

Singularity-free cosmological solutions of the superstring effective action

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We study the cosmological solutions of the one-loop corrected superstring effective action, in a Friedmann–Robertson–Walker background, and in the presence of the dilaton and modulus fields. A particularly interesting class of solutions is found which avoid the initial singularity and are consistent with the perturbative treatment of the effective action.

1. Introduction

Einstein's theory has been very successful as a classical theory of gravitational interactions. However, a quantum theory of gravity would require to incorporate Einstein's theory in a more general framework. Today our best candidate for such a framework is superstring theory which also has the prospect of unification of all other interactions. Superstrings appear to involve a minimum length which is of the order of the Planck scale and, thus, they are expected to lead to drastic modifications of the Einstein action at short distances. These modifications can in principle have cosmological consequences which distinguish string cosmology from the standard model and provide some indications for a stringy origin of the universe. Furthermore, one hopes to find resolutions to some puzzles in Einstein cosmology including the initial singularity problem.

String theory gives rise to two kinds of such modifications. The first is associated with the contribution of the infinite tower of massive string modes and leads to α' -corrections, while the second is due to quantum loop effects. Both these contributions can be studied in the context of an effective field theory which

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involves only the massless string modes. The effective lagrangian can be derived from string theory using a perturbative approach in both the string tension α' and string coupling expansion. An alternative direct way to take into account all α' -corrections at the classical string level is to consider conformal field theories which describe exact string solutions in time-dependent gravitational backgrounds [1,2]. However, this is not in general sufficient for probing the short-distance behavior of the theory since the gravitational coupling is dimensionful and quantum corrections become important in this region.

One of the unique properties of the string effective action is that couplings are field-dependent. For our purposes, the relevant fields are the dilaton which plays the role of the string loop expansion parameter, and the moduli whose vacuum expectation values describe the size and the shape of the internal compactification manifold. Since these fields have no potential their contribution is expected to be important in any cosmological solution of the effective action. Here we restrict ourselves to the simple case of a single modulus field, besides the dilaton, which corresponds to the common compactification radius.

The tree-level string effective action has been calculated up to several orders in the α' -expansion in both the sigma model approach [3], where one considers strings propagating in background fields, and the S -matrix approach, where the effective action is computed directly from string scattering amplitudes [4]. It turns out that there is no moduli dependence of the tree-level couplings. The one-loop corrections to the gravitational couplings have also been calculated recently in the context of orbifold compactifications of the heterotic superstring [5]. It has been shown that there are no moduli-dependent corrections to the Einstein term while there are non-trivial R^2 -contributions. They appear as the Gauss–Bonnet combination multiplied by a function of the modulus field. It is interesting to consider the implications of this term to the cosmological solutions of Einstein equations for many reasons. Firstly, this term is subject to a nonrenormalization theorem which implies that all higher-loop moduli-dependent R^2 -contributions vanish. In addition, it breaks the continuous isometries of the tree-level modulus kinetic terms leaving intact only the duality symmetries. Finally, as has been pointed out in the literature, higher-derivative modifications of the Einstein action could cure the problem of singularities [6,7].

The purpose of this paper is to study the evolution of the equations of motion of the superstring effective lagrangian in the presence of the moduli-dependent R^2 -corrections. We investigate the asymptotic solutions and we show that the one-loop effective action contains a class of interesting cosmological solutions which avoid the initial singularity. This is possible only for a definite sign of the corresponding four-dimensional trace anomaly for which the strong energy conditions [8] related to the modulus energy–momentum tensor can be violated. These solutions start from flat space–time in the infinite past, they pass through an inflationary period and they end up as a slowly expanding universe. Although in

our analysis we omit terms higher than quadratic in the Riemann tensor, we will argue that the above result persists in the full theory under certain assumptions on the moduli dependence of loop corrections to higher-derivative terms.

Nonsingular cosmological solutions were first derived in ref. [6], in the context of higher-derivative extensions of the Einstein action obtained by taking into account the contribution of conformally coupled matter fields. String theory leads to an effective higher-derivative gravitational action already at the classical level. In particular, the Gauss–Bonnet density multiplied by the dilaton field appears at the string tree-level in the next-to-leading order of the α' -expansion, and it has been studied extensively in the literature [9]. As we will show in sect. 3, the addition of this term alone does not lead to violation of the energy conditions and thus it cannot provide any singularity-free solution, in contrast to the modulus-dependent loop correction.

We also examine the possible existence of time-dependent solutions which fix asymptotically the vacuum expectation value of the modulus field. This is motivated from the fact that the Gauss–Bonnet contributions could be viewed as a time-dependent potential for the modulus with an extremum at the self-dual point. We show that this possibility can be realized only at early times and in singular strong coupling solutions.

This paper is organized as follows. In sect. 2 we review the string effective action and the loop corrections to the gravitational couplings and we derive the equations of motion for the coupled system of the graviton, dilaton and modulus field. In sect. 3 we study the cosmological solutions in the simple case where the dilaton is ignored. We classify all asymptotic solutions and we show by numerical integration that two of them can be smoothly joined avoiding the singularity. We justify this behavior by demonstrating the violation of the weak and strong energy conditions related to the singularity theorems. Finally, we explore the parameter space of all initial conditions by the use of the corresponding phase diagram. In sect. 4 we repeat the same analysis in the presence of the dilaton field and show that the previous results remain unaffected. Our conclusions are summarized in sect. 5.

