Nuclear Physics B148 (1979) 283–315

RENORMALIZATION OF THE NAMBU JONA-LASINIO MODEL IN A MEAN FIELD EXPANSION *

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Received 8 August 1978

The Nambu Jona-Lasinio model is expanded in a mean field expansion. All divergences are absorbed in a renormalized fermion mass and a renormalized Yukawa coupling. Consequently, collective boson self-couplings are fixed in terms of this coupling while the boson mass is calculable in terms of the Fermi mass and Yukawa coupling. Equivalence to the σ model in a special limit is demonstrated.

1. Introduction

The Nambu Jona-Lasinio model [1] was proposed sometime ago as a dynamical model of hadrons. Because of supposed problems with renormalizability its study was confined to the Hartree approximation although the basic formulation necessary to prove its renormalizability has existed for many years [2].

Recently the renormalizability of these types of models has been studied [3–6]. It turns out to be possible when they are developed in a mean field expansion in terms of a collective boson field. We show all divergencies occurring in this expansion can be absorbed in a renormalized Yukawa-type coupling and a renormalized fermion mass. The induced boson mass, and cubic and quartic boson self-couplings are not arbitrary, but fixed in terms of the two renormalized parameters. This is achieved employing the Ward identities of the chiral symmetry and the Callan-Symanzik equations. The theory is equivalent to the σ model expanded in a similar way for an appropriate choice of renormalized parameters.

The paper is organized as follows. In sect. 2 we describe our expansion procedure In sect. 3 we study the model to lowest order. In sect. 4 we develop the renormalization of the theory. In sect. 5 we derive equations fixing the boson couplings and mass. Finally, in sect. 6, we demonstrate the equivalence to the σ model. Results are summarized in sect. 7.

* Work supported in part by the US Department of Energy under contract no. EY-76-C-02-3130 A002 (Task A-Theoretical).

Some of what is contained in this paper is complicated by group structure but is otherwise similar to the development of the bound-state mean field expansion [7] for $(\bar{\psi}\psi)^2$. For this reason certain arguments are only outlined here and the reader is referred to ref. [7] for details.

2. The mean field expansion

The Nambu Jona-Lasinio model is described by the Lagrangian

$$L = \bar{\psi}(i\partial\!\!\!/)\,\psi + \tfrac{1}{2}\lambda_0\,[(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5\psi)^2]\,. \tag{2.1}$$

We can write the four-fermion coupling constant as

$$\lambda_0 = g_0^2/\mu_0^2\,,$$

where g_0 is dimensionless and μ_0^2 has dimensions of $(\text{mass})^2$.

The generating functional is given by the functional integral

$$e^{iW(\eta,\bar{\eta})} = N\int d\psi\, d\bar{\psi}\,\exp\left\{i\int d^4x\left(\bar{\psi}(i\partial\!\!\!/)\,\psi + \frac{g_0^2}{2\mu_0^2}\,((\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5\psi)^2) + \bar{\eta}\psi + \bar{\psi}\eta\right)\right\}, \tag{2.2}$$

where $\bar{\eta}$ and η are anticommuting c-number sources.

The Gaussian integral over boson variables σ and π,

$$\int d\sigma\, d\pi\,\exp -i\left\{\tfrac{1}{2}\mu_0^2\int d^4x\left(\sigma - \frac{g_0}{\mu_0^2}\,\bar{\psi}\psi\right)^2 + \tfrac{1}{2}\mu_0^2\int d^4x\left(\pi - i\frac{g_0}{\mu_0^2}\,\bar{\psi}\gamma_5\psi\right)^2\right\},$$

is just a constant, i.e., it does not depend on the fermion variables. Inserting it in (2.2) will only change the normalization constant N. The resulting generating function is

$$e^{iW(\bar{\eta},\eta,S,J)} = N'\int d\psi\, d\bar{\psi}\, d\sigma\, d\pi\,\exp\{i\int d^4x\,[\bar{\psi}(i\partial\!\!\!/ + g_0(\sigma + i\gamma_5\pi))\,\psi - \tfrac{1}{2}\mu_0^2(\sigma^2 + \pi^2) + S\sigma + J\pi + \bar{\psi}\eta + \bar{\eta}\psi\,]\,\}\,. \tag{2.3}$$

We have introduced sources S and J for the scalar and pseudoscalar auxiliary variables σ and π.

The integration over the fermions can be performed exactly to obtain

$$e^{iW(\bar{\eta},\eta,S,J)} = N'\int d\sigma\, d\pi\,\exp\{i\int d^4x\,[-\bar{\eta}G\eta - i\,\mathrm{tr}\,\ln(iG^{-1}) - \tfrac{1}{2}\mu_0^2(\sigma^2 + \pi^2) + S\sigma + J\pi]\}\,. \tag{2.5}$$

Here we have introduced

$$G^{-1}(x,y) \equiv [\gamma^\mu \partial_\mu + g_0(\sigma + i\gamma_5 \pi)]\, \delta^{(4)}(x-y)\,. \tag{2.5}$$

The generating function in Euclidean space is

$$e^W = \int d\sigma\, d\pi\, e^{-F\{\sigma,\pi\}} = N' \int d\sigma\, d\pi\, \exp\{-\int [\bar{\eta} G\, \eta - \mathrm{tr}\ln G^{-1} + \tfrac{1}{2}\mu_0^2(\sigma^2 + \pi^2) - S\sigma - J\pi]\}\,.$$

We can introduce a parameter ϵ as a bookkeeping device to generate an approximation scheme. Thus we make the definition

$$e^{W_\epsilon/\epsilon} \equiv \int d\sigma\, d\pi\, \exp\{-F\{\sigma,\pi\}/\epsilon\}\,. \tag{2.6}$$

At the end of all computations we shall set ϵ equal to one, and W_ϵ becomes W except for an irrelevant constant.

The generating function written in the form (2.6) can be expanded in a bound-state mean field expansion [5]. The mean fields (σ_0, π_0) are defined by the mean field conditions

$$\left.\frac{\delta F}{\delta \sigma}\right|_{\sigma_0,\pi_0} = 0\,, \quad \left.\frac{\delta F}{\delta \pi}\right|_{\pi_0,\sigma_0} = 0\,, \tag{2.7a}$$

$$\left.\frac{\delta^2 F}{\delta\sigma_i \delta\sigma_j}\right|_{\sigma_0,\pi_0} > 0\,. \tag{2.7b}$$

Then the generating function is constructed by the following formula [5]. Here we use the notation $\sigma_i = (\sigma, \pi)$.

$$e^{W_\epsilon/\epsilon} \sim e^{-F\{\boldsymbol{\sigma}_0\}/\epsilon}\, e^{-\mathrm{tr}\ln(A)/2}\{1 - \tfrac{1}{8}\epsilon \iiiint C(x,y,z,w)\, A^{-1}(x,y)\, A^{-1}(z,w)$$
$$+ \tfrac{1}{24}\epsilon \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int B(x,y,z)\, B(a,b,c)\, [2A^{-1}(x,a)\, A^{-1}(y,b)\, A^{-1}(z,c)$$
$$+ 3A^{-1}(x,y)\, A^{-1}(z,a)\, A^{-1}(b,c)] + O(\epsilon^2)\}\,. \tag{2.8}$$

This equation has implicit tensorial multiplication with the components as follows:

$$A_{ij}(x,y) \equiv \left.\frac{\delta^2 F}{\delta\sigma_i(x)\, \delta\sigma_j(y)}\right|_{\boldsymbol{\sigma}_0}\,,$$

$$B_{ijk}(x,y,z) \equiv \left.\frac{\delta^3 F}{\delta\sigma_i(x)\, \delta\sigma_j(y)\, \delta\sigma_k(z)}\right|_{\boldsymbol{\sigma}_0}\,,$$

$$C_{ijkl}(x,y,z,w) \equiv \left.\frac{\delta^4 F}{\delta\sigma_i(x)\, \delta\sigma_j(y)\, \delta\sigma_k(z)\, \delta\sigma_l(w)}\right|_{\boldsymbol{\sigma}_0}\,.$$

$\boldsymbol{\sigma}_0 = (\sigma_0, \pi_0)$ is determined from (2.7a).

3. The lowest order

According to (2.8) the lowest-order connected functional is

$$W_0(\eta, \bar{\eta}, S, J; \sigma_0, \pi_0) = -F\{\eta, \bar{\eta}, S, J; \sigma_0, \pi_0\} . \tag{3.1}$$

The mean fields are defined by the mean field conditions (2.7). Classical fields can be defined as

$$\sigma^\epsilon(x) \equiv \frac{\delta W_\epsilon}{\delta S(x)} \quad \text{and} \quad \pi^\epsilon(x) \equiv \frac{\delta W_\epsilon}{\delta J(x)} .$$

To this order the classical fields coincide with the mean fields

$$\sigma^0(x) = \frac{\delta W_0}{\delta S(x)} = \sigma_0(x) \qquad \text{and} \qquad \pi^0(x) = \frac{\delta W_0}{\delta J(x)} = \pi_0(x) .$$

Classical fermion fields can be defined in the same way

$$\psi_\epsilon(x) \equiv \frac{\delta W_\epsilon}{\delta \bar{\eta}(x)} , \qquad \bar{\psi}_\epsilon(x) \equiv \frac{\delta W_\epsilon}{\delta \eta(x)} .$$

To lowest order the classical fermion fields are

$$\psi_0(x) = \frac{\delta W_0}{\delta \bar{\eta}(x)} = -\int d^4y G_0(x, y)\, \eta(y) \qquad \text{and} \qquad \bar{\psi}_0(x) = -\int d^4y\, \bar{\eta}(y)\, G_0(y, x)$$

The effective action of the composite fields can be defined in the usual way

$$\Gamma_\epsilon\{\sigma^\epsilon, \pi^\epsilon\} = W_\epsilon\{J, S\} - \int d^4x (S(x)\, \sigma^\epsilon(x) + J(x)\, \pi^\epsilon(x)) .$$

To lowest order the effective action is

$$\Gamma_0\{\sigma_0, \pi_0\} = \int [-\bar{\psi}_0 G_0^{-1} \psi_0 + \mathrm{tr} \ln G_0^{-1} - \tfrac{1}{2}\mu_0^2 (\sigma_0^2 + \pi_0^2)] . \tag{3.2}$$

G_0 now refers to

$$G_0^{-1}(x, y) = (i\partial\!\!\!/ + g_0(\sigma_0 + i\gamma_5 \pi_0))\, \delta^{(4)}(x - y) .$$

The graphical interpretation of the trace term is very simple. Translating fields by $(\sigma, \pi) = (s + v, \pi)$ where s is such $\langle 0|s|0\rangle = 0$ in the absence of sources, and adding the constant term $-\mathrm{tr}\ \ln(i\partial\!\!\!/ + g_0 v)$, we obtain

$$\Gamma_0\{\sigma_0, \pi_0\} = \int \left[-\tfrac{1}{2}\mu_0^2(s_0^2 + \pi_0^2) - \mu_0^2 s_0 v + \mathrm{tr} \ln \left(1 + \frac{g_0(s_0 + i\gamma_5\pi_0)}{i\partial\!\!\!/ + g_0 v}\right)\right]$$

$$= \int \left[-\tfrac{1}{2}\mu_0^2 (s_0^2 + \pi_0^2) - \sum_{\nu=2}^{\infty} \frac{(-)^\nu}{\nu} (g_0(s_0 + i\gamma_5\pi_0))^\nu\, \mathrm{tr} \left(\frac{1}{i\partial\!\!\!/_x + g_0 v}\right)^\nu \right] .$$

Fig. 1.

The linear terms do not contribute due to the mean field condition $\delta F/\delta s = 0$. The infinite series corresponds to the infinite set of fermion polygons shown in fig. 1 with vertices $(g_0, ig_0\gamma_5)$.

The (σ, π) effective potential to this order is

$$V(\sigma_0, \pi_0) = \tfrac{1}{2}\mu_0^2\,(\sigma_0^2 + \pi_0^2) - \mathrm{tr}\ln G_0^{-1}(\sigma_0, \pi_0)\,.$$

The mean field conditions (2.7) coincide with the conditions that we are at a minimum of the effective potential. They take the form

$$\frac{\partial V}{\partial \boldsymbol{\sigma}_0} = 0\,, \qquad \left.\frac{\partial^2 V}{\delta\sigma_i \delta\sigma_j}\right|_{\boldsymbol{\sigma}_0} > 0\,.$$

We must keep in mind, however, that this is only demonstrated true to lowest order and in general the mean field conditions (2.7) which must be valid with all sources on might be different than the minimum condition on V. We have not examined this question. If the conditions differ, (2.7) is the more rigorous mathematical statement and should be assumed correct.

