# Metastability of Spherical Membranes in Supermembrane and Matrix Theory

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(July 31, 2013)



Motivated by recent work we study rotating ellipsoidal membranes in the framework of the light-cone supermembrane theory. We investigate stability properties of these classical solutions which are important for the quantization of super membranes. We find the stability modes for all sectors of small multipole deformations. We exhibit an isomorphism of the linearized membrane equation with those of the $SU(N)$ matrix model for every value of $N$. The boundaries of the linearized stability region are at a finite distance and they appear for finite size perturbations.

PACS:

## I. INTRODUCTION

M theory [1] is considered today as the best candidate for the unification of the weak and strong coupling sectors of all known string theories. The most serious attempt up to now to frame M-theory together with all of its ingredients is the Matrix theory [2]. The conjecture of this theory is that all the freedom of various sectors of the five known string theories can be represented by appropriate operators of the Matrix theory using duality properties. Up to now all perturbative checks of this idea although not straight forward have been sucessfull and focus on the nonperturbative sector leads to the study of classical solutions of Matrix theory in general backrounds. Recent progress in this direction is the successfull formulation of Matrix theory in weak gravitational and gauge backgrounds i.e. dynamics of matrix branes in the backround of their mutual and external forces [3].

One of the poorly understood elements of M-Theory is the eleven dimensional classical supermembrane sector. Progress in this direction is important both for the understanding of the strongly coupled string theories as well as for the quantization of the supermembrane. Recent interest for the classical solutions of the Matrix theory representing $D_0$-branes attached to spherical membranes is explained as a first step to a formulation of Matrix theory in weak external gravitational and gauge backrounds [4]. Particular solutions of the classical matrix equations representing rotating ellipsoidal configurations of N $D_0$ branes attached to a membrane which exhibit stability properties have been proposed and their semiclassical spectrum has been studied [5].

In this work we study in detail the stability properties of rotating spherical membrane which are solutions of the bosonic part of the supermembrane equations restricted to six spatial dimensions. They are motivated by the recently found matrix model solutions which represent N $D_0$-branes pinned on the surface of a rotating ellipsoidal membrane [6]. We find stability for all modes of small multipole deformations and we determine explicitly the spectrum and the eigenmodes. There is an interesting isomorphism with the full stability analysis of the Matrix solution which demonstrates that classical membrane excitations can be analyzed in distinct sectors of $D_0$-branes and provide approximations for the quantum mechanical study of the spherical membrane.

## II. MATRIX MODELS VERSUS SUPERMEMBRANE

It is a well known fact that the matrix model was one of the first ideas to study the bosonic membrane in the light cone frame in the approximation of finite number of oscillation modes. The elegant observation of [7] is that $SU(N)$ Yang-Mills mechanics is a consistent mode truncation of the membrane excitations in the light cone frame and moreover in the late eighties when the supermembrane Lagrangian [8] was written down it became clear that the dimensional reduction of the ten dimensional $N = 1$ SUSY Yang-Mills theory to $d = 1$ is the correct supersymmetric extension of the above truncation. [7] The light cone infinite dimensional area preserving symmetry of the supermembrane Hamiltonian [7,8]is truncated by $SU(N)$ Yang-Mills symmetry. This can be represented as the algebra of the corresponding discrete and finite Heisenberg group of a discretized membrane considered as a two dimensional discrete phase space [9]. The large $N$ limit connecting $SU(N)$ Yang-Mills to the membrane Hamiltonian, i.e. commutators with Poisson brackets became clear as an analogy of the passage from quantum to classical mechanics $\hbar = \frac{2\pi}{N} \to 0$, as $N \to \infty$ [10]. With regard to the study of the quantum theory of the membranes it is necessary to understand better this limit both from the point of view of matrix models in general but also from the approximation point of view of the measure for the quantum configuration [11,12].

