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On the existence of singularity-free solutions in quadratic gravity

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Abstract

We study a general field theory of a scalar field coupled to gravitation through a quadratic Gauss-Bonnet term $\xi(\phi) R_{GB}^2$. We show that, under mild assumptions about the function $\xi(\phi)$, the classical solutions in a spatially flat FRW background include singularity-free solutions.

Recently it was found that the loop corrected superstring effective action in a FRW background in the presence of the dilaton and modulus fields admits a particularly interesting class of singularity-free solutions with flat initial asymptotics [1]. Singularity-free solutions have been considered by other authors as well [2,3] and their existence, although linked to the presence of quadratic curvature terms, depends on the form of the gravitational coupling to scalar fields. The purpose of this letter is to study the existence of singularity-free solutions and formulate the necessary conditions that should be imposed on the quadratic gravitational coupling. Although motivated by the concrete case of the superstring the analysis at hand is quite general and is carried out in an effective field theory framework that allows a possible alternative, at present non-existing, quantum gravitational underlying theory.

Consider a scalar field coupled quadratically to gravity through the Langrange density

$$L = \frac{1}{2}R + \frac{1}{2}(D\phi)^2 - \frac{1}{16}\delta\xi(\phi)R_{GB}^2 \quad (1)$$

where $R_{GB}^2 = (R_{\mu\nu\kappa\lambda})^2 - 4(R_{\mu\nu})^2 + R^2$ is the standard Gauss-Bonnet integrand. The function $\xi(\phi)$ is a general smooth function of the scalar field which will be further constrained shortly. δ is just a coupling parameter. Note that we have chosen a Gauss-Bonnet quadratic coupling in accordance with general unitarity arguments as well as the concrete superstring case [4]. In a standard RWF background metric $g_{\mu\nu} = (1, -e^{2\omega}\delta_{ij})$ we obtain the classical equations of motion

$$\phi'' + 3\dot{\omega}\dot{\phi} + \frac{3}{2}\delta\xi'(\phi)\dot{\omega}^2(\omega'' + \dot{\omega}^2) = 0 \quad (2)$$

$$3\dot{\omega}^2 - \frac{1}{2}\dot{\phi}^2 - \frac{3}{2}\delta\xi'(\phi)\dot{\phi}\dot{\omega}^3 = 0 \quad (3)$$

In terms of $x = \dot{\phi}$ and $z = \dot{\omega}$ considered as functions of ϕ , Eq. (3) can be solved as an algebraic equation and give

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$$x = z[-\frac{3}{2}\delta\xi'z^2 \pm \sqrt{6 + (\frac{3}{2}\delta\xi'z^2)^2}] \tag{4}$$

while (2) becomes

$$z' \equiv \frac{dz}{d\phi} = -\frac{z^2}{x}(A/B) \tag{5}$$

where

$$A \equiv 16z^4 + 16z^2x^2 + \frac{20}{9}x^4 - \frac{8}{3}\delta\xi''z^2x^4 \tag{6}$$

$$B \equiv 16z^4 - \frac{16}{3}z^2x^2 + \frac{20}{9}x^4 \tag{7}$$

Let us assume now that $\xi(\phi)$ is a smooth function that possesses one minimum at some point ϕ_0 . For example ξ could be ϕ^2, ϕ^4, \dots

We shall show that under the assumption that $\xi(\phi)$ grows faster than ϕ^2 at $\phi \rightarrow \pm\infty$ the system of equations of motion (2), (3), admits non-singular solutions for the case $\delta > 0$. Let us outline here the basic steps of our proof. We first show that all singularities occur at $|z| \rightarrow \infty$. Expanding asymptotically our equations at this region and analyzing the various possibilities we show that only one singular solution exists $\omega \sim \log t, \phi \sim \phi_s + t^2/(\delta\xi'(\phi_s))$. Then we show that the points $z = 0$ or $x = 0$ cannot be continuously approached for finite ϕ and thus every solution is characterized by a definite sign of z and x . Choosing $z > 0$ (expanding universe) and $x > 0$ we show that the singularity point ϕ_s is related to the ξ minimum ϕ_0 , and we always have $\phi_s > \phi_0$, for $\delta > 0$. Furthermore, we show that a singular solution involves only points $\phi > \phi_s$ and thus any solution that includes a point $\phi < \phi_0$ is non-singular. Since such a point can be always found this is an existence proof of the singularity free solutions. Finally, in order to make the results of our analysis concrete, we present the numerical solutions of the case $\xi = \phi^2/2, \delta = 1$.