The fermion inverse propagator to this order is

$$\left(\frac{\delta W}{\delta\eta\delta\bar{\eta}}\right)^{-1} = G_0^{-1} = \not{p} + g_0(\sigma_0 + i\gamma_5\pi_0)\,.$$

A non-zero mean field, or equivalently a non-zero classical field, implies a massive fermion. Assuming that $\sigma_0 \neq 0$, the mass of the fermion is

$$m = -g_0\sigma_0\,.$$

The expectation value of the operator $\sigma(x) = \bar{\psi}(x)\,\psi(x)$ breaks the chiral invariance spontaneously. The chiral current,

$$j_5^\mu(x) = [\bar{\psi}(x)\,\gamma^\mu\gamma_5, \psi(x)]\,,$$

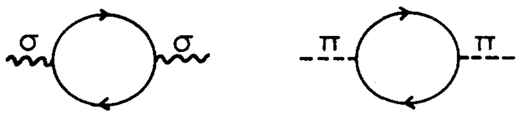
is still conserved but the vacuum state is not invariant under chiral transformations

$$\psi(x) \to \mathrm{e}^{i\alpha\gamma_5}\psi(x)\,.$$

Thus the spontaneously broken Nambu model is characterized to lowest order by the propagation of a massive fermion with a free two-point function.

The inverse scalar propagator to this order, is

$$\left(\frac{\delta W}{\delta S\delta S}\right)^{-1} = \Delta_\sigma^{-1}(p^2) = -\mu_0^2 + g_0^2\Pi_\sigma(p^2)\,.$$



σ self-energy $\qquad$ π self-energy

Fig. 2.

The scalar fermion bubble is (fig. 2.)

$$\Pi_\sigma(p^2) \equiv i \operatorname{tr} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} G_0(k)\, G_0(k+p)$$

$$= 4i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^2 + k\cdot p + m^2}{[(k+p)^2 - m^2]\,[k^2 - m^2]} \,.$$

Note we have shifted back into Minkowski space. The inverse pseudoscalar propagator is

$$\left(\frac{\delta W}{\delta J \delta J}\right)^{-1} = \Delta_\pi^{-1}(p^2) = -\mu_0^2 + g_0^2 \Pi_\pi(p^2)\,.$$

The pseudoscalar fermion bubble is

$$\Pi_\pi(p^2) \equiv -i \operatorname{tr} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \gamma_5 G_0(k)\, \gamma_5 G_0(k+p)$$

$$= 4i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^2 + k\cdot p - m^2}{[(k+p)^2 - m^2]\,[k^2 - m^2]} \,.$$

Direct calculation shows the mixed inverse propagator vanishes.

The fermion bubble is quadratically divergent. Subtracting twice at zero momentum we obtain

$$\Delta_i^{-1} = p^2 \left(g_0^2 \left(\frac{\partial \Pi_i}{\partial p^2}\right)_0 \right) - \psi_0^2 + g_0^2 \Pi_i(0) + g_0^2 \operatorname{sub}_0^2 \Pi_i(p^2)\,.$$

The symbol sub_0^2 stands for

$$\operatorname{sub}_0^2 \Pi_i(p^2) \equiv \Pi_i(p^2) - \Pi_i(0) - p^2 \left(\frac{\partial \Pi_i}{\partial p^2}\right)_0 \,.$$

Next we define renormalization parameters by

$$\Delta_\sigma^{-1}(0) = -\frac{\mu_\sigma^2}{Z_\sigma}\,, \qquad \left(\frac{\partial \Delta_\sigma^{-1}}{\partial p^2}\right)_0 = \frac{1}{Z_\sigma}\,,$$

$$\Delta_\pi^{-1}(0) = -\frac{\mu_\pi^2}{Z_\pi}\,, \qquad \left(\frac{\partial \Delta_\pi^{-1}}{\partial p^2}\right)_0 = \frac{1}{Z_\pi}. \tag{3.3}$$

Thus, we are led to

$$\Delta_\sigma^{-1}(p^2) = Z_\sigma^{-1}\overline{\Delta}_\sigma^{-1}(p^2) = Z_\sigma^{-1}[p^2 - \mu_0^2 + g_\sigma^2 \,\mathrm{sub}_0^2 \Pi_\sigma(p^2)] \,, \tag{3.4}$$

and

$$\Delta_\pi^{-1}(p^2) = Z_\pi^{-1}\overline{\Delta}_\sigma^{-1}(p^2) = Z_\pi^{-1}[p^2 - \mu_\pi^2 + g_\pi^2 \,\mathrm{sub}_0^2 \Pi_\pi(p^2)] \,. \tag{3.5}$$

The renormalized couplings are defined as

$$g_\sigma^2 = g_0^2 Z_\sigma \,, \qquad g_\pi^2 = g_0^2 Z_\pi \,. \tag{3.6}$$

The mean field condition to lowest order leads to the gap equation

$$\mu_0^2 \sigma_0 = -ig_0 \,\mathrm{tr} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} (k^2 + g_0 \sigma_0)^{-1} \,,$$

or

$$\mu_0^2 = 4ig_0^2 \int \frac{\mathrm{d}^4 k}{(4\pi)^4} \frac{1}{[k^2 - m^2]} \,. \tag{3.7}$$

On the other hand, the condition (3.7) for the pseudoscalar mass is

$$\frac{\mu_\pi^2}{Z_\pi} = \mu_0^2 - 4ig_0^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{[k^2 - m^2]} \,.$$

(3.7) immediately implies that

$$\mu_\pi^2 = 0 \,. \tag{3.8}$$

This is equivalent to

$$\Delta_\pi^{-1}(0) = 0 \,. \tag{3.9}$$

Thus, the pseudoscalar propagator has a zero-mass pole. This is obviously the explicit lowest-order realization of the Goldstone theorem. Breaking the symmetry *via* the non-vanishing expectation value of σ has resulted in the presence of a massless pole in the Green function of the related operator π.

Examining the unrenormalized expression for the σ-propagator and using the gap equation to get rid of μ_0^2 we obtain ⋆

$$\Delta_\sigma^{-1}(4m^2) = -4ig_0^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{2m^2(1-2x)}{[k^2 - m^2(1-2x)^2]^2} = 0 \,.$$

⋆ This is the only statement we use that is in any way cutoff dependent. Even so the dependence is absolutely minimal. We only assume that integrals of the form

$$\int \mathrm{d}^4 k \frac{k \cdot p}{(k^2 + A)} = 0 \,.$$

This is satisfied by a rational cutoff scheme. In general we do not use cutoffs, as our renormalization procedure makes them irrelevant. See next footnote.

Thus the pole of the σ propagator appears at threshold. Since we have subtracted at zero, our mass parameter is not the physical σ mass but it is related to it through

$$\mu_\sigma^2 = 4m^2 + g_\sigma^2 \mathrm{sub}_0^2 \Pi_\sigma(4m^2)\ .$$

Evaluating the subtracted bubble we obtain the explicit relation

$$\mu_\sigma^2 = 4m^2 \left(1 + \frac{g_\sigma^2}{12\pi^2}\right)\ . \tag{3.10}$$

The renormalized couplings are to this order

$$\frac{1}{g_\sigma^2} = \frac{Z_\sigma^{-1}}{g_0^2} = \frac{g_0^2}{g_0^2}\frac{2}{3}\int \frac{(\mathrm{d}^4k)_E}{(2\pi)^4}\ \frac{3k_E^2 - m^2}{(k_E^2 + m^2)^3}$$

$$= \frac{1}{24\pi^2}\int \mathrm{d}k^2\ \frac{\mathrm{d}k^2 k^2(3k^2 - m^2)}{(k^2 + m^2)^3}\ ,$$

$$\frac{1}{g_\pi^2} = \frac{Z_\pi}{g_0^2} = \frac{1}{8\pi^2}\int \frac{\mathrm{d}k^2 k^2}{(k^2 + m^2)^2}\ .$$

Thus $1/g_\sigma^2$ and $1/g_\pi^2$ are logarithmically divergent quantities if explicitly calculated and hence require a renormalization procedure to make them meaningful. We shall assume g_π and g_σ are undetermined but finite numbers *. They are independent of the bare coupling g_0 except through the mass m. It is a property of these theories to all orders that the explicit g_0 dependence is absent. In lower dimensions subtraction is not necessary and the couplings can be calculated. We conclude that to this order the renormalized σ and π propagators are finite functions of the fermion mass and the couplings. These couplings, because of the divergences, are considered, in essentially the usual way, as arbitrary renormalized parameters [9]. Higher orders result in further renormalization and conceivably the logarithmic divergences associated with them might cancel. Old work on summing of such divergences suggests that they do not, except in the case of vector theories where a Johnson-Baker-Wiley type eigenvalue condition is the condition for cancellations.

* The reader should be consciously aware of what he might regard as peculiarities in our renormalization procedure. We do not use explicit Langrangian counter term renormalization procedures in our discussions of mean field theories, but instead regard divergent Green functions as defined by either subtracting or differentiating in momentum space until they are finite and then integrating or adding back in the requisite number of powers of momentum multiplied by finite (but undefined except through identities of the theory) constants. This method is obviously fully equivalent to the Lagrangian counter term method but to our taste is somewhat more elegant for dealing with mean field problems as well as more conventional coupling constant renormalization problems. Furthermore we do not need to use an explicit regularization schem(since the results obtained through the above procedure make no reference to divergent quantities. As a very nice alternative example of this type of procedure see ref. [8].

Eq. (3.10) implies that

$$\left.\frac{\partial^2 V}{\partial \sigma^2}\right|_{\sigma_0,\pi_0} = \mu_\sigma^2 = 4m^2\left(1 + \frac{g_\sigma^2}{12\pi^2}\right) > 0 .$$

This goes part way towards satisfying the mean field stability condition. However, the condition (2.7b) is a matrix and since

$$\left.\frac{\delta^2 F}{\delta\pi\delta\sigma}\right|_{\sigma_0,\pi_0} = 0 ,$$

in order to satisfy it we must have

$$\left.\frac{\delta^2 F}{\delta\pi_i\delta\pi_j}\right|_{\sigma_0,\pi_0} = \frac{\mu_\pi^2}{Z_\pi} > 0 .$$

Since $\mu_\pi^2 = 0$ by the Goldstone theorem, this cannot be valid. We can get around the difficulty by arguing that this sort of problem is characteristic of the theories with zero mass and that we will move the mass away from zero by not turning the source $S(x)$ entirely off but instead setting it to a small constant value m_0. Thus we simulate a Fermi bare mass, destroy the chiral symmetry, and allow $\mu_\pi^2 > 0$. At the absolute end of any calculation we will then let $m_0 \to 0$ and let the theory go smoothly to the spontaneously broken chiral symmetric limit. This works fine in lowest order. In higher orders there are infrared divergences which we shall assume, insofar as they need to be, are controlled by this procedure. Consequently, we ignore the problem in formal manipulations. In any event, as will be illustrated in sect. 6, the problem is no worse (or better) than the infrared problems of a normal σ model with massless pions.

Even after satisfying the mean field stability condition by this device we have not guaranteed that our example is free of esoteric behavior. This model as well as all other summed fermion theories have Landau type ghosts induced in the scalar and pseudoscalar propagators. As an example of this it is easily found that the σ propagator has a pole at space-like momentum for $p^2 \sim 2m^2\,e^{(\pi^2/g_\sigma^2)}$. This is an example of the Landau ghost first observed in electrodynamics. In this case, when the infinite series of the fermion bubble graphs is summed, a tachyonic pole appears in QED. near $4m^2\,e^{3\pi/\alpha}$.

It is evident, from the nature of the location of this pole, that for any moderately small values of the coupling constant the problems occur at such large values of momentum as to be beyond any conceivable relevance [9]. This "ghost" will not be any more of a concern to us than it is in any conventional perturbation theory. This is because we will show that the effective expansion parameters of the theory are ϵg_σ^2 and ϵg_π^2. When $\epsilon \to 1$ we therefore must take g_σ^2 and g_π^2 small so they serve the role of expansion parameters. Consequently, the renormalized Green func-

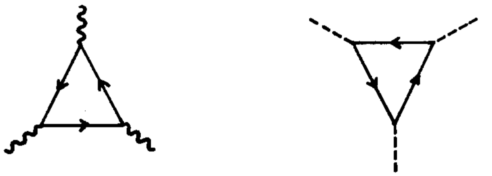


Fig. 3. Three-boson vertices.

tions should be calculated to some fixed order in g_σ^2 and g_π^2. This is easily done systematically (when g_σ^2 and g_π^2 are of the same order) by calculating the unrenormalized theory to order ϵ^n, renormalizing, setting $\epsilon = 1$, expanding in g_σ^2 and g_π^2 and discarding terms of the form $g_\sigma^{2m} g_\pi^{2p}$ where $m + p > n$. This expansion eliminates the ghost which is a property of the infinite summation and puts this theory in a form equivalent to that obtained from a more normal-type perturbation theory. The results can then be resumed in whatever form desired with the usual perils.

