The Effect of the Global Market on Investor's Behaviour

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Abstract

An investor, with a Quadratic Utility Function, can invest in three markets: the Home, the Abroad, and the Global Market. The Global-Market portfolio can be expressed either as a weighted average of the risky assets of the Global Market, or as a weighted average of the Home- and the Abroad-Market portfolios. Defining the 2×2 variance-covariance matrix of the returns on the Home- and Abroad-Market portfolios, enables us to take into account the sign of the correlation of the returns on the two market-portfolios. We examine under what conditions the investor decides to invest in each market. The investor's decision depends on the slope of the Capital Market Line (Capital Market Effect), as well as on the measurement of risk undertaken in the equilibrium (Optimization Effect).

Introduction

In this study we examine an investor with a Quadratic Utility Function, who has the opportunity to invest in three markets: the Home, the Abroad, and the Global Market. We assume that there are N_H assets in the Home Market, N_A assets in the Abroad Market and $N_G = N_H + N_A$ in the Global Market. Also, we assume that there is a risk-free asset, which is the same in all three markets. Under these assumptions, the Global-Market portfolio can be alternatively expressed using the Home-Market and the Abroad-Market portfolios.

The structure of this study is as follows:

In ***Chapter 1***, we prove that whichever the type of the investor's utility function, it can be approximated by the quadratic one. According to that we present the investor's indifference curve.

In ***Chapter 2***, we analyse thoroughly the typical portfolio in any Market with n risky assets and one risk-free asset. We find the Capital Market Line and its slope and we express the typical portfolio using the Portfolio Separation Formula.

In ***Chapter 3***, we use the results from ***Chapters 1*** and ***2*** to calculate the equilibrium of an investor with quadratic utility function and a Market with n risky assets and a risk-free asset.

In ***Chapters 4*** and ***5***, we describe (i) the typical portfolios in the Home and the Abroad Market and (ii) the Home-Market and Abroad-Market portfolios, that follow the properties of the typical portfolio and the market portfolio assumed in ***Chapter 2***, respectively. We, also, find the investor's equilibrium in each Market, and we rewrite the two typical portfolios using the Portfolio Separation Formula.

In ***Chapter 6***, we express the typical portfolio in the Global Market in accordance with the typical portfolio assumed in ***Chapter 2***. Moreover, we find the investor's equilibrium and, using the Portfolio Separation Formula, we rewrite the typical portfolio in the Global Market as a function of the risk-free asset and the typical portfolios in the Home and the Abroad Market. Also, we express the Global-Market portfolio using two alternative methods: On the one hand, we write its equation as in ***Chapter 2***; and on the other hand, we represent it as a weighted average of only two risky assets, the Home-Market and the Abroad-Market portfolios.

In ***Chapter 7***, we make comparisons on where the investor will choose to invest, based on the Capital Market Effect and the Optimization Effect.

Chapter 1

The investor

1.1 Investor's Utility

Without loss of generality, we assume a one-period investment horizon. Let K_t be the initial capital invested at time t in a portfolio P, and K_{t+1} be the final capital at the end of the investment period. Then, the return, r_P, on portfolio P satisfies the equation:

$$x = 1 + r_P \tag{1.1}$$

Using the above assumptions, the investor's final capital can be expressed as follows:

$$\begin{aligned} K_{t+1} &= (1 + r_P)\, K_t && (1.2)\\ \implies (1 + r_P) &= \frac{K_{t+1}}{K_t} \\ \implies x &= \frac{K_{t+1}}{K_t} && (1.3) \end{aligned}$$

where $K_t > 0$ & $K_{t+1} \geq 0$. Therefore,

$$x = \frac{K_{t+1}}{K_t} \geq 0 \tag{1.4}$$

In that case, we assume r_P, x to be normally distributed random variables, and:

$$r_P \sim \mathrm{N}\left(\mu_P, \sigma_P^2\right) \tag{1.5}$$

We also know that:

$$\begin{aligned} x &= 1 + r_P \sim \mathrm{N}\left(\mu_x, \sigma_x^2\right) && (1.6)\\ \text{where: } \mu_x &= 1 + \mu_P && (1.7)\\ \sigma_x^2 &= \sigma_P^2 && (1.8) \end{aligned}$$

Equation (*1.6*) implies the following:

$$\mathrm{E}\left(x-\mu_x\right) = 0 \tag{1.9a}$$

$$\mathrm{E}\left[(x-\mu_x)^2\right] = \sigma_x^2 \tag{1.9b}$$

$$\mathrm{E}\left[(x-\mu_x)^3\right] = 0 \tag{1.9c}$$

$$\mathrm{E}\left[(x-\mu_x)^4\right] = 3\sigma_x^4 \tag{1.9d}$$

$$\mathrm{E}\left[(x-\mu_x)^n\right] = 0, \forall n > 4 \tag{1.9e}$$

1.2 Investor's Expected Utility

We assume that the investor's expected utility function is

$$v(.,.) = \mathrm{E}\left\{u(x)\right\} = \mathrm{E}\left\{u(1+r_P)\right\} \tag{1.10}$$

Let

$$u'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{u(x)\right\} \tag{1.11a}$$

$$u''(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{u(x)\right\} \tag{1.11b}$$

$$u'''(x) = \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left\{u(x)\right\} \tag{1.11c}$$

$$u''''(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{u(x)\right\} \tag{1.11d}$$

be the first-, second-, third- and forth-order derivatives of $u(.)$ with respect to x. Then, the investor's utility function can be approximated by the following Taylor Series expansion around μ_x:

$$u(x) = u(\mu_x) + u'(\mu_x)(x-\mu_x) + \frac{1}{2!}u''(\mu_x)(x-\mu_x)^2 + \frac{1}{3!}u'''(\mu_x)(x-\mu_x)^3 + \frac{1}{4!}u''''(\mu_x)(x-\mu_x)^4 + \ldots \tag{1.12}$$

By using *Equation* (*1.11*), the investor's expected utility can be written as:

$$\begin{aligned}
v(.,.) &= \mathrm{E}\left\{u(x)\right\} \\
&= u(\mu_x) + u'(\mu_x)\,\mathrm{E}(x-\mu_x) + \frac{1}{2}u''(\mu_x)\,\mathrm{E}\left\{(x-\mu_x)^2\right\} \\
&\quad + \frac{1}{6}u'''(\mu_x)\,\mathrm{E}\left\{(x-\mu_x)^3\right\} + \frac{1}{24}u''''(\mu_x)\,\mathrm{E}\left\{(x-\mu_x)^4\right\} + \ldots \\
&= u(\mu_x) + u'(\mu_x)\cdot 0 + \frac{1}{2}u''(\mu_x)\sigma_x^2 + \frac{1}{6}u'''(\mu_x)\cdot 0 + \frac{1}{24}u''''(\mu_x)3\sigma_x^4 + \ldots \\
&= v(\mu_x, \sigma_x) \\
&= v(1+\mu_P, \sigma_P^2) \\
\Rightarrow v(.,.) &= v(\mu_P, \sigma_P^2)
\end{aligned} \tag{1.13}$$

The investor's expected utility function, $v(\mu_P, \sigma_P^2)$, has the following properties:

$$\frac{\partial}{\partial \mu_P}\left\{v(\mu_P, \sigma_P^2)\right\} > 0 \tag{1.14a}$$

$$\frac{\partial}{\partial \sigma_P}\left\{v(\mu_P, \sigma_P^2)\right\} < 0 \tag{1.14b}$$

Equations (*1.14a*), (*1.14b*) imply that the marginal rate of substitution of μ_P for σ_P, is:

$$\frac{\mathrm{d}\mu_P}{\mathrm{d}\sigma_P} = -\frac{\dfrac{\partial v(\mu_P, \sigma_P^2)}{\partial \sigma_P}}{\dfrac{\partial v(\mu_P, \sigma_P^2)}{\partial \mu_P}} > 0 \tag{1.14c}$$

1.3 Investor's Quadratic Utility Function

The Moment Generating Function of $\mathrm{N}(\mu, \sigma^2)$

The Moment Generating Function of the normal distribution is [1, p.71]:

$$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \tag{1.15}$$

The first-, second-, third-, forth-order derivatives of $M(t)$ are:

$$M'(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t) \tag{1.16a}$$

$$M''(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)^2 + e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\sigma^2) \tag{1.16b}$$

$$M'''(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)^3 + e^{\mu t + \frac{1}{2}\sigma^2 t^2}2(\mu + \sigma^2 t)(\sigma^2) + e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)(\sigma^2) \tag{1.16c}$$

$$\begin{aligned} M''''(t) &= e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)^4 + e^{\mu t + \frac{1}{2}\sigma^2 t^2}3(\mu + \sigma^2 t)^2(\sigma^2) + e^{\mu t + \frac{1}{2}\sigma^2 t^2}2(\mu + \sigma^2 t)^2(\sigma^2) \\ &\quad + e^{\mu t + \frac{1}{2}\sigma^2 t^2}2(\sigma^2)^2 + e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)^2(\sigma^2) + e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\sigma^2)^2 \end{aligned} \tag{1.16d}$$

Raw Moments of $\mathrm{N}(\mu, \sigma^2)$

By using *Equations* (*1.16*) with $t = 0$, we can calculate the first four raw momonets of $\mathrm{N}(\mu, \sigma^2)$ [1, p.67]:

$$\mu_1' = M'(0) = \mu \tag{1.17a}$$

$$\mu_2' = M''(0) = \mu^2 + \sigma^2 \tag{1.17b}$$

$$\mu_3' = M'''(0) = \mu^3 + 2\mu\sigma^2 + \mu\sigma^2 = \mu^3 + 3\mu\sigma^2 \tag{1.17c}$$

$$\mu_4' = M''''(0) = \mu^4 + 3\mu^2\sigma^2 + 2\mu^2\sigma^2 + 2\sigma^4 + \mu^2\sigma^2 + \sigma^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \tag{1.17d}$$

Equations (*1.17*) imply that for any random variable $r \sim \mathrm{N}(\mu, \sigma^2)$ we have:

$$\begin{aligned} \mathrm{E}(r) &= \mu'_1 && (1.18a) \\ \mathrm{E}(r^2) &= \mu'_2 && (1.18b) \\ \mathrm{E}(r^3) &= \mu'_3 && (1.18c) \\ \mathrm{E}(r^4) &= \mu'_4 && (1.18d) \\ &\vdots \end{aligned}$$

The Central Moments

For any random variable $r \sim \mathrm{N}(\mu, \sigma^2)$, the i^{th}-order central moment can be calculated as [1, p.64]

$$\mu_i = \mathrm{E}\left\{[r - \mathrm{E}(r)]^i\right\}, \tag{1.19}$$

which implies that the first four central moments are:

$$\begin{aligned} \mu_1 &= \mathrm{E}(r - \mu) = 0 \\ \mu_2 &= \mathrm{E}\left\{(r-\mu)^2\right\} = \sigma^2 = \mu'_2 - (\mu'_1)^2 \\ \mu_3 &= \mathrm{E}\left\{(r-\mu)^3\right\} = 0 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 \\ \mu_4 &= \mathrm{E}\left\{(r-\mu)^4\right\} = 3\sigma^3 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4 \end{aligned}$$

Note that for any random variable $r \sim \mathrm{N}(\mu, \sigma^2)$, the fifth-, sixth-, etc central moment are all equal to zero.