We now make a quick review of the formalism for the bosonic sector of the theory relevant to the present work. After fixing the gauge and using reparametrization invariance of the Nambu-Gotto Lagrangian we find that the eqs of motion for the 9 bosonic coordinates $X_i(t, \sigma_1, \sigma_2)$, $i = 1, 2, ..., 9$ in the light cone frame are:

$$\ddot{X}_i \ = \ \{X_k, \{X_k, X_i\}\} \tag{2.1}$$

where the Poisson bracket of two functions , $f$ and $g$ on $S^2$ is defined as

$$\{f, g\} \ = \ \frac{\partial f}{\partial \cos\theta} \frac{\partial g}{\partial \phi} \ - \ \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \cos\theta} \tag{2.2}$$

and the remaining area preserving symmetry generated by the constraint

$$\left\{X_i, \dot{X}_i\right\} = 0 \tag{2.3}$$

In the matrix model the above coordinates are replaced by $N \times N$ Hermitian and traceless matrices and the corresponding equations of motion and constraint are found by exchanging Poisson brackets with commutators.

The first connection between the $SU(N)$ Susy Yang-Mills truncation of the supermembrane with the recent nonperturbative studies of string theories was discovered by Witten [15] representing the Yang-Mills mechanics as a low energy effective theory of bound states of $N$ $D_0$ branes. The $D_0$ branes carry RR charge. Now it is understood how to couple the $SU(N)$ matrix model with weak background fields either directly using supergravity arguments or truncating supermemmbrane Lagrangians in weak background fields [3]. There is an expectation that taking appropriate limits of $N \to \infty$ for special bound states of $N$ $D_0$ branes one could recover the supermembrane or its magnetic dual, the super-five brane [13]. On the other hand the study of classical solutions of supermembranes or matrix model could provide a nonperturbative information for their dynamics even in the quantum regime. In the next section we turn our attention to the analysis of the stability properties of specific classical solutions which are spherical rotating membranes. Recent work in the matrix model presented such a time dependent solution representing a bound system of $N$ $D_0/D_2$ branes.

## III. SPHERICAL MEMBRANES AS MATRICES

Since we are going to study spherical membranes and their matrix analogs, we begin by reviewing the salient features of the Lie algebra $sDiff(S^2)$ of area preserving diffeomorphisms of the sphere $S^2$ considered as a two dimensional differentiable manifold. We first introduce the canonical or Darboux coordinates on the sphere $\sigma_1 = \phi$ , $\sigma_2 = \cos\theta$. In these coordinates the Poisson bracket on the sphere is defined as

$$\{f, g\} \ = \ \frac{\partial f}{\partial \sigma_1} \frac{\partial g}{\partial \sigma_2} \ - \ \frac{\partial f}{\partial \sigma_2} \frac{\partial g}{\partial \sigma_1} \tag{3.1}$$

where $f, g \in \mathcal{C}^\infty(\mathcal{S}^\in)$. The spherical harmonics $Y_{l,m}(\theta, \phi)$ for $m = -l, \ldots l$ and $l = 0, 1, \ldots, \infty$ give rise to a complete system of generators for $sDiff(S^2)$:

$$L_{l,m} \ = \ \frac{\partial Y_{l,m}}{\partial \cos\theta} \frac{\partial}{\partial \phi} \ - \ \frac{\partial Y_{l,m}}{\partial \phi} \frac{\partial}{\partial \cos\theta} \tag{3.2}$$

They satisfy the algebra

$$[L_{l,m}, L_{l',m'}] \ = \ f_{lm,l'm'}^{l"m"} \ \ L_{l"m"} \tag{3.3}$$

where the structure constants $f_{lm,l'm'}^{l"m"}$ are defined by expanding the Poisson brackets on the basis $Y_{l,m}$ as follows:

$$\{Y_{l,m} \ , \ Y_{l'm'}\} \ = \ f_{lm,l'm'}^{l"m"} \ Y_{l"m"} \tag{3.4}$$

These structure constants have been calculated explicitly by Hoppe in Ref. [7]. In order to get a feeling of the geometrical meaning of these generators , we note that for $l = 1, m = 0, \pm 1$, these are the usual angular momentum generators (up to normalization), $L_z, L_\pm$. Also

$$L_{l,m} \ Y_{l',m'} \ = \ -\{Y_{l,m}, Y_{l',m'}\} \ = \ -f_{lm,l'm'}^{l"m"} \ Y_{l",m"} \tag{3.5}$$

so that

$$[L_{1,\pm 1}, L_{l,m}] \ \cong \ [l(l+1) - m(m+1)]^{1/2} L_{l,m\pm 1} \tag{3.6}$$

