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Octonionic self-duality for supermembranes

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Abstract

In this work we study the recently introduced octonionic duality for membranes. Restricting the self-duality equations to seven space dimensions, we provide various forms for them which exhibit the symmetries of the octonionic and quaternionic structure. These forms may prove to be useful for the question of the integrability of this system. Introducing a consistent quadratic Poisson algebra of functions on the membrane we are able to factorize the time dependence of the self-duality equations. We report further the general linear embeddings of the three-dimensional system into the seven-dimensional system using the invariance of the self-duality equations under the exceptional group G_2. © 1998 Elsevier Science B.V.

1. Introduction

M theory is the leading candidate for the unification of all superstring theories in their perturbative and non-perturbative sector. This theory contains $N = 1$, eleven-dimensional supergravity and at least a sector of supermembranes and their magnetic duals the superfive branes [1,2]. These extended objects exist as solitons of eleven-dimensional supergravity and they are distinguished from fundamental superbranes as solitonic superbranes [3].

Most of the recent work on compactifications of M theory is concentrated on a unified "proof" of various non-perturbative dualities of superstring theories connecting their strong and weak coupling sectors – or small with large volumes – of the compactifying

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manifolds [1,2]. There is a line of attack from the point of view of the 11-d superbranes which either uses double-dimensional reduction to connect with type IIA or heterotic superstrings, or using purely classical world-volume dualities of the superbranes which appears miraculously to explain non-perturbative phenomena (dualities of superstring theories) [4].

From the point of view of superstring theories, supermembranes in non-compact eleven dimensions correspond to the strong coupling regime of the superstrings [1].

Many basic questions concerning supermembrane theories have not been answered. A top priority is the derivation of the eleven-dimensional supergravity theory as a low energy effective action of the supermembrane. To do this one needs to understand the quantum mechanics of the supermembrane, that is define a sensible perturbation theory. This is an extremely hard problem for two reasons. First, the moduli space of three-dimensional Remannian metrics is largely unknown and a representative theory of the three-dimensional diffeomorphism group or (in the light-cone gauge) of the area preserving the diffeomorphism group of the supermembrane is lacking. Second, unlike the string where in the light-cone gauge the theory becomes an infinite set of free transverse oscillators, in the supermembrane case the light-cone gauge does not fully solve the constraints and there is no coupling constant in the interaction term.

Fortunately, the Hamiltonian in the light-cone gauge is of Yang–Mills (YM) type with the gauge group the area preserving diffeomorphisms of the membrane [5–7].

Another issue is the following. During the compactification from eleven dimensions to ten, one is freezing an infinite number of string degrees of freedom of the supermembrane and considering only the Kaluza–Klein dilatonic modes which are supposed to be the infinite tower of superstring solitons which completes the duality picture. It may be possible that taking into account in a controllable way the interaction of the remaining string excitations of the supermembrane, one could define a perturbation theory [8]. Recently, an old mode regularization of the supermembrane through $SU(N)$ matrix super YM mechanics has been re-incarnated as a possible candidate model for M theory [9]

Another possible approach for defining a perturbative expansion for the eleven-dimensional supermembrane is to study various compactifications of the 11-d supergravity where the classical supermembrane has very simple dynamics (it can even be static-stretched) and then study the quantum excitations of the supermembrane around these classical solutions. In this way one hopes to obtain a classical state which could be used as a quantum vacuum state for the membrane. One test would be to find in the excitation spectrum of the supermembrane the 11-d supergravity multiplet around the classical background. The problem is that one has to preserve in one way or another the $N = 1$, 11-d sypersymmetry during these compactifications [3,4]. Following past work on the compactification of 11-d supergravity on the seven sphere [10] there is recent activity on octonionic solitons for strings and supermembranes [11]. In this work, specific background field configurations of the compactified supergravity on the seven sphere are considered as various fiber bundles which are coupled through their singularities to supermembrane sources.