A quadratic approach of the investor's utility function

By using *Equation* (*1.11*), the investor's utility function can be approximated by the following Taylor expansion around 1:

$$\begin{aligned} u(x) &= u(1) + u'(1)(x-1) + \frac{1}{2!}u''(1)(x-1)^2 + \frac{1}{3!}u'''(1)(x-1)^3 + \frac{1}{4!}u''''(1)(x-1)^4 + \ldots \\ &= u(1) + u'(1)x - u'(1) + \frac{1}{2}u''(1)x^2 - \frac{1}{2}2u''x + \frac{1}{2}u''(1)x^2 + \ldots \\ &= \left[u(1) - u'(1) + \frac{1}{2}\right] + [u'(1) - u''(1)]\,x + \frac{1}{2}u''(1)x^2 + \ldots \end{aligned}$$

We define the scalars:

$$\begin{aligned} \gamma_0 &= u(1) - u'(1) + \frac{1}{2}u''(1) \\ \gamma_1 &= u'(1) - u''(1) \\ \gamma_2 &= -u''(1) \end{aligned}$$

Then, following Markowitz [2, p.346] we conclude that the investor's utility function can be approximated by a quadratic utility of the form:

$$u(x) = \gamma_0 + \gamma_1 x - \frac{1}{2}\gamma_2 x^2, \tag{1.20}$$

where:

$$\gamma_1 > 0, \tag{1.21a}$$

$$\gamma_2 > 0, \tag{1.21b}$$

$$0 \leq x \leq \frac{\gamma_1}{\gamma_2} \tag{1.21c}$$

Since, by definition, $\mu_P = \mathrm{E}(r_P) > 0 \Rightarrow 1 + \mu_P > 1$

$$\begin{aligned} \Rightarrow \quad & \mu_x > 1 \\ \Rightarrow \quad & \frac{\gamma_1}{\gamma_2} > 1 \\ \Rightarrow \quad & \gamma_1 > \gamma_2 \end{aligned} \tag{1.21d}$$

The general form of the utility function can be rewritten as follows:

$$\begin{aligned} u(x) &= u(1 + r_P) \\ &= \gamma_0 + \gamma_1(1 + r_P) - \frac{1}{2}\gamma_2(1 + r_P)^2 \\ &= \gamma_0 + \gamma_1 + \gamma_1 r_P - \frac{1}{2}\gamma_2(1 + 2r_P + r_P^2) \\ &= \gamma_0 + \gamma_1 + \gamma_1 r_P - \frac{1}{2}\gamma_2 - \gamma_2 r_P - \frac{1}{2}\gamma_2 r_P^2 \\ &= \gamma_0 + \gamma_1 - \frac{1}{2}\gamma_2 + (\gamma_1 - \gamma_2) r_P - \frac{1}{2}\gamma_2 r_P^2 \\ \Rightarrow u(x) &= \alpha_0 + \alpha_1 r_P - \frac{1}{2}\alpha r_P^2 \end{aligned} \tag{1.22}$$

where:

$$\alpha_0 = \gamma_0 + \gamma_1 - \frac{1}{2}\gamma_2 \tag{1.23a}$$

$$\alpha_1 = \gamma_1 - \gamma_2 \tag{1.23b}$$

$$\alpha_2 = \gamma_2 \tag{1.23c}$$

Equation (*1.23*) implies that:

$$\begin{aligned} \alpha_1 &= \gamma_1 - \alpha_2 \\ \Rightarrow \gamma_1 &= \alpha_1 + \alpha_2 \end{aligned} \tag{1.24}$$

$$\begin{aligned} \alpha_0 &= \gamma_0 + \alpha_1 + \alpha_2 - \frac{1}{2}\alpha_2 \\ &= \gamma_0 + \alpha_1 + \frac{1}{2}\alpha_2 \\ \Rightarrow \gamma_0 &= \alpha_0 - \alpha_1 - \frac{1}{2}\alpha_2 \end{aligned} \tag{1.25}$$

Equations (*1.21b*), (*1.21d*), (*1.23b*), (*1.23c*) imply that:

$$\alpha_1 > 0 \tag{1.26a}$$

$$\alpha_2 > 0 \tag{1.26b}$$

Equations (*1.1*), (*1.21c*), (*1.23c*), (*1.24*) imply that:

$$\begin{gathered} 0 \le x \le \frac{\gamma_1}{\gamma_2} \\ \Rightarrow 0 \le 1 + r_P \le \frac{\gamma_1}{\gamma_2} \\ \Rightarrow -1 \le r_P \le \frac{\gamma_1}{\gamma_2} - 1 \\ \Rightarrow -1 \le r_P \le \frac{\alpha_1 + \alpha_2}{\alpha_2} - 1 \\ \Rightarrow -1 \le r_P \le \frac{\alpha_1}{\alpha_2} \end{gathered} \tag{1.27}$$

Equation (*1.22*) implies that:

$$u(x) = u(1 + r_P) = U(r_P) = \alpha_0 + \alpha_1 r_P - \frac{1}{2}\alpha_2 r_P^2 \tag{1.28}$$

where:

$$\alpha_1 > 0 \tag{1.29}$$

$$\alpha_2 > 0 \tag{1.30}$$

$$-1 \le r_P \le \frac{\alpha_1}{\alpha_2} \tag{1.31}$$

Since, in general, r_P may not be restricted only to values less than 1, it follows that $\frac{\alpha_1}{\alpha_2} > 1$.

$$\alpha_1 > \alpha_2 \tag{1.32}$$

$$\alpha_1 + \alpha_2 > 2\alpha_2 \tag{1.33}$$

$$\gamma_1 > 2\gamma_2 \tag{1.34}$$

1.4 Expected Quadratic Utility Function

Equations (*1.17*), (*1.18*), (*1.28*) imply that the expected quadratic utility function is:

$$
\begin{aligned}
v(.,.) &= \mathrm{E}\{u(x)\} \\
&= \mathrm{E}\{u(1+r_P)\} \\
&= \mathrm{E}\left\{\alpha_0 + \alpha_1 r_P - \frac{1}{2}\alpha_2 r_P^2\right\} \\
&= \alpha_0 + \alpha_1\,\mathrm{E}(r_P) - \frac{1}{2}\alpha_2\,\mathrm{E}(r_P^2) \\
&= \alpha_0 + \alpha_1 \mu_P - \frac{1}{2}\alpha_2(\sigma_P^2 + \mu_P^2) \\
\Rightarrow v(\mu_P, \sigma_P^2) &= \alpha_0 + \alpha_1 \mu_P - \frac{1}{2}\alpha_2(\sigma_P^2 + \mu_P^2) \qquad (1.35)
\end{aligned}
$$

1.5 Indifference Curve of an Investor with Quadratic Utility Function

Let $\mathcal{J}$ be a set of indices and let $j \in \mathcal{J}$ be an index of utility level such that $j = \mathrm{I}, \mathrm{II}, \ldots$

For any specific expected utility level, that is for any specific and constant value of the expected utility function, $\bar{\bar{v}}_{\mathrm{I}}$, we have:

$$
\bar{\bar{v}}_{\mathrm{I}} \equiv \bar{\bar{v}}_{\mathrm{I}}(\mu_P, \sigma_P^2) = \left[\alpha_0 + \alpha_1 \mu_P - \frac{1}{2}\alpha_2 \mu_P^2 - \frac{1}{2}\alpha_2 \sigma_P^2\right]_{\mathrm{I}} \qquad (1.36)
$$

Equation (*1.36*) can be solved either with respect to σ_P^2 or with respect to μ_P, as follows:

1. For any given value of μ_P, we solve *Equation* (*1.36*) with respect to σ_P^2 as follows:

$$
\begin{aligned}
\frac{1}{2}\alpha_2 \sigma_{P_{(\mathrm{I})}}^2 &= \alpha_0 + \alpha_1 \mu_P - \frac{1}{2}\alpha_2 \mu_P^2 - \bar{\bar{v}}_{\mathrm{I}} \\
\Rightarrow \sigma_{P_{(\mathrm{I})}}^2 &= \frac{2}{\alpha_2}\left[\alpha_0 + \alpha_1 \mu_P - \frac{1}{2}\alpha_2 \mu_P^2 - \bar{\bar{v}}_{\mathrm{I}}\right] \\
&= \frac{2\alpha_0 + 2\alpha_1 \mu_P - \alpha_2 \mu_P^2 - 2\bar{\bar{v}}_{\mathrm{I}}}{\alpha_2} \\
&= \frac{2\alpha_1 \mu_P - \alpha_2 \mu_P^2 + 2\,(\alpha_0 + \bar{\bar{v}}_{\mathrm{I}})}{\alpha_2} \\
\Rightarrow \sigma_{P_{(\mathrm{I})}} &= \left[\frac{2\alpha_1 \mu_P - \alpha_2 \mu_P^2 + 2\left(\alpha_0 + \bar{\bar{v}}_{(\mathrm{I})}\right)}{\alpha_2}\right]^{1/2} = \sigma_{P_{(\mathrm{I})}}(\mu_P, \bar{\bar{v}}_{\mathrm{I}}) \qquad (1.37)
\end{aligned}
$$

2. For any given value of σ_P^2, we solve *Equation* (*1.36*) with respect to μ_P as follows:

$$
\begin{aligned}
-\frac{1}{2}\alpha_2 \mu_{P_{\mathrm{I}}}^2 + \alpha_1 \mu_{P_{\mathrm{I}}} + \left(\alpha_0 - \frac{1}{2}\alpha_2 \sigma_P^2 - \bar{\bar{v}}_{\mathrm{I}}\right) &= 0 \\
\Rightarrow -\alpha_2 \mu_{P_{\mathrm{I}}}^2 + 2\alpha_1 \mu_{P_{\mathrm{I}}} - \left[\alpha_2 \sigma_P^2 + 2\,(\bar{\bar{v}}_{\mathrm{I}} - \alpha_0)\right] &= 0
\end{aligned}
$$

Where,

$$\begin{aligned}\Delta &= 4\alpha_1^2 - 4(-a_2)\left\{-\left[a_2\sigma_P^2 + 2\left(\bar{\bar{\upsilon}}_I - \alpha_0\right)\right]\right\}\\ &= 4\alpha_1^2 - 4\alpha_2^2\sigma_P^2 - 8\alpha_2\left(\bar{\bar{\upsilon}}_I - \alpha_0\right)\\ &= 4\alpha_1^2 - 4\alpha_2^2\sigma_P^2 - 8\alpha_2\bar{\bar{\upsilon}}_I + 8\alpha_0\alpha_2\\ \Rightarrow \Delta &= 4\left(\alpha_1^2 - \alpha_2^2\sigma_P^2 - 2\alpha_2\bar{\bar{\upsilon}}_I + 2\alpha_0\alpha_2\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\mu_{P_I} &= \frac{-2\alpha_1 \pm 2\sqrt{\alpha_1^2 - \alpha_2\sigma_P^2 - 2\alpha_2\bar{\bar{\upsilon}}_I + 2\alpha_0\alpha_2}}{-2\alpha_2}\\ \Rightarrow \mu_{P_I} &= \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - \alpha_2\sigma_P^2 - 2\alpha_2\bar{\bar{\upsilon}}_I + 2\alpha_0\alpha_2}}{-\alpha_2} = \mu_{P_I}(\sigma_P^2, \bar{\bar{\upsilon}}_I) \qquad (1.38)\end{aligned}$$

1.6 The slope of the Indifference Curve of a Quadratic Utility Function

By differentiating *Equation* (*1.37*) we can calculate the slope of the i_{th}-indifference curve of the expected quadratic utility function as follows:

$$\begin{aligned}\frac{d\sigma_{P_{(I)}}}{d\mu_P} &= \frac{d}{d\mu_P}\left[\sigma_{P_{(I)}}^2\right]^{1/2}\\ &= \frac{1}{2}\left[\sigma_{P_{(I)}}^2\right]^{-1/2}\frac{d}{d\mu_P}\left[\sigma_{P_{(I)}}^2\right]\\ &= \frac{1}{2}\left[\sigma_{P_{(I)}}^2\right]^{-1/2}\frac{1}{\alpha_2}\left[2\alpha_1 - 2\alpha_2\mu_P\right]\\ &= \left[\sigma_{P_{(I)}}^2\right]^{-1/2}\frac{1}{\alpha_2}\left[\alpha_1 - \alpha_2\mu_P\right]\\ \Rightarrow \frac{d\sigma_{P_{(I)}}}{d\mu_P} &= \frac{1}{\sigma_{P_{(I)}}}\frac{\alpha_1 - \alpha_2\mu_P}{\alpha_2} \qquad (1.39)\end{aligned}$$

Chapter 2

The Typical Portfolio

Basic Statistical Properties

Let M be a market with n risky assets and a risk-free asset. Also, let r_i and r_f be the return on the i^{th} risky asset and on the risk-free asset, respectively. In general, r_i is a random variable with the following statistical properties:

$$\begin{aligned} \mathrm{E}(r_i) &= \mu_i, \forall i = 1, 2, \ldots && (2.1) \\ \mathrm{var}(r_i) &= \mathrm{E}(r_i^2) = \sigma_i^2, \forall i = 1, 2, \ldots, n && (2.2) \\ \mathrm{cov}(r_i, r_j) &= \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j, \forall i, j = 1, 2, \ldots, n \text{ with } i \neq j && (2.3) \\ \rho_{ij} &= \mathrm{corr}(r_i, r_j), \text{ with } -1 \leq \rho_{ij} \leq 1 && (2.4) \end{aligned}$$

Moreover, r_f can be thought of as a degenerate random variable such that:

$$\begin{aligned} \mathrm{E}(r_f) &= r_f && (2.5) \\ \mathrm{var}(r_f) &= \sigma_f^2 = 0 && (2.6) \\ \mathrm{cov}(r_i, r_f) &= \sigma_{if} = 0, \forall i = 1, 2, \ldots, n && (2.7) \end{aligned}$$

