UNIVERSITY OF IOANNINA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

Ph.D. Dissertation

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MINIMAL SUBMANIFOLDS WITH NULLITY IN SPACE FORMS

Ioannina, 2019

The present dissertation was carried out under the Ph.D. program of the Department of Mathematics of the University of Ioannina in order to obtain the degree of Doctor of Philosophy.

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The acceptance of this dissertation by the Department of Mathematics of the University of Ioannina does not imply the approval of the opinions of the author (section 202 par. 2 Law 5343/1932 and section 50 par. 8 Law 1268/1982).

Statutory Declaration

I lawfully declare here with statutorily that the present dissertation was carried out under the international ethical and academical rules and under the protection of intellectual property. According to these rules, I avoided plagiarism of any kind and I made reference to any source which I used in this thesis.

Dedicated to my parents Ioannis Kasioumis and Alexandra Rachioti,
my sister Konstantina Kasioumi and Georgia Petroulea...

Acknowledgements

First, I would like to thank my advisor Professor Theodoros Vlachos for continuous encouragement, motivation, important guidance and mentorship, and all the invaluable advice and insights.

I would also like to thank the other six members of the committee: Professor Marcos Dajczer from IMPA, Reader Giuseppe Tinaglia from King's College London, Associate Professor Andreas Arvanitogeorgos from University of Patras, Professor Stylianos Stamatakis and Assistant Professor Fani Petalidou from University of Thessaloniki for their useful comments and remarks. Furthermore, I would like to extend my sincere gratitude to Assistant Professor Andreas Savas-Halilaj from University of Ioannina for his friendship, continuous motivation, important guidance and many fruitful discussions we had. His contribution was vital in this research.

Moreover, I would like to thank my family: My parents Ioannis Kasioumis and Alexandra Rachioti, my sister Konstantina Kasioumi, as well as Georgia Petroulea, Katerina Rachioti, Niki Rachioti and my cousins for their endless support and for sticking by my side all these years.

Last but not least, I would like to thank my friends and collegues: Nikos Anagnostou, Alexandros Mitrousis, Asterios Mastrogiannis, Olympia Stoli, Vasilis Aliftiras, Mariana Petroulea, Mimis Sabatakakis, Christos Sarakasidis, Katia Voulgari, Kosmas Kosmas, Kostas Liampis, Christos Onti, Kleanthis Polymerakis, Sokratis Zikas, Amalia-Sofia Tsouri, Athina-Eleni Kanellopoulou and all other graduate students for their friendship and valuable conversations.

Theodoros Kasioumis
Ioannina, Greece,
December 2018

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Introduction

A fundamental concept in submanifold theory is the notion of the relative nullity distribution that was introduced by Chern and Kuiper [13]. The relative nullity of a submanifold in a space form is defined as the kernel of the second fundamental form. The index of relative nullity at a point of the submanifold is just the dimension of the kernel of the second fundamental form at that point. The kernels form an integrable distribution along any open subset where the index is constant and the leaves of the foliation are (part of) affine subspaces in the ambient space. Moreover, if the submanifold is complete then the leaves are also complete along the open subset where the index reaches its minimum; see [14].

A frequent theme in submanifold theory is to find geometric conditions for a complete isometric immersion $f\colon M^m \to \mathbb{R}^n$ with positive index of relative nullity $\nu \geq k$ to be a k-cylinder. This means that M^m splits as a Riemannian product $M^m = M^{m-k} \times \mathbb{R}^k$ and there is an isometric immersion $g\colon M^{m-k} \to \mathbb{R}^{n-k}$ such that $f = g \times \mathrm{id}_{\mathbb{R}^k}$. The theory of the relative nullity distribution is an important tool for the characterization of cylindrical submanifolds. In order to conclude that f is a cylinder one has to show that the images under f of the leaves of relative nullity are parallel in the ambient space.

A fundamental result asserting that a complete isometric immersion $f\colon M^m \to \mathbb{R}^n$ with positive index of relative nullity must be a k-cylinder is Hartman's theorem [40] that requires the Ricci curvature of M^m to be nonnegative; see also [52]. A key ingredient for the proof is the famous Cheeger-Gromoll splitting theorem [10], which is used to conclude that the leaves of the minimum relative nullity split intrinsically as a Riemannian factor. Even for hypersurfaces, the same conclusion does not hold if instead we assume that the Ricci curvature is nonpositive. Notice that the latter is always the case if f is a minimal immersion. Counterexamples easy to construct are the complete irreducible ruled hypersurfaces of any dimension discussed in [19, p. 409]. Some of the many papers containing characterizations of submanifolds as cylinders without the requirement of minimality are [15, 17, 38, 40, 52, 54, 58]. When adding the condition of being minimal we have [1, 24, 35, 36, 38, 41, 64, 66].

In this thesis, we aim to extend the aforementioned results for the class of complete minimal immersions $f\colon M^m \to \mathbb{Q}_c^n$ with rank at most two, or equivalently, with index of relative nullity at least $m-2$. We would like to mention that the hypersurface case was treated in [41–43, 59, 60].

The structure of this thesis is as follows: After some background material in submanifold theory introduced in Chapter 1, we present the original results of the thesis in Chapters 2,3,4 and 5.

More precisely, in Chapter 2 we prove a crucial lemma concerning three dimensional minimal submanifolds in space forms with index of relative nullity one. As it turns out, the three dimensional case is the most interesting one.

In Chapter 3, we investigate complete minimal submanifolds $f\colon M^m \to \mathbb{R}^n$ with positive index of relative nullity $\nu \geq m-2$. We prove that the submanifold must be a cylinder over a minimal surface, under the mild assumption that the Omori-Yau maximum principle for the Laplacian holds on M^m; see [21]. The category of complete Riemannian manifolds for which the Omori-Yau maximum principle is valid is quite large. For instance, it contains the manifolds whose Ricci curvature does not decay too fast to $-\infty$. It also contains the class of properly immersed submanifolds in a space form whose norm of the mean curvature vector is bounded [56, Example 1.14]. Our result is truly global in nature, since there are plenty of non complete minimal submanifolds of dimension m having constant index of relative nullity $m-2$ that are not part of a cylinder on any open subset. They can all locally be parametrized in terms of a certain class of elliptic surfaces; see [15, Theorem 22]. Consequently, what remains a challenging open problem is the existence of minimal complete and noncylindrical three dimensional submanifolds with $\nu \geq 1$.

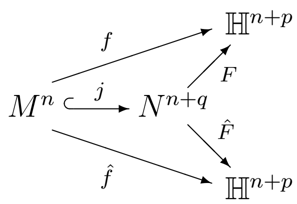
In Chapter 4, we study complete minimal immersions $f\colon M^m \to \mathbb{S}^n$ in Euclidean spheres with positive index of relative nullity at least $m-2$ at any point. These are austere submanifolds in the sense of Harvey and Lawson [44] and were studied by Bryant [7]. For any dimension and codimension there is an abundance of examples of non-complete submanifolds which are fully described by Dajczer and Florit [15] in terms of a class of surfaces, called elliptic, for which the ellipse of curvature of a certain order is a circle at any point. Under the assumption of completeness, it turns out that any minimal submanifold in Euclidean sphere is either totally geodesic or has dimension three. In the latter case, there are plenty of examples, even compact ones. Moreover, under the mild assumption that the Omori-Yau maximum principle holds on M^m we provide a complete local parametric description of such submanifolds in terms of 1-isotropic surfaces in Euclidean space. These are the minimal surfaces for which the standard ellipse of curvature is a circle at any point; see [22]. For these surfaces, there exists a Weierstrass type representation that generates all simply-connected ones.

Chapter 5 is devoted to minimal submanifolds in the hyperbolic space $\mathbb{H}^n$ and will be divided in three parts. In the first part, we study complete minimal submanifolds $f\colon M^m \to \mathbb{H}^n$ having index of relative nullity at least $m-2$ at any point. In contrast to Euclidean and spherical case already being studied, the condition that the index of relative nullity is at least $m-2$ is now quite less restrictive. Nevertheless, we have reasons to believe that the three-dimensional case is still quite special and this is why we obtain a characterization of a class of submanifolds that is contained in the following description. We prove that any complete three dimensional minimal submanifold $f\colon M^3 \to \mathbb{H}^n$ having index of relative nullity at least one at any point, is either totally geodesic or a generalized cone over a complete minimal surface lying in an equidistant submanifold of $\mathbb{H}^n$, under the assumption that the scalar curvature is bounded from below; see [23].

The second part of Chapter 5 is devoted to minimal submanifolds $f\colon M^m \to \mathbb{H}^n$ in arbitrary codimension, whose index of relative nullity is $m-2$ [49]. Our goal is to parametrically describe these submanifolds as subbundles of the normal bundle of certain elliptic spacelike surfaces in the Lorentzian space or in the de Sitter space. Therefore, the assumption of completeness in the characterization of three dimensional manifolds in hyperbolic space in essential, since there exist local examples other than generalized cones. Moreover, using this parametrization, one can construct an abundance of complete submanifolds of any dimension other than generalized cones, as can been seen from the results in [9], [32] and [47]. Another way of constructing complete minimal submanifolds in the hyperbolic space, via regular fibers of harmonic morphisms to Riemann surfaces, was obtained by Gudmundsson in [39].

In the third and last part of the thesis, we introduce a new class of minimal immersions $F\colon M^n \to \mathbb{H}^{n+2}$, $n \geq 3$, that are $(n-2)$-ruled [49]. This means that they carry an integrable tangent distribution of dimension $n-2$, whose leaves are mapped diffeomorphically by F onto open subsets of totally geodesic $(n-2)$-hyperbolic spaces of $\mathbb{H}^{n+2}$. Furthermore, we provide a characterization for them among $(n-2)$-ruled minimal submanifolds of rank 4 (if $n \geq 4$) or 3 (if $n = 3$). If the manifold is simply connected, we show that it allows a one-parameter family of equally ruled minimal isometric deformations that are *genuine*. The deformations are obtained while keeping fixed the normal bundle and the induced connection, but now the second fundamental form relates to the initial one in a much more complex form; in particular, no orthogonal tensor in involved. It is an interesting question if the above associated family of complete ruled minimal submanifolds exhausts all examples in the same class that admit genuine deformations. Of course, a much more challenging classification problem for submanifolds of rank 4 would be to drop one of the conditions, for instance being minimal or ruled.

The notion of *genuine rigidity* was introduced by Dajczer and Florit [16]. This is the right setting to study rigidity problems for higher codimension submanifolds. This concept relies on the idea that, as we discard congruent submanifolds when analyzing rigidity, we should also discard deformations that are induced by deformations of a bigger dimensional submanifold containing the original one. An isometric immersion $\hat{f}\colon M^n \to \mathbb{H}^{n+p}$ is called a *genuine deformation* of a given isometric immersion $f\colon M^n \to \mathbb{H}^{n+p}$, with $p \geq 2$, if there is no open subset $U \subset M^n$ along which $f|_U$ and $\hat{f}|_U$ extend isometrically. That $f\colon M^n \to \mathbb{H}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{H}^{n+p}$ *extend isometrically* means that there is an isometric embedding $j\colon M^n \mapsto N^{n+q}$, $1 \leq q \leq p$, into a Riemannian manifold N^{n+q} and there are isometric immersions $F\colon N^{n+q} \to \mathbb{H}^{n+p}$ and $\hat{F}\colon N^{n+q} \to \mathbb{H}^{n+p}$ such that $f = F \circ j$ and $\hat{f} = \hat{F} \circ j$, i.e., the following diagram commutes:



CHAPTER 1

Background material on submanifold theory

In this chapter, we set up the notation and give a brief overview of the background material needed for the rest of the thesis.

Let $(M^m, \langle \cdot\,, \cdot \rangle)$ be a Riemannian manifold with Levi-Civita connection ∇ and let $\mathfrak{X}(M)$ denote the set of smooth local vector fields of M^m.

The $(1,3)$-curvature tensor $R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The *sectional curvature* $K(X \wedge Y)$ at the point $x \in M^m$ and along the plane spanned by the orthonormal vectors $X, Y \in T_xM$ is defined by

$$K(X \wedge Y) = \langle R(X,Y)Y, X \rangle.$$

A complete and simply-connected n-dimensional Riemannian manifold with constant sectional curvature c is called a *space form* and is denoted by $\mathbb{Q}_c^n$. It is well known that $\mathbb{Q}_c^n$ is the Euclidean space $\mathbb{R}^n$, the Euclidean sphere $\mathbb{S}^n$ or hyperbolic space $\mathbb{H}^n$ according to c being $0, 1$ or -1, respectively. In the sequel, we denote by $\tilde{\nabla}$ the Levi-Civita connection of $\mathbb{Q}_c^n$.

A differentiable map $f\colon M^m \to \mathbb{Q}_c^n$ is called an *immersion* if the differential $f_*(x)\colon T_xM^m \to T_{f(x)}\mathbb{Q}_c^n$ is injective for any point $x \in M^m$. An immersion f is said to be an *isometric immersion* if, moreover,

$$\langle X, Y \rangle_{M^m} = \langle f_*(x)X, f_*(x)Y \rangle_{\mathbb{Q}_c^n},$$

for all $x \in M^m$ and $X, Y \in T_xM$. The number $p = n - m$ is called the *codimension* of f and for simplicity we refer to f as a *submanifold* of $\mathbb{Q}_c^n$.

Given an isometric immersion $f\colon M^m \to \mathbb{Q}_c^n$ we denote by $f^*T\mathbb{Q}_c^n$ the induced bundle over M^m whose fiber at $x \in M^m$ is $T_{f(x)}\mathbb{Q}_c^n$. Moreover, we denote by $\tilde{\nabla}$ the induced connection on $f^*T\mathbb{Q}_c^n$. The orthogonal complement of $f_*(x)T_xM^m$ in $T_{f(x)}\mathbb{Q}_c^n$ is called the *normal space* of f at x and is denoted by $N_fM(x)$. The *normal bundle* N_fM of f is the vector subbundle of $f^*T\mathbb{Q}_c^n$ whose fiber at $x \in M^m$ is $N_fM(x)$. In the sequel, the set of smooth sections of the normal bundle N_fM is denoted by $\Gamma(N_fM)$. Given vector fields $X, Y \in \mathfrak{X}(M)$ we decompose

$$\tilde{\nabla}_X f_*Y = \big(\tilde{\nabla}_X f_*Y\big)^\top + \big(\tilde{\nabla}_X f_*Y\big)^\perp \tag{1.1}$$

with respect to the orthogonal decomposition

$$f^*T\mathbb{Q}_c^n = f_*TM \oplus N_fM.$$

One can easily verify that

$$\big(\tilde{\nabla}_X f_*Y\big)^\top = f_*(\nabla_X Y),$$

where ∇ is the Levi-Civita connection of M^m. Moreover, the map

$$\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(N_fM)$$

defined by

$$\alpha(X, Y) = \big(\tilde{\nabla}_X f_*Y\big)^\perp$$

is called the *second fundamental form* of f.

1.1 The Gauss and Weingarten formulas

From (1.1) we obtain the following first basic formula of the theory of submanifolds, known as the *Gauss formula*

$$\tilde{\nabla}_X f_*Y = \nabla_X Y + \alpha(X, Y). \tag{1.2}$$

For every normal vector field $\xi \in \Gamma(N_fM)$, the endomorphism $A_\xi\colon TM \to TM$ defined by

$$\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle$$

is called the *shape operator* of f in the direction ξ. Now, the second basic formula, known as the *Weingarten formula*, is

$$\tilde{\nabla}_X \xi = -f_*A_\xi X + \nabla_X^\perp \xi,$$

where $\nabla^\perp$ is the *normal connection* of f.

The *mean curvature vector* of f at $x \in M^m$ is the normal vector defined by

$$H(x) = \frac{1}{m}\sum_{i=1}^{m} \alpha(X_i, X_i)$$

where $X_1, \ldots, X_m$ is an orthonormal basis of T_xM. The immersion f is called *minimal* at $x \in M^m$ if $H(x) = 0$. We say that f is a *minimal immersion* if the mean curvature vector vanishes identically, everywhere on M^m. It is well known that any minimal immersion in a space form is real analytic; see [55, Theorem 2.2].

1.2 Gauss-Codazzi-Ricci equations

Using the Gauss-Weigarten formulas and projecting into tangent and normal components we derive the compatibility equations of an isometric immersion $f\colon M^m \to \mathbb{Q}_c^n$. These fundamental equations, called the Gauss-Codazzi-Ricci equations are listed below:

$$\langle R(X,Y)Z,W\rangle = c\langle (X \wedge Y)Z, W\rangle + \langle \alpha(X,W), \alpha(Y,Z)\rangle - \langle \alpha(X,Z), \alpha(Y,W)\rangle,$$
$$(\nabla_Y A_\xi)X - A_{\nabla^\perp_Y \xi}X = (\nabla_X A_\xi)Y - A_{\nabla^\perp_X \xi}Y,$$
$$\langle R^\perp(X,Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle,$$

where $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$ and $R^\perp$ denotes the curvature tensor of the normal bundle N_fM.

The *Ricci tensor* $\mathrm{Ric}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ is defined by

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\,(Z \mapsto R(Z,X)Y)\,.$$

The *Ricci curvature* in the direction of a unit vector field $X \in \mathfrak{X}(M)$ is defined by

$$\mathrm{Ric}(X) = \mathrm{Ric}(X,X).$$

Finally, the *scalar curvature* $\mathrm{s} \in C^\infty(M)$ is defined by

$$\mathrm{s} = \mathrm{trace}\,\mathrm{Ric}\,.$$

Let $\{e_1, \ldots, e_m\}$ be a local orthonormal tangent frame. Using the Gauss equation we derive for $X, Y \in \mathfrak{X}(M)$ that

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \sum_{i=1}^{m} \langle R(e_i,X)Y, e_i\rangle \\ &= c(m-1)\langle X,Y\rangle + \sum_{i=1}^{m} \big(\langle \alpha(e_i,e_i), \alpha(X,Y)\rangle - \langle \alpha(X,e_i), \alpha(Y,e_i)\rangle\big) \\ &= c(m-1)\langle X,Y\rangle + m\langle \alpha(X,Y), H\rangle - \sum_{i=1}^{m} \langle \alpha(X,e_i), \alpha(Y,e_i)\rangle. \qquad (1.3) \end{aligned}$$

Taking traces in (1.3) yields

$$\mathrm{s} = m(m-1)c + m^2\|H\|^2 - \|\alpha\|^2, \tag{1.4}$$

where

$$\|\alpha\|^2 = \sum_{i,j=1}^{m} \|\alpha(e_i, e_j)\|^2$$

is the square of the norm of the second fundamental form.

1.3 Umbilical isometric immersions

An isometric immersion $f\colon M^m \to \mathbb{Q}_c^n$ is said to be *umbilical* at $x \in M^m$ if there exists $\eta \in N_fM(x)$ such that

$$\alpha(X,Y) = \langle X, Y\rangle \eta, \quad \text{for all} \quad X, Y \in T_xM.$$

Then, η is the mean curvature vector $H(x)$ of f at x. Notice, that f being umbilical at x is equivalent to

$$A_\xi = \langle H(x), \xi\rangle I, \quad \text{for all} \quad \xi \in N_fM(x),$$

where I is the identity endomorphism on T_xM. A submanifold is called *totally umbilical* if it is umbilical at every point.

Using the Codazzi and Ricci equations, one can show that an umbilical isometric immersion $f\colon M^m \to \mathbb{Q}_c^n$ has parallel mean curvature vector field with respect to the normal connection $\nabla^\perp$, i.e., $\nabla^\perp_X H = 0$ for all $X \in TM$, and flat normal bundle, i.e., $R^\perp = 0$. Moreover, the Gauss equation yields that M^m has constant sectional curvature $c + \|H\|^2$.

We view the hyperbolic space $\mathbb{H}^n$ inside the Lorentz space $(\mathbb{L}^{n+1}, \langle\cdot\,,\cdot\rangle)$ equipped with the indefinite metric $\langle\cdot\,,\cdot\rangle$ of signature $(1,n)$. Moreover, we denote by $\mathbb{S}_1^n$ the Lorentzian sphere inside $\mathbb{L}^{n+1}$, called de Sitter space. It is well known that every totally umbilical hypersurface Q^{n-1} of $\mathbb{H}^n$ arises as the intersection of $\mathbb{H}^n$ with a hyperplane

$$P(v,d) = \left\{x \in \mathbb{L}^{n+1} \colon \langle x, v\rangle = d\right\},$$

with $\langle v, v\rangle + d^2 > 0$. Its unit normal vector field is

$$\xi = \frac{1}{\langle v, v\rangle + d^2}(v + dx), \quad x \in Q^{n-1}.$$

Using the Gauss and Weingarten formulas if follows that the sectional curvature of Q^{n-1} is given by

$$K = -\frac{\langle v, v\rangle}{\langle v, v\rangle + d^2}.$$

Observe that if $d = 0$, then the hypersurface Q^{n-1} is totally geodesic, i.e., $\alpha(X, Y) = 0$ for $X, Y \in TQ^{n-1}$. Moreover, Q^{n-1} is a *geodesic sphere* if $K > 0$, a *horosphere* if $K = 0$ and an *equidistant hypersurface* if $K < 0$.

The classification of totally umbilical submanifolds of $\mathbb{H}^n$ reduces to the classification of totally umbilical hypersurfaces, see [61, Lemma 25]. Namely, if Q^k is a totally umbilical submanifold of $\mathbb{H}^n$, then Q^k is contained in a totally geodesic submanifold of dimension $k+1$ of $\mathbb{H}^n$. Similar conclusions hold true for totally umbilical submanifolds of de Sitter space. More precisely, if Q^k is a totally umbilical submanifold of $\mathbb{S}^n_1$, then Q^k is contained in the subspace arising as the intersection of $\mathbb{S}^n_1$ with a hyperplane of dimension $k+1$ of $\mathbb{L}^{n+1}$. If in addition this hyperplane passes through the origin of $\mathbb{L}^{n+1}$, then Q^k is a totally geodesic submanifold.

1.4 Relative nullity distribution

Let M^m be a Riemannian manifold and $f\colon M^m \to \mathbb{Q}^n_c$ be an isometric immersion. The *relative nullity* subspace $\mathcal{D}(x)$ of f at any point $x \in M^m$ is the tangent subspace given by

$$\mathcal{D}(x) = \{X \in T_xM : \alpha(X, Y) = 0 \ \text{ for all } \ Y \in T_xM\},$$

i.e., is the kernel of its second fundamental form $\alpha\colon TM \times TM \to N_fM$ with values in the normal bundle. The dimension $\nu(x)$ of $\mathcal{D}(x)$ is called the *index of relative nullity* of f at $x \in M^m$. This notion was introduced by Chern and Kuiper [13] and turned out to be a fundamental concept in the theory of isometric immersions. For simplicity, we call $\rho(x) = m - \nu(x)$ the *rank* of f at $x \in M^m$. Notice that $\rho(x)$ is the rank of the Gauss map of f at $x \in M^m$.

A smooth distribution E of M^m is called *totally geodesic* if $\nabla_X Y \in \Gamma(E)$ whenever $X, Y \in \Gamma(E)$. We recall in the following proposition some well-known results for the relative nullity distribution.

Proposition 1.1. *Let $U \subset M^m$ be an open subset where the index of relative nullity $\nu = s > 0$ is constant. Then*

(i) The index of relative nullity ν is upper semicontinuous. In particular, the subset

$$M_0 = \{x \in M^m \colon \nu(x) = \nu_0\}$$

where ν attains its minimum value ν_0 is open.

(ii) The relative nullity distribution $x \mapsto \mathcal{D}(x)$ is smooth on U.

(iii) The kernels form a totally geodesic and hence integrable distribution $\mathcal{D}$ along U. The leaves of $\mathcal{D}$ are totally geodesic submanifolds of M^m and their images under f are (part of) affine subspaces in the ambient space.

(iv) If $\gamma\colon [0,b] \to M^m$ is a geodesic curve such that $\gamma([0,b))$ is contained in a leaf of relative nullity contained in U, then $\nu(\gamma(b)) = s$.

(v) If M^m is complete then the leaves are also complete along M_0.

Proof: See [14, Proposition 5.2 and 5.3] ∎

1.5 Splitting tensor

In this section we define the notion of the splitting tensor which measures how the conullity distribution is twisting along the tangent bundle of our manifold.

The *conullity subspace* of f at $x \in M^m$ is the orthogonal complement $\mathcal{D}^\perp(x)$ of $\mathcal{D}(x)$ in the tangent space T_xM. We write

$$X = X^v + X^h$$

according to the orthogonal splitting $TM = \mathcal{D} \oplus \mathcal{D}^\perp$ and denote

$$\nabla^h_X Y = (\nabla_X Y)^h \quad \text{and} \quad \nabla^v_X Y = (\nabla_X Y)^v.$$

In the sequel we work on the open subset U of M^m where the index of relative nullity in constant. In order to investigate how $\mathcal{D}^\perp$ is twisting along the tangent space TM we introduce the so-called *splitting tensor* $C\colon \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}^\perp) \to \Gamma(\mathcal{D}^\perp)$ defined by

$$C(T,X) = -\nabla^h_X T$$

for any $T \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}^\perp)$. It is immediate that C is $C^\infty(M)$-linear with respect to the second variable. That is also $C^\infty(M)$-linear with respect to the first variable follows from

$$C(\phi T, X) = -\nabla^h_X(\phi T) = -\phi\nabla^h_X T = \phi C(T,X), \quad \phi \in C^\infty(M).$$

Therefore, the value of $C(T,X)$ at a point $x \in M^m$ depends only on the values of T and X at x. We will write $C_T X$ instead of $C(T,X)$, and also regard C as a map

$$C\colon \Gamma(\mathcal{D}) \to \Gamma(\mathrm{End}(\mathcal{D}^\perp)).$$

Recall that a distribution E is *integrable* if for every $X, Y \in \Gamma(E)$ the Lie bracket $[X,Y]$ lies in $\Gamma(E)$. Hence, the distribution $\mathcal{D}^\perp$ is integrable if and only if C_T is self adjoint. This follows from

$$\begin{aligned}
\langle C_T X, Y\rangle - \langle X, C_T Y\rangle &= -\langle \nabla^h_X T, Y\rangle + \langle X, \nabla^h_Y T\rangle \\
&= -\langle \nabla_X T, Y\rangle + \langle X, \nabla_Y T\rangle \\
&= \langle \nabla_X Y - \nabla_Y X, T\rangle \\
&= \langle [X,Y], T\rangle,
\end{aligned}$$

for $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $T \in \Gamma(\mathcal{D})$. In this case, C_T coincides with the shape operator of the leaves of $\mathcal{D}^\perp$ in M^m with respect to the normal direction T.

Moreover, using the fact that the distribution $\mathcal{D}$ is totally geodesic we obtain that

$$\langle \nabla_T X, S\rangle = -\langle X, \nabla_T S\rangle = 0,$$

where $T, S \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}^\perp)$. Thus, $\nabla_T X \in \Gamma(\mathcal{D}^\perp)$ and one can define the covariant derivative of C_T by

$$(\nabla_S C_T)X = \nabla_S(C_T X) - C_T(\nabla_S X).$$

The next proposition will play a key role in the sequel.

Proposition 1.2. *The following differential equations for the splitting tensor hold:*

(i)
$$\nabla_S C_T = C_T C_S + C_{\nabla_S T} + c\langle S, T\rangle I. \tag{1.5}$$

In particular, the operator $C_{\gamma'}$ along a geodesic γ contained in a leaf of $\mathcal{D}$ satisfies the differential equation

$$\frac{D}{dt} C_{\gamma'} = C_{\gamma'}^2 + cI.$$

(ii)
$$(\nabla^h_X C_T)Y - (\nabla^h_Y C_T)X = C_{\nabla^v_X T}Y - C_{\nabla^v_Y T}X \tag{1.6}$$

and

(iii)
$$\nabla_T A_\xi = A_\xi C_T + A_{\nabla^\perp_T \xi} \tag{1.7}$$

for any $S, T \in \Gamma(\mathcal{D})$, $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $\xi \in \Gamma(N_f M)$, where I stands for the identity endomorphism on $\mathcal{D}^\perp$. In particular, the endomorphism $A_\xi C_T$ of $\mathcal{D}^\perp$ is symmetric, i.e.,

$$A_\xi C_T = C_T^t A_\xi.$$

Proof: The equations (1.5) and (1.6) are derived using the Gauss equation and the fact that $\mathcal{D}$ is totally geodesic, whereas equation (1.7) is an easy consequence of the Codazzi equation. For further details see [14] or [19]. ∎

1.6 Submanifolds with umbilical conullity

Let U be and open subset of M^m where the index of relative nullity of the isometric immersion $f\colon M^m \to \mathbb{Q}_c^n$ is constant. The simplest possible structures of the splitting tensor occur when either C vanishes identically or ImC is spanned by the identity endomorphism I of $\mathcal{D}^\perp$.

A smooth distribution $E \subset TM$ is called *umbilical* if there exists a smooth section V of the orthogonal complement $E^\perp$ of E, called the *mean curvature vector field* of E, such that

$$\langle \nabla_X Y, T\rangle = \langle X, Y\rangle\langle V, T\rangle,$$

for all $X, Y \in \Gamma(E)$ and $T \in \Gamma(E^\perp)$. An umbilical distribution is always integrable and its leaves are umbilical sumbanifolds of M^m.

We call an isometric immersion $f\colon M^m \to \mathbb{R}^n$ a *k-cylinder* over $g\colon M^{m-k} \to \mathbb{R}^{n-k}$ if $M^m = M^{m-k} \times \mathbb{R}^k$ and $f = g \times Id_{\mathbb{R}^k}$, where $Id_{\mathbb{R}^k}\colon \mathbb{R}^k \to \mathbb{R}^k$ is the identity map. The following proposition gives a sufficient condition for an isometric immersion to reduce codimension.

Proposition 1.3. *Let $f\colon M^m \to \mathbb{Q}_c^n$ be an isometric immersion. If there exists a parallel subbundle E of the normal bundle $N_f M$ with rank $p < n - m$ such that $N_1^f(x) \subset E(x)$ for each point $x \in M^m$, then there exists a totally geodesic submanifold $\mathbb{Q}_c^{m+p}$ in $\mathbb{Q}_c^n$ such that $f(M) \subset \mathbb{Q}_c^{m+p}$, i.e., f admits* reduction of codimension *to p.*

Proof: See [14, Proposition 4.1]. ∎

Proposition 1.4. *Let $f\colon M^m \to \mathbb{Q}_c^n$ be an isometric immersion with constant index of relative nullity $\nu(x) = \nu_0 > 0$. If the conullity distribution is totally geodesic, then $c = 0$ and f is locally a k-cylinder over an isometric immersion $g\colon M^{m-\nu_0} \to \mathbb{R}^{m-\nu_0}$.*

Proof: Since the splitting tensor vanishes identically, it follows from (1.5) that $c = 0$. The distribution $\mathcal{D}^\perp$ being totally geodesic yields

$$\tilde{\nabla}_X f_*(T) = f_*(\nabla_X T) + \alpha_f(X, T) = f_*(\nabla_X T) \in f_*(\mathcal{D})$$

for all $X \in \Gamma(\mathcal{D}^\perp)$ and $T \in \Gamma(\mathcal{D})$. Thus, $f_*(\mathcal{D})$ is constant in $\mathbb{R}^n$ along any leaf L of $\mathcal{D}^\perp$. Set $M^{m-\nu_0} = L$ and $g = f \circ i$, where $i\colon L \to M^m$ is the inclusion. Then, due to Proposition 1.3 the immersion g reduces codimension to $n - \nu_0$ and f coincides locally with the cylinder over g. ∎

Corollary 1.5. *Let $f\colon M^m \to \mathbb{R}^n$ be an isometric immersion with constant index of relative nullity $\nu = s > 0$ and complete leaves of relative nullity. If the splitting tensor C vanishes, then f is a s-cylinder.*

Proof: That $C = 0$ is equivalent to $\mathcal{D}$ being parallel in M^m. Consequently, the images via f of the leaves of $\mathcal{D}$ are also parallel in $\mathbb{R}^n$. ∎

Let $Q_{\tilde{c}}^{n-k}$ denote a complete connected submanifold of $\mathbb{Q}_c^n$. For $c = -1$, it is well known that $Q_{\tilde{c}}^{n-k}$ is either a totally geodesic submanifold of a geodesic sphere, or an equidistant hypersurface, or a horosphere, according to whether $\tilde{c} > 0$, $\tilde{c} < 0$ or $\tilde{c} = 0$, respectively.

