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On the GUT scale of F-theory $SU(5)$

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ABSTRACT

In F-theory GUTs, threshold corrections from Kaluza–Klein (KK) massive modes arising from gauge and matter multiplets play an important role in the determination of the weak mixing angle and the strong gauge coupling of the effective low energy model. In this Letter we further explore the induced modifications on the gauge couplings running and the GUT scale. In particular, we focus on the KK-contributions from matter curves and analyze the conditions on the chiral and Higgs matter spectrum which imply a GUT scale consistent with the minimal unification scenario. As an application, we present an explicit computation of these thresholds for matter fields residing on specific non-trivial Riemann surfaces.

1. Introduction

The spectrum of the minimal supersymmetric extension of the Standard Model (SM) being consistent with a gauge couplings unification at a scale $M_{\rm GUT} \sim 2 \times 10^{16}$ GeV, suggests that the gauge group factors emanate from a higher unified gauge symmetry. In the simplest case, the SM gauge symmetry is embedded in the $SU(5)$ Grand Unified Theory (GUT) while the SM matter content is assembled into $SU(5)$ multiplets. In addition, although string theory appears to be the appropriate candidate for incorporating gravity into the unification scenario, one must still confront the mismatch between $M_{\rm GUT}$ and the natural gravitational scale $M_{Pl} \sim 1.2 \times 10^{19}$ GeV. Thus, a plausible implementation of unification, requires a string theory formulation in which the gauge theory decouples from gravity at the desired scale.

Recently, there have been considerable efforts to develop a viable effective field theory model from F-theory [1].[1] This picture consists of a 7-brane wrapping a compact Kähler surface S of two complex dimensions while the gauge theory of a particular model is associated with the geometric singularity of the internal space [5–9]. In this set up it is possible to decouple gauge dynamics from gravity by restricting to compact surfaces S that are of del Pezzo type. The exact determination of the GUT scale however, may depend on the spectrum and other details of the chosen gauge symmetry and on the particular model. Here, we will assume the minimal unified $SU(5)$ GUT.

A reliable computation of the GUT scale should also take into consideration the various threshold corrections. In F-theory $SU(5)$, there exist several sources of threshold effects [10–15]. There are thresholds related to the flux mechanism inducing splitting of the gauge couplings at the GUT scale [10,11], thresholds from heavy KK-massive modes [10,14] and corrections due to the appearance of probe D3-branes [15]. Finally, threshold effects are generated at scales $\mu < M_{\rm GUT}$ when additional light degrees of freedom in particular superpartners are integrated out. The effects of the latter have been extensively studied in the context of supersymmetric and String Grand Unified Theories.[2] In reasonable circumstances (for example when no-extra degrees of freedom remain below $M_{\rm GUT}$) the last two categories can be made consistent with two loop corrections and a unification scale of the order of $M_{\rm GUT} \sim 2 \times 10^{16}$ GeV.

Thresholds induced by the flux mechanism have been extensively analyzed recently [10,11,13]. It has been shown that the $U(1)_Y$-flux induced splitting is compatible with the GUT embedding of the minimal supersymmetric Standard Model, provided that no extra matter other than color triplets is present in the spectrum. Thresholds originating from KK-massive modes have been discussed in [10] and were found to be related to the Ray–Singer analytic torsion [17]. This observation was originally made for the case of manifolds with G_2 holonomy where thresholds were computed and estimates for the GUT scale were given [18]. For F-theory however, the situation is more complicated. Indeed, in M-theory one assumes that massless $SU(5)$ multiplets are generated at singularities of the internal space which are believed to

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[1] For comprehensive reviews see [2–4].

[2] For an incomplete list see [16].

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be conical [18]. Since conical singularities induce no new length, it is expected that no new massive particles are introduced. On the contrary, in F-theory, KK-massive modes exist for both the gauge and the matter fields. In the present context of the $SU(5)$ theory, these come along with massless gauge fields propagating in the bulk, while the chiral matter as well as the Higgs representations reside on two-dimensional Riemann surfaces (matter curves). Both kinds of KK-modes contribute to the gauge coupling running and can in principle modify the unification scale. It is straightforward to estimate the modification induced by the vector supermultiplet, nevertheless the contributions of the matter fields may be model dependent. Here, we aim to revisit this second source of threshold corrections. We will discuss models where chiral matter and Higgs fields occupy complete $SU(5)$ multiplets and we will show that under reasonable assumptions, no further modifications are induced from the corresponding matter KK-massive modes.

2. KK-modes and the GUT scale

In F-theory, threshold corrections associated to KK-massive modes arise from gauge fields as well as from matter fields in the intersections. KK-massive modes from the chiral and the Higgs sectors add up to a common shift of the gauge coupling constants at M_{GUT}. This happens when the charges q_i associated to the matter curves Σ_{q_i} are genuinely embedded into the function $\mathcal{T}(q_i)$ which defines the torsion. Thus, in this respect F-theory looks pretty much the same as M-theory [18]. We will first give a brief account of the gauge thresholds computations adopting the techniques of [18] developed for G_2-manifolds, while we will follow [10] for the case of F-theory. Next, we will compute the KK-thresholds from the chiral matter and the Higgs curves.