2.1 The Typical Portfolio

The typical portfolio P can be written as:

$$P = w_0 f + w_1 \mathrm{a}_1 + w_2 \mathrm{a}_2 + \ldots + w_n \mathrm{a}_n, \quad (2.8)$$

where f is the risk-free asset, and $\mathrm{a}_1, \ldots, \mathrm{a_n}$ are risky asset. We assume that all the initial capital is invested, that is:

$$w_0 + w_1 + w_2 + \ldots + w_n = 1 \quad (2.9)$$

Moreover, the portfolio can be divided into two parts, the **risk-free** portfolio P_F and the **risky** portfolio P_R. Therefore, the portfolio can be expressed as follows:

$$P = P_F + P_R \quad (2.10)$$

Where:

$$P_F = w_0 f \quad (2.10a)$$

$$P_R = w_1 a_1 + w_2 a_2 + ... + w_n a_n \quad (2.10b)$$

The return on portfolio P is:

$$r_P = w_0 r_f + w_1 r_1 + w_2 r_2 + ... + w_n r_n \quad (2.11)$$

$$\Rightarrow r_P = r_{P_F} + r_{P_R} \quad (2.12)$$

where,

$$r_{P_F} = w_0 r_f \quad (2.12a)$$

$$r_{P_R} = w_1 r_1 + w_2 r_2 + ... + w_n r_n \quad (2.12b)$$

Equations (2.1), (2.3), (2.10) imply that:

$$\begin{aligned} \mathrm{E}(r_P) = \mu_P &= \mathrm{E}(w_0 r_f + w_1 r_1 + w_2 r_2 + ... + w_n r_n) \\ &= w_0\,\mathrm{E}(r_f) + w_1\,\mathrm{E}(r_1) + w_2\,\mathrm{E}(r_2) + ... + w_n\,\mathrm{E}(r_n) \\ &= w_0 r_f + w_1 \mu_1 + w_2 \mu_2 + ... + w_n \mu_n \\ \Longrightarrow \mathrm{E}(r_P) &= \mu_{P_f} + \mu_{P_R}, \end{aligned} \quad (2.13)$$

where,

$$\begin{aligned} \mu_{P_F} &= \mathrm{E}(r_{P_F}) \\ &= \mathrm{E}(w_0 r_f) \\ &= w_0\,\mathrm{E}(r_f) \\ \Longrightarrow \mu_{P_F} &= w_0 r_f, \end{aligned} \quad (2.14)$$

and,

$$\begin{aligned} \mu_{P_R} &= \mathrm{E}(r_{P_R}) \\ &= \mathrm{E}(w_1 r_1 + w_2 r_2 + ... + w_n r_n) \\ &= w_1\,\mathrm{E}(r_1) + w_2\,\mathrm{E}(r_2) + ... + w_n\,\mathrm{E}(r_n) \\ \Longrightarrow \mu_{P_R} &= w_1 \mu_1 + w_2 \mu_2 + ... + w_n \mu_n \end{aligned} \quad (2.15)$$

Using *Equations* (2.2), (2.4), (2.6), (2.10), we find:

$$
\begin{aligned}
\sigma_P^2 &= \operatorname{var}(r_P) \\
&= \operatorname{var}(w_0 r_f + w_1 r_1 + w_2 r_2 + \ldots + w_n r_n) \\
&= w_0^2 \operatorname{var}(r_f) + \sum_{i=1}^{N} w_i^2 \operatorname{var}(r_i) + 2\sum_{i=1}^{N} w_0 w_1 \operatorname{cov}(r_f, r_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \operatorname{cov}(r_i, r_j) \\
&= \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
\Longrightarrow \sigma_P^2 &= \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j, \qquad (2.16)
\end{aligned}
$$

where,

$$
\begin{aligned}
\sigma_{P_R}^2 &= \operatorname{var}(r_{P_R}) \\
&= \operatorname{var}(w_1 r_1 + w_2 r_2 + \ldots + w_n r_n) \\
&= \operatorname{var}(r_f) + \sum_{i=1}^{n} w_i^2 \operatorname{var}(r_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \operatorname{cov}(r_i, r_j) \\
&= \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
\Longrightarrow \sigma_{P_R}^2 &= \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j, \qquad (2.17)
\end{aligned}
$$

and,

$$
\sigma_{p_F}^2 = \operatorname{var}(r_{P_F}) = \operatorname{var}(w_0 r_f) = w_0^2 \operatorname{var}(r_f) = 0.
$$

Equation (2.9) implies that:

$$
w_0 = 1 - w_1 - w_2 - \ldots - w_n \qquad (2.18)
$$

Using *Equations* (2.11), (2.18), we can write r_P as follows:

$$
\begin{aligned}
r_P &= (1 - w_1 - w_2 - \ldots - w_n) r_f + w_1 r_1 + w_2 r_2 + \ldots + w_n r_n \\
&= r_f - w_1 r_f - w_2 r_f - \ldots - w_n r_f + w_1 r_1 + w_2 r_2 + \ldots + w_n r_n \\
\Longrightarrow r_P &= r_f + w_1 (r_1 - r_f) + w_2 (r_2 - r_f) + \ldots + w_n (r_n - r_f) \qquad (2.19)
\end{aligned}
$$

Using *Equations* (2.1), (2.19):

$$\begin{aligned} \mu_P &= \mathrm{E}(r_P) \\ \Longrightarrow \mu_P &= r_f + w_1(\mu_1 - r_f) + w_2(\mu_2 - r_f) + ... + w_n(\mu_n - r_f) = \mathrm{E}(r_P) \end{aligned} \tag{2.20}$$

2.2 Vector and matrix notation

We define the $n \times 1$ vectors $\boldsymbol{w}, \boldsymbol{r}, \mathbf{1}$ and $\bar{\boldsymbol{r}}$ as follows:

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \tag{2.21a}$$

$$\boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \tag{2.21b}$$

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{2.21c}$$

$$\bar{\boldsymbol{r}} = \begin{bmatrix} r_1 - r_f \\ r_2 - r_f \\ \vdots \\ r_n - r_f \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} r_f = \boldsymbol{r} - \mathbf{1} r_f \tag{2.21d}$$

Using *Equations* (2.11b), (2.21a), (2.21b):

$$\begin{aligned} r_{P_R} &= w_1 r_1 + w_2 r_2 + ... + w_n r_n \\ &= \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \\ \Longrightarrow r_{P_R} &= \boldsymbol{w}' \boldsymbol{r} \end{aligned} \tag{2.22}$$

Then, using *Equations* (2.19), (2.21a), (2.21b) we can write:

$$\begin{aligned}
r_P &= r_f + w_1(r_1 - r_f) + w_2(r_2 - r_f) + \dots + w_n(r_n - r_f) \\
&= r_f + \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} r_1 - r_f \\ r_2 - r_f \\ \vdots \\ r_n - r_f \end{bmatrix} \\
\Longrightarrow r_P &= r_f + \boldsymbol{w}'\bar{\boldsymbol{r}}
\end{aligned} \tag{2.23}$$

We define the $n \times n$ positive definite matrix $\boldsymbol{\Sigma}$ as follows:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \dots & \sigma_n^2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{12} & \sigma_{22} & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \dots & \sigma_{nn} \end{bmatrix} \tag{2.24}$$

where $\sigma_{ii} = \sigma_i^2$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}'$. Since $\det(\boldsymbol{\Sigma}) \neq 0$, the inverse matrix $\boldsymbol{\Sigma}^{-1}$ exists and can be written as:

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \sigma^{11} & \sigma^{12} & \dots & \sigma^{1n} \\ \sigma^{12} & \sigma^{22} & \dots & \sigma^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^{1n} & \sigma^{2n} & \dots & \sigma^{nn} \end{bmatrix} \Longrightarrow \left(\boldsymbol{\Sigma}^{-1}\right)' = \boldsymbol{\Sigma}^{-1} \tag{2.25}$$

Using *Equation* (2.17), (2.21*a*), (2.24), we can express $\sigma^2_{P_R}$ as a function of $\boldsymbol{w}$ and $\boldsymbol{\Sigma}$ as follows:

$$\begin{aligned}
\sigma^2_{P_R} &= \sigma^2_P = \sum_{i=1}^{n} w_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
&= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + ... + w_n^2\sigma_n^2 \\
&\quad +2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + ... + 2w_1w_n\sigma_{1n} + ... + 2w_{n-1}w_n\sigma_{n-1,n} \\
&= w_1\left(w_1\sigma_1^2 + w_2\sigma_{12} + ... + w_n\sigma_{1n}\right) + w_2\left(w_1\sigma_{12} + w_2\sigma_2^2 + ... + w_n\sigma_{2n}\right) + ... \\
&\quad +w_n\left(w_1\sigma_{1n} + w_2\sigma_{2n} + ... + w_n\sigma_n^2\right) \\
&= \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} \sigma_1^2 w_1 + \sigma_{12}w_2 + ... + \sigma_{1n}w_n \\ \sigma_{12}w_1 + \sigma_2^2 w_2 + ... + \sigma_{2n}w_n \\ \vdots \\ \sigma_{1n}w_1 + \sigma_{2n}w_2 + ... + \sigma_n^2 w_n \end{bmatrix} \\
&= \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \dots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\
\implies \sigma^2_{P_R} &= \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \qquad (2.26)
\end{aligned}$$

We define the scalars b_0, b_1, b_2 and b as follows:

$$\begin{aligned}
b_0 &= \mu'\boldsymbol{\Sigma}^{-1}\mu & (2.27) \\
b_1 &= \mu'\boldsymbol{\Sigma}^{-1}\mathbf{1} = \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mu & (2.28) \\
b_2 &= \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} & (2.29) \\
b &= \bar{\mu}'\boldsymbol{\Sigma}^{-1}\bar{\mu}, & (2.30)
\end{aligned}$$

where *Equation* (2.21*d*) implies that $\bar{\mu} = \mathrm{E}(\bar{r}) = \mathrm{E}(r - \mathbf{1}r_f) = \mu - \mathbf{1}r_f$.

Using *Equations* (2.27), (2.28), (2.29), (2.30) we can write:

$$\begin{aligned}
b &= \bar{\mu}'\boldsymbol{\Sigma}^{-1}\bar{\mu} \\
&= (\mu - \mathbf{1}r_f)'\boldsymbol{\Sigma}^{-1}(\mu - \mathbf{1}r_f) \\
&= (\mu' - \mathbf{1}'r_f)\boldsymbol{\Sigma}^{-1}(\mu - \mathbf{1}r_f) \\
&= (\mu'\boldsymbol{\Sigma}^{-1}\mu) - (\mu'\boldsymbol{\Sigma}^{-1}\mathbf{1})r_f - (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mu)r_f + (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})r_f^2 \\
&= (\mu'\boldsymbol{\Sigma}^{-1}\mu) - 2(\mu'\boldsymbol{\Sigma}^{-1}\mathbf{1})r_f + (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})r_f^2 \\
\implies b &= b_0 - 2b_1r_f + b_2r_f^2 \qquad (2.31)
\end{aligned}$$

Since $\boldsymbol{\Sigma}, \boldsymbol{\Sigma}^{-1} \overset{d}{>} 0$ and $\mu \neq 0$, $\mathbf{1} \neq 0 \implies$

$$b_0 > 0 \quad (2.32a)$$

$$b_2 > 0 \quad (2.32b)$$

$$b > 0 \quad (2.32c)$$

2.3 Capital Market Line (Efficient Frontier)

We know that the typical portfolio can be expressed as: $P = w_0 f + w_1 \mathrm{a}_1 + w_2 \mathrm{a}_2 + ... + w_n \mathrm{a}_n$

Then, following the assumption that the investor invests all his capital, *Equations* (*2.9*), (*2.21c*) imply that:

$$\begin{aligned} w_0 + w_1 + w_2 + ... + w_n &= 1 \implies \\ w_0 + \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} &= 1 \implies \\ w_0 + \boldsymbol{w}'\mathbf{1} &= 1 \end{aligned} \quad (2.33)$$

From *Equation* (*2.23*), we know that the return on the portfolio P is: $r_P = r_f + \boldsymbol{w}'\bar{\boldsymbol{r}}$

Using the above equation and *Equations* (*2.20*), (*2.21a*), we can express the expected returns on the portfolio P as:

$$\begin{aligned} \mathrm{E}(r_P) = \mu_P &= r_f + w_1(\mu_1 - r_f) + w_2(\mu_2 - r_f) + ... + w_n(\mu_n - r_f) \\ &= r_f + \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} \mu_1 - r_f \\ \mu_2 - r_f \\ \vdots \\ \mu_n - r_f \end{bmatrix} \\ &= r_f + \boldsymbol{w}'(\mu - \mathbf{1} r_f) \\ \implies \mathrm{E}(r_P) = \mu_P &= r_f + \boldsymbol{w}'\bar{\mu} \end{aligned} \quad (2.34)$$

Equation (*2.16*), (*2.17*), (*2.26*) imply that:

$$\sigma_P^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \implies \mathrm{var}(r_P) = \sigma_P^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \quad (2.35)$$

The Problem

In order to find the Capital Market Line, we need to minimize the variance of the typical portfolio subject to the linear restriction of *Equation* (2.34).

$$\min_{w} \sigma_P^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$
$$\text{s.t.} \;\; \mu_P = r_f + \boldsymbol{w}'\bar{\mu}$$

The Lagrange function is

$$\mathcal{L} = \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + \lambda\left(\mu_P - r_f - \boldsymbol{w}'\bar{\mu}\right) \tag{2.36}$$

The first-order conditions imply that:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} &= 0 \implies \frac{1}{2}2\boldsymbol{\Sigma}\boldsymbol{w} - \lambda\bar{\mu} = 0 \implies \boldsymbol{\Sigma}\boldsymbol{w} = \lambda\bar{\mu} \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \implies \mu_P - r_f - \boldsymbol{w}'\bar{\mu} = 0 \implies \mu_P = r_f + \boldsymbol{w}'\bar{\mu} \end{aligned} \tag{2.37}$$

From *Equation* (2.37) we find:

$$\boldsymbol{w} = \lambda\boldsymbol{\Sigma}^{-1}\bar{\mu} \implies \boldsymbol{w}' = \lambda\bar{\mu}'\boldsymbol{\Sigma}^{-1} \tag{2.38}$$

Using *Equations* (2.30), (2.35), (2.37), (2.38), (2.39):

$$\begin{aligned} \mu_P &= r_f + \lambda\bar{\mu}'\boldsymbol{\Sigma}^{-1}\bar{\mu} \implies \\ \mu_P - r_f &= \lambda\left(\bar{\mu}'\boldsymbol{\Sigma}^{-1}\bar{\mu}\right) \implies \\ \mu_P - r_f &= \lambda b \implies \\ \lambda &= \frac{\mu_P - r_f}{b} = \frac{\bar{\mu}_P}{b} \end{aligned} \tag{2.39}$$

where $\bar{\mu}_P = \mu_P - r_f$.

