ΠΑΝΕΠΙΣΤΗΜΙΟ ΙΩΑΝΝΙΝΩΝ
ΣΧΟΛΗ ΘΕΤΙΚΩΝ ΕΠΙΣΤΗΜΩΝ
ΤΜΗΜΑ ΦΥΣΙΚΗΣ

Βαρυτοηλεκτρομαγνητισμός:
Βασικές αρχές, νέες προσεγγίσεις και η εφαρμογή τους στον Ηλεκτρομαγνητισμό

Αθανάσιος Μπακόπουλος

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

ΙΩΑΝΝΙΝΑ 2016

UNIVERSITY OF IOANNINA
SCHOOL OF NATURAL SCIENCES
PHYSICS DEPARTMENT

Gravitoelectromagnetism:
Basic principles, novel approaches and their application to Electromagnetism

Athanasios Bakopoulos

MASTER THESIS

IOANNINA 2016

Η τριμελής επιτροπή που εξέτασε την παρούσα μεταπτυχιακή διπλωματική εργασία αποτελείται από τους:

- Π. Καντή, Αναπληρώτρια Καθηγήτρια του Τμήματος Φυσικής του Πανεπιστημίου Ιωαννίνων (επιβλέπουσα)

- Κ. Ταμβάκης Καθηγητής του Τμήματος Φυσικής του Πανεπιστημίου Ιωαννίνων

- Λ. Περιβολαρόπουλος, Καθηγητής του Τμήματος Φυσικής του Πανεπιστημίου Ιωαννίνων

Αφιερώνεται στην οικογένειά μου

Ευχαριστίες

Η παρούσα μεταπτυχιακή διπλωματική εργασία εκπονήθηκε στο Πανεπιστήμιο Ιωαννίνων υπό την επίβλεψη της Καθηγήτριας κα Παναγιώτας Καντή την οποία και ευχαριστώ θερμά για τη βοήθεια, την καθοδήγηση και τη συνεργασία που μου παρείχε όλα αυτά τα χρόνια, για το χρόνο που μου διέθετε καθημερινά καθώς και για την ηθική συμπαράσταση που μου προσέφερε.

Ευχαριστώ θερμά την τριμελή-εξεταστική επιτροπή (κα Π. Καντή, κ. Κ. Ταμβάκη, κ. Λ. Περιβολαρόπουλο) για το χρόνο που διέθεσε στην ανάγνωση της παρούσας εργασίας.

Θα ήθελα τέλος να εκφράσω τις θερμές μου ευχαριστίες στους οικείους μου για την ηθική και οικονομική υποστήριξη που μου παρείχαν καθόλη τη διάρκεια της παρούσας εργασίας.

Abstract

This work is focused on the theory of Gravitoelectromagnetism (GEM). In the first part of this work we present a brief review of gravitoelectromagnetism, we locate and discuss all the problems which appear in this approach. We also try to avoid these problems by proposing new approaches in which we use the additional degrees of freedom of the gravitational field. In the second part of this work, we review our previous work regarding the construction of a tensorial theory, using the formalism of General Relativity, which aims to describe the true electromagnetism. We also extend this theory in order to make it more realistic. Finally in the third part of this work, we investigate the existence of gravitational invariants similar to the electromagnetic ones.

Περίληψη

Στην παρούσα μεταπτυχιακή διπλωματική εργασία μελετάμε την θεωρία του βαρυτοηλεκτρομαγνητισμού GEM. Στο πρώτο μέρος της εργασίας παρουσιάζουμε μια σύντομη ανασκόπηση του βαρυτοηλεκτρομαγνητισμού, επίσης εντοπίζουμε και σχολιάζουμε όλα τα προβλήματα της θεωρίας. Στη συνέχεια προσπαθούμε να διορθώσουμε αυτά τα προβλήματα προτείνοντας νέες προσεγγίσεις στις οποίες χρησιμοποιούμε τους επιπλέον βαθμούς ελευθερίας του βαρυτικού πεδίου. Στο δεύτερο μέρος της εργασίας, αρχικά, παρουσιάζουμε μια ανασκόπηση μιας παλιότερης εργασίας μας σχετικά με την κατασκευή μιας τανυστικής θεωρίας, στην οποία προσπαθούμε να περιγράψουμε την Ηλεκτροδυναμική κάνοντας χρήση του φορμαλισμού της Γενικής Θεωρίας της Σχετικότητας. Στη συνέχεια προσπαθούμε να επεκτείνουμε την θεωρία ώστε να γίνει πιο ρεαλιστική. Τέλος, στο τρίτο μέρος της εργασίας, ερευνούμε την ύπαρξη βαρυτικών αναλλοίωτων ποσοτήτων οι οποίες είναι παρόμοιες με αυτές που εμφανίζονται στην Ηλεκτροδυναμική.

Contents

Contents

Πρόλογος

Ο Βαρυτοηλεκτρομαγνητισμός (GEM) είναι μια προσέγγιση στην οποία το βαρυτικό πεδίο περιγράφεται χρησιμοποιώντας τον φορμαλισμό και την ορολογία της Ηλεκτροδυναμικής. Βασίζεται στην προφανή αναλογία μεταξύ του νόμου του Νεύτωνα για τη Βαρύτητα και του νόμου του Coulomb για την Ηλεκτροδυναμική. Ο Βαρυτοηλεκτρομαγνητισμός έχει μια μεγάλη ιστορία η οποία ξεκινά το 1832 με τον Faraday. Ο Faraday ήταν πεπεισμένος πως το βαρυτικό πεδίο ενοποιείται με το ηλεκτρομαγνητικό και διεξήγαγε πειράματα με σκοπό να ανιχνεύσει αν η κίνηση ενός πηνίου μέσα σε ένα βαρυτικό πεδίο μπορεί να επάγει ηλεκτρικό ρεύμα. Λίγα χρόνια αργότερα ο Maxwell, έχοντας ήδη προτείνει τις εξισώσεις του για την ηλεκτροδυναμική, ωθούμενος από την αναλογία μεταξύ του νόμου του Νεύτωνα για τη Βαρύτητα και του νόμου του Coulomb για την Ηλεκτροδυναμική πρότεινε μία νέα θεωρία για τη Βαρύτητα. Στη νέα αυτή θεωρία, ο Maxwell, εισήγαγε ένα υποθετικό βαρυτικό πεδίο που συμπεριφερόταν όπως το μαγνητικό πεδίο της Ηλεκτροδυναμικής. Η αδυναμία του όμως να ερμηνεύσει ποσότητες όπως η ενέργεια που είναι αποθηκευμένη στο νέο "Βαρυτοηλεκτρομαγνητικό" πεδίο τον έκανε να σταματήσει νωρίς την έρευνα πάνω σε αυτή τη θεωρία. Η πρώτη ουσιαστική ανάπτυξη για τη θεωρία του Βαρυτοηλεκτρομαγνητισμού έγινε την ίδια εποχή όταν πολλοί επιστήμονες όπως οι Holzmuller, Tisserand και Heaviside προσπάθησαν με ένα τρόπο αντίστοιχο με του Maxwell να εξηγήσουν τις ανωμαλίες στην τροχιά του Ερμή.

Ένα πρόβλημα που υπήρχε από τα αρχαία χρόνια ήταν η εξήγηση-πρόβλεψη των πλανητικών τροχιών. Το πρόβλημα αυτό εξηγήθηκε θεωρητικά το δέκατο έβδομο αιώνα από το Νεύτωνα μέσω της θεωρίας του για τη Βαρύτητα. Παρόλο που η θεωρία του Νεύτωνα μπορούσε να εξηγήσει τις πλανητικές τροχιές καθώς και να προβλέψει τη θέση των πλανητών με μεγάλη ακρίβεια, δεν μπορούσε να εξηγήσει τη μετάπτωση του περιηλίου του πλανήτη Ερμή. Ο πλανήτης Ερμής είναι ο πιο κοντινός πλανήτης στον Ήλιο. Ως αποτέλεσμα η βαρυτική αλληλεπίδραση μεταξύ του Ερμή και του Ήλιου είναι ισχυρότερη σε σχέση με την αλληλεπίδραση μεταξύ του Ήλιου και τους άλλους πλανήτες. Αυτό έκανε τους επιστήμονες της εποχής να σκεφτούν ότι η θεωρία του Νεύτωνα δίνει σωστά αποτελέσματα όταν το βαρυτικό πεδίο είναι ασθενές και επιδέχεται διορθώσεις όταν το βαρυτικό πεδίο είναι ισχυρό.

Στον Βαρυτοηλεκτρομαγνητισμό οι διορθώσεις στο βαρυτικό πεδίο μπαίνουν υπό τη μορφή ενός επιπλέον βαρυτικού πεδίου που, ως προς τη μαθηματική του περιγραφή, είναι παρόμοιο με το μαγνητικό πεδίο της Ηλεκτροδυναμικής. Το πλήρες βαρυτικό πεδίο τότε, περιγράφεται από ένα σετ τεσσάρων εξισώσεων που έχουν την ίδια μορφή με τις εξισώσεις του Maxwell για την Ηλεκτροδυναμική. Ωστόσο το 1900 ο Lorentz απόδειξε ότι οι διορθώσεις που εισάγει ένα τέτοιο πεδίο είναι πολύ μικρές για να εξηγήσουν την παρατηρούμενη μετάπτωση του περιηλίου της τροχιάς του Ερμή.

Η μετάπτωση του περιηλίου της τροχιάς του Ερμή εξηγήθηκε θεωρητικά το 1915 από τη Γενική Θεωρία της Σχετικότητας του Einstein. Το 1918 οι Lense και Thirring έδειξαν ότι ακόμη και στο πλαίσιο της Γενικής Θεωρίας της Σχετικότητας μπορεί να υπάρξει μια αναλογία των βαρυτικών εξισώσεων με αυτών της Ηλεκτροδυναμικής. Οι εξισώσεις πεδίου της Γενικής Θεωρίας της Σχετικότητας στη γραμμική προσέγγιση γύρω, αλλά και μέσα, από ένα περιστρεφόμενο έμμαζο σώμα παίρνουν μια μορφή παρόμοια με αυτή των εξισώσεων του Maxwell για την Ηλεκτροδυναμική. Οι διορθώσεις στη μετάπτωση ενός γυροσκοπίου λόγω του βαρυτομαγνητικού πεδίου που δημιουργείται από την περιστροφή του σώματος είναι γνωστές ως *φαινόμενο Lense-Thirring*. Η ανακάλυψη του φαινομένου Lense-Thirring σηματοδοτεί την δεύτερη περίοδο ανάπτυξης του Βαρυτοηλεκτρομαγνητισμού η οποία συνεχίζεται ως σήμερα.

Η πλειοψηφία των προσεγγίσεων του Βαρυτοηλεκτρομαγνητισμού, στην εποχή μας, προέρχονται από τη Γενική Θεωρία της Σχετικότητας και συγκεκριμένα από τη δουλειά των Lense και Thirring. Οι προσεγγίσεις αυτές μπορούν να χωριστούν σε δύο μεγάλες κατηγορίες: αυτές που βασίζονται στη γραμμική προσέγγιση της Γενικής Θεωρίας της Σχετικότητας και σε αυτές που βασίζονται στην αναγωγή του τανυστή του Weyl σε βαρυτοηλεκτρικό και βαρυτομαγνητικό μέρος. Στην πρώτη κατηγορία, που σε αυτήν ανήκει η παρούσα δουλειά, η προσέγγιση του Βαρυτοηλεκτρομαγνητισμού είναι πολύ κοντά στον πραγματικό ηλεκτρομαγνητισμό. Υπάρχει μια πλήρης αντιστοιχία μεταξύ των πεδίων στις δύο θεωρίες και αυτή η αντιστοιχία συνεχίζεται εν μέρει και στις εξισώσεις κίνησης. Παρόλα αυτά η αναλογία μεταξύ των δύο θεωριών είναι εμφανής μόνο στο ασθενές όριο και σε πολύ συγκεκριμένες περιπτώσεις. Στην δεύτερη κατηγορία δεν μπορούμε να έχουμε μια αντιστοιχία μεταξύ των πεδίων αλλά μόνο μια αναλογία μεταξύ των δύο θεωριών. Τα βαρυτοηλεκτρομαγνητικά πεδία, σε αυτή την προσέγγιση, περιγράφονται από δύο "χωρικούς" άιχνους τανυστές δύτερης τάξης οι οποίοι ικανοποιούν τέσσερις γενικευμένες εξισώσεις που έχουν τη μορφή των εξισώσεων του Maxwell. Σε αυτή την περίπτωση μπορεί να μην έχουμε αντιστοιχία μεταξύ των πεδίων, όμως η ισχύς της θεωρίας είναι πιο γενική και δεν ισχύει μόνο στη γραμμική προσέγγιση. Ένα άλλο θετικό στοιχείο αυτής της προσέγγισης είναι ότι τα βαρυτοηλεκτρομαγνητικά πεδία μπορούν να σχηματίσουν βαθμωτές ποσότητες ανάλογες με αυτές που

εμφανίζονται στην Ηλεκτροδυναμική, και με αυτά τα αναλλοίωτα μεγέθη μπορεί να κατασκευαστεί μια λαγκρανζιανή για τον Βαρυτοηλεκτρομαγνητισμό. Τέλος υπάρχουν και κάποιες δουλειές πάνω στον Βαρυτοηλεκτρομαγνητισμό που δεν μπορούμε να τις εντάξουμε στις παραπάνω δύο κατηγορίες όπως η προσέγγιση που αναφέρουμε στην παράγραφο 1.1 της εργασίας, η οποία βασίζεται στη Θεωρία Ομάδων.

Σε αυτή την Εργασία, όπως αναφέραμε και προηγουμένως, ασχολούμαστε με τις προσεγγίσεις του Βαρυτοηλεκτρομαγνητισμού που εμπίπτουν στην πρώτη κατηγορία. Η Εργασία αποτελεί τη συνέχεια μιας προηγούμενης δουλειάς μας. Κύριος σκοπός αυτής της Εργασίας είναι να κάνουμε μια πλήρη ανασκόπηση του Βαρυτοηλεκτρομαγνητισμού, να εντοπίσουμε και να σχολιάσουμε τα προβλήματα που εμφανίζονται στη θεωρία, να προτείνουμε λύσεις για αυτά τα προβλήματα και γενικώς να επεκτείνουμε τη θεωρία. Ένα άλλο θέμα που μας απασχολεί είναι οι εφαρμογές του βαρυτοηλεκτρομαγνητισμού στον πραγματικό Ηλεκτρομαγνητισμό. Τις εφαρμογές στον Ηλεκτρομαγνητισμό τις μελετήσαμε πρώτη φορά στην παλιότερη δουλειά μας, που αναφέρουμε και προηγουμένως, και εδώ προσπαθούμε να τις επεκτείνουμε. Τέλος εδώ ασχολούμαστε και με κάτι νέο, την εύρεση βαρυτικών αναλλοίωτων ποσοτήτων στην γραμμική προσέγγιση, οι οποίες είναι παρόμοιες με αυτές που εμφανίζονται στον Ηλεκτρομαγνητισμό.

Στο πρώτο κεφάλαιο της εργασίας αρχικά δίνουμε έναν ορισμό και μια ιστορική αναδρομή του Βαρυτοηλεκτρομαγνητισμού. Ξεκινάμε από τον Faraday και αναφέρουμε τις σημαντικότερες εργασίες που έχουν δημοσιευτεί σχετικά με τον Βαρυτοηλεκτρομαγνητισμό ως και τις μέρες μας. Στη συνέχεια κάνουμε μια μικρή εισαγωγή στη Γενική Θεωρία της Σχετικότητας. Εκεί αναφέρουμε όλες τις εξισώσεις που θα χρησιμοποιήσουμε στα επόμενα κεφάλαια της εργασίας. Τέλος κάνουμε και μια αναφορά στη γραμμική προσέγγιση στο ασθενές όριο της Γενικής Θεωρίας της Σχετικότητας. Σε όλη την εργασία δουλεύουμε στη γραμμική προσέγγιση. Στη συγκεκριμένη παράγραφο έχουμε υπολογίσει όλες τις βασικές ποσότητες της Γενικής Θεωρίας της Σχετικότητας στη γραμμική προσέγγιση. Συνεπώς αυτή η παράγραφος αποτελεί σημείο αναφοράς για όλες τις εξισώσεις του ασθενούς ορίου.

Στο δεύτερο κεφάλαιο της εργασίας ασχολούμαστε με τον Βαρυτοηλεκτρομαγνητισμό και κάνουμε μια αναφορά στη βιβλιογραφία. Συγκεκριμένα ασχολούμαστε κυρίως με δύο δουλειές, μία του B. Mashhoon που δημοσιεύτηκε το 2005 και μία δική μας που δημοσιεύτηκε το 2014. Αρχικά κάνουμε μια μικρή εισαγωγή στον Βαρυτοηλεκτρομαγνητισμό η οποία είναι βασισμένη στη δουλειά του Mashhoon. Ο Mashhoon το 2005 παρουσίασε μια ανασκόπηση των προηγούμενων εργασιών που βασίζονταν στη γραμμική προσέγγιση και αυτή η εργασία είναι το αρχικό σημείο της ανάλυσής μας. Στη συνέχεια κάνουμε μια αναφορά στην προηγούμενη δουλειά μας. Αν κανείς μελετήσει προσεκτικά την εργασία του Mashhoon θα παρατηρή-

σει ότι η αναλογία μεταξύ Βαρύτητας και Ηλεκτρομαγνητισμού ισχύει μόνο όταν βρισκόμαστε στο μη σχετικιστικό όριο και το βαρυτοηλεκτρομαγνητικό πεδίο είναι στατικό. Επιπλέον αν πάμε στο σχετικιστικό όριο, ακόμη και για στατικά πεδία, στην έκφραση για τις εξισώσεις κίνησης υπάρχουν επιπλέον διορθώσεις πέραν του αντιστοίχου της δύναμης Lorentz. Στην εργασία που είχαμε δημοσιεύσει το 2014 θέλοντας να μελετήσουμε τα παραπάνω προβλήματα της θεωρίας, επαναλάβαμε όλους τους υπολογισμούς από την εργασία του Mashhoon χωρίς να επιβάλουμε κάποια συνθήκη βαθμίδας. Το κάναμε αυτό γιατί αρχικά φαινόταν πως η συνθήκη βαθμίδας μας επέβαλε πως τα πεδία θα πρέπει να είναι στατικά. Αυτό ήταν κάτι που δε θα έπρεπε να συμβαίνει. Γνωρίζουμε πως μια συνθήκη βαθμίδας κάνει πιο απλές τις εξισώσεις που έχουμε να λύσουμε αλλά δεν αλλάζει τη Φυσική. Αφού επαναλάβαμε τους υπολογισμούς δείξαμε πως αυτό που επέβαλε πως τα πεδία θα πρέπει να είναι στατικά ήταν η επιλογή που έκανε ο Mashhoon για τις βαρυτικές διαταραχές. Αφού δείξαμε πως η χρονική εξέλιξη των πεδίων εξαρτάται από τη μορφή των βαρυτικών διαταραχών, στην προηγούμενη δουλειά μας, προτείναμε και μια εναλλακτική μορφή για τις διαταραχές. Η εναλλακτική μορφή που προτείναμε έλυσε το πρόβλημα της χρονικής εξέλιξης αλλά εισήγαγε νέα προβλήματα στη θεωρία όπως το γεγονός ότι η θεωρία ισχύει μόνο στο κενό και πως η επιβολή της συνθήκης βαθμίδας μηδενίζει το Βαρυτοηλεκτρικό πεδίο. Παρόλα τα προβλήματα που εμφανίζονται στην εναλλακτική μορφή, έχει ένα θετικό στοιχείο: οι εξισώσεις κίνησης έχουν την αντίστοιχη μορφή με τη δύναμη Lorentz ακόμη και για υψηλές ταχύτητες. Τέλος, έχοντας μελετήσει δύο διαφορετικές μορφές για τις διαταραχές, καταλήγουμε στο ότι επιλέγοντας κατάλληλα τη μορφή των διαταραχών μπορούμε να φτιάξουμε είτε τη μορφή των εξισώσεων πεδίου είτε τη μορφή των εξισώσεων κίνησης.

