



ELSEVIER

2 May 1996

PHYSICS LETTERS B

Physics Letters B 374 (1996) 49–56

Universal moduli-dependent string thresholds in $Z_2 \times Z_2$ orbifolds

P.M. Petropoulos^{a,1}, J. Rizos^{a,b,c,2}

^a *Theory Division, CERN, 1211 Geneva 23, Switzerland*

^b *Division of Theoretical Physics, Physics Department, University of Ioannina, 45110 Ioannina, Greece*

^c *International School for Advanced Studies, SISSA, Via Beirut 2-4, 34013 Trieste, Italy*

Received 17 January 1996

Editor: L. Alvarez-Gaumé

Abstract

In the context of a recently proposed method for computing exactly string loop corrections regularized in the infra-red, we determine and calculate the universal moduli-dependent part of the threshold corrections to the gauge couplings for the symmetric $Z_2 \times Z_2$ orbifold model. We show that these corrections decrease the unification scale of the underlying effective field theory. We also comment on the relation between this infra-red regularization scheme and other proposed methods.

String theories provide a unified description of gauge and gravitational interactions at a scale close to the Planck mass $M_P = 1/\sqrt{16\pi G_N}$. For four-dimensional vacua of the heterotic string the relation between the running gauge coupling $g_\alpha(\mu)$ of the low-energy effective field theory and the string coupling g_s , assuming the decoupling of massive modes, must have the following form:

$$\frac{16\pi^2}{g_\alpha^2(\mu)} = k_\alpha \frac{16\pi^2}{g_s^2} + b_\alpha \log \frac{M_s^2}{\mu^2} + \Delta_\alpha, \quad (1)$$

where b_α are the usual effective field theory beta-function coefficients of the group factor G_α , and k_α is the level of the associated Kač–Moody algebra. The thresholds Δ_α are due to the infinite tower of string modes and can be calculated at the level of string the-

ory [1–7]. For symmetric-orbifold models they have the general form:

$$\Delta_\alpha = -\hat{b}_\alpha \log (|\eta(T)|^4 |\eta(U)|^4 \text{Im } T \text{Im } U) - k_\alpha Y(T, U) - c_\alpha, \quad (2)$$

where T, U are the VEVs of gauge singlet fields corresponding to the moduli of the internal torus³ and \hat{b}_α are the $(N = 2)$ -sector contributions to the beta-function coefficients. The term $Y(T, U)$ stands for the universal group-independent contribution to the threshold corrections, and c_α are group-dependent constants. These constants are also scheme dependent, that is, they depend on the renormalization scheme in which the running gauge couplings $g_\alpha(\mu)$ are defined.

String unification relates the fundamental string scale $M_s \equiv 1/\sqrt{\alpha'}$ to the Planck scale and to the string coupling constant g_s which is associated with

¹ On leave from Centre National de la Recherche Scientifique, France.

² Work supported by the EEC contract ref. ERBCHBGCT940634.

³ Here $T_i = T, U_i = U$.

the expectation value of the dilaton field. At the tree level this relation reads

$$M_s = g_s M_P. \quad (3)$$

Given the fact that low-energy data, assuming the minimal supersymmetric standard model as the underlying low-energy field theory, indicate gauge unification at a scale $M_X \sim 2 \times 10^{16}$ GeV [8] which is two orders of magnitude less than the Planck scale, threshold corrections (2) play a crucial role in string unification. Their effect has been extensively studied in the literature [9–14]⁴ except for the moduli-dependent universal terms $Y(T, U)$ which have received little attention because they can be formally reabsorbed into a redefinition of g_s . However, such a redefinition alters Eq. (3) in a moduli-dependent way and consequently the relation between string unification scale and Planck mass gets modified. Following this observation, the purpose of the present letter is to evaluate explicitly these terms in the context of the symmetric $Z_2 \times Z_2$ orbifold model and study their effect on the unification of the gauge couplings. We choose this model for several (related) reasons: (i) there are no one-loop corrections to the relation (3) between the Planck scale and the string coupling [6,17], (ii) there is no Green-Schwarz-like threshold $\Delta^{\text{univ}}(T, U)$ i.e. no axion-dilaton-moduli mixing at the one-loop level, and (iii) there are no ($N = 1$)-sector contributions to the beta-functions ($\hat{b}_\alpha = b_\alpha$); the last two points allow us to define the unification scale in a manifest way. As we will see in the sequel, the contribution of the group-factor independent terms $Y(T, U)$ has a decreasing effect on the unification scale. Besides the relevance that the universal contributions $Y(T, U)$ might have in the string unification, we should mention that their non-harmonic part is also related to the one-loop correction of the Kähler potential for the moduli fields [7,18]. The Kähler potential turns out to be related to the one-loop corrections of the Yukawa couplings, and its moduli dependence has attracted much attention in the rapidly growing subject of dualities [19].

