Pseudo-3-branes in a curved 6D bulk

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Abstract

We consider a model involving a 4-brane in a 6D bulk which carries sigma model fields. An axion field on the 4-brane cancels the pressure along one direction leading to an effective codimension-2 3-brane. For a range of parameters of the theory, we get a transverse space which is non-compact, providing a possible solution to the cosmological constant problem. A setup with two branes in a compact space is also treated. In this case, a mild fine-tuning between the radii of the two 4-branes is necessary. Finally, we explore the 4-brane problem in the Gauss–Codazzi formulation and we discuss general aspects of gravity in the presence of additional brane sources.
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1. Introduction

The issue of the cosmological constant appears to be one of the most pressing problems in modern theoretical physics, permeating many areas, ranging from field theory to cosmology. Conventional estimates of its value based on field theoretical approaches are in gross contradiction with observations (around 120 orders of magnitude higher than observed for a Planck-scale cutoff). The apparent failure has led to a number of new and exciting ideas in both basic theory and model building. The emergence of brane models and theories of large extra dimensions [1–3] in recent years has been a major driving force in gravitational physics and a source of rich and interesting phenomenological studies. One of the most interesting feature of these theories is that they provide a possible explanation for the smallness of the cosmological constant [4]. Similarly, the simple observation that a codimension 2 brane is always Ricci-flat, opened another possibility for the cancellation of the cosmological constant. As it was shown in [5], a four-dimensional brane embedded in a six-dimensional bulk space, where the transverse space is compactified using fluxes, induces a conical singularity in the bulk, deforming the space into a rugby-ball configuration.

In this way, the brane cosmological constant (tension) is completely offloaded into the bulk, where it determines the deficit angle of the induced conical singularity. Although this setup only partially solves the problem, as fine-tunings are in the end needed to ensure a flat brane [6–8], it nevertheless provides evidence that extra dimensions may play a role in resolving the cosmological constant conundrum. Extensions and alternatives were consequently investigated by several authors (see [8–18] and references therein).

In this Letter we study a model of an effective codimension-2 3-brane embedded in a 6-dimensional bulk space. It is a pseudo-3-brane in the sense that although the submanifold within which ordinary matter and fields are confined has in fact 5 dimension, i.e., is a 4-brane, the energy–momentum tensor of these fields is selected in such a way to resemble the tensor structure of a 3-brane. The presence of the one additional dimension of the brane is in this way hidden from the exterior space and we can take advantage of the well-known result that branes of codimension-2 only produce conical singularities in their bulk spaces [19–23]. In order to produce the required energy–momentum tensor we will employ the technique discussed in [24,25]. However, we will not assume an empty bulk space, but one endowed with an $O(3)$ sigma model (see [12,14,28]), unlike the usual flux compactification using $U(1)$ fields [32,33]. The transverse space away from the brane will thus be curved,

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but its curvature will only be due to the sigma model fields, since the brane cannot contribute more than a conical singularity at its position. This is a setup similar to the one discussed in [12], the main difference being the existence of a resolved brane, rather than a purely codimension-2 singularity.

The purpose of such a setup is to address the cosmological constant problem. We will see that, by having an appropriate energy content, we can construct an effectively flat 4-geometry on the brane, with all curvature being pushed away into the transverse space. To avoid imposing unsettling fine-tunings, we will allow for a space which is non-compact. We thus have one dimension tangent to the 4-brane which is not compact but can be integrated to a finite volume and a parallel dimension which is compactified in the spirit of Kaluza–Klein theories. As it turns out, for a range of parameters of the model, the 2-dimensional transverse space is normalizable and we thus expect a conventional KK phenomenology, with a zero mode mediating ordinary four-dimensional gravity on the brane and a spectrum of KK excitations as corrections. The cosmological constant of the 3-brane can be set to zero by proper choice of parameters, which induces a corresponding deficit angle in the bulk. Once this choice is made, any variation in one of the model's parameters, e.g., the brane cosmological constant, results to a new value for the deficit angle, without any additional fine-tuning. In the last section of our Letter, we derive Einstein's equations on the 4-brane by employing the Gauss–Codazzi formalism. Based on these equations we discuss general aspects of such models and also the effects of additional brane matter, which will be the source for conventional 4D gravity once the KK reduction of the compact dimension is performed.

