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Uncertainty relation and non-dispersive states in finite quantum mechanics [1]

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Abstract

In this letter, we provide evidence for a classical sector of states in the Hilbert space of Finite Quantum Mechanics (FQM). We construct a subset of states whose the minimum bound of position-momentum uncertainty (equivalent to an effective $\hbar$) vanishes. The classical regime, contrary to standard Quantum Mechanical Systems of particles and fields, but also of strings and branes appears in short distances of the order of the lattice spacing. For linear quantum maps of long periods, we observe that time evolution leads to fast decorrelation of the wave packets, phenomenon similar to the behavior of wave packets in 't Hooft and Susskind holographic picture. Moreoever, we construct explicitly a non-dispersive basis of states in accordance with 't Hooft's arguments about the deterministic behavior of FQM. © 1997 Elsevier Science B.V.

Studies of quantum field and string theories in the vicinity of the horizon of black holes suggest the existence of a minimal length at the string scale [1–5] and consistency with the Bekenstein entropy bounds for black holes [6] requires the finite dimensionality of the Hilbert space of states. These ideas lead 't Hooft [7,8] some years ago, and subsequently Susskind [9] to propose the holographic picture. According to this picture there is a description of the physical world in terms of finite number of Bits of information per Planck unit of area of a two dimensional screen at the boundaries of the world. The particles moving in space-time are represented as two dimensional areas where the number of Bits distributed give information about mass, momentum and quantum numbers of the particles. The black holes are represented by maximum information density and the corresponding number of Bits is proportional to their mass. The various interactions of particles are represented by splitting and joining the representative two dimensional regions. In this picture, it is very natural to represent strings by strings of Bits with length and energy proportional to their number [9,10]. Ideas of information processing by black holes have been introduced many years ago by Wheeler.

Although we are far at the moment from a fundamental theory which encompasses naturally such a picture, recent findings in studies of D-branes, [11–13] concerning microscopic derivations of the Bekenstein entropy formulae and very recent works

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on a specific supersymmetric matrix model approximation of M-theory [14], provide hints that the holographic picture is on the right track. The finite dimensionality of Hilbert space in the holographic picture suggests the existence of a number-theoretic structure with discrete space-time and dynamical variables related to the Bits of information. We remind the reader that the string theory has a number-theoretic (p-adic) nature which is far from being understood [15].

Sometime ago, a discrete version of Quantum Mechanics, called Finite Quantum Mechanics (FQM), using the discrete and finite representations of the Heisenberg group appropriate for toroidal phase spaces, has been introduced by H. Weyl [16] and subsequently studied J. Schwinger [17]. Balian and Itzykson [18] provided explicit expressions for quantum maps of prime integer dimensions, while quantization of a family of maps from the modular group $SL(2,Z)$ has been given by Berry [19] for every finite dimension N.

In this work we study physical properties of FQM, dispersion of wave packets and the modification of the uncertainty relation coming from a sector in the Hilbert space of states which looks classical. According to 't Hooft [7,8] this is expected in any system with finite dimensional Hilbert space and it is the consequence of the existence of bases of states which, although they are not eigenstates of the unitary evolution matrix chosen, they do not disperse under time evolution. For a class of unitary evolution matrices, we construct explicitly their eigenstates which turn out to be completely delocalized. We construct the 't Hooft basis of non-dispersive states and we show that the evolution matrix typically decorrelates Gaussian wave packets, property which is observed in the interaction of wave packets with the horizon of black hole and it is consistent with the existence of a maximum information density.

We also examine the violation of the uncertainty relation which is due to the finiteness of the Heisenberg group. For particular states, we find that the effective Planck constant vanishes. Of course this does not necessarily imply that for these particular states the dispersion in position and momentum simultaneously can become arbitrarily small. Furthermore, in a different subset of states evolution in time does not lead to dispersion in position or in momentum. Obviously, in order to have a real classical behavior both characteristics must simultaneously appear.

We finally note that recently various proposals for the modification of the uncertainty principle implied by the dynamics of D-branes have been put forward which provide evidence that the string scale *is not* the ultimate one and it can be probed by scattering of D-branes [20]. Recent progress on the matrix model interpretation of the M-theory indicates that it is possible to formulate a theory describing these states and their dynamics.

