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ScienceDirect
Nuclear Physics B 812 (2009) 103–127
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Spinor–vector duality in heterotic SUSY vacua

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Received 26 August 2008; accepted 3 December 2008

Available online 11 December 2008

Abstract

We elaborate on the recently discovered spinor–vector duality in realistic free fermionic heterotic vacua. We emphasize the interpretation of the freely-acting orbifolds carried out on the six internal dimensions as coordinate-dependent compactifications; they play a central role in the duality, especially because of their ability to break the right-moving superconformal algebra of the space–time supersymmetric heterotic vacua. These considerations lead to a simple and intuitive proof of the spinor–vector duality, and to the formulation of explicit rules to find the dual of a given model. We discuss the interest of such a duality, notably concerning the structure of the space of vacua of superstring theory.

1. Introduction

Heterotic string theory [1] is a preferred candidate to build realistic string theories. Indeed, its structure allows a large variety of gauge groups, derived from the breaking of the original $SO(32)$ or $E_8 \times E_8$ 10-dimensional gauge group upon compactification [2]. These groups include usual

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doi:10.1016/j.nuclphysb.2008.12.007

grand unification groups such as $SO(10)$ or $SU(5)$, usually arising from the breaking of the E_6 gauge group present in a $N = (2,2)$ Calabi–Yau compactification of heterotic string theories.

One expects a realistic theory to have $\mathcal{N} = 1$ (which is further spontaneously broken) four-dimensional supersymmetry. In our framework, this is achieved by compactifying the six internal dimensions on a $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. This procedure initially breaks supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 1$. The last breaking $\mathcal{N} = 1 \to \mathcal{N} = 0$ is assumed to be realized either by non-perturbative phenomena or by (geometric or non-geometric) fluxes [3]. The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold framework also has the advantage to have three $\mathcal{N} = 2$ twisted sectors, which can lead naturally to a realization of models with three generations [4–8].

The models we are going to be interested in are built using the so-called fermionic construction [5], where the Weyl anomaly is cancelled by inclusion of free fermionic degrees of freedom on the world-sheet. Over the years, several string-derived realistic models have been constructed using this formalism [6]. It is known [7,8] that such models reproduce a wide variety of compactifications, toroidal or more generally Calabi–Yau, at special points of their moduli space. A particular model is specified by a basis of sets of fermions, or more precisely by summation over a set of spin structures authorized for the fermions. In this procedure, standard $\mathbb{Z}_2$ freely-acting and non-freely acting orbifolds are encoded in a very natural way, which arises from the properties of fermionization when the internal manifold is at the extended symmetry point, referred to as the *fermionic point*. Placing ourselves at this specific point of the moduli space of the theory is not very restrictive: indeed, if one chooses to deform these models in order to move away from this point, the form of the twisted sectors, and therefore the chiral matter content of the model, is unchanged as these sectors are insensitive to the geometry of the compactification manifold [4,8,9]. The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold breaking the supersymmetry to $\mathcal{N} = 1$ is realized by means of the introduction of two sets of fermions, that we will call b_1 and b_2. We finally have to specify the value of various discrete torsion coefficients, defining the action of the generalized GSO projections present in the construction; this specification, among other things, encodes the precise effect of all the orbifoldings that have been introduced.

In this paper, we will focus on a duality that has been pointed out in a recent work [4], where several properties of all possible heterotic $\mathbb{Z}_2 \times \mathbb{Z}_2$ models have been detailed, by means of a computerized statistical study of their massless spectra. This study has been restricted to a subclass of models closely resembling the usual three generation realistic string models, where the gauge group yielded by the free fermions include a factor $SO(10)$. This duality exchanges, within the three twisted sectors of the orbifold, the number of vectorial representations of $SO(10)$ with the number of spinorial plus anti-spinorial representations of $SO(10)$. Starting from obviously self-dual cases, namely the cases where the $SO(10)$ gauge is extended to E_6, which can be linked to the usual $N = (2,2)$ compactifications on Calabi–Yau surfaces, we will be able to project out some of the representations of $SO(10)$ by suitable freely-acting orbifolds, therefore explicitly creating dual pairs of models in a straightforward way. We will be able to construct the dual model of some generic model, which will prove the duality. As noted in previous work, this duality is realized internally in each twisted sector. Consequentially, the duality has been shown to hold in $\mathcal{N} = 2$ theories as well (as $\mathcal{N} = 2$ supersymmetry is conserved in each of the twisted sectors). The mechanism of the proof can be adapted in a straightforward way to this case.

The main ingredient of the construction will be to consider the effect of freely-acting orbifolds. These orbifolds, when carried out in the simplest way, correspond to the modding out of a half-shift symmetry $X \to X + \pi R$ on an internal boson X. In this case, the generated twisted sectors are massive; without further hypotheses, the mass shift does not depend on the various representations to which the states belong. However, in a particular framework, the freely-acting orbifold can break a symmetry by lifting the mass degeneracy between the symmetry partners. This happens if, in addition to the translation, we consider modding out a parity operator, discriminating states having different charges under a symmetry group. As a result, states with different charges will undergo different mass shifts, leading to a *spontaneous* breaking of symmetry. This mechanism is the stringy generalization [3] of the field-theoretic Scherk–Schwarz compactification [10]; it can be used to spontaneously break supersymmetry, when the parity operator is chosen to be the space–time helicity of the string state [3]. More generally, various patterns of spontaneous SUSY breaking are obtained by choosing an arbitrary R-symmetry charge (see for example [11] for a recent cosmological application of these constructions).

This enables us also to break an internal superconformal algebra, relating vectorial and spinorial representations of some gauge group of the theory. The current transforming the spinorial representation into the vectorial one and vice-versa is part of the right-hand side of the $N=(2,2)$ superconformal algebra present in the model in the case of an unbroken E_6. By doing a Scherk–Schwarz compactification of an internal direction coupled to the helicity associated to the different representations of the gauge group, one is then able to break this superconformal symmetry, discriminating vectorial and spinorial representations by creating a mass gap.

In the first part of this paper, we will review the free fermionic setup used to construct the class of models we will be interested in. Then we will detail how one can implement freely-acting orbifolds with the sets we introduced, how these freely-acting orbifolds can be used for the spontaneously breaking of some symmetry, and how it can, in our case, lift the mass degeneracy between the spinorial/anti-spinorial representations of $SO(10)$ and the vectorial representations of $SO(10)$. In a third part, we will focus on one twisted plane (that is, one family of twisted sectors) of the theory. We will start by considering one specific model in the first twisted plane, and detail its massless spectrum. Then, we will enunciate the rules to construct the $(S_t \leftrightarrow V)$-dual of a model, and apply them on the model we just constructed. We will also give some tools to perform this duality directly on the partition function of the theory. Finally, we will conclude by some remarks on the significance of this duality, especially regarding the structure of the vacua of $\mathcal{N}=1$ heterotic string theories.

2. Free fermionic construction

2.1. $\mathcal{N}=1$ and $\mathcal{N}=2$ parity set basis and partition function

Starting for a four-dimensional superstring theory made out of free fermions [5], the 20 left-moving fermions are noted, following Refs. [9,12]

$$\left\{\psi^\mu, \chi^{1\ldots6}, y^{1\ldots6}, \omega^{1\ldots6}\right\} \tag{2.1}$$

and the 44 right-moving ones

$$\left\{\bar{y}^{1\ldots6}, \bar{\omega}^{1\ldots6}, \bar{\psi}^{1\ldots5}, \bar{\eta}^{1\ldots3}, \bar{\phi}^{1\ldots8}\right\} \tag{2.2}$$

where the $\bar{\psi}$'s, $\bar{\eta}$'s and $\bar{\phi}$'s are *complex* fermions. These notations fixed, we are considering the sets

$$\begin{aligned}
&F=\left\{\psi^\mu, \chi^{1\ldots6}, y^{1\ldots6}, \omega^{1\ldots6}\middle|\bar{y}^{1\ldots6}, \bar{\omega}^{1\ldots6}, \bar{\psi}^{1\ldots5}, \bar{\eta}^{1\ldots3}, \bar{\phi}^{1\ldots8}\right\},\\
&S=\left\{\psi^\mu, \chi^{1\ldots6}\right\}, \qquad e_i=\left\{y^i, \omega^i|\bar{y}^i, \bar{\omega}^i\right\} \quad [i=1\ldots6],\\
&b_1=\left\{\chi^{3\ldots6}, y^{3\ldots6}\middle|\bar{y}^{3\ldots6}, \bar{\psi}^{1\ldots5}, \bar{\eta}^1\right\},
\end{aligned}$$

$$b_2 = \{\chi^{1,2,5,6}, y^{1,2,5,6} | \bar{y}^{1,2,5,6}, \bar{\psi}^{1\ldots5}, \bar{\eta}^2\},$$
$$z_1 = \{\bar{\phi}^{1\ldots4}\}, \qquad z_2 = \{\bar{\phi}^{5\ldots8}\}. \tag{2.3}$$

Noting additively the usual composition law of the free fermionic formalism, we will use that

$$x = \{\bar{\psi}^{1\ldots5}, \bar{\eta}^{1,2,3}\} = F + S + \sum_i e_i + z_1 + z_2 \tag{2.4}$$

and

$$b_3 = b_1 + b_2 + x = \{\chi^{1\ldots4}, y^{1\ldots4} | \bar{y}^{1\ldots4}, \bar{\psi}^{1\ldots5}, \bar{\eta}^3\} \tag{2.5}$$

are part of the vacua of the theory. Note that the case of a $\mathcal{N} = 2$ theory is treated by considering the previous set, amputated of b_2. This has the effect of considering a $T^4/\mathbb{Z}_2 \times T^2$ orbifold instead of a $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$. The duality also holds in this case, as we will see from the mechanism of construction that the duality holds separately in each twisted sector; and within a twisted sector, $\mathcal{N} = 2$ supersymmetry is preserved.

