# SIMPLE TREATMENT OF THRESHOLD EFFECTS

I. ANTONIADIS
*Centre de Physique Théorique, Ecole Polytechnique, Palaiseau, France*

and

C. KOUNNAS and K. TAMVAKIS
*CERN, Geneva, Switzerland*

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We demonstrate that contributions from threshold effects in coupling constant differences at low energies in GUTs are simply taken into account in a scheme that preserves supersymmetry ($\overline{\text{DR}}$ dimensional reduction). Therefore, automatically in supersymmetric GUTs the naive step approximation at the physical mass associated with the threshold gives the correct result.

From the present-day point of view, the observed strong, electromagnetic and weak interactions are described in terms of a grand unified gauge theory based on a simple Lie group. Perhaps the most striking prediction of grand unification is the mortality of the proton. While experimentalists are now moving ahead to observe proton decay in the near future, there are other important predictions of GUTs testable with existing data or with data obtainable by present-day machines. Needless to say that a precise evaluation of the various phenomenological predictions is necessary in order to confront experiment in an unambiguous way.

The existence of a unique gauge coupling constant in GUTs automatically implies relations between the low energy effective couplings. These relations can be cast in the form

$$1/\alpha_i(\mu) - 1/\alpha_j(\mu) = F_{ij}\,(\text{all independent parameters}; M_X/\mu), \qquad (1)$$

where $M_X$ is the physical mass of the superheavy gauge bosons. If one considers as a prototype GUT the minimal SU(5) model, the standard approach based on the decoupling theorem [1] is to approximate the spontaneously broken SU(5) theory with the following effective theory

$$\begin{aligned} &\mathrm{SU(5)} && \mu \gg M_X, \\ &\mathrm{SU(3)\times SU(2)\times U(1)} && M_W \ll \mu \ll M_X, \\ &\mathrm{SU(3)\times U(1)_{em}} && \mu \ll M_W. \end{aligned} \qquad (2)$$

Extrapolating the validity of (2) to the region $\mu \approx M_W$ and $\mu \approx M_X$ (step approximation) introduces an error. In the approach followed by two of the authors (I.A. and C.K.) and Roiesnel [2], the contribution of threshold effects has been calculated exactly to the order of two loops. A similar estimation of threshold effects has been done by other people as well [3–6]. It should be stressed that the incorporation of threshold effects is crucial since the error introduced by the step approximation leads to an appreciable uncertainty in important quantities as the proton lifetime.

The functions $F_{ij}$ appearing in (1) are, in general [2],

$$F_{ij} = \int_{\mu^2}^{\infty} \frac{d\mu'^2}{\mu'^2} \left( \frac{\beta_i(\alpha_1,\ldots)}{\alpha_i^3} - \frac{\beta_j(\alpha_1,\ldots)}{\alpha_j^3} \right), \qquad (3)$$

where $\beta_i$ are the mass dependent $\beta$ functions. To the one-loop order, (3) becomes

$$F_{ij}^{(0)} = \int_{\mu^2}^{\infty} \frac{d\mu'^2}{\mu'^2} \left[ b_0^i(\mu'^2) - b_0^j(\mu'^2) \right], \qquad (4)$$

with $b_0^i$ the mass dependent coefficients of the $\beta$ functions. In the step approximation, we have [for $i$ corresponding to SU(3), SU(2) or U(1)]

$$b_0^i(\mu^2) = b_0^i \cdot \Theta[M_X^2 - \mu^2],$$

which implies the equality of coupling constants at $M_X$

$$\alpha_1(M_X) = \alpha_2(M_X) = \alpha_3(M_X). \quad (5)$$

In reality, however, we can only have

$$\alpha_1(\mu) = \alpha_2(\mu) = \alpha_3(\mu) \quad \text{for} \quad \mu^2 \gg M_X^2 \quad (6)$$

which is equivalent to

$$4\pi/\alpha_1(M_X) + C_1 = 4\pi/\alpha_2(M_X) + C_2 = 4\pi/\alpha_3(M_X) + C_3$$
$$= 4\pi/\alpha_5(M_X). \quad (7)$$

The constant terms in the minimal SU(5) model, when the $\overline{\text{MS}}$ scheme is used are $C_1 = 0, C_2 = -2/3, C_3 = -1$. Note that these constants are gauge invariant quantities, as we shall show explicitly later on. The error introduced by the naive step approximation is to totally neglect these constant terms in the limit $M_X^2/\mu^2 \to \infty$.

