= \Phi\big(\nabla^F_{e_1} dF(\partial_t), dF(e_2)\big) + \Phi\big(dF(e_1), \nabla^F_{e_2} dF(\partial_t)\big)\\
&\quad +\Phi\big(dF(\nabla_{\partial_t} e_1), dF(e_2)\big) + \Phi\big(dF(e_1), dF(\nabla_{\partial_t} e_2)\big).
\end{aligned}$$

Now observe that

$$\begin{aligned}
\nabla^F_{e_i} dF(\partial_t) + dF(\nabla_{\partial_t} e_i) &= \nabla^F_{e_i} H + \sum_{k,\beta} H^\beta A^\beta_{ik} dF(e_k)\\
&= \nabla^F_{e_i} H - \sum_k \langle \nabla^F_{e_i} H, dF(e_k)\rangle dF(e_k)\\
&= \nabla^\perp_{e_i} H,
\end{aligned}$$

for any $i \in \{1, 2\}$. Hence, putting everything together, we deduce that

$$\partial_t u = \Phi\big(\nabla^\perp_{e_1} H, dF(e_2)\big) + \Phi\big(dF(e_1), \nabla^\perp_{e_2} H\big).$$

Using Lemma 5.3.2, we have

$$\begin{aligned}
\partial_t u &= \Delta u + \sum_{k,l}(A^\alpha_{k1}A^\alpha_{kl}F^*\Phi_{l2} + A^\alpha_{k2}A^\alpha_{kl}F^*\Phi_{1l}) - 2\sum_{k,\alpha,\beta} A^\alpha_{k1}A^\beta_{k2}\Phi_{\alpha\beta}\\
&\quad + \sum_\alpha(\widetilde{R}_{212\alpha}\Phi_{\alpha 2} + \widetilde{R}_{121\alpha}\Phi_{1\alpha})\\
&= \Delta u + |A|^2 u - 2\sum_{k,\alpha,\beta} A^\alpha_{k1}A^\beta_{k2}\Phi_{\alpha\beta} + \sum_\alpha \big(\widetilde{R}_{212\alpha}\Phi_{\alpha 2} + \widetilde{R}_{121\alpha}\Phi_{1\alpha}\big).
\end{aligned}$$

This completes the proof. □

**Lemma 5.3.4** *The gradients of the angle functions $\varphi$, $\vartheta$ at a point $x \in M$ satisfy the equations*

$$\begin{aligned} |\nabla\varphi|^2 &= \big(1-\varphi^2\big)\big((A^3_{11}+A^4_{12})^2+(A^3_{12}+A^4_{22})^2\big), \\ |\nabla\vartheta|^2 &= \big(1-\vartheta^2\big)\big((A^3_{11}-A^4_{12})^2+(A^3_{12}-A^4_{22})^2\big), \end{aligned}$$

*where here $\{e_1, e_2; \xi_3, \xi_4\}$ is a local orthonormal frame.*

*Proof.* We have

$$\begin{aligned} &\Omega_1(e_1,e_2)=u_1, \quad \Omega_1(\xi_3,\xi_4)=u_2, \quad \Omega_1(e_1,\xi_3)=0, \\ &\Omega_2(e_1,e_2)=u_2, \quad \Omega_2(\xi_3,\xi_4)=u_1, \quad \Omega_2(e_1,\xi_3)=0 \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &\Omega_1(\xi_4,e_1)=\mu u_1, \quad \Omega_1(e_2,\xi_3)=\lambda u_1, \quad \Omega_1(e_2,\xi_4)=0, \\ &\Omega_2(e_1,\xi_4)=\lambda u_1, \quad \Omega_2(\xi_3,e_2)=\mu u_1, \quad \Omega_2(e_2,\xi_4)=0. \end{aligned} \tag{5.3}$$

Using (5.2) and (5.3), we compute

$$\begin{aligned} e_k u_1 &= (\nabla_{e_k} F^*\Omega_1)(e_1,e_2) \\ &= \nabla_{e_k} F^*\Omega_1(e_1,e_2) - F^*\Omega_1(\nabla_{e_k} e_1, e_2) - F^*\Omega_1(e_1, \nabla_{e_k} e_2) \\ &= \Omega_1\big(A_{k1}, dF(e_2)\big) + \Omega_1\big(dF(e_1), A_{k2}\big) \\ &= \frac{1}{\sqrt{1+\mu^2}}\big(A^3_{k1}\Omega_1(\xi_3,\alpha_2)+A^3_{k1}\Omega_1(\xi_3,\mu\beta_2)\big) \\ &\quad +\frac{1}{\sqrt{1+\lambda^2}}\big(A^4_{k2}\Omega_1(\alpha_1,\xi_4)+A^4_{k2}\Omega_1(\lambda\beta_1,\xi_4)\big) \end{aligned}$$

and thus, we obtain

$$e_k u_1 = -u_1\big(\lambda A^3_{k1}+\mu A^4_{k2}\big).$$

Therefore,

$$\nabla u_1 = -u_1\big(\lambda A^3_{11}+\mu A^4_{12}\big)e_1 - u_1\big(\lambda A^3_{12}+\mu A^4_{22}\big)e_2.$$

Similarly, we have

$$\nabla u_2 = +u_1\big(\mu A^3_{11}+\lambda A^4_{12}\big)e_1 + u_1\big(\mu A^3_{12}+\lambda A^4_{22}\big)e_2.$$

Combining the above gradient expressions, we get the gradients of the angle functions $\varphi$ and $\vartheta$. This completes the proof. ☐

**Lemma 5.3.5** *The functions $u_1$ and $u_2$ satisfy the following coupled system of parabolic equations*

$$\begin{aligned} \partial_t u_1 - \Delta u_1 &= |A|^2 u_1 + 2\sigma^{\perp} u_2 + \sigma_M(1-u_1^2-u_2^2)u_1 - 2\sigma_N u_1 u_2^2, \\ \partial_t u_2 - \Delta u_2 &= |A|^2 u_2 + 2\sigma^{\perp} u_1 + \sigma_N(1-u_1^2-u_2^2)u_2 - 2\sigma_M u_1^2 u_2. \end{aligned}$$