$$[L_{1,0}, L_{l,m}] \ \cong \ m \ L_{l,m} \tag{3.7}$$

For general l,m , $L_{l,m}$ produce multipole deformations of the spherical membrane. The above two eqs imply that the infinite set of generators $L_{l,m}$ is reduced to an infinite sum of irreducible sets of operators of definite angular momentum l,

$$L_{l,-l}, L_{l,-l+1}, \ldots, L_{l,l}, \qquad l = 1, 2, \ldots \tag{3.8}$$

with respect to the $SU(2)$ Lie algebra, $L_{1,0}, L_{1,\pm 1}$ of solid rotations on the sphere. We now describe the approximation of $sDiff(S^2)$ by $SU(N)$. We choose an appropriate basis of $SU(N)$ describing matrix spherical harmonics [7]. The spherical harmonics $Y_{l,m}(\theta, \phi)$ are harmonic homogeneous polynomials of degree $l$ in the three euclidean coordinates $x_1, x_2, x_3$ of points on $S^2$ where:

$$x_1 = \cos\phi \sin\theta, x_2 = \sin\phi \sin\theta, x_3 = \cos\theta \tag{3.9}$$

and

$$Y_{l,m}(\theta,\phi) = \sum_{i_k=1,2,3} \alpha^{(m)}_{i_1,\ldots,i_l}\; x_{i_1},\ldots,x_{i_l},$$
$$k=1,\ldots,l \;\; k=1,\ldots,l \qquad (3.10)$$

where $\alpha^{(m)}_{i_1,\ldots,i_l}$ is a symmetric and traceless tensor. For fixed $l$ there are $2l+1$ linearly independent tensors $\alpha^{(m)}_{i_1,\ldots,i_l}$, $m=-l,\ldots,l$ [16].

Let $J_1, J_2, J_3$ be $N\times N$ hermitian matrices which form an N-dimensional irreducible representation of the Lie algebra $SU(2)$,

$$[J_i, J_j] \;=\; i\epsilon_{ijk} J_k \qquad (3.11)$$

Hoppe in Ref [7] has shown that the matrix polynomials

$$\hat{Y}^{(N)}_{l,m} \;=\; \sum_{i_k=1,2,3} \alpha^{(N)}_{i_1,\ldots,i_l}\; J_{i_1}\ldots J_{i_l} \qquad (3.12)$$

for $l=1,\ldots,N-1,\; m=-l,\ldots,l$ can be used to construct a basis of $N^2-1$ matrices for the fundamental representation of $SU(N)$ with corresponding structure constants $f^{(N)}$:

$$[\hat{Y}^{(N)}_{l,m}, \hat{Y}^{(N')}_{l',m'}] \;\;=\;\; if^{(N)l^{"}m^{"}}_{lm,l'm'}\,\hat{Y}^{(N)}_{l,m}. \qquad (3.13)$$

There is a normalization of the generators $\hat{Y}^{(N)}_{l,m}$ such that the limit

$$Nf^{(N)l^{"}m^{"}}_{lm,l'm'} \;\stackrel{N\to\infty}{\longrightarrow}\; f^{l^{"}m^{"}}_{lm,l'm'} \qquad (3.14)$$

exists and coincides with the structure constants as defined before in eq:(3.4). After these preliminaries we proceed to establish the relation of the infinite dimensional algebra eq(3.3) ,$sDiff(S^2)$, to the $SU(N)$ algebra as $N\to\infty$ , by an argument which avoids the explicit computation of ref [7] for the structure constants $f^{(N)}$ and $f$ [10]. If we rescale the generators of $SU(2)$ by $1/N$

$$J_i \to T_i \;=\; (1/N)J_i \qquad (3.15)$$

they satisfy the algebra

$$[T_i, T_j] \;=\; (\imath/N)\epsilon_{ijk}T_k \qquad (3.16)$$

and the Casimir element

$$T^2 \;=\; T_1^2+T_2^2+T_3^2 \;\simeq\; 1+1/N \qquad (3.17)$$

has a finite limit for $N\to\infty$. Under the norm

$$|x|^2 \equiv Tr(x^2), \qquad (3.18)$$

for $x\in SU(2)$, the generators $T_i, i=1,2,3$ have definite limits as $N\to\infty$ which are three objects $x_1,x_2,x_3$ which commute and are constrained by eq.(3.17) according to