2. General setup

We will start by describing the action of the setup. It includes a non-linear sigma model targeted on a Kähler manifold inside a six-dimensional bulk space. The bulk cosmological constant is assumed to be zero. A 4-brane is also included, carrying a tension σ and an axion field Σ. The axion is used to counteract the brane tension along the azimuthal direction φ and to ensure that the energy–momentum tensor of the 4-brane mimics a 3-brane outside the resolved brane core, so that we get an effective codimension-2 brane [25]. The metric of the bulk space is considered to be

$$ds_6^2 = n_{\mu\nu}\, dx^\mu\, dx^\nu + d\rho^2 + g_{\varphi\varphi}(\rho)\, d\varphi^2 \tag{1}$$

in Gauss-normal coordinates. Note that only ρ is the direction transverse to the 4-brane, while φ is our compact dimension. The brane is situated at a distance ρ_0 away from the origin of the transverse space in the radial direction. The action of the model is thus of the form

$$S = \int d^6x \sqrt{-g}\left(M^4 R - \frac{1}{2\lambda^2}h_{\alpha\beta}(\phi)\nabla^M\phi^\alpha \nabla_M\phi^\beta\right)$$
$$- \int d^5x\, d\rho\, \delta(\rho - \rho_0)\sqrt{-\gamma}\left(\sigma + \frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\Sigma\partial_\beta\Sigma\right). \tag{2}$$

Uppercase Greek indices run from 0 to 5, while lowercase are indices on the 4-brane that do not include the ρ coordinate and α, β are indices in the Kähler manifold. The first term represents the bulk contributions from the gravitational sector and the sigma model fields ϕ^α, while the second contains the brane contributions for the tension and the axion field. For the moment we assume a zero energy content from ordinary matter on the brane. The energy–momentum tensor of the brane can be written as

$$T_{MN}^{(br)} = \delta(\rho - \rho_0)\frac{\sqrt{-\gamma}}{\sqrt{-g}}\delta_M^\mu\delta_N^\nu$$
$$\times \left(-\sigma\gamma_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\gamma^{\alpha\beta}\partial_\alpha\Sigma\partial_\beta\Sigma + \partial_\mu\Sigma\partial_\nu\Sigma\right). \tag{3}$$

We also have an energy–momentum tensor for the ϕ^α fields

$$T_{MN}^{(\phi)} = \frac{h_{\alpha\beta}}{\lambda^2}\left(\nabla_M\phi^a\nabla_N\phi^\beta - \frac{1}{2}g_{MN}\nabla^\Lambda\phi^a\nabla_\Lambda\phi^\beta\right). \tag{4}$$

Einstein's equations become in this case

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{2M^4}\left(T_{MN}^{(\phi)} + T_{MN}^{(br)}\right) \tag{5}$$

or

$$R_{MN} = \frac{1}{2M^4}\left(T_{MN}^{(\phi)} - \frac{1}{N-2}T^{(\phi)}g_{MN}\right)$$
$$+ \frac{1}{2M^4}\left(T_{MN}^{(br)} - \frac{1}{N-2}T^{(br)}g_{MN}\right), \tag{6}$$

where we denote by N the total bulk dimension and we use n for the dimensions of the brane (in the case we will examine, $N = 6$ and $n = 5$). The energy–momentum tensor contains contributions from both the complex scalar and the brane content. The axion field equations are solved by

$$\Sigma = q\varphi$$

so that Σ has $2\pi q$ jumps as we go around the φ direction. If in addition, the parameter q (the axion charge) is such that

$$q^2 = 2\sigma g_{\varphi\varphi}, \tag{7}$$

the axion contribution completely eliminates any tension along the azimuthal direction φ. Notice that the above condition is not a fine-tuning between q and σ in the usual sense, since the metric component carries integration factors which are not determined yet by other constraints. Thus, the value of q is not fixed against the brane tension. With this choice we get,