In a recent article [6] an interesting method for computing unambiguously the string loop corrections has been proposed. The procedure which is used

⁴ Of course, alternative possibilities are either to modify the low-energy spectrum in order to increase the effective field theory unification scale [14,15], or to choose a non-standard hypercharge normalization [16].

consists of replacing flat four-dimensional space-time with a suitably chosen curved one in a way that preserves gauge symmetries, supersymmetry and modular invariance, and with curvature that induces an infra-red cut-off and can be consistently switched off. In this framework, vertices for space-time fields such as $F_{\mu\nu}^\alpha$ are truly marginal world-sheet operators and therefore deformations induced by the associated background fields are exactly calculable. This allows in particular for the computation of various one-loop correlators. For instance, in the case of the symmetric orbifolds, the vacuum amplitude with two insertions of the magnetic background operator reads

$$Z_2^\alpha \left(\frac{\mu}{M_s} \right) = \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im}\tau} \Gamma \left(\frac{\mu}{M_s} \right) \times \sum_{i=1}^3 \frac{\Gamma_{2,2}(T_i, U_i)}{\bar{\eta}^{24}} \left[\bar{Q}_\alpha^2 - \frac{k_\alpha}{4\pi \text{Im}\tau} \right] \bar{\Omega}. \quad (4)$$

Here $\Gamma_{2,2}(T_i, U_i)$ are the internal two-torus solitonic contributions and \bar{Q}_α is the charge operator associated with the group factor G_α , its square acting as $\frac{i}{\pi} \frac{\partial}{\partial \bar{\tau}}$ on the appropriate part of the model-dependent function $\bar{\Omega}$. The latter is a modular function of degree 10. It deserves stressing here that the radiative corrections (4) include exactly the back-reaction of the gravitationally coupled fields; this accounts for the term ⁵ $\frac{-k_\alpha}{4\pi \text{Im}\tau}$ which is universal and guarantees modular invariance. Finally, the (suitably normalized) $SU(2)_k$ partition function

$$\Gamma(x) = -2x^2 \sqrt{\text{Im}\tau} \frac{\partial}{\partial x} [Z(x) - Z(2x)] \Big|_{x=\frac{1}{\sqrt{k+2}}} \quad (5)$$

with

$$Z(x) = \sum_{m,n} e^{\frac{imx}{2} \left(mx + \frac{n}{x} \right)^2} e^{-\frac{i\bar{m}x}{2} \left(mx - \frac{n}{x} \right)^2} \quad (6)$$

replaces the flat-space contribution and ensures the convergence of the integral at large values of $\text{Im}\tau$ by introducing a universal mass gap $\mu = \frac{M_s}{\sqrt{k+2}}$ to all string excitations. This infra-red regularization of the string (on-shell) loop amplitude vanishes when the flat-space limit is reached since $\lim_{x \rightarrow 0} \Gamma(x) = 1$.

⁵ This universal term was also found in [5] for the three-point function of two gauge bosons and the modulus T .

Before going further in the evaluation of the group-independent terms $Y(T, U)$ we would like to relate the above correlator (4) to the one-loop effective field theory gauge couplings $g_\alpha(\mu)$ renormalized at some scale μ , in a specific ultra-violet scheme, the $\overline{\text{DR}}$ scheme, and show that despite the presence of the curvature-induced infra-red regulator $\Gamma(\frac{\mu}{M_s})$ in (4), the thresholds are infra-red-cutoff independent. This can be done by identifying the string one-loop corrected coupling

$$k_\alpha \frac{16\pi^2}{g_s^2} + Z_2^\alpha \left(\frac{\mu}{M_s} \right), \tag{7}$$

with the corresponding field theory one-loop corrected gauge coupling, similarly regularized in the infra-red

$$\frac{16\pi^2}{g_{\alpha, \text{bare}}^2} + b_\alpha (4\pi)^\epsilon \int_0^\infty \frac{dt}{t^{1-\epsilon}} \Gamma_{\text{FT}} \left(\frac{\mu}{\sqrt{\pi} M} \right). \tag{8}$$

Here we have used dimensional regularization for the ultra-violet and M is an arbitrary mass scale; b_α are the full beta-function coefficients for the group factor G_α and Γ_{FT} is the field theory counterpart of the infra-red regulator (5), obtained by dropping all winding modes⁶. On the other hand, one knows that in the $\overline{\text{DR}}$ scheme the relation between the field theory bare and running coupling is

$$\frac{16\pi^2}{g_{\alpha, \text{bare}}^2} = \frac{16\pi^2}{g_\alpha^2(\mu)} - b_\alpha (4\pi)^\epsilon \int_0^\infty \frac{dt}{t^{1-\epsilon}} e^{-t \frac{\mu^2}{M^2}}. \tag{9}$$