*Moreover, $\varphi$ and $\vartheta$ satisfy the following system of equations*

$$\partial_t\varphi - \Delta\varphi = (|A|^2 - 2\sigma^{\perp})\varphi + \frac{1}{2}\big(\sigma_M(\varphi+\vartheta)+\sigma_N(\varphi-\vartheta)\big)(1-\varphi^2),$$

$$\partial_t\vartheta - \Delta\vartheta = (|A|^2 + 2\sigma^{\perp})\vartheta + \frac{1}{2}\big(\sigma_M(\varphi+\vartheta)-\sigma_N(\varphi-\vartheta)\big)(1-\vartheta^2).$$

*Proof.* Using Lemma 5.3.3, we can see that

$$\begin{aligned}\partial_t u_1 - \Delta u_1 &= |A|^2 u_1 - 2\sum_{k,\alpha,\beta} A^{\alpha}_{k1} A^{\beta}_{k2} \Omega_1(\xi_\alpha, \xi_\beta) \\ &\quad + \sum_{\alpha} \big(\widetilde{R}_{212\alpha}\Omega_1(\xi_\alpha, e_2) + \widetilde{R}_{121\alpha}\Omega_1(e_1, \xi_\alpha)\big).\end{aligned}$$

Computing the curvature terms, with respect to the local adapted frame arising from the singular decomposition and taking into account equations (5.2) and (5.3), we derive the evolution equation of $u_1$. Similarly, we derive the evolution of $u_2$. Now, by adding and subtracting together

$$\partial_t u_1 - \Delta u_1 = |A|^2 u_1 + 2\sigma^{\perp} u_2 + \sigma_M (1 - u_1^2 - u_2^2) u_1 - 2\sigma_N u_1 u_2^2$$

and

$$\partial_t u_2 - \Delta u_2 = |A|^2 u_2 + 2\sigma^{\perp} u_1 + \sigma_N (1 - u_1^2 - u_2^2) u_2 - 2\sigma_M u_1^2 u_2,$$

we obtain

$$\partial_t \varphi - \Delta \varphi = (|A|^2 - 2\sigma^{\perp})\varphi + \frac{1}{2}\big(\sigma_M(\varphi + \vartheta) + \sigma_N(\varphi - \vartheta)\big)(1 - \varphi^2)$$

and

$$\partial_t \vartheta - \Delta \vartheta = (|A|^2 + 2\sigma^{\perp})\vartheta + \frac{1}{2}\big(\sigma_M(\varphi + \vartheta) - \sigma_N(\varphi - \vartheta)\big)(1 - \vartheta^2).$$

This completes the proof. □

The following lemma will be of crucial importance to our analysis.

**Lemma 5.3.6** *Suppose that $M$ and $N$ are compact with the same constant sectional curvature $\sigma$ and that $f: M \to N$ is an area preserving map. Then, the following hold:*

(a) *The evolved by mean curvature flow submanifolds are Lagrangian as long as the flow exists.*

(b) *The mean curvature flow with initial data the graph $\Gamma(f)$ remains graphical as long as the flow exists. In particular, the evolved graphs are generated by area preserving maps.*

(c) *There exists a positive real number $c_0$ such that*

$$1 \geq \vartheta(x,t) \geq \frac{c_0 e^{\sigma t}}{\sqrt{1 + c_0^2 e^{2\sigma t}}},$$

*for any point $(x,t)$ in space-time.*

*Proof.* Without loss of generality, we may assume that initially the map $f\colon M \to N$ is an orientation preserving diffeomorphism. Otherwise, we interchange the roles of the functions $\vartheta$ and $\varphi$.

(a) As in Lemma 5.3.5, we can show that the Kähler angle $\cos a_1$ evolve in time according to

$$\partial_t \cos a_1 - \Delta \cos a_1 = \big(|A|^2 - 2\sigma^{\perp} + \sigma \sin^2 a_1\big) \cos a_1.$$

Since initially we have $\cos a_1 \equiv 0$, we deduce that $\cos a_1(x,t) = 0$, for every point $(x,t)$ in space-time. Consequently, the evolved under mean curvature flow submanifolds are Lagrangian.

(b) By compactness, initially, we have

$$\min_{x\in M} u_1(x,0) = \varepsilon > 0.$$

By continuity, the minimum of $u_1$ stays positive for small values of $t$. However, we will show that the flow remains graphical as long as it exists. As a matter of fact, we will show that

$$\min_{x\in M} u_1(x,t) > 0,$$

as long as the flow exists. Suppose on the contrary, that there exists a first time where the graphical property does not hold. This means that there exists a point $(x_0, t_0)$ in space-time with $t_0 < T$, such that

$$u_1(x_0, t_0) = 0$$

and $u_1(x,t) > 0$, for all $(x,t) \in M \times [0, t_0)$. On $M \times [0, t_0)$ we have $\vartheta \equiv 2u_1$ and, by part (a), that $\varphi \equiv 0$. Moreover, since $|A|^2$ is bounded on $M \times [0, t_0]$, there exists a constant $c(t_0) \in \mathbb{R}$, such that

$$\partial_t \vartheta - \Delta\vartheta \geq c(t_0)\vartheta,$$

for all $(x,t) \in M \times [0, t_0)$. By Theorem 3.2.3, we get

$$\vartheta(x,t) \geq e^{c(t_0)t},$$

for all $(x,t) \in M \times [0, t_0)$. Thus,

$$2u_1(x,t) \geq e^{c(t_0)t},$$

for all $t \in [0, t_0)$. Therefore, we have

$$2u_1(x_0,t_0) = 2\lim_{t\to t_0} u_1(x_0,t) \geq e^{c(t_0)t_0} > 0,$$

which leads to a contradiction.