The generic form of this partition function is quite lengthy but useful. We note, as an index of the various blocks, the corresponding degrees of freedom. Noting for brevity $h_3 = -h_1 - h_2$, it reads:

$$\begin{aligned}
Z_{\mathcal{N}=1} = \int_F \frac{d^2\tau}{\tau_2^2} \frac{\tau_2^{-1}}{\eta^{12}\bar{\eta}^{24}} \frac{1}{2^2}
&\times \sum_{h_i,g_i} \left(\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \vartheta \begin{bmatrix} a+h_1 \\ b+g_1 \end{bmatrix} \vartheta \begin{bmatrix} a+h_2 \\ b+g_2 \end{bmatrix} \vartheta \begin{bmatrix} a+h_3 \\ b+g_3 \end{bmatrix} \right)_{\psi^\mu,\chi} \\
&\times \left(\frac{1}{2} \sum_{\epsilon,\xi} \bar{\vartheta} \begin{bmatrix} \epsilon \\ \xi \end{bmatrix}^5 \bar{\vartheta} \begin{bmatrix} \epsilon+h_1 \\ \xi+g_1 \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \epsilon+h_2 \\ \xi+g_2 \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \epsilon+h_3 \\ \xi+g_3 \end{bmatrix} \right)_{\bar{\psi}^{1\ldots5},\bar{\eta}^{1,2,3}} \\
&\times \left(\frac{1}{2} \sum_{H_1,G_1} \frac{1}{2} \sum_{H_2,G_2} (-)^{H_1G_1+H_2G_2} \bar{\vartheta} \begin{bmatrix} \epsilon+H_1 \\ \xi+G_1 \end{bmatrix}^4 \bar{\vartheta} \begin{bmatrix} \epsilon+H_2 \\ \xi+G_2 \end{bmatrix}^4 \right)_{\bar{\phi}^{1\ldots8}} \\
&\times \left(\sum_{s_i,t_i} \Gamma_{6,6} \begin{bmatrix} h_i|s_i \\ g_i|t_i \end{bmatrix} \right)_{(y\omega\bar{y}\bar{\omega})^{1\ldots6}} e^{i\pi\Phi(\gamma,\delta,s_i,t_i,\epsilon,\xi,h_i,g_i,H_1,G_1,H_2,G_2)},
\end{aligned} \tag{2.6}$$

where the internal twisted/shifted (6, 6) lattice is given by

$$\begin{aligned}
\Gamma_{6,6} \begin{bmatrix} h_i|s_i \\ g_i|t_i \end{bmatrix} = \frac{1}{2^6} \sum_{\gamma_i,\delta_i} &\left(\left| \vartheta \begin{bmatrix} \gamma_1+h_1 \\ \delta_1+g_1 \end{bmatrix} \right| \left| \vartheta \begin{bmatrix} \gamma_1 \\ \delta_1 \end{bmatrix} \right| (-)^{\gamma_1t_1+\delta_1s_1+s_1t_1} \right)_{(y\omega\bar{y}\bar{\omega})^1} \\
&\times \left(\left| \vartheta \begin{bmatrix} \gamma_2+h_1 \\ \delta_2+g_1 \end{bmatrix} \right| \left| \vartheta \begin{bmatrix} \gamma_2 \\ \delta_2 \end{bmatrix} \right| (-)^{\gamma_2t_2+\delta_2s_2+s_2t_2} \right)_{(y\omega\bar{y}\bar{\omega})^2} \\
&\times \left(\left| \vartheta \begin{bmatrix} \gamma_3+h_2 \\ \delta_3+g_2 \end{bmatrix} \right| \left| \vartheta \begin{bmatrix} \gamma_3 \\ \delta_3 \end{bmatrix} \right| (-)^{\gamma_3t_3+\delta_3s_3+s_3t_3} \right)_{(y\omega\bar{y}\bar{\omega})^3} \\
&\times \left(\left| \vartheta \begin{bmatrix} \gamma_4+h_2 \\ \delta_4+g_2 \end{bmatrix} \right| \left| \vartheta \begin{bmatrix} \gamma_4 \\ \delta_4 \end{bmatrix} \right| (-)^{\gamma_4t_4+\delta_4s_4+s_4t_4} \right)_{(y\omega\bar{y}\bar{\omega})^4} \\
&\times \left(\left| \vartheta \begin{bmatrix} \gamma_5+h_3 \\ \delta_5+g_3 \end{bmatrix} \right| \left| \vartheta \begin{bmatrix} \gamma_5 \\ \delta_5 \end{bmatrix} \right| (-)^{\gamma_5t_5+\delta_5s_5+s_5t_5} \right)_{(y\omega\bar{y}\bar{\omega})^5}
\end{aligned}$$

$$\times \left(\left| \vartheta \begin{bmatrix} \gamma_6 + h_3 \\ \delta_6 + g_3 \end{bmatrix} \right| \left| \vartheta \begin{bmatrix} \gamma_6 \\ \delta_6 \end{bmatrix} \right| (-)^{\gamma_6 t_6 + \delta_6 s_6 + s_6 t_6} \right)_{(y\omega\bar{y}\bar{\omega})^6}. \tag{2.7}$$

Here $e^{i\pi\Phi}$ is a global phase whose effect is to implement the various GGSO projections acting on the spectrum of this theory. Following the formalism of [5], these GGSO projections are equivalently defined by the coefficients $C_{(v_i|v_j)} \equiv [v_i|v_j]$, where v_i and v_j are the vectors of (2.3).

This phase is required to satisfy modular invariance constraints, that is, it must be invariant under the following transformations:

$$\tau \to \tau + 1 \Rightarrow \begin{cases} (a,b) \to (a, a+b+1), \\ (\gamma_i, \delta_i) \to (\gamma_i, \gamma_i + \delta_i + 1), \\ (\epsilon, \xi) \to (\epsilon, \epsilon + \xi + 1), \\ (h_i, g_i) \to (h_i, h_i + g_i), \\ (H_i, G_i) \to (H_i, H_i + G_i), \\ (s_i, t_i) \to (s_i, s_i + t_i), \end{cases}$$

$$\tau \to -1/\tau \Rightarrow \begin{cases} (a,b) \to (b, a), \\ (\gamma_i, \delta_i) \to (\delta_i, \gamma_i), \\ (\epsilon, \xi) \to (\xi, \epsilon), \\ (h_i, g_i) \to (g_i, h_i), \\ (H_i, G_i) \to (G_i, H_i), \\ (s_i, t_i) \to (t_i, s_i). \end{cases} \tag{2.8}$$

Here we may make some remarks.

- The global phase Φ does not depend on the spin structure of the space–time fermions, (a, b). This is necessary to preserve $\mathcal{N} = 1$ supersymmetry; otherwise supersymmetry is spontaneously broken, as the gravitini acquire a mass. We will not consider this mechanism here. Note however that the construction of a realistic model also requires such a breaking.
- We want to emphasize the physical meaning of the parameter ϵ in the expression (2.6). As the $\bar{\psi}$ block corresponds to the representations of $SO(10)$, ϵ is the associated chirality: spinorials of $SO(10)$ have $\epsilon = 1$, whereas vectorials have $\epsilon = 0$. We will relate this later to the right-moving SCFT of the model; breaking this SCFT will be done by assuming a non-trivial dependence of the global phase Φ of the spin-structure (ϵ, ξ).
- The inclusion of (s_i, t_i) performs additional shifts on the six (fermionized) internal dimensions compactified on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$. These shifts correspond to the presence of the sets (e_i) in the parity basis; similarly, the twisting parameters (H_i, G_i) account for the presence of the sets z_i. Coupling these parameters to various spin structures by a suitable form of the phase Φ will generate the Scherk–Schwarz symmetry breakings we will consider.

2.2. *SO*(10) *models as Gepner-map duals of Type II models*

The model we have considered above is in fact obtained directly from a Type II model by a map introduced in [13]. This map defines a correspondence between a heterotic model and a Type II model by the following construction.

If we label $B_{\lambda=1,2,3,4}$ the four characters of $SO(8)$ O_8, V_8, S_8, C_8, one can write a generic Type II partition function in the following form

$$Z_{\rm II} = \frac{1}{\tau_2^4 \eta^8 \bar{\eta}^8} \sum_{\lambda,\bar{\lambda}} B_\lambda \bar{B}_{\bar{\lambda}} Z_{\lambda,\bar{\lambda}}. \tag{2.9}$$

Here, $Z_{\lambda,\bar{\lambda}}$ account for the spin–statistics of the model and, in the case of compactified theories, for the internal lattices. The general procedure[2] is then to replace the $SO(2d)$ characters of the right-moving side of the theory by $SO(8+2d) \times E_8$ characters, so that the modular properties of the partition function are preserved. The product only involves the singlet character of E_8, whereas the map for the $SO(2d)$ characters is done as follows:

$$\bar{O}_{2d} \to \bar{V}_{2d+8}, \quad \bar{V}_{2d} \to \bar{O}_{2d+8}, \quad \bar{S}_{2d} \to -\bar{S}_{2d+8}, \quad \bar{C}_{2d} \to -\bar{C}_{2d+8}. \tag{2.10}$$

In particular, for the usual IIA and IIB space–time fermions blocks, $d=4$ and the replacement is done by

$$\frac{1}{2}\sum_{\bar{a},\bar{b}} (-)^{\bar{a}+\bar{b}} \bar{\vartheta}\begin{bmatrix}\bar{a}\\ \bar{b}\end{bmatrix}^4 \to \left[\frac{1}{2}\sum_{\bar{a},\bar{b}} (-)^{\bar{a}\bar{b}} \bar{\vartheta}\begin{bmatrix}\bar{a}\\ \bar{b}\end{bmatrix}^8\right] \times \frac{1}{2}\sum_{\bar{\gamma},\bar{\delta}} \bar{\vartheta}\begin{bmatrix}\bar{\gamma}\\ \bar{\delta}\end{bmatrix}^8, \tag{2.11}$$

$$\frac{1}{2}\sum_{\bar{a},\bar{b}} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} \bar{\vartheta}\begin{bmatrix}\bar{a}\\ \bar{b}\end{bmatrix}^4 \to \left[\frac{1}{2}\sum_{\bar{a},\bar{b}} \bar{\vartheta}\begin{bmatrix}\bar{a}\\ \bar{b}\end{bmatrix}^8\right] \times \frac{1}{2}\sum_{\bar{\gamma},\bar{\delta}} \bar{\vartheta}\begin{bmatrix}\bar{\gamma}\\ \bar{\delta}\end{bmatrix}^8. \tag{2.12}$$

We see that the reversal of the sign of the fermionic characters breaks the usual spin–statistics, so that, from a space–time point of view, this operation has traded a supersymmetric sector for a purely bosonic sector. Following our notations for the free fermionic degrees of freedom and their obvious extension to Type II models, the mapping Type II → Heterotic is done by replacing the free fermions of Type II $\{\bar{\psi}^\mu, \bar{\chi}^{1\ldots6}\}$ by the free fermions of the heterotic $\{\bar{\psi}^{1\ldots5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1\ldots8}\}$. Also note that in both Type IIA and Type IIB cases, the obtained block is in fact a second copy of the singlet of E_8, which signals an enhancement of $SO(16)$ to E_8.

