UNIVERSITY OF IOANNINA
DEPARTMENT OF MATHEMATICS

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Rigidity and Deformability of Immersed Submanifolds

Ph.D. Dissertation

Ioannina, 2019

The present dissertation was carried out under the Ph.D. program of the Department of Mathematics of the University of Ioannina in order to obtain the degree of Doctor of Philosophy.

Accepted by the seven members of the evaluation committee on July 5th, 2019:

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The acceptance of this dissertation by the Department of Mathematics of the University of Ioannina does not imply the approval of the opinions of the author (section 202 par. 2 Law 5343/1932 and section 50 par. 8 Law 1268/1982).

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I lawfully declare here with statutorily that the present dissertation was carried out under the international ethical and academical rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis.

Dedicated to my parents Fotios and Vasiliki
and to my brother Panagiotis.

Acknowledgements

First of all, I would like to express my sincere gratitude to my supervisor Theodoros Vlachos for his continuous encouragement, his endless help and for being a real teacher and mentor for me. He has constantly supported me during all these years, not only at the scientific level, but also in several difficult situations that I faced in my life, and his support was really invaluable for me. I would like to thank him from my heart and to say that it is definitely a great honor for me to be his student.

Furthermore, I would like to thank the other six members of the evaluation committee: Marcos Dajczer from IMPA, Giuseppe Tinaglia from King's College London, Andreas Arvanitogeorgos from the University of Patras, Fani Petalidou from the University of Thessaloniki, Andreas Savas-Halilaj from the University of Ioannina and Stylianos Stamatakis from the University of Thessaloniki, for their useful comments and remarks.

I would also like to to express my gratitude to the Alexander S. Onassis Public Benefit Foundation for the financial support during my Ph.D. studies.

Moreover, I would like to thank my friends and colleagues: Alexandros Stavrou, Tasos Raptis, Giorgos Mitsos, Lefteris Zois, Dimitris Ioannou, Theodoros Kasioumis, Kleio Papadopoulou, Amalia-Sofia Tsouri and Athina-Eleni Kanellopoulou. I would like especially to thank my friends Stelios Frestas and Christos Raent-Onti for sticking by my side all these years.

Last but not least, I would like to thank my parents Fotios Polymerakis and Vasiliki Ziori and my brother Panagiotis Polymerakis, for their endless support during all my life.

Kleanthis Polymerakis
Ioannina, Greece, July 2019

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Chapter 1

Introduction

A basic problem in surface theory is to understand the role and the importance of the mean curvature. Bonnet [8] raised the problem to what extent a surface in a complete simply-connected 3-dimensional space form $\mathbb{Q}_c^3$ of curvature c, is determined (up to congruence) by the metric and the mean curvature. Generically, a surface in $\mathbb{Q}_c^3$ is uniquely determined by these data. The exceptions are the Bonnet surfaces that include the constant mean curvature (CMC) surfaces.

There has been a lot of interest in the following natural problem: given an isometric immersion $f\colon M \to \mathbb{Q}_c^3$ of a 2-dimensional Riemannian manifold M, how many noncongruent isometric immersions of M into $\mathbb{Q}_c^3$ can exist with the same mean curvature with f? Any noncongruent to f such surface is called a Bonnet mate of f. This problem has been studied locally or globally by Bonnet [8], Cartan [10], Lawson [53], Tribuzy [64], Chern [18], Roussos-Hernandez [60] and Kenmotsu [47] among others. It turns out that if $f\colon M \to \mathbb{Q}_c^3$ is a non-compact, simply connected surface then it admits either at most one Bonnet mate, or infinitely many. In the latter case the surface f is called proper Bonnet. Bonnet [8] showed that a proper Bonnet surface is isothermic away from its umbilics. Moreover, Graustein [34] proved that a Bonnet isothermic surface is proper Bonnet. Recently, it has been shown in [43] that a non-compact simply-connected surface which is totally non isothermic, admits a unique Bonnet mate. Lawson and Tribuzy [54] proved that a compact oriented 2-dimensional Riemannian manifold admits at most two noncongruent isometric immersions in $\mathbb{Q}_c^3$, with the same non-constant mean curvature. Their result was strengthened recently in [44], under additional assumptions on the isothermicity of the immersion. On the other hand, Lawson [53] proved that if M is simply-connected and f is a CMC surface in $\mathbb{Q}_c^3$, then the space of isometric immersions with the same mean curvature is the circle $\mathbb{S}^1$, unless f is totally-umbilical. The case of non-simply-connected CMC surfaces has been studied in [2, 6, 62].

Surfaces of constant mean curvature have been extensively studied. Hopf [42] showed the existence of a holomorphic quadratic differential on every CMC surface in $\mathbb{R}^3$, and he proved that a CMC surface of genus zero is a round sphere. His result was extended to nonflat 3-dimensional space forms by Chern [17]. Abresch and Rosenberg [1] proved that every CMC surface in the Riemannian products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ possesses a holomorphic quadratic differential, and they extended Hopf's theorem for such surfaces of genus zero. Their work and an extension of Bonnet's fundamental theorem (cf. [25]), led to the study of the Bonnet problem for surfaces in these spaces (cf. [33]). In codimension greater than one, a generalization of CMC surfaces are the surfaces whose mean curvature vector field is parallel in the normal connection. Existence of holomorphic quadratic differentials, classification results and Hopf-type theorems have been proved for parallel mean curvature surfaces in several ambient spaces, especially in codimension two (cf. [3, 14, 30, 31, 40, 55, 63, 69]). In particular, in [48, 49] has been proved the existence of parallel mean curvature surfaces in $\mathbb{CH}^2$ that admit non-trivial isometric deformations preserving the length of the mean curvature vector field.

As a step towards deciphering the role of the mean curvature in codimension two and inspired by Bonnet's question for surfaces in $\mathbb{Q}_c^3$, we are interested in the following problem: *given an isometric immersion $f\colon M \to \mathbb{Q}_c^4$ of a 2-dimensional Riemannian manifold M, how many noncongruent isometric immersions of M into $\mathbb{Q}_c^4$ can exist with the same mean curvature with f?* Two isometric immersions $f, \tilde{f}\colon M \to \mathbb{Q}_c^4$ are said to have *the same mean curvature* if there exists a parallel vector bundle isometry between their normal bundles that preserves the mean curvature vector fields. A large part of the results of this dissertation is included in [59].

In Chapter 2, we fix the notation and give some preliminaries concerning surfaces in 4-dimensional space forms. For any surface $f\colon M \to \mathbb{Q}_c^4$, we introduce two quadratic differentials with values in the complexified normal bundle of f and we study their relation with the Gauss lifts of f to the twistor bundle of $\mathbb{Q}_c^4$. It is worth noticing that the Gauss lifts will play an important role in the study of the Bonnet problem for surfaces in $\mathbb{Q}_c^4$.

In Chapter 3, we introduce two differential 1-forms associated to a surface in $\mathbb{Q}_c^4$, called the mixed connection forms. For compact surfaces, we prove an index theorem and we provide some applications. We introduce the notion of isotropically isothermic and strongly isotropically isothermic surfaces in 4-dimensional space forms, by requiring the co-closeness either of the one, or both of the mixed connection forms, respectively. A surface is called half totally non isotropically isothermic, if one of the mixed connection forms is nowhere co-closed. It turns out that isotropic isothermicity is a conformally invariant property with a similar effect on the Bonnet problem for surfaces in 4-dimensional space forms, with that of isothermicity on the classical Bonnet problem. The class of isotropically isothermic surfaces in $\mathbb{Q}_c^4$ includes all isothermic surfaces lying in totally umbilical hypersurfaces of $\mathbb{Q}_c^4$, all minimal surfaces in $\mathbb{Q}_c^4$, as well as the higher-codimensional analogues in $\mathbb{Q}_c^4$ of CMC surfaces in 3-dimensional space forms, apart from the totally umbilical ones. These results indicate that isotropic isothermicity is the natural generalization of the notion of isothermicity for surfaces in $\mathbb{Q}_c^3$, to surfaces in $\mathbb{Q}_c^4$ with not necessarily flat normal bundle. However, it is definitely worth mentioning that the class of isotropically isothermic surfaces in $\mathbb{Q}_c^4$ does not seems to contain isothermic surfaces in the sense of Palmer [57] in great abundance; simple examples show that there exist isothermic surfaces in $\mathbb{R}^4$ which are strongly totally non isotropically isothermic, i.e., both mixed connection forms are nowhere co-closed.

In Chapter 4, we set up the framework for the study of the Bonnet problem for non-minimal surfaces in 4-dimensional space forms. We point out that the case of minimal surfaces has been studied in [21, 67]. For a surface $f\colon M \to \mathbb{Q}_c^4$ we denote by $\mathcal{M}(f)$ the moduli space of congruence classes of all isometric immersions of M into $\mathbb{Q}_c^4$, that have the same mean curvature with f. Any nontrivial such class is called a Bonnet mate of f. The surface f is called either a Bonnet, or a proper Bonnet surface, if it admits either at least one, or infinitely many Bonnet mates, respectively.

In Chapter 5, we study the Bonnet problem for non-compact simply-connected surfaces in $\mathbb{Q}_c^4$. We first determine the possible structure of the moduli space of such a surface.

Theorem. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-compact simply-connected, oriented surface.*

(i) If f is not proper Bonnet, then it admits either at most one Bonnet mate, or exactly three.

(ii) If f is proper Bonnet, then the moduli space $\mathcal{M}(f)$ is a space diffeomorphic to a manifold. Moreover, f is characterized according to the structure of $\mathcal{M}(f)$ as follows:

Tight: *The moduli space is 1-dimensional with at most two connected components diffeomorphic to $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$.*

Flexible: *The moduli space is diffeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$.*

In the sequel, we investigate the effect of isotropic isothermicity on the structure of the moduli space and we obtain the following result.

Theorem. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-compact simply-connected oriented surface.*

(i) If f is half totally non isotropically isothermic, then f admits at least one Bonnet mate and it is not flexible. In particular, if f is strongly totally non isotropically isothermic, then it admits exactly three Bonnet mates.

(ii) If f is proper Bonnet, then it is isotropically isothermic on an open, dense and connected subset of M. In particular, if f is flexible, then it is strongly isotropically isothermic away from its isolated pseudo-umbilic points.

As an application of the first part of the above result, we provide examples of isothermic surfaces in $\mathbb{R}^4$ that admit exactly three Bonnet mates. We also prove that a Bonnet surface lying in a totally geodesic hypersurface of $\mathbb{Q}_c^4$ with non-constant mean curvature, always admits at least two Bonnet mates that do not lie in any totally umbilical hypersurface of $\mathbb{Q}_c^4$. In particular, such a surface either admits exactly three Bonnet mates, or it is flexible proper Bonnet.

In Chapter 6, we study the Bonnet problem for compact surfaces in $\mathbb{Q}_c^4$. For such a surface $f\colon M \to \mathbb{Q}_c^4$, we show that the structure of the moduli space is controlled by the behavior of the Gauss lifts $G_+\colon M \to \mathcal{Z}_+$ and $G_-\colon M \to \mathcal{Z}_-$ of f to the twistor bundle of $\mathbb{Q}_c^4$. Here, $\mathcal{Z}_+$ and $\mathcal{Z}_-$ stand for the two connected components of the twistor bundle $\mathcal{Z}$ of $\mathbb{Q}_c^4$.

Theorem. *Let $f\colon M \to \mathbb{Q}_c^4$ be a compact oriented surface. If both Gauss lifts G_+ and G_- of f are not vertically harmonic, then f admits at most three Bonnet mates. In particular, f admits at most one Bonnet mate, if M is homeomorphic to $\mathbb{S}^2$.*

This result implies that compact surfaces in $\mathbb{Q}_c^4$ whose both Gauss lifts are not vertically harmonic, do not allow nontrivial global isometric deformations that preserve the mean curvature. Moreover, in contrast to the results of the previous chapter, we show that additional assumptions involving isotropic isothermicity of a compact surface, turns out to impose strong obstructions for the existence of Bonnet mates.

Theorem. *Let $f\colon M \to \mathbb{Q}_c^4$ be a compact oriented surface. If both Gauss lifts G_+ and G_- of f are not vertically harmonic and f is either isotropically isothermic, or half totally non isotropically isothermic, on an open dense and connected subset V of M, then f admits at most one Bonnet mate. In particular, f does not admit any Bonnet mate, if it is either strongly isotropically isothermic, or strongly totally non isotropically isothermic on V.*

In the last part of this chapter we provide some applications of our results, including a short proof of the theorem of Lawson-Tribuzy [54].

In Chapter 7, we study surfaces in $\mathbb{Q}_c^4$ with a vertically harmonic Gauss lift. Such surfaces have holomorphic mean curvature vector field and they constitute a broader class than parallel mean curvature surfaces. This class contains also non-minimal surfaces with nonflat normal bundle. Extensively studied surfaces with a vertically harmonic Gauss lift are the Lagrangian surfaces in $\mathbb{R}^4$ with conformal or harmonic Maslov form (cf. [12, 13, 39]). Non-minimal superconformal surfaces in the aforementioned class generalize totally umbilical surfaces. We prove that surfaces in $\mathbb{Q}_c^4$ with a vertically harmonic Gauss lift that are neither minimal, nor superconformal, satisfy Ricci-like conditions that extend the Ricci condition for CMC surfaces in 3-dimensional space forms (cf. [53]). We show that non-minimal surfaces in $\mathbb{Q}_c^4$ with a vertically harmonic Gauss lift possess a holomorphic quadratic differential that vanishes identically on superconformal surfaces, yielding thus the following Hopf-type theorem.

Theorem. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-minimal surface. If the Gauss lift $G_\pm$ of f is vertically harmonic and M is homeomorphic to $\mathbb{S}^2$, then f is superconformal. In particular, f is totally umbilical if the Euler number of its normal bundle vanishes.*

We also prove that a non-minimal simply-connected surface in $\mathbb{Q}_c^4$ with a vertically harmonic Gauss lift allows a 1-parameter associated family of isometric deformations with the same mean curvature. This family is trivial only if the surface is superconformal.

Theorem. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-minimal, simply-connected surface. If the Gauss lift $G_\pm$ of f is vertically harmonic, then:*

(i) There exists a one-parameter family of isometric immersions $f_\theta^\pm\colon M \to \mathbb{Q}_c^4$, $\theta \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, which have the same mean curvature with $f_0^\pm = f$.

(ii) *If f is superconformal, then $f_\theta^\pm$ is congruent to f for any θ.*
(iii) *If there exist $\theta \neq \tilde{\theta} \in \mathbb{S}^1$ such that $f_\theta^\pm$ is congruent to $f_{\tilde{\theta}}^\pm$, then f is superconformal.*

For compact surfaces with a vertically harmonic Gauss lift, we determine the possible structure of the moduli space, under appropriate geometric or topological assumptions.

Theorem. *Let $f\colon M \to \mathbb{Q}_c^4$ be a compact oriented surface with vertically harmonic Gauss lift $G_\pm$.*
(i) *If the mean curvature vector field of f is non-parallel, then the moduli space $\mathcal{M}(f)$ is the disjoint union of two sets, each one being either finite, or the circle $\mathbb{S}^1$.*
(ii) *If $c = 0$ and the Euler numbers χ and χ_N of the tangent and normal bundles satisfy $\chi \neq \mp\chi_N$, then $\mathcal{M}(f)$ is a finite set.*

In Chapter 8, we study locally proper Bonnet surfaces in $\mathbb{Q}_c^4$. A surface $f\colon M \to \mathbb{Q}_c^4$ is called locally proper Bonnet if every point of M has a simply-connected neighbourhood, restricted to which f is proper Bonnet. We first show that if M is homeomorphic to the sphere $\mathbb{S}^2$, then f cannot be globally proper Bonnet. We prove that if a locally proper Bonnet surface is non-minimal, then around a point $p \in M$, any continuous isometric deformation that preserves the mean curvature is described by a submanifold $L^n(p), 1 \leq n \leq 2$, of the torus $\mathbb{S}^1 \times \mathbb{S}^1$. We focus on surfaces for which there exists a submanifold $L^n, 1 \leq n \leq 2$, of the torus that gives rise to such a local deformation around every point of M. We call these surfaces uniformly locally proper Bonnet. In particular, such a surface is called locally flexible, if this submanifold is the torus itself. We show that the compact surfaces in $\mathbb{Q}_c^4$, which have a vertically harmonic Gauss lift without being superconformal, are characterized as the only uniformly locally proper Bonnet compact surfaces in $\mathbb{Q}_c^4$. More precisely, we obtain the following result.

Theorem. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-minimal, compact oriented surface. Then, f is uniformly locally proper Bonnet if and only if it has a vertically harmonic, non-conformal Gauss lift.*

We also show that there do not exist compact superconformal surfaces in $\mathbb{Q}_c^4$ that are locally proper Bonnet. Finally, we prove that compact surfaces with parallel mean curvature vector field in $\mathbb{Q}_c^4$ that are not totally umbilical, are characterized as the only locally flexible compact surfaces in $\mathbb{Q}_c^4$.

Chapter 2

Surfaces in 4-Dimensional Space Forms

The aim of this chapter is to set up the notation and to present some aspects of the theory of surfaces in 4-dimensional space forms, which turns out to be special. A consequence of the equality between the dimension and the codimension of a surface in a 4-dimensional space form $\mathbb{Q}_c^4$, is that the twistor theory of $\mathbb{Q}_c^4$ can be used in order to handle the complexity of the normal bundle of the surface, arising by the non-triviality of the Ricci equation. Most of the material presented in this chapter was already known, except from two quadratic differentials associated to a surface in $\mathbb{Q}_c^4$ and their relation with the Gauss lifts of the surface to the twistor bundle.

2.1 Preliminaries

Throughout the manuscript, M is a connected, oriented 2-dimensional Riemannian manifold. A surface $f\colon M \to \mathbb{Q}_c^n, n = 3, 4$, is an isometric immersion into the complete simply-connected n-dimensional space form of curvature c.

Let $f\colon M \to \mathbb{Q}_c^4$ be a surface. Denote by N_fM the normal bundle of f and by $\nabla^\perp, R^\perp$ the normal connection and its curvature tensor, respectively. Let $\alpha\colon TM \times TM \to N_fM$ be the second fundamental form of f. The shape operator A_ξ of f with respect to $\xi \in N_fM$ is the symmetric endomorphism of TM defined by $\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle$, where $\langle\cdot,\cdot\rangle$ stands for the Riemannian metric of $\mathbb{Q}_c^4$. The Gauss, Codazzi and Ricci equations for f are respectively

$$(K - c)\langle (X \wedge Y)Z, W\rangle = \langle \alpha(X, W), \alpha(Y, Z)\rangle - \langle \alpha(X, Z), \alpha(Y, W)\rangle,$$
$$(\nabla^\perp_X \alpha)(Y, Z) = (\nabla^\perp_Y \alpha)(X, Z),$$
$$R^\perp(X, Y)\xi = \alpha(X, A_\xi Y) - \alpha(A_\xi X, Y),$$

where K is the Gaussian curvature, $X, Y, Z, W \in TM$, $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$ and $\xi \in N_fM$.

The orientations of M and $\mathbb{Q}_c^4$ induce an orientation on the normal bundle. The *normal curvature* K_N of f is given by

$$K_N = \langle R^{\perp}(e_1, e_2)e_4, e_3\rangle, \tag{2.1}$$

where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are positively oriented orthonormal frame fields of TM and N_fM, respectively. Notice that if τ is an orientation-reversing isometry of $\mathbb{Q}_c^4$, then f and $\tau \circ f$ have opposite normal curvatures. The surface f is said to have *flat normal bundle*, if $K_N \equiv 0$ on M. This is equivalent to the existence for every $p \in M$ of an orthonormal basis of T_pM that diagonalizes simultaneously all shape operators of f at p. The Gauss and the normal curvatures satisfy the equations

$$d\omega_{12} = -K\omega_1 \wedge \omega_2, \quad d\omega_{34} = -K_N\omega_1 \wedge \omega_2, \tag{2.2}$$

where $\{\omega_j\}$ is the dual frame field of $\{e_j\}, 1 \leq j \leq 4$, and the connection forms $\omega_{kl},\ 1 \leq k, l \leq 4$, are given by

$$d\omega_k = \sum_{m=1}^{4} \omega_{km} \wedge \omega_m,\ 1 \leq k \leq 4. \tag{2.3}$$

If M is compact, the Euler-Poincaré characteristics χ, χ_N of TM and N_fM, are given respectively, by

$$2\pi\chi = \int_M K, \quad 2\pi\chi_N = \int_M K_N.$$

For a symmetric section $\beta \in \Gamma(\mathrm{Hom}(TM \times TM, N_fM))$, the *ellipse associated to* β at each $p \in M$ is defined by

$$\mathcal{E}_\beta(p) = \{\beta(X, X) : X \in T_pM, \|X\| = 1\}.$$

It is indeed an ellipse on $N_fM(p)$ centered at trace$\beta(p)/2$, which may degenerate into a line segment or a point. In particular, the ellipse associated to the second fundamental form is denoted by $\mathcal{E}_f$, is centered at the mean curvature vector H and is called the *curvature ellipse of* f. It is parametrized by

$$\alpha(X_\theta, X_\theta) = H(p) + \cos 2\theta\ \frac{\alpha_{11} - \alpha_{22}}{2} + \sin 2\theta\ \alpha_{12}, \tag{2.4}$$

where $X_\theta = \cos\theta e_1 + \sin\theta e_2,\ \alpha_{ij} = \alpha(e_i, e_j),\ i, j = 1, 2$, and $\{e_1, e_2\}$ is an orthonormal basis of T_pM. The Ricci equation is written equivalently at p as

$$R^{\perp}(e_1, e_2) = (\alpha_{11} - \alpha_{22}) \wedge \alpha_{12}. \tag{2.5}$$

Clearly, the ellipse degenerates into a line segment or a point if and only if the vectors $(\alpha_{11} - \alpha_{22})/2$ and α_{12} are linearly dependent, or equivalently, if $R^{\perp} = 0$ at p. At a point where the curvature ellipse is nondegenerate, K_N is positive if and only if the orientation induced on the ellipse as X_θ traverses positively the unit tangent circle, coincides with the orientation of the normal plane (cf. [35]). Let λ_1, λ_2 be the length of the semiaxes of $\mathcal{E}_f$. Using the Gauss equation and (2.5), we have that (cf. [56])

$$\lambda_1^2 + \lambda_2^2 = \|H\|^2 - (K - c), \quad \lambda_1\lambda_2 = \frac{1}{\pi}A(\mathcal{E}_f) = \frac{1}{2}|K_N| \tag{2.6}$$

at any point, where $A(\mathcal{E}_f)$ is the area of the curvature ellipse. Therefore,

$$\|H\|^2 - (K - c) \geq |K_N|.$$

A point $p \in M$ is called *pseudo-umbilic* if the curvature ellipse is a circle at p. A pseudo-umbilic point is called *umbilic* if the circle degenerates into a point. From (2.6) it follows that the set $M_0(f)$ of pseudo-umbilic points of f is characterized as

$$M_0(f) = \left\{p \in M : \|H\|^2 - (K - c) = |K_N|\right\}.$$

A surface for which any point is pseudo-umbilic is called *superconformal*. By setting

$$M_0^{\pm}(f) = \{p \in M_0(f) : \pm K_N \geq 0\},$$

it is clear that $M_0(f) = M_0^+(f) \cup M_0^-(f)$ and the set $M_1(f)$ of umbilic points is

$$M_1(f) = M_0^+(f) \cap M_0^-(f) = \{p \in M : \|H\|^2 = K - c\}.$$

For later use we need the following elementary fact.

Lemma 2.1. *Let $f\colon M \to \mathbb{Q}_c^4$ be a surface and $\gamma \in \Gamma(Hom(TM \times TM, N_fM))$ a symmetric section. Assume that the ellipse $\mathcal{E}_\gamma$ associated to γ is not a circle at a point $p \in M$. Then, there exist positively oriented orthonormal frame fields $\{e_1, e_2\}$ of TM, $\{e_3, e_4\}$ of N_fM, on a neighbourhood U of p, and $\kappa, \mu \in \mathcal{C}^\infty(U)$ with $\kappa > |\mu|$, such that $\gamma_{11} - \gamma_{22} = 2\kappa e_3$ and $\gamma_{12} = \mu e_4$, where $\gamma_{ij} = \gamma(e_i, e_j)$, $j = 1, 2$.*

Proof: Let $\{\tilde{e}_1, \tilde{e}_2\}$ be a positively oriented orthonormal tangent frame field around p and set $X_t = \cos te_1 + \sin te_2$, $t \in \mathbb{R}$. The ellipse $\mathcal{E}_\gamma(q)$ is parametrized by

$$\gamma(X_t(q), X_t(q)) = trace\gamma(q)/2 + \cos 2tu(q) + \sin 2tv(q),$$

where $u = (\tilde{\gamma}_{11} - \tilde{\gamma}_{22})/2$, $v = \tilde{\gamma}_{12}$ and $\tilde{\gamma}_{ij} = \gamma(\tilde{e}_i, \tilde{e}_j), i, j = 1, 2$. Our assumption implies that at least one of the quantities $\|u\| - \|v\|$, $\langle u, v\rangle$ is non-zero at p. By continuity, we have that either $\|u\| \neq \|v\|$, or $\langle u, v\rangle \neq 0$ everywhere on a neighbourhood U of p. Let $q \in U$. The function $r(t) = \|\mathring{\gamma}(X_t(q), X_t(q))\|^2$, where $\mathring{\gamma}$ is the traceless part of γ, attains its maximum at t_0. Clearly, $\mathring{\gamma}(X_{t_0}(q), X_{t_0}(q))$ is a major semiaxis of $\mathcal{E}_\gamma(q)$ and $\mathring{\gamma}(X_{t_0}(q), X_{t_0+\pi/2}(q))$ is a minor semiaxis. From $r'(t_0) = 0$ and $r''(t_0) \leq 0$, we obtain that

$$\sin 4t_0 \left(\|u\|^2 - \|v\|^2\right)(q) = 2\cos 4t_0 \langle u, v\rangle (q)$$

and

$$\cos 4t_0 \left(\|u\|^2 - \|v\|^2\right)(q) + 2\sin 4t_0 \langle u, v\rangle (q) \geq 0.$$

Define the function $\omega \in \mathcal{C}^\infty(U)$ by

$$\omega = \frac{1}{4}\arctan\left(\frac{2\langle u, v\rangle}{\|u\|^2 - \|v\|^2}\right) \text{ modulo } 2\pi,$$

if $\|u\| \neq \|v\|$ on U, where the branch of arctan is such that $\cos 4\omega \left(\|u\|^2 - \|v\|^2\right) \geq 0$. If $\langle u, v\rangle \neq 0$ on U, then ω is defined by

$$\omega = \frac{1}{4}\operatorname{arccot}\left(\frac{\|u\|^2 - \|v\|^2}{2\langle u, v\rangle}\right) \text{ modulo } 2\pi,$$

where the branch of arccot is such that $\sin 4\omega \langle u, v\rangle \geq 0$. We consider the frame field $e_1 = \cos\omega\tilde{e}_1 + \sin\omega\tilde{e}_2$, $e_2 = -\sin\omega\tilde{e}_1 + \cos\omega\tilde{e}_2$ and the positively oriented orthonormal frame field $\{e_3, e_4\}$ in the normal bundle such that $\mathring{\gamma}(e_1, e_1) = \|\mathring{\gamma}(e_1, e_1)\| e_3$. By the choice of ω, we have that $\mathring{\gamma}(e_1, e_1)$ is a major semiaxis of $\mathcal{E}_\gamma$. Then, the proof follows with $\kappa = \|\mathring{\gamma}(e_1, e_1)\|$ and $\mu = \langle\mathring{\gamma}(e_1, e_2), e_4\rangle$. ∎

2.2 Complexification and Associated Differentials

The complexified tangent bundle $TM \otimes \mathbb{C}$ of a 2-dimensional oriented Riemannian manifold M, decomposes into the eigenspaces of the complex structure J, denoted by $T^{(1,0)}M$ and $T^{(0,1)}M$, corresponding to the eigenvalues i and $-i$, respectively (cf. [50]).

The second fundamental form of a surface $f\colon M \to \mathbb{Q}_c^4$ can be $\mathbb{C}$-bilinearly extended to $TM \otimes \mathbb{C}$ with values in the complexified normal bundle $N_fM \otimes \mathbb{C}$ and then decomposed into its (k, l)-components $\alpha^{(k,l)}$, $k+l = 2$, which are tensors of k many 1-forms vanishing on $T^{(0,1)}M$ and l many 1-forms vanishing on $T^{(1,0)}M$. For a positively oriented orthonormal frame field $\{e_1, e_2\}$ of TM, the *Hopf invariant* $\mathcal{H}(e_1, e_2)$ of f with respect to $\{e_1, e_2\}$ is the local section of $N_fM \otimes \mathbb{C}$ defined by

$$\mathcal{H}(e_1, e_2) = \frac{1}{2}\alpha(e_1 - ie_2, e_1 - ie_2) = \frac{\alpha_{11} - \alpha_{22}}{2} - i\alpha_{12}, \quad \alpha_{ij} = \alpha(e_i, e_j), i, j = 1, 2. \tag{2.7}$$

Let $J^\perp$ be the complex structure of N_fM defined by the metric and the orientation. The complexified normal bundle decomposes as

$$N_fM \otimes \mathbb{C} = N_f^-M \oplus N_f^+M$$

into the eigenspaces N_f^-M and N_f^+M of $J^\perp$, corresponding to the eigenvalues i and $-i$, respectively. Any section $\xi \in N_fM \otimes \mathbb{C}$ is decomposed as $\xi = \xi^- + \xi^+$, with

$$\xi^\pm = \pi^\pm(\xi),$$

where $\pi^{\pm}\colon N_fM \otimes \mathbb{C} \to N_f^{\pm}M$ is given by

$$\pi^{\pm}(\xi) = \frac{1}{2}(\xi \pm iJ^{\perp}\xi), \quad \xi \in N_fM \otimes \mathbb{C}.$$

A section ξ of $N_fM \otimes \mathbb{C}$ is called *isotropic* if at any point of M, either $\xi = \xi^-$, or $\xi = \xi^+$. This is equivalent to $\langle \xi, \xi \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the $\mathbb{C}$-bilinear extension of the metric. Notice that $\langle \zeta, \eta \rangle = 0$ for $\zeta \in N_f^- M$ and $\eta \in N_f^+ M$, implies that $\zeta = 0$ or $\eta = 0$. According to the above decomposition, the Hopf invariant of f with respect to $\{e_1, e_2\}$ splits as $\mathcal{H}(e_1, e_2) = \mathcal{H}^-(e_1, e_2) + \mathcal{H}^+(e_1, e_2)$, where $\mathcal{H}^{\pm}(e_1, e_2)$ is given by

$$\mathcal{H}^{\pm}(e_1, e_2) = \frac{1}{2}\left(\frac{\alpha_{11} - \alpha_{22}}{2} \pm J^{\perp}\alpha_{12} \pm iJ^{\perp}\left(\frac{\alpha_{11} - \alpha_{22}}{2} \pm J^{\perp}\alpha_{12}\right)\right). \tag{2.8}$$

The length of $\mathcal{H}^{\pm}(e_1, e_2)$ is independent of the frame field $\{e_1, e_2\}$, and the function $\|\mathcal{H}^{\pm}\|$ given by

$$\|\mathcal{H}^{\pm}\| = \sqrt{2}\left\|\mathcal{H}^{\pm}(e_1, e_2)\right\| = \sqrt{\|H\|^2 - (K - c) \mp K_N} \tag{2.9}$$

vanishes precisely on $M_0^{\pm}(f)$.

Let E be a complex vector bundle over M equipped with a connection ∇^E. An *E-valued differential* Ψ *of r-order* is an E-valued r-covariant tensor field on M of holomorphic type $(r, 0)$. The r-differential Ψ is called *holomorphic* (cf. [7]) if its covariant derivative $\nabla^E\Psi$ has holomorphic type $(r+1, 0)$. Let $(U, z = x + iy)$ be a local complex coordinate on M. The Wirtinger operators are defined on U by $\partial = \partial_z = (\partial_x - i\partial_y)/2$, $\bar{\partial} = \partial_{\bar{z}} = (\partial_x + i\partial_y)/2$, where $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$. On U, the differential Ψ has the form $\Psi = \psi dz^r$, where $\psi\colon U \to E$ is given by $\psi = \Psi(\partial, \ldots, \partial)$. Then Ψ is holomorphic if and only if

$$\nabla^E_{\bar{\partial}}\psi = 0,$$

i.e., ψ is a holomorphic section. For later use we need the following result (cf. [7, 16]).