$$x_1^2+x_2^2+x_3^2 = 1 \qquad (3.19)$$

If we consider two polynomial functions of three commuting variables $f(x_1,x_2,x_3)$ and $g(x_1,x_2,x_3)$ the corresponding matrix polynomials $f(T_1,T_2,T_3), g(T_1,T_2,T_3)$ have commutation relations for large N which follow from eq(3.16):

$$lim_{N\to\infty}(N/i)[f,g] \;=\; \epsilon^{ijk}x^j\,\frac{\partial f}{\partial x^j}\frac{\partial g}{\partial x^k} \qquad (3.20)$$

This is similar to the passage from quantum mechanics to classical mechanics. This can be generalized to all semisimple Lie groups. If we parametrize $x_i$ by polar coordinates (see eq(3.9)) we see that the right hand side of the previous equation is nothing else but the Poisson brackets. Consider now the basis $T^{(N)}_{l,m}$ of $SU(N)$ obtained by replacing in (3.12) the matrices $J_i$ by the rescaled ones $T_i$. Then according to (3.15) we obtain

$$\lim_{N\to\infty}\frac{N}{\imath}[T^{(N)}_{l,m}, T^{(N)}_{l',m'}] \;=\; \{Y_{l,m}, Y_{l',m'}\} \qquad (3.21)$$

If we replace the left hand side with eq (3.13) we obtain eq (3.14). From the above discussion it is obvious that the membrane equations of motion and constraint are the semiclassical limit $N\to\infty$ of the corresponding matrix equations. Going from the membranes to the matrix model is analogous to the correspondence of classical with quantum mechanics. The various observables of the classical membrane correspond to $N\times N$ matrices but there are ordering ambiguities.

Below we present an explicit construction of a completely symmetrized basis of observables in the matrix model which corresponds to the basis of spherical harmonics as was pointed out by eq.(3.10). This method was first developed by Schwinger [14].

Let $\alpha$ a complex null three dimensional vector i.e. $\alpha^2=\alpha\cdot\alpha=0$ parametrized by two complex numbers $z_+, z_-$

$$\alpha_1 \;=\; -z_+^2+z_-^2, \quad \alpha_2 = -\imath(z_+^2+z_-^2), \alpha_3 = 2z_+z_- \qquad (3.22)$$

If $\mathbf{r}$ as the position vector then $(a\cdot r)^k$ is a spherical harmonic of order k given by :

$$\frac{(a\cdot r)^k}{2^k k!} \;=\; \left[\frac{4\pi}{2k+1}\right]^{1/2}\sum_m \Phi_{jm}(z)Y_{jm}(r) \qquad (3.23)$$

where

$$\Phi_{jm}(z) \;=\; \frac{z_+^{j+m} z_-^{j-m}}{[(j+m)!(j-m)!]^{1/2}} \tag{3.24}$$

and also $Y_{jm}(\mathbf{r})$ usually designates a spherical harmonic. Here it includes a factor $r^k$. Accordingly

$$\frac{(a\cdot J)^k}{2^k k!} \;=\; \left[\frac{4\pi}{2k+1}\right]^{1/2} \sum_m \Phi_{jm}(z)\hat{Y}_{jm}(J) \tag{3.25}$$

in which $\hat{Y}_{jm}(J)$ differs from the analogous $Y_{jm}(r)$ only in that the order of factors is significant.

These operators have the following properties :

$$\hat{Y}_{jm}(\mathbf{J})^\dagger \;=\; (-1)^m \hat{Y}_{j-m}(\mathbf{J}). \tag{3.26}$$

If J belongs to the spin j representation they also satisfy an orthogonality and tracelessness property given by eqs.

$$\frac{1}{2j+1} tr\hat{Y}_{j_1m_1}(\mathbf{J})^\dagger \hat{Y}_{j_2m_2}(\mathbf{J}) \;=\; \frac{1}{4\pi}[j(j+1)]^{j_1}\delta_{j_1j_2}\delta_{m_1m_2} \tag{3.27}$$

$$\frac{1}{2j+1} tr\hat{Y}_{jm}(J) \;=\; \delta_{j0} \tag{3.28}$$

The $N\times N$ matrices $\hat{Y}_{lm}$ are nothing else but the standard spherical tensor operators of Quantum Mechanics [17]