Carrying out the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold on both of these models, we see that the heterotic model we consider in this paper is no other than the Gepner-map of a Type II $\mathcal{N}_4 = 2$ model, *via* the mapping

$$\begin{aligned}&\frac{1}{2}\sum_{\bar{a},\bar{b}} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} \bar{\vartheta}\begin{bmatrix}\bar{a}\\ \bar{b}\end{bmatrix} \bar{\vartheta}\begin{bmatrix}\bar{a}+h_1\\ \bar{b}+g_1\end{bmatrix} \bar{\vartheta}\begin{bmatrix}\bar{a}+h_2\\ \bar{b}+g_2\end{bmatrix} \bar{\vartheta}\begin{bmatrix}\bar{a}+h_3\\ \bar{b}+g_3\end{bmatrix} \\ &\quad\longrightarrow \left[\frac{1}{2}\sum_{\bar{a},\bar{b}} \bar{\vartheta}\begin{bmatrix}\bar{a}\\ \bar{b}\end{bmatrix}^5 \bar{\vartheta}\begin{bmatrix}\bar{a}+h_1\\ \bar{b}+g_1\end{bmatrix} \bar{\vartheta}\begin{bmatrix}\bar{a}+h_2\\ \bar{b}+g_2\end{bmatrix} \bar{\vartheta}\begin{bmatrix}\bar{a}+h_3\\ \bar{b}+g_3\end{bmatrix}\right] \times \frac{1}{2}\sum_{\bar{\gamma},\bar{\delta}} \bar{\vartheta}\begin{bmatrix}\bar{\gamma}\\ \bar{\delta}\end{bmatrix}^8.\end{aligned} \tag{2.13}$$

One recognizes the block of (2.6) corresponding to the $\bar{\psi}$'s and $\bar{\eta}$'s. The second block accounts for an E_8 gauge group formed by the complex fermions $\bar{\phi}^{1\ldots8}$; generically, this group will be broken due to the inclusion of the sets z_1 and z_2 in our construction.

Out of the two four-dimensional supersymmetries of the Type II model, only the left-moving one is still present in the heterotic; however, the right-moving superconformal algebra survives the mapping. This is nothing but the embedding of the spin connection of Type II models into the

[2] There exists a second solution, which is the replacement by $SO(32)$ characters.

connection of the corresponding heterotic ones. Then, this superconformal algebra does not give birth to a space–time SUSY, but relates spinors to vectors, belonging to representations which are now of the internal $SO(10)$ spanned by the $\bar{\psi}$'s. The survival of this symmetry will guarantee the existence at the massless level of what were formerly right-moving gravitinos and are now gauge bosons in a spinorial of $SO(10)$: then, $SO(10) \times U(1)^3$ gets enhanced to $E_6 \times U(1)^2$. This enhancement comes as no surprise from the Calabi–Yau point of view: the general embedding of spin-connection into gauge connection singles out a subalgebra $SU(3)$ inside the first E_8, corresponding to the holonomy of the compactification manifold. The anomaly cancellation mechanism [14] then requires that we switch on background values for this $SU(3)$, and the surviving gauge group is E_6, coming from the embedding $SU(3) \times E_6 \subset E_8$. Of course, the Cartans of $SU(3)$ still define a gauge group $U(1)^2$, so that, in the presence of a right-moving $N = 2$ SCFT, we indeed find a gauge group $E_6 \times U(1)^2 \times E_8$. This is realized explicitly in our constructions.

Note that this procedure underlines the naturalness of the appearance of a gauge group $SO(10)$ in $\mathcal{N} = 1$ realistic theories: the Type II right-moving fermionic block made out of S, V, C representations of the Lorentz group $SO(8)$ is traded for a block made out of E_8 characters. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold required to break the four-dimensional supersymmetry $\mathcal{N} = 4 \to 1$ is forced by consistency to act on this E_8, generically breaking it to $E_6 \times U(1)^2$.

We will now enumerate the sectors from which we will be able to build massless states, and identify their interpretation as twisted sectors of the $\mathcal{N} = 4 \to \mathcal{N} = 1 \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

3. Spectrum of the model; superconformal x-map and its spontaneous breaking

3.1. $\mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sectors

It is pretty straightforward to check that the $\mathcal{N} = 1$ supersymmetric partner of a state built on some vacuum $|\alpha\rangle$ will come from the vacuum $|\alpha + S\rangle$. Here, we will therefore restrain our enumeration to the bosonic vacua. Apart form the pure NS vacuum, states can be built from the following sets:

- the 16 twisted sectors $|\mathcal{B}^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle = |b_1 + \sum_{i=3}^{6} \lambda_i e_i\rangle$, where $\lambda_i = 0$ or 1;
- the 16 twisted sectors $|\mathcal{B}^2_{\lambda_1\lambda_2\lambda_5\lambda_6}\rangle = |b_2 + \sum_{i=1,2,5,6} \lambda_i e_i\rangle$, where $\lambda_i = 0$ or 1;
- the 16 twisted sectors $|\mathcal{B}^3_{\lambda_1\lambda_2\lambda_3\lambda_4}\rangle = |b_3 + \sum_{i=1}^{4} \lambda_i e_i\rangle$, where $\lambda_i = 0$ or 1;
- the sectors $|\alpha + x\rangle$, where α is any of the sectors described above;
- the sectors $|z_1\rangle$, $|z_2\rangle$, $|z_1 + z_2\rangle$.

To properly distinguish a particle from its anti-particle, it will be handy to consider instead the fermionic sectors $B \equiv S + \mathcal{B}$, so as the space–time chirality appears in a clear way. We will then restrain ourselves to considering positive ψ^μ-helicity states. In the following, we will denote $|B^1\rangle$ (and similarly for $|B^2\rangle$, $|B^3\rangle$) a generic sector $|B^1_{\lambda_1\lambda_2\lambda_3\lambda_4}\rangle$, and more generally $|B\rangle$ an arbitrary twisted sector. The $|B\rangle$ sectors are in one-to-one correspondence with the fixed points of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold transformation.

Let us make some comments:

- In the following, we will pay no attention to the sectors $|z_1\rangle$, $|z_2\rangle$, $|z_1 + z_2\rangle$, which can lead to additional gauge bosons. The minimal gauge group is $SO(8) \times SO(8)$; as pointed out in [4], appropriate choice of the GGSO phases ensures that this gauge group is not enhanced, and that no mixed[3] massless states appear. In the following, we will assume these no-enhancement hypotheses, which state that there exists e_i and e_j, $i \neq j$, such as $[e_i|z_1] = -1$ and $[e_j|z_2] = -1$. This choice projects out any would-be gauge bosons that would enhance $SO(8) \times SO(8) \to SO(16)$; the largest enhancement one can have in that case is a $SO(8) \to SO(9)$, which can also be eliminated by allowing one more i such as $[e_i|z_1] = -1$; at any rate, there is no mixing between the "observable" gauge and the "hidden" gauge.

- The spinor–vector duality finds its root from the fact that if $|\alpha\rangle$ is a relevant vacuum to build massless states, so is $|\alpha + x\rangle$. This correspondence is the superconformal "x-map" $|B\rangle \mapsto |B + x\rangle$ pointed out in [15]. It is obvious that if (the excitations of) $|\alpha\rangle$ are in the vectorial of the $SO(10)$ induced by the 5 complex fermions $\bar{\psi}^{1\ldots5}$, then $|\alpha + x\rangle$ will belong to a spinorial of the same group; the x-map being an involution, the converse is also true. What is at stake is then to find, given a set of GGSO projections, which sectors will survive; and for each theory, describe the dual theory in terms of the effects of its various GGSO projections.
- An important case of figure brings a self-dual case. When preserving the $N = (0, 2)$ superconformal field theory, the $SO(10)_{\bar{\psi}} \times U(1)_{\bar{\eta}}$, where $U(1)_{\bar{\eta}}$ is the diagonal $U(1)$ induced by $\bar{\eta}^{1,2,3}$, is lifted to E_6. In this case, the vectorial **10** and the spinorial **16** of $SO(10)$ (resp. the anti-spinorial $\overline{\mathbf{16}}$) are grouped in the fundamental **27** (resp. $\overline{\mathbf{27}}$) of E_6, which decomposes as $\mathbf{27} \to \mathbf{10} \oplus \mathbf{16} \oplus \mathbf{1}$ (resp. $\overline{\mathbf{27}} \to \mathbf{10} \oplus \overline{\mathbf{16}} \oplus \mathbf{1}$).

3.2. The x-map and superconformal algebra in representations of $SO(10)$

To begin with, we will restrain ourselves to consider only one twisted sector, namely $B^1_{0000} = S + b_1$. We will note the associated ground state $|B^1_{0000}\rangle$. Our results will easily be extended to any of the 48 twisted sectors detailed above. The untwisted sector, built out of the pure Neveu–Schwarz ground state, gives the gauge bosons of the gauge group, but not the spinorial/vectorial representations we are interested in.