2.1. The gauge multiplet

The decomposition of the $SU(5)$ gauge multiplet under the SM symmetry is

$$24 \to R_0 + R_{-5/6} + R_{5/6}$$

with

$$R_0 = (8,1)_0 + (1,3)_0 + (1,1)_0, \qquad R_{-5/6} = (3,2)_{-5/6},$$
$$R_{5/6} = (\bar{3},2)_{5/6}. \tag{1}$$

Massless fields in the bulk are given by the Euler characteristic $\mathcal{X}$, thus, the condition $\mathcal{X}(S, L^{5/6}) = 0$ avoids the massless exotics $R_{\pm 5/6}$. Massive modes in representations (1) induce threshold effects to the running of the gauge couplings. At the one-loop level we write

$$\frac{16\pi^2}{g_a^2(\mu)} = \frac{16\pi^2 k_a}{g_s^2} + b_a \log\frac{\Lambda^2}{\mu^2} + \mathcal{S}_a^{(g)}, \quad a = 3, 2, Y. \tag{2}$$

Here, Λ is the gauge theory cutoff scale, $k_a = (1, 1, 5/3)$ are the normalization coefficients for the usual embedding of the Standard Model into $SU(5)$, g_s is the value of the gauge coupling at the high scale and $\mathcal{S}_a^{(g)}$ stand for the gauge fields thresholds. The one-loop β-function coefficients b_a for the massless spectrum (in the notation of [18]) are

$$b_a = 2\,\mathrm{Str}_{M=0}\, Q_a^2\left(\frac{1}{12} - \chi^2\right) \tag{3}$$

where χ is the helicity operator and Q_a stands for the three generators of the Standard Model gauge group $SU(3) \times SU(2) \times U(1)_Y$. In computing the supertrace Str we count bosonic contributions with weight $+1$ and fermionic with -1. Similarly, the one-loop threshold corrections from the KK-massive modes in R_i are

$$S_a^{(g)} = 2\sum_i \mathrm{Tr}_{R_i}\, Q_a^2\, \mathrm{Str}_{M\neq 0}\left(\frac{1}{12} - \chi^2\right)\log\frac{\Lambda^2}{M^2}. \tag{4}$$

The KK-modes squared masses in the threshold formula correspond to the massive spectrum of the Laplacian Δ_{k,R_i} acting on each k-form of the representation R_i. We recall [7] that the spectrum consists of zero, one and two form multiplets. Each eigenvector of the zero-form Laplacian Δ_{0,R_i} contributes a vector multiplet with helicities $1, -1, \frac{1}{2}, -\frac{1}{2}$, while the one-form Laplacian Δ_{1,R_i} gives a chiral multiplet with helicities $0, 0, \frac{1}{2}, -\frac{1}{2}$. Finally, Δ_{2,R_i} is associated to anti-chiral multiplets. The sum of all the contributions to the gauge fields thresholds is

$$\mathcal{S}_a^{(g)} = 2\sum_i \mathrm{Tr}_{R_i}\left(Q_a^2\right)\mathcal{K}_i \tag{5}$$

with [10]

$$\mathcal{K}_i = \frac{3}{2}\log\det{}'\frac{\Delta_{0,R_i}}{\Lambda^2} - \frac{1}{2}\log\det{}'\frac{\Delta_{1,R_i}}{\Lambda^2} - \frac{1}{2}\log\det{}'\frac{\Delta_{2,R_i}}{\Lambda^2} \tag{6}$$

where the prime on $\det'$ means that zero modes are omitted. Using the well-known properties characterizing the massive spectra of the Laplacians Δ_{k,R_i}, it has been shown [10] that expression (6) is the Ray–Singer analytic torsion $\mathcal{T}_i$ [17]; more precisely,

$$2\mathcal{T}_i = \mathcal{K}_i = 2\log\det{}'\frac{\Delta_{0,R_i}}{\Lambda^2} - \log\det{}'\frac{\Delta_{1,R_i}}{\Lambda^2}. \tag{7}$$

Note that for the trivial representation R_0 there exist zero-modes and the torsion differs from $\mathcal{K}_0$ by a scaling dependent part $\propto 2\log(V_S^{1/2}\Lambda^2)$ where V_S is the volume of the compact surface S. Details on the scaling dependence can be found in [10]. Using (4) we compute the traces and since $\mathcal{K}_{5/6} = \mathcal{K}_{-5/6}$ we get

$$\left(\mathcal{S}_Y^{(g)}, \mathcal{S}_2^{(g)}, \mathcal{S}_3^{(g)}\right) = \left(\frac{50}{3}\mathcal{K}_{5/6}, 6\mathcal{K}_{5/6} + 4\mathcal{K}_0, 4\mathcal{K}_{5/6} + 6\mathcal{K}_0\right). \tag{8}$$

Using the torsion $\mathcal{T}_i$ and the β-functions $b_a^{(g)} = (0, -6, -9)$, we deduce that

$$\mathcal{S}_a^{(g)} = \frac{4}{3}b_a^{(g)}(\mathcal{T}_{5/6} - \mathcal{T}_0) + 20k_a\mathcal{T}_{5/6}. \tag{9}$$

Absorbing the term proportional to k_a into a redefinition of g_s we may now write the one loop equation (2) for the running of the gauge couplings [14] as

$$\frac{16\pi^2}{g_a^2(\mu)} = \left(\frac{16\pi^2}{g_s^2} + 20\mathcal{T}_{5/6}\right)k_a + b_a^{(g)}\log\frac{\exp[4/3(\mathcal{T}_{5/6} - \mathcal{T}_0)]}{\mu^2 V_S^{1/2}}. \tag{10}$$