Lemma 2.2. *Assume that the E-valued differential Ψ is holomorphic and let $p \in M$ be such that $\Psi(p) = 0$. Let (U, z) be a local complex coordinate with $z(p) = 0$. Then either $\Psi \equiv 0$ on U; or $\Psi = z^m\Psi^*$, where m is a positive integer and $\Psi^*(p) \neq 0$.*

Of particular importance for our results are two quadratic differentials associated to a surface in $\mathbb{Q}^4_c$, as well as their relation with the Gauss lifts of the surface to the twistor bundle. Let $f\colon M \to \mathbb{Q}^4_c$ be an oriented surface. In terms of a local complex coordinate $(U, z = x + iy)$, the metric ds^2 of M is written as $ds^2 = \lambda^2|dz|^2$, where $\lambda > 0$ is the conformal factor. Setting $e_1 = \partial_x/\lambda$ and $e_2 = \partial_y/\lambda$, the components of α are given by

$$\alpha^{(2,0)} = \alpha(\partial, \partial)dz^2, \quad \alpha^{(0,2)} = \overline{\alpha^{(2,0)}}, \quad \alpha^{(1,1)} = \alpha(\partial, \bar{\partial})(dz \otimes d\bar{z} + d\bar{z} \otimes dz),$$

where

$$\alpha(\partial, \partial) = \frac{\lambda^2}{2}\mathcal{H}(e_1, e_2), \quad \text{and} \quad \alpha(\partial, \bar{\partial}) = \frac{\lambda^2}{2}H. \tag{2.10}$$

The *Hopf differential of* f is the quadratic $N_fM \otimes \mathbb{C}$-valued differential $\Phi = \alpha^{(2,0)}$ with local expression $\Phi = \alpha(\partial, \partial)dz^2$. According to the decomposition of $N_fM \otimes \mathbb{C}$, the Hopf differential splits as

$$\Phi = \Phi^- + \Phi^+, \quad \text{where} \quad \Phi^\pm = \pi^\pm \circ \Phi.$$

On (U, z) the differential $\Phi^\pm$ has the expression

$$\Phi^\pm = \phi^\pm dz^2 \tag{2.11}$$

and the compatibility equations for f can be written as

$$\text{(Gauss)} \quad (\log \lambda^2)_{z\bar{z}} - \frac{2}{\lambda^2}\left(\langle \phi^-, \overline{\phi^-}\rangle + \langle \phi^+, \overline{\phi^+}\rangle\right) + \frac{\lambda^2}{2}(\|H\|^2 + c) = 0, \tag{2.12}$$

$$\text{(Codazzi)} \quad \nabla^\perp_{\bar\partial}\phi^- = \frac{\lambda^2}{2}\nabla^\perp_{\partial}H^-, \quad \nabla^\perp_{\bar\partial}\phi^+ = \frac{\lambda^2}{2}\nabla^\perp_{\partial}H^+, \tag{2.13}$$

$$\text{(Ricci)} \quad R^\perp(\partial, \bar\partial) = \frac{2}{\lambda^2}(\phi^- \wedge \overline{\phi^-} + \phi^+ \wedge \overline{\phi^+}), \tag{2.14}$$

where $R^\perp$ is the $\mathbb{C}$-trilinear extension of the normal curvature tensor and $(\xi \wedge \zeta)\eta = \langle \zeta, \eta\rangle\xi - \langle \xi, \eta\rangle\zeta$, for $\xi, \zeta, \eta \in N_fM \otimes \mathbb{C}$. It follows from (2.11) and (2.13) that Φ is holomorphic if and only if the mean curvature vector field H is parallel in the normal connection.

Lemma 2.3. *(i) The zero-sets of* $\Phi^\pm$ *and* Φ*, are* $M_0^\pm(f)$ *and* $M_1(f)$*, respectively.*
(ii) The surface f *is superconformal with normal curvature* $\pm K_N \geq 0$ *if and only if* $\Phi^\pm \equiv 0$*. In particular, if* f *is superconformal, then* K_N *vanishes precisely on* $M_1(f)$*.*

Proof: In terms of a local complex coordinate z around a point p, from (2.11), (2.10) and (2.9) it follows that $\Phi^\pm(p) = 0$ if and only if $\|\mathcal{H}^\pm\|(p) = 0$, or equivalently, if $p \in M_0^\pm(f)$. Obviously, Φ vanishes precisely at the points where both Φ^- and Φ^+ vanish, i.e., the umbilic points. This proves part (i), and the first assertion of part (ii) follows immediately. If f is superconformal, then the second equation in (2.6) implies that the normal curvature vanishes precisely at the umbilic points. ∎

2.3 Absolute Value Type Functions

We will need some facts about absolute value type functions (cf. [27] or [28]). Let M be a 2-dimensional oriented Riemannian manifold. A smooth function $u\colon M \to [0, +\infty)$ is called of *absolute value type*, if for all $p \in M$ and any complex coordinate z around p, there exists a non-negative integer m and a smooth positive function u_0 on a neighbourhood U of p such that

$$u = |z - z(p)|^m u_0, \quad \text{on} \quad U.$$

If $m > 0$ then p is called *a zero of u of multiplicity m*. It is clear that if an absolute value type function u does not vanish identically, then its zeros are isolated and they have well-defined multiplicities. Furthermore, the Laplacian $\Delta \log u$ is still defined and smooth at the zeros. If u does not vanish identically, then we denote by $N(u)$ the number of its zeros, counted with multiplicities. The following has been proved in [27].

Lemma 2.4. *Let M be a compact oriented 2-dimensional Riemannian manifold and u an absolute value type function on M. If u does not vanish identically, then*

$$\int_M \Delta \log u = -2\pi N(u).$$

A smooth complex function t on M is called of *holomorphic type* if locally it is expressed as $t = t_0 t_1$, where t_0 is holomorphic and t_1 is smooth without zeros. Clearly, if t is of holomorphic type then $u = |t|$ is of absolute value type.

2.4 Twistor Spaces and Gauss Lifts

Let $f\colon M \to \mathbb{R}^4$ be an oriented surface. We recall that the Grassmannian $Gr(2,4)$ of oriented 2-planes in $\mathbb{R}^4$, is isometric to the product $\mathbb{S}^2_+ \times \mathbb{S}^2_-$ of two spheres of radius $1/\sqrt{2}$ (we refer to Section 7.3.3 for details). Accordingly, the Gauss map $g\colon M \to Gr(2,4)$ of f, decomposes into a pair of maps as $g = (g_+, g_-)\colon M \to \mathbb{S}^2_+ \times \mathbb{S}^2_-$. For surfaces in not necessarily flat space forms $\mathbb{Q}^4_c$, the geometric information encoded in the components g_+ and g_- of the Gauss map of a surface in $\mathbb{R}^4$, is encoded in the Gauss lifts of the surface to the twistor bundle of $\mathbb{Q}^4_c$.

We recall some known facts about the twistor theory of 4-dimensional space forms. The reader may consult [26, 32], although the paper of Jensen and Rigoli [45] is closer to our approach. Let $O(\mathbb{Q}^4_c)$ be the principal $O(4)$-bundle of orthonormal frames in $\mathbb{Q}^4_c$, which has two connected components denoted by $O_+(\mathbb{Q}^4_c)$ and $O_-(\mathbb{Q}^4_c)$, corresponding to the two connected components of $O(4)$. The twistor bundle $\mathcal{Z}$ of $\mathbb{Q}^4_c$ is defined as the set of all pairs $(p, \tilde{J})$, where $p \in \mathbb{Q}^4_c$ and $\tilde{J}$ is an orthogonal complex structure on $T_p\mathbb{Q}^4_c$. The twistor projection $\varrho\colon \mathcal{Z} \to \mathbb{Q}^4_c$ is defined by $\varrho(p, \tilde{J}) = p$, and $\mathcal{Z}$ is an $O(4)/U(2)$-fiber bundle over $\mathbb{Q}^4_c$, which is associated to $O(\mathbb{Q}^4_c)$. Indeed, at a point $p \in \mathbb{Q}^4_c$ and for any orthonormal frame $e = (e_1, e_2, e_3, e_4)$ of $T_p\mathbb{Q}^4_c$, define an orthogonal complex structure $\tilde{J}_e$ by

$$\tilde{J}_e e_1 = e_2,\ \tilde{J}_e e_3 = e_4,\ \tilde{J}_e^2 = -I.$$

Any orthogonal complex structure on $T_p\mathbb{Q}^4_c$ is equal to $\tilde{J}_e$ for some orthonormal frame e of $T_p\mathbb{Q}^4_c$ and $\tilde{J}_e = \tilde{J}_{\tilde{e}}$ if and only if $\tilde{e} = eA$ for some $A \in U(2)$. Thus, the set of all orthogonal complex structures on $T_p\mathbb{Q}^4_c$ is $O(4)/U(2)$ and has two connected components isomorphic to $SO(4)/U(2) = \{\tilde{J}_e : e \text{ is a} \pm \text{oriented frame of } T_p\mathbb{Q}^4_c\}$. Hence, the twistor bundle is

$$\mathcal{Z} = O(\mathbb{Q}^4_c) \times_{O(4)} O(4)/U(2) = O(\mathbb{Q}^4_c)/U(2)$$

and its two connected components are denoted by $\mathcal{Z}_+$ and $\mathcal{Z}_-$. Each projection $\varrho_\pm\colon \mathcal{Z}_\pm \to \mathbb{Q}^4_c$ is a $P^1(\mathbb{C}) \simeq \mathbb{S}^2$-fiber bundle over $\mathbb{Q}^4_c$.

A one-parameter family of Riemannian metrics $g_t,\ t > 0$, is defined on $\mathcal{Z}$ in a natural way, making ϱ_+ and ϱ_- Riemannian submersions. With respect to the (common) decomposition of the tangent bundle of $\mathcal{Z}_\pm$ induced by the Levi-Civitá connection of g_t

$$T\mathcal{Z}_\pm = T^h\mathcal{Z}_\pm \oplus T^v\mathcal{Z}_\pm$$

into horizontal and vertical subbundles, the metric g_t is given by the pull-back of the metric of $\mathbb{Q}^4_c$ to the horizontal subspaces and by adding the t^2-fold of the metric of the fibers.

Denote by $Gr_2(T\mathbb{Q}^4_c)$ the Grassmann bundle of oriented 2-planes tangent to $\mathbb{Q}^4_c$. There are projections

$$\Pi_+\colon Gr_2(T\mathbb{Q}^4_c) \to \mathcal{Z}_+ \quad \text{and} \quad \Pi_-\colon Gr_2(T\mathbb{Q}^4_c) \to \mathcal{Z}_-$$

defined as follows; if $\zeta \subset T_p\mathbb{Q}^4_c$ is an oriented 2-plane, then $\Pi_\pm(p,\zeta)$ is the complex structure on $T_p\mathbb{Q}^4_c$ corresponding to the rotation by $+\pi/2$ on ζ and the rotation by $\pm\pi/2$ on $\zeta^\perp$. The Gauss lift $G_f\colon M \to Gr_2(T\mathbb{Q}^4_c)$, of an oriented surface $f\colon M \to \mathbb{Q}^4_c$ is defined by $G_f(p) = (f(p), f_*T_pM)$. The *Gauss lifts of f to the twistor bundle* are the maps

$$G_+\colon M \to \mathcal{Z}_+ \ \text{ and } \ G_-\colon M \to \mathcal{Z}_-, \ \text{ where } \ G_\pm = \Pi_\pm \circ G_f.$$

At any point $p \in M$, we obviously have $G_\pm(p) = (f(p), \tilde{J}_\pm(f(p)))$, where

$$\tilde{J}_\pm(f(p)) = \begin{cases} f_* \circ J(p), & \text{on } f_*T_pM, \\ \pm J^\perp(p), & \text{on } N_fM(p). \end{cases}$$

Let $\{e_j\}, 1 \le j \le 4$, be a $\pm$ oriented, local adapted orthonormal frame field of $\mathbb{Q}^4_c$, where $\{e_1, e_2\}$ is in the orientation of TM. Denote by $\{\omega_j\}, 1 \le j \le 4$, the corresponding coframe and by $\omega_{kl},\ 1 \le k,l \le 4$, the connection forms given by (2.3). The pull-back of g_t on M under $G_\pm$, is related to the metric ds^2 of M as follows

$$G^*_\pm(g_t) = ds^2 + \frac{t^2}{4}\left((\omega_{13} - \omega_{24})^2 + (\omega_{14} - \omega_{23})^2\right).$$

The covariant differential of the mean curvature vector field $H = H^3e_3 + H^4e_4$ is given by

$$\nabla^\perp H = \sum_{a=3}^{4}\Big(dH^a + \sum_{b=3}^{4} H^b\omega_{ba}\Big) \otimes e_a = \sum_{j=1}^{2}\sum_{a=3}^{4} H^a_j\omega_j \otimes e_a. \tag{2.15}$$

The Gauss lift $G_\pm\colon M \to (\mathcal{Z}_\pm, g_t)$ is called *conformal* if the metric $G^*_\pm(g_t)$ is conformal to ds^2. The following has been proved in [45].

Proposition 2.5. *Let $f\colon M \to \mathbb{Q}^4_c$ be an oriented surface. The Gauss lift $G_\pm\colon M \to (\mathcal{Z}_\pm, g_t)$ of f is conformal if and only if either f is minimal, or superconformal with normal curvature $\pm K_N \geq 0$.*

The Gauss lift $G_\pm\colon M \to (\mathcal{Z}_\pm, g_t)$ is called *vertically harmonic* if its tension field (cf. [68]) has vanishing vertical component with respect to the decomposition $T\mathcal{Z}_\pm = T^h\mathcal{Z}_\pm \oplus T^v\mathcal{Z}_\pm$. The squared length of the vertical component of the tension field of $G_\pm$ is computed in the following proposition. Its proof is a slight modification of the proof of Theorem 8.1. in [45] for space forms, where the scalar curvature of $\mathbb{Q}^4_c$ is normalized to be equal to c.

Proposition 2.6. *Let $f\colon M \to \mathbb{Q}^4_c$ be an oriented surface with mean curvature vector field H. Then, the squared length of the vertical component $\tau^v(G_\pm)$ of the tension field of the Gauss lift $G_\pm\colon M \to (\mathcal{Z}_\pm, g_1)$ of f is given by*

$$\|\tau^v(G_\pm)\|^2 = 4\left((H^3_1 \mp H^4_2)^2 + (H^3_2 \pm H^4_1)^2\right),$$

where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are positively oriented orthonormal frame fields of TM and N_fM, respectively, and H^a_j, $j = 1,2$, $a = 3,4$, is given by (2.15).

Proof: Let $H = H^{3\pm}e^\pm_3 + H^{4\pm}e^\pm_4$, where $\{e^\pm_3, e^\pm_4\}$ is a $\pm$ oriented orthonormal frame field of N_fM. The tension field of $G_\pm$, in terms of an appropriate orthonormal frame field $\{E^\pm_k,\ 1 \leq k \leq 6\}$ of $(\mathcal{Z}_\pm, g_t)$, is given by (cf. [45])

$$\tau(G_\pm) = \sum_{k=1}^{6} B^\pm_k E^\pm_k,$$

where

$$\begin{aligned} B^\pm_j &= 0 \text{ for } j = 1,2;\ \ B^\pm_a = 2H^{a\pm}(1 - ct^2) \text{ for } a = 3,4,\\ B^\pm_5 &= 2t(H^{4\pm}_2 - H^{3\pm}_1),\ \ B^\pm_6 = -2t(H^{4\pm}_1 + H^{3\pm}_2). \end{aligned}$$

Its vertical component is given by

$$\tau^v(G_\pm) = B^\pm_5 E^\pm_5 + B^\pm_6 E^\pm_6.$$

By setting $e^\pm_3 = e_3, e^\pm_4 = \pm e_4$, it follows that

$$g_t\left(\tau^v(G_\pm), \tau^v(G_\pm)\right) = 4t^2\left((H^3_1 \mp H^4_2)^2 + (H^3_2 \pm H^4_1)^2\right) \tag{2.16}$$

and this completes the proof. ∎

The following proposition relates the vertical harmonicity of the Gauss lift $G_\pm$ with the holomorphicity of the differential $\Phi^\pm$ and the holomorphicity of the section $H^\pm$. The equivalence of (i) and (iv) below, was proved by Hasegawa [36] who studied surfaces with a vertically harmonic Gauss lift.

Proposition 2.7. *Let $f\colon M \to \mathbb{Q}_c^4$ be a surface with mean curvature vector field H. The following are equivalent:*
(i) The Gauss lift $G_\pm\colon M \to (\mathcal{Z}_\pm, g_t)$ of f is vertically harmonic.
(ii) The differential $\Phi^\pm$ is holomorphic.
(iii) The section $H^\pm$ of $N_f^\pm M$ is anti-holomorphic.
(iv) $\nabla^\perp_{JX} H = \pm J^\perp \nabla^\perp_X H$, for any $X \in TM$.

Proof: The equivalence of (ii), (iii) and (iv) is an immediate consequence of the Codazzi equation (2.13). By virtue of (2.16), it follows that (i) is equivalent to (iv). ∎

From the proof of Proposition 2.6 it follows that if $t^2 = 1/c$, then $G_\pm$ is vertically harmonic if and only if it is harmonic.

It is clear from Proposition 2.7 that both Gauss lifts are vertically harmonic if and only if the surface has parallel mean curvature vector field in the normal connection. For surfaces in $\mathbb{R}^4$ this result is due to Ruh and Vilms [61].

Proposition 2.7 and Lemma 2.3(ii) imply that any superconformal surface $f\colon M \to \mathbb{Q}_c^4$ with $\pm K_N \geq 0$ has vertically harmonic Gauss lift $G_\pm$. The Gauss lift $G_\pm$ of such surfaces is holomorphic with respect to a complex structure $\mathcal{J}$ on $\mathcal{Z}$, that makes $(\mathcal{Z}, g_t)$ a Hermitian manifold (cf. [26, 45]). The following proposition shows that the converse is also true for non-minimal superconformal surfaces.

Proposition 2.8. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-minimal superconformal surface. If the Gauss lift $G_\pm$ of f is vertically harmonic, then $\Phi^\pm \equiv 0$.*

Proof: Arguing indirectly, assume that $\Phi^\pm \not\equiv 0$. From Proposition 2.7, we know that $\Phi^\pm$ is holomorphic and Lemma 2.2 implies that its zeros are isolated. From Lemma 2.3(ii) it follows that $\Phi^\mp \equiv 0$ and consequently Φ is holomorphic. Then, the mean curvature vector field of f is parallel. Hence, $K_N \equiv 0$ on M and Lemma 2.3(ii) implies that f is totally umbilical, a contradiction. ∎

Remark 2.9. In the case of $\mathbb{R}^4$, $(\mathcal{Z}_\pm, g_t)$ is isometric to the product $\mathbb{R}^4 \times \mathbb{S}^2(t)$. The Grassmann bundle is trivial $Gr_2(\mathbb{R}^4) \simeq \mathbb{R}^4 \times Gr(2,4)$ and the Gauss lift of f to the Grassmann bundle is given by $G_f = (f, g)$, where $g = (g_+, g_-)\colon M \to \mathbb{S}^2_+ \times \mathbb{S}^2_-$ is the Gauss map of f. The Gauss lift $G_\pm$ of f to the twistor bundle is then given by $G_\pm = (f, \sqrt{2}tg_\pm)$ and it is vertically harmonic if and only if $g_\pm$ is harmonic.

Chapter 3

The Mixed Connection Forms on Surfaces in $\mathbb{Q}_c^4$

In this chapter, we introduce two differential 1-forms Ω^- and Ω^+ associated to an oriented surface in $\mathbb{Q}_c^4$, called the mixed connection forms. Both forms are defined away from pseudo-umbilic points and at least one of them is defined away from umbilics. It turns out that the mixed connection forms on surfaces in $\mathbb{Q}_c^4$ generalize the connection form corresponding to principal frame fields of surfaces in $\mathbb{Q}_c^3$. This allows us to obtain an index theorem that extends the Poincaré-Hopf index theorem for surfaces with isolated umbilic points in $\mathbb{Q}_c^3$, which will be used for our results in the last chapter. We also introduce the notion of isotropically isothermic surfaces in $\mathbb{Q}_c^4$ as a generalization of the notion of isothermic surfaces in $\mathbb{Q}_c^3$. The notion of isothermicity for surfaces in $\mathbb{Q}_c^3$, has been extended for surfaces with flat normal bundle in arbitrary codimension by Palmer [57], and also for discrete surfaces in $\mathbb{R}^3$ (cf. [5, 52]). It turns out that in any case, isothermicity is a conformally invariant property. We show that isotropic isothermicity is also a conformally invariant property that extends the notion of isothermicity for surfaces in $\mathbb{Q}_c^3$, to surfaces in $\mathbb{Q}_c^4$ with not necessarily flat normal bundle.

3.1 An Index Theorem

Let $f\colon M \to \mathbb{Q}_c^4$ be an oriented surface with $M_0^\pm(f) = \emptyset$ and consider a local orthonormal frame field $\{e_1, e_2 = Je_1\}$ on an open $U \subset M$. By virtue of (2.8) and (2.9), the frame field $\{e_1, e_2\}$ determines a unique orthonormal frame field $\{e_3^\pm, e_4^\pm\}$ of N_fU such that

$$\mathcal{H}^\pm(e_1, e_2) = \frac{1}{2}\|\mathcal{H}^\pm\|(e_3^\pm \pm ie_4^\pm), \tag{3.1}$$

where

$$e_3^\pm = \|\mathcal{H}^\pm\|^{-1}\left(\frac{\alpha_{11} - \alpha_{22}}{2} \pm J^\perp\alpha_{12}\right), \quad e_4^\pm = J^\perp e_3^\pm, \tag{3.2}$$

and $\alpha_{ij} = \alpha(e_i, e_j), i, j = 1, 2$. Define the 1-form $\Omega^{\pm}(e_1, e_2)$ on U by

$$\Omega^{\pm}(e_1, e_2) = 2\omega_{12} \pm \omega_{34}^{\pm}, \tag{3.3}$$

where the connection forms ω_{12} and $\omega_{34}^{\pm}$, correspond to the dual frame field of $\{e_1, e_2, e_3^{\pm}, e_4^{\pm}\}$ and are given by (2.3).

Proposition 3.1. *Let $f\colon M \to \mathbb{Q}_c^4$ be an oriented surface with $M_0^{\pm}(f)$ isolated. Then:*
(i) There exists a 1-form $\Omega^{\pm}$ on $M \smallsetminus M_0^{\pm}(f)$ such that

$$\Omega^{\pm}|_U = \Omega^{\pm}(e_1, e_2) \tag{3.4}$$

for every positively oriented orthonormal frame field $\{e_1, e_2\}$ defined on an open $U \subset M \smallsetminus M_0^{\pm}(f)$.
(ii) The exterior derivative of $\Omega^{\pm}$ is globally defined on M and satisfies

$$d\Omega^{\pm} = -(2K \pm K_N)dM, \tag{3.5}$$

where dM is the volume element of M.
(iii) For every point $p \in M_0^{\pm}(f)$ the limit

$$I^{\pm}(p) = \lim_{r \to 0} \frac{1}{2\pi} \int_{S_r(p)} \Omega^{\pm} \tag{3.6}$$

exists, where $S_r(p)$ is a positively oriented geodesic circle of radius r centered at p.

Proof: (i) Let $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ be positively oriented orthonormal frame fields on an open, simply-connected $U \subset M \smallsetminus M_0^{\pm}(f)$. Since U is simply-connected, it follows that

$$\tilde{e}_1 - i\tilde{e}_2 = e^{i\tau}(e_1 - ie_2), \tag{3.7}$$

for some $\tau \in \mathcal{C}^{\infty}(U)$. This implies that

$$\tilde{\omega}_{12} = \omega_{12} + d\tau. \tag{3.8}$$

Consider the frame fields $\{e_3^{\pm}, e_4^{\pm}\}$ and $\{\tilde{e}_3^{\pm}, \tilde{e}_4^{\pm}\}$ of N_fU determined by $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$, respectively, from (3.2). From (2.7), (3.7) and (3.2) it follows that $\tilde{e}_3^{\pm} \pm i\tilde{e}_4^{\pm} = e^{2i\tau}(e_3^{\pm} \pm ie_4^{\pm})$. Therefore,

$$\tilde{\omega}_{34}^{\pm} = \omega_{34} \mp 2d\tau. \tag{3.9}$$

Using (3.8) and (3.9), from (3.3) we obtain that

$$\Omega^{\pm}(\tilde{e}_1, \tilde{e}_2) = \Omega^{\pm}(e_1, e_2).$$

By virtue of the above, we define $\Omega^{\pm}$ by (3.4), for an arbitrary positively oriented orthonormal frame field $\{e_1, e_2\}$ on a simply-connected $U \subset M \smallsetminus M_0^{\pm}(f)$. Clearly, $\Omega^{\pm}$ is globally defined on $M \smallsetminus M_0^{\pm}(f)$. From the definition of $\Omega^{\pm}$ it follows that (3.4) also holds for frame fields defined on non-simply-connected subsets $U \subset M \smallsetminus M_0^{\pm}(f)$.

(ii) Using part (i) and (2.2), exterior differentiation of (3.3) yields that (3.5) holds on $M \smallsetminus M_0^{\pm}(f)$. Since the right-hand side of (3.5) is defined globally on M, the proof follows.

(iii) Let $p \in M_0^{\pm}(f)$. Consider positively oriented geodesic circles $S_{r_1}(p)$, $S_{r_2}(p)$, $r_2 < r_1$, centered at p, and denote by D the annular region bounded by $S_{r_1}(p)$ and $S_{r_2}(p)$. Stokes' theorem implies that

$$\int_{S_{r_1}(p)} \Omega^{\pm} - \int_{S_{r_2}(p)} \Omega^{\pm} = \int_D d\Omega^{\pm}.$$

From part (ii) it follows that the right hand side of the above tends to zero as $r_1, r_2 \to 0$. This implies that any sequence $\int_{S_{r_n}(p)} \Omega^{\pm}$ with $r_n \to 0$, is a Cauchy sequence and thus, it converges. The proof now follows. ∎

Remark 3.2. Let $F\colon M \to \mathbb{Q}_c^3$ be an umbilic-free oriented surface with shape operator A and corresponding principal curvatures k_1, k_2, with $k_1 > k_2$. Every point $p \in M$ has a neighbourhood U at which there exists a principal frame field $\{e_1, e_2\}$ of F, i.e., a positively oriented orthonormal frame field of TU such that $Ae_l = k_l e_l, l = 1, 2$. Since a principal frame field of F is unique up to sign in its domain, it follows that there exists a 1-form Ω on M such that $\Omega|_U = \omega_{12}$, where ω_{12} is the connection form corresponding to the dual coframe of a principal frame field $\{e_1, e_2\}$ of F on $U \subset M$. We call Ω *the principal connection form of* F.

The following proposition shows that the mixed connection forms Ω^- and Ω^+ are the natural generalizations to surfaces in 4-dimensional space forms, of the principal connection form Ω of surfaces in 3-dimensional space forms.

Proposition 3.3. *Let $f\colon M \to \mathbb{Q}_c^4$ be the composition of an umbilic-free oriented surface $F\colon M \to \mathbb{Q}_{\tilde c}^3, \tilde c \geq c$, with a totally umbilical inclusion $j\colon \mathbb{Q}_{\tilde c}^3 \to \mathbb{Q}_c^4$. Then, $\Omega^- = \Omega^+ = 2\Omega$, where Ω is the principal connection form of F.*

Proof: Let ξ be the unit normal vector field of F in $\mathbb{Q}_{\tilde c}^3$ and A be the shape operator of F with respect to ξ. As in the Remark 3.2, let k_1, k_2, with $k_1 > k_2$ be the corresponding principal curvatures of F and consider a principal frame field $\{e_1, e_2\}$ of F on $U \subset M$. Proposition 3.1(i) and (3.3) imply that $\Omega^{\pm}|_U = \Omega^{\pm}(e_1, e_2) = 2\omega_{12} \pm \omega_{34}^{\pm}$. Moreover, for the second fundamental form α of f we have that $\alpha_{11} - \alpha_{22} = (k_1 - k_2) j_* \xi$ and $\alpha_{12} = 0$, where $\alpha_{kl} = \alpha(e_k, e_l), k, l = 1, 2$. Then, from (3.2) it follows that $e_3^- = e_3^+ = j_*\xi$. Since $j_*\xi$ is parallel in the normal connection of f, we obtain that $\omega_{34}^- = \omega_{34}^+ = 0$. Then, Proposition 3.1(i) and Remark 3.2 imply that $\Omega^-|_U = \Omega^+|_U = 2\Omega|_U$ and this completes the proof. ∎

Assume that $f\colon M \to \mathbb{Q}_c^4$ is a surface with $M_0^{\pm}(f)$ isolated. Proposition 3.1(i) allows us to express locally the mixed connection form $\Omega^{\pm}$ by (3.3), for an orthonormal frame field of the tangent bundle. In the sequel such a frame field will often arise from the basic vectors fields corresponding to a complex coordinate. Let $(U, z = x + iy)$ be a local complex coordinate on M and set $e_1 = \partial_x/\lambda$, $e_2 = \partial_y/\lambda$, where $\lambda > 0$ is the conformal factor. Then, the connection form ω_{12} of the corresponding coframe is given by $\omega_{12} = \star d \log \lambda$, where $\star$ is the Hodge star operator. In particular, exterior differentiation gives $d\omega_{12} = \Delta \log \lambda \omega_1 \wedge \omega_2$, where $\Delta = 4\lambda^{-2}\partial\bar{\partial}$ is the Laplacian on M, and (2.2) implies that the Gaussian curvature is given by $K = -\Delta \log \lambda$. Moreover, for the Hopf differential Φ of f, from (2.11), (2.10) and (3.1) it follows that

$$\phi^{\pm} = \frac{\lambda^2}{2}\mathcal{H}^{\pm}(e_1, e_2) = \frac{\lambda^2}{4}\|\mathcal{H}^{\pm}\|(e_3^{\pm} \pm ie_4^{\pm}) \quad \text{on} \quad U \smallsetminus M_0^{\pm}(f), \tag{3.10}$$

where $e_3^{\pm}, e_4^{\pm}$ are given by (3.2). Therefore, by virtue of Proposition 3.1(i), *the expression of* $\Omega^{\pm}$ *in terms of the complex coordinate* z is

$$\Omega^{\pm} = \star d \log \lambda^2 \pm \omega_{34}^{\pm} \quad \text{on} \quad U \smallsetminus M_0^{\pm}(f). \tag{3.11}$$

Proposition 3.4. *Let $f\colon M \to \mathbb{Q}_c^4$ be an oriented surface with $M_0^{\pm}(f)$ isolated. Let $p \in M_0^{\pm}(f)$ and (U, z) a simply-connected complex chart with $U \cap M_0^{\pm}(f) = \{p\}$ and $z(p) = 0$. If there exists a positive integer m such that the differential $\Phi^{\pm}$ is written as*

$$\Phi^{\pm} = z^m\hat{\Phi}^{\pm} \quad \textit{on} \quad U, \quad \hat{\Phi}^{\pm}(p) \neq 0, \tag{3.12}$$

then $I^{\pm}(p) = -m$.

Proof: Let $\Phi^{\pm} = \phi^{\pm}dz^2$ on U, where $\phi^{\pm}$ is given by (3.10) on $U \smallsetminus \{p\}$. For $r > 0$, consider a positively oriented geodesic circle $S_r(p) = \partial B_r(p) \subset U$. Stokes' theorem implies that $\int_{S_r(p)} \star d \log \lambda = -\int_{B_r(p)} K\omega_1 \wedge \omega_2$, and since the Gaussian curvature is bounded on $B_r(p)$, from (3.11) we obtain that

$$\lim_{r\to 0}\int_{S_r(p)} \Omega^{\pm} = \pm \lim_{r\to 0}\int_{S_r(p)} \omega_{34}^{\pm}. \tag{3.13}$$

Assume that $\hat{\Phi}^{\pm}$ is given by $\hat{\Phi}^{\pm} = \hat{\phi}^{\pm}dz^2$ on U. Since $\hat{\phi}^{\pm} \in N_f^{\pm}U$ and $\hat{\phi}^{\pm} \neq 0$ everywhere on U, there exist $R \in \mathcal{C}^{\infty}(U; (0, +\infty))$ and an orthonormal frame field $\{e_3, e_4\}$ of N_fU, such that $\hat{\phi}^{\pm} = R(e_3 \pm ie_4)$. Then, from (3.10) and (3.12) it follows that

$$\frac{\lambda^2}{2}\|\mathcal{H}^{\pm}\|(e_3^{\pm} \pm ie_4^{\pm}) = z^m R(e_3 \pm ie_4), \quad \text{on} \quad U \smallsetminus \{p\}. \tag{3.14}$$

Let $c(s), s \in [0, 2\pi]$, be a parametrization of $S_r(p)$ as a simple closed curve. Then, there exists a smooth function $\tau(s), s \in [0, 2\pi]$, such that

$$e_3^{\pm}(s) \pm ie_4^{\pm}(s) = e^{\mp i\tau(s)}(e_3(s) \pm ie_4(s)) \tag{3.15}$$

along c, and therefore

$$\frac{1}{2\pi}\int_{S_r(p)} \omega_{34}^{\pm} - \frac{1}{2\pi}\int_{S_r(p)} \omega_{34} = \frac{1}{2\pi}\int_{S_r(p)} d\tau. \tag{3.16}$$

We argue that the right hand side of (3.16) is equal to $\mp m$. From (3.14) and (3.15) it follows that along c we have

$$\frac{(\lambda(s))^2\|\mathcal{H}^{\pm}\|(s)}{2R(s)} = (z(s))^m e^{\pm i\tau(s)}.$$

Let $k(s)$ be the function at the left hand side of the above. Then $k(s) > 0, s \in [0, 2\pi]$, and $k(0) = k(2\pi)$. Hence, we have

$$\log k(s) = \log((z(s))^m e^{\pm i\tau(s)}).$$

Differentiating the above with respect to s and then integrating from 0 to 2π we obtain

$$0 = \log k(2\pi) - \log k(0) = m\int_0^{2\pi} \frac{z'(s)}{z(s)}ds \pm i\int_0^{2\pi} \tau'(s)ds,$$

or, equivalently

$$\frac{1}{2\pi}\int_{S_r(p)} d\tau = \mp\frac{m}{2\pi i}\int_{z(S_r(p))} \frac{dw}{w} = \mp m. \tag{3.17}$$

Since ω_{34} is defined everywhere on U and K_N is bounded on $B_r(p)$, by using (2.2) we obtain $\lim_{r\to 0}\int_{S_r(p)}\omega_{34} = \lim_{r\to 0}\int_{B_r(p)} d\omega_{34} = -\lim_{r\to 0}\int_{B_r(p)} K_N\omega_1\wedge\omega_2 = 0$. The proof follows by taking limits in (3.16) and using (3.13), (3.17) and (3.6). ∎

Theorem 3.5. *Let $f\colon M \to \mathbb{Q}^4_c$ be a compact oriented surface with $M_0^{\pm}(f)$ isolated. Then,*

$$2\chi \pm \chi_N = \sum_{p\in M_0^{\pm}(f)} I^{\pm}(p).$$

Proof: Let $M_0^{\pm}(f) = \{p_1, \dots, p_k\}$, where k is a nonnegative integer. For a sufficiently small $r > 0$, let $M_r = M \setminus (B_r(p_1)\cup\cdots\cup B_r(p_k))$, where $B_r(p_j)$ is the geodesic ball of radius r, centered at $p_j, j = 1, \dots, k$. Stokes' theorem implies that

$$\int_{M_r} d\Omega^{\pm} = -\sum_{j=1}^{k}\int_{S_r(p_j)}\Omega^{\pm},$$

where $\Omega^{\pm}$ is the form of Proposition 3.1(i), and $S_r(p_j) = \partial B_r(p_j)$ is positively oriented with respect to its interior. The above and (3.5) imply that

$$2\chi \pm \chi_N = -\frac{1}{2\pi}\lim_{r\to 0}\int_{M_r} d\Omega^{\pm} = \sum_{j=1}^{k}\frac{1}{2\pi}\lim_{r\to 0}\int_{S_r(p_j)}\Omega^{\pm}$$

and the proof follows from (3.6). ∎

In the sequel, we provide some applications of Theorem 3.5. The first one is a short proof of the following result due to Asperti [4].