In this work, we want to move in a different direction which exploits some aspects of the non-perturbative structure of the supermembrane vacuum in flat space time, studying classical Euclidean time equations which describe quantum tunneling processes between classical configurations of the supermembrane which could be considered as vacua of different topological sectors. Although there is an extensive article [3], where essentially the background field equations are solved, as far as we know the question of the Euclidean membrane as an extended object connecting different topological sectors has not been addressed except in Refs. [12–14]. The topological charge and the Bogomol'nyi bound known from supersymmetric YM theory can be extended to Euclidean supermembranes in $(4+1)$ [12,13] and, as has been shown recently, in $(8+1)$ dimensions [14]. In Section 2 we recall the main results of the reports [12,13] where the self-dual bosonic membrane in $(2+1)$ and $(4+1)$ dimensions was introduced. In Section 3, the generalization reported in Ref. [14] in $(8+1)$ dimensions is described in a compact form and possible factorizations of the time dependence are discussed. In Section 4, the same equations in octonionic and quartenionic representations are introduced which exhibit specific properties of the self-duality equations. Finally, in Section 5 the general formulation for embedding the three-dimensional solutions into seven dimensions is described and the constraint equations are derived. Some examples of specific embeddings of the $(4+1)$-dimensional system into $(8+1)$ dimensions are also analysed.

2. $SU(N)$ Yang–Mills and membranes

To start, we recall that it has been known for some time that the supermembrane Hamiltonian in the light-cone gauge is a very close relative of Yang–Mills (YM) theories in the gauge $A_0 = 0$ and in one space dimension less [5,6]. To describe this relationship in more detail, we restrict our discussion to the bosonic part of the Hamiltonian of the supermembrane in the light-cone gauge and to spherical topology for the membrane [6,15,13]. In Ref. [15], using results of Ref. [5], it was pointed out that, in the large-N limit, $SU(N)$ YM theories have, at the classical level, a simple geometrical structure with the $SU(N)$ matrix potentials $A_\mu(X)$ replaced by c-number functions of two additional coordinates, θ and ϕ, of an internal sphere S^2 at every space-time point, while the $SU(N)$ symmetry is replaced by the infinite-dimensional algebra of area-preserving diffeomorphisms of the sphere S^2 called SDiff(S^2). The $SU(N)$ fields ($N \times N$ matrices)

$$A_\mu(X) = A_\mu^\alpha(X) t^\alpha, \quad t^\alpha \in SU(N),\ \alpha = 1,2,\dots,N^2-1,\ \mu = 0,1,\dots d-1 \tag{1}$$

in the large-N limit become c-number functions of an internal sphere S^2

$$A_\mu(X,\theta,\phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} A_\mu^{lm}(X) Y_{lm}(\theta,\phi), \tag{2}$$

where $Y_{lm}(\theta,\phi)$ are the spherical harmonics on S^2. The local gauge transformations

$$\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega], \quad \omega = \omega^\alpha t^\alpha, \tag{3}$$

and

$$\delta F_{\mu\nu} = [F_{\mu\nu}, \omega], \tag{4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{5}$$

are replaced by

$$\delta A_\mu(X,\theta,\phi) = \partial_\mu \omega(X,\theta,\phi) + \{A_\mu, \omega\}, \tag{6}$$

$$\delta F_{\mu\nu}(X,\theta,\phi) = \{F_{\mu\nu}, \omega\}, \tag{7}$$

where

$$F_{\mu\nu}(X,\theta,\phi) = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\mu\}, \tag{8}$$

and the Poisson bracket on S^2 is defined as

$$\{f,g\} = \frac{\partial f}{\partial\phi}\frac{\partial g}{\partial\cos\theta} - \frac{\partial g}{\partial\phi}\frac{\partial f}{\partial\cos\theta}. \tag{9}$$

So that the commutators are replaced by Poisson brackets according to

$$\lim_{N\to\infty} N[A_\mu, A_\nu] = \{A_\mu, A_\nu\}. \tag{10}$$

Then the YM action in the large-N limit becomes [15]

$$S_\infty = \frac{1}{16\pi g^2}\int_{S^2} d\Omega \int d^d X\, F_{\mu\nu}(X,\theta,\phi) F^{\mu\nu}(X,\theta,\phi), \tag{11}$$

where

$$g = \lim_{N\to\infty} \frac{g_N}{N^{3/2}}. \tag{12}$$

This large-N limit of $SU(N)$ YM theories was found by making use of the relation between the $SU(N=2s+1)$ algebra in a particular basis (up to spin s $SU(2)$ tensor $N\times N$ matrices) and SDiff(S^2) in the basis of the spherical harmonics $Y_{lm}(\theta,\phi)$. In present-day language, this SDiff(S^2) YM theory corresponds to the effective theory of infinite number, $N\to\infty$, $d-1$-dimensional Dirichlet branes [16,17]. Similar considerations hold for membranes of different topologies, torus, double torus, etc. [18]. Here, we note that the recently proposed matrix theory, which is claimed to be the long sought formulation of M theory, is nothing but the $SU(N)$ supersymmetric YM mechanics which was used as a consistent truncation of the supermembrane [5,6,19].