2. The loop corrected effective action and the equations of motion

Let us consider the universal part of the effective action of any four-dimensional heterotic superstring model which describes the dynamics of the graviton, the dilaton S and the common modulus field T . At the string tree-level, and up to first order in the α' -expansion, it takes the form [3,4]

$$\mathcal{L}_{\text{eff}} = \frac{1}{2\kappa^2} R + \frac{DS D\bar{S}}{(S + \bar{S})^2} + 3 \frac{DT D\bar{T}}{(T + \bar{T})^2} + \frac{1}{8} (\text{Re } S) R_{\text{GB}}^2 + \frac{1}{8} (\text{Im } S) R\tilde{R}, \quad (2.1)$$

where D denotes covariant differentiation, R is the scalar curvature, $\kappa = \sqrt{8\pi G_N}$ with G_N the Newton constant, R_{GB}^2 is the Gauss–Bonnet integrand

$$R_{\text{GB}}^2 = R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \quad (2.2)$$

and $R\tilde{R} = \eta^{\mu\nu\kappa\lambda} R_{\mu\nu}^{\sigma\tau} R_{\kappa\lambda\sigma\tau}^*$. The inverse of $\text{Re } S$ plays the role of the string coupling constant squared while $\text{Im } S$ is a pseudoscalar axion. Finally, the real part of the complex modulus field T corresponds to the square of the compactification radius.

At the one-loop level the moduli dependence of the gravitational couplings in the case of the heterotic string compactified on a symmetric orbifold has been studied in ref. [5]. It is shown that there are no moduli-dependent corrections to the Einstein term, while the contributions to the four derivative gravitational terms take the form

$$\Delta\mathcal{L}_{\text{eff}} = \Delta(T, \bar{T}) R_{\text{GB}}^2 + \Theta(T, \bar{T}) R\tilde{R}. \quad (2.3)$$

The moduli-dependent functions are defined as

$$\Delta(T, \bar{T}) = \frac{\hat{b}_{\text{gr}}}{32\pi^2} \ln[(T + \bar{T}) |\eta(iT)|^4] \quad (2.4)$$

and $\Theta(T, \bar{T}) = -i\Delta(T, \bar{T})$, where $\eta(\tau) = q^{1/12} \prod_{n \geq 1} (1 - q^{2n})$, with $q = e^{i\pi\tau}$, is the Dedekind η -function. $\Delta(T, \bar{T})$ is invariant under the duality $\text{SL}(2, \mathbb{Z})$ transformations $T \rightarrow 1/T$ and $T \rightarrow T + i$ which are the discrete subgroup of the continuous $\text{SL}(2, \mathbb{R})$ isometry group of the modulus kinetic terms in (2.1). The coefficient $\hat{b}_{\text{gr}}$ is proportional to the four-dimensional trace anomaly of the $N = 2$ sectors of the theory

$$\frac{1}{6}(-3N_V + N_S) - \frac{11}{3}(-3 + N_{3/2}), \quad (2.5)$$

where N_S , N_V and $N_{3/2}$ denote the number of chiral, vector and spin- $\frac{3}{2}$ massless supermultiplets.

We now consider the effective action (2.1) + (2.3) in a spatially flat homogeneous and isotropic Robertson–Walker background. Since $R\tilde{R}$ vanishes identically in this background, it is consistent with the equations of motion to assume for simplicity $\text{Im } S = \text{const.}$ and $\text{Im } T = 0$. The latter is required in order to have a vanishing derivative of $\Delta(T, \bar{T})$ with respect to $\text{Im } T$. Setting the length scale $\kappa = 1$ and defining $\text{Re } S = e^\Phi/g^2$, where g is the four-dimensional string coupling, and $\text{Re } T = e^\sigma$, the effective action takes the form

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2}R + \frac{1}{4}(D\Phi)^2 + \frac{3}{4}(D\sigma)^2 + \frac{1}{16}[\lambda e^\Phi - \delta\xi(\sigma)] R_{\text{GB}}^2 \right\}, \quad (2.6)$$

where $\lambda = 2/g^2$, $\delta = \hat{b}_{\text{gr}}/2\pi^2$ and $\xi(\sigma) = \ln[2e^\sigma \eta^4(ic^\sigma)]$.

* We use the conventions $R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda$, $R_{\mu\nu\kappa}^\lambda = \partial_\kappa \Gamma_{\mu\nu}^\lambda + \dots$, and $\eta^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta}/\sqrt{-g}$ with $\epsilon^{0ijk} = -\epsilon_{ijk}$.

The equations of motion derived from the action (2.6) by varying with respect to the metric, the dilaton and the modulus field have the form

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}}{\delta g^{\mu\nu}} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} = 0, \quad (2.7)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{S}}{\delta \sigma} = -\frac{3}{2} D^\mu D_\mu \sigma + \frac{\partial f}{\partial \sigma} R_{\text{GB}}^2 = 0, \quad (2.8)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{S}}{\delta \Phi} = -\frac{1}{2} D^\mu D_\mu \Phi + \frac{\partial f}{\partial \Phi} R_{\text{GB}}^2 = 0, \quad (2.9)$$

where

$$\begin{aligned} T_{\mu\nu}^{(1)} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{3}{2} D_\mu \sigma D_\nu \sigma - \frac{3}{4} g_{\mu\nu} (D\sigma)^2 + \frac{1}{2} D_\mu \Phi D_\nu \Phi - \frac{1}{4} g_{\mu\nu} (D\Phi)^2, \\ T_{\mu\nu}^{(2)} &= (g_{\mu\rho} g_{\nu\lambda} + g_{\nu\rho} g_{\mu\lambda}) \eta^{\kappa\lambda\alpha\beta} D_\gamma \left(\tilde{R}_{\alpha\beta}^{\gamma\rho} D_\kappa f \right) \end{aligned} \quad (2.10)$$

and $f = \frac{1}{16} [\lambda e^\Phi - \delta\xi(\sigma)]$.