Plugging (9) into (8) and identifying the latter with (7) leads to

$$\frac{16\pi^2}{g_\alpha^2(\mu)} = k_\alpha \frac{16\pi^2}{g_s^2} + Z_2^\alpha \left(\frac{\mu}{M_s} \right) - b_\alpha (2\gamma + 2). \tag{10}$$

Eq. (10) has been obtained by using an explicitly infra-red-regularized string loop amplitude. It is worthwhile to compare this expression to the one derived in [1] without any infra-red regulator. In order to do so we first isolate the contribution of the massless states

$$\lim_{\text{Im } \tau \rightarrow \infty} \sum_{i=1}^3 \frac{\Gamma_{2,2}(T_i, U_i)}{\bar{\eta}^{24}} \bar{Q}_\alpha^2 \bar{\Omega} \equiv b_\alpha, \tag{11}$$

⁶ The extra $\sqrt{\pi}$ in the argument of Γ_{FT} accounts for the identification of the (dimensionless in the above convention) Schwinger proper-time parameter t with $\pi \text{Im } \tau$.

responsible for the non-trivial infra-red behaviour of the integral in (4). We then rewrite (10) in the form

$$\begin{aligned} \frac{16\pi^2}{g_\alpha^2(\mu)} &= k_\alpha \frac{16\pi^2}{g_s^2} + \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \Gamma \left(\frac{\mu}{M_s} \right) \\ &\times \left(\sum_{i=1}^3 \frac{\Gamma_{2,2}(T_i, U_i)}{\bar{\eta}^{24}} \left[\bar{Q}_\alpha^2 - \frac{k_\alpha}{4\pi \text{Im } \tau} \right] \bar{\Omega} - b_\alpha \right) \\ &+ b_\alpha \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \Gamma \left(\frac{\mu}{M_s} \right) - b_\alpha (2\gamma + 2), \end{aligned} \tag{12}$$

where we have subtracted and added back a $b_\alpha \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \Gamma \left(\frac{\mu}{M_s} \right)$ term, a manipulation perfectly well defined thanks to the presence of the regulator $\Gamma \left(\frac{\mu}{M_s} \right)$. Inserting the result

$$\int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \Gamma \left(\frac{\mu}{M_s} \right) = \log \frac{M_s^2}{\mu^2} + \log \frac{2e^{\gamma+3}}{\pi\sqrt{27}} + \mathcal{O} \left(\frac{\mu}{M_s} \right) \tag{13}$$

into (12) and taking the limit $\mu \rightarrow 0$ in the remaining integral since it does not suffer any longer from divergences at $\text{Im } \tau \rightarrow \infty$, we finally obtain

$$\begin{aligned} \frac{16\pi^2}{g_\alpha^2(\mu)} &= k_\alpha \frac{16\pi^2}{g_s^2} + \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \\ &\times \left(\sum_{i=1}^3 \frac{\Gamma_{2,2}(T_i, U_i)}{\bar{\eta}^{24}} \left[\bar{Q}_\alpha^2 - \frac{k_\alpha}{4\pi \text{Im } \tau} \right] \bar{\Omega} - b_\alpha \right) \\ &+ b_\alpha \log \frac{M_s^2}{\mu^2} + b_\alpha \log \frac{2e^{1-\gamma}}{\pi\sqrt{27}}. \end{aligned} \tag{14}$$

As far as the group-factor dependent terms are concerned, expression (14) agrees, including the constant contribution, with the one obtained in [1] also in the $\overline{\text{DR}}$ scheme. Hence, the relation between the running gauge couplings of the low-energy field theory and the string coupling does not depend on the infra-red regularization prescription. This result could have been anticipated as a consequence of the cancellation of the infra-red divergences between the fundamental and the effective theory since they have the same massless spectrum. However it could only be proved in the presence of a consistent infra-red regulator, similar in both theories. Moreover, it is important to emphasize that (14) contains rigorously all universal terms that were missing in previous approaches [1,2] and that we will now determine.

The symmetric $Z_2 \times Z_2$ orbifold model has a $E_8 \times E_6 \times U(1)^2$ gauge group⁷ and $\bar{\Omega} = \bar{\Omega}_8 \bar{\Omega}_6$ where

$$\begin{aligned}\Omega_8 &= \frac{1}{2} \sum_{a,b} \vartheta^8 \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right], \\ \Omega_6 &= \frac{1}{4} (\vartheta_2^4 + \vartheta_3^4) (\vartheta_3^4 + \vartheta_4^4) (\vartheta_2^4 - \vartheta_4^4).\end{aligned}\quad (15)$$