The form (10) suggests that we can define M_{GUT} as [14]

$$M_{\mathrm{GUT}}^2 = \frac{\exp[4/3(\mathcal{T}_{5/6} - \mathcal{T}_0)]}{V_S^{1/2}} \tag{11}$$

and a gauge coupling g_U at the GUT scale shifted by

$$\frac{16\pi^2}{g_U^2} = \frac{16\pi^2}{g_s^2} + 20\mathcal{T}_{5/6}. \tag{12}$$

Associating the world volume factor $V_S^{-1/4}$ with the characteristic F-theory compactification scale M_C, we can write

$$M_{\rm GUT} = e^{2/3(\mathcal{T}_{5/6}-\mathcal{T}_0)} M_C. \tag{13}$$

Thus, as far as the gauge fields thresholds are concerned, $M_{\rm GUT}$ is given in terms of M_C through an elegant relation involving only the torsion.

2.2. The chiral matter

We will now discuss contributions arising from chiral matter, the Higgs fields and the possible exotic representations. In F-theory constructions, these fields arise in the intersections of the GUT-brane with other 7-branes as well as from the decomposition of the adjoint representation in the bulk. We have already imposed the conditions which avoid the exotic bulk zero modes $R_{-5/6} = (3, 2)_{-5/6}$ and $R_{5/6} = (\bar{3}, 2)_{5/6}$, so we are only left with light matter fields at the intersections. In the $SU(5)$ case, these correspond to the standard 10, $\overline{10}$ and 5, $\bar{5}$ non-trivial representations and contribute to the RG running a term of the form $b_a^x \log \Lambda'^2/\mu^2$ where b_a^x are the β-function coefficients for the matter fields, and Λ' a cutoff scale which may differ from the gauge cutoff Λ.

We should mention that the $U(1)_Y$-flux introduced in order to break $SU(5)$ might eventually lead to incomplete $SU(5)$ representations, spoiling thus the gauge coupling unification. However, it is still possible to work out realistic cases [19,20,14] where the matter fields add up to complete $SU(5)$ multiplets, so that the b_a^x-functions contribute in proportion to the coefficients k_a. Then, as in the case of the gauge contributions discussed earlier, we can absorb the logarithmic Λ'-dependence into a redefinition of the gauge coupling. Nevertheless, the color triplet pair descending from the $5_H + \bar{5}_H$ Higgs quintuplets must receive a mass at a relatively high scale $M_X \leqslant M_{\rm GUT}$ so to avoid rapid proton decay. Taking all into account, we write (10) in the form

$$\frac{16\pi^2}{g_a^2(\mu)} = k_a \frac{16\pi^2}{g_{\rm GUT}^2} + \left(b_a^{(g)} + b_a\right) \log \frac{M_{\rm GUT}^2}{\mu^2} + b_a^T \log \frac{M_{\rm GUT}^2}{M_X^2} \tag{14}$$

where we have split $b_a^x = b_a + b_a^T$ with b_a denoting the MSSM β-functions and b_a^T the color triplet part.

In the context of F-theory constructions, in addition to the light degrees of freedom on matter curves, one also has to include contributions from Kaluza–Klein massive modes. This is in contrast to the case of G_2 manifolds, where no new contributions are introduced to the gauge coupling running apart from the massless states [18]. Threshold contributions arise from the massive states along the $\Sigma_{\bar{5}}$ and Σ_{10} matter curves. To compute them we write down the decompositions of the corresponding representations

$$10 \to (3,2)_{\frac{1}{6}} + (\bar{3},1)_{-\frac{2}{3}} + (1,1)_1, \qquad \bar{5} \to (\bar{3},1)_{\frac{1}{3}} + (1,2)_{-\frac{1}{2}}.$$

For each of the above matter curves we consider the Laplacian acting on the representations with eigenvalues corresponding to chiral and anti-chiral fields. Thus, for the massive modes of Σ_{10} we have

$$\mathcal{K}_{\Sigma_{10}} = -\frac{1}{2} \log {\det}' \frac{\Delta_{0,Y}}{\Lambda'^2} - \frac{1}{2} \log {\det}' \frac{\Delta_{1,Y}}{\Lambda'^2}$$

and similarly for the $\Sigma_{\bar{5}}$. Denoting by $S_{a=3,2,Y}$ the thresholds to the three gauge factors of the SM, for a representation r we then have

$$S_a^r = \sum_i 2\,{\rm Tr}\big(Q_{a,r}^2\big)\mathcal{K}_i.$$

Table 1
Threshold corrections $S_a^{\bar{5}}$, S_a^{10} to the three gauge couplings from Kaluza–Klein massive modes along the matter curves.

Thresholds	$SU(3)$	$SU(2)$	$U(1)$
$S_a^{\bar{5}}$	$\mathcal{K}_{1/3}$	$\mathcal{K}_{-1/2}$	$\mathcal{K}_{-1/2} + 2/3\mathcal{K}_{1/3}$
S_a^{10}	$2\mathcal{K}_{1/6} + \mathcal{K}_{-2/3}$	$3\mathcal{K}_{1/6}$	$1/3\mathcal{K}_{1/6} + 2\mathcal{K}_1 + 8/3\mathcal{K}_{-2/3}$

Computing the traces we readily find the KK-thresholds shown in Table 1.