$$T_{MN}^{(br)} - \frac{1}{N-2}g_{MN}T^{(br)}$$
$$= \delta(\rho - \rho_0)\frac{\sqrt{-\gamma}}{\sqrt{-g}}\left[\sigma\left(-\gamma_{\mu\nu}\delta_M^\mu\delta_N^\nu + \frac{n}{N-2}g_{MN}\right)\right.$$
$$+ \frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\Sigma\partial_\beta\Sigma\left(-\gamma_{\mu\nu}\delta_M^\mu\delta_N^\nu + \frac{n-2}{N-2}g_{MN}\right)$$
$$\left. + \delta_M^\mu\delta_N^\nu\partial_\mu\Sigma\partial_\nu\Sigma\right] \tag{8}$$

and similarly for the scalar fields

$$T_{MN}^{(\phi)} - \frac{1}{N-2}g_{MN}T^{(\phi)} = \frac{h_{\alpha\beta}}{\lambda^2}\nabla_M\phi^\alpha\nabla_N\phi^\beta. \tag{9}$$

Since we are using Gauss-normal coordinates, the determinants of the bulk and the induced metric are the same and their ratio cancels. We also have the equation of motion for the scalar fields, which reads

$$\nabla^M \nabla_M \phi^\alpha + \Gamma^\alpha_{\beta\gamma} \nabla^\Gamma \phi^\beta \nabla_\Gamma \phi^\gamma = 0. \tag{10}$$

3. Single and double brane solution

In order to proceed in solving the equations in the bulk, we must choose an ansatz for the bulk fields and the transverse metric. We will assume an $O(3)$ sigma model and a metric for the Kähler manifold, which is now an S^2 sphere, of the form

$$h_{\alpha\beta} = \frac{4}{(1+\phi^2)^2} \delta_{\alpha\beta}, \tag{11}$$

where $\phi^2 = (\phi^1)^2 + (\phi^2)^2$. This particular choice of the sigma model is entirely it ad hoc and has been taken as a simple example. Another choice of the sigma model target space of the form appearing in supergravity like $SL(2)/U(1)$ would be equally good. In order to investigate the resulting geometry of the transverse 2D space, we will reparametrize it in a conformally flat fashion as

$$ds_2^2 = d\rho^2 + g_{\phi\phi}\, d\varphi^2 = \psi(r)\big(dr^2 + r^2\, d\varphi^2\big)$$
$$= \psi(r)\delta_{mn}\, dy^m\, dy^n. \tag{12}$$

This change of coordinates implies that $d\rho^2 = \psi(r)\, dr^2$, which leads to

$$\Rightarrow \delta(\rho - \rho_0) = \frac{1}{\sqrt{\psi(r)}} \delta(r - r_0)$$

for the transformation of delta function, where r_0 is the position corresponding to ρ_0 in the new conformal coordinates. We will now adopt an ansatz for the sigma model fields of the form $\phi^\alpha = y^\alpha$, such that $\phi^2 = r^2$. This ansatz solves the equations for the scalar fields without any further constraints and from the expressions for the scalar energy–momentum tensor we see that only the $T^{(\phi)}_{\mu\nu}$ components survive. Einstein's equations reduce to

$$R_{\mu\nu} = 0, \tag{13}$$

$$R_{mn} = \frac{2}{M^4\lambda^2(1+r^2)^2} \delta_{\alpha\beta} \nabla_m \phi^\alpha \nabla_n \phi^\beta$$
$$+ \frac{2\sigma}{M^4} \frac{g^{(2)}_{mn}}{\sqrt{\psi(r)}} \delta(r - r_0), \tag{14}$$

where $\mu, \nu = 0, 1, 2, 3$ are coordinates on the effective 3-brane and m, n are coordinates of the 2-dimensional transverse space. Note again that only r is truly transverse to the brane. The first set of equations ensures a flat 4-dimensional space. The transverse 2-space will be curved. Taking the trace of the last equation yields

$$R^{(2)} = \frac{4}{M^4\lambda^2} \frac{1}{\psi(r)(1+r^2)^2} + \frac{4\sigma}{M^4} \frac{1}{\sqrt{\psi(r)}} \delta(r - r_0), \tag{15}$$

where $R^{(2)} = -\frac{1}{\psi}\nabla^2 \ln\psi$ is the Ricci scalar of the 2D transverse space. Since the conformal factor $\psi(r)$ depends only on

the radial coordinate, when it multiplies the delta function, it becomes just a constant $\psi(r_0)$. The solution in the absence of the brane is known to be

$$\psi(r) = C_2 \frac{r^{\frac{2}{M^4\lambda^2}+C_1}}{(1+r^2)^{\frac{1}{M^4\lambda^2}}}. \tag{16}$$