These are proportional to the Eisenstein modular-covariant functions G_2 and G_3 respectively [20]. They are both related to the modular invariant j (we use the standard normalizations, namely $j(\tau) = \frac{1}{q} + 744 + \mathcal{O}(q)$, $q = \exp(2\pi i\tau)$):

$$\left(\frac{\Omega_8}{\eta^8}\right)^3 = j, \quad \left(\frac{\Omega_6}{\eta^{12}}\right)^2 = \frac{1}{4}(j - j(i)), \quad (16)$$

with $j(i) = 12^3$. The operator \bar{Q}_α^2 acts as $\frac{i}{8\pi} \frac{\partial}{\partial \bar{\tau}}$ on $\bar{\Omega}_8$ for $\alpha = E_8$ while for $\alpha = E_6$ it acts similarly on the factors $\bar{\vartheta}_2^8$, $\bar{\vartheta}_3^8$ and $\bar{\vartheta}_4^8$ of $\bar{\Omega}_6$. One can use the above relations (16) as well as⁸

$$\frac{1}{\eta^4} \frac{\partial \log j}{\partial \log q} = \frac{(j - j(i))^{1/2}}{j^{1/3}} \quad (17)$$

to show that

$$\begin{aligned}\frac{1}{\eta^{24}} \left[\bar{Q}_\alpha^2 - \frac{1}{4\pi \text{Im} \tau} \right] \bar{\Omega} &= \frac{b_\alpha}{3} \\ &+ 2 \left[\bar{\Omega}_2 - \frac{1}{8\pi \text{Im} \tau} \right] \frac{\bar{\Omega}}{\eta^{24}} + \frac{\bar{j}}{24} - 42,\end{aligned}\quad (18)$$

where

$$\Omega_2 = \frac{\partial \log \eta}{\partial \log q} \quad (19)$$

is a non-modular-covariant function which plays a role in string gravitational anomalies. Notice, however, that the non-holomorphic combination $\Omega_2 - \frac{1}{8\pi \text{Im} \tau}$ is modular covariant of degree 2; once multiplied by $\frac{\bar{\Omega}}{\eta^{24}}$, the latter is proportional to the gravitational R^2 -term renormalization. If we introduce (18) into (14), we can perform the integral corresponding to the group-

factor dependent part by using the result (for $T_i = T$, $U_i = U$)

$$\begin{aligned}\int_{\mathcal{F}} \frac{d^2 \tau}{\text{Im} \tau} (\Gamma_{2,2}(T, U) - 1) \\ = -\log (|\eta(T)|^4 |\eta(U)|^4 \text{Im} T \text{Im} U) \\ - \log \frac{8\pi e^{1-\gamma}}{\sqrt{27}},\end{aligned}\quad (20)$$

first established in [2] and recently generalized in [4]. Finally, comparison with Eqs. (1) and (2) leads to the universal part of the thresholds for the $Z_2 \times Z_2$ orbifold model,

$$\begin{aligned}Y(T, U) &= \int_{\mathcal{F}} \frac{d^2 \tau}{\text{Im} \tau} \Gamma_{2,2}(T, U) \\ &\times \left(-6 \left[\bar{\Omega}_2 - \frac{1}{8\pi \text{Im} \tau} \right] \frac{\bar{\Omega}}{\eta^{24}} - \frac{\bar{j}}{8} + 126 \right),\end{aligned}\quad (21)$$

as well as to the constants c_α in the $\overline{\text{DR}}$ scheme:

$$c_\alpha = b_\alpha \log 4\pi^2. \quad (22)$$

A few remarks are in order here. We observe that, apart from the expected universal threshold induced by the back-reaction term $\frac{-1}{4\pi \text{Im} \tau}$, there are other universal contributions originated by the group-trace factor $\frac{1}{\eta^{24}} \bar{Q}_\alpha^2 \bar{\Omega}$. It is quite remarkable that the only group-factor dependence of the latter (see (18)) is a constant proportional to the beta-function coefficients while the other pieces are all universal. These play a very specific role and could almost have been guessed: combined with the back-reaction term they ensure modular invariance and finiteness of $Y(T, U)$ everywhere in the moduli space. Indeed, by using Eqs. (16) and the Fourier expansion of j it appears that

$$6 \frac{\bar{\Omega}_2 \bar{\Omega}}{\eta^{24}} = -\frac{1}{8} \frac{1}{\bar{q}} + 33 + \mathcal{O}(\bar{q})$$

and

$$\frac{\bar{j}}{8} - 126 = \frac{1}{8} \frac{1}{\bar{q}} - 33 + \mathcal{O}(\bar{q}).$$

The cancellation of constant and tachyonic terms avoids large- $\text{Im} \tau$ divergences in (21) even when the gauge group gets enlarged.

⁷ In this case $k_\alpha = 1$ and $\hat{b}_\alpha = b_\alpha$ with $b_{E_8} = -90$ and $b_{E_6} = 126$.