A "singular solution" is one for which $|z| \rightarrow \infty$ or $|\dot{\phi}| \rightarrow \infty$ at some finite time. Note that we could have $z \rightarrow \infty$ for $\phi \rightarrow \infty$ and still have a singularity at a finite time. We shall restrict the coupling function $\xi(\phi)$ to increase at least as fast as ϕ^2 at infinity. Then, we can show that $x(\phi)$ cannot have a singularity while z is maintained at a finite value. In order to prove it we first solve Eq. (4) for large x and obtain $x \sim -3\delta\xi'z^3$. Obviously x has a singularity while z is finite, only if $\delta\xi'(\phi)$ has one. Since $\xi(\phi)$ is supposed to be a smooth function this can occur only at $\phi \rightarrow \pm\infty$. Next, we go to Eq. (5) which takes the form

$$z' \sim (\frac{1}{3\delta\xi'z})(1 - \frac{6}{5}\delta\xi''z^2) \tag{8}$$

We can proceed considering three separate cases. Namely $\delta\xi''z^2 \gg O(1), \delta\xi''z^2 \sim O(1)$ and $\delta\xi''z^2 \ll O(1)$ while always $\delta\xi'z^2 \gg O(1)$. Solving (8) in the first case leads to the contradiction $x \sim (\delta\xi')^{-\frac{1}{3}} \rightarrow 0$. The second case can be treated by setting $\delta\xi''z^2 = b$ and getting from (8) the solution $z \sim \phi^{\frac{1}{2}(1-\kappa)}, x \sim \phi^{\frac{3}{2}-\frac{\kappa}{2}}, \delta\xi \sim \phi^{1+\kappa}$ for $\phi \rightarrow \pm\infty$ with $\kappa \equiv (\frac{1}{5} + \frac{2}{3b})^{-1}$. In order to have the assumed behavior of z, x and $\delta\xi$ we have to constrain $1 \leq \kappa \leq 3$. Solving in terms of time we obtain $\phi \sim (t + \dots)^{\frac{2}{1-\kappa}}$. Thus, the point $\phi \rightarrow \pm\infty$ is not approached at a finite time and we do not have a true singularity in this case. Finally, in the last case $\delta\xi''z^2 \ll O(1)$ the Eq. (8) becomes $z' \sim \frac{1}{3\delta\xi'z}$ which is satisfied by $z^2\delta\xi' \sim \frac{2}{3}\phi$ for $\phi \rightarrow \pm\infty$. Expressing (8) solely in terms of z gives $z^2 \sim \phi, \delta\xi' \sim \text{const.}$ which contradicts our assumptions.

Let us now proceed by deriving the asymptotic form of the singular solutions. We can start from Eq. (4) and solve it around the singular point ϕ_s for which $z(\phi_s) \rightarrow \infty$. Three different possibilities arise in principle for z near the singularity, namely $\delta\xi'z^2 \sim O(1), \delta\xi'z^2 \ll O(1)$ and $\delta\xi'z^2 \gg O(1)$. They can be analyzed as follows:

1) $\delta\xi'z^2 \sim O(1)$. In this case we set $\frac{3}{2}\delta\xi'z^2 = a$ near ϕ_s and obtain from (4) $x = \lambda z$ with $a = \frac{3}{\lambda} - \frac{1}{2}$. Eq. (5) gives

$$\frac{z'}{z} = \frac{-(1 + \lambda^2 + \frac{5\lambda^4}{36} - \frac{\lambda^4\delta\xi''z^2}{6})}{\lambda(1 - \frac{\lambda^2}{3} + \frac{5\lambda^4}{36})} \tag{9}$$

or $\frac{z'}{z} = -\frac{A}{\lambda B}$. For $\delta\xi''z^2 = b \sim O(1)$ we get $z \sim \exp(-\frac{A\phi}{\lambda B})$ which is singular at $\phi \rightarrow \pm\infty$ and leads to $\delta\xi' \sim b \exp(-\frac{2A\phi}{\lambda B})$ in contradiction to our assumptions for the behavior of $\xi(\phi)$ at infinity. For $\delta\xi''z^2 \gg O(1)$ we get the solution $z^{-2} \sim \frac{\lambda^3\delta\xi'}{3B} + c$ which requires $c = 0$ and $\frac{1}{2} - \frac{3}{\lambda} = 9(1 - \frac{\lambda^2}{3} + \frac{5\lambda^4}{36})/2\lambda^3$. The last relation is impossible since there are no real solutions for λ . Finally, the case $\delta\xi''z^2 \ll O(1)$ is just like the first case.