The B^1_{0000} vacuum is then written as

$$B^1_{0000}\colon \quad \mathrm{Spin}\big(\psi^\mu, \chi^{1,2}, y^{3\ldots6}\big) \otimes \mathrm{Spin}\big(\bar{y}^{3\ldots6}, \bar{\psi}^{1\ldots5}, \bar{\eta}^1\big) \tag{3.1}$$

and the addition of the sector x brings the vacuum

$$B^1_{0000} + x\colon \quad \mathrm{Spin}\big(\psi^\mu, \chi^{1,2}, y^{3\ldots6}\big) \otimes \mathrm{Spin}\big(\bar{y}^{3\ldots6}, \bar{\eta}^{2,3}\big). \tag{3.2}$$

Here, one may make a few remarks, which will be valid for any of the 48 twisted sectors. Firstly, due to the presence of 8 left-moving and 16 right-moving real fermions obeying Ramond boundary conditions, the sector $|B^1_{0000}\rangle$ is massless by itself, and contains spin-fields made out of the $SO(10)$ fermions $\bar{\psi}$; it therefore induces a spinorial of $SO(10)$. On the other hand, the sector $|B^1_{0000} + x\rangle$ has 8 left-moving and 8 right-moving Ramond real fermions, so that its ground energies read

$$M_L^2 = 0, \qquad M_R^2 = -\frac{1}{2}.$$

[3] By mixed states, we mean states charged under both the "observable" $SO(10)$ or E_6 and the "hidden" gauge group containing the $SO(8) \times SO(8)$.

A massless state will then be reached when exciting this ground state by a weight 1/2 right-moving fermionic oscillator. If we wish to consider states charged under $SO(10)$, this excitation has to be taken to be $\bar{\psi}^i_{-1/2}$, and the resulting state lies in a vectorial representation of $SO(10)$. Therefore, the x-map links vectorials to spinorials of $SO(10)$. Obviously, the x-map arises as the right-moving part of the $N = (2, 2)$ superconformal field theory that is still present after the Type II → Heterotic Gepner-map, and acts inside the gauge group, due to the embedding of the spin connection into the gauge connection.

As in the case of spontaneous breaking of supersymmetry, a spontaneous breaking of the x-map will amount to projecting out from the spectrum spinorial or vectorial representations of $SO(10)$, giving different masses to the two partners. In terms of the free fermionic construction, this situation is reflected in the fact that states from the massless sector $|B\rangle$ (resp. $|B + x\rangle$) will be projected out, whereas states from the sectors $|B + e_i\rangle$ (resp. $|B + x + e_i\rangle$) will be preserved. These sectors are massive and are naturally interpreted as the twisted sector of the freely-acting orbifold based on the half-shift of the coordinate X^i. We see that the net effect of this action is that the sectors $|B\rangle$ (resp. $|B + x\rangle$) will get a mass, whereas the sectors $|B + x\rangle$ (resp. $|B\rangle$) will remain massless. We carry out an explicit example of such a mass lift in the next subsection; as one can expect, it crucially relies on a careful choice of the GGSO projections.

3.3. Implementing the e_i-generated freely-acting orbifolds

In this subsection, we briefly recall some useful results about twisted/shifted lattices. The usual equivalence between a compact boson taken at the fermionic point and two left-moving plus two right-moving real fermions is easily extended to orbifold partition functions of each theory.

When we consider two internal dimensions, the ϑ-function form of a zero-mode lattice $\Gamma_{2,2}$, taken at the enhanced symmetry (or fermionic) point (denoted f.p.)

$$\Gamma_{2,2}\big|_{\text{f.p.}} = \left(\frac{1}{2}\sum_{\gamma,\delta}\left|\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\right|^2\right)^2 \tag{3.3}$$

is generalized to the orbifold version of the theory. When one implements the non-freely-acting $\mathbb{Z}_2$ orbifold $X^{1,2} \to -X^{1,2}$, whose twisting parameters will be denoted (h, g), as well as the two freely-acting $\mathbb{Z}_2$ orbifolds $X^{1,2} \to X^{1,2} + \pi$, whose shifting parameters will be noted (s_1, t_1, s_2, t_2), the lattice sum is modified as

$$\Gamma_{2,2}\begin{bmatrix}h|s_1,s_2\\ g|t_1,t_2\end{bmatrix}\Bigg|_{\text{f.p.}} = \frac{1}{4}\sum_{\gamma_{1,2},\delta_{1,2}}(-)^{\gamma_1 t_1+\delta_1 s_1+s_1 t_1}(-)^{\gamma_2 t_2+\delta_2 s_2+s_2 t_2}$$
$$\times\left|\vartheta\begin{bmatrix}\gamma_1+h\\ \delta_1+g\end{bmatrix}\vartheta\begin{bmatrix}\gamma_1\\ \delta_1\end{bmatrix}\vartheta\begin{bmatrix}\gamma_2+h\\ \delta_2+g\end{bmatrix}\vartheta\begin{bmatrix}\gamma_2\\ \delta_2\end{bmatrix}\right|. \tag{3.4}$$

Therefore, implementing in the above partition function the freely-acting orbifolds (in this case, half-way shifts) corresponding to the sets e_i only amounts to inserting the phases $(-)^{\gamma t+\delta s+st}$. For now, we have just shifted the internal $\Gamma_{6,6}$ lattice, independently of the rest of the spectrum. The corresponding orbifold is the $\mathbb{Z}_2$-translation along each circle of the internal space.

If we wish to couple this shift to other states of the theory, we must introduce a phase relating the shift parameters (s_i, t_i) to the spin structures of the states we want to act on. Such a freely-acting orbifold takes the form $(-)^Q \cdot T^i$, where T^i is the $\mathbb{Z}_2$-translation of the ith coordinate $X^i \mapsto X^i + \pi R^i$, and $(-)^Q$ is the parity operator associated to the spin structure we are considering (generalizing the usual fermion counting operator $(-)^F$, which would correspond to coupling to the spin-structure of the space–time fermion spin structure (a, b)).

One can carry out the calculation of the partition function corresponding to this orbifold, by inserting the projection operator in the computation of the trace over physical states and adding the contribution of the twisted sector. The result is that this orbifold is done by simply adding a cocycle in the partition function. As an example, if we consider a $\Gamma_{1,1}$ lattice coupled to some spin structure (ϵ, ξ), the modification is made as follows:

$$\begin{aligned} Z &= [\cdots] \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m},n} \exp\left[-\frac{\pi R^2}{\tau_2} |\tilde{m} + n\tau|^2\right] \\ &\longrightarrow [\cdots] \times \frac{1}{2} \sum_{h,g} (-)^{\epsilon g + \xi h + gh} \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m},n} \exp\left[-\frac{\pi R^2}{\tau_2} \left|\left(\tilde{m} + \frac{g}{2}\right) + \left(n + \frac{h}{2}\right)\tau\right|^2\right] \\ &= [\cdots] \times \sum_{h,g} (-)^{\epsilon g + \xi h + gh} \Gamma_{1,1} \begin{bmatrix} h \\ g \end{bmatrix} \left(\frac{R}{2}\right), \end{aligned} \tag{3.5}$$

where $\Gamma_{1,1}\left[{h \atop g}\right]$ is the shifted $\Gamma_{1,1}$ lattice

$$\Gamma_{1,1} \begin{bmatrix} h \\ g \end{bmatrix} = \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m},n} \exp\left[-\frac{\pi R^2}{\tau_2} \left|(2\tilde{m} + g) + (2n + h)\tau\right|^2\right] \tag{3.6}$$

and the overall $[\cdots]$ refers to all the other blocks of the partition function, which are unchanged in the process.

Setting $R_{\mathrm{SS}} = R/2$, we recover the well-known fact that this mechanism is equivalent to performing a stringy Scherk–Schwarz compactification, which is done by coupling the internal dimension to the $SO(10)$ helicity current [3]

$$\oint (\bar{\psi}^1)^\dagger \bar{\psi}^1.$$

Such a task is achieved by inserting in the concerned partition function block the cocycle

$$(-)^{\epsilon \tilde{m} + \xi n + \tilde{m} n}, \tag{3.7}$$

where now $\tilde{m}$ and n are the momentum/winding numbers of the string state along the radius R_{SS} [3]. Looking at the expressions (3.4) and (3.5), one sees that, since the internal shift parameters of the internal dimensions are no other than (s_i, t_i) that the coupling of the internal shifted lattice to the $SO(10)$ spin-structure (ϵ, ξ) will be done by inserting a phase of the form

$$(-)^{\epsilon t_i + \xi s_i + s_i t_i}. \tag{3.8}$$

It is worth noting that this coupling indeed lifts the mass of the states according to their chirality ϵ: by considering the insertion of the Scherk–Schwarz cocycle (3.7), a Poisson resummation of the modified lattice

$$\frac{R_{\mathrm{SS}}}{\sqrt{\tau_2}} \sum_{\tilde{m},n} (-)^{\epsilon \tilde{m} + \xi n + \tilde{m} n} \exp\left[-\frac{\pi R_{\mathrm{SS}}^2}{\tau_2} |\tilde{m} + n\tau|^2\right] \tag{3.9}$$

shows that the string states now have momentum and winding numbers

$$\left(m - \frac{\epsilon}{2} - \frac{n}{2}, n\right) \tag{3.10}$$

which signals a mass lifting in the $\epsilon = 1$ sector. This procedure is of course encoded in the basic form of the fermionic construction and does not require further elaboration: it is related to the values of the discrete torsions $[e_i|B]$ and $[e_i|B + x]$, where B is an arbitrary twisted sector of the theory.

3.4. Breaking the x-symmetry with the freely-acting orbifold e_i

We start by considering the two sectors already written above, which read, in terms of spin-fields

$$\begin{aligned} B^1_{0000}:&\quad \mathrm{Spin}\big[(\psi^\mu)_+, (\chi^{12})_{\epsilon_2}, (y^{34})_{\epsilon_3}, (y^{56})_{\epsilon_4}\big] \\ &\quad \otimes \mathrm{Spin}\big[(\bar{y}^{34})_{\bar{\epsilon}_1}, (\bar{y}^{56})_{\bar{\epsilon}_2}, (\bar{\psi}^{1\ldots5})_{\bar{\epsilon}_3}, (\bar{\eta}^1)_{\bar{\epsilon}_4}\big], \end{aligned} \tag{3.11}$$

$$\begin{aligned} B^1_{0000}+x:&\quad \mathrm{Spin}\big[(\psi^\mu)_+, (\chi^{12})_{\epsilon_2}, (y^{34})_{\sigma_3}, (y^{56})_{\sigma_4}\big] \\ &\quad \otimes \mathrm{Spin}\big[(\bar{y}^{34})_{\bar{\sigma}_1}, (\bar{y}^{56})_{\bar{\sigma}_2}, (\bar{\eta}^2)_{\bar{\sigma}_3}, (\bar{\eta}^3)_{\bar{\sigma}_4}\big], \end{aligned} \tag{3.12}$$

where the $\epsilon_i, \bar{\epsilon}_i, \sigma_i, \bar{\sigma}_i$ are the helicities of the spin-fields.