Theorem 3.6. *If a compact 2-dimensional Riemannian manifold immerses isometrically into $\mathbb{Q}_c^4$ with everywhere non-vanishing normal curvature, then it is homeomorphic either to the sphere $\mathbb{S}^2$, or to the real projective space $\mathbb{R}P^2$.*

Proof: Let $\tilde{M}$ be a compact 2-dimensional Riemannian manifold and $f\colon \tilde{M} \to \mathbb{Q}_c^4$ an isometric immersion with $K_N \neq 0$ everywhere. Assume that $\tilde{M}$ is oriented and that $\pm K_N > 0$. Then, $M_0^{\mp}(f) = \emptyset$ and Theorem 3.5 implies that $2\chi = \pm\chi_N$. Since $\pm\chi_N > 0$, it follows that $\chi > 0$ and thus, $\tilde{M}$ is homeomorphic to $\mathbb{S}^2$. If $\tilde{M}$ is non-orientable, then we apply the previous procedure to the lift of f to the orientable double covering of $\tilde{M}$, and the proof follows. ∎

We mention here that a long-standing open problem posed by S.S. Chern [15, p. 45] is to investigate the existence of compact surfaces of negative Gaussian curvature in $\mathbb{R}^4$. In this direction, we obtain the following result.

Theorem 3.7. *Let M be a compact oriented 2-dimensional Riemannian manifold and $f\colon M \to \mathbb{Q}_c^4$ an isometric immersion. If $c \geq 0$ and the normal curvature of f does not change sign, then the Gaussian curvature K of M satisfies* $\max K \geq 0$.

Proof: Arguing indirectly, suppose that $\max K < 0$. Since $c \geq 0$, this implies that $M_1(f) = \emptyset$. Since K_N does not change sign, we may assume that $\pm K_N \geq 0$. Then $M_0^{\mp}(f) = \emptyset$ and as in the proof of Theorem 3.6 we obtain that M is homeomorphic to $\mathbb{S}^2$. The theorem of Gauss-Bonnet then implies that there exist points of M with positive Gaussian curvature and this is a contradiction. ∎

Immediate consequences of the above theorem are the following corollaries. The first one has been proved by Peng and Tang [58] for surfaces in $\mathbb{R}^4$.

Corollary 3.8. *Let M be a compact oriented 2-dimensional Riemannian manifold and $f\colon M \to \mathbb{Q}_c^4, c \geq 0$, an isometric immersion. If the normal curvature of f is constant, then there exists a point of M with nonnegative Gaussian curvature.*

Corollary 3.9. *Let M be a compact oriented 2-dimensional Riemannian manifold with Gaussian curvature $K < 0$. If there exists an isometric immersion $f\colon M \to \mathbb{Q}_c^4, c \geq 0$, then its normal curvature satisfies* $\min K_N < 0 < \max K_N$.

3.2 Isotropically Isothermic Surfaces

We introduce here the notion of isotropically isothermic surfaces in 4-dimensional space forms, as a generalization of the notion of isothermic surfaces in 3-dimensional space forms. We recall that an umbilic-free surface $F\colon M \to \mathbb{Q}_c^3$ is called isothermic if it admits conformal curvature line parametrization around every point. This is equivalent (see for instance [43]) with the co-closeness of the principal connection form Ω of F. Inspired by Proposition 3.3 we give the following definitions.

Let $f\colon M \to \mathbb{Q}_c^4$ be an oriented surface with $M_0^{\pm}(f) = \emptyset$. A point $p \in M$ is called a $\pm$ *isotropically isothermic point* for f if $d \star \Omega^{\pm}(p) = 0$. The surface $f\colon M \to \mathbb{Q}_c^4$ is called $\pm$ *(totally non) isotropically isothermic* if every point is $\pm$ (non) isotropically isothermic. Moreover, f is called *strongly (totally non) isotropically isothermic* if it is both $+$ and $-$ (totally non) isotropically isothermic. In the sequel, a $\pm$ isotropically isothermic surface is simply called *isotropically isothermic* in every case that we do not need to distinguish between the signs. In such a case, a $\pm$ totally non isotropically isothermic surface is called *half totally non isotropically isothermic.*

The following lemma provides a characterization of $\pm$ isotropically isothermic points in terms of a complex coordinate. Notice that if $f\colon M \to \mathbb{Q}_c^4$ is a surface with $M_0^{\pm}(f) = \emptyset$, then for every complex chart (U, z) on M there exists a smooth complex function $h^{\pm}$ on U such that the Hopf differential Φ of f satisfies

$$\nabla^{\perp}_{\bar{\partial}} \phi^{\pm} = h^{\pm} \phi^{\pm}, \tag{3.18}$$

where $\phi^{\pm}$ is given by (2.11) on U.

Lemma 3.10. *Let $f\colon M \to \mathbb{Q}_c^4$ be an oriented surface with $M_0^{\pm}(f) = \emptyset$. A point $p \in M$ is a $\pm$ isotropically isothermic point for f if and only if*

$$\operatorname{Im} h_z^{\pm}(p) = 0$$

for every complex chart (U, z) around p.

Proof: Let $(U, z = x + iy)$ be a complex chart around p and set $e_1 = \partial_x/\lambda$, $e_2 = \partial_y/\lambda$, where $\lambda > 0$ is the conformal factor. Consider the frame field $\{e_3^{\pm}, e_4^{\pm}\}$ of N_fU determined by $\{e_1, e_2\}$ from (3.1). Then (3.10) and (3.11) hold on U. From (3.18) and (3.10) it follows that

$$\nabla^{\perp}_{\bar{\partial}} \phi^{\pm} = \frac{\lambda^2}{4} \|\mathcal{H}^{\pm}\| h^{\pm} (e_3^{\pm} \pm i e_4^{\pm}) \quad \text{on} \quad U. \tag{3.19}$$

By differentiating (3.10) with respect to $\bar{\partial}$ in the normal connection, we obtain

$$\nabla^{\perp}_{\bar{\partial}} \phi^{\pm} = \frac{1}{4} \left(\bar{\partial}(\lambda^2 \|\mathcal{H}^{\pm}\|) \mp i\lambda^2 \|\mathcal{H}^{\pm}\| \omega_{34}^{\pm}(\bar{\partial}) \right) (e_3^{\pm} \pm i e_4^{\pm}).$$

The above and (3.19) yield

$$h^{\pm} = \bar{\partial} \log(\lambda^2 \|\mathcal{H}^{\pm}\|) \mp i\omega_{34}^{\pm}(\bar{\partial}). \tag{3.20}$$

By differentiating (3.20) with respect to z, and taking the imaginary part yields

$$\frac{4}{\lambda^2} \operatorname{Im} h_z^{\pm} = \mp \left(e_1(\log \lambda)\omega_{34}^{\pm}(e_1) + e_2(\log \lambda)\omega_{34}^{\pm}(e_2) + e_1(\omega_{34}^{\pm}(e_1)) + e_2(\omega_{34}^{\pm}(e_2)) \right).$$

From (3.11) and the above we obtain that

$$d \star \Omega^{\pm} = -\frac{4}{\lambda^2} \operatorname{Im} h_z^{\pm} \omega_1 \wedge \omega_2$$

and the proof follows. ∎

Proposition 3.11. *Let $f\colon M \to \mathbb{Q}_c^4$ be an oriented surface with $M_0^{\pm}(f) = \emptyset$. Then, f is $\pm$ isotropically isothermic if and only if for every simply-connected complex chart (U, z), the section $\phi^{\pm}$ given by (2.11) has the form*

$$\phi^{\pm} = D^{\pm}\xi^{\pm}, \tag{3.21}$$

where $D^{\pm}$ is a smooth positive function on U and $\xi^{\pm}$ is a nowhere vanishing holomorphic local section.

Proof: Let (U, z) be a simply-connected complex chart. Appealing to Proposition 3.1(i), we express $\Omega^{\pm}$ on U in terms of z as in (3.11).

Assume that f is $\pm$ isotropically isothermic. By virtue of (3.11) we have that $d\star\omega_{34}^{\pm} = 0$ and thus, there exists a smooth positive function $r^{\pm}$ on U such that

$$\omega_{34}^{\pm} = \mp \star d \log r^{\pm}. \tag{3.22}$$

We define $D^{\pm}$ and $\xi^{\pm}$ by

$$D^{\pm} = \frac{\lambda^2 \|\mathcal{H}^{\pm}\|}{4r^{\pm}} \quad \text{and} \quad \xi^{\pm} = r^{\pm}(e_3^{\pm} \pm i e_4^{\pm}), \tag{3.23}$$

respectively. By differentiating $\xi^{\pm}$ with respect to $\bar{\partial}$ in the normal connection yields

$$\nabla^{\perp}_{\bar{\partial}}\xi^{\pm} = \frac{1}{r^{\pm}}\left((\log r^{\pm})_{\bar{z}} \mp i\omega_{34}^{\pm}(\bar{\partial})\right)(e_3^{\pm} \pm i e_4^{\pm}). \tag{3.24}$$

From the above and (3.22), it follows that $\xi^{\pm}$ is holomorphic.

Conversely, assume that (3.21) holds on U. By setting $r^{\pm} = \|\xi^{\pm}\|/\sqrt{2}$, from (3.21) it follows that $\xi^{\pm}$ is given by (3.23). Therefore, (3.24) holds. Since $\xi^{\pm}$ is holomorphic, from (3.24) we obtain (3.22). Hence, $\omega_{34}^{\pm}$ is co-closed and (3.11) implies that $\star\Omega^{\pm}$ is closed on U. Since U is arbitrary, it follows that f is $\pm$ isotropically isothermic. ∎

It is clear that the characterization of $\pm$ isotropic isothermicity provided by Proposition 3.11 also makes sense for oriented surfaces immersed in orientable 4-dimensional Riemannian manifolds of not necessarily constant sectional curvature.

Proposition 3.12. *Let N be a Riemann surface and $F\colon N \to \mathbb{Q}_c^4$ a conformal immersion. The property of F equipped with its induced metric being isotropically isothermic is invariant under conformal changes of the metric of $\mathbb{Q}_c^4$. In particular, if F is $\pm$ isotropically isothermic and $\tau\colon \mathbb{Q}_c^4 \to \mathbb{Q}_c^4$ is an orientation-preserving conformal transformation, then the surface $\tau \circ F$ is also $\pm$ isotropically isothermic.*

Proof: Let $f\colon M \to \mathbb{Q}_c^4$ be the isometric immersion induced by F, where $M = (N, ds^2)$ and $ds^2 = F^*\langle\cdot,\cdot\rangle$. Consider the Riemannian manifold $\tilde{\mathbb{Q}}_c^4$, obtained from $\mathbb{Q}_c^4$ by the conformal change $\langle\cdot,\cdot\rangle_\mu = \mu^2\langle\cdot,\cdot\rangle$ of its metric, where $\mu \in \mathcal{C}^\infty(\mathbb{Q}_c^4; (0, +\infty))$, equipped with the same orientation with $\mathbb{Q}_c^4$. Then, F induces an isometric immersion $\tilde{f}\colon \tilde{M} \to \tilde{\mathbb{Q}}_c^4$, where $\tilde{M} = (N, d\tilde{s}^2)$ and $d\tilde{s}^2 = F^*\langle\cdot,\cdot\rangle_\mu = \mu^2 ds^2$.

Assume that f is $\pm$ isotropically isothermic. We argue that $\tilde{f}$ is also $\pm$ isotropically isothermic. It is clear that the normal bundles of f and $\tilde{f}$ coincide as vector bundles over N and they differ only in their bundle metric. In particular, they have the same complex structure $J^\perp$. It follows easily (see for instance [23]) that the second fundamental forms $\alpha, \tilde{\alpha}$ and the normal connections $\nabla^\perp, \tilde{\nabla}^\perp$, of $f, \tilde{f}$, respectively, are related by

$$\tilde{\alpha}(X,Y) = \alpha(X,Y) - \frac{1}{\mu}\langle X,Y\rangle(\operatorname{grad}\mu)^\perp, \tag{3.25}$$

and

$$\tilde{\nabla}^\perp_X\eta = \nabla^\perp_X\eta + \frac{1}{\mu}\langle \operatorname{grad}\mu, X\rangle\eta, \tag{3.26}$$

for all $X,Y \in TN$ and $\eta \in N_fM = N_{\tilde{f}}\tilde{M}$, where grad denotes the gradient with respect to $\langle\cdot,\cdot\rangle$. Let (U,z) be a complex chart on $\tilde{M}$ with conformal factor $\tilde{\lambda}$. Then, (U,z) is also a complex chart on M with conformal factor $\lambda = \tilde{\lambda}/\mu$. From (3.25) it follows that the Hopf differentials $\Phi, \tilde{\Phi}$ of $f, \tilde{f}$, respectively, coincide. In particular, if $\Phi^\pm$ is given by (2.11) and $\tilde{\Phi}^\pm = \tilde{\phi}^\pm dz^2$ on U, then $\phi^\pm = \tilde{\phi}^\pm$. Proposition 3.11 implies that $\phi^\pm = D^\pm\xi^\pm$, where $D^\pm$ is a smooth positive function on U and $\xi^\pm$ a nowhere vanishing $\nabla^\perp$-holomorphic local section. Then, we have that

$$\tilde{\phi}^\pm = \phi^\pm = \tilde{D}^\pm\tilde{\xi}^\pm, \quad \text{where} \quad \tilde{D}^\pm = \mu D^\pm \ \text{ and } \ \tilde{\xi}^\pm = \frac{1}{\mu}\xi^\pm.$$

Since $\xi^\pm$ is $\nabla^\perp$-holomorphic, from (3.26) we obtain that $\tilde{\xi}^\pm$ is $\tilde{\nabla}^\perp$-holomorphic. Therefore, Proposition 3.11 implies that $\tilde{f}$ is $\pm$ isotropically isothermic. The rest of the proof follows immediately. ∎

3.2.1 Examples

We provide here some classes of isotropically isothermic surfaces in $\mathbb{Q}_c^4$. When a surface $f\colon M \to \mathbb{Q}_c^4$ in some of the following classes is $\pm$ isotropically isothermic, it is always assumed that $M_0^\pm(f) = \emptyset$.

1. *Non-superconformal surfaces with a vertically harmonic Gauss lift are isotropically isothermic.*

Let $f\colon M \to \mathbb{Q}_c^4$ be a surface with $M_0^\pm(f) = \emptyset$. If the Gauss lift $G_\pm$ of f is vertically harmonic, then Proposition 2.7 implies that $\Phi^\pm$ is holomorphic. From Proposition 3.11 it follows that f is $\pm$ isotropically isothermic. Moreover, Proposition 3.12 implies that the composition of f with an orientation-preserving conformal transformation of $\mathbb{Q}_c^4$ which is not an isometry, gives rise to a $\pm$ isotropically isothermic surface $\tilde{f}$ whose corresponding Gauss lift $\tilde{G}_\pm$ is not vertically harmonic.

2. *Minimal superconformal surfaces are isotropically isothermic.*

Let $f\colon M \to \mathbb{Q}_c^4$ be a minimal superconformal surface with $M_0^{\pm}(f) = \emptyset$. Then, for the Hopf differential Φ of f we have $\Phi^{\mp} \equiv 0$ and thus, $\Phi = \Phi^{\pm}$. The Codazzi equation implies that Φ is holomorphic and from Proposition 3.11 it follows that f is $\pm$ isotropically isothermic. Moreover, Proposition 3.12 implies that the composition of f with an orientation-preserving conformal transformation of $\mathbb{Q}_c^4$ which is not an isometry, gives rise to a $\pm$ isotropically isothermic surface $\tilde{f}$ which is clearly non-minimal. In particular, since the property of the ellipse of curvature being a circle is conformally invariant, it follows that $\tilde{f}$ is superconformal.

3. *Non-superconformal minimal surfaces are strongly isotropically isothermic.*

Let $f\colon M \to \mathbb{Q}_c^4$ be a minimal surface with $M_0(f) = \emptyset$. The Codazzi equation implies that the Hopf differential of f is holomorphic and Proposition 3.11 yields that f is strongly isotropically isothermic. From Proposition 3.12 it follows that the composition of f with an orientation-preserving conformal transformation of $\mathbb{Q}_c^4$ which is not an isometry, determines a non-minimal, strongly isotropically isothermic surface $\tilde{f}$. Moreover, since the flatness of the normal bundle of a surface in $\mathbb{Q}_c^4$ is a conformally invariant property, it follows that if f has non-flat normal bundle, then the normal bundle of $\tilde{f}$ is also non-flat.

We recall that a surface $f\colon M \to \mathbb{Q}_c^4$ is called isothermic (cf. [57]) if around every point of M there exists a complex chart $(U, z = x + iy)$ with the property that its corresponding basic vector fields ∂_x, ∂_y diagonalize at every point of U all shape operators. By setting $e_1 = \partial_x/\lambda, e_2 = \partial_y/\lambda$, where λ is the conformal factor, it is straightforward to show that such a complex chart is characterized by the property that $\alpha(e_1, e_2) = 0$ at every point of U, where α is the second fundamental form of f.

4. *Isothermic surfaces lying in totally umbilical hypersurfaces of $\mathbb{Q}_c^4$ are strongly isotropically isothermic.*

Let $f\colon M \to \mathbb{Q}_c^4$ be an umbilic-free isothermic surface lying in $\mathbb{Q}_{\tilde{c}}^3, \tilde{c} \geq c$. Clearly, f is the composition of an isothermic surface $F\colon M \to \mathbb{Q}_{\tilde{c}}^3$ with a totally umbilical inclusion. Proposition 3.3 then implies that f is strongly isotropically isothermic.

5. *Examples of isothermic surfaces in $\mathbb{R}^4$ that are strongly totally non isotropically isothermic.*

Let $\gamma_j : I_j \to \mathbb{R}^2$ be a smooth curve parametrized by its arc length s_j, where I_j is an open interval, $j = 1, 2$. We denote by n_j the normal vector of γ_j such that $\{t_j = \dot{\gamma}_j, n_j\}$ is positively oriented, where the dot denotes the derivative with respect to s_j, $j = 1, 2$. By setting $M = I_1 \times I_2$ and $z = s_1 + is_2$, it is clear that z is a global complex coordinate on M with basic vector fields e_1, e_2, where $e_j = \partial/\partial s_j, j = 1, 2$. Moreover, the connection form of the corresponding coframe of $\{e_1, e_2\}$ satisfies $\omega_{12} = 0$. We consider the product surface $f\colon M \to \mathbb{R}^4$, $f = \gamma_1 \times \gamma_2$. Then the adapted to f frame field

$$\{f_* e_1 = (t_1, 0), N_1 = (n_1, 0), f_* e_2 = (0, t_2), N_2 = (0, n_2)\}$$

is positively oriented in $\mathbb{R}^4$. Therefore, $J^\perp N_1 = -N_2$. Let k_j be the curvature of γ_j, $j = 1, 2$. Then, for the second fundamental form α of f we have $\alpha_{11} = k_1 N_1$, $\alpha_{22} = k_2 N_2$ and $\alpha_{12} = 0$, where $\alpha_{ij} = \alpha(e_i, e_j), i, j = 1, 2$. Since $\alpha_{12} = 0$ it follows that f is isothermic.

Assume furthermore that f is umbilic-free, or equivalently, that there do not exist points (s_1, s_2) on M such that $k_1(s_1) = k_2(s_2) = 0$. We set

$$e_3 = \frac{\alpha_{11} - \alpha_{22}}{\|\alpha_{11} - \alpha_{22}\|} = \frac{1}{\sqrt{k_1^2 + k_2^2}}(k_1 N_1 - k_2 N_2), \quad e_4 = J^\perp e_3.$$

Then, (3.2) implies that $e_3 = e_3^- = e_3^+$. Since $\omega_{12} = 0$, from Proposition 3.1 and (3.3) it follows that f is strongly isotropically isothermic if and only if ω_{34} is co-closed. An easy computation shows that $d \star \omega_{34} = 0$ is equivalent to the differential equation

$$k_1 \ddot{k}_2 - \ddot{k}_1 k_2 + 2k_1 k_2 \frac{(\dot{k}_1)^2 - (\dot{k}_2)^2}{k_1^2 + k_2^2} = 0, \tag{3.27}$$

for the curvatures of γ_1 and γ_2, at every point of M, where each dot denotes a derivative of k_j with respect to $s_j, j = 1, 2$. Clearly, if $k_j(s_j) = c_j s_j, c_j \neq 0, j = 1, 2$, and $c_1 \neq c_2$, then for $s_1 s_2 > 0$ it follows from (3.27) that f is strongly totally non isotropically isothermic.

Chapter 4

Surfaces with the Same Mean Curvature

We develop here the required theory for the study of the Bonnet problem for non-minimal surfaces in 4-dimensional space forms. The case of minimal surfaces has been studied in [21] and [67]. In this chapter, we assume that all surfaces under consideration are non-minimal.

4.1 The Distortion Differential

Let M be a 2-dimensional oriented Riemannian manifold and $f, \tilde{f}\colon M \to \mathbb{Q}_c^4$ be isometric immersions with second fundamental forms $\alpha, \tilde{\alpha}$ and mean curvature vector fields $H, \tilde{H}$, respectively. The surfaces $f, \tilde{f}$ are said to have *the same mean curvature*, if there exists a parallel vector bundle isometry $T\colon N_fM \to N_{\tilde{f}}M$ such that $TH = \tilde{H}$.

Suppose that $f, \tilde{f}\colon M \to \mathbb{Q}_c^4$ have the same mean curvature and let $T\colon N_fM \to N_{\tilde{f}}M$ be a parallel vector bundle isometry satisfying $TH = \tilde{H}$. After an eventual composition of one of the surfaces with an orientation-reversing isometry of $\mathbb{Q}_c^4$, we may hereafter suppose that T is orientation-preserving. To such a pair $(f, \tilde{f})$ we assign a holomorphic differential which is going to play a fundamental role in the sequel. The section of $\mathrm{Hom}(TM \times TM, N_fM)$ given by

$$D^T_{f,\tilde{f}} = \alpha - T^{-1} \circ \tilde{\alpha}$$

measures how far the surfaces deviate from being congruent. Since $D^T_{f,\tilde{f}}$ is traceless, its $\mathbb{C}$-bilinear extension decomposes into its (k,l)-components, $k + l = 2$, as

$$D^T_{f,\tilde{f}} = (D^T_{f,\tilde{f}})^{(2,0)} + (D^T_{f,\tilde{f}})^{(0,2)}, \text{ where } (D^T_{f,\tilde{f}})^{(0,2)} = \overline{(D^T_{f,\tilde{f}})^{(2,0)}}.$$

We are interested into the $(2,0)$-part which is given by

$$Q^T_{f,\tilde{f}} = (D^T_{f,\tilde{f}})^{(2,0)} = \Phi - T^{-1} \circ \tilde{\Phi},$$

where $\Phi, \tilde{\Phi}$ stand for the Hopf differentials of $f, \tilde{f}$, respectively.

Lemma 4.1. *Let $f, \tilde{f} \colon M \to \mathbb{Q}_c^4$ be non-minimal surfaces and $T \colon N_fM \to N_{\tilde{f}}M$ an orientation-preserving, parallel vector bundle isometry satisfying $TH = \tilde{H}$. Then:*
(i) The quadratic differential $Q_{f,\tilde{f}}^T$ is holomorphic and independent of T.
(ii) The normal curvatures of the surfaces are equal and the curvature ellipses $\mathcal{E}_f$, $\mathcal{E}_{\tilde{f}}$ are congruent at any point of M. In particular, $M_0^{\pm}(f) = M_0^{\pm}(\tilde{f})$.

Proof: (i) From our assumption it follows that the section $T^{-1} \circ \tilde{\alpha}$ of $\mathrm{Hom}(TM \times TM, N_fM)$ satisfies the Codazzi equation for the data on N_fM and thus, $Q_{f,\tilde{f}}^T$ is holomorphic by (2.13).

Suppose that there exists another orientation-preserving parallel vector bundle isometry $S \colon N_fM \to N_{\tilde{f}}M$ with $SH = \tilde{H}$. We argue that $Q_{f,\tilde{f}}^T \equiv Q_{f,\tilde{f}}^S$. Set $L = T^{-1} \circ S$ and $U = \{p \in M : H(p) \neq 0\}$. On N_fU, L preserves both of H and $J^{\perp}H$ and thus, $T = S$ on N_fU. Therefore, the holomorphic differential $Q_{f,\tilde{f}}^T - Q_{f,\tilde{f}}^S$ vanishes identically on the open subset U of M. Then by Lemma 2.2, we obtain that $Q_{f,\tilde{f}}^T \equiv Q_{f,\tilde{f}}^S$ on M.

(ii) The vector bundle isometry T preserves the normal curvature tensors. Since it is orientation-preserving, (2.1) implies that the normal curvatures of $f, \tilde{f}$ are equal. The fact that the curvature ellipses are congruent, now follows from (2.6) and this completes the proof. ∎

Lemma 4.1(i) allows us to assign to each pair of surfaces $(f, \tilde{f})$ with the same mean curvature, a holomorphic differential denoted by $Q_{f,\tilde{f}}$, which is called *the distortion differential of the pair* and is given by

$$Q_{f,\tilde{f}} = \Phi - T^{-1} \circ \tilde{\Phi}.$$

Obviously, $Q_{f,\tilde{f}} \equiv 0$ if and only if f and $\tilde{f}$ are congruent. To simplify the notation, we denote the distortion differential associated to the pair $(f, \tilde{f})$ by Q, whenever there is no danger of confusion. A pair $(f, \tilde{f})$ of noncongruent surfaces with the same mean curvature is called a *pair of Bonnet mates*. In this case, the zero-set of Q is denoted by Z and according to Lemmas 2.2 and 4.1(i), consists of isolated points only.

With respect to the decomposition $N_fM \otimes \mathbb{C} = N_f^-M \oplus N_f^+M$, the distortion differential Q splits as

$$Q = Q^- + Q^+, \quad \text{where} \quad Q^{\pm} = \pi^{\pm} \circ Q.$$

It follows from Lemma 4.1(i) that each differential

$$Q^{\pm} = \Phi^{\pm} - T^{-1} \circ \tilde{\Phi}^{\pm} \tag{4.1}$$

is holomorphic. According to Lemma 2.2, either $Q^{\pm} \equiv 0$, or its zero-set $Z^{\pm}$ consists of isolated points only.

4.2 The Decomposition of the Moduli Space

Let $f\colon M \to \mathbb{Q}_c^4$ be a non-minimal oriented surface. We denote by $\mathcal{M}(f)$ *the moduli space of congruence classes of all isometric immersions of* M *into* $\mathbb{Q}_c^4$*, that have the same mean curvature with* f. Since the distortion differential of a pair of Bonnet mates does not vanish identically, the moduli space can be written as

$$\mathcal{M}(f) = \mathcal{N}^-(f) \cup \mathcal{N}^+(f) \cup \{f\},$$

where

$$\mathcal{N}^\pm(f) = \{\tilde{f}:\ Q^\pm_{f,\tilde{f}} \not\equiv 0\}/\mathrm{Isom}^+(\mathbb{Q}_c^4),$$

$\{f\}$ is the trivial congruence class and $\mathrm{Isom}^+(\mathbb{Q}_c^4)$ is the group of orientation-preserving isometries of $\mathbb{Q}_c^4$. Moreover, the moduli space decomposes into disjoint components as

$$\mathcal{M}(f) = \mathcal{M}^*(f) \cup \mathcal{M}^-(f) \cup \mathcal{M}^+(f) \cup \{f\},$$

where

$$\mathcal{M}^\pm(f) = \mathcal{N}^\pm(f) \smallsetminus \mathcal{N}^\mp(f) = \{\tilde{f}:\ Q_{f,\tilde{f}} \equiv Q^\pm_{f,\tilde{f}}\}/\mathrm{Isom}^+(\mathbb{Q}_c^4),$$

and

$$\mathcal{M}^*(f) = \mathcal{N}^-(f) \cap \mathcal{N}^+(f) = \{\tilde{f}:\ Q^-_{f,\tilde{f}} \not\equiv 0 \ \text{ and } \ Q^+_{f,\tilde{f}} \not\equiv 0\}/\mathrm{Isom}^+(\mathbb{Q}_c^4).$$

In order to simplify the notation in the sequel, we set $\bar{\mathcal{M}}^\pm(f) = \mathcal{M}^\pm(f) \cup \{f\}$.

Hereafter, whenever we refer to a surface in the moduli space we mean its congruence class. A surface $f\colon M \to \mathbb{Q}_c^4$ is called a *Bonnet surface* if $\mathcal{M}(f) \smallsetminus \{f\} \neq \emptyset$. Any $\tilde{f} \in \mathcal{M}(f) \smallsetminus \{f\}$ is called a *Bonnet mate* of f. A Bonnet surface f is called *proper Bonnet* if it admits infinitely many Bonnet mates.

4.3 Bonnet Mates

In view of Lemma 4.1(ii), we denote by $M_0 = M_0^- \cup M_0^+$ and M_1 the set of pseudo-umbilic and umbilic points of a pair of Bonnet mates, respectively.

Proposition 4.2. *If $\tilde{f} \in \mathcal{N}^\pm(f)$, then there exists $\theta^\pm \in \mathcal{C}^\infty(M \smallsetminus M_0^\pm; (0, 2\pi))$, such that the distortion differential of the pair $(f, \tilde{f})$ satisfies on $M \smallsetminus M_0^\pm$ the relation*

$$Q^\pm = (1 - e^{\mp i\theta^\pm})\Phi^\pm. \tag{4.2}$$

Moreover, $M_0^\pm = Z^\pm$ consists of isolated points only.

Proof: We first prove that there exists $\theta^\pm \in \mathcal{C}^\infty(M \smallsetminus Z^\pm; (0, 2\pi))$ such that (4.2) is valid on $M \smallsetminus Z^\pm$. Since $\tilde{f} \in \mathcal{N}^\pm(f)$, it follows that $Z^\pm$ is isolated. Lemma 4.1(ii) and (4.1) imply that $M_0^\pm \subset Z^\pm$ and thus, $M_0^\pm$ is isolated. We set $\beta = T^{-1} \circ \tilde{\alpha}$, where

$T\colon N_fM \to N_{\tilde{f}}M$ is an orientation and mean curvature vector field-preserving, parallel vector bundle isometry.