Consider an isometric immersion $g\colon L^{m-k} \to Q_{\tilde{c}}^{n-k}$ and $i\colon Q_{\tilde{c}}^{n-k} \to \mathbb{Q}_c^n$ the umbilical inclusion. Then, the normal bundle of $h = i \circ g\colon L^{m-k} \to \mathbb{Q}_c^n$ splits orthogonally as

$$N_h L = i_* N_g L \oplus N_i Q_c^{n-k},$$

where $L = L^{m-k}$ and $N_i Q_c^{n-k}$ is regarded as a subbundle of $N_h L$.

Let $G\colon N_i Q_c^{n-k} \to \mathbb{Q}_c^n$ be defined by

$$G(x, w) = \exp_{g(x)} w,$$

where exp denotes the exponential map of $\mathbb{Q}_c^n$. We denote by M^m the open subset of $N_i Q_c^{n-k}$ where G is an immersion, endowed with the metric induced by the map G. The *generalized cone* in $\mathbb{Q}_c^n$ over $g\colon L^{m-k} \to Q_{\tilde{c}}^{n-k}$ is the isometric immersion $F_g\colon M^m \to \mathbb{Q}_c^n$, defined by $F_g = G|_{M^m}$.

Proposition 1.6. *Let $f\colon M^m \to \mathbb{Q}_c^n$ be an isometric immersion with constant index of relative nullity $\nu(x) = \nu_0 > 0$. If the conullity distribution is umbilical, then f coincides locally with the generalized cone over $g\colon M^{m-\nu_0} \to Q_{\tilde{c}}^{n-\nu_0}$ into an umbilical submanifold $Q_{\tilde{c}}^{n-\nu_0}$ of $\mathbb{Q}_c^n$. Moreover, the submanifold is globally a generalized cone if the relative nullity leaves are complete.*

Proof: Let $j\colon L \to M^m$ be the inclusion of a leaf L of $\mathcal{D}^\perp$ into M^m and let $h = f \circ j$. Then, the normal bundle $N_h L$ of h splits as

$$N_h L = f_*(N_j L) \oplus N_f M = f_*(\mathcal{D}) \oplus N_f M.$$

By assumption, there exists $S \in \Gamma(\mathcal{D})$ such that

$$C_T = \langle T, S \rangle I,$$

for all $T \in \Gamma(\mathcal{D})$. Hence,

$$\begin{aligned} \tilde{\nabla}_X f_*(T) &= f_*(\nabla_X T) + \alpha_f(X, T) \\ &= -f_*(C_T X) + f_*(\nabla_X^v T) \\ &= -\langle T, S \rangle f_*(X) + f_*(\nabla_X^v T) \end{aligned}$$

for all $T \in \Gamma(\mathcal{D})$, where $\tilde{\nabla}$ is the induced connection on $f^*T\mathbb{Q}_c^n$. It follows that the subbundle $L = f_*(\mathcal{D})$ of N_hL is parallel with respect to the normal connection and that the shape operator of h with respect to any section $\eta = f_*(T)$ of L, where $T \in \Gamma(\mathcal{D})$, is given by

$$A_\eta^h = \langle T, S\rangle I.$$

Hence, $h(L)$ is contained in an umbilical submanifold $Q_{\tilde{c}}^{n-\nu_0}$ of $\mathbb{Q}_c^n$. This means that there exists an umbilical inclusion $i\colon Q_{\tilde{c}}^{n-\nu_0} \to \mathbb{Q}_c^n$ and an isometric immersion $g\colon M^{m-\nu_0} = L \to Q_{\tilde{c}}^{n-\nu_0}$ such that $h = i \circ g$. Moreover, at any point $x \in L$ the fiber $L(x) = f_*(\mathcal{D})(x)$ coincides with the normal space of i at $i(x)$. Therefore the generalized cone over g coincides locally with f. The global statement is immediate. ∎

1.7 Elliptic surfaces

Denote by $\mathbb{Q}_{c,\varepsilon}^n$ the space form of constant sectional curvature c with index (signature) $\varepsilon = 0, 1$. If $\varepsilon = 0$ then the metric on $\mathbb{Q}_{c,0}^n$ is Riemannian, meaning that all eigenvalues of the real symmetric matrix g_{ij} of the metric tensor are positive, whereas if $\varepsilon = 1$ then the metric g_{ij} has one negative eigenvalue and the rest positive. Hence, $\mathbb{Q}_{c,1}^n$ is either the Lorentz space $\mathbb{L}^n$ or the de Sitter space $\mathbb{S}_1^n$ according to $c = 0$ or $c = 1$ respectively, whereas $\mathbb{Q}_{c,0}^n = \mathbb{Q}_c^n$ stands for the Euclidean space, the n-sphere or the hypebolic space, with respect to the sectional curvature c being 0, 1 or -1.

Throughout this section let $g\colon L^2 \to \mathbb{Q}_{c,\varepsilon}^n$ be a *substantial* isometric immersion of a 2-dimensional Riemannian manifold L^2, where by substantial we mean that the codimension cannot be reduced.

We recall from [61] the notion of the s^{th}*-osculating space* $\mathrm{Osc}_x^s g$ of an immersion g at a point $x \in L^2$. It is the subspace of the tangent space $T_{g(x)}\mathbb{Q}_{c,\varepsilon}^n$ defined as

$$\mathrm{Osc}_x^s g = \mathrm{span}\left\{ g_*X_1|_x, \tilde{\nabla}_{X_2} g_*X_1|_x, \ldots, \tilde{\nabla}_{X_s} \cdots \tilde{\nabla}_{X_2} g_*X_1|_x \,:\, X_1, \ldots, X_s \in \mathfrak{X}(L^2) \right\},$$

where $\tilde{\nabla}$ stands for the induced connection on $g^*T\mathbb{Q}_{c,\varepsilon}^n$. Hence, the first osculating space $\mathrm{Osc}_x^1 g$ coincides with the tangent space $g_*(T_xL)$.

An isometric immersion $g\colon L^2 \to \mathbb{Q}_{c,\varepsilon}^n$ is called *k-regular* if all osculating spaces $\mathrm{Osc}_x^s g$ for $s \le k+1$ have constant dimension and the metric induced from $\mathbb{Q}_{c,\varepsilon}^n$ is Riemannian. We call g *regular* if all osculating spaces have constant dimension and Riemannian induced metric.

The s^{th}*-normal space* $N_s^g(x)$ of a k-regular immersion g at $x \in L^2$ is defined as the orthogonal complement of $\mathrm{Osc}_x^s g$ in $\mathrm{Osc}_x^{s+1} g$, i.e.,

$$\mathrm{Osc}_x^{s+1} g = \mathrm{Osc}_x^s g \oplus N_s^g(x) \quad \text{for} \quad 1 \le s \le k+1.$$

Notice that the s^{th}-normal space can be interpreted as the subspace

$$N_s^g(x) = \text{span}\big\{\alpha_g^{s+1}(X_1, \ldots, X_{s+1}) : X_1, \ldots, X_{s+1} \in T_xL\big\}.$$

Here $\alpha_f^2 = \alpha_f$ and for $s \geq 3$ the symmetric tensor $\alpha_g^s\colon TL \times \cdots \times TL \to N_gL$, called the s^{th}-*fundamental form*, is defined inductively by

$$\alpha_g^s(X_1, \ldots, X_s) = \pi^{s-1}\left(\nabla^\perp_{X_s} \cdots \nabla^\perp_{X_3} \alpha_g(X_2, X_1)\right),$$

where π^s denotes taking the projection onto the normal subspace $(N_1^g \oplus \cdots \oplus N_{s-1}^g)^\perp$.

Following [15], a surface $g\colon L^2 \to \mathbb{Q}^n_{c,\varepsilon}$ is called *elliptic* if there exists a (necessary unique up to a sign) almost complex structure $J\colon TL \to TL$ such that the second fundamental form satisfies

$$\alpha_g(X, X) + \alpha_g(JX, JX) = 0$$

for all $X \in TL$. Notice that J is orthogonal if and only g is minimal, i.e., has zero mean curvature.

Minimal surfaces are elliptic, but the class of elliptic surfaces is much larger. Equivalently, that a surface $h\colon L^2 \to \mathbb{R}^n$ is elliptic means that given a basis X, Y of the tangent plane T_xL at any $x \in L^2$ the second fundamental form $\alpha_h\colon TL \times TL \to N_hL$ of h satisfies

$$a\alpha_h(X, X) + 2b\alpha_h(X, Y) + c\alpha_h(Y, Y) = 0,$$

where $a, b, c \in \mathbb{R}$ verify $ac - b^2 > 0$. Equivalently, in any local system of coordinates (u, v) of L^2, any coordinate function of h is a solution of the elliptic PDE of the type

$$a\frac{\partial^2}{\partial u^2} + 2b\frac{\partial^2}{\partial u \partial v} + c\frac{\partial^2}{\partial v^2} + d\frac{\partial}{\partial u} + e\frac{\partial}{\partial v} = 0,$$

where the smooth functions a, b, c, d, e satisfy $ac - b^2 > 0$. The reason that are named elliptic surfaces is exactly because they satisfy the latter elliptic PDE.

Assume from now on that $g\colon L^2 \to \mathbb{Q}^n_{c,\varepsilon}$ is a k-regular elliptic surface. We consider the bundle

$$\Lambda_k^g = \left(N_1^g \oplus \cdots \oplus N_k^g\right)^\perp$$

and the corresponding timelike unit bundle

$$U_1\Lambda_k^g = \big\{(x, w) \in \Lambda_k^g : \langle w, w\rangle = -1\big\}.$$

The normal bundle decomposes orthogonally as

$$N_gL = N_1^g \oplus \cdots \oplus N_k^g \oplus \Lambda_k^g,$$

where all N_s^g are plane bundles for $1 \leq s \leq k$. The induced bundle $g^*T\mathbb{Q}^n_{c,\varepsilon}$ splits as

$$g^*T\mathbb{Q}^n_{c,\varepsilon} = N_0^g \oplus N_1^g \oplus \cdots \oplus N_k^g \oplus \Lambda_k^g,$$

with $N_0^g = g_*(TL)$. The almost complex structure J on TL induces an almost complex structure J_s on each N_s^g, $0 \leq s \leq k$, defined by

$$J_s\alpha_g^{s+1}(X_1,\ldots,X_s,X_{s+1}) = \alpha_g^{s+1}(X_1,\ldots,X_s,JX_{s+1}),$$

where $\alpha_g^1 = g_*$ stands for the differential of the immersion g. The s^{th}*-order ellipse of curvature* $\mathcal{E}_s^g(x) \subset N_s^g(x)$ of g at $x \in L^2$ for $0 \leq s \leq k$ is

$$\mathcal{E}_s^g(x) = \left\{\alpha_g^{s+1}(Z_\theta,\ldots,Z_\theta) : Z_\theta = \cos\theta Z + \sin\theta JZ \;\text{ and }\; \theta \in [0,\pi)\right\},$$

where $Z \in T_xL$ has unit length and satisfies $\langle Z, JZ\rangle = 0$. From ellipticity assumption such a Z always exists.

We say that the curvature ellipse $\mathcal{E}_s^g$ of a k-regular elliptic surface g is a *circle* for some $0 \leq s \leq k$ if all ellipses $\mathcal{E}_s^g(x)$ are circles. Notice that $\mathcal{E}_0^g$ is a circle if and only if g is minimal. Furthermore, a surface $g\colon L^2 \to \mathbb{Q}^n_{c,\varepsilon}$ will be called *k-isotropic* if all ellipses of curvature $\mathcal{E}_s^g$ for $0 \leq s \leq k$, are circles.

Moreover, there exists a Weierstrass type representation from [20] that generates all simply-connected k-isotropic surfaces $h\colon U \subset \mathbb{C} \to \mathbb{R}^{2k+4}$. Start with any nonzero holomorphic map $\alpha_0\colon U \to \mathbb{C}^2$ in a simply-connected domain U. Assuming inductively that $\alpha_r\colon U \to \mathbb{C}^{2r+2}$ has been defined, for $0 \leq r \leq k$, set

$$\alpha_{r+1} = \beta_{r+1}\left(1-\phi_r^2,\, i(1+\phi_r^2),\, 2\phi_r\right) \tag{1.8}$$

where $\phi_r = \int^z \alpha_r(z)dz$ and $\beta_{r+1} \neq 0$ is any holomorphic function. Then, we have that $h = \operatorname{Re}\{\alpha_{k+1}\}$ is a k-isotropic surface in $\mathbb{R}^{2k+4}$.

1.8 Elliptic submanifolds and polar surfaces

In this section, we recall from [27] the notion of elliptic submanifolds into a space form as well as several of their basic properties.

Let $f\colon M^m \to \mathbb{Q}^n_c$ be a rank two isometric immersion of a Riemannian manifold M^m. The relative nullity subspaces $\mathcal{D} \subset TM$ form a tangent subbundle of codimension two. We can assume that f is locally the saturation of a fixed cross section $L^2 \subset M^m$ to the relative nullity foliation.

An isometric immersion $f\colon M^m \to \mathbb{Q}^n_c$ of rank two is called *elliptic* if there exists a (necessary unique up to a sign) almost complex structure $J\colon \mathcal{D}^\perp \to \mathcal{D}^\perp$ such that

$$\alpha_f(X,X) + \alpha_f(JX,JX) = 0 \quad \text{for each} \quad X \in \Gamma(\mathcal{D}^\perp).$$

Notice that J is orthogonal if and only f is minimal.

The s^{th}-*normal space* $N_s^f(x)$ of f at $x \in M^m$ for $s \geq 1$ is defined as

$$N_s^f(x) = \text{span}\big\{\alpha_f^{s+1}(X_1, \dots, X_{s+1}) : X_1, \dots, X_{s+1} \in T_xM\big\},$$

where $\alpha_f^2 = \alpha_f$ and for $s \geq 3$ the symmetric tensor $\alpha_f^s \colon TM \times \cdots \times TM \to N_fM$ is given by

$$\alpha_f^s(X_1, \dots, X_s) = \pi^{s-1}\left(\nabla^\perp_{X_s} \cdots \nabla^\perp_{X_3} \alpha_f(X_2, X_1)\right)$$

and π^s being the projection onto $(N_1^f \oplus \cdots \oplus N_{s-1}^f)^\perp$. Notice that due to Proposition 1.7 the normal spaces N_s^f form subbundles of the normal bundle, along connected components of an open and dense subset of M^m. Then, along that subset the normal bundle splits orthogonally as

$$N_fM = N_1^f \oplus \cdots \oplus N_{\tau_f}^f,$$

where all N_s^f's have rank two, except possibly the last one $N_{\tau_f}^f$ that has rank one in the case the codimension is odd.

We call an elliptic submanifold $f \colon M^m \to \mathbb{Q}_c^n$ *nicely curved* if all normal subspaces N_ℓ^f's have constant dimension and thus form normal subbundles. According to the following proposition, any elliptic immersion is nicely curved along connected components of an open and dense subset of M^m.

Proposition 1.7. *Let $f \colon M^m \to \mathbb{Q}_c^n$ be a substantial, elliptic immersion. Denote by U_k the set of points where the dimension of N_k^f is maximal, except the last one U_{τ_f} that may have dimension one if the codimension is odd. Then, the subsets U_k, $1 \leq k \leq \tau_f$ are open and dense in M^m.*

Proof: Let $f : M^m \to \mathbb{Q}_c^n$ be a substantial elliptic immersion. At first, we are going to prove that the subset

$$U_1 = \Big\{x \in M^m / \text{dim} N_1^f = 2\Big\}$$

is open and dense in M^m. Due to ellipticity of f, there exists a unit tangent vector field Z such that $\langle Z, JZ\rangle = 0$ and $\{e_1 = Z,\, e_2 = JZ/\|JZ\|\}$ constitute a local orthonormal frame of N_1^f. Set for simplicity $\alpha_{11} = \alpha_f(Z, Z)$ and $\alpha_{12} = \alpha_f(Z, JZ)$.

Since α_{11} and α_{12} are linearly independent at a point $x \in U_1$, they stay linearly independent in a neighborhood of x. This shows that U_1 is open.

To prove that the set U_1 is dense, suppose to the contrary that is not. Then, its complement $V_1 = U_1^c$ has non empty interior. Hence, there exists and open subset W of M^m such that $\text{dim} N_1^f \leq 1$ on W, that is,

$$\alpha_{12} = \lambda \alpha_{11}, \quad \lambda \in C^\infty(W).$$

We aim to prove that N_1^f is parallel and because is has dimension less than its maximum it reduces codimension in view of Proposition 1.3. We compute using the symmetric of the third fundamental form α_f^3 that

$$\begin{aligned}\alpha_f^3(e_1,e_1,e_2) &= \left(\nabla_{e_1}^\perp \alpha_{12}\right)_{(N_1^f)^\perp}\\ &= \left(\nabla_{e_1}^\perp \lambda\alpha_{11}\right)_{(N_1^f)^\perp}\\ &= \left(e_1(\lambda)\alpha_{11} + \lambda\nabla_{e_1}^\perp\alpha_{11}\right)_{(N_1^f)^\perp}\\ &= \lambda\alpha_f^3(e_1,e_1,e_1),\end{aligned}$$

and using the ellipticity of f we obtain

$$\begin{aligned}\alpha_f^3(e_1,e_1,e_1) &= -\alpha_f^3(e_2,e_2,e_1)\\ &= -\left(\nabla_{e_2}^\perp \alpha_{12}\right)_{(N_1^f)^\perp}\\ &= -\left(e_2(\lambda)\alpha_{11} + \lambda\nabla_{e_1}^\perp\alpha_{11}\right)_{(N_1^f)^\perp}\\ &= -\lambda\alpha_f^3(e_1,e_1,e_2)\\ &= -\lambda^2\alpha_f^3(e_1,e_1,e_1).\end{aligned}$$

Hence

$$\alpha_f^3(e_1,e_1,e_1) = \alpha_f^3(e_1,e_1,e_2) = 0.$$

This means that N_1^f is a parallel subbundle of the normal bundle and $\dim N_1^f=1$ hence by Proposition 1.3 it reduces codimension. We have reached a contradiction in view of f being substantial. The results for higher normal spaces are obtained similarly by showing that $N_1^f \oplus \cdots \oplus N_k^f$ are parallel for $2 \le k \le \tau_f$. Thus, by Proposition 1.3 f reduces codimension which contradicts the assumption of f being substantial. ∎

From the last proposition it is immediate that the normal bundle of f can be decomposed orthogonally as

$$N_f M = N_1^f \oplus \cdots \oplus N_{\tau_f}^f, \tag{1.9}$$

where all N_ℓ^f's have rank 2, except possibly the last one $N_{\tau_f}^f$ that has rank 1 in case the codimension is odd. Thus, the induced bundle $f^*T\mathbb{Q}_c^n$ splits as

$$f^*T\mathbb{Q}_c^n = f_*\mathcal{D} \oplus N_0^f \oplus N_1^f \oplus \cdots \oplus N_{\tau_f}^f,$$

where $N_0^f = f_*\mathcal{D}^\perp$. Set

$$\tau_f^o = \begin{cases} \tau_f & \text{if } n-m \text{ is even}\\ \tau_f - 1 & \text{if } n-m \text{ is odd.}\end{cases}$$

It turns out that the almost complex structure J on $\mathcal{D}^\perp$ induces an almost complex structure J_ℓ on each N^f_ℓ, $0 \le \ell \le \tau^o_f$, defined by

$$J_\ell \alpha_f^{\ell+1}(X_1,\dots,X_\ell,X_{\ell+1}) = \alpha_f^{\ell+1}(X_1,\dots,X_\ell,JX_{\ell+1}),$$

where $\alpha^1_f = f_*$. Moreover it holds, see [15],

$$J_s\big(\tilde\nabla_X\xi\big)_{N^f_s} = \big(\tilde\nabla_X J_{s-1}\xi\big)_{N^f_s} = \big(\tilde\nabla_{JX}\xi\big)_{N^f_s}, \quad \forall\xi\in N^f_{s-1},\ X\in\mathcal{D}^\perp$$

and

$$J^t_{s-1}\big(\tilde\nabla_X\eta\big)_{N^f_{s-1}} = \big(\tilde\nabla_X J^t_s\eta\big)_{N^f_{s-1}} = \big(\tilde\nabla_{JX}\eta\big)_{N^f_{s-1}}, \quad \forall\eta\in N^f_s,\ X\in\mathcal{D}^\perp,$$

where J^t_{s-1} denotes the transpose of J_{s-1}.

The ℓ^{th}*-order curvature ellipse* $\mathcal{E}^f_\ell(x)\subset N^f_\ell(x)$ of f at $x\in M^m$ for $0\le\ell\le\tau^o_f$ is

$$\mathcal{E}^f_\ell(x) = \big\{\alpha_f^{\ell+1}(Z_\theta,\dots,Z_\theta) : Z_\theta = \cos\theta Z + \sin\theta JZ \ \text{ and } \ \theta\in[0,\pi)\big\},$$

where $Z\in\mathcal{D}^\perp(x)$ has unit length and satisfies $\langle Z, JZ\rangle = 0$. From ellipticity such a Z always exists.

We say that the curvature ellipse $\mathcal{E}^f_\ell$ of an elliptic submanifold f is a *circle* for some $0\le\ell\le\tau^o_f$ if all ellipses $\mathcal{E}^f_\ell(x)$ are circles. That the curvature ellipse $\mathcal{E}^f_\ell$ is a circle is equivalent to the almost complex structure J_ℓ being orthogonal. Notice that $\mathcal{E}^f_0$ is a circle if and only if f is minimal.

An elliptic submanifold f is called ℓ*-isotropic* if all curvature ellipses up to order ℓ are circles. Then f is called *isotropic* if the curvature ellipses of any order are circles.

Substantial isotropic surfaces in $\mathbb{R}^{2n}$ are holomorphic curves in $\mathbb{C}^n\equiv\mathbb{R}^{2n}$. Isotropic surfaces in spheres are also referred to as *pseudoholomorphic* surfaces. For this class of surfaces a Weierstrass type representation was given in [28].

Let $f\colon M^m\to\mathbb{Q}^{n-c}_c$, $c=0,1$, be a substantial and nicely curved elliptic submanifold. Assume that M^m is the saturation of a fixed cross section $L^2\subset M^m$ to the relative nullity foliation. The subbundles in the orthogonal splitting (1.9) are parallel in the normal connection (and thus in $\mathbb{Q}^{n-c}_c$) along $\mathcal{D}$. Hence each N^f_ℓ can be seen as a vector bundle along the surface L^2.

A *polar surface* to an elliptic immersion f is an immersion g of a cross section L^2 to the relative nullity foliation defined as follows:

(a) If $n-c-m$ is odd, then the polar surface $g\colon L^2\to\mathbb{S}^{n-1}$, $c=0,1$ (resp. $g\colon L^2\to\mathbb{S}^{n+1}_1$ if $c=-1$) is the spherical image of the unit normal field spanning the last normal bundle $N^f_{\tau_f}$.

(b) If $n-c-m$ is even, then the polar surface $g\colon L^2 \to \mathbb{R}^n$, $c = 0,1$ (resp. $g\colon L^2 \to \mathbb{L}^{n+2}$ if $c=-1$) is any surface such that $g_* T_x L = N^f_{\tau_f}(x)$ up to parallel identification in $\mathbb{R}^n$ (resp. $\mathbb{L}^{n+2}$).

Polar surfaces always exist since, in case (b), any elliptic submanifold admits locally many polar surfaces.

Since our work is local, we may assume that an elliptic submanifold f is the saturation of a fixed cross section $L^2 \subset M^m$ to the relative nullity foliation. The almost complex structure J on $\mathcal{D}^\perp$ induces an almost complex structure $\tilde{J}$ on TL defined by

$$P\tilde{J} = JP,$$

where $P\colon TL \to \mathcal{D}^\perp$ is the orthogonal projection.

The following proposition ensures that associated to any elliptic submanifold of rank two, there is an elliptic surface that "integrates" its k-th normal space and relates the complex structure and normal spaces between them.

Proposition 1.8. *Any elliptic submanifold $f\colon M^m \to \mathbb{Q}^n_c$ admits locally a polar surface. Moreover, in substantial codimension, any polar surface g to f is k-regular for appropriate k and elliptic with respect to $J_0^g = \tilde{J}$ such that*

$$N^g_s = N^f_{k-s} \quad \text{and} \quad J^g_s = J^t_{k-s}, \tag{1.10}$$

for each $0 \le s \le k$ and $x \in L^2$, up to parallel identification in $\mathbb{R}^{n+1}$ if $c = 0,1$, or in $\mathbb{L}^{n+1}$ if $c=-1$.

Proof: See Proposition 8 in [15]. ∎

A *bipolar surface* to f is any polar surface to a polar surface to f. In particular, if we are in case $f\colon M^3 \to \mathbb{S}^{n-1}$, then a bipolar surface to f is a nicely curved elliptic immersion $g\colon L^2 \to \mathbb{R}^n$.

1.9 Omori-Yau maximum principle

A main ingredient in the proof of our results is the Omori-Yau maximum principle. The *Omori-Yau maximum principle* is said to hold on a complete Riemannian manifold M^m if for any function $\varphi \in C^2(M)$ bounded from above there exists a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ such that

$$\varphi(x_j) > \sup\varphi - 1/j, \quad \|\nabla\varphi\|(x_j) \le 1/j \quad \text{and} \quad \Delta\varphi(x_j) \le 1/j,$$

for any $j \in \mathbb{N}$. The category of complete Riemannian manifolds for which the principle is valid is quite large. For instance, it contains the complete manifolds whose Ricci curvature satisfies Ric $\geq -K(1 + r^2 \log^2(r + 2))$, where r is the geodesic distance function from a fixed point of the manifold and K is a non negative constant. It also contains the class of properly immersed submanifolds in a space form whose norm of the mean curvature vector is bounded; see [3] or [56, Example 1.14].

In the sequel, we recall the elementary strong maximum principle and two results that are consequences of the Omori-Yau maximum principle and will be crucial in the proof of our main results.

Proposition 1.9. *If a harmonic function attains maximum value in an interior point of M then it must be constant.*

Proof: See [57, Theorem 5 in Chapter 2]. ∎

The following proposition is a consequence of a result due to Cheng and Yau [11].

Proposition 1.10. *Let M^m be a Riemannian manifold for which the Omori-Yau maximum principle holds. If $\varphi \in C^\infty(M)$ satisfies that $\Delta\varphi \geq 2\varphi^2$ then* $\sup \varphi = 0$.

Proof: See [3] or [42, Lemma 4.1]. ∎

The next proposition was proved by Yau in his attempt to generalize the classic Liouville Theorem of complex analysis to complete Riemmanian manifolds.

Proposition 1.11. *Let M^m be a Riemannian manifold with Ricci curvature bounded from below by $-K$ for some constant $K \geq 0$. If $\varphi \in C^\infty(M)$ is a harmonic function which is bounded from below, then*

$$\|\nabla\varphi\| \leq \sqrt{(m-1)K}(\varphi - \inf \varphi).$$

Proof: See [67, Theorem 3″]. ∎

1.10 Removable singularities of harmonic maps

In this section, we state some well known results, regarding extensions of harmonic maps between Riemannian manifolds.

Let $F\colon M \to \tilde{M}$ be a C^2 mapping between smooth Riemannian manifolds $M, \tilde{M}$ with Levi-Civita connections ∇ and $\tilde{\nabla}$ respectively. Pick a local orthonormal tangent frame field $\{e_1, \ldots, e_m\}$ around a point $x \in M$. We define the *tension field* of F at a point $x \in M$ to be

$$\tau(F)(x) = \sum_{j=1}^{m} \big(\tilde{\nabla}_{F_* e_j} F_* e_j - F_* \nabla_{e_j} e_j\big)(x)$$

We say that F is a *harmonic map* if the tension field vanishes identically.

The following proposition allows us to extend analytically a harmonic map between Riemannian manifolds. For a proof, we refer to Eells-Sampson [30, Proposition p. 117].

Proposition 1.12. *Every harmonic map $F\colon M \to \tilde{M}$ of class C^2, where M and $\tilde{M}$ are complete Riemannian manifolds, is smooth. If M and $\tilde{M}$ are both analytic Riemannian manifolds, then every such map is analytic.*

In order to state the next theorem, we need to introduce the notion of the relative 2-capacity of a compact set K of $\mathbb{R}^n$. We denote by $B_{r_0}(0)$ the open ball of center 0 and radius $r_0 > 0$ in $\mathbb{R}^n$. If $K \subset B_{r_0}(0)$, then $\mathrm{Cap}_{2,r_0}(K)$ is defined by

$$\mathrm{Cap}_{2,r_0}(K) = \inf\left\{\int_{\mathbb{R}^n} \|D\psi\|^2 dx \colon \psi \in C_c^\infty(B_{r_0}(0),\mathbb{R}),\ \ \psi \geq 1 \ \text{ on } \ K\right\}.$$

By $C_c^\infty(B_{r_0}(0),\mathbb{R})$ we denote the space of smooth functions $\psi\colon B_{r_0}(0) \to \mathbb{R}$ with compact support. For $n \geq 3$, we set $\mathrm{Cap}_2(K)$ $=\mathrm{Cap}_{2,\infty}(K)$. If B is open subset of $\mathbb{R}^n$ then

$$\mathrm{Cap}_2(B) = \sup\left\{\mathrm{Cap}_2(K) : \ \text{ for each } \ K \subset B,\ K \ \text{ compact}\right\}.$$

If E is an arbitrary subset of $\mathbb{R}^n$ then

$$\mathrm{Cap}_2(E) = \inf\left\{\mathrm{Cap}_2(B) : \ \text{ for each } \ E \subset B,\ B \ \text{ open}\right\}.$$

If M is a Riemannian manifold and $\Sigma \subset M$ then by $\mathrm{Cap}_2(\Sigma) = 0$ we mean that $\mathrm{Cap}_2\big(\phi(\Sigma \cap U)\big) = 0$ for each chart (U,ϕ) of M. It is well known that if K is a smooth curve, then its 2-capacity is zero, see (cf. [33, Theorem 3]) or (cf. [45, p. 37]).