We will now attempt to recast the corrections as a sum of two different pieces, one being proportional to k_a. The KK-thresholds induced by the $\bar{5}$ can be written as follows:

$$S_a^{\bar{5}} = -\frac{2}{3}\beta_a^{\bar{5}}(\mathcal{K}_{-1/2} - \mathcal{K}_{1/3}) + k_a \cdot (\mathcal{K}_{-1/2}) \tag{15}$$

where we have introduced the "β"-coefficients

$$\beta_{3,2,1}^{\bar{5}} = \left\{\frac{3}{2}, 0, 1\right\}$$

and, as usually, $k_a = (1, 1, 5/3)$. For the Σ_{10} we can write the thresholds related to $U(1)_Y$ in the form

$$\begin{aligned} S_1^{10} &= \frac{1}{3}\mathcal{K}_{1/6} + \frac{8}{3}\mathcal{K}_{-2/3} + 2\mathcal{K}_1 \\ &= \frac{8}{3}(\mathcal{K}_{-2/3} - \mathcal{K}_{1/6}) - 2(\mathcal{K}_{1/6} - \mathcal{K}_1) + \frac{15}{3}\mathcal{K}_{1/6}. \end{aligned} \tag{16}$$

In the two parentheses, the $U(1)_Y$ charge differences obey the relation $q_i - q_j = -\frac{5}{6}$. This suggests that a non-trivial line bundle structure could be sought with the 'periodicity' property $\mathcal{K}_{q_i} - \mathcal{K}_{q_j} = f(q_i - q_j)$ so that

$$\mathcal{K}_{1/6} - \mathcal{K}_1 = \mathcal{K}_{-2/3} - \mathcal{K}_{1/6}.$$

Adopting this assumption, in straight analogy with (15) we finally get

$$S_a^{10} = \frac{2}{3}\beta_a^{10}(\mathcal{K}_{-2/3} - \mathcal{K}_{1/6}) + k_a \cdot (3\mathcal{K}_{1/6})$$

with $\beta_a^{10} = \beta_a^{\bar{5}}$. Recalling the Ray–Singer torsion $\mathcal{T}_i$ we may write threshold terms for both matter curves as follows

$$S_a^{\bar{5}} = -\frac{4}{3}\beta_a^{\bar{5}}(\mathcal{T}_{-1/2} - \mathcal{T}_{1/3}) + k_a(2 \cdot \mathcal{T}_{-1/2}), \tag{17}$$

$$S_a^{10} = +\frac{4}{3}\beta_a^{10}(\mathcal{T}_{-2/3} - \mathcal{T}_{1/6}) + k_a(6 \cdot \mathcal{T}_{1/6}). \tag{18}$$

The hypercharge assignments in both Σ_{10} and $\Sigma_{\bar{5}}$ satisfy the same condition $q_i - q_j = -\frac{5}{6}$. Given that and employing the torsion invariance, one could assume the existence of bundle structures for Σ_{10} and $\Sigma_{\bar{5}}$ matter curves characterized by the same topological properties so that we may envisage a specific embedding of the hypercharge generator implying

$$\mathcal{T}_{-1/2} - \mathcal{T}_{1/3} = \mathcal{T}_{-2/3} - \mathcal{T}_{1/6} = 0. \tag{19}$$

In this limit, threshold contributions which are not proportional to k_a cancel in both Σ_{10} and $\Sigma_{\bar{5}}$ curves.

In general, matter curves accommodating different representations of the gauge group do not necessarily bear the same bundle structure. In particular, in the case of $SU(5)$ it often happens that the $\Sigma_{\bar{5}}$ curve is of higher genus than the Σ_{10} for example. One of course could not exclude the possibility that the condition (19) can be separately satisfied for surfaces of different genera. However, we mention that in the recent literature one can find several examples where Σ_{10} and $\Sigma_{\bar{5}}$ curves are of the same genus and the required property holds true. To further support our argument, we will briefly present a model discussed in Ref. [8]. Bearing in mind that in order to decouple gauge dynamics from gravity and allow for the possibility $M_{\rm GUT} \ll M_{Planck}$, we choose the surface S to be one of the del Pezzo type dP_n with $n = 1, 2, \dots, 8$. We choose dP_8 which is generated by the hyperplane divisor H from $\mathbb{P}^2$ and the exceptional divisors $E_{1,\dots,8}$ with intersection numbers

$$H \cdot H = 1, \qquad H \cdot E_i = 0, \qquad E_i \cdot E_j = -\delta_{ij}. \tag{20}$$

We also note that the canonical divisor for dP_8 is

$$K_S = -c_1(dP_8) = -3H + \sum_{i=1}^{8} E_i. \tag{21}$$

Then, denoting with C and g the class and the genus of a matter curve respectively, we have $C \cdot (C + K_S) = 2g - 2$. In the particular example of Section 17 in Ref. [8] the 10_M chiral matter of the three generations resides on one Σ_{10}, with $C = 2H - E_1 - E_5$ and the three $\bar{5}_M$ on a single Σ_5^1 curve with $C = H$. Higgs fields 5_H and $\bar{5}_{\bar{H}}$ are localized on different $\Sigma_5^{2,3}$ matter curves with classes $C = H - E_1 - E_3$ and $H - E_2 - E_4$ respectively. Checking the relevant intersections, one readily finds that all the above matter curves are of the same genus $g = 0$ and therefore the criterion is fulfilled.