Substituting the spatially flat Robertson–Walker ansatz for the metric

$$g_{\mu\nu} = (1, -e^{2\omega} \delta_{ij}) \quad (2.11)$$

in (2.7)–(2.10) and considering only time-dependent fields, we end up with the equations

$$3\dot{\omega}^2 - \frac{3}{4}\dot{\sigma}^2 - \frac{1}{4}\dot{\Phi}^2 + 24\dot{f}\dot{\omega}^3 = 0, \quad (2.12)$$

$$2\ddot{\omega} + 3\dot{\omega}^2 + \frac{3}{4}\ddot{\sigma}^2 + \frac{1}{4}\ddot{\Phi}^2 + 16\dot{f}\dot{\omega}^3 + 8\ddot{f}\dot{\omega}^2 + 16\dot{f}\dot{\omega}\ddot{\omega} = 0, \quad (2.13)$$

$$\ddot{\sigma} + 3\dot{\omega}\dot{\sigma} + \delta \frac{\partial \xi}{\partial \sigma} \dot{\omega}^2 (\dot{\omega}^2 + \ddot{\omega}) = 0, \quad (2.14)$$

$$\ddot{\Phi} + 3\dot{\omega}\dot{\Phi} - 3\lambda e^\Phi \dot{\omega}^2 (\dot{\omega}^2 + \ddot{\omega}) = 0, \quad (2.15)$$

where (2.12) and (2.13) correspond to the T_{00} and T_{ii} components of (2.7), respectively. These two equations are not functionally independent because of the Bianchi identity related to the conservation of the total energy–momentum tensor. In fact the linear combination $\dot{\omega} \times (2.13) - \dot{\omega} \times (2.12) - \frac{1}{2}\dot{\sigma} \times (2.14) - \frac{1}{6}\dot{\Phi} \times (2.15)$ yields the time derivative of (2.12). Thus, we can reject (2.13) and consider the system of (2.12), (2.14) and (2.15) as the independent equations of motion.

Finally, let us also define here the energy density (ρ) and pressure (p) of the dilaton–modulus matter system. Assuming a perfect fluid form for their energy–momentum tensor $T_{00} = \rho$, $T_{ii} = p e^{2\omega}$, and using (2.12), (2.13), we get

$$\rho = 3\dot{\omega}^2, \quad p = -(2\ddot{\omega} + 3\dot{\omega}^2). \quad (2.16)$$

3. Analysis of the metric–modulus system

For simplicity we start our analysis by neglecting all the dilaton related terms. Although $\Phi = \text{const.}$ is not a solution of the dilaton equation of motion (2.15), the metric–modulus system provides a simple model to study the effect of the Gauss–Bonnet loop correction (2.3). This simplification will be justified in sect. 4, where the full system will be examined, and we will show that the main results of the following analysis remain valid even in the presence of the dilaton.

The system of equations of motion (2.12)–(2.15) is now reduced to

$$4\dot{\omega}^2 - \dot{\sigma}^2 - 2\delta \frac{\partial \xi}{\partial \sigma} \dot{\sigma} \dot{\omega}^3 = 0, \quad (3.1)$$

$$\ddot{\sigma} + 3\dot{\omega} \dot{\sigma} + \delta \frac{\partial \xi}{\partial \sigma} \dot{\omega}^2 (\dot{\omega}^2 + \ddot{\omega}) = 0. \quad (3.2)$$

Note that the absolute value of the trace anomaly coefficient δ can be absorbed by a time rescaling

$$t \rightarrow t' = \sqrt{|\delta|} t, \quad (3.3)$$

which implies that we can replace δ by its sign. From the expression (2.5) one sees that in any theory with $N \leq 4$ supersymmetry the sign of δ tends to be positive unless there is a considerable excess of vector bosons.

The nonlinear system (3.1), (3.2) can be integrated using numerical methods. Given initial values for σ , ω and $\dot{\omega}$, the value of $\dot{\sigma}$ can be determined from (3.1). Then, the time derivative of (3.1) together with (3.2) form a second-order system of differential equations which can be numerically solved given the above boundary conditions.

Before proceeding to the numerical integration, we shall first derive analytically the asymptotic solutions in the limits $t \rightarrow \infty$ and $t \rightarrow 0$. For this purpose we will use the asymptotic expansion of $\xi'(\sigma) \equiv \partial \xi / \partial \sigma$ for $\sigma \rightarrow \pm \infty$,

$$\xi'(\sigma) \sim -\text{sign}(\sigma) \frac{1}{3} \pi e^{|\sigma|}. \quad (3.4)$$

In the limit $t \rightarrow \infty$ we find two kind of solutions. The first is obtained using the ansatz

$$\omega = \omega_0 + \alpha \ln t, \quad \sigma = \sigma_0 + \beta \ln t, \quad (3.5)$$

with ω_0 and σ_0 constants, and it leads to two possibilities,

$$(A_\infty): \quad \alpha = \frac{1}{3}, \quad |\beta| = \frac{2}{3} \quad (3.6)$$

and

$$(B_\infty): \quad |\beta| = 2, \quad \alpha^3 - \alpha^2 + 5\alpha - 1 = 0, \quad \exp[\text{sign}(\beta)\sigma_0] = \frac{3(1 - \alpha^2)}{\delta\pi\alpha^3}, \quad (3.7)$$

which leads to $\delta > 0$, $\alpha \sim 0.207$ and $\text{sign}(\beta)\sigma_0 \sim 4.64 - \ln \delta$. A_∞ is actually the asymptotic solution of the tree-level system ($\delta = 0$), while B_∞ is a new asymptotic solution where the Gauss–Bonnet term is important at late times. They both describe a slowly expanding universe, while the radius $e^{\sigma/2}$ is either expanding or contracting due to the duality symmetry $\sigma \rightarrow -\sigma$ of eqs. (3.1), (3.2).

The second kind of solutions is obtained using the flat space ansatz

$$\omega = \omega_0 + \omega_1 t^a, \quad \sigma = \sigma_0 + \beta \ln t, \quad (3.8)$$

which leads to

$$(C_\infty): \quad \alpha = -1, \quad |\beta| = 5, \quad \exp[\text{sign}(\beta)\sigma_0] = -\frac{15}{2\delta\pi\omega_1^3}. \quad (3.9)$$

This describes an asymptotically flat universe with slowly expanding (or contracting) radius.

In the limit $t \rightarrow 0$ we find only one asymptotic solution,

$$(A_0): \quad \omega = \omega_0 + \alpha \ln t, \quad \sigma = \sigma_0 + \sigma_1 t^\beta, \quad (3.10)$$

with

$$\alpha = 1, \quad \beta = 2, \quad \sigma_1 = \frac{1}{\delta\xi'(\sigma_0)}. \quad (3.11)$$

This is a singular solution which also fixes the modulus field to a constant (but arbitrary) value.