Next, we examine the three- and four-boson vertex functions. The non-vanishing cubic vertices are the $\pi\pi\sigma$ and $\sigma\sigma\sigma$. The surviving quartic vertices are $\pi\pi\pi\pi$, $\sigma\sigma\sigma\sigma$ and $\pi\pi\sigma\sigma$.

The three-boson vertex is just a fermion triangle, to this order, (fig. 3.)

$$\Gamma^0_{\pi\pi\sigma}(p_1, p_2) \equiv \frac{\delta\Delta^{-1}_{\pi 0}}{\delta\sigma_0} = ig_0^3 \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} G_0(k)\gamma_5 G_0(k+p_1)\gamma_5 G_0(k+p_1+p_2) + (2\mathrm{xT}),$$

$$\Gamma^0_{\sigma\sigma\sigma}(p_1, p_2) \equiv \frac{\delta\Delta^{-1}_{\sigma 0}}{\delta\sigma_0} = -ig_0^3 \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} G_0(k)\, G_0(k+p_1)\, G_0(k+p_1+p_2) + (2\mathrm{xT}).$$

The symbol xT is used here and in what follows as a shorthand for cross terms. The fermion triangles are logarithmically divergent. We subtract them once at zero and introduce renormalized cubic boson self-couplings by

$$\Gamma^0_{\pi\pi\sigma}(0,0) = \frac{m\lambda_{\pi\pi\sigma}}{3Z_\sigma^{1/2} Z_\pi}, \qquad \Gamma^0_{\sigma\sigma\sigma}(0,0) = \frac{m\lambda_{\sigma\sigma\sigma}}{Z_\sigma^{3/2}}.$$

We then obtain

$$\Gamma^0_{\pi\pi\sigma}(p_1,p_2) = Z_\pi^{-1} Z_\sigma^{-1/2} \overline{\Gamma}^0_{\pi\pi\sigma}(p_1,p_2) = Z_\pi^{-1} Z_\sigma^{-1/2} \times \left[\tfrac{1}{3} m\lambda_{\pi\pi\sigma} + ig_\pi^2 g_\sigma \operatorname{sub}_0^1 \left(\operatorname{tr} \int \frac{d^4k}{(2\pi)^4} G_0(k+p_1)\gamma_5 G_0(k+p_1+p_2)\gamma_5 G_0(k) + (2\mathrm{xT})\right)\right], \tag{3.11}$$

$$\Gamma^0_{\sigma\sigma\sigma}(p_1,p_2) = Z_\sigma^{-3/2} \overline{\Gamma}_{\sigma\sigma\sigma}(p_1,p_2) = Z_\sigma^{-3/2} \left[m\lambda_{\sigma\sigma\sigma} - ig_\sigma^3 \operatorname{sub}_0^1 \left(\operatorname{tr} \int \frac{d^4k}{(2\pi)^4} G_0(k+p_1)\, G_0(k+p_1+p_2) G_0(k)\right) + (2\mathrm{xT})\right]. \tag{3.12}$$

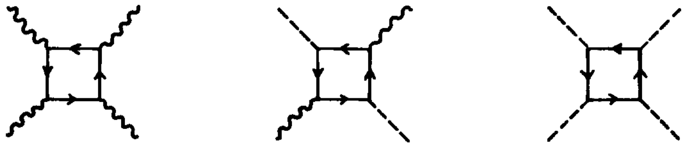


Fig. 4. Four-boson vertices.

g_π, $\lambda_{\pi\pi\sigma}$, and $\lambda_{\sigma\sigma\sigma}$ can be calculated in terms of g_σ and m. We defer the lowest-order case to sect. 5 where the lowest-order results are derived as a special case of the all-order results. Details of the lowest-order calculations are contained in ref. [7].

The four-boson vertex to this order is just the fermion quadrangle (fig. 4). These quadrangles are logarithmically divergent and need one subtraction. Subtracting at zero, as before, and introducing the renormalized quartic self-couplings by

$$\Gamma^0_{\pi\pi\pi\pi}(0,0,0) = -\frac{\lambda_{\pi\pi\pi\pi}}{Z_\pi^2}\ ,$$

$$\Gamma^0_{\sigma\sigma\sigma\sigma}(0,0,0) = -\frac{\lambda_{\sigma\sigma\sigma\sigma}}{Z_\sigma^2}\,,\qquad \Gamma^0_{\pi\pi\sigma\sigma}(0,0,0) = -\frac{\lambda_{\sigma\sigma\pi\pi}}{3Z_\sigma Z_\pi}\,,$$

we obtain

$$\Gamma^0_{\pi\pi\pi\pi}(p_1,p_2,p_3) = Z_\sigma^{-2}\overline{\Gamma}^0_{\sigma\sigma\sigma\sigma}(p_1,p_2,p_3) = Z_\sigma^{-2}\Big[-\lambda_{\sigma\sigma\sigma\sigma}$$
$$+ ig^4 \operatorname{sub}_0^1\Big(\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}G_0(k)G_0(k+p_1)G_0(k+p_1+p_2)G_0(k+p_1+p_2+p_3)$$
$$+ (6\mathrm{xT})\Big)\Big]\,, \tag{3.14}$$

$$\Gamma^0_{\pi\pi\sigma\sigma}(p_1,p_2,p_3) = Z_\pi^{-1}Z_\sigma^{-1}\overline{\Gamma}^0_{\pi\pi\sigma\sigma}(p_1,p_2,p_3) = Z_\pi^{-1}Z_\sigma^{-1}\Big[-\tfrac{1}{3}\lambda_{\pi\pi\sigma\sigma}$$
$$- g_\pi^2 g_\sigma^2 \operatorname{sub}_0^1\Big(\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}G_0(k)\gamma_5 G_0(k+p_1)\gamma_5 G_0(k+p_1+p_2)$$
$$G_0(k+p_1+p_2+p_3) + (6\mathrm{xT})\Big)\Big]\,. \tag{3.15}$$

Just as the renormalized three-boson couplings are not independent renormalized quantities, so the four-boson self-couplings will be shown to be determined to every order as functions of the renormalized Yukawa-type couplings and the fermion mass m. No other Green functions need subtractions since they are superficially finite. For example, the pure Fermi four-point function to lowest order is (fig. 5)

$$\frac{\delta^2 G_0}{\delta\eta\delta\overline{\eta}} = g_0^2 G_0 G_0 \Delta_{\sigma_0} G_0 G_0 + g_0^2 G_0 i\gamma_5 G_0 \Delta_{\pi_0} G_0 i\gamma_5 G_0 + (\mathrm{xT})$$
$$= g_\sigma^2 \overline{G}_0\overline{G}_0\overline{\Delta}_{\sigma_0}\overline{G}_0\overline{G}_0 + g_\pi^2\overline{G}_0 i\gamma_5\overline{G}_0\overline{\Delta}_{\pi_0}\overline{G}_0 i\gamma_5\overline{G}_0 + (\mathrm{xT})\,.$$

Fig. 5. Fermi-Fermi scattering.

The important observation to be made from this is that there is no remnant of the original contact four Fermi interaction. It is this property which makes the model renormalizable in the mean field approximation.

4. Renormalization to all orders

The first step in our renormalization program is to recognize the superficially divergent diagrams of the theory. It is clear that, in the bound-state mean field expansion, expression (2.8) gives the every-order vacuum-to-vacuum amplitude as a function of the lowest-order quantities G_0, σ_0, and π_0 in the presence of sources. It is tedious but straightforward to demonstrate that by differentiation of the log of this amplitude with respect to the sources that any graph to any order can be constructed out of two basic vertices [5–7] after the sources are turned off. One is the trilinear coupling of form

$$g_0^2 \Delta_{\sigma_0}(x, y)\, G_0(z, y)\, G_0(y, w)\,,$$

and the other is

$$g_0^2 \Delta_{\pi_0}(x, y)\, G_0(z, y)\, \gamma_5 G_0(y, w)\,.$$

Fig. 6 shows all the basic vertices with sources on. It further can be shown that the order of the expansion in ϵ^n of any graph is determined by adding the number of its independent momentum integrations which have at least one σ propagator or π

Fig. 6. Basic vertices.

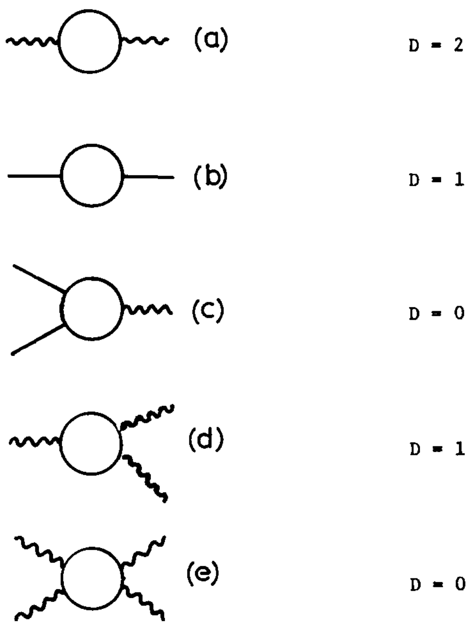


Fig. 7. Classes of superficially divergent graphs.

propagator. Since from the results of sect. 3 it follows that

$$\Delta_{\sigma_0}(p^2) \overset{p^2\to\infty}{\to} \left(\frac{1}{p^2}\right) , \qquad \Delta_{\pi_0}(p^2) \overset{p^2\to\infty}{\to} \left(\frac{1}{p^2}\right),$$

$$G_0(p^2) \overset{p^2\to\infty}{\to} \left(\frac{1}{p}\right),$$

we can analyze the large momentum behavior of any combination of the propagation functions. The superficial degree of divergence corresponding to a graph with B external σ or π lines and F external fermion lines is found to be

$$D = 4 - B - \tfrac{3}{2} F . \tag{4.1}$$

According to the above formula, vacuum graphs will have maximal $D = 4$. This is irrelevant since vacuum bubbles are always divided out of any Green function. Also graphs with one external σ or π line, having $D = 3$, will always be absorbed into the fermion mass and need not be discussed further.

The superficially divergent graphs that occur in our expansion, i.e., those with $D \geqslant 0$ are the following (fig. 7):

(a) graphs with two external (σ, π) lines ($D = 2$) ;
(b) graphs with two external ψ lines ($D = 1$) ;

(c) graphs with two external ψ lines and a (σ, π) line $(D = 0)$;
(d) graphs with three external (σ, π) lines $(D = 1)$;
(e) graphs with four external (σ, π) lines $(D = 0)$.

The analysis is the same for other four-fermion theories [6,7] like $(\bar{\psi}\psi)^2$ or $(\bar{\psi}\lambda^a\psi)^2$ and is not altered by the group structure of the Nambu model.

Graphs having two external meson lines have a maximal D of 2. The leading behavior of the lowest-order inverse (σ, π) propagator is

$$\int \frac{d^4k}{k^2} \sim \Lambda^2 .$$

As we have already seen, these graphs require two subtractions.

Graphs with two external fermion lines are linearly divergent, thus requiring two subtractions. They behave like

$$\int \frac{d^4k}{(k^2)^2}\, k \sim \gamma \cdot \Lambda .$$

The vertex graphs having two external fermion lines and one (σ, π) line have $D = 0$ They are logarithmically divergent behaving like

$$\int \frac{d^4k}{(k^2)^2} \sim \ln \Lambda .$$

Graphs with three external (σ, π) lines have $D = 1$. Their true superficial divergence is logarithmic. Having dimensions of mass, the $3(\sigma, \pi)$ 1PI functions must be proportional to the fermion mass since they vanish when the chiral symmetry is preserved. The other dimensional parameter of the model μ_σ^2, as we shall show, is always determined in terms of m and the couplings.

The $3(\sigma, \pi)$ graphs behave like

$$m_0 \int \frac{d^4k}{(k^2)^2} \sim m_0 \ln \Lambda .$$

Graphs with four external (σ, π) lines have $D = 0$ and are logarithmic. They behave like

$$\int \frac{d^4k}{(k^2)^2} \sim \ln \Lambda .$$

No other graphs have superficial divergences.