Στο τρίτο κεφάλαιο της εργασίας προσπαθούμε να επεκτείνουμε την προηγούμενη δουλειά μας σχετικά με τον Βαρυτοηλεκτρομαγνητισμό. Αυτό που κάνουμε είναι να χρησιμοποιήσουμε τους επιπλέον βαθμούς ελευθερίας του βαρυτικού πεδίο. Το βαρυτικό πεδίο, ως γνωστόν, στα πλαίσια της Γενικής Θεωρίας της Σχετικότητας περιγράφεται από τον μετρικό τανυστή $g_{\mu\nu}$. Οι συνιστώσες του μετρικού τανυστή μπορούν να αναχθούν σε τρία μέρη: ένα βαθμωτό g_{00}, ένα διανυσματικό g_{0i}, και ένα τανυστικό g_{ij}. Στην Ηλεκτροδυναμική έχουμε μόνο τέσσερις βαθμούς ελευθερίας: το βαθμωτό δυναμικό Φ και το διανυσματικό δυναμικό $\vec{A}$. Επειδή στην Ηλεκτροδυναμική υπάρχουν μόνο τέσσερις βαθμοί ελευθερίας, στο δεύτερο κεφάλαιο είχαμε χρησιμοποιήσει μόνο τους αντίστοιχους βαθμούς ελευθερίας του βαρυτικού πεδίου και θεωρούσαμε πως οι υπόλοιποι, οι g_{ij}, μηδενίζονται. Σε αυτό το κεφάλαιο υποθέτουμε πως οι συνιστώσες g_{ij} είναι μη μηδενικές αλλά σημαντικά μικρότερης τάξης μεγέθους από τις g_{00} και g_{0i} και για τις δύο μορφές των διαταραχών. Με αυτό τον τρόπο, και στις δύο περιπτώσεις, μπορούμε να λύσουμε το πρόβλημα με τη χρονική εξάρτηση των πεδίων. Επίσης

14

διάφορες συνθήκες που στο προηγούμενο κεφάλαιο περιόριζαν τη μορφή των g_{00} και g_{0i}, τώρα είναι συνθήκες που περιορίζουν μόνο τη μορφή των g_{ij}. Όσον αφορά τις εξισώσεις κίνησης, η μορφή που χρησιμοποιεί ο Mashhoon για τις βαρυτικές διαταραχές συνεχίζει να περιέχει επιπλέον διορθώσεις στο σχετικιστικό όριο. Όμως στην εναλλακτική μορφή που προτείναμε εμείς, στην παλιότερη δουλειά μας, όλες οι επιπλέον διορθώσεις μπορούν να απλοποιηθούν διότι ακόμη και στο σχετικιστικό όριο είναι μικρότερης τάξης από τους υπόλοιπους όρους. Συνεπώς η εναλλακτική μορφή για τις βαρυτικές διαταραχές είναι η καλύτερη δυνατή για να φανεί η αναλογία μεταξύ Βαρύτητας και Ηλεκτροδυναμικής.

Στο τέταρτο κεφάλαιο της εργασίας κάνουμε μια εφαρμογή του Βαρυτοηλεκτρομαγνητισμού στον πραγματικό Ηλεκτρομαγνητισμό. Πρόκειται για μια προσπάθεια περιγραφής της Ηλεκτροδυναμικής χρησιμοποιώντας μια τανυστική θεωρία που χρησιμοποιεί το φορμαλισμό της Γενικής Θεωρίας της Σχετικότητας στην προσέγγιση του Βαρυτοηλεκτρομαγνητισμού. Θα πρέπει εδώ να σημειώσουμε πως στην πραγματικότητα δεν προσπαθούμε να βρούμε μια πιο γενική θεωρία η οποία μπορεί να αντικαταστήσει τον Ηλεκτρομαγνητισμό. Γνωρίζουμε εξάλλου ότι ο Ηλεκτρομαγνητισμός είναι μια από τις πιο ακριβείς θεωρίες της Φυσικής, οι προβλέψεις της οποίας συμφωνούν με εκπληκτική ακρίβεια με τα πειράματα. Αυτό που προσπαθούμε να κάνουμε είναι να δούμε πόσο μακριά μπορεί να φτάσει η αναλογία μεταξύ Βαρύτητας και Ηλεκτρομαγνητισμού. Η εφαρμογή αυτή στον Ηλεκτρομαγνητισμό ξεκίνησε στην παλιότερη δουλειά μας που αναφέραμε και πιο πάνω. Στις πρώτες παραγράφους του κεφαλαίου κάνουμε μια ανασκόπηση αυτής της δουλειάς και στην τελευταία παράγραφο χρησιμοποιούμε τα αποτελέσματα του προηγούμενου κεφαλαίου ώστε να την επεκτείνουμε. Τα αποτελέσματα αυτού του κεφαλαίου είναι πως όντως σε κάποιες ειδικές περιπτώσεις η Ηλεκτροδυναμική μπορεί να περιγραφεί από μια θεωρία σαν αυτή, η ισχύς όμως της θεωρίας δεν είναι γενική.

Τέλος στο πέμπτο κεφάλαιο της εργασίας προσπαθούμε να βρούμε βαρυτικές αναλλοίωτες ποσότητες, στο γραμμικό όριο, οι οποίες να είναι παρόμοιες με αυτές που εμφανίζονται στην Ηλεκτροδυναμική. Όλες οι βαθμωτές ποσότητες που υπάρχουν στην Βαρύτητα προέρχονται από τον τανυστή του Riemann ο οποίος περιέχει δεύτερης τάξης παραγώγους του μετρικού τανυστή. Οι βαθμωτές ποσότητες της Ηλεκτροδυναμικής προέρχονται από τον ηλεκτρομαγνητικό τανυστή ο οποίος περιέχει πρώτης τάξης παραγώγους των δυναμικών. Για να λύσουμε αυτό το πρόβλημα ορίσαμε ένα νέο τανυστή $F_{\alpha\mu\nu}$ που περιέχει πρώτης τάξης παραγώγους του μετρικού τανυστή και εκφράσαμε τον τανυστή του Einstein χρησιμοποιώντας παραγώγους του $F_{\alpha\mu\nu}$. Από τη στιγμή που ο τανυστής $F_{\alpha\mu\nu}$ περιέχει πρώτης τάξης παραγώγους της μετρικής θεωρούμε πως τα βαθμωτά που προκύπτουν από αυτόν θα έχουν παρόμοια μορφή με αυτά της Ηλεκτροδυναμικής. Στις τελευταίες δύο παραγράφους του κεφαλαίου υπολογίζουμε το βαθμωτό $F \equiv F_{\alpha\mu\nu}F^{\alpha\mu\nu}$

για τις δύο μορφές των βαρυτικών διαταραχών που χρησιμοποιούμε στον Βαρυτοηλεκτρομαγνητισμό. Το βαθμωτό F για τις δύο μορφές τον διαταραχών περιέχει έναν όρο αντίστοιχο με αυτόν που εμφανίζεται στην Ηλεκτροδυναμική ($B^2 - E^2$) αλλά και πολλούς επιπλέον όρους τους οποίους δεν μπορούμε να διώξουμε. Η εργασία τελειώνει με τον υπολογισμό του βαθμωτού F όμως η δουλειά μας σχετικά με τα βαθμωτά που προκύπτουν από τον τανυστή $F_{\alpha\mu\nu}$ είναι ακόμη σε εξέλιξη. Προσπαθούμε να βρούμε μια βαθμωτή ποσότητα παρόμοια με την $\vec{E} \cdot \vec{B}$ της Ηλεκτροδυναμικής. Επίσης επειδή εμπλέκονται βαθμωτά, προσπαθούμε να κατασκευάσουμε μια λνγκρανζιανή για το ασθενές όριο παρόμοια με της Ηλεκτροδυναμικής χρησιμοποιώντας τον τανυστή $F_{\alpha\mu\nu}$.

Preface

A problem which existed from the ancient times was the explanation of the planetary orbits. In the 17th century this problem was theoretically explained by Newton's theory for Gravitation. Although Newton's theory seemed to explain everything regarding the planetary orbits, it could not explain a few anomalies in Mercuries orbit. In order to explained these anomalies, many new theories were proposed and one of them is the gravitoelectromagnetism.

Gravitoelectromagnetism (GEM) is an approach in which the gravitation field is described using the formulation and the terminology of electromagnetism. It is based on the profound analogy between the Newton's law for gravitation and Coulomb's law for electricity. Gravitoelectromagnetism has a long history, which starts in 1832, when Faraday –who was convinced that the gravitational field unifies with the electromagnetic one– conducted some experiments in order to detect if the gravitational field can induct electric current in a circuit, and which continues until our days.

For about two centuries many papers and new approximations of Gravitoelectromagnetism have been proposed. However, during the last years, the majority of the scientific articles focusing on gravitoelectromagnetism are based on Einstein's General Relativity and especially on the work of Lense and Thirring. Lense and Thirring showed that in the context of General Relativity a rotating massive body can create a gravitomagnetic field. This effect is known as the Lense-Thirring effect. These works are split in two categories. The first one uses the linearised form of General Relativity field equations in the weak field approximation, while the other uses the decomposition of the Weyl tensor into gravitoelectromagnetic parts.

In this project we follow the first category, and work in the linear approximation. The main achievements of this approach have been reviewed by B. Mashhoon. However, the last years many researchers have noticed some problems in this approach. Several scientific articles, including one of ours, have been published in which solutions to the above problems have been proposed. Our objective for this project is to review our previous work and to propose new solutions to the problems which have been reported.

Another topic, also covered in our published work, is an attempt to describe true electromagnetism using the formalism of General Relativity. This attempt is based on the results of our work regarding gravitoelectromagnetism. We should mention that our goal is not to replace electromagnetism with another theory but to see how far the analogy with GEM extends and where and why it breaks down. In this thesis we continue our work in this subject by proposing new approaches.

In the last chapter of this thesis, we also cover the search for invariant quantities similar to the electromagnetic ones, in the linear approach of gravitoelectromagnetism. This work is still in progress and we only present some of our results. We hope that this work would contribute to a better understanding of gravitoelectromagnetism and, maybe, lead to a lagrangian formulation.

The outline of this thesis is as follows: in chapter 1 we present a historical overview of gravitoelectromagnetism and a short introduction to General Relativity and its linear approximation. In chapter 2 we give a brief review of gravitoelectromagnetism based on Mashhoon's papers and our previous work. In this chapter we locate and discuss all the problems which appear in this approach. In chapter 3 we try to solve the problems which appear in gravitoelectromagnetism by proposing new approaches in which we use the additional degrees of freedom of the gravitational field. In chapter 4 we review our previous work regarding the construction of a tensorial theory, using the formalism of General Relativity, which aims to describe the true electromagnetism. We also extend this theory in order to make it more realistic. In chapter 5 we investigate the existence of gravitational invariants similar to the electromagnetic ones. Finally in chapter 6 we present our conclusions.

Chapter 1

Introduction

1.1 A historical review of Gravitoelectromagnetism

Gravitoelectromagnetism is a theory in which the main goal is to describe gravity using the formulation of electromagnetism. Gravitoelectromagnetism as a theory has a long history.

In 1832 Faraday conducted a series of experiments in order to detect the induction of electric current in a coil falling through the gravitational field of the Earth [1]. 33 years later, Maxwell, motivated from the analogy between the Newton's law of gravitation and Coulomb's law of electricity, tried to develop a theory for gravitation using the formulation of electromagnetism [2]. At the same time, Holzmuller [3] and Tisserand [4] presented their attempts to explain the excess advance in the perihelion of Mercury in terms of a gravitomagnetic field generated by the Sun. In 1893, Heaviside [5] developed a full set of Maxwell-like equations for the gravitational field. Heaviside's field equations in the vacuum can give wave solutions which propagate with the speed of light. These attempts were very promising for the explanation of the excess advance in the perihelion of Mercury, however, in 1900, Lorentz [6] proved that a gravitomagnetic force would be too weak to explain this phenomenon.

In 1915, Einstein published the General Theory of Relativity [7]. General Relativity is a modern theory of gravitation which has extended Newton's theory. General Relativity has, also, provided an explanation to the problem of the perihelion of Mercury. A few years later, Lense and Thirring, using General Relativity, showed that a rotating massive body can create a gravitomagnetic field [8]; this is known as the Lense-Thirring effect. Thirring also proved that geodesics equation can be expressed in terms of a gravitoelectric and a gravitomagnetic field in the same way as the Lorentz force for electromagnetism

[9].

In the 1950's, Matte [10], Bel [11] and Debever [12] explored the analogy of gravity and electromagnetism based on the decomposition of the Weyl tensor into traceless tensors in the weak approximation limit. In the 1960's, Forward [13] extended the above model and obtained a Maxwell structure for the perturbation; he also proposed experiments to detect these fields [14]. In this decade, a new approach appeared, presented by Scott [15], in which the gravitational field is expressed by a vector field in strict analogy to the electromagnetic field. Similar vector theories have been proposed by Coster and Shepanski [16] and by Schwebel [17]. In the Seventies, another vector theory was presented by Spieweck [18] in which a gravitomagnetic vector potential was introduced. Also, in order to explain the strange behavior of astrophysical objects under extreme physical conditions, new vector approaches of gravitoelectromagnetism were published [19][20][21]. The above approaches have many applications in Astrophysics.

During the last thirty years many approaches of gravitoelectromagnetism, based on General Relativity, appeared in the literature [22][23]. We can say that the gravitoelectromagnetic formulations can be split into two categories. The first, which is the one we use here, is an approach based in the linearised form of General Relativity field equations around a massive rotating body. A review of the linear approach has been published by Mashhoon [24] and it is the starting point for many papers. The second is an approach based on tidal tensors and on the decomposition of the Weyl tensor into gravitomagnetic and gravitoelectric parts. This approach has been extensively reviewed by Costa, Natario and Herdeiro [25][26]. However, new approaches are still appearing in the literature, such as the formulation based on group theory by Ramos [27] or the Langrangian formulation of gravitoelectromagnetism published by Ramos, Montigny and Khanna [28]. Many interesting phenomena are associated to gravitoelectromagnetism: Ruggiero and Tartaglia have published a review of the gravitoelectromagnetic effects [29] which also contains references to experimental attempts to detect them.

In the following two sections we give a brief introduction to General Relativity and to the linear approach. We consider this specifically important in order to give the context in which our work is based and to provide the reader with the information needed to understand better our project.

1.2   General Relativity

The General Theory of Relativity is the modern theory of gravitation. It was published by A. Einstein in 1915 [7], and is a theory that connects the gravita-

tional force with the space-time geometry.

The geometry in General Relativity is described by the metric tensor $g_{\mu\nu}$ which is a second rank symmetric tensor. The metric tensor is appearing in the well-known "first fundamental form" of the Differential Geometry which is also known as the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{1.1}$$

where $x^\mu = (ct, \vec{x})$. The above equation is a generalized form of the well-known line element of the Special Theory of Relativity [30]

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu, \tag{1.2}$$

where $\eta_{\mu\nu}$ is the metric tensor of the flat Minkowski space-time[1].

The metric tensor is also related with the curvature of the space which is described by the Riemann's –curvature– tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}, \tag{1.3}$$

where $\Gamma^\rho{}_{\mu\nu}$ is the Christoffel's symbols given by the following combination of the metric tensor and its derivatives

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right). \tag{1.4}$$

Two useful geometrical quantities for the General Relativity, which can be extracted from the Riemann tensor, are the Ricci tensor which is defined as

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\mu \Gamma^\rho{}_{\rho\nu} + \Gamma^\rho{}_{\rho\lambda}\Gamma^\lambda{}_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\rho\nu}, \tag{1.5}$$

and its trace defined as

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{1.6}$$

The trace of the Ricci tensor is also known as the Ricci scalar.

The field equations of the General Relativity are

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.7}$$

where the tensor $G_{\mu\nu}$ is called Einstein's tensor and is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{1.8}$$

---

[1]Throughout this work, we will use the $(+1, -1, -1, -1)$ signature for the Minkowski tensor $\eta_{\mu\nu}$.

The tensor $T_{\mu\nu}$ is the so-called energy-momentum tensor which describes the distribution of mass and energy in the region of space where we solve the field equations.

In General Relativity the equations of motion of a particle which moves only under the gravitational force is given by the geodesics equation

$$\frac{d^2 x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0. \tag{1.9}$$

As in the Newton's theory of gravity the motion is independent of the particle's mass.