Let M be a connected Riemannian manifold and let $\tilde{M}$ be a complete Riemannian manifold without boundary. Let A be a relatively closed subset of M. Meier proved in [53, Theorem 1] the following removable singularity result:

Theorem 1.13. *Let $F\colon M \setminus A \to \tilde{M}$ be a bounded harmonic map and $Cap_2(A) = 0$. Suppose $F(M \setminus A)$ is contained in some closed geodesic ball $B_r(\tilde{x_0})$ of $\tilde{M}$ which does not meet the cut locus of its center $\tilde{x_0}$, and for which $r \leq \pi/(2\sqrt{\kappa})$, where $\kappa \geq 0$ is an upper bound for the sectional curvature of $\tilde{M}$ on $B_r(\tilde{x_0})$. If $r = \pi/(2\sqrt{\kappa})$, assume in addition that $F(M \setminus A)$ is not completely contained in the boundary of $B_r(\tilde{x_0})$. Then F extends to a harmonic map of class C^2 in all of M.*

Using Proposition 1.12 and Theorem 1.13 we obtain the following lemma:

Lemma 1.14. *Let $F\colon M^3 \setminus A \to \tilde{M}$ be a bounded harmonic map, where A is a smooth curve and $M, \tilde{M}$ are complete Riemannian manifolds. If F has continuous extension over A, then F can be extended analytically over A.*

1.11 Real analytic subvarieties

A function $H(x_1, \dots, x_{k-1}; x_k)$ of k real variables is called a *distinguished polynomial* or *Weierstrass polynomial* if it has the form

$$\begin{aligned} H(x_1, \dots, x_{k-1}; x_k) &= x_k^m + A_1(x_1, \dots, x_{k-1})x_k^{m-1} + \cdots \\ &\quad + A_{m-1}(x_1, \dots, x_{k-1})x_k + A_m(x_1, \dots, x_{k-1}), \end{aligned}$$

where each A_i vanishes at $(x_1, \dots, x_{k-1}) = (0, \dots, 0)$. It is an important fact that any analytic function is locally, up to an invertible factor, a distinguished polynomial, see for instance Weierstrass preparation theorem [50, Theorem 6.3.1].

Every distinguished polynomial H admits a unique decomposition into irreducible distinguished polynomials. The *discriminant* $D(H)(x_1, \dots, x_{k-1})$ of a distinguished polynomial H vanishes if and only if $H(x_1, \dots, x_{k-1}; x_k)$ has a repeated irreducible factor, see [6].

Recall that a closed set $X \subset M$ is called a *real analytic subvariety* of M, if for each point $p \in X$ there exists a neighborhood V and a set $\mathcal{F}$ of real analytic functions defined in V such that

$$X \cap V = \left\{p \in V / f(p) = 0 \ \text{ for all } \ f \in \mathcal{F}\right\}.$$

By a careful analysis of symmetric functions of the roots of a distinguished polynomials, Lojasiewicz was able to prove in [50, Theorem 6.3.3], the following stratification theorem for real analytic subvarieties.

Theorem 1.15. *Let $\Phi(x_1, \dots, x_n)$ be a real analytic function in a neighborhood of the origin. After a rotation of the coordinates $x_1, \dots, x_{n-1}$ one has that there exist numbers $\delta_j > 0$, $j = 1, \dots, n$, and a system of distinguished polynomials*

$$H_\ell^k(x_1, \dots, x_k; x_\ell), \ \ 0 \le k \le n, \ \ k+1 \le \ell \le n,$$

defined on $Q_k = \{|x_j| < \delta_j,\ 1 \le j \le k\}$, such that the discriminant D_ℓ^k of H_ℓ^k does not vanish on Q_k and the following properties are satisfied:

(i) Each root ζ of $H_\ell^k(x_1, \dots, x_k; \cdot)$ on Q_k satisfies $\|\zeta\| < \delta_\ell$.

(ii) The set

$$Z := \{x = (x_1, \dots, x_n) : \|x_j\| < \delta_j \ \textit{ for each } \ j \in \{1, \dots, n\} \ \textit{ and } \ \Phi(x) = 0\}$$

has a decomposition

$$Z = V^0 \cup \cdots \cup V^{n-1}.$$

The set V^0 is either empty or consists of the origin alone. For $1 \leq k \leq n-1$, we may write V^k as a finite, disjoint union

$$V^k = \cup_j \Gamma_j^k$$

of k-dimensional subvarieties which have the following explicit description:

(a) *(Analytic Parametrization) Each Γ_j^k is defined by a system of $n-k$ equations*

$$\begin{aligned} x_{k+1} &= \phi_{j,k+1}^k(x_1, \ldots, x_k), \\ &\vdots \\ x_n &= \phi_{j,n}^k(x_1, \ldots, x_k), \end{aligned}$$

where each function $\phi_{j,\ell}^k$ is real analytic on an open subset $\Omega_j^k \subset Q_k \subset \mathbb{R}^k$,

$$H_\ell^k(x_1, \ldots, x_k; \phi_{j,\ell}^k) \equiv 0,$$

and

$$D_\ell^k(x_1, \ldots, x_k) \neq 0$$

for all $(x_1, \ldots, x_k) \in \Omega_j^k$, $\ell = k+1, \ldots, n$.

(b) *(Non-Redundancy) For any integers k, i, j, either $\Omega_i^k = \Omega_j^k$ or $\Omega_i^k \cap \Omega_j^k = \emptyset$. In the second instance one has, for any $\ell = k+1, \ldots, n$, either $\phi_{i,\ell}^k \equiv \phi_{j,\ell}^k$ on Ω_i^k or $\phi_{i,\ell}^k(x_1, \ldots, x_k) \neq \phi_{j,\ell}^k(x_1, \ldots, x_k)$ for all $x = (x_1, \ldots, x_k) \in \Omega_i^k$.*

(c) *(Stratification) For each k the closure of V^k contains all the subsequent V^m's, that is, $V^0 \cup \cdots \cup V^{k-1} \subset Q \cap V^k$. The lower dimensional varieties V^m, $m < n-1$, do not occur as isolated sets; they are in fact the zero sets of certain discriminants and (in a sence) form the boundaries of the components Γ_j^{m+1} of $V^{m+1}, \ldots, V^{n-1}$.*

A point $x_0 \in Z$ is called a *regular point of dimension d* if there is a neighborhood Ω of x_0 such that $\Omega \cap Z$ is a d-dimensional real analytic submanifold of Ω. If otherwise x_0 is said to be a *singular* point. The set of singular points is locally a finite union of submanifolds.

We now turn our attention to the Cauchy-Kowalewski theorem. It concerns the existence and uniqueness of a real analytic solution of a Cauchy problem, given real analytic initial data on a hypersruface S. Before we state this theorem, we introduce some notation and definitions.

An n-tuple $a = (a_1, \ldots, a_n)$ of nonnegative integers will be called a *multi-index*. We define

$$|a| = \sum_{j=1}^{n} a_j, \qquad a! = a_1!a_2! \cdots a_n!,$$

and for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

We will use the shorthand

$$\partial_{x_j} = \frac{\partial}{\partial x_j}$$

for derivatives in $\mathbb{R}^n$ and for higher order derivatives we use the following convention:

$$\partial^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}.$$

If ν is a vector field on an open set $\Omega \subset \mathbb{R}^n$ we define the directional derivative ∂_ν by

$$\partial_\nu = \langle \nu, \nabla \rangle,$$

where $\langle \cdot\,, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. For any differentiable function f in Ω, we have that

$$\partial_\nu f(x) = \langle \nu(x), \nabla f(x) \rangle = \sum_{i=1}^{n} \nu_j(x) \partial_{x_i} f(x).$$

Let

$$\mathcal{L} = \sum_{|a| \leq k} \lambda_a \partial^a$$

be a linear differential operator of order k on $\Omega \subset \mathbb{R}^n$. Then, its *characteristic form* at $x \in \Omega$ is the homogeneous polynomial of degree k on $\mathbb{R}^n$ defined by

$$Q(x, \zeta) = \sum_{|a|=k} \lambda_a(x) \zeta^a, \quad \zeta \in \mathbb{R}^n.$$

A nonzero vector ζ is called *characteristic* for $\mathcal{L}$ at x if $Q(x, \zeta) = 0$ and the set of all such ζ is called the *characteristic variety* of $\mathcal{L}$ at x and is denoted by $\mathrm{char}_x(\mathcal{L})$. Thus, the condition $\zeta \in \mathrm{char}_x(\mathcal{L})$ means that, in some sence, $\mathcal{L}$ fails to be "genuinely kth order" in the direction ζ at x. The operator $\mathcal{L}$ is said to be *elliptic* at x if $\mathrm{char}_x(\mathcal{L}) = \emptyset$ and *elliptic* on Ω if it is elliptic at every point $x \in \Omega$.

A hypersurface S in Ω is called *characteristic* for $\mathcal{L}$ at a point $x \in S$, if the unit normal vector $\vec{\eta}(x)$ to S at x is in $\mathrm{char}_x(\mathcal{L})$. The hypersurface S is called *non-characteristic* if it is not characteristic at any point.

Theorem 1.16. *Consider the Cauchy problem*

$$\begin{cases} \mathcal{L}(u) = F\left(x, (\partial^a u)_{|a|\leq k}\right) = 0, \\ \partial_{\vec{\eta}}^{j}\, u = \phi_j \quad on \quad S \quad for \quad 0 \leq j < k, \end{cases} \tag{1.11}$$

where the functions $F, \phi_0, \ldots, \phi_{k-1}$ *are near analytic near the origin. If the hypersurface* S *is non-characteristic, then there exists a neighborhood of the origin on which the Cauchy problem* (1.11) *has a unique analytic solution.*

Proof: See [37, Theorem 1.25]. ∎

CHAPTER 2

A fundamental lemma

The ideas in this chapter will play a crucial role in the proofs of our main results in Chapters 3,4 and 5.

Let $f\colon M^3 \to \mathbb{Q}_c^n$ be a substantial minimal isometric immersion with index of relative nullity $\nu(x) \geq 1$ at any point of M^3, that is, the index is either 1 or 3. Let $U \subset M^3$ be an open subset where $\nu = 1$ and the line bundle of relative nullity is trivial. Fix a smooth unit section e spanning the relative nullity distribution along U and let J denote the unique, up to sign, almost complex structure acting on the conullity distribution $\mathcal{D}^\perp = \{e_3\}^\perp = \operatorname{span}\{e_1, e_2\}$, where $e_3 = e$ and e_1, e_2 are orthonormal. Moreover, set for simplicity $\mathcal{C} = C_{e_3}$. The following lemma is of crucial importance.

Lemma 2.1. *There are harmonic functions $u, v \in C^\infty(U)$ such that*

$$\mathcal{C} = vI - uJ \tag{2.1}$$

where I stands for the identity map on the conullity distribution. Moreover, the integral curves of e are geodesics and the functions u and v satisfy the following differential equations:

$$e(v) = v^2 - u^2 + c, \quad e(u) = 2uv \tag{2.2}$$

and

$$e_1(u) = e_2(v), \quad e_2(u) = -e_1(v). \tag{2.3}$$

Proof: We may assume that the immersion f is substantial, that is, it does not reduce codimension. Let A_ξ be the shape operator of f with respect to the normal direction ξ, i.e.,

$$\langle A_\xi \,\cdot\,, \cdot\rangle = \langle \alpha(\,\cdot\,, \cdot), \xi\rangle.$$

From the Codazzi equation for $A_\xi|_{\mathcal{D}^\perp}$ restricted to $\mathcal{D}^\perp$ we have that

$$\nabla_e A_\xi|_{\mathcal{D}^\perp} = A_\xi|_{\mathcal{D}^\perp} \circ \mathcal{C} + A_{\nabla_e^\perp \xi}|_{\mathcal{D}^\perp}$$

for any normal vector field $\xi \in \Gamma(N_f M)$. Thus $A_\xi|_{\mathcal{D}^\perp} \circ \mathcal{C}$ has to be symmetric, and hence

$$A_\xi|_{\mathcal{D}^\perp} \circ \mathcal{C} = \mathcal{C}^t \circ A_\xi|_{\mathcal{D}^\perp}. \tag{2.4}$$

On the other hand, the minimality condition is equivalent to

$$A_\xi|_{\mathcal{D}^\perp} \circ J = J^t \circ A_\xi|_{\mathcal{D}^\perp}. \tag{2.5}$$

First we consider the hypersurface case $n = m + 1$. Take a local orthonormal tangent frame e_1, e_2, e_3 that diagonalizes the shape operator of f such that

$$Je_1 = e_2 \quad \text{and} \quad e_3 = e$$

and let ξ be a unit normal along the hypersurface. Set

$$u = \langle \nabla_{e_2} e_1, e \rangle \ \text{ and } \ v = \langle \nabla_{e_1} e_1, e \rangle.$$

From the Codazzi equation

$$(\nabla_{e_i} A_\xi) e = (\nabla_e A_\xi) e_i,$$

where $1 \leq i \leq 2$, we have that $\langle \nabla_{e_2} e_2, e \rangle = v$ and $\langle \nabla_e e, e_1 \rangle = \langle \nabla_e e, e_2 \rangle = 0$. The latter shows that the integral curves of e are geodesics, i.e., $\nabla_e e = 0$. Moreover, from

$$\langle (\nabla_{e_1} A_\xi) e_2, e \rangle = \langle (\nabla_{e_2} A_\xi) e_1, e \rangle,$$

we obtain that $\langle \nabla_{e_1} e_2, e \rangle = -u$. Now we can readily see that (2.1) holds true.

Assume now that f does not reduce codimension to one. Due to minimality assumption, we have that $\dim N_k^f \leq 2$. If $\dim N_k^f = 1$ on an open subset $V \subset M^3$, a simple argument using the Codazzi equation shows that N_1^f is parallel in the normal bundle along V and thus according to Proposition 1.3, $f|_V$ reduces codimension to a hypersurface. Due to real analyticity the same would hold globally, and that has been excluded. Hence, there is an open dense subset W of M^3 where $\dim N_1^f = 2$. We conclude from (2.4) and (2.5) that $\mathcal{C} \in \operatorname{span}\{I, J\}$ on $U \cap W$. By continuity, we then get that $\mathcal{C} \in \operatorname{span}\{I, J\}$ on U. Therefore, also in this case there are functions $u, v \in C^\infty(U)$ such that (2.1) holds.

It remains to show that u, v are harmonic. From (1.5) and (1.6) we have

$$\nabla_e^h \mathcal{C} = \mathcal{C}^2 + cI \tag{2.6}$$

and

$$(\nabla^h_X \mathcal{C})Y = (\nabla^h_Y \mathcal{C})X \tag{2.7}$$

for any $X, Y \in \Gamma(\mathcal{D}^\perp)$. For a local orthonormal tangent frame e_1, e_2, e_3 such that $Je_1 = e_2$ and $e_3 = e$, it follows from (2.1) that

$$v = \langle \nabla_{e_1} e_1, e \rangle = \langle \nabla_{e_2} e_2, e \rangle \tag{2.8}$$

and

$$u = -\langle \nabla_{e_1} e_2, e \rangle = \langle \nabla_{e_2} e_1, e \rangle. \tag{2.9}$$

It is easily seen that (2.6) is equivalent to (2.2), whereas (2.7) is equivalent to (2.3). The Laplacian of v is given by

$$\Delta v = \sum_{j=1}^{3} e_j e_j(v) + \omega_{12}(e_2)e_1(v) - \omega_{12}(e_1)e_2(v) - (\omega_{13}(e_1) + \omega_{23}(e_2))e(v), \tag{2.10}$$

where $\omega_{ij}(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle$, for $1 \le i, j, k \le 3$. Using (2.9) and (2.3), we have that

$$\begin{aligned} e_1e_1(v) + e_2e_2(v) &= -e_1e_2(u) + e_2e_1(u) = [e_2, e_1](u) \\ &= \nabla_{e_2}e_1(u) - \nabla_{e_1}e_2(u) \\ &= \omega_{12}(e_1)e_1(u) + \omega_{12}(e_2)e_2(u) + (\omega_{13}(e_2) - \omega_{23}(e_1))e(u) \\ &= \omega_{12}(e_1)e_2(v) - \omega_{12}(e_2)e_1(v) + 2ue(u). \end{aligned}$$

Inserting the last equality into (2.10) and using (2.8) and (2.2) yields

$$\Delta v = ee(v) + 2ue(u) - 2ve(v) = 0.$$

Also, that the function u is harmonic is proved in a similar manner. ∎

CHAPTER 3

Minimal immersions with relative nullity in Euclidean space

A frequent theme in submanifold theory is to find geometric conditions for an isometric immersion of a complete Riemannian manifold into Euclidean space $f\colon M^m \to \mathbb{R}^n$ with index of relative nullity $\nu \geq k > 0$ at any point to be a k-cylinder. This means that the manifold M^m splits as a Riemannian product $M^m = M^{m-k} \times \mathbb{R}^k$ and there is an isometric immersion $g\colon M^{m-k} \to \mathbb{R}^{n-k}$ such that $f = g \times \mathrm{id}_{\mathbb{R}^k}$.

A fundamental result asserting that an isometric immersion $f\colon M^m \to \mathbb{R}^n$ with positive index of relative nullity must be a k-cylinder is Hartman's theorem [40] that requires the Ricci curvature of M^m to be nonnegative; see also [52]. A key ingredient for the proof of this result is the famous Cheeger-Gromoll splitting theorem [10] used to conclude that the leaves of minimum relative nullity split intrinsically as a Riemannian factor. Even for hypersurfaces, the same conclusion does not hold if instead we assume that the Ricci curvature is nonpositive. Notice that the latter is always the case if f is a minimal immersion. Counterexamples easy to construct are the complete irreducible ruled hypersurfaces of any dimension discussed in [19, p. 409].

Some of the many papers containing characterizations of submanifolds as cylinders without the requirement of minimality are [15, 17, 38, 40, 52, 54, 58]. When adding the condition of being minimal we have [1, 24, 35, 36, 38, 41, 64, 66].

3.1 The main result

In this section, we extend a result for hypersurfaces due to Savas-Halilaj [60] to the situation of arbitrary codimension.

Theorem 3.1. *Let M^m be a complete Riemannian manifold and $f\colon M^m \to \mathbb{R}^n$ be a minimal isometric immersion with index of relative nullity $\nu \geq m-2$ at any point of M^m. If the Omori-Yau maximum principle holds on M^m, then f is a cylinder over a minimal surface.*

Corollary 3.2. *Let M^m be a complete Riemannian manifold and $f\colon M^m \to \mathbb{R}^n$ be a minimal isometric immersion with index of relative nullity $\nu \geq m-2$ at any point of M^m. Assume that either the scalar curvature* s *of M^m satisfies* $\mathrm{s} \geq -c(d\log d)^2$ *outside a compact set, where $c > 0$ and $d = d(\cdot, o)$ is the geodesic distance to a reference point $o \in M^m$, or that f is proper. Then f is a cylinder over a minimal surface.*

Theorem 3.1 is truly global in nature since there are plenty of examples of noncomplete minimal submanifolds of any dimension m with constant index $\nu = m-2$ that are not part of a cylinder on any open subset. They can be all locally parametrically described in terms of a certain class of elliptic surfaces; see Theorem 22 in [15]. In particular, there is a Weierstrass type representation for these submanifolds when the manifold possesses a Kähler structure; see Theorem 27 in [15]. On the other hand, after the results of this chapter what remains as a challenging open problem is the existence of a minimal complete noncylindrical submanifold $f\colon M^3 \to \mathbb{R}^n$ with index of relative nullity $\nu \geq 1$.

The main difficulty in the proof of Theorem 3.1 arises from the fact that the index of relative nullity ν is allowed to vary. Consequently, one has to fully understand the structure of the set of points $\mathcal{A} \subset M^m$ where f is totally geodesic in order to conclude that the relative nullity foliation on $M^m \smallsetminus \mathcal{A}$ extends smoothly to $\mathcal{A}$.

Recently Jost, Yang and Xin [48] proved various Bernstein type results for complete m-dimensional minimal graphical submanifolds in Euclidean space with index $\nu \geq m-2$. We observe that from a result in [19] it follows that the submanifolds considered in [48, Theorem 1.1] are cylinders over 3-dimensional complete minimal submanifolds with $\nu \geq 1$. Moreover, from Corollary 3.2 it follows that the submanifolds considered in [48, Theorem 1.2] are just cylinders over complete minimal surfaces, since entire graphs are proper submanifolds. Thus, to prove a Bernstein theorem for such submanifolds is equivalent to show a Bernstein theorem for entire minimal 2-dimensional graphs in Euclidean space.

3.2 The proofs

Let M^m be a Riemannian manifold. An isometric immersion $f\colon M^m \to \mathbb{R}^n$ is called *ruled* if M admits a continuous codimension one foliation such that f maps each leaf (*ruling*) onto an open subset of an affine subspace of $\mathbb{R}^n$. We say that f is *completely ruled* if all rulings are complete. Observe that in this case, the leaves in each connected component of M (called a *ruled strip*) form an affine vector bundle over a curve with or without endpoints. We then say f is a *cylinder* if $M = L^1 \times \mathbb{R}^{m-1}$ and $f = f_1 \times \mathrm{id}$ splits.

The possible structures of an isometric immersion $f\colon M^m \to \mathbb{R}^n$ when M^m is complete and the index of relative nullity of f satisfies $\nu \geq m-2$ at any point was completely described by Dajczer and Gromoll in [19, Proposition 2.1]. Among other results they proved the following:

Proposition 3.3. *Let $f\colon M^m \to \mathbb{R}^n$, $m \geq 3$, be an isometric immersion of a complete Riemannian manifold which does not contain an open set $L^3 \times \mathbb{R}^{m-3}$ with L^3 unbounded, and ρ the rank of the Gauss map. Suppose that $\rho \leq 2$ everywhere, and let M^* be the open subset of all points in M with $\rho = 2$. Then the following hold:*

(i) M^* is a union of smoothly ruled strips.

(ii) If f is completely ruled on M^*, then it is completely ruled everywhere, and a cylinder on each component of the complement of the closure of M^*.

Consequently, if f is real analytic, then either $M = L^3 \times \mathbb{R}^{m-3}$ and $f = f_1 \times id$ splits, or f is completely ruled.

In the case of minimal (hence elliptic) submanifolds, Dajczer and Florit proved in [15, Theorem 16] the following:

Theorem 3.4. *Let $f\colon M^m \to \mathbb{R}^n$ be a complete submanifold elliptic on a dense subset of M^m. Then, each connected component of an open dense subset of M^m is isometric to $L^3 \times \mathbb{R}^{m-3}$ and f splits accordingly. Moreover, the splitting is global if M^m is simply connected and does not contain an open subset $L^2 \times \mathbb{R}^{m-2}$.*

From the latter it is obvious that the interesting case occurs when $m = 3$. Hence, we only have to consider the case of a nontrivial minimal $f\colon M^3 \to \mathbb{R}^n$ with $\nu \geq 1$ at any point of M^3.

Let $\mathcal{A}$ denote the set of totally geodesic points of f. From Proposition 1.1(iv) the relative nullity foliation $\mathcal{D}$ is a line bundle on $M^3 \smallsetminus \mathcal{A}$. Due to the real analyticity of the submanifold, the square of the norm of the second fundamental form is a real analytic function. It follows that $\mathcal{A}$ is a real analytic set. According to Lojasewicz's structure Theorem 1.15 the set $\mathcal{A}$ locally decomposes as

$$\mathcal{A} = \mathcal{V}^0 \cup \mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3,$$

where each $\mathcal{V}^d$, $0 \leq d \leq 3$, is either empty or a disjoint finite union of d-dimensional real analytic subvarieties.

Our goal now is to show that $\mathcal{A} = \mathcal{V}^1$, unless f is just an affine subspace in $\mathbb{R}^n$ in which case Theorem 3.1 trivially holds. After excluding the latter trivial case, we have from the real analyticity of f that $\mathcal{V}^3$ is empty.

Lemma 3.5. *The set $\mathcal{V}^2$ is empty.*

Proof: We only have to show is that there is no regular point in $\mathcal{V}^2$. Suppose to the contrary that such a point do exist. Let $\Omega \subset M^3$ be an open neighborhood of a regular point $x_0 \in \mathcal{V}^2$ such that $L^2 = \Omega \cap \mathcal{A}$ is an embedded surface. Let $e_1, e_2, e_3, \xi_1, \ldots, \xi_{n-3}$ be an orthonormal frame adapted to M^3 along Ω near x_0. The coefficients of the second fundamental form are

$$h_{ij}^a = \langle \alpha(e_i, e_j), \xi_a \rangle,$$

where from now on $1 \leq i, j, k \leq 3$ and $1 \leq a, b \leq n-3$.

The Gauss map $\gamma\colon M^3 \to Gr(3, n)$ of f as a map into the Grassmannian of oriented 3-dimensional linear subspaces in $\mathbb{R}^n$ is defined by $\gamma(x) = f_*(T_x M^3) \subset \mathbb{R}^n$, up to parallel translation in $\mathbb{R}^n$ to the origin. Regarding $Gr(3, n)$ as a submanifold in $\wedge^3 \mathbb{R}^n$ via the map for the Plücker embedding, we have that $\gamma = f_* e_1 \wedge f_* e_2 \wedge f_* e_3$. Then

$$\gamma_* e_i = \sum_{j,a} h_{ij}^a e_{ja}, \tag{3.1}$$

where e_{ja} is obtained by replacing $f_* e_j$ with ξ_a in $f_* e_1 \wedge f_* e_2 \wedge f_* e_3$. Then

$$\sum_i \langle \gamma_* e_i, \gamma_* e_i \rangle = \sum_{i,j,a} (h_{ij}^a)^2 = \|\alpha\|^2,$$

where the inner product of two simple 3-vectors in $\wedge^3 \mathbb{R}^n$ is defined by

$$\langle a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3 \rangle = \det\big(\langle a_i, b_j \rangle\big).$$

For a fixed simple 3-vector $A = a_1 \wedge a_2 \wedge a_3$ in $\wedge^3 \mathbb{R}^n$, let $w_A : M^3 \to \mathbb{R}$ be the function defined by

$$w_A = \langle \gamma, A \rangle.$$

Because the immersion f is minimal, the height function w_A satisfies

$$\Delta w_A = -\|\alpha\|^2 w_A + \sum_{i, a \neq b, j \neq k} h_{ij}^a h_{ik}^b \langle e_{ja,kb}, A \rangle,$$

where $e_{ja,kb}$ is obtained by replacing $f_* e_j$ with ξ_a and $f_* e_k$ with ξ_b in $f_* e_1 \wedge f_* e_2 \wedge f_* e_3$ (cf. [65, p. 36]). Let $\varepsilon_1, \ldots, \varepsilon_n$ be an orthonormal basis of $\mathbb{R}^n$. The set

$$\{\varepsilon_{j_1} \wedge \varepsilon_{j_2} \wedge \varepsilon_{j_3} : 1 \leq j_1 < j_2 < j_3 \leq n\}$$

of 3-vectors is an orthonormal basis of $\wedge^3\mathbb{R}^n$ by means of which identify $\wedge^3\mathbb{R}^n$ with $\mathbb{R}^{\binom{n}{3}} = \mathbb{R}^N$. Denoting by $\{A_J\}_{J\in\{1,\dots,N\}}$ the corresponding base in $\mathbb{R}^N$, we have

$$\gamma = \sum_{J=1}^{N} w_J A_J \quad \text{where} \quad w_J = \langle \gamma, A_J \rangle.$$

From $h^a_{ij} = \langle \gamma_* e_i, e_{ja}\rangle$, we obtain

$$h^a_{ij} = \sum_J \langle e_{ja}, A_J\rangle e_i(w_J). \tag{3.2}$$

Moreover, for any $J \in \{1,\dots,N\}$, it holds

$$\Delta w_J = -\|\alpha\|^2 w_J + \sum_{i,a\neq b,j\neq k} h^a_{ij} h^b_{ik} \langle e_{ja,kb}, A_J\rangle. \tag{3.3}$$

Take a local chart $\phi\colon U \to \mathbb{R}^3$ of coordinates $x = (x_1, x_2, x_3)$ on an open subset U of Ω and set

$$e_i = \sum_j \mu_{ij} \partial_{x_j}. \tag{3.4}$$

Setting $\theta_J = w_J \circ \phi^{-1}$, we obtain the map $\theta\colon \phi(U) \subset \mathbb{R}^3 \to \mathbb{R}^N$ given by

$$\theta = \sum_J \theta_J A_J = (\theta_1, \dots, \theta_N).$$

Note that $\theta = \gamma \circ \phi^{-1}$, i.e., θ is just the representation of the Gauss map with respect to the above mentioned charts. From (3.2) and (3.4) we have

$$h^a_{ij} = \sum_{k,J} \mu_{ik} \langle e_{ja}, A_J\rangle (\theta_J)_{x_k} \tag{3.5}$$

and

$$\|\alpha\|^2 = \sum_{i,j,a} \Big(\sum_{k,J} \mu_{ik} \langle e_{ja}, A_J\rangle (\theta_J)_{x_k} \Big)^2. \tag{3.6}$$

The Laplacian of M^3 is given by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_{x_i} \Big(\sqrt{g} g^{ij} \partial_{x_j} \Big)$$

where g^{ij} are the components of inverse of the metric g_{ij} of M^3 and $g = \det(g_{ij})$. Using (3.5) and (3.6) we see that (3.3) is of the form

$$\sum_{i,j} g^{ij} (\theta_J)_{x_i x_j} + C_J\big(x, \theta, \theta_{x_1}, \theta_{x_2}, \theta_{x_3}\big) = 0,$$

where $C_J\colon \phi(U)\times\mathbb{R}^{4N}\to\mathbb{R}$ is given by

$$\begin{aligned} C_J(x,y,z_1,z_2,z_3) &= \frac{1}{\sqrt{g}}\sum_{i,j}(\sqrt{g}g^{ij})_{x_i}z_{jJ} + y_J\sum_{i,j,a}\Big(\sum_{k,I}\mu_{ik}\langle e_{ja},A_I\rangle z_{kI}\Big)^2 \\ &\quad -\sum_{I,K}\sum_{\substack{i,l,m\\ a\neq b,j\neq k}}\mu_{il}\mu_{im}\langle e_{ja,kb},A_J\rangle\langle e_{ja},A_K\rangle\langle e_{kb},A_I\rangle z_{mI}z_{lK}\end{aligned}$$

with $y=(y_1,\ldots,y_N)$, $z_i=(z_{i1},\ldots,z_{iN})$, $i,m,l\in\{1,2,3\}$ and $I,J,K\in\{1,\ldots,N\}$. Therefore, we have that the vector valued map $\theta=(\theta_1,\ldots,\theta_N)$ satisfies the elliptic equation

$$\mathcal{L}\theta=\sum_{i,j}A_{ij}(x)\theta_{x_ix_j}+C\big(x,\theta,\theta_{x_1},\theta_{x_2},\theta_{x_3}\big)=0,$$

where $A_{ij}=g^{ij}I_N$, I_N being the identity $N\times N$ matrix and $C=(C_1,\ldots,C_N)$. Moreover, we have from (3.1) that θ is constant on $\phi(L^2)$ and $\vec{n}(\theta)=0$ on $\phi(L^2)$ where $\vec{n}$ is a unit normal field to the surface $\phi(L^2)$ in $\mathbb{R}^3$.