⁸ Eq. (17) can be proved very easily. Both sides are holomorphic modular-invariant functions that have the same analyticity properties. Therefore they must be proportional. The proportionality factor is found by comparing the corresponding power expansions.

Expression (21) can be further simplified if one uses a generalization of (20) (see [4]), valid for more general modular-invariant functions, to integrate the last terms⁹:

$$Y(T, U) = \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im}\tau} \Gamma_{2,2}(T, U) \times \left(-6 \left[\bar{\Omega}_2 - \frac{1}{8\pi \text{Im}\tau} \right] \frac{\bar{\Omega}}{\eta^{24}} + 33 \right) + \frac{1}{2} \log |j(T) - j(U)|. \quad (23)$$

Hence, while $Y(T, U)$ is finite and remains finite along the whole line of enhanced symmetry $T = U$, both terms of (23) are logarithmic divergent when $T \rightarrow U$ as a consequence of the extra massless states. This divergence is precisely the one that is responsible for the non-trivial monodromy properties of the prepotential in $N = 2$ models [4,21,22].

Finally, it is interesting to note that Eqs. (16) and (17) allow us to recast the function Ω into the form

$$\frac{\Omega}{\eta^{24}} = \frac{j - j(i)}{2} \left(\frac{\partial \log j}{\partial \log q} \right)^{-1}. \quad (24)$$

Although this expression strictly holds for the model under consideration, it seems that its validity could be extended (up to a factor) to more general string vacua with $N = 2$ supersymmetry [21]. Again, advocating modular invariance and infra-red finiteness, one could draw the conclusion that for these models the universal thresholds are proportional to those of the $Z_2 \times Z_2$ orbifold. At the present stage of investigation, however, this observation is by no means to be considered as a claim.

Let us now proceed to the numerical evaluation of $Y(T, U)$ for the model at hand. We will concentrate on the case $\text{Re}T = \text{Re}U = 0$ ¹⁰ and express $\text{Im}T, \text{Im}U$ in terms of the internal radii: $\text{Im}T = R_1 R_2, \text{Im}U = R_2/R_1$. Our starting point is Eq. (21) which reads now

$$Y(R_1, R_2) = \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im}\tau} \Gamma_{1,1}(R_1) \Gamma_{1,1}(R_2) \bar{\rho}, \quad (25)$$

where $\Gamma_{1,1}(R)$ is the soliton contribution of a compactified single boson,

$$\Gamma_{1,1}(R) = \sum_{m,n} e^{-\pi\tau_2 \left(\frac{m^2}{R^2} + n^2 R^2 \right)} e^{2\pi i \tau_1 m n} \quad (26)$$

(we set $\tau_1 \equiv \text{Re}\tau, \tau_2 \equiv \text{Im}\tau$), and

$$\bar{\rho} = -6 \left[\bar{\Omega}_2 - \frac{1}{8\pi \text{Im}\tau} \right] \frac{\bar{\Omega}}{\eta^{24}} - \frac{\bar{j}}{8} + 126. \quad (27)$$

One can readily derive the large-radius behaviour of the universal corrections. For $R_1 = R_2 = R \gg 1$ we can neglect the windings, setting $n = 0$ in Eq. (26), and split the integral (25) into two terms:

$$Y(R, R) = \int_{\mathcal{F}_1} \frac{d\tau_1 d\tau_2}{\tau_2} \vartheta_3^2 \left(i \frac{\tau_2}{R^2} \right) \bar{\rho} + \int_1^\infty \frac{d\tau_2}{\tau_2} \vartheta_3^2 \left(i \frac{\tau_2}{R^2} \right) \int_{-1/2}^{+1/2} d\tau_1 \bar{\rho} + \mathcal{O} \left(e^{-\pi R^2} \right), \quad (28)$$

where \mathcal{F}_1 corresponds to the part of the fundamental domain with $\tau_2 < 1$. We can now perform a Poisson resummation in the first term. Then, by using the results

$$\int_{\mathcal{F}_1} \frac{d\tau_1 d\tau_2}{\tau_2^2} \bar{\rho} = 6\pi - \frac{45}{\pi} \quad (29)$$

and

$$\int_{-1/2}^{+1/2} d\tau_1 \bar{\rho} = \frac{90}{\pi\tau_2},$$

$$\int_1^\infty \frac{dx}{x^2} \vartheta_3^2 \left(i \frac{x}{R^2} \right) = \frac{R^2}{2} + \frac{\kappa}{R^2} + \mathcal{O} \left(e^{-\pi R^2} \right) \quad (30)$$

in the first and second terms of (28) respectively, we finally obtain

$$Y(R, R) = 6\pi R^2 + \frac{90\kappa}{\pi R^2} + \mathcal{O} \left(e^{-\pi R^2} \right), \quad (31)$$

⁹ Using the method of orbits of the modular group, the remaining integral in (23) can also be reduced to a multiple series expansion [4] but such a manipulation is not useful for our purpose.