1.3 The linear approach

Einstein's field equations are non linear and thus, quite often, difficult to solve. A usual technique is to adopt the linear approach valid in the limit of the weak gravitational field. In this the metric tensor may be written as

$$g_{\mu\nu}(x^\mu) = \eta_{\mu\nu} + h_{\mu\nu}(x^\mu), \tag{1.10}$$

where $h_{\mu\nu}$ are small perturbations over the Minkowski space-time. In the context of General Relativity, the metric perturbations $h_{\mu\nu}$ are sourced by gravitating bodies and obey the inequality $|h_{\mu\nu}| \ll 1$. As a result a linear-approximation analysis may be followed. Here we will briefly review the corresponding formalism and we will present the equations of General Relativity in the linear-order approximation (for a more detailed analysis, see for example [31]). If we use Eq. (1.10) and keep only terms linear in the perturbation $h_{\mu\nu}$, we easily find that the Christoffel symbols take the form

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \eta^{\alpha\rho} \left(h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho} \right). \tag{1.11}$$

Using the above equation, the Ricci tensor (Eq. (1.5)) assumes the form

$$R_{\mu\nu} = \frac{1}{2} \left(h^\rho{}_{\mu,\nu\rho} + h^\rho{}_{\nu,\mu\rho} - \partial^2 h_{\mu\nu} - h_{,\mu\nu} \right), \tag{1.12}$$

where $\partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu$. In the linear approximation the tensor indices are raised and lowered by the Minkowski metric $\eta_{\mu\nu}$. In the above equation, we have also defined the trace of the metric perturbations $h = \eta^{\mu\nu} h_{\mu\nu}$. Correspondingly, the trace of the Ricci tensor (Eq. (1.6)) is found to be

$$R = h^{\mu\nu}{}_{,\mu\nu} - \partial^2 h. \tag{1.13}$$

If we combine the Ricci tensor and its trace, the Einstein tensor (Eq. (1.8)) takes the form

$$G_{\mu\nu} = \frac{1}{2} \left(h^{\alpha}{}_{\mu,\nu\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - \partial^2 h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu}\, h^{\alpha\beta}{}_{,\alpha\beta} + \eta_{\mu\nu}\, \partial^2 h \right). \qquad (1.14)$$

Usually in gravity, in the linear approximation, one may define the new perturbations $\tilde{h}_{\mu\nu}$ in terms of the original perturbations $h_{\mu\nu}$:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu}\, h\,. \qquad (1.15)$$

In terms of the new perturbations the Einstein tensor simplifies to:

$$G_{\mu\nu} = \frac{1}{2} \left(\tilde{h}^{\alpha}{}_{\mu,\nu\alpha} + \tilde{h}^{\alpha}{}_{\nu,\mu\alpha} - \partial^2 \tilde{h}_{\mu\nu} - \eta_{\mu\nu}\, \tilde{h}^{\alpha\beta}{}_{,\alpha\beta} \right). \qquad (1.16)$$

The above tensor satisfies Einstein's field equations Eq. (1.7) which now are written as

$$\tilde{h}^{\alpha}{}_{\mu,\nu\alpha} + \tilde{h}^{\alpha}{}_{\nu,\mu\alpha} - \partial^2 \tilde{h}_{\mu\nu} - \eta_{\mu\nu}\, \tilde{h}^{\alpha\beta}{}_{,\alpha\beta} = 2k\, T_{\mu\nu}\,. \qquad (1.17)$$

The value of the proportionality constant k will be $\frac{8\pi G}{c^4}$, as usual, except in chapter 4 where it will be different. If we make a coordinate transformation of the form $x^{\mu} \to x'^{\mu} = x^{\mu} - \gamma^{\mu}$, we can easily see that the metric perturbations transform as $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \gamma_{\mu,\nu} + \gamma_{\nu,\mu}$. This transformation leaves Einstein tensor unchanged and hence the field equation. Transformations of this form, which do not alter observable quantities, are called *gauge transformations*. When we solve a problem, it is often convenient to reduce the gauge freedom by imposing constraints to the perturbations $h_{\mu\nu}$. These constraints are called *gauge conditions*. Usually, in the linear approximation, we use the so-called *transverse* gauge condition $\tilde{h}^{\mu\nu}{}_{,\nu} = 0$. Under the imposition of the transverse gauge condition the Einstein tensor takes the simple form

$$G_{\mu\nu} = -\frac{1}{2}\partial^2 \tilde{h}_{\mu\nu}, \qquad (1.18)$$

while the field equations are

$$\partial^2 \tilde{h}_{\mu\nu} = -2k\, T_{\mu\nu}\,. \qquad (1.19)$$

If we multiply Eqs. (1.19) with $\eta^{\mu\nu}$, we get $\partial^2 \tilde{h} = -2k\, T$ or $\partial^2 h = 2k\, T$ and we can write the field equations as:

$$\partial^2 h_{\mu\nu} = -2k \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T \right). \qquad (1.20)$$

In the vacuum we have $T_{\mu\nu} = 0$ and the field equations take the form:

$$\partial^2 \tilde{h}_{\mu\nu} = 0 \qquad or \qquad \partial^2 h_{\mu\nu} = 0\,. \tag{1.21}$$

In what follows we will assume that the distribution of energy in the system is described by the expression $T_{\mu\nu} = \rho\, u_\mu u_\nu$, where ρ is the mass density and $u^\mu = (u^0, u^i) = (c, \vec{u})$ is the velocity of the source. If we define the mass current density as $j_\mu = \rho\, u_\mu$, the energy-momentum tensor takes the form $T_{\mu\nu} = j_\mu u_\nu$.

Chapter 2

GravitoElectroMagnetism

2.1 The traditional ansatz for the perturbations

In the context of the theory of gravitoelectromagnetism, Eqs. (1.19) are Einstein's linearised gravitational field equations, with $k = 8\pi G/c^4$, under the imposition of the transverse gauge condition. The components of the metric perturbations $\tilde{h}_{\mu\nu}$ have the form [24]

$$\tilde{h}_{00} = \frac{4\Phi}{c^2}\,, \qquad \tilde{h}_{0i} = \frac{2A_i}{c^2}\,, \qquad \tilde{h}_{ij} = \mathcal{O}(c^{-4}) \tag{2.1}$$

or

$$\tilde{h}_{0\mu} = \frac{4}{c^2}A_\mu\,, \qquad \tilde{h}_{ij} = \mathcal{O}(c^{-4}) \tag{2.2}$$

where $A^\mu = \left(\Phi, \frac{\vec{A}}{2}\right)$ is the gravitoelectromagnetic 4-potential. If we contract Eq. (1.15) by $\eta^{\mu\nu}$ we find $h = -\tilde{h}$; and we can write the inverse relation between the original and the new perturbations as

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\tilde{h}\,. \tag{2.3}$$

Then, using the definition (1.10), the spacetime line-element takes the form

$$ds^2 = c^2\left(1 + \frac{2\Phi}{c^2}\right)dt^2 - \frac{4}{c}\left(\vec{A}\cdot d\vec{x}\right)dt - \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j\,. \tag{2.4}$$

The perturbations are thus expressed in terms of the so-called gravitoelectromagnetic potentials, a scalar $\Phi(x^\mu)$ and a vector potential $\vec{A}(x^\mu)$. The $\tilde{h}_{00}$ component yields the Newtonian potential Φ, while the $\tilde{h}_{0i}$ component is associated to the "vector" potential $\vec{A}$ generated by a rotating massive body; the

$\tilde{h}_{ij}$ component of the metric perturbations is usually assumed to be negligible due to the suppression of the corresponding source by a $1/c^4$ factor.

The transverse gauge, $\tilde{h}^{\mu\nu}{}_{,\nu} = 0$, gives:

- For $\mu = 0$, the analogue to the Lorentz gauge condition of the electromagnetism :

$$\frac{1}{c}\,\partial_t\Phi + \partial_i\left(\frac{A^i}{2}\right) = 0 \qquad or \qquad A^\mu{}_{,\mu} = 0. \tag{2.5}$$

- And for $\mu = i$, an additional condition that connects the $\tilde{h}_{ij}$ component with the time derivative of the vector potential $\vec{A}$:

$$\partial_j\tilde{h}^{ij} \ = \ -\,\partial_t\,A^i. \tag{2.6}$$

Due to the tensorial structure of gravity, from the gauge condition we get three more equations, Eq. (2.6), that do not appear in the true electromagnetism. These equations were ignored in many scientific articles because they involve $\mathcal{O}(c^{-4})$ terms [24][32] and lead to static vector potential $\partial_t\,\vec{A} = 0$ in some others [33][25]. Here we are following Mashhoon's papers [24] and we are ignoring the presence of these equations. In the following sections we will discuss further the presence and importance of Eq (2.6).

The field equations are:

- For $\mu = 0$ and $\nu = 0$:

$$\partial^2\Phi \ = \ 4\pi G\,\rho, \tag{2.7}$$

- And for $\mu = 0$ and $\nu = i$:

$$\partial^2\left(\frac{A^i}{2}\right) \ = \ \frac{4\pi G}{c}\,j^i, \tag{2.8}$$

where we have ignored all the equations involving $\mathcal{O}(c^{-4})$ terms.

In analogy with electromagnetism, we can define the GEM fields $\vec{E}$ and $\vec{B}$ in terms of the GEM potentials [24][33]

$$\vec{E} \equiv -\frac{1}{c}\,\partial_t\left(\frac{\vec{A}}{2}\right) - \vec{\nabla}\Phi\,, \qquad \vec{B} \equiv \vec{\nabla}\times\left(\frac{\vec{A}}{2}\right). \tag{2.9}$$

According to [24], if we use vector notation, Eqs. (2.7) and (2.8) reduce to two Maxwell-like equations for the Gravitoelectromagnetic field

$$\vec{\nabla}\cdot\vec{E} = 4\pi G\rho\,, \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\,\partial_t\vec{E} + \frac{4\pi G}{c}\,\vec{j}\,. \tag{2.10}$$

The definitions of the fields Eqs. (2.9) reduce to the remaining Maxwell equations

$$\vec{\nabla} \times \vec{E} = \frac{1}{c} \partial_t \vec{B}, \qquad \vec{\nabla} \cdot \vec{B} = 0. \tag{2.11}$$

A more detailed analysis, that will be presented in the following section, will give more information on how Eqs (2.7-2.8) actually reduce to a set of Maxwell-like equations.

Finally we can find the equation of motion of a test particle which propagates in the background (2.4). Here, and in the following sections, we will keep all corrections in the equation of motion and we discuss their importance. We should also mention that we work in the non relativistic limit; this means that we can write

$$ds^2 = c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = c^2 dt^2 \left(1 - \frac{|\vec{u}|^2}{c^2}\right) \simeq c^2 dt^2 \,. \tag{2.12}$$

In the non-relativistic limit, the spatial components of the geodesics equation

$$\frac{d^2 x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

take the form

$$\frac{d^2 x^i}{dt^2} + c^2 \, \Gamma^i_{00} + 2c \, \Gamma^i_{0j} \frac{dx^j}{dt} + \Gamma^i_{kj} \frac{dx^k}{dt} \frac{dx^j}{dt} = 0 \,. \tag{2.13}$$

In the linear approximation, the Christoffel symbols are given by Eq. (1.11). Reading the form of the initial perturbations $h_{\mu\nu}$ from the line-element (2.4), we find:

$$\Gamma^i_{00} = -\frac{1}{c^2} \, \partial^i \Phi + \frac{2}{c^3} \, \partial_t A^i \,, \tag{2.14}$$

$$\Gamma^i_{0j} = \frac{1}{c^2} \, F_j{}^i - \delta^i_j \frac{1}{c^3} \partial_t \Phi, \tag{2.15}$$

$$\Gamma^i_{kj} = -\frac{1}{c^2} \left(\delta^i_j \, \partial_k \Phi + \delta^i_k \, \partial_j \Phi - \eta_{kj} \, \partial^i \Phi \right). \tag{2.16}$$

In the second of the above equations we use the definition $F_{ij} \equiv \partial_i A_j - \partial_j A_i$. Substituting the above into Eq. (2.13) [34][33][25] we find

$$\ddot{x}^i = E^i + \frac{2}{c} F^{ij} u_j + \frac{1}{c^2} \left(2u^i \partial_t \Phi + 2u^i u^k \partial_k \Phi - u^k u_k \partial^i \Phi \right), \tag{2.17}$$

which, if we are in the non relativistic limit and the scalar and the vector potential are stationary i.e. $\partial_t \Phi = 0$ and $\partial_t \vec{A} = 0$, reduces to the analogue of the Lorentz force law for gravitation:

$$\ddot{x}^i = E^i + \frac{2}{c} F^{ij} u_j \,, \tag{2.18}$$

with "electric charge" equal to m and "magnetic charge" equal to $2m$. According to the literature the coefficient 2 in the "magnetic charge" is related to the spin of the gravitational field [24]. Actually this coefficient depends on the definition of the magnetic field and on the h_{0i} component of the metric perturbations as well. From the definition of the magnetic field Eq. (2.9), we can see that the F_{ij} components are $F_{ij} = -2\varepsilon_{ijk} B_k$ and, as a result, the coefficient in front of the magnetic field is $4m$. In the above equation ε_{ijk} is the Levi-Civita symbol.

With the traditional ansatz (Eq. (2.1)) one can easily see the analogy between gravity and electromagnetism and, by solving the field equations, one may study many interesting phenomena [8] [35]. However, this ansatz "works", i.e. it keeps the analogy with true electromagnetism, only if the potentials are stationary and, as it is mentioned before, some times –through the gauge condition– the ansatz dictates that the vector potential $\vec{A}$ is time-independent. If the potentials are time-dependent, the theory gives correct results, from the point of view of General Relativity, but the analogy with the electromagnetism breaks down. The truth is that, in reality, the majority of the gravitoelectromagnetic effects are stationary and well described by the traditional ansatz. But we are also interested in time-dependent effects and we want to see if there is any way to preserve the analogy. In the literature a number of works have been found attempting to solve this problem [36] but they are dealing only with special cases. On the other hand, another big problem with gravitoelectromagnetism is that, even in the stationary case, we get the correct form of the Lorentz force law only in the non relativistic limit. It has been claimed [24] that we can keep the analogy ether with the field equations or the Lorentz force but not both with the same ansatz [25][1]. If we choose an ansatz that fixes the Lorentz force law, we do not obtain the correct field equations [33]. In the following sections we present different ansatzes that attempt to fix the problems appearing in the traditional one that we presented in this section.

---

[1]See Section 2.3.

2.2 A detailed analysis of the traditional ansatz without the imposition of a gauge condition

As we saw in the previous section, the traditional ansatz demonstrates the analogy with the true electromagnetism but it has some problems. In order to shed light to the problems of the traditional ansatz, in a previous work of ours [33], we performed the analysis again without the imposition of a gauge condition. However, there is a difference between the ansatz we are using in this section and the one employed in section (2.1). Here the $\tilde{h}_{ij}$ components of the metric perturbations $\tilde{h}_{\mu\nu}$ are set equal to zero [33][25]. Actually there is not a big difference since in the previous ansatz [24] these components were treated as null. However, as we will see in the following chapter, the presence of the $\tilde{h}_{ij}$ is in fact important for the solution of the problems which arise here.

The ansatz we use in this section is

$$\tilde{h}_{00} = \frac{4\Phi}{c^2}, \qquad \tilde{h}_{0i} = -\frac{2A^i}{c^2}, \qquad \tilde{h}_{ij} = 0, \tag{2.19}$$

and again the metric perturbations are expressed in terms of the so-called Gravitoelectromagnetic potentials.

For the evaluation of the field equations we need the components of $\tilde{h}_{\mu\nu}$ in mixed form - these are:

$$\tilde{h}^0{}_0 = \frac{4\Phi}{c^2}, \qquad \tilde{h}^0{}_i = -\frac{2A^i}{c^2}, \qquad \tilde{h}^i{}_0 = \frac{2A^i}{c^2}, \qquad \tilde{h}^i{}_j = 0. \tag{2.20}$$

The field equations, Eq. (1.17), for the above ansatz take the form

- For $\mu = 0$ and $\nu = 0$:
$$\frac{\delta^{ij}}{c^2} \partial_i \partial_j \Phi = \frac{k}{2} \rho\, u_0\, u_0. \tag{2.21}$$

- For $\mu = 0$ and $\nu = i$:
$$\frac{1}{c^2} \partial_i \left(\frac{1}{2} \partial_k A^k + \frac{1}{c} \partial_t \Phi \right) - \frac{1}{2c^2} \delta^{kl} \partial_k \partial_l A^i = \frac{k}{2} \rho\, u_0\, u_i. \tag{2.22}$$

- And finally for $\mu = i$ and $\nu = j$:
$$-\frac{1}{2c^3} \partial_t \left(\partial_i A^j + \partial_j A^i \right) + \delta_{ij} \left[\frac{1}{c^4} \partial_t^2 \Phi + \frac{1}{c^3} \partial_t (\partial_k A^k) \right] = \frac{k}{2} \rho\, u_i\, u_j. \tag{2.23}$$

As in the previous section, the constant k is $8\pi G/c^4$. If we use vector notation, Eq. (2.21) readily takes the analogue of Poisson's law

$$\nabla^2 \Phi = 4\pi G\rho \,, \tag{2.24}$$

while Eq. (2.22) in turn can be rewritten as

$$\vec{\nabla}\left[\vec{\nabla}\cdot\left(\frac{\vec{A}}{2}\right) + \frac{1}{c}\partial_t\Phi\right] - \nabla^2\left(\frac{\vec{A}}{2}\right) = \frac{4\pi G}{c}\,\rho\,\vec{u}\,. \tag{2.25}$$

Equations (2.24) and (2.25) differ from (2.7 - 2.8) since here, we have not imposed the transverse gauge condition. We can define again the GEM fields $\vec{E}$ and $\vec{B}$ in terms of the GEM potentials [24] [33]:

$$\vec{E} \equiv -\frac{1}{c}\partial_t\left(\frac{\vec{A}}{2}\right) - \vec{\nabla}\Phi\,, \qquad \vec{B} \equiv \vec{\nabla}\times\left(\frac{\vec{A}}{2}\right)\,. \tag{2.26}$$

Then, Eqs. (2.24) and (2.25),using vector notation, take the form

$$\vec{\nabla}\cdot\vec{E} = 4\pi G\rho\,, \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\partial_t\vec{E} + \frac{4\pi G}{c}\,\vec{j}\,, \tag{2.27}$$

only if the vector potential $\vec{A}$ has not a time-dependence, i.e.

$$\partial_t\,\vec{A}\,=\,0\,. \tag{2.28}$$

The definitions (2.26) then take the form

$$\vec{\nabla}\times\vec{E} = \frac{1}{c}\partial_t\vec{B}, \qquad \vec{\nabla}\cdot\vec{B} = 0. \tag{2.29}$$

Here we can see that the choice we made for the form of the metric perturbations dictates that the vector potential is static (for different approaches regarding the role of this constraint in the context of gravitoelectromagnetism, see [23][25][26][33][37][38]). This does not fix the problems of the traditional ansatz; it actually makes them worst, however, it shows us that the time dependence of the vector potential depends on the ansatz. This is an important result, because in the previous section the constraint regarding the time dependence of the vector potential deduced from the gauge condition. If we impose the transverse gauge condition we obtain the same results. Therefore, we can conclude that the only way to solve the problems regarding the time dependence of the vector potential is to seek for new ansatzes.