Consider the Cauchy problem $\mathcal{L}\theta=0$ with the following initial conditions: θ is constant on $\phi(L^2)$ and $\vec{n}(\theta)=0$ on $\phi(L^2)$. According to the Cauchy-Kowalewsky theorem (cf. [62]) the problem has a unique solution if the surface $\phi(L^2)$ is noncharacteristic. This latter is satisfied if $Q(\vec{n})\neq 0$, where Q is the characteristic form given by

$$Q(\zeta)=\det(\Lambda(\zeta)),$$

where $\zeta=(\zeta_1,\zeta_2,\zeta_3)$ and

$$\Lambda(\zeta)=\sum_{i,j}g^{ij}\zeta_i\zeta_jI_N$$

is the symbol of the differential operator $\mathcal{L}$. That the surface $\phi(L^2)$ is noncharacteristic follows from

$$Q(\zeta)=\Big(\sum_{i,j}g^{ij}\zeta_i\zeta_j\Big)^N.$$

Because $C(x,y,0,0,0)=0$ the constant maps are solutions to the Cauchy problem. From the uniqueness part of the Cauchy-Kowalewsky theorem we conclude that the Gauss map γ is constant on an open subset of M^3, and that is not possible. ∎

Lemma 3.6. *The set $\mathcal{V}^0$ is empty.*

Proof: Suppose that $x_0\in\mathcal{V}^0$ and let Ω be an open neighborhood around x_0 such that $\nu=1$ on $\Omega\smallsetminus\{x_0\}$. Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in $\Omega\smallsetminus\{x_0\}$ converging to x_0. Let $e_j=e(x_j)\in T_{x_j}M$ be the sequence of unit vectors contained in the relative nullity distribution of f. By passing to a subsequence, if necessary, there is a unit vector $e_0\in T_{x_0}M$ such that $\lim e_j=e_0$. By continuity, the geodesic tangent to e_0 at x_0 is a leaf of relative nullity outside x_0. But this is clearly impossible in view of Proposition 1.1(iv). ∎

Lemma 3.7. *The foliation $\mathcal{F}$ of the nullity distribution extends analytically over the regular points of $\mathcal{A}$.*

Proof: First observe that the relative nullity distribution extends continuously over the regular points of $\mathcal{A}$. In fact, by the previous lemmas it remains to consider the case when Ω is an open subset of M^3 such that $\Omega \cap \mathcal{A}$ is a open segment in a straight line in the ambient space. But in this situation the result follows by a argument of continuity similar than in the proof of Lemma 3.6.

Let Ω be an open subset of $M^3 \setminus \mathcal{A}$ and let $\{e_1, e_2, e_3 = e\}$ be a local frame on Ω as in the proof of Lemma 2.1. Consider the map $F\colon \Omega \to \mathbb{S}^{n-1}$ into the unit sphere given by $F = f_* e$. A straightforward computation using (2.8), (2.9) and (2.3) gives that its tension field

$$\tau(F) = \sum_{j=1}^{3} \left(\overline{\nabla}_{F_* e_j} F_* e_j - F_* \nabla_{e_j} e_j\right)$$

vanishes, where $\overline{\nabla}$ denotes the Levi-Civita connection of $\mathbb{S}^{n-1}$. Hence F is a harmonic map. Since $\mathcal{A} = \mathcal{V}^1$ we obtain that F is real analytic in view Lemma 1.14. ∎

Lemma 3.8. *The set $\mathcal{A}$ has no singular points.*

Proof: According to Lemmas 3.5 and 3.6 the set $\mathcal{A}$ only contains subvarieties of dimension one with possible isolated singular points. Thus, by Lemma 3.7, the set of regular points of $\mathcal{A}$ just contains segments of straight lines.

Assume that the set S of singular points of $\mathcal{A}$ is not empty. Then, a singular point $x_0 \in S \subset \mathcal{A}$ should be the intersection of transversal regular arcs γ_1, γ_2 of $\mathcal{A}$. We know from previous lemmas that the line bundle $\mathcal{D}|_{M \setminus \mathcal{A}}$ extends to a line bundle on $M \setminus S$ which we denote again by $\mathcal{D}$. Take a local section e of the extended line bundle $\mathcal{D}$ on an open subset U of $M \setminus S$.

We claim that the integral curves of e are geodesics. Indeed, we know that the integral curves of e on $U \cap (M \setminus \mathcal{A})$ are geodesics, hence by continuity we have that $\nabla_e e = 0$ on $U \cap (M \setminus S)$. Now we claim that e is tangent to the regular arcs of $\mathcal{A}$. Assume to the contrary that there exists a point x on a regular arc c such that $e(x)$ is transversal to c at that point x and let γ be the geodesic passing through x with speed $e(x)$. Since $\mathcal{A} = \mathcal{V}^1$, there exists $\epsilon > 0$ such that $\gamma(s) \in M \setminus \mathcal{A}$, for $s \in (0, \epsilon]$. This means that $\nu\big(\gamma(s)\big) = 1$, for $s \in (0, \epsilon]$. On the other hand $\nu\big(\gamma(0)\big) = \nu(x) = 3$, that contradicts Proposition 1.1(iv).

In this way, we obtain a geodesic flow tangent to $\mathcal{D}$ on $M \setminus S$, hence $U \setminus \{x_0\}$ foliates by geodesics having e as tangent vector field. Therefore, there exists a geodesic γ that intersects either γ_1 or γ_2 at a point y and the image $\gamma \setminus \{y\}$ lies in $M \setminus \mathcal{A}$. This contradicts again with Proposition 1.1(iv). ∎

The proof of our main result relies heavily on Proposition 1.10 that is a consequence of the Omori-Yau maximum principle; see [3, Theorem 28] or [42, Lemma 4.1].

Proof of Theorem 3.1: Without loss of generality we may assume that M^3 is oriented by passing to the oriented double cover if necessary, see for details see [59, pp. 95-101]. It follows from Lemma 3.7 and 3.8 that the almost complex structure J is globally defined and that $\|\mathcal{C}\|^2 = u^2 + v^2$ is real analytic on M^3. From Lemma 2.1 and (2.2) it follows that

$$\begin{aligned}\Delta(u^2+v^2) &= 2\|\nabla u\|^2 + 2\|\nabla v\|^2 \\ &\geq 2\big(e(u)\big)^2 + 2\big(e(v)\big)^2 \\ &= 8u^2v^2 + 2(u^2-v^2)^2 \\ &= 2(u^2+v^2)^2.\end{aligned}$$

Hence, the following differential inequality holds

$$\Delta\|\mathcal{C}\|^2 \geq 2\|\mathcal{C}\|^4.$$

Using Proposition 1.10 we derive that $\mathcal{C} = 0$ and from Corollary 1.5 we obtain the desired splitting result. ∎

Proof of Corollary 3.2: The Omori-Yau maximum principle holds on M^m under the assumption on the scalar curvature (see [2] or [3, Theorem 2.4]) or if the immersion f is proper (see [3, Theorem 2.5]). Hence we can apply the same arguments as in the proof of Theorem 3.1, due to the validity of the Omori-Yau maximum principle on M^m. ∎

CHAPTER 4

Minimal immersions with relative nullity in Euclidean spheres

In this chapter, we investigate minimal isometric immersions $f\colon M^m \to \mathbb{S}^n$, $m \geq 3$, into Euclidean spheres with index of relative nullity at least $m-2$ at any point. These are austere submanifolds in the sense of Harvey and Lawson [44] and were studied by Bryant [7]. Austerity is a pointwise algebraic condition on the second fundamental form. It requires that the nonzero principal curvatures in any normal direction occur in oppositely signed pairs, hence, the austerity condition is, aside from surfaces, much stronger than minimality.

For any dimension and codimension there is an abundance of examples of non-complete minimal isometric immersions $f\colon M^m \to \mathbb{S}^n$ fully described by Dajczer and Florit [15] in terms of a class of surfaces, called elliptic, for which the ellipse of curvature of a certain order is a circle at any point. Under the assumption of completeness, it turns out that any submanifold is either totally geodesic or has dimension three. In the latter case there are plenty of examples, even compact ones. Under the mild assumption that the Omori-Yau maximum principle holds on the manifold, a trivial condition in the compact case, we provide a complete local parametric description of the submanifolds in terms of 1-isotropic surfaces in Euclidean space. These are the minimal surfaces for which the standard ellipse of curvature is a circle at any point. For these surfaces, there exists a Weierstrass type representation that generates all simply-connected ones.

4.1 The main result

The completeness of M^m imposes f to be totally geodesic unless $m = 3$. On the other hand, it follows from the subsequent results of this chapter that any example for $m = 3$ can locally be constructed as follows:

Let $g\colon L^2 \to \mathbb{R}^{n+1}$, $n \geq 4$, be an elliptic surface whose first curvature ellipse is a circle. Then, the map $\psi_g\colon T^1L \to \mathbb{S}^n$ defined on the unit tangent bundle of L^2

$$T^1L = \{(x, w)\colon x \in L,\ w \in TL,\ \|w\| = 1\}$$

and given by

$$\psi_g(x, w) = g_* w \tag{4.1}$$

parametrizes (outside singular points) a minimal immersion $f\colon M^3 \to \mathbb{S}^n$ with index of relative nullity at least one at any point. More precisely, we prove the following:

Theorem 4.1. *Let $f\colon M^m \to \mathbb{S}^n$, $m \geq 3$, be a minimal isometric immersion with index of relative nullity at least $m-2$ at any point. If M^m is complete, then f is totally geodesic unless $m = 3$. Moreover, if the Omori-Yau maximum principle holds on M^3, then, along an open dense subset, f is locally parametrized by (4.1) where $g\colon L^2 \to \mathbb{R}^{n+1}$ is a minimal surface whose first curvature ellipse is always a circle.*

A minimal surface $g\colon L^2 \to \mathbb{R}^n$ whose first curvature ellipse is a circle at any point is called a 1-isotropic surface. The above result should be complemented by the fact that there is a Weierstrass type representation, see (1.8), that generates all simply-connected 1-isotropic surfaces.

Examples: There are plenty of compact examples of three-dimensional minimal submanifolds in spheres with index of relative nullity at least one at any point:

(i) Hopf lifts: If $g\colon L^2 \to \mathbb{CP}^n$, $n \geq 2$, is a substantial holomorphic curve, then the Hopf fibration $\mathcal{H}\colon \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ induces a circle bundle M^3 over L^2. This lifting induces an immersion $f\colon M^3 \to \mathbb{S}^{2n+1}$ such that $g \circ \pi = \mathcal{H} \circ f$, where $\pi\colon M^3 \to L^2$ is the projection map. Such submanifolds (called Hopf lifts) are minimal with index of relative nullity at least 1 if $n = 2$ (see [29]) or $n = 3$ (see [51]). Moreover, if L^2 is compact, then M^3 is also compact.

(ii) Tubes over minimal 2-spheres: Due to the work of Calabi, Chern, Barbosa and others, see [8], [12], [4], it is known that minimal 2-spheres in spheres are pseudoholomorphic (isotropic) in substantial even codimension. Calabi [8] proved that any such surface in $\mathbb{S}^{2n}$ is nicely curved if its area is $2\pi n(n+1)$, and Barbosa showed [4] that the space of these surfaces is diffeomorphic to $SO(2n+1,\mathbb{C})/SO(2n+1,\mathbb{R})$. According to Proposition 4.4 below such surfaces produce examples of compact three-dimensional minimal submanifolds in $\mathbb{S}^{2n}$ with index of relative nullity one.

Recall that a triple $(M, \langle\cdot\,,\cdot\rangle, J)$ is called *almost Hermitian* if $(M, \langle\cdot\,,\cdot\rangle)$ is an even dimensional Riemannian manifold and J is an almost complex structure on M that is orthogonal with respect to the metric, i.e.,

$$\langle X, Y\rangle = \langle JX, JY\rangle, \quad \text{for} \quad X, Y \in \mathfrak{X}(M).$$

An almost Hermitian manifold $(M^{2m}, \langle\cdot\,,\cdot\rangle, J)$ is called *nearly Kähler* if ∇J is a skew bilinear form with values on TM, i.e.,

$$(\nabla_X J)X = 0, \quad \text{for} \quad X \in TM.$$

The 6-sphere $\mathbb{S}^6$ inherits a nearly Kähler structure from its natural inclusion to imaginary Octonions. It is thus endowed with an almost symplectic structure, given by a non-degenerate 2-from ω which is not closed. The canonical almost complex structure J on $\mathbb{S}^6$, is compatible with ω in the sence that

$$\langle X, Y\rangle = \omega(X, JY).$$

We define *Lagrangian* submanifolds of $\mathbb{S}^6$ as 3-dimensional submanifolds on which ω vanishes.

Among the second family of examples given above in (ii), there are the submanifolds produced from pseudoholomorphic surfaces $g\colon \mathbb{S}^2 \to \mathbb{S}^6$ with area 24π which are holomorphic with respect to the nearly Kähler structure in $\mathbb{S}^6$. For instance, this is the situation of the Veronese surface in $\mathbb{S}^6$. In this case, the compact submanifolds M^3 are Lagrangian (also called totally real) in $\mathbb{S}^6$; see [29].

Corollary 4.2. *Let $f\colon M^3 \to \mathbb{S}^6$ be an isometric immersion with index of relative nullity at least one at any point. Assume that f is Lagrangian with respect to the nearly Kähler structure in $\mathbb{S}^6$. If M^3 is complete and the Omori-Yau maximum principle holds, then f is locally parametrized by (4.1) along an open dense subset of M^3 where g is a 2-isotropic surface in $\mathbb{R}^6$ (respectively, $\mathbb{R}^7$) and f is substantial in $\mathbb{S}^5$ (respectively, $\mathbb{S}^6$).*

That the surface g is 2-isotropic means that it is 1-isotropic and that the second ellipse of curvature is also a circle at any point. Hence, in the case of $\mathbb{R}^6$ we have that g is congruent to a holomorphic curve in $\mathbb{C}^3 \equiv \mathbb{R}^6$.

It follows from the results in [27] that the universal cover of any of the complete three-dimensional submanifolds considered in Theorem 4.1 admits a one-parameter associated family of isometric immersions of the same type. Moreover, that family is trivial if and only if the (local) generating minimal surface is congruent to a holomorphic curve. We refer to Lotay [51] for a discussion about the existence of such an associated family in the case of yet another family of examples.

4.2 The local case

We discuss next two alternative ways to parametrically describe, at least locally, all spherical three-dimensional minimal submanifolds of rank two in spheres. This follows from the results in [15] bearing in mind that a submanifold is minimal in a sphere if and only if the cone shaped over it is minimal in the Euclidean space.

Let $g\colon L^2 \to \mathbb{R}^{n+1}$, $n \geq 4$, be an elliptic surface and let T^1L denote its unit tangent bundle.

Proposition 4.3. *If the ellipse $\mathcal{E}_1^g$ is a circle, then the map $\psi_g\colon T^1L \to \mathbb{S}^n$ given by*

$$\psi_g(x, w) = g_* w$$

is a minimal immersion with index of relative nullity $\nu \geq 1$ outside the subset of singular points, which correspond to points where $\dim N_1^g = 0$. Moreover, a regular point $(x, w) \in T^1L$ is totally geodesic for ψ_g if and only if $\dim N_2^g(x) = 0$. Conversely, any three-dimensional minimal submanifold in the sphere with $\nu = 1$ at any point can be at least locally parametrized in this way.

The above parametrization (used for Theorem 4.1) is called the *bipolar parametrization* in [15] because g is a bipolar surface to ψ_g. The parametrization in the sequel (used for the examples discussed above) was called in [15] the *polar parametrization*.

Let $g\colon L^2 \to \mathbb{Q}_{1-c}^{2n+2c}$ $(c = 0, 1)$, $n \geq 2$, be a nicely curved elliptic surface and let $M^3 = UN_{\tau_g}^g$ stand for the unit bundle of $N_{\tau_g}^g$.

Proposition 4.4. *If the ellipse $\mathcal{E}_{\tau_g-1}^g$ is a circle, then $\phi_g\colon M^3 \to \mathbb{S}^{2n+c}$ given by $\phi_g(x, w) = w$ is a minimal immersion of rank two and polar surface g. Conversely, any minimal submanifold M^3 in $\mathbb{S}^{2n+c}$ of rank two can locally be parametrized in this way.*

4.3 The complete case

We first observe that for complete submanifolds of rank at most two the interesting case is the three-dimensional one. The remaining of the section is devoted to the study of the latter case.

Proposition 4.5. *Let $f\colon M^m \to \mathbb{S}^n$, $m \geq 3$, be a minimal isometric immersion with index of relative nullity $\nu \geq m - 2$ at any point. If M^m is complete, then f is totally geodesic unless $m = 3$.*

The above is an immediate consequence of the following result due to Ferus [34] (see [14, Lemma 6.16] where the proof holds regardless the codimension) since due to minimality we cannot have points with index of relative nullity $m-1$. For the sake of completeness we will provide an alternative proof of this but first we need the following result.

Lemma 4.6. *Let $f\colon M^m \to \mathbb{S}^n$ be an isometric immersion and let $U \subset M^m$ be an open subset where the index of relative nullity is constant and the leaves of the relative nullity distribution $\mathcal{D}$ are complete. Then, for any $x_0 \in U$ and $T_0 \in \mathcal{D}(x_0)$ the only possible real eigenvalue of C_{T_0} is zero.*

Proof: If C_{T_0} has nonzero real eigenvalues $\lambda_1, \ldots, \lambda_k$, set

$$\tan\phi = \min_{1\le j\le k} |\lambda_j^{-1}|,$$

where $\phi \in (-\pi/2, \pi/2)$. Let $\gamma\colon \mathbb{R} \to M^m$ be the geodesic such that $\gamma(0) = x_0$ and $\gamma'(0) = T_0$. From Proposition 1.2(i) we have that

$$\nabla_{\gamma'} C_{\gamma'} = C_{\gamma'}^2 + I. \tag{4.2}$$

The endomorphism $I - \tan t\, C_{T_0}$ is invertible for any $t \in (-\phi, \phi)$. The unique solution of the differential equation (4.2) for $t \in (-\phi, \phi)$ with initial data $C_{\gamma'(0)} = C_{T_0}$ is

$$C_{\gamma'(t)} = \mathcal{P}_0(t)(\tan t\, I + C_{T_0})(I - \tan t\, C_{T_0})^{-1}\mathcal{P}_0^{-1}(t),$$

where $\mathcal{P}_0(t)$ denotes the parallel transport along γ from the point $\gamma(0) = x_0$ to $\gamma(t)$. If follows that either $1/\tan(\phi - t)$ or $-1/\tan(\phi + t)$ must be an eigenvalue of $C_{\gamma'(t)}$ for $t \in (-\phi, \phi)$. On one hand, these quantities become unbounded when the parameter t tends to ϕ and $-\phi$, respectively. On the other hand, by our completeness assumption $C_{\gamma'(t)}$ is well defined for any $t \in \mathbb{R}$, and this is a contradiction. ∎

Proof of Proposition 4.5: Let $U \subset M^m$ be the open subset where f has rank two. Clearly, if U is empty the minimality condition implies that f is totally geodesic. Thus, we may assume that f is not totally geodesic in which case the leaves of relative nullity in U are complete.

The codimension of $\ker C$ in $\mathcal{D}$ satisfies $\operatorname{codim}\ker C \le 1$. If otherwise, we have from $\dim \operatorname{End}(\mathcal{D}^\perp) = 4$ and $\dim \operatorname{Sym}(\mathcal{D}^\perp) = 3$ that the image $\operatorname{Im}(C) \subset \operatorname{End}(\mathcal{D}^\perp)$ contains a non trivial self-adjoint endomorphism, in contradiction to Lemma 4.6. From (1.5) we obtain

$$\nabla_S C_S = C_S^2 + C_{\nabla_S S} + I. \tag{4.3}$$

In particular, it follows that $\operatorname{codim}\ker C = 1$.

Suppose that $m > 3$. By the above there exists a unit vector field $T_0 \in \mathcal{D}$ spanning $(\ker C)^\perp$. This implies that $\dim\ker C = m-3 > 0$. Hence, the tangent bundle splits as

$$TM = \mathcal{D}^\perp \oplus \operatorname{span}\{T_0\} \oplus \ker C.$$

Moreover there exists a unit vector field $S \in \ker C$. Then (4.3) takes the form

$$\langle \nabla_S S, T_0\rangle C_{T_0} + I = 0$$

which contradicts Lemma 4.6. ∎

Let $\mathcal{A}$ denote the set of totally geodesic points of f. By Proposition 1.1(iv) the relative nullity distribution $\mathcal{D}$ is a line bundle on $M^3 \smallsetminus \mathcal{A}$. Being f real analytic, the square of the norm of the second fundamental form is a real analytic function and hence $\mathcal{A}$ is a real analytic set. According to Theorem 1.15 the set $\mathcal{A}$ locally decomposes as

$$\mathcal{A} = \mathcal{V}^0 \cup \mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3$$

where each $\mathcal{V}^d$, $0 \le d \le 3$, is either empty or a disjoint finite union of d-dimensional real analytic subvarieties.

We want to show that $\mathcal{A} = \mathcal{V}^1$ unless f is just a totally geodesic three-sphere in $\mathbb{S}^n$. After excluding the latter case, we have from the real analyticity of f that $\mathcal{V}^3$ is empty. We will proceed now following ideas as the ones developed in Euclidean case. In fact, we only sketch the proof of the following fact, which is similar to the proof of Lemma 3.5.

Lemma 4.7. *The set $\mathcal{V}^2$ is empty.*

Proof: We only have to show that there are no regular points in $\mathcal{V}^2$. Suppose that a regular point $x_0 \in \mathcal{V}^2$ exists. Let $\Omega \subset M^3$ be an open neighborhood of x_0 such that $L^2 = \Omega \cap \mathcal{A}$ is an embedded surface. Let $e_1, e_2, e_3, \xi_1, \ldots, \xi_{n-3}$ be an orthonormal frame adapted to M^3 along Ω near x_0. The coefficients of the second fundamental form are

$$h^a_{ij} = \langle \alpha_f(e_i, e_j), \xi_a\rangle,$$

where $1 \le i, j, k \le 3$ and $1 \le a, b \le n-3$.

The Gauss map $\gamma\colon M^3 \to Gr(4, n+1)$ of f is a map into the Grassmannian of oriented 4-dimensional subspaces in $\mathbb{R}^{n+1}$ defined by

$$\gamma = f \wedge f_* e_1 \wedge f_* e_2 \wedge f_* e_3.$$

We can regard $Gr(4, n+1)$ as a submanifold in $\wedge^4\mathbb{R}^{n+1}$ via the map for the Plücker embedding. Then

$$\gamma_* e_i = \sum_{j,a} h^a_{ij} f \wedge e_{ja},$$

where e_{ja} is taken by replacing f_*e_j with ξ_a in $e_1 \wedge e_2 \wedge e_2$. Moreover, it easy to see that the Gauss map satisfies the partial differential equation

$$\Delta\gamma + \|\alpha_f\|^2\gamma = \sum_{i,a\neq b,j\neq k} h^a_{ij}h^b_{ik} f \wedge e_{ja,kb},$$

where $e_{ja,kb}$ is obtained by replacing f_*e_j with ξ_a and f_*e_k with ξ_b in $f_*e_1 \wedge f_*e_2 \wedge f_*e_3$. Hence, we may write the latter equation in the form

$$\Delta\gamma(x) + \|\gamma_*(x)\|^2\gamma(x) + G(x, \gamma_*) = 0,$$

where G is real analytic with $G(\cdot\,, 0) = 0$. Clearly, we have that γ is constant along L^2 and $\gamma_*(\vec{\eta}) = 0$ on L^2, where $\vec{\eta}$ is a unit normal of $L^2 \subset M^3$. Then, from the uniqueness part of the Cauchy-Kowalewsky theorem (cf. [62]) we deduce that the Gauss map γ must be constant. This would imply that $f(M)$ is a three-dimensional totally geodesic sphere which contradicts our assumption. ∎

Lemma 4.8. *The set $\mathcal{V}^0$ is empty.*

Proof: The proof is identical to the one give in Euclidean space, see Lemma 3.6. ∎

Lemma 4.9. *The foliation of relative nullity extends analytically over the regular points in the set $\mathcal{A}$.*

Proof: Observe that the distribution $\mathcal{D}$ extends continuously over the regular points of $\mathcal{A}$. In fact, by the previous lemmas it remains to consider the case when Ω is an open subset of M^3 such that $\Omega \cap \mathcal{A}$ is a open piece of a great circle in the ambient space. But in this situation the result follows by a argument of continuity similar than in the proof of Lemma 4.8. The rest of the proof is similar to Euclidean case, see Lemma 3.7 for details.

Lemma 4.10. *The set $\mathcal{A}$ has no singular points.*

Proof: Let $x_0 \in \mathcal{A}$ be a singular point. From Lemmas 4.7 and 4.8 the set $\mathcal{A}$ contains subvarieties of dimension one. It is well known that the singular points of such curves are isolated. Moreover, according to Lemma 4.9 the set of regular points of $\mathcal{A}$ contains geodesic curves of the relative nullity foliation. Hence x_0 is an intersection of such geodesic curves. The rest of the proof is carried out as in Lemma 3.8.

4.4 The proofs

Proof of Theorem 4.1: By Proposition 4.5 we only have to consider the case $m = 3$. We distinguish two cases.

Case $\mathcal{A} = \emptyset$. At first suppose that the line bundle $\mathcal{D}$ is trivial with e a unit global section. By Lemma 2.1 there exist harmonic functions $u, v \in C^\infty(M)$ such that $\mathcal{C} = vI - uJ$.

We claim that u is nowhere zero. To the contrary suppose that $u(x_0) = 0$ at $x_0 \in M^3$. Let $\gamma\colon \mathbb{R} \to M^3$ the maximal integral curve of e emanating from x_0. The second equation in (2.2) gives that u must vanish along γ. Thus the first equation in (2.2) reduces to $v'(s) = v^2(s) + 1$, where $v(s) = v(\gamma(s))$ is an entire function. But this is a contradiction since this equation has no entire solutions. In the sequel, we assume that $u > 0$. Using (2.2) and $u > 0$, one can easily see that

$$\begin{aligned}\Delta\big((u-1)^2 + v^2\big) &= 2(\|\nabla u\|^2 + \|\nabla v\|^2) \geq 2((e(u))^2 + (e(v))^2)\\ &\geq 2\big((u-1)^2 + v^2\big)^2,\end{aligned}$$

where in the last inequality we used that $u > 0$. Hence, from Proposition 1.10 we obtain

$$\mathcal{C} = -J.$$

Let $\mathcal{U} \subset M^3$ be the open dense subset where f is nicely curved. Let $U \subset \mathcal{U}$ be an open connected subset $\mathcal{U}$ that is the saturation of a simply connected cross section $L^2 \subset U$ to the relative nullity foliation. Hereafter we work on U where f is nicely curved. Hence polar and bipolar surfaces of $f|_U$ are well defined.

Let h be a polar surface to $f|_U$. We have seen that the almost complex structure J on $\mathcal{D}^\perp$ induces an almost complex structure $\tilde{J}$ on TL defined by $P\tilde{J} = JP$, where $P\colon TL \to \mathcal{D}^\perp$ is the orthogonal projection. Moreover, h is elliptic with respect to $\tilde{J}$ and (1.10) holds. In addition, it follows from Proposition 4.4 that $\mathcal{E}^h_{\tau_h - 1}$ is a circle.

We claim that the last curvature ellipse $\mathcal{E}^h_{\tau_h}$ of h is also a circle. In that case the bipolar surface $g\colon L^2 \to \mathbb{R}^{n+1}$ to f is 1-isotropic, and we are done. Observe that

$$N^h_{\tau_h} = \operatorname{span}\{\xi, \eta\},$$

where $\xi = f_* e|_{L^2}$ and $\eta = f|_{L^2}$. Using $\mathcal{C} = -J$, we obtain that

$$\xi_* = f_*|_{\mathcal{D}^\perp} \circ J \circ P. \tag{4.4}$$

Consider vector fields $X_1, \ldots, X_{\tau_h}, Y \in TL$. Since $N^h_{\tau_h - 1} = N^f_0 = f_*(\mathcal{D}^\perp)$, we have

$$\alpha_h^{\tau_h}(X_1, \ldots, X_{\tau_h}) = f_* Z$$

for some $Z \in \mathcal{D}^\perp$. Keeping in mind the bundle isometries, we obtain that

$$\alpha_h^{\tau_h + 1}(X_1, \ldots, X_{\tau_h}, Y) = \big(\nabla^{h\perp}_Y \alpha_h^{\tau_h}(X_1, \ldots, X_{\tau_h})\big)_{N^h_{\tau_h}} = \big(\tilde{\nabla}_Y f_* Z\big)_{N^h_{\tau_h}}.$$

Taking into account (4.4) we see that

$$\begin{aligned}\alpha_h^{\tau_h+1}(X_1,\ldots,X_{\tau_h},Y) &= \langle\tilde{\nabla}_Y f_*Z,\xi\rangle\xi + \langle\tilde{\nabla}_Y f_*Z,\eta\rangle\eta\\ &= -\langle f_*Z,\xi_*Y\rangle\xi - \langle f_*Z,\eta_*Y\rangle\eta\\ &= -\langle Z,JPY\rangle\xi - \langle Z,PY\rangle\eta.\end{aligned}$$

Recall that the almost complex structure $J^h_{\tau_h}$ on $N^h_{\tau_h}$ is given by

$$J^h_{\tau_h}\alpha_h^{\tau_h+1}(X_1,\ldots,X_{\tau_h},Y) = \alpha_h^{\tau_h+1}(X_1,\ldots,X_{\tau_h},\tilde{J}Y).$$

Since

$$\alpha_h^{\tau_h+1}(X_1,\ldots,X_{\tau_h},Y) = -\langle Z,JPY\rangle\xi - \langle Z,PY\rangle\eta$$

and

$$\alpha_h^{\tau_h+1}(X_1,\ldots,X_{\tau_h},\tilde{J}Y) = \langle Z,PY\rangle\xi - \langle Z,JPY\rangle\eta,$$

we have that $\alpha_h^{\tau_h+1}(X_1,\ldots,X_{\tau_h},Y)$ and $\alpha_h^{\tau_h+1}(X_1,\ldots,X_{\tau_h},\tilde{J}Y)$ are perpendicular of the same length. Thus $J^h_{\tau_h}$ is orthogonal, and proves the claim.

Finally, if the line bundle $\mathcal{D}$ is not trivial, it suffices to argue for a 2-fold covering $\Pi\colon \tilde{M}^3 \to M^3$ such that the nullity distribution $\tilde{\mathcal{D}}$ of $\tilde{f} = f\circ\Pi$ is a trivial line bundle and $\Pi_*\tilde{\mathcal{D}} = \mathcal{D}$, see [59, pp. 95-101] for details.