As discussed above, the physical states of the sector $B^1_{0000} + x$ we are interested in are obtained by exciting the vacuum with a weight $1/2\bar{\psi}$ oscillator:

$$\mathrm{Spin}\big[(\psi^\mu)_+, (\chi^{12})_{\epsilon_2}, (y^{34})_{\sigma_3}, (y^{56})_{\sigma_4}\big] \otimes \big[\bar{\psi}^i_{-1/2}\big]\,\mathrm{Spin}\big[(\bar{y}^{34})_{\bar{\sigma}_1}, (\bar{y}^{56})_{\bar{\sigma}_2}, (\bar{\eta}^2)_{\bar{\sigma}_3}, (\bar{\eta}^3)_{\bar{\sigma}_4}\big]. \tag{3.13}$$

The relevant GGSO projections to carry out in this example as those arising from the sets S, $S + b_1$, b_2, $(e_i)_{i=1\ldots6}$. The F-projection is redundant with the $(S + b_1)$-one. The z_i-projections do not change the features of the spectrum in the sector B^1_{0000} as soon as we assume that they do not project the whole sector out; we will, for now, neglect them.

Equivalently, we will find it handy to consider instead, on a sector $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}$ the projections induced by the sets

$$S, \quad S + b_1, \quad \tilde{b}_2 = S + b_2 + (1 - \lambda_5)e_5 + (1 - \lambda_6)e_6, \quad (e_i)_{i=1\ldots6}. \tag{3.14}$$

Recall that, as $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle$ are fermionic sectors, the constraints to be met are $(-)^\alpha = -(\alpha|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6})$, where α is one of the sets above.

Initially, the sectors $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}$ have 2^{12} degrees of freedom. Carrying out the S, $S + b_1$, $\tilde{b}_2$, $(e_{3\ldots6})$ projections cut the number of physical states down to $2^5 = 32$. Noticing that

$$\tilde{b}_2 \cap B^1_{\lambda_3\lambda_4\lambda_5\lambda_6} = \{\psi^\mu|\bar{\psi}^{1\ldots5}\}, \tag{3.15}$$

we see that, as the ψ^μ helicity has been fixed, this GGSO projection implies that the spectrum of states inside the sectors $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}$ is chiral with respect to the group $SO(10)$. Such a feature crucially depends on the presence of the set b_2 in our construction; this is consistent with the fact that the presence of a chiral matter spectrum requires $\mathcal{N} = 1$ space–time supersymmetry.

Now we look at the effect of the e_1 and e_2 projections, first restricting our attention to B^1_{0000}. The latter survives the e_1 projection if $[e_1|B^1_{0000}] = -1$; otherwise the entire sector $|B^1_{0000}\rangle$ is projected out. However, in the latter case, as mentioned earlier, one has to consider the massive sector $|B^1_{0000} + e_1\rangle$. The spin field accounting for this Ramond ground now has an initial degeneracy of 2^{14}; carrying out the S, $S + b_1$, $\tilde{b}_2$, $(e_{1,3\ldots6})$ projections cut the number of degrees of freedom to 2^6. This time, the various projections are not able to fix the $SO(10)$-chirality of the massive state, since

$$\tilde{b}_2 \cap \left(B^1_{\lambda_3\lambda_4\lambda_5\lambda_6} + e_1\right) = \left\{\psi^\mu, \omega^1|\bar{\omega}^1, \bar{\psi}^{1\ldots5}\right\}. \tag{3.16}$$

This is consistent with the fact that when fixing the space–time spin, we still have a degeneracy in the representations **16** and $\overline{\mathbf{16}}$ of $SO(10)$, which is mandatory for these representations to be massive.

The superconformal partner of $|B^1_{0000}\rangle$ is $|B^1_{0000} + x\rangle$; this sector contains vectorial representations of $SO(10)$. Let us recall that, from the usual constraints of the free fermionic models, the discrete torsion coefficients we are interested in obey, for $i = 1, 2$:

$$\left[B^1 + x|e_i\right] = \left[B^1|e_i\right][x|e_i]. \tag{3.17}$$

Therefore, if we set $[x|e_i] = 1$, the sector $|B^1_{0000} + x\rangle$ will behave in the same way as $|B^1_{0000}\rangle$ with respect to the e_i projections. If $[B^1_{0000}|e_i] = 1$, the twisted sector will be projected out as a whole, regardless of the spinorial/vectorial character of the representations; if $[B^1_{0000}|e_i] = -1$, both spinors and vectors will survive.

Up to now, we have thus not been able to discriminate between spinorial and vectorial representations of $SO(10)$ lying in the same twisted sector. As one can expect, this will be done by acting on the value of the discrete torsion $[x|e_i]$. Indeed, let us again place ourselves in the twisted sector $|B^1_{0000}\rangle$, and its vectorial counterpart $|B^1_{0000} + x\rangle$. The same reasoning as before, and the use of Eq. (3.17), yields the following rules of survival (we recall that $\delta_B = -1$ for any fermionic twisted sector):

- when $[B^1_{0000}|e_i] = -1$ and $[x|e_i] = 1$, both sectors $|B^1_{0000}\rangle$ and $|B^1_{0000} + x\rangle$ survive at the massless level;
- when $[B^1_{0000}|e_i] = 1$ and $[x|e_i] = 1$, both sectors $|B^1_{0000}\rangle$ and $|B^1_{0000} + x\rangle$ are projected out;
- when $[B^1_{0000}|e_i] = -1$ and $[x|e_i] = -1$, $|B^1_{0000}\rangle$ survives and $|B^1_{0000} + x\rangle$ is projected out;
- when $[B^1_{0000}|e_i] = 1$ and $[x|e_i] = -1$, $|B^1_{0000}\rangle$ is projected out and $|B^1_{0000} + x\rangle$ survives.

Now that we know how to manipulate each twisted sector, we can start to explore the duality. Note that the list of ingredients at our disposal is quite simple and handy.

We are dealing with three twisted planes, in which four left-moving and four right-moving fermions picked among the fermionized coordinates $(y^i\omega^i)(\bar{y}^i\bar{\omega}^i)$ are in Ramond boundary conditions. These fermions carry indices $(i_1, i_2, i_3, i_4) = (3, 4, 5, 6)$ for the B^1 family, $(1, 2, 5, 6)$ for the B^2 family, and $(1, 2, 3, 4)$ for the B^3 family. We can act on these twisted sectors by making the freely-acting orbifold generated by the set e_i act in a non-trivial way on them. Then one sees that, to be able to project out states, one must consider the action of the sets e_i and e_j, where i and j are different from $i_{1\ldots4}$; otherwise, the e_i-projection's effect is to choose the internal chiralities of the corresponding spin-field. Moreover, if i is one of the four indices $i_{1\ldots4}$, the sector $B + e_i$ is not massive, but rather another twisted sector of the same plane.

Then two projections have to be considered for each twisted plane. In the following, we will be interested in the B^1 plane, so that we will consider the orbifolds induced by e_1 and e_2. This fact is not surprising: in the B^1 plane, the physics is independent of the volume of the four internal coordinates corresponding to the fermions $(y\omega|\bar{y}\bar{\omega})^{3456}$; therefore, a spontaneous breaking of symmetry in this plane must be constructed out of the two last internal coordinates, as the value of the mass gap will depend on the size of these coordinates. Of course, in this paper we will encounter no such dependence, as all moduli are set at the fermionic point; however, a deformation of these models would make this feature clear.

Finally, to compute the action of the orbifolds e_1 and e_2 on one arbitrary sector of the first twisted plane $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle$, we remark that the usual constraints of the fermionic construction impose

$$\left[B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}|e_i\right] = [b_1 + S|e_i] \prod_{j=3}^{6} [e_j|e_i]^{\lambda_j}, \quad i = 1, 2. \tag{3.18}$$

Knowing all the coefficients $[e_i|e_j]$, which are part of the definition of the model, we are then able to repeat the above reasoning to deduce the action of e_1 and e_2 projections on $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle$ and $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6} + x\rangle$.

3.5. The z_i projections

The case of the z_i projections is in many ways similar to the case of the e_i's. This time, as we have, for any twisted sector B of the theory $B \cap z_1 = B \cap z_2 = \varnothing$, any non-trivial discrete torsion turned on for the z sets will have an effect on the three twisted planes. One can derive all the rules in a similar way as for the e_i's: the z_i projections can be taken to break the x superconformal CFT or not, and various combinations of hypotheses on the GGSO yields various cuts in the spectrum of the theory. As this case is identical to the $e_{1,2}$ orbifolds, the rules of the previous subsection apply.

We will often omit the z_i projections, to which most of the rules we derive for the (e_i) projections similarly apply. We will actually specifically need them to perform further cuts in the spectrum, giving us the possibility to restrain the number of representations present in our models.

4. Construction of dual pairs of models

4.1. A class of self-dual models: the E_6 models

As we mentioned previously, since in E_6 models the spectrum arranges itself in fundamental representations **27** and $\overline{\mathbf{27}}$, these models are trivially self-dual.