Taking into consideration the delta function term, we obtain the solution

$$\psi(r) = C_2 \frac{r^{\frac{2}{M^4\lambda^2}+C_1}}{(1+r^2)^{\frac{1}{M^4\lambda^2}}} e^{-4\frac{\sigma r_0}{M^4}\sqrt{\psi(r_0)}\Theta(r-r_0)\ln(\frac{r}{r_0})}, \tag{17}$$

which obviously reduces to the smooth case when we take the limit $r_0 \to 0$. We note that the value $\psi(r_0)$, which enters in the exponent is just a constant factor. This factor enters in (7) and, thus, the integration constant C_2 prevents any fine-tuning. We also see that the transverse space retains a non-vanishing curvature inside the brane core. There are also, in general, conical singularities at $r = 0$ and $r \to \infty$. However, we can eliminate the singularity at the origin, by imposing that $C_1 = -\frac{2}{M^4\lambda^2}$. In this case the 2D metric is regular at $r = 0$ and we only have a singularity at infinity, which signifies a non-compact geometry. We can also check the existence of a deficit angle at infinity and demonstrate that no singularity occurs as we cross the boundary of the resolved brane. To simplify our discussion, we define $b = 4\sigma r_0 M^{-4}\sqrt{\psi(r_0)}$ and $c = M^{-4}\lambda^{-2}$. Expanding the conformal factor around $r = r_0$, we obtain

$$\psi(r) \sim \frac{C_2}{(1+r_0^2)^c}, \tag{18}$$

so that the two radial coordinates ρ and r are proportional near the brane and no deficit angle is involved. However, for $r \to \infty$ we get

$$\psi(r) \sim r^{-2c}\left(\frac{r_0}{r}\right)^b \tag{19}$$

and the coordinate transformation yields a transverse space metric of the form

$$ds_2^2 = d\rho^2 + k^2\rho^2\, d\varphi^2, \tag{20}$$

where

$$k = 1 - c - \frac{b}{2}. \tag{21}$$

The associated deficit angle is

$$\delta = 2\pi(1 - k) = 2\pi\left(c + \frac{b}{2}\right). \tag{22}$$

The combination $c + \frac{b}{2}$, which enters the expression for the deficit angle, is indeed the Euler number of the transverse space, as we can easily verify

$$\chi = \frac{1}{4\pi} \int dr\, r \int d\varphi\, \psi(r) R^{(2)}$$
$$= \frac{1}{2} \int dr\, r \left(\frac{4}{M^4\lambda^2} \frac{1}{(1+r^2)^2} + \frac{4\sigma}{M^4}\sqrt{\psi(r_0)}\delta(r - r_0)\right)$$

$$= \frac{1}{M^4 \lambda^2} + \frac{2\sigma r_0 \sqrt{\psi(r_0)}}{M^4} = c + \frac{b}{2} = \frac{\delta}{2\pi}. \tag{23}$$

In order for the space to have a finite volume, we must have

$$\frac{1}{M^4 \lambda^2} + \frac{2\sigma r_0}{M^4} \sqrt{\psi(r_0)} > 1, \tag{24}$$

which, given (23), is equivalent to $\chi > 1$. The corresponding finite volume of the 2D space turns out to be

$$V_2 = \pi C_2 \frac{(1 + r_0^2)^{1-c}}{c - 1}$$

$$- i^{-b-2c} \pi C_2 r_0^b B\left(-\frac{1}{r_0^2}, c + \frac{b}{2} - 1, 1 - c\right). \tag{25}$$

B is the incomplete Beta function. As a result of the positivity of parameters and the constraint (24), the volume is real and positive. Given the finite volume of the transverse space [26–31], this setup will exhibit a four-dimensional gravitational behavior at low energies compared to the compactification scale, mediated by a graviton zero mode on the brane and followed by a KK tower at higher energy scales. Apparently, the space remains non-compact for the entire range of parameter values for which the relation $1 < \frac{1}{M^4 \lambda^2} + \frac{2\sigma r_0}{M^4} \sqrt{\psi(r_0)} < 2$ is satisfied. Eventually, for appropriate values, the Euler number of the transverse space reaches the value $\chi = 2$ and the space compactifies into a sphere.