The large-N limit is a very specific one which depends on the appropriate basis of $SU(N)$ generators convenient for the topology of the membrane and it has nothing to do, at least in a direct way, with the planar approximation of YM theories. Also, it is different from the large-N limit used in matrix theory.

In the case of spherical membranes the SDiff(S^2) YM theory describes the dynamics of an infinite number of $D0$-branes forming a topological two sphere. In the light-cone gauge the transverse coordinates X_i $(i = 1,\dots,9)$ of the 11-d bosonic part of the supermembrane satisfy the following equations

$$\ddot{X}_i = \{X_k, \{X_k, X_i\}\}, \quad i,k = 1,\dots 9, \tag{13}$$

where summation over repeated indices is implied. The corresponding Gauss law, which is the generator of the SDiff(S^2) group, is given by the constraint

$$\{X_i, \dot{X}_i\} = 0. \tag{14}$$

In Ref. [12] Euclidean bosonic membranes in three-dimensional target space have been introduced, defining the topological charge density to be

$$\Omega(X) = \frac{1}{3!}\epsilon^{abc} f_{ijk} X^i_a X^j_b X^k_c, \tag{15}$$

where a, b, c run from 1 to 3 and i, j, k from 1 to d space time dimensions,

$$X^i_a = \partial_{\xi_a} X^i \tag{16}$$

and $\xi_{1,2,3}$ are the world-volume coordinates. The self-duality equations are defined as

$$P^a_i = \pm\tfrac{1}{2}\epsilon^{abc} f_{ijk} X^j_b X^k_c. \tag{17}$$

Here, P^a_i are the canonical momenta

$$P^a_i = T\frac{\delta}{\delta X^i_a}(\mathrm{Det}[X^i_a X^i_b])^{1/2}. \tag{18}$$

The self-duality equations for the case $d = 3$ and $f_{ijk} = \epsilon_{ijk}$ were shown to satisfy both the constraints and the equations of motion. Solutions were given for the case of sphere and torus. In Ref. [13] 3-d Euclidean self-duality equations in the light-cone gauge (that is $4+1$-dimensional target space) for the bosonic part of the supermembrane could be written in analogy with the 3-d Nahm equations of self-dual BPS YM monopoles. In the light-cone gauge this means that one had to fix six of the nine transverse coordinates to be constants. This constraint solves the second-order Eqs. (13) for the six coordinates.

Then the self-duality equations are

$$\dot{X}_i = \tfrac{1}{2}\epsilon_{ijk}\{X_j, X_k\}, \quad i, j = 1, 2, 3. \tag{19}$$

The self-duality equations solve automatically the second-order Euclidean time equations as well as the Gauss law due to the Jacobi identity for ϵ and its well-known properties. The above system has a Lax pair and an infinite number of conservation laws [13]. In order to see this, first we rewrite Eqs. (19) in the form

$$\dot{X}_+ = i\{X_3, X_+\}, \qquad \dot{X}_- = i\{X_3, X_-\}, \qquad \dot{X}_3 = \tfrac{1}{2}i\{X_+, X_-\}, \tag{20}$$

where

$$X_\pm = X_1 \pm iX_2. \tag{21}$$

There exists a linear system corresponding to Eq. (20):

$$\dot{\psi} = L_{X_3+\lambda X_-}\psi, \qquad \dot{\psi} = L_{(1/\lambda)X_+-X_3}\psi, \tag{22}$$

where the differential operators L_f are defined as

$$L_f \equiv i\left(\frac{\partial f}{\partial\phi}\frac{\partial}{\partial\cos\theta} - \frac{\partial f}{\partial\cos\theta}\frac{\partial}{\partial\phi}\right). \tag{23}$$

The compatibility condition of Eq. (22) is

$$[\partial_t - L_{X_3+\lambda X_-}, \partial_t - L_{(1/\lambda)X_+-X_3}] = 0, \tag{24}$$

from which, comparing the two sides for the coefficients of the powers $1/\lambda$, λ^0, λ^1 of the spectral parameter λ, we find Eq. (20). From the linear system (22) using the inverse scattering method one could, in principle, construct all solutions of the self-duality equations.