The next step in our renormalization program is to write down a set of suitable renormalization conditions for the superficially divergent functions at some arbitrary point p_0 in momentum space. For the time being, we wish to avoid $p_0 = 0$ because the Goldstone boson in $D_{\pi 0}$ can lead to complications at this point as previously mentioned. We define the following (valid after all sources are turned off):

$$G^{-1}(p_0) = \left(\frac{\delta^2 W_\epsilon}{\delta\eta\delta\bar{\eta}}\right)^{-1}\Bigg|_{p_0} = \frac{\not{p}_0 - m}{Z_2}\,,$$

$$\Delta_\sigma^{-1}(p_0^2) = \left(\frac{\delta^2 W_\epsilon}{\delta S\delta S}\right)^{-1}\Bigg|_{p_0} = \frac{p_0^2 - u_0^2}{Z_\sigma}\,,\qquad \Delta_\pi^{-1}(p_0^2) = \left(\frac{\delta^2 W_\epsilon}{\delta J\delta J}\right)^{-1}\Bigg|_{p_0} = \frac{p_0^2 - \mu_\pi^2}{Z_\pi}\,,$$

$$\left(\frac{\partial\Delta_\sigma^{-1}}{\partial p^2}\right)\Bigg|_{p_0} = \frac{1}{Z_\sigma}\,,\qquad \left(\frac{\partial\Delta_\pi^{-1}}{\partial p^2}\right)\Bigg|_{p_0} = \frac{1}{Z_\pi}\,,$$

$$\Gamma_\sigma(p_0, -p_0) = \left(\frac{\delta G^{-1}}{\delta g_0\sigma}\right)\Bigg|_{p_0} = \frac{1}{Z_\sigma(1)}\,,\qquad \Gamma_\pi(p_0, -p_0) = \left(\frac{\delta G^{-1}}{\delta g_0\pi}\right)\Bigg|_{p_0} = \frac{i\gamma_5}{Z_\pi(1)}\,,$$

$$\Gamma_{\sigma\sigma\sigma}(p_0, -p_0, 0) = \frac{\delta\Delta_\sigma^{-1}}{\delta\sigma}\Bigg|_{p_0} = \frac{m\lambda_{\sigma\sigma\sigma}}{Z_\sigma^{3/2}}\,,$$

$$\Gamma_{\pi\pi\pi}(p_0, -p_0, 0) = \frac{\delta\Delta_{\pi\pi}^{-1}}{\delta\sigma}\Bigg|_{p_0} = \frac{m\lambda_{\pi\pi\sigma}}{3Z_\pi Z_\sigma^{1/2}}\,,$$

$$\Gamma_{\sigma\sigma\sigma\sigma}(p_0, -p_0, 0, 0) = \frac{\delta^2\Delta_\sigma^{-1}}{\delta\sigma^2}\Bigg|_{p_0} = -\frac{\lambda_{\sigma\sigma\sigma\sigma}}{Z_\sigma^2}\,,$$

$$\Gamma_{\pi\pi\pi\pi}(p_0, -p_0, 0,0) = \frac{\delta^2\Delta_\pi^{-1}}{\delta\pi^2}\Bigg|_{p_0} = \frac{-\lambda_{\pi\pi\pi\pi}}{Z_\pi^2}\,,$$

$$\Gamma_{\pi\pi\sigma\sigma}(p_0, -p_0, 0, 0) = \frac{\delta^2\Delta_\sigma^{-1}}{\delta\pi\delta\pi}\Bigg|_{p_0} = \frac{-\lambda_{\pi\pi\sigma\sigma}}{3Z_\pi Z_\sigma}\,. \tag{4.2}$$

The renormalized parameters of the theory as will be explicitly demonstrated from the Schwinger-Dyson equations are:

(a) a renormalized fermion mass m;

(b) two renormalized boson masses μ_σ^2 and μ_π^2;

(c) a set of dimensionless Yukawa-type couplings

$$g_\sigma^2 \equiv g_0^2 Z_\sigma\left(\frac{Z(2)}{Z_\sigma(1)}\right)^2\,,\qquad \hat{g}_\sigma^2 \equiv g_0^2 Z_\sigma\left(\frac{Z(2)}{Z_\pi(1)}\right)^2\,,$$

$$g_\pi^2 \equiv g_0^2 Z_\pi\left(\frac{Z(2)}{Z_\pi(1)}\right)^2\,,\qquad \hat{g}_\pi^2 \equiv g_0^2 Z_\pi\left(\frac{Z(2)}{Z_\sigma(1)}\right)^2\,;$$

(d) two dimensionless cubic boson self-couplings

$\lambda_{\pi\pi\sigma}$, $\lambda_{\sigma\sigma\sigma}$;

(e) three dimensionless quartic boson self-couplings

$\lambda_{\pi\pi\pi\pi}$, $\lambda_{\sigma\sigma\sigma\sigma}$, $\lambda_{\pi\pi\sigma\sigma}$.

As we have already seen to lowest order, the boson masses are always determined in terms of the fermion mass and the dimensionless couplings. We shall show that this is true of all orders of the mean field expansion. The plethora of coupling constants resulting from the group structure will be reduced to one Yukawa coupling and one quartic coupling through the use of Ward identities of the spontaneously broken chiral symmetry. In addition, we shall show that the quartic coupling is not an independent renormalization parameter but is always a function of the renormalized Yukawa coupling. This is a consequence of the absence of a bare quartic coupling. In order to demonstrate that we shall employ the Callan-Symanzik equations.

Thus, finally, we shall be left with only two independent renormalized parameters, a renormalized fermion mass and a renormalized dimensionless Yukawa coupling.

Postponing proof of this until sect. 5, we now outline how all divergences can be absorbed in the renormalized parameters to every order in our expansion.

The Schwinger-Dyson equations of the model can be derived in a straightforward way from the eq. (2.6) in Minkowski space. By differentiating (2.6) with respect to the sources using the explicit form of $F(\sigma, \pi)$ we find the following Green function equations

$$\left(i\partial\!\!\!/ + g_0(\sigma(x) + i\gamma_5\pi(x)) + \epsilon g_0\left(\left(-\frac{i\delta}{\delta S(x)}\right) + i\gamma_5 - \left(\frac{i\delta}{\delta J(x)}\right)\right)\right)\psi(x) + \eta(x) = 0 \,, \tag{4.4a}$$

$$\mu_0^2\sigma(x) = S(x) + g_0 i \operatorname{tr} G(x, x) \,, \tag{4.4b}$$

$$\mu_0^2\pi(x) = J(x) + g_0 i \operatorname{tr} i\gamma_5 G(x, x) \,. \tag{4.4c}$$

Eqs. (4.4b) and (4.4c) are displayed with Fermi sources off. We have lost no generality from this as all Green functions with the sources off can be derived by differentiating these equations [5]. We find for the Green functions with superficial divergence the following:

$$G^{-1}(p) = p\!\!\!/ + g_0\sigma + ig_0^2\epsilon\int\frac{d^4k}{(2\pi)^4}G(k+p)\,\Delta_\sigma(k)\,\Gamma_\sigma(k+p, k)$$

$$+ ig_0^2\epsilon\int\frac{d^4k}{(2\pi)^4}i\gamma_5 G(k+p)\,\Delta_\pi(k)\,\Gamma_\pi(k+p, k) \,, \tag{4.5}$$

$$\Delta_\sigma^{-1}(p^2) = -u_0^2 + ig_0^2 \operatorname{tr}\int\frac{d^4k}{(2\pi)^4}G(k+p)\,\Gamma_\sigma(k+p, k)\,G(k) \,, \tag{4.6}$$

$$\Delta_\pi^{-1}(p^2) = -\mu_0^2 + ig_0^2 \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} i\gamma_5 G(k+p)\, \Gamma_\pi(k+p,k)\, G(k)\,, \tag{4.7}$$

$$\Gamma_\sigma(k,k+p) = 1 + ig_0^2 \epsilon \frac{\delta}{\delta g_0 \sigma} \left[\int G\Delta_\sigma\Gamma_\sigma + \int i\gamma_5 G\Delta_\pi\Gamma_\pi \right]\,, \tag{4.8}$$

$$\Gamma_\pi(k,k+p) = i\gamma_5 + ig_0^2 \epsilon \frac{\delta}{\delta g_0 \pi} \left[\int G\Delta_\sigma\Gamma_\sigma + \int i\gamma_5 G\Delta_\pi\Gamma_\pi \right]\,, \tag{4.9}$$

$$\Gamma_{\pi\pi\sigma} = ig_0^3 \frac{\delta}{\delta g_0 \sigma} \left[\operatorname{tr} \int i\gamma_5 G\Gamma_\pi G \right]\,, \tag{4.10}$$

$$\Gamma_{\sigma\sigma\sigma} = ig_0^3 \frac{\delta}{\delta g_0 \sigma} \left[\operatorname{tr} \int G\Gamma_\sigma G \right]\,, \tag{4.11}$$

$$\Gamma_{\sigma\sigma\sigma\sigma} = ig_0^4 \frac{\delta^2}{\delta (g_0 \sigma)^2} \left[\operatorname{tr} \int G\Gamma_\sigma G \right]\,, \tag{4.12}$$

$$\Gamma_{\pi\pi\pi\pi} = ig_0^4 \frac{\delta^2}{\delta (g_0 \pi)^2} \left[\operatorname{tr} \int i\gamma_5 G\Gamma_\pi G \right]\,, \tag{4.13}$$

$$\Gamma_{\pi\pi\sigma\sigma} = ig_0^4 \frac{\delta^2}{\delta (g_0 \sigma)^2} \left[\operatorname{tr} \int i\gamma_5 G\Gamma_\pi G \right]\,. \tag{4.14}$$

The sources S and J are off in this set of equations.

Next we start our subtraction procedure. The number of necessary subtractions is dictated by the degree of superficial divergence.

The inverse fermion propagator (4.5) is of the form

$$G^{-1}(p) = \not{p} + g_0\sigma + \sum(p)\,.$$

The fermion self energy is in general

$$\sum(p) = \not{p}A(p^2) + B(p^2)\,.$$

Subtracting once at momentum p_0 and consulting (4.2) leads to the following

$$G^{-1}(p) = [\not{p} - m + Z_2 \operatorname{sub}^2_{p_0} \sum(p)]\, Z_2^{-1}\,,$$

where

$$\operatorname{sub}^2_{p_0} \sum(p) \equiv \not{p}(A(p^2) - A(p_0^2)) + (B(p^2) - B(p_0^2))\,.$$

Thus the renormalized fermion propagator,

$$\bar{G}^{-1} = Z_2 G^{-1}\,,$$

is defined as

$$\bar{G}^{-1}(p) = \not{p} - m + Z_2 i g_0^2 \,\mathrm{sub}^2_{p_0}\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4} G\Delta_\sigma\Gamma_\sigma + \int \frac{\mathrm{d}^4k}{(2\pi)^4} i\gamma_5 G\Delta_\pi\Gamma_\pi\right] . \tag{4.15}$$

The inverse σ-propagator $\Delta_\sigma^{-1}(p^2) = -\mu_0^2 + \Pi_\sigma(p^2)$ must be subtracted twice since it contains quadratic superficial divergence. Thus, consulting (4.2) we obtain

$$\Delta_\sigma(p^2) = Z_\sigma^{-1}[p^2 - \mu^2 + Z_\sigma \,\mathrm{sub}^2_{p_0^2}\Pi_\sigma(p^2)] ,$$

where

$$\mathrm{sub}^2_{p_0^2}\, \Pi_\sigma(p^2) = \Pi_\sigma(p^2) - \Pi_\sigma(p_0^2) - (p^2 - p_0^2)\left(\frac{\partial \Pi_\sigma}{\partial p^2}\right)_{p_0^2} .$$

Thus, the renormalized σ-propagator

$$\bar{\Delta}_\sigma^{-1} = Z_\sigma \Delta_\sigma^{-1}$$

is found to be

$$\bar{\Delta}_\sigma^{-1}(p^2) = p^2 - \mu_\sigma^2 + Z_\sigma i g_0^2 \,\mathrm{sub}^2_{p_0^2}\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{tr}\, G(k)\,\Gamma_\sigma(k, k+p)\, G(k+p)\right] . \tag{4.16}$$

The π-propagator is subtracted in the same way. The renormalized π-propagator

$$\bar{\Delta}_\pi^{-1} = Z_\pi \Delta_\pi^{-1}$$

is

$$\bar{\Delta}_\pi^{-1}(p^2) \equiv p^2 - \mu_\pi^2 + Z_\pi i g_0^2 \,\mathrm{sub}^2_{p_0^2}\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{tr}\, i\gamma_5 G(k)\,\Gamma_\pi(k, k+p)\, G(k+p)\right] \tag{4.17}$$