(c) Since

$$\begin{aligned} |A|^2 + 2\sigma^{\perp} = {} & (A^3_{11})^2 + 2(A^3_{12})^2 + (A^3_{22})^2 + (A^4_{11})^2 + 2(A^4_{12})^2 + (A^4_{22})^2 \\ & -2A^3_{11}A^4_{12} + 2A^3_{12}A^4_{11} - 2A^3_{12}A^4_{22} + 2A^3_{22}A^4_{12} \\ = {} & (A^3_{11} - A^4_{12})^2 + (A^4_{11} + A^3_{12})^2 + (A^3_{22} + A^4_{12})^2 + (A^4_{22} - A^3_{12})^2 \\ \geq {} & 0, \end{aligned}$$

from the evolution equation of $\vartheta$, we get

$$\partial_t \vartheta - \Delta \vartheta \geq \sigma \vartheta (1 - \vartheta^2).$$

According to Theorem 3.2.3, there exist a positive real number $c_0$ such that

$$\vartheta(x,t) \geq \frac{c_0 e^{\sigma t}}{\sqrt{1 + c_0^2 e^{2\sigma t}}},$$

for any $(x,t)$ in space-time. □

### 5.3.3 Decay estimate for the mean curvature

In the sequel, we provide an important decay estimate for the mean curvature, due to Wang [45].

**Lemma 5.3.7** *Let $M$ and $N$ compact Riemannian manifolds with the same constant sectional curvature $\sigma$ and that $f : M \to N$ is an area preserving map. Then, the following decay estimate holds:*

$$\frac{d}{dt} \int \frac{|H|^2}{\vartheta} d\mu \leq \sigma \int \frac{|H|^2}{\vartheta} d\mu,$$

*where $d\mu$ is the volume element of the induced metric.*

*Proof.* For arbitrary time-dependent smooth functions $f$ and $g$ it holds

$$\partial_t \left( \frac{f}{g} \right) - \Delta \left( \frac{f}{g} \right) = \frac{1}{g} (\partial_t f - \Delta f) - \frac{f}{g^2} (\partial_t g - \Delta g) + \frac{2}{g^2} \langle \nabla f, \nabla g \rangle - 2 \frac{f}{g^3} |\nabla g|^2.$$

Setting $f = |H|^2$ and $g = \vartheta$, we have

$$\begin{aligned} \partial_t \left( \frac{|H|^2}{\vartheta} \right) - \Delta \left( \frac{|H|^2}{\vartheta} \right) &= \frac{1}{\vartheta} (\partial_t |H|^2 - \Delta |H|^2) - \frac{|H|^2}{\vartheta^2} (\partial_t \vartheta - \Delta \vartheta) \\ &\quad + \frac{2}{\vartheta^2} \langle \nabla |H|^2, \nabla \vartheta \rangle - 2 \frac{|H|^2}{\vartheta^3} |\nabla \vartheta|^2. \end{aligned}$$

But from Lemma 5.3.5, we obtain

$$\begin{aligned} \partial_t \left( \frac{|H|^2}{\vartheta} \right) - \Delta \left( \frac{|H|^2}{\vartheta} \right) = \frac{1}{\vartheta} \Big( &- 2 |\nabla^\perp H|^2 + 2 \sum_{k,\alpha,\beta} H^\alpha H^\beta \widetilde{R}_{\alpha k \beta k} + 2 \sum_{i,j} (A^H_{ij})^2 \Big) \\ &- \frac{|H|^2}{\vartheta^2} \Big( (|A|^2 + 2\sigma^\perp) \vartheta + \sigma \vartheta (1 - \vartheta^2) \Big) \\ &+ \frac{2}{\vartheta^2} \langle \nabla |H|^2, \nabla \vartheta \rangle - 2 \frac{|H|^2}{\vartheta^3} |\nabla \vartheta|^2. \end{aligned}$$

Since the graph is Lagrangian, using (5.1), we have

$$\begin{aligned}\partial_t\left(\frac{|H|^2}{\vartheta}\right)-\Delta\left(\frac{|H|^2}{\vartheta}\right) &= \frac{2\vartheta\langle\nabla\vartheta,\vartheta\nabla|H|^2-|H|^2\nabla\vartheta\rangle}{\vartheta^4}\\ &-\frac{2|\nabla^\perp H|^2}{\vartheta}+\frac{2\sum_{i,j}(A^H_{ij})^2-2|H|^2|A|^2+|H|^4}{\vartheta}\\ &+\frac{2\sum_{k,\alpha,\beta}H^\alpha H^\beta\widetilde{R}_{\alpha k\beta k}}{\vartheta}-\sigma\frac{|H|^2}{\vartheta}(1-\vartheta^2). \qquad (5.4)\end{aligned}$$

Also,

$$\sum_{k,\alpha,\beta}H^\alpha H^\beta\widetilde{R}_{\alpha k\beta k}=\sigma\left(1-\frac{\vartheta^2}{2}\right)|H|^2. \qquad (5.5)$$

Therefore, combining (5.5) with (5.4) and using

$$\nabla|H|^2=2|H|\nabla|H|,$$

we obtain

$$\begin{aligned}\partial_t\left(\frac{|H|^2}{\vartheta}\right)-\Delta\left(\frac{|H|^2}{\vartheta}\right) &= \frac{4\vartheta|H|\langle\nabla\vartheta,\nabla|H|\rangle-2|\nabla\vartheta|^2|H|^2-2\vartheta^2|\nabla^\perp H|^2}{\vartheta^3}\\ &+\frac{2\sum_{i,j}(A^H_{ij})^2-2|H|^2|A|^2+|H|^4}{\vartheta}+\sigma\frac{|H|^2}{\vartheta}.\end{aligned}$$

Integrating this inequality, we have

$$\begin{aligned}\frac{d}{dt}\int\frac{|H|^2}{\vartheta}d\mu-\int\frac{|H|^2}{\vartheta}\frac{d}{dt}d\mu=2\int &\frac{2\vartheta|H|\langle\nabla\vartheta,\nabla|H|\rangle-|\nabla\vartheta|^2|H|^2-\vartheta^2|\nabla^\perp H|^2}{\vartheta^3}d\mu\\ &+\int\frac{2\sum_{i,j}(A^H_{ij})^2-2|H|^2|A|^2+|H|^4}{\vartheta}d\mu\\ &+\int\sigma\frac{|H|^2}{\vartheta}d\mu.\end{aligned}$$