Case $\mathcal{A}\neq\emptyset$. We have seen that the relative nullity distribution $\mathcal{D}$ can be extended analytically to a line bundle on M^3, denoted again by $\mathcal{D}$, over the set of totally geodesic points $\mathcal{A}$. Without loss of generality, we may assume that there is a global unit section e spanning $\mathcal{D}$, since otherwise we can pass to the 2-fold covering space

$$\tilde{M}^3 = \big\{(x,w) : x\in M^3, w\in\mathcal{D}(x) \text{ and } \|w\| = 1\big\}$$

and argue as in the previous case. From Lemma 2.1, we know that there exist harmonic functions $u,v\in C^\infty(M^3\smallsetminus\mathcal{A})$ such that (2.1) holds on $M^3\smallsetminus\mathcal{A}$. By previous results the functions u and v can be extended analytically to harmonic functions on the entire M^3. Moreover, since u is positive on $M^3\smallsetminus\mathcal{A}$ and $\mathcal{A}$ consists of geodesic curves, by continuity we get that $u\geq 0$ on M^3. Then $\|\mathcal{C}+J\|^2$ is globally well defined and, arguing as in the previous case, we conclude again that $\mathcal{C} = -J$ on M^3. The remaining of the proof now goes as before. ∎

Proof of Corollary 4.2: By a result of Ejiri [31, Theorem 1] we have that f is minimal. Let e_1,e_2,e_3 be a local orthonormal tangent frame such that $e_3\in\mathcal{D}$. Since f is Lagrangian, we have that Je_1,Je_2,Je_3 is an orthonormal frame in the normal bundle of f. Moreover, it is well known that the 3-linear tensor h given by

$$h(e_i,e_j,e_k) = \langle\alpha_f(e_i,e_j),Je_k\rangle,\quad i,j,k\in\{1,2,3\},$$

is fully symmetric. Away from the totally geodesic points, using the symmetry of h and the minimality of f we obtain that $\alpha_f(e_1, e_1)$ and $\alpha_f(e_1, e_2)$ are perpendicular to each other and have the same length. Hence the ellipse $\mathcal{E}_1^f$ is a circle.

Suppose at first that f is substantial in $\mathbb{S}^6$. Assume that the submanifold is the saturation of a fixed cross section L^2 to the relative nullity foliation and denote by $h\colon L^2 \to \mathbb{S}^6$ the polar surface to f. From Proposition 1.8 we obtain that h is 1-isotropic. Proceeding as in the proof of Theorem 4.1, we deduce that the second ellipse of h is also a circle. Therefore, h is pseudoholomorphic and any bipolar surface g to f is 2-isotropic in $\mathbb{R}^7$.

Now we consider the case where f is substantial in $\mathbb{S}^5$. Consider a fixed cross section L^2 to the relative nullity foliation and let $h\colon L^2 \to \mathbb{R}^6$ be a polar surface to f. As in the previous case, we obtain that h must be isotropic. Therefore, any bipolar surface g to f is an isotropic surface in $\mathbb{R}^6$. ∎

CHAPTER 5

Minimal immersions with relative nullity in hyperbolic space

This chapter will be divided in three parts.

At first, we study complete minimal isometric immersions $f\colon M^m \to \mathbb{Q}_c^n$ in space forms initiated in [21] for sectional curvature $c = 0$ and continued in [22] for $c > 0$ (see Chapters 3 and 4). The basic hypothesis is that the index of relative nullity satisfies $\nu \geq m-2$ everywhere. The goal is to conclude that under some reasonable assumption the submanifold has to be of a simple geometric type other than totally geodesic. For instance, under the hypothesis that the Omori-Yau maximum principle holds on the manifold, we showed in Chapter 3 that the Euclidean submanifold has to be a $(m-2)$-cylinder.

In the second section, we provide a parametrization of all minimal submanifolds M^m of rank two lying in hyperbolic space $\mathbb{H}^n$ through k-regular elliptic surfaces. Using this parametrization, and the results in [9], [32] and [47] one can construct many complete examples of any dimension other than generalized cones.

The last section, is devoted to minimal submanifolds in hyperbolic space with rank three or four. Explicit examples of minimal submanifolds in the hyperbolic space are rare and new examples are certainly welcome. In this direction, we introduce a new class of minimal submanifolds $F\colon M^n \to \mathbb{H}^{n+2}$, $n \geq 3$, in the hyperbolic space that are $(n-2)$-ruled. If in addition the manifold is simply connected, then we prove that F allows a one-parameter associate family of equally ruled minimal isometric deformations that are genuine.

5.1 Minimal immersions of three dimensional submanifolds

In any of the two cases already studied, namely the Euclidean and spherical case, the proofs reduce to analyze the situation of the three dimensional submanifolds. In fact, for submanifolds in spheres only this case turned out to be possible. For complete minimal immersions $f\colon M^m \to \mathbb{H}^n$ the condition that the index of relative nullity satisfies $\nu \geq m-2$ turns out to be quite less restrictive than in the previously studied cases. Nevertheless, we have reasons to believe that the manifold being three-dimensional is still quite special and this is why this case allows a characterization of a class of submanifolds that is contained in the following description. In fact, in this section we prove the following result for complete minimal three dimensional submanifolds in hyperbolic space $f\colon M^3 \to \mathbb{H}^n$ under the assumption that $\nu \geq 1$.

Theorem 5.1. *Let $f\colon M^3 \to \mathbb{H}^n$ be a minimal isometric immersion with index of relative nullity at least $\nu \geq 1$ at any point. Assume that M^3 is complete with scalar curvature bounded from below. Then f is either totally geodesic or a generalized cone over a complete minimal surface with bounded Gauss curvature lying in an equidistant submanifold of $\mathbb{H}^n$.*

Notice that generalized cones over minimal surfaces contained in the other two types of umbilical submanifolds are not part of the theorem. In fact, if the surface lies inside a geodesic sphere then the generalized cone is never complete, whereas if it lies in a horosphere then the scalar curvature of the cone is unbounded.

Like it happens for $c \geq 0$, in the present case where $c < 0$ there are plenty of local examples other than generalized cones. As a matter of fact, a local parametrization of all minimal submanifolds $f\colon M^m \to \mathbb{H}^n$ with index of relative nullity $\nu = m-2$ was given in [49] in terms of certain elliptic spacelike surfaces in either the de Sitter space or the Lorentzian flat space according to $n-m$ being even or odd, respectively (see section 5.2). Moreover, from the results in [9], [32] and [47] it is clear that this parametrization can be used to construct complete examples of any dimension other than generalized cones.

5.1.1 Generalized cones

In this section, we find sufficient conditions for an isometric immersion into the hyperbolic space to be globally a generalized cylinder. First, we recall from Subsection 1.6 the definition of generalized cone.

Let $g\colon L^{m-k} \to Q_c^{n-k}$ be an isometric immersion into a totally umbilical submanifold Q_c^{n-k} of the hyperbolic space and $i\colon Q_c^{n-k} \to \mathbb{H}^n$ the umbilical inclusion. The normal bundle of $h = i \circ g \colon L^{m-k} \to \mathbb{H}^n$ splits orthogonally as

$$N_h L = i_* N_g L \oplus N_i Q_c^{n-k},$$

where $L = L^{m-k}$ and $N_i Q_c^{n-k}$ is regarded as a subbundle of $N_h L$. Consider $G \colon N_i Q_c^{n-k} \to \mathbb{H}^n$ the map defined by

$$G(x, w) = \exp_{g(x)} w,$$

where exp denotes the exponential map of $\mathbb{H}^n$. We denote by M^m the open subset of $N_i Q_c^{n-k}$ where G is an immersion, endowed with the metric induced by the map G. The *generalized cone* in $\mathbb{H}^n$ over $g \colon L^{m-k} \to Q_c^{n-k}$ is the isometric immersion $F_g \colon M^m \to \mathbb{H}^n$, defined by $F_g = G|_{M^m}$.

The following proposition characterizes generalized cones over a minimal surface lying into an umbilical submanifold.

Proposition 5.2. *Let $g \colon L^2 \to Q_c^{n-\nu}$ be a minimal surface into an umbilical submanifold $Q_c^{n-\nu}$ of $\mathbb{H}^n$. Then*

(i) The generalized cone $F_g \colon M^m \to \mathbb{H}^n$, $m = 2 + \nu$, over g is a minimal immersion with index of relative nullity at least ν at any point.

(ii) The map G is an immersion if and only if $Q_c^{n-\nu}$ is a totally geodesic submanifold of either an equidistant hypersurface or a horosphere in $\mathbb{H}^n$. In that case M^m is complete if and only if L^2 is complete. Moreover, if $Q_c^{n-\nu}$ is contained in an equidistant (respectively, horosphere) hypersurface then the scalar curvature of M^m is bounded (respectively, unbounded) along each fiber of the normal bundle of the umbilical inclusion $i \colon Q_c^{n-\nu} \to \mathbb{H}^n$.

Proof: Let $i \colon Q_c^{n-\nu} \to \mathbb{H}^n$ be a complete simply connected umbilical submanifold. Then let $\eta_1, \eta_2, \ldots, \eta_\nu$ be a global orthonormal frame of the normal bundle of i such that η_1 points in the direction of the mean curvature vector field H.

Since the normal bundle $N_i Q_c^{n-\nu}$ is a trivial vector bundle we have that the map $G \colon L^2 \times \mathbb{R}^\nu \to \mathbb{H}^n$, is given parametrically by

$$G(x, t_1, t_2, \ldots, t_\nu) = \cosh t_\nu \, f_{\nu-1}(x) + \sinh t_\nu \, \eta_\nu(x),$$

where f_j are defined inductively by $f_0 = g$ and

$$f_j = \cosh t_j f_{j-1} + \sinh t_j \eta_j, \quad 1 \leq j \leq \nu.$$

Set

$$h_j = \prod_{k=j+1}^{\nu} \cosh t_k, \quad 1 \leq j \leq \nu - 1$$

and

$$r = h_1(\cosh t_1 - \|H\| \sinh t_1).$$

A straightforward computation gives

$$\begin{aligned} G_*(X) &= r g_*(X), \quad X \in TL, \\ G_*(\partial_{t_j}) &= h_j(\sinh t_j\, f_{j-1} + \cosh t_j\, \eta_j), \quad 1 \le j \le \nu - 1, \\ G_*(\partial_{t_\nu}) &= \sinh t_\nu\, f_{\nu-1} + \cosh t_\nu\, \eta_\nu. \end{aligned}$$

It is clear that the map G is an immersion if and only if $\|H\| \le 1$, which in turn is equivalent to $Q_c^{n-\nu}$ being a totally geodesic submanifold of either an equidistant hypersurface or a horosphere in $\mathbb{H}^n$. Moreover, its second fundamental form is given by

$$\alpha_G(X, Y) = r\alpha_g(X, Y)$$

if $X, Y \in TL$, and the fact that the vectors $\partial_{t_1}, \dots, \partial_{t_\nu}$ belong to the relative nullity subspace. This proves part (i).

The induced metric on $L^2 \times \mathbb{R}^\nu$ is given by

$$\langle \cdot\,, \cdot \rangle_G = r^2 \langle \cdot\,, \cdot \rangle_g + \langle \cdot\,, \cdot \rangle_0,$$

where the Euclidean space $\mathbb{R}^\nu$ is equipped with the complete Riemannian metric

$$\langle \cdot\,, \cdot \rangle_0 = h_1^2 dt_1^2 + \cdots + h_{\nu-1}^2\, dt_{\nu-1}^2 + dt_\nu^2.$$

It follows from Lemma 7.2 in [5] that the manifold M^m is complete if and only if L^2 is complete.

Finally, the Gauss equation yields that the scalar curvature s of M^m is given by

$$\mathrm{s} = -m(m-1) - \frac{1}{r^2}\|\alpha_g\|^2.$$

This implies that the scalar curvature of M^m is bounded (respectively, unbounded) along each fiber of the normal bundle of the umbilical inclusion $i\colon Q_c^{n-\nu} \to \mathbb{H}^n$ if $Q_c^{n-\nu}$ is a totally geodesic submanifold of an equidistant hypersurface (respectively, horosphere). ∎

5.1.2 The proofs

Next we make use of the real analytic structure of a minimal submanifold in order to extend smoothly the relative nullity distribution to the totally geodesic points.

Let $\mathcal{A}$ denote the set of totally geodesic points of f. By Proposition 1.1, the relative nullity distribution $\mathcal{D}$ is a line bundle on $M^3 \setminus \mathcal{A}$. Since f is real analytic we have that $\mathcal{A}$ is a real analytic set. According to Theorem 1.15, it follows that $\mathcal{A}$ locally decomposes as

$$\mathcal{A} = \mathcal{V}^0 \cup \mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3,$$

where each set $\mathcal{V}^k$, $0 \leq k \leq 3$, is either empty or a disjoint finite union of k-dimensional real analytic subvarieties.

We can assume that $\mathcal{V}^3$ is empty since, otherwise, we already have by real analyticity that f is a totally geodesic submanifold.

Lemma 5.3. *The set $\mathcal{V}^0$ is empty.*

Proof: The proof goes as in Euclidean case, see Lemma 3.6. ∎

Lemma 5.4. *The set $\mathcal{V}^2$ is empty.*

Proof: The proof is similar to the spherical case. All we have to show is that $\mathcal{V}^2$ does not contain regular points. Suppose to the contrary and let $\Omega \subset M^3$ be an open neighborhood of a regular point $x_0 \in \mathcal{V}^2$ such that $L^2 = \Omega \cap \mathcal{A}$ is an embedded surface. Let $e_1, e_2, e_3, \xi_1, \ldots, \xi_{n-3}$ be an orthonormal frame adapted to M^3 along Ω near x_0.

The Gauss map $\gamma\colon M^3 \to Gr(4, n+1)$ takes values into the Grassmannian of oriented spacelike 4-dimensional subspaces in the Lorentzian space $\mathbb{L}^{n+1}$. Regarding $Gr(4, n+1)$ as a submanifold in $\wedge^4\mathbb{L}^{n+1}$ via the map for the Plücker embedding, we have that

$$\gamma = f \wedge f_*e_1 \wedge f_*e_2 \wedge f_*e_3.$$

The coefficients of the second fundamental form are

$$h^a_{ij} = \langle \alpha(e_i, e_j), \xi_a \rangle,$$

where from now on $1 \leq i, j, k \leq 3$ and $1 \leq a, b \leq n-3$. It is easy to see that

$$\gamma_* e_i = \sum_{j,a} h^a_{ij} f \wedge e_{ja}, \tag{5.1}$$

where e_{ja} is obtained by replacing f_*e_j with ξ_a in $f \wedge f_*e_1 \wedge f_*e_2 \wedge f_*e_3$. Then

$$\sum_i \langle \gamma_* e_i, \gamma_* e_i \rangle = \sum_{i,j,a} (h^a_{ij})^2 \langle f \wedge e_{ja}, f \wedge e_{ja} \rangle = -\|\alpha\|^2,$$

where the inner product of two simple 4-vectors in $\wedge^4\mathbb{L}^{n+1}$ is defined by

$$\langle a_1 \wedge a_2 \wedge a_3 \wedge a_4, b_1 \wedge b_2 \wedge b_3 \wedge b_4 \rangle = \det\big(\langle a_i, b_j \rangle\big).$$

A long but straightforward computation using the Codazzi equation yields

$$\Delta\gamma = -\|\alpha\|^2\gamma + \sum_{i,a\neq b,j\neq k} h^a_{ij}h^b_{ik}\, f\wedge e_{ja,kb}, \tag{5.2}$$

where $e_{ja,kb}$ is obtained by replacing f_*e_j with ξ_a and f_*e_k with ξ_b in $f_*e_1\wedge f_*e_2\wedge f_*e_3$.

We identify $\wedge^4\mathbb{L}^{n+1}$ with $\mathbb{L}^N_S$ where $N=\binom{n+1}{4}$ and $S=\binom{n}{3}$ and regard γ as a map from M^3 into $\mathbb{L}^N_S$. Denoting by $\{A_J\}_{J\in\{1,\dots,N\}}$ the corresponding base in $\mathbb{L}^N_S$, where $A_1,\dots,A_S$ are timelike and the remaining vectors spacelike, we have that

$$\gamma = \sum_{J=1}^N w_J A_J,$$

where $w_J = -\langle\gamma, A_J\rangle$ for $1\leq J\leq S$ and $w_J = \langle\gamma, A_J\rangle$ for $S+1\leq J\leq N$.

We obtain from (5.2) that

$$\Delta w_J = -\|\alpha\|^2 w_J - \epsilon_J \sum_{i,a\neq b,j\neq k} h^a_{ij}h^b_{ik}\langle f\wedge e_{ja,kb}, A_J\rangle, \tag{5.3}$$

where

$$\epsilon_J = \begin{cases} +1, & 1\leq J\leq S \\ -1, & S+1\leq J\leq N. \end{cases}$$

Take a local chart $\phi\colon U\to\mathbb{R}^3$ of coordinates $x=(x_1,x_2,x_3)$ on an open subset U of Ω and set

$$e_i = \sum_j \mu_{ij}\partial_{x_j}. \tag{5.4}$$

Setting $\theta_J = w_J\circ\phi^{-1}$, we obtain the map $\theta\colon \phi(U)\subset\mathbb{R}^3\to\mathbb{L}^N_S$ given by

$$\theta = \sum_{J=1}^N \theta_J A_J = (\theta_1,\dots,\theta_N).$$

Thus $\theta=\gamma\circ\phi^{-1}$ is the representation of the Gauss map with respect to the above mentioned charts. From (5.4) and

$$h^a_{ij} = \sum_J \langle f\wedge e_{ja}, A_J\rangle e_i(w_j)$$

we derive that

$$h^a_{ij} = \sum_{k,J} \mu_{ik}\langle f\wedge e_{ja}, A_J\rangle (\theta_J)_{x_k}. \tag{5.5}$$

Thus

$$\|\alpha\|^2 = \sum_{i,j,a}\Big(\sum_{k,J}\mu_{ik}\langle f\wedge e_{ja}, A_J\rangle(\theta_J)_{x_k}\Big)^2. \tag{5.6}$$

The Laplacian of M^3 is given by

$$\Delta = \frac{1}{\sqrt{g}}\sum_{i,j}\partial_{x_i}\Big(\sqrt{g}g^{ij}\partial_{x_j}\Big),$$

where g^{ij} are the components of the inverse of the metric g_{ij} of M^3 and $g = \det(g_{ij})$. Using (5.5) and (5.6), we see that (5.3) is of the form

$$\sum_{i,j} g^{ij}(\theta_J)_{x_ix_j} + C_J\big(x,\theta,\theta_{x_1},\theta_{x_2},\theta_{x_3}\big) = 0,$$

where $C_J\colon \phi(U)\times\mathbb{R}^{4N}\to\mathbb{R}$ is given by

$$C_J(x,y,z_1,z_2,z_3) = \frac{1}{\sqrt{g}}\sum_{i,j}(\sqrt{g}g^{ij})_{x_i}z_{jJ} + y_J\sum_{i,j,a}\Big(\sum_{k,I}\mu_{ik}\langle f\wedge e_{ja}, A_I\rangle z_{kI}\Big)^2$$
$$+\epsilon_J\sum_{I,K}\sum_{\substack{i,l,m\\ a\neq b, j\neq k}}\mu_{il}\mu_{im}\langle f\wedge e_{ja,kb}, A_J\rangle\langle f\wedge e_{ja}, A_K\rangle\langle f\wedge e_{kb}, A_I\rangle z_{mI}z_{lK}$$

with $y = (y_1,\ldots,y_N)$, $z_i = (z_{i1},\ldots,z_{iN})$, $i,m,l\in\{1,2,3\}$ and $I,J,K\in\{1,\ldots,N\}$. Let $A_{ij} = g^{ij}I_N$, I_N being the identity $N\times N$ matrix, $C = (C_1,\ldots,C_N)$ and $\vec{\eta}$ the unit normal field to the surface $\phi(L^2)$ in $\mathbb{R}^3$. Then, the vector valued map $\theta = (\theta_1,\ldots,\theta_N)$ satisfies the elliptic equation

$$\mathcal{L}\theta = \sum_{i,j}A_{ij}(x)\theta_{x_ix_j} + C\big(x,\theta,\theta_{x_1},\theta_{x_2},\theta_{x_3}\big) = 0$$

with initial conditions: θ is constant on $\phi(L^2)$ and $\theta_*(\vec{\eta}) = 0$ on $\phi(L^2)$, where $\vec{\eta}$ is a unit normal of $\phi(L^2)\subset R^3$.

According to the Cauchy-Kowalewsky theorem (cf. [62]) the above system has a unique solution if the surface $\phi(L^2)$ is noncharacteristic. This latter condition is satisfied if $Q(\vec{\eta})\neq 0$, where Q is the characteristic form given by

$$Q(\zeta) = \det(\Lambda(\zeta))$$

with $\zeta = (\zeta_1,\zeta_2,\zeta_3)$ and

$$\Lambda(\zeta) = \sum_{i,j}g^{ij}\zeta_i\zeta_jI_N$$

is the symbol of the differential operator $\mathcal{L}$. That the surface $\phi(L^2)$ is noncharacteristic follows from

$$Q(\zeta) = \Big(\sum_{i,j} g^{ij}\zeta_i\zeta_j\Big)^N.$$

Since $C(x, y, 0, 0, 0) = 0$ the constant maps satisfy the system. Due to uniqueness of solutions to the Cauchy problem, we deduce that the Gauss map γ is constant on an open subset of M^3 and that is not possible. ∎

Lemma 5.5. *The relative nullity distribution can be extended analytically over the regular points of the set $\mathcal{A}$.*

Proof: Clearly $\mathcal{D}$ extends continuously over the regular points of $\mathcal{A}$. Let $e_1, e_2, e_3 = e$ be a local orthonormal tangent frame on an open subset U of $M^3 \smallsetminus \mathcal{A}$ as in Lemma 2.1. We view e as a map $F\colon U \to T^1M$ into the unit tangent bundle of M^3 endowed with the Riemannian metric inherited from the Sasaki metric on TM. We argue that the map $F = e$ is harmonic. In fact, from (2.8), (2.9), (2.2) and (2.3) we obtain that

$$\begin{aligned}
\Delta F &= \sum_{i=1}^{3}\big(\nabla_{e_i}\nabla_{e_i}e - \nabla_{\nabla_{e_i}e_i}e\big)\\
&= -2\left(u^2+v^2\right)e\\
&= -\left(\|\nabla_{e_1}e\|^2 + \|\nabla_{e_2}e\|^2\right)e.
\end{aligned}$$

Hence the map F satisfies the differential equation

$$\Delta F + \|\nabla F\|^2 F = 0,$$

which is precisely the Euler-Lagrange equation for the energy functional of F (cf. [63, Proposition 1.1]). Thus $F\colon U \to T^1M$ is harmonic. Since $\mathcal{A} = \mathcal{V}^1$ we obtain that F is real analytic in view Lemma 1.14. ∎

Lemma 5.6. *The set $\mathcal{A}$ has no singular points.*

Proof: The proof is the same as in Lemma 3.8. ∎

Proof of Theorem 5.1: We have seen that the relative nullity distribution $\mathcal{D}$ extends to a global line bundle, also denoted by $\mathcal{D}$. By passing to the 2-fold covering, if necessary, we have that this line bundle is trivial. Thus it is spanned by a globally defined unit section e. Hence, there is a unique, up to sign, orthogonal almost complex structure $J\colon \mathcal{D}^\perp \to \mathcal{D}^\perp$. By Lemma 2.1 there are harmonic functions $u, v \in C^\infty(M)$ such that

$$C = vI - uJ.$$

To obtain the proof of the theorem all we have to show is that u vanishes. In fact, if that is the case then the result will follow from Propositions 1.6 and 5.2.

Making use of the equations (2.2) and that the functions u, v are harmonic, we obtain that

$$\begin{aligned}\Delta(u^2+v^2-1) &= 2\|\nabla u\|^2+2\|\nabla v\|^2\\ &\geq 2(e(u))^2+2(e(v))^2\\ &= 8u^2v^2+2(v^2-u^2-1)^2\\ &\geq 2(u^2+v^2-1)^2.\end{aligned}$$

Since the Ricci curvature of M^3 is bounded from below, then Proposition 1.10 applies and gives that $u^2+v^2\leq 1$. Hence u and v are bounded functions.

We claim that $v^2<1$. Suppose to the contrary that there is $x_0\in M^3$ such that $|v(x_0)|=1$. The maximum principle for harmonic functions (see Proposition 1.9) yields that $v=1$ or $v=-1$ everywhere. Hence $C=\pm I$. We have using (1.7) that

$$e(\|\alpha\|^2)=e\Big(\sum_{j=1}^{n-3}\mathrm{tr}(A^2_{\xi_j})\Big)=\sum_{j=1}^{n-3}\mathrm{tr}(\nabla_e A^2_{\xi_j})=2\sum_{j=1}^{n-3}\mathrm{tr}(A_{\xi_j}\circ C\circ A_{\xi_j})=\pm 2\|\alpha\|^2,$$

where $\xi_1,\dots,\xi_{n-3}$ is an orthonormal normal frame parallel along a geodesic integral curve γ of e. Thus

$$\|\alpha(\gamma(t))\|^2=ce^{\pm t}$$

where $c>0$ is a constant. Therefore $\|\alpha\|$ is unbounded along γ. Using equation (1.4) and the minimality of f we derive that the scalar curvature is given by

$$\mathrm{s}=-6-\|\alpha\|^2.$$

This clearly contradicts the assumption on the scalar curvature and proves the claim.

Let $\gamma\colon\mathbb{R}\to M^3$ be a unit speed geodesic contained in a leaf of the relative nullity foliation. Since $v^2<1$, we have from the first equation in (2.2) that

$$(v\circ\gamma)'=(v\circ\gamma)^2-(u\circ\gamma)^2-1\leq(v\circ\gamma)^2-1.$$

Hence the function $v\circ\gamma\colon\mathbb{R}\to(-1,1)$ is strictly decreasing and satisfies $\sup v\circ\gamma=1$ and $\inf v\circ\gamma=-1$. Thus the function v changes sign only once along each leaf of the relative nullity foliation. From the first equation in (2.2) and $v^2<1$ it follows that

$$e(v)=v^2-u^2-1<0.$$

Since 0 is a regular value of v, the level set $L^2=v^{-1}(0)$ is a 2-dimensional connected submanifold of M^3 and the map $\rho\colon L^2\times\mathbb{R}\to M^3$ defined by

$$\rho(x,t)=\exp_x te(x)$$

is a diffeomorphism. Notice that for $x \in v^{-1}(0)$ the integral curve of e, passing through $x = \rho(x, 0)$, is given by $\rho(x, t)$, $t \in \mathbb{R}$. Thus, $\rho_* \left(\frac{d}{dt}\right) = e$.

Consider the smooth function $\phi \colon L^2 \times \mathbb{R} \to \mathbb{R}$ given by

$$\phi \circ \rho^{-1} = \frac{-2v}{1 + u^2 + v^2 + \sqrt{(1 + u^2 + v^2)^2 - 4v^2}}.$$

Setting $\psi = \phi \circ \rho^{-1}$, we have that

$$\frac{\psi}{1 + \psi^2} = \frac{-v}{1 + u^2 + v^2}. \tag{5.7}$$

Differentiating (5.7) using (2.2) yields

$$\begin{aligned} \frac{e(\psi)(1 - \psi^2)}{(1 + \psi^2)^2} &= \frac{(1 + u^2 + v^2)^2 - 4v^2}{(1 + u^2 + v^2)^2} \\ &= 1 - 4\left(\frac{-v}{1 + u^2 + v^2}\right)^2 \\ &= 1 - 4\left(\frac{\psi}{1 + \psi^2}\right)^2. \end{aligned}$$

Hence,

$$e(\psi) = 1 - \psi^2. \tag{5.8}$$

Since ϕ vanishes on L^2 we obtain that $\phi(x, t) = \tanh t$. Thus ψ is bounded on M^3. Hence $\theta \in C^\infty(M)$ given by

$$\theta = u^2 + (v + \psi)^2$$

is also bounded. Using (2.2) and (5.8) we readily see that

$$\begin{aligned} e(\theta) &= 2ue(u) + 2(v + \psi)(e(v) + e(\psi)) \\ &= 4u^2 v + 2(v + \psi)(v^2 - u^2 - \psi^2) \\ &= 2(v - \psi)\theta. \end{aligned} \tag{5.9}$$

Since u and v are harmonic functions, we obtain that

$$\begin{aligned} \Delta\theta &= 2\|\nabla u\|^2 + 2(v + \psi)\Delta\psi + 2\|\nabla(v + \psi)\|^2 \\ &\geq 8u^2 v^2 + 2(v + \psi)\Delta\psi + 2(e(v) + e(\psi))^2 \\ &= 8u^2 v^2 + 2(v + \psi)\Delta\psi + 2(v^2 - u^2 - \psi^2)^2. \end{aligned} \tag{5.10}$$

On the other hand, it follows from (5.7) that

$$\frac{(1 - \psi^2)(1 + u^2 + v^2)^2}{(1 + \psi^2)^2}\nabla\psi = 2uv\nabla u - (1 + u^2 - v^2)\nabla v. \tag{5.11}$$

Using the harmonicity of u and v again, a straightforward computation gives

$$\begin{aligned}\frac{1-\psi^2}{2(1+\psi^2)^2}\Delta\psi &= \frac{v(1-3u^2+v^2)}{(1+u^2+v^2)^3}\|\nabla u\|^2 + \frac{2u(1+u^2-3v^2)}{(1+u^2+v^2)^3}\langle\nabla u,\nabla v\rangle \\ &\quad + \frac{v(3+3u^2-v^2)}{(1+u^2+v^2)^3}\|\nabla v\|^2 + \frac{3\psi-\psi^3}{(1+\psi)^3}\|\nabla\psi\|^2. \qquad (5.12)\end{aligned}$$

Since θ is bounded, by the Omori-Yau maximum principle there is a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in M^3 such that

$$(i)\ \lim\theta(x_j)=\sup\theta,\ \ (ii)\ \|\nabla\theta(x_j)\|\leq 1/j\ \text{ and }\ (iii)\ \Delta\theta(x_j)\leq 1/j. \qquad (5.13)$$

Taking a subsequence, we have that $\lim u(x_j)=u_0$, $\lim v(x_j)=v_0$ and $\lim\psi(x_j)=\psi_0$. Estimating at x_j and letting $j\to\infty$, we obtain from (i) and (ii) of (5.13) and (5.9) that

$$(v_0-\psi_0)\sup\theta=0.$$

We conclude that u has to vanish unless $v_0=\psi_0$.

Suppose now that $v_0=\psi_0$. We have from (5.7) that $v_0=\psi_0=0$. On the other hand, since the Ricci curvature of M^3 is bounded from below it follows from Proposition 1.11 that $\|\nabla u\|$ and $\|\nabla v\|$ are bounded. Hence, from (5.11), (5.12) and since $\psi_0=0$, we have that $\Delta\psi(x_j)$ is bounded. Passing to the limit and using part (iii) of (5.13), we obtain from (5.10) that $u_0=0$. It follows using part (i) of (5.13) that $\sup\theta=0$. Thus the function u vanishes, and this concludes the proof. ∎

5.2 Local parametrization

In this section, we locally parametrize all minimal submanifolds M^m of rank two lying in hyperbolic space $\mathbb{H}^n$ through k-regular elliptic surfaces. It turns out that rank two minimal submanifolds of odd codimension are parametrized via timelike bundles over k-regular elliptic surfaces lying in de Sitter space, whereas the ones lying in even codimension are parametrized by k-regular elliptic surfaces lying in Lorentz space.