In what follows we shall show that in the dimensional reduction scheme ($\overline{\text{DR}}$) [7] all constant contributions vanish and the naive condition (5) is effectively valid. The dimensional reduction scheme, designed to preserve supersymmetry, consists in keeping the algebra of fields in four dimensions while performing dimensional continuation in the momentum loop integrals. Integrations over mass dependent coefficient functions in expressions (4) generally lead to

$$\int \frac{d\mu'^2}{\mu'^2}\, b_0^i(\mu'^2) = b_0^i \ln(\mu^2/M_X^2) + 1/\alpha_5(M_X) + \phi^i(\mu^2/M_X^2),$$

where

$$\phi^i(\mu^2/M_X^2) = -C_i \qquad \text{for } \mu^2/M_X^2 \to 0,$$
$$= (b_0^5 - b_0^i)\ln(\mu^2/M_X^2) \quad \text{for } \mu^2/M_X^2 \to \infty.$$

Thus, in general

$$F_{ij}(\mu^2) \xrightarrow[\mu^2/M_X^2 \to 0]{} (b_0^i - b_0^j)\ln(\mu^2/M_X^2) + (C_j - C_i).$$

As an example we have calculated the one-loop contributions of superheavy gauge particles together with the corresponding unphysical Goldstone and ghost particles, i.e., the total physically meaningful combination (see table 1). Although individual graphs in the limit $M_X^2/\mu^2 \to \infty$ have gauge dependent constant terms, their sum gives a gauge independent constant contributions [6]. Defining $g^{\mu\nu}g_{\mu\nu} = 4 - 2\epsilon'$ and performing integrations over $\int dk^{4-2\epsilon}$ we obtain in the low energy regime [in the minimal SU(5) model]

$$[-\tfrac{7}{2}\ln(M_X^2/\mu^2) + \tfrac{1}{3}\epsilon'/\epsilon]\,[C(5) - C(i)], \quad (8)$$

where the Casimir coefficients are $C(1) = 0, C(2) = 2, C(3) = 3, C(5) = 5$. Heavy scalars give no contributions to the constant term in either the $\overline{\text{MS}}$ or $\overline{\text{DR}}$ scheme. No superheavy fermions are present in our minimal SU(5). In case we had superheavy fermions and followed Weinberg's prescription [3] of extrapolating the spinor algebra in $2^{2-\epsilon'}$ dimensions, we would find a $-2/3(\ln 2)\epsilon'$ constant piece. In the $\overline{\text{MS}}$ scheme $\epsilon' = \epsilon$ and a gauge independent constant term survives. On the contrary, in the $\overline{\text{DR}}$ scheme in which we maintain the algebra in four dimensions, $\epsilon' = 0$ and *no constant term appears*.

It is not difficult to see how this comes about in general in the $\overline{\text{DR}}$ scheme. Since the total constant term is gauge independent at low energies, it suffices to consider just the Feynman gauge. Then, two-point functions with external legs corresponding to light gauge bosons (gluons, $W_\pm$, $Z_0$, $\gamma$) will be transverse and only logarithmically divergent. They will be of the form

$$(g_{\mu\nu}p^2 - p_\mu p_\nu)\int d^{4-2\epsilon}k \frac{F(p^2/M^2)}{[(p+k)^2 - M^2](k^2 - M^2)}, \quad (9)$$

where in the denominator only the same mass appears. In the limit $p^2/M^2 \to 0$, the integral will not give any constant term but just a $\ln M^2$. It should be stressed that this is due to keeping the algebra in four dimensions ($\epsilon' = 0$) on the one hand, and gauge invariance on the other. Similarly, one could argue about three-point functions.

It is not too hard to see that this property of the $\overline{\text{DR}}$ scheme will be preserved to the next order of perturbation theory. In the next order of the loop expansion the gauge couplings will be corrected by terms of the form

$$\ln[\alpha^{(i)}_{\text{one-loop}}(\mu^2)].$$

It is evident that constant terms cannot arise since they are not present at the one-loop level.

The alert reader has probably already noticed that

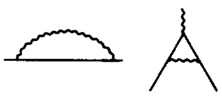
Table 1

$$= \tfrac{11}{6} + \tfrac{1}{2}\alpha + \tfrac{1}{3}\epsilon' + \epsilon\{[(1-\alpha^3)/3\alpha^3]\ln(1-\alpha) - [(3-4\alpha+\alpha^2)/2\alpha]\ln(1-\alpha)$$
$$- \tfrac{25}{18} + \tfrac{1}{4}\alpha + 1/6\alpha + 1/3\alpha^2\}$$