Using

$$|\nabla|H||\leq|\nabla^\perp H|$$

in the first term on the right hand side of the above equation and completing the square, we have

$$\frac{2\vartheta|H|\langle\nabla\vartheta,\nabla|H|\rangle-|\nabla\vartheta|^2|H|^2-\vartheta^2|\nabla|H||^2}{\vartheta^3}=-\frac{||H|\nabla\vartheta-\vartheta\nabla|H||^2}{\vartheta^3}\leq 0.$$

Moreover, from Lemma 4.2.1, we have

$$\partial_t(d\mu)=-|H|^2d\mu.$$

Also, by Cauchy-Schwarz inequality, we have

$$\sum_{i,j}(A^H_{ij})^2\leq\sum_{i,j}|A_{ij}|^2|H|^2=|A|^2|H|^2.$$

Therefore,

$$\frac{d}{dt}\int \frac{|H|^2}{\vartheta} d\mu \leq \sigma \int \frac{|H|^2}{\vartheta} d\mu$$

and by integration, we obtain

$$\int \frac{|H|^2}{\vartheta} d\mu \leq e^{\sigma t}.$$

This completes the proof. □

## 5.4 Singular analysis

In this section, we present how one can built smooth singularity models for the mean curvature flow by rescaling properly around points, where the second fundamental form attains its maximum. The proof relies on a compactness theorem of Cheeger–Gromov-Taylor [8] for pointed Riemannian manifolds and on the standard compactness theorem for immersions; see for example [13].

### 5.4.1 Cheeger-Gromov compactness for metrics

Let us recall here the basic notions and definitions. For more details, see the books [1, Chap. 9], [11, Chap. 3] and [31, Chap. 5].

**Definition 5.4.1** *Let $(E, \pi, \Sigma)$ be a vector bundle endowed with a Riemannian metric $g$ and a metric connection $\nabla$ and suppose that $\{\xi_k\}_{k\in\mathbb{N}}$ is a sequence of sections of E. Let U be an open subset of $\Sigma$ with compact closure $\bar{U}$ in $\Sigma$. Fix a natural number $p \geq 0$. We say that $\{\xi_k\}_{k\in\mathbb{N}}$ converges in $C^p$ to $\xi_\infty \in \Gamma(E|_{\bar{U}})$, if for every $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon)$, such that*

$$\sup_{0\leq\alpha\leq p}\ \sup_{x\in\bar{U}} |\nabla^\alpha(\xi_k - \xi_\infty)| < \varepsilon$$

*where $k \geq k_0$. We say that $\{\xi_k\}_{k\in\mathbb{N}}$ converges in $C^\infty$ to $\xi_\infty \in \Gamma(E|_{\bar{U}})$ if $\{\xi_k\}_{k\in\mathbb{N}}$ converges in $C^p$ to $\xi_\infty \in \Gamma(E|_{\bar{U}})$, for any $p \geq 0$.*

**Definition 5.4.2** *Let $(E, \pi, \Sigma)$ be a vector bundle endowed with a Riemannian metric $g$ and a metric connection $\nabla$. Let $\{U_n\}_{n\in\mathbb{N}}$ be an exhaustion of $\Sigma$ and $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence of sections of E defined on open sets $A_k$ of $\Sigma$. We say that $\{\xi_k\}_{k\in\mathbb{N}}$ converges smoothly on compact sets to $\xi_\infty \in \Gamma(E)$ if:*

(a) *For every $n \in \mathbb{N}$ there exists $k_0$ such that $\bar{U}_n \subset A_k$, for all natural numbers $k \geq k_0$.*

(b) *The sequence $\{\xi|_{\bar{U}_k}\}_{k\geq k_0}$ converges in $C^\infty$ to the restriction of the section $\xi_\infty$ on $\bar{U}_n$.*

In the next definitions, we recall the notion of Cheeger-Gromov convergence of sequences of Riemannian manifolds.

**Definition 5.4.3** *A pointed Riemannian manifold $(\Sigma, g, x)$ is a Riemannian manifold $(\Sigma, g)$ with a choice of origin or base point $x \in \Sigma$. If the metric g is complete, we say that $(\Sigma, g, x)$ is a complete pointed Riemannian manifold.*

**Definition 5.4.4** *We say that a sequence $\{(\Sigma_k, g_k, x_k)\}_{k\in\mathbb{N}}$ of complete, pointed Riemannian manifolds smoothly converges in the sense of Cheeger-Gromov into a complete pointed Riemannian manifold $(\Sigma_\infty, g_\infty, x_\infty)$, if there exists:*

(a) *An exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of $\Sigma_\infty$ with $x_\infty \in U_k$, for all $k \in \mathbb{N}$.*

(b) *A sequence of diffeomorphisms $\Phi_k \colon U_k \to \Phi_k(U_k) \subset \Sigma_k$, with $\Phi_k(x_\infty) = x_k$ and such that $\{\Phi_k^* g_k\}_{k\in\mathbb{N}}$ smoothly converges in $C^\infty$ to $g_\infty$ on compact sets in $\Sigma_\infty$.*

The family $\{(U_k, \Phi_k)\}_{k\in\mathbb{N}}$ is called a *family of convergence pairs* of $\{(\Sigma_k, g_k, x_k)\}_{k\in\mathbb{N}}$, with respect to the limit $(\Sigma_\infty, g_\infty, x_\infty)$.

When we say *smooth convergence*, we will always mean smooth convergence in the sense of Cheeger-Gromov. We would like to mention that the family of convergence pairs is not unique. Two such families $\{(U_k, \Phi_k)\}_{k\in\mathbb{N}}$,$\{(W_k, \Psi_k)\}_{k\in\mathbb{N}}$ are equivalent in the sense that there exists an isometry $I$ of the limit $(\Sigma_\infty, g_\infty, x_\infty)$, such that for every compact subset $K$ of $\Sigma_\infty$, there exists a natural number $k_0$, such that for any natural $k \geq k_0$:

(a) The mapping $\Phi_k^{-1} \circ \Psi_k$ is well defined over $K$.