2.2. A detailed analysis of the traditional ansatz without the imposition of a gauge condition

Now we can return to the linearised field equations. Their spatial component gives us a third set of constraints Eq. (2.23). The diagonal components of Eq. (2.23) (i.e. for $i = j$) reduce to the relation

$$\partial_t^2 \Phi = -\frac{\pi}{3} \rho \, |\vec{u}|^2 \, , \tag{2.30}$$

with $(u^1)^2 = (u^2)^2 = (u^3)^2$, and the off-diagonal ones (for $i \neq j$) give us

$$\partial_0 \left(\partial_i A^j + \partial_j A^i \right) = 8\pi G \rho \, \frac{u_i u_j}{c^2} \, . \tag{2.31}$$

Therefore Eq. (2.30) demands that the distribution of sources is isotropic (i.e. the current vector has the same magnitude along all three spatial directions); it also restricts the magnitude of $\partial_t^2 \Phi$, which is a quantity that does not appear in the other two derived equations. The other equation (2.31) imposes that the velocity of the source is absolutely non-relativistic, $|\vec{u}| \ll c$: it is only then that the time variation of the vector potential is extremely small, due to the suppression factor $1/c^2$ on its right-hand-side, and the consistency of the complete set of derived equations is guaranteed.

Finally, we can give the equation of motion of a test particle that moves in the field area. First of all the line element has the same form as in the previous section Eq. (2.4)

$$ds^2 = c^2 \left(1 + \frac{2\Phi}{c^2} \right) dt^2 - \frac{4}{c} \left(\vec{A} \cdot d\vec{x} \right) dt - \left(1 - \frac{2\Phi}{c^2} \right) \delta_{ij} dx^i dx^j.$$

The Christoffel symbols are also the same as in the previous section:

$$\Gamma^i_{00} = -\frac{1}{c^2} \partial^i \Phi + \frac{2}{c^3} \partial_t A^i \, ,$$

$$\Gamma^i_{0j} = \frac{1}{c^2} F_j{}^i - \delta^i_j \frac{1}{c^3} \partial_t \Phi,$$

$$\Gamma^i_{kj} = -\frac{1}{c^2} \left(\delta^i_j \partial_k \Phi + \delta^i_k \partial_j \Phi - \eta_{kj} \partial^i \Phi \right) .$$

In the second of the above equations, we use the definition $F_{ij} \equiv \partial_i A_j - \partial_j A_i$. Then, the equations on motion, in a familiar vector notation, are [33]:

$$m \, \vec{a} = \vec{F} = m \, \vec{E} \left(1 + \frac{|\vec{u}|^2}{c^2} \right) + \frac{4m}{c} \, \vec{u} \times \vec{B} + 2m \left[\frac{\vec{u}}{c} \frac{\partial_t \Phi}{c} - \frac{\vec{u}}{c} \left(\frac{\vec{u}}{c} \cdot \vec{E} \right) \right] \tag{2.32}$$

where, we have set $\partial_t \vec{A} = 0$ and thus $\vec{E} = -\vec{\nabla}\Phi$. The Gravitomagnetic Field ($\vec{B}$) and the gravitoelectromagnetic tensor ($F_{\mu\nu}$) are related through the equation

$F_{ij} = -2\,\varepsilon_{ijk}B_k$. If we are in the non relativistic limit and the scalar potential is static, the above equation reduces to the analogue of the Lorentz force law:

$$m\,\vec{a} = \vec{F} = m\,\vec{E} + \frac{4m}{c}\,\vec{u} \times \vec{B}\,. \tag{2.33}$$

In this section, we have performed again all the calculations of the traditional ansatz without the imposition of a gauge condition in order to investigate in depth the problems that arise. The nature of the problems forces us to seek modified ansatzes for the metric perturbations. In the following section we present an alternative ansatz and in the following chapter we will add corrections to both ansatzes.

2.3   An alternative ansatz

As we saw in the previous section (2.2), a possible remedy for the problems which appear at the traditional ansatz is the adoption of a new one. In this section we follow the literature [33][25] and present an additional ansatz for the metric perturbations. The ansatz we use here has been carefully chosen in order to fix the Lorentz force law.

The metric perturbations in this section will be:

$$\tilde{h}_{00} = \frac{\Phi}{c^2}\,, \qquad \tilde{h}_{0i} = \frac{A_i}{c^2}\,, \qquad \tilde{h}_{ij} = -\frac{\Phi}{c^2}\,\eta_{ij}\,, \tag{2.34}$$

with the basic difference being the assumption of a non-vanishing (or subdominant) $\tilde{h}_{ij}$.

Working as before, we first derive the components of $\tilde{h}_{\mu\nu}$ in mixed form

$$\tilde{h}_0{}^0 = \frac{\Phi}{c^2}\,, \qquad \tilde{h}_0{}^i = \frac{A^i}{c^2}\,, \qquad \tilde{h}_i{}^0 = \frac{A_i}{c^2}\,, \qquad \tilde{h}_i{}^j = -\frac{\Phi}{c^2}\,\delta_i{}^j\,, \tag{2.35}$$

and then, from the field equations (1.17), we obtain the following system

- For $\mu = 0$ and $\nu = 0$:

$$0 = 2k\rho\,u_0\,u_0\,, \tag{2.36}$$

- For $\mu = 0$ and $\nu = i$:

$$\frac{1}{c^2}\,\partial_i\left(\partial_k A^k\right) - \frac{1}{c^2}\,\delta^{kl}\,\partial_k\partial_l A^i = 2k\rho\,u_0\,u_i\,. \tag{2.37}$$

- Finally for $\mu = i$ and $\nu = j$:

$$-\frac{1}{c^3}\,\partial_t\left(\partial_i A^j + \partial_j A^i\right) - \frac{2}{c^2}\,\partial_i\partial_j\Phi \tag{2.38}$$

$$+\delta_{ij}\left[\frac{2}{c^2}\,\delta^{kl}\,\partial_k\partial_l\Phi + \frac{2}{c^3}\,\partial_t(\partial_k A^k)\right] = 2k\rho\,u_i\,u_j\,.$$

Setting again $k \equiv 8\pi G/c^4$, Eq. (2.37) takes a form similar to that of the fourth Maxwell's equation for both a static scalar and vector potential, $\partial_t\Phi = \partial_t\vec{A} = 0$,

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A} = 16\pi G\rho\,\frac{\vec{u}}{c}\,. \tag{2.39}$$

This equation differs from the exact Maxwell equation by a factor of 4 on the right-hand-side but, as we will see, this will be irrelevant. Equation (2.36), that in the previous case gave us Poisson's law, is now reduced to the trivial result $\rho = 0$, which, when combined with the fact that $\partial_t\vec{A} = 0$, leads to the demand that Φ satisfies the equation

$$\nabla^2\,\Phi = 0\,. \tag{2.40}$$

We conclude that this choice for the gravitational perturbations leads to a static model of gravity in vacuum. For $\rho = 0$, the right-hand-side of Eq. (2.39) also vanishes making the numerical factor irrelevant. In retrospect, our assumption of non-vanishing $\tilde{h}_{ij}$ seems justified: although this component is indeed significantly suppressed in the presence of an energy-momentum tensor of the form $T_{\mu\nu} = \rho u_\mu u_\nu$, in vacuum all components are of the same order.

Imposing the transverse gauge condition $\tilde{h}^{\mu\nu}{}_{,\nu} = 0$ results into two constraints, namely:

- For $\mu = 0$, the Lorentz gauge condition:

$$\frac{1}{c}\,\partial_t\Phi + \vec{\nabla}\cdot\vec{A} = 0 \quad . \tag{2.41}$$

- For $\mu = i$, the gauge condition dictates that the gravitoelectric field is vanishing:

$$\frac{1}{c}\,\partial_t\vec{A} + \vec{\nabla}\Phi = 0\,, \tag{2.42}$$

We note that the different numerical coefficients that appear in the ansatz, compared to the traditional one (2.1) used in GEM, allow us to define the GEM fields $\vec{E}$ and $\vec{B}$ in an exact analogy with the electromagnetism

$$\vec{E} \equiv -\frac{1}{c}\,\partial_t\vec{A} - \vec{\nabla}\Phi\,, \qquad \vec{B} \equiv \vec{\nabla}\times\vec{A}\,. \tag{2.43}$$

Although this case seems to be not particularly rich in content compared to the one studied in the previous section, it is in contrast free of additional constraints. The set of equations (2.38) – both the diagonal and off-diagonal components – are trivially satisfied if one uses the aforementioned constraints Eqs. (2.41-2.42) following from the gauge conditions. However, we have encountered a big problem in this ansatz, the vanishing of the gravitoelectric field $\vec{E}$. Taking in consideration that this ansatz is valid only in vacuum, we can conclude that it is very limited. We are only left with the analogue of the magnetostatic field in the Coulomb gauge: if the gravitoelectric field $\vec{E}$ vanishes, the scalar potential Φ is a constant; then, Eq. (2.41) reduces to $\vec{\nabla} \cdot \vec{A} = 0$, which is the well known –from the electromagnetism– Coulomb gauge.

In this ansatz the spacetime line-element assumes the simplified form

$$ds^2 = c^2 \left(1 + \frac{2\Phi}{c^2} \right) dt^2 - \frac{2}{c} \left(\vec{A} \cdot d\vec{x} \right) dt - \delta_{ij} dx^i dx^j \,. \tag{2.44}$$

Turning finally to the geodesics equation, by using the expressions for the initial perturbations $h_{\mu\nu}$ as these are read in the line-element (2.44), we arrive at particularly simple forms for the Christoffel symbols

$$\Gamma^i_{00} = \frac{1}{c^2} \, \partial_i \Phi + \frac{1}{c^3} \, \partial_t A^i \,, \quad \Gamma^i_{0j} = \frac{1}{2c^2} \, F_{ij} \,, \qquad \Gamma^i_{kj} = 0 \,. \tag{2.45}$$

Substituting these into the geodesics equation (2.13), we obtain the exact functional analogue of the Lorentz force

$$m \, \vec{a} = \vec{F} = m \, \vec{E} + \frac{m}{c} \, \vec{u} \times \vec{B} \,. \tag{2.46}$$

We have seen that the gravitoelectric field $\vec{E}$ in this ansatz vanishes and thus the scalar potential is a constant. Having that in mind, in Eq. (2.45) the Γ^i_{00} should vanish and consequently Eq. (2.46) should be:

$$m \, \vec{a} = \frac{m}{c} \, \vec{u} \times \vec{B} \,. \tag{2.47}$$

In short, we have given the equations of motion in the form of Eq. (2.46) in order to show that in this ansatz we can get the correct form of the Lorentz force law. In this case additional terms do not emerge, not even sub-dominant ones in the non-relativistic limit. We can also see that the same coefficient appears in front of the gravitoelectric and gravitomagnetic terms, in exact analogy to electromagnetism. Therefore we conclude that this particular ansatz for the gravitational perturbations may lead to another class of phenomena observed in vacuum in the context of gravitoelectromagnetism: the field equations in

conjunction to the gauge condition lead to a self-consistent set of fundamental equations with no additional constraints and a remarkable similarity to the corresponding formulae of electromagnetism.

However this ansatz has two serious problems; the gravitoelectric field vanishes and it works only in vacuum. One could claim that the vanishing of the gravitoelectric field is the most important problem but actually the most important problem is the latter. Usually when we solve a problem in vacuum, we have information about the source in the form of boundary conditions. In other words, we solve the field equations inside the source and outside (in the vacuum) and we demand that on the boundary (the surface that encloses the source) the two solutions coincide. In the gravitoelectromagnetism theory, we solve the field equations of General Relativity around a rotating massive body, which means that again we have a source. However, in this ansatz we do not have any information about the source; we only know that the field exists, and possibly propagates, in vacuum but we are unable to connect it with its source. Despite the problems we have encountered, in this model we have obtained some promising results such as the fact that we get the exact form of the Lorentz force law. We will continue presenting corrections for the two ansatzes. In fact, as we will see in the following chapter (3.2), the work we presented in this section will lead us to the best ansatz for the metric perturbation, in which we can have both time-dependent fields and equations of motion very close to the form of the Lorentz force law.

Chapter 3

Improving Gravitoelectromagnetism

3.1 A correction to the traditional ansatz for the metric perturbations

In this section we will try to fix the problems we have encountered in the study of the traditional ansatz. We stay in the traditional ansatz and assume non-vanishing $\tilde{h}_{ij}$ components of the metric perturbations. If we look more carefully at the spatial component of the gauge condition, (Eq. (2.6)), in section (2.1), we see that the use of these components of the metric perturbations can instantly solve the problem of the time dependence of the vector potential $\vec{A}$. In Mashhoon's works [24][32][34][36], the spatial components of the gauge condition $(\tilde{h}^{i\mu}{}_{,\mu})$ are not taken into account and the vector potential is assumed to be time dependent. However in these articles the $\tilde{h}_{ij}$ components are not zero and, indeed, the vector potential may have time dependence, although nothing is said about the $\tilde{h}^{i\mu}{}_{,\mu}$ condition and its importance on the time dependence of the vector potential. What was only mentioned here [32] is that these equations involve $\mathcal{O}(c^{-4})$ terms and for this reason they are negligible.

Now let us think about the form of the $\tilde{h}_{ij}$ components. According to the usual assumptions of General Relativity, the scalar potential Φ is associated only with the $\tilde{h}_{00}$ component of the metric perturbations. To preserve the analogy with electromagnetism, the vector potential should appear linearly in the expression of $\tilde{h}_{\mu\nu}$, and thus can only be accommodated by the $\tilde{h}_{0i}$ component. We could also introduce a scalar potential Θ in the $\tilde{h}_{0i}$ components of the metric perturbations through the relation:

$$\tilde{h}_{0i} = \frac{2\,\Theta}{c^2} A_i. \tag{3.1}$$

However, as we can easily see by making the change $A^i \longrightarrow \Theta A^i$ in all equations of the previous chapter involving the A^i potential, that would make the situation worse. As for the $\tilde{h}_{ij}$ components, they can be either zero, as they were in section (2.2), or associated with a scalar potential and a tensor potential. The scalar potential could be a general scalar potential or the gravitoelectric scalar potential Φ as we will see in the next section. Although this may seem peculiar at first, in the course of our analysis it will be justified and shown to exhibit interesting features.

We make the following changes to the traditional ansatz for the metric perturbations $\tilde{h}_{\mu\nu}$ [25][26][39][40]

$$\tilde{h}_{00} = \frac{4\Phi}{c^2}, \qquad \tilde{h}_{0i} = \frac{2A_i}{c^2}, \qquad \tilde{h}_{ij} = \frac{2\lambda}{c^4}\,\eta_{ij} \,+\, \frac{2}{c^4}\,d_{ij}\,. \qquad (3.2)$$

In the above ansatz the $\tilde{h}_{00}$ and $\tilde{h}_{0i}$ components of the metric perturbations are again expressed in terms of Φ and $\vec{A}$ potentials, while the $\tilde{h}_{ij}$ component consists of a scalar field λ and a traceless symmetric tensor field d_{ij}. Obviously the potential λ is connected with the trace of the component $\tilde{h}_{ij}$

$$\eta^{ij}\tilde{h}_{ij} = \frac{6}{c^4}\lambda. \qquad (3.3)$$

We can associate the traceless symmetric tensor field d_{ij} with two perpendicular 3-vectors $\vec{C}$ and $\vec{D}$ through the relation

$$d_{ij} \,=\, C_i D_j \,+\, C_j D_i. \qquad (3.4)$$

Then we could easily write our final equations in a familiar vector notation, but for now we keep the tensor formulation and we study the most general case. In some equations, as the equations of motion, the $\tilde{h}_{ij}$ components will still be negligible, due to the $1/c^4$ factor, especially in the non relativistic limit, but they will not be negligible everywhere, as we will shortly see.

From this section we will start using again the gauge condition $\tilde{h}^{\mu\nu}{}_{,\nu} = 0$. The transverse gauge now gives:

- For $\mu = 0$, the analogue of the Lorentz gauge condition of the electromagnetism :

$$\frac{1}{c}\,\partial_t\Phi + \partial_i\left(\frac{A^i}{2}\right) = 0. \qquad (3.5)$$

- And for $\mu = i$, an additional condition that connects the scalar λ and the tensor field d_{ij} with the time derivative of the vector potential $\vec{A}$:

$$\frac{1}{c}\,\partial_j\left(\lambda\eta^{ij} + d^{ij}\right) \,=\, -\,\partial_t\,A^i. \qquad (3.6)$$

The field equations are:

- For $\mu = 0$ and $\nu = 0$:

$$\partial^2 \Phi = 4\pi G \, \rho. \tag{3.7}$$

- For $\mu = 0$ and $\nu = i$:

$$\partial^2 \left(\frac{A^i}{2} \right) = \frac{4\pi G}{c} \, j^i. \tag{3.8}$$

- Finally for $\mu = i$ and $\nu = j$:

$$\frac{1}{2} \partial^2 \left(\lambda \eta^{ij} + d^{ij} \right) = 4\pi G \, j^i \, u^j. \tag{3.9}$$

Equations (3.7) and (3.8) have the same form as the corresponding equations of electromagnetism. Equation (3.9) is the Poisson's equation that gives us the additional fields λ and d_{ij}. These are important, despite the fact that they are suppressed due to the $1/c^4$ factor, because without them Eq. (3.6) takes the form $\partial_t A^i = 0$. We are going to discuss again the importance of the fields λ and d^{ij} at the end of the section.