Using *Equations* (2.35), (2.37), (2.38), (2.39)

$$\begin{aligned} \sigma_P^2 &= \boldsymbol{w}'\lambda\bar{\mu} \\ &= \lambda\left(\boldsymbol{w}'\bar{\mu}\right) \\ &= \frac{\mu_P - r_f}{b}(\mu_P - r_f) \\ &= \frac{(\mu_P - r_f)^2}{b} \implies \\ \sigma_P &= \frac{\mu_P - r_f}{\sqrt{b}} \implies \\ \sigma_P\sqrt{b} &= \mu_P - r_f \end{aligned} \tag{2.40}$$

Therefore, the Capital Market Line (CML) is given by the following equation:

$$\mu_P = r_f + \sigma_P \sqrt{b}, \tag{2.41}$$

and its slope is given by:

$$\sqrt{b} = \frac{\mu_P - r_f}{\sigma_P}. \tag{2.42}$$

To be certain that the above solution is the minimum, we calculate the second-order condition. Since

$$\frac{\partial^2 \mathcal{L}}{\partial w \partial w'} = \frac{\partial}{\partial w'}\left(\frac{\partial \mathcal{L}}{\partial w}\right) = \frac{\partial}{\partial w'}\left(\boldsymbol{\Sigma} w - \lambda \bar{\mu}\right) = \boldsymbol{\Sigma} \overset{d}{>} 0, \tag{2.43}$$

the weights in *Equation* (2.38) give the minimum σ_P^2.

Moreover, *Equation* (2.40) implies that:

$$\begin{aligned}
\mu_P - r_f &= \sigma_P \sqrt{b} \implies \\
\bar{\mu}_P &= \sigma_P \sqrt{b} \implies \\
\sqrt{b} &= \frac{\bar{\mu}_P}{\sigma_P} \implies \\
b &= \left(\frac{\bar{\mu}_P}{\sigma_P}\right)^2 \implies \\
b &= \bar{\mu}_P^2 \sigma_P^{-2}
\end{aligned} \tag{2.44}$$

Equation (2.44) is a very useful result, which enables us to compare Home-, Abroad- and Global-equilibrium portfolios, as we shall see in ***Chapter 7***.

2.4 The slope of the Capital Market Line

By differentiating *Equation* (2.42) we can calculate the slope of the CML:

$$\begin{aligned}
\frac{\mathrm{d}\sigma_P}{\mathrm{d}\mu_P} &= \frac{d}{\mathrm{d}\mu_P}\left(\sigma_P^2\right)^{1/2} \\
&= \frac{1}{2}\left(\sigma_P^2\right)^{-1/2} \frac{\mathrm{d}\sigma_P^2}{\mathrm{d}\mu_P} \\
&= \frac{1}{2}\left(\sigma_P^2\right)^{-1/2} \frac{\mathrm{d}}{\mathrm{d}\mu_P}\left[\frac{1}{b}\left(\mu_P - r_f\right)^2\right] \\
&= \frac{1}{2}\left(\sigma_P^2\right)^{-1/2} 2\frac{1}{b}\left(\mu_P - r_f\right)(-1) \\
\implies \frac{\mathrm{d}\sigma_P}{\mathrm{d}\mu_P} &= \frac{1}{\sigma_P}\frac{1}{b}\left(\mu_P - r_f\right)
\end{aligned} \tag{2.45}$$

2.5 The Market Portfolio (M)

If

$$w_0 = 0 \tag{2.46}$$

then the investor's constraint, from *Equation* (2.8), implies that:

$$\begin{aligned} w_1 + w_2 + ... + w_n &= 1 \implies \\ w_{1M} + w_{2M} + ... + w_{nM} &= 1, \end{aligned} \tag{2.47}$$

where:

$$\begin{aligned} w_{1M} &= w_1 \\ w_{2M} &= w_2 \\ &\vdots \\ w_{nM} &= w_n. \end{aligned}$$

Therefore, according to *Equation* (2.8), we can write the market portfolio M as follows:

$$M = w_{1M}\mathrm{a}_1 + w_{2M}\mathrm{a}_2 + ... + w_{nM}\mathrm{a}_n \tag{2.48}$$

We define the $n \times 1$ vector $\boldsymbol{w}_M$, as follows:

$$\boldsymbol{w}_M = \begin{bmatrix} w_{1M} \\ w_{2M} \\ \vdots \\ w_{nM} \end{bmatrix} \tag{2.49}$$

Then, using *Equations* (2.47), (2.49) we take:

$$\begin{aligned} \begin{bmatrix} w_{1M} & w_{2M} & \dots & w_{nM} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} &= 1 \implies \\ \boldsymbol{w}'_M \mathbf{1} &= 1 \end{aligned} \tag{2.50}$$

Using *Equations* (2.11), (2.21b), (2.46), (2.49), we can calculate the return on portfolio M as follows:

$$\begin{aligned} r_M &= w_{1M}r_1 + w_{2M}r_2 + ... + w_{nM}r_{nM} \\ &= \begin{bmatrix} w_{1M} & w_{2M} & \dots & w_{nM} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \\ \implies r_M &= \boldsymbol{w}'_M \boldsymbol{r} \end{aligned} \tag{2.51}$$

Alternatively, using *Equations* (2.11), (2.21d), (2.46), (2.49), r_M can be written as:

$$\begin{aligned} r_M &= r_f + w_{1M}(r_1 - r_f) + w_{2M}(r_2 - r_f) + ... + w_{nM}(r_n - r_f) \\ &= r_f + \begin{bmatrix} w_{1M} & w_{2M} & \dots & w_n \end{bmatrix} \begin{bmatrix} r_1 - r_f \\ r_2 - r_f \\ \vdots \\ r_n - r_f \end{bmatrix} \\ \implies r_M &= r_f + \boldsymbol{w}'_M \bar{\boldsymbol{r}} \end{aligned} \tag{2.52}$$

Using *Equations* (2.1), (2.51), we can calculate the expected return of the portfolio M as:

$$\begin{aligned} \mu_M &= \mathrm{E}(\boldsymbol{r}_M) \\ &= \mathrm{E}(\boldsymbol{w}'_M \boldsymbol{r}) \\ &= \boldsymbol{w}'_M \, \mathrm{E}(r) \\ \implies \mu_M &= \boldsymbol{w}'_M \boldsymbol{r} \end{aligned} \tag{2.53}$$

Alternatively, using *Equations* (2.1), (2.52), μ_M can be written as:

$$\begin{aligned} \mu_M &= \mathrm{E}(\boldsymbol{r}_M) \\ &= \mathrm{E}(r_f + \boldsymbol{w}'_M \bar{\boldsymbol{r}}) \\ &= \mathrm{E}(\boldsymbol{r}_f) + \boldsymbol{w}'_M \, \mathrm{E}(\bar{\boldsymbol{r}}) \\ \implies \mu_M &= r_f + \boldsymbol{w}'_M \bar{r} \end{aligned} \tag{2.54}$$

Equations (2.26), implies that:

$$\sigma_M^2 = \boldsymbol{w}'_M \boldsymbol{\Sigma} \boldsymbol{w}_M \tag{2.55}$$

Using *Equations* (2.38), (2.50) we find that:

$$\begin{aligned}
\boldsymbol{w}_M &= \lambda_M \boldsymbol{\Sigma}^{-1}\bar{\mu} \\
&= \frac{\lambda_M \boldsymbol{\Sigma}^{-1}\bar{\mu}}{1} \\
&= \frac{\lambda_M \boldsymbol{\Sigma}^{-1}\bar{\mu}}{\boldsymbol{w}_M' \mathbf{1}} \\
&= \frac{\lambda_M \boldsymbol{\Sigma}^{-1}\bar{\mu}}{\lambda_M \bar{\mu}' \boldsymbol{\Sigma}^{-1}\mathbf{1}} \\
\Rightarrow \boldsymbol{w}_M &= \frac{\boldsymbol{\Sigma}^{-1}\bar{\mu}}{\bar{\mu}' \boldsymbol{\Sigma}^{-1}\mathbf{1}} \qquad (2.56)
\end{aligned}$$

Also:

$$\boldsymbol{\Sigma}^{-1}\bar{\mu} = \boldsymbol{\Sigma}^{-1}\left(\mu - \mathbf{1}r_f\right) \qquad (2.57)$$

Then, *Equations* (2.27), (2.28), (2.29), (2.57) imply that:

$$\begin{aligned}
\bar{\mu}'\boldsymbol{\Sigma}^{-1}\mu &= \left(\mu' - \mathbf{1}r_f\right)\boldsymbol{\Sigma}^{-1}\mu = \mu'\boldsymbol{\Sigma}^{-1}\mu - \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mu\right)r_f = b_0 - b_1 r_f \qquad (2.58) \\
\bar{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} &= \left(\mu' - \mathbf{1}r_f\right)\boldsymbol{\Sigma}^{-1}\mathbf{1} = \mu'\boldsymbol{\Sigma}^{-1}\mathbf{1} - \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)r_f = b_1 - b_2 r_f \qquad (2.59)
\end{aligned}$$

Therefore, by using *Equations* (2.53), (2.56), (2.58), (2.59) we find that:

$$\begin{aligned}
\mu_M &= \frac{\bar{\mu}'\boldsymbol{\Sigma}^{-1}}{\bar{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mu \\
&= \frac{\bar{\mu}'\boldsymbol{\Sigma}^{-1}\mu}{\bar{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \\
\Longrightarrow \mu_M &= \frac{b_0 - b_1 r_f}{b_1 - b_2 r_f}, \qquad (2.60)
\end{aligned}$$

and by using *Equations* (2.30), (2.55), (2.56), (2.59) we find that:

$$\begin{aligned}
\sigma_M^2 &= \frac{\bar{\mu}'\boldsymbol{\Sigma}^{-1}}{\bar{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\boldsymbol{\Sigma}\frac{\boldsymbol{\Sigma}^{-1}\bar{\mu}}{\bar{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \\
&= \frac{\bar{\mu}'\boldsymbol{\Sigma}^{-1}\bar{\mu}}{(\bar{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2} \\
\Longrightarrow \sigma_M^2 &= \frac{b}{\left(b_1 - b_2 r_f\right)^2} \qquad (2.61) \\
\Longrightarrow \sigma_M &= \frac{\sqrt{b}}{b_1 - b_2 r_f} \qquad (2.62)
\end{aligned}$$

2.6 The Portfolio Separation Formula

Equations (2.39), (2.40) imply that:

$$\begin{aligned} \lambda_P &= \frac{\mu_P - r_f}{b} \\ &= \frac{\sigma_P \sqrt{b}}{b} \\ \Longrightarrow \lambda_P &= \frac{\sigma_P}{\sqrt{b}} \end{aligned} \tag{2.63}$$

Also, *Equations* (2.39), (2.40) imply that:

$$\begin{aligned} \lambda_M &= \frac{\mu_M - r_f}{b} \\ &= \frac{\sigma_M \sqrt{b}}{b} \\ \Longrightarrow \lambda_M &= \frac{\sigma_M}{\sqrt{b}} \end{aligned} \tag{2.64}$$

By dividing *Equation* (2.63) by *Equation* (2.64) we find:

$$\frac{\lambda_P}{\lambda_M} = \frac{\frac{\sigma_P}{\sqrt{b}}}{\frac{\sigma_M}{\sqrt{b}}} \Longrightarrow \frac{\lambda_P}{\lambda_M} = \frac{\sigma_P}{\sigma_M} = \phi \tag{2.65}$$

Using *Equations* (2.38), (2.50), (2.56) we can write $\boldsymbol{w}_P$ as follows:

$$\begin{aligned} \boldsymbol{w}_P &= \lambda_P \boldsymbol{\Sigma}^{-1} \bar{\mu} \\ &= \frac{\lambda_P \boldsymbol{\Sigma}^{-1} \bar{\mu}}{1} \\ &= \frac{\lambda_P \boldsymbol{\Sigma}^{-1} \bar{\mu}}{\boldsymbol{w}_M' \mathbf{1}} \\ &= \frac{\lambda_P \boldsymbol{\Sigma}^{-1} \bar{\mu}}{\lambda_M \bar{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{1}} \\ &= \frac{\lambda_P}{\lambda_M} \frac{\boldsymbol{\Sigma}^{-1} \bar{\mu}}{\bar{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{1}} \\ \Longrightarrow \boldsymbol{w}_P &= \frac{\lambda_P}{\lambda_M} \boldsymbol{w}_M, \end{aligned} \tag{2.66}$$

which implies that:

$$w_{1P} = \frac{\lambda_P}{\lambda_M} w_{1M} \tag{2.67a}$$
$$w_{2P} = \frac{\lambda_P}{\lambda_M} w_{2M} \tag{2.67b}$$
$$\vdots$$
$$w_{nP} = \frac{\lambda_P}{\lambda_M} w_{nM}. \tag{2.67c}$$