(b) The sequence $\{\Phi_k^{-1} \circ \Psi_k\}_{k\geq k_0}$ smoothly converges to $I$ on $K$.

The limiting pointed Riemannian manifold $(\Sigma_\infty, g_\infty, x_\infty)$ of the Definition 5.4.4 is unique up to isometries.

**Definition 5.4.5** *Let $M$ be a Riemannian manifold. The injectivity radius at $x \in M$ is the supremum of all values r, such that the expotential map from the unit ball $B_r(x)$ in $T_xM$, to the manifold M, is injective.*

**Definition 5.4.6** *A complete Riemannian manifold $(\Sigma, g)$ is said to have bounded geometry, if the following conditions are satisfied:*

(a) *For any integer $j \geq 0$, there exists a uniform positive constant $C_j$, so that $|\nabla^j R| \leq C_j$.*

(b) *The injectivity radius satisfies $inj_g(\Sigma) > 0$.*

The following proposition is standard and will be useful in the proof of the long-time existence of the graphical mean curvature flow.

**Proposition 5.4.7** *Suppose $(\Sigma, g)$ is a complete Riemannian manifold with bounded geometry. Suppose that $\{\alpha_k\}_{k\in\mathbb{N}}$ is an increasing sequence of real numbers that tends to $+\infty$ and let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of points on $\Sigma$. Then, the sequence $\{(\Sigma, \alpha_k^2 g, x_k)\}_{k\in\mathbb{N}}$ smoothly subconverges to the Euclidean space $(\mathbb{R}^m, g_{euc}, 0)$.*

We will use the following definition of uniformly bounded geometry for a sequence of pointed Riemannian manifolds.

**Definition 5.4.8** *We say that a sequence $\{(\Sigma_k, g_k, x_k)\}_{k\in\mathbb{N}}$ of complete pointed Riemannian manifolds has uniformly bounded geometry, if the following conditions are satisfied:*

(a) *For any integer $j \geq 0$, there exists a uniform constant $C_j$, such that for each $k \in \mathbb{N}$ it holds $|\nabla^j R_k| \leq C_j$, where $R_k$ is the curvature operator of $g_k$.*

(b) *There exists a uniform constant $c_0$, such that $inj_{g_k}(\Sigma_k) \geq c_0 > 0$.*

In the next result, we state the Cheeger-Gromov compactness theorem for sequences of complete pointed Riemannian manifolds. The version that we present here is due to Hamilton.

**Theorem 5.4.9** *Let $\{(\Sigma_k, g_k, x_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds with uniformly bounded geometry. Then, the sequence $\{(\Sigma_k, g_k, x_k)\}_{k\in\mathbb{N}}$ subconverges smoothly to a complete pointed Riemannian manifold $(\Sigma_\infty, g_\infty, x_\infty)$.*

**Remark 5.4.1** We would like to mention here that due to an estimate from Cheeger, the above compactness theorem still holds under the weaker assumption that the injectivity radius is uniformly bounded from below by a positive constant, only along the base points $\{x_k\}_{k\in\mathbb{N}}$, thereby avoiding the assumption of the uniform lower bound for $inj_{g_k}(\Sigma_k)$.

### 5.4.2 Convergence of immersions

**Definition 5.4.10** *Let $F_k\colon (\Sigma_k, g_k, x_k) \to (P_k, h_k, y_k)$ be a sequence of isometric immersions, such that $F(x_k) = y_k$, for any $k \in \mathbb{N}$. We say that the sequence $\{F_k\}_{k\in\mathbb{N}}$ converges smoothly to an isometric immersion $F_\infty\colon (\Sigma_\infty, g_\infty, x_\infty) \to (P_\infty, h_\infty, y_\infty)$ if the following conditions are satisfied:*

(a) *The sequence $\{(\Sigma_k, g_k, x_k)\}_{k\in\mathbb{N}}$ smoothly converges to $(\Sigma_\infty, g_\infty, x_\infty)$.*

(b) *The sequence $\{(P_k, h_k, y_k)\}_{k\in\mathbb{N}}$ smoothly converges to $(P_\infty, h_\infty, y_\infty)$.*

(c) *If* $\{(U_k, \Phi_k)\}_{k\in\mathbb{N}}$ *is a family of convergence pairs of the sequence* $\{(\Sigma_k, g_k, x_k)\}_{k\in\mathbb{N}}$ *and* $\{(W_k, \Psi_k)\}_{k\in\mathbb{N}}$ *is a family of convergence pairs of* $\{(P_k, h_k, y_k)\}_{k\in\mathbb{N}}$*, then for each* $k \in \mathbb{N}$*, we have* $F_k \circ \Phi_k(U_k) \subset \Psi_k(W_k)$ *and* $\Psi_k^{-1} \circ F \circ \Phi_k$ *smoothly converges to* $F_\infty$ *on compact sets.*

**Lemma 5.4.11** *Suppose that* $(P, h)$ *is a complete Riemannian manifold with bounded geometry. Then, for any* $C > 0$*, there exists a positive constant* $r > 0$*, such that* $inj_g(\Sigma) > r$*, for any isometric immersion* $F\colon (\Sigma, g) \to (P, h)$ *such that the norm* $|A_F|$ *of its second fundamental form satisfies* $|A_F| \le C$*.*

The last lemma and the Cheeger-Gromov compactness theorem allow us to deduce a compactness theorem in the category of sequences of immersions.