We review first the structure of Finite Quantum Mechanics(FQM). The classical phase space for one degree of freedom is the discrete square torus T^2 of dimensions $N \times N$ and the classical time is discrete [2]. Such a phase space can be realized physically as the configuration space of the center of the electronic circular orbits in a transverse magnetic field periodically closing the plane to form a torus.

The simplest classical canonical transformations are the linear ones

$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q_n \\ p_n \end{pmatrix} \tag{1}$$

with $a,b,c,d,$ integers modulo N (mod N), with determinant $ad - bc = 1 \bmod N$ (here we differ from Berry et al. [19,22] who take $ad - bc = 1$). Thus the linear canonical transformations form the group $SL(2,N)$. For $N = p^n$ where p is a prime integer, the quantum mechanical representation has been studied by [23]. For general $N = \prod_{i=1}^{k} p_i^{n_i}$ it is known that $SL(2,N) = \prod_{i=1}^{k} SL(2,p_i^{n_i})$ and the quantization is reduced with tensor products to the previous case.

The classical harmonic oscillator corresponds to the mapping $a = d = 0$ and $-b = c = 1$ but this mapping commutes with elements of $SL(2,N)$ of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a^2 + b^2 = 1 \bmod N.$$

These elements form a subgroup of $SL(2,N)$ which we call $O_2(N)$. For N power of a prime, this is a

[2] for the Heisenberg group of phase spaces of arbitrary arbitrary genus see [21].

cyclic group and we call the corresponding generator the Balian-Itzykson (BI) oscillator [18,24,23]. When $N=\prod_{i=1}^{k} p_i^{n_i}$ the group is the product of k cyclic abelian groups.

On the N dimensional Hilbert space the basic operators are the shift and clock matrices $Q_{ij}=\omega^{s+1-i}\delta_{ij}$ and $P_{ij}=\delta_{s+1-i,j}$ where δ_{ij} is the Kronecker delta with indices $\mathrm{mod}(N)$ and $N=2s+1$ where ω is the N-th root of unity, $\omega=e^{\frac{2\pi i}{N}}$. These matrices satisfy the relation

$$QP=\omega PQ \tag{2}$$

and generate the discrete and finite Heisenberg-Weyl group $HW(N)$ [17]

$$J_{rs}=\omega^{rs/2}P^rQ^s, \quad r,s=\{1,\dots N\}. \tag{3}$$

The matrices Q and P are connected by the finite Fourier transform

$$F_{kl}=\frac{1}{\sqrt{N}}\omega^{(s+1-k)(s+1-l)} \tag{4}$$

with the properties, $QF=FP$ and $F^4=1$. The monomials J_{rs} satisfy the relations

$$J_{rs}J_{r's'}=\omega^{(r's-s'r)/2}J_{r's'}J_{rs} \tag{5}$$

with $J_{rs}^N=I$, and so they form a complex basis for $SU(N)$ [25]. One can show [17] that $HW(N)=\prod_{i=1}^{k} HW(p_i^{n_i})$ where the prime factors belong to the factorization of N. In FQM the time is discrete and we cannot define unambiguously the Hamiltonian of dynamical systems but only the unitary evolution operator of a unit time step. In the case of linear maps, i.e. elements of $SL(2,N)$, the dynamical equation which replaces the Heisenberg equation of motion is provided by the metaplectic representation of $SL(2,N)$ [18,24]

$$U^\dagger(A)J_{rs}U(A)=J_{r's'}$$

where $(r',s')=(r,s)A$. Explicit construction of the evolution matrix $U(A)$ is given in [24] for a matrix A of $SL(2,N)$ and N a prime integer:

$$U(A)_{kl} = \frac{1}{\sqrt{N}}(-2c|N) \times \left\{ \begin{matrix} 1 \\ -\iota \end{matrix} \right\} \omega^{-[a(k-1)^2+d(l-1)^2-2(k-1)(l-1)]/(2c)} \tag{6}$$

where

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7}$$

and $(a|p)=\pm 1$ depending on whether a is (or is not) the square of an integer mod p and the upper (lower) value, $1(-\iota)$, in the curly bracket corresponds to $N=4k\pm 1$. The exponent has to be calculated $\mathrm{mod}\,N$ so the inverse $1/(2c)$ has to exist $\mathrm{mod}\,N$.