Specific solutions could be obtained due to the existence of an $SU(2)$ subalgebra of SDiff(S^2) which happens to be its only finite-dimensional subalgebra. Using this $SU(2)$ subalgebra, for spherically symmetric solutions it can be shown that the system reduces to the Toda $SU(2)$ equations. Another method for finding solutions of the integrable system (20) has been proposed in Ref. [20], where the system is linearized by considering the target space variables as world-volume variables and vice versa. More recently, there have been discussions of the same issue by other authors [14,21]. In Ref. [22] the connection with self-dual Einstein equations has been discussed. Before closing this section we would like to note that the Euclidean membrane configurations which are solutions of the self-duality equations are expected to interpolate between classical vacuum configurations of the membrane that is, points or strings. Also, the case of the membrane is the first in the series of extended objects where there is a gauge principle to define the interactions and the possibility arises for topology change through gauge interactions. The case of the string has an ad hoc interaction which is not enforced uniquely by any gauge principle. Moreover, the classical vacua of string are points. [12]

3. The octonionic structure of the self-duality equations

An obvious way to generalize duality for super p-branes is to use Poincaré duality. For the fundamental supermembranes, in particular, this has been done by Duff et al. [23] and has been exploited to prove various conjectures of string–string, string–membrane and membrane–membrane dualities [24–26]. Another type of duality has recently been investigated [14,27] which is based on the existence of the last real division algebra, the octonionic or Cayley algebra [28]. The work of Ref. [14] is based on the similarity between the supermembrane and the super YM theories referred to previously and the work on eight-dimensional YM instantons many years ago [29]. Another way of considering the work of Ref. [14] is as an extension of the quaternionic case [13] using the possibility of defining a cross product of two vectors in eight dimensions through the multiplication rule of octonions.

In this section we restrict the self-duality equations of Ref. [14] to seven dimensions by choosing fixed values for eight and nine membrane coordinates. Then, the self-duality equations [14] become

$$\dot{X}_i = \tfrac{1}{2}\Psi_{ijk}\{X_j, X_k\}, \tag{25}$$

where Ψ_{ijk} is the completely antisymmetric tensor which defines the multiplications of octonions [28]. The Gauss law results automatically by making use of the Ψ_{ijk} cyclic symmetry

$$\{\dot{X}_i, X_i\} = 0. \tag{26}$$

The Euclidean equations of motion are obtained as

$$\ddot{X}_i = \tfrac{1}{2}\Psi_{ijk}\left(\{\dot{X}_j, X_k\} + \{X_j, \dot{X}_k\}\right) \tag{27}$$

$$= \{X_k, \{X_i, X_k\}\}, \tag{28}$$

where use has been made of the identity

$$\Psi_{ijk}\Psi_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} + \phi_{ijlm}, \tag{29}$$

and of the cyclic property of ϕ_{ijlm} [28].

As in the case of the 3-d system we may try to factorize the time dependence. We assume the following factorization

$$X_i = Z_{ij}(t)f_j(\xi). \tag{30}$$

Then, from Eq. (25) we obtain

$$\dot{Z}_{im}f_m = \tfrac{1}{2}\Psi_{ijk}Z_{jl}Z_{kn}\{f_l, f_n\}. \tag{31}$$

We observe that if we make the ansatz for the 7×7 matrix

$$\dot{Z}_{im}(t)\Psi_{mln} = \Psi_{ijk}Z_{jl}(t)Z_{kn}(t), \tag{32}$$

then the equation

$$f_i = \tfrac{1}{2}\Psi_{ijk}\{f_j, f_k\} \tag{33}$$

is automatically satisfied, while at the same time we have succeeded in disentangling the time dependence from the self-duality equation. Therefore, the problem is reduced to finding solutions for $f_i(\xi)$ and Z_{kl} equations separately.