**Theorem 5.4.12** *Let* $\{(\Sigma_k, g_k, x_k)\}_{k\in\mathbb{N}}$ *and* $\{(P_k, h_k, y_k)\}_{k\in\mathbb{N}}$ *be two sequences of complete pointed Riemannian manifolds with dimensions m and l, respectively. Suppose further that* $F_k\colon (\Sigma_k, g_k, x_k) \to (P_k, h_k, y_k)$ *is a family of isometric immersions, where* $F_k(x_k) = y_k$*. Assume that:*

(a) *Each* $\Sigma_k$ *is compact.*

(b) *The sequence* $\{(P_k, h_k, y_k)\}_{k\in\mathbb{N}}$ *has uniformly bounded geometry.*

(c) *For any nonnegative integer* $j \ge 0$*, there exists a uniform constant* $C_j$*, such that* $|(\nabla^{F_k})^j A_{F_k}| \le C_j$*, for any natural number* $k \in \mathbb{N}$*. Here,* $A_{F_k}$ *stands for the second fundamental form of the immersion* $F_k$*.*

*Then, the sequence of immersions* $\{F_k\}_{k\in\mathbb{N}}$ *subconverges smoothly to a complete isometric immersion* $F_\infty\colon (\Sigma_\infty, g_\infty, x_\infty) \to (P_\infty, h_\infty, y_\infty)$*.*

### 5.4.3 Modeling the singularities

In the following theorem, we describe a method of rescaling around points, where the second fundamental form attains its maximum.

**Theorem 5.4.13** *Let* $\Sigma$ *be a compact manifold and let* $F\colon \Sigma \times [0, T) \to (P, h)$ *be a solution of mean curvature flow, where* $P$ *is a Riemannian manifold with bounded geometry and* $T \le \infty$ *is the maximal time of existence. Suppose that there exists a sequence of points* $\{(x_k, t_k)\}_{k\in\mathbb{N}}$ *in* $\Sigma \times [0, T)$ *with* $\lim t_k = T$ *and such that the sequence* $\{\alpha_k\}_{k\in\mathbb{N}}$*, where*

$$\alpha_k := \max_{(x,t)\in\Sigma\times[0,t_k]} |A(x, t)| = |A(x_k, t_k)|,$$

*tends to infinity. Then:*

(a) *The maps* $F_k\colon \Sigma\times[-\alpha_k^2 t_k,0]\to(P,\alpha_k^2 h)$*,*$k\in\mathbb{N}$*, given by*

$$F_k(x,s):=F_{k,s}(x):=F(x,s/\alpha_k^2+t_k),$$

*form a sequence of mean curvature flow solutions. Moreover, we have* $|A_{F_k}|\le 1$ *and* $|A_{F_k}(x_k,0)|=1$*, for any* $k\in\mathbb{N}$*.*

(b) *For any time* $s\le 0$*, the sequence* $\{(\Sigma,F_{k,s}^*(\alpha_k^2 h),x_k)\}_{k\in\mathbb{N}}$ *smoothly subconverges to a complete pointed Riemannian manifold* $(\Sigma_\infty,g_\infty,x_\infty)$ *which does not depend on the choice of s. Moreover, the sequence* $\big\{\big(P,\alpha_k^2 h,F_k(x_k,s)\big)\big\}_{k\in\mathbb{N}}$ *smoothly subconverges to the standard Euclidean space* $(\mathbb{R}^l,g_{euc},0)$*.*

(c) *There is a mean curvature flow* $F_\infty\colon\Sigma_\infty\times(-\infty,0]\to\mathbb{R}^l$*, such that for each fixed time* $s\le 0$*, the sequence* $\{F_{k,s}\}_{k\in\mathbb{N}}$ *smoothly subconverges to the map* $F_{\infty,s}$*. This convergence is uniform, with respect to the parameter s. Additionally,* $|A_{F_\infty}|\le 1$ *and* $|A_{F_\infty}(x_\infty,0)|=1$*.*

(d) *If* $\dim\Sigma=2$ *and* $H_{F_\infty}=0$*, then the limiting Riemann surface* $\Sigma_\infty$ *has finite total curvature. In the matter of fact, the limiting surface* $\Sigma_\infty$ *is conformally diffeomorphic to a compact Riemann surface minus a finite number of points and is of parabolic type.*

*Proof.* The proof of this theorem is deep and goes beyond the scopes of this thesis. For this proof we refer to [10] and [13]. However, part $(a)$ is simple and for that reason we provide the proof.

Let $\{e_1,\dots,e_m\}$ be an orthonormal frame with respect to the induced metric $F^*h$. Then,

$$\left\{\tilde e_1=\frac{e_1}{\alpha_k},\dots,\tilde e_m=\frac{e_m}{\alpha_k}\right\}$$

is an orthonormal frame with respect to $F^*(\alpha_k^2 h)$. We compute,

$$\begin{aligned}
A_{F_k}(\tilde{e_i},\tilde{e_j}) &= \nabla^{F_k}_{\tilde{e_i}}dF(\tilde{e_j})-dF\Big(\nabla^{F_k^*(\alpha_k^2 h)}_{\tilde{e_i}}\tilde{e_j}\Big)\\
&= \nabla^{\alpha_k^2 h}_{\frac{1}{\alpha_k}dF(e_i)}\frac{1}{\alpha_k}dF(e_j)-dF\Big(\nabla^{\alpha_k^2 g}_{\frac{1}{\alpha_k}e_i}\frac{1}{\alpha_k}e_j\Big)\\
&= \frac{1}{\alpha_k^2}\nabla^h_{dF(e_i)}dF(e_j)-\frac{1}{\alpha_k^2}dF\Big(\nabla^g_{e_i}e_j\Big)\\
&= \frac{1}{\alpha_k^2}\Big(\nabla^F_{e_i}dF(e_j)-dF\Big(\nabla^g_{e_i}e_j\Big)\Big)\\
&= \frac{1}{\alpha_k^2}A(e_i,e_j),
\end{aligned}$$

since, for any $v_1,v_2\in T_xM$, it holds

$$\nabla^{\alpha_k^2 h}_{v_1}v_2=\nabla^h_{v_1}v_2+\frac{1}{\alpha_k^2}\Big(v_1(\alpha_k^2)v_2+v_2(\alpha_k^2)v_1-h(v_1,v_2)\nabla\alpha_k^2\Big)=\nabla^h_{v_1}v_2.$$

Moreover,

$$H_{F_k} = \operatorname{tr} A_{F_k} = \frac{1}{\alpha_k^2} \operatorname{tr} A = \frac{1}{\alpha_k^2} H.$$