## IV. STABILITY

The equation of motion for the supermembrane in six dimensions may be written as

$$\ddot{X}_i = \{X_j, \{X_j, X_i\}\} \tag{4.1}$$

where summation is implied in the j indices and {} stands for the Poisson bracket with respect to the angular coordinates $\theta,\ \phi$. The Gauss constraint that also needs to be satisfied is

$$\left\{\dot{X}_i, X_i\right\} = 0 \tag{4.2}$$

where $i,j = 1,2..6$. We now define $Y_i \equiv X_{i+3}$ with $i = 1,2,3$. This constraint is preserved by the equations of motion and therefore if it is initially obeyed (as is the case in what follows) it will be obeyed at all times. The equations of motion are

$$\begin{aligned} \ddot{X}_i &= \{X_j, \{X_j, X_i\}\} + \{Y_j, \{Y_j, X_i\}\} \\ \ddot{Y}_i &= \{X_j, \{X_j, Y_i\}\} + \{Y_j, \{Y_j, Y_i\}\} \end{aligned} \tag{4.3}$$

We now use the ansatz of a rotating spherical membrane in analogy with the matrix membrane ansatz given in [5]:

$$\begin{aligned} X_i &= r_i(t)e_i(\theta,\phi) \\ Y_i &= s_i(t)e_i(\theta,\phi) \end{aligned} \tag{4.4}$$

where the generators $e_i(\theta,\phi)$ are defined as

$$\begin{aligned} e_1 &= \sin\theta\ \sin\phi \\ e_2 &= \sin\theta\ \sin\phi \\ e_3 &= \cos\theta \end{aligned} \tag{4.5}$$

satisfy the relations

$$\{e_i, e_j\} = -\epsilon_{i,j,k}e_k \tag{4.6}$$

Using now the ansatz (4.4) in the equations of motion (4.3) we obtain the differential equations obeyed by the functions $r(t),\ s(t)$

$$\ddot{r}_i = -(r^2+s^2-r_i^2-s_i^2)r_i \tag{4.7}$$
$$\ddot{s}_i = -(r^2+s^2-r_i^2-s_i^2)s_i \tag{4.8}$$

where $r^2 = r_1^2+r_2^2+r_3^2$, $s^2 = s_1^2+s_2^2+s_3^2$. The solution of (4.7) is of the form

$$r_i = R_i\cos(\omega_i t+\phi_i) \tag{4.9}$$
$$s_i = R_i\sin(\omega_i t+\phi_i) \tag{4.10}$$

with

$$\omega_i^2 = R^2 - R_i^2 \tag{4.11}$$

where $R^2 = R_1^2+R_2^2+R_3^2$.

We observe that all the relations we obtained for the ansatz (4.4) are identical with those of ref. [5] for the matrix model solution of a bound state of $ND_0/D_2$-branes where the three functions $e_i(\theta,\phi)$ are replaced by N-dimensional representational matrices $J_\imath(\imath = 1,2,3)$ of $SU(2)$. This unique isomorphism is due to the existence of an $SU(2)$ subgroup of the infinite dimensional area preserving group of the sphere $(sDiff(S^2))$. It is known that there is no other finite dimensional subalgebra of $(sDiff(S^2)$. As we shall see the stability analysis of the spherical membrane solution follows an isomorphic pattern with the matrix model solution . We point out that in ref [5] the matrix solution was found to be stable under a restricted set of the $l=1$ pertutbations. In the following we extend their analysis for every value of $l$ and we complete also the case $l=1$. The variational equations that correspond to the splitting in eq. (4.3) between $X_i$ and $Y_i$ are:

$$\begin{aligned} \delta\ddot{X}_i \;=\; & \{\delta X_j, \{X_j, X_i\}\} + \{X_j, \{\delta X_j, X_i\}\} \\ & + \{X_j, \{X_j, \delta X_i\}\} + \{\delta Y_j, \{Y_j, X_i\}\} \\ & + \{Y_j, \{\delta Y_j, X_i\}\} + \{Y_j, \{Y_j, \delta X_i\}\} \end{aligned} \tag{4.12}$$