Another equivalent form of the previous equation for the matrices Z_{ij} is

$$\dot{Z}_{ij} = \tfrac{1}{6}\Psi_{ikl}\Psi_{jmn}Z_{km}Z_{ln}. \tag{34}$$

In the case of diagonal matrices $Z_{ij} = \delta_{ij}R_j(t)$, we have

$$\dot{R}_i = \frac{1}{6}\Psi^2_{ikl}R_kR_l. \tag{35}$$

We now make some observations about the symmetries of Eqs. (25) and (33). If X_i is a solution of Eq. (25) then for every matrix R of the group G_2, which is a subgroup of $SO(7)$, then

$$Y_i = R_{ij}X_j \tag{36}$$

is automatically a solution of the same equation because the elements of G_2 preserve the structure constants Ψ_{ijk}. In components

$$\Psi_{ijk}R_{kl} = \Psi_{imn}R_{mj}R_{nl}. \tag{37}$$

The above relation shows how to define G_2 group elements starting from two orthonormal seven-vectors. The equation is obviously covariant under SDiff$(S2)$ transformations. One can define combined G_2 and SDiff$(S2)$ transformations to obtain $SO(3)$ spherically symmetric solutions since $SO(3)$ can be realized as a subalgebra of SDiff$(S2)$.

We note that, in principle, it is possible to look for non-linear symmetries of the self-duality equations generalizing Eq. (36)

$$Y_i = f_i(X), \tag{38}$$

where $f_i(X)$ must satisfy the equation

$$\Psi_{ijk}\frac{\partial f_k}{\partial X_l} = \Psi_{imn}\frac{\partial f_m}{\partial X_j}\frac{\partial f_n}{\partial X_l}. \tag{39}$$

In the following we examine the self-consistency of Eq. (33). Multiplying by Ψ_{ilm} we obtain

$$\Psi_{ilm}f_i = \{f_l, f_m\} + \frac{1}{2}\phi_{lmjk}\{f_j, f_k\}. \tag{40}$$

Then, since the Poisson brackets satisfy the Jacobi identity, the above equation is constrained to satisfy the identity

$$\frac{1}{3}\phi_{ijkl}f_l = \Psi_{ijm}\{f_m, f_k\} + \text{cyclic perm. of } (ijk). \tag{41}$$

This system of equations is exactly the same as Eq. (33).

Another check for the self-consistency of f_i equations can be found as follows. Define the tensors

$$X^{ij}{}_{kl}(u) = \Delta^{ij}{}_{kl} + \frac{u}{4}\phi^{ij}{}_{kl}, \tag{42}$$

where $\Delta^{ij}{}_{kl} = \frac{1}{2}(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k)$ and $\phi^{ij}{}_{kl} \equiv \phi_{ijkl}$. Then, Eq. (33) can be written as

$$\Psi_{ijk}f_k = X^{ij}{}_{lm}(2)\{f_l, f_m\}. \tag{43}$$

Using now the algebra of the $X^{ij}{}_{kl}(u)$ tensors discussed in detail in Appendix A we can prove that both the identities

$$\Psi_{ijk}X^{jk}{}_{lm}(-1) = 0 \tag{44}$$

and

$$X^{ij}{}_{mn}(-1)X^{mn}{}_{kl}(2) = 0 \tag{45}$$

hold, and this terminates the second consistency check.

4. Octonionic and quaternionic formulation of the self-duality equations

The octonionic or Cayley algebra is the appropriate structure to organize the seven self-duality equations [28,29]. The octonionic units o_i satisfy the algebra

$$o_i o_j = -\delta_{ij} + \Psi_{ijk} o_k, \tag{46}$$

where $i = 1, \dots, 7$ are the seven octonionic imaginary units with the property

$$\{o_i, o_j\} = -2\delta_{ij}. \tag{47}$$

We choose the multiplication table [28]

$$\Psi_{ijk} = \begin{cases} 1\ 2\ 4\ 3\ 6\ 5\ 7 \\ 2\ 4\ 3\ 6\ 5\ 7\ 1 \\ 3\ 6\ 5\ 7\ 1\ 2\ 4. \end{cases} \tag{48}$$