Now we come to the implications of FQM. The basic features of quantum mechanical behavior, is the uncertainty relation, the dispersion of the wave-packets and the superposition principle. Any violation of the above features is considered as violation of Quantum Mechanics. Field Theories share the same characteristics plus the transformation of energy into matter and vice versa subject to Lorenz invariance and conservation of Quantum numbers. The first departure from Quantum Mechanical particle behavior is observed in string theory due to the non-locality of the string [26]. The well known correction of the uncertainty relation found in Refs. [1–3,5] in the high energy scattering of strings for fixed impact parameter has the form

$$\Delta x\Delta p \ge \hbar\left(1+\alpha'(\Delta p)^2\right) \tag{8}$$

and implies a minimum distance of the order of the string scale $\ell_s=g/\sqrt{\alpha'}$. Another equivalent form of the same fact has been proposed by Yoneya [27] which is more intuitive

$$\Delta x\Delta t \ge \ell_s^2. \tag{9}$$

In this relation, the size of the string of energy E contrary to the particle behavior, is $\Delta x \propto \ell_s^2 E$ [9] and the time resolution $\Delta t \propto 1/E$. This form of the uncertainty relation, has the benefit to apply also to the semi-classical D-branes as recently shown by Yoneya. In the recent literature there is an extended study of possible violation of the uncertainty relation coming from string defects [28].

In the following, we wish to investigate the form of the violation of the uncertainty relation in the context of FQM. Therefore we must calculate the commutator of the position and momentum operators. We define the position operator $\hat{q}$, $Q = e^{\imath\hat{q}}$, and so $\hat{q}$ is defined modulo diagonal integer matrices. We choose

$$\hat{q}_{ij} = \frac{2\pi}{N}(s+1-i)\delta_{ij} \tag{10}$$

By Fourier transform, the corresponding momentum operator is found to be $P = e^{-\imath\hat{p}}$, with

$$\hat{p}_{ij} = -\imath\frac{\pi}{N}\frac{(-1)^{(i-j)}}{\sin\frac{\pi}{N}(i-j)} \tag{11}$$

when $i \neq j$ and zero when $i = j$.

The position and momentum operators chosen have definite properties under the parity operator $S = F^2$, i.e., $S\hat{q}S = -\hat{q}$ and the same holds for $\hat{p}$, as it follows from $\hat{p} = -F^{-1}\hat{q}F$. The matrix $\hat{p}$ is a discrete version of the derivative of the δ-function.

Incidentally, we note here that for continuous time, one particle hamiltonians can be written as $N \times N$ matrices

$$\mathscr{H} = \frac{\hat{p}^2}{2m} + \mathscr{V}(\hat{q}) \tag{12}$$

which can be easily treated numerically with fast convergence to the continuum standard Quantum Mechanical results [29]. We have checked the case of harmonic oscillator for the ground state as well as for excited states. The eigenvalues come very close to the exact ones (exept the end of the spectrum) with an effective $\hbar$ being equal to $2\pi/N$. We hope to come back in the future on this point.

Having defined explicitly the form of the position and momentum operators in FQM, we calculate the uncertainty relation for these operators $(\hat{q},\hat{p})$,

$$\Delta\hat{q}\Delta\hat{p} \geq \tfrac{1}{2}|\langle[\hat{q},\hat{p}]\rangle| \tag{13}$$

where the dispersion of an observable A is defined as $\Delta A = \sqrt{\langle(A - \langle A\rangle)^2\rangle}$.

The commutator on the right hand side(RHS) can be explicitely calculated

$$-\imath[\hat{q},\hat{p}]_{ij} = \frac{2\pi}{N}\frac{\frac{\pi}{N}(i-j)(-1)^{(i-j)}}{\sin\frac{\pi}{N}(i-j)} \equiv \hat{C}_{ij} \tag{14}$$

when $i \neq j$ and zero when $i = j$. In Ref. [30] it has been shown that the above commutator has the correct continuum limit. It will prove important to know the eigenvalues of this hermitian matrix. Although it is difficult to find explicit expressions for eigenvectors and eigenvalues, we observe that the commutator matrix $\hat{C}$ for fixed $(i-j)$ goes in the large N limit to the matrix $\hat{C}_\infty$

$$\hat{C}_{ij} \to \hat{C}_\infty \equiv \frac{2\pi}{N}(-1)^{(i-j)}\delta_{ij}. \tag{15}$$