Note that the modulus equation of motion (3.2) for slowly varying $\dot{\omega}$ can be considered as describing the motion of σ in the presence of a potential proportional to $\xi(\sigma)$. This potential has an extremum at the self-dual point $\sigma = 0$ which corresponds to a minimum for $\delta < 0$, providing a possible mechanism to fix the

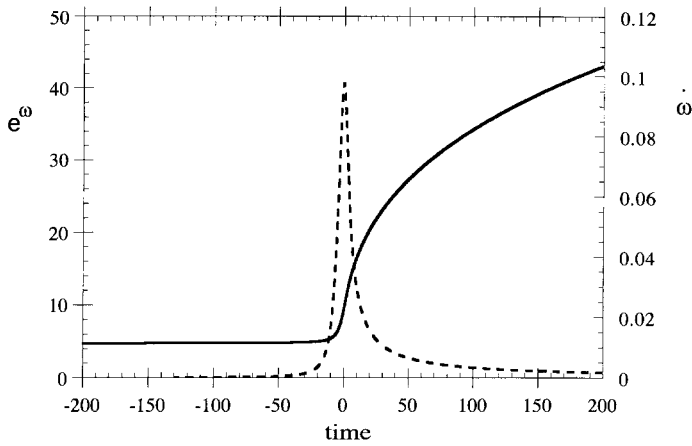


Fig. 1. The scale factor e^ω (continuous line) and the Hubble expansion rate $\dot{\omega}$ (dashed line) for the nonsingular solution ($\delta < 0$).

value of the compactification radius. Unfortunately, the first equation (3.1) requires $\dot{\omega} = 0$ when $\sigma = 0$ which leads to a vanishing potential for σ . The resulting solution $\sigma = 0$ and $\omega = \text{const.}$ cannot be continuously approached in the asymptotic region. In the next section we will see that in the presence of the dilaton, (3.1) has a slowly expanding solution ($\omega \sim \ln t$) which leads to a realization of this mechanism at early times. In addition, the possibility of fixing the modulus through the same mechanism at finite time, in a region where $\dot{\omega}$ exhibits an extremum, remains open.

Numerical integration of the system (3.1), (3.2) verifies the existence of the above list of asymptotic solutions but it also reveals another very interesting characteristic: For $\delta < 0$, there exists a region of boundary conditions, for which the two asymptotic solutions A_∞ and C_∞ are smoothly joined avoiding the singularity. In fact, starting from the asymptotically flat solution C_∞ at the infinite past, one is always continuously driven to the slowly expanding solution A_∞ at the infinite future. For the rest of the boundary conditions, as well as for $\delta > 0$, the singular solution A_0 is recovered. A typical nonsingular solution is presented in figs. 1, 2^{*}. Fig. 1 shows that the expansion rate of the universe $\dot{\omega}$ starts from a zero value (flat space time) at $t \rightarrow -\infty$, grows up to a maximum value, and then falls down again as $1/t$ at $t \rightarrow \infty$. The scale factor, correspondingly, starts from a constant value, goes through a period of rapid expansion (inflation) and ends up as a slowly expanding universe. On the other hand, fig. 2 shows that the modulus field starts from $-\infty$ corresponding to a zero compactification radius at the remote past, passes through the self-dual point $\sigma = 0$ during the inflationary period, and ends up as a slowly expanding regime at the infinite future^{**}. The vanishing of σ in the

^{*} In all plots we have used $|\delta| = 64$.

^{**} The duality symmetry implies also the existence of the dual solution $\sigma \rightarrow -\sigma$.

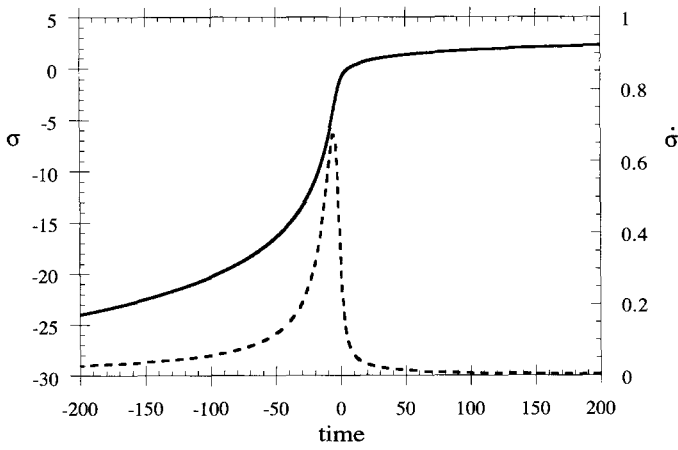


Fig. 2. The modulus field σ (continuous line) and its derivative $\dot{\sigma}$ (dashed line) for the nonsingular solution.

region where $\dot{\omega}$ is maximum is a general feature of this kind of solution, which is due to the form of the Gauss–Bonnet loop correction as described above.

Figs. 1, 2 also show that the obtained nonsingular solutions can have all time derivatives of ω and σ less than one in Planck units, which is consistent with our approximation of neglecting higher-derivative terms in the effective action. This is in general sufficient provided that the moduli-dependent coefficients of the higher-order terms are not large enough to compensate the derivative suppression. Under this assumption the main features of these solutions are expected to survive after higher-order corrections are taken into account.