In terms of these units an octonion can be written as

$$X = x_0 o_0 + \sum_{i=1}^{7} x_i o_i, \tag{49}$$

with o_0 the identity element. The conjugate is

$$\bar{X} = x_0 o_0 - \sum_{i=1}^{7} x_i o_i. \tag{50}$$

The octonions over the real numbers can also be defined as pairs of quaternions

$$X = (x_1, x_2), \tag{51}$$

where $x_1 = x_1^\mu \sigma_\mu$, $x_2 = x_2^\mu \sigma_\mu$ and the indices μ run from 0 to 3, while $x_{1,2}^0$ are real numbers and $x_{1,2}^i$, $i = 1, 2, 3$, are imaginary numbers. Finally, σ_0 is the Identity 2×2 matrix and σ_i are the three standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{52}$$

If $q = (q_1, q_2)$ and $r = (r_1, r_2)$ are two octonions, the multiplication law is defined as

$$q * r \equiv (q_1, q_2) * (r_1, r_2) = (q_1 r_1 - \bar{r}_2 r_2, r_2 q_1 + q_2 \bar{r}_1), \tag{53}$$

where $q_1 = q_1^0 + q_1^i\sigma_i$ and $\bar{q}_1 = q_1^0 - q_1^i\sigma_i$. One can also define a conjugate operation for an octonion as

$$\bar{q} \equiv \overline{(q_1, q_2)} = (\bar{q}_1, -q_2), \tag{54}$$

and we obtain the possibility of defining the norm and the scalar product q and r

$$q\bar{q} = (q_1\bar{q}_1 + \bar{q}_2 q_2, 0) \tag{55}$$

$$= \sum_{\mu=0}^{3} (x_1^{\mu 2} + x_2^{\mu 2}), \tag{56}$$

$$\langle q|r\rangle = \tfrac{1}{2}(q\bar{r} + \bar{q}r). \tag{57}$$

In terms of the above formalism, the self-duality equations can be written as

$$\dot{X} = \tfrac{1}{2}\{X, X\}, \tag{58}$$

where now $X = X^i o_i$ with $i = 1, \ldots, 7$ and the Poisson bracket for two octonions is defined as

$$\{X, Y\} = \frac{\partial X}{\partial \xi_1}\frac{\partial Y}{\partial \xi_2} - \frac{\partial X}{\partial \xi_2}\frac{\partial Y}{\partial \xi_1}. \tag{59}$$

Using now Eq. (51) and the multiplication rule (53) we can write Eq. (58) as

$$\dot{x}_1 = \tfrac{1}{2}(\{x_1, x_1\} + \{\bar{x}_2, x_2\}), \tag{60}$$

$$\dot{x}_2 = -\{x_2, x_1\} \equiv \{x_2, \bar{x}_1\}, \tag{61}$$

where $x_1 = x_1^\mu \sigma_\mu$ and $x_2 = x_2^\mu \sigma_\mu$. Defining the octonionic units

$$o_0 = (1,0), \quad o_1 = (i\sigma_1, 0), \quad o_2 = (i\sigma_2, 0), \quad o_3 = (-i\sigma_3, 0),$$

$$o_4 = (0,1), \quad o_5 = (0, i\sigma_3), \quad o_6 = (0, i\sigma_2), \quad o_7 = (0, i\sigma_1), \tag{62}$$

we can easily check that the chosen multiplication table for the octonions (48) is satisfied and the seven coordinates X_i are now grouped as $(x_1^0 = 0)$:

$$x_1^i = iX_1, iX_2, iX_3, \tag{63}$$

$$x_2^\mu = X_4, iX_7, iX_6, iX_5 \tag{64}$$

and

$$x_1 = \begin{pmatrix} X_3 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 \end{pmatrix}, \tag{65}$$

$$x_2 = \begin{pmatrix} X_4 + iX_5 & X_6 + iX_7 \\ -(X_6 - iX_7) & X_4 - iX_5 \end{pmatrix}. \tag{66}$$

The organization in Eqs. (65) and (66) of the seven X_i components obtained from the quaternionic formulation will prove very useful for identifying specific classes of solutions, as we will see in the next section.