Therefore, since

$$\partial_s F_k(x,s) = \frac{1}{\alpha_k^2} \frac{\partial F(x, s/\alpha_k^2 + t_k)}{\partial(s/\alpha_k^2 + t_k)} = \frac{1}{\alpha_k^2} H(x, s/\alpha_k^2 + t_k) = H_{F_k}(x,s),$$

for any $(x,s) \in M \times [-\alpha_k^2 t_k, 0]$, we conclude that $F_k$ is a sequence of mean curvature flow solutions. Also,

$$\begin{aligned} |A_{F_k}(\tilde{e_i}, \tilde{e_j})|^2 &= \langle A_{F_k}(\tilde{e_i}, \tilde{e_j}), A_{F_k}(\tilde{e_i}, \tilde{e_j}) \rangle_{\alpha_k^2 h} \\ &= \alpha_k^2 \langle A_{F_k}(\tilde{e_i}, \tilde{e_j}), A_{F_k}(\tilde{e_i}, \tilde{e_j}) \rangle_h = \frac{1}{\alpha_k^2} |A(e_i, e_j)|^2 \\ &\leq 1 \end{aligned}$$

and thus, $|A_{F_k}| = \alpha_k^{-2}|A| \leq 1$. From the above computation, at a time in $t \in [0, t_k]$, we can obtain

$$|A_{F_k}|^2(x_k, 0) = \frac{1}{\alpha_k^2} |A|^2(x_k, t_k) = \frac{\alpha_k^2}{\alpha_k^2} = 1,$$

for any $k \in \mathbb{N}$. This completes the proof. □

### 5.4.4 Long-time existence

In the next theorem, we show that under the assumptions of the main theorem, the graphical mean curvature flow exists for all times.

**Theorem 5.4.14** *Let $(M, g_M)$ and $(N, g_N)$ be two compact Riemann surfaces, with the same constant sectional curvature. Also, let $f\colon M \to N$ be an area preserving map. Evolve the graph of $f$ under the mean curvature flow. Then, the norm of the second fundamental form of the evolved graphs stays uniformly bounded in time and so the graphical mean curvature flow exists for all times.*

*Proof.* On the contrary, suppose that $|A|$ is not uniformly bounded. Then, there exists a sequence $\{(x_k, t_k)\}_{k \in \mathbb{N}}$ in $M \times [0, T)$ with

$$\lim t_k = T, \quad \alpha_k := \max_{(x,t) \in M \times [0, t_k]} |A(x,t)| = |A(x_k, t_k)|,$$

such that $\{\alpha_k\}_{k \in \mathbb{N}}$ tends to infinity. Let $F_k\colon M \times [-\alpha_k^2 t_k, 0] \to (M \times N, \alpha_k^2 g_M \times g_N)$ as in Theorem 5.4.13 be the graph of the rescaled function

$$f\colon (M, \alpha_k^2 g_M) \to (N, \alpha_k^2 g_N).$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame with respect to $g_M$. Then $\tilde{e_i} = e_i/\alpha_k$, for every $i = 1, \ldots, n$ with respect to $\alpha_k^2 g_M$. Therefore, the singular values of the rescaled function $f$ can be obtained by

$$df(\tilde{e_i}) = \frac{1}{\alpha_k} df(e_i) = \frac{1}{\alpha_k}\lambda\beta_1 = \lambda\frac{\beta_1}{\alpha_k} = \lambda\tilde{\beta}_1$$

and

$$df(\tilde{e_j}) = \frac{1}{\alpha_k} df(e_j) = \frac{1}{\alpha_k}\mu\beta_2 = \mu\frac{\beta_2}{\alpha_k} = \mu\tilde{\beta}_2.$$

Thus, $\vartheta_{F_k} = \vartheta$. Also, from Theorem 5.4.13 (a) we have

$$H_{F_k}(x, s) = \frac{1}{\alpha_k^2} H(x, s/\alpha_k^2 + t_k),$$

for any $(x, s) \in M \times [-\alpha_k^2 t_k, 0]$. Therefore,

$$\frac{|H_{F_k}|^2}{\vartheta_{F_k}} = \frac{1}{\alpha_k^2}\frac{|H|^2}{\vartheta}.$$

We distinguish two cases:

**Case** $\sigma \leq 0$: Using Lemma 5.3.7, we compute

$$\int \frac{|H_{F_k}|^2}{\vartheta_{F_k}} d\mu_k = \frac{1}{\alpha_k^2}\int \frac{|H|^2}{\vartheta} d\mu \leq \frac{1}{\alpha_k^2} e^{\sigma(s/\alpha_k^2 + t_k)} \leq \frac{1}{\alpha_k^2} c,$$

where $c > 0$. Since the convergence is smooth, we have

$$0 = \lim_{k\to\infty}\int \frac{|H_{F_k}|^2}{\vartheta_{F_k}} d\mu = \int \lim_{k\to\infty}\frac{|H_{F_k}|^2}{\vartheta_{F_k}} d\mu = \int \frac{|H_{F_\infty}|^2}{\vartheta_\infty} d\mu.$$

Therefore, $H_{F_\infty} = 0$, which means that $F_\infty$ in Theorem 5.4.13 is a complete minimal Lagrangian immersion in $\mathbb{R}^4$ of parabolic type. Hence, any nonnegative superharmonic function must be constant.

Since the convergence is smooth, the corresponding Kähler angle $\vartheta_\infty$ of $F_\infty$, with respect to the complex structure $J_2 = J_{\mathbb{R}^2} \oplus J_{\mathbb{R}^2}$ of $\mathbb{R}^4$, is nonnegative. Therefore, from Lemma 5.3.5, we obtain

$$\Delta\vartheta_\infty + (|A_{F_\infty}|^2 + 2\sigma^\perp_{F_\infty})\vartheta_\infty = 0, \tag{5.6}$$

where $-\sigma^\perp_{F_\infty}$ is the normal curvature of $F_\infty$. Moreover, from Lemma 5.3.4, we have

$$|\nabla\vartheta_\infty|^2 = (1 - \vartheta_\infty^2)\Big(\big((A_{F_\infty})^3_{11} - (A_{F_\infty})^4_{12}\big)^2 + \big((A_{F_\infty})^3_{12} + (A_{F_\infty})^4_{11}\big)^2\Big).$$

Also, we can easily derive the inequalities

$$|A_{F_\infty}|^2 \pm 2\sigma^\perp_{F_\infty} \geq 0.$$

From (5.6) we deduce that $\vartheta_\infty$ is superharmonic and consequently must be constant. Thus, the function $(u_1)_\infty > 0$ is also constant.