The above matrix has eigenvectors

$$v_k = \left\{\frac{1}{\sqrt{N}}(-1)^l\omega^{l\cdot k}\right\}_{l:0,\dots,N-1} \tag{16}$$

and eigenvalues, $-\frac{2\pi}{N}$ for $k = 1,\dots,N-1$ and $\frac{2\pi}{N}(N-1)$ for $k = 0$. Except from the last eigenvalue, this reminds the situation in the standard quantum mechanics if we identify $\hbar = \frac{2\pi}{N}$. For finite N, the matrix $\hat{C}$ commutes with the finite Fourier transform so they share common eigenvectors. Numerical investigation shows that for relatively large $N \sim 100$, there are few big negative eigenvalues and most of the remaining eigevalues are very close to $2\pi/N$.

We discuss now the implications of the new uncertainty relation (13). For a general wave function

$$\psi = \sum_{i=1}^{N} c_i\psi_i, \quad \sum_{i=1}^{N}|c_i|^2 = 1 \tag{17}$$

where ψ_i are the normalised eigenvectors of the hermitian commutator matrix $\hat{C}$,

$$\hat{C}\psi_i = \epsilon_i\psi_i \tag{18}$$

we find that

$$\Delta\hat{q}\Delta\hat{p} \geq \tfrac{1}{2}\left|\sum_{i=1}^{N}|c_i|^2\epsilon_i\right| \equiv \tfrac{1}{2}\hbar(\psi) \tag{19}$$

It would be desirable to write the RHS of (19), which is an effective Planck constant as a function of $\Delta\hat{q}$ and $\Delta\hat{p}$, but this is not in general possible. For the limiting matrix $\hat{C}_\infty$

$$\hbar(\psi) = \frac{2\pi}{N}\left|1 - N|c_N|^2\right|. \tag{20}$$

Because some of the ϵ_i's have to be negative, we can find an $(N-2)$-dimensional subset $\mathscr{H}_{\mathrm{cl}}$ of the N-dimensional complex unit sphere for which $\hbar(\psi)$ vanishes. To get some feeling of the geometry of $\mathscr{H}_{\mathrm{cl}}$

we observe that the latter is invariant under the group of complex matrices which preserve $\hat{C}$, i.e., $U^{\dagger}\hat{C}U=\hat{C}$. Such an example is the Finite Fourier Transform matrix. Defining a non-positive definite inner product

$$(\psi_1,\psi_2)=\langle\psi_1,\hat{C}\psi_2\rangle \tag{21}$$

we find that if $\psi_{1,2}$ belong to $\mathscr{H}_{\rm cl}$ and moreover they are orthogonal with respect to this inner product, then any linear combination of them belongs to $\mathscr{H}_{\rm cl}$. The set of matrices which preserve $\hat{C}$ form a group which is isomorphic to the non-compact group $U(n,m)$ where $n(m)$ are the numbers of positive (negative) eigenvalues of $\hat{C}$. On the other hand, in order to stay on the unit complex N- dimensional sphere, we have to restrict to the diagonal compact subgroup $U(n)\oplus U(m)$.

Numerical study of the $\hat{C}$ eigenstates shows smooth localization structure indicating possible analytic expression for their exact forms. In Figs. 1, 2 we plot some eigenstates of the commutator $\hat{C}$ for $N=47$ which correspond to non-degenerate eigenvalues. The parity symmetry is realized in all eigenstates and we observe that the simplest structure appears for the most negative eigenvalue. Also, plotted is the eigenvector corresponding to the smallest positive eigenvalue which happens to be smaller than $2\pi/N$.

The commutation relation (13) breaks explicitly translation invariance since the shift operator P does not commute with $\hat{C}$. The violation of translation invariance implies a non-trivial dependence of the RHS of the uncertainty relation on the momentum spectrum of the wavefunction ψ. One possible way to probe the dependence on the width of the momentum spectrum is to saturate the RHS for a class of Gaussians ψ_G of various widths.