The avoidance of the singularity is accompanied, as expected, by a violation of the weak and strong energy conditions [8] related to the energy–momentum tensor of the modulus field, which are illustrated in fig. 3. This violation can be demonstrated analytically in the following way. The system of equations consisting

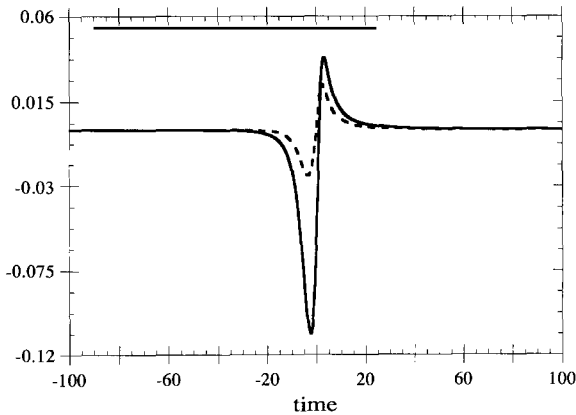


Fig. 3. $\rho + p$ (dashed line) and $\rho + 3p$ (continuous line) for the nonsingular solution.

of the time derivative of (3.1) together with (3.2) can be solved for $\ddot{\omega}$, and then $\xi'(\sigma)$ terms can be removed using (3.1). Using (2.16) we obtain

$$\begin{aligned}\rho + p &= -2\ddot{\omega} = 2\dot{\omega}^2 \frac{16\dot{\omega}^4 + 24\dot{\omega}^2\dot{\sigma}^2 + 5\dot{\sigma}^4 - 4\delta\xi''(\sigma)\dot{\omega}^2\dot{\sigma}^4}{16\dot{\omega}^4 - 8\dot{\omega}^2\dot{\sigma}^2 + 5\dot{\sigma}^4}, \\ \rho + 3p &= -6(\dot{\omega}^2 + \ddot{\omega}) = 24\dot{\omega}^4\dot{\sigma}^2 \frac{8 - \delta\xi''(\sigma)\dot{\sigma}^2}{16\dot{\omega}^4 - 8\dot{\omega}^2\dot{\sigma}^2 + 5\dot{\sigma}^4},\end{aligned}\quad (3.12)$$

where $\xi''(\sigma) \equiv \partial^2\xi/\partial\sigma^2$. It is easy to see that since $\xi''(\sigma) < 0$, eq. (3.12) implies that $\rho + p > 0$ and $\rho + 3p > 0$ for $\delta > 0$. Thus, the energy conditions can never be violated in this case and we cannot avoid the initial singularity. This is also the result of the numerical integration, for $\delta > 0$, which always leads to the singular solution A_0 at every times.

On the contrary, as seen from (3.12), $\delta < 0$ allows for solutions which explicitly violate both energy conditions. Of course this is in general a necessary but not sufficient requirement in order to avoid singularities. In our case, the numerical integration of the system (3.1), (3.2) has shown that the violation of the energy conditions at some instant is also sufficient. In fact, the nonsingular solutions can be obtained by imposing $\ddot{\omega} \sim \rho + p = 0$, which implies $\rho + 3p < 0$, at the starting point of our integration. The initial values for σ and $\dot{\omega}$ at this point obey a constraint which can be derived using eqs. (3.12) and (3.1). For a specific value of the modulus σ , the expansion rate $\dot{\omega}$ is given by

$$4\nu^4 z^5 - 5\nu^4 z^4 + 40\nu^2 z^3 - 8(2 + 7\nu^2)z^2 + 96z - 144 = 0, \quad z > \frac{5}{4}, \quad (3.13)$$

where $z \equiv \delta\xi''(\sigma)\dot{\omega}_{\max}^2$ and $\nu \equiv \xi'(\sigma)/\xi''(\sigma)$ satisfying $-1 < \nu < 1$ due to the properties of $\xi(\sigma)$. One can show that the above equation has at least one solution for z for any value of ν . In terms of our original variables σ and $\dot{\omega}$, these solutions are plotted in fig. 4.

The entire region of initial conditions which lead to singular or nonsingular solutions for $\delta < 0$ can be explored by the use of the corresponding phase diagram. In fact, the system of second-order differential equations (3.1), (3.2) can be reduced to a single first-order equation for the variable $\dot{\omega}$ as a function of σ . First (3.1) can be solved for $\ddot{\sigma}$ in terms of $\dot{\omega}$ and σ ,

$$\dot{\sigma} = -\delta\xi'(\sigma)\dot{\omega}^3 \pm |\dot{\omega}| \sqrt{\delta^2\xi'(\sigma)^2\dot{\omega}^4 + 4}. \quad (3.14)$$

On the other hand, the time derivative of (3.1) together with (3.2) leads to (3.12) which can be written in the form

$$\frac{d\dot{\omega}}{d\sigma} = -\frac{\dot{\omega}^2}{\dot{\sigma}} \frac{16\dot{\omega}^4 + 24\dot{\omega}^2\dot{\sigma}^2 + 5\dot{\sigma}^4 - 4\delta\xi''(\sigma)\dot{\omega}^2\dot{\sigma}^4}{16\dot{\omega}^4 - 8\dot{\omega}^2\dot{\sigma}^2 + 5\dot{\sigma}^4}. \quad (3.15)$$

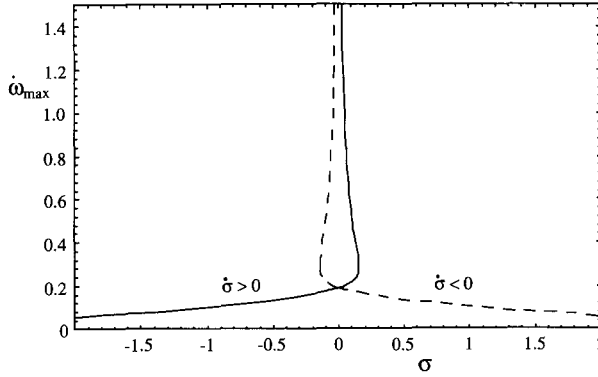


Fig. 4. The maximum of the expansion rate $\dot{\omega}_{\max}$ as a function of the modulus σ for $\dot{\sigma} > 0$ (continuous line) and $\dot{\sigma} < 0$ (dashed line).

Inspection of (3.14) shows that the two branches defined by the two signs of the square root correspond to two disconnected classes of solutions associated with positive or negative values of $\dot{\sigma}$. These are related by a duality transformation $\sigma \rightarrow -\sigma$, and thus we can restrict to the case $\dot{\sigma} > 0$.