We could also place two branes in our setup, that would ensure a compact transverse geometry from the beginning. To include the second brane of tension σ', situated at some position $\rho_1 > \rho_0$ away from the origin, we add to the action the term

$$\int d^5 x \, d\rho \, \delta(\rho - \rho_1) \sqrt{-\gamma} \left(\sigma' + \frac{1}{2} \gamma^{\alpha\beta} \partial_\alpha \bar{\Sigma} \partial_\beta \bar{\Sigma}\right), \tag{26}$$

where the induced metric γ is to be evaluated now at $\rho = \rho_1$. With this addition, both branes remain flat as long as we impose the condition

$$\sigma' = \frac{\bar{q}^2}{2} g^{\phi\phi}, \tag{27}$$

relating the brane tension σ' and the charge of the axion $\bar{q}$ on the second brane. The metric component $g^{\phi\phi}$ is evaluated at $\rho = \rho_1$. Eq. (15) now becomes

$$R^{(2)} = \frac{4}{M^4 \lambda^2} \frac{1}{\psi(r)(1 + r^2)^2} + \frac{4\sigma}{M^4} \frac{1}{\sqrt{\psi(r)}} \delta(r - r_0)$$

$$+ \frac{4\sigma'}{M^4} \frac{1}{\sqrt{\psi(r)}} \delta(r - r_1) \tag{28}$$

and the corresponding solution for the conformal factor reads

$$\psi(r) = C_2 \frac{r^{2c+C_1}}{(1 + r^2)^c} e^{-b\Theta(r-r_0) \ln \frac{r}{r_0} - b_1 \Theta(r-r_1) \ln \frac{r}{r_1}}, \tag{29}$$

where, in addition, we have defined $b_1 = 4M^{-4} \sigma' r_1 \sqrt{\psi(r_1)}$. Again, the metric is regular at the origin if we set $C_1 = -2c$. No deficit angle is encountered around $r = 0$ or as we cross each of the two branes. The total deficit angle of the space is deduced by checking the metric at infinity,

$$\psi(r) \sim r_0^b r_1^{b_1} r^{2-2c-b-b_1} \tag{30}$$

from which the corresponding deficit angle

$$\delta = 2\pi(1 - k) = 2\pi \left(c + \frac{b + b_1}{2}\right) \tag{31}$$

may immediately be obtained. To ensure that the space has the topology of a sphere, we must impose the condition

$$\chi = c + \frac{b + b_1}{2} = 2, \tag{32}$$

which relates the tensions and positions of the two branes.

Let us take a moment here to discuss the way in which the cosmological constant problem is addressed in the context of our model. As we already pointed out, the effective 3-brane appears Ricci-flat, with its tension inducing a deficit angle in the non-compact transverse space. The relations connecting the various physical constants in the case of the single brane setup are (7) and (22). The pitfall of unwanted fine-tunings may originate from these equations. To see if this is the case, we imagine a situation where a flat brane solution has been found for a specific brane tension σ. We then change the value of the tension and check whether it induces a shift in the deficit angle, which is unobservable, while leaving other physical constants of the model unchanged. We see that such a change may affect through (7) the value of the axion field charge q. However, as we previously stressed, the metric component $g^{\varphi\varphi}$ carries the undetermined integration constant C_2, which in turn enters the constant $\psi(r_0)$ and consequently b. Thus, changing the brane tension leaves the axion charge unaltered and only affects the deficit angle through (22), which involves b. In this way, fine-tuning of physical constants in this setup is avoided. The new brane tension could also lead to an additional change in the radius of the 4-brane.