Returning to the threshold contributions (17), (18), once the parts proportional to $\beta_a^{\bar{5}}, \beta_a^{10}$ cancel out we observe that the remaining contributions from KK-thresholds are just those proportional to the coefficients k_a and consequently, they only induce a shift of the gauge coupling value at $M_{\rm GUT}$. We finally get

$$\frac{16\pi^2}{g_a^2(\mu)} = \left(\frac{16\pi^2}{g_s^2} + 20\mathcal{T}_{5/6} + 6\mathcal{T}_{1/6} + 2\mathcal{T}_{1/3}\right)k_a + \left(b_a^{(g)} + b_a\right)\log\frac{M_{\rm GUT}^2}{\mu^2} + b_a^T \log\frac{M_{\rm GUT}^2}{M_X^2}. \tag{22}$$

Thus, matter thresholds leave the GUT scale $M_{\rm GUT}$ intact, their only net effect amounts to a further shift of the common gauge coupling. The value of the latter at the GUT scale is defined by

$$\frac{16\pi^2}{g_{\rm GUT}^2} = \frac{16\pi^2}{g_s^2} + 20\mathcal{T}_{5/6} + 6\mathcal{T}_{1/6} + 2\mathcal{T}_{1/3}. \tag{23}$$

In the case where KK-modes from the gauge multiplet are associated to a bundle with different properties, we denote $\mathcal{T}_{5/6} \to \mathcal{T}'_{5/6}$ while the above analysis still holds.

We observe that (22) are just the one-loop renormalization group equations for the minimal $SU(5)$ GUT, with extra color triplets becoming massive at a scale $M_X \leqslant M_{\rm GUT}$. We further note that in F-theory constructions, a $U(1)_Y$ flux mechanism is employed to break the $SU(5)$ symmetry, inducing a splitting of the gauge couplings at the GUT scale. This gauge coupling splitting is still consistent with a unification scale $M_{\rm GUT} \sim 2 \times 10^{16}$ GeV provided that the triplets receive a mass at a scale determined by consistency conditions [11,13].

2.2.1. Example: the case of non-trivial line bundle

We now present an example of non-trivial $\Sigma_{10}, \Sigma_{\bar{5}}$ matter curves and a genus $g = 1$ Riemann surface. We will use the torsion results of [17] to compute the KK-matter contributions. The masses of the KK-modes are the eigenvalues of the Laplacian on a complex $d = 1$ Riemann surface. Thresholds are given as functions of the torsion which is expressed in terms of the eigenvalues through the associated zeta function for the Laplacian Δ_k

$$\Delta_{k,R(V)} = \left(\bar{\partial} + \bar{\partial}^\dagger\right)^2 = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}. \tag{24}$$

If we collectively denote ψ_k^n as the k-form eigenfunction, then

$$\Delta_{k,R(V)}\psi_k^n = \lambda_n^k \psi_k^n \tag{25}$$

where λ_n^k is the corresponding eigenvalue which in four dimensions corresponds to a mass squared. The associated zeta function is given by

$$\zeta_{\Delta_k}(s) = \sum_n \frac{1}{\lambda_n^s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,{\rm Tr}\left(e^{-\Delta_k t}\right)t \tag{26}$$

so that

$$\ln(\mathrm{Det}\,\Delta_k) = -\frac{d\zeta_{\Delta_k}(s)}{ds}\bigg|_{s=0}.$$

The torsion is written as

$$\mathcal{T} = \sum_k (-1)^{k+1} k \frac{d\zeta_{\Delta_k}(s)}{ds}\bigg|_{s=0}. \tag{27}$$

For our application, we have already assumed a Riemann surface of genus $g = 1$ and a character given by $\chi = \exp\{2\pi i(mu + nv)\}$ with the identification $\chi \leftrightarrow u - \tau v$. The eigenvalues are

$$\lambda_n = \frac{4\pi^2}{\mathrm{Im}\,\tau}\left|u + m - \tau(v + n)\right|^2. \tag{28}$$

The eigenfunctions are

$$\psi_n = \exp\left\{\frac{2\pi i}{\mathrm{Im}\,\tau}\mathrm{Im}\left[z\left(u + m - \bar{\tau}(v + n)\right)\right]\right\}.$$

Given the eigenvalues (28), the torsion can be computed [17] using (27) and (26). In the following, we present the basic steps of its derivation, adapting the notation [17] into our formalism. Let us assume that $\tau = \tau_1 + i\tau_2$ and let us define $S_1 = {\rm Tr}(e^{-\Delta_k t})$ which amounts to the calculation of the following double sum:

$$S_1 = \sum_{m,n=-\infty}^{\infty} \exp\left[-\frac{4\pi^2 t}{\tau_2^2}\left((u + m)^2 + \tau^2(v + n)^2 - 2\tau_1(u + m)(v + n)\right)\right]. \tag{29}$$

Applying the Poisson summation formula we get

$$S_1 = \frac{\tau_2}{4\pi t}\sum_{m,n=-\infty}^{\infty} \exp\left[-\frac{1}{4t}\left(m^2\tau^2 + n^2 + 2\tau_1 mn\right) + 2\pi i(mu + nv)\right]. \tag{30}$$