The gauge group E_6 is present in a model if and only if the x-map is unbroken. This is equivalent to requiring that the freely-acting orbifolds do not break the right-moving part of the $N = (2, 2)$ superconformal algebra of the initial model. In terms of discrete torsion coefficients, this condition is encoded in the equality

$$\forall i = 1 \ldots 6, \quad [x|e_i] = 1, \qquad [x|z_{1,2}] = 1. \tag{4.1}$$

From the considerations of the previous section, it is then obvious that if the above equalities are met, in any twisted sector $|B\rangle$, the representations $(S, V) \subset \mathbf{27}$ and $(\bar{S}, V) \subset \overline{\mathbf{27}}$ will be either simultaneously conserved or simultaneously destroyed, depending on the value of the GGSO coefficients $[B|e_i], [B|z_i]$. Explicitly building the spectrum and counting the states surviving after the application of the various GGSO projections confirms the self-duality; we find that a given twisted sector $|B\rangle$ possesses one $SO(10)$-spinor (chiral or anti-chiral, its chirality being fixed by the $\tilde{b}_2$-projection), one $SO(10)$-vector and one singlet under $SO(10)$, but charged with respect to the additional $U(1)$ of $SO(10) \times U(1) \subset E_6$:

$$|B\rangle: \quad (S, V) \subset \mathbf{27} \quad \text{or} \quad (\bar{S}, V) \subset \overline{\mathbf{27}}. \tag{4.2}$$

When the action of all z_i-induced and e_i-induced freely-acting orbifolds are trivial on the twisted sectors, we find therefore that the model possesses N_+ **27** and N_- $\overline{\mathbf{27}}E_6$ representations, with $N_+ + N_- = 48$. As the various orbifolds act, they are able to cut in each twisted sector, either the vectorial, or the spinorial, or the whole sector. As an example, we consider the twisted sectors $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle$. Depending on the values of the GGSO coefficients $[b_1|e_i]$, $i = 1, 2$, and $[e_j|e_i]$, $i = 1, 2$, $j = 3, 4, 5, 6$, we are able, thanks to the identities

$$\left[B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\middle|e_i\right] = \left[B^1_{0000}\middle|e_i\right]\prod_{j=3}^{6}[e_j|e_i]^{\lambda_j}, \tag{4.3}$$

$$\left[B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\middle|e_i\right] = \left[B^1_{0000}\middle|z_i\right]\prod_{j=3}^{6}[e_j|z_i]^{\lambda_j}, \tag{4.4}$$

to determine the effect of the e_i- and z_i-projections on each one of the twisted sectors of the B^1 plane. In particular, if $[e_k|e_i] = -1$, one sees that the e_i-projection has opposite effects on the sectors $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle$ and $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6} + e_k\rangle$.

4.2. Duality inside the $\mathcal{N} = 2$ sectors

As the classification in [4,9] shows, one can create several kinds of non-self-dual models, in which, in a given twisted plane generated by the sectors $|B_1\rangle$ and $|B_1 + x\rangle$, one has either only spinorials of $SO(10)$ (with either positive or negative chirality; moreover, the number of spinors and antispinors do not have to be equal) or only vectorials. For a non-self dual model, as the x-superconformal map is broken, there exists at least one $i \in \{1 \dots 6\}$ such that $[x|e_i] = -1$ or (inclusive) one $i \in \{1, 2\}$ such that $[x|z_i] = -1$.

Let us start by considering a breaking by e_i. First we argue that the condition $[x|e_i] = -1$ is able to break the self-duality only in the sectors where the freely-acting orbifold e_i has the possibility to project out entire representations of $SO(10)$: namely $i = 1, 2$ for B^1 sectors, $i = 3, 4$ for B^2 sectors, and $i = 5, 6$ for B^3 sectors. Indeed, let us suppose that $[x|e_1] = -1$ while the others $[x|e_i] = 1$, and investigate the consequences on the spectrum. In the B^1 sectors, we have seen in a previous section that this breaking of x-map can project out spinors and/or vectors of $SO(10)$. However, in B^2 and B^3 sectors, due to the intersections

$$\forall\lambda_i \in \{0, 1\}, \quad B^2_{\lambda_1\lambda_2\lambda_5\lambda_6} \cap e_1 = \left(B^2_{\lambda_1\lambda_2\lambda_5\lambda_6} + x\right) \cap e_1 \neq \varnothing \tag{4.5}$$

and

$$\forall\lambda_i \in \{0, 1\}, \quad B^3_{\lambda_1\lambda_2\lambda_3\lambda_4} \cap e_1 = \left(B^3_{\lambda_1\lambda_2\lambda_3\lambda_4} + x\right) \cap e_1 \neq \varnothing \tag{4.6}$$

the e_1-projection only kills helicities, having a similar action in the sectors $B^{2,3}$ and their superconformal partners $B^{2,3} + x$; it is not able to annihilate entire representations. Then the duality spinor–vector is still valid in these sectors.

With this in mind, we focus on a case where the x-map is only broken in the first plane, that is by e_1 and/or e_2. The duality map is then the following: *the $(S_t \leftrightarrow V)$-dual of a model where the x-map is broken only in the first twisted plane is constructed by reversing the signs of the discrete torsion coefficients $[B^1_{0000}|e_i]$ and $[B^1_{0000}|z_j]$ for every e_i, $i = 1, 2$, satisfying $[x|e_i] = -1$, and for every z_j satisfying $[x|z_j] = -1$.* This procedure is easily seen to be in agreement with the rules given in [4], where the general form of the duality transformation is formulated as the exchange of the ranks of the matrices $[\Delta^{(1)}, Y_{16}^{(1)}]$ and $[\Delta^{(1)}, Y_{10}^{(1)}]$; this particular set of rules actually exchanges the vectors $Y_{16}^{(1)}$ and $Y_{10}^{(1)}$.

To prove this, let us suppose that $[x|e_1] = -1$ and consider the action of the e_1 projection on a given sector $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle$.

- Since one has

$$\left[B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\middle|e_1\right] = \left[B^1_{0000}\middle|e_1\right] \times \underbrace{[e_3|e_1]^{\lambda_3}[e_4|e_1]^{\lambda_4}[e_5|e_1]^{\lambda_5}[e_6|e_1]^{\lambda_6}}_{=\varepsilon}, \tag{4.7}$$

  we conclude that the sector $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}$ survives the e_1 projection iff $[B^1_{0000}|e_i] = -\varepsilon$, and is projected out iff $[B^1_{0000}|e_i] = \varepsilon$.
- Then, since $[x|e_1] = -1$, we see that the sector $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6} + x$ survives iff $[B^1_{0000}|e_i] = \varepsilon$, and is projected out iff $[B^1_{0000}|e_i] = -\varepsilon$.
- Therefore, the case $[B^1_{0000}|e_i] = \varepsilon$ corresponds to keeping only the spinorial of $SO(10)$ arising from $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}$, whereas $[B^1_{0000}|e_i] = -\varepsilon$ preserves only the vectorial representation from this sector.
- Then, it is obvious to see that reversing the sign of $[B^1_{0000}|e_1]$ will bring the dual model, since the factor $\varepsilon = [e_3|e_1]^{\lambda_3}[e_4|e_1]^{\lambda_4}[e_5|e_1]^{\lambda_5}[e_6|e_1]^{\lambda_6}$ has not been changed in the process.

One must also look at the case where both e_1 and e_2 are breaking the x-map. It is easy to convince oneself that one must reverse *the two* discrete torsions $[B^1_{0000}|e_1]$ and $[B^1_{0000}|e_2]$ to get the dual model. Indeed, supposing that we start from a configuration where only the spinorial representation survive from the sector $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}$ after the two projections, one sees that reversing only one of the two GGSO coefficients annihilates the whole sector $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}$; whereas reversing both coefficients brings back the vectorial of the sector.

Using similar arguments, one shows that, in the case of a breaking of the x-map by a set z_i, the dual model is obtained by also switching the sign of the corresponding GGSO coefficient $[B^1_{0000}|z_j]$. Indeed, the z_i are never, in all three planes, part of the spin-fields giving the vacuum, and then we can derive rules for them which are similar to the rules we have for $e_{1,2}$ when acting on the first plane, $e_{3,4}$ on the second plane and $e_{5,6}$ on the third plane. We note that, since the coefficients $(S|e_i)$ and $(S|z_i)$ are set to preserve $\mathcal{N} = 1$ supersymmetry, we may replace in the above rules $[B^1_{0000}|\ldots]$ by $[b_1|\ldots]$. We recover the fact that the spinor–vector duality is realized *within* each $\mathcal{N} = 2$ twisted plane $B^{1,2,3}$.

Note that the rule we gave for the duality is not unique. One can check that, if we perform the duality in the first plane, a dual model can be obtained by reversing the sign of $[B^1_{0000}|e_i]$ for every i, $i = 1, \ldots, 6$, satisfying $[x|e_i] = -1$ (that is, we do not restrain ourselves to the two "relevant" projections in the first twisted plane which are e_1 and e_2). As a consequence, a given model admits more than one dual. We will give additional arguments to this point at the end of this section.

When the x-map is broken in more than one plane, some subtleties arise, that require finer details. Consider a x-map-breaking set α, that is, $[\alpha|x] = -1$. α may be one of the e_i or one of the z_i. The duality operation has to be carried out in the three planes, by reversing the GGSO coefficients $[b_1|\alpha]$, $[b_2|\alpha]$, and $[b_3|\alpha]$. However, the third twisted plane is not independent from the two others, since $b_3 = b_1 + b_2 + x$. Having carried out the two first steps of the duality, we see that the two reversals

$$[b_1|\alpha] \to -[b_1|\alpha], \qquad [b_2|\alpha] \to -[b_2|\alpha] \tag{4.8}$$

entail, since $[b_3|\alpha] = [b_1|\alpha] \cdot [b_2|\alpha] \cdot [x|\alpha]$:

$$[b_3|\alpha] \to [b_3|\alpha]. \tag{4.9}$$

This situation arises if a set α is able to break the spinor–vector duality in all three planes. This is not the case for the e_i's: as we have seen, e_1 and e_2 can only break the duality in the first plane B^1, e_3 and e_4 in the second plane B^2, and e_5 and e_6 in the third plane B^3.

It is however problematic when α is equal to z_1 and z_2. In that case, the duality is restored if we assume the existence of e_i and e_j, $i \neq j$, such as:

$$[e_i|z_1] = -1 \quad \text{and} \quad [e_j|z_2] = -1. \tag{4.10}$$

These conditions are precisely the no-enhancements hypotheses we assumed to define the class of models in which we demonstrate the duality.

Indeed, when (4.10) is verified, the transformation (4.8) for $\alpha = z_1$ entails[4]

$$[b_3|z_1] \to -[b_3 + e_i|z_1]. \tag{4.11}$$

This feature has the following effect. In the two first twisted planes, the transformations (4.8) imply that if, in a model, the sector $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle$ contains a spinorial representation, it will contain a vectorial representation in the dual model. However, due to the transformation (4.11), we learn that if, in a model, the sector $|B^3_{\lambda_1\lambda_2\lambda_3\lambda_4}\rangle$ contains a spinorial representation, *the sector* $|B^3_{\lambda_1\lambda_2\lambda_3\lambda_4} + e_i\rangle$ will contain a vectorial representation in the dual model. Then, in the third plane, we have a modified the x-map: instead of linking a sector $|B^3_{\lambda_1\lambda_2\lambda_3\lambda_4}\rangle$ to $|B^3_{\lambda_1\lambda_2\lambda_3\lambda_4} + x\rangle$, we have linked it to $|B^3_{\lambda_1\lambda_2\lambda_3\lambda_4} + (x + e_i)\rangle$. In this respect, the duality in the third plane can also be viewed as being a sector-by-sector correspondence.