If $\mathrm{int}(M_0) \neq \emptyset$, then since $M_0^{\pm}$ is isolated, we obtain that $\mathrm{int}(M_0) \subset M_0^{\mp}$. Lemma 2.3(ii) implies that $\pm K_N < 0$, $\Phi^{\mp} \equiv 0$ and $\Phi^{\pm} \neq 0$ on $\mathrm{int}(M_0 \smallsetminus Z^{\pm})$. Let z be a local complex coordinate defined on a simply-connected neighbourhood $V \subset \mathrm{int}(M_0 \smallsetminus Z^{\pm})$. From Lemma 4.1(ii), it follows that the isotropic sections $\alpha(\partial, \partial)$ and $\beta(\partial, \partial)$ have the same length. Hence, there exists $\tau \in \mathcal{C}^{\infty}(V)$ with values in $(0, 2\pi)$, such that

$$\beta(\partial, \partial) = J_{\tau}^{\perp} \alpha(\partial, \partial),$$

where the rotation $J_{\tau}^{\perp} = \cos\tau I + \sin\tau J^{\perp}$ satisfies $J_{\tau}^{\perp} = e^{\mp i\tau} I$ on $N_f^{\pm}M$. Since $\Phi^{\pm} \neq 0$ on $\mathrm{int}(M_0 \smallsetminus Z^{\pm})$, the function τ is well-defined modulo 2π on $\mathrm{int}(M_0 \smallsetminus Z^{\pm})$. Moreover, it is non-vanishing modulo 2π on $\mathrm{int}(M_0 \smallsetminus Z^{\pm})$ and thus, there exists a branch in $\mathcal{C}^{\infty}(\mathrm{int}(M_0 \smallsetminus Z^{\pm}))$ with values in $(0, 2\pi)$. By setting $\theta^{\pm} = \tau$, we have that (4.2) holds on $\mathrm{int}(M_0 \smallsetminus Z^{\pm})$. In particular, the assertion is obvious if $M = M_0$.

Assume that $M \neq M_0$ and let $p \in M \smallsetminus M_0$. According to Lemma 2.1, there exist smooth frame fields $\{e_1, e_2, e_3, e_4\}, \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ on a neighbourhood $U \subset M \smallsetminus M_0$ of p, such that

$$\alpha_{11} - \alpha_{22} = 2\kappa e_3, \;\; \alpha_{12} = \mu e_4, \;\; \text{where} \;\; \alpha_{ij} = \alpha(e_i, e_j), \; j = 1, 2,$$

and

$$\beta_{11} - \beta_{22} = 2\tilde{\kappa}\tilde{e}_3, \;\; \beta_{12} = \tilde{\mu}\tilde{e}_4, \;\; \text{where} \;\; \beta_{ij} = \beta(\tilde{e}_i, \tilde{e}_j), \; j = 1, 2.$$

Lemma 4.1(ii) yields that the ellipses $\mathcal{E}_f(q)$ and $\mathcal{E}_{\beta}(q)$ are congruent at any point $q \in U$ and consequently, $\kappa = \tilde{\kappa}$. Using (2.1) and (2.14), we obtain that $K_N = 2\kappa\mu$ and $\tilde{K}_N = 2\tilde{\kappa}\tilde{\mu}$. Then, Lemma 4.1(ii) implies that $\mu = \tilde{\mu}$. Setting $\tilde{e}_3 - i\tilde{e}_4 = e^{i\theta}(e_3 - ie_4)$ for some $\theta \in \mathcal{C}^{\infty}(U)$, we have that

$$J_{\theta}^{\perp}(\alpha_{11} - \alpha_{22}) = \beta_{11} - \beta_{22} \;\; \text{and} \;\; J_{\theta}^{\perp}\alpha_{12} = \beta_{12} \;\; \text{on} \;\; U,$$

where $J_{\theta}^{\perp} = \cos\theta I + \sin\theta J^{\perp}$. This gives

$$\beta(\tilde{e}_1 - i\tilde{e}_2, \tilde{e}_1 - i\tilde{e}_2) = J_{\theta}^{\perp}\left(\alpha(e_1 - ie_2, e_1 - ie_2)\right).$$

Setting $\tilde{e}_1 - i\tilde{e}_2 = e^{i\sigma}(e_1 - ie_2)$ for some $\sigma \in \mathcal{C}^{\infty}(U)$, the above is written equivalently as

$$T^{-1} \circ \tilde{\Phi} = e^{i\theta^-}\Phi^- + e^{-i\theta^+}\Phi^+, \;\; \text{where} \;\; \theta^{\pm} = \theta \pm 2\sigma.$$

Since Φ^- and Φ^+ are everywhere non-vanishing on $M \smallsetminus M_0$, the functions θ^- and θ^+ are well-defined modulo 2π on $M \smallsetminus M_0$. From the assumption $\tilde{f} \in \mathcal{N}^{\pm}(f)$, it follows that $\theta^{\pm}$ is non-vanishing modulo 2π on $M \smallsetminus (M_0 \cup Z^{\pm})$ and thus, there exists a branch in $\mathcal{C}^{\infty}(M \smallsetminus (M_0 \cup Z^{\pm}))$ with values in $(0, 2\pi)$. Obviously, (4.2) holds on $M \smallsetminus (M_0 \cup Z^{\pm})$.

Lemma 4.1(ii) implies that for a point $q \in M_0 \smallsetminus (\mathrm{int}(M_0) \cup Z^{\pm})$, there exists a unique number $l(q) \in (0, 2\pi)$ such that

$$T^{-1} \circ \tilde{\Phi}(q) = J^{\perp}_{l(q)} \Phi(q),$$

where the rotation is given by $J^{\perp}_{l(q)} = e^{\mp i l(q)} I$, since $q \in M_0^{\mp}$. We extend $\theta^{\pm}$ on $M \smallsetminus Z^{\pm}$ by setting $\theta^{\pm}(q) = l(q)$. Then, (4.2) holds on $M \smallsetminus Z^{\pm}$. Since $Q^{\pm}$ and $\Phi^{\pm}$ are everywhere non-vanishing on $M \smallsetminus Z^{\pm}$, from (4.2) it follows that $\theta^{\pm}$ is smooth.

It remains to prove that $M_0^{\pm} = Z^{\pm}$. Arguing indirectly, assume that there exists $p \in Z^{\pm} \smallsetminus M_0^{\pm}$. From Lemma 2.3(i) it follows that $\Phi^{\pm}(p) \neq 0$. Since $Q^{\pm}$ and $\Phi^{\pm}$ are smooth, (4.2) implies that the function $k = e^{\mp i\theta^{\pm}}$ can be smoothly extended at p, with $k(p) = 1$.

We claim that $\theta^{\pm}$ can be continuously extended at p. Assume to the contrary that there exist sequences $p_n, q_n \in M \smallsetminus Z^{\pm}, n \in \mathbb{N}$, converging at p, such that $\theta^{\pm}(p_n) \to 0$ and $\theta^{\pm}(q_n) \to 2\pi$. Since $\theta^{\pm}$ is continuous on $M \smallsetminus Z^{\pm}$, it follows that for every $r > 0$ there exists $s_r \in B_r(p) \smallsetminus \{p\}$ such that $\theta^{\pm}(s_r) = \pi$, or equivalently, $k(s_r) = -1$. On the other hand, since k is continuous at p, there exists $r' > 0$ such that $|k - 1| < 1/2$ on $B_{r'}(p)$, which is a contradiction. Therefore, the limit of $\theta^{\pm}$ at p exists and the claim follows. Since $\theta^{\pm}$ is continuous and k is smooth on $M \smallsetminus M_0^{\pm}$, it follows that $\theta^{\pm}$ is also smooth on on $M \smallsetminus M_0^{\pm}$.

Let (U, z) be a complex chart with $U \cap Z^{\pm} = \{p\}$. From Lemmas 4.1(i) and 2.2 it follows that there exists a positive integer m such that $Q^{\pm} = z^m \Psi^{\pm}$ on U, and $\Psi^{\pm}(p) \neq 0$. Using (4.2), this is equivalent to

$$(1 - e^{\mp i\theta^{\pm}})\phi^{\pm} = z^m \psi^{\pm}, \quad \psi^{\pm}(p) \neq 0, \tag{4.3}$$

where $\phi^{\pm}$ is given by (2.11), and $\Psi^{\pm} = \psi^{\pm} dz^2$ on U. By differentiating (4.2) with respect to $\bar{\partial}$ in the normal connection and using the holomorphicity of $Q^{\pm}$ yields

$$\left(h^{\pm}(1 - e^{\mp i\theta^{\pm}}) \pm i e^{\mp i\theta^{\pm}} \theta^{\pm}_{\bar{z}}\right) \phi^{\pm} = 0,$$

where $h^{\pm}$ is given by (3.18). Since $\phi^{\pm} \neq 0$ everywhere on U, the above implies that

$$\theta^{\pm}_{\bar{z}} = \mp i h^{\pm}(1 - e^{\pm i\theta^{\pm}}), \quad \theta^{\pm}_{z} = \pm i \overline{h^{\pm}}(1 - e^{\mp i\theta^{\pm}}).$$

Since $\theta^{\pm}(p) = 0$, or 2π, from the above we obtain that all derivatives of $\theta^{\pm}$ vanish at p. Then, by differentiating (4.3) m-times with respect to ∂ in the normal connection, we obtain that $m! \psi^{\pm}(p) = 0$, which is a contradiction. Therefore, $M_0^{\pm} = Z^{\pm}$ and the proof follows. ∎

The following lemma is essential for our results.

Lemma 4.3. *Let M be a simply-connected oriented, 2-dimensional Riemannian manifold with a global complex coordinate z, and $f\colon M \to \mathbb{Q}^4_c$ an isometric immersion with $M_0^{\pm}(f)$ isolated. We consider the system*

$$\theta^{\pm}_{\bar z} = \mp i h^{\pm}(1 - e^{\pm i\theta^{\pm}}), \quad \theta^{\pm}_{z} = \pm i\overline{h^{\pm}}(1 - e^{\mp i\theta^{\pm}}), \tag{4.4}$$

where $h^{\pm}$ is given by (3.18) *on $M \smallsetminus M_0^{\pm}(f)$, and $\theta^{\pm} \in \mathcal{C}^{\infty}(M \smallsetminus M_0^{\pm}(f); \mathbb{R})$. Then, the graph of any solution of* (4.4) *is an integral surface of the 2-dimensional distribution $D^{\pm}$ on $\mathbb{R} \times (M \smallsetminus M_0^{\pm}(f))$, defined by the 1-form*

$$\rho^{\pm} = d\theta^{\pm} \mp i\overline{h^{\pm}}(1 - e^{\mp i\theta^{\pm}})dz \pm ih^{\pm}(1 - e^{\pm i\theta^{\pm}})d\bar z. \tag{4.5}$$

We have that:

(i) A function $\theta^{\pm} \in \mathcal{C}^{\infty}(M \smallsetminus M_0^{\pm}(f); \mathbb{R})$ satisfies (4.4) *if and only if*

$$A^{\pm}e^{\pm 2i\theta^{\pm}} - 2i(\operatorname{Im} A^{\pm})e^{\pm i\theta^{\pm}} - \overline{A^{\pm}} = 0, \tag{4.6}$$

where

$$A^{\pm} = i\left(h^{\pm}_z - |h^{\pm}|^2\right) = -\operatorname{Im} h^{\pm}_z + i(\operatorname{Re} h^{\pm}_z - |h^{\pm}|^2). \tag{4.7}$$

Moreover, if $\theta^{\pm}$ satisfies (4.4) *then*

$$\theta^{\pm}_{z\bar z} = \mp A^{\pm}(1 - e^{\pm i\theta^{\pm}}). \tag{4.8}$$

(ii) Assume that $h^{\pm}$ can be smoothly extended on M. Then, $D^{\pm}$ is involutive on $\mathbb{R} \times M$ if and only if $A^{\pm} \equiv 0$ on M. If $D^{\pm}$ is involutive then its maximal integral surfaces are graphs of solutions of (4.4) *on M. In particular, any solution of* (4.4) *on M is equivalent modulo 2π, either to a harmonic function $\theta^{\pm} \in \mathcal{C}^{\infty}(M; (0, 2\pi))$, or to the constant function $\theta^{\pm} \equiv 0$, and the space of the distinct modulo 2π solutions can be smoothly parametrized by $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$.*

(iii) If (4.4) *has a harmonic solution $\theta^{\pm} \in \mathcal{C}^{\infty}(M \smallsetminus M_0^{\pm}(f); (0, 2\pi))$, then $h^{\pm}$ can be smoothly extended on M and $A^{\pm} \equiv 0$.*

Proof: It is clear that the graph of any solution of (4.4) is an integral surface of $D^{\pm}$.

(i) Assume that $\theta^{\pm} \in \mathcal{C}^{\infty}(M \smallsetminus M_0^{\pm}(f); \mathbb{R})$ satisfies (4.4). Since its graph $\Sigma \subset \mathbb{R} \times (M \smallsetminus M_0^{\pm}(f))$ is an integral surface of $D^{\pm}$, the Frobenius condition yields that $\rho^{\pm} \wedge d\rho^{\pm} = 0$ on Σ, or equivalently, $\theta^{\pm}_{\bar z z} = \theta^{\pm}_{z\bar z}$ on $M \smallsetminus M_0^{\pm}(f)$. From (4.4) it follows that

$$\theta^{\pm}_{\bar z z} = \mp A^{\pm}(1 - e^{\pm i\theta^{\pm}}) \quad \text{and} \quad \theta^{\pm}_{z\bar z} = \mp\overline{A^{\pm}}(1 - e^{\mp i\theta^{\pm}}),$$

where $A^{\pm}$ is given by (4.7). The above implies (4.6) and (4.8). Conversely, if $\theta^{\pm}$ satisfies (4.6), then we have that $\rho^{\pm} \wedge d\rho^{\pm} = 0$ on its graph Σ. Therefore, Σ is an integral surface of $D^{\pm}$ and thus, (4.5) implies that $\theta^{\pm}$ satisfies (4.4) on $M \smallsetminus M_0^{\pm}(f)$.

(ii) From (4.5) and (4.7) it follows that $\rho^{\pm}$ and $A^{\pm}$ can be smoothly extended on $\mathbb{R}\times M$ and M, respectively. The Frobenius Theorem implies that $D^{\pm}$ is involutive if and only if $\rho^{\pm}\wedge d\rho^{\pm}\equiv 0$ on $\mathbb{R}\times M$, or equivalently, $A^{\pm}\equiv 0$ on M.

Assume that $D^{\pm}$ is involutive on $\mathbb{R}\times M$ and let Σ be a maximal integral surface. Then $\rho^{\pm}=0$ on Σ. Since M is simply-connected and $\rho^{\pm}$ is defined globally on $\mathbb{R}\times M$, from (4.5) it follows that Σ is the graph of a solution of (4.4) on M.

Let $\theta^{\pm}\in\mathcal{C}^{\infty}(M;\mathbb{R})$ be a solution of (4.4) on M. Since $A^{\pm}\equiv 0$ on M, from (4.8) it follows that $\theta^{\pm}$ is harmonic. It is clear that $\theta^{\pm}+2k\pi$ also satisfies (4.4) for every $k\in\mathbb{Z}$. Therefore, if $\theta^{\pm}\not\equiv 0 \mod 2\pi$, we may assume that $\theta^{\pm}(p)\in(0,2\pi)$ at some $p\in M$. Then, the graph of $\theta^{\pm}$ must lie between the graphs of the constant solutions 0 and 2π and thus, $\theta^{\pm}$ takes values in $(0,2\pi)$. Therefore, any solution of (4.4) on M is equivalent modulo 2π, either to a harmonic function $\theta^{\pm}\in\mathcal{C}^{\infty}(M;(0,2\pi))$, or to the constant function $\theta^{\pm}\equiv 0$.

Since $\mathbb{R}\times M$ is foliated by maximal integral surfaces of $D^{\pm}$, which are graphs over M of solutions of (4.4), it follows that the space of these surfaces can be parametrized by a smooth curve $\gamma(t)=(t,p), t\in\mathbb{R}$, where $p\in M$ is an arbitrary point. Obviously, the space of the distinct modulo 2π solutions of (4.4) can be smoothly parametrized by $\mathbb{S}^1\simeq\mathbb{R}/2\pi\mathbb{Z}$.

(iii) Let $\theta^{\pm}\in\mathcal{C}^{\infty}(M\smallsetminus M_0^{\pm}(f);(0,2\pi))$ be a harmonic function satisfying (4.4). Since $\theta^{\pm}$ is bounded with isolated singularities, it can be extended to a harmonic function $\theta^{\pm}\in\mathcal{C}^{\infty}(M;[0,2\pi])$. We claim that $\theta^{\pm}$ does not attain the values 0 and 2π on M. Arguing indirectly, assume that there exists a point at which $\theta^{\pm}$ attains the value 0 or 2π. Then $\theta^{\pm}$ has an interior minimum or maximum, respectively, and the maximum principle implies that $\theta^{\pm}\equiv 0$ or 2π, respectively, on M. This is a contradiction, since $\theta^{\pm}(p)\in(0,2\pi)$ for every $p\in M\smallsetminus M_0^{\pm}(f)$. Therefore, $\theta^{\pm}\in\mathcal{C}^{\infty}(M;(0,2\pi))$. From (4.4), it follows that $h^{\pm}$ can be smoothly extended at every point of $M_0^{\pm}(f)$. Since $\theta^{\pm}$ is harmonic, from (4.8) it follows that $A^{\pm}\equiv 0$ on M. ∎

Lemma 4.4. *(i) If $f_1\in\mathcal{M}^{-}(f_3)$ and $f_2\in\mathcal{M}^{+}(f_3)$, then $f_1\in\mathcal{M}^{*}(f_2)$.*
(ii) If $f_1,f_2\in\mathcal{M}^{\pm}(f_3)$, then $f_1\in\mathcal{M}^{\pm}(f_2)$.

Proof: Let $T_{jk}\colon N_{f_j}M\to N_{f_k}M$, $1\le j,k\le 3$, $j\neq k$, be orientation and mean curvature vector field-preserving, parallel vector bundle isometries. Denote by Q_{jk} and Φ_j the distortion differential of the pair (f_j,f_k) and the Hopf differential of f_j, respectively. From Lemma 4.1(i), we know that Q_{jk} is independent of T_{jk}. Hence,

$$Q_{12}=\Phi_1-T_{12}^{-1}\circ\Phi_2=\Phi_1-(T_{31}\circ T_{32}^{-1})\circ\Phi_2,$$

or equivalently,

$$T_{31}^{-1}\circ Q_{12}=T_{31}^{-1}\circ\Phi_1-T_{32}^{-1}\circ\Phi_2=Q_{31}-Q_{32}.$$

Therefore,

$$Q_{12}^{\pm}=T_{31}\circ(Q_{31}^{\pm}-Q_{32}^{\pm}) \tag{4.9}$$

and the results follow immediately. ∎

Proposition 4.5. *If $\tilde{f} \in \mathcal{N}^{\pm}(f)$, then the function $\theta^{\pm}$ of Proposition 4.2 satisfies* (4.4) *on $U \smallsetminus M_0^{\pm}$ for every complex chart (U, z) on M. Moreover, if one of the following holds, then it extends to a harmonic function $\theta^{\pm} \in \mathcal{C}^{\infty}(M; (0, 2\pi))$.*

(i) There exists $\hat{f} \in \mathcal{N}^{\pm}(f) \cap \mathcal{N}^{\pm}(\tilde{f})$.

(ii) The surface f is $\pm$ isotropically isothermic on $M \smallsetminus M_0^{\pm}$.

Proof: Let (U, z) be a complex chart on M. In the proof of Proposition 4.2 it was shown that $\theta^{\pm}$ satisfies (4.4) on $U \smallsetminus M_0^{\pm}$. We claim that if (i) or (ii) holds, then $\theta^{\pm}$ is harmonic on $U \smallsetminus M_0^{\pm}$.

(i) To unify the notation, set $f_1 = \tilde{f}$, $\theta_1^{\pm} = \theta^{\pm}$ and $f_2 = \hat{f}$. Proposition 4.2 implies that there exists $\theta_j^{\pm} \in \mathcal{C}^{\infty}(M \smallsetminus M_0^{\pm}; (0, 2\pi))$ such that the distortion differential Q_j of the pair (f, f_j), $j = 1, 2$, satisfies

$$Q_j^{\pm} = (1 - e^{\mp i\theta_j^{\pm}})\Phi^{\pm} \quad \text{on} \quad M \smallsetminus M_0^{\pm}, \tag{4.10}$$

where Φ is the Hopf differential of f. On the other hand, (4.9) implies that the distortion differential Q of the pair (f_1, f_2) satisfies

$$Q^{\pm} = T \circ (Q_1^{\pm} - Q_2^{\pm}),$$

where $T \colon N_f M \to N_{f_1} M$ is an orientation and mean curvature vector field-preserving, parallel vector bundle isometry. Therefore, from (4.10) and the above, it follows that

$$Q^{\pm} = (e^{\mp i\theta_2^{\pm}} - e^{\mp i\theta_1^{\pm}}) T \circ \Phi^{\pm} \quad \text{on} \quad M \smallsetminus M_0^{\pm}.$$

Since $f_2 \in \mathcal{N}^{\pm}(f_1)$, it is clear that $f_1 \in \mathcal{N}^{\pm}(f_2)$. Proposition 4.2 implies that $Q^{\pm}$ vanishes precisely on $M_0^{\pm}$ and from the above it follows that $\theta_1^{\pm} \neq \theta_2^{\pm}$ everywhere on $M \smallsetminus M_0^{\pm}$. Since $\theta_j^{\pm}, j = 1, 2$, satisfies (4.4) on $U \smallsetminus M_0^{\pm}$, from Lemma 4.3(i) it follows that it also satisfies (4.6). At every point of $U \smallsetminus M_0^{\pm}$, equation (4.6) viewed as a polynomial equation, has the distinct roots $1, e^{\mp i\theta_1^{\pm}}, e^{\mp i\theta_2^{\pm}}$. Hence, $A^{\pm} \equiv 0$ on $U \smallsetminus M_0^{\pm}$ and the claim follows by virtue of (4.8).

(ii) Arguing indirectly, assume that $\theta^{\pm}$ is not harmonic on $U \smallsetminus M_0^{\pm}$. Appealing to Lemma 4.3(i), equation (4.8) implies that there exists $p \in U \smallsetminus M_0^{\pm}$ such that $A^{\pm}(p) \neq 0$. On the other hand, Lemma 3.10 and (4.7) yield that $\operatorname{Re} A^{\pm} \equiv 0$ on $U \smallsetminus M_0^{\pm}$. Therefore $\operatorname{Re} A^{\pm}(p) = 0 \neq \operatorname{Im} A^{\pm}(p)$. Then, (4.6) implies that $e^{\pm i\theta^{\pm}(p)} = 1$. This is a contradiction, since $\theta^{\pm}$ takes values in $(0, 2\pi)$, and this proves the claim.

Since $\theta^{\pm}$ is a harmonic function satisfying (4.4) on $U \smallsetminus M_0^{\pm}$, Lemma 4.3(iii) implies that $h^{\pm}$ extends smoothly on U and $A^{\pm} \equiv 0$ on U. From Lemma 4.3(ii) it follows that $\theta^{\pm}$ extends to a harmonic function on U with values in $(0, 2\pi)$, satisfying (4.4) on U. Since U was arbitrary, this completes the proof. ∎

Chapter 5

Simply-Connected Surfaces

In this chapter, we study the Bonnet problem for surfaces $f\colon M \to \mathbb{Q}_c^4$, where M is a non-compact, simply-connected and oriented 2-dimensional Riemannian manifold. From the Uniformization Theorem it follows that M is conformally equivalent either to the complex plane, or to the unit disk. Therefore, in what follows in this chapter, M always admits a global complex coordinate z.

We point out that the non-compactness assumption is not restrictive for the most of our results at all. This will become clear in Chapter 8.

5.1 The Structure of the Moduli Space

The following theorem determines the possible structure of the moduli space $\mathcal{M}(f)$ for non-compact simply-connected surfaces $f\colon M \to \mathbb{Q}_c^4$.

Theorem 5.1. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-compact simply-connected, oriented surface.*

(i) If f is not proper Bonnet, then it admits either at most one Bonnet mate, or exactly three.

(ii) If f is proper Bonnet, then the moduli space $\mathcal{M}(f)$ is a space diffeomorphic to a manifold. Moreover, f is characterized according to the structure of $\mathcal{M}(f)$ as follows:

Tight: *The moduli space is 1-dimensional with at most two connected components diffeomorphic to $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$.*

Flexible: *The moduli space is diffeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$.*

For the proof of the above theorem, we need some auxiliary results.

Proposition 5.2. *Let M be a simply-connected oriented, 2-dimensional Riemannian manifold with a global complex coordinate z, and $f\colon M \to \mathbb{Q}_c^4$ a non-minimal surface with $M_0^{\pm}(f)$ isolated.*

(i) *If* $\tilde{f} \in \mathcal{M}^{\pm}(f)$ *and* $\mathcal{M}^{\pm}(f) \smallsetminus \{\tilde{f}\} \neq \emptyset$, *then there exists a harmonic function* $\theta^{\pm} \in \mathcal{C}^{\infty}(M;(0,2\pi))$ *satisfying* (4.4) *on* M, *such that the distortion differential of the pair* $(f,\tilde{f})$ *is given by* (4.2) *on* M.

(ii) *If* $h^{\pm}$ *can be smoothly extended on* M, *then the distinct modulo* 2π *solutions of* (4.4) *on* M *determine noncongruent surfaces in* $\bar{\mathcal{M}}^{\pm}(f)$. *In particular, any solution* $\theta^{\pm} \in \mathcal{C}^{\infty}(M;(0,2\pi))$ *determines a unique Bonnet mate* $\tilde{f} \in \mathcal{M}^{\pm}(f)$ *such that the distortion differential of the pair* $(f,\tilde{f})$ *is given by* (4.2) *on* M.

Proof: (i) Propositions 4.2 and 4.5 imply that there exists $\theta^{\pm} \in \mathcal{C}^{\infty}(M \smallsetminus M_0^{\pm};(0,2\pi))$ satisfying (4.4) on $M \smallsetminus M_0^{\pm}$, such that the distortion differential Q of the pair $(f,\tilde{f})$ is given by (4.2) on $M \smallsetminus M_0^{\pm}$. Let $\hat{f} \in \mathcal{M}^{\pm}(f) \smallsetminus \{\tilde{f}\}$. Lemma 4.4(ii) yields that $\hat{f} \in \mathcal{M}^{\pm}(f) \cap \mathcal{M}^{\pm}(\tilde{f})$. From Proposition 4.5 it follows that $\theta^{\pm}$ extends to a harmonic function $\theta^{\pm} \in \mathcal{C}^{\infty}(M;(0,2\pi))$. In particular, from the proof of Proposition 4.5 it follows that $\theta^{\pm}$ satisfies (4.4) on M. Proposition 4.2 implies that Q vanishes precisely on $M_0^{\pm}$. Then, from Lemma 2.3(i) it follows that Q is given by (4.2) on M.

(ii) Assume that $h^{\pm}$ can be smoothly extended on M. For a solution $\theta^{\pm}$ of (4.4), consider the quadratic differential

$$\Psi = \Phi^{\mp} + e^{\mp i\theta^{\pm}}\Phi^{\pm}. \tag{5.1}$$

By using (2.11), it is straightforward to check that Ψ satisfies equations (2.12) and (2.14) with respect to $\nabla^{\perp}, R^{\perp}, H$. Since $\theta^{\pm}$ satisfies (4.4), by using (2.11) it follows that $\Phi - \Psi$ is holomorphic. Therefore, Ψ satisfies the Codazzi equation. By the fundamental theorem of submanifolds, there exists a unique (up to congruence) isometric immersion $\tilde{f}\colon M \to \mathbb{Q}_c^4$, and an orientation-preserving parallel vector bundle isometry $T\colon N_fM \to N_{\tilde{f}}M$, such that the Hopf differential $\tilde{\Phi}$ and the mean curvature vector field $\tilde{H}$ of $\tilde{f}$ are given by $\tilde{\Phi} = T \circ \Psi$ and $\tilde{H} = TH$, respectively. Clearly, $\tilde{f}$ is congruent to f if and only if $\theta^{\pm} \equiv 0$ mod 2π. Furthermore, if $\tilde{f}$ is noncongruent to f then the distortion differential of the pair $(f,\tilde{f})$ satisfies $Q^{\mp} \equiv 0$ and thus, $\tilde{f} \in \mathcal{M}^{\pm}(f)$. In particular, if $\theta^{\pm} \in \mathcal{C}^{\infty}(M;(0,2\pi))$ then $\tilde{f} \in \mathcal{M}^{\pm}(f)$ and (5.1) implies that the distortion differential of the pair $(f,\tilde{f})$ is given by (4.2) on M. ∎

Theorem 5.3. *Let* M *be a non-compact simply-connected oriented, 2-dimensional Riemannian manifold, and* $f\colon M \to \mathbb{Q}_c^4$ *a non-minimal surface. Then:*

(i) *Either there exists at most one Bonnet mate of* f *in* $\mathcal{M}^{\pm}(f)$, *or the space* $\bar{\mathcal{M}}^{\pm}(f)$ *is diffeomorphic to* $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$.

(ii) *We have that* $\mathcal{M}^*(f) \neq \emptyset$ *if and only if* $\mathcal{M}^-(f) \neq \emptyset \neq \mathcal{M}^+(f)$. *If* $\mathcal{M}^*(f) \neq \emptyset$, *then there is a one-to-one correspondence between Bonnet mates* $\tilde{f} \in \mathcal{M}^*(f)$ *and pairs* f^-, f^+ *with* $f^{\pm} \in \mathcal{M}^{\pm}(f)$, *such that the distortion differential of the pair* $(f,\tilde{f})$ *is given by*

$$Q = Q_{f,f^-} + Q_{f,f^+},$$

where $Q_{f,f^{\pm}}$ *is the distortion differential of the pair* $(f,f^{\pm})$.

(iii) The surface f is proper Bonnet if and only if either $\bar{\mathcal{M}}^-(f) = \mathbb{S}^1$, or $\bar{\mathcal{M}}^+(f) = \mathbb{S}^1$.

(iv) The moduli space $\mathcal{M}(f)$ can be parametrized by the product $\bar{\mathcal{M}}^-(f) \times \bar{\mathcal{M}}^+(f)$. In particular, if f is proper Bonnet then $\mathcal{M}(f)$ is a smooth manifold.

Proof: Let z be a global complex coordinate on M.

(i) Assume that f admits at least two Bonnet mates in $\mathcal{M}^\pm(f)$ and let $\tilde{f} \in \mathcal{M}^\pm(f)$. Proposition 4.2 implies that $M_0^\pm$ is isolated. Since $\mathcal{M}^\pm(f) \smallsetminus \{\tilde{f}\} \neq \emptyset$, from Proposition 5.2(i) it follows that (4.4) has a harmonic solution $\theta^\pm \in \mathcal{C}^\infty(M \smallsetminus M_0^\pm; (0, 2\pi))$. Then, Lemma 4.3(iii) yields that $h^\pm$ can be smoothly extended on M and $A^\pm \equiv 0$. From Lemma 4.3(ii) it follows that the space of the distinct modulo 2π solutions of (4.4) can be smoothly parametrized by $\mathbb{S}^1$. The proof follows by virtue of Proposition 5.2(ii).

(ii) Assume that there exists $\tilde{f} \in \mathcal{M}^*(f)$ and consider the quadratic differentials

$$\Psi_{f^-} = \Phi - Q^- \quad \text{and} \quad \Psi_{f^+} = \Phi - Q^+,$$

where Φ is the Hopf differential of f and Q is the distortion differential of the pair $(f, \tilde{f})$. We argue that Ψ_{f^-} and Ψ_{f^+} satisfy the compatibility equations with respect to $\nabla^\perp, R^\perp, H$. From Lemma 4.1(i), it follows that $Q^\pm$ is holomorphic and thus, the differential $\Psi_{f^\pm}$ satisfies the Codazzi equation. Lemma 2.3(i) and Proposition 4.2 yield that $\Phi^\pm$ and $Q^\pm$ vanish precisely on $M_0^\pm$. Hence, $\Psi_{f^\pm}(p) = \Phi(p)$ at any point $p \in M_0^\pm$ and therefore $\Psi_{f^\pm}$ satisfies the algebraic equations (2.12) and (2.14) on $M_0^\pm$. Moreover, since $\tilde{f} \in \mathcal{M}^*(f)$, Proposition 4.2 implies that there exist θ^-, θ^+ with $\theta^\pm \in \mathcal{C}^\infty(M \smallsetminus M_0^\pm; (0, 2\pi))$ such that $Q^\pm$ is given by (4.2) on $M \smallsetminus M_0^\pm$. Using (4.2) and (2.11) it follows that $\Psi_{f^\pm}$ satisfies the equations (2.12) and (2.14) on $M \smallsetminus M_0^\pm$. The fundamental theorem of submanifolds implies that there exist unique Bonnet mates $f^-, f^+ \colon M \to \mathbb{Q}_c^4$ of f, such that the Hopf differential $\Phi_{f^\pm}$ of $f^\pm$ is given by $\Phi_{f^\pm} = T^\pm \circ \Psi_{f^\pm}$, where $T^\pm \colon N_f M \to N_{f^\pm} M$ is an orientation-preserving parallel vector bundle isometry. From Lemma 4.1(i), it follows that the distortion differential of the pair $(f, f^\pm)$ is $Q^\pm$ and thus, $f^\pm \in \mathcal{M}^\pm(f)$.