The corresponding perturbation for $\delta Y_i$s and $Y_i$s satisfy equations that are obtained by exchanging $\delta X_i \leftrightarrow \delta Y_i, X_i \leftrightarrow Y_i$ in eq(4.12). The equations of motion imply the validity of the constraint at all times

$$\{\dot{X}_i, X_i\} + \{\dot{Y}_i, Y_i\} = 0 \tag{4.13}$$

This is obtained by taking the time derivative of eq.(4.13) and by applying the equations of motion and the Jacobi identity. By expanding a configuration which at $t = 0$ is consistent with the constraint (4.13) around any classical solution we see (by using only the linearized eqs.(4.12)) that the variation $\delta X_i$ and $\delta Y_i$ satisfy the constraint

$$\{\delta\dot{X}_i, X_i\} + \{\dot{X}_i, \delta X_i\} + \{\delta\dot{Y}_i, Y_i\} + \{\dot{Y}_i, \delta Y_i\} = 0 \tag{4.14}$$

for all times. In order to proceed with the study of these variational equations we observe that:

$$\{e_i, Y_{lm}\} = i\hat{L}_i Y_{lm} \tag{4.15}$$

where $\hat{L}_i$ are the angular momentum operators of Quantum Mechanics in spherical coordinates. In the N-dim. representation ($N = 2l + 1$) the right hand side is given by :

$$\hat{L}_i Y_{lm} = \sum_{m'} (L_i)_{mm'} Y_{lm'} \tag{4.16}$$

where $(L_i)_{mm'}$ are the matrix representations of $SU(2)$. In what follows on the basis of the previous argument it is enough to consider the specific variation

$$\delta X_i(t) = \sum_m \epsilon_i^m(t) Y_{lm},\ \delta Y_i(t) = \sum_m \zeta_i^m(t) Y_{lm} \tag{4.17}$$

with initial conditions $\epsilon_i(0) = 0, \zeta_i(0) = 0$ but with $\dot{\epsilon}_i(0) \neq 0,\ \dot{\zeta}_i(0) \neq 0$. As a result the constraint equation is satisfied at $t = 0$.

In order to study the stability of this solution we consider the following general form of perturbations

$$\begin{aligned} \delta X_i(t) &= \sum_{l,m} \epsilon_i^{lm}(t) Y_{lm}(\theta, \phi) \\ \delta Y_i(t) &= \sum_{l,m} \zeta_i^{lm}(t) Y_{lm}(\theta, \phi) \end{aligned} \tag{4.18}$$

We now use the fact that

$$\{e_i, Y_{lm}(\theta, \phi)\} = i\hat{L}_i Y_{lm}(\theta, \phi) \tag{4.19}$$

where $\hat{L}_i$ is the angular momentum differential operator. This implies that

$$\{e_i, Y_{lm}(\theta, \phi)\} = \sum_{m'} a_{lm'} Y_{lm'}(\theta, \phi) = i \sum_{m'} (L_i)_{mm'} Y_{lm'} \tag{4.20}$$

where $L_\imath$ are the angular momenta in the representation $l = (N - 1)/2$. A crucial observation is that the sum involves spherical harmonics of the same $l$ as the spherical harmonic in the Poisson bracket. This decouples the various $l$ fluctuation modes and simplifies the differential equations obeyed by the modes $\epsilon_i^{lm}$ and $\zeta_i^{lm}$. This feature is specific to the particular background solution of the spherical membrane.

The equations obeyed by the fluctuation modes $\epsilon$ and $\zeta$ may be written as

$$\begin{aligned} \ddot{\epsilon}_i + R^2 l(l+1)\epsilon_i &= R^2 \cos\omega t T_{ij}[\epsilon_j\ cos\omega t + \zeta_j\ sin\omega t] \\ \ddot{\zeta}_i + R^2 l(l+1)\zeta_i &= R^2 \sin\omega t T_{ij}[\epsilon_j\ cos\omega t + \zeta_j\ sin\omega t] \end{aligned} \tag{4.21}$$

where

$$T_{ij} = L_i L_j - 2i\epsilon_{ijk} L_k \tag{4.22}$$

and $L_i$ is the angular momentum operator.