2) $\delta\xi'z^2 \ll O(1)$. In this case from Eq. (4) we obtain $x \sim \pm z\sqrt{6}$. Eq. (5) becomes

$$\frac{z'}{z} \sim \mp 3(1 - \frac{\delta\xi'' z^2}{2})/\sqrt{6} \tag{10}$$

If $\delta\xi'' z^2 = b \sim O(1)$ we obtain $z \sim \exp(\mp\sqrt{\frac{3}{2}}(1 - \frac{b}{2})\phi)$ which corresponds to a singularity only for $\phi \rightarrow \pm\infty$. This implies $\delta\xi'' \sim \exp(\pm 2\sqrt{\frac{3}{2}}(1 - \frac{b}{2})\phi)$ which is not in agreement with our assumptions for $\xi(\pm\infty)$. Similarly for the case $\delta\xi'' z^2 \ll O(1)$. Finally, if $\delta\xi'' z^2 \gg O(1)$, (10) gives $z^{-2} \sim \pm\sqrt{\frac{3}{2}}\delta\xi'$ which is incompatible with $\delta\xi' z^2 \ll O(1)$.

3) $\delta\xi' z^2 \gg O(1)$. In this case we get two different solutions for x , namely $x_- \sim -3\delta\xi' z^3$ and $x_+ \sim \frac{2}{\delta\xi' z}$.

For $x = x_-$, necessarily $|x| \gg |z|$ and Eq. (5) becomes

$$z' \sim \frac{(1 - \frac{6\delta\xi'' z^2}{5})}{3\delta\xi' z} \tag{11}$$

which is equivalent to $(\frac{3}{2}\delta\xi' z^2)' \sim 1 + \frac{3}{10}\delta\xi'' z^2$. If $\delta\xi'' z^2 = b \sim O(1)$ we obtain $\frac{3}{2}\delta\xi' z^2 \sim (1 + \frac{3b}{10})\phi + \dots$ which implies that the singularity appears for infinite ϕ . Then, we get $\delta\xi' \sim \phi^{3b/(2+\frac{3b}{10})}$ and $z^2 \sim \phi^{(1-\frac{3b}{10})/(1+\frac{3b}{10})}$, with $b < \frac{5}{6}$ in order to have $|z| \rightarrow \infty$. The constraint $b < \frac{5}{6}$ leads to a function $\delta\xi$ that does not increase at infinity as fast as ϕ^2 , in contradiction to our assumptions. If $\delta\xi'' z^2 \ll O(1)$, we obtain $\frac{3}{2}\delta\xi' z^2 \sim \phi$ which also requires $\phi \rightarrow \pm\infty$. Then, we are led to $\delta\xi'' \phi/\delta\xi' \ll O(1)$ which is not acceptable. Finally, for $\delta\xi'' z^2 \gg O(1)$ we get from (11) $z \sim (\delta\xi')^{-\frac{2}{3}}$ which is not consistent with $z^2\delta\xi' \gg 1$ at $z \rightarrow \infty$. Thus, the case of x_- is impossible.

For $x = x_+ \sim \frac{2}{\delta\xi' z}$ we must have $|x/z| \ll 1$. Then, Eq. (5) becomes

$$z' \sim -\frac{1}{2}\delta\xi' z^3(1 - \frac{\frac{8}{3}\delta\xi''}{z^6(\delta\xi')^4}) \tag{12}$$

If $\delta\xi''/z^6(\delta\xi')^4 \ll O(1)$, we obtain

$$z^2 \sim \frac{1}{\delta\xi(\phi) - \delta\xi(\phi_s)} \tag{13}$$

which is a singular solution at some finite point ϕ_s . If $\delta\xi''/(\delta\xi')^4 \gg z^6$, we can get from Eq. (12) $z^4 \sim -\frac{2}{3}/((\delta\xi')^2 + \dots)$, which is impossible as $z \rightarrow \infty$.