In analogy with electromagnetism, we can define the GEM fields $\vec{E}$ and $\vec{B}$ in terms of the GEM potentials [24][33][25]:

$$\vec{E} \equiv -\frac{1}{c} \partial_t \left(\frac{\vec{A}}{2} \right) - \vec{\nabla}\Phi, \qquad \vec{B} \equiv \vec{\nabla} \times \left(\frac{\vec{A}}{2} \right). \tag{3.10}$$

One may easily see that Eqs. (2.7) and (2.8) along with Eqs. (2.9) reduce to a set of four Maxwell-like equations for the Gravitoelectromagnetic fields:

$$\vec{\nabla} \cdot \vec{E} = 4\pi G \rho, \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c} \partial_t \vec{E} + \frac{4\pi G}{c} \, \vec{j}, \tag{3.11}$$

and

$$\vec{\nabla} \times \vec{E} = \frac{1}{c} \partial_t \vec{B}, \qquad \vec{\nabla} \cdot \vec{B} = 0. \tag{3.12}$$

The above equations along with Eq. (3.9) describe the gravitational field of a rotating massive body in the weak-field (linear) approximation.

The spacetime line-element, in this ansatz, assumes the form

$$ds^2 = c^2 \left(1 + \frac{2\Phi}{c^2} - \frac{3\lambda}{c^4} \right) dt^2 - \frac{4}{c} \left(\vec{A} \cdot d\vec{x} \right) dt - \left(1 - \frac{2\Phi}{c^2} + \frac{3\lambda}{c^4} \right) \delta_{ij} dx^i dx^j$$

$$+ \frac{2}{c^4} d_{ij} dx^i dx^j. \tag{3.13}$$

Finally, we can find the equation of motion of a test particle propagating in the background (3.13). This has been derived and discussed in the literature [24][33][25][34], but only in a very simplified form. In [24] the equation of motion is presented only in the non-relativistic limit. In [33] and [34], although the corrections for large velocities are given, these do not contain the corrections for the $\tilde{h}_{ij}$ components. In [25] the authors start with an ansatz, different from the one we are using here, which contains the components $\tilde{h}_{ij}$ but at the end of their analysis they assume that d_{ij} vanish. Here, we keep all corrections and comment on their importance at the final stage of our calculation.

In the linear approximation, the Christoffel symbols are given by Eq. (1.11). Reading the form of the initial perturbations $h_{\mu\nu}$ from the line-element (3.13), we find:

$$\Gamma^i_{00} = -\frac{1}{c^2}\,\partial^i\Phi + \frac{2}{c^3}\,\partial_t A^i \ - \frac{3}{2c^4}\partial^i\lambda, \tag{3.14}$$

$$\Gamma^i_{0j} = \ \frac{1}{c^2}\,F_j{}^i - \delta^i_j\left(\frac{1}{c^3}\partial_t\Phi + \frac{3}{2c^5}\,\partial_t\lambda\right) + \frac{1}{c^5}\partial_t\,d^i{}_j\,, \tag{3.15}$$

$$\begin{aligned}\Gamma^i_{kj} = &-\frac{1}{c^2}\left(\delta^i_j\,\partial_k\Phi + \delta^i_k\,\partial_j\Phi - \eta_{kj}\,\partial^i\Phi\right)\\ &-\frac{3}{2c^4}\left(\delta^i_j\,\partial_k\lambda + \delta^i_k\,\partial_j\lambda - \eta_{kj}\,\partial^i\lambda\right)\\ &+\frac{1}{c^4}\left(\partial_k d^i{}_j + \partial_j d^i{}_k - \partial^i d_{jk}\right).\end{aligned} \tag{3.16}$$

Substituting the above in the geodesics equation Eq. (2.13) we find:

$$\begin{aligned}\ddot{x}^i = \ &E^i + \frac{2}{c}F^{ij}u_j + \frac{1}{c^2}\left(\frac{5}{2}\partial^i\lambda + \partial_j\,d^{ij} + 2u^i\partial_t\Phi + 2u^iu^k\partial_k\Phi\right)\\ &-\frac{1}{c^2}u^ku_k\partial^i\Phi - \frac{1}{c^4}\left(3u^i\partial_t\lambda - 3u^iu^k\partial_k\lambda + u^ku_k\partial^i\lambda\right)\\ &-\frac{1}{c^4}\left(2u_j\partial_t d^{ij} + 2u^ku^j\partial_k d^i{}_j - u^ju^k\partial^i d_jk\right).\end{aligned} \tag{3.17}$$

The above equation in the non relativistic limit takes the well-known form of the Lorentz force law:

$$\ddot{x}^i = \ E^i + \frac{2}{c}F^{ij}u_j, \tag{3.18}$$

in which the gravitoelectromagnetic fields can be time dependent.

In this section we have made a correction to the traditional ansatz for the metric perturbations. We have used the $\tilde{h}_{ij}$ components, which in previous

works were supposed to be negligible due to the $1/c^4$ factor, and we propose that this fixes the time dependence of the potentials. We have also seen that this ansatz keeps the analogy with the true electromagnetism but with the presence of additional fields, i.e the fields which are produced from the potentials λ and d_{ij}. The additional potentials are necessary and we should have used them from the start of our analysis. Actually the existence of these potentials comes from the fact that Gravity is a Spin 2 field instead of Electromagnetism which is a Spin 1 field. Consequently we need more degrees of freedom to describe gravity instead of electromagnetism, in which we need initially only four degrees of freedom. When we take a theory of many degrees of freedom, like Gravity, and we use only some of them, we do not use all the information we may get; for this reason we have encountered many problems in the previous chapter. However, this ansatz is still giving us extra terms in the equation of motion. In the next section we will try to fix the problem of the extra terms in the equation of motion using a modified version of the ansatz we introduced in section (2.3).

3.2 A correction to the alternative ansatz for the metric perturbations

In this section we will try to make some corrections to the ansatz that we presented in section (2.3). This section is split into two subsections. In the first one we present the most general case of the ansatz and in the second we give a special case from which we obtain some interesting results.

3.2.1 The general case

In section (2.3) we worked with an ansatz that may give the correct form of the analogue of the Lorentz force law for the equations of motion. However we have encountered many important problems such as the vanishing of both the source and the gravitoelectric field and of the time dependence of the fields. In this section we try to modify the ansatz of section (2.3) in order to fix its problems. If we think as in the previous section, the only thing that we have to do is to involve in our analysis the additional degrees of freedom of the gravitational field. In the ansatz of section (2.3) the components $\tilde{h}_{ij}$ are not zero, see Eq. (2.34); however from the line element in this case, Eq. (2.44), we see that the h_{ij} components again vanish. But these are the components that we should involve, in our analysis and associate them with the scalar potential λ and the traceless tensor potential d_{ij}.

The ansatz we will use here is the following:

$$\tilde{h}_{00} = \frac{\Phi}{c^2} - \frac{3\lambda}{c^4}\,, \qquad \tilde{h}_{0i} = \frac{A_i}{c}\,, \qquad \tilde{h}_{ij} = -\left(\frac{\Phi}{c^2} + \frac{\lambda}{c^4}\right)\,\eta_{ij} \,+\, \frac{2}{c^4}\,d_{ij}\,. \quad (3.19)$$

In this ansatz the magnitude of the potentials Φ, λ and d_{ij} is in principle the same. The different $1/c$ coefficients, first, are connected with the T_{ij} components of the energy-momentum tensor and, second, will result in simplified forms of the equations of motion. The line-element in this ansatz is:

$$ds^2 = c^2\left(1 + \frac{2\Phi}{c^2}\right)dt^2 - 2\left(\vec{A}\cdot d\vec{x}\right)dt$$

$$+ \left[\left(1 + \frac{2\lambda}{c^4}\right)\,\eta_{ij} + \frac{2}{c^4}\,d_{ij}\right]dx^i dx^j\,, \qquad (3.20)$$

We start our analysis from the gauge condition. The transverse gauge condition $\tilde{h}^{\mu\nu}{}_{,\nu} = 0$ gives:

- For $\mu = 0$, an equation similar to the Lorentz gauge condition:

$$\partial_t\left(\frac{\Phi}{c^3} - \frac{3\lambda}{c^5}\right) + \partial_i\frac{A^i}{c} = 0. \qquad (3.21)$$

- And for $\mu = i$, an equation which relates the gravitoelectric field with the gradient of the scalar λ and divergence of the tensor d_{ij}:

$$\partial_t A^i - \partial^i\Phi = E^i = \frac{1}{c^2}\left(\partial^i\lambda - 2\,\partial_j d^{ij}\right). \qquad (3.22)$$

From Eq. (3.21) we can get the Lorentz gauge condition if we demand that the scalar potential λ is not depending on time $\partial_t\lambda = 0$. Since the Lorentz gauge condition is important for our analysis, from here we conclude that the scalar potential λ must be static. Although this may seem a problem, considering that we have inserted the scalar λ and the tensor d_{ij} potentials in order to avoid the problems with the Φ and $\vec{A}$ potentials, it is not a serious one. From Eq. (3.22) we get an additional constraint for the scalar λ and the tensor d_{ij} potentials; they must be related with the gravitoelectric field. The last equation saves both the gravitoelectric field and the time dependence of the potentials.

We can now continue with the field equations Eq. (1.19), which inside the source are

- For $\mu = 0$ and $\nu = 0$:

$$\partial^2\left(\frac{\Phi}{c^2} - \frac{3\lambda}{c^4}\right) = -\frac{16\pi G}{c^2}\,\rho. \qquad (3.23)$$

- For $\mu = 0$ and $\nu = i$:
$$\partial^2 A^i = -\frac{16\pi G}{c^2} j^i. \tag{3.24}$$

- Finally for $\mu = i$ and $\nu = j$:
$$\partial^2 \left[-\left(\frac{\Phi}{c^2} + \frac{\lambda}{c^4} \right) \eta_{ij} + \frac{2}{c^4} d_{ij} \right] = -\frac{16\pi G}{c^4} j^i u^j, \tag{3.25}$$

and if we use Eq. (3.21), Eq. (3.25) takes the form:
$$\partial^2 \left(2\lambda \eta^{ij} + d^{ij} \right) = -8\pi G \left(j^i u^j + \rho c^2 \eta^{ij} \right). \tag{3.26}$$

While, in vacuum the field equations take the form:

- For $\mu = 0$ and $\nu = 0$:
$$\partial^2 \Phi = 0. \tag{3.27}$$

- For $\mu = 0$ and $\nu = i$:
$$\partial^2 A^i = 0. \tag{3.28}$$

- Finally for $\mu = i$ and $\nu = j$:
$$\partial^2 \lambda = 0 \quad and \quad \partial^2 d_{ij} = 0. \tag{3.29}$$

The above equations are the field equations which describe the gravitational field around (and inside) a rotating massive body. Here we face the only real problem with this ansatz: in spite of the fact that we have restored the source, as we can see from Eqs. (3.23-3.26), the field equations inside it have a different form from the true electromagnetic ones. However, in vacuum the field equations (3.27-3.29) have the correct form. As we see in the next subsection, we can easily fix this problem.

Finally we can give the equation of motion of a test particle moving in the background (3.20). Here, as always, we keep all the corrections and we comment on their importance at the final stage of the calculation. The Christoffel symbols for the line-element (3.20) are:

$$\Gamma^i_{00} = \frac{1}{c^2} \left(\partial_t A^i - \partial^i \Phi \right). \tag{3.30}$$

$$\Gamma^i_{0j} = \frac{1}{2} \eta^{ik} \left[\frac{1}{c} F_{jk} + \frac{2}{c^5} \partial_t \left(\lambda \eta_{jk} + d_{jk} \right) \right]. \tag{3.31}$$

$$\Gamma^i_{kj} = \frac{1}{c^4} \left(\delta^i_j \partial_k \lambda + \delta^i_k \partial_j \lambda - \eta_{jk} \partial^i \lambda \right)$$
$$+ \frac{1}{c^4} \left(\partial_k d_j{}^i + \partial_j d_k{}^i - \partial^i d_{jk} \right). \tag{3.32}$$

Substituting these into the geodesics equation (2.13), we obtain the equations of motion

$$\ddot{x}^i = E^i + F^{ij}u_j - \frac{1}{c^4}\left(2\,u^i\,\partial_t\lambda + 2\,u^i u^k\,\partial_k\,\lambda - u^k u_k\,\partial^i\lambda\right)$$
$$- \frac{1}{c^4}\left(2\,u^j\,\partial_t d^i{}_j + 2\,u^i u^k\,\partial_k\,d^i{}_j - u^j u^k\,\partial^i\,d_{jk}\right). \tag{3.33}$$

In the above equation we define the field $\vec{E}$ as:

$$E^i = -\partial_t A^i - \partial^i\Phi, \tag{3.34}$$

and the field $\vec{B}$ as

$$F_{ij} = 4\,\varepsilon_{ijk}B_k. \tag{3.35}$$

We did not define the fields earlier because the analogy with the electromagnetism is broken in this ansatz. We have defined the above fields in the usual way, but we can not recognize them as the gravitoelectromagnetic fields, i.e the analogues of the electromagnetic ones. In the following section we will say more about the fields. However, as we can easily see, equation (3.33) contains additional terms but even at high velocities, due to the presence of the $1/c^4$ factor, it can take a form similar to the Lorentz force law:

$$\ddot{x}^i = E^i + F^{ij}u_j. \tag{3.36}$$

In this subsection we presented a modified version of the ansatz we used in section (2.3). With the new ansatz, we have managed to solve the problems that we have encountered in section (2.3). We have obtained time-dependent potentials, and we have avoided the vanishing of both the gravitoelectric potential and the source. Also this ansatz is the first which gives us the correct form of the equations of motion even at high velocities. However, the ansatz we have used here has still a serious problem: it does not give us the correct field equations inside the source, and as a result the analogy with the true electromagnetism is broken. In the following subsection we see how we can solve this problem.

3.2.2 An interesting special case

The ansatz we have used in the previous subsection fixes all the problems we had encountered in section (2.3) but it still has an important problem. Inside the source, as already mentioned, the analogy with the true electromagnetism is broken. We can easily fix this problem and this subsection presents how we can do it.

If we look more carefully at Eqs. (3.23-3.26), we can see that the vanishing of the scalar potential λ may fix the problem. In this subsection we take the ansatz of the previous one and we demand that $\lambda = 0$. This means that the trace of the spatial part of the metric perturbations h_{ij} will be vanishing, i.e. $\eta_{ij} h^{ij} = 0$. In the previous subsection, we discussed the general case of non-vanishing λ. In this subsection we disccuss separately the special case with $\lambda = 0$ as it leads to important results that will also be used in the following chapter.

The ansatz we will use here for the metric perturbations is the following

$$\tilde{h}_{00} = \frac{\Phi}{c^2}, \qquad \tilde{h}_{0i} = \frac{A_i}{c}, \qquad \tilde{h}_{ij} = -\frac{\Phi}{c^2}\,\eta_{ij} \,+\, \frac{2}{c^4}\,d_{ij}. \tag{3.37}$$

As in the previous subsection, the above ansatz is expressed in terms of the GEM potentials, $\Phi(x^\mu)$ and $\vec{A}(x^\mu)$, plus one additional potential, that comes from the h_{ij} components, the traceless tensor potential $d_{ij}(x^\mu)$.

The transverse gauge condition $\tilde{h}^{\mu\nu}{}_{,\nu} = 0$ gives:

- For $\mu = 0$, an equation similar to the Lorentz gauge condition:

$$\partial_t \frac{\Phi}{c^3} + \partial_i \frac{A^i}{c} \,=\, 0. \tag{3.38}$$

- And for $\mu = i$, an equation which relates the gravitoelectric field with the divergence of the tensor potential d_{ij}:

$$\partial_t A^i - \partial^i \Phi \,=\, E^i \,=\, \frac{2}{c^2}\,\partial_j d^{ij}. \tag{3.39}$$

Eq. (3.38) has the well-known form of the Lorentz gauge condition. From Eq. (3.39) we get an additional constraint for the tensor potential d_{ij}: it must be related with the gravitoelectromagnetic field. The last equation, as in the previous subsection, saves both the gravitoelectric field and the time-dependence of the potentials.

Now we can continue with the field equations Eq. (1.19), which in the presence of source have the form

- For $\mu = 0$ and $\nu = 0$:
$$\partial^2 \Phi = -16\pi G\,\rho. \tag{3.40}$$

- For $\mu = 0$ and $\nu = i$:
$$\partial^2 A^i \,=\, -\frac{16\pi G}{c}\,j^i. \tag{3.41}$$

- Finally for $\mu = i$ and $\nu = j$:

$$\partial^2 \left(-\frac{\Phi}{c^2}\, \eta_{ij} + \frac{2}{c^4}\, d_{ij} \right) = -\frac{16\pi G}{c^4}\, j^i\, u^j \,, \tag{3.42}$$

and if we use Eq. (3.38), Eq. (3.42) takes the form:

$$\partial^2 d^{ij} = -8\,\pi G \left(j^i u^j + \rho c^2 \eta^{ij} \right). \tag{3.43}$$

Whereas in vacuum the field equations are

- For $\mu = 0$ and $\nu = 0$:

$$\partial^2 \Phi = 0. \tag{3.44}$$

- For $\mu = 0$ and $\nu = i$:

$$\partial^2 A^i = 0. \tag{3.45}$$

- Finally for $\mu = i$ and $\nu = j$:

$$\partial^2 d_{ij} = 0. \tag{3.46}$$

The above equations are the field equations which describe the gravitational field around (and inside) a rotating massive body. As in the previous subsection we have restored the source but, as we can see from Eqs. (3.40-3.41), the coefficient on the right-hand side is 16π, instead of 4π which is the coefficient appearing in the true electromagnetism. Despite that, in vacuum the field equations (3.44-3.46) have the correct form. We can fix the problem of the wrong coefficient easily by redefining the gravitoelectromagnetic potentials as:

$$\Phi \longrightarrow 4\Phi^*,$$
$$\vec{A} \longrightarrow 4\vec{A}^*, \tag{3.47}$$

where Φ^* and $\vec{A}^*$ are the true potentials. If we employ the new potentials, Eq. (3.40) takes the form:

$$\partial^2 \Phi^* = -4\pi G\, \rho, \tag{3.48}$$

while Eq. (3.41) takes the form:

$$\partial^2 A^{*i} = -\frac{4\pi G}{c}\, j^i. \tag{3.49}$$

The above equations have the correct form and along with Eq. (3.43) are the full system of the field equations.