Renormalized scalar and pseudoscalar fields are defined to be

$$\sigma = Z_\sigma^{1/2}\bar{\sigma} , \tag{4.18a}$$

$$\pi = Z_\pi^{1/2}\,\bar{\pi} . \tag{4.18b}$$

The renormalized fermion vertices are subtracted once to obtain

$$\begin{aligned}\Gamma_\sigma(p_1, p_2) &= 1 + i g_0^2\, \epsilon \frac{\delta}{\delta g_0 \sigma}\left[\int G\Delta_\sigma\Gamma_\sigma + \int i\gamma_5 G\Delta_\pi\Gamma_\pi\right] \\ &= \frac{1}{Z_\sigma(1)} + i g_0^2 \epsilon \,\mathrm{sub}^1_{p_0, -p_0}\left\{\frac{\delta}{\delta g_0 \sigma}\left[\int G\Delta_\sigma\Gamma_\sigma + \int i\gamma_5 G\Delta_\pi\Gamma_\pi\right]\right\} \\ &= Z_\sigma^{-1}(1)\,\bar{\Gamma}_\sigma(p_1, p_2) = Z_\sigma^{-1}(1)\left[1 + i g_0^2 \epsilon Z_\sigma(1)\,\mathrm{sub}^1_{p_0, -p_0}\left\{\frac{\delta}{\delta g_0 \sigma}\left[\int G\Gamma_\sigma\Delta_\sigma\right.\right.\right. \\ &\quad \left.\left.\left. + \int i\gamma_5 G\Delta_\pi\Gamma_\pi\right]\right\}\right] .\end{aligned} \tag{4.19}$$

Similarly

$$\Gamma_\pi = Z_\pi(1)^{-1}\bar{\Gamma}_\pi = Z_\pi(1)^{-1}\Big[i\gamma_5 + ig_0^2\epsilon Z_\pi(1)\,\mathrm{sub}^1_{p_0,-p_0}\Big\{\frac{\delta}{\delta g_0\pi}\,[\textstyle\int G\Gamma_\sigma\Delta_\sigma$$

$$+\textstyle\int i\gamma_5 G\Gamma_\pi\Delta_\pi]\Big\}\Big]\,. \tag{4.20}$$

The three- and four-boson vertex functions have logarithmic superficial divergence and must be subtracted once. Thus, we have

$$\Gamma_{\pi\pi\sigma} = ig_0^3\,\frac{\delta}{\delta g_0\sigma}\,\mathrm{tr}(i\gamma_5 G\Gamma_\pi G)$$

$$= \frac{m\lambda_{\pi\pi\sigma}}{3Z_\pi Z_\sigma^{1/2}} + ig_0^3\,\mathrm{sub}^1_{p_0,-p_0}\Big\{\frac{\delta}{\delta g_0\sigma}\mathrm{tr}(i\gamma_5 G\Gamma_\pi G)\Big\}$$

$$= Z_\pi^{-1}Z_\sigma^{-1/2}\bar{\Gamma}_{\pi\pi\sigma} = Z_\pi^{-1}Z_\sigma^{-1/2}\,\tfrac{1}{3}m\lambda_{\pi\pi\sigma}$$

$$+ ig_0^3 Z_\pi Z_\sigma^{1/2}\,\mathrm{sub}^1_{p_0,-p_0}\Big\{\frac{\delta}{\delta g_0\sigma}\,\mathrm{tr}(i\gamma_5 G\Gamma_\pi G)\Big\}\,. \tag{4.21}$$

Similarly:

$$\Gamma_{\sigma\sigma\sigma} = Z_\sigma^{-3/2}\bar{\Gamma}_{\sigma\sigma\sigma} = Z_\sigma^{-3/2}\Big[m\lambda_{\sigma\sigma\sigma} + ig_0^3 Z_\sigma^{3/2}\,\mathrm{sub}^1_{p_0,-p_0}\Big\{\frac{\delta}{\delta g_0\sigma}\,\mathrm{tr}\,(G\Gamma_\sigma G)\Big\}\Big]\,, \tag{4.22}$$

$$\Gamma_{\pi\pi\pi\pi} = Z_\pi^{-2}\bar{\Gamma}_{\pi\pi\pi\pi} = Z_\pi^{-2}\Big[-\lambda_{\pi\pi\pi\pi} + ig_0^4 Z_\pi^2\,\mathrm{sub}^1_{p_0,-p_0,0}\Big\{\frac{\delta^2}{\delta(g_0\pi)^2}\,\mathrm{tr}\,(i\gamma_5 G\Gamma_\pi G)\Big\}\Big]\,, \tag{4.23}$$

$$\Gamma_{\sigma\sigma\sigma\sigma} = Z_\sigma^{-2}\bar{\Gamma}_{\sigma\sigma\sigma\sigma} = Z_\sigma^{-2}\Big[-\lambda_{\sigma\sigma\sigma\sigma} + ig_0^4 Z_\sigma^2\,\mathrm{sub}^1_{p_0,-p_0,0}\Big\{\frac{\delta^2}{\delta(g_0\sigma)^2}\,\mathrm{tr}\,(G\Gamma_\sigma G)\Big\}\Big]\,, \tag{4.24}$$

$$\Gamma_{\sigma\sigma\pi\pi} = Z_\sigma^{-1}Z_\pi^{-1}\bar{\Gamma}_{\sigma\sigma\pi\pi} = Z_\sigma^{-1}Z_\pi^{-1}\Big[-\tfrac{1}{3}\lambda_{\pi\pi\sigma\sigma}$$

$$+ ig_0^4 Z_\sigma Z_\pi\,\mathrm{sub}^1_{p_0,-p_0,0}\Big\{\frac{\delta^2}{\delta(g_0\sigma)^2}\,\mathrm{tr}(i\gamma_5 G\Gamma_\pi G)\Big\}\Big]\,. \tag{4.25}$$

Our next major step will be to indicate that (4.15)–(4.17) and (4.19)–(4.25) are finite functions of the momenta and the renormalized parameters, i.e., all the cutoff dependence has been absorbed in the renormalized parameters. Our proof will only be lacking in a detailed discussion of the overlapping divergences through displaying the renormalized Green functions in a form manifestly free of overlaps. We will go most of the way towards this goal by removing the bare vertices 1, and $i\gamma_5$ from our Green functions. This will, in fact, produce functions which are adequate for calculational purposes and properly display the parameterization of the theory. We will provide more details following the procedures of ref. [8] in a work more specifically aimed at the study of the Schwinger Dyson equations [10].

We observe that (4.19) can be written as

$$\Gamma_\sigma = 1 + ig_0^2\epsilon \frac{\delta}{\delta(g_0\sigma)}[\int G\Gamma_\sigma\Delta_\sigma + \int i\gamma_5 G\Gamma_\pi\Delta_\pi] .$$

Introducing quantities I_σ and I_π this becomes

$$\Gamma_\sigma(\xi) \equiv 1(\xi')[\delta(\xi' - \xi) - \epsilon g_0^2[\Delta_\sigma(\xi'\xi'')\, GI_\sigma(\xi''\xi) + \Delta_\pi(\xi'\xi'')\, i\gamma_5 GI_\pi(\xi''\xi)]] . \tag{4.26a}$$

Here we have explicitly displayed the coordinate associated with the propagation of the bound states. We have used an extended matrix notation and suppressed the coordinate and spin indices associated with the Fermi field. The only information relevant to our discussion at this point is that I_σ and I_π, which are functions of four coordinate and two Fermi matrix indices, are related to meson and fermion scattering to meson and fermion and hence by our earlier arguments are superficially finite. The only divergences they contain come from the divergences of other Green functions. It is the use of this fact and the similar set of conclusions from an analysis of $\Gamma_\pi(\epsilon)$ which allows us to eliminate the overlaps in a full analysis. For our purposes here we observe that we can write

$$\Gamma_\sigma = 1 + ig_\sigma^2 \frac{\delta}{\delta(g_\sigma\bar{\sigma})}[\int \bar{G}\bar{\Gamma}_\sigma\bar{\Delta}_\sigma]$$
$$+ \frac{ig_\pi^2 g_\sigma\epsilon}{g_\sigma}\frac{\delta}{\delta(g_\sigma\bar{\sigma})}[\int i\gamma_5\bar{G}\bar{\Gamma}_\pi\bar{\Delta}_\pi] ,$$

which implies that

$$1 = Z_\sigma(1)^{-1}\bar{\Gamma}_\sigma\bar{B}_\sigma , \tag{4.26}$$

where we have made the definition

$$\bar{B}_\sigma \equiv \left\{1 + ig_\sigma^2\frac{\delta}{\delta(g_\sigma\bar{\sigma})}(\bar{G}\bar{\Gamma}_\sigma\bar{\Delta}_\sigma) + \frac{ig_\pi^2 g_\sigma}{\hat{g}_\sigma}\frac{\delta}{\delta(g_\sigma\bar{\sigma})}(i\gamma_5\bar{G}\bar{\Gamma}_\pi\bar{\Delta}_\pi)\right\}^{-1} ,$$

with $\bar{B}_\sigma$ only a function of renormalized quantities. Of course, B_σ is not finite but its divergences are associated with those of Γ_σ and will be removed in a subtraction and so cause no harm. In an analogous fashion we can obtain

$$i\gamma_5 = Z_\pi(1)_\pi^{-1}\overline{\Gamma}_\pi\overline{B}_\pi \,. \tag{4.27}$$

We are now ready to examine the renormalized functions and prove what we have promised, i.e., their dependence only on renormalized parameters.
The fermion propagator is

$$\overline{G}^{-1}(p) = p - m + ig_0^2\epsilon Z_2 \,\mathrm{sub}^2_{p_0}\left\{Z_2 Z_\sigma Z_\sigma(1)^{-2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\overline{\Gamma}_\sigma\overline{B}_\sigma\overline{G}\overline{\Delta}_\sigma\overline{\Gamma}_\sigma\right.$$
$$\left.+ Z_2 Z_\pi Z_\pi(1)^{-2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\overline{\Gamma}_\pi\overline{B}_\pi\overline{G}\overline{\Delta}_\pi\overline{\Gamma}_\pi\right\} ,$$

or using the definition of the renormalized parameter

$$\overline{G}^{-1}(p) = \not p - m + ig_\sigma^2\epsilon\,\mathrm{sub}^2_{p_0}\left\{\int\overline{\Gamma}_\sigma\overline{B}_\sigma\overline{G}\overline{\Delta}_\sigma\overline{\Gamma}_\sigma\right\}$$
$$+ ig_\pi^2\epsilon\,\mathrm{sub}^2_{p_0}\left\{\int\overline{\Gamma}_\pi\overline{B}_\pi\overline{G}\overline{\Delta}_\pi\overline{\Gamma}_\pi\right\} . \tag{4.28}$$

The σ-propagator is

$$\overline{\Delta}_\sigma^{-1} = p^2 - \mu_\sigma^2 + ig_0^2 Z_\sigma Z_2^2 Z_\sigma(1)^{-2}\,\mathrm{sub}^2_{p_0^2}\left[\mathrm{tr}\int\overline{\Gamma}_\sigma\overline{B}_\sigma\overline{G}\overline{\Gamma}_\sigma\overline{G}\right]$$

or

$$\overline{\Delta}_\sigma^{-1}(p^2) = p^2 - \mu_\sigma^2 + ig_\sigma^2\,\mathrm{sub}^2_{p_0^2}\left[\mathrm{tr}\int\overline{\Gamma}_\sigma\overline{B}_\sigma\overline{G}\overline{\Gamma}_\sigma\overline{G}\right] . \tag{4.29}$$

Similarly the π-propagator is

$$\overline{\Delta}_\pi^{-1}(p^2) = p^2 - \mu_\pi^2 + ig_\pi^2\,\mathrm{sub}^2_{p_0^2}\left\{\mathrm{tr}(\overline{\Gamma}_\pi\overline{B}_\pi\overline{G}\overline{\Gamma}_\pi\overline{G})\right\} . \tag{4.30}$$