In Fig. 3 we plot the RHS of the uncertainty relation (19) for Gaussians of various widths in units of lattice spacings. We observe that for widths bigger than a few lattice spacings the RHS becomes $2\pi/N$ which is the equivalent of $\hbar$ for FQM. Going to smaller widths we find a sharp transition to classical behavior because $\hbar(\psi_G)$ goes to zero. Thus, the classical regime appears in the order of the lattice spacing where the FQM exhibits deviation from the standard uncertainty relation. The classical sector $\mathscr{H}_{\rm cl}$, must disappear in the appropriate limit where the standard quantum mechanics is recovered, but for every finite N it signals a completely different behavior from particle string or D-brane uncertainty relation.

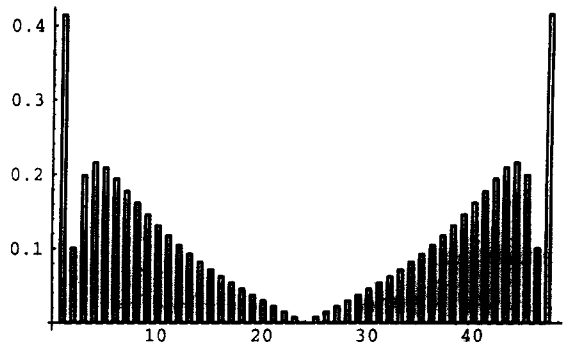

Fig. 2. The eigenvector of the commutator for $N=47$ corresponding to the smallest positive eigenvalue.

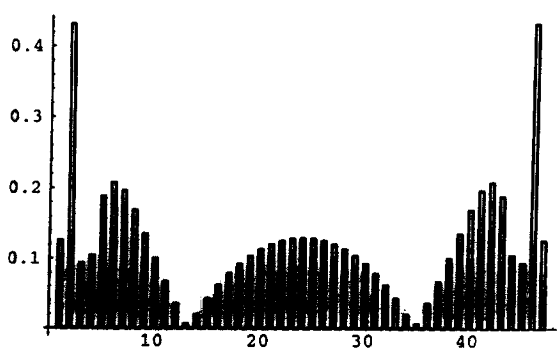

Fig. 1. The eigenvector of the commutator for $N=47$ corresponding to the largest negative eigenvalue.

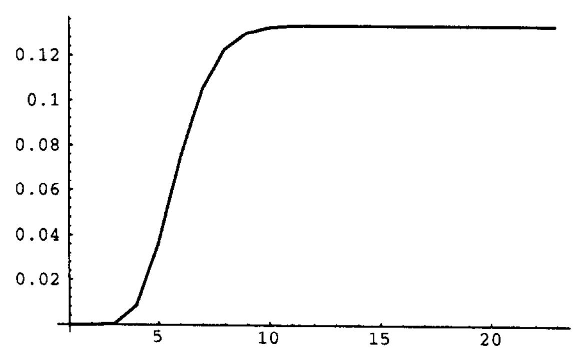

Fig. 3. $\hbar$ for Gaussian widths from 1 to 4 lattice spacings for the case $N=47$.

We now construct the 't Hooft non-dispersive states for a class of specific evolution matrices of FQM. The BI harmonic oscillator group of matrices

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{22}$$

$a^2 + b^2 = 1 \mod N$ is cyclic abelian group with $4k$ elements and its generator $R_0 = \begin{pmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{pmatrix}$ for primes up to 20000 were determined by a search algorithm in Ref. [24]. For $N = 4k+1$, R_0 is diagonalized by a matrix in $SL(2,N)$

$$R_0 = T \begin{pmatrix} a_0 - tb_0 & 0 \\ 0 & a_0 + tb_0 \end{pmatrix} T^{-1} \tag{23}$$

$$T = \frac{1}{1+t} \begin{pmatrix} 1 & 1 \\ -t & t \end{pmatrix} \tag{24}$$

with $t = g^k \mod N$. Since the $4k$-period of R_0 and g coincide we see that $a_0 - tb_0$ turns out to be a primitive element g. Thus it is simple to construct the generator R_0,

$$R_0 = \begin{pmatrix} \dfrac{g + g^{-1}}{2} & \dfrac{g^{-1} - g}{2t} \\ \dfrac{g - g^{-1}}{2t} & \dfrac{g + g^{-1}}{2} \end{pmatrix}. \tag{25}$$

Now, since $U(A)$ defines a representation of $SL(2,N)$, we have

$$U(R_0) = U(T) U \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} U^{-1}(T). \tag{26}$$