¹⁰ We have verified numerically that as far as the universal thresholds are concerned, switching on the $\text{Re}T$ and $\text{Re}U$ fields leads to small dumping oscillations (for increasing radii) with maximum width $< 3\%$ around the $\text{Re}T = \text{Re}U = 0$ results.

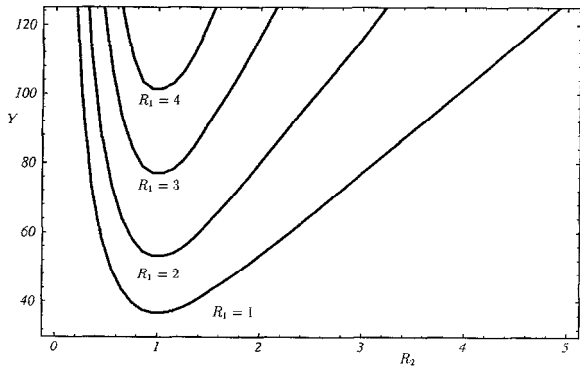


Fig. 1. Plots of the universal thresholds $Y(R_1, R_2)$ as a function of R_2 for $R_1 = 1, 2, 3, 4$.

where

$$\kappa = \frac{2}{\pi^2} \zeta(4) + \sum_{j>0} \left(\frac{\coth \pi j}{\pi} \frac{1}{j^3} + \frac{1}{\sinh^2 \pi j} \frac{1}{j^2} \right)$$

is a constant that takes the numerical value $\kappa \approx 0.6106$. Thus the asymptotic form reads

$$Y(R, R) \sim 18.85 \times R^2 + 17.49 \times \frac{1}{R^2} \quad \text{for } R \gg 1. \quad (32)$$

It is possible to compute explicitly the universal thresholds for arbitrary values of the radii R_1, R_2 by numerically evaluating (25). Problems related to the loss of numerical accuracy caused by the pole part $e^{2\pi\tau_2}$ in $\bar{\rho}(\tau_1, \tau_2)$ can be cured by power expanding the integrand to the required accuracy, before performing the numerical integration [13]. Plots of the numerical results for $Y(R_1, R_2)$ as a function of R_2 for $R_1 = 1, 2, 3, 4$ are given in Fig. 1 and a contour plot of $Y(R_1, R_2)$ is given in Fig. 2. One clearly sees that the minimum value of Y is obtained for the radii at the self-dual point and it is $Y^{\min} = Y(1, 1) \approx 36.4$ while for the fermionic point we have $Y^{\text{fermionic}} = Y(1, \frac{1}{2}) \approx 53.0$. We also notice that the asymptotic formula (32) reproduces the numerical results for $R_1 = R_2 \gtrsim \frac{3}{2}$, which is consistent with the fact that (32) is almost invariant under the duality transformation $R \rightarrow 1/R$.

Let us now analyze the effect of the universal thresholds on the unification scale of the low-energy effective field theory. In the model under consideration, the $E_8 \times E_6$ gauge group couplings will run according to (1) with c_α and $Y(R_1, R_2)$ given by (22) and (25)

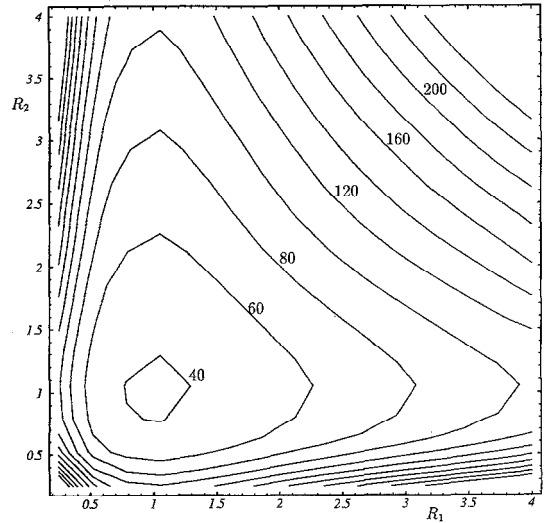


Fig. 2. Contour plots of the universal thresholds $Y(R_1, R_2)$ as a function of the internal radii R_1 and R_2 .