The phase diagram of (3.15) is presented in fig. 5. The asymptotic solutions C_∞ and A_∞ are located in the regions $\sigma \rightarrow -\infty$ and $\sigma \rightarrow +\infty$, respectively, while the singular A_0 appears in the region $\dot{\omega} \rightarrow \infty$, $\sigma > 0$. As one sees, for negative initial values of σ , nonsingular solutions are obtained for all values of $\dot{\omega}$. They always interpolate between an asymptotically flat and a slowly expanding universe. The maxima of $\dot{\omega}$ correspond to points of the curve ($\dot{\sigma} > 0$) shown in fig. 4. On the other hand, for positive initial values of σ , singular solutions are obtained for all values of $\dot{\omega}$ lying above a critical curve (bold line). Note that they never end up as an asymptotically flat space-time. Compared to other known nonsingular solutions

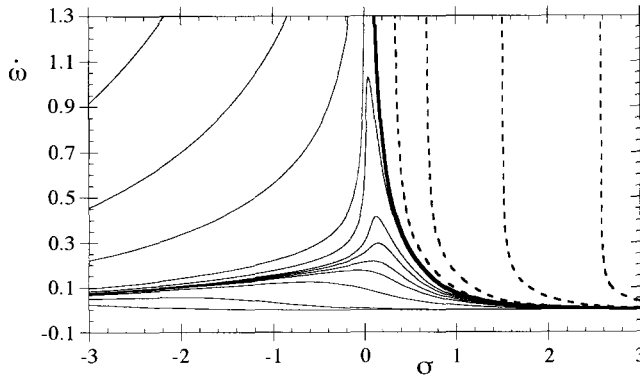


Fig. 5. The phase diagram of $\dot{\omega}$ as a function of σ ; continuous lines correspond to nonsingular solutions and dashed lines to singular ones. The boundary of the two regions is plotted by a bold line.

[6,7] derived from higher-derivative effective gravity theories, our solutions extend over an infinite region of the phase space where the maximum value of curvature is not bounded.

As already mentioned, another interesting characteristic of these nonsingular solutions is that they contain an inflationary regime *. This is expected at least during the period when the energy conditions are violated due to the development of negative pressure. The numerical analysis shows that the amount of inflation increases with rising values of $\delta^{1/2}\dot{\omega}_{\max}$ given in fig. 4. For values of order one only a few e-foldings are obtained (see fig. 1), while larger values of this parameter break the validity of the perturbative treatment of the effective action. In any case, these solutions provide an example of a cosmological model which is driven to an inflationary era and then exits in a finite time. This is in contrast to the behavior of other nonsingular solutions which start from de-Sitter space in the remote past [6,7]. In fact the solutions presented here start asymptotically from flat space-time, which is welcome as it has recently been argued that eternal de-Sitter inflation may not be possible without a beginning [11]. Moreover, their characteristics are very similar to those arising in the context of a “pre-big-bang” scenario motivated by generalized scale-factor duality symmetries of string theory [2].

The properties of the nonsingular solutions depend crucially on the form of the modulus-dependent function $\xi(\sigma)$ arising from the string loop corrections to the dimensionless gravitational couplings (2.6). For instance if $\xi(\sigma) \sim \sigma^2$, de-Sitter space is obtained asymptotically. On the other hand, if one considers the dilaton field instead of the modulus in the presence of the first order in α' Gauss–Bonnet interaction (2.6), the singularity can never be avoided. In fact, the analysis is equivalent to that of the metric–modulus system (3.1), (3.2) with the substitutions $\sigma \rightarrow \frac{1}{3}\Phi$, $\delta\xi'(\sigma) \rightarrow -\lambda e^\Phi$. Then, the energy conditions (3.12) have also the same form with $-\delta\xi''(\sigma)$ replaced by λe^Φ and they cannot be violated since $\lambda = 2/g^2$ is positive.

4. Analysis including the dilaton terms

We now extend our analysis including the dilaton contributions in the effective action (2.6), which leads to the equations of motion (2.12), (2.14) and (2.15). A time dilatation combined with a shift in the dilaton field,

$$t \rightarrow t' = \sqrt{|\delta|} t, \quad \Phi \rightarrow \Phi' = \Phi + 2 \ln |\delta| - \ln \lambda, \quad (4.1)$$

can be used to eliminate both λ and the absolute value of δ . In analogy with the analysis of the previous section we shall first derive the asymptotic solutions. As

* A discussion of the contribution of the string loop moduli-dependent corrections to the gravitational couplings in connection with the inflationary solutions of ref. [6] was reported in ref. [10].

one can see from (3.4), the asymptotic behavior of the modulus-dependent Gauss–Bonnet coefficient $\xi(\sigma)$ for large σ is proportional to $e^{|\sigma|}$, while the dilaton-dependent coefficient is proportional to e^Φ . It follows that the asymptotic solutions of the metric–modulus system derived in sect. 3 will survive in this case, with the dilaton either being negligible or behaving similarly to the modulus. In addition, some new solutions are expected for dominant large asymptotic values of the dilaton field.

In the limit $t \rightarrow \infty$ the asymptotic solutions A_∞ and B_∞ of sect. 3 are extended with the dilaton being of the form $\Phi = \phi_0 + \gamma \ln t$ to the solutions

$$(A'_\infty): \quad \alpha = \frac{1}{3}, \quad 9\beta^2 + 3\gamma^2 = 4 \quad (4.2)$$

and

$$(B'_\infty): \quad |\beta| = \gamma = 2, \quad \alpha \sim 0.205, \\ \text{sign}(\beta)\sigma_0 \sim 4.67 - \ln \delta, \quad \delta > 0, \quad \phi_0 \sim 3.62. \quad (4.3)$$

Furthermore, A_∞ can also be extended with the dilaton of the form $\Phi = \phi_0 + \phi_1 t^\gamma$,

$$(A''_\infty): \quad \alpha = \frac{1}{3}, \quad |\beta| = \frac{2}{3}, \quad \gamma = -2, \quad \phi_1 = -\frac{1}{54}\exp(\phi_0). \quad (4.4)$$

In the above solutions the dilaton field either grows to plus or minus infinity (A'_∞ or A''_∞ and B'_∞) corresponding to weak or strong string coupling, respectively, or it asymptotically reaches an arbitrary constant value (A''_∞). As in the metric–modulus case, in the solution A'_∞ the Gauss–Bonnet terms become irrelevant at large times.