As it was also discussed in [12], having a non-compact transverse space is crucial. In the case of the resolved double brane setup, however, there is still room for a solution which does not require more than a mild fine-tuning, despite the fact that the transverse space has necessarily the topology of a sphere. This is due to relation (32), which fixes the value of the previously undetermined constant C_2. Taking the ratio of (7) and (27), we see that altering the value of either σ or σ' will induce a change in the ratio of the corresponding axion fields. This can be avoided if the change in brane tension is compensated by a change in the ratio of the metric components $g^{\varphi\varphi}$ at r_0 and r_1. Since both carry the same overall constant C_2, which cancels, the only way to satisfy this requirement is for the brane radii to shift. Thus, a fine-tuning between the two brane radii must be imposed to have flat branes for arbitrary varying tensions. The four-dimensional cosmological constant is again affecting the value of the deficit angle through (31).

4. Gauss–Codazzi formulation

An alternative way of investigating the problem explored above is by using the Gauss–Codazzi formalism discussed in [34]. The equations on a codimension-1 brane, when the bulk

space is n-dimensional turn out to be[1]

$$
\begin{aligned}
{}^{(n-1)}G_{\mu\nu} &= \frac{n-3}{n-2}\kappa_n^2\left(T_{\rho\sigma}q_\mu^\rho q_\nu^\sigma + T_{\rho\sigma}n^\rho n^\sigma q_{\mu\nu} - \frac{1}{n-1}Tq_{\mu\nu}\right) \\
&\quad + KK_{\mu\nu} - K_\mu^\sigma K_{\nu\sigma} - \frac{1}{2}q_{\mu\nu}\left(K^2 - K^{\alpha\beta}K_{\alpha\beta}\right) - E_{\mu\nu},
\end{aligned}
\tag{33}
$$

where $E_{\mu\nu}$ is the projection of the Weyl tensor of the bulk,

$$
E_{\mu\nu} = {}^{(n)}C^\alpha_{\beta\rho\sigma}n_\alpha n^\rho q_\mu^\beta q_\mu^\sigma
\tag{34}
$$

and $K_{\mu\nu}$ is the extrinsic curvature tensor of the brane. The constant κ_n is related to the n-dimensional Planck mass of the theory. We will assume Gauss-normal coordinates, with the coordinate normal to the brane denoted by ρ and an energy–momentum tensor of the form

$$
T_{\mu\nu} = -\Lambda g_{\mu\nu} + T_{\mu\nu}^{(B)} + S_{\mu\nu}\delta(\rho),
\tag{35}
$$

where $T_{MN}^{(B)}$ is the bulk energy–momentum tensor, not including the bulk cosmological constant. $S_{\mu\nu}$ is the energy–momentum content of the brane and it is further decomposed in our case according to

$$
S_{\mu\nu} = -\sigma q_{\mu\nu} + \tilde{\tau}_{\mu\nu} + \bar{\tau}_{\mu\nu}.
\tag{36}
$$

The first term represents brane tension, the second is the axion energy–momentum tensor and the third accounts for additional matter content on the brane. This way we can also treat cases with an arbitrary brane content in addition to the axion field which is only used to eliminate the azimuthal brane pressure and is effectively hidden. We proceed by further assuming a Z_2 symmetry in the ρ direction. From Israel's junction conditions, this constraint fixes uniquely the extrinsic curvature of the brane in terms of its matter content, with the resulting expression being

$$
K_{\mu\nu} = -\frac{\kappa_n^2}{2}\left(S_{\mu\nu} - \frac{1}{n-2}q_{\mu\nu}S\right).
\tag{37}
$$

Substituting this expression into (33) and using the above mentioned energy–momentum tensors yields the effective Einstein equation's on the brane

$$
\begin{aligned}
{}^{(n-1)}G_{\mu\nu} &= -\Lambda_{(n-1)}q_{\mu\nu} + 8\pi\bar{G}_{(n-1)}\bar{T}_{\mu\nu} \\
&\quad + 8\pi G_{(n-1)}(\tilde{\tau}_{\mu\nu} + \bar{\tau}_{\mu\nu}) + \kappa_n^4(\tilde{\pi}_{\mu\nu} + \bar{\pi}_{\mu\nu} + \kappa_{\mu\nu}) \\
&\quad - E_{\mu\nu}.
\end{aligned}
\tag{38}
$$