Theorem 5.7. *Let $g\colon L^2 \to \mathbb{Q}^n_{c,1}$, $c = 0, 1$, be a k-regular elliptic surface. Assume that the ellipse $\mathcal{E}^g_k$ is circular. Then, the map $\psi_g\colon U_1\Lambda^g_k \to \mathbb{H}^{n+c-1}$ given by*

$$\psi_g(x, v) = v,$$

parametrizes at regular points, a rank two minimal submanifold in $\mathbb{H}^{n+c-1}$. The converse is also true, i.e., any nicely curved rank two minimal submanifold of hyperbolic space can be parametrized in this way at least locally.

We can construct k-regular elliptic surfaces in de Sitter space by the following procedure: Start with a substantial minimal isometric immersion $g\colon L^2 \to Q^{2k+3}$ lying into an umbilical submanifold Q^{2k+3} of hyperbolic space $\mathbb{H}^{2k+4}$. Clearly, the last normal space of g is a line bundle. Consider the polar surface $h : L^2 \to \mathbb{S}^{2k+4}_1$ associated with g. According to Proposition 1.8, h is k-regular elliptic surface and the ellipse of curvature $\mathcal{E}^h_k$ is circular.

Proof of Theorem 5.7: We first deal with the direct statement. Let $g\colon L^2 \to \mathbb{Q}^n_{c,1}$ be a k-regular elliptic surface with circular curvature ellipse $\mathcal{E}^g_k$.

We assume that $n = 2m + 2$ and set $\ell = m - k > 0$. For a unit $Z \in TL$ satisfying $\langle Z, JZ\rangle = 0$, consider the local tangent orthonormal frame $\{e_1 = Z,\, e_2 = JZ/\|JZ\|\}$. Let $\{\xi_1, \dots, \xi_{2\ell}\}$ be a local orthonormal frame of Λ^g_k, such that $\langle \xi_1, \xi_1\rangle = -1$ and $\langle \xi_j, \xi_j\rangle = 1$ for $2 \le j \le 2\ell$. We denote by $\tilde{\nabla}$ the connection on the induced bundle

$$g^*T\mathbb{Q}^n_{c,1} = N^g_0 \oplus \cdots \oplus N^g_k \oplus \Lambda^g_k,$$

with $N^g_0 = g_*TL$. We define recursively the following sequence of normal vector fields

$$\begin{aligned} f_0 &= \xi_1, \\ f_j &= \cosh t_j\, f_{j-1} + \sinh t_j\, \xi_{j+1} \quad \text{for} \quad 1 \le j \le 2\ell - 1. \end{aligned}$$

Then, the map $\psi_g\colon U_1\Lambda^g_k \to \mathbb{H}^{2m+c+1}$ can be parametrized by

$$\psi_g(x, t_1, \dots, t_{2\ell-1}) = f_{2\ell-1} = \cosh t_{2\ell-1}\, f_{2\ell-2} + \sinh t_{2\ell-1}\, \xi_{2\ell}.$$

We set

$$h_j = \prod_{s=j+1}^{2\ell-1} \cosh t_s \quad \text{for} \quad 1 \leq j \leq 2\ell - 2$$

and $h_{2\ell-1} = 1$. We can readily verify that the differential of ψ_g satisfies

$$\psi_{g_*}(\partial/\partial t_j) = h_j(\sinh t_j\, f_{j-1} + \cosh t_j\, \xi_{j+1}) \quad \text{for} \quad 1 \leq j \leq 2\ell - 1.$$

Notice that due to dimension reasons Λ_k^g is spanned by the linearly independent vector fields $\big\{f_{2\ell-1}, \psi_{g_*}(\partial/\partial t_1), \ldots, \psi_{g_*}(\partial/\partial t_{2\ell-1})\big\}$. We consider for $1 \leq s \leq \ell - 1$, the orthogonal projections

$$\mathcal{P}_s \colon g^*T\mathbb{Q}_{c,1}^n \to N_0^g \oplus \cdots \oplus N_s^g,$$

$$\mathcal{P}_{s-1}^\perp \colon g^*T\mathbb{Q}_{c,1}^n \to \big(N_0^g \oplus \cdots \oplus N_{s-1}^g\big)^\perp.$$

Set also for simplicity $w = f_{2\ell-1}$. Using that $\mathcal{P}_k^\perp(\tilde{\nabla}_X w)$ and $\psi_{g_*}(\partial/\partial t_i)$'s are both perpendicular to w for $1 \leq i \leq 2\ell-1$ and $X \in TL$, we obtain that there exist 1-forms λ_i such that

$$\mathcal{P}_k^\perp(\tilde{\nabla}_X w) = \sum_{i=1}^{2\ell-1} \lambda_i(X)\psi_{g_*}(\partial/\partial t_i).$$

For any $X \in TL$ we have

$$\begin{aligned} \psi_{g_*}(X) &= \tilde{\nabla}_X w = \mathcal{P}_k(\tilde{\nabla}_X w) + \mathcal{P}_k^\perp(\tilde{\nabla}_X w) \\ &= (\tilde{\nabla}_X w)_{N_k^g} + \sum_{i=1}^{2\ell-1} \lambda_i(X)\psi_{g_*}(\partial/\partial t_i), \end{aligned}$$

where $(\tilde{\nabla}_X w)_{N_k^g}$ denotes the orthogonal projection of $\tilde{\nabla}_X w$ to N_k^g. Hence,

$$\psi_{g_*}\big(X - \sum_{i=1}^{2\ell-1} \lambda_i(X)\partial/\partial t_i\big) = (\tilde{\nabla}_X w)_{N_k^g}.$$

Assume that $\ell \leq m-2$. At regular points, ψ_g parametrizes a $(2\ell+1)$-dimensional submanifold of hyperbolic space whose normal bundle is given by

$$N_{\psi_g} = \text{span}\,\{cg\} \oplus N_0^g \oplus \cdots \oplus N_{k-1}^g.$$

Choose a local orthonormal frame $\{\eta_1, \ldots, \eta_{2k}\}$ such that

$$N_s^g = \text{ span}\,\{\eta_{2s-1}, \eta_{2s}\}\,, \quad 1 \leq s \leq k.$$

Since the ellipse $\mathcal{E}_k^g$ is circular, the vector fields $\alpha_g^{k+1}(Z,\dots,Z)$ and $\alpha_g^{k+1}(JZ,Z,\dots,Z)$ are perpendicular to each other and they have the same non-zero length ρ. Hence, $\{\eta_{2k-1},\eta_{2k}\}$ can be chosen such that

$$\eta_{2k-1}=\frac{1}{\rho}\alpha_g^{k+1}(Z,\dots,Z) \quad\text{and}\quad \eta_{2k}=\frac{1}{\rho}\alpha_g^{k+1}(JZ,Z,\dots,Z).$$

For any $X\in TL$ we have

$$\begin{aligned}(\tilde\nabla_X w)_{N_k^g} &= \langle\tilde\nabla_X w,\eta_{2k-1}\rangle\eta_{2k-1}+\langle\tilde\nabla_X w,\eta_{2k}\rangle\eta_{2k}\\ &= -\langle w,\tilde\nabla_X\eta_{2k-1}\rangle\eta_{2k-1}-\langle w,\tilde\nabla_X\eta_{2k}\rangle\eta_{2k}\\ &= -\frac{1}{\rho}\langle w,\alpha_g^{k+2}(X,Z,\dots,Z)\rangle\eta_{2k-1}-\frac{1}{\rho}\langle w,\alpha_g^{k+2}(X,JZ,Z,\dots,Z)\rangle\eta_{2k}.\end{aligned}$$

Using the ellipticity of g and the above, we derive that $(\tilde\nabla_Z w)_{N_k^g}$ and $(\tilde\nabla_{JZ} w)_{N_k^g}$ are perpendicular to each other and they have the same non-zero length, say r. Consequently, the vector fields

$$\begin{aligned}X_1 &= \frac{1}{r}\big(Z-\sum_{i=1}^{2\ell-1}\lambda_i(Z)\partial/\partial t_i\big),\\ X_2 &= \frac{1}{r}\big(JZ-\sum_{i=1}^{2\ell-1}\lambda_i(JZ)\partial/\partial t_i\big),\end{aligned}$$

are orthonormal with respect to the induced metric of ψ_g and perpendicular to $\psi_{g*}(\partial/\partial t_i)$'s for $1\le i\le 2\ell-1$. The vector fields $\{\partial/\partial t_1,\dots,\partial/\partial t_{2\ell-1}\}$ belong to the relative nullity distribution of ψ_g in view of

$$\tilde\nabla_{\partial/\partial t_i}\psi_{g*}(\partial/\partial t_j)\in\Lambda_k^g \quad\text{for}\quad 1\le i,j\le 2\ell-1$$

and

$$\langle\tilde\nabla_{X_s}\psi_{g*}(\partial/\partial t_i),\eta\rangle=-\langle\psi_{g*}(\partial/\partial t_i),\tilde\nabla_{X_s}\eta\rangle=0,\quad s=1,2,$$

for $\eta\in N_{\psi_g}$, where in the last equality we have used that

$$\tilde\nabla_{X_s}\eta\in N_0^g\oplus\cdots\oplus N_k^g.$$

Notice that for $\xi\in\operatorname{span}\{cg\}\oplus N_0^g\oplus\cdots\oplus N_{k-2}^g$, the shape operators A_ξ of ψ_g vanish since

$$\begin{aligned}\langle A_\xi X_i,X_j\rangle_{\psi_g} &= \langle\alpha_{\psi_g}(X_i,X_j),\xi\rangle\\ &= \langle\tilde\nabla_{X_i}\psi_{g*}(X_j),\xi\rangle\\ &= \langle\tilde\nabla_{X_i}(\tilde\nabla_{X_j}w)_{N_k^g},\xi\rangle=0,\quad i,j=1,2,\end{aligned}\tag{5.14}$$

where in the last equality we have used that

$$\tilde{\nabla}_{X_i}(\tilde{\nabla}_{X_j} w)_{N_k^g} \in N_{k-1}^g \oplus N_k^g \oplus N_{k+1}^g.$$

We compute now the shape operators of ψ_g with respect to $\eta_{2k-3} = \alpha_g^k(Y_1, \dots, Y_k)$ and $\eta_{2k-2} = \alpha_g^k(Z_1, \dots, Z_k)$, where $Y_i, Z_i \in TL$, $i = 1, \dots, k$.

$$\begin{aligned}
\langle A_{\eta_{2k-3}} X_2, X_2 \rangle_{\psi_g} &= \langle \psi_{g*} A_{\eta_{2k-3}} X_2,\ \psi_{g*} X_2 \rangle \\
&= -\frac{1}{r^2} \langle \tilde{\nabla}_{JZ}\ \eta_{2k-3},\ (\tilde{\nabla}_{JZ} w)_{N_k^g} \rangle \\
&= \frac{1}{r^2} \langle \tilde{\nabla}_{JZ} \tilde{\nabla}_{JZ}\ \eta_{2k-3}, w \rangle \\
&= \frac{1}{r^2} \langle \tilde{\nabla}_{JZ} \tilde{\nabla}_{JZ}\ \alpha_g^k(Y_1, \dots, Y_k), w \rangle \\
&= \frac{1}{r^2} \langle \alpha_g^{k+2}(JZ, JZ, Y_1, \dots, Y_k), w \rangle \\
&= -\frac{1}{r^2} \langle \alpha_g^{k+2}(Z, Z, Y_1, \dots, Y_k), w \rangle \\
&= -\frac{1}{r^2} \langle \tilde{\nabla}_Z \tilde{\nabla}_Z\ \alpha_g^k(Y_1, \dots, Y_k), w \rangle \\
&= \frac{1}{r^2} \langle \tilde{\nabla}_Z\ \eta_{2k-3},\ (\tilde{\nabla}_Z w)_{N_k^g} \rangle \\
&= -\langle A_{\eta_{2k-3}} X_1, X_1 \rangle_{\psi_g}.
\end{aligned}$$

Similarly for η_{2k-2} we derive

$$\begin{aligned}
\langle A_{\eta_{2k-2}} X_2, X_2 \rangle_{\psi_g} &= -\frac{1}{r^2} \langle \tilde{\nabla}_{JZ}\ \eta_{2k-2},\ (\tilde{\nabla}_{JZ} w)_{N_k^g} \rangle \\
&= \frac{1}{r^2} \langle \tilde{\nabla}_{JZ} \tilde{\nabla}_{JZ}\ \alpha_g^k(Z_1, \dots, Z_k), w \rangle \\
&= \frac{1}{r^2} \langle \alpha_g^{k+2}(JZ, JZ, Z_1, \dots, Z_k), w \rangle \\
&= -\frac{1}{r^2} \langle \alpha_g^{k+2}(Z, Z, Z_1, \dots, Z_k), w \rangle \\
&= \frac{1}{r^2} \langle \tilde{\nabla}_Z\ \eta_{2k-2},\ (\tilde{\nabla}_Z w)_{N_k^g} \rangle \\
&= -\langle A_{\eta_{2k-2}} X_1, X_1 \rangle_{\psi_g}.
\end{aligned}$$

Hence,

$$\text{trace} A_{\eta_{2k-3}} = \text{trace} A_{\eta_{2k-2}} = 0.$$

If $\ell = m-1$, then the normal bundle of ψ_g is $N_{\psi_g} = \text{span}\{cg\} \oplus N_0^g$. If $c=1$ it follows from (5.14) that $A_g = 0$. Moreover for $i=1,2$ we have that

$$\begin{aligned}
\langle A_{g_*e_i} X_2, X_2\rangle_{\psi_g} &= -\frac{1}{r^2}\langle \alpha_g(JZ,e_i),\ (\tilde{\nabla}_{JZ} w)_{N_1^g}\rangle \\
&= \frac{1}{r^2}\langle \tilde{\nabla}_{JZ}\alpha_g(JZ,e_i), w\rangle \\
&= \frac{1}{r^2}\langle \alpha_g^3(JZ,JZ,e_i), w\rangle \\
&= -\frac{1}{r^2}\langle \tilde{\nabla}_Z\alpha_g(Z,e_i), w\rangle \\
&= \frac{1}{r^2}\langle \tilde{\nabla}_Z g_*e_i, (\tilde{\nabla}_Z w)_{N_1^g}\rangle \\
&= -\langle A_{g_*e_i} X_1, X_1\rangle_{\psi_g}
\end{aligned}$$

The latter imply that

$$\text{trace} A_{g_*e_1} = \text{trace} A_{g_*e_2} = 0.$$

As for the case where $\ell = m$ and $c=1$, we have that $N_{\psi_g} = \text{span}\{g\}$ and

$$\begin{aligned}
\langle A_g X_2, X_2\rangle_{\psi_g} &= -\frac{1}{r^2}\langle g_*JZ,\ (\tilde{\nabla}_{JZ} w)_{N_0^g}\rangle \\
&= \frac{1}{r^2}\langle \alpha_g(JZ,JZ), w\rangle \\
&= -\frac{1}{r^2}\langle \alpha_g(Z,Z), w\rangle \\
&= -\frac{1}{r^2}\langle \tilde{\nabla}_Z g_*Z, w\rangle \\
&= \frac{1}{r^2}\langle g_*Z,\ (\tilde{\nabla}_Z w)_{N_0^g}\rangle \\
&= -\langle A_g X_1, X_1\rangle_{\psi_g}.
\end{aligned}$$

The proof for the case where $n = 2m+1$ is carried out in a similar manner, the only difference being that Λ_k^g is now spanned locally by an orthonormal frame $\{\xi_1, \dots, \xi_{2\ell-1}\}$.

The proof of the converse will be divided in two parts, according to the parity of the codimension. At first let $f\colon M^m \to \mathbb{H}^{m+2k+1}$ be a substantial and nicely curved minimal isometric immersion with index of relative nullity $\nu = m-2$. Let L^2 be a cross section to the relative nullity foliation and consider the polar surface $g\colon L^2 \to \mathbb{S}_1^{m+2k+1}$ to f given by $g = \xi_{2k+1}$, where ξ_{2k+1} is a unit section of the last normal bundle of f. Then,

$$g_*(X) = \nabla_X^{\perp}\xi_{2k+1}, \quad X \in TL.$$

Moreover, according to Proposition 1.8, g is k-regular elliptic with respect to the almost complex structure $J_0^g = J_k^f$. The Lorentzian bundle Λ_k^g is given by

$$\Lambda_k^g = \operatorname{span}\{f\} \oplus f_*(\mathcal{D}).$$

We parametrize locally the manifold M^m via the map

$$T : M^m \to U_1\Lambda_k^g$$

given by

$$T(x) = \big(\pi(x), f(x)\big),$$

where $\pi\colon M^m \to L^2$ is the natural projection. Since

$$\psi_g\colon U_1\Lambda_k^g \to \mathbb{H}^{m+2k+1}$$

is given by

$$\psi_g(x, v) = v,$$

it follows that

$$f = \psi_g \circ T.$$

It remains to consider the case of a substantial and nicely curved minimal isometric immersion $f\colon M^m \to \mathbb{H}^{m+2k}$ with index of relative nullity $\nu = m-2$. Let L^2 be a cross section to the relative nullity foliation and consider a polar surface $g\colon L^2 \to \mathbb{L}^{m+2k+1}$ to f given by $g_*T_xL = N_{\tau_f}^f(x)$ up to parallel identification in $\mathbb{L}^{n+1}$, where $N_{\tau_f}^f(x)$ stands for the last normal plane bundle of f. From Proposition 1.8 we have that $J_k^g = J_0^f$. Using the minimality of f we deduce that the ellipse $\mathcal{E}_k^g$ is a circle. Moreover the Lorentzian bundle Λ_k^g is given by

$$\Lambda_k^g = \operatorname{span}\{f\} \oplus f_*(\mathcal{D}).$$

Define the maps

$$T\colon V \subset M^m \to U_1\Lambda_k^g$$
$$T(x) = \big(\pi(x), f(x)\big)$$

and

$$\psi_g\colon U_1\Lambda_k^g \to \mathbb{H}^{m+2k}$$
$$\psi_g(x, v) = v.$$

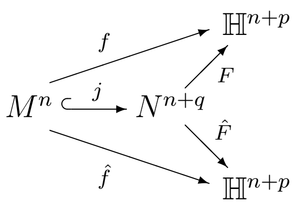
Then, $f = \psi_g \circ T$ completing the proof. ∎

5.3 Minimal submanifolds with rank three or four

In this section, we introduce a new class of minimal submanifolds $F\colon M^n \to \mathbb{H}^{n+2}$, $n \geq 3$, in the hyperbolic space that are $(n-2)$-ruled. This means that they carry an integrable tangent distribution of dimension $n-2$, whose leaves are mapped diffeomorphically by F onto open subsets of totally geodesic $(n-2)$-hyperbolic spaces of $\mathbb{H}^{n+2}$. Furthermore, we provide a characterization for them among $(n-2)$-ruled minimal submanifolds of rank 4 ($n \geq 4$) or 3 ($n = 3$). If the manifold is simply connected we show that it allows a one-parameter associate family of equally ruled minimal isometric deformations that are genuine. These results may be considered as a continuation of those in [25] and [26].

The notion of *genuine rigidity* was introduced in [16] and it is the proper setting to study rigidity problems for submanifolds of higher codimension. This concept relies on the idea that, as we discard congruent submanifolds when analyzing rigidity, we should also discard deformations that are induced by deformations of a bigger dimensional submanifold containing the original one.

An isometric immersion $\hat{f}\colon M^n \to \mathbb{H}^{n+p}$ is a *genuine deformation* of a given isometric immersion $f\colon M^n \to \mathbb{H}^{n+p}$, $p \geq 2$, if there is no open subset $U \subset M^n$ along which $f|_U$ and $\hat{f}|_U$ extend isometrically. That $f\colon M^n \to \mathbb{H}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{H}^{n+p}$ *extend isometrically* means that there is an isometric embedding $j\colon M^n \hookrightarrow N^{n+q}$, $1 \leq q < p$, into a Riemannian manifold N^{n+q} and there are isometric immersions $F\colon N^m \to \mathbb{H}^{n+q}$ and $\hat{F}\colon N^m \to \mathbb{H}^{n+q}$ such that $f = F \circ j$ and $\hat{f} = \hat{F} \circ j$, i.e., the following diagram commutes:



5.3.1 A class of ruled submanifolds

Let $g\colon L^2 \to \mathbb{H}^{n+2}$ be a substantial surface whose first normal space N_1^g is a plane bundle. Let $\pi\colon \Sigma_g \to L^2$ denote the vector bundle of rank $n-2$ whose fibers are the orthogonal complement in the normal bungle N_gL of g of its first normal bundle N_1^g.

Define $F_g\colon \Sigma_g \to \mathbb{H}^{n+2}$ to be the submanifold of $\mathbb{H}^{n+2}$ associated to g constructed by attaching at each point of the surface g the totally geodesic hyperbolic space $\mathbb{H}^{n-2}$ whose tangent space at that point is the fiber of Σ_g, that is,

$$(x, v) \in \Sigma_g \mapsto F_g(x, v) = \exp_{g(x)} v, \tag{5.15}$$

where exp is the exponential map of the hyperbolic space. By definition F_g is a $(n-2)$-ruled submanifold.

Consider the map $G\colon N^{n+1} = \mathbb{R} \times \Sigma_g \to \mathbb{L}^{n+3}$ given by

$$G(s, x, v) = sg(x) + v. \tag{5.16}$$

It is clear now that $F_g = G|_{M^n}$, where

$$M^n = \left\{(s, x, v) \in \mathbb{R} \times \Sigma_g\colon\ -s^2 + \|v\|^2 = -1\right\}.$$

We can locally parametrize M^n with $L^2 \times \mathbb{R}^{n-2}$ via the map $F_g\colon L^2 \times \mathbb{R}^{n-2} \to \mathbb{H}^{n+2}$ given by

$$F_g(x, \varphi, t_1, \ldots, t_{n-3}) = \cosh\varphi\, g(x) + \sinh\varphi\, w, \tag{5.17}$$

where $w = w(x, t_1, \ldots, t_{n-3})$ is a parametrization of the unit sphere inside the fiber of Σ_g at x.

If in addition g is 1-isotropic, then $L^2 \smallsetminus L_0$ consists of isolated points, where L_0 is the open subset of L^2 where the first normal space N_1^g is a plane bundle. It was shown in [25] that the vector bundle $N_1^g|_{L_0}$ extends smoothly to a plane bundle over L^2 that will be denoted by the same symbol N_1^g. Moreover, from the results in [46], there exists a method for constructing isotropic (superconformal) surfaces in hyperbolic space. In the sequel denote by $\mathcal{V}$ the vertical bundle of $\pi\colon \Sigma_g \to L^2$ given by $\mathcal{V} = \ker\pi_*$.

Lemma 5.8. *Let $g\colon L^2 \to \mathbb{H}^{n+2}$, $n \geq 3$, be a substantial oriented minimal surface. Then, $F_g\colon \Sigma_g \to \mathbb{H}^{n+2}$ is an immersion. In addition Σ_g, equipped with the induced metric, is complete if and only if L^2 is complete. Moreover for any $(x, v) \in \Sigma_g$ we have that*

$$F_{g*}(\mathcal{V})|_{(x,v)} = d(\exp_{g(x)})_v\big(\Sigma_g(x)\big)$$

holds up to parallel identification in $\mathbb{L}^{n+3}$.

Proof: Fix a point $(x_0, v_0) \in \Sigma_g$ and take $V \in T_{(x_0,v_0)}\Sigma_g$. Let $V = \gamma'(0)$, where $\gamma(t) = \big(c(t), v(t)\big)$ is a curve in Σ_g with $\gamma(0) = (x_0, v_0)$. Then,

$$F_g \circ \gamma(t) = \exp_{g\circ c(t)} v(t) = \begin{cases} g \circ c(t), & \text{if } \ v(t) = 0, \\ \cosh\|v(t)\|\, g\circ c(t) + \sinh\|v(t)\| \dfrac{v(t)}{\|v(t)\|}, & \text{if } \ v(t) \neq 0. \end{cases}$$

Observe that

$$\begin{aligned}\frac{dv}{dt}(t) &= \frac{\tilde{\nabla} v}{dt}(t) - \langle g_* c'(t), v(t)\rangle\, g\circ c(t)\\ &= -g_* A_{v(t)} c'(t) + \frac{\nabla^\perp v}{dt}(t) = \frac{\nabla^\perp v}{dt}(t),\end{aligned} \tag{5.18}$$

where $\tilde{\nabla}$ stands for the induced connection on the induced bundle on g.

We claim that

$$F_{g_*}(V) = \begin{cases} g_{*x_0} c'(0) + \dfrac{\nabla^\perp v}{dt}(0), & \text{if } \ v_0 = 0,\\ \cosh\|v_0\|\ g_{*x_0} c'(0) + d(\exp_{g(x_0)})_{v_0}\left(\dfrac{\nabla^\perp v}{dt}(0)\right), & \text{if } \ v_0 \neq 0. \end{cases} \tag{5.19}$$

To prove this claim we distinguish two cases.

Case a: We assume that $v_0 \neq 0$. Then $v(t) \neq 0$ for all $t \in (-\epsilon, \epsilon)$, where $\epsilon > 0$ is small enough. Using (5.18) and

$$\frac{d}{dt}\Big|_0 \big(\|v\|\big) = \frac{1}{\|v_0\|}\langle v_0, \frac{\nabla^\perp v}{dt}(0)\rangle,$$

we obtain that

$$\begin{aligned}(F_g\circ\gamma)'(0) &= \frac{\sinh\|v_0\|}{\|v_0\|}\langle v_0, \frac{\nabla^\perp v}{dt}(0)\rangle\, g(x_0) + \cosh\|v_0\|\, g_* c'(0) \\ &\quad + \frac{\|v_0\|\cosh\|v_0\| - \sinh\|v_0\|}{\|v_0\|^3}\langle v_0, \frac{\nabla^\perp v}{dt}(0)\rangle v_0 + \frac{\sinh\|v_0\|}{\|v_0\|}\frac{\nabla^\perp v}{dt}(0).\end{aligned} \tag{5.20}$$

For any $w \in T_{g(x_0)}\mathbb{H}^{n+2}$, we set

$$W(t) = v_0 + tw, \quad t \in (-\epsilon, \epsilon).$$

Then,

$$\big(\exp_{g(x_0)}\circ W\big)(t) = \cosh\|v_0 + tw\|\, g(x_0) + \frac{\sinh\|v_0 + tw\|}{\|v_0 + tw\|}\,(v_0 + tw).$$

Hence,

$$\begin{aligned}d\big(\exp_{g(x_0)}\big)_{v_0}(w) &= \frac{d}{dt}\Big|_0\big(\exp_{g(x_0)}\circ W\big) = \frac{\sinh\|v_0\|}{\|v_0\|}\langle v_0, w\rangle\, g(x_0)\\ &\quad + \big(\|v_0\|\cosh\|v_0\| - \sinh\|v_0\|\big)\frac{\langle v_0, w\rangle}{\|v_0\|^3} v_0 + \frac{\sinh\|v_0\|}{\|v_0\|} w.\end{aligned}$$

Our claim now follows from (5.20) and the above.

Case b: We assume that $v_0 = 0$. If $v(t) = 0$ for all $t \in (-\epsilon, \epsilon)$, then $F_{g_*}(V) = g_* c'(0)$.

Suppose now that $v(t) \neq 0$ for $t \neq 0$ and t small enough. Then we obtain

$$\begin{aligned}
\big(F_g \circ \gamma\big)'(0) &= \lim_{t\to 0} \frac{1}{t}\left\{\cosh\|v(t)\|\, g\big(c(t)\big) + \frac{\sinh\|v(t)\|}{\|v(t)\|} v(t) - g\big(c(0)\big)\right\} \\
&= \lim_{t\to 0}\left\{\frac{\cosh\|v(t)\|\, g\big(c(t)\big) - \cosh\|v(0)\|\, g\big(c(0)\big)}{t} + \frac{\sinh\|v(t)\|}{\|v(t)\|}\frac{v(t)}{t}\right\} \\
&= \frac{d}{dt}\Big|_0 \big(\cosh\|v\|\, g\circ c\big) + \frac{dv}{dt}(0) \\
&= g_{*x_0} c'(0) + \frac{d}{dt}\Big|_0 \big(\cosh\|v\|\big) g(x_0) + \frac{\nabla^\perp v}{dt}(0).
\end{aligned}$$

However, we have that

$$\begin{aligned}
\frac{d}{dt}\Big|_0 \big(\cosh\|v\|\big) &= \lim_{t\to 0}\frac{\cosh\|v(t)\| - 1}{t} \\
&= \lim_{t\to 0}\frac{\sinh\|v(t)\|}{\|v(t)\|}\langle v(t), \frac{\nabla^\perp v}{dt}(t)\rangle \\
&= \langle v_0, \frac{\nabla^\perp v}{dt}(0)\rangle = 0.
\end{aligned}$$

Hence, our claim follows.

Observe that $g_{*x_0} c'(0)$ is perpendicular to $d(\exp_{g(x_0)})_{v_0}\big(\frac{\nabla^\perp v}{dt}(0)\big)$. Then, we have that $V \in \ker F_{g*}|_{(x_0,v_0)}$ if and only if

$$c'(0) = 0 \quad \text{and} \quad \nabla^\perp_{d/dt} v(0) = 0.$$

This proves that F_g is an immersion.