Similarly, the asymptotically flat solution C_∞ can be extended with the dilaton being either of the form $\Phi = \phi_0 + \phi_1 t^\gamma$,

$$(C'_\infty): \quad \alpha = -1, \quad |\beta| = -\gamma = 5, \\ \exp[\text{sign}(\beta)\sigma_0] = -\frac{15}{2\pi\omega_1^3\delta}, \quad \phi_1 = \frac{1}{5}\exp(\phi_0)\omega_1^3, \quad (4.5)$$

or of the form $\Phi = \phi_0 + \gamma \ln t$,

$$(C''_\infty): \quad \alpha = -1, \quad |\beta| = \gamma = 5, \\ \exp(\phi_0) = -\frac{5}{6\omega_1^3}, \quad \delta > 0, \quad \exp[\text{sign}(\beta)\sigma_0] = -\frac{15}{2\pi\omega_1^3\delta} \quad (4.6)$$

In C'_∞ the dilaton goes to a constant, while C''_∞ is a weak coupling solution.

There are also three new solutions where the modulus field goes asymptotically to an arbitrary constant. They are obtained from A''_∞ (D_∞ , D'_∞) or C'_∞ (E_∞) by interchanging the role of σ and Φ , $\sigma = \sigma_0 + \sigma_1 t^\beta$ and $\Phi = \phi_0 + \gamma \ln t$,

$$(D_\infty): \quad \alpha = \frac{1}{3}, \quad |\gamma| = \frac{2}{\sqrt{3}}, \quad \beta = -2, \quad \sigma_1 = \frac{1}{162} \delta \xi'(\sigma_0), \quad (4.7)$$

$$(D'_\infty): \quad \alpha \sim 0.223, \quad \beta = -\gamma = -2, \\ \sigma_1 \sim 0.002 \delta \xi'(\sigma_0), \quad \phi_0 \sim 3.24, \quad (4.8)$$

$$(E_\infty): \quad \alpha = -1, \quad -\beta = \gamma = 5, \\ \exp(\phi_0) = -\frac{5}{6\omega_1^3}, \quad \xi'(\sigma_0) = -\frac{15\sigma_1}{\delta\omega_1^3}, \quad (4.9)$$

with $\sigma_0 \neq 0$.

The same procedure can be followed to derive the singular solutions in the limit $t \rightarrow 0$. The solution A_0 is extended with the dilaton field going to a constant, $\Phi = \phi_0 + \phi_1 t^\gamma$,

$$(A'_0): \quad \alpha = 1, \quad \beta = \gamma = 2, \\ \sigma_1 = \frac{\delta \xi'(\sigma_0)}{\delta^2 \xi'(\sigma_0)^2 + 3 \exp(2\phi_0)}, \quad \phi_1 = -\frac{3 \exp(\phi_0)}{\delta^2 \xi'(\sigma_0)^2 + 3 \exp(2\phi_0)}. \quad (4.10)$$

Furthermore, for the same form of ω and $\delta < 0$, two new solutions can be obtained with the modulus behaving logarithmically, $\sigma = \sigma_0 + \beta \ln t$, and the dilaton being either of the form $\Phi = \phi_0 + \gamma \ln t$,

$$(B_0): \quad \alpha = 1, \quad |\beta| = -\gamma, \quad \exp(\phi_0) = -\frac{\delta\pi}{3\delta} \exp[-\text{sign}(\beta)\sigma_0], \quad (4.11)$$

or of the form $\Phi = \phi_1 t^\gamma$,

$$(C_0): \quad \alpha = 1, \quad |\beta| = -\gamma = -2, \quad \phi_1 = -\sqrt{-\pi\delta} \exp[-\frac{1}{2}\text{sign}(\beta)\sigma_0]. \quad (4.12)$$

B_0 is a weak coupling solution, while C_0 is a strong coupling one.

Finally, for the same logarithmic behavior of ω and $\delta > 0$, there is one more singular solution,

$$(D_0): \quad \Phi = \phi_0 + \gamma \ln t, \quad \sigma = \sigma_0 \exp \frac{-\beta}{t}, \quad (4.13)$$

with

$$\gamma = 2, \quad 3\alpha^3 - 3\alpha^2 + 5\alpha - 1 = 0, \quad \beta = \delta \xi''(0) \alpha^3 (\alpha - 1) > 0,$$

$$\exp(\phi_0) = \frac{2(3\alpha - 1)}{3(1 - \alpha)\alpha^3}, \quad (4.14)$$

which leads to $\alpha \sim 0.223$, $\beta \sim 0.019\delta$, $\phi_0 \sim 3.24$. This is a strong coupling asymptotic solution where the modulus field approaches the self-dual point in a nonanalytic way. It provides an example of realizing the mechanism described in sect. 3 according to which the Gauss–Bonnet term in (2.14) acts as a potential for the modulus with a minimum at the self-dual point.

The integration of the nonlinear system (2.12), (2.14), (2.15) can be performed numerically following a similar procedure we used in the previous section for the metric–modulus case. However, in the presence of the dilaton the phase space is enlarged considerably since two more initial values are required for Φ and $\dot{\Phi}$ in addition to σ , ω and $\dot{\omega}$. Instead of presenting a detailed numerical investigation which will not be very illuminating, we concentrate to the analysis of some physically interesting cases. In the case $\delta > 0$, we verified that the fixed modulus singular solution D_0 (4.14) is obtained for a large region of the parameter space. It turns out that the modulus field at $t = 0$ approaches its self-dual point by dumping oscillations consistently with the effective potential interpretation (see fig. 6).