Quantities with tilde refer to the axion field contributions, while barred quantities come from brane matter, the exception being the second term which comes from the bulk content. The definitions we use are

$$
\Lambda_{(n-1)} = \left(\frac{n-3}{n-1}\right)\kappa_n^2\left(\Lambda + \frac{n-1}{8(n-2)}\kappa_n^2\sigma^2\right),
\tag{39}
$$

$$
G_{(n-1)} = \left(\frac{n-3}{n-2}\right)\frac{\kappa_n^4\sigma}{32\pi},
\tag{40}
$$

$$
\bar{G}_{(n-1)} = \left(\frac{n-3}{n-2}\right)\frac{\kappa_n^2}{8\pi},
\tag{41}
$$

$$
\bar{T}_{\mu\nu} = T_{\rho\sigma}^{(B)}q_\mu^\rho q_\nu^\sigma + T_{\rho\sigma}^{(B)}n^\rho n^\sigma q_{\mu\nu} - \frac{1}{n-1}T^{(B)}q_{\mu\nu},
\tag{42}
$$

$$
\begin{aligned}
\pi_{\mu\nu} &= -\frac{1}{4}\tau_\mu^\rho\tau_{\nu\rho} + \frac{1}{4(n-2)}\tau\tau_{\mu\nu} + \frac{1}{8}\tau^{\alpha\beta}\tau_{\alpha\beta} \\
&\quad - \frac{1}{8(n-2)}\tau^2 q_{\mu\nu},
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\kappa_{\mu\nu} &= -\frac{1}{2}\tilde{\tau}_{(\mu}^\rho\bar{\tau}_{\nu)\rho} + \frac{1}{4(n-2)}(\tilde{\tau}\bar{\tau}_{\mu\nu} + \bar{\tau}\tilde{\tau}_{\mu\nu}) \\
&\quad + \frac{1}{4}\tilde{\tau}^{\alpha\beta}\bar{\tau}_{\alpha\beta}q_{\mu\nu} - \frac{1}{4(n-2)}\tilde{\tau}\bar{\tau}q_{\mu\nu}.
\end{aligned}
\tag{44}
$$

As a cross check of the formulas above, we note that by setting $n = 5$ and $\bar{T}_{\mu\nu} = \tilde{\tau}_{\mu\nu} = 0$, we recover the same equation derived in [34]. Eq. (38) is supplemented by Codazzi's equation

$$
D_\nu K_\mu^\nu - D_\mu K = \kappa_n^2 T_{\rho\sigma}n^\rho n^\sigma q_\mu^\rho.
\tag{45}
$$

Using (38), we can investigate the evolution on the hypersurface of dimensionality $n - 1$. Using (38), we can investigate the evolution an $(n-1)$-dimensional hypersurface by applying the formalism described above. We first consider the model presented in [24], where we have an effectively codimension-2 brane situated in a 6D bulk space. There is a 4-brane and the axion field, without any additional matter in the bulk or the 4-brane and we also assume a Z_2 symmetry. The resulting space is flat everywhere, except from the position of the brane, where the tension induces a deficit angle in the bulk. Interestingly, it turns out that the imposed condition (7), relating the axion charge and the brane tension forces all terms to be quadratic in σ, so that the cosmological constant and $\tilde{\tau}_{\mu\nu}$ terms cancel against the $\tilde{\pi}_{\mu\nu}$ term. The equation of motion (38) reduces in this case to Einstein's equations in five dimensions, without any matter content,

$$
{}^{(5)}G_{\mu\nu} = 0.
\tag{46}
$$

We must stress here that $E_{\mu\nu}$ is evaluated near the brane and not on it, so in the above mentioned space, which is flat outside the brane we get $E_{\mu\nu} = 0$. By looking at this equation and taking into consideration the fact that the angular coordinate is compact, one is prematurely lead to think of this model as a regular Kaluza–Klein theory, with a zero mode graviton and a tower of massive modes. This is however misleading, since (46) carries no information of the fact that a dimensional reduction has already been performed on a non-compact dimension. In the presence of matter the resulting equation is

$$
{}^{(5)}G_{\mu\nu} = 8\pi G_5\bar{\tau}_{\mu\nu} + \kappa_6^4\bar{\pi}_{\mu\nu} + \kappa_6^4\kappa_{\mu\nu} - E_{\mu\nu}.
\tag{47}
$$

Notice that in this case the empty space solution no longer holds, so the bulk space will be curved in general and we can no longer neglect the $E_{\mu\nu}$ term, which depends on the bulk curvature.