The fermion vertices are

$$\overline{\Gamma}_\sigma = 1 + ig_\sigma^2\epsilon\,\mathrm{sub}^1_{p_0,-p_0}\left\{\frac{\delta}{\delta(g_\sigma\overline{\sigma})}(\overline{\Gamma}_\sigma\overline{B}_\sigma\overline{G}\overline{\Delta}_\sigma\overline{\Gamma}_\sigma)\right\}$$
$$+ ig_\pi\hat{g}_\pi\epsilon\,\mathrm{sub}^1_{p_0,-p_0}\left\{\frac{\delta}{\delta(g_\sigma\overline{\sigma})}(\overline{\Gamma}_\pi\overline{B}_\pi\overline{G}\overline{\Delta}_\pi\overline{\Gamma}_\pi)\right\} , \tag{4.31}$$

$$\overline{\Gamma}_\pi = i\gamma_5 + ig_\pi^2\epsilon\,\mathrm{sub}^1_{p_0,-p_0}\left\{\frac{\delta}{\delta(g_\pi\overline{\pi})}(\overline{\Gamma}_\pi\overline{B}_\pi\overline{G}\overline{\Delta}_\pi\overline{\Gamma}_\pi)\right\}$$
$$+ ig_\sigma^2\epsilon\,\mathrm{sub}^1_{p_0,-p_0}\left\{\frac{\delta}{\delta(g_\pi\overline{\pi})}(\overline{\Gamma}_\sigma\overline{B}_\sigma\overline{G}\overline{\Gamma}_\sigma\overline{\Delta}_\sigma)\right\} . \tag{4.32}$$

Finally, the boson vertex functions are:

$$\overline{\Gamma}_{\pi\pi\sigma} = \tfrac{1}{3}m\lambda_{\pi\pi\sigma} + ig_\pi^2 g_\sigma\,\mathrm{sub}^1_{p_0,-p_0}\left\{\frac{\delta}{\delta(g_\sigma\overline{\sigma})}\ \mathrm{tr}(\overline{\Gamma}_\pi\overline{B}_\pi\overline{G}\overline{\Gamma}_\pi\overline{G}\,)\right\} , \tag{4.33}$$

$$\bar{\Gamma}_{\sigma\sigma\sigma} = m\lambda_{\sigma\sigma\sigma} + ig_\sigma^3 \, \mathrm{sub}^1_{p_0,-p_0} \left\{ \frac{\delta}{\delta(g_\sigma\bar{\sigma})} \, \mathrm{tr}(\bar{\Gamma}_\sigma \bar{B}_\sigma \bar{G} \bar{\Gamma}_\sigma \bar{G}) \right\} , \tag{4.34}$$

$$\bar{\Gamma}_{\sigma\sigma\sigma\sigma} = -\lambda_{\sigma\sigma\sigma\sigma} + ig_\sigma^4 \, \mathrm{sub}^1_{p_0,-p_0,0} \left\{ \frac{\delta^2}{\delta(g_\sigma\bar{\sigma})^2} \, \mathrm{tr}(\bar{\Gamma}_\sigma \bar{B}_\sigma \bar{G} \bar{\Delta}_\sigma \bar{G}) \right\} , \tag{4.35}$$

$$\bar{\Gamma}_{\sigma\sigma\pi\pi} = -\tfrac{1}{3}\lambda_{\pi\pi\sigma\sigma} + ig_\sigma^2 g_\pi^2 \, \mathrm{sub}^1_{p_0,-p_0,0} \left\{ \frac{\delta^2}{\delta(g_\pi\bar{\pi})^2} \, \mathrm{tr}(\bar{\Gamma}_\sigma \bar{B}_\sigma \bar{G} \bar{\Delta}_\sigma \bar{G}) \right\} , \tag{4.36}$$

$$\bar{\Gamma}_{\pi\pi\pi\pi} = -\lambda_{\pi\pi\pi\pi} + ig_\pi^4 \, \mathrm{sub}^1_{p_0,-p_0,0} \left\{ \frac{\delta^2}{\delta(g_\pi\bar{\pi})^2} \, \mathrm{tr}(\bar{\Gamma}_\pi \bar{B}_\pi \bar{G} \bar{\Gamma}_\pi \bar{G}) \right\} . \tag{4.37}$$

Thus, we have succeeded in expressing the renormalized functions only in terms of renormalized quantities. This shows that the renormalized functions are finite functions of the momenta and the renormalized parameter if the overlaps are not a problem as in fact is the case.

A further important observation that we can make here is that the parameter ϵ in these Green function equations always multiplies one of the coupling parameters. Thus, if we let $\epsilon \to 1$, we can keep an effective small expansion parameter by making all the g's approach zero. Note that with $\epsilon = 1$ and expanding with the g's small, the renormalized Green functions look identical to the renormalized coupling constant expansion of a conventional σ model. The esoteric aspects of the mean field expansion caused by the re-ordering of terms relative to the coupling constant expansion are gone. What remains and is exciting are the relations among renormalized parameters which are a consequence of how the theory was generated. We will display these relationships in sect. 5 while sect. 6 will develop the relationship to conventional theories further.

5. The renormalized parameters

The renormalized parameters of the theory are not all independent. As we promised we shall show that the independent renormalized parameters are only a mass parameter and a dimensionless coupling, for example m and g_σ^2.

Let us consider the identity (A.12) that is derived in the appendix

$$(g_0\sigma)\,\Gamma_{\pi\sigma}(p,-p) = \tfrac{1}{2}\{i\gamma_5, \Gamma_\sigma(p,-p)\} - \Gamma_\pi(p,-p) .$$

At the subtraction momentum the above gives

$$(g_\sigma\bar{\sigma}) \left(\frac{\delta \bar{G}^{-1}}{\delta(g_\sigma\bar{\sigma})\,\delta(g_\pi\bar{\pi})} \right)_{p_0,-p_0} = i\gamma_5 \left(\frac{Z_\pi(1)}{Z_\sigma(1)} - 1 \right) , \tag{5.1}$$

which leads to

$$\frac{Z_\pi(1)}{Z_\sigma(1)} = 1 - i\gamma_5 \left(\frac{\delta \bar{G}^{-1}}{\delta(g_\sigma\bar{\sigma})\,\delta(g_\pi\bar{\pi})} \right)_{p_0,-p_0} (g_\sigma\bar{\sigma}) . \tag{5.2}$$

Thus, the ratio of the two vertex renormalization factors is a function of the renormalized parameters of the theory and the subtraction momentum. Since $\delta G^{-1}/\delta\sigma\delta\pi$ is superficially finite, this ratio is a finite function of the renormalized parameters:

$$\frac{Z_\pi(1)}{Z_\sigma(1)} \equiv 1 + \alpha(m, \mu_\sigma^2, g^2 \dots, \lambda \dots; p_0^2)\,. \tag{5.3}$$

Next let us examine (A.11) at the subtraction momentum:

$$2(g_0\sigma)\,\Gamma_\pi(p_0, -p_0) = \{G^{-1}(p_0), i\gamma_5\}\,.$$

The renormalization conditions imply that

$$2\frac{Z_\sigma(1)}{Z_\pi(1)}(g_\sigma\bar{\sigma})\,i\gamma_5 = \{\not{p}_0 - m, i\gamma_5\} = ip_0^\mu\{\gamma_\mu, \gamma_5\} - 2mi\gamma_5\,.$$

Hence,

$$g_\sigma\bar{\sigma} = -m(1+\alpha)\,. \tag{5.4}$$

The $\pi\pi\sigma$ vertex obeys the following identity (see appendix)

$$\sigma\Gamma_{\pi\pi\sigma}(p, -p) = \Delta_\sigma^{-1}(p^2) - \Delta_\pi^{-1}(p^2)\,.$$

Expanding both sides around the subtraction momentum, we obtain

$$\sigma\left(\Gamma_{\pi\pi\sigma}(p_0, -p_0) + \left(\frac{\partial\Gamma_{\pi\pi\sigma}}{\partial p^2}\right)_{p_0^2}(p^2 - p_0^2) + \dots\right)$$

$$= \Delta_\sigma^{-1}(p_0^2) - \Delta_\pi^{-1}(p_0^2) + \left(\left(\frac{\partial\Delta_\sigma^{-1}}{\partial p^2}\right)_{p_0^2} - \left(\frac{\partial\Delta_\pi^{-1}}{\partial p^2}\right)_{p_0^2}\right)(p^2 - p_0^2) + \dots\,.$$

Equating term by term we are led to the following equations:

$$\sigma\Gamma_{\pi\pi\sigma}(p_0, -p_0) = \Delta_\sigma^{-1}(p_0^2) - \Delta_\pi^{-1}(p_0^2)\,, \tag{5.5}$$

$$\sigma\left(\frac{\partial\Gamma_{\pi\pi\sigma}}{\partial p^2}\right)_{p_0^2} = \left(\frac{\partial\Delta_\sigma}{\partial p^2}\right)_{p_0^2} - \left(\frac{\partial\Delta_\pi^{-1}}{\partial p^2}\right)_{p_0^2}\,. \tag{5.6}$$

Consulting the renormalization conditions we can write the second equation as

$$\frac{Z_\pi}{Z_\sigma} = 1 + \bar{\sigma}\left(\frac{\partial\bar{\Gamma}_{\pi\pi\sigma}}{\partial p^2}\right)_{p_0^2}\,. \tag{5.7}$$

This means that the ratio of the two renormalization factors is always a function of renormalized parameters. It is going to be a finite function since $\partial\Gamma_{\pi\pi\sigma}/\partial p^2$ is superficially finite. Thus, we can write

$$\beta \equiv \frac{Z_\pi}{Z_\sigma} = 1 - \frac{m}{g_\sigma}(1+\alpha)\left(\frac{\partial\bar{\Gamma}_{\pi\pi\sigma}}{\partial p^2}\right)_{p_0^2}\,. \tag{5.8}$$

At this point it is obvious that there is only one independent Yukawa coupling since

$$\frac{g_\sigma^2}{g_\pi^2} = (1+\alpha)^2\beta^{-1} , \qquad \frac{g_\sigma^2}{\hat{g}_\sigma^2} = \frac{\hat{g}_\pi^2}{g_\pi^2} = (1+\alpha)^2 .$$

Eq. (5.5), on the other hand, gives

$$\bar{\sigma}\frac{m\lambda_{\pi\pi\sigma}}{3Z_\pi} = \frac{p_0^2-\mu_\sigma^2}{Z_\sigma} - \frac{p_0^2-\mu_\pi^2}{Z_\pi} ,$$

which becomes, using (5.4) and (5.8),

$$\frac{-m^2(1+\alpha)\,\lambda_{\pi\pi\sigma}}{3g_\sigma} = (p_0^2-\mu_0^2)\,\beta - (p_0^2-\mu_\pi^2) . \tag{5.10}$$

This equation can serve to fix the mass μ_σ^2 in terms of the fermion mass and the dimensionless couplings. The mass μ_π^2 is always fixed by the Goldstone theorem (A.5). If we subtract at zero momentum ($p_0^2 = 0$), we have $\mu_\pi^2 = 0$ and

$$\mu_\sigma^2 = \frac{m^2(1+\alpha)\,\lambda_{\pi\pi\sigma}}{3g_\sigma\beta} . \tag{5.11}$$

The Ward identities (A.7)–(A.10), at the subtraction momentum p_0, take the form

$$\sigma\Gamma_{\pi\pi\sigma\sigma}(p_0, -p_0, 0) = \Gamma_{\sigma\sigma\sigma}(p_0, -p_0) - 2\Gamma_{\pi\pi\sigma}(p_0, -p_0) ,$$

$$\sigma\Gamma_{\pi\pi\pi\pi}(p_0, -p_0, 0) = 3\Gamma_{\pi\pi\sigma}(p_0, -p_0) ,$$

$$\sigma\Gamma_{\pi\pi\pi\pi\sigma}(p_0, -p_0, 0, 0) = 3\Gamma_{\pi\pi\sigma\sigma}(p_0, -p_0, 0) - \Gamma_{\pi\pi\pi\pi}(p_0, -p_0, 0) ,$$

$$\sigma\Gamma_{\pi\pi\sigma\sigma\sigma}(p_0, -p_0, 0, 0) = \Gamma_{\sigma\sigma\sigma\sigma}(p_0, -p_0, 0) - 3\Gamma_{\pi\pi\sigma\sigma}(p_0, -p_0, 0) .$$

Using the renormalization conditions we immediately obtain from these:

$$\frac{(1+\alpha)}{3g_\sigma} = \lambda_{\sigma\sigma\sigma}\beta - \tfrac{2}{3}\lambda_{\pi\pi\sigma} , \tag{5.12}$$

$$\lambda_{\pi\pi\pi\pi} = \frac{g_\sigma\beta}{(1+\alpha)}\,\lambda_{\pi\pi\sigma} , \tag{5.13}$$

$$\lambda_{\pi\pi\sigma\sigma} - \beta\lambda_{\sigma\sigma\sigma\sigma} = -\frac{m}{g_\sigma}(1+\alpha)\,\bar{\Gamma}_{\pi\pi\sigma\sigma\sigma}(0, 0, p_0, -p_0) , \tag{5.14}$$

$$\Gamma_{\pi\pi\pi\pi}\beta\lambda_{\pi\pi\sigma\sigma} = -\frac{m}{g_\sigma}(1+\alpha)\,\bar{\Gamma}_{\pi\pi\pi\pi\sigma}(0, 0, p_0, -p_0) . \tag{5.15}$$

The last four equations leave us with one independent boson self-coupling. Let us choose $\lambda_{\pi\pi\pi\pi}$ as the independent boson self-coupling. We shall show that it is in fact a function of the Yukawa couplings.