For any local section η of N_1^g we have

$$\langle \frac{dv}{dt}(0), \eta(x_0)\rangle = -\langle v_0, \frac{\nabla^\perp(\eta\circ c)}{dt}(0)\rangle = -\langle v_0, \nabla^\perp_{c'(0)}\eta\rangle.$$

In particular, for any $V \in \mathcal{V}(x_0, v_0)$ it follows from (5.19) that

$$F_{g*}|_{(x_0,v_0)}(V) = d(\exp_{g(x_0)})_{v_0}\big(\frac{\nabla^\perp v}{dt}(0)\big),$$

with $\frac{\nabla^\perp v}{dt}(0) \in \Sigma_g(x_0)$. Hence, $F_{g_*}(\mathcal{V})|_{(x_0,v_0)} \subset d(\exp_{g(x_0)})_{v_0}\big(\Sigma_g(x_0)\big)$ and the proof follows by dimension reasons. ∎

The tangent bundle of the manifold Σ_g splits orthogonally as

$$T\Sigma_g = \mathcal{H} \oplus \mathcal{V},$$

where $\mathcal{H}$ is the horizontal bundle and $\mathcal{V}$ the vertical bundle. In addition, $\mathcal{V}$ can be orthogonally decomposed as $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$ where $\mathcal{V}^1$ denotes the plane bundle determined by

$$F_{g_*}(\mathcal{V}^1)|_{(x,v)} = d(\exp_{g(x)})_v \big(N_2^g(x)\big).$$

Hereafter, we assume that $g\colon L^2 \to \mathbb{H}^{n+2}$ is a substantial and nicely curved minimal surface. We choose positively oriented local orthonormal frames $\{e_1,\, e_2 = Je_1\}$, where J is the complex structure of L^2 induced by orientation in TL and $\{e_3, e_4\}$ of N_1^g such that

$$\alpha_g(e_1, e_1) = \kappa\, e_3 \quad \text{and} \quad \alpha_g(e_1, e_2) = \mu\, e_4,$$

where κ, μ are the semi-axis of the first ellipse of curvature. Let $\{e_5, \dots, e_{n+2}\}$ be a local orthonormal frame of Σ_g such that $\{e_{2s+1}, e_{2s+2}\}$ spans N_s^g for $1 \le s \le \tau_g - 1$, and the last normal bundle N_{τ_g} is spanned by $\{e_{n+1}, e_{n+2}\}$ for n even, whereas for odd n it holds $N^g_{\tau_g} = \operatorname{span}\{e_{n+2}\}$. We refer to $\{e_1, \dots, e_{n+2}\}$ as an *adapted frame* of g and consider the 1-forms ω_{ij} by

$$\omega_{ij}(X) = \langle \tilde\nabla_X e_i, e_j\rangle \quad X \in TL,$$

for $1 \le i, j \le n+2$, where $\tilde\nabla$ stands for the connection on the induced bundle $g^*T\mathbb{H}^{n+2}$. To simplify the notation we set $\omega_{ij}^k = \omega_{ij}(e_k)$, $\;k = 1, 2$.

Using the minimality of g and the symmetry of the third fundamental form α_g^3, we obtain

$$\alpha_g^3(e_1, e_1, e_1) = -\alpha_g^3(e_2, e_1, e_2).$$

This implies that

$$\omega_{45} = -\frac{1}{\lambda} * \omega_{35} \quad \text{and} \quad \omega_{46} = -\frac{1}{\lambda} * \omega_{36}, \tag{5.21}$$

where the quantity $\lambda = \mu/\kappa$ measures how much g deviates from being 1-isotropic and $*$ denotes the Hodge operator, i.e., $*\omega(X) = -\omega(JX)$, $X \in TL$. Set for simplicity

$$a_i = \omega_{35}^i \quad \text{and} \quad b_i = \omega_{36}^i \quad \text{for} \quad i = 1, 2.$$

In the sequel, we provide several proofs for $n \ge 4$, but similar arguments take care of the case $n = 3$. We choose a parametrization for the unit sphere in the fibers of Σ_g with parameters $t_1, \dots, t_{n-4} \in (0, \pi)$, $t_{n-3} \in (0, 2\pi)$, as follows

$$w = \sum_{j=1}^{n-2} u_j e_{j+4}, \tag{5.22}$$

where

$$u_1 = \sin t_1, \quad u_i = \prod_{j=1}^{i-1} \cos t_j \sin t_i \quad \text{for} \quad i = 2, \dots, n-3,$$

and

$$u_{n-2} = \prod_{j=1}^{n-3} \cos t_j.$$

Then, we have the following parametrization for F_g

$$F_g(x, \varphi, t_1, \dots, t_{n-3}) = \cosh\varphi\, g(x) + \sinh\varphi\, w, \tag{5.23}$$

with $\varphi \neq 0$. The differential of F_g satisfies

$$\begin{aligned} F_{g*}(\partial/\partial\varphi) &= \sinh\varphi\, g + \cosh\varphi\, w, \\ F_{g*}(\partial/\partial t_j) &= \sinh\varphi \sum_{i=1}^{n-2} \frac{\partial u_i}{\partial t_j} e_{i+4}, \quad 1 \leq j \leq n-3. \end{aligned}$$

Notice that

$$g_{ij} = \langle F_{g*}(\partial/\partial t_i), F_{g*}(\partial/\partial t_j)\rangle = \begin{cases} \sinh^2\varphi, & \text{if} \quad i = j = 1 \\ \sinh^2\varphi \displaystyle\prod_{k=1}^{j-1} \cos^2 t_k, & \text{if} \quad i = j \geq 2 \\ 0, & \text{if} \quad i \neq j. \end{cases}$$

Denote by $\mathcal{P}_1\colon N_gL \to N_1^g$ the orthogonal projection onto the first normal bundle and by $\mathcal{P}_2\colon N_gL \to \Sigma_g$ the orthogonal projection onto Σ_g. We define the functions

$$\phi_i = \sin t_1 \omega_{35}^i + \cos t_1 \sin t_2 \omega_{36}^i \quad \text{and} \quad \lambda_j(e_i) = \frac{1}{g_{jj}} \langle \nabla_{e_i}^\perp w, \frac{\partial w}{\partial t_j}\rangle \tag{5.24}$$

for $i = 1, 2$ and $1 \leq j \leq n-3$. From

$$\begin{aligned} \mathcal{P}_1(\nabla_{e_i}^\perp w) &= \sin t_1 \nabla_{e_i}^\perp e_5 + \cos t_1 \sin t_2 \nabla_{e_i}^\perp e_6 \\ &= -(\sin t_1 \omega_{35}^i + \cos t_1 \sin t_2 \omega_{36}^i) e_3 - (\sin t_1 \omega_{45}^i + \cos t_1 \sin t_2 \omega_{46}^i) e_4, \end{aligned}$$

we obtain

$$\mathcal{P}_1(\nabla_{e_1}^\perp w) = -\phi_1 e_3 - \frac{\phi_2}{\lambda} e_4, \quad \mathcal{P}_1(\nabla_{e_2}^\perp w) = -\phi_2 e_3 + \frac{\phi_1}{\lambda} e_4$$

and

$$\mathcal{P}_2(\nabla_{e_i}^\perp w) = \sum_{j=1}^{n-3} \lambda_j(e_i) \frac{\partial w}{\partial t_j}.$$

Consequently, from

$$F_{g*}(e_i) = \cosh\varphi\, g_*(e_i) + \sinh\varphi\, \mathcal{P}_1(\nabla^\perp_{e_i} w) + \sinh\varphi\, \mathcal{P}_2(\nabla^\perp_{e_i} w),$$

we derive that

$$\begin{aligned} r_1 F_*(X_1) &= \cosh\varphi\, g_*(e_1) - \phi_1 \sinh\varphi\, e_3 - \frac{\phi_2}{\lambda}\sinh\varphi\, e_4, \\ r_2 F_*(X_2) &= \cosh\varphi\, g_*(e_2) - \phi_2 \sinh\varphi\, e_3 + \frac{\phi_1}{\lambda}\sinh\varphi\, e_4, \end{aligned}$$

where

$$X_i = \frac{1}{r_i}(e_i - \sum_{j=1}^{n-3} \lambda_j(e_i)\partial/\partial t_j), \quad i = 1, 2, \tag{5.25}$$

and

$$r_1^2 = \cosh^2\varphi + (\phi_1^2 + \frac{\phi_2^2}{\lambda^2})\sinh^2\varphi, \quad r_2^2 = \cosh^2\varphi + (\frac{\phi_1^2}{\lambda^2} + \phi_2^2)\sinh^2\varphi.$$

If g is 1-isotropic, then we have $r_1 = r_2 = r$. We set

$$h_i = \begin{cases} (\sinh\varphi)^{-1}, & \text{if } \ i = 1 \\ (\sinh\varphi \prod_{j=1}^{i-1} \cos t_j)^{-1}, & \text{if } \ 2 \le i \le n-3. \end{cases}$$

The vector fields $\{X_1, X_2,\, \partial/\partial\varphi,\, h_1\,\partial/\partial t_1,\, \ldots,\, h_{n-3}\,\partial/\partial t_{n-3}\}$ constitute a local orthonormal frame with respect to the induced metric of F_g. Moreover, the normal space N_{F_g} of F_g is spanned by the orthogonal vector fields

$$\xi = \phi_1 \sinh\varphi\, g_*(e_1) + \phi_2 \sinh\varphi\, g_*(e_2) + \cosh\varphi\, e_3, \tag{5.26}$$

$$\eta = \frac{\phi_2}{\lambda}\sinh\varphi\, g_*(e_1) - \frac{\phi_1}{\lambda}\sinh\varphi\, g_*(e_2) + \cosh\varphi\, e_4. \tag{5.27}$$

Denote by $\mathcal{H}$ the distribution $\mathcal{H} = \operatorname{span}\{X_1, X_2\}$. Observe that

$$F_{g*}(\mathcal{H}) \oplus N_{F_g} M = \operatorname{span}\{g_* e_1, g_* e_2, e_3, e_4\}$$

and

$$F_{g*}(\mathcal{V}) \oplus \operatorname{span}\{F_g\} = \Sigma_g \oplus \operatorname{span}\{g\}.$$

Lemma 5.9. *Let* $g\colon L^2 \to \mathbb{H}^{n+2}$ *be a 1-isotropic surface. Then the second fundamental form of* F_g *vanishes on* $\operatorname{span} V \oplus \operatorname{span}\{\partial/\partial t_3, \ldots, \partial/\partial t_{n-3}\}$, *where*

$$V = (\psi_1\chi_2 - \psi_2\chi_1)\partial/\partial\varphi + (\phi_1\chi_2 - \phi_2\chi_1)h_1\,\partial/\partial t_1 + (\phi_1\psi_2 - \phi_2\psi_1)h_2\,\partial/\partial t_2,$$

and

$$\psi_i = \cosh\varphi(b_i \sin t_1 \sin t_2 - a_i \cos t_1), \quad \chi_i = b_i \cosh\varphi \cos t_2.$$

Moreover, the second fundamental form of F_g *restricted to* $\mathcal{H} \oplus \mathit{span}\{\partial/\partial\varphi\} \oplus \mathcal{V}^1$ *is given by*

$$rA_\xi = \begin{bmatrix} r(\kappa+\zeta_1) & r\,\zeta_2 & -\phi_1 & \psi_1 & -\chi_1 \\ r\,\zeta_2 & -r(\kappa+\zeta_1) & -\phi_2 & \psi_2 & -\chi_2 \\ -\phi_1 & -\phi_2 & 0 & 0 & 0 \\ \psi_1 & \psi_2 & 0 & 0 & 0 \\ -\chi_1 & -\chi_2 & 0 & 0 & 0 \end{bmatrix},$$

$$rA_\eta = \begin{bmatrix} r\,\zeta_2 & r(\kappa-\zeta_1) & -\phi_2 & \psi_2 & -\chi_2 \\ r(\kappa-\zeta_1) & -r\,\zeta_2 & \phi_1 & -\psi_1 & \chi_1 \\ -\phi_2 & \phi_1 & 0 & 0 & 0 \\ \psi_2 & -\psi_1 & 0 & 0 & 0 \\ -\chi_2 & \chi_1 & 0 & 0 & 0 \end{bmatrix},$$

with respect to the frame $\{X_1, X_2, \partial/\partial\varphi, h_1\partial/\partial t_1, h_2\partial/\partial t_2\}$, *where*

$$\begin{aligned}\zeta_i \;=\; & \frac{\sinh 2\varphi}{2r^2}\big(\sin t_1(-e_i(a_1) + a_2 B_i + b_1\omega_{56}^i) + \cos t_1 \sin t_2(-e_i(b_1) + b_2 B_i - a_1\omega_{56}^i) \\ & - \cos t_1 \cos t_2 \sin t_3(a_1\omega_{57}^i + b_1\omega_{67}^i) - \cos t_1 \cos t_2 \cos t_3 \sin t_4(a_1\omega_{68}^i + b_1\omega_{68}^i)\big),\end{aligned}$$

and $B_i = \omega_{12}^i + \omega_{34}^i$, $i = 1, 2$.

Proof: By a straightforward computation, we can verify that the Ricci equations

$$\langle R^\perp(e_1, e_2)e_a, e_b\rangle = \langle [A_{e_a}, A_{e_b}]e_1, e_2\rangle = 0 \tag{5.28}$$

for $a = 3, 4$ and $b = 5, 6$ are equivalent to

$$\begin{aligned} e_1(a_1) + e_2(a_2) - a_2B_1 + a_1B_2 - b_1\omega_{56}^1 - b_2\omega_{56}^2 &= 0, \\ e_1(a_2) - e_2(a_1) + a_1B_1 + a_2B_2 - b_2\omega_{56}^1 + b_1\omega_{56}^2 &= 0, \\ e_1(b_1) + e_2(b_2) - b_2B_1 + b_1B_2 + a_1\omega_{56}^1 + a_2\omega_{56}^2 &= 0, \\ e_1(b_2) - e_2(b_1) + b_1B_1 + b_2B_2 + a_2\omega_{56}^1 - a_1\omega_{56}^2 &= 0, \end{aligned}$$

whereas for $a = 3, 4$ and $b = 7, 8$, are equivalent to

$$\begin{aligned} a_1\omega_{57}^1 + a_2\omega_{57}^2 + b_1\omega_{67}^1 + b_2\omega_{67}^2 &= 0, \\ a_2\omega_{57}^1 - a_1\omega_{57}^2 + b_2\omega_{67}^1 - b_1\omega_{67}^2 &= 0, \\ a_1\omega_{58}^1 + a_2\omega_{58}^2 + b_1\omega_{68}^1 + b_2\omega_{68}^2 &= 0, \\ a_2\omega_{58}^1 - a_1\omega_{58}^2 + b_2\omega_{68}^1 - b_1\omega_{68}^2 &= 0. \end{aligned}$$

The vector fields $\{\partial/\partial t_3,\dots,\partial/\partial t_{n-3}\}$ belong to the relative nullity distribution of F_g since

$$\tilde\nabla_T F_{g*}(\partial/\partial t_s)\in\Gamma(\Sigma_g)$$

for $3\le s\le n-3$ and $T\in TM$.

Furthermore, a direct computation shows that

$$\alpha_{F_g}(\partial/\partial\varphi,\partial/\partial t_j)=\alpha_{F_g}(\partial/\partial t_i,\partial/\partial t_j)=\alpha_{F_g}(\partial/\partial\varphi,\partial/\partial\varphi)=0 \tag{5.29}$$

for $1\le i,j\le n-3$ and

$$\alpha_{F_g}(\partial/\partial\varphi,X_1)=-\frac{\phi_1}{r}\xi-\frac{\phi_2}{r}\eta,\qquad \alpha_{F_g}(\partial/\partial\varphi,X_2)=-\frac{\phi_2}{r}\xi+\frac{\phi_1}{r}\eta.$$

Moreover, using the Gauss and Weingarten formula's we obtain

$$\begin{aligned}\alpha_{F_g}(h_1\,\partial/\partial t_1,X_1)&=\frac{\psi_1}{r}\xi+\frac{\psi_2}{r}\eta,\qquad \alpha_{F_g}(h_1\,\partial/\partial t_1,X_2)=\frac{\psi_2}{r}\xi-\frac{\psi_1}{r}\eta,\\ \alpha_{F_g}(h_2\,\partial/\partial t_2,X_1)&=-\frac{\chi_1}{r}\xi-\frac{\chi_2}{r}\eta,\qquad \alpha_{F_g}(h_2\,\partial/\partial t_2,X_2)=-\frac{\chi_2}{r}\xi+\frac{\chi_1}{r}\eta,\end{aligned}$$

whereas, using the Ricci equations (5.28) we have that

$$\begin{aligned}\alpha_{F_g}(X_1,X_1)&=(\kappa+\zeta_1)\xi+\zeta_2\eta,\\ \alpha_{F_g}(X_1,X_2)&=\zeta_2\xi+(\kappa-\zeta_1)\eta,\\ \alpha_{F_g}(X_2,X_2)&=-(\kappa+\zeta_1)\xi-\zeta_2\eta.\end{aligned}$$

Observe that

$$\alpha_{F_g}(V,X_j)=0,\quad j=1,2.$$

Combining the latter with (5.29), we have that the relative nullity distribution of F_g is $\operatorname{span}V\oplus\operatorname{span}\{\partial/\partial t_3,\dots,\partial/\partial t_{n-3}\}$. The result is a direct consequence of the latter computations. ∎

For a $(n-2)$-ruled submanifold $F\colon M^n\to\mathbb{H}^{n+2}$ we denote by $\mathcal{H}$ the tangent distribution orthogonal to the rulings. An embedded surface $j\colon L^2\to M^n$ is called an *integral surface* of $\mathcal{H}$ if $j_*(T_xL)=\mathcal{H}(j(x))$ at every point $x\in L^2$. The following theorem describes locally all rank three or four minimal submanifolds M^n of the hyperbolic space $\mathbb{H}^{n+2}$ that are $(n-2)$-ruled.

5.3.2 Main results and proofs

Theorem 5.10. *Let $g\colon L^2\to\mathbb{H}^{n+2}$, $n\ge 3$, be a 1-isotropic substantial surface. Then the map $F_g\colon\Sigma_g\to\mathbb{H}^{n+2}$ parametrizes a $(n-2)$-ruled minimal submanifold M^n with* $\operatorname{rank}\rho=4$ *(unless $n=3=\rho$) on an open and dense subset of Σ_g.*

Conversely, let $F\colon M^n \to \mathbb{H}^{n+2}$ *be a* $(n-2)$*-ruled minimal immersion with* $\operatorname{rank}\rho = 4$ *(unless* $n = 3 = \rho$*) on an open and dense subset of* M^n*. Assume that* $j\colon L^2 \to M^n$ *is a totally geodesic integral surface of the 2-dimensional distribution* $\mathcal{H}$ *which is a global cross section to the rulings. Then the surface* $g = F \circ j\colon L^2 \to \mathbb{H}^{n+2}$ *is 1-isotropic and* F *can be parametrized by* (5.15)*.*

Proof: We first deal with the direct statement. Let $g\colon L^2 \to \mathbb{H}^{n+2}$ be a 1-isotropic and substantial surface. Notice that $g = F_g \circ j$, where $j\colon L^2 \to \Sigma_g$ is the inclusion given by $j(x) = (x, 0)$. Clearly, we have that j is an integral surface of the distribution orthogonal to the rulings that is totally geodesic and a global cross section to the rulings. The rest of the proof is an immediate consequence of Lemma 5.9.

We now prove the converse statement. Let $F\colon M^n \to \mathbb{H}^{n+2}$, $n \geq 4$ be a $(n-2)$-ruled minimal immersion with Gauss map of rank four everywhere. Denote by $\mathcal{H}$ the distribution which is orthogonal to the rulings and $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ in TM, i.e., $TM = \mathcal{H} \oplus \mathcal{V}$. The distribution $\mathcal{V}$ splits as $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$, with the fibers of $\mathcal{V}^0$ being the $(n-4)$-dimensional relative nullity leaves.

The normal space of $g = F \circ j$ at a point $x \in L^2$ is given by

$$N_g L(x) = F_*(j(x))\mathcal{V} \oplus N_F M(j(x)).$$

Using the Gauss equation and the fact that j is totally geodesic, we obtain

$$\alpha_g(X, Y) = \alpha_F\big(j_*(Y), j_*(X)\big) \tag{5.30}$$

for any $X, Y \in TL$. The latter and our assumptions imply that g is minimal.

Consider the subbundle $\pi\colon \Sigma_g \to L^2$ of the normal bundle $N_g L$, whose fiber at $x \in L^2$ is $F_*(j(x))\mathcal{V}$, and introduce the cone $\mathcal{C}F\colon \mathbb{R} \times M^n \to \mathbb{L}^{n+3}$ given by

$$\mathcal{C}F(t, p) = tF(p).$$

Pick $p \in M^n$, $x = \pi(p)$ and define

$$u(t, p) = -t\langle F(p), g \circ \pi(p)\rangle.$$

Then,

$$\mathcal{C}F(t, p) - \mathcal{C}F\big(u(t, p), j(x)\big) = \mathcal{C}F(t, p) - u(t, p) g \circ \pi(p) \in F_*(j(x))\mathcal{V},$$

because p and $j(x)$ belong to the same leaf of $\mathcal{V}$. Since $\mathcal{C}F$ maps locally diffeomorphically the leaves of the vertical bundle $\mathcal{V}$ onto affine subspaces, it follows that the map $T\colon \mathbb{R} \times M^n \to \mathbb{R} \times \Sigma_g$, given by

$$T(t, p) = \big(u(t, p), \pi(p), \mathcal{C}F(t, p) - u(t, p) g \circ \pi(p)\big),$$

is a local diffeomorphism. Therefore, the immersion $G=\mathcal{C}F\circ T^{-1}$ satisfies

$$G(s,x,v)=sg(x)+v,$$

i.e., G is of the form (5.16). The horizontal and vertical bundles satisfy

$$G_{*(s,x,v)}\mathcal{V}=\big(N_1^g(x)\big)^\perp\subset N_gL(x),\quad G_{*(s,x,v)}\mathcal{H}^G\subset g_*(T_xL)\oplus\big(\Sigma_g(x)\big)^\perp,$$
$$N_G\mathbb{R}\times\Sigma_g(s,x,v)\subset g_*(T_xL)\oplus\big(\Sigma_g(x)\big)^\perp$$

and (5.30) yields $N_1^g=\Sigma_g^\perp$. It is clear now that $F=G|_{M^n}$, where

$$M^n=\big\{(s,p,v)\in\mathbb{R}\times\Sigma_g\colon\ -s^2+\|v\|^2=-1\big\}\,,$$

through the local identification of $\mathbb{R}\times M^n$ with $\mathbb{R}\times\Sigma_g$ via the map T. Consequently, j is the zero section of Σ_g and F can be parametrized as $F_g\colon L^2\times\mathbb{R}^{n-2}\to\mathbb{H}^{n+2}$ given by (5.17).

It remains to prove that g is 1-isotropic. Let $\{e_1,e_2,e_3,\ldots,e_{n+2}\}$ be an adapted orthonormal frame along g and denote by

$$g_{ij}=\langle F_*(X_i),F_*(X_j)\rangle,\quad i,j=1,2,$$

the metric components of M^n, where X_i given by (5.25). Let

$$\beta^\xi_{ij}=\langle\tilde\nabla_{X_i}\xi,F_*(X_j)\rangle,\quad \beta^\eta_{ij}=\langle\tilde\nabla_{X_i}\eta,F_*(X_j)\rangle,$$

be the components of the second fundamental form of F, with respect to the normal directions ξ and η given by (5.26) and (5.27). Then, we have that

$$\begin{aligned}
g_{11}&=\cosh^2\varphi+\sinh^2\varphi\,(\phi_1^2+\frac{\phi_2^2}{\lambda^2}),\\
g_{12}&=\phi_1\phi_2\sinh^2\varphi\,(1-\frac{1}{\lambda^2}),\\
g_{22}&=\cosh^2\varphi+\sinh^2\varphi\,(\phi_2^2+\frac{\phi_1^2}{\lambda^2}).
\end{aligned}$$

Define for $i=1,2$ the functions

$$\begin{aligned}
G_i&=\cos t_1\sin t_2\omega^i_{56}+\cos t_1\cos t_2\sin t_3\omega^i_{57}+\cos t_1\cos t_2\cos t_3\sin t_4\omega^i_{58},\\
H_i&=\sin t_1\omega^i_{56}+\cos t_1\cos t_2\sin t_7\omega^i_{57}+\cos t_1\cos t_2\cos t_3\sin t_4\omega^i_{58}.
\end{aligned}$$

Using the Ricci equations we compute

$$\begin{aligned}
2\beta^\xi_{11}&=\sinh 2\varphi\big(e_1(\phi_1)-\phi_2\omega^1_{12}-\frac{\phi_2}{\lambda}\omega^1_{34}+a_1G_1+b_1H_1\big)-\kappa\big((\phi_1^2+\phi_2^2)\sinh^2\varphi+\cosh^2\varphi\big),\\
2\beta^\xi_{12}&=\sinh 2\varphi\big(e_1(\phi_2)+\phi_1\omega^1_{12}+\frac{\phi_1}{\lambda}\omega^1_{34}+a_2G_1+b_2H_1\big),\\
2\beta^\xi_{21}&=\sinh 2\varphi\big(e_2(\phi_1)-\phi_2\omega^2_{12}-\frac{\phi_2}{\lambda}\omega^2_{34}+a_1G_2+b_1H_2\big),\\
2\beta^\xi_{22}&=\sinh 2\varphi\big(e_2(\phi_2)+\phi_1\omega^2_{12}+\frac{\phi_1}{\lambda}\omega^2_{34}+a_2G_2+b_2H_2\big)+\kappa(\phi_1^2+\phi_2^2)\sinh^2\varphi
\end{aligned}$$

and

$$\begin{aligned}
2\beta^\eta_{11} &= \sinh 2\varphi\big(e_1(\psi_1) - \psi_2\omega^1_{12} + \phi_1\omega^1_{34} + \frac{1}{\lambda}a_2G_1 + \frac{1}{\lambda}b_2H_1\big),\\
2\beta^\eta_{12} &= \sinh 2\varphi\big(e_1(\psi_2) + \psi_1\omega^1_{12} + \phi_2\omega^1_{34} - \frac{1}{\lambda}a_1G_1 - \frac{1}{\lambda}b_1H_1\big) - \mu\cosh^2\varphi - \frac{\kappa}{\lambda}(\phi_1^2+\phi_2^2),\\
2\beta^\eta_{21} &= \sinh 2\varphi\big(e_2(\psi_1) - \psi_2\omega^2_{12} + \phi_1\omega^2_{34} + \frac{1}{\lambda}a_2G_2 + \frac{1}{\lambda}b_2H_2\big) - \mu\cosh^2\varphi - \frac{\kappa}{\lambda}(\phi_1^2+\phi_2^2),\\
2\beta^\eta_{22} &= \sinh 2\varphi\big(e_2(\psi_2) + \psi_1\omega^2_{12} + \phi_2\omega^2_{34} - \frac{1}{\lambda}a_1G_2 - \frac{1}{\lambda}b_1H_2\big).
\end{aligned}$$

Since F is minimal, we obtain

$$g_{11}\beta^\xi_{22} - g_{12}(\beta^\xi_{12}+\beta^\xi_{21}) + g_{22}\beta^\xi_{11} = 0 \quad\text{and}\quad g_{11}\beta^\eta_{22} - g_{12}(\beta^\eta_{12}+\beta^\eta_{21}) + g_{22}\beta^\eta_{11} = 0.$$

Moreover, the coefficients of $\sin^4 t_1$, $\cos^4 t_1 \sin^4 t_2$ and $\cos^2 t_1 \sin^2 t_1 \sin^2 t_2$ must vanish, thus we obtain

$$(\lambda^2-1)(a_1^2-a_2^2)(a_1^2+a_2^2) = 0 = (\lambda^2-1)(b_1^2-b_2^2)(b_1^2+b_2^2)$$

and

$$(\lambda^2-1)a_1a_2(a_1^2+a_2^2) = 0 = (\lambda^2-1)b_1b_2(b_1^2+b_2^2).$$

Consequently $\lambda = 1$ since, otherwise, the latter equations together with (5.21) would imply $\omega^i_{35} = \omega^i_{36} = \omega^i_{45} = \omega^i_{46} = 0$ for $i = 1, 2$, which contradicts our assumption that F has rank four. ∎

We recall some well known results regarding the *associated family* of a substantial simply connected oriented minimal surface $g\colon L^2 \to \mathbb{H}^{n+2}$. The associated family is obtained by rotating the second fundamental form while keeping fixed the normal bundle and the induced normal connection. More explicitly, for $\theta \in \mathbb{S}^1 = [0,\pi)$ consider the orthogonal parallel tensor field

$$J_\theta = \cos\theta I + \sin\theta J,$$

where I is the identity endomorphism and J denotes the complex structure on TL determined by the metric and orientation. Define on the bundle $Hom(TL\times TL, N_gL)$ the symmetric section $\alpha_g(J_\theta\cdot,\cdot)$ which satisfies the Gauss, Codazzi and Ricci equations with respect to the same induced normal connection; see [18]. Then, according to the fundamental theorem of submanifolds, there exists an isometric minimal immersion $g_\theta\colon L^2 \to \mathbb{H}^{n+2}$ whose second fundamental form is

$$\alpha_{g_\theta}(X,Y) = \phi_\theta\, \alpha_g(J_\theta X, Y),$$

where $\phi_\theta\colon N_gL \to N_{g_\theta}L$ is the parallel vector bundle isometry that identifies the normal bundles as well as each of the normal subbundles N^g_s with $N^{g_\theta}_s$ for $1 \le s \le \tau_g$.

In the sequel, we assume without loss of generality that g is 2-regular, i.e., N_1^g and N_2^g are subbundles of the normal bundle. Then, the vertical bundle $\mathcal{V} = \ker \pi_*$ of the submersion $\pi\colon \Sigma_g \to L^2$, can be orthogonally decomposed as

$$\mathcal{V} = \mathcal{V}^0 \oplus \mathcal{V}^1$$

on an open dense subset of L^2, where $\mathcal{V}^1$ denotes the plane bundle determined by

$$F_{g*}(\mathcal{V}^1)|_{(x,v)} = d(\exp_{g(x)})_v\big(N_2^g(x)\big).$$

Furthermore, consider the orthogonal decomposition of the tangent bundle of M^n given by

$$TM = \mathcal{H} \oplus \mathcal{V},$$

where we identify isometrically the subbundle $\mathcal{V}$ tangent to the rulings with the corresponding normal subbundle of g. It is a direct consequence from the proof that the relative nullity leaves of F can be identified with the fibers of $\mathcal{V}^0$.

Let $\mathcal{J}$ be the endormorphism of $T\Sigma_g$ such that restricted to $\mathcal{H}$ is the almost complex structure $\mathcal{J}|_{\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}$ determined by the orientation, whereas restricted to $\mathcal{V}$ is the identity. Moreover, we set

$$\mathcal{J}_\theta = \cos\theta I + \sin\theta \mathcal{J}.$$

The next theorem ensures the existence of genuine deformations and describes the relation between the second fundamental forms of the associated family members.

Theorem 5.11. *Let $g\colon L^2 \to \mathbb{H}^{n+2}$, $n \geq 3$, be a simply-connected 1-isotropic substantial surface. Then F_g allows a smooth one-parameter family of genuine minimal isometric deformations $F_\theta\colon \Sigma_g \to \mathbb{H}^{n+2}$, $\theta \in \mathbb{S}^1$, such that $F_0 = F_g$ and each F_θ carries the same rulings and relative nullity leaves as F_g.*

Moreover, there is a parallel vector bundle isometry $T_\theta : N_{F_g}\Sigma_g \to N_{F_\theta}\Sigma_g$ such that the relation between the second fundamental forms is given by

$$\alpha_{F_\theta}(X,Y) = T_\theta\big(R_{-\theta}\alpha_{F_g}(X,Y) + 2\kappa\sin(\theta/2)\mathcal{B}(\mathcal{J}_{-\theta/2}X,Y)\big), \tag{5.31}$$

where R_θ is the rotation of angle θ on $N_{F_g}\Sigma_g$ that preserves orientation, κ is the radius of the ellipse of curvature of g and $\mathcal{B}$ is the traceless bilinear form defined by (5.38).

Proof: Consider the one-parameter family $F_\theta\colon M^n \to \mathbb{H}^{n+2}$ of isometric immersions, defined by

$$F_\theta(x,\varphi,v) = \cosh\varphi\, g_\theta(x) + \sinh\varphi\ \phi_\theta v,$$

where $\theta \in \mathbb{S}^1$, $(x,v) \in U_1\Sigma_g$ and $\phi_\theta\colon N_gL \to N_{g_\theta}L$ is the parallel vector bundle isometry that identifies the normal subbundles of g and g_θ.