5. Embeddings of the three-dimensional system

An obvious observation is that any three-dimensional solution is also a solution of the seven-dimensional system discussed here. However, there are various ways to embed a three-dimensional solution in a seven-dimensional system. In this section we discuss solutions of the self-duality equations where the coordinates X_i are linear functions of the $SU(2)$ basis of functions on the sphere, which are the components of the unit vector in three dimensions written in spherical coordinates [13]

$$\{e_a, e_b\} = -\epsilon_{abc}e_c. \tag{67}$$

Thus, our ansatz is

$$X_i(\xi_1, \xi_2, t) = A_i^a(t)e_a(\xi_1, \xi_2), \tag{68}$$

and implies a generalised form of Nahm's equations

$$\dot{A}_i^a = -\tfrac{1}{2}\Psi_{ijk}A_j^bA_k^c\epsilon_{abc}, \tag{69}$$

where a, b, c take the values 1,2,3. This ansatz contains all the embeddings of the three-dimensional system with $SU(2)$ symmetry which can be written explicitly as a G_2 rotation R_{ij} of a seven-vector with the first three non-zero components

$$A_i^a = R_{ij}B_j^a, \tag{70}$$

where B_j^a is defined through the three-dimensional $SU(2)$ solution

$$B_i^a = (T_1^a, T_2^a, T_3^a, 0, 0, 0, 0). \tag{71}$$

Here, the matrix T_a^b, $a, b = 1, 2, 3$, satisfies the three-dimensional Nahm equations.

Let us now present some simple cases. The grouping of coordinates in relations (65) and (66) suggests the writing of the self-duality equations in terms of the complex coordinates $X_\pm = X_1 \pm iX_2$, $Y_\pm = X_4 \pm iX_5$ and $Z_\pm = X_6 \pm iX_7$. In terms of the latter the system can be written as

$$\dot{X}_+ = i(\{X_3, X_+\} + \{Y_+, Z_-\}), \tag{72}$$

$$\dot{Y}_+ = i(\{Y_+, X_3\} + \{X_+, Z_+\}), \tag{73}$$

$$\dot{Z}_+ = i(\{X_3, Z_+\} + \{X_-, Y_+\}), \tag{74}$$

$$\dot{X}_3 = i\tfrac{1}{2}(\{X_+, X_-\} + \{Z_+, Z_-\} - \{Y_+, Y_-\}). \tag{75}$$

We can easily obtain some simple solutions of the system in five or seven dimensions. In five dimensions, in particular, we set $X_+ = iY_-$ and $Z_+ = 0$. Then we find that the system is reduced in the three-dimensional case [13] with the identifications

$$X_\pm \to A_\pm/\sqrt{2}, \quad X_3 \to A_3. \tag{76}$$

Another solution which embeds every solution of the three-dimensional case in seven dimensions can be obtained by the identifications $X_+ = Z_+ = iY_-$. This solution is reduced to that of the three-dimensional one [13] with the following rescaling

$$X_\pm \to A_\pm/\sqrt{3}, \quad X_3 \to A_3. \tag{77}$$

An explicit construction shows that the two solutions are connected with the orthogonal transformation $|\xi_7\rangle = \mathcal{O}|\xi_3\rangle$, where the matrix $\mathcal{O}$ is

$$\mathcal{O} = \begin{pmatrix} a & 0 & 0 & -b & 0 & b & a \\ 0 & a & 0 & 0 & -b & a & -b \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & -b & -a & b \\ a & 0 & 0 & -b & 0 & -b & -a \\ a & 0 & 0 & 2b & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 2b & 0 & 0 \end{pmatrix}, \tag{78}$$

where $a = 1/\sqrt{3}$, $b = 1/\sqrt{6}$, $\langle\xi_3| = (X_1, X_2, X_3, 0, 0, 0, 0)$ and $|\xi_7\rangle$ is the seven-dimensional vector.

We conclude this work by summarizing our results. The relation of the octonionic algebra with quaternions gives a useful formulation of the self-duality equations which extends in a natural way the three-dimensional system and the corresponding generalized Nahm's equations for SDiff S_2. By introducing in place of $SU(2)$ algebra of functions on the sphere, a quadratic algebra of seven functions with G_2 symmetry, we succeeded in factorizing the time dependence in a simple way which may facilitate the study of solutions of the self-duality equations. Although the general system of self-duality equations in seven dimensions does not seem to have a Lax pair, at least in a direct way, due to the non-associativity of the octonionic algebra, it may happen that there is a generalization of the zero-curvature condition under which this system is integrable. In the case of three dimensions the restriction of the solutions to the $SU(2)$ subalgebra of functions on the spherical membrane reduces the problem to the study of Nahm's $SU(2)$ equations. In the same way, in seven dimensions the introduction of the quadratic algebra of functions on the sphere reduces the problem to the generalization of Nahm's equations with similar scaling properties with respect to time. This gives indications that the specific system may be relevant for the study of the monopole-type of configurations of membranes.