Conversely, assume that there exist f^-, f^+ with $f^\pm \in \mathcal{M}^\pm(f)$ and consider the quadratic differential Ψ with

$$\Psi^- = \Phi^- - Q_{f,f^-} \quad \text{and} \quad \Psi^+ = \Phi^+ - Q_{f,f^+},$$

where $Q_{f,f^\pm}$ is the distortion differential of the pair $(f, f^\pm)$. Lemma 4.1(i) implies that Q_{f,f^-} and Q_{f,f^+} are both holomorphic and thus, Ψ satisfies the Codazzi equation. From Lemma 2.3(i) and Proposition 4.2 it follows that $\Psi^\pm$ vanishes precisely on $M_0^\pm$. Furthermore, Proposition 4.2 implies that there exist θ^-, θ^+ with $\theta^\pm \in \mathcal{C}^\infty(M \smallsetminus M_0^\pm; (0, 2\pi))$ such that

$$Q_{f,f^\pm} = (1 - e^{\mp i\theta^\pm})\Phi^\pm \quad \text{on} \quad M \smallsetminus M_0^\pm.$$

Using the above and (2.11) it follows that Ψ satisfies (2.12) and (2.14) on $M \smallsetminus M_0$. Taking into account that $\Psi^\pm(p) = 0$ at any point $p \in M_0^\pm$, from the above and (2.11) we obtain that Ψ also satisfies (2.12) and (2.14) at any point of M_0. The fundamental theorem of submanifolds and Lemma 4.1(i) imply that there exists a unique Bonnet mate $\tilde{f}$ of f, such that the distortion differential of the pair $(f, \tilde{f})$ is $Q = Q_{f,f^-} + Q_{f,f^+}$. Clearly, $\tilde{f} \in \mathcal{M}^*(f)$.

If $\mathcal{M}^*(f) \neq \emptyset$, the above correspondence is obviously one-to-one.

(iii) Assume that f is proper Bonnet. Then at least one of the disjoint components of $\mathcal{M}(f)$ is infinite. From part (ii) it follows that at least one of $\mathcal{M}^-(f)$ and $\mathcal{M}^+(f)$ is infinite. If $\mathcal{M}^{\pm}(f)$ is infinite, then part (i) implies that $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$. The converse is obvious.

(iv) From Proposition 4.5 and the proof of part (i) it follows that $\bar{\mathcal{M}}^{\pm}(f)$ is parametrized by the space of the distinct modulo 2π solutions of (4.4). Then, Proposition 4.2 and part (ii) imply that the moduli space can be parametrized by pairs of functions (θ^-, θ^+), where $\theta^{\pm} \in \mathcal{C}^\infty(M \smallsetminus M_0^{\pm}, [0, 2\pi))$ satisfies (4.4). Moreover, according to this parametrization, $\theta^{\mp} \equiv 0$ correspond to $\bar{\mathcal{M}}^{\pm}(f)$. It is now clear that $\mathcal{M}(f)$ can be parametrized by $\bar{\mathcal{M}}^-(f) \times \bar{\mathcal{M}}^+(f)$. In particular, if f is proper Bonnet then parts (iii) and (i) imply that the moduli space is a smooth manifold. ∎

Remark 5.4. From the proof of Theorem 5.3(i) it follows that if $\bar{\mathcal{M}}^{\pm}(f)$ can be smoothly parametrized by $\mathbb{S}^1$, then its parametrization is induced by the parametrization of the space of the distinct modulo 2π solutions of (4.4). In the proof of Lemma 4.3(ii) the parametrization $\theta_t^{\pm}, t \in \mathbb{S}^1$, of these solutions is such that

$$\theta_t^{\pm}(p) = t, \quad t \in \mathbb{S}^1, \tag{5.2}$$

at a point $p \in M$. Obviously, this parametrization depends on p and is not unique, unless the solutions of (4.4) are constant. In this case from (4.4) it follows that $h^{\pm} \equiv 0$ on M. Then, (3.18) and Proposition 2.7 imply that the Gauss lift $G_{\pm}$ of f is vertically harmonic.

Proof of Theorem 5.1: Assume that f is non-minimal.

(i) If f is not proper Bonnet, then Theorem 5.3(iii) and (i) imply that f admits at most one Bonnet mate in each one of $\mathcal{M}^-(f)$ and $\mathcal{M}^+(f)$. If $\mathcal{M}^-(f) \neq \emptyset \neq \mathcal{M}^+(f)$, then Theorem 5.3(ii) yields that f admits exactly three Bonnet mates.

(ii) If f is proper Bonnet, then Theorem 5.3(iii) implies that either $\bar{\mathcal{M}}^-(f) = \mathbb{S}^1$, or $\bar{\mathcal{M}}^+(f) = \mathbb{S}^1$. Assume that $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$. From Theorem 5.3(i) and (iv) it follows that f is either tight, or flexible, if there exist either at most one, or infinitely many Bonnet mates of f in $\mathcal{M}^{\mp}(f)$, respectively.

If f is minimal then it is known that (cf. [20]) either $\mathcal{M}(f) = \{f\}$, or $\mathcal{M}(f) = \mathbb{S}^1$. ∎

5.2 Proper Bonnet Surfaces

We study here non-minimal proper Bonnet surfaces $f\colon M \to \mathbb{Q}_c^4$. By virtue of Theorem 5.3(iii-iv), we focus on surfaces with $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$. For such a surface, Proposition 4.2 implies that $M_0^{\pm}(f)$ consists of isolated points only.

Proposition 5.5. *Let $f\colon M \to \mathbb{Q}^4_c$ be a simply-connected surface with $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$. Let $p \in M_0^{\pm}(f)$ and consider a complex chart (U, z) with $U \cap M_0^{\pm}(f) = \{p\}$ and $z(p) = 0$. Then:*

(i) The differential $\Phi^{\pm}$ is written as

$$\Phi^{\pm} = z^m \hat{\Phi}^{\pm} \quad on \quad U, \quad \hat{\Phi}^{\pm}(p) \neq 0, \tag{5.3}$$

where m is a positive integer.

(ii) The function $\|\mathcal{H}^{\pm}\|$ is of absolute value type on M. The multiplicity of its zero $p \in M_0^{\pm}(f)$ is the integer m given by (5.3).

Proof: (i) Let $\tilde{f} \in \mathcal{M}^{\pm}(f)$. From Proposition 5.2(i) it follows that there exists $\theta^{\pm} \in \mathcal{C}^{\infty}(U; (0, 2\pi))$ such that the distortion differential Q of the pair $(f, \tilde{f})$ is given by

$$Q = (1 - e^{\mp i\theta^{\pm}})\Phi^{\pm} \quad \text{on} \quad M.$$

Proposition 4.2 implies that p is the only zero of Q in U. From Lemmas 4.1(i) and 2.2 it follows that there exists a positive integer m such that

$$Q = z^m \hat{\Psi}^{\pm} \quad \text{on} \quad U, \quad \hat{\Psi}^{\pm}(p) \neq 0.$$

The proof follows from the above expressions of Q, by setting $\hat{\Phi}^{\pm} = (1 - e^{\mp i\theta^{\pm}})^{-1}\hat{\Psi}^{\pm}$.

(ii) Let $z = x + iy$ and set $e_1 = \partial_x/\lambda, e_2 = \partial_y/\lambda$, where λ is the conformal factor. Let $\hat{\Phi}^{\pm} = \hat{\phi}^{\pm} dz^2$ on U. Part (i) implies that $\phi^{\pm} = z^m \hat{\phi}^{\pm}$, where $\phi^{\pm}$ is given by (2.11) on U. Then, from (3.10) it follows that

$$\|\mathcal{H}^{\pm}\| = |z|^m u, \ \text{ where } \ u = \sqrt{2}\lambda^{-2}\|\hat{\phi}^{\pm}\| \ \text{ is smooth and positive.}$$

Clearly, the multiplicity of p is m. ∎

Lemma 5.6. *Let M be an oriented, 2-dimensional Riemannian manifold with a global complex coordinate z, and $f\colon M \to \mathbb{Q}^4_c$ a surface with $M_0^{\pm}(f) = \emptyset$. Then, the 1-forms $a_1^{\pm}, a_2^{\pm}$ on M given by*

$$a_1^{\pm} = d \log \|\mathcal{H}^{\pm}\| - \star\Omega^{\pm}, \ \ a_2^{\pm} = \star a_1^{\pm}, \tag{5.4}$$

vanish precisely at the points where the Gauss lift $G_{\pm}$ of f is vertically harmonic. Moreover:

(i)

$$da_2^{\pm} = \Big(\Delta \log \|\mathcal{H}^{\pm}\| - 2K \mp K_N\Big)\, dM = \frac{4}{\lambda^2} \operatorname{Re} h_z^{\pm} dM,$$

(ii)

$$a_1^{\pm} \wedge a_2^{\pm} = \frac{\|\tau^v(G_{\pm})\|^2}{4\|\mathcal{H}^{\pm}\|^2} dM = \frac{4}{\lambda^2}|h^{\pm}|^2 dM,$$

where λ is the conformal factor and $h^{\pm}$ is given by (3.18) *on M.*

Proof: Let $z = x + iy$ and set $e_1 = \partial_x/\lambda$, $e_2 = \partial_y/\lambda$. Consider the frame field $\{e_3^{\pm}, e_4^{\pm}\}$ of $N_f U$ determined by $\{e_1, e_2\}$ from (3.1). Then (3.10) and (3.11) hold on M and as in the proof of Lemma 3.10 we obtain (3.19) and (3.20). Using (3.11), from (3.20) it follows that

$$a_1^{\pm} = \frac{2}{\lambda}\left(\operatorname{Re} h^{\pm}\omega_1 + \operatorname{Im} h^{\pm}\omega_2\right), \tag{5.5}$$

where $\{\omega_1, \omega_2\}$ is the dual frame field of $\{e_1, e_2\}$.

Proposition 2.7 and (3.18) imply that $h^{\pm}(p) = 0$ if and only if the Gauss lift $G_{\pm}$ of f is vertically harmonic at p. Therefore, from (5.5) it follows that $a_1^{\pm}$ vanishes precisely at the points where $G_{\pm}$ is vertically harmonic.

(i) Appealing to Proposition 3.1(ii), exterior differentiation of (5.4) gives

$$da_2^{\pm} = \left(\Delta \log \|\mathcal{H}^{\pm}\| - 2K \mp K_N\right) dM.$$

Differentiating the relation $\omega_{34}^{\pm} = \omega_{34}^{\pm}(e_1)\omega_1 + \omega_{34}^{\pm}(e_2)\omega_2$ and using (2.2) and the fact that $\omega_{12} = \star d \log \lambda$, we obtain

$$K_N = \mp\left(e_1(\log\lambda)\omega_{34}^{\pm}(e_2) - e_2(\log\lambda)\omega_{34}^{\pm}(e_1) + e_1(\omega_{34}^{\pm}(e_2)) - e_2(\omega_{34}^{\pm}(e_1))\right).$$

By differentiating (3.20) with respect to z, taking the real part, using the above and that $\Delta \log\lambda = -K$ yields

$$\frac{4}{\lambda^2}\operatorname{Re} h_z^{\pm} \;=\; \Delta \log \|\mathcal{H}^{\pm}\| - 2K \mp K_N$$

and the proof follows.

(ii) Let $H = H^{3\pm}e_3^{\pm} + H^{4\pm}e_4^{\pm}$ be the mean curvature vector field. Then,

$$H^{\pm} = \frac{1}{2}(H \pm iJ^{\perp}H) = \frac{1}{2}(H^{3\pm} \mp iH^{4\pm})(e_3^{\pm} \pm ie_4^{\pm}). \tag{5.6}$$

By differentiating (5.6) with respect to ∂ in the normal connection, we obtain from (2.13) that

$$\nabla_{\bar\partial}^{\perp}\phi^{\pm} = \frac{\lambda^2}{4}\left(\partial(H^{3\pm} \mp iH^{4\pm}) \mp i\omega_{34}^{\pm}(\partial)(H^{3\pm} \mp iH^{4\pm})\right)(e_3^{\pm} \pm ie_4^{\pm}).$$

From (3.19) and the above it follows that

$$h^{\pm} = \frac{\lambda}{2}\left(\frac{H_1^{3\pm} \mp H_2^{4\pm}}{\|\mathcal{H}^{\pm}\|} - i\frac{H_2^{3\pm} \pm H_1^{4\pm}}{\|\mathcal{H}^{\pm}\|}\right), \tag{5.7}$$

where $H_j^{a\pm}$, $j = 1, 2,\ a = 3, 4$, is given by (2.15). Then, (5.5) implies that

$$a_1^{\pm} = u^{\pm}\omega_1 + v^{\pm}\omega_2, \quad \text{where} \quad u^{\pm} = \frac{H_1^{3\pm} \mp H_2^{4\pm}}{\|\mathcal{H}^{\pm}\|},\;\; v^{\pm} = -\frac{H_2^{3\pm} \pm H_1^{4\pm}}{\|\mathcal{H}^{\pm}\|}. \tag{5.8}$$

From (5.7) and (5.8) it follows that

$$a_1^\pm \wedge a_2^\pm = \Big((u^\pm)^2 + (v^\pm)^2\Big)\, dM = \frac{4}{\lambda^2}|h^\pm|^2 dM,$$

where $dM = \omega_1 \wedge \omega_2$. On the other hand, from Proposition 2.6 we obtain

$$\|\tau^v(G_\pm)\|^2 = g_1(\tau^v(G_\pm), \tau^v(G_\pm)) = 4\|\mathcal{H}^\pm\|^2 \Big((u^\pm)^2 + (v^\pm)^2\Big)$$

and the proof follows. ∎

Theorem 5.7. *Let $f\colon M \to \mathbb{Q}_c^4$ be a simply-connected surface. If $\bar{\mathcal{M}}^\pm(f) = \mathbb{S}^1$, then:*
(i) The Gauss lift $G_\pm$ of f is vertically harmonic at any point of $M_0^\pm(f)$.
(ii) The surface f is $\pm$ isotropically isothermic on $M \smallsetminus M_0^\pm(f)$, and the differential equation

$$\Delta \log \|\mathcal{H}^\pm\| - 2K \mp K_N = \frac{\|\tau^v(G_\pm)\|^2}{4\|\mathcal{H}^\pm\|^2} \tag{5.9}$$

is valid on M.
(iii) The forms $a_1^\pm, a_2^\pm$ of Lemma 5.6 satisfy on $M \smallsetminus M_0^\pm(f)$ the relations

$$da_1^\pm = 0, \tag{5.10}$$
$$da_2^\pm = a_1^\pm \wedge a_2^\pm. \tag{5.11}$$

Conversely, if $M_0^\pm(f) = \emptyset$ and (ii) or (iii) holds, then $\bar{\mathcal{M}}^\pm(f) = \mathbb{S}^1$.

Proof: Let $\tilde f \in \mathcal{M}^\pm(f)$. Proposition 4.2 yields that $M_0^\pm$ is isolated. From Proposition 5.2(i) it follows that there exists a harmonic function $\theta^\pm \in \mathcal{C}^\infty(M;(0,2\pi))$ satisfying (4.4) on M. Lemma 4.3(iii) implies that $h^\pm$ can be smoothly extended on M and $A^\pm \equiv 0$. Then, from (4.7) it follows that

$$\operatorname{Im} h_z^\pm \equiv 0 \quad \text{and} \quad |h^\pm|^2 \equiv \operatorname{Re} h_z^\pm \quad \text{on} \quad M. \tag{5.12}$$

(i) Since $h^\pm$ extends smoothly on M, it follows that (3.18) holds on M. From Lemma 2.3(i) and (3.18) we obtain that

$$\nabla^\perp_{\bar\partial}\phi^\pm(p) = 0 \quad \text{for any} \quad p \in M_0^\pm(f).$$

Appealing to Proposition 2.7, this is equivalent with the vertical harmonicity of $G_\pm$ at p.

(ii) By virtue of Lemma 3.10, the first equation in (5.12) implies that f is $\pm$ isotropically isothermic on $M \smallsetminus M_0^\pm$. Using Lemma 5.6, the second equation in (5.12) yields that (5.9) holds on $M \smallsetminus M_0^\pm$. From Proposition 5.5(ii) it follows that the left-hand side of (5.9) can be smoothly extended on M. Therefore, (5.9) is valid on M.

(iii) From (5.4) it follows that (5.10) is equivalent with the fact that f is $\pm$ isotropically isothermic on $M \smallsetminus M_0^\pm$, and Lemma 5.6 implies that (5.11) is equivalent with the second equation in (5.12) on $M \smallsetminus M_0^\pm$.

Conversely, assume that $M_0^{\pm}(f) = \emptyset$. As in the proofs of (ii) and (iii), we obtain that (ii) and (iii) are both equivalent to (5.12). Then, from (4.7) it follows that $A^{\pm} \equiv 0$ on M and Lemma 4.3(ii) implies that the space of the distinct modulo 2π solutions of (4.4) on M is parametrized by $\mathbb{S}^1$. From Proposition 5.2(ii) and Theorem 5.3(i) it follows that $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$. ∎

Corollary 5.8. *Let $f\colon M \to \mathbb{Q}_c^4$ be a simply-connected surface. If $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$ and $\tau^v(G_{\pm}) \neq 0$ everywhere, then the conformal metric*

$$d\hat{s}^2 = \frac{\|\tau^v(G_{\pm})\|^2}{4\|\mathcal{H}^{\pm}\|^2} ds^2 \tag{5.13}$$

has Gaussian curvature $\hat{K} = -1$.

Proof: By virtue of Theorem 5.7(i), it follows that $M_0^{\pm}(f) = \emptyset$. Consider the forms $a_1^{\pm}, a_2^{\pm}$ of Lemma 5.6. Proposition 2.6 and (5.8) yield that

$$d\hat{s}^2 = a_1^{\pm} \otimes a_1^{\pm} + a_2^{\pm} \otimes a_2^{\pm} \quad \text{on} \quad M.$$

Let $a_{12}^{\pm}$ be the connection form associated to the coframe $\{a_1^{\pm}, a_2^{\pm}\}$. Then,

$$da_2^{\pm} = a_1^{\pm} \wedge a_{12}^{\pm} \quad \text{and} \quad da_{12}^{\pm} = -\hat{K} a_1^{\pm} \wedge a_2^{\pm}.$$

Since $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$, the first equation of the above and (5.11) yield that $a_{12}^{\pm} = a_2^{\pm}$. Using the second equation of the above, this implies that $da_2^{\pm} = -\hat{K} a_1^{\pm} \wedge a_2^{\pm}$, and the proof follows by virtue of (5.11). ∎

Remark 5.9.

(i) Theorems 5.3(i) and 5.7(i) imply that a surface f admits at most one Bonnet mate in $\mathcal{M}^{\pm}(f)$, if there exists a point $p \in M_0^{\pm}(f)$ at which the Gauss lift $G_{\pm}$ of f is not vertically harmonic. By virtue of Theorem 5.3(iv), it follows that f admits at most three Bonnet mates if there exists an umbilic point at which H is not parallel. This extends a result of Roussos-Hernandez [60, Thm. 1B].

(ii) For umbilic-free surfaces in $\mathbb{R}^3$, integrability conditions similar to (5.10) and (5.11) are due to Chern [18] and the analogue of equation (5.9) is due to Colares and Kenmotsu [19].

5.3 The Effect of Isotropic Isothermicity

The structure of the moduli space $\mathcal{M}(f)$ is seriously affected by the property of isotropic isothermicity for f, as the following theorem shows.

Theorem 5.10. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-compact simply-connected oriented surface.*

(i) If f is half totally non isotropically isothermic, then f admits at least one Bonnet mate and it is not flexible. In particular, if f is strongly totally non isotropically isothermic, then it admits exactly three Bonnet mates.

(ii) If f is proper Bonnet, then it is isotropically isothermic on an open, dense and connected subset of M. In particular, if f is flexible, then it is strongly isotropically isothermic away from its isolated pseudo-umbilic points.

Proof: (i) Assume that f is totally non $\pm$ isotropically isothermic. From the examples 2 and 3 of Section 3.2.1, it follows that f is non-minimal. Let z be a global complex coordinate on M. Lemma 3.10 implies that $\operatorname{Im} h_z^{\pm} \neq 0$ everywhere on M and therefore, from (4.7) it follows that $A^{\pm} \neq 0$ everywhere on M. Then, Lemma 4.3(i) yields that the solution $e^{\pm i\theta^{\pm}} = -\overline{A^{\pm}}/A^{\pm}$ of equation (4.6), determines the unique solution $\theta^{\pm} \in \mathcal{C}^{\infty}(M;(0,2\pi))$ of (4.4) on M. Proposition 5.2(ii) implies that there exists a unique Bonnet mate of f in $\mathcal{M}^{\pm}(f)$. From Theorem 5.3(iv) it follows that f is not flexible. In particular, if f is strongly totally non isotropically isothermic, then it admits a unique Bonnet mate in each one of $\mathcal{M}^{-}(f)$ and $\mathcal{M}^{+}(f)$. Then, from Theorem 5.3(iii) it follows that there exists exactly one Bonnet mate of f in $\mathcal{M}^{*}(f)$, and the proof follows.

(ii) Assume that f is non-minimal and appealing to Theorem 5.3(iii), let $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$. Proposition 4.2 implies that $M_0^{\pm}(f)$ is isolated. From Theorem 5.7(ii) it follows that f is $\pm$ isotropically isothermic on $M \smallsetminus M_0^{\pm}(f)$, which is an open, dense and connected subset of M. In particular, if f is flexible, then Theorem 5.3(i) and (iv) implies that $\bar{\mathcal{M}}^{-}(f) = \mathbb{S}^1$ and $\bar{\mathcal{M}}^{+}(f) = \mathbb{S}^1$. Then, from Theorem 5.7(ii) it follows that f is strongly isotropically isothermic on $M \smallsetminus M_0(f)$. If f is minimal, the proof follows from the examples 2 and 3 of Section 3.2.1. ∎

The following proposition shows that a Bonnet, strongly isotropically isothermic surface is proper Bonnet.

Proposition 5.11. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-compact, simply-connected oriented surface. If f is $\pm$ isotropically isothermic and non-minimal, then either $\bar{\mathcal{M}}^{\pm}(f) = \{f\}$, or $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$. In particular, if f is Bonnet and strongly isotropically isothermic then either $\mathcal{M}(f) = \mathbb{S}^1$, or $\mathcal{M}(f) = \mathbb{S}^1 \times \mathbb{S}^1$.*

Proof: Assume that there exists $\tilde{f} \in \mathcal{M}^{\pm}(f)$. Proposition 4.2 implies that there exists $\theta^{\pm} \in \mathcal{C}^{\infty}(M;(0,2\pi))$, such that the distortion differential of the pair $(f,\tilde{f})$ is given by (4.2) on M. Let z be a global complex coordinate on M. From Proposition 4.5 it follows that $\theta^{\pm}$ is harmonic and satisfies (4.4) on M. Then, Lemma 4.3(iii-ii) implies that the space of the distinct modulo 2π solutions of (4.4) is parametrized by $\mathbb{S}^1$. From Proposition 5.2(ii) it follows that $\bar{\mathcal{M}}^{\pm}(f) = \mathbb{S}^1$. In particular, if f is non-minimal, Bonnet and strongly isotropically isothermic, the proof follows from Theorem 5.3(iv). If f is Bonnet and minimal, the proof follows from [21]. ∎

Example 5.12. *Isothermic surfaces in $\mathbb{R}^4$ that admit exactly three Bonnet mates.*

By virtue of example 5 of Section 3.2.1, there exist isothermic surfaces in $\mathbb{R}^4$ that are strongly totally non isotropically isothermic. Then, Theorem 5.10(ii) implies that every simply-connected such surface admits exactly three Bonnet mates.

5.4 Bonnet Surfaces in $\mathbb{Q}_c^3 \subset \mathbb{Q}_c^4$

The following theorem shows that there exist Bonnet surfaces lying fully in $\mathbb{Q}_c^4$, arising as Bonnet mates of surfaces lying in totally geodesic hypersurfaces of $\mathbb{Q}_c^4$.

Theorem 5.13. *Let $f\colon M \to \mathbb{Q}_c^4$ be a simply-connected oriented surface, which is the composition of a non-minimal Bonnet surface $F\colon M \to \mathbb{Q}_c^3$, with a totally geodesic inclusion. Then, any Bonnet mate of F in $\mathbb{Q}_c^3$, determines two Bonnet mates f^-, f^+ of f in $\mathbb{Q}_c^4$. The surface $f^\pm$ lies in some totally umbilical $\mathbb{Q}_{\tilde{c}}^3 \subset \mathbb{Q}_c^4, \tilde{c} \geq c$, if and only if F has constant mean curvature. Moreover, either f admits exactly three Bonnet mates, or it is a flexible proper Bonnet surface.*

For the proof of the above theorem we need the following lemma.

Lemma 5.14. *Let $f\colon M \to \mathbb{Q}_c^4$ be an oriented surface which is the composition of a non-minimal Bonnet surface $F\colon M \to \mathbb{Q}_c^3$ with a totally geodesic inclusion $j\colon \mathbb{Q}_c^3 \to \mathbb{Q}_c^4$. Then, for every Bonnet mate $\tilde{F}$ of F in $\mathbb{Q}_c^3$ we have that $\tilde{f} = j \circ \tilde{F} \in \mathcal{M}^*(f)$.*

Proof: Let $\tilde{F}\colon M \to \mathbb{Q}_c^3$ be a Bonnet mate of F. Denote by ξ and $\tilde{\xi}$ the unit normal vector fields of F and $\tilde{F}$ in $\mathbb{Q}_c^3$, respectively, and by h their common mean curvature function. Then, the mean curvature vector fields of f and $\tilde{f}$, are given by $H = hj_*\xi$ and $\tilde{H} = hj_*\tilde{\xi}$, respectively. The parallel vector bundle isometry $T\colon N_fM \to N_{\tilde{f}}M$ given by $Tj_*\xi = j_*\tilde{\xi}$, $T(J^\perp j_*\xi) = \tilde{J}^\perp j_*\tilde{\xi}$ preserves the mean curvature vector fields, where $J^\perp$ and $\tilde{J}^\perp$ are the complex structures of the normal bundles of f and $\tilde{f}$, respectively. Therefore, $\tilde{f} \in \mathcal{M}(f)$. Since the image of the second fundamental form of $f, \tilde{f}$ is contained in the line bundle spanned by $j_*\xi, j_*\tilde{\xi}$, respectively, from Lemma 4.1(i) and the definition of T it follows that the zeros of the distortion differential of the pair $(f, \tilde{f})$ satisfy $Z^- = Z^+ = Z$. Hence, $\tilde{f} \in \mathcal{M}^*(f)$. ∎

Proof of Theorem 5.13: Let $f = j \circ F$, where $j\colon \mathbb{Q}_c^3 \to \mathbb{Q}_c^4$ is a totally geodesic inclusion and denote by ξ the unit normal of F in $\mathbb{Q}_c^3$. Since M is simply-connected and F is a Bonnet surface, the theorem of Lawson-Tribuzy [54] implies that M is non-compact. Let $\tilde{F}\colon M \to \mathbb{Q}_c^3$ be a Bonnet mate of F. From Lemma 5.14 it follows that $j \circ \tilde{F} \in \mathcal{M}^*(f)$ and Theorem 5.3(ii) implies that there exist Bonnet mates f^- and f^+ of f with $f^\pm \in \mathcal{M}^\pm(f)$. In particular, since any Bonnet mate of f lying in some totally geodesic $\mathbb{Q}_c^3 \subset \mathbb{Q}_c^4$ belongs to $\mathcal{M}^*(f)$, the surface $f^\pm$ does not lie in any totally geodesic hypersurface of $\mathbb{Q}_c^4$.

Assume that $f^\pm$ lies in some totally umbilical $\mathbb{Q}_{\tilde{c}}^3 \subset \mathbb{Q}_c^4, \tilde{c} > c$. Proposition 4.2 implies that M_1 is isolated. Let (U, z) be a complex chart on M with $U \cap M_1 = \emptyset$. Then, there exist $\varphi, \varphi^{\pm} \in \mathcal{C}^{\infty}(U)$ such that the Hopf differentials Φ and $\Phi_{f^{\pm}}$ of f and $f^{\pm}$, respectively, are given by

$$\Phi = \frac{\lambda^2}{2} e^{i\varphi} \sqrt{\|H\|^2 - K} e_3 dz^2, \quad \Phi_{f^{\pm}} = \frac{\lambda^2}{2} e^{i\varphi^{\pm}} \sqrt{\|H\|^2 - K} \tilde{e}_3^{\pm} dz^2, \quad \text{on} \quad U, \tag{5.14}$$

where λ is the conformal factor, $e_3 = j_* \xi$, and $\tilde{e}_3^{\pm} \in N_{f^{\pm}} M$ is a smooth unit vector field, parallel to the line segment that the ellipse of curvature of $f^{\pm}$ degenerates. Let $T_{\pm} \colon N_f M \to N_{f^{\pm}} M$ be an orientation and mean curvature vector field-preserving, parallel vector bundle isometry. Appealing to Lemma 4.1(i) and using (5.14), it follows that the distortion differential $Q_{f,f^{\pm}}$ of the pair $(f, f^{\pm})$ is given by

$$Q_{f,f^{\pm}} \equiv Q^{\pm}_{f,f^{\pm}} = \frac{\lambda^2}{4} \sqrt{\|H\|^2 - K} \left(e^{i\varphi}(e_3 \pm i e_4) - e^{i\varphi^{\pm}} (\hat{e}_3^{\pm} \pm i \hat{e}_4^{\pm}) \right) dz^2 \quad \text{on} \quad U, \tag{5.15}$$

where $e_4 = J^{\perp} e_3, \hat{e}_3^{\pm} = T_{\pm}^{-1} \tilde{e}_3^{\pm}, \hat{e}_4^{\pm} = J^{\perp} \hat{e}_3^{\pm}$. Proposition 4.2 implies that there exists $\theta^{\pm} \in \mathcal{C}^{\infty}(U; (0, 2\pi))$ such that $Q_{f,f^{\pm}}$ is given by (4.2) on U. Substituting $\Phi^{\pm}$ from (5.14) into (4.2), and using (5.15) we obtain that

$$\hat{e}_3^{\pm} \pm i \hat{e}_4^{\pm} = e^{i(\varphi - \varphi^{\pm} \mp \theta^{\pm})} (e_3 \pm i e_4) \quad \text{on} \quad U.$$

On the other hand, since $Q^{\mp}_{f,f^{\pm}} \equiv 0$, from Lemma 4.1(i) and (5.14) it follows that

$$\hat{e}_3^{\pm} \mp i \hat{e}_4^{\pm} = e^{i(\varphi - \varphi^{\pm})} (e_3 \mp i e_4) \quad \text{on} \quad U.$$

From the last two equations we obtain that $\theta^{\pm} = \pm 2(\varphi - \varphi^{\pm}) \mod 2\pi$. Then, the above implies that

$$\hat{\omega}_{34}^{\pm} = \frac{1}{2} d\theta^{\pm} + \omega_{34}, \tag{5.16}$$

where ω_{34} and $\hat{\omega}_{34}^{\pm}$ are the connection forms associated to the dual frame fields of $\{e_3, e_4\}$ and $\{\hat{e}_3^{\pm}, \hat{e}_4^{\pm}\}$, respectively. Since f and $f^{\pm}$ lie in totally umbilical hypersurfaces and $T_{\pm}$ is parallel, it follows that the vector fields e_3 and $\hat{e}_3^{\pm}$ are parallel in the normal connection of f. Therefore, (5.16) yields that $\theta^{\pm}$ is constant on U. Proposition 5.2(i) implies that $\theta^{\pm}$ satisfies (4.4) on U. From (4.4) it follows that $h^{\pm} \equiv 0$ on U. Then, (3.18) and Proposition 2.7 yield that the section $H^{\pm}$ is anti-holomorphic on U. Since $H = h e_3$, where h is the mean curvature function of F, this implies that h is constant on U. Since U is arbitrary and M_1 is isolated, it follows that the mean curvature function of F is constant on M.

Conversely, if F has constant mean curvature function, then f and its Bonnet mates have non-vanishing parallel mean curvature vector field. From [14, 69] it follows that $f^{\pm}$ lies in some totally umbilical hypersurface of $\mathbb{Q}_c^4$.

Moreover, from Theorem 5.3(i) and (iv) it is clear that either f admits exactly three Bonnet mates, or it is flexible proper Bonnet. ∎

Remark 5.15. If the surface F in Theorem 5.13 is proper Bonnet with non-constant mean curvature then f is flexible and both of its Gauss lifts are not vertically harmonic. Moreover, every surface in $\mathcal{M}^{\pm}(f)$ is flexible and does not lie in any totally umbilical hypersurface of $\mathbb{Q}_c^4$. From Theorem 5.10(ii) it follows that every such surface is strongly isotropically isothermic away from its isolated umbilic points. Therefore, there exist strongly isotropically isothermic surfaces in $\mathbb{Q}_c^4$ with flat normal bundle, that do not lie in any totally umbilical hypersurface of $\mathbb{Q}_c^4$ and whose both Gauss lifts are not vertically harmonic.

CHAPTER 6

Compact Surfaces

In this chapter, we study compact oriented surfaces $f\colon M \to \mathbb{Q}_c^4$. We show that there are obstructions on the structure of the moduli space $\mathcal{M}(f)$, imposed by the behavior of the Gauss lifts of f to the twistor bundle. Moreover, stronger obstructions are imposed by additional assumptions involving isotropic isothermicity. Our main results are presented in the second section and they concern surfaces whose both Gauss lifts are not vertically harmonic. In the third section we show that the theorem of Lawson-Tribuzy [54] follows as a consequence of our results, and we give some applications concerning superconformal surfaces in $\mathbb{Q}_c^4$ and Lagrangian surfaces in $\mathbb{R}^4$.

6.1 Obstructions on the Structure of the Moduli Space

The following theorem shows that the structure of the moduli space $\mathcal{M}(f)$ is controlled by the behavior of the Gauss lifts of f.