This also points out that the duality operation is not unique: one can choose to modify the x-map $\alpha \mapsto \alpha + x$ into $\alpha \mapsto \alpha + x + e_i$ in the two first planes, for appropriate sets e_i, i.e. such as $\alpha + e_i$ is massless, and e_i satisfies a condition of the type (4.10). This observation is connected to the fact that the duality operation is viewed in [4] as an exchange of the rank of the matrices

$$\text{rank}\big[\Delta^{(I)}, Y^{(I)}_{16}\big] \leftrightarrow \text{rank}\big[\Delta^{(I)}, Y^{(I)}_{10}\big]; \tag{4.12}$$

this rank being constant under linear combinations on the columns of $\Delta^{(I)}$.

Also note that when we will detail in Section 4.4 the duality procedure, in the no-enhancement framework, in terms of cocycle insertions, it will be sufficient to insert cocycles relative to the twist parameters h_1 and h_2; the effect on the third plane will automatically follow.

4.3. *Explicit realization of the duality in the first twisted plane*

We consider a model given by the following discrete torsion coefficients:

$$\big[B^1_{0000}\big|e_1\big] = 1, \qquad [x|e_1] = 1, \qquad [x|e_2] = -1; \tag{4.13}$$

and

[4] We suppose here that $e_i \neq e_5, e_6$. If not, one adapts the proof in the straightforward way by exchanging the roles of b_1, b_2, b_3.

[.|.]	e_1	e_2
e_3	-1	1
e_4	1	-1
e_5	1	1
e_6	-1	1

Then the action of e_1 and e_2 projections on the B^1 twisted plane and the resulting spectrum are summarized in Table 1. This table gives, for a model and its dual, the discrete torsion accounting for the effect of the projections e_1 and e_2 for each of the 16 sectors of the first twisted plane, and the corresponding surviving representations. The left part of the table assumes $[B^1_{0000}|e_2] = 1$ while the right part is for $[B^1_{0000}|e_2] = -1$. As we discussed, a coefficient 1 relatively to e_1 projects out spinors and vectors altogether; a coefficient 1 with respect to e_2 projects out spinors and a -1 projects out vectors

Note that in fact, this model is already self-dual; however, the duality operation is non-trivial, as it exchanges spinorial and vectorial representations inside each twisted sector $B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}$, and we find it more instructive to detail the duality procedure in this model rather than in a purely vectorial or purely spinorial model (recall from [4] that in one twisted plane, one has either a purely vectorial, purely spinorial/anti-spinorial or half-vectorial half-spinorial – i.e., self-dual – spectrum). Obviously, under a duality transformation, a model having only spinorial representations (which can be specifically obtained, for example, by setting $[e_{3,4,5,6}|e_2] = 1$) will be related to a model having only vectorial representations, the transformation being done sector by sector. We present an explicit example of such a duality transformation in Appendix A.

Table 1
GGSO coefficients for the first twisted plane and corresponding surviving representation, for the choice of coefficients (4.13).

[.|.]	e_1	e_2	rep.	e_1	e_2	rep.
B^1_{0000}	1	1	∅	1	-1	∅
B^1_{0001}	-1	1	V	-1	-1	S
B^1_{0010}	1	1	∅	1	-1	∅
B^1_{0100}	1	-1	∅	1	1	∅
B^1_{1000}	-1	1	V	-1	-1	S
B^1_{1100}	-1	-1	S	-1	1	V
B^1_{1010}	-1	1	V	-1	-1	S
B^1_{1001}	1	1	∅	1	-1	∅
B^1_{0101}	-1	-1	S	-1	1	V
B^1_{0110}	1	-1	∅	1	1	∅
B^1_{0011}	-1	1	V	-1	-1	S
B^1_{1110}	-1	-1	S	-1	1	V
B^1_{1101}	1	-1	∅	1	1	∅
B^1_{1011}	1	1	∅	1	-1	∅
B^1_{0111}	-1	-1	S	-1	1	V
B^1_{1111}	1	-1	∅	1	1	∅

We have not mentioned here the chirality of the spinorial representations; these depend on the $\tilde{b}_2$ projection, which in turn depends on the discrete torsions

$$\left[B^1_{0000}\middle|\tilde{b}^2\right]; \qquad [e_i|\tilde{b}_2], \quad i=3,4,5,6. \tag{4.14}$$

We will fix $[B^1_{0000}|\tilde{b}^2]=-1$ and consider two cases of figure for the other four GGSO coefficients:

(1):

[.|.]	$\tilde{b}_2$
e_3	1
e_4	1
e_5	1
e_6	1

and (2):

[.|.]	$\tilde{b}_2$
e_3	1
e_4	-1
e_5	1
e_6	-1

Extracting the spinorial representations from the previous model, we find that for case (1), before and after duality, all $SO(10)$ spinors have positive chirality. For case (2), we find that, before and after duality, we have 2 chiral and 2 anti-chiral spinors.

Note that to put in evidence more features of the construction, we have taken non-trivial values for the coefficients $[e_{3,4,5,6}|e_{1,2}]$. Had we not done this, the remaining model would have had more generations. One sees that within a twisted plane, arbitrary values of the coefficients $[e_{3,4,5,6}|e_i]$, where e_i does not break the x-map, are only able to project out half of the twisted sectors; only 8 sectors out of 16 contribute, giving either a purely spinorial, purely vectorial, or half-vectorial and half-spinorial spectrum.

Further projections in the spectrum can then be performed by acting with the orbifolds generated by z_1 and z_2. Indeed, we can obtain the formula

$$\left[B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\middle|z_{1,2}\right]=\left[B^1_{0000}\middle|z_{1,2}\right]\times\prod_{i=3}^{6}[e_i|z_{1,2}]^{\lambda_i} \tag{4.15}$$

and the survival condition of the sector $|B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\rangle$ is

$$\left[B^1_{\lambda_3\lambda_4\lambda_5\lambda_6}\middle|z_{1,2}\right]=-1. \tag{4.16}$$

Setting, for some $(i,j)\in\{3,4,5,6\}\times\{1,2\}$, some discrete torsions

$$[e_i|z_j]=-1 \tag{4.17}$$

gives one access to models in which only 4 sectors or only 2 sectors out of the 16 survive at the massless level.

To conclude this subsection, let us note that the explicit model we constructed above is self-dual; however the E_6 gauge symmetry has been broken. Breaking $E_6\to SO(10)\times U(1)$, as it makes an Abelian factor $U(1)$ appear in the gauge group, is generally believed to lead to anomalies. However, in a class of self-dual models, $U(1)$ anomalies can be evaded when summing on the contribution of the three twisted planes. We provide an explicit example of this property in the Appendix B.

4.4. Plane by plane insertions of discrete torsion coefficients, and their overall effects

In this subsection, we want to indicate how these constructions can be translated in terms of modifications of the overall phase Φ introduced in the general form of the partition function (2.6).

Again, we focus on the first twisted plane; the generalization for the simultaneous action on the three planes will be addressed at the end of this subsection. We are then considering the internal dimensions e_i, $i = 1, 2$. The term of the partition function representing the first twisted plane is obtained when the four space–time fermions $\chi^{3,4,5,6}$ are twisted: therefore, $h_1 = 0$ and, $h_2 = h_3 = h$ is the relevant twisting parameter.

Remembering that the freely-acting orbifolds are conveniently represented by the insertion of cocycles in the partition function, we find the following rules.

First, in the absence of superconformal symmetry breaking, one is able to project out a whole sector of the twisted plane (that is, both the spinorial and the vectorial coming from this sector) by adding a phase

$$(-)^{ht_i+gs_i}, \qquad (-)^{hG_i+gH_i}, \tag{4.18}$$

depending on the breaking being done by a e_i or a z_i projection. As we discussed earlier, such a coupling renders the $h = 1$ sectors massive, which is the case in the plane that we are considering. Furthermore, as we have explained before, the effect the different sectors of the plane is dictated by the values of the coefficients $(e_i|e_j)$. These discrete torsions are controlled by the insertion of the cocycles

$$(-)^{s_it_j+s_jt_i}. \tag{4.19}$$

One is then able to construct a variety of self-dual models using these rules. Similarly, one is able to control the value of the coefficient $(e_i|z_j)$ by means of the insertion of

$$(-)^{s_iG_j+H_jt_i}. \tag{4.20}$$

The superconformal x-map is broken as soon as we couple a freely-acting orbifold to the $SO(10)$ spin-structure (ϵ, ξ). In the first twisted plane, such a breaking requires the action of at least one of the sets (e_1, e_2, z_1, z_2); the corresponding cocycles to be inserted then read, respectively:

$$(-)^{\epsilon t_i+\xi s_i+s_it_i}, \quad i = 1, 2, \qquad (-)^{\epsilon G_i+\xi H_i+H_iG_i}, \quad i = 1, 2. \tag{4.21}$$

Coupling the two previous effects now allow us to control which representation (spinorial or vectorial) survives at the massless level in the model. Starting from a case where both spinors and vectors survive, the addition of one of the SCFT-breaking phases (4.21) lifts the spinorials of $SO(10)$, so that only the vectorials survive. If instead we start from a case where the whole sector has been projected out, by the insertion of a cocycle of the form (4.18), adding a cocycle (4.21) recovers the spinorials, while the vectorials remain massive. The phase we inserted in this case is then the product of (4.18) and (4.21). We may summarize the possibilities as follows:

- no cocycle introduced: S and V stay at the massless level;
- $(-)^{ht_i+gs_i}$: both S and V become massive;
- $(-)^{\epsilon t_i+\xi s_i+s_it_i}$: S becomes massive, V stays massless;
- $(-)^{(\epsilon+h)t_i+(\xi+g)s_i+s_it_i}$: S stays massless, V becomes massive.