respectively. There is some arbitrariness in the definition of the unification scale M_U but in the specific model there is a manifest way to define it [9], in the $\overline{\text{DR}}$ scheme:

$$M_U(R_1, R_2) = \frac{M_p g_U}{2\pi \left| \eta \left(i \frac{R_2}{R_1} \right) \right|^2 \left| \eta \left(i R_1 R_2 \right) \right|^2 R_2} \times \frac{1}{\sqrt{1 + g_U^2 \frac{Y(R_1, R_2)}{16\pi^2}}}; \quad (33)$$

here $g_U \equiv g_\alpha(M_U) = g_s / \sqrt{1 - g_s^2 \frac{Y}{16\pi^2}}$ for any group factor, and we have used explicitly (3) in order to express the unification scale in terms of the effective field theory parameters M_p and g_U . The last factor in (33) is due to the existence of the universal terms which lead to a shift of the dilaton field in order to reabsorb the universal contributions into the string coupling. It is interesting to observe that since $Y(R_1, R_2) > 0$ this extra factor always gives a lower unification scale with respect to the case where these terms are neglected. One can consider the minimum value of the unification scale M_U^{\min} with respect to the radii R_1, R_2 . Since $M_U(R_1, R_2)$ possesses target-space duality properties, it has an extremum at $R_1 = R_2 = 1$ independently of the value of g_U . On the other hand the first factor in (33) monotonically increases for radii moving away from the self-dual point, while the second one mono-

tonically decreases. In the asymptotic limit $R_1 = R_2 = R \gg 1$, the universal thresholds become large and the g_U dependence of M_U cancels between the two factors of (33). Moreover, the first factor dominates and using (32) we find that M_U grows exponentially:

$$M_U \sim 0.78 \times M_P \times \frac{e^{\frac{\pi}{6} R^2}}{R^2}$$

. Numerical evaluation of (33) shows that for perturbative values of g_U the $R_1 = R_2 = 1$ extremum is a minimum and that there is no other minimum apart from that. Thus for the specific model, in the $\overline{\text{DR}}$ scheme,

$$M_U^{\text{min}} = \frac{M_P g_U}{2\pi |\eta(i)|^4} \times \frac{1}{\sqrt{1 + g_U^2 \frac{Y(1,1)}{16\pi^2}}}. \quad (34)$$

Using explicit values for the various quantities that appear in this formula we obtain

$$M_U^{\text{min}} \approx 7.87 \times 10^{17} \times g_U \times \frac{1}{\sqrt{1 + 0.23 \times g_U^2}} \text{ GeV}, \quad (35)$$

where the last factor in the product represents the effect of the universal thresholds.

We would like to conclude this note with a few comments. We have determined the complete one-loop threshold corrections (see (14)) for general symmetric-orbifold models, in the $\overline{\text{DR}}$ scheme, by using a method introduced in [6] that allows us to handle the infra-red problems. The group-factor dependent parts of these thresholds were obtained previously following a different procedure [1,2]. The two results for the group-factor dependent contributions (constant and moduli-dependent) are in agreement, when evaluated within the same ultra-violet renormalization scheme. Put differently, this shows that the relation between the running gauge couplings of the low-energy field theory in the $\overline{\text{DR}}$ scheme and the string coupling does not depend on the infra-red regularization prescription. This amounts to the decoupling of the (infinite tower of) massive states and allows for an unambiguous definition of string effective theory. Such a conclusion could not have been drawn without using a consistent infra-red regulator. Although our result has been established in the framework of an infra-red regulator induced by a particular $N = 4$ four-dimensional curved background, we would have reached the same conclusions within

any other background possessing similar properties, such as those listed in [23].

Going beyond what has been achieved in previous studies, we have determined and calculated the moduli-dependent universal part $Y(T, U)$ of the thresholds for the symmetric $Z_2 \times Z_2$ orbifold model, Eq. (21). We have found that $Y(R_1, R_2)$ is strictly positive with a minimum $Y^{\text{min}} \approx 36.4$ at the self-dual point $R_1 = R_2 = 1$, and is monotonically growing away from it. For large radii the asymptotic form of the universal term is $Y(R, R) \sim 18.85 \times R^2 + 17.49 \times R^{-2}$. It would be interesting to compute the universal thresholds in models such as the Z_3 orbifold, where the $N = 1$ sectors do contribute, and check whether this contribution is indeed as small as one generally believes by looking at the group-factor dependent terms [12].

Finally, we have studied the effect of the universal thresholds on the unification scale of the underlying field theory. As we stressed in the introduction, the universal thresholds cannot be reabsorbed into a redefinition of the coupling constant without affecting the relation between the Planck scale and the unification scale, and we have actually found that the existence of these terms leads to a decrease of that scale. The minimum of the unification scale is obtained for radii at the self-dual point but the specific value depends on the value of the effective field theory gauge coupling g_U at this scale. For example for $g_U^2 = \frac{1}{2}$ we have a 5% decrease while for $g_U^2 = 1$ we can reach 10%, with respect to the case where these corrections are not taken into account. Of course one could argue that this unification scale might be lowered further in some model other than the $Z_2 \times Z_2$ orbifold. However, one should be aware that the above scale concerns the $E_8 \times E_6$ symmetry breaking and that one has somehow to introduce a lower scale where E_6 breaks down to some subgroup, eventually leading to the standard model. In order to describe such a realistic situation in the framework of strings, it seems difficult to avoid the introduction of y -fields and Wilson lines [11,12]. Those will enhance the moduli space and allow for a better exploration of the various symmetry-breaking possibilities.