Now we can define a gravitomagnetic and a gravitoelectric field as:

$$\vec{E} \equiv -\partial_t \vec{A}^* - \vec{\nabla}\Phi^*, \qquad \vec{B} \equiv \vec{\nabla} \times \vec{A}^*. \tag{3.50}$$

One may easily see that Eqs. (3.48) and (3.49) along with Eqs. (3.50) reduce to a set of four Maxwell-like equations for the Gravitoelectromagnetic fields:

$$\vec{\nabla} \cdot \vec{E} = 4\pi G\rho, \qquad \vec{\nabla} \times \vec{B} = \partial_t \vec{E} + \frac{4\pi G}{c}\, \vec{j}, \tag{3.51}$$

and

$$\vec{\nabla} \times \vec{E} = \partial_t \vec{B}, \qquad \vec{\nabla} \cdot \vec{B} = 0. \tag{3.52}$$

Consequently, this ansatz gives us the correct form of the field equations.

The line-element in this ansatz is:

$$ds^2 = c^2 \left(1 + \frac{2\Phi}{c^2}\right) dt^2 - 2\left(\vec{A} \cdot d\vec{x}\right) dt + \left(\eta_{ij} + \frac{2}{c^4}\, d_{ij}\right) dx^i dx^j. \tag{3.53}$$

Finally we can give the equations of motion of a test particle moving in the background (3.53). The Christoffel symbols for the line element (3.53) are:

$$\Gamma^i_{00} = \frac{1}{c^2}\left(\partial_t A^i - \partial^i \Phi\right). \tag{3.54}$$

$$\Gamma^i_{0j} = \frac{1}{2}\eta^{ik}\left(\frac{1}{c}\, F_{jk} + \frac{2}{c^5}\, \partial_t\, d_{jk}\right). \tag{3.55}$$

$$\Gamma^i_{kj} = \frac{1}{c^4}\left(\partial_k\, d_j{}^i + \partial_j\, d_k{}^i - \partial^i\, d_{jk}\right). \tag{3.56}$$

Substituting these into the geodesics equation (2.13), we obtain the equations of motion

$$\ddot{x}^i = 4\,E^i + F^{ij}u_j - \frac{1}{c^4}\left(2\,u^j\,\partial_t d^i{}_j + 2\,u^i u^k\,\partial_k\, d^i{}_j - u^j u^k\,\partial^i\, d_{jk}\right). \tag{3.57}$$

As we can see, the above equation contains some corrections but, due to the presence of the $1/c^4$ factor even at high velocities, it can take a form similar to the Lorentz force law:

$$\ddot{x}^i = 4\,E^i + F^{ij}u_j, \tag{3.58}$$

or in a more familiar vector notation:

$$\ddot{\vec{x}} = 4\left(\vec{E} + \vec{u} \times \vec{B}\right). \tag{3.59}$$

As we can see from the above equation, in this ansatz, the equations of motion are very close to the Lorentz force law. The only difference is an overall factor 4 on the right hand side of the equation.

In this subsection we have solved the problems that we have encountered in the previous one by –simply– setting $\lambda = 0$. Although this may be a special case, it has led to the correct form of both the field equations and the Lorentz force law. Moreover, this ansatz does not generate any corrections to the Lorentz force, even at high velocities. Concluding, the ansatz that we have used here is the optimum as it preserves the analogy between gravitation and electromagnetism.

Chapter 4

True Electromagnetism

4.1 An alternative expression for the Lorentz force law

The usual procedure when someone solves an electromagnetic problem, in the Lorentz gauge, is first to find the potential through the relation

$$\partial^2 A^\mu = \frac{4\pi C}{c^2} J^\mu,$$
(4.1)

where the electromagnetic 4-potential is defined as $A^\mu = \left(\Phi/c, \vec{A}\right)$ and C is the Coulomb's constant ($C \equiv \frac{1}{4\pi\varepsilon_0}$). Then one is able to calculate the electric and magnetic fields

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A} \quad and \quad \vec{B} = \vec{\nabla} \times \vec{A},$$
(4.2)

and finally to find the equation of motion of a test particle

$$m\ddot{\vec{x}} = q\left(\vec{E} + \vec{u} \times \vec{B}\right).$$
(4.3)

However, in the previous chapters, we have seen that the Christoffel symbols can give us the definitions of both the Electric and the Magnetic field and that the geodesics equation contains the expression for the Lorentz force law. Consequently, instead of Eq. (4.3), we could use a modified version of the geodesics equation in order to describe the equations of motion.

First of all we use the potential from Eq. (4.1) in order to define an approximately flat "space-time" $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ with line element

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2 dt^2 - 2\left(\vec{A} \cdot \vec{dx}\right)dt + \eta_{ij}dx^i dx^j.$$
(4.4)

Next we claim that the equation

$$m\,\frac{d^2 x^\rho}{ds^2} + q\,\Gamma^\rho_{\mu\nu}\,\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \tag{4.5}$$

describes the motion of a test particle with charge q and mass m. The above equation is not anymore a geodesics equation, it gives geodesic curves only if $q = m$. Eq. (4.5) takes the explicit form

$$\frac{m}{q}\,\frac{d^2 x^i}{dt^2} + c^2\,\Gamma^i_{00} + 2c\,\Gamma^i_{0j}\,\frac{dx^j}{dt} + \Gamma^i_{kj}\,\frac{dx^k}{dt}\frac{dx^j}{dt} = 0\,. \tag{4.6}$$

The Christoffel symbols Eq. (1.11) for the above line element are

$$\Gamma^i_{00} = -\frac{E^i}{c^2}. \tag{4.7}$$

$$\Gamma^i_{0j} = \frac{1}{2c}\varepsilon_{ijk}B_k\,, \tag{4.8}$$

$$\Gamma^i_{0j} = 0, \tag{4.9}$$

where we have used the definitions for the fields (4.2). Finally if we use Eqs. (4.7-4.9) we can see that the modified geodesics equation (4.5) is equivalent to the Lorentz force law (4.3).

In this chapter we suggest that we can use the potential of the electromagnetism in order to define a "geometry", and then from a modified geodesics equation we can get the Lorentz force law. This is just an additional way for someone to find the equations of motion. However, if we use Eq. (4.5), we do not actually need to define the electromagnetic fields. We can start from Eq. (4.1), then we use the potential to define the line element Eq. (4.4) –bypassing the definition of the fields– and finally we can find the equation of motion using Eq. (4.5). With this new definition of the equations of motion we have connected electromagnetism with a "geometry". Nevertheless, the potential is still calculated in the usual way –the well-known equations of electromagnetism– and we can only construct the geometry (Eq. (4.4)) by hand. In the following sections we will try to find an alternative formulation for electrodynamics in which the metric tensor is the solution of the field equations. In this point we must say that our objective is not to replace Maxwell's theory of electromagnetism by a tensorial theory but rather to investigate how far the analogy between the two dynamics can go.

4.2 An alternative description of electromagnetism

In the previous section we have presented an alternative expression for the Lorentz force law which premises the construction of a "geometry" using the electromagnetic potential. In this section we try to find a new formulation for electrodynamics in which the metric tensor is a fundamental quantity.

A careful inspection of Eq. (4.4) shows that the electromagnetic potential is related with the metric perturbations as follows:

$$h_{00} = \frac{2\Phi}{c^2}, \quad h_{0i} = \frac{A_i}{c} \quad and \quad h_{ij} = 0. \tag{4.10}$$

The above, along with the analysis in Chapters 2 and 3, dictates that we should use the formulation of General Relativity in the weak field approximation. We use, as field equations, Eq. (1.17)

$$\tilde{h}^{\alpha}_{\mu,\nu\alpha} + \tilde{h}^{\alpha}_{\nu,\mu\alpha} - \partial^2 \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \tilde{h}^{\alpha\beta}_{,\alpha\beta} = 2k\, T_{\mu\nu}\,,$$

in which the proportionality constant k is going to be defined later and the energy-momentum tensor has the form $T_{\mu\nu} = \rho\, u_\mu u_\nu$, with ρ the electric charge density.

However if we try to describe electromagnetism with a "gravity-like" theory a big problem arises. Electromagnetism is a theory with 4 degrees of freedom while gravity has 10. We get a theory very rich in content and we suppress it.

In a previous work of ours [33] we have used the following ansatz for the metric perturbations $\tilde{h}_{\mu\nu}$

$$\tilde{h}_{00} = \frac{\alpha\hat{\Phi}}{c^2}\,, \qquad \tilde{h}_{0i} = \frac{\beta\hat{A}_i}{c^2}\,, \qquad \tilde{h}_{ij} = \frac{\gamma\hat{\Phi}}{c^2}\,\delta_{ij}\,, \tag{4.11}$$

where α, β and γ are arbitrary numerical coefficients while $\hat{\Phi}$ and $\hat{\vec{A}}$ are the electromagnetic potentials and, as we can easily see, the potentials have the same magnitude. In the previous chapters we have showed that the form of the metric perturbation affects the derived equations for the scalar and vector potential. In order to study simultaneously a large collection of choices we have inserted the α, β and γ coefficients. We can see that the ansatz of section (2.2) corresponds to the choice ($\alpha = \beta = 1, \gamma = 0$) while the ansatz of section (2.3) corresponds to ($\alpha = \beta = \gamma = 1$). Our goal is to reduce the field equations of General Relativity to Maxwell's equations. The only way to do this is to demand that the choice of $\tilde{h}_{\mu\nu}$ has to be linear in the electromagnetic potentials and should not contain any derivatives. Also, we have mentioned that we need

only four degrees of freedom, as a result, the additional potentials we have used in Chapter 3 should apparently vanish here. We thus conclude that the scalar Φ and the vector $\vec{A}$ potentials can be accommodated in the metric perturbations $\tilde{h}_{\mu\nu}$ in a limited number of distinct ways. With the insertion of the coefficients α, β and γ in the above ansatz Eq. (4.11) we have included all possible choices.

As in the previous chapters we start our analysis from the linearised form of Einstein's field equations (1.17). The field equations for the ansatz (4.11) take the form

- For $\mu = 0$ and $\nu = 0$

$$\frac{(\alpha - \gamma)}{c^2} \, \delta^{ij} \, \partial_i \partial_j \hat{\Phi} = 2k\rho \, u_0 \, u_0 \,, \tag{4.12}$$

- For $\mu = 0$ and $\nu = i$

$$\frac{(\alpha - \gamma)}{c^3} \, \partial_i \partial_t \hat{\Phi} + \frac{\beta}{c^2} \, \partial_i \left(\partial_k \hat{A}^k \right) - \frac{\beta}{c^2} \, \delta^{kl} \, \partial_k \partial_l \hat{A}^i = 2k\rho \, u_0 \, u_i \,, \tag{4.13}$$

- And finally for $\mu = i$ and $\nu = j$

$$-\frac{\beta}{c^3} \, \partial_t \left(\partial_i \hat{A}^j + \partial_j \hat{A}^i \right) - \frac{2\gamma}{c^2} \, \partial_i \partial_j \hat{\Phi}$$
$$+ \, \delta_{ij} \left[\frac{2\gamma}{c^2} \, \delta^{kl} \, \partial_k \partial_l \hat{\Phi} + \frac{2\beta}{c^3} \, \partial_t (\partial_k \hat{A}^k) + \frac{(\alpha - \gamma)}{c^4} \, \partial_t^2 \hat{\Phi} \right] = 2k\rho \, u_i \, u_j \,. \tag{4.14}$$

From the above we can see that for $\alpha = \gamma$, Eq. (4.12) reduces to $\rho = 0$ and we only get a model of electro-magnetism in vacuum, a model similar to the one that we have found in section (2.3) in the context of Gravitoelectromagnetism. On the other hand, for $\alpha \neq \gamma$, if we identify the proportionality constant as:

$$k \equiv -\frac{2\pi C}{c^4} \, (\alpha - \gamma) \,, \tag{4.15}$$

we obtain the Poisson's equation for the scalar potential $\hat{\Phi}$

$$\nabla^2 \, \hat{\Phi} = -4\pi C \rho \,. \tag{4.16}$$

For the above value for k, and using the more familiar vector notation, Eq. (4.13) takes the form

$$\vec{\nabla} \, (\vec{\nabla} \cdot \hat{\vec{A}} + \frac{1}{c} \, \partial_t \hat{\Phi}) - \nabla^2 \hat{\vec{A}} = 4\pi C \rho \, \frac{\vec{u}}{c} \,, \tag{4.17}$$

where we have assumed that $\beta = \alpha - \gamma$. Consequently, there is an infinite number of choices one could make for the numerical coefficients appearing in the metric perturbations $\tilde{h}_{\mu\nu}$, which could derive the electromagnetic equations from the gravitational field equations (1.17). Now we can define the electric $\hat{\vec{E}}$ and magnetic field $\hat{\vec{B}}$, in terms of the scalar $\hat{\Phi}$ and the vector $\hat{\vec{A}}$ potentials, in the usual way

$$\hat{\vec{E}} \equiv -\frac{1}{c}\,\partial_t\hat{\vec{A}} - \vec{\nabla}\hat{\Phi}\,, \qquad \hat{\vec{B}} \equiv \vec{\nabla}\times\hat{\vec{A}}\,. \tag{4.18}$$

The above definitions are equivalent to the following Maxwell's equations

$$\vec{\nabla}\cdot\hat{\vec{B}} = 0\,, \qquad \vec{\nabla}\times\hat{\vec{E}} = -\,\partial_t\hat{\vec{B}}\,, \tag{4.19}$$

while, it is easy to show that Eqs. (4.16) and (4.17) are the remaining Maxwell's equations,

$$\vec{\nabla}\cdot\hat{\vec{E}} = 4\pi C\rho\,, \qquad \vec{\nabla}\times\hat{\vec{B}} = \frac{1}{c}\,\partial_t\hat{\vec{E}} + \frac{4\pi C}{c}\,\vec{j}\,, \tag{4.20}$$

under the constraint that the vector potential is not depending on time, $\partial_t\hat{\vec{A}} = 0$. One can easily see that due to the coefficient $(\alpha - \gamma)$ in Eqs. (4.12) and (4.13), Maxwell's equations do not change under the simultaneous changes $(\alpha \leftrightarrow \gamma)$ and $\hat{\Phi} \to -\hat{\Phi}$. Therefore, instead of what happens in Gravitoelectromagnetism, where $\tilde{h}_{00}$ must always be connected with the Newtonian potential, and as a result it should always be non-vanishing, in the case that we study here, even if the $\tilde{h}_{00}$ component of the metric perturbations is zero and the potential $\hat{\Phi}$ is introduced only through the $\tilde{h}_{ij}$ component, we could also recover the whole set of Maxwell's equations. Nevertheless, as in Gravitoelectromagnetism, along with the Maxwell's equations, the field equations give an additional set of equations, Eqs. (4.14). For the above choice of k and β, these equations take the form

$$\frac{1}{c}\,\partial_t\left(\partial_i\hat{A}^j + \partial_j\hat{A}^i\right) + \frac{2\gamma}{\alpha-\gamma}\,\partial_i\partial_j\hat{\Phi}$$

$$-\,\delta_{ij}\left[\frac{2\gamma}{\alpha-\gamma}\,\delta^{kl}\,\partial_k\partial_l\hat{\Phi} + \frac{2}{c}\,\partial_t(\partial_k\hat{A}^k) + \frac{1}{c^2}\,\partial_t^2\hat{\Phi}\right] = 4\pi C\rho\,u_i\,u_j/c^2\,. \tag{4.21}$$

For $i = j$, the above reduces to the relation

$$\partial_t^2\hat{\Phi} = -\frac{4\pi C}{3}\,\rho\,|\vec{u}|^2\,, \tag{4.22}$$

with $(u^1)^2 = (u^2)^2 = (u^3)^2$, for any value of α and γ. For $i \neq j$, the term proportional to δ_{ij} in Eq. (4.21) vanishes and, under the assumption that

$\alpha = 2\gamma$, the remaining terms can be combined to form the components of the electric field. In that case, Eq. (4.21) takes the form

$$\partial_i \hat{E}_j + \partial_j \hat{E}_i = -4\pi C \rho\, u_i\, u_j / c^2\,. \tag{4.23}$$

In the case where $\alpha \neq 2\gamma$, Eq. (4.21) is formed of time and space-derivatives of the electromagnetic potentials $\hat{\vec{A}}$ and $\hat{\Phi}$.

Therefore, due to the tensorial structure of the formalism, the four Maxwell's equations are accompanied by additional constraints. Equivalent constraints have also arisen in the analysis of the previous chapters. In chapter 3 we have showed that, if we use the additional degrees of freedom of the gravitational field, we can solve all the problems related to the time-dependence of the fields. Here, we use the formalism of General Relativity in order to describe the Electromagnetism and, as we have already mentioned, we can not use more than four of the available degrees of freedom. Consequently, in this case we can not avoid the presence of these constraints and, thus, with this formalism we can only describe a special part of Electromagnetism and not the general case. We can only describe magnetostatics since the vector potential $\vec{A}$ does not depend on time and the distribution of charges should be isotropic as in section (2.2).

If we impose a gauge condition, we get additional constraints to the model; some of these constraints complete the theory, while others give unnecessary restrictions to the electromagnetic potentials. As in the previous chapters, we will use the transverse gauge condition $\tilde{h}^{\mu\nu}{}_{,\nu} = 0$. The time-component of the gauge condition takes the form

$$\frac{\alpha}{c}\,\partial_t \hat{\Phi} + \beta\,\partial_i \hat{A}^i = 0\,. \tag{4.24}$$

If we demand $\alpha = \beta$, the above reduces to the well known Lorentz condition

$$\frac{1}{c}\,\partial_t \hat{\Phi} + \vec{\nabla} \cdot \hat{\vec{A}} = 0 \quad or \quad A^\mu{}_{,\mu} = 0, \tag{4.25}$$

where we have defined the electromagnetic 4-potential as $A^\mu = \left(\Phi,\, \vec{A} \right)$. The spatial components of the gauge condition, though, lead to an additional constraint

$$\frac{\beta}{c}\,\partial_t \hat{\vec{A}} + (\alpha - \beta)\,\vec{\nabla}\hat{\Phi} = 0\,, \tag{4.26}$$

where we have used the relation $\beta = \alpha - \gamma$. The choice $\alpha = \beta$ leads to the time-independence of the vector potential $\hat{\vec{A}}$; in this case we do not get a new constraint, because we already know –from the field equations– that the vector potential should be static. If we choose $\alpha = 2\beta$, we find that the electric

field $\hat{\vec{E}} = -\partial_t \hat{\vec{A}}/c - \vec{\nabla}\hat{\Phi}$ should vanish, a result similar to the one of section (2.3). For all the remaining cases, Eq. (4.26) imposes an additional constraint between the scalar and vector potentials which does not exist in the traditional electromagnetism.