We will not display the calculations here but it is fairly straightforward to calculate the lowest-order Green functions at zero momentum using (4.4b) and (4.4c) with $S(x)$ and $J(x)$ constant. Careful analysis then shows there are only two independent renormalized quantities, say g_π^2 and m to this order. The detailed method to do this calculation is developed in ref. [7]. We exploit this observation to construct the argument that to all orders this theory has only two independent parameters in what follows.

We start by considering

$$\delta\Gamma_{\pi\pi\pi\pi}/\delta\sigma = \Gamma_{\pi\pi\pi\pi\sigma} \,.$$

For constant σ field, the variational derivative becomes an ordinary derivative and we obtain the differential equation

$$\sigma\frac{\mathrm{d}\Gamma_{\pi\pi\pi\pi}(...)}{\mathrm{d}\sigma} = \sigma\Gamma_{\sigma\pi\pi\pi\pi}(0; ...) \,. \tag{5.16}$$

At the subtraction momentum p_0, the renormalization conditions imply

$$\sigma\frac{\mathrm{d}}{\mathrm{d}\sigma}(-\lambda_{\pi\pi\pi\pi}Z_\pi^{-2}) = Z_\pi^2\bar{\sigma}\bar{\Gamma}_{\sigma\pi\pi\pi\pi}(0, 0, p_0 - p_0) \,,$$

or

$$\left(4\gamma_\pi + \sigma\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)\lambda_{\pi\pi\pi\pi} = \bar{\sigma}\bar{\Gamma}_{\sigma\pi\pi\pi\pi}(0, 0, p_0, p_0) \,. \tag{5.17}$$

We have introduced the "anomalous dimension" of the pseudoscalar field defined as

$$\gamma_\pi \equiv \sigma\,\partial \ln Z_\pi/\partial\sigma \,.$$

γ_π is a finite function of the dimensionless couplings. The operator $\sigma\mathrm{d}/\mathrm{d}\sigma$ appearing in (5.17) can be expanded in terms of the independent renormalized parameters m and g_π^2. It is more convenient here to use g_π^2 instead of any other Yukawa coupling as our fundamental coupling. Defining the β function of the g_π^2 coupling as usual as

$$\left(4\gamma_\pi + \alpha m\frac{\partial}{\partial m} + \hat{\alpha}p_0^2\frac{\partial}{\partial p_0^2} - \beta_{g_\pi}(g_\pi^2)\frac{\partial}{\partial g_\pi^2}\right)\lambda_{\pi\pi\pi\pi} = \bar{\sigma}\bar{\Gamma}_{\sigma\pi\pi\pi\pi}(0, 0, p_0, -p_0) \,,$$

we have

$$\beta(g_\pi^2) \equiv \sigma\,\partial g_\pi^2/\partial\sigma \,.$$

Note that we did not put in a term of the form

$$-\beta(\lambda')\frac{\partial}{\partial\lambda'}\lambda_{\pi\pi\pi\pi}$$

on the left-hand side of this equation. Let us explain this with $p_0 = 0$. This term is not present because in lowest order only g_π and m are independent parameters. Because γ, β, etc., are finite functions of the renormalized quantities and because we have an iteration scheme, we can always construct one more order out of a given order in which γ, β, etc., depend only on g_π and m. Thus we find that in this next order γ, β, etc., depend only on g_π and m. Then by an induction argument on the above equation with the omitted term added we can conclude to any order that $\lambda_{\pi\pi\pi\pi}$ is fixed in terms of g_π^2. If our subtraction point is zero, since $\lambda_{\pi\pi\pi\pi}$ is dimensionless, the mass derivative does not contribute and we have

$$\left(4\gamma_\pi - \beta_g(g_\pi^2)\frac{\partial}{\partial g_\pi^2}\right)\lambda_{\pi\pi\pi\pi} = \bar{\sigma}\bar{\Gamma}_{\sigma\pi\pi\pi\pi}(0,0,0,0)\,. \tag{5.18}$$

To lowest order in g_π it is easy to show that

$$\beta_{g_\pi} = \frac{g_\pi^4}{4\pi^2}\,, \qquad \gamma_\pi = \frac{1}{2g_\pi^2}\beta_{g_\pi} = \frac{g_\pi^2}{8\pi^2}\,, \qquad \Gamma^{(4\pi,\sigma)}(0,0,0,0) = -\frac{3g_0^5}{m\pi^2}\,.$$

Thus, our equation becomes

$$\left(1 - \tfrac{1}{2}g_\pi^2\frac{\partial}{\partial g_\pi^2}\right)\lambda_{\pi\pi\pi\pi} = 6g_\pi^2\,. \tag{5.19}$$

A solution is immediately obtained *

$$\lambda_{\pi\pi\pi\pi} = 12g_\pi^2\,. \tag{5.20}$$

This is, of course, identical to the result obtained if we had calculated directly [7]. The other couplings are easily computed from the Ward identities. For example (5.13) gives to lowest order

$$\lambda_{\pi\pi\sigma} = \frac{\lambda_{\pi\pi\pi\pi}}{\sigma^\beta} = \frac{12g_\pi^2}{g_\sigma\beta}\,.$$

To lowest order

$$\beta = \frac{g_\pi^2}{g_\sigma^2} = 1 - \frac{g_\pi^2}{12\pi^2} = \left(1 + \frac{g_\sigma^2}{12\pi^2}\right)^{-1}\,.$$

* The authors are grateful to the referee for pointing out that eq. (5.20) which relates the Yukawa and quartic boson couplings is exactly the same as in supersymmetry (see, for example, ref. [13]) in which case it has been proven that the relation is true in the bare theory and is preserved in the renormalized theory. In the case of supersymmetry this has been shown to hold for a single Majorana fermion while for our case it follows directly that such a relation holds for any number of fermion species.

Thus

$$\lambda_{\pi\pi\sigma} = 12 g_\sigma \, . \tag{5.21}$$

Eq. (5.11) for the bound-state mass becomes to lowest order

$$\mu_\sigma^2 = \frac{m^2 \lambda_{\pi\pi\sigma}}{3 g_\sigma \beta} = \frac{4m^2}{\beta} = 4m^2 \left(1 + \frac{g_\sigma^2}{12\pi^2}\right).$$

This is exactly what we obtained by explicit calculation in sect. 3.

It is clear now that the independent renormalized parameters of the theory are a fermion mass m and a Yukawa-type dimensionless coupling. By iteration of our equations we can calculate the other parameters to any order.

6. Equivalence with the σ model

Let us consider the theory described by the Lagrangian

$$L(\psi, \overline{\psi}, \sigma, \pi) = \overline{\psi}(i\partial\!\!\!/ + g_0(\sigma + i\gamma_5 \pi))\, \psi + \tfrac{1}{2}(\partial_\mu \sigma)^2 + \tfrac{1}{2}(\partial_\mu \pi)^2$$
$$- \tfrac{1}{2}\mu_0^2(\sigma^2 + \pi^2) - \frac{\lambda_0}{4!}(\sigma^2 + \pi^2)^2 \, . \tag{6.1}$$

The connected vacuum-to-vacuum amplitude after the integration of the fermion variables is, in Euclidean space,

$$e^W = \int d\sigma\, d\pi \exp\left\{- \int (\overline{\eta} G \eta - \mathrm{tr}\ln G^{-1} + \tfrac{1}{2}\sigma(-\partial^2 + \mu_0^2)\, \sigma\right.$$
$$\left. + \tfrac{1}{2}\pi(-\partial^2 + \mu_0^2)\, \pi + \frac{\lambda_0}{4!}(\sigma^2 + \pi^2)^2 - S\sigma - J\pi)\right\} . \tag{6.2}$$

Again, for convenience, we have introduced

$$G^{-1} = i\partial\!\!\!/ + g_0(\sigma + i\gamma_5 \pi) \, .$$

The reader should not confuse this G with the exact Green function. In context no confusion is likely.

The mean field expansion is defined in the same way as for the Nambu Jona-Lasinio model. A parameter ϵ is introduced. The connected functional becomes

$$e^{W_\epsilon/\epsilon} \equiv \int d\sigma\, d\pi \exp\{-F(\sigma, \pi)/\epsilon\} \, .$$

At the end of all computations ϵ must be set equal to one. The mean field conditions are the same as (2.7). The expanded functional is given again by (2.8). Of course, F is different.

The lowest-order functional is

$$W_0\{\sigma_0, \pi_0, \bar{\eta}, \eta, S, J\} = -F\{\sigma_0, \pi_0, \bar{\eta}, \eta, S, J\} = \int \Big[-\bar{\eta} G_0 \eta + \mathrm{tr}\ln G_0^{-1}$$

$$+ \tfrac{1}{2}\sigma_0(\partial^2 - \mu_0^2)\,\sigma_0 + \tfrac{1}{2}\pi_0(\partial^2 - \mu_0^2)\,\pi_0$$

$$- \frac{\lambda_0}{4!}(\sigma_0^2 + \pi_0^2)^2 + S\sigma_0 + J\pi_0 \Big] \,. \tag{6.3}$$

Classical fields can be defined in the usual way. To lowest order the boson classical fields coincide with the mean fields.

An effective action can be defined by the Legendre transformation

$$\Gamma_\epsilon\{\boldsymbol{\sigma}\} = W_\epsilon\{\boldsymbol{S}\} - \int \mathrm{d}^4 x \boldsymbol{\sigma}(x)\cdot \boldsymbol{S}(x)\,.$$

For convenience we have introduced

$$\boldsymbol{\sigma} = (\sigma, \pi)\,, \qquad \boldsymbol{\tau} = (1, i\gamma_5)\,, \qquad \boldsymbol{S} = (S, J)\,.$$

To lowest order we have

$$\Gamma_0\{\boldsymbol{\sigma}\} = \Gamma_0\{\boldsymbol{\sigma}_0\} = \int [\bar{\psi}_0 G_0^{-1}\psi_0 + \mathrm{tr}\ln(i\partial\!\!\!/ + g_0 \boldsymbol{\sigma}_0\cdot\boldsymbol{\tau}))$$

$$+ \tfrac{1}{2}\boldsymbol{\sigma}_0\ (\partial^2 - \mu_0^2)\,\boldsymbol{\sigma}_0 - \frac{\lambda_0}{4!}(\boldsymbol{\sigma}_0\cdot\boldsymbol{\sigma}_0)^2 \quad . \tag{6.4}$$

Translating fields by $\boldsymbol{\sigma}_0 = \boldsymbol{s}_0 + \boldsymbol{\upsilon}$, where $\boldsymbol{\upsilon} + (\upsilon, 0)$, we obtain

$$\Gamma_0\{\boldsymbol{s}_0\} = \int \Big[\bar{\psi}_0(i\partial\!\!\!/ + g_0\,\boldsymbol{\tau}\cdot\boldsymbol{\upsilon})\,\psi_0 - \sum_{j=2}^{\alpha} \frac{(-)^j}{j}\,\mathrm{tr}\Big(\frac{g_0 \boldsymbol{s}_0\cdot\boldsymbol{\tau}}{i\partial\!\!\!/ + g_0\,\boldsymbol{\tau}\cdot\boldsymbol{\upsilon}}\Big)^j$$

$$+ \tfrac{1}{2}\boldsymbol{s}_0\cdot(\partial^2 - \mu_0^2)\cdot\boldsymbol{s}_0 - \frac{\lambda_0}{4!}(\boldsymbol{s}_0\cdot\boldsymbol{s}_0)^2 + 2(\boldsymbol{s}_0\cdot\boldsymbol{s}_0)\,\upsilon^2 + 4(\boldsymbol{s}_0\cdot\boldsymbol{\upsilon})^2 \Big] \quad .$$

The linear terms cancel out because of the mean field condition

$$\delta\Gamma_0\{\boldsymbol{s}_0\}/\delta\boldsymbol{s}_0 = 0\,.$$

Thus,

$$\Gamma_0\{\boldsymbol{s}_0\} = \int \Big[\bar{\psi}_0(i\partial\!\!\!/ - m)\,\psi_0 + \tfrac{1}{2}g_0^2\, s_0^i \Delta_{ij}^{-1} s_0^j + \tfrac{1}{4}g_0^4\, s_0^i s_0^j s_0^k s_0^l \Gamma_{ijkl}^{(4s)}$$

$$+ \tfrac{1}{3}g_0^3\, s_0^i s_0^j s_0^k \Gamma_{ijk}^{(3s)} - \sum_{j=5}^{\infty} \frac{(-)^j}{j}\ \mathrm{tr}\Big(\frac{g_0 \boldsymbol{s}_0\cdot\boldsymbol{\tau}}{i\partial\!\!\!/ - m}\Big)^j \Big].$$