Putting $a = (m^2\tau^2 + n^2 + 2\tau_1 mn)$ and substituting into (26), we get

$$\zeta(s) = \frac{\tau_2}{4\pi}\frac{1}{\Gamma(s)}\sum_{m,n=-\infty}^{\infty}\int_0^\infty dt\, t^{s-2} e^{-\frac{a}{4t}}\exp\left[2\pi i(mu + nv)\right]. \tag{31}$$

For $s > 1$ the integration gives

$$\zeta(s) = \frac{\tau_2}{4\pi}\frac{\Gamma(1 - s)}{\Gamma(s)}\sum_{m,n=-\infty}^{\infty}\left(\frac{4}{a}\right)^{1-s}\exp\left(2\pi i(mu + nv)\right). \tag{32}$$

We readily now find that

$$\zeta'(0) = \frac{\tau_2}{\pi}\sum_{m,n=-\infty}^{\infty}\frac{\exp[2\pi i(mu + nv)]}{(m^2\tau^2 + n^2 + 2\tau_1 mn)}. \tag{33}$$

According to Kronecker's second limit theorem, the singular term $m = 0$, $n = 0$ has to be omitted [21]. This way we get

$$\zeta'(0) = \frac{\tau_2}{\pi} \sum_{n \neq 0} \frac{\exp[2\pi i n v]}{n^2} + \frac{\tau_2}{\pi} \sum_{m \neq 0} e^{2i\pi m u} \sum_{n=-\infty}^{\infty} \frac{e^{2i\pi n v}}{m^2 \tau^2 + n^2 + 2\tau_1 m n}. \tag{34}$$

The first sum is [22]

$$\sum_{n \neq 0} \frac{\exp[2\pi i n v]}{n^2} = 2 \sum_{n=1}^{\infty} \frac{\cos 2\pi v n}{n^2} = \frac{3(2\pi v)^2 - 6\pi(2\pi v) + 2\pi^2}{6} = 2\pi^2 \left(v^2 - v + \frac{1}{6} \right)$$

where $0 < v < 1$. The n sum in the second term of (34) can be evaluated by means of the Poisson formula

$$\sum_{n=-\infty}^{\infty} f(-n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i n x} f(x)\, dx. \tag{35}$$

The denominator can be written as

$$m^2\tau^2 + x^2 + 2\tau_1 m x = (m\tau_1 + x)^2 + m^2 \tau_2^2 \tag{36}$$

so that

$$I = \int_{-\infty}^{\infty} dx \frac{e^{2i\pi(n+v)x}}{(m\tau_1 + x)^2 + m^2\tau_2^2} = \int_{-\infty}^{\infty} dx \frac{e^{-2i\pi(n+v)m\tau_1} e^{2i\pi(n+v)x}}{x^2 + m^2\tau_2^2} = \pi \frac{e^{-2i\pi(n+v)m\tau_1} e^{-2\pi|v+n||m\tau_2|}}{|m\tau_2|}. \tag{37}$$

Restricting to the upper plane so that $\tau_2 = \operatorname{Im}\tau > 0$, we finally get

$$\zeta'(0) = 2\pi\tau_2 \left(v^2 - v + \frac{1}{6} \right) + \sum_{n=-\infty}^{\infty} \sum_{m \neq 0} \frac{1}{|m|} e^{-2|m||v+n|\pi\tau_2 - 2i\pi(n+v)m\tau_1 + 2i\pi m u}.$$

The sum over m gives

$$\sum_{m \neq 0} \frac{1}{|m|} e^{-2a\pi|m| + 2i\pi b m} = -\ln\left(1 - e^{-2\pi(a+bi)}\right) - \ln\left(1 - e^{-2\pi(a-bi)}\right) \tag{38}$$

or

$$\zeta'(0) = 2\pi\tau_2 \left(v^2 - v + \frac{1}{6} \right) - \sum_{n=-\infty}^{\infty} \ln\left|1 - e^{-2|v+n|\pi\tau_2 + 2i\pi(n+v)\tau_1 - 2i\pi u}\right|^2.$$

Take now the exponent

$$2i\pi\left[|v+n| i\tau_2 + (n+v)\tau_1 - u\right]. \tag{39}$$

Looking at the contributions for $n = 0$, $n > 1$ and $n < -1$ we can write everything in a compact form as follows:

$$\zeta'(0) = 2\pi\tau_2 \left(v^2 - v + \frac{1}{6} \right) - \sum_{n=-\infty}^{\infty} \ln\left|1 - e^{2i\pi(|n|\tau - \varepsilon_n(u - \tau v))}\right|^2$$

where we have introduced the sign convention $\varepsilon_n = \operatorname{sign}(n + \frac{1}{2})$.