Since:

$$\begin{aligned} w_{0P} &= 1 - \boldsymbol{w}'_P \mathbf{1} \\ &= 1 - \frac{\lambda_P}{\lambda_M} \boldsymbol{w}'_M \mathbf{1} \\ \Longrightarrow w_{0P} &= 1 - \frac{\sigma_P}{\sigma_M}. \end{aligned} \tag{2.68}$$

Equations (2.8), (2.48), (2.65), (2.67), (2.68) imply that the typical portfolio P can be written as:

$$\begin{aligned} P &= w_{0P} f + w_{1P} \mathrm{a}_1 + w_{2P} \mathrm{a}_2 + ... + w_{NP} \mathrm{a}_\mathrm{N} \\ &= \left(1 - \frac{\sigma_P}{\sigma_M}\right) f + \frac{\lambda_P}{\lambda_M} w_{1M} \mathrm{a}_1 + \frac{\lambda_P}{\lambda_M} w_{2M} \mathrm{a}_2 + .. + \frac{\lambda_P}{\lambda_M} w_{NM} \mathrm{a}_\mathrm{N} \\ &= \left(1 - \frac{\sigma_P}{\sigma_M}\right) f + \frac{\lambda_P}{\lambda_M} \left(w_{1M} \mathrm{a}_1 + w_{2M} \mathrm{a}_2 + ... + w_{NM} \mathrm{a}_\mathrm{N}\right) \\ &= \left(1 - \frac{\sigma_P}{\sigma_M}\right) f + \frac{\sigma_P}{\sigma_M} \left(w_{1M} \mathrm{a}_1 + w_{2M} \mathrm{a}_2 + ... + w_{NM} \mathrm{a}_\mathrm{N}\right) \\ &= \left(1 - \frac{\sigma_P}{\sigma_M}\right) f + \frac{\sigma_P}{\sigma_M} M \\ \Rightarrow P &= \left(1 - \phi\right) f + \phi M, \end{aligned} \tag{2.69}$$

This means that any portfolio P is the weighted average of the risk-free portfolio f and the market portfolio M, with weights $(1-\phi)$ and ϕ, respectively, where $\phi = \frac{\sigma_P}{\sigma_M}$.

If $0 < \phi < 1$, the portfolio P is located on the CML between the risk-free portfolio f and the market portfolio M, and $0 = \sigma_f < \sigma_P < \sigma_M$. On the other hand, if $\phi > 1$ the portfolio P is located on the CML to the right of M, and $\sigma_P > \sigma_M$.

Equation (2.69) $\Longrightarrow$ $r_P = (1-\phi) r_f + \phi r_M$, which implies that:

$$\begin{aligned} \mu_P &= \mathrm{E}(\boldsymbol{r}_P) = \mathrm{E}\left[\left(1-\phi\right) r_f + \phi \boldsymbol{r}_M\right] = \left(1-\phi\right) \mathrm{E}(r_f) + \phi\, \mathrm{E}(\boldsymbol{r}_M) \Longrightarrow \\ \mu_P &= \left(1-\phi\right) r_f + \phi \mu_M \end{aligned} \tag{2.70}$$

It also implies that:

$$\begin{aligned} \sigma_P^2 &= \operatorname{var}(\boldsymbol{r}_P) = \operatorname{var}\left[\left(1-\phi\right) r_f + \phi \boldsymbol{r}_M\right] = \left(1-\phi\right)^2 \operatorname{var}(r_f) + \phi^2 \operatorname{var}(\boldsymbol{r}_M) + 2\phi\left(1-\phi\right)\operatorname{cov}(r_f, \boldsymbol{r}_M) \implies \\ \sigma_P^2 &= \phi^2 \sigma_M^2 \end{aligned} \tag{2.71}$$

Chapter 3

Investor's Equilibrium

In previous chapters we have assumed an investor with Quadratic Utility Function, and a market with n risky assets and one risk-free asset. Under those assumptions, we proved the following:

- The slope of the investor's indifference curve is:

$$\frac{\mathrm{d}\sigma_P}{\mathrm{d}\mu_P} = \frac{1}{\sigma_P}\frac{\alpha_1 - \alpha_2\mu_P}{\alpha_2}$$

- The slope of the CML is:

$$\frac{\mathrm{d}\sigma_P}{\mathrm{d}\mu_P} = \frac{1}{\sigma_P}\frac{1}{b}\left(\mu_P - r_f\right)$$

At the equilibrium point E, the slope of the investor's indifference curve is equal to the slope of the CML, therefore, at point E we have the equilibrium portfolio E, with μ_E and σ_E. Using the above equations:

$$\begin{aligned}
\frac{1}{\sigma_E}\frac{\alpha_1 - \alpha_2\mu_E}{\alpha_2} &= \frac{1}{\sigma_E}\frac{1}{b}\left(\mu_E - r_f\right) \\
\Rightarrow \alpha_2\left(\mu_E - r_f\right) &= b\left(\alpha_1 - \alpha_2\mu_E\right) \\
\Rightarrow \alpha_2\mu_E - \alpha_2 r_f &= \alpha_1 b - b\alpha_2\mu_E \\
\Rightarrow \mu_E\alpha_2\left(1+b\right) &= \alpha_1 b + \alpha_2 r_f \\
\Rightarrow \mu_E &= \frac{\alpha_1 b + \alpha_2 r_f}{\alpha_2\left(1+b\right)} \\
\Rightarrow \mu_E &= \frac{1}{1+b}\left(\frac{\alpha_1}{\alpha_2}b + r_f\right) \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \mu_E - r_f &= \frac{1}{1+b}\left(\frac{\alpha_1}{\alpha_2}b + r_f\right) - r_f \\
\Rightarrow \mu_E - r_f &= \frac{1}{1+b}\left[\frac{\alpha_1}{\alpha_2}b + r_f - \left(1+b\right)r_f\right] \\
\Rightarrow \mu_E - r_f &= \frac{1}{1+b}\left[\frac{\alpha_1}{\alpha_2}b + r_f - r_f - br_f\right] \\
\Rightarrow \mu_E - r_f &= \frac{b}{1+b}\left(\frac{\alpha_1}{\alpha_2} - r_f\right) \qquad (3.2)
\end{aligned}$$

Using *Equations* (2.40), (3.2):

$$
\begin{aligned}
\sigma_E &= \frac{1}{\sqrt{b}}\left(\mu_E - r_f\right) \\
&= \frac{1}{\sqrt{b}}\frac{b}{b+1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right) \\
\Rightarrow \sigma_E &= \frac{\sqrt{b}}{b+1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right) \qquad (3.3)
\end{aligned}
$$

Equation (1.37) implies that:

$$
\begin{aligned}
\bar{\bar{v}}_E &= \alpha_0 + \alpha_1\mu_E - \frac{1}{2}\alpha_2\mu_E^2 - \frac{1}{2}\alpha_2\sigma_E^2 \\
\Rightarrow \bar{\bar{v}}_E &= \alpha_0 + \alpha_1\mu_E - \frac{1}{2}\alpha_2\left(\mu_E^2 + \sigma_E^2\right) \qquad (3.4)
\end{aligned}
$$

3.1 Equilibrium Portfolio weights (w_E)

Using *Equations* (2.38), (2.39), (3.2):

$$
\begin{aligned}
\boldsymbol{w}_E = \begin{bmatrix} w_{1E} \\ w_{2E} \\ \vdots \\ w_{NE} \end{bmatrix} &= \lambda_E \boldsymbol{\Sigma}^{-1}\bar{\mu} \\
&= \frac{1}{b}\left(\mu_E - r_f\right)\boldsymbol{\Sigma}^{-1}\bar{\mu} \\
&= \frac{1}{b}\frac{b}{b+1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right)\boldsymbol{\Sigma}^{-1}\bar{\mu} \\
&= \frac{1}{b+1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right)\boldsymbol{\Sigma}^{-1}\bar{\mu} \qquad (3.5)
\end{aligned}
$$

Chapter 4

The Equilibrium Portfolio in the Home Market

In the previous chapters we analysed a general market with n assets and one risk-free asset and we solved the optimization problem of an investor with a Quadratic Utility Function. In this chapter we apply those results on the assumed **Home Market**.

The typical portfolio in the Home Market is:

$$P_H = w_{H0} f + w_{H1} \mathrm{a}_{H1} + w_{H2} \mathrm{a}_{H2} + \ldots + w_{HN_H} \mathrm{a}_{HN_H} \tag{4.1}$$

where f is the risk-free asset, a_{H1}, a_{H2}, ..., a_{HN_H} are the risky assets of the Home Market, w_{H0} is the weight of the risk-free asset and w_{H1}, w_{H2}, ..., w_{HN_H} are the weights of the risky assets. Moreover, the weights obey the following constraint:

$$w_{H0} + w_{H1} + w_{H2} + \ldots + w_{HN_H} = 1, \tag{4.2}$$

For the Home-Market portfolio M_H we write:

$$\begin{aligned}
w_{H0M} &= w_{H0} && (4.3a)\\
w_{H1M} &= w_{H1} && (4.3b)\\
w_{H2M} &= w_{H2} && (4.3c)\\
&\vdots \\
w_{HN_HM} &= w_{HN_H} && (4.3d)
\end{aligned}$$

Since, by definition, $w_{H0M} = w_{H0} = 0$, the above constraint in *Equation* (*4.2*) becomes:

$$w_{H1M} + w_{H2M} + \ldots + w_{HN_HM} = 1 \tag{4.4}$$

Therefore, the Home-Market Portfolio can be written as:

$$M_H = w_{H1M} \mathrm{a}_{H1} + w_{H2M} \mathrm{a}_{H2} + \ldots + w_{HN_HM} \mathrm{a}_{HN_H} \tag{4.5}$$

According to ***Chapter 1***, the investor's indifference curve is:

$$\bar{\bar{v}}_H \equiv \bar{\bar{v}}_H\left(\mu_{P_H}, \sigma^2_{P_H}\right) = \left[\alpha_0 + \alpha_1\mu_{P_H} - \frac{1}{2}\alpha_2\mu^2_{P_H} - \frac{1}{2}\alpha_2\sigma^2_{P_H}\right]_{\mathrm{I}} \tag{4.6}$$

And its slope is:

$$\frac{\mathrm{d}\sigma_{P_H}}{\mathrm{d}\mu_{P_H}} = \frac{1}{\sigma_{P_H}}\frac{\alpha_1 - \alpha_2\mu_{P_H}}{\alpha_2} \tag{4.7}$$

According to ***Chapter 2***, the CML is:

$$\mu_{P_H} = r_f + \sigma_{P_H}\sqrt{b_H} \tag{4.8}$$

And its slope is:

$$\frac{\mathrm{d}\sigma_{P_H}}{\mathrm{d}\mu_{P_H}} = \frac{1}{\sigma_{P_H}}\frac{1}{b_H}(\mu_{P_H} - r_f) \tag{4.9}$$

Equations (*4.7*), (*4.9*) imply that in the equilibrium:

$$\mu_{E_H} = \frac{1}{1+b_H}\left(\frac{\alpha_1}{\alpha_2}b_H + r_f\right) \tag{4.10}$$

$$\Rightarrow \mu_{E_H} - r_f = \frac{b_H}{1+b_H}\left(\frac{\alpha_1}{\alpha_2} - r_f\right), \tag{4.11}$$

and

$$\sigma_{E_H} = \frac{\sqrt{b_H}}{b_H+1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right) \tag{4.12}$$

Therefore, the equilibrium investor's expected utility is:

$$v_{E_H} = \alpha_0 + \alpha_1\mu_{E_H} - \frac{1}{2}\alpha_2\left(\mu^2_{E_H} + \sigma^2_{E_H}\right), \tag{4.13}$$

and the equilibrium vector of weights is:

$$\boldsymbol{w}_{E_H} = \frac{1}{b_H+1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right)\boldsymbol{\Sigma}_H^{-1}\bar{\mu}_H. \tag{4.14}$$

Moreover,

$$b_H = \bar{\mu}'_H\boldsymbol{\Sigma}_H^{-1}\bar{\mu}_H \tag{4.15a}$$
$$b_{H0} = \mu'_H\boldsymbol{\Sigma}_H^{-1}\mu_H \tag{4.15b}$$
$$b_{H1} = \bar{\mu}'_H\boldsymbol{\Sigma}_H^{-1}\mathbf{1} \tag{4.15c}$$
$$b_{H2} = \mathbf{1}'\boldsymbol{\Sigma}_H^{-1}\mathbf{1} \tag{4.15d}$$

and

$$b_H = b_{H0} - 2b_{H1}r_f + b_{H2}r_f^2 \tag{4.16}$$

Portfolio Separation

The Portfolio Separation Theorem of ***Chapter 2*** can also be applied in the case of the **Home Market**. More specifically, using *Equations* (*2.69*), (*2.70*), (*2.71*) we can write:

$$P_H = \left(1 - \phi_H\right) f + \phi_H M_H \tag{4.17}$$

$$\mu_{P_H} = \left(1 - \phi_H\right) r_f + \phi_H \mu_{M_H} = r_f + \phi_H \left(\mu_{M_H} - r_f\right)$$

$$\Rightarrow \bar{\mu}_{P_H} = \phi_H \left(\mu_{M_H} - r_f\right) = \phi_H \bar{\mu}_{M_H} \tag{4.18}$$

$$\sigma_{P_H}^2 = \phi_H^2 \sigma_{M_H}^2 \tag{4.19}$$

Chapter 5

The Equilibrium Portfolio in the Abroad Market

In the previous chapters we analysed a market with n risky assets and one risk-free asset and we solved the optimization problem of an investor with a Quadratic Utility Function. Here, we apply those results on the assumed **Abroad Market**.