Now we can find the equations of motion of a test particle which moves in the electromagnetic field. As in the previous section, we assume that the equations of motion are given from the modified geodesics equation (4.5):

$$m \frac{d^2 x^\rho}{ds^2} + q\, \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0\,.$$

In the linear approximation, the components of the Christoffel symbols that appear in Eqs. (4.5-4.6) have the form

$$\Gamma^i_{00} = \frac{(\alpha + 3\gamma)}{4c^2} \partial_i \hat{\Phi} + \frac{\beta}{c^3} \partial_t \hat{A}^i\,, \tag{4.27}$$

$$\Gamma^i_{0j} = \frac{\beta}{2c^2} \hat{F}_{ij} - \frac{(\alpha - \gamma)}{4c^3} \delta^i_j \partial_t \hat{\Phi}\,, \tag{4.28}$$

$$\Gamma^i_{kj} = -\frac{(\alpha - \gamma)}{4c^2} \left(\delta^i_j \partial_k \hat{\Phi} + \delta^i_k \partial_j \hat{\Phi} - \delta_{kj} \partial_i \hat{\Phi} \right)\,, \tag{4.29}$$

where we have defined the spatial part of the electromagnetic tensor as $\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i$. The electromagnetic tensor is connected with the magnetic field through the relation $\hat{F}_{ij} = -\varepsilon_{ijk} \hat{B}_k$. Substituting the above Christoffel symbols into Eq. (4.6), we find

$$m\, a^i + q \left[\frac{\beta}{c} \partial_t \hat{A}^i + \frac{(\alpha + 3\gamma)}{4} \partial_i \hat{\Phi} \right] - \frac{\beta q}{c} \epsilon_{ijk}\, u^j \hat{B}_k - \frac{(\alpha - \gamma)q}{2c^2} \left(\partial_t \hat{\Phi} \right) u^i$$

$$+ \frac{(\alpha - \gamma)q}{4c^2} \left[\partial_i \hat{\Phi} \left(u^k u^k \right) - 2(\partial_j \hat{\Phi}\, u^j)\, u^i \right] = 0\,. \tag{4.30}$$

Under the assumption that we can construct a theory of electromagnetism using the formalism of General Relativity, the above equation describes the motion of a massive, charged test particle. The above equation is a generalized form of the Lorentz force of the system. This general form is obviously far away from what we would expect from a realistic electromagnetic theory.

Now let us examine some special cases. If $\alpha = \gamma$, we already know that we get a static theory in vacuum. In this case, the last two terms on the left hand side of Eq. (4.30) and all the time derivatives trivially vanish and we get the more familiar form

$$m\, \vec{a} = \vec{F} = q\alpha\, \hat{\vec{E}} + \frac{q\beta}{c}\, \vec{u} \times \hat{\vec{B}}\,. \tag{4.31}$$

From the above equation we see that we must choose $\alpha = \beta$, because we know that we need a common numerical factor in front of the electric and magnetic terms in the Lorentz force. Actually, we should choose $\alpha = \beta = 1$ in order to get the correct expression for the Lorentz force, but for any other value of the coefficients α, β we could easily fix the problem by modifying the equation of motion Eq. (4.5).

On the other hand, if $\alpha \neq \gamma$, we have seen that we get a more realistic theory of electromagnetism in terms of fields, but in the expression of the Lorentz force extra terms appear. If we apply the constraint $\beta = \alpha - \gamma$, which we know is necessary to restore the form of Maxwell's equations, Eq. (4.30) takes the form

$$
m\,\vec{a} = \vec{F} = \frac{q}{4}\,\hat{\vec{E}}\left[(4\alpha - 3\beta) + \frac{\beta\,|\vec{u}|^2}{c^2}\right] + \frac{\beta q}{c}\,\vec{u} \times \hat{\vec{B}} + \frac{\beta q}{2}\left[\frac{\vec{u}}{c}\frac{\partial_t \hat{\Phi}}{c} - \frac{\vec{u}}{c}\left(\frac{\vec{u}}{c} \cdot \hat{\vec{E}}\right)\right].
\tag{4.32}
$$

The additional terms which appear in the above expression are similar to that arising in section (2.2) in the context of Gravitoelectromagnetism. In the non relativistic limit and if we also assume that the potentials are static, the above equation takes the form

$$
\vec{F} = \frac{q}{4}(4\alpha - 3\beta)\,\hat{\vec{E}} + \frac{\beta q}{c}\,\vec{u} \times \hat{\vec{B}},
\tag{4.33}
$$

which for $\alpha = 7\beta/4$ takes the simple form

$$
\vec{F} = \beta q\,\hat{\vec{E}} + \frac{\beta q}{c}\,\vec{u} \times \hat{\vec{B}}.
\tag{4.34}
$$

The above equation for $\beta = 1$, takes the correct form of the Lorentz force, but it can take the correct form, even for a general value of β, if we redefine the equation of motion Eq. (4.5). However, this model can not be realistic since it contains corrections for high velocities which we know that do not exist in the theory of electromagnetism.

In this section we have showed that the formalism of the General Relativity is able to describe electromagnetism in some special cases. This happens because the tensorial structure of gravity adds additional unnecessary constraints in the theory or/and extra terms in the Lorentz force law. In the following section we will try to fix these problems by using the additional degrees of freedom that the theory contains.

4.3 Employing the additional degrees of freedom

In this section we use the additional degrees of freedom of the theory (i.e. the h_{ij} components of the metric perturbations) in order to examine if we can get a more realistic theory of electromagnetism. We expect that the use of the additional degrees of freedom will lead to the appearance of new fields. Therefore, the theory we present here is a generalized theory of electromagnetism which unifies the classical electromagnetism with the new fields which we expect they arise. If we want this theory to be realistic, the new fields should not have any contribution into observable quantities or, if they contribute, their contribution should be negligible.

In subsection (3.2.2) we showed that the ansatz which we used gives us the best results in the context of Gravitoelectromagnetism. Here we use the results of the subsection (3.2.2) in order to describe the generalized theory of electromagnetism. We use again as field equation the linearized equations of General Relativity, Eq. (1.19), under the imposition of the transverse gauge condition $\partial_\nu \tilde{h}^{\mu\nu} = 0$, namely

$$\partial^2 \tilde{h}_{\mu\nu} = -2k\, T_{\mu\nu}\,.$$

The proportionality constant is taken to be $k = -\frac{2\pi C}{c^4}$ and the energy-momentum tensor is $T_{\mu\nu} = \rho\, u_\mu u_\nu$, with ρ the electric charge density.

The ansatz which we use here is the same as in the subsection (3.2.2)

$$\tilde{h}_{00} = \frac{\Phi}{c^2}\,, \qquad \tilde{h}_{0i} = \frac{A_i}{c^2}\,, \qquad \tilde{h}_{ij} = -\frac{\Phi}{c^2}\, \eta_{ij} \,+\, \frac{2}{c^4}\, d_{ij}\,. \qquad (4.35)$$

Nevertheless, here Φ and $\vec{A}$ are recognized as the true electromagnetic potentials, while the potential d_{ij} denotes the extra degrees of freedom and it should obviously be related with the new fields. In the above ansatz the potentials have the same magnitude and we can also define the electromagnetic 4-potential as $A^\mu = \left(\Phi\,, \vec{A} \right)$.

The transverse gauge condition gives:

- For $\mu = 0$, the Lorentz gauge condition:

$$\partial_t \frac{\Phi}{c^3} + \partial_i \frac{A^i}{c^2} \,=\, 0 \quad or \quad A^\mu{}_{,\mu} = 0. \qquad (4.36)$$

- And for $\mu = i$, an equation which relates the electric field with the divergence of the tensor potential d_{ij}:

$$\partial_t \frac{A^i}{c} - \partial^i \Phi \,=\, E^i \,=\, \frac{2}{c^2}\, \partial_j d^{ij}. \qquad (4.37)$$

Equation (4.36) has the well-known form of the Lorentz gauge condition. From Eq. (4.37) we get an additional constraint for the tensor potential d_{ij}; it must be related to the electric field. In the last equation we can see the usefulness of the potentials d_{ij}: the problematic constraints for the electromagnetism are now constraints regarding the additional fields.

The field equations take the form:

- For $\mu = 0$ and $\nu = 0$:
$$\partial^2 \frac{\Phi}{c^2} = \frac{4\pi C}{c^2}\,\rho. \tag{4.38}$$

- For $\mu = 0$ and $\nu = i$:
$$\partial^2 A^i = \frac{4\pi C}{c}\,j^i. \tag{4.39}$$

- Finally for $\mu = i$ and $\nu = j$:
$$\partial^2 \left(-\frac{\Phi}{c^2}\,\eta_{ij} + \frac{2}{c^4}\,d_{ij} \right) = \frac{4\pi C}{c^4}\,j^i\,u^j\,, \tag{4.40}$$

and if we use Eq. (4.38), Eq. (4.40) takes the form:
$$\partial^2 d^{ij} = 2\,\pi C \left(j^i u^j + \rho c^2 \eta^{ij} \right). \tag{4.41}$$

Equations (4.38-4.39) are the correct equations for the electromagnetic potential and along with Eq. (4.41) are the full system of the field equations.

Now we can define the electric and magnetic fields as usual:
$$\vec{E} \equiv -\,\partial_t \frac{\vec{A}}{c} - \vec{\nabla}\Phi\,, \qquad \vec{B} \equiv \vec{\nabla} \times \vec{A}. \tag{4.42}$$

Then one may easily see that Eqs. (4.38) and (4.39) along with Eqs. (4.42) reduce to the four Maxwell equations for the electromagnetic fields:
$$\vec{\nabla} \cdot \vec{E} = 4\pi C \rho\,, \qquad \vec{\nabla} \times \vec{B} = \partial_t \vec{E} + \frac{4\pi C}{c}\,\vec{j}\,, \tag{4.43}$$

and
$$\vec{\nabla} \times \vec{E} = \partial_t \vec{B}, \qquad \vec{\nabla} \cdot \vec{B} = 0. \tag{4.44}$$

The new fields can be defined as:
$$M_{ij} = -2\,\partial_t\,d_{ij} \tag{4.45}$$

and
$$K_{ijk} = -\left(\partial_j\,d_{ik} + \partial_k\,d_{ij} - \partial_i\,d_{jk} \right). \tag{4.46}$$

The above definitions will become clear when we give the expression for the Lorentz force.

The line element for this ansatz is:

$$ds^2 = c^2 \left(1 + \frac{2\Phi}{c^2} \right) dt^2 - \frac{2}{c} \left(\vec{A} \cdot d\vec{x} \right) dt + \left(\eta_{ij} + \frac{2}{c^4} d_{ij} \right) dx^i dx^j \,. \tag{4.47}$$

We can now give the equations of motion of a test particle moving in the background (4.47). Keeping all the corrections the Christoffel symbols for the line element (4.47) are:

$$\Gamma^i_{00} = \frac{1}{c^2} \left(\partial_0 A^i - \partial^i \Phi \right) , \tag{4.48}$$

$$\Gamma^i_{0j} = \frac{1}{2} \eta^{ik} \left(\frac{1}{c^2} F_{jk} + \frac{2}{c^5} \partial_t d_{jk} \right) , \tag{4.49}$$

$$\Gamma^i_{kj} = \frac{1}{c^4} \left(\partial_k d_j{}^i + \partial_j d_k{}^i - \partial^i d_{jk} \right) , \tag{4.50}$$

Substituting these into the geodesics equation (4.6), we obtain the equations of motion

$$\frac{m}{q} \ddot{x}^i = E^i + \frac{1}{c} F^{ij} u_j - \frac{1}{c^4} \left(2\, u^j\, \partial_t d^i{}_j + 2\, u^i u^k\, \partial_k d^i{}_j - u^j u^k\, \partial^i d_{jk} \right) . \tag{4.51}$$

If we use the definitions for the new fields, the above equation takes the form:

$$\frac{m}{q} \ddot{x}^i = E^i + \frac{1}{c} F^{ij} u_j + \frac{1}{c^4} M^{ij} u_j + \frac{1}{c^4} K^i{}_{jk} u^j u^k \tag{4.52}$$

As we can see, the above equation contains additional corrections; nevertheless, due to the presence of the $1/c^4$ factor, even at high velocities, it can take the form of the Lorentz force law:

$$\frac{m}{q} \ddot{x}^i = E^i + F^{ij} u_j , \tag{4.53}$$

or in a more familiar vector notation:

$$m\ddot{\vec{x}} = q \left(\vec{E} + \vec{u} \times \vec{B} \right) . \tag{4.54}$$

In this section we have showed that, while the use of the additional degrees of freedom of the theory has as a result the appearance of new fields, the contribution of these new fields in observable quantities is negligible. The only field that remains even at high velocities is the electromagnetic field. This means that the theory which we study here could be considered as a realistic theory of electromagnetism [40].

Chapter 5

Seeking for invariants

5.1 The theoretical framework

In this chapter we focus on the invariant quantities we can get from the General Relativity in the context of gravitoelectromagnetism. Actually, we are interested in invariants similar to the ones appearing in the true Electromagnetism i.e $(B^2 - E^2$ and $\vec{E} \cdot \vec{B})$. The first one comes from the inner product of the electromagnetic tensor

$$F^{\mu\nu} F_{\mu\nu} = 2 \left(B^2 - E^2 \right), \tag{5.1}$$

and appears in the expression of the Lagrangian of the electromagnetic field

$$\mathcal{L}_{ED} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu, \tag{5.2}$$

and also, in the expression of the stress-energy tensor of electromagnetism

$$T^{\mu\nu} = F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}. \tag{5.3}$$

The second one comes from the product of the electromagnetic tensor with its dual tensor

$$\tilde{F}^{\mu\nu} F_{\mu\nu} = -4 \, \vec{E} \cdot \vec{B}, \tag{5.4}$$

and is obviously a pseudoscalar since the dual tensor of the electromagnetic tensor is defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{5.5}$$

In the above equations, for simplicity, we have set $c = \varepsilon_0 = \mu_0 = 1$.

We already know that we can construct invariants similar to the electromagnetic ones from the Weyl tensor [25], but from a different approach of gravitoelectromagnetism. Here we try to get invariants similar to the electromagnetic ones from the "linearised" theory that we have used in the previous chapters.

We know that we can get many invariants from the General Relativity, such as the scalar curvature

$$R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}, \tag{5.6}$$

the Ricci scalar

$$R = g_{\mu\nu}R^{\mu\nu}, \tag{5.7}$$

or any other scalar that we can construct using the Riemann and Ricci tensors. The problem is that the above tensors contain second derivatives of the metric tensor. More specificaly, in the definition of the fields only first derivatives of the metric tensor are involved, and, as a result, we can not connect the fields with the gravitational scalars which we have mentioned above. Therefore, the only way to get invariants similar to the electromagnetic ones is to define a new gravitational tensor, which is going to contain only first derivatives of the metric tensor, and to construct scalars from this tensor.

In the weak field approximation, we can easily construct a tensor which contains only first derivatives of the metric tensor from the Einstein's tensor

$$G_{\mu\nu} = \frac{1}{2}\left(\tilde{h}^{\alpha}{}_{\mu,\nu\alpha} + \tilde{h}^{\alpha}{}_{\nu,\mu\alpha} - \partial^2\tilde{h}_{\mu\nu} - \eta_{\mu\nu}\,\tilde{h}^{\alpha\beta}{}_{,\alpha\beta}\right).$$

If we "pull" a derivative out, the Einstein's tensor takes the form

$$G_{\mu\nu} = \frac{1}{2}\partial^{\alpha}F_{\alpha\mu\nu}, \tag{5.8}$$

where the tensor $F_{\alpha\mu\nu}$ is defined as

$$F_{\alpha\mu\nu} = \partial_{\mu}\tilde{h}_{\alpha\nu} + \partial_{\nu}\tilde{h}_{\alpha\mu} - \partial_{\alpha}\tilde{h}_{\mu\nu} - \eta_{\mu\nu}\partial^{\beta}\tilde{h}_{\alpha\beta}, \tag{5.9}$$

and is obviously symmetric in the last two indices. Using the $F_{\alpha\mu\nu}$ tensor the field equations (1.17) take a form very close to the form of the field equations of electrodynamics $(\partial_{\mu}F^{\mu\nu} = \frac{4\pi C}{c^2}\,J^{\nu})$

$$\partial^{\alpha}F_{\alpha\mu\nu} = 2kT_{\mu\nu}. \tag{5.10}$$

We, thus, conclude that we can use the tensor $F_{\alpha\mu\nu}$ in order to construct a number of scalars in which the expressions of the fields should appear. However, having in mind the analogy between the field equations of the two theories, we are expecting that the scalar

$$F = F^{\alpha\mu\nu}F_{\alpha\mu\nu} \tag{5.11}$$

will contain a term similar to the $B^2 - E^2$ term of the true electromagnetism. In the following sections we are going to calculate the scalar F for the two ansatzes of chapter 3 and to see that for both ansatzes we can get the requested term.

Our work on the derivation of the second electromagnetic scalar $(\vec{E} \cdot \vec{B})$ is still in progress [40]. We have not yet managed to construct a scalar which contains a term similar to the $\vec{E} \cdot \vec{B}$ term of the true electromagnetism, but we are expecting that a scalar of the form

$$\tilde{F}^{\alpha\mu\nu} F_{\alpha\mu\nu} \tag{5.12}$$

will contain such a term. In the above expression we have defined the dual tensor of $F_{\alpha\mu\nu}$ as

$$\tilde{F}^{\alpha\mu}{}_\nu = \varepsilon^{\alpha\mu\lambda\rho} F_{\lambda\rho\nu}. \tag{5.13}$$

The dual tensor is antisymmetric in the first two indices and, if the analogy with the true electromagnetism is still persistent, we are expecting that the expression

$$\partial^\alpha \tilde{F}_{\alpha\mu\nu} = 0 \tag{5.14}$$

will contain the other two "Maxwell-like" equations; the ones that we have used to get straight from the definitions of the fields and not from a fundamental equation.