Theorem 6.1. *Let $f\colon M \to \mathbb{Q}_c^4$ be a compact oriented surface.*

(i) If the Gauss lift $G_\pm$ of f is not vertically harmonic, then there exists at most one Bonnet mate of f in $\mathcal{M}^\pm(f)$. Moreover, if $\tilde{f} \in \mathcal{M}^*(f)$ then $\mathcal{M}^*(f) \cup \mathcal{M}^\pm(f) = \bar{\mathcal{M}}^\mp(\tilde{f})$.

(ii) If both Gauss lifts of f are not vertically harmonic, then there exists at most one Bonnet mate of f in $\mathcal{M}^*(f)$. In particular, $\mathcal{M}^*(f) = \emptyset$ if M is homeomorphic to $\mathbb{S}^2$.

Proof: We claim that if there exist $f_1, f_2 \in \mathcal{N}^\pm(f)$ with $f_1 \in \mathcal{N}^\pm(f_2)$, then the Gauss lift $G_\pm$ of f is vertically harmonic. From Proposition 4.2, it follows that there exists $\theta^\pm \in \mathcal{C}^\infty(M \smallsetminus M_0^\pm; (0, 2\pi))$ such that the distortion differential Q of the pair (f, f_1) satisfies (4.2) on $M \smallsetminus M_0^\pm$. Since $f_2 \in \mathcal{N}^\pm(f) \cap \mathcal{N}^\pm(f_1)$, Proposition 4.5(i) implies that $\theta^\pm$ extends to a bounded harmonic function on M, which has to be constant by the maximum principle. From Lemma 4.1(i) and (4.2) it follows that $\Phi^\pm$ is holomorphic. Then, Proposition 2.7 implies that the Gauss lift $G_\pm$ is vertically harmonic and this proves the claim.

(i) Arguing indirectly, assume that there exist Bonnet mates $f_1, f_2 \in \mathcal{M}^{\pm}(f) \subset \mathcal{N}^{\pm}(f)$. From Lemma 4.4(ii), we have that $f_1 \in \mathcal{M}^{\pm}(f_2) \subset \mathcal{N}^{\pm}(f_2)$. Therefore, the Gauss lift $G_{\pm}$ is vertically harmonic, a contradiction.

For the second assertion, assume that there exists $f_1 \in \mathcal{M}^{\mp}(\tilde{f})$. If $f_1 \in \mathcal{M}^{\mp}(f)$, then Lemma 4.4(ii) implies that $f \in \mathcal{M}^{\mp}(\tilde{f})$, which is a contradiction. Therefore, $f_1 \notin \mathcal{M}^{\mp}(f)$ and thus, $\mathcal{M}^{\mp}(\tilde{f}) \subset \mathcal{N}^{\pm}(f)$, which obviously holds if $\mathcal{M}^{\mp}(\tilde{f}) = \emptyset$. The converse inclusion is obvious if $\mathcal{N}^{\pm}(f) = \{\tilde{f}\}$. Assume that there exists $f_1 \in \mathcal{N}^{\pm}(f) \smallsetminus \{\tilde{f}\}$. From the claim proved above, it follows that $f_1 \in \mathcal{M}^{\mp}(\tilde{f})$ and thus, $\mathcal{N}^{\pm}(f) \subset \mathcal{M}^{\mp}(\tilde{f})$.

(ii) Arguing indirectly, assume that there exist Bonnet mates $f_1, f_2 \in \mathcal{M}^{*}(f)$. Since both G_+ and G_- are not vertically harmonic, from the above claim we obtain that $f_1 \notin \mathcal{N}^{+}(f_2) \cup \mathcal{N}^{-}(f_2)$, which is a contradiction since f_1 is a Bonnet mate of f_2.

If M is homeomorphic to the sphere, then for any $\tilde{f} \in \mathcal{M}(f) \smallsetminus \{f\}$, the fourth-order differential $\langle Q^-, Q^+\rangle$ is holomorphic with zero-set $Z^- \cup Z^+$, where Q is the distortion differential of the pair $(f, \tilde{f})$. From the Riemann-Roch theorem we have that $\langle Q^-, Q^+\rangle \equiv 0$. Hence, either $Q^- \equiv 0$ or $Q^+ \equiv 0$ and consequently $\mathcal{M}^{*}(f) = \emptyset$. ∎

Theorem 5.10 shows that for non-compact simply-connected surfaces, the property of half totally non isotropic isothermicity implies the existence of Bonnet mates, whereas isotropic isothermicity characterizes proper Bonnet surfaces. The following result implies that both properties are very obstructive for the existence of Bonnet mates for compact surfaces.

Theorem 6.2. *Let $f\colon M \to \mathbb{Q}_c^4$ be a compact oriented surface and V an open and dense subset of M. If one of the following holds, then $\mathcal{N}^{\pm}(f) = \emptyset$.*
(i) The Gauss lift $G_{\pm}$ of f is not vertically harmonic and f is $\pm$ isotropically isothermic on V.
(ii) The set V is connected and f is totally non $\pm$ isotropically isothermic on V.

Proof: Arguing indirectly, assume that there exists $\tilde{f} \in \mathcal{N}^{\pm}(f)$. Proposition 4.2 implies that $M_0^{\pm}$ is isolated and that there exists $\theta^{\pm} \in \mathcal{C}^{\infty}(M \smallsetminus M_0^{\pm}; (0, 2\pi))$, such that the distortion differential Q of the pair $(f, \tilde{f})$ satisfies (4.2) on $M \smallsetminus M_0^{\pm}$.

(i) Since V is dense, it follows that f is $\pm$ isotropically isothermic on $M \smallsetminus M_0^{\pm}$. Then, Proposition 4.5(ii) implies that $\theta^{\pm}$ extends to a bounded harmonic function on M, which has to be constant by the maximum principle. By virtue of Lemma 4.1(i), from (4.2) it follows that the differential $\Phi^{\pm}$ is holomorphic. Then, Proposition 2.7 implies that the Gauss lift $G_{\pm}$ of f is vertically harmonic, a contradiction.

(ii) From the definition of non $\pm$ isotropically isothermic points it follows that $M_0^{\pm} \subset M \smallsetminus V$. Therefore, $\theta^{\pm}$ is defined everywhere on V. Let (U, z) be a complex chart with $U \subset V$. Proposition 4.5 implies that $\theta^{\pm}$ satisfies (4.4) on U. From Lemma 3.10 it follows that $\operatorname{Im} h_z^{\pm} \neq 0$ everywhere on U. Appealing to Lemma 4.3(i), (4.7) and (4.8) yield that $\Delta\theta^{\pm}$ is nowhere vanishing on U. Since V is connected and U is an arbitrary subset of V, we deduce that either $\Delta\theta^{\pm} > 0$, or $\Delta\theta^{\pm} < 0$ on V. Since V is dense in $M \smallsetminus M_0^{\pm}$, it follows by continuity that either $\Delta\theta^{\pm} \geq 0$, or $\Delta\theta^{\pm} \leq 0$ on $M \smallsetminus M_0^{\pm}$. As in the proof of [44, Thm.(2)], it can be shown that either $\theta^{\pm}$, or $-\theta^{\pm}$ can be extended to a subharmonic function on M which attains a maximum and thus, it has to be constant by the maximum principle for subharmonic functions. As in the proof of part (i), it follows that the Gauss lift $G_{\pm}$ of f is vertically harmonic. Then, the first example of Section 3.2.1 implies that f is $\pm$ isotropically isothermic on V, which is a contradiction. ∎

6.2 Surfaces whose both Gauss Lifts are not Vertically Harmonic

The following result is a Lawson-Tribuzy type theorem [54], and implies that compact surfaces in $\mathbb{Q}_c^4$ whose both Gauss lifts are not vertically harmonic, do not allow nontrivial global isometric deformations that preserve the mean curvature.

Theorem 6.3. *Let $f\colon M \to \mathbb{Q}_c^4$ be a compact oriented surface. If both Gauss lifts G_+ and G_- of f are not vertically harmonic, then f admits at most three Bonnet mates. In particular, f admits at most one Bonnet mate, if M is homeomorphic to $\mathbb{S}^2$.*

Proof: Theorem 6.1 implies that f admits at most three Bonnet mates. Assume that M is homeomorphic to $\mathbb{S}^2$. Theorem 6.1 shows that $\mathcal{M}^*(f) = \emptyset$ and that there exists at most one Bonnet of f in each one of $\mathcal{M}^+(f)$ and $\mathcal{M}^-(f)$. Suppose that there exist $f_1 \in \mathcal{M}^+(f)$ and $f_2 \in \mathcal{M}^-(f)$. From Lemma 4.4(i) it follows that $f_1 \in \mathcal{M}^*(f_2)$, which contradicts Theorem 6.1(ii). Therefore, f admits at most one Bonnet mate. ∎

In the particular case of surfaces $f\colon M \to \mathbb{R}^4$, the above theorem can be stated in terms of the Gauss map $g = (g_+, g_-)\colon M \to \mathbb{S}^2_+ \times \mathbb{S}^2_-$ of the surface f.

Corollary 6.4. *Let $f\colon M \to \mathbb{R}^4$ be a compact oriented surface. If both components g_+ and g_- of the Gauss map of f are not harmonic, then f admits at most three Bonnet mates. In particular, f admits at most one Bonnet mate, if M is homeomorphic to $\mathbb{S}^2$.*

Proof: Follows immediately from Theorem 6.3, by virtue of Remark 2.9. ∎

The following theorem extends a recent result due to Jensen, Musso and Nicolodi [44]. It shows that the conclusion of Theorem 6.3 can be strengthened, under additional assumptions involving isotropic isothermicity.

Theorem 6.5. *Let $f\colon M \to \mathbb{Q}_c^4$ be a compact oriented surface. If both Gauss lifts G_+ and G_- of f are not vertically harmonic and f is either isotropically isothermic, or half totally non isotropically isothermic, on an open dense and connected subset V of M, then f admits at most one Bonnet mate. In particular, f does not admit any Bonnet mate, if it is either strongly isotropically isothermic, or strongly totally non isotropically isothermic on V.*

Proof: Since $G_\pm$ is not vertically harmonic and f is either $\pm$ isotropically isothermic, or totally non $\pm$ isotropically isothermic, on an open dense and connected subset V of M, Theorem 6.2 implies that $\mathcal{N}^\pm(f) = \emptyset$. On the other hand, since $G_\mp$ is not vertically harmonic, from Theorem 6.1(i) it follows that there exists at most one Bonnet mate of f in $\mathcal{M}^\mp(f)$. Therefore, f admits at most one Bonnet mate. In particular, if f is either strongly isotropically isothermic, or strongly totally non isotropically isothermic, on V, Theorem 6.2 implies that $\mathcal{N}^-(f) = \mathcal{N}^+(f) = \emptyset$ and thus, f does not admit any Bonnet mate. ∎

Remark 6.6. In the proof of Theorem 6.1, compactness is only required for the use of the maximum principle. This theorem and also Theorem 6.3 and Theorems 6.7, 6.9 and 6.11 of the next section still hold true if M is parabolic. In particular, this includes the case where M is complete with non-negative Gaussian curvature.

6.3 Applications to Certain Classes of Surfaces

The theorem of Lawson-Tribuzy [54], follows as an application of Theorem 6.1.

Theorem 6.7. *Let M be a compact oriented 2-dimensional Riemannian manifold and $h \in \mathcal{C}^\infty(M)$. If h is not constant, then there exist at most two congruence classes of isometric immersions of M into $\mathbb{Q}_c^3$ with mean curvature h. In particular, there exists at most one congruence class if M is homeomorphic to $\mathbb{S}^2$.*

Proof: Assume that there exists an isometric immersion $F\colon M \to \mathbb{Q}_c^3$ with mean curvature function h unit normal vector field ξ. Consider a totally geodesic inclusion $j\colon \mathbb{Q}_c^3 \to \mathbb{Q}_c^4$ and set $f = j \circ F$. Then, f has non-parallel mean curvature vector field $hj_*\xi$ and Proposition 2.7 implies that both Gauss lifts of f are not vertically harmonic. Assume that there exists a Bonnet mate $\tilde{F}\colon M \to \mathbb{Q}_c^3$ of F and set $\tilde{f} = j \circ \tilde{F}$. Lemma 5.14 implies that $\tilde{f} \in \mathcal{M}^*(f)$ and the proof follows from Theorem 6.1(ii). ∎

The result of Jensen, Musso and Nicolodi [44] follows from Theorem 6.5.

Theorem 6.8. *Let $F\colon M \to \mathbb{Q}_c^3$ be a compact oriented surface with not constant mean curvature. If F is either isothermic, or totally non isothermic, on an open dense and connected subset V of M, then it does not admit any Bonnet mate.*

Proof: Let $j\colon \mathbb{Q}_c^3 \to \mathbb{Q}_c^4$ be a totally geodesic inclusion and set $f = j \circ F$. Proposition 3.3 implies that F is (totally non) isothermic on V if and only if f is strongly (totally non) isotropically isothermic on V. By virtue of Theorem 6.5, our assumption implies that f does not admit any Bonnet mate. Then, the proof follows from Lemma 5.14. ∎

Theorem 6.9. *Let $f\colon M \to \mathbb{Q}_c^4$ be a compact oriented superconformal surface. Then f admits at most one Bonnet mate.*

Proof: Assume that f is non-minimal and let $\tilde{f}$ be a Bonnet mate of f. Then either $\tilde{f} \in \mathcal{N}^-(f)$, or $\tilde{f} \in \mathcal{N}^+(f)$. Since $M_1 = M_0^- \cap M_0^+$, in any case from Proposition 4.2 it follows that M_1 is isolated. Then, Lemma 2.3(ii) yields that the normal curvature is everywhere non-vanishing on $M \smallsetminus M_1$. Assume that $\pm K_N > 0$ on $M \smallsetminus M_1$. Therefore, $\pm K_N \geq 0$ on M. Lemma 2.3(ii) implies that $\Phi^{\pm} \equiv 0$ and thus, $\tilde{f} \in \mathcal{M}^{\mp}(f)$.

We claim that $G_{\mp}$ is not vertically harmonic. Arguing indirectly, assume that $G_{\mp}$ is vertically harmonic. Then, Proposition 2.8 yields that $\Phi^{\mp} \equiv 0$ and Lemma 2.3(ii) implies that f is totally umbilical. This contradicts the fact that M_1 is isolated, and the proof of the claim follows. Hence, from Theorem 6.1(i) we obtain that $\mathcal{M}^{\mp}(f) = \{\tilde{f}\}$ and consequently, $\mathcal{M}(f) = \{f, \tilde{f}\}$. In the case where f is minimal, the result follows from [46] or [66]. ∎

We give an application to Lagrangian surfaces in $\mathbb{R}^4$. Let $\widetilde{J}$ be a canonical complex structure on $\mathbb{R}^4$ which is compatible with the orientation, i.e., for orthonormal vectors $e_1, e_2 \in \mathbb{R}^4$, the oriented orthonormal basis $\{e_1, e_2, \widetilde{J}e_1, \widetilde{J}e_2\}$ is in the orientation of $\mathbb{R}^4$. Denote by $\Omega(\cdot,\cdot) = \langle \cdot, \widetilde{J}\,\cdot \rangle$ the associated Kähler form. A surface $f\colon M \to \mathbb{R}^4$ is called Lagrangian if $f^*\Omega = 0$. In such a case, from $(\widetilde{J} \circ f_*) \circ \nabla = \nabla^\perp \circ (\widetilde{J} \circ f_*)$ we have that $\widehat{J}_f = \widetilde{J} \circ f_*\colon TM \to N_fM$ is a parallel vector bundle isometry and the second fundamental form of f satisfies $\alpha(X,Y) = \widehat{J}_f A_{\widehat{J}_f X} Y$, $X, Y \in TM$. Thus, the trilinear map C_f on TM given by

$$C_f(X,Y,Z) = \Omega(\alpha(X,Y), f_*Z)$$

is symmetric. Associated to f are its *mean curvature form* Υ_f and the cubic differential Θ_f, given by

$$\Upsilon_f = \Omega(H, f_*\partial)dz, \quad \Theta_f = \Omega(\alpha(\partial,\partial), f_*\partial)dz^3,$$

in terms of a local complex coordinate z, where Ω and $\widehat{J}_f$ have been extended $\mathbb{C}$-linearly. Since $\widetilde{J}$ is compatible with the orientation, $\widehat{J}_f\colon TM \otimes \mathbb{C} \to N_fM \otimes \mathbb{C}$ satisfies

$$\widehat{J}_f T^{(1,0)}M = N_f^- M \quad \text{and} \quad \widehat{J}_f T^{(0,1)}M = N_f^+ M.$$

The *Maslov form* ϖ_f of f, is the 1-form on M defined by $\varpi_f(X) = (1/\pi)\Omega(f_*X, H)$. The Gauss map of a Lagrangian surface is $g = (g_+, g_-)\colon M \to \mathbb{S}^2_+ \times \mathbb{S}^1_-$, i.e., its second component lies in a great circle of $\mathbb{S}^2_-$. Lagrangian surfaces with conformal (respectively, harmonic) Maslov form provide examples of surfaces in $\mathbb{R}^4$ with harmonic g_+ (respectively, g_-). Indeed, the following was proved in [12].

Proposition 6.10. *Let $f\colon M \to (\mathbb{R}^4, \widetilde{J})$ be a Lagrangian surface. The following are equivalent:*

(i) The Maslov form ϖ_f is conformal (respectively, harmonic).

(ii) The differential Θ_f (respectively, Υ_f) is holomorphic.

(iii) The component g_+ (respectively, g_-) is harmonic.

Using Theorem 6.1, we are able to give a short proof of the following result due to He, Ma and Wang [38].

Theorem 6.11. *Let $f\colon M \to (\mathbb{R}^4, \widetilde{J})$ be a compact, oriented Lagrangian surface with mean curvature form Υ. If its Maslov form is not conformal, then there exists at most one nontrivial congruence class of Lagrangian isometric immersions of M into $(\mathbb{R}^4, \widetilde{J})$, with mean curvature form Υ.*

Proof: Suppose that $f, \tilde{f}\colon M \to (\mathbb{R}^4, \widetilde{J})$ are noncongruent Lagrangian surfaces with mean curvature forms $\Upsilon = \tilde{\Upsilon}$. It follows that $T = \widehat{J}_{\tilde{f}} \circ \widehat{J}_f^{-1}\colon N_f M \to N_{\tilde{f}} M$ is an orientation and mean curvature vector field-preserving, parallel vector bundle isometry. Let (U, z) be a complex chart. From our assumption, we have that $C_f(\partial, \partial, \bar{\partial}) = C_{\tilde{f}}(\partial, \partial, \bar{\partial})$. Hence,

$$\langle \phi_f^- - T^{-1} \circ \phi_{\tilde{f}}^-, \widehat{J}_f \bar{\partial} \rangle \equiv 0 \text{ on } U,$$

where ϕ_f^- and $\phi_{\tilde{f}}^-$ are given by (2.11). Since $\phi_f^- - T^{-1} \circ \phi_{\tilde{f}}^- \in N_f^- U$ and $\widehat{J}_f \bar{\partial} \in N_f^+ U$, it follows that $\phi_f^- - T^{-1} \circ \phi_{\tilde{f}}^- \equiv 0$ on U. Therefore, $\tilde{f} \in \mathcal{M}^+(f)$ and the proof follows from Theorem 6.1(i) and Proposition 6.10. ∎

In [12] it was proved that if $f\colon M \to \mathbb{R}^4$ is a compact, oriented Lagrangian surface with conformal (respectively, harmonic) Maslov form, then $\operatorname{genus}(M) \leq 1$ (respectively, $\operatorname{genus}(M) \geq 1$). The classification of compact oriented Lagrangian surfaces in $\mathbb{R}^4$ with conformal Maslov form, was given in [12]. It turns out that there exist Lagrangian tori in $\mathbb{R}^4$ with non-parallel mean curvature vector field and conformal Maslov form. Lagrangian surfaces with harmonic Maslov form are Hamiltonian minimal. Examples of Hamiltonian minimal Lagrangian tori in $\mathbb{R}^4$, with non-parallel mean curvature vector field, were constructed in [13] and the complete classification was given in [39]. Furthermore, it was proved in [11] that the only compact, orientable superconformal Lagrangian surface in $\mathbb{R}^4$ is the Whitney sphere. Therefore, there exist compact, oriented non-superconformal surfaces in $\mathbb{R}^4$, whose only one of the components g_+, g_- of their Gauss map is harmonic.

Chapter 7

Surfaces with a Vertically Harmonic Gauss Lift

The results of the previous chapter indicate that it is interesting to study surfaces in $\mathbb{Q}^4_c$ with a vertically harmonic Gauss lift. It turns out that such surfaces share common properties both with minimal surfaces in $\mathbb{Q}^4_c$ and with CMC surfaces in 3-dimensional space forms.

7.1 A Hopf-type Theorem

In the following proposition, we show that surfaces with a vertically harmonic Gauss lift possess a holomorphic quadratic differential and they satisfy Ricci-like conditions that extend the well-known Ricci condition (cf. [53]) for CMC surfaces in 3-dimensional space forms.

Proposition 7.1. *Let $f\colon M \to \mathbb{Q}^4_c$ be a non-minimal surface with mean curvature vector field H and vertically harmonic Gauss lift $G_\pm$. Then:*

(i) The quadratic differential $\Psi^\pm = \langle \Phi^\pm, H^\mp\rangle$ is holomorphic with zero-set $Z(\Psi^\pm) = M_0^\pm(f) \cup \{p \in M : H(p) = 0\}$.

(ii) The functions $\|H\|$ and $\|\mathcal{H}^\pm\|$ are of absolute value type.

(iii) We have that

$$\Delta \log \|H\| = \mp K_N, \tag{7.1}$$

$$\Delta \log \|\mathcal{H}^\pm\| = 2K \pm K_N \quad \textit{if} \quad \Psi^\pm \not\equiv 0. \tag{7.2}$$

Proof: (i) The holomorphicity of $\Psi^\pm$ follows from Proposition 2.7. The zeros of $\Psi^\pm$ are precisely the points where $\langle \Phi^\pm, H\rangle = 0$, which is equivalent to $\Phi^\pm = 0$ at points where $H \neq 0$.

(ii) Let (U, z) be a complex chart. From Proposition 2.7(iv) we have

$$\nabla^\perp_{\bar\partial} H = \pm i J^\perp \nabla^\perp_{\bar\partial} H.$$

This is equivalently written as

$$(H^3 \pm iH^4)_{\bar{z}} = \mp i\omega_{34}(\bar{\partial})(H^3 \pm iH^4),$$

where $H = H^3 e_3 + H^4 e_4$, and $\{e_3, e_4 = J^{\perp} e_3\}$ is a local orthonormal frame field of $N_f M$. From [27, Lemma 9.1.] it follows that the function $H^3 \pm iH^4$ is of holomorphic type and this proves our claim for $\|H\|$.

From part (i), the function $\langle \phi^{\pm}, H^{\mp} \rangle$ is holomorphic, where $\phi^{\pm}$ is given by (2.11). Moreover,

$$|\langle \phi^{\pm}, H^{\mp} \rangle|^2 = \frac{\lambda^4}{16} \|H\|^2 \|\mathcal{H}^{\pm}\|^2, \tag{7.3}$$

where λ is the conformal factor. Clearly, the function

$$t = \frac{4\langle \phi^{\pm}, H^{\mp} \rangle}{\lambda^2 (H^3 \pm iH^4)}$$

can be smoothly extended to the zeros of H as a holomorphic type function. Since $|t| = \|\mathcal{H}^{\pm}\|$, this completes the proof.

(iii) Away from the zeros of H, we consider the local orthonormal frame field $\{e_3 = H/\|H\|, e_4 = J^{\perp} e_3\}$ of the normal bundle. Using Proposition 2.7(iv), we find that the normal connection form is given by

$$\omega_{34} = \pm * d \log \|H\|.$$

Then (7.1) follows from (2.2) and the above.

We choose a complex chart with coordinate z, away from the zeros of $\Psi^{\pm}$. From the holomorphicity of $\Psi^{\pm}$ we have that

$$\Delta \log |\langle \phi^{\pm}, H^{\mp} \rangle|^2 = 0.$$

Equation (7.2) follows from (7.3) and the fact that $\Delta \log \lambda = -K$. ∎

Proposition 7.2. *Let $f \colon M \to \mathbb{Q}^4_c$ be a compact surface with mean curvature vector field H and vertically harmonic Gauss lift $G_{\pm}$.*
(i) If f is non-minimal, then

$$\chi_N = \pm N(\|H\|).$$

(ii) If f is neither minimal nor superconformal, then

$$2\chi \pm \chi_N = -N\left(\|\mathcal{H}^{\pm}\|\right).$$

Proof: By virtue of Lemma 2.4, the proofs of (i) and (ii) follow immediately from Proposition 7.1(ii), by integrating (7.1) and (7.2), respectively. ∎

The following result is a Hopf-type theorem for non-minimal surfaces with a vertically harmonic Gauss lift.

Theorem 7.3. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-minimal surface. If the Gauss lift $G_\pm$ of f is vertically harmonic and M is homeomorphic to $\mathbb{S}^2$, then f is superconformal. In particular, f is totally umbilical if the Euler number of its normal bundle vanishes.*

Proof: From the assumption and Proposition 7.1(i) we obtain that $\Psi^\pm \equiv 0$. Since f is non-minimal, Proposition 7.1(i-ii) implies that $\Phi^\pm \equiv 0$. From Lemma 2.3(ii) it follows that f is superconformal with normal curvature $\pm K_N \geq 0$. Therefore, the Euler number of the normal bundle of f satisfies $\pm\chi_N \geq 0$, and it vanishes if and only if $K_N = 0$ on M. If $\chi_N = 0$, then Lemma 2.3(ii) implies that f is totally umbilical. ∎

Clearly, the theorem of Hopf-Chern [17, 42] is an immediate consequence of the above theorem. This result can be also seen as an extension to the case of non-minimal surfaces, of the well-known theorem of Calabi [9] that a minimal surface of genus zero in the 4-sphere is superminimal. For surfaces in $\mathbb{R}^4$, an alternative proof was given by Hasegawa [37], with essential use of the Hyperkähler structure of $\mathbb{R}^4$.

7.2 The Associated Family

Dajczer and Gromoll [20] proved that any simply-connected minimal surface admits a 1-parameter associated family of isometric deformations through minimal surfaces. This family is trivial if and only if the surface is superconformal. Extending their result, we are able to produce a new 1-parameter family of isometric deformations that preserve the mean curvature, for any non-minimal surface in $\mathbb{Q}_c^4$ with a vertically harmonic Gauss lift. It is worth noticing that the second fundamental form of any surface in this family relates to the initial one in a more involved way than in [20].

Theorem 7.4. *Let $f\colon M \to \mathbb{Q}_c^4$ be a non-minimal, simply-connected surface. If the Gauss lift $G_\pm$ of f is vertically harmonic, then:*

(i) There exists a one-parameter family of isometric immersions $f_\theta^\pm\colon M \to \mathbb{Q}_c^4$, $\theta \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, which have the same mean curvature with $f_0^\pm = f$.

(ii) If f is superconformal, then $f_\theta^\pm$ is congruent to f for any θ.

(iii) If there exist $\theta \neq \tilde{\theta} \in \mathbb{S}^1$ such that $f_\theta^\pm$ is congruent to $f_{\tilde{\theta}}^\pm$, then f is superconformal.

In particular, $f_\theta^\pm \in \bar{\mathcal{M}}^\pm(f), \theta \in \mathbb{S}^1$, and $\bar{\mathcal{M}}^\pm(f) = \mathbb{S}^1$ if f is not superconformal.

Proof: (i) For any $\theta \in \mathbb{R}$ define the symmetric section $\beta_\theta^\pm \in \Gamma(\mathrm{Hom}(TM \times TM, N_f M))$ by

$$\beta_\theta^\pm(X,Y) = J_{\theta/2}^\perp\Big(\alpha(J_{\mp\theta/4}X, J_{\mp\theta/4}Y) - \langle X,Y\rangle H\Big) + \langle X,Y\rangle H,$$

where $X, Y \in TM$, $J_\theta^\perp = \cos\theta I + \sin\theta J^\perp$ and $J_\theta = \cos\theta I + \sin\theta J$. We argue that $\beta_\theta^\pm$ satisfies the Gauss, Codazzi and Ricci equations. Clearly, we have that

$$(\beta_\theta^+)^{(2,0)} = \Phi^- + e^{-i\theta}\Phi^+, \quad (\beta_\theta^-)^{(2,0)} = e^{i\theta}\Phi^- + \Phi^+ \tag{7.4}$$

and

$$(\beta_\theta^\pm)^{(1,1)} = \alpha^{(1,1)}.$$

In terms of a local complex coordinate z with conformal factor λ, by using (2.11) and (7.4) it is straightforward to check that $\beta_\theta^\pm$ satisfies the Gauss, Codazzi and Ricci equations (2.12)-(2.14). By the fundamental theorem of submanifolds, for every $\theta \in \mathbb{R}$ there exists an isometric immersion $f_\theta^\pm \colon M \to \mathbb{Q}_c^4$ and an orientation-preserving parallel vector bundle isometry $T_\theta \colon N_f M \to N_{f_\theta^\pm} M$ such that $\alpha_{f_\theta^\pm} = T_\theta \circ \beta_\theta^\pm$. Clearly, $T_\theta H$ is the mean curvature vector field of $f_\theta^\pm$, for any $\theta \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and $f_0^\pm = f$.

(ii) From Proposition 2.8 it follows that $\Phi^\pm \equiv 0$. Then, (7.4) yields that $(\beta_\theta^\pm)^{(2,0)} = \Phi$ for any $\theta \in \mathbb{S}^1$. This implies that each T_θ preserves the Hopf differential and the mean curvature vector field and consequently, it preserves the second fundamental form as well. This shows that the family is trivial.

(iii) Without loss of generality, we may assume that $\tilde{\theta} = 0$. The distortion differential of the pair $(f, f_\theta^\pm)$ vanishes identically. Lemma 4.1(i) implies that any orientation and mean curvature vector field-preserving parallel vector bundle isometry $T \colon N_f M \to N_{f_\theta^\pm} M$ preserves the Hopf differential, and consequently the second fundamental form as well. Hence, $\beta_\theta^\pm = T_\theta^{-1} \circ \alpha_{f_\theta} = \alpha$ and (7.4) implies that $(1 - e^{\mp i\theta})\Phi^\pm \equiv 0$. Since $\theta \neq 0$, the last relation yields $\Phi^\pm \equiv 0$ and thus, f is superconformal.

In particular, if f is not superconformal, then from (7.4) it follows that the distortion differential Q_θ of the pair $(f, f_\theta^\pm)$, $\theta \neq 0$, satisfies $Q_\theta^\mp \equiv 0$ and therefore $f_\theta^\pm \in \mathcal{M}^\pm(f)$. The proof now follows. ∎

The following proposition determines the moduli space of simply-connected surfaces with parallel mean curvature vector field. The two-parameter family given here, coincides up to a parameter transformation, with the one given by Eschenburg-Tribuzy [29].

Proposition 7.5. *Let $f \colon M \to \mathbb{Q}_c^4$ be a simply-connected surface with parallel mean curvature vector field $H \neq 0$. Then:*

(i) There exists a two-parameter family of isometric immersions $f_{\theta,\varphi} \colon M \to \mathbb{Q}_c^4$, $(\theta, \varphi) \in \mathbb{S}^1 \times \mathbb{S}^1$, which have the same mean curvature with $f_{0,0} = f$.

(ii) The family is trivial if and only if f is totally umbilical.

(iii) If f is not totally umbilical, then $\mathcal{M}(f) = \mathbb{S}^1 \times \mathbb{S}^1$.

Proof: (i) Since both Gauss lifts are vertically harmonic, from Theorem 7.4 we may consider the two-parameter family $f_{\theta,\varphi} = (f_\theta^-)_\varphi^+, \theta, \varphi \in \mathbb{S}^1$. Clearly, $f_{\theta,\varphi}$ has the same mean curvature with f.

(ii) From Theorem 7.4, it is clear that $f_{\theta,\varphi}$ is congruent to $f_{\tilde{\theta},\tilde{\varphi}}$ for $(\theta, \varphi) \neq (\tilde{\theta}, \tilde{\varphi}) \in \mathbb{S}^1 \times \mathbb{S}^1$ if and only if f is superconformal. Since H is parallel, this can only occur if f is totally umbilical.

(iii) Since f is not superconformal, Theorem 7.4 implies that $\bar{\mathcal{M}}^-(f) = \mathbb{S}^1$ and $\bar{\mathcal{M}}^+(f) = \mathbb{S}^1$. The proof follows by virtue of Theorem 5.3(iv). ∎

Remark 7.6. We recall (cf. [14, 69]) that any surface with parallel mean curvature vector field $H \neq 0$ splits as $f = j \circ f'$, where $j\colon \mathbb{Q}^3_{c'} \to \mathbb{Q}^4_c, c' \geq c$, is a totally umbilical inclusion and $f'\colon M \to \mathbb{Q}^3_{c'}$ is a CMC-h' surface with $h' = \pm(\|H\|^2 - (c' - c))^{1/2}$. It is known that there exists locally a bijective correspondence (the so-called Lawson correspondence [53, Theorem 8]) between CMC surfaces in 3-dimensional space forms. Since $f_{\theta,\varphi} = \hat{j} \circ \hat{f}_{\theta,\varphi}$ and $\|H_{f_{\theta,\varphi}}\| = \|H\|$, the surfaces f' and $\hat{f}_{\theta,\varphi}$ are in Lawson correspondence for any $\theta, \varphi \in \mathbb{S}^1$. In particular, $f_{\theta,2\pi-\theta}$ is congruent to $j \circ f'_\theta$, where f'_θ, $\theta \in \mathbb{S}^1$, is the associated family of f' in $\mathbb{Q}^3_{c'}$ as a CMC surface.