Of course, if one considers a breaking by z_i, one has to replace (s_i, t_i) by (H_i, G_i).

We then learn how to engineer the duality map directly on the partition function. We have stated that it has to be done by reversing the GGSO projections $[B^1|e_i]$, $[B^1|z_i]$ for each x-breaking projections e_i, z_i. But these values are encoded in cocycles

$$(-)^{ht_i+gs_i}, \qquad (-)^{hG_i+gH_i}, \tag{4.22}$$

where h is the orbifold parameter relevant for the plane we are interested in. Therefore, to carry out the duality map, one has to insert a cocycle (4.18) for each projection breaking the x-map (i.e. such that a cocycle of the form (4.21) is present in the partition function).

5. Conclusion and discussion

In this paper, we gave a new demonstration of the spinor–vector duality that was shown to hold among the $\mathcal{N} = 2\mathbb{Z}_2$ and the $\mathcal{N} = 1, \mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic-string vacua obtained *via* the free fermionic construction. We interpreted the freely-acting orbifolds present in the model in terms of stringy Scherk–Schwarz mechanisms; these have been used to give a non-vanishing mass to some sectors of the theory, and/or to perform a spontaneous breaking of the right-moving superconformal algebra (also called x-map) which is responsible of the gauge enhancement $SO(10) \times U(1) \to E_6$. Such a breaking creates non-self-dual models, where we do not have the same number of spinorial and vectorial representations of $SO(10)$ at the massless level of the theory. We described the procedure used to construct the dual of a given model. Moreover, we explicitly constructed self-dual models in which E_6 gauge is broken.

Such models may, or may not, be free from all Abelian and mixed anomalies. The cases in which the self-dual models are particularly interested, as in such models one does not need to resort to field theory arguments to shift the vacuum to a stable supersymmetric vacuum. Finally, we have given rules on how to perform this duality directly on the expression of the 1-loop partition function of the model.

One may ask what are the implications of such a duality. Firstly, we can see it as a symmetry in the space of vacua of string theory, whose study has been of great interest over the past years [16]. Furthermore, the duality is exhibited in the space of free fermionic models that have also given rise to some of the most realistic string models constructed to date. The geometrical structure underlying the free fermionic models is that of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, and a natural question is whether it extends to other orbifolds. The spinor–vector duality can be thought of as being of the same kind as mirror symmetry [17]. Indeed, mirror symmetry is manifest in this model as the symmetry exchanging spinorials of $SO(10)$ into anti-spinorials of $SO(10)$. This is due to the Type II $\leftrightarrow$ Heterotic correspondence being related to the embedding of the spin-connection in the gauge connection. Therefore, changing the chirality of the $SO(10)$ spinors amounts, on the Type II side, to change the GSO projection on the right-hand side of the theory. This Type IIA $\leftrightarrow$ Type IIB switch is known [18] to be equivalent to the substitution of the compactification manifold by its mirror. Our constructions displays this mirror symmetry: this relies on the choice of the coefficients $[b_1|\tilde{b}_2]$ and $[e_i|\tilde{b}_2]$, as we have shown that the $\tilde{b}_2$ projection imposes the chirality of the massless spinorial representations (if any). The mirror symmetry implies a change in the topology of the compactification manifold, as the Euler characteristic is taken to its opposite. Spinor–vector duality can, as well, be thought of as another topology-changing duality. Note that its range of application is wider than the mirror case. Here, non-self-dual points correspond to $N = (2, 0)$ compactifications. Just as mirror symmetry can be thought of as a manifestation of T-duality [18] also the spinor–vector duality may be regarded as such, but with the added action on the bundle representing the gauge degrees of freedom of the heterotic string, induced by the breaking of the $N = 2$ world-sheet superconformal symmetry on the right-moving bosonic side of the heterotic string. Thus, just as mirror symmetry have led to the notion of topology changing transition between mirror manifolds, the spinor–vector duality suggests that the web of connections is far more complex, and further demonstrating that our understanding of string theory is truly only rudimentary. Furthermore, what we may find is that the distinction of particles into spinor and vector representation is a mere low energy organisation. What the string truly cares about is its internal consistency, characterized by the modular invariance of the partition function. One should note, however, that from the point of view of the effective field theory limit of the string vacua, the spinor–vector duality map is between two different theories, with differing spectra. Therefore, it is clear that dual pairs are not equivalent in the field theory approximation. Nevertheless, as we demonstrated here there exist a map between them. As regards the couplings, we may anticipate, in general in a stringy set up, some relations between couplings in respective dual pairs. Such possible stringy relations are naturally of further interest and we hope to address such issues in the future.

Acknowledgements

A.E.F. and J.R. would like to thank the École Normale Supérieure and the École Polytechnique for hospitality. A.E.F. work is supported in part by STFC under contract PP/D000416/1 and by the EU under contract MRTN-CT-2006-035863-1. C.K. is supported in part by the EU under the contracts MRTN-CT-2004-005104, MRTN-CT-2004-512194, MRTN-CT-2004-503369, MEXT-CT-2003-509661, ANR (CNRS-USAR) contract 05-BLAN-0079-01. J.R. work is supported in part by the EU under contracts MRTN-CT-2006-035863-1 and MRTN-CT-2004-503369.

Appendix A. A dual pair of models with spectrum in the first twisted plane

We consider the model given by the following GGSO coefficients matrix:

$$[v_i|v_j] = e^{i\pi(v_i|v_j)} \tag{A.1}$$

$$(v_i|v_j) = \begin{array}{c} \\ 1 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 1 & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 \end{array} \\ \left(\begin{array}{cccccccccccc} 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&0&1&0&0&0&1&0&0&1&1\\ 1&1&1&0&0&0&1&1&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&1&1&1\\ 1&1&0&1&0&0&0&0&0&0&1&0\\ 1&1&1&1&0&0&0&0&1&1&1&0\\ 1&0&0&0&0&0&0&1&1&0&0&1\\ 1&0&0&0&0&1&0&1&1&1&0&0\\ 1&1&1&0&0&1&1&1&0&0&1&0\\ 1&1&1&0&0&1&0&0&1&0&0&1 \end{array}\right) \end{array}. \tag{A.2}$$

As far as the $SO(10)$ representations are concerned, this model contains two vectorials **10**, one in the sector $S+b_1+e_5+x$, and one in the sector $S+b_1+e_3+e_5+x$. The spectrum is therefore contained in the first twisted plane; we will only need to carry out the duality in this plane.

We apply the duality procedure as follows.

First, we notice that, since

$$x = 1 + S + \sum_{i=1}^{6} e_i + z_1 + z_2,$$

we have

$$\begin{aligned} &(x|e_1) = 0, \qquad &&(x|e_2) = 1, \qquad &&(x|e_3) = 0, \\ &(x|e_4) = 0, \qquad &&(x|e_5) = 0, \qquad &&(x|e_6) = 1, \\ &(x|z_1) = 1, \qquad &&(x|z_2) = 1. \end{aligned}$$

The method we exposed then consists in reversing the GGSO coefficients $(b_1|e_2)$, $(b_1|z_1)$ and $(b_1|z_2)$. The resulting matrix is therefore (the coefficients we changed are in bold):

$$(v_i|v_j) = \begin{array}{c} \\ 1 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccccccccc} 1 & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 \end{array} \\ \left(\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & \mathbf{1} & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 1 & 1 & 0 & \mathbf{1} & \mathbf{0} \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & \mathbf{1} & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 \end{array}\right) \end{array}. \tag{A.3}$$

When explicitly computing the spectrum of this new model, we find indeed that two spinors $\overline{\mathbf{16}}$ of $SO(10)$ arise from the first plane, in the sectors $S + b_1 + e_5$ and $S + b_1 + e_3 + e_5$. We see then that in this simple case, the duality transformation occurs sector by sector in the first twisted plane, like described in Section 4.

Appendix B. A self-dual, anomaly-free model without E_6 enhancement

We are considering the model given by the matrix which coefficients $(v_i|v_j) \in \{0, 1\}$ are defined by

$$[v_i|v_j] = e^{i\pi(v_i|v_j)}, \tag{B.1}$$

$$
(v_i|v_j) = \begin{array}{c} \\ 1 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 \end{array} \\
\left(\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}. \tag{B.2}
$$

We see that since $(z_1|x) = (z_1|1) + (z_1|S) + \sum_{i=1}^{6}(z_1|e_i) + (z_1|z_1) + (z_1|z_2) \equiv 1$ mod. 2, the gauge group E_6 is broken. Moreover, the conditions $(e_1|z_2) = (e_4|z_1) = 1$ ensure that the "hidden" gauge group is minimal and the full gauge group is $SO(10) \times U(1)^3 \times SO(8) \times SO(8)$. The spectrum of this model contains (we note as an index the three charges under the $U(1)_{\bar{\eta}^i}$, $i = 1, 2, 3$):

- three spinors **16** of $SO(10)$, one for each twisted plane,

$$\mathbf{16}_{(1/2,0,0)}, \quad \mathbf{16}_{(0,-1/2,0)}, \quad \mathbf{16}_{(0,0,-1/2)},$$

- three vectors **10** of $SO(10)$, one for each twisted plane,

$$\mathbf{10}_{(0,1/2,1/2)}, \quad \mathbf{10}_{(-1/2,0,1/2)}, \quad \mathbf{10}_{(-1/2,1/2,0)},$$

- six non-Abelian gauge group singlets, two for each twisted plane,

$$\mathbf{1}_{(1,-1/2,-1/2)}, \mathbf{1}_{(1/2,1,-1/2)}, \mathbf{1}_{(1/2,-1/2,1)},$$
$$\mathbf{1}_{(-1,-1/2,-1/2)}, \mathbf{1}_{(1/2,-1,-1/2)}, \mathbf{1}_{(1/2,-1/2,-1)}.$$

By verifying the identities $\sum q_i = \sum q_i^3 = 0$ for the three Abelian factors of the gauge group, we see that the observable spectrum is anomaly-free. Note that this anomaly does not occur plane by plane, but results from a cancellation between the three planes.

One can also check that in this model, the contributions of the $(\mathbf{8}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{8})$ multiplets of $SO(8) \times SO(8)$ to the $U(1)$ anomalies cancel.

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