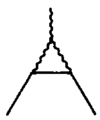
$$= -\tfrac{1}{6} + \tfrac{1}{6}\epsilon\ln(1-\alpha)$$

$$= -\tfrac{1}{6} + \tfrac{1}{6}\epsilon\ln(1-\alpha)$$

$$= -\epsilon\,[(1/3\alpha^3)\ln(1-\alpha) + 1/3\alpha^2 + 1/6\alpha + 1/9]$$



$$= -1 + \alpha + \epsilon' + \epsilon\,[-\tfrac{1}{2} + \alpha + (1-\alpha)\ln(1-\alpha)]$$



$$= 3 - \tfrac{3}{2}\alpha - \epsilon' + \epsilon\,[2 - \tfrac{5}{4}\alpha + \tfrac{3}{2}\alpha^{-1}(1-\alpha)^2\ln(1-\alpha)]$$

gauge propagator $= [1/(k^2 - M^2)]\,\{g_{\mu\nu} - \alpha K_\mu K_\nu/[k^2 - (1-\alpha)M^2]\}$

$g^{\mu\nu}g_{\mu\nu} = 4 - 2\epsilon'$

integrations over $\int dp^{4-2\epsilon}$

this property of the $\overline{\mathrm{DR}}$ scheme has important implications for supersymmetric GUTs. In a supersymmetric theory, in order to preserve supercurrent conservation, i.e., supersymmetry, the algebra must be kept in four dimensions [7]. Therefore, one is forced to use the $\overline{\mathrm{DR}}$ scheme or perhaps some derivative subtraction scheme. But in the $\overline{\mathrm{DR}}$ scheme, as we have shown, the supernaive step approximation at $M_X$ (not $2M_X$ or elsewhere) gives the correct result. Thus, all naive renormalization group computations in SUSY GUTs, as long as they have been done in the $\overline{\mathrm{DR}}$ scheme, do not have to be corrected with threshold effects. Of course, this property of the $\overline{\mathrm{DR}}$ scheme is just a property of the scheme and is not connected with supersymmetry. Hence, although this scheme might break supersymmetry at higher loops and perhaps should be abandoned for SUSY GUTs, its legitimate use in ordinary GUTs is unquestionable.

As an application, let us consider the renormalization group equations for the three gauge couplings of a minimal supersymmetric SU(5) theory (with two Higgs doublets). They are (to the order of two loops)

$$1/\alpha_3(M_W) = 1/\alpha_5(M_X) - (b_3^{(0)}/2\pi)\ln(M_X/M_W) - \frac{1}{4\pi}\sum_{j=1}^{3}(b_{3j}^{(1)}/b_j^{(0)})\ln[\alpha_j(M_W)/\alpha_5(M_X)],$$

$$[1/\alpha(M_W)]\sin^2\theta_W(M_W) = 1/\alpha_5(M_X) - (b_2^{(0)}/2\pi)\ln(M_X/M_W) - \frac{1}{4\pi}\sum_{j=1}^{3}(b_{2j}^{(1)}/b_j^{(0)})\ln[\alpha_j(M_W)/\alpha_5(M_X)], \quad (10)$$

$$\tfrac{3}{5}[1/\alpha(M_W)]\cos^2\theta_W(M_W)$$
$$= 1/\alpha_5(M_X) - (b_1^{(0)}/2\pi)\ln(M_X/M_W)$$
$$-\frac{1}{4\pi}\sum_{j=1}^{3}(b_{3j}^{(1)}/b_j^{(0)})\ln[\alpha_j(M_W)/\alpha_5(M_X)], \quad (10\text{ con'd})$$

where

$$b_3^{(0)} = 9 - 2N_G, \quad b_2^{(0)} = 5 - 2N_G, \quad b_\Lambda^{(0)} = -2N_G - \tfrac{3}{5},$$

and

$$b_{ij}^{(1)} = \begin{bmatrix} \frac{199}{25} & \frac{27}{5} & \frac{88}{5} \\ \frac{9}{5} & 25 & 24 \\ \frac{11}{5} & 9 & 14 \end{bmatrix} \quad (\text{for } N_G = 3),$$

For $\Lambda_{\overline{MS}} = 0.1$ GeV, we obtain $M_X/M_W = 8.7 \times 10^{13}$ and $\sin^2\theta_W(M_W) = 0.233$.

In conclusion, we would like to restate our result. In the $\overline{\text{DR}}$ scheme, where we keep the algebra in four dimensions while performing momentum loop integrations in $4-2\epsilon$ dimensions, constant terms in coupling constant differences that correspond to the contribution of threshold effects are zero. Thus, the naive step approximation gives the correct result. It is evident that our result implies a great simplification in computations. By performing all calculations in the $\overline{\text{DR}}$ scheme, the naive step approximation at $M_X$ contains all threshold effects.

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