We now perform a rotation and define the new variables $\theta_i$ and $\eta_i$

$$\theta_i \equiv \epsilon_i \cos\omega t + \zeta_i \sin\omega t \tag{4.23}$$

$$\eta_i \equiv -\epsilon_i \sin\omega t + \zeta_i \cos\omega t \tag{4.24}$$

The equations obeyed by the new variables $\theta$ and $\eta$ may now be shown to be

$$\ddot{\theta}_i - 2\omega\dot{\eta}_i + [R^2 l(l+1) - \omega^2]\theta_i - R^2 T_{ij}\theta_j = 0 \tag{4.25}$$

$$\ddot{\eta}_i - 2\omega\dot{\theta}_i + [R^2 l(l+1) - \omega^2]\eta_i = 0 \tag{4.26}$$

where from the equation of motion of the background solution we have $\omega^2 = 2R^2$. Using this relation and defining the rescaled time variable $\tau = Rt$ we obtain the system

$$\begin{aligned} \ddot{\theta} - 2\sqrt{2}\dot{\eta} + [l(l+1) - 2]\theta &= T\theta \\ \ddot{\eta} - 2\sqrt{2}\dot{\theta} + [l(l+1) - 2]\eta &= 0 \end{aligned} \tag{4.27}$$

where the time derivative is with respect to the new variable $\tau$ and we have suppressed indices. To investigate the stability we now use the ansatz

$$\begin{pmatrix} \theta \\ \eta \end{pmatrix} = e^{i\lambda\tau} \begin{pmatrix} a \\ b \end{pmatrix} \tag{4.28}$$

in the system (4.27) to obtain the equations for $a$ and $b$

$$b = \frac{2\sqrt{2}\lambda a}{i[l(l+1) - 2 - \lambda^2]} \tag{4.29}$$

and

$$Ta = [l(l+1) - 2 - \lambda^2 - \frac{8\lambda^2}{l(l+1) - 2 - \lambda^2}]a \quad (4.30)$$

Therefore the problem of finding if $\lambda$ has an imaginary part (which would imply instability) has been reduced to solving the eigenvalue problem of the $3(2l+1) \times 3(2l+1)$ Hermitian matrix $T$. In order to solve this problem we will use the spectral theorem of algebra as follows: We expand the matrix $T$ into a complete set of three projector matrices and read from the coefficient of each term the eigenvalues which have degeneracy equal to the trace of each projector. The total number of eigenvalues should add up to $3(2l+1)$ which is the dimensionality of $T$. Eigenverctors can also be found by acting with each one of the projectors on any vector on the large space $3(2l+1)$. The derivation of the explicit form of a *complete set* of eigenvectors however is a non-trivial task.

Defining the two $3(2l+1) \times 3(2l+1)$ Hermitian matrices

$$P = \frac{1}{l(l+1)} L_i L_j$$
$$Q = i\epsilon_{ijk} L_k \quad (4.31)$$

we observe that $P$ is a projector *ie* $P^2 = P$ and $Q$ satisfies

$$Q^2 = l(l+1)(I - P) + Q \quad (4.32)$$

It is straightforward to show that $T$ may be expressed as:

$$T = [l(l+1) - 2]P + 2lR_+ - 2(l+1)R_- \quad (4.33)$$

where $P$, $R_+$ and $R_-$ are orthocanonical projectors with the usual properties $R_+R_- = PR_+ = PR_- = 0$ and $R_+^2 = R_+$, $R_-^2 = R_-$. They are defined as

$$R_- \equiv \frac{1}{(2l+1)}[(l+1)(I-P) - (I-Q)] \quad (4.34)$$
$$R_+ \equiv \frac{1}{(2l+1)}[l(I-P) + I - Q] \quad (4.35)$$

From the spectral expansion of T (4.33) it becomes clear that its eigenvalues are $q_1 = [l(l+1) - 2]$, $q_2 = 2l$ and $q_3 = -2(l+1)$ with multiplicities given by the traces of the corresponding projectors ie $2l+1$, $2l+3$ and $2l-1$. The corresponding eigenvectors are given by the set of $Pv$, $R_+v$ and $R_-v$ where $v$ runs over $\mathcal{R}^{3(2l+1)}$