The relevance of the self-duality membrane equations in seven dimensions for the spectrum of instantons of the eleven-dimensional supermembrane is an open problem as well as the number of supersymmetries surviving these solutions.

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Appendix A

In this appendix we derive the properties of the tensors $X^{ij}{}_{kl}(u)$ used in Section 3 to make consistency checks of our ansatz. Consider the generalized matrices $\mathcal{P}^{ij}{}_{kl}(u,v)$

$$\mathcal{P}^{ij}{}_{kl}(u,v) = u\Delta^{ij}{}_{kl} + \frac{v}{4}\phi^{ij}{}_{kl}. \tag{A.1}$$

Using the properties

$$\Delta^{ij}{}_{kl}\Delta^{mn}{}_{ij} = \Delta^{mn}{}_{kl}, \tag{A.2}$$

$$\phi^{ij}{}_{mn}\phi^{mn}{}_{kl} = 8(\Delta^{ij}{}_{kl} + \tfrac{1}{4}\phi^{ij}{}_{kl}), \tag{A.3}$$

$$\Delta^{ij}{}_{kl}\phi^{kl}{}_{mn} = \phi^{ij}{}_{mn}, \tag{A.4}$$

we derive the following multiplication rule

$$\mathcal{P}^{ij}{}_{kl}(u_1,v_1)\mathcal{P}^{ij}{}_{kl}(u_2,v_2) = \mathcal{P}^{ij}{}_{kl}(u_3,v_3), \tag{A.5}$$

where

$$u_3 = u_1u_2 + \frac{v_1v_2}{2},$$

$$v_3 = u_1v_2 + u_2v_1 + \frac{v_1v_2}{2}.$$

We observe that this is a group structure which can be realized as a subgroup of the general linear group in two dimensions through the matrices

$$G(u,v) = \begin{pmatrix} u & v/2 \\ v & u+v/2 \end{pmatrix}. \tag{A.6}$$

For the existence of the inverse, one should restrict the parameters u and v inside the angular regions

$$v = u, \tag{A.7}$$

$$v = -2u. \tag{A.8}$$

For the case of $u = 1$, we restrict ourselves to $\mathcal{P}^{ij}{}_{kl}(1,v) \equiv X^{ij}{}_{kl}(v)$ matrices. Using the above, we find the multiplication law

$$\begin{aligned} X^{ij}{}_{kl}(u)X^{mn}{}_{ij}(v) &= \left(1 + \frac{uv}{2}\right)\Delta^{ij}{}_{kl} + \frac{1}{4}\left(u + v + \frac{uv}{2}\right)\phi^{ij}{}_{kl} \\ &\equiv \left(1 + \frac{uv}{2}\right)X^{mn}{}_{kl}(w), \end{aligned} \tag{A.9}$$

where

$$w = \frac{u + v + uv/2}{1 + uv/2}. \tag{A.10}$$

From the properties of Ψ_{ijk}, we find

$$\Psi_{ijk} X^{jk}{}_{lm}(u) = (1 + u)\Psi_{ilm}, \tag{A.11}$$

so for $u = -1$, the $X^{jk}(u)$ antisymmetric matrices satisfy the constraint for the G_2 algebra

$$\Psi_{ijk} X^{jk}{}_{lm}(-1) = 0. \tag{A.12}$$

Finally, we observe the following interesting projective properties for the end points of the group parameter u

$$X^{ij}{}_{mn}(u) X^{mn}{}_{kl}(2) = (1 + u) X^{ij}{}_{kl}(2), \tag{A.13}$$

$$X^{ij}{}_{mn}(u) X^{mn}{}_{kl}(-1) = \left(1 - \frac{u}{2}\right) X^{ij}{}_{kl}(-1). \tag{A.14}$$

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