Example 7.7. *Tight proper Bonnet surfaces in $\mathbb{R}^4$ with a vertically harmonic Gauss lift, that are strongly isotropically isothermic.*

We consider the product in $\mathbb{R}^4$ of two plane curves γ_1, γ_2, as in the example 5 of Section 3.2.1 and we adopt the notation used there. Assume that the curvature of the curve γ_j is $k_j(s_j) = cs_j, j = 1, 2$, with $c \neq 0$, and we restrict the product surface f such that $f\colon M \to \mathbb{R}^4$ is simply-connected and umbilic-free. Clearly, f has flat normal bundle and does not lie in any totally umbilical hypersurface of $\mathbb{R}^4$. Moreover, from (3.27) it follows that f is strongly isotropically isothermic.

Hasegawa [36] proved that the Gauss lift G_- of f is vertically harmonic. Since f is neither minimal, nor superconformal, from Theorem 7.4 it follows that $\bar{\mathcal{M}}^-(f) = \mathbb{S}^1$. Therefore, f is proper Bonnet.

Moreover, since f is + isotropically isothermic, from Proposition 5.11 it follows that either $\bar{\mathcal{M}}^+(f) = \{f\}$, or $\bar{\mathcal{M}}^+(f) = \mathbb{S}^1$. We claim that $\bar{\mathcal{M}}^+(f) = \{f\}$. Arguing indirectly, assume that $\bar{\mathcal{M}}^+(f) = \mathbb{S}^1$. Then, Theorem 5.7 implies that

$$\Delta \log \|\mathcal{H}^+\| - 2K = \frac{\|\tau^v(G_+)\|^2}{4\|\mathcal{H}^+\|^2}.$$

On the other hand, since $\bar{\mathcal{M}}^-(f) = \mathbb{S}^1$, Theorem 5.7 yields that

$$\Delta \log \|\mathcal{H}^-\| - 2K = 0.$$

Since $K_N = 0$ everywhere, it follows that $\|\mathcal{H}^-\| = \|\mathcal{H}^+\|$ and the above two relations imply that the Gauss lift G_+ of f is vertically harmonic. Then, the mean curvature vector field of f is parallel and thus, f lies in some totally umbilical hypersurface of $\mathbb{R}^4$. This is a contradiction and the claim follows. Then, Theorem 5.3(iv) implies that $\mathcal{M}(f) = \mathbb{S}^1$.

7.3 The Structure of the Moduli Space

7.3.1 Compact Surfaces: The Main Result

Although non-minimal surfaces in $\mathbb{Q}^4_c$ with a vertically harmonic Gauss lift share common properties with both minimal surfaces in $\mathbb{Q}^4_c$ and CMC surfaces in 3-dimensional space forms, an essential difference between them is that the associated family of Theorem 7.4 does not necessarily coincide with the whole moduli space $\mathcal{M}(f)$. However, for compact surfaces with a vertically harmonic Gauss lift, we are able to determine the structure of the moduli space under appropriate geometric or topological assumptions.

Theorem 7.8. *Let $f\colon M \to \mathbb{Q}^4_c$ be a compact oriented surface with vertically harmonic Gauss lift $G_\pm$.*

(i) If the mean curvature vector field of f is non-parallel, then the moduli space $\mathcal{M}(f)$ is the disjoint union of two sets, each one being either finite, or the circle $\mathbb{S}^1$.

(ii) If $c = 0$ and the Euler numbers χ and χ_N of the tangent and normal bundles satisfy $\chi \neq \mp\chi_N$, then $\mathcal{M}(f)$ is a finite set.

For the proof of the above theorem, we need a series of auxiliary results, that we present in the following two subsections.

7.3.2 Non-Simply-Connected Surfaces

Let M be a 2-dimensional oriented Riemannian manifold with nontrivial fundamental group and $f\colon M \to \mathbb{Q}^4_c$ a non-minimal surface. Consider the universal cover $(\tilde{M}, \tilde{\pi})$ of M, equipped with metric and orientation that make the covering map $\tilde{\pi}\colon \tilde{M} \to M$ an orientation-preserving local isometry. Then, $\tilde{f} = f \circ \tilde{\pi}\colon \tilde{M} \to \mathbb{Q}^4_c$ is an isometric immersion. It is clear that the Gauss lift $\tilde{G}_\pm$ of $\tilde{f}$ is vertically harmonic if and only if the Gauss lift $G_\pm$ of f is vertically harmonic.

If $(f = f_1, f_2)$ is a pair of Bonnet mates, then $(\tilde{f}_1, \tilde{f}_2)$ is also a pair of Bonnet mates, where $\tilde{f}_j = f_j \circ \tilde{\pi}$, $j = 1, 2$. Moreover, $f_2 \in \mathcal{N}^\pm(f_1)$, if and only if $\tilde{f}_2 \in \mathcal{N}^\pm(\tilde{f}_1)$. If $G_\pm$ is vertically harmonic and $f_2 \in \mathcal{M}^\pm(f_1)$, then from Theorem 7.4 it follows that $\tilde{f}_2$ is congruent to some $\tilde{f}^\pm_\theta$ in the associated family of $\tilde{f}_1$. Therefore $\bar{\mathcal{M}}^\pm(f)$ can be parametrized by the set

$$\left\{\theta \in \mathbb{S}^1 : \text{there exists } \ f_\theta\colon M \to \mathbb{Q}^4_c \ \text{ such that } \ \tilde{f}^\pm_\theta = f_\theta \circ \tilde{\pi}\right\}.$$

In particular, if H is parallel, then by Proposition 7.5, the moduli space $\mathcal{M}(f)$ can be parametrized by the set

$$\left\{(\theta, \varphi) \in \mathbb{S}^1 \times \mathbb{S}^1 : \text{there exists } \ f_{\theta,\varphi}\colon M \to \mathbb{Q}^4_c \ \text{ such that } \ \tilde{f}_{\theta,\varphi} = f_{\theta,\varphi} \circ \tilde{\pi}\right\}.$$

The following is essential for the proof of Theorem 7.8. For its proof we adopt techniques used in [24, 62, 67].

Proposition 7.9. *Let $f\colon M \to \mathbb{Q}^4_c$ be a non-minimal surface with mean curvature vector field H.*

(i) If the Gauss lift $G_\pm$ of f is vertically harmonic, then $\bar{\mathcal{M}}^\pm(f)$ is either a finite set, or the circle $\mathbb{S}^1$.

(ii) If H is parallel, then either $\mathcal{M}(f) = \mathbb{S}^1 \times \mathbb{S}^1$, or it locally decomposes as $\mathcal{M}(f) = \mathcal{V}_0 \cup \mathcal{V}_1$, where each $\mathcal{V}_d, d = 0, 1$, is either empty, or a disjoint finite union of d-dimensional real-analytic varieties.

Proof: (i) We claim that for any $\sigma \in \mathcal{D}$ in the group of deck transformations of the universal cover $\tilde{\pi}\colon \tilde{M} \to M$, the surfaces $\tilde{f}_\theta^{\pm}\colon \tilde{M} \to \mathbb{Q}_c^4$ in the associated family of $\tilde{f} = f \circ \tilde{\pi}$ and $\tilde{f}_\theta^{\pm} \circ \sigma$ are congruent for any $\theta \in \mathbb{S}^1$. It is sufficient to show the existence of a parallel vector bundle isometry between the normal bundles of $\tilde{f}_\theta^{\pm}$ and $\tilde{f}_\theta^{\pm} \circ \sigma$ that preserves the second fundamental forms. Let T_θ be the parallel vector bundle isometry between the normal bundles of $\tilde{f}$ and $\tilde{f}_\theta^{\pm}$ such that

$$\alpha_{\tilde{f}_\theta^{\pm}}(X,Y) = T_\theta \left(\tilde{J}_{\theta/2}^{\perp}\Big(\alpha_{\tilde{f}}(\tilde{J}_{\mp\theta/4}X, \tilde{J}_{\mp\theta/4}Y) - \langle X,Y\rangle H_{\tilde{f}}\Big) + \langle X,Y\rangle H_{\tilde{f}}\right)$$

for any $X, Y \in T\tilde{M}$, where $\tilde{J}_\theta^{\perp} = \cos\theta \tilde{I} + \sin\theta \tilde{J}^{\perp}$, $\tilde{J}_\theta = \cos\theta\tilde{I} + \sin\theta\tilde{J}$ and $\tilde{J}^{\perp}$, $\tilde{J}$ stand for the complex structures of $N_{\tilde{f}}\tilde{M}$ and $T\tilde{M}$, respectively. Since σ is a deck transformation, we have that $\tilde{f} \circ \sigma = \tilde{f}$ and thus, the normal spaces satisfy $N_{\tilde{f}}\tilde{M}(p) = N_{\tilde{f}}\tilde{M}(\sigma(p))$ at any $p \in \tilde{M}$. We define the vector bundle isometry $\Sigma_\theta\colon N_{\tilde{f}_\theta^{\pm}}\tilde{M} \to N_{\tilde{f}_\theta^{\pm}\circ\sigma}\tilde{M}$ which is given pointwise by

$$\Sigma_\theta|_p(\xi) = T_\theta|_{\sigma(p)} \circ \left(T_\theta|_p\right)^{-1}(\xi), \;\; \xi \in N_{\tilde{f}_\theta}\tilde{M}(p).$$

The second fundamental forms of $\tilde{f}_\theta^{\pm}$ and $\tilde{f}_\theta^{\pm} \circ \sigma$ are related at $p \in \tilde{M}$ by

$$\begin{aligned}
\alpha_{\tilde{f}_\theta^{\pm}\circ\sigma}|_p(X,Y) &= \alpha_{\tilde{f}_\theta^{\pm}}|_{\sigma(p)}(\sigma_*X, \sigma_*Y)\\
&= T_\theta|_{\sigma(p)}\left(\tilde{J}_{\theta/2}^{\perp}\Big(\alpha_{\tilde{f}}|_{\sigma(p)}(\tilde{J}_{\mp\theta/4}\sigma_*X, \tilde{J}_{\mp\theta/4}\sigma_*Y) - \langle X,Y\rangle H_{\tilde{f}}|_{\sigma(p)}\Big)\right.\\
&\qquad \left. + \langle X,Y\rangle H_{\tilde{f}}|_{\sigma(p)}\right)\\
&= T_\theta|_{\sigma(p)}\left(\tilde{J}_{\theta/2}^{\perp}\Big(\alpha_{\tilde{f}\circ\sigma}|_p(\tilde{J}_{\mp\theta/4}X, \tilde{J}_{\mp\theta/4}Y) - \langle X,Y\rangle H_{\tilde{f}\circ\sigma}|_p\Big)\right.\\
&\qquad \left. + \langle X,Y\rangle H_{\tilde{f}\circ\sigma}|_p\right)\\
&= T_\theta|_{\sigma(p)}\left(\tilde{J}_{\theta/2}^{\perp}\Big(\alpha_{\tilde{f}}|_p(\tilde{J}_{\mp\theta/4}X, \tilde{J}_{\mp\theta/4}Y) - \langle X,Y\rangle H_{\tilde{f}}|_p\Big) + \langle X,Y\rangle H_{\tilde{f}}|_p\right)\\
&= \Sigma_\theta|_p \circ \alpha_{\tilde{f}_\theta^{\pm}}|_p(X,Y)
\end{aligned}$$

for any $X, Y \in T\tilde{M}$ and thus, Σ_θ preserves the second fundamental forms. For any section ξ of $N_{\tilde{f}_\theta^{\pm}}\tilde{M}$ we have $\Sigma_\theta\xi = T_\theta(\eta \circ \sigma^{-1}) \circ \sigma$, where $\xi = T_\theta\eta$ for a section η of $N_{\tilde{f}}\tilde{M}$. Using the fact that for any section δ of $N_{\tilde{f}}\tilde{M}$ and any deck transformation σ we have that $\nabla_X^{\perp}(\delta \circ \sigma) = \nabla_{\sigma_*X}^{\perp}\delta \circ \sigma$, we obtain

$$\begin{aligned}
(\nabla_X^{\perp}\Sigma_\theta)\xi &= \nabla_X^{\perp}\Big(T_\theta(\eta\circ\sigma^{-1})\circ\sigma\Big) - T_\theta\Big(\nabla_X^{\perp}\eta\circ\sigma^{-1}\Big)\circ\sigma\\
&= \Big(\nabla_{\sigma_*X}^{\perp}T_\theta(\eta\circ\sigma^{-1})\Big)\circ\sigma - T_\theta\Big(\nabla_X^{\perp}\eta\circ\sigma^{-1}\Big)\circ\sigma\\
&= T_\theta\Big(\nabla_{\sigma_*X}^{\perp}(\eta\circ\sigma^{-1}) - \nabla_X^{\perp}\eta\circ\sigma^{-1}\Big)\circ\sigma,
\end{aligned}$$

where, by abuse of notation, $\nabla^\perp$ stands for the normal connection of $\tilde{f}, \tilde{f}_\theta^\pm$ and $\tilde{f}_\theta^\pm \circ \sigma$. Observe that

$$\nabla^\perp_{\sigma_* X}(\eta \circ \sigma^{-1}) = \nabla^\perp_X \eta \circ \sigma^{-1},$$

and thus Σ_θ is parallel and the claim has been proved.

This allows us to define a homomorphism $S_\theta \colon \mathcal{D} \to \mathrm{Isom}(\mathbb{Q}_c^4)$ for each $\theta \in [0, 2\pi]$, such that

$$\tilde{f}_\theta^\pm \circ \sigma = S_\theta(\sigma) \circ \tilde{f}_\theta^\pm, \quad \sigma \in \mathcal{D}.$$

Thus, $\theta \in \bar{\mathcal{M}}^\pm(f)$ if and only if $S_\theta(\mathcal{D}) = \{I\}$. Assume that $\bar{\mathcal{M}}^\pm(f)$ is infinite and let $\{\theta_m\}$ be a sequence in $\bar{\mathcal{M}}^\pm(f)$ which converges to some $\theta_0 \in [0, 2\pi]$. From $S_{\theta_m}(\mathcal{D}) = \{I\}$ for all $m \in \mathbb{N}$ we obtain that $S_{\theta_0}(\mathcal{D}) = \{I\}$. Let $\sigma \in \mathcal{D}$. By the mean value theorem applied to each entry $(S_\theta(\sigma))_{jk}$ of the corresponding matrix, we have

$$\frac{d}{d\theta}(S_\theta(\sigma))_{jk}(\mathring{\theta}_m) = 0 \tag{7.5}$$

for some $\mathring{\theta}_m$ which lies between θ_0 and θ_m. By continuity it follows that

$$\frac{d}{d\theta}(S_\theta(\sigma))_{jk}(\theta_0) = 0.$$

Consider the sequence $\{\mathring{\theta}_m\}$ that converges to θ_0 and observe that in view of (7.5), a similar argument gives

$$\frac{d^2}{d\theta^2}(S_\theta(\sigma))_{jk}(\theta_0) = 0.$$

Repeating the argument yields

$$\frac{d^n}{d\theta^n}(S_\theta(\sigma))_{jk}(\theta_0) = 0$$

for any integer $n \geq 1$. From the definition of the associated family, it is clear that $f_\theta^\pm$ depends on the parameter θ in a real-analytic way. Since $S_\theta(\sigma)$ is an analytic curve in $\mathrm{Isom}(\mathbb{Q}_c^4)$, we conclude that $S_\theta(\sigma) = I$ for each $\sigma \in \mathcal{D}$, and thus $\bar{\mathcal{M}}^\pm(f) = \mathbb{S}^1$.

(ii) We claim that for any $\sigma \in \mathcal{D}$, the surfaces $\tilde{f}_{\theta,\varphi} \colon \tilde{M} \to \mathbb{Q}_c^4$ and $\tilde{f}_{\theta,\varphi} \circ \sigma$ in $\mathcal{M}(\tilde{f})$ are congruent for any $(\theta, \varphi) \in \mathbb{S}^1 \times \mathbb{S}^1$. Let $T_{\theta,\varphi}$ be the parallel vector bundle isometry between the normal bundles of $\tilde{f}$ and $\tilde{f}_{\theta,\varphi}$ such that

$$\alpha_{\tilde{f}_{\theta,\varphi}}(X,Y) = T_{\theta,\varphi}\left(\tilde{J}^\perp_{(\theta+\varphi)/2}\Big(\alpha_{\tilde{f}}(\tilde{J}_{(\theta-\varphi)/4}X, \tilde{J}_{(\theta-\varphi)/4}Y) - \langle X, Y\rangle H_{\tilde{f}}\Big) + \langle X, Y\rangle H_{\tilde{f}}\right)$$

for any $X, Y \in T\tilde{M}$. We define the vector bundle isometry $\Sigma_{\theta,\varphi} \colon N_{\tilde{f}_{\theta,\varphi}}\tilde{M} \to N_{\tilde{f}_{\theta,\varphi}\circ\sigma}\tilde{M}$ which is given pointwise by

$$\Sigma_{\theta,\varphi}|_p(\xi) = T_{\theta,\varphi}|_{\sigma(p)} \circ \left(T_{\theta,\varphi}|_p\right)^{-1}(\xi), \quad \xi \in N_{\tilde{f}_{\theta,\varphi}}\tilde{M}(p).$$

As in the proof of part (i) above, it can be shown that $\Sigma_{\theta,\varphi}$ is parallel and preserves the second fundamental forms, and the claim follows. This allows us to define a homomorphism $S_{\theta,\varphi}\colon \mathcal{D} \to \mathrm{Isom}(\mathbb{Q}^4_c)$ for each $\theta, \varphi \in [0, 2\pi]$, such that

$$\tilde{f}_{\theta,\varphi} \circ \sigma = S_{\theta,\varphi}(\sigma) \circ \tilde{f}_{\theta,\varphi}, \quad \sigma \in \mathcal{D}.$$

Clearly, $(\theta, \varphi) \in \mathcal{M}(f)$ if and only if $S_{\theta,\varphi}(\mathcal{D}) = \{I\}$. Since $\tilde{f}_{\theta,\varphi}$ is real-analytic with respect to (θ, φ), it follows that $\mathcal{M}(f)$ is a real-analytic set. According to Lojacewisz's structure theorem [51, Theorem 6.3.3.], $\mathcal{M}(f)$ locally decomposes as

$$\mathcal{M}(f) = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2,$$

where each $\mathcal{V}_d, 0 \leq d \leq 2$, is either empty, or a disjoint finite union of d-dimensional real-analytic subvarieties. If $\mathcal{M}(f) \neq \mathbb{S}^1 \times \mathbb{S}^1$, then $\mathcal{V}_2 = \emptyset$ and this completes the proof. ∎

7.3.3 Surfaces in $\mathbb{R}^4$

In the sequel, we deal with surfaces in $\mathbb{R}^4$ whose one component of the Gauss map is harmonic. We regard the Grassmannian $Gr(2,4)$ of oriented 2-planes in $\mathbb{R}^4$ as a submanifold in $\Lambda^2\mathbb{R}^4$ via the Plücker embedding. The inner product of two simple 2-vectors in $\Lambda^2\mathbb{R}^4$ is given by

$$\langle\langle v_1 \wedge v_2, w_1 \wedge w_2 \rangle\rangle = \det(\langle v_j, w_k \rangle).$$

Then, $\Lambda^2\mathbb{R}^4$ splits orthogonally into the eigenspaces of the Hodge star operator $\star$, denoted by $\Lambda^2_+\mathbb{R}^4$ and $\Lambda^2_-\mathbb{R}^4$, corresponding to the eigenvalues 1 and -1, respectively. An element $a \wedge b$ of $Gr(2,4)$, where a, b are orthonormal vectors in $\mathbb{R}^4$, decomposes as

$$a \wedge b = (a \wedge b)_+ + (a \wedge b)_-, \quad \text{where} \quad (a \wedge b)_\pm = \frac{1}{2}\Big(a \wedge b \pm \star(a \wedge b)\Big).$$

Therefore, $Gr(2,4)$ can be identified with the product $\mathbb{S}^2_+ \times \mathbb{S}^2_-$, where $\mathbb{S}^2_\pm$ is the sphere of radius $1/\sqrt{2}$ in $\Lambda^2_\pm\mathbb{R}^4$, centered at the origin.

Let $f\colon M \to \mathbb{R}^4$ be a non-minimal surface, with mean curvature vector field H and Gauss map $g = (g_+, g_-)\colon M \to \mathbb{S}^2_+ \times \mathbb{S}^2_-$. In terms of a local complex coordinate z away from the zeros of H, the components of the Gauss map are given by

$$g_\pm = -\frac{i}{\lambda^2} f_*\partial \wedge f_*\bar{\partial} \mp \frac{i}{\|H\|^2} H^- \wedge H^+, \tag{7.6}$$

where λ is the conformal factor. The differential $\Psi^\pm$ is written as

$$\Psi^\pm = \psi^\pm dz^2, \quad \text{where} \quad \psi^\pm = \langle \phi^\pm, H^\mp \rangle \tag{7.7}$$

and $\phi^{\pm}$ is given by (2.11). The Gauss and Weingarten formulas become respectively,

$$\begin{aligned}\widetilde{\nabla}_{\partial} f_* \partial &= (\log \lambda^2)_z f_* \partial + \frac{2\psi^-}{\|H\|^2} H^- + \frac{2\psi^+}{\|H\|^2} H^+, &(7.8)\\ \widetilde{\nabla}_{\partial} f_* \bar{\partial} &= \frac{\lambda^2}{2}(H^- + H^+), &(7.9)\\ \widetilde{\nabla}_{\partial} H^{\pm} &= -\frac{\|H\|^2}{2} f_* \partial - \frac{2\psi^{\mp}}{\lambda^2} f_* \bar{\partial} + \frac{2\langle \nabla^{\perp}_{\bar{\partial}} H^{\pm}, H^{\mp}\rangle}{\|H\|^2} H^{\pm}, &(7.10)\end{aligned}$$

where $\widetilde{\nabla}$ is the induced connection on the induced bundle $f^*T\mathbb{R}^4$.

Lemma 7.10. *Let $f\colon M \to \mathbb{R}^4$ be a non-minimal surface. If the component $g_{\pm}$ of the Gauss map of f is harmonic, then its height functions in $\Lambda^2_{\pm}\mathbb{R}^4$ are eigenfunctions of the elliptic operator $\Delta + 2\left(\|H\|^2 + \|\mathcal{H}^{\pm}\|^2\right)$, corresponding to the zero eigenvalue.*

Proof: Let z be a local complex coordinate away from the isolated zeros of H (see Proposition 7.1(ii)). By using (7.8)-(7.10), equation (7.6) yields

$$(g_{\pm})_z = \frac{4i\psi^{\pm}}{\lambda^2\|H\|^2} f_* \bar{\partial} \wedge H^{\pm} - i f_* \partial \wedge H^{\mp}. \tag{7.11}$$

Differentiating (7.11) with respect to $\bar{z}$, we obtain that the normal component of $(g_{\pm})_{z\bar{z}}$ with respect to $\mathbb{S}^2_{\pm}$ is given by

$$((g_{\pm})_{z\bar{z}})^{\perp} = -\frac{\lambda^2}{2}\left(\|H\|^2 + \|\mathcal{H}^{\pm}\|^2\right) g_{\pm}.$$

For an arbitrary vector $v_{\pm} \in \Lambda^2_{\pm}\mathbb{R}^4$ we have

$$\Delta\langle\langle g_{\pm}, v_{\pm}\rangle\rangle = \langle\langle \tau(g_{\pm}) + \frac{4}{\lambda^2}\left((g_{\pm})_{z\bar{z}}\right)^{\perp}, v_{\pm}\rangle\rangle,$$

where $\tau(g_{\pm})$ is the tension field of $g_{\pm}$. The result follows from the above and the harmonicity of $g_{\pm}$. ∎

Lemma 7.11. *Let $f\colon M \to \mathbb{R}^4$ be a surface, which is neither minimal nor superconformal. Assume that $g_{\pm}$ is harmonic and that there exist surfaces $f_j \in \mathcal{M}^{\pm}(f)$ with $\tilde{f}^{\pm}_{\theta_j} = f_j \circ \tilde{\pi}$, and vectors $v^j_{\pm} \in \Lambda^2_{\pm}\mathbb{R}^4 \smallsetminus \{0\}$, $j = 1, \dots, n$, such that the Gauss maps $g^j = (g^j_+, g^j_-)$ of f_j satisfy*

$$\sum_{j=1}^{n} \langle\langle g^j_{\pm}, v^j_{\pm}\rangle\rangle = 0. \tag{7.12}$$

Then:

(i) The differential $\mathcal{U}^{\pm} = u^{\pm}dz$ *is holomorphic, where*

$$u^{\pm} = \sum_{j=1}^{n} \langle\langle f_{j*}\partial \wedge H_j^{\mp}, v_{\pm}^{j}\rangle\rangle \tag{7.13}$$

and H_j *is the mean curvature vector field of* f_j*.*

(ii) If $\mathcal{U}^{\pm} \equiv 0$*, then*

$$\sum_{j=1}^{n} e^{i\theta_j}\langle\langle g_{\pm}^{j}, v_{\pm}^{j}\rangle\rangle = 0. \tag{7.14}$$

Proof: From (7.4) and since by definition $\Psi_{f_j}^{\pm} = \langle \Phi_{f_j}^{\pm}, H_j^{\mp}\rangle$, we have that

$$\Psi_{f_j}^{\pm} = e^{\mp i\theta_j}\Psi^{\pm}, \;\; j = 1, \ldots, n.$$

Let (U, z) be a complex chart. On $U \smallsetminus Z(\Psi^{\pm})$, (7.11) yields

$$(g_{\pm}^{j})_z = e^{\mp i\theta_j}\frac{4i\psi^{\pm}}{\lambda^2\|H\|^2} f_{j*}\bar{\partial} \wedge H_j^{\pm} - i f_{j*}\partial \wedge H_j^{\mp}, \;\; j = 1, \ldots, n. \tag{7.15}$$

Differentiating (7.12) with respect to z and using (7.15), we find that

$$u^{\pm} = \frac{4\psi^{\pm}}{\lambda^2\|H\|^2}\sum_{j=1}^{n} e^{\mp i\theta_j}\langle\langle f_{j*}\bar{\partial} \wedge H_j^{\pm}, v_{\pm}^{j}\rangle\rangle \;\text{ on }\; U \smallsetminus Z(\Psi^{\pm}). \tag{7.16}$$

From Proposition 2.7 it follows that $H_j^{\pm}$ is an anti-holomorphic section. Hence,

$$\nabla_{\partial}^{\perp} H_j^{\pm} = 0 \quad\text{and}\quad (\|H\|^2)_z = 2\langle \nabla_{\partial}^{\perp} H_j^{\mp}, H_j^{\pm}\rangle, \;\; j = 1, \ldots, n. \tag{7.17}$$

Differentiating (7.16) with respect to z, and using (7.9), (7.10), (7.17), (7.6) and (7.16), we obtain that

$$u_z^{\pm} = u^{\pm}\left(\log\frac{\psi^{\pm}}{\lambda^2\|H\|^2}\right)_z + 2i\psi^{\pm}\sum_{j=1}^{n} e^{\mp i\theta_j}\langle\langle g_{\pm}^{j}, v_{\pm}^{j}\rangle\rangle \;\text{ on }\; U \smallsetminus Z(\Psi^{\pm}). \tag{7.18}$$

On the other hand, differentiating (7.13) with respect to z, and using (7.8), (7.10), (7.17), (7.6) and (7.13), we find that

$$u_z^{\pm} = u^{\pm}\left(\log\left(\lambda^2\|H\|^2\right)\right)_z - 2i\psi^{\pm}\sum_{j=1}^{n} e^{\mp i\theta_j}\langle\langle g_{\pm}^{j}, v_{\pm}^{j}\rangle\rangle \;\text{ on }\; U \smallsetminus Z(\Psi^{\pm}). \tag{7.19}$$

(i) From (7.9), (7.10), (7.17) and (7.6), we have that

$$u_{\bar{z}}^{\pm} = \frac{\lambda^2\|H\|^2}{2i}\sum_{j=1}^{n}\langle\langle g_{\pm}^{j}, v_{\pm}^{j}\rangle\rangle \;\text{ on }\; U \tag{7.20}$$

and the claim follows from (7.12).

(ii) Using (7.18) and (7.19), we obtain that

$$\sum_{j=1}^{n} e^{\mp i\theta_j}\langle\langle g_{\pm}^{j}, v_{\pm}^{j}\rangle\rangle = \frac{iu^{\pm}}{4\psi^{\pm}}\left(\log\frac{\psi^{\pm}}{\lambda^4\|H\|^4}\right)_z \quad \text{on } U\smallsetminus Z(\Psi^{\pm}) \tag{7.21}$$

and (7.14) follows from (7.21). ∎

Theorem 7.12. *Let $f\colon M\to\mathbb{R}^4$ be a non-superconformal isometric immersion of a compact, oriented 2-dimensional Riemannian manifold, with mean curvature vector field H and Gauss map $g=(g_+,g_-)\colon M\to\mathbb{S}^2_+\times\mathbb{S}^2_-$.*
(i) If $g_\pm$ is harmonic and $\chi\neq\mp\chi_N$, then $\mathcal{M}^{\pm}(f)$ is a finite set.
(ii) If H is parallel and $\chi\neq 0$, then $\mathcal{M}(f)$ is a finite set.

Proof: (i) Suppose that $\mathcal{M}^{\pm}(f)$ is infinite and consider surfaces $f_j\in\mathcal{M}^{\pm}(f)$ such that $\tilde{f}^{\pm}_{\theta_j}=f_j\circ\tilde{\pi}$, $j=1,\dots,n$, with $0<\theta_1<\cdots<\theta_n<\pi$ or $\pi<\theta_1<\cdots<\theta_n<2\pi$. We prove that the height functions of the $\Lambda^2_{\pm}\mathbb{R}^4$-component of the Gauss maps of f_j are linearly independent. Suppose to the contrary that (7.12) holds for vectors $v^j_{\pm}\in\Lambda^2_{\pm}\mathbb{R}^4\smallsetminus\{0\}$, $j=1,\dots,n$.

We claim that $\mathcal{U}^{\pm}\equiv 0$. Arguing indirectly, assume that $\mathcal{U}^{\pm}\not\equiv 0$. From Lemmas 2.2 and 7.11(i), it follows that its zero-set $Z(\mathcal{U}^{\pm})$ is isolated. Let z be a complex coordinate in a connected neighbourhood $U\subset M\setminus(Z(\Psi^{\pm})\cup Z(\mathcal{U}^{\pm}))$. From (7.18) and (7.19), we obtain

$$\left(\log\frac{\psi^{\pm}}{(u^{\pm})^2}\right)_z=0.$$

Using Proposition 7.1(i) and Lemma 7.11(i), we have

$$\left(\log\frac{\psi^{\pm}}{(u^{\pm})^2}\right)_{\bar z}=0.$$

Therefore,

$$\psi^{\pm}=c(u^{\pm})^2 \tag{7.22}$$

on U, for a non-zero constant $c\in\mathbb{C}$. It is easy to see that c is independent of the complex coordinate and thus, $\Psi^{\pm}=c\,\mathcal{U}^{\pm}\otimes\mathcal{U}^{\pm}$ on M. We argue that $Z(\Psi^{\pm})=Z(\mathcal{U}^{\pm})\neq\emptyset$. Indeed, if $Z(\Psi^{\pm})=\emptyset$, then the holomorphic differential $\Psi^{\pm}$ is everywhere nonvanishing and by the Riemann-Roch theorem we obtain that $\chi=0$. On the other hand, Proposition 7.1(i) implies that H is everywhere nonvanishing and Proposition 7.2(i) gives $\chi_N=0$. This contradicts our assumption. Let $Z(\Psi^{\pm})=\{p_1,\dots,p_k\}$ and consider a complex chart (U,z) around $p_r, r=1,\dots,k$, with $z(p_r)=0$. Since $\mathcal{U}^{\pm}=u^{\pm}dz$ is holomorphic, there exists a positive integer m_r such that around p_r we have

$$u^{\pm}=z^{m_r}\hat{u},\ \text{ where } \hat{u} \text{ is holomorphic with } \hat{u}(0)\neq 0. \tag{7.23}$$

Hence, from (7.22) we have that $|\psi^{\pm}|^2 = |z|^{4m_r}|c|^2|\hat{u}|^4$, or equivalently, bearing in mind (7.3) and (7.7)

$$\|H\|^2\|\mathcal{H}^{\pm}\|^2 = |z|^{4m_r}u_1, \quad \text{where } \ u_1 \ \text{ is smooth and positive.}$$

Proposition 7.1(ii) implies that there exist non-negative integers l_r, s_r such that

$$\|H\|^2 = |z|^{2l_r}u_2 \quad \text{and} \quad \|\mathcal{H}^{\pm}\|^2 = |z|^{2s_r}u_3,$$

where u_2, u_3 are smooth and positive. It is clear that $s_r = 2m_r - l_r$. From (7.21), by using (7.22), (7.23) and the above, on $U \smallsetminus Z(\Psi^{\pm})$ we have that

$$\sum_{j=1}^{n} e^{\mp i\theta_j}\langle\langle g^j_{\pm}, v^j_{\pm}\rangle\rangle = \frac{i\lambda^2\|H\|^2}{2c(u^{\pm})^2}\left(\frac{u^{\pm}}{\lambda^2\|H\|^2}\right)_z = \frac{i\lambda^2 z^{l_r}\bar{z}^{l_r}u_2}{2cz^{2m_r}\hat{u}^2}\left(\frac{z^{m_r}\hat{u}}{\lambda^2 z^{l_r}\bar{z}^{l_r}u_2}\right)_z,$$

or equivalently

$$\sum_{j=1}^{n} e^{\mp i\theta_j}\langle\langle g^j_{\pm}, v^j_{\pm}\rangle\rangle = \frac{1}{z^{m_r+1}}\frac{i\lambda^2 u_2}{2c\hat{u}^2}\left((m_r - l_r)\frac{\hat{u}}{\lambda^2 u_2} + z\Big(\frac{\hat{u}}{\lambda^2 u_2}\Big)_z\right).$$

If $m_r \neq l_r$ for some $r = 1, \ldots, k$, then the right-hand side of the above has a pole at $z = 0$, whereas the left-hand side is bounded. Hence, $m_r = l_r = s_r$ for any $r = 1, \ldots, k$. Then, Proposition 7.2 implies that $\chi = \mp\chi_N$, which is a contradiction. Therefore, $\mathcal{U}^{\pm} \equiv 0$ and this proves the claim.