The typical portfolio in the Abroad Market is:

$$P_A = w_{A0} f + w_{A1} \mathrm{a}_{A1} + w_{A2} \mathrm{a}_{A2} + \ldots + w_{AN_A} \mathrm{a}_{AN_A} \tag{5.1}$$

where f is the risk-free asset, a_{A1}, a_{A2}, ..., a_{AN_A} are the risky assets of the Abroad Market and w_{A0} is the weight of the risk-free asset and w_{A1}, w_{A2}, ..., w_{AN_A} are the weights of the risky assets. Moreover, the weights obey the following constraint:

$$w_{A0} + w_{A1} + w_{A2} + \ldots + w_{AN_A} = 1 \tag{5.2}$$

For the Abroad-Market portfolio M_A we can write:

$$w_{A0M} = w_{A0} \tag{5.3a}$$

$$w_{A1M} = w_{A1} \tag{5.3b}$$

$$w_{A2M} = w_{A2} \tag{5.3c}$$

$$\vdots$$

$$w_{AN_AM} = w_{AN_A} \tag{5.3d}$$

Since, by definition, $w_{A0M} = w_{A0} = 0$, the constraint in *Equation* (*5.2*) becomes:

$$w_{A1M} + w_{A2M} + \ldots + w_{AN_AM} = 1 \tag{5.4}$$

Therefore, the Abroad-Market portfolio can be written as:

$$M_A = w_{A1M} \mathrm{a}_{A1} + w_{A2M} \mathrm{a}_{A2} + \ldots + w_{AN_AM} \mathrm{a}_{AN_A} \tag{5.5}$$

According ***Chapter 1***, the investor's indifference curve is:

$$\bar{\bar{v}}_A \equiv \bar{\bar{v}}_A \left(\mu_{P_A}, \sigma^2_{P_A}\right) = \left[\alpha_0 + \alpha_1 \mu_{P_A} - \frac{1}{2}\alpha_2 \mu^2_{P_A} - \frac{1}{2}\alpha_2 \sigma^2_{P_A}\right]_{\mathrm{I}} \tag{5.6}$$

And its slope is:

$$\frac{d\sigma_{P_A}}{d\mu_{P_A}} = \frac{1}{\sigma_{P_A}} \frac{\alpha_1 - \alpha_2\mu_{P_A}}{\alpha_2} \tag{5.7}$$

According ***Chapter 2***, the CML is:

$$\mu_{P_A} = r_f + \sigma_{P_A}\sqrt{b_A} \tag{5.8}$$

And its slope is:

$$\frac{d\sigma_{P_A}}{d\mu_{P_A}} = \frac{1}{\sigma_{P_A}} \frac{\alpha_1 - \alpha_2\mu_{P_A}}{\alpha_A} \tag{5.9}$$

Equations (*5.7*), (*5.9*) imply that in the equilibrium:

$$\mu_{E_A} = \frac{1}{1+b_A}\left(\frac{\alpha_1}{\alpha_2}b_A + r_f\right) \tag{5.10}$$

$$\Rightarrow \mu_{E_A} - r_f = \frac{b_A}{1+b_A}\left(\frac{\alpha_1}{\alpha_2} - r_f\right), \tag{5.11}$$

and

$$\sigma_{E_A} = \frac{\sqrt{b_A}}{b_A+1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right) \tag{5.12}$$

Therefore, the equilibrium investor's expected utility is:

$$v_{E_A} = \alpha_0 + \alpha_1\mu_{E_A} - \frac{1}{2}\alpha_2\left(\mu_{E_A}^2 + \sigma_{E_A}^2\right), \tag{5.13}$$

and the equilibrium vector of weights is:

$$\boldsymbol{w}_{E_A} = \frac{1}{b_A+1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right)\boldsymbol{\Sigma}_A^{-1}\bar{\mu}_A. \tag{5.14}$$

Moreover,

$$b_A = \bar{\mu}_A'\boldsymbol{\Sigma}_A^{-1}\bar{\mu}_A \tag{5.15a}$$

$$b_{A0} = \mu_A'\boldsymbol{\Sigma}_A^{-1}\mu_A \tag{5.15b}$$

$$b_{A1} = \bar{\mu}_A'\boldsymbol{\Sigma}_A^{-1}\mathbf{1} \tag{5.15c}$$

$$b_{A2} = \mathbf{1}'\boldsymbol{\Sigma}_A^{-1}\mathbf{1} \tag{5.15d}$$

and

$$b_A = b_{A0} - 2b_{A1}r_f + b_{A2}r_f^2 \tag{5.16}$$

Portfolio Separation

The Portfolio Separation Theorem of ***Chapter 2*** can also be applied in the case of the **Abroad Market**. More specifically, using *Equations* (2.69), (2.70), (2.71) we can write:

$$P_A = \left(1 - \phi_A\right) f + \phi_A M_A \tag{5.17}$$

$$\mu_{P_A} = \left(1 - \phi_A\right) r_f + \phi_A \mu_{M_A} = r_f + \phi_A \left(\mu_{M_A} - r_f\right)$$

$$\Rightarrow \bar{\mu}_{P_A} = \phi_A \left(\mu_{M_A} - r_f\right) = \phi_A \bar{\mu}_{M_A} \tag{5.18}$$

$$\sigma^2_{P_A} = \phi^2_A \sigma^2_{M_A} \tag{5.19}$$

Chapter 6

The Equilibrium Portfolio in the Global Market

In the previous chapters we analysed a market with n risky assets and one risk-free asset and we solved the optimization problem of an investor with a Quadratic Utility Function. We can now apply those results on the assumed **Global Market**.

The typical portfolio in the Global Market is:

$$P_G = w_{G0}f + w_{G1}\mathrm{a}_{G1} + w_{G2}\mathrm{a}_{G2} + ... + w_{GN}\mathrm{a}_{GN} \tag{6.1}$$

where f is the risk-free asset, a_{G1}, a_{G2}, ..., a_{GN} are the risky assets of the Global Market, w_{G0} is the weight of the risk-free asset and w_{G1}, w_{G2}, ..., w_{GN} are the weights of the risky assets. Moreover, the weights obey the following constraint:

$$w_{G0} + w_{G1} + w_{G2} + ... + w_{GN} = 1 \tag{6.2}$$

For the Global-Market portfolio M_G we write:

$$w_{G0M} = w_{G0} \tag{6.3a}$$

$$w_{G1M} = w_{G1} \tag{6.3b}$$

$$w_{G2M} = w_{G2} \tag{6.3c}$$

$$\vdots$$

$$w_{GNM} = w_{GN} \tag{6.3d}$$

Since, by definition, $w_{G0M} = w_{G0} = 0$, the constraint in *Equation* (*6.2*) becomes:

$$w_{G1M} + w_{G2M} + ... + w_{GNM} = 1 \tag{6.4}$$

Therefore, the Global-Market portfolio can be written as:

$$M_G = w_{G1M}\mathrm{a}_{G1} + w_{G2M}\mathrm{a}_{G2} + ... + w_{GNM}\mathrm{a}_{GN} \tag{6.5}$$

According to ***Chapter 1***, the investor's indifference curve is:

$$\bar{\bar{v}}_G \equiv \bar{\bar{v}}_G\left(\mu_{P_G}, \sigma^2_{P_G}\right) = \left[\alpha_0 + \alpha_1 \mu_{P_G} - \frac{1}{2}\alpha_2 \mu^2_{P_G} - \frac{1}{2}\alpha_2 \sigma^2_{P_G}\right]_{\mathrm{I}} \tag{6.6}$$

And its slope is:

$$\frac{\mathrm{d}\sigma_{P_G}}{\mathrm{d}\mu_{P_G}} = \frac{1}{\sigma_{P_G}} \frac{\alpha_1 - \alpha_2 \mu_{P_G}}{\alpha_2} \tag{6.7}$$

As proved in ***Chapter 3***, the Global CML is:

$$\mu_{P_G} = r_f + \sigma_{P_G}\sqrt{b_G} \tag{6.8}$$

And its slope is:

$$\frac{\mathrm{d}\sigma_{P_G}}{\mathrm{d}\mu_{P_G}} = \frac{1}{\sigma_{P_G}} \frac{1}{b_G}\left(\mu_{P_G} - r_f\right) \tag{6.9}$$

Equations (6.7), (6.9) imply that in the equilibrium:

$$\mu_{E_G} = \frac{1}{1+b_G}\left(\frac{\alpha_1}{\alpha_2} b_G + r_f\right) \tag{6.10}$$

$$\Rightarrow \mu_{E_G} - r_f = \frac{b_G}{1+b_G}\left(\frac{\alpha_1}{\alpha_2} - r_f\right), \tag{6.11}$$

and

$$\sigma_{E_G} = \frac{\sqrt{b_G}}{b_G + 1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right) \tag{6.12}$$

Therefore, the equilibrium investor's expected utility is:

$$v_{E_G} = \alpha_0 + \alpha_1 \mu_{E_G} - \frac{1}{2}\alpha_2\left(\mu^2_{E_G} + \sigma^2_{E_G}\right), \tag{6.13}$$

and the equilibrium vector of weights is

$$\boldsymbol{w}_{E_G} = \frac{1}{b_G + 1}\left(\frac{\alpha_1}{\alpha_2} - r_f\right)\boldsymbol{\Sigma}_G^{-1}\bar{\mu}_G \tag{6.14}$$

Moreover,

$$b_G = \bar{\mu}_G' \boldsymbol{\Sigma}_G^{-1} \bar{\mu}_G \quad (6.15a)$$

$$b_{G0} = \mu_G' \boldsymbol{\Sigma}_G^{-1} \mu_G \quad (6.15b)$$

$$b_{G1} = \bar{\mu}_G' \boldsymbol{\Sigma}_G^{-1} \mathbf{1} \quad (6.15c)$$

$$b_{G2} = \mathbf{1}' \boldsymbol{\Sigma}_G^{-1} \mathbf{1} \quad (6.15d)$$

and

$$b_G = b_{G0} - 2b_{G1} r_f + b_{G2} r_f^2 \quad (6.16)$$

Portfolio Separation

The Portfolio Separation Theorem of ***Chapter 2*** can also be applied in the case of the **Global Market**. More specifically, using *Equations* (2.63), (2.64), (2.65), (2.66) we can write:

$$\lambda_{P_G} = \frac{\sigma_{P_G}}{\sqrt{b_G}} \quad (6.17)$$

$$\lambda_{M_G} = \frac{\sigma_{M_G}}{\sqrt{b_G}} \quad (6.18)$$

$$\phi_G = \frac{\sigma_{P_G}}{\sigma_{M_G}} = \frac{\lambda_{P_G}}{\lambda_{M_G}} \quad (6.19)$$

$$\boldsymbol{w}_{P_G} = \frac{\lambda_{P_G}}{\lambda_{M_G}} \boldsymbol{w}_{M_G} \quad (6.20)$$

Therefore:

$$\boldsymbol{w}_{HP_G} = \frac{\lambda_{P_G}}{\lambda_{M_G}} \boldsymbol{w}_{HM_G} \quad (6.21a)$$

$$\boldsymbol{w}_{AP_G} = \frac{\lambda_{P_G}}{\lambda_{M_G}} \boldsymbol{w}_{AM_G} \quad (6.21b)$$

Also, using *Equation* (2.68):

$$w_{0P_G} = 1 - \frac{\sigma_{P_G}}{\sigma_{M_G}} \quad (6.22)$$

The typical portfolio P_G on the Global CML can be expressed as follows:

$$P_G = \left(1 - \frac{\sigma_{P_G}}{\sigma_{M_G}}\right) f + \frac{\sigma_{P_G}}{\sigma_{M_G}} M_G$$

$$\Rightarrow P_G = \left(1 - \phi_G\right) f + \phi_G M_G \quad (6.23)$$

The return on the portfolio P_G is:

$$r_{P_G} = \left(1 - \phi_G\right) r_f + \phi_G r_{M_G}, \quad (6.24)$$

which implies that:

$$\begin{aligned} \mu_{P_G} &= \left(1-\phi_G\right) r_f + \phi_G \mu_{M_G} \\ &= r_f + \phi_G \left(\mu_{M_G} - r_f\right) \\ \Rightarrow \bar{\mu}_{P_G} &= \phi_G \bar{\mu}_{M_G}, \end{aligned} \tag{6.25}$$

and

$$\sigma^2_{P_G} = \phi^2_G \sigma^2_{M_G}. \tag{6.26}$$

Using *Equations* (2.44), (6.25), (6.26) we can write:

$$\begin{aligned} b_G &= \bar{\mu}^2_{P_G} \left(\sigma^2_{P_G}\right)^{-1} \\ &= \left(\phi_G \bar{\mu}_{M_G}\right)^2 \left(\phi^2_G \sigma^2_{M_G}\right)^{-1} \\ \Rightarrow b_G &= \bar{\mu}^2_{M_G} \sigma^{-2}_{P_G}. \end{aligned} \tag{6.27}$$