5.2 The scalar F for the general form of the traditional ansatz

In this section we calculate the F scalar for the general form of the traditional ansatz

$$\tilde{h}_{00} = \frac{4\Phi}{c^2}, \qquad \tilde{h}_{0i} = \frac{2A_i}{c^2}, \qquad \tilde{h}_{ij} = \frac{2\lambda}{c^4}\, \eta_{ij} \,+\, \frac{2}{c^4}\, d_{ij}\,.$$

The components of the $F_{\alpha\mu\nu}$ tensor for the above ansatz are

$$F_{000} = -\frac{2}{c^2}\partial^i A_i, \tag{5.15}$$

$$F_{00i} = \frac{4}{c^2}\partial_i\Phi, \tag{5.16}$$

$$F_{i00} = \frac{4}{c^2}E_i - \frac{2}{c^4}\left(\partial_i\lambda + \partial^j d_{ij}\right), \tag{5.17}$$

$$F_{ij0} = -\frac{2}{c^2}F_{ij} + \frac{2}{c^5}\partial_t\left(\lambda\eta_{ij} + d_{ij}\right), \tag{5.18}$$

$$F_{0ij} = -\frac{2}{c^2} \left[\eta_{ij} \left(2\partial_0\Phi + \partial^k A_k \right) - \left(\partial_i A_j + \partial_j A_i \right) \right] - \frac{2}{c^5}\partial_t \left(\lambda\eta_{ij} + d_{ij} \right), \quad (5.19)$$

$$F_{ijk} = -\frac{2}{c^4} \left[\partial_i \left(\lambda\eta_{jk} + d_{jk} \right) - \partial_j \left(\lambda\eta_{ik} + d_{ik} \right) - \partial_k \left(\lambda\eta_{ij} + d_{ij} \right) \right],$$

$$-\frac{2}{c^3}\eta_{jk} \left[\partial_t A_i + \frac{1}{c}\partial^l \left(\lambda\eta_{il} + d_{il} \right) \right], \quad (5.20)$$

where the fields are defined as in Eq. (3.10)

$$E_i \equiv \frac{1}{c}\partial_t \left(\frac{A_i}{2} \right) - \partial_i\Phi, \qquad F_{ij} \equiv \partial_i A_j - \partial_j A_i = -2\varepsilon_{ijk}B_k. \quad (5.21)$$

Then, the scalar $F = F^{\alpha\mu\nu}F_{\alpha\mu\nu}$ given in Eq. (5.11), can be written as

$$F = F^{000}F_{000} + 2F^{00i}F_{00i} + F^{0ij}F_{0ij} + F^{i00}F_{i00} + 2F^{ij0}F_{ij0} + F^{ijk}F_{ijk}. \quad (5.22)$$

The dominant terms of order $\mathcal{O}(c^{-4})$ are

$$c^4 F = -16E^2 + 64B^2 + 4\left(\partial^i A_i \right)^2 + 32\left(\partial_i\Phi \right)^2 + 4\left(\partial_i A_j + \partial_j A_i \right)^2$$

$$-16\left(\partial^i A_i \right)\left(2\partial_0\Phi + \partial^j A_j \right) + 12\left(2\partial_0\Phi + \partial^j A_j \right)^2. \quad (5.23)$$

We do not give the full expression of the scalar F here because the terms which we want are appearing only in the lowest order. However, it is easy for someone to calculate the full expression for the scalar F from Eq. (5.22) by using the components of the $F_{\alpha\mu\nu}$ tensor (5.15-5.20). We should also mention that in the above calculations we do not use any gauge condition. If we use the transverse gauge condition $\partial^\nu \tilde{h}_{\mu\nu} = 0$, the last term in the definition of the $F_{\alpha\mu\nu}$, Eq. (5.9), disappears and as a result the last two terms of the scalar F, Eq. (5.23), also disappear.

Concluding, we have managed to construct a scalar combination in terms of gravitational quantities in which a term like the electromagnetic one $(B^2 - E^2)$ appears. However, this term is followed by additional terms due to the tensorial structure of the theory.

5.3 The scalar F for the general form of the alternative ansatz

In this section we calculate the F scalar for the general form of the alternative ansatz

$$\tilde{h}_{00} = \frac{\Phi}{c^2} - \frac{3\lambda}{c^4}, \qquad \tilde{h}_{0i} = \frac{A_i}{c}, \qquad \tilde{h}_{ij} = -\left(\frac{\Phi}{c^2} + \frac{\lambda}{c^4} \right)\eta_{ij} + \frac{2}{c^4}d_{ij}.$$

5.3. The scalar F for the general form of the alternative ansatz

The components of the $F_{\alpha\mu\nu}$ tensor for the above ansatz are

$$F_{000} = -\frac{1}{c^2}\partial^i A_i, \tag{5.24}$$

$$F_{00i} = \frac{1}{c^2}\partial_i\Phi - \frac{3}{c^4}\partial_i\lambda, \tag{5.25}$$

$$F_{i00} = \frac{1}{c^2}\left(4E_i + \partial_i\Phi\right) + \frac{2}{c^4}\partial^j\left(\lambda\eta_{ij} + d_{ij}\right), \tag{5.26}$$

$$F_{ij0} = -\frac{1}{c^2}F_{ij} - \frac{1}{c^3}\partial_t\left[\left(\Phi + \frac{1}{c^2}\lambda\right)\eta_{ij} - \frac{2}{c^2}d_{ij}\right], \tag{5.27}$$

$$F_{0ij} = \frac{1}{c^2}\left[(\partial_i A_j + \partial_j A_i) - \eta_{ij}\left(\partial_0\left(\Phi - \frac{3}{c^2}\lambda\right) + \partial^l A_l\right)\right]$$
$$+ \frac{1}{c^3}\partial_t\left[\left(\Phi + \frac{1}{c^2}\lambda\right)\eta_{ij} - \frac{2}{c^2}d_{ij}\right], \tag{5.28}$$

$$F_{ijk} = -\frac{1}{c^2}\left(\eta_{ij}\partial_k\Phi + \eta_{ik}\partial_j\Phi - 2\eta_{jk}\partial_i\Phi\right) + \frac{1}{c^4}\eta_{jk}\partial^l\left(\lambda\eta_{il} - 2d_{il}\right)$$
$$- \frac{1}{c^4}\left[\partial_k\left(\lambda\eta_{ij} - 2d_{ij}\right) + \partial_j\left(\lambda\eta_{ik} - 2d_{ik}\right) - \partial_i\left(\lambda\eta_{jk} - 2d_{jk}\right)\right]$$
$$- \frac{1}{c^3}\eta_{jk}\partial_t A_i, \tag{5.29}$$

where the fields are defined as in Eq. (3.50)

$$E_i \equiv \frac{1}{4}\left(\frac{1}{c}\partial_t A_i - \partial_i\Phi\right), \qquad F_{ij} \equiv \partial_i A_j - \partial_j A_i = -4\varepsilon_{ijk}B_k. \tag{5.30}$$

However, as we have already seen in section (3.2), the above fields are recognized as the gravitoelectromagnetic fields, only if the scalar potential λ vanishes.

Finally the lowest order $\mathcal{O}(c^{-4})$ terms of the scalar F are

$$c^4 F = -16E^2 + 32B^2 + \left(\partial^i A_i\right)^2 + 3\left((\partial_i\Phi)^2 + 8E_i\partial^i\Phi\right.$$
$$+ \left[(\partial_i Aj + \partial_j A_i) - \eta_{ij}\left(\partial_0\left(\Phi - \frac{3}{c^2}\lambda\right) + \partial^l A_l\right)\right]^2$$
$$+ \left(\eta_{ij}\partial_k\Phi + \eta_{ik}\partial_j\Phi - 2\eta_{jk}\partial_i\Phi\right)^2. \tag{5.31}$$

Here, as in the previous section, we give the dominant terms of the scalar F without the imposition of a gauge condition. If we use the transverse gauge

condition, the term $\left[\partial_0\left(\Phi - \frac{3}{c^2}\lambda\right) + \partial^l A_l\right]$ in Eq. (5.31) vanishes. In this ansatz the scalar F also contains extra terms. In the case of gravitoelectromagnetism the extra terms do not create any problem; they just show us where the analogy with the true electromagnetism stops. However, in chapter 4 we have used the above ansatz in order to give an alternative description of the true electromagnetism. If we have alternative descriptions of a theory, they should all give us the same results. The presence of the additional terms in the scalar F –for the analysis of chapter 4– is then problematic.

Chapter 6

Conclusions

There is undoubtedly a striking similarity between the gravitational and electromagnetic forces at classical level. These similarities, which persist even in the context of the General Theory of Relativity, led to the development of gravitoelectromagnetism. In our days the majority of the gravitoelectromagnetism approaches are split in two large categories. In the first one the analogy between gravitation and electromagnetism is shown from the linearised form of Einstein's field equations, in the weak field approximation, around a rotating massive body. In the second, the analogy is shown from the decomposition of the Weyl tensor into gravitoelectric and gravitomagnetic parts. The main purpose of this thesis was to fully review the first approach of gravitoelectromagnetism, to shed light to particular problematic aspects of the theory and, also, to propose solutions.

In chapter 1 we gave a historical review of gravitoelectromagnetism and also a brief introduction to General Relativity and to the linear approach. We considered this as particularly important in order to give the context in which our work was based and to provide the reader with all the information needed to understand better our project.

In chapter 2 we reviewed the linear approach of gravitoelectromagnetism. In the first section of the chapter, we started from the perturbed Einstein's equations in the linear approximation and, employing the so-called traditional ansatz for the metric perturbations, we showed the analogies between gravity and electromagnetism. However, these analogies stand only if the gravitoelectromagnetic potentials are static; this along with the presence of additional terms in the equations of motion consist the two big problems of gravitoelectromagnetism. In the second section, in order to investigate further the problems of the traditional ansatz, we did again all the calculations without the imposition of a gauge condition. We showed that the problem of the time dependence of the vector potential is a problem of the ansatz which can not be solved by

67

choosing an alternative gauge condition. The above, along with the presence of extra terms in the Lorentz force law, led us to the search for new ansatzes for the metric perturbations. In the third section, we proposed an alternative ansatz; this ansatz was chosen carefully, in order not to give extra terms in the equations of motion. However, in this ansatz we encountered a serious problem, that of the vanishing of the source. Of course this ansatz could be used in a vacuum problem, but in gravitoelectromagnetism the presence of the source is necessary.

In chapter 3 we tried to resolve the problems appearing in gravitoelectromagnetism. In chapter 2 we showed that in order to solve these problems we must try new ansatzes for the metric perturbations. However, from the equation $\partial_j \tilde{h}^{ij} = -\partial_t A^i$, we understood that the use of the –suppressed– additional degrees of freedom of the gravitational field could solve some of our problems. In this chapter we used the extra degrees of freedom in both ansatzes. In the first section we tried the most general form of the traditional ansatz. We showed that the condition, which in the previous chapter was leading to a static vector potential, in this one, became a condition which connects the vector potential with the additional degrees of freedom of the gravitational field. However, extra terms in the equation of motion were still present. In the second section we did the same with the alternative ansatz, the one introduced in chapter 2. This section was split in two subsections. In the first one we considered the most general ansatz. Although this ansatz did not lead to a valid model of gravitoelectromagnetism, we included it in our analysis because it could be used in other aspects of gravitational problems. In the second we studied a special case of the above ansatz, which turned out to be the best one in the context of gravitoelectromagnetism because it gives both the correct form of the field equations and the equations of motion.

In chapter 4 we covered a topic we initiated in a previous work of ours. In this chapter we attempted to describe true electromagnetism using the formalism of General Relativity. This attempt was based on the results of our work regarding gravitoelectromagnetism. In the first section we presented a short introduction of electromagnetism and we gave an alternative expression for the Lorentz force law using geometrical quantities. In the second section we reviewed our previous work. In the third section we presented some improvements to our previous work based on the results of chapter 2.

Finally, in chapter 5 we covered the search for invariant quantities similar to the electromagnetic ones, in the linear approach of gravitoelectromagnetism. In the first section of the chapter we presented the theoretical framework and we defined a gravitoelectromagnetic tensor similar to the electromagnetic field tensor $F_{\mu\nu}$. In the following two sections of this chapter we presented some

calculations of scalars constructed by the gravitoelectromagnetic tensor we have defined. However, this work is still in progress and we only present some of our results. We hope that this work would contribute to a better understanding of gravitoelectromagnetism and, maybe, lead to a lagrangian formulation.

Bibliography

[1] G. Cantor, Phys. Ed. **26** (1991) 289.

[2] J. C. Maxwell, Phil. Trans. Roy. Soc. London **155** (1865) 492.

[3] G. Holzmuller, Z. Math. Phys. **15** (1870) 69.

[4] F. Tisserand, Compt. Rend. **75** (1872) 760; **110** (1890) 313.

[5] O. Heaviside, The Electrician **31** (1893) 281; **31** (1893) 259.

[6] H. A. Lorentz, Proc. Acad. Amsterdam **2** (1900) 559.

[7] A. Einstein, Annalen Phys. **49** (1916) 769 [Annalen Phys. **14** (2005) 517].

[8] H. Thirring, Phys. Z. **19** (1918) 33; **22** (1921) 29;
J. Lense and H. Thirring, Phys. Z. **19** (1918) 156;
B. Mashhoon, F.W. Hehl and D.S. Theiss, Gen. Rel. Grav. **16** (1984) 711.

[9] H. Thiring, Phys. Z. **19** (1918) 204.

[10] A. Matte, Canad. J. Math. **5** (1953) 1.

[11] L. Bel, Compt. Rend. **247** (1958) 1094.

[12] R. Debever, Bull. Soc. Math. Bel. **10** (1958) 112.

[13] R. L. Forward, Proc. of the IRE **49** (1961) 892.

[14] R. L. Forward, Am. J. Phys. **31** (1963) 166.

[15] J. W. C. Scott, Canad. J. Phys. **44** (1966) 1147; **44** (1966) 1639;
J. W. C. Scott, Nature **213** (1967) 767.

[16] H. G. L. Coster and J. R. Shepanski, J. Phys. A **2** (1969) 22.

[17] S. L. Schwebel, Int. J. Theor. Phys. **3** (1970) 315; **4** (1971) 87.

[18] F. Spieweck, Astron. & Astrophys. **12** (1971) 287.

[19] V. Majernik, Astrophys. Space. Sci. **14** (1971) 265; **15** (1972) 375;
C. C. Leiby, Jr., Astrophys. Space. Sci. **17** (1972) 368.

[20] L. Brillouin, *"Relativity Reexamined"*, Academic Press, New York (1970).

[21] D. P. Whitmire, Lett. Nuovo Cimento **7** (1973) 305.

[22] O. Gron and Knutsen, Eur. J. Phys. **10** (1989) 200;
P. Dolan and C. W. Kim, Proc. Royal Soc. A **447** (1994) 557;
R. Jantzen, P. Carini and D. Bini, Ann. Phys. **215** (1992) 1;
W. B. Bonnor, Class. Quantum Frav. **12** (1995) 1483;
I. Ciufolini and J. A. Wheeler, *"Gravitation and Inertia"*, Princeton Univercity Press, Princeton (1995);
M. A. G. Bonilla and J. M. M. Senovilla, Phys. Rev. Lett. **78** (1997) 783;
R Maartens and B. A. Basset, Class. Quantum Grav. **15** (1998) 705;
S. Clark and R. W. Tucker, Class. Quantum Grav. **17** (2000) 4125;
J. M. M. Senovilla, Mod. Phys. Lett. A **15** (2000) 159; Class. Quantum Grav. **17** (2000) 2799;
L. Iorio and D. M. Luchessi, Class. Quantum Grav. **20** (2003) 2477.

[23] E. G. Harris, Am. J. Phys. **59(5)** (1991) 421.

[24] B. Mashhoon, Phys. Lett. A **173** (1993) 347; Class. Quant. Grav. **17** (2000) 2399; Int. J. Mod. Phys. D **14** (2005) 2025; gr-qc/0311030.
S. Kopeikin and B. Mashhoon, Phys. Rev. D **65** (2002) 064025 [gr-qc/0110101].

[25] L. Filipe Costa and C. A. R. Herdeiro, Phys. Rev. D **78** (2008) 024021 [gr-qc/0612140]; arXiv:0912.2146 [gr-qc].

[26] L. F. O. Costa and J. Natario, arXiv:1207.0465 [gr-qc].

[27] J. Ramos, *Gravitoelectromagnetism (GEM): A group Theoretical Approach"*, PHD Thesis, Drexel University (2006) http://hdl.handle.net/1860/1123.

[28] J. Ramos, M. de Montigny and F. C. Khanna, Gen. Rel. Grav. **42** (2010) 2403.

[29] M. L. Ruggiero and A. Tartaglia, Nuovo Cimento B **117** (2002) 743. [arXiv:gr-qc/0207065]

[30] A. Einstein, Annalen Phys. **17** (1905) 891 [Annalen Phys. **14** (2005) 194].

[31] L.D. Landau and E.M. Lifshitz, *"The Classical Theory of Fields"*, Butterworth-Heinemann Editions, 2003.

[32] B. Mashhoon, F. Gronwald and H. I. M. Lichtenegger, Lect. Notes Phys. **562** (2001) 83 [gr-qc/9912027].

[33] A. Bakopoulos and P. Kanti, Gen. Rel. Grav. **46** (2014) 1742 [arXiv:1405.0265 [gr-qc]]

[34] B. Mashhoon, Int. J. Mod. Phys. D **14** (2005) 2025 [astro-ph/0510002].

[35] P. Teyssandier Phys. Rev. D **16** (1977) 946;
P. Teyssandier Phys. Rev. D **18** (1978) 1037.

[36] B. Mashhoon, Class. Quant. Grav. **25** (2008) 085014
[arXiv:0802.1356 [gr-qc]].

[37] V. B. Braginsky, C. M. Caves and K. S. Thorne, Phys. Rev. D **15** (1977) 2047.

[38] J. F. Pascual-Sanchez, Nuovo Cim. B **115** (2000) 725 [gr-qc/0010075].

[39] S. M. Carroll, *"Space Time and Geometry, An Introduction to General Relativity"*, Addison Weslay, San Francisco (2004).

[40] A. Bakopoulos and P. Kanti, work in progress.