According to Lemma 7.11(ii), (7.14) is valid, or equivalently

$$\sum_{j=1}^{n} \cos\theta_j\langle\langle g^j_{\pm}, v^j_{\pm}\rangle\rangle = 0 \quad \text{and} \quad \sum_{j=1}^{n} \sin\theta_j\langle\langle g^j_{\pm}, v^j_{\pm}\rangle\rangle = 0.$$

Eliminating $\langle\langle g^n_{\pm}, v^n_{\pm}\rangle\rangle$, we obtain

$$\sum_{j=1}^{n-1}\langle\langle g^j_{\pm}, w^j_{\pm}\rangle\rangle = 0,$$

where $w^j_{\pm} = \sin(\theta_n - \theta_j)v^j_{\pm} \neq 0$, $j = 1, \ldots, n-1$. By induction, we finally find that $\langle\langle g^n_{\pm}, w_{\pm}\rangle\rangle = 0$ for some non-zero vector $w_{\pm} \in \Lambda^2_{\pm}\mathbb{R}^4$. Therefore, $g^n_{\pm}$ takes values in a great circle of $\mathbb{S}^2_{\pm}$ and thus, its Jacobian $\mathcal{J}_{g^n_{\pm}}$ vanishes. On the other hand, we know that (cf. [41, Proposition 4.5.])

$$K = \mathcal{J}_{g^n_+} + \mathcal{J}_{g^n_-} \quad \text{and} \quad K_N = \mathcal{J}_{g^n_+} - \mathcal{J}_{g^n_-}.$$

Hence, we conclude that $K = \mp K_N$, which contradicts our topological assumption. Therefore, we have proved that the height functions of the $\Lambda^2_{\pm}\mathbb{R}^4$-component of the Gauss maps of f_j are linearly independent. This contradicts Lemma 7.10, since the eigenspaces of an elliptic operator are finite dimensional. Hence, $\mathcal{M}^{\pm}(f)$ is a finite set.

(ii) Assume that $\mathcal{M}(f)$ is infinite. Then there exists a sequence $f_k \in \mathcal{M}(f)$ such that $\tilde{f}_{\theta_k,\varphi_k} = f_k \circ \tilde{\pi}$, for which $(\theta_l, \varphi_l) \neq (\theta_m, \varphi_m)$ for $l \neq m$. Without loss of generality, we may assume that either $0 < \theta_l < \theta_m < \pi$, or $\pi < \theta_l < \theta_m < 2\pi$, for $l, m \in \mathbb{N}$ with $l < m$. We prove that the height functions of the $\Lambda^2_-\mathbb{R}^4$-component of the Gauss maps of f_j, $j = 1, \dots, n$, are linearly independent. Suppose to the contrary that (7.12) holds for vectors $v^j_- \in \Lambda^2_-\mathbb{R}^4 \smallsetminus \{0\}$, $j = 1, \dots, n$. From the construction of the associated family in Proposition 7.5 it follows that $\Psi^-_{f_j} = e^{i\theta_j}\Psi^-$. Consequently, the relations (7.15)-(7.21) are valid and thus, the conclusion of Lemma 7.11 also holds. Taking into account that $K_N = 0$, we can prove as in the proof of part (i) that our topological assumption implies $\mathcal{U}^- \equiv 0$. The remaining of the proof is the same with the one of part (i). ∎

7.3.4 Proof of the Main Result

We are now ready to give the proof of our result for compact surfaces.

Proof of Theorem 7.8: (i) From Proposition 7.9(i) we know that $\bar{\mathcal{M}}^{\pm}(f)$ is either finite, or the circle $\mathbb{S}^1$. We show that the same holds true for the set $\mathcal{M}^*(f) \cup \mathcal{M}^{\mp}(f)$.

Suppose that $\mathcal{M}^*(f) \cup \mathcal{M}^{\mp}(f)$ is infinite. Since $G_{\mp}$ is not vertically harmonic, from Theorem 6.1(i) it follows that there exists at most one Bonnet mate in $\mathcal{M}^{\mp}(f)$ and thus, $\mathcal{M}^*(f)$ is infinite. For $\tilde{f} \in \mathcal{M}^*(f)$, Theorem 6.1(i) implies that $\mathcal{M}^*(f) \cup \mathcal{M}^{\mp}(f) = \bar{\mathcal{M}}^{\pm}(\tilde{f})$ and the proof follows from Proposition 7.9(i) applied to the surface $\tilde{f}$.

(ii) By virtue of Theorem 6.9, we assume that f is non-superconformal. The case where H is parallel has been proved in Theorem 7.12(ii). Assume that H is non-parallel and suppose to the contrary that $\mathcal{M}(f)$ is infinite. From Theorem 7.12(i) it follows that $\mathcal{M}^{\pm}(f)$ is finite. Since $g_{\mp}$ is not harmonic, Theorem 6.1(i) implies that there exists at most one Bonnet mate in $\mathcal{M}^{\mp}(f)$ and therefore, $\mathcal{M}^*(f)$ is infinite. Theorem 6.1(i) yields that $\mathcal{M}^*(f) \cup \mathcal{M}^{\mp}(f) = \bar{\mathcal{M}}^{\pm}(\tilde{f})$ for any $\tilde{f} \in \mathcal{M}^*(f)$, which contradicts Theorem 7.12(i) for $\tilde{f}$. ∎

Chapter 8

Locally Proper Bonnet Surfaces

An oriented surface $f\colon M \to \mathbb{Q}_c^4$ is called *locally proper Bonnet*, if every point of M has a simply-connected neighbourhood U such that $f|_U$ is proper Bonnet. The following proposition shows that there do not exist compact simply-connected surfaces in $\mathbb{Q}_c^4$ that are globally proper Bonnet.

Proposition 8.1. *Let $f\colon M \to \mathbb{Q}_c^4$ be an oriented surface. If M is homeomorphic to $\mathbb{S}^2$, then f admits at most one Bonnet mate.*

Proof: If both Gauss lifts of f are not vertically harmonic, then Theorem 6.3 implies that f admits at most one Bonnet mate. Assume that f has a vertically harmonic Gauss lift. We claim that f is superconformal. Indeed, if f is non-minimal then Theorem 7.3 yields that it is superconformal. If f is minimal, the claim follows by a well-known result of Calabi [9]. Then, Theorem 6.9 implies that f admits at most one Bonnet mate. ∎

Remark 8.2. From the above proposition it follows that Theorem 5.1 holds for any simply-connected surface. Moreover, for our results in Chapter 5 concerning proper Bonnet surfaces, the non-compactness assumption is not restrictive at all.

In order to state the following proposition, we recall that if $f\colon M \to \mathbb{Q}_c^4$ is a non-minimal, simply-connected proper Bonnet surface, then Theorem 5.3(iv) implies that $\mathcal{M}(f)$ is a smooth manifold.

Proposition 8.3. *Let $f\colon M \to \mathbb{Q}_c^4$ be a locally proper Bonnet surface. Then:*

(i) *Either f is minimal, or* $\operatorname{int}\{p \in M : H(p) = 0\} = \emptyset$.

(ii) *If f is non-minimal, then for every $p \in M$ there exists a submanifold $L^n(p), 1 \leq n \leq 2$, of the torus $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, with the property that $L^n(p)$ is also a submanifold of $\mathcal{M}(f|_U)$ for every sufficiently small simply-connected neighbourhood U of p. In particular, for every point of M, a submanifold of the torus with this property is either $\mathbb{S}^1_- = \mathbb{S}^1 \times \{0\}$, or $\mathbb{S}^1_+ = \{0\} \times \mathbb{S}^1$.*

Proof: (i) Arguing indirectly, assume that f is locally proper Bonnet, non-minimal and $\mathrm{int}\{p \in M : H(p) = 0\} \neq \emptyset$. Let $\bar{p}$ be a boundary point of $\{p \in M : H(p) = 0\}$. Then, there exists a simply-connected complex chart (U, z) around $\bar{p}$ such that $f|_U$ is proper Bonnet and non-minimal. By virtue of Theorem 5.3(iii), we may assume that $\bar{\mathcal{M}}^{\pm}(f|_U) = \mathbb{S}^1$. Let $\tilde{f} \in \mathcal{M}^{\pm}(f|_U)$. From Proposition 4.2 it follows that $M_0^{\pm}(f|_U)$ is isolated. Since $M_0^{\pm}(f|_U) = M_0^{\pm}(f) \cap U$, we may assume that $\bar{p}$ and U are such that $M_0^{\pm}(f|_U) = \emptyset$. Then, the Codazzi equation and (3.18) imply that

$$h^{\pm} \equiv 0 \quad \text{on} \quad U \cap \mathrm{int}\{p \in M : H(p) = 0\}. \tag{8.1}$$

Appealing to Proposition 5.2(i), there exists a harmonic function $\theta^{\pm} \in \mathcal{C}^{\infty}(U; (0, 2\pi))$ satisfying (4.4) on U, such that the distortion differential of the pair $(f|_U, \tilde{f})$ is given by (4.2) on U. From (8.1) and (4.4) it follows that the harmonic function $\theta^{\pm}$ is constant on $U \cap \mathrm{int}\{p \in M : H(p) = 0\}$ and thus, constant on U. Then, (3.18) and Proposition 2.7 imply that the Gauss lift $G_{\pm}$ of f is vertically harmonic on U. From Proposition 7.1(ii) it follows that $\|H\|$ is an absolute value type function on U. Since $\|H\|$ vanishes on an open subset of U, this implies that $H \equiv 0$ on U. This is a contradiction, since $f|_U$ is non-minimal.

(ii) Assume that f is non-minimal and let $p \in M$. There exists a simply-connected complex chart (V, z) around p such that $f|_V$ is proper Bonnet. From part (i) it follows that $f|_V$ is non-minimal and Theorem 5.3(iii) implies that either $\bar{\mathcal{M}}^{-}(f|_V) = \mathbb{S}^1$, or $\bar{\mathcal{M}}^{+}(f|_V) = \mathbb{S}^1$. Assume that $\bar{\mathcal{M}}^{\pm}(f|_V) = \mathbb{S}^1$. By virtue of Remark 5.4, we parametrize $\bar{\mathcal{M}}^{\pm}(f|_V)$ such that (5.2) is valid at p and we write $\bar{\mathcal{M}}_p^{\pm}(f|_V) = \mathbb{S}^1$. For every sufficiently small simply-connected neighbourhood U of p we have that $U \subset V$ and therefore, $\bar{\mathcal{M}}_p^{\pm}(f|_U) = \mathbb{S}^1$. Appealing to Theorem 5.3(iv), it is clear that $\mathbb{S}^1_{\pm}$ is a submanifold of $\mathcal{M}(f|_U)$. ∎

Let $f\colon M \to \mathbb{Q}_c^4$ be a non-minimal locally proper Bonnet surface. By virtue of Proposition 8.3(ii) we give the following definition; the surface f is called *uniformly locally proper Bonnet* if there exists a submanifold $L^n, 1 \leq n \leq 2$, of the torus $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, with the property that for every $p \in M$, L^n is also a submanifold of $\mathcal{M}(f|_U)$ for every sufficiently small simply-connected neighbourhood U of p. In this case, L^n is called *a deformation manifold for f*. Moreover, f is called *locally flexible proper Bonnet* if the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is a deformation manifold for f.

Lemma 8.4. *A surface $f\colon M \to \mathbb{Q}_c^4$ is uniformly locally proper Bonnet with deformation manifold $\mathbb{S}^1_{\pm}$ if and only if every point of M has a simply-connected neighbourhood U such that $\bar{\mathcal{M}}^{\pm}(f|_U) = \mathbb{S}^1$. Moreover, if $\mathbb{S}^1_{\pm}$ is a deformation manifold for f, then the set $M_0^{\pm}(f)$ is isolated.*

Proof: Assume that $\mathbb{S}^1_{\pm}$ is a deformation manifold for f. Then, every point of M has a simply-connected neighbourhood U such that $\mathbb{S}^1_{\pm}$ is a submanifold of $\mathcal{M}(f|_U)$. From Theorem 5.3(iv) it follows that $\bar{\mathcal{M}}^{\pm}(f|_U) = \mathbb{S}^1$. The converse follows in a similar manner with the proof of Proposition 8.3(ii).

Suppose now that that $\mathbb{S}^1_\pm$ is a deformation manifold for f and arguing indirectly, assume that $M_0^\pm(f)$ has an accumulation point p. Then, there exists a neighbourhood U of p such that $\bar{\mathcal{M}}^\pm(f|_U) = \mathbb{S}^1$. Proposition 4.2 implies that $M_0^\pm(f|_U)$ is isolated. This is a contradiction, since $M_0^\pm(f|_U) = M_0^\pm(f) \cap U$. ∎

From Theorem 7.4 and the above lemma, it follows that surfaces in $\mathbb{Q}^4_c$ that are neither minimal, nor superconformal and whose Gauss lift $G_\pm$ is vertically harmonic, are uniformly locally proper Bonnet with deformation manifold $\mathbb{S}^1_\pm$. The following theorem shows that the converse is also true for compact surfaces.

Theorem 8.5. *Let $f\colon M \to \mathbb{Q}^4_c$ be a non-minimal compact oriented surface. Then, f is uniformly locally proper Bonnet with deformation manifold $\mathbb{S}^1_\pm$ if and only if the Gauss lift $G_\pm$ of f is vertically harmonic and non-conformal.*

Proof: Assume that $\mathbb{S}^1_\pm$ is a deformation manifold for f. Lemma 8.4 implies that $M_0^\pm(f)$ is isolated. From Lemma 2.3(ii) and Proposition 2.5 it follows that the Gauss lift $G_\pm$ of f is non-conformal. By virtue of Lemma 8.4 and Theorem 5.7, it follows that equation (5.9) is valid at every point of M. By integrating (5.9) on M yields

$$\int_M \Delta \log \|\mathcal{H}^\pm\| - \int_M (2K \pm K_N) = \int_M \frac{\|\tau^v(G_\pm)\|^2}{4\|\mathcal{H}^\pm\|^2}. \tag{8.2}$$

Moreover, Lemma 8.4 and Proposition 5.5(ii) imply that $\|\mathcal{H}^\pm\|$ is an absolute value function on M. Therefore, from Lemma 2.4 it follows that

$$\int_M \Delta \log \|\mathcal{H}^\pm\| = -2\pi N(\|\mathcal{H}^\pm\|).$$

On the other hand, Theorem 3.5 and Propositions 3.4 and 5.5(i) yield that

$$\int_M (2K \pm K_N) = -2\pi N(\|\mathcal{H}^\pm\|).$$

From the above two relations it follows that the left hand side of (8.2) vanishes and thus $\|\tau^v(G_\pm)\| \equiv 0$ on M. Therefore, the Gauss lift $G_\pm$ of f is vertically harmonic.

Conversely, assume that the Gauss lift $G_\pm$ of f is vertically harmonic and non-conformal. By virtue of Lemma 2.3(ii), Proposition 2.5 implies that f is non-minimal and $M \neq M_0^\pm(f)$. From Proposition 7.1(ii) it follows that $M_0^\pm(f)$ is isolated. Then, Theorem 7.4 implies that every point in M has a simply-connected neighbourhood U such that $\bar{\mathcal{M}}^\pm(f|_U) = \mathbb{S}^1$. The proof now follows from Lemma 8.4. ∎

Theorem 8.6. *Let $f\colon M \to \mathbb{Q}^4_c$ be a non-minimal, compact oriented surface. Then, f is uniformly locally proper Bonnet if and only if it has a vertically harmonic, non-conformal Gauss lift.*

Proof: By virtue of Lemma 8.3, either $\mathbb{S}^1_-$, or $\mathbb{S}^1_+$, is a deformation manifold for f. The proof follows immediately from Theorem 8.5. ∎

Theorem 8.7. *There do not exist compact oriented superconformal surfaces in $\mathbb{Q}^4_c$ that are locally proper Bonnet.*

Proof: For superminimal surfaces, the proof follows from [46, 67]. Let $f\colon M \to \mathbb{Q}^4_c$ be a non-minimal, compact superconformal surface. Arguing indirectly, assume that f is locally proper Bonnet.

We claim that the normal curvature of f does not change sign. By virtue of Lemma 8.3(ii) and Theorem 5.3(iii), every point of M has a neighbourhood U such that either $\bar{\mathcal{M}}^-(f|_U) = \mathbb{S}^1$, or $\bar{\mathcal{M}}^+(f|_U) = \mathbb{S}^1$. Then, Proposition 4.2 implies that either $M_0^-(f|_U)$, or $M_0^+(f|_U)$ is isolated. Since $M_0^\pm(f|_U) = M_0^\pm(f) \cap U$ and $M_1(f) = M_0^-(f) \cap M_0^+(f)$, we deduce that $M_1(f)$ is isolated. From Lemma 2.3(ii) it follows that the normal curvature of f vanishes at isolated points only, and this proves the claim.

Assume that $\pm K_N \geq 0$. Lemma 2.3(ii) implies that $\Phi^\pm \equiv 0$. Therefore, $\mathcal{M}^\pm(f|_U) = \emptyset$ for every $U \subset M$. Since f is locally proper Bonnet, from Theorem 5.3(iii) and Lemma 8.4 it follows that f is uniformly locally proper Bonnet with deformation manifold $\mathbb{S}^1_\mp$. Then, Theorem 8.5 implies that the Gauss lift $G_\mp$ is vertically harmonic and non-conformal. On the other hand, since $\Phi^\pm \equiv 0$, from Proposition 2.7 it follows that $G_\pm$ is vertically harmonic. Since both Gauss lifts of f are vertically harmonic, the mean curvature vector field of f is parallel in the normal connection. Therefore, $K_N \equiv 0$ on M. Proposition 2.5 then implies that $G_\mp$ is conformal, which is a contradiction. ∎

Corollary 8.8. *There do not exist uniformly locally proper Bonnet surfaces in $\mathbb{Q}^4_c$ of genus zero.*

Proof: Arguing indirectly, assume that M is homeomorphic to $\mathbb{S}^2$ and let $f\colon M \to \mathbb{Q}^4_c$ be a uniformly locally proper Bonnet surface. By virtue of Lemma 8.3(ii), assume that $\mathbb{S}^1_\pm$ is a deformation manifold for f. Theorem 8.5 implies that the Gauss lift $G_\pm$ of f is vertically harmonic. Then, from Theorem 7.3 it follows that f is superconformal. This contradicts Theorem 8.7. ∎

In Chapter 5, we have shown the existence of flexible proper Bonnet surfaces in $\mathbb{Q}^4_c$ that do not lie in any totally umbilical hypersurface of $\mathbb{Q}^4_c$ (see Remark 5.15). The following theorem shows that for compact surfaces, local flexibility characterizes surfaces with parallel mean curvature vector field. Therefore, from [14, 69] it follows that a compact locally flexible proper Bonnet surface lies in some totally umbilical hypersurface of the ambient space.

Theorem 8.9. *A compact oriented surface $f\colon M \to \mathbb{Q}^4_c$ is locally flexible proper Bonnet if and only if it has non-vanishing parallel mean curvature vector field and genus$(M) > 0$.*

Proof: Assume that f is locally flexible proper Bonnet. Then, f is non-minimal and both of $\mathbb{S}^1_-$ and $\mathbb{S}^1_+$ are deformation manifolds for f. Theorem 8.5 implies that both Gauss lifts of f are vertically harmonic. Therefore, f has non-vanishing parallel mean curvature vector field. Moreover, Corollary 8.8 yields that $genus(M) > 0$.

Conversely, assume that f has non-vanishing parallel mean curvature vector field and $genus(M) > 0$. Since M is not homeomorphic to $\mathbb{S}^2$, it follows that f is not totally umbilical. Then, Lemma 2.3(i) yields that the Hopf differential Φ of f does not vanish identically on M. On the other hand, the Codazzi equation implies that Φ is holomorphic. Therefore, from Lemmas 2.2 and 2.3(i) it follows that the umbilic points of f are isolated. Then, Proposition 7.5(iii) implies that every point of M has a simply-connected neighbourhood U such that $\mathcal{M}(f|_U) = \mathbb{S}^1 \times \mathbb{S}^1$. Therefore, f is locally flexible proper Bonnet. ∎

An immediate consequence of Theorems 5.13 and 8.9 is the following result due to Umehara [65].

Theorem 8.10. *Let $F\colon M \to \mathbb{Q}^3_c$ be a compact oriented surface with $genus(M) > 0$. If F is locally proper Bonnet, then it has constant mean curvature.*

Proof: Let $j\colon \mathbb{Q}^3_c \to \mathbb{Q}^4_c$ be a totally geodesic inclusion and set $f = j \circ F$. From Theorem 5.13 it follows that f is locally flexible. Theorem 8.9 implies that f has parallel mean curvature vector field and therefore, the mean curvature of F is constant. ∎

Abstract

We study the Bonnet problem for surfaces in 4-dimensional space forms $\mathbb{Q}_c^4$. Two isometric surfaces are said to have the same mean curvature if there exists a parallel vector bundle isometry between their normal bundles that preserves the mean curvature vector fields. Noncongruent surfaces with the same mean curvature are called Bonnet mates. A surface in $\mathbb{Q}_c^4$ is called a Bonnet, or a proper Bonnet surface, if it admits either at least one, or infinitely many Bonnet mates, respectively.

We introduce the notions of isotropically isothermic and strongly isotropically isothermic surfaces in $\mathbb{Q}_c^4$ as a generalization of the notion of isothermic surfaces in $\mathbb{Q}_c^3$ and we show that isotropic isothermicity is a conformally invariant property.

We show that if a non-compact simply connected surface $f\colon M \to \mathbb{Q}_c^4$ is not proper Bonnet, then it admits either at most one Bonnet mate, or exactly three. If such a surface is proper Bonnet, then the moduli space $\mathcal{M}(f)$ of congruence classes of all isometric immersions of M into $\mathbb{Q}_c^4$ that have the same mean curvature with f, is diffeomorphic to a manifold. Proper Bonnet surfaces are distinguished in two categories: the tight surfaces whose moduli space is 1-dimensional with at most two connected components diffeomorphic to $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, and the flexible ones whose moduli space is diffeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$. We prove that isotropic isothermicity characterizes proper Bonnet surfaces and in particular, strong isotropic isothermicity characterizes the flexible surfaces. Moreover, we show that a half totally non isotropically isothermic surface is always a Bonnet surface which in particular, admits exactly three Bonnet mates if it is furthermore strongly totally non isotropically isothermic. We also prove that a Bonnet surface lying in a totally geodesic hypersurface of $\mathbb{Q}_c^4$ with non-constant mean curvature, admits at least two Bonnet mates that do not lie in any totally umbilical hypersurface of $\mathbb{Q}_c^4$.

We prove that if both Gauss lifts of a compact surface to the twistor bundle are not vertically harmonic, then the surface admits at most three Bonnet mates. In particular, we show that such a surface admits at most one Bonnet mate, under additional assumptions involving isotropic isothermicity.

We show that non-minimal surfaces with a vertically harmonic Gauss lift possess a holomorphic quadratic differential, yielding thus a Hopf-type theorem. We prove that such surfaces allow locally a 1-parameter family of isometric deformations with the same mean curvature. This family is trivial only if the surface is superconformal. For such compact surfaces with non-parallel mean curvature, we prove that the moduli space is the disjoint union of two sets, each one being either finite, or a circle. In particular, for surfaces in $\mathbb{R}^4$ we prove that the moduli space is a finite set, under a condition on the Euler numbers of the tangent and normal bundles.

We study locally proper Bonnet surfaces in $\mathbb{Q}_c^4$. A surface $f\colon M \to \mathbb{Q}_c^4$ is called locally proper Bonnet if every point of M has a simply-connected neighbourhood, restricted to which f is proper Bonnet. We prove that if a locally proper Bonnet surface is non-minimal, then around a point $p \in M$, any continuous isometric deformation that preserves the mean curvature is described by a submanifold $L^n(p), 1 \leq n \leq 2$, of the torus $\mathbb{S}^1 \times \mathbb{S}^1$. We focus on surfaces for which there exists a submanifold $L^n, 1 \leq n \leq 2$, of the torus that gives rise to such a local deformation around every point of M. We call these surfaces uniformly locally proper Bonnet. We prove that a compact surface in $\mathbb{Q}_c^4$ is uniformly locally proper Bonnet if and only if it has a vertically harmonic Gauss lift, without being superconformal. We also show that there do not exist compact superconformal surfaces in $\mathbb{Q}_c^4$ that are locally proper Bonnet. Finally, we prove that compact surfaces with parallel mean curvature vector field in $\mathbb{Q}_c^4$ that are not totally umbilical, are characterized as the only locally flexible compact surfaces in $\mathbb{Q}_c^4$.

Περίληψη

Μελετάμε το πρόβλημα Bonnet για επιφάνειες σε τετραδιάστατους χώρους μορφής $\mathbb{Q}_c^4$. Δύο ισομετρικές επιφάνειες λέγεται ότι έχουν την ίδια μέση καμπυλότητα, εάν υπάρχει μια παράλληλη ισομετρία διανυσματικών δεσμών μεταξύ των καθέτων δεσμών τους, η οποία διατηρεί τα διανυσματικά πεδία μέσης καμπυλότητας. Μη γεωμετρικά ισότιμες επιφάνειες με την ίδια μέση καμπυλότητα καλούνται Bonnet mates. Μια επιφάνεια στον $\mathbb{Q}_c^4$ καλείται επιφάνεια Bonnet, ή γνήσια επιφάνεια Bonnet, εάν δέχεται τουλάχιστον μία, ή άπειρες το πλήθος Bonnet mates, αντίστοιχα.

Εισάγουμε τις έννοιες των ισοτροπικά ισοθερμικών και ισχυρά ισοτροπικά ισοθερμικών επιφανειών στον $\mathbb{Q}_c^4$, ως γενίκευση της έννοιας των ισοθερμικών επιφανειών στον $\mathbb{Q}_c^3$ και αποδεικνύουμε ότι η ισοτροπική ισοθερμικότητα είναι μια σύμμορφα αναλλοίωτη ιδιότητα.

Αποδεικνύουμε ότι εάν μια μη-συμπαγής, απλά συνεκτική επιφάνεια $f\colon M \to \mathbb{Q}_c^4$ δεν είναι γνήσια επιφάνεια Bonnet, τότε δέχεται είτε το πολύ μία, είτε ακριβώς τρεις Bonnet mates. Εάν μια τέτοια επιφάνεια είναι γνήσια επιφάνεια Bonnet, τότε ο moduli space των κλάσεων γεωμετρικής ισοτιμίας όλων των ισομετρικών εμβαπτίσεων του M στον $\mathbb{Q}_c^4$ που έχουν την ίδια μέση καμπυλότητα με την f, είναι διαφορομορφικός με ένα πολύπτυγμα. Οι γνήσιες επιφάνειες Bonnet χωρίζονται σε δύο κατηγορίες: τις tight επιφάνειες που ο moduli space είναι μονοδιάστατος με το πολύ δύο συνεκτικές συνιστώσες διαφορομορφικές με τον κύκλο $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, και τις flexible επιφάνειες που ο moduli space είναι διαφορομορφικός με τον τόρο $\mathbb{S}^1 \times \mathbb{S}^1$. Αποδεικνύουμε ότι η ισοτροπική ισοθερμικότητα χαρακτηρίζει τις γνήσιες επιφάνειες Bonnet και ειδικότερα, η ισχυρή ισοτροπική ισοθερμικότητα, χαρακτηρίζει τις flexible επιφάνειες. Επιπλέον, δείχνουμε ότι μια ολικά μη ημι-ισοτροπικά ισοθερμική επιφάνεια είναι πάντα μια επιφάνεια Bonnet η οποία ειδικότερα, δέχεται ακριβώς τρεις Bonnet mates αν είναι επιπροσθέτως ισχυρά ολικά μη ισοτροπικά ισοθερμική. Επίσης, αποδεικνύουμε ότι μια επιφάνεια Bonnet που κείται σε ολικά γεωδαισιακή υπερεπιφάνεια του $\mathbb{Q}_c^4$ με μη-σταθερή μέση καμπυλότητα, δέχεται τουλάχιστον δύο Bonnet mates οι οποίες δεν κείνται σε καμία ολικά ομφαλική υπερεπιφάνεια του $\mathbb{Q}_c^4$.

Αποδεικνύουμε ότι αν και τα δύο Gauss lifts μιας συμπαγούς επιφάνειας στην twistor bundle δεν είναι vertically harmonic, τότε η επιφάνεια δέχεται το πολύ τρεις Bonnet ma-

tes. Ειδικότερα, δείχνουμε ότι μια τέτοια επιφάνεια δέχεται το πολύ μία Bonnet mate, υπό πρόσθετες υποθέσεις που αφορούν την ισοτροπική ισοθερμικότητα.

Δείχνουμε ότι οι μη-ελαχιστικές επιφάνειες με ένα vertically harmonic Gauss lift δέχονται ένα ολόμορφο τετραγωνικό διαφορικό, κι έτσι προκύπτει ένα θεώρημα τύπου Hopf. Αποδεικνύουμε ότι τέτοιες επιφάνειες δέχονται τοπικά μια μονοπαραμετρική οικογένεια ισομετρικών παραμορφώσεων που διατηρούν τη μέση καμπυλότητα. Η οικογένεια αυτή είναι τετριμμένη μόνο εάν η επιφάνεια είναι superconformal. Για τέτοιες συμπαγείς επιφάνειες με μη-παράλληλο διανυσματικό πεδίο μέσης καμπυλότητας, αποδεικνύουμε ότι ο moduli space είναι η ξένη ένωση δύο συνόλων, το καθένα από τα οποία είναι είτε πεπερασμένο είτε ο κύκλος. Ειδικότερα, για επιφάνειες στον $\mathbb{R}^4$ αποδεικνύουμε ότι ο moduli space είναι ένα πεπερασμένο σύνολο, υπό μία συνθήκη για τους αριθμούς Euler της εφαπτόμενης και της κάθετης δέσμης.

Μελετούμε επίσης επιφάνειες που είναι τοπικά γνήσια Bonnet. Μια επιφάνεια $f\colon M \to \mathbb{Q}^4_c$ λέγεται τοπικά γνήσια Bonnet εάν κάθε σημείο του M έχει μια περιοχή, περιορισμένη στην οποία η f είναι γνήσια Bonnet. Δείχνουμε ότι αν μια τοπικά γνήσια επιφάνεια Bonnet είναι μη-ελαχιστική, τότε γύρω από κάθε σημείο $p \in M$, κάθε συνεχής ισομετρική παραμόρφωση που διατηρεί τη μέση καμπυλότητα περιγράφεται από ένα υποπολύπτυγμα $L^n(p), 1 \leq n \leq 2$, του τόρου $\mathbb{S}^1 \times \mathbb{S}^1$. Επικεντρωνόμαστε στις επιφάνειες για τις οποίες υπάρχει ένα υποπολύπτυγμα $L^n, 1 \leq n \leq 2$ του τόρου τέτοιο ώστε να περιγράφει μια τέτοια ισομετρική παραμόρφωση γύρω από κάθε σημείο του M. Καλούμε αυτές τις επιφάνειες ομοιόμορφα τοπικά γνήσια Bonnet. Αποδεικνύουμε ότι μια συμπαγής επιφάνεια στον $\mathbb{Q}^4_c$ είναι ομοιόμορφα τοπικά γνήσια Bonnet αν και μόνο αν έχει ένα vertically harmonic Gauss lift, χωρίς να είναι superconformal. Επίσης, δείχνουμε ότι δεν υπάρχουν συμπαγείς superconformal επιφάνειες στον $\mathbb{Q}^4_c$ που είναι τοπικά γνήσια Bonnet. Τελικώς, αποδεικνύουμε ότι οι συμπαγείς επιφάνειες με παράλληλο διανυσματικό πεδίο μέσης καμπυλότητας στον $\mathbb{Q}^4_c$, οι οποίες δεν είναι ολικά ομφαλικές, χαρακτηρίζονται ως οι μόνες συμπαγείς τοπικά flexible επιφάνειες στον $\mathbb{Q}^4_c$.

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