According to a result of Aiyama [3] the minimal Lagrangian immersion $F_\infty$ is locally of the form

$$F_\infty = \frac{1}{\sqrt{2}} e^{i\beta/2}(\mathcal{F}_1 - i\bar{\mathcal{F}}_2, \mathcal{F}_2 + i\bar{\mathcal{F}}_1),$$

where $\beta$ is a constant and $\mathcal{F}_1, \mathcal{F}_2 \colon \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ are holomorphic functions defined in a simply connected domain $\mathbb{D}$, such that

$$|(\mathcal{F}_1)_z|^2 + |(\mathcal{F}_2)_z|^2 > 0.$$

The Gauss image of $F_\infty$ lies in the slice $\mathbb{S}^2 \times \{(e^{i\beta}, 0)\}$ of the product $\mathbb{S}^2 \times \mathbb{S}^2$. As a matter of fact, all the information on the Gauss image of $F_\infty$ is encloded in the map $\mathcal{G} \colon \mathbb{D} \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ given by

$$\mathcal{G} = (\mathcal{F}_1)_z / (\mathcal{F}_2)_z.$$

Since

$$(u_1)_\infty = const > 0$$

we get that $F_\infty$ is the graph of an area preserving map $h$. Then,

$$\mathcal{F}_1 = (z + i\bar{h})/2, \;\; \mathcal{F}_2 = (-i\bar{z} + h)/2 \;\text{ and }\; |h_z|^2 - |h_{\bar{z}}|^2 = 1.$$

Therefore,

$$\mathcal{G} = (1 - ih_{\bar{z}})/h_z.$$

An easy computation shows that

$$|\mathcal{G}|^2 = \frac{|1 + i\bar{h}_{\bar{z}}|^2}{|h_z|^2} = \frac{1 + |h_{\bar{z}}|^2 + i(\bar{h}_{\bar{z}} - h_{\bar{z}})}{1 + |h_{\bar{z}}|^2} = 1 + \frac{2\,\mathrm{Im}(h_{\bar{z}})}{1 + |h_{\bar{z}}|^2} \leq 2.$$

Hence, the image of $\mathcal{G}$ is contained in a bounded subset of $\mathbb{C} \cup \{\infty\}$. But then, due to a result of Osserman [32], the immersion $F_\infty$ must be flat, which is a contradiction. Therefore, the norm of the second fundamental form is uniformly bounded in time. This completes the proof of the case $\sigma \leq 0$.

**Case** $\sigma > 0$: We will show at first that $T = \infty$. To show this, assume in contrary that $T < +\infty$. Then,

$$\int \frac{|H|^2}{\vartheta} d\mu \leq e^{\sigma t} \leq e^{\sigma T} < +\infty.$$

As in the previous case, we deduce that

$$|H_{F_\infty}| = 0.$$

Performing exactly the same procedure as above, we get a contradiction. Therefore, there is no finite time singularity and the flow exists for all times. It remains to show that $|A|^2 \leq C$, where $C$ is time independent. Indeed, from Lemma 5.3.6 and since $\lambda\mu = 1$, we obtain

$$1 \geq \frac{2\lambda}{1 + \lambda^2} = \vartheta \geq \frac{c_0 e^t}{\sqrt{1 + c_0^2 e^{2t}}},$$

which tends to 1 as $t \to \infty$. Therefore, $\vartheta_\infty = 1$ and $\lambda_\infty = 1$. Therefore, the map $f_\infty$ is an isometry and, thus, $F_\infty$ must be totally geodesic. The latter implies $|A_{F_\infty}| = 0$ and this is a contradiction. □

## 5.5 Proof of the main theorem

We are ready to prove the main theorem stated in the introduction of this thesis. We will show that the graphical mean curvature flow of an area preserving map converges to an isometry in the positive case, to an affine map in the zero case and to a minimal surface in the negative case. Recall that from Theorem 5.4.14, we already know that the norm of the second fundamental form stays uniformly bounded in time. Since

$$\partial_t d\mu = -\int_M |H|^2 d\mu$$

and since the graphical flow exists for all time we have that there exists a time-independent constant $C$, such that

$$\int_0^\infty \left( \int_M |H|^2 d\mu \right) dt \leq C.$$

Therefore, there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$, such that

$$\lim_{k \to \infty} \int_M |H|^2 d\mu = 0. \tag{5.7}$$

From Theorem 5.4.14, the norms of the second fundamental forms of the evolving submanifolds and their derivatives are uniformly bounded in time. Since the product manifold $M \times N$ is compact, after passing to a subsequence of $\{t_k\}_{k \in \mathbb{N}}$ if necessary, we deduce that the flow subconverges smoothly to a smooth surface $M_\infty$ of $M \times N$; see for example [5, Theorem 1.1]. From (5.7) $M_\infty$ should be minimal. Due to a deep result of Simon [40], it follows that the flow converges smoothly and uniformly to a minimal surface $M_\infty \subset M \times N$. Additionally, we have the following situations:

(a) If $\sigma > 0$, then from Lemma 5.3.6 $(c)$, we have $\vartheta \to 1$, as $t \to \infty$. Therefore, $M_\infty$ is the graph of an isometry $f_\infty \colon M \to N$.

(b) If $\sigma = 0$, then from Lemma 5.3.6 $(c)$, we have that $\vartheta \geq c_0 > 0$. Hence, the surface $M_\infty$ is the graph of a map $f_\infty \colon M \to N$. From Lemma 5.3.4 and the fact that $2\sigma_\infty^\perp = |A_\infty|^2$, we have

$$-\Delta \vartheta_\infty = 2|A_\infty|^2 \vartheta_\infty \geq 0.$$

By Hopf's strong maximum principle [22], we get $|A_\infty|^2 = 0$. Hence, $M_\infty$ is totally geodesic.

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