Now consider the function

$$g(w, \tau) = \prod_{n=-\infty}^{\infty} \left(1 - \exp\left[2i\pi\left(|n|\tau - \varepsilon_n w\right)\right]\right). \tag{40}$$

Separating out the zero mode we may write

$$g(w, \tau) = \left(1 - \exp[-2i\pi w]\right) \prod_{n=1}^{\infty} \left(1 - \exp\left[2i\pi(n\tau - w)\right]\right) \times \prod_{n=1}^{\infty} \left(1 - \exp\left[2i\pi(n\tau + w)\right]\right). \tag{41}$$

Using the nome $q = e^{i\pi\tau}$ we get

$$g(w, \tau) = 2i \sin\pi w e^{-i\pi w} \prod_{n=1}^{\infty} \left(1 - 2q^{2n} \cos 2\pi w u + q^{4n}\right). \tag{42}$$

The elliptic function ϑ_1 is defined as

$$\vartheta_1(w, \tau) = 2q^{\frac{1}{4}} \sin\pi w \prod_{n=1}^{\infty} \left(1 - 2q^{2n} \cos 2\pi w + q^{4n}\right)\left(1 - q^{2n}\right). \tag{43}$$

Using the Dedekind eta function $\eta(\tau) = q^{\frac{1}{12}} \prod_{n=1}^{\infty}(1 - q^{2n})$ we deduce that

$$\vartheta_1(w, \tau) = -i e^{i\pi(w + \frac{\tau}{6})} \eta(\tau) g(w, \tau). \tag{44}$$

This way,

$$\sum_{n=-\infty}^{\infty} \ln\left|1 - e^{2i\pi(|n|\tau - \varepsilon_n(u - \tau v))}\right|^2 = \ln\left|\frac{\vartheta_1(u - \tau v, \tau)}{\eta(\tau)}\right|^2 + \ln\left(e^{-i\pi(u - \tau(v - \frac{1}{6}))} e^{i\pi(u - \tau^*(v - \frac{1}{6}))}\right) = 2\ln\left|\frac{\vartheta_1(u - \tau v, \tau)}{\eta(\tau)}\right| - 2\pi\tau_2\left(v - \frac{1}{6}\right). \tag{45}$$

Finally, collecting all the terms we get

$$\zeta'(0) = 2\pi\tau_2 \left(v^2 - v + \frac{1}{6} \right) - 2\ln\left|\frac{\vartheta_1(u - \tau v, \tau)}{\eta(\tau)}\right| + 2\pi\tau_2\left(v - \frac{1}{6}\right) = -2\ln\left|e^{i\pi\tau v^2} \frac{\vartheta_1(u - \tau v, \tau)}{\eta(\tau)}\right| \tag{46}$$

Therefore, the analytic torsion is

$$\mathcal{T}_z = \ln\left|\frac{e^{\pi i v^2 \tau} \vartheta_1(z, \tau)}{\eta(\tau)}\right|, \quad z = u - \tau v. \tag{47}$$

In order to use this result, we need to make a proper identification of the hypercharge q_i. Let us first recall the following identity for $\vartheta_1(z, \tau)$:

$$\vartheta_1(z + \tau, \tau) = -e^{-\pi i \tau} e^{-2\pi i z} \vartheta_1(z, \tau). \tag{48}$$

For $z = u - \tau v$ this becomes

$$\vartheta_1(u - \tau v + \tau, \tau) = -e^{\pi i(2v-1)} e^{-2\pi i u} \vartheta_1(u - \tau v). \tag{49}$$

In terms of the variables u, v, the transformation is essentially equivalent to the shift $v \to v - 1$, i.e. the left part can be rewritten as $\vartheta_1(u - \tau(v-1), \tau)$. Consequently, for two different points v, $v - 1$ the torsion reads

$$\mathcal{T}_v \equiv \mathcal{T}_{z=u-\tau v} = \ln\left|\frac{e^{\pi i \tau v^2} \vartheta_1(u - \tau v, \tau)}{\eta(\tau)}\right|, \tag{50}$$

$$\mathcal{T}_{v-1} \equiv \mathcal{T}_{z=u-\tau(v-1)} = \ln\left|\frac{e^{\pi i \tau (v-1)^2} \vartheta_1(u - \tau(v-1), \tau)}{\eta(\tau)}\right|. \tag{51}$$

Using the identity (49) the numerator in the logarithmic quantity (51) becomes

$$e^{\pi i \tau (v-1)^2} \vartheta_1\big(u - \tau(v-1), \tau\big) = -e^{-2\pi i u} e^{\pi i \tau v^2} \vartheta_1(u - \tau v, \tau). \tag{52}$$

Now, substituting into the torsion formula and taking into account that u is real, we obtain

$$\mathcal{T}_{z=u-\tau(v-1)} = \ln\left|-e^{-2\pi i u} e^{\pi i \tau v^2} \vartheta_1(u - \tau v, \tau)\right| = \mathcal{T}_{z=u-\tau v}. \tag{53}$$

Considering now two successive hypercharge values q_i, q_j such that $|q_i - q_j| = \frac{5}{6}$ and using the association

$$v_i = \frac{q_i}{|q_i - q_j|}, \tag{54}$$

we get the identification $\mathcal{T}_{u-\tau v_i} \leftrightarrow \mathcal{T}_{q_i}$. With this embedding we can easily see that the differences $\mathcal{T}_{-2/3} - \mathcal{T}_{1/6}$ and $\mathcal{T}_{-1/2} - \mathcal{T}_{1/3}$ vanish and the result (22) is readily obtained.