The case $\delta < 0$ is more interesting since it admits nonsingular solutions. They could in principle be derived by the method described in sect. 3 which consists of

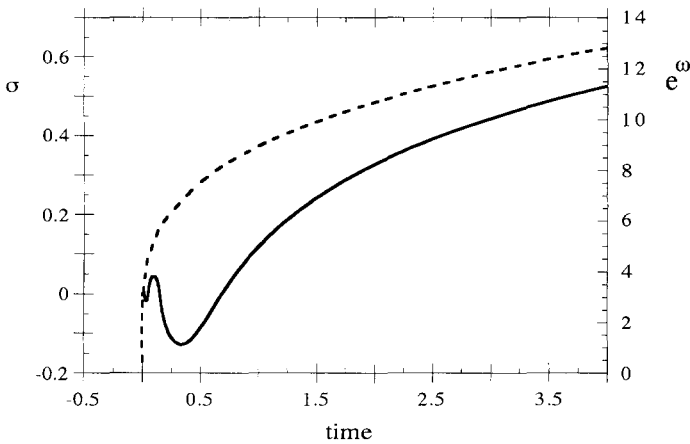


Fig. 6. The scale factor e^ω (dashed line) and the modulus σ (continuous line) in a singular solution ($\delta > 0$) where the modulus approaches its self-dual point ($\sigma = 0$) at early times.

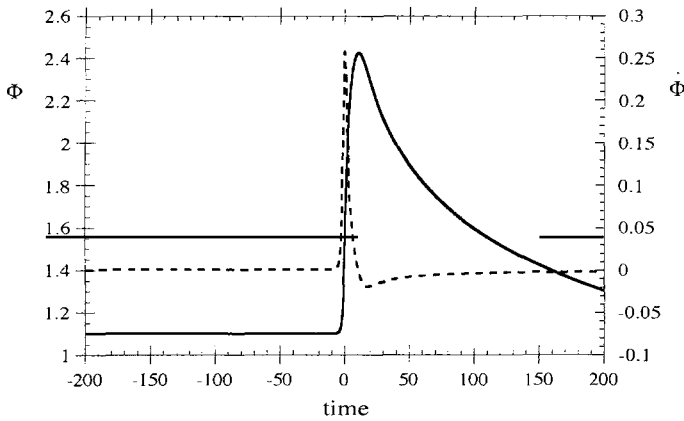


Fig. 7. The dilaton field Φ (continuous line) and its derivative $\dot{\Phi}$ (dashed line) for the nonsingular solution.

starting the numerical integration at a point where the energy conditions are violated, for instance when $\ddot{\omega} = 0$. It turns out that this method is not very useful because, in this case, the violation of energy conditions although necessary is not sufficient to avoid the initial singularity [8]. An alternative way would be to start the integration with an asymptotically flat solution in the infinite past, which in the metric-modulus case was shown to lead always to nonsingular solutions. This turns out to be the case even in the presence of the dilaton. Starting from C'_∞ , which extends C_∞ with a negligible dilaton, one is smoothly driven to A'_∞ in the infinite future. This confirms that the main characteristics of such solutions are a consequence of the modulus-dependent string loop correction to the Gauss-Bonnet term and they do not depend on the existence of the dilaton.

In a typical nonsingular solution the scale factor and the internal radius behave similarly to those obtained in the absence of the dilaton (figs. 1, 2). The dilaton evolution is presented in fig. 7; it starts from a constant value in the remote past and grows logarithmically towards a strong coupling in the future. During the inflationary period when the modulus passes through its self-dual point, the dilaton also jumps to its maximum value. This can be easily understood by inspection of its equation of motion (2.15), where the Gauss-Bonnet term plays the role of a runaway potential for $\dot{\omega} \sim \text{const}$. In the limit $t \rightarrow \infty$, although a weak coupling solution of the form A'_∞ or an asymptotically constant dilaton of the form A''_∞ are not a priori excluded, strong coupling seems to be preferred at least in the restricted region of the parameter space we scanned. In any case, at late times the effective action (2.6) should probably be modified by the addition of a dilaton and modulus potential arising from supersymmetry breaking or other nonperturbative effects, which would stabilize these fields.

5. Conclusions

In this work we have examined the cosmological implications of the moduli-dependent loop corrections to the gravitational couplings of the superstring effective action in the case of orbifold compactifications. These corrections consist of the Gauss–Bonnet integrand multiplied by a universal nontrivial function of the moduli fields and a numerical coefficient δ which depends on the massless spectrum of every particular model. We first derived the equations of motion for the metric, the dilaton and the modulus corresponding to the common compactification radius, and we classified all asymptotic solutions. Among them, there is one where the internal radius is fixed at its self-dual point as the universe approaches the initial singularity.

In the case of negative sign for the parameter δ , we have shown that the strong energy condition $\rho + p > 0$ associated to the stress–energy tensor of the modulus can be violated, leading to an inflationary period and providing the possibility of avoiding the initial singularity. In fact a numerical analysis of the system has verified the existence of a class of nonsingular solutions which interpolate between an asymptotically flat and a slowly expanding universe with a period of rapid expansion, when the modulus field passes through its self-dual value. This is in contrast to the behavior of other nonsingular solutions which have been proposed in the literature, where de-Sitter space was always obtained at “early” times [6,7], but it shares the properties of a “pre-big-bang” phase proposed in ref. [2].

Our solutions depend crucially on the form of the modulus-dependent string loop corrections, while the dilaton contribution is negligible and can be ignored. Furthermore all time derivatives can remain bounded in the perturbative regime, which is consistent with our approximation of neglecting higher-derivative terms in the effective action, provided the moduli-dependent coefficients of the higher-order terms are not large enough to compensate the derivative suppression. Finally, the whole parameter range has been explored in the absence of the dilaton by the study of the corresponding phase diagram and we found that the class of nonsingular solutions extends over an infinite region of the phase space.

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