The only remaining case $\frac{\delta\xi''}{(\delta\xi')^4} \sim O(z^6)$ can be studied by putting $\delta\xi'' = bz^6(\delta\xi')^4$. Then, from (12) we get $z^{-2} \sim (1 - \frac{8b}{3})(c + \delta\xi(\phi))$. Solving for $\delta\xi'$ we also get

$$(\delta\xi')^{-2} \sim \text{const.} + \frac{b}{(1 - \frac{8b}{3})^3(c + \delta\xi)},$$

which is equivalent to

$$\frac{b(\delta\xi' z^2)^2}{(1 - \frac{8b}{3})} \sim 1.$$

This is however, for any non-zero b , contradictory to our assumptions.

Thus, assuming that $\xi(\phi)$ is a smooth function that grows as fast as ϕ^2 at both infinities $\phi \rightarrow \pm\infty$, we have found only one singular solution. Near the singular point ϕ_s ($\xi'(\phi_s) \neq 0$) it behaves as

$$z \sim (\delta\xi(\phi) - \delta\xi(\phi_s))^{-\frac{1}{2}} \tag{14a}$$

$$x \sim \frac{2}{\delta\xi'(\phi)z} \tag{14b}$$

In terms of the time, (14) becomes

$$\phi \sim \phi_s + \frac{t^2}{\delta\xi'(\phi_s)} \tag{15a}$$

$$\omega \sim \ln t \tag{15b}$$

where we have shifted the time variable so that the singular point ϕ_s corresponds to $t=0$. Note that while $z \rightarrow \infty$, x goes to zero as long as $\delta\xi'(\phi_s) \neq 0$.

The limit $x \rightarrow 0$ of the system (2), (3) deserves some more attention. Indeed, as can be seen in Eq. (4), when $x \rightarrow 0$ for finite ϕ the only possibility for z is to go to zero too while $x \sim \pm\sqrt{6}z$. Substituting the last relation in Eq. (5), we obtain $z'/z \sim \mp\sqrt{\frac{3}{2}}$ or $z \sim \exp \mp\sqrt{\frac{3}{2}}\phi$ which does not go to zero for finite ϕ in contradiction with our hypothesis. Inversely, when $z \rightarrow 0$ for finite ϕ the only possibility read off from Eq. (4) is $x \sim \pm\sqrt{6}z$. Again, substituting that in Eq. (5) we reach the same contradiction. The only conclusion we can draw is that the points $x = 0$ or $z = 0$ cannot be reached analytically. Therefore, the signs of x and z must be independently conserved. Then, the following statement is true for the singular solution

we found: Every connected piece of the solution will be characterized by a fixed sign for each of x and z . We have assumed single-valuedness of $z(\phi)$. If in a connected piece of the solution $z(\phi)$ were not single-valued then at a non-singular point ϕ_1 we would have $\frac{d\phi}{dz} = 0$ at $z_1 = z(\phi_1)$. This is equivalent to $x B / A z_1^2 = 0$ and it can happen only when either $x \rightarrow 0$ or $A \rightarrow \infty$ in contradiction to $z_1 < \infty$.

Consider now a singular solution with the singularity occurring at a point ϕ_s . Each connected piece of the solution will be characterized by a given sign of x and z . Take the piece with $x > 0, z > 0$. Then,

$$x = x_- \sim \frac{2}{z\delta\xi'} \tag{16}$$

implies that $\delta\xi' > 0$ and therefore, if $\delta > 0$,

$$\phi_s > \phi_0$$

where ϕ_0 is the minimum of $\xi(\phi)$. Because of $z' \sim -\frac{1}{2}z^3\delta\xi' < 0$ and the single-valuedness of z we come to the conclusion that these solutions cover at best the full quarter-plane $z > 0, \phi > \phi_0$ but they do not contain any points of the $z > 0, \phi < \phi_0$ region. We are now in a position to state the following “theorem”:

For $\delta > 0$, any solution containing a point $\phi^ < \phi_0$ such that $x(\phi^*) > 0, z(\phi^*) > 0$ must be a non-singular solution.*

Since the solution is controlled by one first order differential equation, namely (5), while (4) is just a soluble algebraic equation, it depends only on z, x at some initial point and we can always satisfy the requirements of the theorem starting with values $z^* > 0, x^* > 0$ at some initial point $\phi^* < \phi_0$.