Given now the eigenvalues of $T$ $q_i$ we are in position to use equation (4.30) and find the form of the corresponding eigenfrequencies $\lambda_i$. We must solve the algebraic equation

$$l(l+1) - 2 - \lambda_i^2 - \frac{8\lambda_i^2}{l(l+1) - 2 - \lambda_i^2}] = q_i \quad (4.36)$$

which leads to a pair of solutions for each $\lambda_i^2$. These solutions are

$$\lambda_{1a}^2 = 0, \qquad \lambda_{1b}^2 = l^2 + l + 6 \quad (4.37)$$
$$\lambda_{2a}^2 = l^2 - 3l + 2, \quad \lambda_{2b}^2 = l^2 + 3l + 2 \quad (4.38)$$
$$\lambda_{3a}^2 = l^2 - l, \quad \lambda_{3b}^2 = l^2 + 5l + 6 \quad (4.39)$$

It is obvious that all $\lambda_i^2$ are non-negative and therefore the eigenfrequencies $\lambda_i$ are all real. This implies that the membrane solution studied is stable to first order in perturbation theory.

Perturbations of the classical solutions along the $7, 8, 9$ dimensions can be parametrized as

$$\delta Z_i \equiv \delta X_{i+6} \quad (4.40)$$

with $i = 1, 2, 3$ and

$$\delta \ddot{Z}_{ii} = \{X_j, \{X_j, \delta Z_i\}\} + \{Y_j, \{Y_j, \delta Z_i\}\} \quad (4.41)$$

For the spherically symmetric membrane using the ansatz

$$\delta Z_i = \rho_i^m(t) Y_l^m(\theta, \phi) \quad (4.42)$$

we find

$$\ddot{\rho_i^m} + R^2 l(l+1)\rho_i^m = 0 \quad (4.43)$$

that is stable harmonic motion.

The above considered fluctuations are more general than the constraint (2.3) would allow. This however does not invalidate our analysis since we have shown that even those generalized fluctuations do not include an instability mode and therefore this will also be true for the physical fluctuations.

These results however can not be valid to all orders. It is well known that the ground state of the system studied is a string or point configuration and therefore finite size fluctuations will eventually lead to a decay to the vacuum. This metastability may also be seen by considering higher orders in perturbation theory where non-linear effects start to show up.

We conclude our work by stressing the analogy between the membrane stability analysis presented above and the corresponding one for the matrix model $ND_0/D_2$ spherical solution of ref [5]. As discussed before spherical matrix and membrane solutions are isomorphic [4]. Moreover the linearized problems for the fluctuations preserve the same isomorphism due to the specific spin 1 form of the solution. The matrix solution is a bound state of N $D_0$ branes attached on a $D_2$ brane whose stability properties is obtained using our continuous membrane investigation. Indeed one only has to replace the perturbations $\delta X_i = \sum_m \epsilon_i^m Y_{lm}$ by the matrix fluctuations $\delta \hat{X}_i = \sum_m \epsilon_i^m \hat{Y}_{lm}$. Here $\hat{Y}_{lm}$ are the $SU(2)$

tensor spherical harmonics defined in eq(3.25). One can easily check that the 2l+1 dimensional vectors $\epsilon_i(\zeta_i)$ satisfy the same eqs.(4.21). On the other hand higher order perturbations differ by terms of order $1/N$ for every $N$.

At the linearized level small fluctuations of various multipole deformations described by $l = 1, 2, ...$ do not destabilize the calssical solution. On the other hand we know that the potential term in the Hamiltonian of the supermembrane has as global minimum configurations tensionless strings and points. Therefore there are finite size deformations which can lead to instabilities of the rotating classical solution. In order to determine the modes of instability one has to study the potential of the moduli space of finite size deformations.

Upon completion of our work we were informed by K. G. Savvidy of analogous results in the matrix model [19]

## V. ACKNOWLEDGEMENTS

We are grateful to K.G.Savvidy for informing us about his work. We also acknowledge G. Athanasiou, I. Kiritsis and T. Tomaras for useful discussions.

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