The Global-Market portfolio can also be expressed as weighted average of two risky assets only, namely the Home-Market and the Abroad-Market portfolios, i.e.

$$M_G = w_{GH} M_H + w_{GA} M_A \tag{6.28}$$

where M_H and M_A are the Home-Market and the Abroad-Market portfolios, respectively, and w_{GH} and w_{GA} are the corresponding weights of M_H and M_A in the Global Market; which implies that:

$$\mu_{M_G} = w_{GH} \mu_{M_H} + w_{GA} \mu_{M_A}, \tag{6.29}$$

and

$$\begin{aligned} \sigma^2_{M_G} &= \mathrm{var}(r_{M_G}) \\ &= \mathrm{var}(w_{GH} r_{M_H} + w_{GA} r_{M_A}) \\ &= w^2_{GH} \sigma^2_{M_H} + w^2_{M_{GA}} \sigma^2_{M_A} + 2 w_{GH} w_{GA} \sigma_{M_H M_A} \\ \Rightarrow \sigma^2_{M_{GH}} &= w^2_{GH} \sigma^2_{M_H} + w^2_{M_{GA}} \sigma^2_{M_A} + 2 w_{GH} w_{GA} \rho_{M_H M_A} \sigma_{M_H} \sigma_{M_A} \end{aligned} \tag{6.30}$$

We define the 2×1 vector of weights:

$$\boldsymbol{w}_* = \begin{bmatrix} w_{GH} \\ w_{GA} \end{bmatrix} \tag{6.31}$$

and the 2×2 variance-covariance matrix:

$$\boldsymbol{\Sigma}_* = \begin{bmatrix} \sigma^2_{M_H} & \sigma_{M_H M_A} \\ \sigma_{M_H M_A} & \sigma^2_{M_A} \end{bmatrix} = \begin{bmatrix} \sigma^2_{M_H} & \rho_{M_H M_A}\sigma_{M_H}\sigma_{M_A} \\ \rho_{M_H M_A}\sigma_{M_H}\sigma_{M_A} & \sigma^2_{M_A} \end{bmatrix} \tag{6.32}$$

Since $\boldsymbol{\Sigma}_*$ is positive definite, its inverse exists and can be calculated as follows:

$$\begin{aligned} \boldsymbol{\Sigma}_*^{-1} &= \frac{1}{\sigma^2_{M_H}\sigma^2_{M_A} - \rho^2_{M_H M_A}\sigma^2_{M_H}\sigma^2_{M_A}} \begin{bmatrix} \sigma^2_{M_A} & -\rho_{M_H M_A}\sigma_{M_H}\sigma_{M_A} \\ -\rho_{M_H M_A}\sigma_{M_H}\sigma_{M_A} & \sigma^2_{M_H} \end{bmatrix} \\ &= \frac{1}{(1-\rho^2_{M_H M_A})\sigma^2_{M_H}\sigma^2_{M_A}} \begin{bmatrix} \sigma^2_{M_A} & -\rho_{M_H M_A}\sigma_{M_H}\sigma_{M_A} \\ -\rho_{M_H M_A}\sigma_{M_H}\sigma_{M_A} & \sigma^2_{M_H} \end{bmatrix} \\ \Rightarrow \boldsymbol{\Sigma}_*^{-1} &= \frac{1}{(1-\rho^2_{M_H M_A})} \begin{bmatrix} \left(\sigma^2_{M_H}\right)^{-1} & -\rho_{M_H M_A}\left(\sigma_{M_H}\sigma_{M_A}\right)^{-1} \\ -\rho_{M_H M_A}\left(\sigma_{M_H}\sigma_{M_A}\right)^{-1} & \left(\sigma^2_{M_A}\right)^{-1} \end{bmatrix} \end{aligned} \tag{6.33}$$

Also, define the 2×1 vector:

$$\bar{\mu}_* = \mu_* - \mathbf{1}r_f, \tag{6.34}$$

where:

$$\mu_* = \begin{bmatrix} \mu_{M_H} \\ \mu_{M_A} \end{bmatrix}. \tag{6.35}$$

Then, by using *Equations* (2.30), (2.44), (6.33), (6.34), we can express b_G as:

$$\begin{aligned} b_G &= \bar{\mu}_*'\boldsymbol{\Sigma}_*^{-1}\bar{\mu}_* \\ &= \begin{bmatrix} \bar{\mu}_{M_H} & \bar{\mu}_{M_A} \end{bmatrix} \frac{1}{(1-\rho^2_{M_H M_A})} \begin{bmatrix} \left(\sigma^2_{M_H}\right)^{-1} & -\rho_{M_H M_A}\left(\sigma_{M_H}\sigma_{M_A}\right)^{-1} \\ -\rho_{M_H M_A}\left(\sigma_{M_H}\sigma_{M_A}\right)^{-1} & \left(\sigma^2_{M_A}\right)^{-1} \end{bmatrix} \begin{bmatrix} \bar{\mu}_{M_H} \\ \bar{\mu}_{M_A} \end{bmatrix} \\ &= \frac{1}{(1-\rho^2_{M_H M_A})} \begin{bmatrix} \bar{\mu}_{M_H} & \bar{\mu}_{M_A} \end{bmatrix} \begin{bmatrix} \left(\sigma^2_{M_H}\right)^{-1}\bar{\mu}_{M_H} - \rho_{M_H M_A}\left(\sigma_{M_H}\sigma_{M_A}\right)^{-1}\bar{\mu}_{M_A} \\ -\rho_{M_H M_A}\left(\sigma_{M_H}\sigma_{M_A}\right)^{-1}\bar{\mu}_{M_H} + \left(\sigma^2_{M_A}\right)^{-1}\bar{\mu}_{M_A} \end{bmatrix} \\ &= \frac{1}{(1-\rho^2_{M_H M_A})} \left[\bar{\mu}_{M_H}\left(\sigma^2_{M_H}\right)^{-1}\bar{\mu}_{M_H} - 2\bar{\mu}_{M_H}\bar{\mu}_{M_A}\rho_{M_H M_A}\left(\sigma_{M_H}\sigma_{M_A}\right)^{-1} + \bar{\mu}_{M_A}\left(\sigma^2_{M_A}\right)^{-1}\bar{\mu}_{M_A}\right] \\ \Rightarrow b_G &= \frac{1}{(1-\rho^2_{M_H M_A})}\left(b_H + b_A - 2\rho_{M_H M_A}\sqrt{b_H b_A}\right), \end{aligned} \tag{6.36}$$

where:

$$b_H = \bar{\mu}^2_{M_H}\sigma^{-2}_{M_H}, \tag{6.37}$$

$$b_A = \bar{\mu}^2_{M_A}\sigma^{-2}_{M_A}. \tag{6.38}$$

For *Equation* (6.36) we can discriminate four different cases according to the value of the correlation coefficient $\rho_{M_H M_A}$.

1. If $\rho_{M_H M_A} = 0$, *Equations* (*6.27*), (*6.36*), (*6.37*), (*6.38*) imply that:

$$b_G = b_H + b_A \tag{6.39}$$

and

$$\begin{aligned} \bar{\mu}^2_{M_G}\sigma^{-2}_{M_G} &= \bar{\mu}^2_{M_H}\sigma^{-2}_{M_H} + \bar{\mu}^2_{M_A}\sigma^{-2}_{M_A} \\ \bar{\mu}^2_{M_G} &= \bar{\mu}^2_{M_H}\left(\frac{\sigma^2_{M_G}}{\sigma^2_{M_H}}\right) + \bar{\mu}^2_{M_A}\left(\frac{\sigma^2_{M_G}}{\sigma^2_{M_A}}\right) \end{aligned} \tag{6.40}$$

2. If $0 < \rho_{M_H M_A} < 1$, we have:

$$b_H + b_A - 2\rho_{M_H M_A}\sqrt{b_H b_A} < b_H + b_A$$

and since,

$$\frac{1}{(1-\rho^2_{M_H M_A})} > 1,$$

we cannot derive any definitive conclusions about the direction of the inequality between $b_G\big|_{0<\rho_{M_H M_A}<1}$ and $b_G\big|_{\rho_{M_H M_A}=0}$

3. If $-1 < \rho_{M_H M_A} < 0$, we have:

$$b_H + b_A + 2|\rho_{M_H M_A}|\sqrt{b_H b_A} > b_H + b_A$$

and,

$$\frac{1}{(1-\rho^2_{M_H M_A})} > 1,$$

which implies that

$$b_G\big|_{-1<\rho_{M_H M_A}<0} > b_G\big|_{\rho_{M_H M_A}=0}$$

4. If $\rho_{M_H M_A} \pm 1$, b_G cannot be defined.

However, the typical portfolio of the Global Market portfolio is a weighted average of the typical portfolio of the Home and the Abroad Markets, together with the risk-free. Therefore, the equation of that portfolio can be written as follows:

$$P_G = w_0 f + w_H P_H + w_A P_A, \tag{6.41}$$

where from *Equations* (*4.17*), (*5.17*) we know that:

$$P_H = (1-\phi_H)f + \phi_H M_H, \tag{6.42a}$$

$$P_A = (1-\phi_A)f + \phi_A M_A. \tag{6.42b}$$

Using *Equations* (*6.41*), (*6.42a*), (*6.42b*):

$$\begin{aligned} P_G &= w_0 f + w_H \left[(1-\phi_H)f + \phi_H M_H\right] + w_A \left[(1-\phi_A)f + \phi_A M_A\right] \\ \Rightarrow P_G &= \left[w_0 + w_H(1-\phi_H) + w_A(1-\phi_A)\right] f + (w_H\phi_H)M_H + (w_A\phi_A)M_A \end{aligned} \tag{6.43}$$

Using *Equations* (*6.23*), (*6.28*):

$$\begin{aligned} P_G &= (1-\phi_G)f + \phi_G(w_{GH}M_H + w_{GA}M_A) \\ \Rightarrow P_G &= (1-\phi_G)f + \phi_G w_{GH} M_H + \phi_G w_{GA} M_A \end{aligned} \tag{6.44}$$

Using *Equations* (*6.43*), (*6.44*) imply the following results:

$$w_0 + w_H(1-\phi_H) + w_A(1-\phi_A) = 1 - \phi_G, \tag{6.45}$$

$$\begin{aligned} w_H\phi_H &= \phi_G w_{GH} \\ &\Rightarrow w_H = \frac{\phi_G}{\phi_H} w_{GH}, \end{aligned} \tag{6.46}$$

$$\begin{aligned} w_A\phi_A &= \phi_G w_{GA} \\ &\Rightarrow w_A = \frac{\phi_G}{\phi_A} w_{GA}, \end{aligned} \tag{6.47}$$

Using *Equation* (*6.41*), (*6.46*), (*6.47*) we can write:

$$P_G = w_0 f + \frac{\phi_G}{\phi_H} w_{GH} P_H + \frac{\phi_G}{\phi_A} w_{GA} P_A \tag{6.48}$$

Chapter 7

Comparisons

In this chapters we shall compare the following four portfolios:

- Portfolio E_H on the CML_H is the Home Equilibrium Portfolio, with variance σ_{E_H} and expected return

$$\mu_{E_H} = r_f + \sigma_{E_H} \sqrt{b_H} \tag{7.1}$$

- Portfolio E_{H*} on the CML_A with variance σ_{E_H} and expected return

$$\mu_{E_{H*}} = r_f + \sigma_{E_H} \sqrt{b_A} \tag{7.2}$$

- Portfolio E_A on the CML_A is the Abroad Equilibrium Portfolio, with variance σ_{E_A} and expected return

$$\mu_{E_A} = r_f + \sigma_{E_A} \sqrt{b_A} \tag{7.3}$$

- Portfolio E_{A*} on the CML_H with variance σ_{E_A} and expected return

$$\mu_{E_{A*}} = r_f + \sigma_{E_A} \sqrt{b_H} \tag{7.4}$$

When we move from the one CML to the other we can calculate the **Capital Market Effect**, and when we move on the same CML we calculate the **Optimization Effect**.