The existence of the non-singular solution depends crucially on the fact that the singular solutions do not cross over to the $\phi < \phi_0$ region. Notice that if we keep $x > 0, z > 0$ and chose $\delta < 0$ equation (16) demands $\phi_s < \phi_0$. Then, since $z' < 0$ still, the whole upper plane is filled with singular solutions and there are no singularity-free solution in this case. This behavior is in agreement with the violation of the energy conditions required by singularity theorems [5] which cannot occur in the $\delta < 0$ case since in this case $\rho + p$ and $\rho + 3p$ are positive definite.

In order to make our analysis concrete we have also studied numerically the case $\xi(\phi) = \phi^2/2, \delta = 1$. The phase diagram for $z = z(\phi) > 0$ and $x > 0$ is

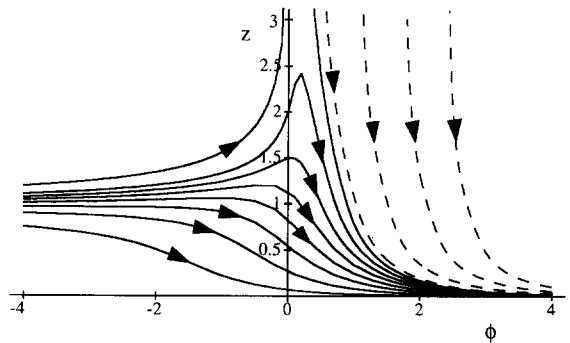


Fig. 1. The phase space diagram $z(\phi)$ for the case $\xi(\phi) = \frac{1}{2}\phi^2, \delta = 1$ and $z > 0, x > 0$. The dashed lines correspond to singular solutions and the continuous lines to non-singular ones. Arrows indicate the direction of the time.

presented in Fig. 1. As expected the singular solutions can occur only for $\phi_s > \phi_0 = 0$ and points with $\phi < \phi_0 = 0$ belong to non-singular solutions. It can also be shown that in this specific case the non-singular solutions interpolate between a DeSitter space universe at the remote past ($t \rightarrow -\infty$) and a slowly expanding one at the future ($t \rightarrow -\infty$).

The general behavior of the non-singular solutions at late times ($t \rightarrow \infty$), assuming an expanding universe ($z > 0$), is that of slow expansion, while the behavior at early times ($t \rightarrow -\infty$) depends crucially on the particular choice of $\xi(\phi)$. In the case of the loop corrected superstring effective action [1] which has served as a motivation for the present investigation³ we obtain at early times a flat-space. In contrast, in the case $\xi(\phi) = \frac{1}{2}\phi^2$, analyzed above, we have obtained DeSitter space.

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References

[1] I. Antoniadis, J. Rizos and K. Tamvakis, preprint hep-

³ In the superstring effective action case [1], the modulus dependent coefficient $\xi(\phi)$ of the R_{GB}^2 term, satisfies the requirements of our proof, since it has a unique minimum at $\phi = 0$ and it grows as $\xi \sim e^{|\phi|}$ at $\phi \rightarrow \pm\infty$.

- th/9305025, CPTH-A239.0593, IOA-291 (May 1993), to appear in Nucl. Phys. B.
- [2] A.A. Starobinsky, Phys. Lett. B 91 (1980) 99.
- [3] V. Mukhanov and R. Brandenberger, Phys. Rev. Lett. 68 (1992) 1969;
R. Brandenberger, V. Mukhanov and A. Sornborger, Phys. Rev. D 48 (1993) 1629.
- [4] I. Antoniadis, E. Gava and K.S. Narain, Phys. Lett. B 283 (1992) 209; Nucl. Phys. B 393 (1992) 93.
- [5] R. Penrose, Phys. Rev. Lett. 14 (1956) 57;
S. Hawking, Proc. R. Soc. London A 300 (1967) 182;
S. Hawking and R. Penrose, Proc. R. Soc. London A 314 (1970) 529.