This example, although not fully realistic (since we have restricted our investigation to the flat torus) is sufficient to support the aforementioned ideas. In proposing the above identification we relied on the assumption that a $U(1)$ symmetry is naturally associated with the one cycle of the torus, while the hypercharge identification seems to be in accordance with the notion of $U(1)$ fluxes piercing the matter curves. Indeed, we know that when the $U(1)$ fluxes are turned on they affect the multiplicity of the various massless representations along the matter curves. For example, assuming the Σ_5 matter curve, the number of 5's and/or $\bar{5}$'s is determined by the fluxes of $U(1)_i$'s corresponding to some Cartan generators of the commutant gauge group inside E_8 (here being $SU(5)_\perp$). Furthermore, $U(1)_Y \in SU(5)_{\rm GUT}$ determines in a similar manner the splitting of the Standard Model representations obtained from the decomposition of 10 and $\bar{5}$'s. Indeed, in the presence of $U(1)_Y \in SU(5)_{\rm GUT}$ flux, we can express for example the splitting of the massless spectrum for n units of hyperflux for $5 \to (3,1)_{1/3} + (1,2)_{-1/2}$ as $\#(3,1)_{1/3} - \#(1,2)_{-1/2} = (v_d - v_l)n = n$. Notice that Eq. (28) and the hypercharge association assumed in (54) imply also the same v-dependence of the corresponding massive modes.

2.2.2. On matter curves with higher genera

So far we have presented simple examples where threshold corrections from KK-states associated to genus one matter curves do not alter the unification scale. For $g = 1$ the properties of the determinants are well understood and (at least in the case of flat torus) we can corroborate our assumption for the $U(1)_Y$ embedding with an explicit computation. However, in F-theory, we deal quite often with examples involving matter curves of higher genera ($g \geqslant 2$). In this case a natural extension of the $\bar\partial$-torsion can be possibly related to the Selberg's zeta function [23]. Then one has to seek for specific realistic cases where the required properties are satisfied. Here, we will only give a brief account on the possibility of implementing our analysis for $g > 1$, leaving a more detailed consideration for future work.

We first note that the compact Riemannian manifold (for $g > 1$) can be written as $\mathcal{H}/\Gamma$, that is, it can be identified as the quotient of the upper half plane $\mathcal{H}$ by the group of isometries Γ of $\mathcal{H}$ with elements

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma :\to \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

with the condition $|a+d| > 2$.[3] An element $\gamma \in \Gamma$ is called primitive if it is not a power of some other element in Γ. An element γ' is said to be conjugate to another γ if there exists an element γ_1 in Γ such that

$$\gamma' = \gamma_1 \gamma \gamma_1^{-1}.$$

We denote $\{\gamma\}$ the set of elements which are conjugate to γ. This way, Γ is the union of disjoint conjugacy classes. If γ_0 is the primitive element of $\{\gamma\}$, then any other element in the same class can be written as $\gamma = \gamma_0^n$ for some integer power n. We mention that for a compact manifold the element $\gamma \in \Gamma$ can also be written as

$$\gamma \in \Gamma : \frac{z' - z_0}{z' - z_1} = e^{2\rho_\gamma} \frac{z - z_0}{z - z_1}$$

for two real fixed points $z_{0,1}$ and $\rho_\gamma > 0$. For given finite unitary representation $\chi(\gamma)$, the Selberg zeta function is defined [17] as

$$Z(s,\chi) = \prod_{\{\gamma\}} \prod_{n=0}^{\infty} \det\big(1 - \chi(\gamma) e^{-\rho_\gamma (s+n)}\big) \tag{55}$$

with ${\rm Re}(s) > 1$. Hence, any properties of the torsion could be investigated with respect to its relation to the Selberg zeta function given by the general formula (55). For example, for two non-trivial unitary representations $\chi(\gamma)$ and $\chi'(\gamma')$ of Γ and for a compact Riemann surface of $g > 1$, according to a theorem by Ray and Singer [17] the difference $\ln(\mathcal{T}_0(\chi)) - \ln(\mathcal{T}_0(\chi'))$ is proportional to $\ln(Z(\chi)) - \ln(Z(\chi'))$. Several studies [24] have revealed interesting properties of Selberg's function. It is envisaged that one can find examples where the required quantities exhibit periodicity properties and an appropriate hypercharge embedding could also be feasible. We plan to return to these issues in a future publication.

3. Conclusions

In unified theories emerging in the context of F-theory compactification, threshold corrections from Kaluza–Klein massive modes play a decisive role in gauge coupling unification and the determination of the GUT scale. In this work, we have revisited this issue in the context of a specific minimal unification scenario, the F-theory $SU(5)$ GUT. Although the problem of KK-thresholds is in general quite complicated, in the model under consideration it gets remarkably simplified using the fact that these thresholds can be expressed in terms of a topologically invariant quantity, the Ray–Singer analytic torsion. Previous considerations have shown that the KK-modes from the gauge multiplets can be absorbed into a redefinition of the effective GUT mass scale and the

[3] This is a space with hyperbolic geometry with metric $ds^2 = y^{-2}(dx^2 + dy^2)$ and constant negative curvature $R = -1$.

string gauge coupling. However, for KK-mode contributions emerging from the matter curves the situation is less clear. We have pursued this issue one step further, and analyzed the conditions to be imposed on the matter spectrum and the nature of bundle structure where matter resides, in order to ensure that the emerging F-theory GUT comply with low energy phenomenological expectations. We have given examples where matter resides on genus one matter curves with chiral matter forming complete $SU(5)$ multiplets consistent with the minimal unification scenario, so that the low energy values for the weak mixing angle and the strong gauge coupling can be reproduced. A short discussion on the prospects of models with higher genus matter curves is also included.

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