The eigenvalue problem for $U(R_0)$ was solved analytically for all primes of the form $N = 4k+1$ in Ref. [24,23]. The observation is simply that $U\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ is by construction the matrix

$$D_{k,\ell} = -\delta_{k-1,g(\ell-1)}, \quad k,\ell = 1,\dots,N \tag{27}$$

which has eigenvalues $-e^{\frac{\pi\imath}{2k}\ell}, \ell = 1,\dots,4k$ and -1. The eigenvectors are determined by the multiplicative characters of the finite field $F_p = \{0,1,\dots,p-1\}$ and these are the $(4k+1)$-dimensional vectors $\Pi_j = \{0, \Pi_j(g^n)\}$ where

$$\Pi_j(g^n) = \frac{e^{\frac{\imath\pi}{2k} jn}}{\sqrt{4k}}, \quad j,n = 1,2,\dots,4k \tag{28}$$

whilst the $(4k+1)$th eigenvector is defined to be $\{1,0,\dots,0\}$. To proceed, we observe that the circulant matrix D, in (27) permutes the axes vectors e^i,

$$e^i_j = \delta_{ij} \tag{29}$$

as follows:

$$D^m e^i = (-)^m e^{1 + g^m(i-1)}. \tag{30}$$

So the non-dispersive 't Hooft states for $G = U(R_0) = U(T) C U^{-1}(T)$ in (26) are $u^i = U(T) e^i$ and they evolve as

$$D^m u^i = (-)^m u^{1 + g^m(i-1)}. \tag{31}$$

Using (6) we find explicitely the $U(T)$ and u^i's are found to be

$$(u^i)_k = \frac{1}{\sqrt{N}} (((1+t)|N)) \times \omega^{\frac{1}{2}(t(k-1)^2 + (i-1)^2 + 2(1-t)(k-1)(i-1))}. \tag{32}$$

We observe that all the components of u^i's are pure phases. The evolution law given by (31) implies that the dispersion of the position operator in the states u^i,

$$(\Delta \hat{q})_i^2 = u^{i\dagger} (\hat{q} - \langle \hat{q} \rangle_i)^2 u^i, \quad \langle \hat{q} \rangle_i = u^{i\dagger} \hat{q} u^i \tag{33}$$

remains constant, since all $(u^i)_\ell$ are pure phases and they remain so under evolution with $U(R_0)$. The same happens to the dispersion of momentum operator because $U(R_0)$ commutes with the finite Fourier transform, $F = \imath^k U(R_0)^k$.

According to 't Hooft, the set of the non-dispersive states is not observable by localized experiments but only the effects of linear combinations of a big number of them which produce localized states (particles). In terms of the non-dispersive basis, FQM is a deterministic theory. For the quantum linear maps the non- dispersive states permute among themselves during evolution in a random way. This implies that Gaussian wave packets seen as linear combinations of the non-dispersive states basis will decorrelate quickly and become delocalized. This can be understood by the following semi-classical argument: Since linear maps of long period produce long random trajectories and the corresponding wavefunctions have to be extended along them, Gaussian wave packets at time t_0 will disperse under time evolution along random the trajectories [23,31].

This property is reminiscent of the behavior of wavepackets falling on the horizon of a black hole which become extended along the horizon due to a minimum information density per Planck area. This is also true for strings which become extended and are wrapped around the horizon [7,9].

We conclude our discussion with the following observations. The classical sector of FQM is characterized by the following two features:

1. There is a subset of states ψ_i where the effective Planck constant $\hbar(\psi_i)$ vanishes. This however, does not necessarily imply that for the particular (Gaussian) states the dispersion in position and momentum simultaneously can become arbitrarily small. In fact the product of the position momentum uncertainty is bounded to be bigger than $1/16\pi^2$ due to the property of the Fourier transform. The intringuing property here, is the vanishing of the commutator for very narrow Gaussians.
2. In a different subset of states evolution in time does not lead to dispersion in position or in momentum.

The above two characteristics must simultaneously appear in order to have a real classical behavior. In other words, one should determine the intersection of the two different subsets of the Hilbert space. This problem remains open for future investigation.

Another novel feature of FQM is the existence of quantum maps which decorrelate Gaussian initial states and distribute the wave packet information equally among the points of the configuration space. This is a desirable property according to the holographic picture which preserving unitarity randomizes the initial information in a retrievable way.

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