Let us define renormalized s fields by $\bar{s} = Z_\pi^{-1/2}\pi$, then

$$\Gamma\{s\} = \int \Big[\bar{\psi}_0 (i\partial\!\!\!/ - m)\psi_0 + \tfrac{1}{2}\bar{s}_i \bar{\Delta}_{ij}^{-1} \bar{s}_j + \tfrac{1}{3}\bar{s}_i\bar{s}_j\bar{s}_k \bar{\Gamma}_{ijk}^{(3s)} + \tfrac{1}{4}\bar{s}_i\bar{s}_j\bar{s}_k\bar{s}_l \bar{\Gamma}_{ijkl}^{(4s)} - \sum_{n=5}^{\infty} \frac{(-)^n}{n} \operatorname{tr}\Big(\frac{g\bar{s}\cdot\tau}{i\partial\!\!\!/ - m}\Big)^n \Big]. \tag{6.5}$$

The renormalized couplings are defined as

$$g = \begin{pmatrix} g_\sigma & 0 \\ 0 & g_\pi \end{pmatrix} = \begin{pmatrix} g_0 Z_\sigma^{1/2} & 0 \\ 0 & g_0 Z_\pi^{1/2} \end{pmatrix}.$$

The renormalized 1PI functions are given by

$$\bar{\Delta}_{ij}^{-1} = Z_i^{1/2} Z_j^{1/2} \Delta_{ij}^{-1},$$

$$\Gamma_{ijk}^{(3s)} = Z_i^{-1/2} Z_j^{-1/2} Z_k^{-1/2} \bar{\Gamma}_{ijk}^{(3s)},$$

$$\Gamma_{ijkl}^{(4s)} = Z_i^{-1/2} Z_j^{-1/2} Z_k^{-1/2} Z_l^{-1/2} \bar{\Gamma}_{ijkl}^{(4s)}.$$

Let us consider first the inverse s propagator

$$\Delta_s^{-1}(p^2) = p^2 - \mu_0^2 - \tfrac{1}{2}\lambda_0 v^2 + i g_0^2 \operatorname{tr}\int \frac{d^4k}{(2\pi)^4} (k\!\!\!/ + p\!\!\!/ - m)^{-1} (k\!\!\!/ + p\!\!\!/ - m)^{-1}.$$

Imposing the renormalization conditions

$$\Delta_s^{-1}(0) = -\frac{\mu_s^2}{Z_s}, \qquad \left(\frac{\partial \Delta_s^{-1}}{\partial p^2}\right)_0 = \frac{1}{Z_s},$$

and subtracting twice, we are led to the following definition for the renormalized propagator

$$\Delta_s^{-1}(p^2) = Z_s^{-1}\bar{\Delta}_s^{-1}(p^2) = Z_s^{-1}\Big[p^2 - \mu_s^2 + i g_s^2 \operatorname{sub}_0^2 \operatorname{tr}\int \frac{d^4k}{(2\pi)^4} (k\!\!\!/ + p\!\!\!/ - m)^{-1} \times (k\!\!\!/ - m)^{-1}\Big].$$

The renormalized s propagator is exactly the same as the renormalized propagator of the Nambu model. This is true for the other 1PI functions $\bar{\Gamma}_{ijk}^{(3s)}$ and $\bar{\Gamma}_{ijk}^{(4s)}$. They are exactly the same functions of the momenta and the renormalized parameters.

The Ward identities ((A.6)–(A.10)) of the broken chiral symmetry are the same for both models. Thus, the relations between the parameters obtained through them are valid in the case of the σ model. On the other hand the quartic boson coup ling is independent because the Callan-Symanzik equation that was used to fix it turns into an identity

$$\left(4\gamma_\pi - \beta_{g_\pi}(g_\pi^2)\frac{\partial}{\partial g_\pi^2} - \beta_{\lambda_{4\pi}}(g_\pi^2, \lambda_{4\pi})\frac{\partial}{\partial \lambda_{4\pi}}\right)\lambda_{4\pi} = \frac{3g_\pi^4}{\pi^2}. \tag{6.6}$$

The β-function of the quartic coupling is defined as usual

$$\beta_{\lambda_{4\pi}} \equiv \sigma \frac{\partial \lambda_{4\pi}}{\partial \sigma} = \text{to lowest order} = \frac{g_\pi^2}{2\pi^2}(\lambda_{4\pi} - 6g_\pi^2).$$

Substituting in (6.6) we obtain

$$0 = 0 .$$

Thus the independent renormalized parameters of the σ model are m, g_π^2 and $\lambda_{4\pi}$. All other parameters are determined. For example the s mass is

$$\frac{\mu_s^2}{4m^2} = \frac{\lambda_{\pi\pi s}}{12g_s}\left(1 + \frac{g_s^2}{12\pi^2}\right). \qquad (6.7)$$

If we restrict the σ model by imposing the condition $\lambda_{\pi\pi\pi\pi} = 12g_\pi^2$, to lowest order, all the 1PI Green functions will be exactly the same for both models. In fact, the effective action will be the same function of momentum, m and g_π^2.

The models will consequently be equivalent to all orders in the mean field expansion, because according to formula (2.8) if the generating functionals are the same to lowest order they will be the same to all orders since they are iterated in the same way. Further the identities that set the values of the renormalized parameters iterated in a similar manner as discussed in sect. 5.

The equivalence happens only in a special limit since the equations defining the renormalized parameters in terms of the bare parameters are different. For example in the σ model we have

$$\frac{1}{g_s^2} = \frac{1}{g_0^2} + \left(\frac{\partial \Pi}{\partial p^2}\right)_0 .$$

On the other hand in the Nambu model

$$\frac{1}{(g_s^2)_\mathrm{N}} = \left(\frac{\partial \Pi}{\partial p^2}\right)_0 ,$$

where Π is exactly the same function. True equivalence is achieved when

$$Z = \frac{g_s^2}{g_0^2} = 1 - g_s^2\left(\frac{\partial \Pi}{\partial p^2}\right)_0 = 1 - \frac{g_s^2}{(g_s^2)_\mathrm{N}} = 0 .$$

This condition is a typical compositeness condition. Careful examination of these conditions leads to the conclusion that the bare coupling parameter λ_0 becomes irrelevant. That is, no matter what value λ_0 takes, the renormalized Green function of the theory in the $Z = 0$ limit is unaffected [5,11].

7. Conclusions

In this paper we studied the Nambu Jona-Lasinio model expanded in a mean field expansion. The connected generating functional was expressed as an integral over boson variables and then expanded in a Laplace expansion. All divergences can be absorbed to all orders in a renormalized fermion mass and a renormalized Yukawa type coupling. The renormalized Yukawa coupling becomes the new effective expansion parameter. The boson self-couplings required for renormalization are determined to every order as functions of the renormalized Yukawa coupling. Similarly the σ mass is fixed in every order by the coupling and Fermi mass.

We expanded the σ model in the same way. If a restriction is imposed on the renormalized lowest-order quartic boson self-couplings, the renormalized effective action of the σ model to lowest order becomes equivalent to the renormalized effective action of the Nambu model. Equivalence to lowest order is enough to guarantee equivalence to all orders in accordance with formula (2.8) and through iteration of the Ward identities and Callan-Symanzik equations. This equivalence corresponds to a $Z = 0$ limit. In this limit the bare quartic coupling becomes irrelevant.

The authors wish to thank Neil Snyderman and Fred Cooper for many useful conversations. We would also like to thank Michael Ogilvie, Bill Caswell, and Desmond Fitzpatrick for helpful conversations and P.J. Stiles for encouragement.

Appendix

Generalized Ward identities ★

The Lagrangian of the Nambu Jona-Lasinio model is invariant under the transformations of the form

$$\psi \to e^{i\alpha\gamma_5}\psi ,$$

where α is a constant. This symmetry is a consequence of the absence of a bare fermion mass.

Every continuous symmetry implies the existence of a conserved current. The current associated with the chiral invariance is

$$j_5^\mu(x) \equiv [i\bar{\psi}(x)\,\gamma_5\gamma^\mu,\ \psi(x)]\ . \tag{A.1}$$

★ Ward identities have also been used for bound-state mean field expansions in ref. [12].

The divergence of this current in the presence of sources is

$$\partial_\mu j_5^\mu = 2J\sigma - 2S\pi - \bar{\eta}\gamma_5\psi - \bar{\psi}\gamma_5\eta\ . \tag{A.2}$$

The vacuum expectation value of (A.2) can be used to obtain a set of generalized Ward identities. Taking a variational derivative of the vacuum expectation value of (A.2) with respect to the classical pseudoscalar field, defined as

$$\pi(x) = \frac{\delta W}{\delta J(x)}\ ,$$

we obtain

$$\frac{\delta}{\delta\pi(y)}\{\langle\partial_\mu j_5(x)\rangle\} = 2\Delta_{\pi\pi}^{-1}(x, y)\,\sigma(x) - 2\Delta_{\sigma\pi}^{-1}(x, y)\,\pi(x)$$
$$- 2S(x)\,\delta(x-y) - \frac{\delta}{\delta\pi(y)}(\bar{\eta}\gamma_5\psi(x)$$
$$+ \bar{\psi}(x)\,\gamma_5\eta(x))\ . \tag{A.3}$$

σ, π, ψ, $\bar{\psi}$ stand for the vacuum expectation values of the corresponding operators (classical fields). $\Delta_{ij} \equiv \delta W/\delta S_i \delta S_j$ is the boson propagator. Integrating (A.3) with respect to x, turning off the fermion sources and taking the limit J, $S \to$ constant, we obtain

$$\int d^4x\partial_u \frac{\delta}{\delta\pi(y)}\ \{j_5^\mu(x)\} = 2\sigma\Delta_{\pi\pi}^{-1}(0) - 2\pi\Delta_{\sigma\pi}^{-1}(0) - 2S\ . \tag{A.4}$$

$\Delta_{ij}(0)$ stands for the momentum-space propagator at zero momentum.

Taking the limit of zero sources we end up with

$$\sigma\Delta_\pi^{-1}(0) = 0\ . \tag{A.5}$$

This is Goldstone's theorem. It states that, in the case $\sigma \neq 0$, the exact pseudoscalar propagator will have a zero-mass pole. Taking successive variational derivatives with respect to the classical fields and applying the same limiting procedure we obtain the following set of identities:

$$\sigma\Gamma_{\pi\pi\sigma}(p, -p) = \Delta_\sigma^{-1}(p^2) - \Delta_\pi^{-1}(p^2)\ , \tag{A.6}$$

$$\sigma\Gamma_{\pi\pi\sigma\sigma}(p, -p, 0) = \Gamma_{\sigma\sigma\sigma}(p, -p) - 2\Gamma_{\pi\pi\sigma}(p, -p)\ , \tag{A.7}$$

$$\sigma\Gamma_{\pi\pi\pi\pi}(p, -p, 0) = 3\Gamma_{\pi\pi\sigma}(p, -p)\ , \tag{A.8}$$

$$\sigma\Gamma_{\pi\pi\sigma\sigma\sigma}(p, -p, 0, 0) = \Gamma_{\sigma\sigma\sigma\sigma}(p, -p, 0) - 3\Gamma_{\pi\pi\sigma\sigma}(p, -p, 0)\ , \tag{A.9}$$

$$\sigma\Gamma_{\pi\pi\pi\pi\sigma}(p, -p, 0, 0) = 3\Gamma_{\pi\pi\sigma\sigma}(p, -p, 0) - \Gamma_{\pi\pi\pi\pi}(p, -p, 0)\ . \tag{A.10}$$

There are two additional identities involving the fermion Green functions that will be of use to us. They can be derived from the expectation value of (A.2), if we vary the classical fermion fields:

$$2g_0\sigma\Gamma_\pi(p, -p) = \{G^{-1}(p), i\gamma_5\}\,, \tag{A.11}$$

$$g_0\sigma\Gamma_{\pi\sigma}(p, -p, 0) = \tfrac{1}{2}\{\Gamma_\sigma(p, -p), i\gamma_5\} - \Gamma_\pi(p, -p)\,. \tag{A.12}$$

By definition

$$\Gamma_{\pi\sigma} \equiv \frac{\delta G^{-1}}{\delta(g_0\pi)\,\delta(g_0\sigma)} = \frac{\delta\Gamma_\sigma}{\delta(g_0\pi)}\,.$$

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