7.1 Home VS Abroad

The investor prefers to invest in the Home Market

Under the assumption that the investor prefers to invest in the Home Market the slope of the CML_H must be greater than the slope of CML_A, i.e.,

$$b_H > b_A. \tag{7.5}$$

This result can be proven by comparing the expected utilities of the investor in the Abroad Market at points (E_A) and (E_{A*}). We know that

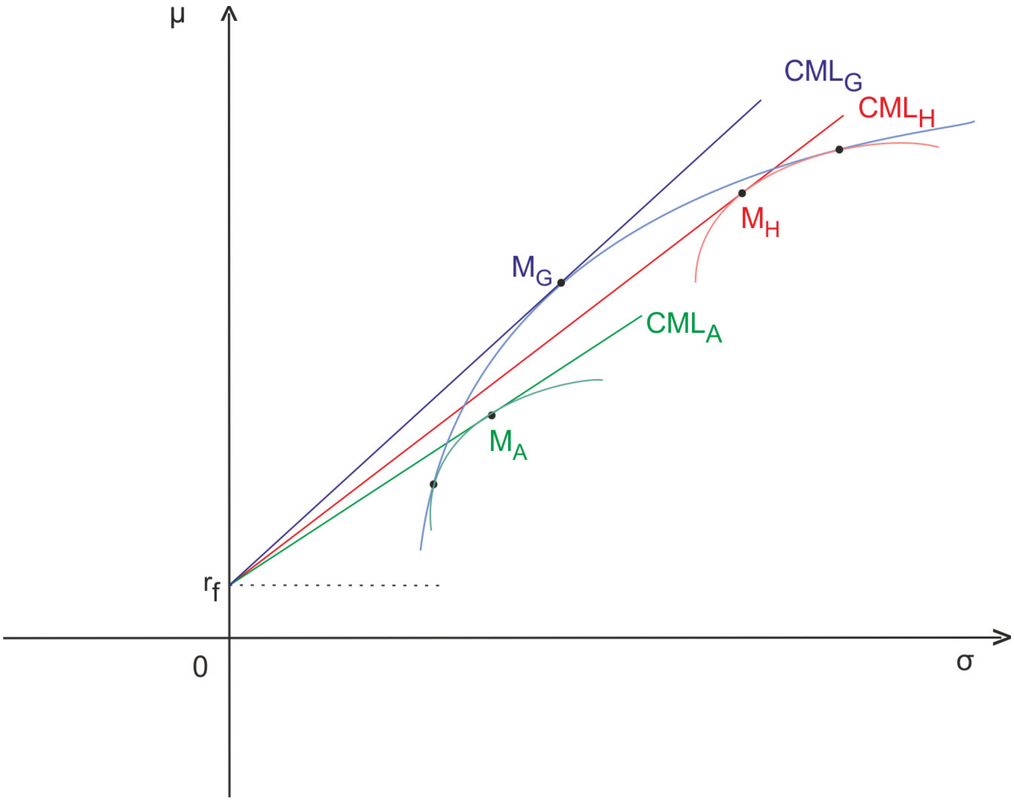

Figure 7.1: Efficient Frontiers

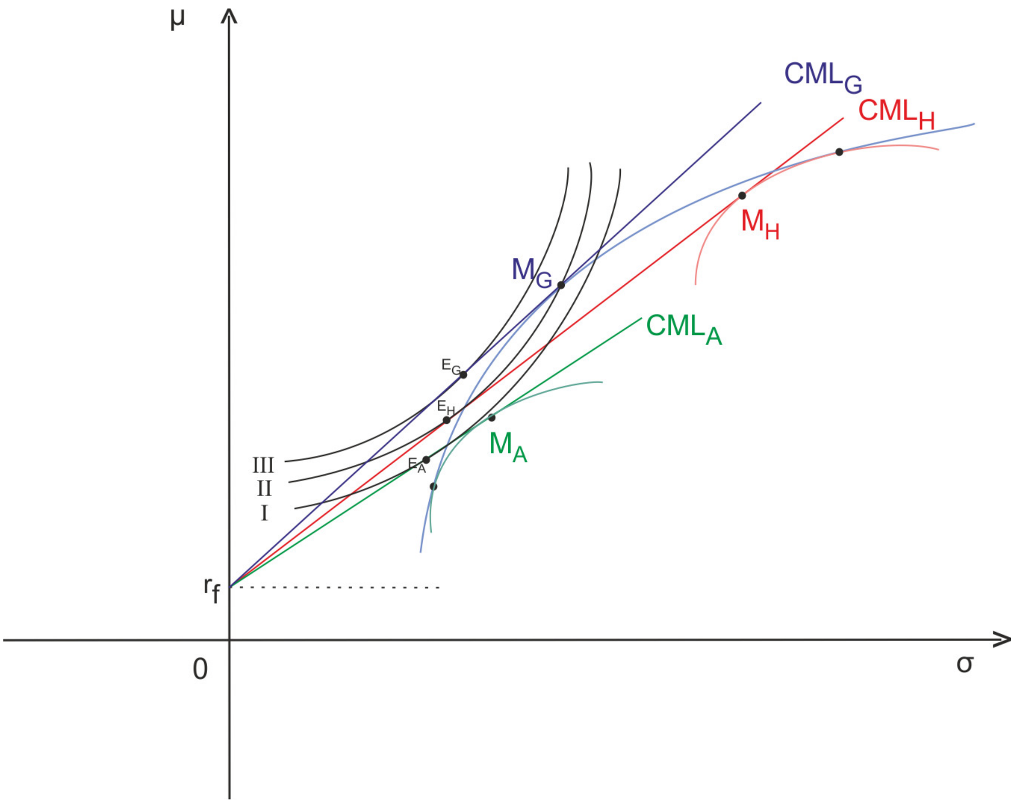

Figure 7.2: Equilibria

$$\begin{aligned}
\bar{\bar{v}}_{E_{A*}} &> \bar{\bar{v}}_{E_A} \\
\alpha_0 + \alpha_1 \mu_{E_{A*}} - \frac{1}{2}\alpha_2(\mu^2_{E_{A*}} + \sigma^2_{E_A}) &> \alpha_0 + \alpha_1 \mu_{E_A} - \frac{1}{2}\alpha_2(\mu^2_{E_A} + \sigma^2_{E_A}) \\
\alpha_1(\mu_{E_{A*}} - \mu_{E_A}) - \frac{1}{2}\alpha_2(\mu^2_{E_{A*}} - \mu^2_{E_A}) &> 0 \\
\alpha_1(\mu_{E_{A*}} - \mu_{E_A}) - \frac{1}{2}\alpha_2(\mu_{E_{A*}} - \mu_{E_A})(\mu_{E_{A*}} + \mu_{E_A}) &> 0 \\
(\mu_{E_{A*}} - \mu_{E_A})\left[\alpha_1 - \frac{1}{2}\alpha_2(\mu_{E_{A*}} + \mu_{E_A})\right] &> 0 \qquad (7.6)
\end{aligned}$$

In the previous chapters it is proven that:

$$\begin{aligned}
r_P &\leq \frac{\alpha_1}{\alpha_2} \\
\Rightarrow \mu_{E_{A*}} &\leq \frac{\alpha_1}{\alpha_2} \\
&\text{and} \\
\Rightarrow \mu_{E_A} &\leq \frac{\alpha_1}{\alpha_2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu_{E_{A*}} + \mu_{E_A} &\leq 2\frac{\alpha_1}{\alpha_2} \\
\frac{1}{2}\alpha_2(\mu_{E_{A*}} + \mu_{E_A}) &\leq \alpha_1 \\
\Rightarrow \alpha_1 - \frac{1}{2}\alpha_2(\mu_{E_{A*}} + \mu_{E_A}) &\geq 0 \qquad (7.7)
\end{aligned}$$

Equations (7.3), (7.4), (7.6), (7.7) imply that:

$$\begin{aligned}
\mu_{E_{A*}} - \mu_{E_A} &> 0 \Rightarrow \\
r_f + \sigma_{E_A}\sqrt{b_H} - r_f - \sigma_{E_A}\sqrt{b_A} &> 0 \Rightarrow \\
\sqrt{b_H} &> \sqrt{b_A} \Rightarrow \\
b_H &> b_A
\end{aligned}$$

This is the **Capital Market Effect**, since we move from CML_A to CML_H.

Since the indifference curve is an increasing function of σ, and the slope of the indifference curve at point (E_H) is greater than the slope of the indifference curve at point (E_A), we can intuitively assume that

$$\sigma_{E_H} > \sigma_{E_A}. \qquad (7.8)$$

This result can be proven by comparing the expected utility of the investor at points (E_H) and (E_{A*}). Since the investor prefers to invest in the Home Market,

$$\begin{aligned}
\bar{\bar{v}}_{E_H} &> \bar{\bar{v}}_{E_{A*}} \\
\alpha_0 + \alpha_1\mu_{E_H} - \frac{1}{2}\alpha_2(\mu_{E_H}^2 + \sigma_{E_H}^2) &> \alpha_0 + \alpha_1\mu_{E_{A*}} - \frac{1}{2}\alpha_2(\mu_{E_{A*}}^2 + \sigma_{E_A}^2) \\
\alpha_1(\mu_{E_H} - \mu_{E_{A*}}) - \frac{1}{2}\alpha_2(\mu_{E_H}^2 - \mu_{E_{A*}}^2) &> 0 \\
\alpha_1(\mu_{E_H} - \mu_{E_{A*}}) - \frac{1}{2}\alpha_2(\mu_{E_H} - \mu_{E_{A*}})(\mu_{E_H} + \mu_{E_{A*}}) &> 0 \\
(\mu_{E_H} - \mu_{E_{A*}})\left[\alpha_1 - \frac{1}{2}\alpha_2(\mu_{E_H} + \mu_{E_{A*}})\right] &> 0. \qquad (7.9)
\end{aligned}$$

In the previous chapters it is proven that:

$$\begin{aligned}
r_P &\leq \frac{\alpha_1}{\alpha_2} \\
\Rightarrow \mu_{E_H} &\leq \frac{\alpha_1}{\alpha_2} \\
&\text{and} \\
\Rightarrow \mu_{E_{A*}} &\leq \frac{\alpha_1}{\alpha_2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu_{E_H} - \mu_{E_{A*}} &> 0 \\
r_f + \sigma_{E_H}\sqrt{b_H} - r_f - \sigma_{E_A}\sqrt{b_H} &> 0 \\
\sigma_{E_H} &> \sigma_{E_A} \qquad (7.10)
\end{aligned}$$

This is the **Optimization Effect**, since we move on the CML_H.

The investor prefers to invest in the Abroad Market

Similarly, the **Capital Market Effect** is:

$$b_H < b_A \qquad (7.11)$$

and the **Optimization Effect** is:

$$\sigma_{E_H} < \sigma_{E_G} \qquad (7.12)$$

7.2 Home VS Global

The investor prefers to invest in the Home Market

Similarly, the **Capital Market Effect** is:

$$b_H > b_G \qquad (7.13)$$

and the **Optimization Effect** is:

$$\sigma_{E_H} > \sigma_{E_G} \tag{7.14}$$

The investor prefers to invest in the Global Market

Similarly, the **Capital Market Effect** is:

$$b_H < b_G \tag{7.15}$$

and the **Optimization Effect** is:

$$\sigma_{E_H} < \sigma_{E_G} \tag{7.16}$$

7.3 Abroad VS Global

The investor prefers to invest in the Abroad Market

Similarly, the **Capital Market Effect** is:

$$b_A > b_G \tag{7.17}$$

and the **Optimization Effect** is:

$$\sigma_{E_A} > \sigma_{E_G} \tag{7.18}$$

The investor prefers to invest in the Global Market

Similarly, the **Capital Market Effect** is:

$$b_A < b_G \tag{7.19}$$

and the **Optimization Effect** is:

$$\sigma_{E_A} < \sigma_{E_G} \tag{7.20}$$

Conclusion

In this study we examined an investor with a Quadratic Utility Function, who can invest in three markets: the Home, the Abroad, and the Global Market; with N_H, N_A and $N_G = N_H + N_A$ risky assets, respectively, and a risk-free asset.

However, we expressed the Global-Market portfolio, M_G, using two different techniques:

1. As a weighted average of the N_G risky assets that the Global Market consists of.
2. As a weighted average of only two risky assets, the Home-Market, M_H, and the Abroad-Market, M_A, portfolios.

In the first case, we followed the classic technique used to analyse such portfolios.

In the second case, we needed to define one new 2×1 vector of weights, $\boldsymbol{w}_*$, instead of the $N_G \times 1$ $\boldsymbol{w}_G$.

Additionally, we defined a new 2×2 variance-covariance matrix, $\boldsymbol{\Sigma}_*$, which consists of the σ of M_H and M_A and their covariance. This matrix was defined so as to substitute the $N_G \times N_G$ matrix $\boldsymbol{\Sigma}_G$, which consists of all the σ of the N_G risky assets. This substitution gave us the opportunity to include in our analysis **the sign** of the correlation of the Home and the Abroad Market, which we could not do by using the off-diagonal elements of the original $\boldsymbol{\Sigma}_G$ matrix.

Finally, we examined under what conditions the investor decides to invest in each market.

We proved that the investor invests on the market with the greater b, that is the market which Capital Market Line has the greater slope; we named this effect **Capital Market Effect**.

Moreover, comparing the utility functions of the investor in each market we showed that the investor decides to invest in the market in which σ, in the equilibrium, is greater; we named this effect **Optimization Effect**.

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