We would like to thank E. Kiritsis and C. Kounnas for stimulating discussions. One of us (J.R.) would like to thank the CERN Theory Division for hospitality and acknowledges financial support from the EEC

contract ref. ERBCHBGCT940634.

References

- [1] V. Kaplunovsky, *Nucl. Phys. B* 307 (1988) 145; *B* 382 (1992) 436 (E).
- [2] L. Dixon, V. Kaplunovsky and J. Louis, *Nucl. Phys. B* 355 (1991) 649.
- [3] I. Antoniadis, K.S. Narain and T.R. Taylor, *Phys. Lett. B* 267 (1991) 37;
J.-P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, *Nucl. Phys. B* 372 (1992) 145; *Phys. Lett. B* 271 (1991) 307;
S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, *Nucl. Phys. B* 365 (1991) 431;
P. Mayr and S. Stieberger, *Nucl. Phys. B* 407 (1993) 725;
V. Kaplunovsky and J. Louis, *Nucl. Phys. B* 444 (1995) 191;
D. Bailin, A. Love, W.A. Sabra and S. Thomas, *Mod. Phys. Lett. A* 9 (1994) 67; *A* 10 (1995) 337.
- [4] J.A. Harvey and G. Moore, hep-th/9510182.
- [5] I. Antoniadis, E. Gava and K.S. Narain, *Phys. Lett. B* 283 (1992) 209; *Nucl. Phys. B* 383 (1992) 93.
- [6] E. Kiritsis and C. Kounnas in the proceedings of STRINGS 95, Future Perspectives in String Theory, Los Angeles, CA, 13–18 March 1995; *Nucl. Phys. B* 442 (1995) 472.
- [7] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Nucl. Phys. B* 407 (1993) 706; *B* 413 (1994) 162.
- [8] J. Ellis, S. Kelley and D.V. Nanopoulos, *Phys. Lett. B* 249 (1990) 441;
U. Amaldi, W. de Boer and H. Fürstenaу, *Phys. Lett. B* 260 (131) 1991;
P. Langacker and M.X. Luo, *Phys. Rev. D* 44 (1991) 817.
- [9] L.E. Ibáñez, D. Lüst and G. Ross, *Phys. Lett. B* 272 (1991) 251.
- [10] I. Antoniadis, J. Ellis, R. Lacaze and D.V. Nanopoulos, *Phys. Lett. B* 268 (1991) 188;
L. Dolan and J.T. Liu, *Nucl. Phys. B* 387 (1992) 86;
M. Chemtob, hep-th/9506178.
- [11] C. Kounnas in the proceedings of the International Workshop in Supersymmetry and Unification of Fundamental Interactions Palaiseau, France, 15–19 May, 1995.
- [12] P. Mayr and S. Stieberger, *Phys. Lett. B* 355 (1995) 107;
H.P. Nilles and S. Stieberger, hep-th/9510009.
- [13] P. Mayr, H.P. Nilles and S. Stieberger, *Phys. Lett. B* 317 (1993) 53.
- [14] K.R. Dienes and A.E. Faraggi, *Phys. Rev. Lett.* 75 (1995) 2646; hep-th/9505046.
- [15] I. Antoniadis, J. Ellis, S. Kelley and D.V. Nanopoulos, *Phys. Lett. B* 281 (1992) 374;
A.E. Faraggi, *Phys. Lett. B* 302 (1993) 202;
M.K. Gaillard and R. Xiu, *Phys. Lett. B* 296 (1992) 71;
C. Bachas, C. Fabre and T. Yanagida, hep-th/9510094.
- [16] L.E. Ibáñez, *Phys. Lett. B* 318 (1993) 73;
K.R. Dienes, A.E. Faraggi and J. March-Russell, hep-th/9510223.
- [17] E. Kiritsis and C. Kounnas, to appear.
- [18] P.M. Petropoulos in the proceedings of the 5th Hellenic School and Workshops on Elementary Particle Physics Corfu, Greece, 3–24 September, 1995.
- [19] See, [4], and references therein.
- [20] S. Lang, *Elliptic Functions* Addison-Wesley, Reading, Mass. (1973).
- [21] I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, *Nucl. Phys. B* 447 (1995) 35.
- [22] G. Lopes Cardoso, D. Lüst and T. Mohaupt, *Nucl. Phys. B* 450 (1995) 115;
B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Nucl. Phys. B* 451 (1995) 53.
- [23] E. Kiritsis, C. Kounnas and D. Lüst, *Int. J. Mod. Phys. A* 9 (1994) 1361;
C. Kounnas, *Phys. Lett. B* 321 (1994) 26.