We now turn to the model we considered earlier, with the sigma model fields in the bulk. By inspecting the energy–momentum tensor for these fields, we see that the $T_{mn}^{(B)}$ components are zero in the coordinate system (12). Notice that

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[1] While this paper was in preparation, a similar treatment of the brane equations was presented in [35], although in a cosmological context.

these coordinates are not Gauss-normal, but are related to them through a conformal transformation. Since this transformation does not mix $T_{mn}^{(B)}$ and $T_{\mu\nu}^{(B)}$, the former will also be zero when we turn back to the Gauss-normal coordinates used to derive Eq. (38). On the other hand, this transformation will not affect the later components, which will be the same in both cases. Assuming no additional bulk or brane matter besides the axion and the ϕ^a, the equations on the brane yield

$$^{(5)}G_{\alpha\beta} = \frac{3\kappa_6^2}{4}\left(\frac{16}{5\lambda^2(1+r^2)^2\psi}q_{\alpha\beta}\right. $$
$$\left. -\frac{4}{\lambda^2(1+r^2)^2\psi}n_{\mu\nu}\delta_\alpha^\mu\delta_\beta^\nu\right) - E_{\alpha\beta}. \qquad (48)$$

Here, α, β denote coordinates on the 4-brane, while μ, ν are coordinates on the 3-brane. The components of the bulk energy–momentum tensor are taken as the limiting values near the brane position. From this expression we immediately deduce that $^{(5)}R = 0$, which means that the 4-brane appears Ricci-scalar-flat, but not Ricci-flat. In fact, the fields ϕ^a act on the brane as a form of perfect fluid with anisotropic pressure, since we see that the fifth diagonal element of the energy–momentum tensor is different. However, we still have contributions from the Weyl tensor of the bulk space, which for the metric (12) takes the form

$$E_{\alpha\beta} = \frac{3\kappa_6^2}{5\lambda^2}\frac{1}{\psi(1+r^2)^2}\text{Diag}\big(1, -1, -1, -1, 4\psi r^2\big). \qquad (49)$$

Once this is taken into account, we interestingly find that the bulk field contribution is cancelled by the projected Weyl tensor and the 4-brane has

$$^{(5)}R_{\mu\nu} = 0, \qquad (50)$$

and thus, it is Ricci-flat in five dimensions.

Inclusion of matter on the brane is also straightforward in this setup. In fact, the only additional contribution compared to the previous model in empty bulk space comes from the sigma model fields. The resulting equation is

$$^{(5)}G_{\alpha\beta} = \frac{3\kappa_6^2}{4}\left(\frac{16}{5\lambda^2(1+r^2)\psi}q_{\alpha\beta} - \frac{4}{\lambda^2(1+r^2)\psi}n_{\mu\nu}\delta_\alpha^\mu\delta_\beta^\nu\right) $$
$$+ 8\pi G_5\bar{\tau}_{\alpha\beta} + \kappa_6^4\bar{\pi}_{\alpha\beta} + \kappa_6^4\kappa_{\alpha\beta} - E_{\alpha\beta}. \qquad (51)$$

To first order, we expect the first and last terms to cancel as before. In addition, we will have residual contributions in $E_{\alpha\beta}$ due to the brane content.

5. Conclusions

We presented a model where effective codimension-2 branes are embedded in a 6D bulk space endowed with an $O(3)$ sigma model. The introduction of resolved branes in such a background provides a more realistic approach compared to purely codimension-2 branes, which are known to be plagued by technical problems [11]. We presented solutions in the case of a single brane and non-compact transverse geometry and for a double brane setup in a sphere-compactified space. It turns out

that the single brane setup can account for a flat 3-brane without requiring any fine-tunings of the physical parameters of the model. In the double brane scenario, a mild fine-tuning between the brane radii is needed. The Einstein equations on the effective codimension-2 brane for arbitrary dimensionality were also derived and applied in our model to study the dynamics on the 4-brane. The model seems to admit a straightforward interpretation after KK reduction of the compact dimension of the brane. A more detailed future treatment of perturbations in this setup will help clarify further aspects of 4D gravity and its possible modifications.

Acknowledgements

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