In the sequel, corresponding quantities of F_θ will be denoted by the same symbol used for F_g marked with θ. Using the fact that ϕ_θ is a parallel vector bundle isometry we can easily prove that F_θ is isometric to F_g. Let $\{e_1, e_2, e_3, \ldots, e_{n+2}\}$ be an adapted orthonormal frame along g. Then, for the adapted frames of g_θ we have that

$$e_3^\theta = \phi_\theta \circ R_\theta^1 e_3 \ \text{ and } \ e_4^\theta = \phi_\theta \circ R_\theta^1 e_4, \tag{5.32}$$

where R_θ^1 is the rotation of angle θ on N_1^g. We complete the adapted frame for g_θ by choosing

$$e_j^\theta = \phi_\theta e_j, \quad 5 \leq j \leq n+2. \tag{5.33}$$

We can readily verify that $\omega_{34}^\theta = \omega_{34}$ and $\omega_{ij}^\theta = \omega_{ij}$ for $i, j \geq 5$. Furthermore, we have that

$$\omega_{35}^\theta = \omega_{35} \circ \mathcal{J}_\theta \quad \text{and} \quad \omega_{36}^\theta = \omega_{36} \circ \mathcal{J}_\theta,$$

which implies

$$a_1^\theta := \omega_{35}^\theta(e_1) = a_1 \cos\theta + a_2 \sin\theta, \quad a_2^\theta := \omega_{35}^\theta(e_2) = a_2 \cos\theta - a_1 \sin\theta,$$

$$b_1^\theta := \omega_{36}^\theta(e_1) = b_1 \cos\theta + b_2 \sin\theta, \quad b_2^\theta := \omega_{36}^\theta(e_2) = b_2 \cos\theta - b_1 \sin\theta.$$

We parametrize the unit sphere in the fiber of Σ_g as in (5.22). Then, we have the following parametrization for F_θ

$$F_\theta(x, \varphi, t_1, \ldots, t_{n-3}) = \cosh\varphi\, g_\theta(x) + \sinh\varphi\, \phi_\theta w,$$

where w given in (5.22). The differential of F_θ is given by

$$\begin{aligned} F_{\theta*}(\partial/\partial\varphi) &= \sinh\varphi\, g_\theta + \cosh\varphi\, \phi_\theta w, \\ F_{\theta*}(\partial/\partial t_j) &= \sinh\varphi \sum_{i=1}^{n-2} \frac{\partial u_i}{\partial t_j} e_{i+4}, \quad 1 \leq j \leq n-3. \end{aligned}$$

Set $g_{ij} = \langle F_{\theta*}(\partial/\partial t_i), F_{\theta*}(\partial/\partial t_j)\rangle$. Then, we obtain

$$\begin{aligned} rF_{\theta*}(X_1) &= \cosh\varphi\, g_{\theta*}(e_1) - \phi_1^\theta \sinh\varphi\, e_3^\theta - \phi_2^\theta \sinh\varphi\, e_4^\theta, \\ rF_{\theta*}(X_2) &= \cosh\varphi\, g_{\theta*}(e_2) - \phi_2^\theta \sinh\varphi\, e_3^\theta + \phi_1^\theta \sinh\varphi\, e_4^\theta, \end{aligned}$$

where

$$X_i = \frac{1}{r}(e_i - \sum_{j=1}^{n-3} \frac{1}{g_{jj}} \langle \nabla_{e_i}^\perp w, \frac{\partial w}{\partial t_j}\rangle \partial/\partial t_j), \quad \text{for} \quad i = 1, 2,$$

$$r^2 = \cosh^2\varphi + \sinh^2\varphi\left((\phi_1^\theta)^2 + (\phi_2^\theta)^2\right) = \cosh^2\varphi + \sinh^2\varphi(\phi_1^2 + \phi_2^2),$$

$$g_{\theta*}(X) = g_*(J_\theta X) = \cos\theta\, g_*(X) + \sin\theta\, g_*(JX), \quad X \in TL \tag{5.34}$$

and

$$\phi_1^\theta = \phi_1 \cos\theta + \phi_2 \sin\theta, \quad \phi_2^\theta = -\phi_1 \sin\theta + \phi_2 \cos\theta, \tag{5.35}$$

with ϕ_i, $i = 1, 2$ defined in (5.24).

The vector fields $\{X_1, X_2, \partial/\partial\varphi, h_1\,\partial/\partial t_1, \ldots, h_{n-3}\,\partial/\partial t_{n-3}\}$ constitute a local orthonormal frame with respect to the induced metric of F_θ. Moreover, the normal space of F_θ is spanned by the vector fields

$$\xi_\theta = \phi_1^\theta \sinh\varphi\, g_{\theta*}(e_1) + \phi_2^\theta \sinh\varphi\, g_{\theta*}(e_2) + \cosh\varphi\, e_3^\theta, \tag{5.36}$$

$$\eta_\theta = \phi_2^\theta \sinh\varphi\, g_{\theta*}(e_1) - \phi_1^\theta \sinh\varphi\, g_{\theta*}(e_2) + \cosh\varphi\, e_4^\theta. \tag{5.37}$$

The map $\Psi_\theta\colon N_{F_g}\Sigma_g \to N_{F_\theta}\Sigma_g$ given by

$$\Psi_\theta \xi = \xi_\theta \quad \text{and} \quad \Psi_\theta \eta = \eta_\theta$$

is a parallel vector bundle isometry. The shape operators $A_{\xi_\theta}, A_{\eta_\theta}$ of F_θ vanish on $\mathcal{V}^0$ and restricted to the subspace $\mathcal{H} \oplus \operatorname{span}\{\partial/\partial\varphi\} \oplus \mathcal{V}^1$ spanned by the vectors $\{X_1, X_2, \partial/\partial\varphi, h_1\partial/\partial t_1, h_2\partial/\partial t_2\}$, are given by

$$rA_{\xi_\theta} = \begin{bmatrix} r(\kappa+\zeta_1^\theta) & r\,\zeta_2^\theta & -\phi_1^\theta & \psi_1^\theta & -\chi_1^\theta \\ r\,\zeta_2^\theta & -r(\kappa+\zeta_1^\theta) & -\phi_2^\theta & \psi_2^\theta & -\chi_2^\theta \\ -\phi_1^\theta & -\phi_2^\theta & 0 & 0 & 0 \\ \psi_1^\theta & \psi_2^\theta & 0 & 0 & 0 \\ -\chi_1^\theta & -\chi_2^\theta & 0 & 0 & 0 \end{bmatrix}$$

and

$$rA_{\eta_\theta} = \begin{bmatrix} r\,\zeta_2^\theta & r(\kappa-\zeta_1^\theta) & -\phi_2^\theta & \psi_2^\theta & -\chi_2^\theta \\ r(\kappa-\zeta_1^\theta) & -r\,\zeta_2^\theta & \phi_1^\theta & -\psi_1^\theta & \chi_1^\theta \\ -\phi_2^\theta & \phi_1^\theta & 0 & 0 & 0 \\ \psi_2^\theta & -\psi_1^\theta & 0 & 0 & 0 \\ -\chi_2^\theta & \chi_1^\theta & 0 & 0 & 0 \end{bmatrix},$$

where

$$\zeta_1^\theta = \zeta_1 \cos\theta + \zeta_2 \sin\theta. \quad \zeta_2^\theta = -\zeta_1 \sin\theta + \zeta_2 \cos\theta,$$

and ζ_1, ζ_2 are defined in Lemma 5.9. This proves that there exists a one-parameter family of minimal isometric immersions $F_\theta\colon M^n \to \mathbb{H}^{n+2}$, $\theta \in \mathbb{S}^1$, associated to F_g, such that $F_0 = F_g$ and each F_θ carries the same $(n-2)$-dimensional rullings and same $(n-4)$-dimensional relative nullity leaves as F_g. Observe also that since the shape operators of F_θ have rank 4, the isometric deformations F_θ of F are genuine.

Finally, consider the operator $L_\theta\colon TM \to TM$ such that $L_\theta|_{\operatorname{span}\{\partial/\partial\varphi\}\oplus\mathcal{V}} = 0$ and $L_\theta|_{\mathcal{H}^{F_g}}\colon \mathcal{H}^{F_g} \to \mathcal{H}^{F_g}$ is the reflection given by

$$L_\theta|_{\mathcal{H}^{F_g}} = \begin{bmatrix} -\sin(\theta/2) & \cos(\theta/2) \\ \cos(\theta/2) & \sin(\theta/2) \end{bmatrix},$$

with respect to the tangent frame $\{X_1, X_2\}$. It follows easily that

$$A_{\xi_\theta} = A_{R_\theta \xi} - 2\kappa \sin(\theta/2) L_\theta \quad \text{and} \quad A_{\eta_\theta} = A_{R_\theta \eta} - 2\kappa \sin(\theta/2) \mathcal{J} \circ L_\theta.$$

A straightforward computation shows that

$$\alpha_{F_\theta}(X, Y) = \Psi_\theta \left(R_{-\theta} \alpha_{F_g}(X, Y) - \frac{2\kappa}{r^2} \sin(\theta/2)(\langle L_\theta X, Y \rangle \xi + \langle L_\theta \mathcal{J} X, Y \rangle \eta) \right).$$

Consider the symmetric section $\mathcal{B}$ of $Hom(TM \times TM, N_{F_g} \Sigma_g)$ with corresponding nullity distribution $\mathcal{V}$, defined by

$$\mathcal{B}(X_1, X_1) = \frac{1}{r^2} \xi = -\mathcal{B}(X_2, X_2), \quad \mathcal{B}(X_1, X_2) = -\frac{1}{r^2} \eta. \tag{5.38}$$

Then, (5.31) follows immediately. ∎

Abstract

In this thesis, we investigate complete minimal isometric immersions $f\colon M^m \to \mathbb{Q}_c^n$ into space forms with positive index of relative nullity. The index of relative nullity was introduced by Chern and Kuiper [13] and turned out to be a fundamental concept in submanifold theory. At a point of M^m the index is just the dimension of the kernel of the second fundamental form of an isometric immersion $f\colon M^m \to \mathbb{Q}_c^n$ at that point. The kernels form an integrable distribution, the so called relative nullity distribution denoted by $\mathcal{D}$, along any open subset where the index is constant and the images under f of the leaves of the foliation are (part of) affine subspaces in the ambient space.

At first, we consider complete minimal isometric immersions $f\colon M^m \to \mathbb{Q}_c^n$ into space forms $\mathbb{Q}_c^n$, $c = -1, 0, 1$, with index of relative nullity at least $m-2$. Our technique for classifying the latter immersions consists of studing a tensor, the so called splitting tensor $\mathcal{C}$, that describes how the conullity distribution $\mathcal{D}^\perp$ is twisting inside the manifold M^m. We employ tools from geometric analysis, among them is the Omori-Yau maximum principle and the gradient estimate of Yau, in order to describe the structure of the splitting tensor as an endomorphism of the conullity distribution. The main difficulty arises from the fact that we allow the index of the relative nullity to vary. In order to extend the splitting tensor over the real analytic set $\mathcal{A}$ of totally geodesic points, it is essential to analyze the structure of the set $\mathcal{A}$. This is accomplished by employing regularity extension theorems for harmonic maps.

For minimal isometric immersions into Euclidean space $\mathbb{R}^n$, we prove that the immersion f must be a cylinder over a minimal surface, under the mild assumption that the Omori-Yau maximum principle is satisfied for the Laplacian. The category of complete Riemannian manifolds for which the Omori-Yau maximum principle is valid is quite large. For instance, it contains the manifolds whose Ricci curvature is bounded from below. It also contains the class of properly immersed submanifolds in a space form whose norm of the mean curvature vector is bounded [56, Example 1.14]. The aforementioned result is truly global in nature, since there are plenty of non complete minimal submanifolds of dimension m having constant index of relative nullity $m-2$ that are not part of a cylinder on any open subset. They can all locally be parametrized in terms of a certain class of elliptic surfaces [15, Theorem 22]. Consequently, what remains as a challenging open problem is the existence of minimal complete and noncylindrical submanifolds with index of relative nullity $\nu \geq m-2$.

It is worth noticing that many authors where interested into finding geometric conditions for an isometric immersion $f\colon M^m \to \mathbb{R}^n$ of a complete Riemannian manifold with positive index of relative nullity to be a cylinder. Some of the many papers containing characterizations of submanifolds as cylinders without the requirement for the immersion to be minimal are [15, 17, 38, 40, 52, 54, 58]. When adding the condition of being minimal we have [1, 24, 35, 36, 38, 41, 64, 66].

For complete minimal immersions $f\colon M^m \to \mathbb{S}^n$ in Euclidean spheres, we prove that any such submanifold M^m is either totally geodesic or has dimension three. In the latter case, there are plenty of examples, even compact ones. For any dimension and codimension there is an abundance of examples of non-complete submanifolds fully described by Dajczer and Florit [15] in terms of a class of surfaces, called elliptic, for which the ellipse of curvature of a certain order is a circle at any point. Under the mild assumption that the Omori-Yau maximum principle holds on the manifold, a trivial condition in the compact case, we provide a complete local parametric description of the submanifolds in terms of 1-isotropic surfaces in Euclidean space. These are the minimal surfaces for which the standard ellipse of curvature is a circle at any point. For these surfaces, there exists a Weierstrass type representation that generates all simply-connected ones.

In any of the two cases already studied, namely the Euclidean and spherical case, the proofs reduced to analyze the situation of the three dimensional submanifolds. In fact, for submanifolds in spheres only this case turned out to be possible. For minimal immersions $f\colon M^m \to \mathbb{H}^n$ in hyperbolic space of complete Riemannian manifolds M^m, the condition that the index of relative nullity satisfies $\nu \geq m-2$ turns out to be quite less restrictive than in the previously studied cases. Nevertheless, we have reasons to believe that the manifold being three-dimensional is still quite special and this is why this case allows a characterization of a class of submanifolds that is contained in the following description. We prove that any three dimensional minimal submanifold $f\colon M^3 \to \mathbb{H}^n$ having index of relative nullity at least one at any point, is either totally geodesic or a generalized cone over a complete minimal surface lying in an equidistant submanifold of $\mathbb{H}^n$, under the assumption that the scalar curvature is bounded from below, see [23].

Furthermore, we parametrically describe all minimal immersions $f\colon M^m \to \mathbb{H}^n$, whose index of relative nullity is $m-2$, as subbundles of the normal bundle of certain elliptic spacelike surfaces in the Lorentzian space or in the de Sitter space [49]. From this parametrization it is straightforward than there exist a plethora of examples of non-complete minimal submanifolds with index of relative nullity $m-2$. Additionally, using this parametrization, one can construct an abundance of complete minimal submanifolds of any dimension other than generalized cones, as can been seen from the results in [9], [32] and [47].

Finally we introduce a new class of minimal immersions $F\colon M^n \to \mathbb{H}^{n+2}$, $n \geq 3$, in the hyperbolic space that are $(n-2)$-ruled [49]. This means that they carry an integrable tangent distribution of dimension $n-2$, whose leaves are mapped diffeomorphically by F onto open subsets of totally geodesic $(n-2)$-hyperbolic spaces of $\mathbb{H}^{n+2}$. If the manifold is simply connected, we show that it allows a one-parameter family of equally ruled minimal isometric deformations that are genuine. The deformations are obtained while keeping fixed the normal bundle and the induced connection, but now the second fundamental form relates to the initial one in a much more complex form, in particular, no orthogonal tensor in involved. It is an interesting question if the above associated family of complete ruled minimal submanifolds exhausts all examples in the same class that admit genuine deformations. Of course, a much more challenging classification problem of complete submanifolds of rank four would be to drop one of the conditions, for instance being minimal or ruled.

Περιληψη

Μια από τις σημαντικότερες έννοιες στην θεωρία υποπολυπτυγμάτων είναι αυτή της μηδενοκατανομής, η οποία εισήχθη από τους Chern και Kuiper [13]. Η μηδενοκατανομή ενός υποπολυπτύγματος μέσα σε έναν χώρο σταθερής καμπυλότητος ορίζεται ως ο πυρήνας της δεύτερης θεμελιώδους μορφής. Ο δείκτης της μηδενοκατανομής σε ένα σημείο του υποπολυπτύγματος ορίζεται ως η διάσταση του πυρήνα της δεύτερης θεμελιώδους μορφής στο σημείο αυτό. Οι πυρήνες αυτοί συνιστούν μια ολοκληρώσιμη κατανομή κατά μήκος κάθε ανοικτού υποσυνόλου του υποπολυπτύγματος όπου ο δείκτης είναι σταθερός και τα φύλλα της μηδενοκατανομής συνιστούν ολικά γεωδαιτικά υποπολυπτύγματα στον περιβάλλοντα χώρο. Εάν επιπλέον το υποπολύπτυγμα είναι πλήρες, τότε αποδεικνύεται ότι τα φύλλα της μηδενοκατανομής είναι επίσης πλήρη στο ανοικτό υποσύνολο όπου ο δείκτης λαμβάνει την ελάχιστη τιμή του (δείτε [14]).

Συχνό αντικείμενο μελέτης στην θεωρία υποπολυπτυγμάτων είναι η εύρεση γεωμετρικών υποθέσεων, ώστε μια πλήρης ισομετρική εμβάπτιση $f\colon M^m \to \mathbb{R}^n$ με θετικό δείκτη μηδενοκατανομής $\nu \geq k > 0$ να είναι k-κύλινδρος. Αυτό σημαίνει ότι το πολύπτυμα M^m διασπάται ως γινόμενο Riemann $M^m = M^{m-k} \times \mathbb{R}^k$ και η ισομετρική εμβάπτιση f διασπάται ως $f = g \times \mathrm{id}_{\mathbb{R}^k}$. Η μηδενοκατανομή αποτελεί σημαντικό εργαλείο για τον χαρακτηρισμό των κυλίνδρων, διότι προκειμένου να αποδειχθεί ότι μια ισομετρική εμβάπτιση είναι k-κύλινδρος αρκεί να δειχθεί ότι οι εικόνες των φύλλων της μηδενοκατανομής μέσω της f είναι παράλληλες στον περιβάλλοντα χώρο.

Σημαντικό αποτέλεσμα σε αυτή την κατεύθυνση είναι το θεώρημα του Hartman [40], σύμφωνα με το οποίο κάθε ισομετρική εμβάπτιση $f\colon M^m \to \mathbb{R}^n$ με θετικό δείκτη μηδενοκατανομής $\nu \geq k > 0$ και μη-αρνητική καμπυλότητα Ricci είναι k-κύλινδρος. Βασικό εργαλείο στην απόδειξη είναι το θεώρημα διάσπασης των Cheeger-Gromoll [10] το οποίο χρησιμοποιείται για να αποδειχτεί ότι τα φύλλα που αντιστοιχούν στον ελάχιστο δείκτη μηδενοκατανομής διασπώνται ως γινόμενο Riemann. Το ανωτέρω αποτέλεσμα δεν αληθεύει στην περίπτωση όπου η καμπυλότητα Ricci είναι μη-θετική, κάτι που συμβαίνει πάντα για ελαχιστικές εμβάπτισεις. Αντιπαραδείγματα αποτελούν οι ευθειογενείς υπερεπιφάνειες οποιασδήποτε συνδιάστασης στο [19, σελ. 409].

Αξίζει να σημειωθεί ότι πολλά άρθρα χαρακτηρίζουν την κλάση των κυλίνδρων, εκ των οποίων τα [15,17,38,40,52,54,58] δεν αναφέρονται σε ελαχιστικά υποπολυπτύγματα, ενώ τα [1,24,35,36,38,41,64,66] περιγράφουν την κλάση των κυλίνδρων τα οποία είναι ελαχιστικά υποπολυπτύγματα.

Στην παρούσα διατριβή στόχος μας είναι να επεκτείνουμε τα ανωτέρω αποτελέσματα στην κλάση των ελαχιστικών ισομετρικών εμβαπτίσεων $f\colon M^m \to \mathbb{Q}_c^n$ με δείκτη μηδενοκατανομής τουλάχιστον $m-2$. Η τεχνική μας είναι να κάνουμε χρήση του λεγόμενου τανυστή διάσπασης, ο οποίος περιγράφει πως καμπυλώνεται το ορθοσυμπλήρωμα της μηδενοκατανομής εντός του πολυπτύγματος M^m. Χρησιμοποιούμε εργαλεία από γεωμετρική ανάλυση, όπως η αρχή μεγίστου Omori-Yau και η εκτίμηση κλίσης του Yau ώστε να κατανοηθεί η δομή του τανυστή διάσπασης. Μια από τις σημαντικότερες τεχνικές δυσκολίες στην απόδειξη προέρχεται από το γεγονός ότι επιτρέπουμε στον δείκτη της μηδενοκατανομής να μεταβάλλεται από σημείο σε σημείο. Επομένως, προκειμένου να επεκτείνουμε τον τανυστή διάσπασης υπεράνω του αναλυτικού συνόλου $\mathcal{A}$ των ολικά γεωδαιτικών σημείων, χρησιμοποιούμε θεωρήματα επέκτασης για αρμονικές απεικονίσεις.

Η παρούσα διδακτορική διατριβή διαρθρώνεται ως εξής: Αρχικά αναφέρουμε μερικές εισαγωγικές έννοιες στο Κεφάλαιο 1 και στα Κεφάλαια 2,3,4 και 5 περιέχονται τα πρωτότυπα αποτελέσματα της διατριβής.

Πιο συγκεκριμένα, στο Κεφάλαιο 2 μελετούμε την δομή του τανυστή διάσπασης για τριδιάστατα ελαχιστικά υποπολυπτύγματα σε χώρους μορφής με δείκτη μηδενοκατανομής ένα. Αξίζει να σημειωθεί ότι η τριδιάστατη περίπτωση είναι ουσιώδους σημασίας για τα αποτελέσματα της παρούσας διατριβής.

Στο Κεφάλαιο 3, εξετάζουμε πλήρη ελαχιστικά υποπολυπτύγματα M^m στον Ευκλείδειο χώρο με θετικό δείκτη μηδενοκατανομής τουλάχιστον $m-2$. Αποδεικνύουμε ότι κάθε τέτοιο υποπολύπτυγμα είναι κύλινδρος υπεράνω μιας ελαχιστικής επιφάνειας, υπό την ασθενή υπόθεση ότι ισχύει η αρχή μεγίστου Omori-Yau για τη Λαπλασιανή. Η κλάση των πλήρων πολυπτυγμάτων για τα οποία ισχύει η αρχή μεγίστου είναι ευρεία, αφού περιλαμβάνει τα πολυπτύγματα Riemann των οποίων η καμπυλότητα Ricci δεν φθίνει ταχέως στο μείον άπειρο καθώς και τα proper υποπολυπτύγματα των οποίων η νόρμα του διανύσματος μέσης καμπυλότητας είναι φραγμένη. Το αποτέλεσμά μας είναι ολικό εκ φύσεως, καθώς υπάρχει πληθώρα παραδειγμάτων μη-πλήρων ελαχιστικών υποπολυπτυγμάτων σε οποιασδήποτε συνδιάσταση, με σταθερό δείκτη μηδενοκατανομής, τα οποία δεν είναι τμήματα κυλίνδρου σε κανένα ανοικτό υποσύνολό τους. Όλα αυτά τα υποπολυπτύγματα, μπορούν να παραμετρηθούν τοπικά ως διανυσματικές υποδέσμες της κάθετης δέσμης μιας κλάσης ελλειπτικών επιφανειών για τις οποίες μια συγκεκριμένη έλλειψη καμπυλότητος είναι κύκλος σε κάθε σημείο (δείτε Θεώρημα 22 στο [15]). Αξίζει να αναφερθεί ότι ένα απαιτητικό ανοικτό πρόβλημα το οποίο αποτελεί πρόκληση, είναι η ύπαρξη ενός μη-κυλινδρικού πλήρους ελαχιστικού υποπολυπτύγματος με δείκτη μηδενοκατανομής $\nu \geq 1$.

Στο Κεφάλαιο 4, μελετούμε πλήρεις ελαχιστικές εμβαπτίσεις $f\colon M^m \to \mathbb{S}^n$ με δείκτη μηδενοκατανομής τουλάχιστον $m-2$ σε κάθε σημείο. Τα ανωτέρω υποπολυπτύγματα είναι austere υπό την έννοια των Harvey και Lawson [44] και μελετήθηκαν από τον Bryant [7]. Υπάρχει μεγάλη ποικιλία από μη-πλήρη ελαχιστικά υποπολυπτύγματα, σε κάθε συνδιάσταση, τα όποια έχουν παραμετρηθεί από τους Dajczer και Florit [15] ως διανυσματικές υποδέσμες της κάθετης δέσμης μιας κλάσης ελλειπτικών επιφανειών, για τις οποίες μια συγκεκριμένη έλλειψη καμπυλότητος είναι κύκλος σε κάθε σημείο. Εάν υποθέσουμε ότι το υποπολύπτυγμα είναι πλήρες, τότε συνάγουμε ότι είτε είναι ολικά γεωδαιτικό, είτε η διάστασή του είναι τρία. Στην τελευταία περίπτωση υπάρχουν πολλά παραδείγματα, μεταξύ αυτών και συμπαγή. Υπό την ασθενή υπόθεση ότι ισχύει η αρχή μεγίστου Omori-Yau, μια τετριμμένη υπόθεση όταν το πολύπτυγμα είναι συμπαγές, παρέχουμε μια πλήρη τοπική περιγραφή των ανωτέρω υποπολυπτυγμάτων ως μοναδιαίες εφαπτόμενες υποδέσμες της κάθετης δέσμης 1-ισοτροπικών επιφανειών στον Ευκλείδειο χώρο. Οι 1-ισοτροπικές επιφάνειες είναι ελαχιστικές επιφάνειες για τις οποίες η συνήθης έλλειψη καμπυλότητος είναι κύκλος σε κάθε σημείο. Για αυτές της επιφάνειες υπάρχει Weierstrass αναπαράσταση που παράγει όλες όσες είναι απλά συνεκτικές.

Τέλος, το Κεφάλαιο 5 αναφέρεται σε ελαχιστικά υποπολυπτύγματα του υπερβολικού χώρου και διαιρείται σε τρία μέρη. Στο πρώτο μέρος, μελετούμε πλήρεις ελαχιστικές ισομετρικές εμβαπτίσεις $f\colon M^m \to \mathbb{H}^n$ με δείκτη μηδενοκατανομής τουλάχιστον $m-2$. Σε αντιδιαστολή με τις περιπτώσεις του Ευκλειδείου χώρου και της σφαίρας, η υπόθεση ότι ο δείκτης της μηδενοκατανομής είναι τουλάχιστον $m-2$ είναι λιγότερο περιοριστική στον υπερβολικό χώρο. Έχουμε ισχυρές ενδείξεις ότι η τριδιάστατη περίπτωση $m = 3$ διαφοροποιείται της περίπτωσης $m \geq 4$, γεγονός που μας οδηγεί στον χαρακτηρισμό πλήρων ελαχιστικών εμβαπτίσεων $f\colon M^3 \to \mathbb{H}^n$ με δείκτη μηδενοκατανομής τουλάχιστον ένα σε κάθε σημείο. Υπό την υπόθεση ότι η αριθμητική καμπυλότητα είναι φραγμένη από κάτω, αποδεικνύουμε ότι το υποπολύπτυγμα M^3 είναι είτε ολικά γεωδαιτικό, είτε γενικευμένος κώνος υπεράνω μιας πλήρους ελαχιστικής επιφάνειας που κείται σε ισαπέχον υποπολύπτυγμα του $\mathbb{H}^n$ (δείτε [23]). Η υπόθεση της πληρότητας είναι απαραίτητη στην ανωτέρω περιγραφή, καθώς υπάρχει πληθώρα παραδειγμάτων μη-πλήρων υποπολυπτυγμάτων τα οποία δεν ανήκουν στην κλάση των γενικευμένων κώνων.

Στο δεύτερο μέρος, μελετάμε m-διάστατα ελαχιστικά υποπολυπτύγματα του υπερβολικού χώρου σε οποιασδήποτε συνδιάσταση, με δείκτη μηδενοκατανομής $m-2$ [49]. Στόχος μας είναι να παραμετρήσουμε τοπικά αυτά τα υποπολυπτύγματα, ως διανυσματικές υποδέσμες της κάθετης δέσμης μιας κατηγορίας ελλειπτικών επιφανειών του χώρου Lorentz ή του χώρου de Sitter. Είναι πλέον εμφανές ότι η υπόθεση της πληρότητας στον χαρακτηρισμό των τριδιάστατων ελαχιστικών υποπολυπτυγμάτων είναι αναγκαία, καθώς υπάρχουν πολλά τοπικά παραδείγματα που δεν ανήκουν στην κλάση των γενικευμένων κώνων. Επιπλέον, η παραμέτρηση αυτή μπορεί να χρησιμοποιηθεί για την κατασκευή πλήρων υποπολυπτυγμάτων τυχούσας συνδιάστασης, όπως φαίνεται με χρήση των αποτελεσμάτων στα [9], [32] και [47]. Αξίζει να αναφερθεί ότι ένας εναλλακτικός τρόπος κατασκευής πλήρων ελαχιστικών υποπολυπτυγμάτων στον υπερβολικό χώρο, μέσω κανονικών δεσμών από αρμονικούς μορφισμούς σε επιφάνειες Riemann, δόθηκε από τον Gudmundsson στο άρθρο [39].

Στο τρίτο και τελευταίο μέρος της διατριβής κατασκευάζουμε μια νέα κλάση ελαχιστικών εμβαπτίσεων $F\colon M^n \to \mathbb{H}^{n+2}$, $n \geq 3$, στον υπερβολικό χώρο οι οποίες είναι $(n-2)$-ευθειογενείς [49]. Αυτό σημαίνει ότι το M^n επιδέχεται μια ολοκληρώσιμη κατανομή διάστασης $n-2$ της εφαπτόμενης δέσμης, της οποίας τα φύλλα απεικονίζονται διαφορομορφικά μέσω της F σε ανοικτά υποσύνολα ολικά γεωδαιτικών $(n-2)$-διάστατων υπερβολικών χώρων. Εάν το M^n είναι απλά συνεκτικό τότε αποδεικνύουμε ότι η εμβάπτιση F επιδέχεται μια μονοπαραμετρική οικογένεια από ελαχιστικές παραμορφώσεις οι οποίες είναι "γνήσιες". Οι παραμορφώσεις αυτές αποκτώνται κρατώντας σταθερή την κάθετη δέσμη και την επαγόμενη κάθετη συνοχή, αλλά η δεύτερη θεμελιώδη μορφή συνδέεται με την αρχική με περίπλοκο τρόπο. Ενδιάφερον ερώτημα είναι αν η ανωτέρω κλάση περιέχει όλες τις γνήσιες παραμορφώσεις. Πρόκληση αποτελεί επίσης και το πρόβλημα ταξινόμησης των πλήρων υποπολυπτυγμάτων βαθμίδος τέσσερα, αφαιρώντας την υπόθεση της πληρότητας ή της ευθειογένειας.

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